On Stability and Stabilization of Hybrid Systems

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

The thesis addresses the stability, input-to-state stability (ISS), and stabilization problems for deterministic and stochastic hybrid systems with and without time delay. The stabilization problem is achieved by reliable, state feedback controllers, i.e., controllers experience possible faulty in actuators and/or sensors. The contribution of this thesis is presented in three main parts.

Firstly, a class of switched systems with time-varying norm-bounded parametric uncertainties in the system states and an external time-varying, bounded input is addressed. The problems of ISS and stabilization by a robust reliable H_∞ control are established by using multiple Lyapunov function technique along with the average dwell-time approach. Then, these results are further extended to include time delay in the system states, and delay systems subject to impulsive effects. In the latter two results, Razumikhin technique in which Lyapunov function, but not functional, is used to investigate the qualitative properties.

Secondly, the problem of designing a decentralized, robust reliable control for deterministic impulsive large-scale systems with admissible uncertainties in the system states to guarantee exponential stability is investigated. Then, reliable observers are also considered to estimate the states of the same system. Furthermore, a time-delayed large-scale impulsive system undergoing stochastic noise is addressed and the problems of stability and stabilization are investigated. The stabilization is achieved by two approaches, namely a set of decentralized reliable controllers, and impulses.

Thirdly, a class of switched singularly perturbed systems (or systems with different time scales) is also considered. Due to the dominant behaviour of the slow subsystem, the stabilization of the full system is achieved through the slow subsystem. This approach results in reducing some unnecessary sufficient conditions on the fast subsystem. In fact, the singular system is viewed as a large-scale system that is decomposed into isolated, low order subsystems, slow and fast, and the rest is treated as interconnection. Multiple Lyapunov functions and average dwell-time switching signal approach are used to establish the stability and stabilization. Moreover, switched singularly perturbed systems with time-delay in the slow system are considered.

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Dedicated to..

My beloved parents, dear husband, and precious children, Abdulmalik, Adam, and Batoul.

Table of Contents

List of Figures

List of Symbols

$\mathbb{N}$ The set of natural numbers

$\mathbb{R}$ The set of real numbers

$\mathbb{R}_+$ The set of non-negative real numbers

$\mathbb{R}^n$ n-dimensional Euclidean space

$\mathbb{R}^{n\times m}$ The set of all $n \times m$ real matrices

$\mathcal{C}(A,B)$ The space of all continuous functions mapping A into B

$\mathcal{C}^{1,2}((\mathbb{R}_+,A),B)$ The space of all functions that are continuously differentiable once in the first argument and twice in the second argument

$\mathcal{PC}(A,B)$ The space of all piecewise continuous functions mapping A into B

$\mathcal{C}_r$ The space of all continuous functions defined from $[-r,0]$ to $\mathbb{R}^n$

$\|\cdot\|$ The Euclidean norm: $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$

$\|\cdot\|_r$ $\|x_t\|_r = \sup_{t-r\leq\theta\leq t} \|x(\theta)\|$

$\|\cdot\|_2$ The L_2 norm: $\|x(t)\|_2 = \left(\int_{t_0}^\infty \|x(t)\|^2\, dt \right)^{\frac{1}{2}}$

A^T	Transpose of a matrix A
A^{-1}	Inverse of a (nonsingular) matrix A
x^T	Transpose of a vector x
sup	Supremum, the least upper bound
inf	Infimum, the greatest lower bound
$\lambda_{\max}(P)$	The maximum eigenvalue of the matrix P
$\lambda_{\min}(P)$	The minimum eigenvalue of the matrix P
Δx	$\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ where $x(t^+),\ x(t^-)$ are the right and left limits at t_k respectively

Chapter 1

Introduction

The term "Hybrid" has been widely used recently. Generally, it means a composite of heterogeneous sources, while, mathematically, a hybrid system is a dynamical system having behaviours modelled by differential equations describing the continuous evolution, and difference equations representing the discrete events. This coexistence of the continuous and discrete dynamics is frequently encountered in practice which makes it the focus of researchers' attention for the last few decades. Some typical examples that exhibit both continuous and discrete dynamics in their behaviour are listed as follows

- **Bouncing ball**, in which an abrupt changes occur to the velocity direction. The state variables are continuous during the discrete events of the instance changes (the impacts with the surface) [136].

- **Biological systems**, in which sudden changes (discrete events) occur during a continuous state, such as periodic vaccinations or treatments in epidemic models

[107, 155].

- **Classical mechanics**, in which the velocities of a multi-body system are disturbed by sudden changes either due to an external force, a stop object [162], or a collision or due to change in the length (in a constrained pendulum case) [37].

- **Communication systems**, in which the communications can be achieved by abrupt impulses. The process implies that, at discrete times, the receiver's state is updated by sampling all state variables together and sending them from the transmitter to the receiver [77, 78].

- **Power converter**, where in the boost circuit with clamping diode, there are four modes and two switch positions that are on and off. The transitions from a mode to another between the on and off switches are controlled, but the transitions from a section to another of the diode remain unknown [162].

- **Water level monitor**, in which two continuous and two discrete variables are presented. The continuous variables are the water level $y(t)$, and the time passed since last signal $x(t)$. the discrete variables are the pump status $P(t) \in \{\text{on, off}\}$, and the last sent signal from the monitor $S(t) \in \{\text{on, off}\}$. When the water reaches specific levels, the sensor will send a signal to the pump to switch it on or off, and these dynamics represent a hybrid system [3].

Hybrid systems have found applications in various areas such as mechanical and electrical engineering, industrial robotics, aerospace industry and traffic control, population dynamics, epidemic disease models, control systems, neural networks, secure communications and much more. Further examples and references in hybrid systems can be found in [4, 24, 25, 54, 91] and the references therein.

The combination of continuous and discrete behaviours leads to more valuable and significant dynamical phenomena that can not be achieved by exclusively continuous or exclusively discrete dynamics [54]. This combination commonly arises in two contexts: either a family of subsystems and a logic-based discrete law to jump amongst them or a continuous system to experience some abrupt changes or sudden jumps. The first class is known as ***switched systems*** and the latter one is known as ***impulsive systems***. ***Impulsive switched systems*** are another type of hybrid systems where jumps occur whenever switchings occur.

1.1   Switched Systems

Switched systems describe phenomena that are characterized by a finite number of dynamical subsystems (modes) and a logic-based switching rule (signal) that governs the switchings between the modes to achieve a desired performance of the system. Switched systems appear in two frameworks. Either by the nature of the system as many natural and engineering systems natures are changing dynamics according to certain environmental

factors, or by controlling the system as many continuous systems are stabilized by several control signals.
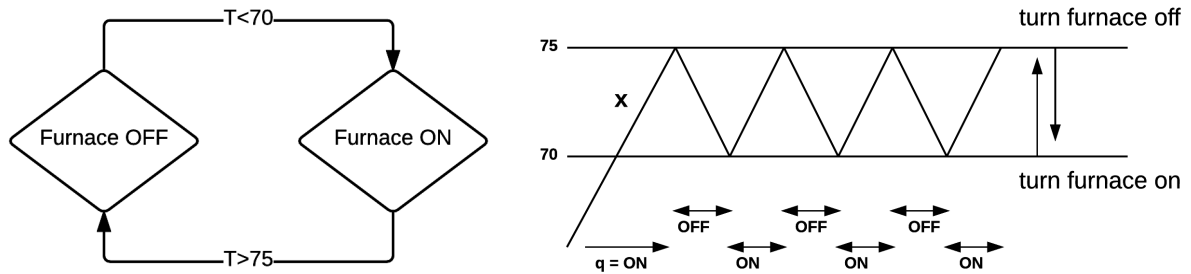
Figure 1.1.1: Automatic Heater Control.

Example 1.1.1. *[63] An automatic heater control which is controlled by a furnace is designed to respond to different temperatures, e.g.* $70 < T < 75$, *and is an example of the first framework described earlier. The switching is determined by environmental factors (the temperature). Here we have two modes* $q = \{ON, OFF\}$, *see Figure 1.1.1.*

Example 1.1.2. *[63] An example of the second framework is the supervisory switching control system which is shown in Figure 1.1.2. The stability of the process is achieved by several controllers each of them is designed to accomplish a specific task. In this case, the supervisory control organizes the switching among them.*

Example 1.1.3. *(Manual Transmission Gear Control) [113] Consider a car with a manual four gear transmission. The motion is determined by the position of the car* $x(t)$ *and the velocity* $v(t)$. *The system has two control signals, the gear* $\textbf{gear} \in \{1, 2, 3, 4\}$ *and the position of the accelerator pedal* u. $x(t), v(t)$ *are both continuous state variables while*

4
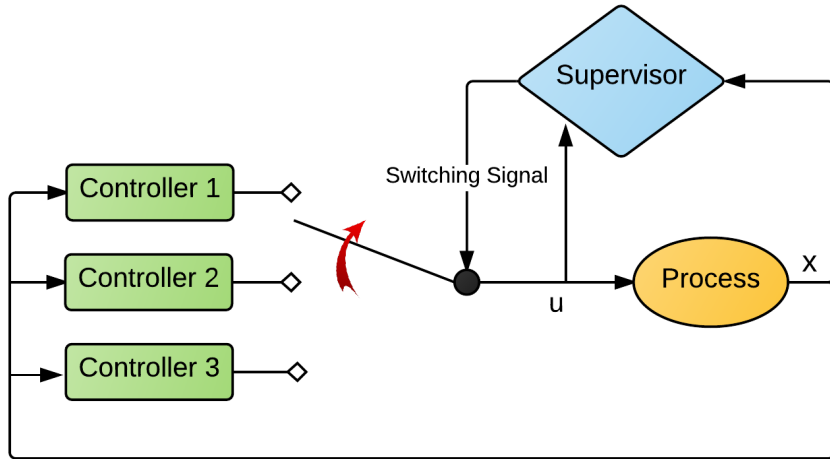
Figure 1.1.2: Logic-based supervisory switching control.

*the gear **gear** is discrete. Every gear represents a mode, and the driver is the decision maker who takes the action of switchings between the gears (modes). See Figure 1.1.3 for illustration.*
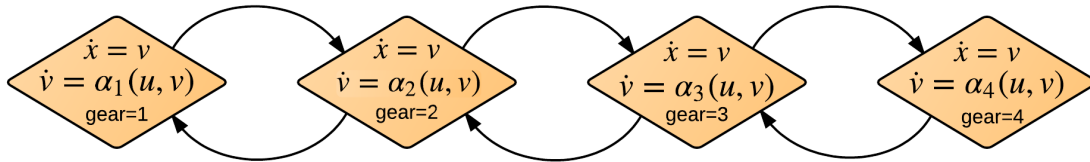
Figure 1.1.3: Gear Shift Control.

Many real world applications are modelled as switched systems. Applications include automotive industry, aircraft control, switching power converters and many other fields (see for example [93, 94, 124, 147, 162] and the references therein).

A remarkable feature of switched systems is that the stability properties are not inher-

ited. In other words, even if all the subsystems are stable, the switched system may not be stable. However, stability of such system can be attained if a proper switching rule is applied. The stability of switched systems has received much attention, and been studied using either common Lyapunov function method [44, 132, 166], or multiple Lyapunov function method [35, 188]. A number of authors discussed the stability using the first approach, but due to complexity, restrictions, or even nonexistence of the common Lyapunov function, the latter one is more convenient and practical to handle switched systems [93]. The main focus of stability study is to design a suitable switching rule that guarantees the stability property. In [131], it has been shown that the stability of the switched system composed entirely of stable subsystems can be preserved if the *dwell-time* τ_d, the time between any two consecutive switchings, is sufficiently large, i.e. every subsystem must be activated for at least τ_d time. From practical perspective, the latter dwell-time condition may not hold in some situations, such as aging systems or systems with finite escape time. However, one can get the same stability result if the *average dwell time* τ_a is satisfied [64]. These two switching rules were utilized to fulfil the stability purpose for switched systems involving some unstable modes among the subsystems [66, 94, 189]. Intuitively, to restrain the growth effect of the unstable modes, the stable ones must be activated longer. A more general switching law, *Markovian switching*, has also been used where the switchings between the system's modes follow a random rule [94, 120]. The *dwell-time* and *average dwell-time* approaches have been intensively exploited in literature to obtain stability criteria for large class of, linear, nonlinear switched systems, with and without time-delay, with

6

and without the perturbation effects, and switched systems with control. For the review of the switched system literature, see [4, 26, 33, 35, 40, 54, 67, 91, 93–95, 129, 147, 192–194, 197] and the references therein.

1.2   Impulsive Systems

In practice, many dynamical processes encounter some abrupt jumps (impulses) at certain moments during the continuous physical evolutions. These systems are referred to as *impulsive systems*, a special type of hybrid systems, in which the durations of these jumps (impulses) are often negligible and thus can be approximated as instantaneous impulses. Motivation examples of impulsive systems include

Example 1.2.1 (Optimal control)**.** *[134]*

   Suppose an optimal control problem that represents a certain physical process and given by

$$x' = f(t, x, u) \tag{1.1}$$

The problem is to choose u in a given set of controls such that the solution x has a desired behaviour in a time interval $[t_0, T]$ to minimize some cost functional. If this control function has to be chosen from a set of integrable functions defined on $[t_0, T]$, then the solution x of the control system may have discontinuities (impulses).

Example 1.2.2 (A bouncing ball)**.** *[136] Consider a bouncing ball that is jumping on a horizontal surface (Fig. 1.2.1). Here, Newton's law of motion governs the continuous*

motion of the falling ball, which is dropped from an initial height h_0. We consider the ball as a point mass. The friction of the surface decreases the ball's energy μ. This process is
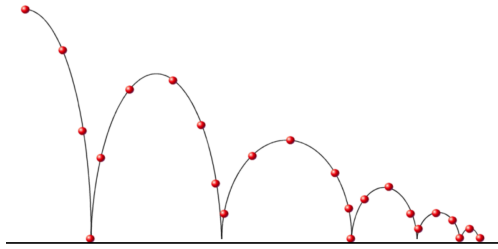
Figure 1.2.1: A bouncing ball

modelled by a second order differential equation

$$m\frac{d^2x}{dt^2} = F \tag{1.2}$$

where m is the ball's mass, $F = -mg$, is the force, and g is the acceleration. When the ball reaches the surface, its vertical velocity rv reversed and decreased, where v is the incoming velocity before an impact, and $0 \leq r \leq 1$. The dynamics of this process are given by

$$\dot{x}_1 = x_2, \ \dot{x}_2 = -g, \ \dot{x}_3 = v_0 \tag{1.3}$$

where x_1 represents the vertical position of the ball, x_2 the vertical velocity, and x_3 the horizontal position of the ball with initial condition $x^T = (x_1, x_2, x_3) = (h_0, 0, 0)$, and the impulse condition

$$I_k^T = (x_1, -\mu x_2, x_3), \quad for \ u(x) = x_1 = 0 \tag{1.4}$$

8

Each bounce displays continuous motion while at each impact the velocity undergoes a sudden change .

Example 1.2.3 (Price expectations and governmental price adjustments in marketing)**.**
[53] Consider market model with n merchandises. The vectors $Q_s^T = (Q_s^1, ..., Q_s^n)$, $Q_d^T = (Q_d^1, ..., Q_d^n)$ and $P^T = (P^1, ..., P^n)$ represent the supply, demand, and price, respectively. Assume that

$$Q_s = S(t, P(t), P'(t), P''(t))$$

and

$$Q_d = D(t, P(t), P'(t), P''(t))$$

where $P'(t)$, $P''(t)$ are the changes of prices and the expectation of the price rising respectively. Assume that the price adjustment process is given by

$$P' = g(Q_s, Q_d)$$

Solving for P'' gives

$$P'' = F(t, P(t), P'(t))$$

So, the dynamics of the prices remain continuous, but at certain times, say, $0 < t_1 < t_2 < ... < t_n < T$, the government will adjust the prices, according to certain law, creating impulses (sudden jumps in the prices) during the continuous evolution at which the price $P(t)$ is replaced by the new price $I_k(P(t_k))$, $k = 1, 2, ..., n$. This will also affect the actual changes of the price $P'(t)$ and replace it by $L_k(P(t_k), P'(t, k))$ creating another jumps.

9

Example 1.2.4 (Insulin treatment). *For diabetics, the sugar level in the blood should be maintained in a certain interval, say $[a, b]$. During the day, due to having foods, the sugar level increases slowly in the blood (this is the continuous evolution) approaching the upper bound of the interval b, say $b - \epsilon$. So, at certain times during the day, say, 6 am, 2 pm, and 10 pm, the insulin should be injected to keep the sugar level under b. So, at these times, the sugar level suddenly jumps to some value near a, say, $a + \epsilon$ creating impulses.*
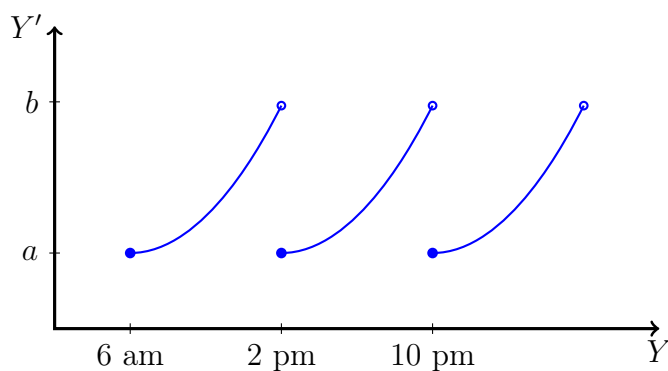
Impulsive systems have many applications in control systems [31, 100], population dynamics [36, 101, 103], neural networks [88, 108], secure communications [77, 78], physics, biology, robotic industry, aeronautics, and many others. The stability of impulsive systems have received a great deal of attention, and have been intensively investigated in the literature [2, 27, 28, 32, 85, 91, 100, 167, 183].

Impulses have , sometimes, a significant role in maintaining the continuation of the solutions of the models [85] or stabilizing the system in some unstable non impulsive sys-

10

tems [99, 100]. Theoretically, dealing with impulsive systems is more challenging than the one of classical continuous ordinary differential equation systems, that is the solution is piecewise continuous which leads to a number of difficulties. In some cases, impulses are considered as a disturbance to the system, and stable systems may lose stability due to impulsive effects. On the other hand, if impulses are applied to unstable systems logically, they may stabilize them. Consequently, the classical qualitative properties such as existence, continuous dependence on initial data, stability results on the continuous systems may be violated and/or need some extra conditions or even a totally new interpretation. Moreover, in the state-dependent impulses, a beating phenomenon, pulses are happening infinitely many times on a hyper-surface, could arise.

1.3   Impulsive Switched Systems

Switched systems with impulsive effects generate a wider class of applications that are rich of significant dynamical phenomena. Stability analysis of these systems has attracted many authors in the last three decades (see, e.g. [91, 92, 175, 180, 186]). The applications include biological systems since there are continuous changes in the regular operations and abrupt changes occur caused by the impulse factor. Further examples in population dynamics can be found in [36]. The following are simple examples where this combination of switchings and impulses occurs

Example 1.3.1 (Population control). *[91] Suppose that the state vector $X(t)$ is the pop-*

ulation of a country, and $U(t)$ is the policy of that country which is the control. Let Ω indicates the whole set of the population, and is divided into n subsets. The problem can be described as

$$\tilde{U}(t) = K_i(X(t)), \ \ where \ X(t) \in \Omega_i. \tag{1.5}$$

It means that if a country is in need of population, the policy will encourage people to have more children, and the reverse is true.

Example 1.3.2 (A switched Server System with Arrival Rate Less than Service Rate)**.**
[91] Consider a system of four buffers and one server, After the server removes the work from any buffer $i = 1, 2, 3, 4$, it switches to the next buffer with a positive reset time t that the server removes work at a unit rate and gives work to next buffer at a constant rate of p_i ($\Sigma_{i=1}^{4} p_i < 1$). The switching process between the buffers forms a cycle which repeats itself in a closed-loop manner. This system is composed of five continuous variable systems (CVS) given as follows

$$CVS1 = \begin{cases} \dot{X}_1(t) = p_1 - 1 \\ \dot{X}_2(t) = p_2 \\ \dot{X}_3(t) = p_3 \\ \dot{X}_4(t) = p_4 \end{cases} , CVS2 = \begin{cases} \dot{X}_1(t) = p_1 \\ \dot{X}_2(t) = p_2 - 1 \\ \dot{X}_3(t) = p_3 \\ \dot{X}_4(t) = p_4 \end{cases} , CVS3 = \begin{cases} \dot{X}_1(t) = p_1 \\ \dot{X}_2(t) = p_2 \\ \dot{X}_3(t) = p_3 - 1 \\ \dot{X}_4(t) = p_4 \end{cases} ,$$

$$CVS4 = \begin{cases} \dot{X}_1(t) = p_1 \\ \dot{X}_2(t) = p_2 \\ \dot{X}_3(t) = p_3 \\ \dot{X}_4(t) = p_4 - 1 \end{cases} , CVS = \begin{cases} \dot{X}_1(t) = p_1 \\ \dot{X}_2(t) = p_2 \\ \dot{X}_3(t) = p_3 \\ \dot{X}_4(t) = p_4 \end{cases} ,$$

where $CVSi$ ($i = 1, 2, 3, 4$) denotes the process that the server removes work from buffer i, and CVS denotes the process that the server switches from one buffer to another, $X_i(t)$

12

*denotes the work in buffer i at time t. Then, the relation between these CVSs can be seen
in Figure 1.3.1.*
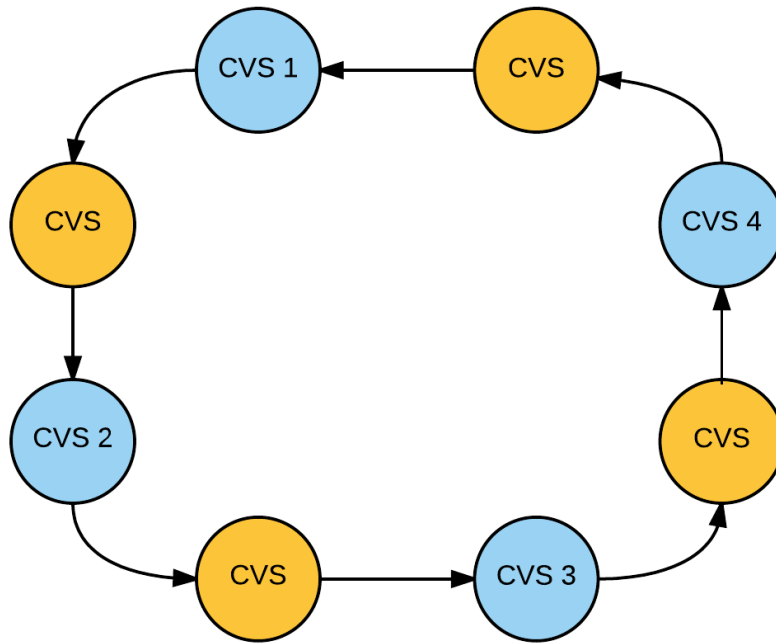
Figure 1.3.1: The Switchings between the buffers

1.4 Hybrid Systems with Time-Delay

The classical stability results of hybrid systems used ***ordinary differential equations
(ODEs)***, which depend on the present states only ignoring any information from their his-
tory. As a matter of fact, many real world phenomena and man-made plants are subject to
some time-delay. Namely, the past information contribute in forming a better idea about
the future system behaviour. This class of equations is referred to as ***delay differential***

equations (DDEs)*.* Since time-delay is unavoidable in practice, considering time-delay in hybrid systems is more realistic and practical. Time-delay in secure communication systems, population dynamics, ship stabilization and many other application has an important role and has motivated many researchers to implement time-delay in the models under study. Stability analysis of DDEs, compared with the stability analysis of ODEs, is more challenging. For instance, the presence of a small time-delay may cause undesirable behaviour such as discontinuities, loss of uniqueness, chaos. On the other hand, delay can improve the performance [38]. Fortunately, some of useful tools such as ***Lyapunov-Razumikhin technique, Lyapunov Krasovskii functional (LKF) method, Halanay inequalities*** have been developed to deal with a large class of hybrid systems with time-delay. A considerable progression has been achieved in this field since 1980's and up until now, for more readings, one may refer to [8, 38, 51, 55, 59–61, 71, 84] and many references therein. Implementing time-delay to hybrid systems, gives rise to three classes of delayed hybrid systems, which are ***switched systems with time-delay, impulsive systems with time-delay***, and ***switched impulsive systems with time-delay***. Some applications of switched delay systems are epidemic disease models and consensus in dynamical networks. Moreover, many significant applications have motivated researchers to consider impulsive delayed systems. The most interesting applications include secure communications and cellular neural networks (CNNs).

Stability criteria for dynamical systems has attracted researcher's attention for a long time. The universally most efficient used stability tool is ***Lyapunov second method***,

which is named after Aleksandr Lyapunov (1892). A feature of this method is that one may analyze the stability of the systems without any knowledge of the solution. Later, another stability concept, **bounded-input, bounded-output (BIBO)**, was developed for linear systems. This technique implies that the boundedness of the output of the system is maintained if a bounded input is applied to the system states. The connection between these two stability methods is known as **input-to-state stability (ISS)**, which was first introduced by Sontag in 1989, in which an input function is included in the system model and no measured output is present. The ISS notion deals with the system response to a norm (particularly L_2)-bounded disturbance when the unforced system is asymptotically stable. ISS is an efficient tool to investigate stability-like criteria of nonlinear systems that are subject to input disturbance, which is frequently encountered in practice. As a result, it becomes important in the modern nonlinear control theory and design. When that input disturbance is identically zero, ISS concept reduced to the conventional Lyapunov stability of the system. For further readings and applications on ISS property, one may consult [151–154, 160, 161]. Numerous research works have discussed the ISS criteria of hybrid systems with and without time-delay. In [169], the ISS property for switched nonlinear systems with time-delay has been achieved using piecewise LKF approach and average dwell time scheme. Furthermore, the delay was considered in the state and the switchings, i.e., in system switchings and controller switchings. Later in [171], a new piecewise LKF was constructed for the nonlinear switched systems with time-varying input delay to satisfy ISS assuming that the Lyapunov function for the nominal system is available and using a

15

mode-dependent average dwell-time scheme. ISS for impulsive systems has gained much attention in the literature [21, 42, 43]. The authors in [42] have used Razumikhin-type technique to guarantee ISS for impulsive systems with time delay. They have considered the situation when the continuous dynamics are ISS but the discrete dynamics are not and the converse case. In [43], the same authors addressed the problem of ISS for networked control systems by an impulsive system method. Here, the system was viewed as an interconnected system of impulsive subsystems and the method of LKF was adopted, and thus, sufficient conditions, based on linear matrix inequalities (LMI), were derived to guarantee ISS for the proposed systems. Sufficient conditions to achieve ISS property for impulsive switched systems with time-delay were stablished in [98]. A number of papers addressed the problem of ISS for stochastic hybrid systems (see e.g. [17, 19, 21, 158, 191]).

1.5   Reliable Control

The reliable control is the controller that tolerates actuator and/or sensor failures while maintaining a desired performance. The control components failure is frequently encountered in reality, yet the immediate repair may not be available such as in aerospace or submarine vehicles. Therefore, designing a reliable controller to guarantee an acceptable level performance becomes crucial. The trend to design reliable controllers has increased, see for instance [46,143,163,165,182]. In most of the available results about reliable control, the faulty actuators are modelled as outages i.e., the output is assumed to be zero. In [182],

a more general failure model was adopted which consists of a scaling factor with upper and lower bounds to the control action. In [19,112,143,165], the output signal is considered as a disturbance signal with bounded magnitude that is augmented with the system disturbance signal. Many research articles have addressed the problem of reliable control for a various types of systems. For switched systems, [67] handled the problem of designing robust H_∞ reliable control for uncertain switched systems using Schur's complement and LMI, while for positive switched systems with time-delay where both stable and unstable modes are involved, LKF approach and LMI together with average dwell-time signal were employed in [174] to accomplish exponential stability via reliable control. The latter approach was also adopted in [139] for uncertain mechanical systems to guarantee asymptotic stability. Recently, robust reliable control for neutral-type systems with time-delay was considered in [123,142]. There are also many results of reliable control of deterministic and stochastic systems, one may refer to [50,52,56,65,69,116,173] and the references therein.

1.6 Large-Scale Systems (LSS)

In real world systems, it has been realized that for many control systems, either the system is naturally modelled as an interconnected system or the system cannot be analyzed via the known simple approaches due to its complexity. This complexity may be due to high dimensionality, delays, uncertainties, or data structure restrictions. The notion of LSS[1] represents a dynamical system that is characterized by several dynamics, or a system

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[1]Also known as interconnected or composite systems.

that needs to be split up into independent manageable subsystems. There are mainly two structures of LSS, multi-channel systems, and interconnected systems (see Figure 1.6.1). Motivating applications arise in interconnected power systems, computer and telecommunication networks, economic systems, nuclear reactors, mobile robotics, multiagent systems, traffic systems, to name a few [22, 23, 75, 117, 144, 187].

(a) n-channel System.

(b) interconnected Systems.
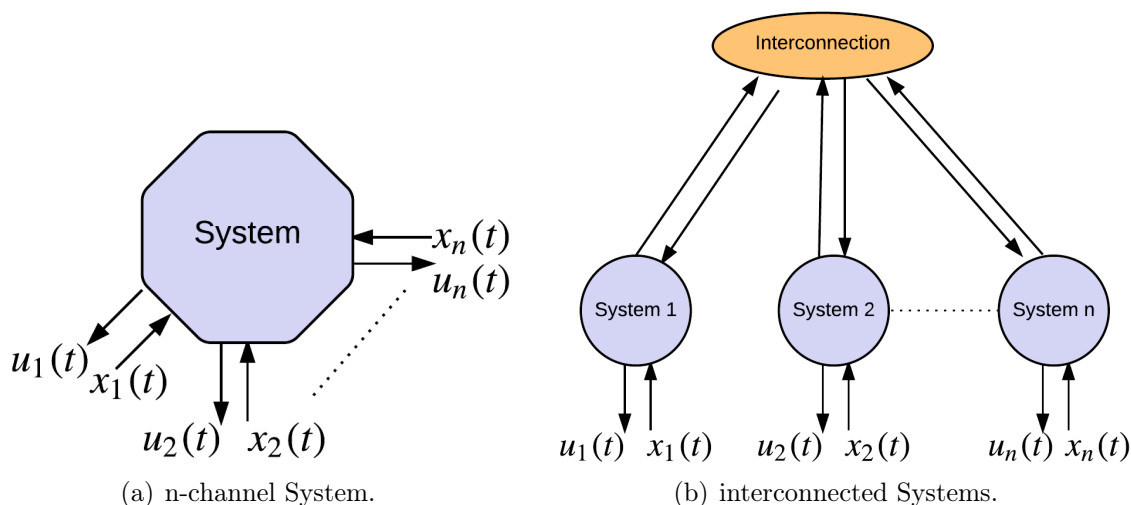
Figure 1.6.1: Large-Scale System.

A common approach to analyze the stability of LSS is to decompose the system into lower order, isolated subsystems and establish the stability of each subsystem ignoring the interconnection part. Then, this available information is used together with the interconnection, which is treated as a perturbation, to get a conclusion about the stability of the interconnected system. LSS problem has received a great deal of attention over the past few decades. Interested readers may refer to [16, 29, 48, 57, 70, 72, 79, 97, 102, 106, 115, 122,

18

126, 128, 140, 148, 185]. In [97], robust exponential stabilization of large-scale uncertain impulsive with coupling time delay was studied, and Lyapunov method and Razumikhin technique where used to design controller in terms of linear matrix inequalities. In [102], a comparison method was used to discuss robust stability of large-scale dynamical systems of ODEs. This method was developed later in [106] to study the same problem with time delay.

1.7 Singularly Perturbed Systems (SPS)

Systems involving multiple time-scale dynamics arise in a large class of applications in science and engineering such as celestial mechanics, many-particle dynamics, and climate systems, mechanical systems, and many other areas [190], and known as singularly perturbed systems. They can be viewed as a class of LSS in which two or more time-scale dynamics are coexisting and interacting. They are characterized by small parameters multiplied by the highest derivatives. The stability problem of these systems has attracted many researchers; see [5, 7–9, 76, 81, 82, 104, 135, 137, 138, 141] and some references therein. The exponential stability for linear and nonlinear singularly perturbed systems with time-delay and uncertainties has been addressed in [76]. In this work, the KLF method has been employed to prove the exponential stability. In [90], the global asymptotic stability criteria for a class of impulsive singularly perturbed systems were proved using the KLF and free weighting approach. The sufficient conditions were formulated in terms of LMIs.

19

Motivated by what have been discussed earlier, the focus of the present thesis is on studying the stability properties of the presented types of hybrid systems with and without time-delay via reliable feedback control technique.

The contents of this thesis is outlined as follows: Chapter 2 has the basic definitions and problem formulations. The rest of this thesis is displayed in three parts. In Part I, the problem of robust and reliable H_∞ control and ISS has been introduced for uncertain switched systems with and without time delay. The same problem for the impulsive systems with time-delay and for the impulsive switched systems with time-delay is investigated. Part II addresses the exponential stability criteria for the Impulsive LSS via robust and reliable feedback control and the state estimation problem. In part III, the stability analysis of switched singularly perturbed systems with and without time-delay is illustrated. Chapter 7 has the stochastic one. Conclusions and some future study directions.

1.8 Summary of Contribution

The research contribution of the present thesis is shown below

- **Robust and reliable H_∞ control and ISS for uncertain hybrid systems with and without time-delay (Part I):** The novelty here is to develop new sufficient conditions that guarantee the input-to-state stabilization and H_∞ performance of the hybrid system in the presence of the disturbance, state uncertainties, and nonlinear lumped perturbation not only when all the actuators are operational, but also when

some of them experience failure. To accomplish this goal, we assume that every single mode is input-to-state stabilized by a robust reliable controller. The methodology of multiple Lyapunov functions and average dwell-time signals are used to analyze the input-to-state stabilization.

For the systems with time-delay, i.e., switched system, impulsive systems, and impulsive switched systems, Razumikhin type technique is employed to obtain the ISS results.

- **Robust and reliable control for impulsive LSS (Part II)** This part addresses the problem of exponential stabilization of impulsive large-scale systems (ILSS) with admissible uncertainties in the system states via a robust reliable decentralized control. Furthermore, reliable observers are also considered to estimate the states of the system under consideration. The faulty actuator/sensor outputs are assumed to be zero. Moreover, the input-to-state stability via reliable controller and stabilization via impulses problems are considered for the stochastic ILSS with time delay. The results are achieved using a scalar Lyapunov function.

- **Reliable control stabilization of switched singularly perturbed systems (Part III)** The exponential stability for a class of switched singularly perturbed systems not only when all the control actuators are operational, but also when some of them experience failures is discussed. Multiple Lyapunov functions and average dwell-time switching signal approach are used to establish the stability criteria for

the proposed systems. A full access to all the system modes is assumed to be available, though the mode-dependent, slow-state feedback controllers experience faulty actuators of an outage type. The system under study is viewed as an interconnected system that has been decomposed into isolated, lower order, slow and fast subsystems, and the interconnection between them. Moreover, time-delay is considered for this system. Halanay inequalities are utilized to prove the exponential stability result for the delayed systems.

Chapter 2

Mathematical Background and Formulations

In this chapter, we present some basic materials that will be needed in the rest of this thesis.

2.1   Preliminaries and Basic Concepts

Denote by $\mathbb{N}$ the set of all natural numbers, $\mathbb{R}_+$ the set of all non-negative real numbers, $\mathbb{R}^n$ the n-dimensional Euclidean space and its norm $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$ for every $x \in \mathbb{R}^n$, $\mathbb{R}^{n \times m}$ the set of all $n \times m$ real matrices. Let $\mathcal{C}([a,b], \mathcal{D})$ $(\mathcal{PC}([a,b], \mathcal{D}))$ denote the space of continuous (piecewise continuous) functions mapping $[a,b]$, with $a < b$ for any $a, b \in \mathbb{R}_+$, into $\mathcal{D}$, for some open set $\mathcal{D} \subseteq \mathbb{R}^n$. Also, denote by $\mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+)$ the space of all real-valued functions $V(t,x)$ defined on $\mathbb{R}_+ \times \mathbb{R}^n$ such that they are continuously differentiable once in t and twice in x.

Consider the following system of time-variant ordinary differential equations (ODE),

$$\dot{x}(t) = f(t, x(t)), \tag{2.1}$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$, and $f : I \times \mathcal{D} \to \mathbb{R}^n$ is continuous on $I \times \mathcal{D}$ with $I \subseteq \mathbb{R}_+$ and $\mathcal{D} \subseteq \mathbb{R}^n$ such that it contains the origin. For a given $(t_0, x_0) \in I \times \mathcal{D}$ with $x_0 = x(t_0)$, the corresponding initial-value problem (IVP) related to equation (2.1) can be written as

$$\begin{cases} \dot{x}(t) = f(t, x(t)), \\ x(t_0) = x_0. \end{cases} \tag{2.2}$$

A continuously differentiable function $\phi(t)$ defined on an interval $I \subset \mathbb{R}_+$ such that $\phi(t) \in \mathcal{D}$ for all $t \in I$ is said to be a solution of the IVP (2.2) if $\dot{\phi}(t) = f(t, \phi(t))$ for all $t \in I$, and $\phi(t_0) = x_0$ with $t_0 \in I$.

Theorem 2.1.1. *If f is continuous on $I \times \mathcal{D}$, then for any $(t_0, x_0) \in I \times \mathcal{D}$ there exists at least one solution of the IVP (2.2) in I.*

Definition 2.1.2. *A function $f(t, x)$ defined on $I \times \mathcal{D}$ is said to be locally Lipschitz in x if there exists a constant $L > 0$, called Lipschitz constant, such that*
$$\|f(t, x_1) - f(t, x_2)\| \leq L\|x_1 - x_2\|$$
for any points $(t, x_1) \in I \times \mathcal{D}$ and $(t, x_2) \in I \times \mathcal{D}$. Moreover, if this inequality holds for all $x \in \mathbb{R}^n$, then f is said to be globally Lipschitz in x.

Theorem 2.1.3. *If $f(t, x)$ is continuous with respect to first variable and locally Lipschitz continuous with respect to second variable, then for any $(t_0, x_0) \in I \times \mathcal{D}$ there exists a unique solution of the IVP (2.2).*

A point $x_{eq} \in \mathbb{R}^n$ is said to be an equilibrium point of the differential equation in (2.1), or trivial solution of system (2.2) if $f(t, x_{eq}) = 0$ for all $t \geq t_0$. Since any equilibrium point can be shifted to the origin, throughout this thesis we will deal $x_{eq} \equiv 0$ (or $x \equiv 0$, for simplicity of notation). In the following, we state the definitions of some stability concepts.

Definition 2.1.4 (Stability). *Let $x(t) = x(t, t_0, x_0)$ be the solution of IVP (2.2) for any $t \geq t_0$, then the trivial solution $x \equiv 0$, is said to be*

(i) **stable** *if, for any $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that*

$$\|x_0\| < \delta \qquad implies \qquad \|x(t)\| < \epsilon \quad for \ any \quad t \geq t_0;$$

(ii) **uniformly stable** *if (i) holds with $\delta = \delta(\epsilon)$;*

(iii) **asymptotically stable** *if (i) holds and there exists a positive constant $\delta = \delta(t_0)$ such that*

$$\|x_0\| < \delta \qquad implies \qquad \lim_{t \to \infty} x(t) = 0;$$

(iv) **globally asymptotically stable** *if (iii) holds with an arbitrary large constant δ;*

(v) **uniformly asymptotically stable** *if it is uniformly stable and there is a positive constant δ, independent of t_0, such that, for all $\|x_0\| < \delta$, $\lim_{t \to \infty} x(t) \to 0$, uniformly in t_0; that is, for any $\eta > 0$, there is $T = T(\eta) > 0$ such that, for all $\|x_0\| < \delta$,*

$$\|x(t)\| < \eta, \qquad \forall t \geq t_0 + T;$$

(vi) **globally uniformly asymptotically stable** *if (v) holds with an arbitrary large constant δ;*

(vii) **exponentially stable** *if there exist positive constants δ, k and λ such that*

$$\|x(t)\| \le k\|x_0\|e^{-\lambda(t-t_0)}, \qquad whenever \quad \|x_0\| < \delta, \quad and \quad t \ge t_0;$$

(viii) **globally exponentially stable** *if (vii) holds with an arbitrary large constant δ;*

(ix) **unstable** *if (i) does not hold.*

In the following, we define some important classes of function that will be used in rest of the thesis.

Definition 2.1.5. *Let $\mathcal{D} \subset \mathbb{R}^n$ be an open set containing $x = 0$. A function $W : \mathcal{D} \to \mathbb{R}$ is said to be* **positive-definite** *on $\mathcal{D}$ if it is continuous on $\mathcal{D}$, $W(0) = 0$, $W(x) > 0$ for $x \in \mathcal{D}\backslash\{0\}$; it is said to be* **radially unbounded** *if it is positive-definite and $W(x) \to \infty$ as $\|x\| \to \infty$.*

Definition 2.1.6. *The upper right-hand Dini derivative of $V(t, x)$ that is continuous in t and locally Lipschitz in x along the solution of* (2.1) *is defined by*

$$D^+V(t, x) = \lim_{h \to 0^+} \sup \frac{1}{h}\big[V(t + h, x + hf(t, x)) - V(t, x)\big].$$

Furthermore, if $V(t, x)$ has continuous partial derivatives with respect to t and x, This derivative becomes

$$\dot{V}(t, x) = \frac{\partial V(t, x)}{\partial t} + \nabla_x V(t, x) \cdot f(t, x)$$

26

where $\nabla_x V(t, x)$ is the gradient vector of V with respect to x.

Definition 2.1.7. *A continuously differentiable function* $V : \mathcal{D} \to \mathbb{R}$ *is said to be a* **Lyapunov function** *if it is positive-definite and non increasing in its domain, i.e.,*

$$V(0) = 0, \ V(x) > 0, \quad for \ x \in \mathcal{D}\backslash\{0\} \ \ and \ \dot{V} \leq 0 \ \ in \ \ \mathcal{D}. \tag{2.3}$$

The following theorem provides sufficient conditions to guarantee stability and asymptotic stability of the autonomous system

$$\dot{x}(t) = f(x), \tag{2.4}$$

Theorem 2.1.8. *Let* $x = 0$ *be an equilibrium point of* (2.4)*, and* $V : \mathcal{D} \to \mathbb{R}$ *where* $\mathcal{D}$ *contains* $x = 0$ *be a continuously differentiable function satisfying* (2.3)*. Then,* $x = 0$ *is stable. It is said to be asymptotically stable if*

$$\dot{V} < 0 \qquad in \ \mathcal{D}\backslash\{0\}.$$

Consider the positive-definite function $V(x) = x^T P x$, where P is a positive-definite matrix. Then, the following inequalities hold

$$\lambda_{\min}(P)\|x\|^2 \leq x^T P x \leq \lambda_{\max}(P)\|x\|^2 \tag{2.5}$$

where $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the maximum and minimum eigenvalues of P, respectively. Consider the linear system

$$\begin{cases} \dot{x} = Ax, \ \ t \geq 0 \\ x(0) = x_0 \end{cases} \tag{2.6}$$

where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$. The stability properties can be analyzed by Lyapunov stability theory as follows. Define $V(x) = x^T P x$ as a Lyapunov function candidate of system (2.6), where P is a positive-definite matrix satisfies the Lyapunov equation

$$A^T P + P A = -Q$$

with Q being a positive-definite matrix. Then, the derivative of $V(x)$ along the trajectories of (2.6) is given by

$$\dot{V} = \dot{x}^T P x + x^T P \dot{x} = x^T (A^T P + P A) x = -x^T Q x < 0,$$

which implies that the equilibrium point of system is asymptotic stable.

In practice, transforming physical phenomena into mathematical models often includes uncertain factors due to modelling mismatches, linearization, approximations or measurement errors, etc. It has been realized that considering such uncertainties results in more accurate systems; see for instance [16, 46, 57, 97, 176]. Consider the uncertain system

$$\begin{cases} \dot{x} = (A + \Delta A)x, & t \geq t_0 \\ x(t_0) = x_0, \end{cases} \tag{2.7}$$

where ΔA is a piecewise continuous function representing parameter uncertainty with bounded norm, we always assume the following assumption holds throughout this thesis

Assumption A. The admissible parameter uncertainties are defined by

$$\Delta A(t) = D \mathcal{U}(t) H, \quad \forall t \in \mathbb{R}_+,$$

with D, H being known real matrices with appropriate dimensions that give the structure of the uncertainty, and $\mathcal{U}(t)$ being unknown real time-varying matrix representing the uncertain parameter and satisfying $\|\mathcal{U}(t)\| \leq 1$.

To further analyze the stability properties of nonlinear systems, the following class functions, also known as *comprison functions*, are needed [79].

Definition 2.1.9. *A function $\alpha \in \mathcal{C}([0, a], \mathbb{R}_+)$ is said to be in class $\mathcal{K}$ if $\alpha(0) = 0$, and it is strictly increasing. It is said to be in class $\mathcal{K}_\infty$ if it is in class $\mathcal{K}$, $a = \infty$, and $\alpha(r) \to \infty$ as $r \to \infty$.*

Definition 2.1.10. *A function $\beta \in \mathcal{C}([0, a] \times \mathbb{R}_+, \mathbb{R}_+)$ is said to belong to class $\mathcal{KL}$ if $\beta(\cdot, s) \in \mathcal{K}$ for each fixed s, $\beta(r, \cdot)$ is decreasing for each fixed r, and $\beta(r, s) \to 0$ as $s \to \infty$.*

Assume that $x \equiv 0$ of the nonlinear system (2.2) is asymptotically stable. If this system undergoes a bounded-energy disturbance input $w \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R}^m)$, what can be said about the qualitative behaviour of the output x of the forced system

$$\begin{cases} \dot{x} = f(t, x, w), & t \geq t_0 \\ x(t_0) = x_0 \end{cases} \tag{2.8}$$

where $f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x and w. The input $w(t)$ is a piecewise continuous bounded function of t for all $t \geq 0$.

The following definition gives an answer to this question.

Definition 2.1.11 (Input-to-State Stability [79])**.** *System* (2.8) *is said to be ISS if there exist functions* $\beta \in \mathcal{KL}$ *and* $\gamma \in \mathcal{K}$ *such that, for any* x_0 *and* w, *the solution* $x(t)$ *exists for all* $t \geq t_0$ *and satisfies*

$$\|x(t)\| \leq \beta(\|x_0\|, t - t_0) + \gamma(\sup_{t_0 \leq s \leq t} \|w(s)\|). \tag{2.9}$$

Clearly, for a large enough t, $\beta \to 0$ and the solution will eventually be bounded by a class $\mathcal{K}$ function γ, which depends on the input. One can easily notice that if input $w(t) = 0$, for all $t \geq t_0$, the ISS property reduces to the classical asymptotic stability of the trivial solution of the corresponding unforced system.

To analyze the ISS of (2.8), one can use Lyapunov-type theorem to provide a set of sufficient conditions as follows:

Theorem 2.1.12. *[79] Let* $x(t) = x(t, t_0, x_0)$ *be the solution of* (2.8). *Let* $V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ *be a continuously differentiable function such that the following conditions hold for any* $(t, x, w) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m$

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|) \tag{2.10}$$

$$\frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} f(t, x, u) \leq -W(x), \qquad \forall \|x\| \geq \rho(\|w\|) > 0 \tag{2.11}$$

for all $(t, x, w) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m$ *where* α_1, α_2 *are class* $\mathcal{K}_\infty$ *functions,* ρ *is a class* $\mathcal{K}$ *function, and* $W(x)$ *is a continuous positive definite function on* $\mathbb{R}^n$. *Then, system* (2.8) *is ISS with* $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

2.2 Switched Systems

Consider the following control system

$$\begin{cases} \dot{x} = f(t, x) + u(t), \\ x(t_0) = x_0, \end{cases} \tag{2.12}$$

where $u : \mathbb{R}_+ \to \mathbb{R}^n$ is the control input given by

$$u(t) = \sum_{k=1}^{\infty} B_k(x(t)) l_k(t),$$

where B_k is the control gain matrix and $l_k(\cdot)$ is the ladder function defined by

$$l_k(t) = \begin{cases} 1, & \tau_{k-1} \leq t < \tau_k \\ 0, & \text{otherwise.} \end{cases}$$

Then, $u(t)$ can be written as

$$u(t) = B_k(x(t)), \qquad t \in [\tau_{k-1}, \tau_k), \qquad k \in \mathbb{N},$$

that is the controller changes its values at each time instant $t = \tau_k$, which means that u is a switched controller. Therefore, the closed loop system (2.12) takes the form

$$\begin{cases} \dot{x} = f(t, x) + B_k x(t), & t \in [\tau_{k-1}, \tau_k), \ k \in \mathbb{N}, \\ x(t_0) = x_0, \end{cases}$$

This system is called a switched system. Generally, a non-autonomous switched system may take the form

$$\begin{cases} \dot{x} = f_{\varrho(t)}(t, x), & t \geq t_0, \\ x(t_0) = x_0, \end{cases}$$

where $f_{\varrho(t)} : \mathbb{R}_+ \times \mathcal{D} \to \mathbb{R}^n$ represents a family of non-autonomous ODEs, ϱ is the switching signal which is a piecewise constant function defined by $\varrho : [t_0, \infty) \to \mathcal{S} = \{1, 2, \cdots, N\}$, for some $N \in \mathbb{N}$. The role of ϱ is to switch among the system modes. For each $i \in \mathcal{S}$, $f_i : \mathbb{R}_+ \times \mathcal{D} \to \mathbb{R}^n$, and $\{f_i : i \in \mathcal{S}\}$ is a family of sufficiently smooth functions. The index set $\mathcal{S}$ is assumed to be finite in the present thesis. The switching moments $\{\tau_k\}_{k=0}^{\infty}$ satisfy $\tau_0 < \tau_1 < \cdots < \tau_k < \cdots$ with $\lim_{k \to \infty} \tau_k = \infty$. It is worth mentioning that there are three types of switching signals: time dependent [64, 131], state dependant, including the initial state [17], Markovian switching [94]. In the present thesis, we are mainly concerned with the time-dependent switching signals.

The switched system can be rewritten as

$$\begin{cases} \dot{x} = f_i(t, x), & i \in \mathcal{S}, \ t \in [\tau_{k-1}, \tau_k), \\ x(t_0) = x_0, & k \in \mathbb{N}. \end{cases} \tag{2.13}$$

The solution of system (2.13) evolves according to the continuous dynamics of the active continuous mode, while at the switching moments τ_k, the switching rule changes from $f_{\varrho(\tau_{k-1})}$ in $[\tau_{k-1}, \tau_k)$ to $f_{\varrho(\tau_k)}$ in $[\tau_k, \tau_{k+1})$.

In the time-dependent switching case, the existence and uniqueness results of (2.13) are analogous to those from the fundamental theory of ODEs with the **method of steps**, where the initial value for each mode operating on the subinterval $[\tau_k, \tau_{k+1})$ is $x(\tau_k)$.

On the other hand, the stability properties of switched systems are not inherited from the single-mode systems. As mentioned in Chapter 1, the stability properties of system

modes is neither sufficient nor necessary unless the activation times of the modes follow a certain switching law. In fact, in analyzing the stability or stabilization of switched systems, there are three main problems [94], namely

1. Stability under Arbitrary Switching.

2. Stability under Slow or Constrained Switching.

3. Stability under Switching Control.

In the present thesis, we are mainly concerned about the stability under constrained switching. As mentioned earlier, the stability of a switched system is not inherited, i.e., a switched system may be unstable even if all the individual subsystems are stable.

The method of common Lyapunov function, i.e., a single Lyapunov function whose derivative decreases along the solutions of all the individual subsystems of (2.13), has been firstly used to analyze the stability of the switched system, yet it is found to be very restrictive because finding one Lyapunov function for all modes may be difficult to find or even does not exist. Another useful tool can be used here, known as **multiple Lyapunov function** approach. The idea of the latter notion is to have a decreasing Lyapunov function along the solution of each mode and, moreover, these Lyapunov functions form a decreasing sequence at the switching moments, i.e., $V_{i+1}(x(t)) \leq V_i(x(t))$. Adopting this approach, stability of a switched system composed entirely of stable subsystems can be guaranteed if the switching between the subsystems is sufficiently slow. This is known as **dwell-time** (τ_d) switching, in which the time between any two consecutive switching moments

33

is sufficiently large [131], i.e., $\inf\{t_{k+1} - t_k : k \in \mathbb{Z}_+\} \geq \tau_d$, fore a given $\tau_d > 0$. From a practical perspective, the dwell-time condition might be restrictive and might not hold in some physical situations, such as aging systems or systems with finite escape time. However, one can get the stability result if the more general concept called the **average dwell-time** τ_a is satisfied [64]. If the number of switches $N(t_0, t)$ in the interval (t_0, t) for a finite t satisfies

$$N(t_0, t) \leq N_0 + \frac{t - t_0}{\tau_a}, \tag{2.14}$$

where N_0 is the chatter bound, then the switching signal ϱ is said to satisfy the average dwell-time condition τ_a. The average dwell-time condition (2.14) allows fast switchings on some intervals and compensate for it by dwelling more on some other intervals. For more information on the stability of switched systems under slow switchings, see [39, 64, 93, 147].

2.3   Impulsive Systems

An impulsive system, which is another type hybrid systems, describes phenomena that experience abrupt changes in the system state during the system continuous evolution. Consider the control system

$$\begin{cases} \dot{x} = f(t, x) + u(t), \\ x(t_0) = x_0, \end{cases} \tag{2.15}$$

where $u : \mathbb{R}_+ \to \mathbb{R}^n$ is the control input given by

$$u(t) = \sum_{k=1}^{\infty} I_k(x(t))\delta(t - \tau_k),$$

with I_k being the impulsive effects and $\delta(\cdot)$ being the Dirac delta function defined by

$$\delta(t - \tau_k) = \begin{cases} 0, & t \neq \tau_k \\ \infty, & t = \tau_k. \end{cases}$$

The sequence of times $\{\tau_k\}_{k=1}^{\infty}$, also known as impulsive moments, is a strictly increasing sequence with $\lim_{k\to\infty} \tau_k = \infty$. Moreover, when $t \neq \tau_k$, the system has continuous dynamics, while at τ_k's, the system evolution encounters instantaneous changes (impulses).

Integrating (2.15) over $[\tau_k, \tau_k + h]$, for a small h, yields

$$x(\tau_k + h) - x(\tau_k) = \int_{\tau_k}^{\tau_k+h} \left(f(s, x(s)) + \sum_{k=1}^{\infty} I_k(x(t))\delta(t - \tau_k) \right) ds$$

This implies that, as $h \to 0^+$,

$$\Delta x(t)|_{\tau_k} = x(\tau_k^+) - x(\tau_k) = I_k(x(\tau_k)),$$

where $x(\tau_k^+) = \lim_{h\to 0^+} x(\tau_k + h)$, and $x(\tau_k) = x(\tau_k^-)$, i.e., the solution is assumed to be left-continuous. Thus, (2.15) can be written as

$$\dot{x} = f(t, x), \qquad\qquad t \neq \tau_k, \qquad\qquad (2.16a)$$

$$\Delta x(t) = I_k(x(t)), \qquad\qquad t = \tau_k, \qquad\qquad (2.16b)$$

$$x(t_0) = x_0, \qquad\qquad k \in \mathbb{N}. \qquad\qquad (2.16c)$$

35

where $f : \mathbb{R}_+ \times \mathcal{D} \to \mathbb{R}^n$ and $\mathcal{D} \subset \mathbb{R}^n$ is an open set. System (2.16) is referred to as an *impulsive system*. A function $\phi(t) = \phi(t, t_0, x_0)$ on some interval I containing t_0 is said to be a solution of (2.16) if [28]

1. if $(t_0, x_0) \in \mathbb{R}_+ \times \mathcal{D}$, then $\phi(t_0) = x_0$ and $(t, \phi(t)) \in \mathbb{R}_+ \times \mathcal{D}$ for all $t \in I$;

2. for $t \in I$ and $t \neq \tau_k$, $\dot{\phi}(t) = f(t, \phi(t))$;

3. at $t = \tau_k \in I$, $\phi(\tau_k^+) = \phi(\tau_k) + I_k(\phi(\tau_k))$, and $\phi(t)$ is continuous from the left.

In fact, the solution of the impulsive systems evolves as follows: Let $\phi(t) = \phi(t, t_0, x_0)$ be the solution of $\dot{\phi} = f(t, \phi)$ starting at the initial point (t_0, x_0), the point $P_t(t, \phi(t))$ starts the motion, governed by the ODE (2.16a), from the initial point (t_0, x_0) along the solution curve $\{(t, \phi) : t \geqslant t_0, \phi = \phi(t)\}$ until the time $t = \tau_1 > t_0$. At this moment (i.e., at $t = \tau_1$), the evolution undergoes a sudden jump by some amount $I_k(x(t))$, given by the deference equation (2.16b), transferring the point $P_{\tau_1} = (\tau_1, \phi_1 = \phi(\tau_1))$ to the point $P_{\tau_1^+} = (t, \phi_1^+(t))$. Then, the point P_t continues its movement along the solution curve $\phi(t) = \phi(t, t_1, \phi_1^+)$ in the same way as previous starting from the initial point (t, ϕ_1^+) until the second impulsive moment at $t = \tau_2$, then another jump occurs transferring P_{τ_2} to the point $P_{\tau_2^+}$ and the process proceeds in the same manner as long as the solution exists.

Definition 2.3.1 (**Existence and Uniqueness**). *[28] If the function $f \in \mathcal{C}^1(\mathbb{R}_+ \times \mathcal{D}; \mathbb{R}^n)$ and $x + I_k(x(t)) \in \mathcal{D}$ for each $x \in \mathcal{D}$ and $k \in \mathbb{N}$, then the IVP (2.16) has a unique solution for each $(t_0, x_0) \in \mathbb{R}_+ \times \mathcal{D}$.*

Based on the impulsive moments types, impulsive systems can be classified as follows

1. Systems with impulses at fixed times, i.e., $t = \tau_k$.

2. Systems with impulses at variable times, i.e., $t = \tau_k(x)$.

3. Systems with impulses satisfying the spatio-temporal relation $\kappa(t, x) = 0$.

In the case of fixed-time impulses, the solutions starting at different initial time jump at the same impulsive moments. Other than this case, a challenge that might arise is that solutions evolving from different initial times/states jump at different impulsive moments Furthermore, a "pulse phenomena" in which the solution hits a hyper-surface infinitely many times, or a "confluence" in which different solutions merge after some time may be encountered in the systems with variable time impulses. In the present thesis, we are mainly concerned with systems experience impulsive actions at fixed times.

2.4   Delay Differential Equations (DDEs)

For $r > 0$, let $\mathcal{C}_r$ be the space of all continuous functions that are defined from $[-r, 0]$ to $\mathbb{R}^n$. For any $t \in \mathbb{R}_+$, let $x(t)$ be a function defined on $[t_0, \infty)$. Then, we define a new function $x_t : [-r, 0] \to \mathbb{R}^n$ by $x_t(s) = x(t + s)$ for all $s \in [-r, 0]$, and its norm by $\|x_t\|_r = \sup_{t-r \leq \theta \leq t} \|x(\theta)\|$. The positive r represents the time delay.

A general nonlinear *functional differential equation* may have the form

$$\dot{x} = f(t, x_t), \tag{2.17}$$

where, for $\mathcal{D} \subset \mathcal{C}_r$, the functional $f : \mathbb{R}_+ \times \mathcal{D} \to \mathbb{R}^n$.

Given $t_0 \in \mathbb{R}_+$ and an initial continuous function $\phi(s)$, the corresponding IVP is given by

$$\begin{cases} \dot{x} = f(t, x_t), \\ x_{t_0}(s) = \phi(s), \quad s \in [-r, 0]. \end{cases} \tag{2.18}$$

A function $x(t) = x(t, t_0, \phi)$ is said to be a solution of (2.18) on $[t_0 - r, t_0 + a)$ for $a > 0$ if $x \in \mathcal{C}([t_0 - r, t_0 + a), \mathbb{R}^n)$, $x(t)$ satisfies (2.18) for $t \in [t_0, t_0 + a)$ and $x(t_0 + s) = \phi(s)$ for $s \in [-r, 0]$. In (2.18), the delay is finite. In this case, the continuity of x on $[t_0 - r, t_0 + a]$ implies the continuity of x_t on $[t_0, t_0 + a]$ for $a > 0$.

Theorem 2.4.1 (Existence). *If $f \in \mathcal{C}(\mathbb{R}_+ \times \mathcal{D}; \mathbb{R}^n)$ where $\mathcal{D} \subset \mathcal{C}_r$ is an open set, then for any $(t_0, \phi) \in \mathbb{R}_+ \times \mathcal{D}$ there exists at least one solution of the IVP (2.18).*

Definition 2.4.2. *A function $f(t, x)$ defined on $\mathbb{R}_+ \times \mathcal{D}$ is said to be Lipschitz in ψ if there exists a constant $L > 0$ such that*

$$\|f(t, \psi_1) - f(t, \psi_2)\| \le L\|\psi_1 - \psi_2\|_r \quad \text{for all} \quad (t, \psi_1), (t, \psi_2) \in \mathbb{R}_+ \times \mathcal{D}.$$

Theorem 2.4.3 (Uniqueness). *If f is continuous in t and Lipschitz in ψ, then, for any $(t_0, \phi) \in \mathbb{R}_+ \times \mathcal{D}$, there exists a unique solution of the IVP (2.18).*

In the following, we state the definitions of the stability notions of delay systems.

Definition 2.4.4 (Stability). *Suppose $f(t, 0) = 0$ for all $t \in \mathbb{R}_+$. The trivial solution $x(t) = 0$ of IVP (2.18) is said to be*

(i) **stable** *if, for any $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that*

$$\|\phi\|_r < \delta \qquad implies \qquad \|x(t)\| < \epsilon \quad for \ any \quad t \geq t_0 - r;$$

(ii) **uniformly stable** *if (i) holds with $\delta = \delta(\epsilon)$;*

(iii) **asymptotically stable** *if (i) holds and there exists a positive constant $\delta = \delta(t_0)$ such that*

$$\|\phi\|_r < \delta \qquad implies \qquad \lim_{t \to \infty} x(t) = 0;$$

(iv) **globally asymptotically stable** *if (iii) holds with an arbitrary large constant δ;*

(v) **uniformly asymptotically stable** *if it is uniformly stable and there is a positive constant δ, independent of t_0, such that, for all $\|\phi\|_r < \delta$, $\lim_{t \to \infty} x(t) \to 0$, uniformly in t_0; that is, for any $\eta > 0$, there is $T = T(\eta) > 0$ such that, for all $\|\phi\|_r < \delta$,*
$$\|x(t)\| < \eta, \qquad \forall t \geq t_0 + T;$$

(vi) **globally uniformly asymptotically stable** *if (v) holds with an arbitrary large constant δ;*

(vii) **exponentially stable** *if there exist positive constants δ, k and λ such that*

$$\|x(t)\| \leq k\|\phi\|_r e^{-\lambda(t - t_0)}, \qquad whenever \ \ \|\phi\|_r < \delta, \ \ and \ \ t \geq t_0;$$

(viii) **globally exponentially stable** *if (vii) holds with an arbitrary large constant δ;*

(ix) **unstable** *if (i) does not hold.*

There are two main methods to analyze stability of delay systems. One is called Lyapunov-krasovskii functional method, which uses Lyapunov functional, and the other method is Razumikhin type technique, which uses Lyapunov function. In the present thesis, Razumikhin method is adopted to analyze the stability of delay systems. Razumikhin approach deals with the delay by assuming the delay terms are bounded by some non delay terms which leads to cases similar to those of ODEs.

2.4.1   Razumikhin-Type Theorem

The contents of this subsection are taken from [84]. Razumikhin-type theorem is an effective method to analyze the stability of DDEs. It explores the possibility of using the derivative of a function on $\mathbb{R}^n$ to generate sufficient conditions that guarantee stability. A powerful feature of this approach is to have a control on the relationship between $\|x(t)\|$ and $\|x(t+s)\|$, $s \in [-r, 0]$.

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a positive definite continuously differentiable function. Then, its derivative along the solution of (2.17) is given by

$$\dot{V}(x(t)) = \frac{\partial V(x(t))}{\partial x} \cdot f(x_t) \tag{2.19}$$

In order for $\dot{V}(x(t))$ to be nonpositive, it is required that $x(t)$ dominates $x(t+s)$. In fact,

40

by the uniform stability definition, if x_t is initially in a ball $B = B(0, \delta)$ in C. Then, in order to leave this ball, it has to reach the boundary of B at some time t^*. Thus, at time t^*, we have $\|x(t^*)\| = \delta$, and $\|x(t^* + s)\| < \delta$ for $s \in [-r, 0)$, and so $d/dx\|x(t^*)\| \geq 0$. Thus, one can get the stability result by showing this is impossible.

Theorem 2.4.5. *Suppose $f : \mathbb{R} \times C \to \mathbb{R}^n$ maps $\mathbb{R} \times$ (bounded sets of C) into bounded sets of $\mathbb{R}^n$, u, v, $w : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous, nondecreasing functions satisfying $u(0) = v(0) = w(0) = 0$, and $u(s)$, $v(s)$ are positive for $s > 0$. Assume that there is a continuous function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that*

$$u(\|x\|) \leq V(t, x) \leq v(\|x\|), \qquad t \in \mathbb{R}, \qquad x \in \mathbb{R}^n \qquad (2.20)$$

The following statements are true:

(i) the solution $x=0$ of (2.17) is uniformly stable if

$$\dot{V}(t, x(t)) \leq -w(|x(t)|) \quad \text{whenever} \quad V(t+s, x(t+s)) \leq V(t, x(t)), \quad s \in [-r, 0] \quad (2.21)$$

(ii) the solution $x=0$ of (2.17) is uniformly asymptotically stable if $w(s) > 0$ for $s > 0$ and there is a continuous decreasing function $p(s) > s$ for $s > 0$ such that

$$\dot{V}(t, x(t)) \leq -w(|x(t)|) \quad \text{whenever} \quad V(t+s, x(t+s)) < p(V(t, x(t))), \quad s \in [-r, 0]$$

$$(2.22)$$

If $u(s) \to \infty$ as $s \to \infty$, then $x = 0$ is globally asymptotically satble.

2.5 Hybrid Systems with Time Delay

Considering a time delay in the switched system (2.13) leads to the so-called *switched system with time delay*, which may have the form

$$\begin{cases} \dot{x} = f_i(t, x_t), & i \in \mathcal{S}, \\ x_{t_0}(s) = \phi(s), & s \in [-r, 0], \end{cases} \qquad (2.23)$$

where $f_i : \mathbb{R}_+ \times \mathcal{D} \to \mathbb{R}^n$ for all i, and $\phi \in \mathcal{C}([-r, 0], \mathcal{D})$. A special class of (2.23) is obtained when $s = -r$. In this case, we have $x_t = x(t - r)$. Accordingly, the delayed differential equation becomes

$$\dot{x} = f_i(t, x(t - r)), \qquad i \in \mathcal{S}. \qquad (2.24)$$

Assumption A, mentioned earlier, can be adjusted to suit the switched systems with time delay. Consider the linear switched systems with time delay

$$\begin{cases} \dot{x} = (A_i + \Delta A_i)x + (\bar{A}_i + \Delta \bar{A}_i)x(t - r), \\ x_{t_0}(s) = \phi(s), & s \in [-r, 0], \quad r > 0 \end{cases} \qquad (2.25)$$

Assumption A. For any $i \in \mathcal{S}$ and $\forall\, t \in \mathbb{R}_+$,

$$\Delta A_i(t) = D_i \mathcal{U}_i(t) H_i \quad \text{and} \quad \Delta \bar{A}_i(t) = \bar{D}_i \bar{\mathcal{U}}_i(t) \bar{H}_i,$$

with D_i, H_i, $\bar{D}_i$, $\bar{H}_i$ being known real matrices with appropriate dimensions, and $\mathcal{U}_i(t), \bar{\mathcal{U}}_i(t)$ being unknown real time-varying matrices and satisfying

$$\|\mathcal{U}_i(t)\| \leq 1 \quad \text{and} \quad \|\bar{\mathcal{U}}_i(t)\| \leq 1.$$

Similarly, considering a time delay in the system state and impulsive function of the impulsive system (2.16) leads to the *impulsive system with time delay*

$$\dot{x} = f(t, x_t), \qquad t \neq \tau_k, \quad k \in \mathbb{N} \tag{2.26a}$$

$$\Delta x(t) = I_k(t, x_{t^-}), \qquad t = \tau_k, \tag{2.26b}$$

$$x_{t_0}(s) = \phi(s), \qquad s \in [-r, 0]. \tag{2.26c}$$

where $f : \mathbb{R}_+ \times \mathcal{PC}([-r, 0]; \mathcal{D}) \to \mathbb{R}^n$ and $\phi \in \mathcal{PC}([-r, 0]; \mathcal{D})$ equipped with the norm $\|\phi\|_r = \sup_{-r \leq s \leq 0} \|\phi(s)\|$ with $\mathcal{PC}$ being a class of piecewise-continuous function stated in the following definition. We should mention that the solution of (2.26) is assumed to be right-continuous, and the impulsive function in (2.26b) is taken to be time-varying.

Definition 2.5.1. *[30] For any $a, b \in \mathbb{R}$ with $a < b$ and for some set $\mathcal{D} \subset \mathbb{R}^n$, define the following classes of functions*

$$\mathcal{PC}([a, b], \mathcal{D}) = \Big\{ \psi : [a, b] \to \mathcal{D} | \psi(t^+) = \psi(t), \forall t \in [a, b); \ \psi(t^-) \text{ exists in } \mathcal{D}, \forall t \in (a, b];$$

$$\text{and } \psi(t^-) \text{ for all but at most a finite number of points } t \in (a, b] \Big\},$$

$$\mathcal{PC}([a, b), \mathcal{D}) = \Big\{ \psi : [a, b) \to \mathcal{D} | \psi(t^+) = \psi(t), \forall t \in [a, b); \ \psi(t^-) \text{ exists in } \mathcal{D}, \forall t \in (a, b);$$

$$\text{and } \psi(t^-) \text{ for all but at most a finite number of points } t \in (a, b) \Big\},$$

and

$$\mathcal{PC}([a, \infty), \mathcal{D}) = \Big\{ \psi : [a, \infty) \to \mathcal{D} | \forall c > a, \ \psi|_{[a,c]} \in \mathcal{PC}([a, c], \mathcal{D}) \Big\}.$$

In this definition, on a finite interval, the jump discontinuities form a finite set, while they form a countably infinite set on an infinite interval.

In the following, we state the definitions of some stability concepts of impulsive systems with time delay.

Definition 2.5.2 (**Stability**). *[30] Let $x(t, t_0, \phi)$ be any solution of* (2.26). *The trivial solution $x(t) = 0$ of IVP* (2.26) *is said to be*

(i) **stable** *if, for any $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that if*

$$\|\phi\|_r < \delta \qquad with \ \phi \in \mathcal{PC}([-r, 0], \mathcal{D})$$

implies

$$\|x(t, t_0, \phi)\| \le \epsilon \quad for \ any \quad t \ge t_0 - r;$$

(ii) **uniformly stable** *if* (i) *holds with $\delta = \delta(\epsilon)$;*

(iii) **asymptotically stable** *if* (i) *holds and for each $t_0 \in \mathbb{R}_+$ there exists a positive constant $\delta = \delta(t_0)$ such that*

$$\|\phi\|_r < \delta \qquad with \ \phi \in \mathcal{PC}([-r, 0], \mathcal{D})$$

implies

$$\lim_{t \to \infty} x(t, t_0, \phi) = 0;$$

(iv) **globally asymptotically stable** *if* (iii) *holds with an arbitrary large constant δ;*

44

(v) **uniformly asymptotically stable** *if it is uniformly stable and there is a positive constant δ, such that for any η > 0, there is $T = T(\eta, \delta) > 0$ such that, if*

$$\|\phi\|_r < \delta \qquad with \ \phi \in \mathcal{PC}([-r, 0], \mathcal{D})$$

implies

$$\|x(t, t_0, \phi)\| \leq \eta, \qquad \forall t \geq t_0 + T;$$

(vi) **globally uniformly asymptotically stable** *if (vi) holds with an arbitrary large constant δ;*

(vii) **exponentially stable** *if there exist positive constants δ, k and λ such that, if*

$$\|\phi\|_r < \delta \qquad with \ \phi \in \mathcal{PC}([-r, 0], \mathcal{D})$$

implies

$$\|x(t)\| \leq k\|\phi\|_r e^{-\lambda(t-t_0)}, \quad t \geq t_0;$$

(viii) **globally exponentially stable** *if (vii) holds with an arbitrary large constant δ;*

(ix) **unstable** *if (i) does not hold.*

In the same manner, one can define the *impulsive switched systems with time delay* as follows:

$$\begin{cases} \dot{x}(t) = f_i(t, x_t), & t \neq \tau_k, \quad i \in \mathcal{S}, \\ \Delta x(t) = I_{i_k}(t, x_{t^-}), & t = \tau_k, \\ x_{t_0}(s) = \phi(s), & s \in [-r, 0], \end{cases} \qquad (2.27)$$

where $f_i : \mathbb{R}_+ \times \mathcal{PC}([-r, 0]; \mathcal{D}) \to \mathbb{R}^n$ for any $i \in \mathcal{S}$, and $\phi \in \mathcal{PC}([-r, 0]; \mathcal{D})$.

2.6 Interconnected or Large-Scale Systems (LSS)

Due to complexity, some systems are characterized by LSS, which is an interconnection of lower order subsystems. A LSS also known as an interconnected system or a composite system. A common way to analyze such high order systems is by decomposing them into isolated subsystems and establish the stability of each subsystem ignoring the interconnection part. Then, this available information is used together with the interconnection, which is treated as a perturbation, to get a conclusion about the stability of the interconnected system. For the LSS to be (exponentially) stable, it is required that the degree of stability of the isolated subsystems as a whole be greater than the interconnection strength. This type of relation is represented by the so-called M-matrix. The results in this section are taken from [79]. Consider the following nth order interconnected system

$$\dot{w}^i = f_i(t, w^i) + g_i(t, w^1, w^2, \cdots, w^i, \cdots, w^l)$$

$$w^i(t_0) = w_0^i, \tag{2.28}$$

where $i = 1, 2, \cdots, l$, $w^i \in \mathbb{R}^{n_i}$ is the ith subsystem state, such that $\Sigma_{i=1}^l n_i = n$, and $x^T = (w^{1T} \ w^{2T} \ \cdots \ w^{lT})$. Assume that the trivial solution $x \equiv 0$ is an equilibrium point of system (2.28), i.e.,

$$f_i(t, 0) = 0, \qquad g_i(t, 0) = 0, \qquad \forall \, t \geq 0, \text{ and } \forall \, i.$$

The corresponding l isolated subsystems are given by

$$\dot{w}^i = f_i(t, w^i), \qquad i = 1, 2, \cdots, l \tag{2.29}$$

To analyze the stability of (2.28), assume that all these systems have uniformly asymptotically stable equilibrium points, and there are l corresponding Lyapunov functions $V^i(t, w^i)$. For $\beta_i > 0$, define

$$V(t, x) = \sum_{i=1}^{l} \beta_i V^i(t, w^i) \tag{2.30}$$

as a composite Lyapunov function candidate for the interconnected system. Then, the derivative of $V(t, x)$ along the trajectories of (2.28) is given by

$$\dot{V}(t, x) = \sum_{i=1}^{l} \beta_i \left[\frac{\partial V^i}{\partial t} + \frac{\partial V^i}{\partial w^i} f_i(t, w^i) \right] + \sum_{i=1}^{l} \beta_i \frac{\partial V^i}{\partial w^i} g_i(t, x) \tag{2.31}$$

Since the isolated subsystems are assumed to be uniformly asymptotically stable, the first term in the right hand side of (2.31) is bounded above by a strictly negative term. Thus, the second term, which is indefinite, is assumed to be bounded by some nonnegative upper bound.

Now, assume that for $\|x\| < c$ with $c > 0$, $V^i(t, w^i)$ and $g_i(t, x)$ satisfy the following conditions

(i) $\dfrac{\partial V^i}{\partial t} + \dfrac{\partial V^i}{\partial w^i} f_i(t, w^i) \leq -a_i \phi_i^2(w^i)$ for $t \geq 0$;

(ii) $\left\| \dfrac{\partial V^i}{\partial w^i} \right\| \leq b_i \phi_i(w^i)$;

(iii) $\|g_i(t, x)\| \leq \displaystyle\sum_{j=1}^{l} \gamma_{ij} \phi_j(w^j)$, for $i = 1, 2, \cdots, l$ and $t \geq 0$,

47

where a_i and b_i are positive constants, ϕ_i is a positive-definite function, and γ_{ij} are non-negative constants. From (2.31), one gets

$$\dot{V}(t,x) \leq \sum_{i=1}^{l} \beta_i \left[-a_i \phi_i^2(w^i) \right] + \sum_{i=1}^{l} \beta_i b_i \gamma_{ij} \phi_i(w^i) \phi_j(w^j)$$

which can be written as

$$\dot{V}(t,x) \leq -\frac{1}{2}\phi^T(B\mathcal{S} + \mathcal{S}^T B)\phi$$

where $\phi = (\phi_1, \phi_2, \cdots, \phi_m)^T$, $B = diag(\beta_1, \beta_2, \cdots, \beta_l)$, and $\mathcal{S} = [s_{ij}]$ is an $l \times l$ matrix such that

$$s_{ij} = \begin{cases} a_i - b_i\gamma_{ij}, & i = j \\ -b_i\gamma_{ij}, & i \neq j \end{cases}. \tag{2.32}$$

According to Lyapunov stability theory, the asymptotic stability of the composite system (2.28) can be achieved if there exists a matrix B such that

$$B\mathcal{S} + \mathcal{S}^T B > 0. \tag{2.33}$$

The following lemma gives a sufficient condition to guarantee the existence of matrix B

Lemma 2.6.1. *There exists a diagonal matrix $B > 0$ satisfying (2.33) if and only if $\mathcal{S}$ is an M-matrix; that is, all its leading successive principle minors are positive, i.e.,*

$$det \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1k} \\ s_{21} & s_{22} & \cdots & s_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ s_{k1} & s_{k2} & \cdots & s_{kk} \end{bmatrix} > 0, \quad k = 1, 2, \cdots, l.$$

The M-matrix has an important role here. The diagonal elements represent the stability degree for the isolated subsystems, while the nonpositive off-diagonal elements represent the strengths of the interconnections.

Theorem 2.6.2. *Consider the LSS* (2.28). *Suppose that for all* $t \geq 0$ *and* $\|x\| < c$ *with* $c > 0$, $V^i(t, w^i)$ *and* $g_i(t, x)$ *satisfy the conditions (i)-(ii), where* $V^i(t, w^i)$ *is a positive-definite decrescent Lyapunov functions for the ith isolated subsystem. Suppose further that the matrix* $\mathcal{S}$ *defined in* (2.32) *is an M-matrix. Then, the trivial solution is uniformly asymptotically stable. It is, moreover, globally uniformly asymptotically stable if all assumptions hold globally and* $V^i(t, w^i)$ *are radially unbounded.*

2.7 Singularly Perturbed Systems (SPS)

Systems involving multiple time-scale dynamics are known as SPS. They can be viewed as a class of LSS where the multi time-scale subsystems are the isolated subsystems, and the interaction between them is the perturbation to the system. SPS are characterized by small parameters multiplied by the highest derivatives creating the fast and slow subsystems. The contents of this section are taken from [79] unless otherwise specified. Consider the following autonomous SPS

$$\dot{x} = f(x, z), \tag{2.34a}$$

$$\epsilon \dot{z} = g(x, z), \tag{2.34b}$$

where $x \in \mathbb{R}^m$, $z \in \mathbb{R}^n$ are, respectively, the system slow and fast states, $0 < \epsilon \ll 1$, and f and g are locally Lipschitz in a domain containing the origin. Assume that $(x, z) = (0, 0)$ is an isolated equilibrium point. Thus, we have

$$f(0,0) = 0, \qquad g(0,0) = 0.$$

In order to analyze the stability properties, we set $\epsilon = 0$. This reduces the dimension of the system from $m + n$ to m because the differential equation (2.34b) declines into the following algebraic equation

$$0 = g(x, z).$$

The foregoing equation is assumed to have the isolated real root

$$z = h(x).$$

For simplicity, we shift the equilibrium point to the origin by considering the transformation

$$y = z - h(x).$$

Then, the SPS is

$$\dot{x} = f(x, y + h(x)), \tag{2.35a}$$

$$\epsilon \dot{y} = g(x, y + h(x)) - \epsilon \frac{\partial h}{\partial x} f(x, y + h(x)). \tag{2.35b}$$

The corresponding slow reduced subsystem is

$$\dot{x} = f(x, h(x)) \tag{2.36}$$

50

has an equilibrium $x = 0$, and

$$\frac{dy}{ds} = g(x, y + h(x)),$$

where $s = t/\epsilon$, and x is treated as a fixed parameter, has an equilibrium point at $y = 0$. Sufficient conditions for SPS (2.34) to be asymptotically stable have been provided in the following theorem.

Theorem 2.7.1. *Consider the SPS (2.34). Assume there exist two Lyapunov functions $V(x)$ and $W(x, y)$ for the slow and fast subsystems respectively, positive-definite functions ψ_1, ψ_2, W_1, W_2, and positive constants $a_1, a_2, b_1, b_2, \gamma$ such that the following conditions hold*

$$\frac{\partial V}{\partial x} f(x, h(x)) \leq -a_1 \psi_1^2(x);$$

$$\frac{\partial W}{\partial y} g(x, y + h(x)) \leq -a_2 \psi_2^2(y);$$

$$W_1(y) \leq W(x, y) \leq W_2(y);$$

$$\frac{\partial V}{\partial x} \left[f(x, y + h(x)) - f(x, h(x)) \right] \leq b_1 \psi_1(x) \psi_2(x);$$

$$\left[\frac{\partial W}{\partial x} - \frac{\partial W}{\partial y} \frac{\partial h}{\partial x} \right] f(x, y + h(x)) \leq b_2 \psi_1(x) \psi_2(x) + \gamma \psi_2^2(y).$$

Then, there is a positive constant $\epsilon^* = \dfrac{a_1 a_2}{a_1 \gamma + b_1 b_2}$ such that the origin $(x, z) = (0, 0)$ is asymptotically stable for all $0 < \epsilon < \epsilon^*$.

2.8   Reliable Control

The contents of this section are taken from [149, 165]. Consider the linear control system

$$\begin{cases} \dot{x} = Ax + Bu, \\ x(t_0) = x_0, \end{cases} \qquad (2.37)$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^q$ is the control input of the form $u = Kx$ with $K \in \mathbb{R}^{q \times n}$ being the control gain matrix, $A \in \mathbb{R}^{n \times n}$ is a non Hurwitz[1] matrix, $B \in \mathbb{R}^{n \times q}$. The matrix pair (A, B) is assumed to be stabilizable (i.e., $A + BK$ is Hurwitz). The closed-loop system is

$$\begin{cases} \dot{x} = (A + BK)x, \\ x(t_0) = x_0. \end{cases} \qquad (2.38)$$

To analyze the reliable stabilization with respect to actuator failures, let $u \in \mathbb{R}^q$. The q control actuators are divided into two sets. $\Sigma \subseteq \{1, 2, ..., q\}$ the set of actuators that are susceptible to failure, i.e., they may occasionally fail, and $\overline{\Sigma} \subseteq \{1, 2, ..., q\} - \Sigma$ the other set of actuators which are robust to failures and essential to stabilize the given system. The elements of Σ are redundant in terms of the stabilization but necessary to improve the system performance, while the elements of $\overline{\Sigma}$ are required to stabilize the system and assumed that they never fail, i.e., the pair $(A, B_{\overline{\Sigma}})$ is assumed to be stabilizable.

Consider the decomposition of the control matrix

$$B = B_\Sigma + B_{\overline{\Sigma}},$$

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[1] A Hurwitz matrix is a matrix in which all eigenvalues have negative real parts.

where B_Σ and $B_{\overline{\Sigma}}$ are the control matrices associated with Σ and $\overline{\Sigma}$, respectively, and B_Σ and $B_{\overline{\Sigma}}$ are generated by zeroing out the columns corresponding to $\overline{\Sigma}$ and Σ, respectively. Let $\sigma \subseteq \Sigma$ corresponds to some of the actuators that experience failure. Then, the decomposition becomes

$$B = B_\sigma + B_{\overline{\sigma}},$$

where B_σ and $B_{\overline{\sigma}}$ have the same definition of B_Σ and $B_{\overline{\Sigma}}$, respectively. The closed-loop system with reliable control becomes

$$\begin{cases} \dot{x} = (A + B_{\overline{\sigma}}K)x, \\ x(t_0) = x_0. \end{cases} \tag{2.39}$$

Consider the following input/output system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Gw(t), \\ z(t) = Cx(t) + Fu(t) \\ x(t_0) = x_0, \end{cases} \tag{2.40}$$

where w is an external disturbance such that $w(t) = K_w x(t)$, and the control input is of the form $u(t) = K_u x(t)$. In analyzing the system output behaviour using the $H_\infty-$ norm, we are mainly interested in an optimization problem of the form

$$\inf_u \sup_w J(u, w) < \infty, \tag{2.41}$$

where

$$J(u, w) = \int_{t_0}^{\infty} (z^T z - \gamma^2 w^T w) dt \tag{2.42}$$

for some positive constant γ. Then, by the definitions of u and w, we get

$$J(u, w) = \int_{t_0}^{\infty} x^T (C_c^T C_c - \gamma^2 K_w^T K_w) x dt, \qquad (2.43)$$

where $C_c = C + F K_u$.

Consider

$$d(x^T P x) = (\dot{x}^T P x + x^T P \dot{x}) dt \qquad (2.44)$$

$$= x^T [(A + B K_u + G K_w)^T P] x + x^T [P(A + B K_u + G K_w)] x$$

where P is a symmetric matrix.

Adding and subtracting $d(x^T P x)$ to J leads to

$$J(u, w) = \int_{t_0}^{\infty} x^T \Big[(A + B K_u + G K_w)^T P + P(A + B K_u + G K_w) + C_c^T C_c - \gamma^2 K_w^T K_w \Big] x dt$$

$$- x(\infty)^T P x(\infty) + x^T(t_0) P x(t_0)$$

If we assume that $A + B K_u + G K_w$ is stable, $x(\infty) = 0$. Then

$$J(u, w) = x^T(t_0) P x(t_0) \qquad (2.45)$$

where P satisfies the Riccati-like equation

$$(A + B K_u + G K_w)^T P + P(A + B K_u + G K_w) + C_c^T C_c - \gamma^2 K_w^T K_w = 0. \qquad (2.46)$$

The maximization condition for J with respect to K_w in (2.45) is

$$\nabla_{K_w} P = 0, \qquad (2.47)$$

where the gradient $\nabla_{K_w} P$ is defined as follows:

$$(\nabla_{K_w} P)_{ij} = \frac{\partial P}{\partial K_{w_{ij}}}. \qquad (2.48)$$

So that, from the Ricattti-like equation, we have

$$(A + BK_u + GK_w)^T \nabla_{K_w} P + G^T P + G^T P + \nabla_{K_w} P(A + BK_u + GK_w) - 2\gamma^2 K_w = 0. \tag{2.49}$$

By (2.47), we get

$$K_w = \frac{1}{\gamma^2} G^T P. \tag{2.50}$$

Similarly, the minimization condition of J is

$$\nabla_{K_u} P = 0. \tag{2.51}$$

So that, one can get

$$K_u = -B^T P \tag{2.52}$$

Substituting K_u and K_w in the Ricatti-like equation gives

$$A^T P + PA + C_c^T C_c - P(B^T B - \frac{1}{\gamma^2} G^T G)P = 0. \tag{2.53}$$

The H_∞ has received a great deal of attention in control theory [68, 69, 80, 143, 156, 176, 177, 179, 180]. It is a useful measure used to guarantee the performance of the plant when dealing with control problems that involve robust design. However, in the event of control component failures, the stability or performance of the plant may not be achieved by such designs. Therefore, it would be advantageous if it is associated with a reliable control design to handle such failures when they occur. As a result, many researchers have considered the robust reliable H_∞ control since 1992 up until now. Interested readers may refer to [67, 112, 143, 182]. In [143], the authors discussed uncertain linear systems

with disturbances where the norm-bounded uncertainty occurs in the system state. The main objective was designing a robust feedback-reliable controller so that the system is quadratically stabilized with an acceptable performance level not only when the actuators are operational, but also when failure occurs in some control components; moreover, a state feedback control design was used in that work. In [112], a more general design was established for uncertain switched linear systems with norm-bounded uncertainties in both the system state and the output. The authors used the linear matrix inequality approach and convex combination technique to design a robust reliable H_∞ controller and a switching rule so that the system maintains the global quadratic stability with a good performance not only when the actuators are operational but also for the faulty case.

2.9   Stochastic Differential Equations

In this section, we present some basic concepts that will be used later.

Definition 2.9.1. *Let $I \subset \mathbb{R}_+$ and Ω be a sample space of an experiment. A Stochastic process $X(t, \omega)$ (or $X(t)$, for notation simplicity) is a family of random variables $\{X_t(\omega) : t \in I \text{ and } \omega \in \Omega\}$.*

Definition 2.9.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. An almost surely (a.s.) continuous stochastic process $W(t)$ for all $t \in \mathbb{R}_+$ is said to be* Wiener (or Brownian motion) process *if*

1. $\mathbb{P}\{\omega : W(0) = 0\} = 1;$

56

2. *for any $0 \leq s < t < \infty$, the increment $W(t) - W(s)$ is independent of $W(s) - W(u)$ for all $0 \leq u < s$;*

3. *for all $t \in \mathbb{R}_+$ and $h > 0$, the increment $W(t+h) - W(t)$ is Gaussian (or normally) distributed with*

$$\mathbb{E}[W(t+h) - W(t)] = \mu h;$$

$$\mathbb{E}[(W(t+h) - W(t))^2] = \sigma^2 h,$$

where the mean $\mu \in \mathbb{R}$ and the variance σ^2 is a positive constant.

Particularly, W is said to be a standard Wiener process *if $\mu = 0$ and $\sigma^2 = 1$.*

A typical nonlinear *stochastic systems with time delay* or *systems with stochastic functional differential equations* may be defined by

$$\begin{cases} dx(t) = f(t, x_t)dt + g(t, x_t)dW(t), & t \in [t_0, T], \\ x_{t_0}(s) = \phi(s), & s \in [-r, 0], \end{cases} \tag{2.54}$$

for any $t_0, T \in \mathbb{R}_+$ with $T \geq t_0$, where $x \in \mathbb{R}^n$ is the delayed system state, $W(t) = (W_1(t), W_2(t), \cdots, W_m(t))^T$ is an $m-$dimensional Wiener process, $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is the drift coefficient of the process x, $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^m$ is the diffusion coefficient of the process x, and $\phi : [-r, 0] \to \mathbb{R}^n$ is the initial function process.

The stochastic integral equation corresponding to the IVP in (2.54) is

$$x(t) = \phi(0) + \int_{t_0}^t f(s, x_s)ds + \int_{t_0}^t g(s, x_s) \, dW(s), \tag{2.55}$$

where $t \geq t_0$. The first integral is a Riemann integral almost surely (a.s.) and the second one is called an Itô integral satisfying

$$\mathbb{E}\left[\int_{t_0}^t g(s, x_s)\, dW(s)\right] = 0,$$

$$\mathbb{E}\left\|\int_{t_0}^t g(s, x_s)\, dW(s)\right\|^2 = \int_{t_0}^t \mathbb{E}\|g(s, x_s)\|^2\, ds.$$

Considering impulse effects in (2.54) leads to the following *stochastic impulsive systems with time delay* (SISD)

$$dx(t) = f(t, x_t)dt + g(t, x_t)\, dW(t), \quad t \neq \tau_k, \tag{2.56}$$

$$\Delta x(t) = \mathcal{I}(t, x_{t^-}), \qquad\qquad t = \tau_k,$$

$$x_{t_0}(s) = \phi(s), \quad s \in [-r, 0],$$

where τ_k represents an impulsive moment, for $k = 0, 1, 2, \cdots$, and satisfies $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$ and $\lim_{k\to\infty} \tau_k = \infty$.

Itô formula. For any $t_0 \in \mathbb{R}_+$ and $t \geq t_0$, assume that $x(t)$ is an $\mathbb{R}^n$-dimensional stochastic process satisfying

$$dx(t) = f(t, x(t))dt + g(t, x(t))dW(t), \qquad \text{(a.s.)}, \tag{2.57}$$

Let $V \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+)$. Then, for any $t \geq t_0$, V is a stochastic process satisfying

$$dV(t, x) = \mathcal{L}V(t, x)f(t, x)dt + V_x(t, x)g(t, x)dW(t), \qquad \text{(a.s.)}$$

where operator $\mathcal{L}$ (or $\mathcal{L}V$ as a single notation) is defined by

$$\mathcal{L}V(t, x) = V_t(t, x) + V_x(t, x)f(t, x) + \frac{1}{2}\text{tr}[g^T(t, x)V_{xx}(t, x)g(t, x)],$$

where "tr" stands for the trace of a matrix.

In the above formula, $\mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+)$ denotes the space of all real-valued functions $V(t, x)$ defined on $\mathbb{R}_+ \times \mathbb{R}^n$ such that they are continuously differentiable once in t and twice in x. For instance, if $V(t, x) \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+)$, then we have

$$V_t(t, x) = \frac{\partial V(t, x)}{\partial t}, V_x(t, x) = \left(\frac{\partial V(t, x)}{\partial x_1}, \cdots, \frac{\partial V(t, x)}{\partial x_n}\right)^T, V_{xx}(t, x) = \left(\frac{\partial^2 V(t, x)}{\partial x_i \partial x_j}\right)_{n \times n}.$$

Definition 2.9.3. *Let $x = x(t, t_0, \phi)$ be the solution of system (2.56). The trivial solution $x \equiv 0$ is said to be locally exponentially stable in the pth moment if there exist positive constants λ, $\bar{\lambda}$ and c such that*

$$\mathbb{E}\big[\|x(t)\|^p\big] \leq \bar{\lambda}\mathbb{E}\big[\|\phi\|_r^p\big]e^{-\lambda(t-t_0)}, \qquad \forall t \geq t_0,$$

for any initial function ϕ such that $\mathbb{E}\big[\|\phi\|_r^p\big] < c$, and $t_0 \in \mathbb{R}_+$. It is said to be globally exponentially stable if c is chosen arbitrarily large.

Part I

Robust Reliable H_∞ Control and Input-to-State Stabilization for Uncertain Hybrid Systems

This part discusses input-to-state stabilization (ISS) of a class of hybrid systems with time-varying norm-bounded parametric uncertainties in the system states. The main objective is to design a robust reliable H_∞ control that guarantees ISS not only when all the actuators are operational, but also when some of them experience failure. The faulty actuator output is assumed to be nonzero, which is treated as a disturbance signal that is augmented with the system disturbance input.

The input disturbance of the system is assumed to be time varying with norm-bounded energy. The faulty output can be treated either as an outage (i.e., zero output) or a nonzero disturbance that augmented with the system input disturbance. The latter case is more practical because most of the control component failures occur unexpectedly, and at the same time an immediate repair may not be feasible. Therefore, designing a reliable controller to guarantee an acceptable level of performance becomes crucial. We also assume the system jumps amongst a finite set of modes.

Thus, new sufficient conditions have been developed here to guarantee the input to state stabilization and H_∞ performance of the hybrid system in the presence of the input disturbance, state uncertainties, and nonlinear lumped perturbation not only when all the actuators are operational, but also when some of them experience failure.

To achieve this result, we assume that every individual subsystem is input-to-state stabilized by a robust reliable controller. As well known, a peculiar phenomenon of a switched system is that the stability or boundedness of each individual mode does not

guarantee the boundedness of the switched system unless the switching moments are ruled by a logic-based switching signal.

The methodology of multiple Lyapunov functions is used to analyze the input-to-state stabilization. This approach results in solving a finite number of Riccati-like matrix equations to obtain the feedback control laws for each mode, which includes some tuning parameters to reduce the conservativeness of the control design. Simultaneously, to properly orchestrate the jump among the system modes, the dwelling times or switching moments are evaluated by the average dwell-time switching rule, where it is ensured that the averaged dwell times of all modes should be sufficiently large.

Finally, some numerical examples with simulations are presented to clarify the theoretical results.

Chapter 3

Switched Systems

The main contribution of this part is designing a robust reliable H_∞ control that guarantees exponential input-to-state stabilization of uncertain hybrid systems. The system under study is a switched system that has a time-varying, norm-bounded uncertainty in the system state, a nonlinear term that is linearly bounded, and a disturbance that belongs to $L_2[0, \infty)$. Two cases of the control actuators have been considered which are operational actuators and faulty actuators. In the latter case, the output is treated as a disturbance signal that is augmented with the system disturbance. In addition, multiple Lyapunov functions, which lead to solving Riccati-like equations, and the average dwell-time condition are used to provide sufficient conditions to guarantee ISS property of the system. An illustrative numerical example with simulation is presented to show both cases. The material of this chapter forms the basis of [10].

3.1 Problem Formulation and Preliminaries

$L_2[t_0, \infty)$ denotes the space of square integrable vector-valued functions on $[t_0, \infty)$ and $\|\cdot\|_2$ denotes $L_2[t_0, \infty)$-norm (i.e., $w \in L_2[t_0, \infty)$ means $\|w\|_2^2 = \int_{t_0}^{\infty} \|w(t)\|^2 \, dt < \infty$).

Consider a class of uncertain switched systems given by

$$\begin{cases} \dot{x} = (A_{\varrho(t)} + \Delta A_{\varrho(t)})x + B_{\varrho(t)}u + G_{\varrho(t)}w + f_{\varrho(t)}(x), \\ z = C_{\varrho(t)}x + F_{\varrho(t)}u, \\ x(t_0) = x_0, \end{cases} \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^q$ is the control input, $w \in \mathbb{R}^p$ is an input disturbance, which is assumed to be in $L_2[t_0, \infty)$, and $z \in \mathbb{R}^r$ is the controlled output. ϱ is the switching law which is a piecewise constant function defined by $\varrho : [t_0, \infty) \to \mathcal{S} = \{1, 2, \cdots, N\}$. The role of ϱ is to switch among the system modes. For each $i \in \mathcal{S}$, A_i is a non Hurwitz matrix, $K_i \in \mathbb{R}^{q \times n}$ is the control gain matrix such that $u = K_i x$, where (A_i, B_i) is assumed to be stabilizable, $f_i(x) \in \mathbb{R}^n$ is some nonlinearity, A_i, B_i, G_i, C_i and F_i are known real constant matrices, and ΔA_i is a piecewise continuous function representing parameter uncertainty with bounded norm. For any $i \in \mathcal{S}$, the closed-loop system is

$$\begin{cases} \dot{x} = (A_i + \Delta A_i + B_i K_i)x + G_i w + f_i(x), \\ z = C_{ic}x, \\ x(t_0) = x_0, \end{cases} \tag{3.2}$$

where $C_{ic} = C_i + F_i K_i$.

As mentioned in Chapter 2, for the reliability analysis, we have

$$B_i = B_{i\sigma} + B_{i\bar{\sigma}}.$$

Furthermore, the augmented disturbance input to the system becomes

$$w_\sigma^F = (w^T \ (u_\sigma^F)^T)^T,$$

where $u_\sigma^F \in \mathbb{R}^q$ is the failure vector whose elements corresponding to the set of faulty actuators σ, and F here stands for "failure". Since the control input u is applied to the system through the normal actuators, and the outputs of the faulty actuators are assumed to be arbitrary signals, the closed-loop system becomes

$$\begin{cases} \dot{x} = (A_i + \Delta A_i + B_{i\bar\sigma} K_i)x + G_{ic}w_\sigma^F + f_i(x), & i \in \mathcal{S} = \{1, 2, ..., N\}, \\ z = C_{ic}x, \\ x(t_0) = x_0, \end{cases} \tag{3.3}$$

where $G_{ic} = (G_i \ B_{i\sigma})$. In the following, we define the concept of ISS.

Definition 3.1.1 (Exponential Input-to-State Stability). *System* (3.2) *is said to be robustly globally exponentially ISS if there exist positive constants* λ, $\overline{\lambda}$ *and a function* $\rho \in \mathcal{K}$ *such that*

$$\|x\| \le \overline{\lambda}\|x_0\|e^{-\lambda(t-t_0)} + \rho(\sup_{t_0 \le \tau \le t} \|w(\tau)\|), \quad \forall\, t \ge t_0,$$

for any solution $x(t) = x(t, t_0, x_0)$ *of* (3.2).

Definition 3.1.2 (input-to-state stability with an H_∞- norm (ISS-H_∞)). *Given a constant* $\gamma > 0$, *system* (3.2) *is said to be ISS-H_∞ if there exists a state feedback law* $u(t) = K_i x(t)$, *such that, for any admissible parameter uncertainties* ΔA_i, *the closed loop system* (3.2) *is globally exponentially ISS, and the controlled output* z *satisfies*

$$\|z\|_2^2 = \int_{t_0}^{\infty} \|z\|^2\, dt \le \gamma^2 \|w\|_2^2 + m_0,$$

65

for some positive constant m_0.

Lemma 3.1.3. *[6] For any arbitrary positive constants ξ_i, $i = 1, \cdots, 6$, and a positive definite matrix P, we have*

(i) $2x^T P(\Delta A)x \le x^T(\xi_1 PDD^T P + \frac{1}{\xi_1}H^T H)x.$

(ii) $2x^T PGw \le x^T(\xi_2 PGG^T P)x + \frac{1}{\xi_2}w^T w.$

(iii) $2x^T Pf(x) \le x^T(\xi_3 P^2 + \frac{1}{\xi_3}\delta I)x$ *such that* $\|f(x)\|^2 \le \delta\|x\|^2$ *with* $\delta > 0$.

   Moreover, if $x(t - r) \in \mathcal{C}_r$, $\|x(t-r)\|_r^2 \le q\|x\|^2$ *with* $q > 1$, *then*

(iv) $2x^T P\bar{A}x(t - r) \le x^T(\xi_4 P\bar{A}(\bar{A})^T P + \frac{q}{\xi_4}I)x.$

(v) $2x^T P(\Delta\bar{A})x(t - r) \le x^T(\xi_5 P\bar{D}\bar{D}^T P + \frac{q}{\xi_5}\|\bar{H}\|^2)x.$

(vi) $2x^T Pf(x(t - r)) \le x^T(\xi_6 P^2 + \frac{1}{\xi_6}\delta qI)x$, *where* $\delta > 0$ *such that* $\|f(x(t-r))\|^2 \le \delta\|x(t-r)\|_r^2.$

3.2   Main Results

In this section, we present and prove two theorems. The first theorem discusses the robust reliable H_∞ controller for system (3.2) to guarantee the globally exponentially ISS when all the actuators are operational while the second theorem deals with the faulty actuator case, namely, system (3.3).

Theorem 3.2.1. *Let the controller gain K_i and the constant $\gamma_i > 0$ be given, and assume that Assumption A holds. Then, the switched control system* (3.2) *is robustly globally exponentially ISS with an H_∞-norm bound γ if the average dwell-time condition holds, and there exist positive constants ξ_{1i}, ξ_{2i}, ξ_{3i}, and a positive definite matrix P_i satisfying the Riccati-like equation*

$$(A_i + B_i K_i)^T P_i + P_i(A_i + B_i K_i) + \xi_{1i} P_i D_i D_i^T P_i + \frac{1}{\xi_{1i}} H_i^T H_i + C_{ic}^T C_{ic} + \xi_{2i} P_i G_i G_i^T P_i$$

$$+ \xi_{3i} P_i^2 + \frac{1}{\xi_{3i}} \delta_i I + \alpha_i P_i = 0, \tag{3.4}$$

where δ_i is a positive constant such that

$$\|f_i(x)\|^2 \leq \delta_i \|x\|^2. \tag{3.5}$$

Proof. Let $x(t) = x(t, t_0, x_0)$ be the solution of system (3.2). For any $i \in \mathcal{S}$, define $V_i(x) = x^T P_i x$ as a Lyapunov function candidate for the *ith* mode. Then, the derivative of $V_i(x)$ along the trajectory of (3.2) is
$$\dot{V}_i(x) = \dot{x}^T P_i x + x^T P_i \dot{x}$$

$$= [(A_i + \Delta A_i + B_i K_i)x + G_i w + f_i(x)]^T P_i x + x^T P_i [(A_i + \Delta A_i + B_i K_i)x + G_i w + f_i(x)]$$

$$= x^T [(A_i + B_i K_i)^T P_i + P_i(A_i + B_i K_i)]x + 2x^T P_i(\Delta A_i)x + 2x^T P_i G_i w + 2x^T P_i f_i(x)$$

$$\leq x^T [(A_i + B_i K_i)^T P_i + P_i(A_i + B_i K_i) + \xi_{1i} P_i D_i D_i^T P_i + \xi_{2i} P_i G_i G_i^T P_i + \frac{1}{\xi_{1i}} H_i^T H_i$$

$$+ \xi_{3i} P_i^2 + \frac{1}{\xi_{3i}} \delta_i I]x + \frac{1}{\xi_{2i}} w^T w$$

$$\leq -\alpha_i V_i(x) + \frac{1}{\xi_{2i}} w^T w,$$

where we used (3.5) and Lemma 3.1.3 in the second bottom line, and condition (3.4) in the last line. Hence, for each subinterval $[t_{k-1}, t_k)$ we have

$$
\begin{aligned}
\dot{V}_i(x) &\leq -(\alpha_i - \theta_i)V_i(x) - \theta_i V_i(x) + \frac{1}{\xi_{2i}} w^T w \\
&= -\overline{\alpha}_i V_i(x) - \theta_i V_i(x) + \frac{1}{\xi_{2i}} w^T w,
\end{aligned}
$$

where

$$
\overline{\alpha}_i = \alpha_i - \theta_i \qquad \text{and} \quad 0 < \theta_i < \alpha_i.
$$

The foregoing inequality implies that

$$
\dot{V}_i(x) \leq -\overline{\alpha}_i V_i(x), \quad \text{for all } t \in [t_{k-1}, t_k),
$$

provided that

$$
V_i(x) > \frac{1}{\theta_i \xi_{2i}} \|w\|^2, \tag{3.6}
$$

by (2.5),

$$
\|x\| > \frac{\|w\|}{\sqrt{\theta_i c_2 \xi_{2i}}} =: \rho_i(\|w\|).
$$

Then, for all $t \in [t_{k-1}, t_k)$,

$$
V_i(x(t)) \leq V_i(x(t_{k-1}))e^{-\overline{\alpha}_i(t - t_{k-1})} \qquad \text{provided that } \|x\| > \rho(\|w\|), \tag{3.7}
$$

where

$$
\rho(\|w\|) = \max_{i \in \mathcal{S}}\{\rho_i(\|w\|)\}.
$$

68

From (2.5), we have for any $i, j \in \mathcal{S}$

$$V_j(x(t)) \le \mu V_i(x(t)), \quad \mu = \frac{c_2}{c_1}, \tag{3.8}$$

where $c_1 = \min_{i \in \mathcal{S}} \{\lambda_{min}(P_i)\}$ and $c_2 = \max_{i \in \mathcal{S}} \{\lambda_{max}(P_i)\}$.

Activating modes 1 and 2 on the first and second intervals, respectively, we have

$$V_1(x(t)) \le V_1(x_0)e^{-\overline{\alpha}_1(t-t_0)}, \ t \in [t_0, t_1) \ \text{ provided that } \ \|x\| > \rho(\|w\|)$$

and

$$\begin{aligned}
V_2(x(t)) &\le V_2(x(t_1))e^{-\overline{\alpha}_2(t-t_1)}, \ t \in [t_1, t_2) \ \text{ provided that } \ \|x\| > \rho(\|w\|) \\
&\le \mu V_1(x(t_1))e^{-\overline{\alpha}_2(t-t_1)} \ \text{ provided that } \ \|x\| > \rho(\|w\|) \\
&\le \mu e^{-\overline{\alpha}_2(t-t_1)}e^{-\overline{\alpha}_1(t_1-t_0)}V_1(x_0) \ \text{ provided that } \ \|x\| > \rho(\|w\|).
\end{aligned}$$

Generally, for $i \in \mathcal{S}$ and $t \in [t_{k-1}, t_k)$, we have

$$V_i(x(t)) \le \mu^{k-1} e^{-\overline{\alpha}_i(t-t_{k-1})}e^{-\overline{\alpha}_{i-1}(t_{k-1}-t_{k-2})} \cdots e^{-\overline{\alpha}_1(t_1-t_0)}V_1(x_0)$$

provided that $\|x\| > \rho(\|w\|)$.

Letting $\alpha^* = \min\{\overline{\alpha}_i; \ i \in \mathcal{S}\}$, one may get

$$\begin{aligned}
V_i(x(t)) &\le \mu^{k-1} e^{-\alpha^*(t-t_0)}V_1(x_0) \\
&= e^{(k-1)\ln\mu - \alpha^*(t-t_0)}V_1(x_0) \ \text{ provided that } \ \|x\| > \rho(\|w\|).
\end{aligned}$$

Using the average dwell-time condition with $N_0 = \frac{\eta}{\ln\mu}$, $\tau_a = \frac{\ln\mu}{\alpha^* - \nu}$, $(\nu < \alpha^*)$, for some arbitrary positive constant η, we get

$$V_i(x(t)) \le e^{\eta - \nu(t-t_0)}V_1(x_0) \qquad \text{provided that } \ \|x\| > \rho(\|w\|).$$

69

This implies by Theorem 2.1.12 that

$$\|x\| \le b\|x_0\|e^{-\nu(t-t_0)/2} + \gamma(\sup_{t_0 \le \tau \le t} \|w(\tau)\|), \quad t \ge t_0,$$

where $b = \sqrt{e^\eta c_2 / c_1}$, and $\gamma(s) = \sqrt{\frac{c_2}{c_1}}\rho(s)$. This completes the proof of exponential ISS.

To prove the upper bound on the output magnitude $\|z\|$, for any $i \in \mathcal{S}$, we introduce the performance function

$$J_i = \int_{t_0}^{\infty} (z^T z - \gamma_i^2 w^T w)dt.$$

Then,

$$J_i = \int_{t_0}^{\infty} (z^T z - \gamma_i^2 w^T w)\,dt + \int_{t_0}^{\infty} \dot{V}_i\,dt - V_i(\infty) + V_i(x_0)$$

$$\le \int_{t_0}^{\infty} (z^T z - \gamma_i^2 w^T w)\,dt + \int_{t_0}^{\infty} \dot{V}_i\,dt + V_i(x_0)$$

$$\le \int_{t_0}^{\infty} (z^T z - \gamma_i^2 w^T w)\,dt + V_i(x_0) + \int_{t_0}^{\infty} \{x^T[(A_i + B_iK_i)^T P_i + P_i(A_i + B_iK_i)$$

$$+ \xi_{1i}P_iD_iD_i^T P_i + \frac{1}{\xi_{1i}}H_i^T H_i + \xi_{3i}P_i^2 + \frac{1}{\xi_{3i}}\delta_i I - \gamma_i^{-2}P_iG_iG_i^T P_i + \gamma_i^{-2}P_iG_iG_i^T P_i]x$$

$$+ 2x^T P_iG_iw\}\,dt$$

$$= V_i(x_0) + \int_{t_0}^{\infty} \{x^T[(A_i + B_iK_i)^T P_i + P_i(A_i + B_iK_i) + \xi_{1i}P_iD_iD_i^T P_i + \frac{1}{\xi_{1i}}H_i^T H_i$$

$$+ \xi_{3i}P_i^2 + \frac{1}{\xi_{3i}}\delta_i I + \gamma_i^{-2}P_iG_iG_i^T P_i + C_{ic}^T C_{ic}]x\}\,dt$$

$$- \int_{t_0}^{\infty} \gamma_i^2(w - \gamma_i^{-2}G_i^T P_ix)^T(w - \gamma_i^{-2}G_i^T P_ix)\,dt.$$

70

The last term is strictly negative, so, using condition (3.4) with $\gamma_i^{-2} = \xi_{2i}$, we get

$$J_i \leq V_i(x_0) - \alpha_i P_i \leq V_i(x_0)$$

which leads to

$$\|z\|_2^2 \leq \gamma^2 \|w\|_2^2 + m_0,$$

where $m_0 = \max_{i \in \mathcal{S}} \{V_i(x_0)\}$, and $\gamma = \max_{i \in \mathcal{S}} \{\gamma_i\}$. $\qquad\square$

Remark 3.2.2. *Theorem 3.2.1 provides sufficient conditions to ensure robust global exponential ISS of uncertain switched systems with norm-bounded uncertainty in the system state. The algebraic Riccati-like equation given in (3.4) is to guarantee the existence of the positive-definite matrix P_i (for all $i \in \mathcal{S}$), which implies that the solution trajectories of the subsystems are decreasing outside a certain neighbourhood of the disturbance $w(t)$. The role of average dwell time condition is to organize the switching among the system modes which eventually guarantees the exponential ISS. Condition (3.5) is made to ensure that nonlinear perturbation f is bounded by a linear growth bound. The positive tuning parameters ξ_1, ξ_2 are presented to reduce the conservativeness of the Riccati equation.*

Theorem 3.2.3 (Reliability)**.** *Let the constant $\gamma_i > 0$ be given. Assume that Assumption A holds, the switched control system (3.3) is robustly globally exponentially ISS-H_∞ if the average dwell-time condition holds, the controller gain $K_i = -\frac{1}{2}\epsilon_i B_{i\bar{\sigma}}^T P_i$, for some constants $\epsilon_i > 0$, and positive definite matrix P_i, and there exist positive constants ξ_{1i}, ξ_{2i}, ξ_{3i}, ϵ_i, and*

a positive definite matrix P_i satisfying the Riccati-like equation

$$A_i^T P_i + P_i A_i + P_i(\xi_{1i} D_i D_i^T + \xi_{2i} G_{ic} G_{ic}^T - \epsilon_i B_{i\overline{\Sigma}} B_{i\overline{\Sigma}}^T + \xi_{3i} I) P_i + \frac{1}{\xi_{1i}} H_i^T H_i + C_{ic}^T C_{ic}$$

$$+ \frac{1}{\xi_{3i}} \delta_i I + \alpha_i P_i = 0, \tag{3.9}$$

where δ_i is a positive constant such that

$$\|f_i(x)\|^2 \leq \delta_i \|x\|^2. \tag{3.10}$$

Proof. Let $x(t) = x(t, t_0, x_0)$ be the solution of system (3.3). For any $i \in \mathcal{S}$, define $V_i(x) = x^T P_i x$ as a Lyapunov function candidate for the *ith* mode. Then, the derivative of $V_i(x)$ along the trajectory of (3.3) is

$$\dot{V}_i(x) = x^T [A_i^T P_i + P_i A_i + 2P_i(\Delta A_i) + (B_{i\overline{\sigma}} K_i)^T P_i + P_i B_{i\overline{\sigma}} K_i] x + 2x^T P_i G_{ic} w_\sigma^F + 2x^T P_i f_i(x)$$

$$= x^T [A_i^T P_i + P_i A_i + 2P_i(\Delta A_i) - \epsilon_i P_i(B_{i\overline{\sigma}})(B_{i\overline{\sigma}})^T P_i] x + 2x^T P_i G_{ic} w_\sigma^F + 2x^T P_i f_i(x)$$

$$\leq x^T [A_i^T P_i + P_i A_i + \xi_{1i} P_i D_i D_i^T P_i + \xi_{2i} P_i G_{ic} G_{ic}^T P_i + \frac{1}{\xi_{1i}} H_i^T H_i + \xi_{3i} P_i^2 + \frac{1}{\xi_{3i}} \delta_i I$$

$$- \epsilon_i P_i(B_{i\overline{\sigma}})(B_{i\overline{\sigma}})^T P_i] x + \frac{1}{\xi_{2i}} (w_\sigma^F)^T w_\sigma^F$$

$$\leq x^T [A_i^T P_i + P_i A_i + P_i(\xi_{1i} D_i D_i^T + \xi_{2i} G_{ic} G_{ic}^T - \epsilon_i B_{i\overline{\Sigma}} B_{i\overline{\Sigma}}^T + \xi_{3i} I) P_i + \frac{1}{\xi_{1i}} H_i^T H_i$$

$$+ \frac{1}{\xi_{3i}} \delta_i I] x + \frac{1}{\xi_{2i}} (w_\sigma^F)^T w_\sigma^F$$

$$= -\alpha_i V_i(x) + \frac{1}{\xi_{2i}} (w_\sigma^F)^T w_\sigma^F,$$

where we used (3.10) and Lemma 3.1.3 in the third bottom line, the fact that [143]

$$B_{i\overline{\Sigma}}(B_{i\overline{\Sigma}})^T \leq B_{i\overline{\sigma}}(B_{i\overline{\sigma}})^T,$$

and condition (3.4) in the last line. Then, for all $t \in [t_{k-1}, t_k)$, we have

$$\dot{V}_i(x) \leq -\overline{\alpha}_i V_i(x) - \theta_i V_i(x) + \frac{1}{\xi_{2i}}(w_\sigma^F)^T w_\sigma^F,$$

where $\overline{\alpha}_i = \alpha_i - \theta_i$ and $0 < \theta_i < \alpha_i$. This implies that

$$\dot{V}_i(x) \leq -\overline{\alpha}_i V_i(x), \quad \text{for all } t \in [t_{k-1}, t_k)$$

provided that

$$\|x\| > \frac{\|w_\sigma^F\|}{\sqrt{\theta_i c_2 \xi_{2i}}} =: \rho_i(\|w_\sigma^F\|).$$

As done in Theorem 3.2.1, one may get

$$V_i(x(t)) \leq e^{\eta - \nu(t-t_0)} V_1(x_0) \quad \text{provided that} \quad \|x\| > \rho(\|w\|),$$

where $\rho(\|w\|) = \max_{i \in \mathcal{S}}\{\rho_i(\|w\|)\}$. This also implies that

$$\|x\| \leq b\|x_0\|e^{-\nu(t-t_0)} + \gamma(\sup_{t_0 \leq \tau \leq t} \|w_\sigma^F(\tau)\|), \quad t \geq t_0,$$

where $b = \sqrt{e^\eta c_2/c_1}$, $\gamma(s) = \sqrt{\frac{c_2}{c_1}}\rho(s)$. This completes the proof of exponential ISS.

As for the upper bound on $\|z\|$, one may follow the same steps in Theorem 3.2.1, where

$$J_i = \int_{t_0}^\infty (z^T z - \gamma_i^2(w_\sigma^F)^T w_\sigma^F)dt,$$

73

to obtain

$$\|z\|_2^2 \leq \gamma^2 \|w_\sigma^F\|_2^2 + m_0,$$

where $m_0 = \max_{i \in \mathcal{S}}\{V_i(x_0)\}$, and $\gamma = \max_{i \in \mathcal{S}}\{\gamma_i\}$, which completes the proof. $\square$

Example 3.2.4. *Consider system* (3.2) *where* $\mathcal{S} = \{1, 2\}$,

$$A_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0 & -6 \end{bmatrix}, B_1 = \begin{bmatrix} -7 & 1 \\ 0.1 & 0.2 \end{bmatrix}, C_1 = \begin{bmatrix} 2 & 0.1 \\ 0 & 2 \end{bmatrix}, F_1 = \begin{bmatrix} 0.1 & -2 \\ 0.1 & 0 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, H_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}, G_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, f_1 = 0.01 \begin{bmatrix} \sin(x_1) \\ \sin(x_2) \end{bmatrix}, \mathcal{U}_1 = \sin(t),$$

$\epsilon_1 = 2$, $\xi_{11} = 0.2$, $\gamma_1 = 0.1$, $\alpha_1 = 2$, $\xi_{21} = \gamma_1^{-2}$, $\xi_{31} = 1$, *and* $\theta_1 = 1$ *with* $t_0 = 0$.

From (3.5) *one may get* $\delta_1 = 0.01$. *As for the second mode, we take*

$$A_2 = \begin{bmatrix} -9 & 0.2 \\ 0 & 0.1 \end{bmatrix}, B_2 = \begin{bmatrix} 0.1 & 0.5 \\ 0.1 & -8 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, F_2 = \begin{bmatrix} 0.1 & 0 \\ -3 & 0.1 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, H_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, G_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, f_2 = 0.01 \begin{bmatrix} \sin(x_1) \\ \sin(x_2) \end{bmatrix}, \mathcal{U}_2 = \sin(t),$$

$\epsilon_2 = 0.5$, $\xi_{12} = 0.3$, $\gamma_2 = 0.15$, $\alpha_2 = 2.5$, $\xi_{22} = \gamma_2^{-2}$, $\xi_{32} = 1$, *and* $\theta_2 = 1.5$.

From (3.5), *one may get that* $\delta_2 = 0.01$. *Let the system input disturbance be defined by*

$$w(t) = \begin{bmatrix} \sin(t) \\ \sin(t) \end{bmatrix}.$$

Case 1. *[All the actuators are operational]*

When all the control actuators are operational, we have from Riccati-like equation,

$$P_1 = \begin{bmatrix} 1.6437 & 0.0149 \\ 0.0149 & 0.2499 \end{bmatrix}, P_2 = \begin{bmatrix} 0.1633 & 0.0859 \\ 0.0859 & 0.2724 \end{bmatrix},$$

with $c_{11} = \lambda_{\min}(P_1) = 0.2498$, $c_{12} = \lambda_{\max}(P_1) = 1.6439$, $c_{21} = \lambda_{\min}(P_2) = 0.1161$, $c_{22} = \lambda_{\max}(P_2) = 0.3197$, so, $c_1 = 0.1161$, $c_2 = 1.6439$, and the control gain matrices are

$$K_1 = \begin{bmatrix} 11.5047 & 0.0796 \\ -1.6467 & -0.0649 \end{bmatrix}, K_2 = \begin{bmatrix} -0.0062 & -0.0090 \\ 0.1514 & 0.5342 \end{bmatrix}.$$
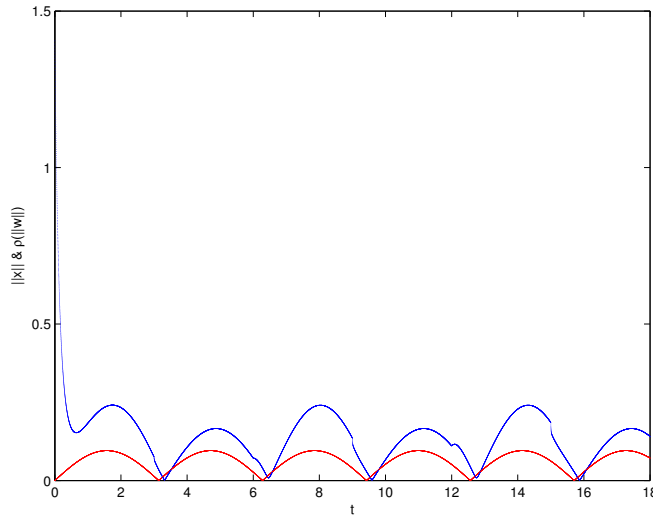

Figure 3.2.1: Input-to-state stabilization: Operational actuators.

Thus, the matrices

$$A_1 + B_1 K_1 = \begin{bmatrix} -81.9796 & -0.5220 \\ 0.8211 & -6.0050 \end{bmatrix}, \text{ and } A_2 + B_2 K_2 = \begin{bmatrix} -8.9249 & 0.4662 \\ -1.2121 & -4.1741 \end{bmatrix}$$

are Hurwitz. The average dwell time is $\tau_a = \frac{\ln \mu}{\alpha^* - \nu} = 2.7898$, with $\nu = 0.05$. Figure 3.2.1

shows the simulation results of $\|x\|$ (top) and $\gamma(\|w\|) = \sqrt{c_2/c_1}\rho(\|w\|)$ (bottom), where

$\rho(s) = \max\{\rho_1(s), \rho_2(s)\}$ *and* $\rho_i(s) = s/\sqrt{c_2\theta_i\xi_{2i}}$, *and* $\tau_a = 3$.

Case 2. *[Failure in the second actuator in the first mode and first actuator in the second mode]*

When there is a failure in the second actuator, i.e., $B_{1\Sigma} = \{2\}$ and $B_{1\overline{\Sigma}} = \begin{bmatrix} -7 & 0 \\ 0.1 & 0 \end{bmatrix}$,

and $B_{2\Sigma} = \{1\}$ and $B_{2\overline{\Sigma}} = \begin{bmatrix} 0 & 0.5 \\ 0 & -8 \end{bmatrix}$, we have from Riccati-like equation,

$$P_1 = \begin{bmatrix} 1.1265 & -0.1913 \\ -0.1913 & 0.3110 \end{bmatrix}, P_2 = \begin{bmatrix} 0.1676 & 0.0980 \\ 0.0980 & 0.2436 \end{bmatrix},$$

with $c_{11} = \lambda_{\min}(P_1) = 0.2683$, $c_{12} = \lambda_{\max}(P_1) = 1.1691$, $c_{21} = 0.1005$, $c_{22} = 0.3107$, so

$c_1 = 0.1005$, $c_2 = 1.1691$, and the control gain matrices

$$K_1 = \begin{bmatrix} 7.9046 & -1.3703 \\ 0 & 0 \end{bmatrix}, K_2 = \begin{bmatrix} 0 & 0 \\ 0.1751 & 0.4750 \end{bmatrix}.$$

Thus, the matrices

$$A_1 + B_1 K_1 = \begin{bmatrix} -55.1320 & 9.6920 \\ 0.7905 & -6.1370 \end{bmatrix}, \text{ and } A_2 + B_2 K_2 = \begin{bmatrix} -8.9125 & 0.4375 \\ -1.4006 & -3.7000 \end{bmatrix}$$

are Hurwitz, and $\tau_a = 2.5834$.
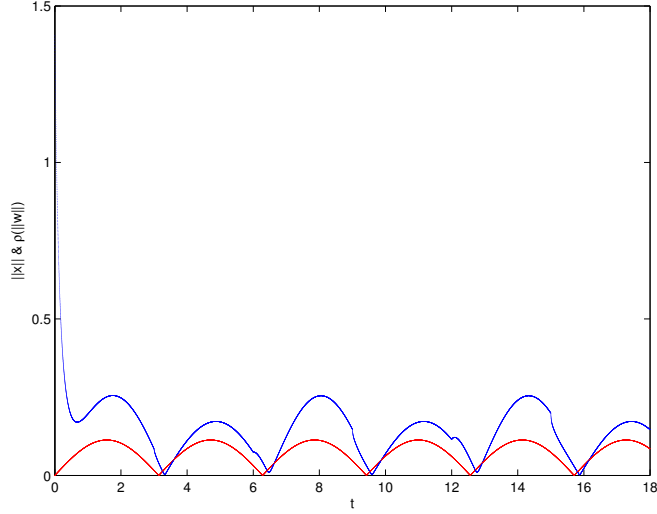
Figure 3.2.2: Input-to-state stabilization: Faulty in the second actuator.

Figure *3.2.2* shows the simulation results of $\|x\|$ (top) and $\gamma(\|w\|) = \sqrt{c_2/c_1}\rho(\|w\|)$ (bottom), where $\rho(s) = \max\{\rho_1(s),\,\rho_2(s)\}$ and $\rho_i(s) = s/\sqrt{c_2\theta_i\xi_{2i}}$, $\tau_a = 3$.

If we consider the system disturbance input

$$w(t) = \left[\begin{array}{c} e^{-0.2t}\sin(t) \\ e^{-0.2t}\sin(t) \end{array}\right],$$

we get the same result, and this shows that the system state is decaying following the decayed disturbance. The simulation results of $\|x\|$ (top) and $\gamma(\|w\|) = \sqrt{c_2/c_1}\rho(\|w\|)$ (bottom) are shown in Figures *3.2.3* and *3.2.4*.
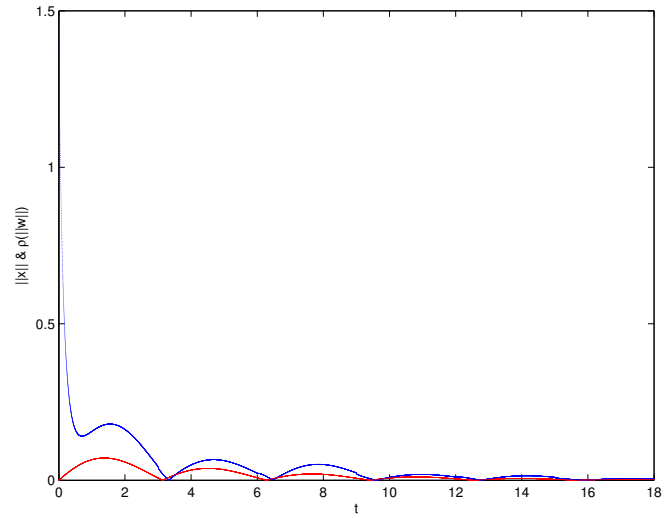
Figure 3.2.3: Input-to-state stabilization with a decaying disturbance: Operational case.
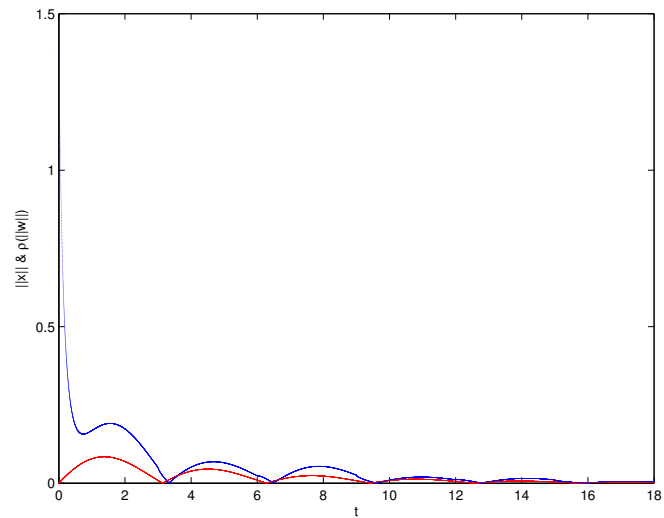
Figure 3.2.4: Input-to-state stabilization with a decaying disturbance: Faulty actuators.

78

3.3 Conclusion

This chapter has addressed the problem of designing a robust reliable H_∞ controller that guaranteed the global exponential ISS to uncertain switched systems. We have considered a time-varying parameter uncertainty in the system state, an L_2 norm-bounded input disturbance, and a linearly bounded nonlinear term. The output of the faulty actuators has been treated as a disturbing signal that has been augmented with the system disturbance. We have shown that, using the average dwell-time to organize the switching among the system modes, and multiple Lyapunov functions, the switched system is exponentially input-to-state stabilizable, when every individual mode is exponentially input-to-state stabilized by a reliable feedback controller so long as the average dwell-time is sufficiently large.

Chapter 4

Switched Systems with Time Delay

In this chapter, time-delayed switched systems that is subject to external disturbance are considered. The main focus is to establish the problem of ISS of the system, which is analyzed by using Lyapunov-Razumikhin approach. The jump among the system modes follow the average dwell-time switching law. Some numerical examples are considered to illustrate the results of this Chapter. The contents of this chapter forms the basis of [156].

4.1   Problem Formulation and Preliminaries

Consider the following switched system

$$
\begin{cases}
\dot{x} = f_{\varrho(t)}(x_t, w(t)), \\
x_{t_0}(s) = \phi(s), \quad s \in [-r, 0], \quad r > 0,
\end{cases}
\tag{4.1}
$$

where $x \in \mathbb{R}^n$ is the system state, and $w \in \mathbb{R}^p$ is an input disturbance, which is assumed to be in $L_2[t_0, \infty)$. For $r > 0$, let $\mathcal{C}_r$ be the space of all continuous functions that are defined

from $[-r, 0]$ to $\mathbb{R}^n$. For any $t \in \mathbb{R}_+$, let $x(t)$ be a function defined on $[t_0, \infty]$. Then, we define the function $x_t : [-r, 0] \to \mathbb{R}^n$ by $x_t(s) = x(t+s)$ for all $s \in [-r, 0]$, and its norm by $||x_t||_r = \sup_{t-r \leq \theta \leq t} ||x(\theta)||$, where $r > 0$ is the time delay. ϱ is the switching rule which is a piecewise constant function defined by $\varrho : [t_0, \infty) \to \mathcal{S} = \{1, 2, \cdots, N\}$. So system (4.1) can be expressed as follows

$$\begin{cases} \dot{x} = f_i(x_t, w(t)), & i \in \mathcal{S} \\ x_{t_0}(s) = \phi(s), & s \in [-r, 0], \quad r > 0, \end{cases} \tag{4.2}$$

Definition 4.1.1. *System (4.2) is said to be globally exponentially ISS if there exist $\lambda > 0$, $\bar{\lambda} > 0$ and a function $\gamma \in \mathcal{K}$ such that the solution $x(t)$ exists $\forall t \geq t_0$ and satisfies*

$$||x|| \leq \bar{\lambda} ||\phi||_r e^{-\lambda(t-t_0)} + \gamma \big(\sup_{t_0 \leq \tau \leq t} ||w(\tau)|| \big).$$

4.2 Main Results

In this section, we shall state and prove our main results. The following theorem gives sufficient conditions of global exponential ISS property of the system.

Theorem 4.2.1. *For any $i \in \mathcal{S}$, let K_i and a differentiable class $\mathcal{K}$ function γ be given. Assume that there exist positive constants c_1, c_2, r, β, and a continuously differentiable function $V_i : \mathbb{R}^n \to \mathbb{R}_+$ such that*

(i) $c_1 ||x||^2 \leq V_i(x) \leq c_2 ||x||^2$ *for all $t \geq t_0 - r$;*

81

(ii) $\dot{V}_i(\psi(0)) < -\lambda V_i(\psi(0))$ *whenever* $V_i(\psi(s)) \leq q V_i(\psi(0))$, *and* $\gamma(\sup_{t_0 \leq \theta \leq t_k} |w(\theta)|) \leq$

$V_i(\psi(0))$ *for* $\psi \in \mathcal{C}_r$, $s \in [-r, 0]$ *and* $t \in [t_{k-1}, t_k)$, *where* $q = \max\{\mu h, e^{\lambda r}\} > 1$ *and*

$\mu = c_2/c_1$;

(iii) for all k, $r \leq t_k - t_{k-1} \leq \beta$ *and the average dwell time condition holds, and* $\beta > 0$;

(iv) for $s \in [-r, 0]$ *and* $h > 1$, $V_i(x(t + s)) \leq h V_j(x(t))$ *for any* $i, j \in \mathcal{S}$ *and any* $t \geq t_0$.

Then, system (4.2) *is globally exponentially ISS.*

Proof. Let $x(t, t_0, \phi)$ be any solution of system (4.2) with $x_{t_0} = \phi$ and $v_i(t) = V_i(x(t))$. First, we want to show that every mode is globally exponentially ISS using conditions (*i*) and (*ii*). For any $i \in \mathcal{S}$, and $k \in \mathbb{N}$, $t \in [t_{k-1}, t_k)$, we shall show that

$$v_i(t) \leq c_2 ||x_{t_{k-1}}||_r^2 e^{-\lambda(t - t_{k-1})} + \gamma(\sup_{t_0 \leq s \leq t} ||w(s)||). \tag{4.3}$$

Let

$$Q_i(t) = \begin{cases} v_i(t) - c_2 ||x_{t_{k-1}}||_r^2 e^{-\lambda(t - t_{k-1})} - \gamma(\sup_{t_0 \leq s \leq t} ||w(s)||), & t \in [t_{k-1}, t_k), \ k \in \mathbb{N} \\ v_i(t) - c_2 ||x_{t_0}||_r^2 e^{-\lambda(t - t_0)}, & t \in [t_0 - r, t_0). \end{cases}$$

We need to show that $Q_i(t) \leq 0$ for all $t \geq t_0 - r$. For $t \in [t_0 - r, t_0]$, it is clear that $Q_i(t) \leq 0$. By condition (*i*),

$$v_i(t) \leq c_2 ||x||^2$$

$$\leq c_2 ||x_{t_0}||_r^2$$

$$\leq c_2 ||x_{t_0}||_r^2 e^{-\lambda(t - t_0)} \qquad \text{since } -\lambda(t - t_0) > 0 \text{ for } t \in [t_0 - r, t_0]. \tag{4.4}$$

82

So, we have

$$Q_i(t) = v_i(t) - c_2\|x_{t_0}\|_r^2 e^{-\lambda(t-t_0)} \leq 0$$

Step 1, for $t \in [t_0, t_1)$, we need to show

$$Q_i(t) = v_i(t) - c_2\|x_{t_0}\|_r^2 e^{-\lambda(t-t_0)} - \gamma(\sup_{t_0 \leq \theta \leq t_1} \|w(\theta)\|) \leq 0. \tag{4.5}$$

For any $i \in \mathcal{S}$, let $\alpha_i > 0$ be arbitrary, and we show $Q_i(t) \leq \alpha_i$ for $[t_0, t_1)$. If not, then there would exist some $t \in [t_0, t_1)$ so that $Q_i(t) > \alpha_i$. Let

$$t_i^* = \inf\{t \in [t_0, t_1) : Q_i(t) > \alpha_i, \ i \in \mathcal{S}\}.$$

We also have

$$Q_i(t_0) \leq v_i(t_0) - c_2\|x_{t_0}\|_r^2 \leq c_2(\|x(t_0)\|^2 - \|x_{t_0}\|_r^2) \leq 0.$$

Since we have $Q_i(t) \leq 0 < \alpha_i$ for $t \in [t_0 - r, t_0]$, then $t_i^* \in (t_0, t_1)$. Also, since $Q_i(t)$ is continuous on $[t_0, t_1)$, then we have

$$Q_i(t_i^*) = \alpha_i \qquad \text{and } Q_i(t) \leq \alpha_i \ \text{ for } [t_0 - r, t_i^*].$$

Then, we have

$$v_i(t_i^*) = Q_i(t_i^*) + c_2\|x_{t_0}\|_r^2 e^{-\lambda(t_i^*-t_0)} + \gamma(\sup_{t_0 \leq \theta \leq t_i^*} \|w(\theta)\|) \tag{4.6}$$

and for $s \in [-r, 0]$, we have

$$v_i(t_i^* + s) = Q_i(t_i^* + s) + c_2\|x_{t_0}\|_r^2 e^{-\lambda(t_i^*+s-t_0)} + \gamma(\sup_{t_0 \leq \theta \leq t_i^*+s} \|w(\theta)\|)$$

$$\leq \alpha_i + c_2\|x_{t_0}\|_r^2 e^{-\lambda(t_i^*-t_0)} e^{\lambda r} + \gamma(\sup_{t_0 \leq \theta \leq t_i^*} \|w(\theta)\|)$$

$$\leq \big[\alpha_i + c_2 \|x_{t_0}\|_r^2 e^{-\lambda(t_i^* - t_0)} + \gamma(\sup_{t_0 \leq \theta \leq t_i^*} \|w(\theta)\|)\big] e^{\lambda r}$$

$$= e^{\lambda r} v_i(t_i^*)$$

$$\leq q v_i(t^*), \tag{4.7}$$

where from (4.6), we use

$$\gamma(\sup_{t_0 \leq \theta \leq t_i^*} \|w(\theta)\|) \leq v_i(t_i^*).$$

Thus, from condition (ii), we have

$$\dot{v}_i(t_i^*) \leq -\lambda v_i(t_i^*)$$

which implies

$$\dot{Q}_i(t_i^*) = \dot{v}_i(t_i^*) + \lambda c_2 \|x_{t_0}\|_r^2 e^{-\lambda(t_i^* - t_0)} - \dot{\gamma}(\sup_{t_0 \leq \theta \leq t_i^*} \|w(\theta)\|)$$

$$\leq -\lambda v_i(t_i^*) + \lambda c_2 \|x_{t_0}\|_r^2 e^{-\lambda(t_i^* - t_0)} - \dot{\gamma}(\sup_{t_0 \leq \theta \leq t_i^*} \|w(\theta)\|)$$

$$\leq -\lambda \big[v_i(t_i^*) - c_2 \|x_{t_0}\|_r^2 e^{-\lambda(t_i^* - t_0)} - \gamma(\sup_{t_0 \leq \theta \leq t_i^*} \|w(\theta)\|)\big]$$

$$= -\lambda \alpha_i. \tag{4.8}$$

Then, $Q_i(t)$ is decreasing at t_i^* which contradicts it being increasing at t^* according to the definition of t^*. Thus, we get $Q_i(t) \leq \alpha_i$ for all $t \in [t_0, t_1)$. Let $\alpha_i \to 0^+$, then we have $Q_i(t) \leq 0$ for $t \in [t_0, t_1)$.

Step 2, for any $i \in \mathcal{S}$ assume $Q_i(t) \leq 0$ for all $t \in [t_{k-1}, t_k)$ for $k = 1, \cdots m$.

$$Q_i(t_m) = v_i(t_m) - c_2 \|x_{t_m}\|_r^2 - \gamma(\sup_{t_0 \leq \theta \leq t_{m+1}} \|w(\theta)\|)$$

$$\leq c_2 \big(\|x(t_m)\|^2 - \|x_{t_m}\|_r^2\big) - \gamma(\sup_{t_0 \leq \theta \leq t_{m+1}} \|w(\theta)\|)$$

$$\leq 0.$$

Step 3, we will show that $Q_i(t) \leq 0$ for all $t \in [t_m, t_{m+1})$, i.e., we need to show that

$$v_i(t) \leq c_2 \|x_{t_m}\|_r^2 e^{-\lambda(t-t_m)} + \gamma(\sup_{t_0 \leq s \leq t} \|w(s)\|).$$

To do so, one needs to prove that $Q_i(t) \leq \alpha_i$ for all $t \in [t_m, t_{m+1})$ and any $i \in \mathcal{S}$. If this were not true, then there would exist some $t \in [t_m, t_{m+1})$ such that for any $i \in \mathcal{S}$ we have $Q_i(t) > \alpha_i$. Let

$$t_i^* = \inf\{t \in [t_m, t_{m+1}) : Q_i(t) > \alpha_i, \ i \in \mathcal{S}\}$$

by the continuity, we have $Q_i(t_i^*) = \alpha_i$ and $Q_i(t) \leq \alpha_i$ for all $t \in [t_m, t_i^*)$, i.e., $\dot{Q}_i(t_i^*) > 0$. Thus, we have

$$v_i(t_i^*) = \alpha_i + c_2 \|x_{t_m}\|_r^2 e^{-\lambda(t_i^* - t_m)} + \gamma(\sup_{t_0 \leq \theta \leq t_i^*} \|w(\theta)\|). \tag{4.9}$$

We want to show $v_i(t_i^* + s) \leq v_i(t_i^*)$ for $s \in [-r, 0]$.

Case 1. If $t_i^* + s \in [t_m, t_{m+1})$, then we have for each $i \in \mathcal{S}$

$$v_i(t_i^* + s) = Q_i(t_i^* + s) + c_2 \|x_{t_m}\|_r^2 e^{-\lambda(t_i^*+s-t_m)} + \gamma(\sup_{t_0 \leq \theta \leq t_i^*+s} \|w(\theta)\|)$$

$$\leq \alpha_i + c_2 \|x_{t_m}\|_r^2 e^{-\lambda(t_i^*-t_m)} e^{\lambda r} + \gamma(\sup_{t_0 \leq \theta \leq t_i^*} \|w(\theta)\|)$$

$$\leq \left[\alpha_i + c_2 \|x_{t_m}\|_r^2 e^{-\lambda(t_i^*-t_m)} + \gamma(\sup_{t_0 \leq \theta \leq t_i^*} \|w(\theta)\|)\right] e^{\lambda r}$$

$$= e^{\lambda r} v_i(t_i^*) \leq q v_i(t_i^*). \tag{4.10}$$

Case 2. If $t_i^* + s \in [t_m - r, t_m)$. Then, since for any $i, j \in \mathcal{S}$ and for any $t \geq t_0 - r$,

$$v_i(t) \leq \mu v_j(t), \quad \mu = \frac{c_2}{c_1} \geq 1. \tag{4.11}$$

Then, we have from the foregoing inequality and condition (iv)

$$v_i(t_i^* + s) \leq \mu v_j(t_i^* + s)$$

$$\leq \mu h v_i(t_i^*)$$

$$\leq q v_i(t_i^*), \tag{4.12}$$

where $q = \max\{e^{\lambda r}, \mu h\}$. Also, from (4.9), we have that

$$\gamma(\sup_{t_0 \leq \theta \leq t_i^*} \|w(\theta)\|) \leq v_i(t_i^*).$$

Thus, from condition (ii), we have

$$\dot{v}_i(t_i^*) \leq -\lambda v_i(t_i^*)$$

86

which implies

$$\dot{Q}_i(t_i^*) = \dot{v}_i(t_i^*) + \lambda c_2 \|x_{t_m}\|_r^2 e^{-\lambda(t_i^* - t_m)} - \dot{\gamma}(\sup_{t_0 \le \theta \le t_i^*} \|w(\theta)\|)$$

$$\le -\lambda v_i(t_i^*) + \lambda c_2 \|x_{t_m}\|_r^2 e^{-\lambda(t_i^* - t_m)} - \dot{\gamma}(\sup_{t_0 \le \theta \le t_i^*} \|w(\theta)\|)$$

$$\le -\lambda \big[v_i(t_i^*) - c_2 \|x_{t_m}\|_r^2 e^{-\lambda(t_i^* - t_m)} - \gamma(\sup_{t_0 \le \theta \le t_i^*} \|w(\theta)\|) \big]$$

$$= -\lambda \alpha_i. \tag{4.13}$$

Then, $Q_i(t)$ is decreasing at t_i^* which contradicts it being increasing at t_i^* according to the definition of t_i^*. Thus, we get $Q_i(t) \le \alpha_i$ for all $t \in [t_m, t_{m+1})$. Let $\alpha_i \to 0^+$, then we have $Q_i(t) \le 0$ for $t \in [t_m, t_{m+1})$. By induction, we have $Q_i(t) \le 0$ for all $t \ge t_0 - r$. Thus, we have for $t \in [t_{k-1}, t_k)$,

$$v_i(t) \le c_2 \|x_{t_{k-1}}\|_r^2 e^{-\lambda(t - t_{k-1})} + \gamma(\sup_{t_0 \le s \le t} \|w(s)\|). \tag{4.14}$$

By condition (i), one can show

$$\|x\| \le \sqrt{\mu} \|x_{t_{k-1}}\|_r e^{-\lambda(t - t_{k-1})/2} + \sqrt{\frac{1}{c_1} \gamma(\sup_{t_0 \le s \le t} \|w(s)\|)}. \tag{4.15}$$

This proves that every mode is globally exponentially ISS. Second, we shall show that the whole switched system is globally exponentially ISS. Since condition (i) is assumed to hold for all $t \ge t_0 - r$, then we have from (4.14)

$$V_i(x(t)) \le \mu V_i(x(t_{k-1} - r)) e^{-\lambda(t - t_{k-1})} + \gamma(\sup_{t_0 \le s \le t} \|w(s)\|), \tag{4.16}$$

87

Activating modes i, j and l on the first, second and third intervals, respectively, we have for $t \in [t_0, t_1)$

$$V_i(x(t)) \leq \mu V_i(x_{t_0})e^{-\lambda(t-t_0)} + \gamma(\sup_{t_0 \leq s \leq t_1} ||w(s)||),$$

and for $t \in [t_1, t_2)$

$$V_j(x(t)) \leq \mu V_j(x_{t_1})e^{-\lambda(t-t_1)} + \gamma(\sup_{t_0 \leq s \leq t_2} ||w(s)||)$$

$$\leq \mu^2 V_i(x_{t_1})e^{-\lambda(t-t_1)} + \gamma(\sup_{t_0 \leq s \leq t_2} ||w(s)||)$$

$$\leq \mu^2 \Big[\frac{c_2}{c_1}V_i(x_{t_0})e^{-\lambda(t_1-r-t_0)} + \gamma(\sup_{t_0 \leq s \leq t_1-r} ||w(s)||)\Big]e^{-\lambda(t-t_1)} + \gamma(\sup_{t_0 \leq s \leq t_2} ||w(s)||)$$

$$\leq \mu^3 e^{\lambda r}V_i(x_{t_0})e^{-\lambda(t_1-t_0)}e^{-\lambda(t-t_1)} + \mu^2\gamma(\sup_{t_0 \leq s \leq t_1} ||w(s)||)e^{-\lambda(t-t_1)}$$

$$+ \gamma(\sup_{t_0 \leq s \leq t_2} ||w(s)||)$$

$$\leq \mu^3 e^{\lambda r}V_i(x_{t_0})e^{-\lambda(t-t_0)} + (\mu^2 + 1)\gamma(\sup_{t_0 \leq s \leq t_2} ||w(s)||),$$

and for $t \in [t_2, t_3)$

$$V_l(x(t)) \leq \mu V_l(x_{t_2})e^{-\lambda(t-t_2)} + \gamma(\sup_{t_0 \leq s \leq t_3} ||w(s)||)$$

$$\leq \mu^2 V_j(x_{t_2})e^{-\lambda(t-t_2)} + \gamma(\sup_{t_0 \leq s \leq t_3} ||w(s)||)$$

$$\leq \mu^2 \Big[\mu^3 e^{\lambda r}V_i(x_{t_0})e^{-\lambda(t_2-r-t_0)} + (\mu^2 + 1)\gamma(\sup_{t_0 \leq s \leq t_2-r} ||w(s)||)\Big]e^{-\lambda(t-t_2)} + \gamma(\sup_{t_0 \leq s \leq t_3} ||w(s)||)$$

88

$$\leq \mu^5 e^{2\lambda r} V_i(x_{t_0}) e^{-\lambda(t_2-t_0)} e^{-\lambda(t-t_2)} + \left((\mu^2)^2 + \mu^2\right) \gamma(\sup_{t_0 \leq s \leq t_2-r} ||w(s)||) e^{-\lambda(t-t_2)}$$

$$+ \gamma(\sup_{t_0 \leq s \leq t_3} ||w(s)||)$$

$$\leq \mu^5 e^{2\lambda r} V_i(x_{t_0}) e^{-\lambda(t-t_0)} + (\mu^4 + \mu^2 + 1) \gamma(\sup_{t_0 \leq s \leq t_3} ||w(s)||).$$

Generally, for $i \in \mathcal{S}$ and $t \in [t_{k-1}, t_k)$, we have

$$V_i(x(t)) \leq \mu^{2k-1} e^{(k-1)\lambda r} e^{-\lambda(t-t_0)} V_i(x_{t_0}) + \left(\sum_{j=0}^{k-1} (\mu^2)^j\right) \gamma(\sup_{t_0 \leq s \leq t_k} ||w(s)||)$$

$$\leq \mu^{2k-1} e^{(k-1)\lambda r} e^{-\lambda(t-t_0)} V_i(x_{t_0}) + k(\mu^2)^{k-1} \gamma(\sup_{t_0 \leq s \leq t_k} ||w(s)||)$$

$$\leq \mu^k (\mu e^{\lambda r})^{k-1} e^{-\lambda(t-t_0)} V_i(x_{t_0}) + k(\mu^2)^{k-1} \gamma(\sup_{t_0 \leq s \leq t_k} ||w(s)||)$$

$$\leq \mu^k (\varrho)^{k-1} e^{-\lambda(t-t_0)} V_i(x_{t_0}) + k(\mu^2)^{k-1} \gamma(\sup_{t_0 \leq s \leq t_k} ||w(s)||)$$

$$\leq (\mu\varrho)^k \varrho^{-1} e^{-\lambda(t-t_0)} V_i(x_{t_0}) + k(\mu^2)^{k-1} \gamma(\sup_{t_0 \leq s \leq t_k} ||w(s)||)$$

$$\leq e^{k \ln(\mu\varrho) - \ln(\varrho) - \lambda(t-t_0)} V_i(x_{t_0}) + \Gamma(t),$$

where $\varrho = \mu e^{\lambda r}$ and $\Gamma(t) = k(\mu^2)^{k-1} \gamma(\sup_{t_0 \leq s \leq t_k} ||w(s)||)$ is class $\mathcal{K}$ function. Using the average dwell-time condition with $N_0 = \frac{\eta}{\ln(\mu\varrho)}$, $\tau_a = \frac{\ln(\mu\varrho)}{\lambda-\nu}$, $(0 < \nu < \lambda)$, for some arbitrary positive constant η, we get

$$V_i(x(t)) \leq e^{\eta + \ln \mu - \nu(t-t_0)} V_i(x_{t_0}) + \Gamma(t)$$

$$\leq D e^{-\nu(t-t_0)} ||x_{t_0}||_r^2 + \Gamma(t)$$

where $D = c_2 \mu e^\eta$. This implies that

$$\|x\| \le b\|x_{t_0}\|_r e^{-\nu(t-t_0)/2} + \bar{\gamma}(t), \quad t \ge t_0,$$

where $b = \mu\sqrt{e^\eta}$, and $\bar{\gamma}(t) = \sqrt{\Gamma(t)/c_1}$ is class $\mathcal{K}$. This completes the proof of exponential ISS. $\qquad\square$

As a special case, consider the following uncertain switched systems with time delay

$$\begin{cases} \dot{x} = (A_{\varrho(t)} + \Delta A_{\varrho(t)})x + (\bar{A}_{\varrho(t)} + \Delta \bar{A}_{\varrho(t)})x(t-r) + B_{\varrho(t)}u + G_{\varrho(t)}w + f_{\varrho(t)}(x(t-r)), \\ z = C_{\varrho(t)}x + F_{\varrho(t)}u, \\ x_{t_0}(s) = \phi(s), \quad s \in [-r, 0], \quad r > 0, \end{cases} \tag{4.17}$$

where $x \in \mathbb{R}^n$ is the system state, , $u \in \mathbb{R}^l$ is the control input, and $w \in \mathbb{R}^p$ is an input disturbance, which is assumed to be in $L_2[t_0, \infty)$ and $z \in \mathbb{R}^r$ is the controlled output. $\varrho : [t_0, \infty) \to \mathcal{S} = \{1, 2, \cdots, N\}$ is the switching rule and $r > 0$ is the time delay. For each $i \in \mathcal{S}$, A_i is a non Hurwitz matrix, $K_i \in \mathbb{R}^{l \times n}$ is the control gain matrix such that $u = K_i x$, where (A_i, B_i) is assumed to be stabilizable, $f_i(\cdot) \in \mathbb{R}^n$ is some nonlinearity, A_i, B_i, G_i, C_i and F_i are known real constant matrices, and ΔA_i, $\Delta \bar{A}_i$ are piecewise continuous functions representing system parameter uncertainties. For any $i \in \mathcal{S}$ the closed-loop system is

$$\begin{cases} \dot{x} = (A_i + \Delta A_i + B_i K_i)x + (\bar{A}_i + \Delta \bar{A}_i)x(t-r) + G_i w + f_i(x(t-r)), \\ z = C_{ic}x, \quad C_{ic} = C_i + F_i K_i \\ x_{t_0}(s) = \phi(s), \quad s \in [-r, 0], \quad r > 0, \end{cases} \tag{4.18}$$

Thus, the closed-loop system in the faulty case becomes

$$\begin{cases} \dot{x} = (A_i + \Delta A_i + B_{i\bar{\sigma}}K_i)x + (\bar{A}_i + \Delta \bar{A}_i)x(t-r) + G_{ic}w_\sigma^F + f_i(x(t-r)), \\ z = C_{ic}x, \\ x_{t_0}(s) = \phi(s), \quad s \in [-r, 0], \quad r > 0, \end{cases} \tag{4.19}$$

90

where $G_{ic} = (G_i \quad B_{i\sigma})$.

Then we have the following results

Corollary 4.2.1. *For any $i \in \mathcal{S}$, let K_i and $\gamma_i > 0$ be given. Assume that Assumption A holds and there exist positive constants ξ_{ji} ($j = 1, 2, 4, 5, 6$), α_i, a positive-definite matrix P_i satisfying*

$$(A_i + B_i K_i)^T P_i + P_i(A_i + B_i K_i) + P_i(\xi_{1i} D_i D_i^T + \xi_{2i} G_i G_i^T + \xi_{4i} \bar{A}_i(\bar{A}_i)^T + \xi_{5i} \bar{D}_i(\bar{D}_i)^T$$

$$+ \xi_{6i}I)P_i + \frac{1}{\xi_{1i}}H_i^T H_i + (\frac{q_i}{\xi_{4i}} + \frac{q_i}{\xi_{5i}}\|\bar{H}_i\|^2 + \frac{\delta_i q_i}{\xi_{6i}})I + C_{ic}^T C_{ic} + \alpha_i P_i = 0 \qquad (4.20)$$

where $\delta_i > 0$ such that $\|f_i(\psi)\|^2 \leq \delta_i \|\psi\|_r^2$. Assume further that $\|w\|^2 \leq \xi_{2i}\alpha_i^* V_i(x)$ with $\alpha_i^* < \alpha_i$ and for all k, $r \leq t_k - t_{k-1} \leq \beta$ where $\beta > 0$, and the average dwell time condition holds. Then, system (4.18) is robustly globally exponentially ISS-H_∞.

Proof. For all $t \in [t_0 - r, \infty)$, let $x(t) = x(t, t_0, \phi)$ be the solution of system (4.18). For any $i \in \mathcal{S}$, define $V_i(x) = x^T P_i x$ as a Lyapunov function candidate. We need to check if the conditions of Theorem 4.2.1 hold. It is clear that condition (i) holds as

$$\lambda_{\min}(P_i)\|x\|^2 \leq V_i(x) \leq \lambda_{\max}(P_i)\|x\|^2$$

and so

$$c_1\|x\|^2 \leq V_i(x) \leq c_2\|x\|^2$$

where $c_1 = \min_{i \in \mathcal{S}}\{\lambda_{\min}(P_i)\}$ and $c_2 = \max_{i \in \mathcal{S}}\{\lambda_{\max}(P_i)\}$.

For condition (ii), we have

$$\dot{V}_i(x) = [(A_i + \Delta A_i + B_i K_i)x + (\bar{A}_i + \Delta \bar{A}_i)x(t-r) + G_i w + f_i(x(t-r))]^T P_i x$$

$$+ x^T P_i[(A_i + \Delta A_i + B_i K_i)x + (\bar{A}_i + \Delta \bar{A}_i)x(t-r) + G_i w + f_i(x(t-r))]$$

$$= x^T[(A_i + B_i K_i)^T P_i + P_i(A_i + B_i K_i)]x + 2x^T P_i(\Delta A_i)x + 2x^T P_i G_i w$$

$$+ 2x^T P_i f_i((t-r)) + 2x^T P_i(\Delta \bar{A}_i)x(t-r) + 2x^T P_i \bar{A}_i x(t-r)$$

$$\leq x^T[(A_i + B_i K_i)^T P_i + P_i(A_i + B_i K_i) + P_i(\xi_{1i} D_i D_i^T + \xi_{2i} G_i G_i^T + \xi_{4i} \bar{A}_i(\bar{A}_i)^T$$

$$+ \xi_{5i} \bar{D}_i(\bar{D}_i)^T + \xi_{6i} I)P_i + \frac{1}{\xi_{1i}} H_i^T H_i + (\frac{q_i}{\xi_{4i}} + \frac{q_i}{\xi_{5i}} \|\bar{H}_i\|^2 + \frac{\delta_i q_i}{\xi_{6i}})I]x + \frac{1}{\xi_{2i}} w^T w$$

$$\leq -\alpha_i V_i(x) + \frac{1}{\xi_{2i}} w^T w$$

$$\leq -\lambda_i V_i(x) \leq -\lambda V_i(x),$$

where $\lambda_i = \alpha_i - \alpha_i^*$, $\lambda = \min_{i \in \mathcal{S}}\{\lambda_i\}$ and we used Lemma 3.1.3, and condition (4.20). Hence, condition (ii) holds. The rest of the proof is similar to the proof of Theorem 4.2.1. Thus, we have

$$\|x\| \leq b\|x_{t_0}\|_r e^{-\alpha(t-t_0)/2} + \bar{\gamma}(t), \quad t \geq t_0,$$

where $b = \mu\sqrt{e^\eta}$, and $\bar{\gamma}(t) = \sqrt{\Gamma(t)/c_1}$ is class $\mathcal{K}$ such that $\Gamma(s) = k(\mu^2)^{k-1}\frac{\|w(s)\|^2}{\xi_2 \alpha^*}$ and $\xi_2 \alpha^* = \min_{i \in \mathcal{S}}\{\xi_{2i}\alpha_i^*\}$. This completes the proof of globally exponentially ISS.

To prove the upper bound on $\|z\|$, for any $i \in \mathcal{S}$, let $J_i = \int_{t_0}^{\infty}(z^T z - \gamma_i^2 w^T w)dt$. Then,

$$
J_i = \int_{t_0}^{\infty}(z^T z - \gamma_i^2 w^T w)\, dt + \int_{t_0}^{\infty} \dot{V}_i\, dt - V_i(\infty) + V_i(x_0)
$$

$$
\leq \int_{t_0}^{\infty}(z^T z - \gamma_i^2 w^T w)\, dt + V_i(x_0) + \int_{t_0}^{\infty}\Big\{x^T[(A_i + B_i K_i)^T P_i + P_i(A_i + B_i K_i)
$$

$$
+ P_i(\xi_{1i} D_i D_i^T + \xi_{4i}\bar{A}_i(\bar{A}_i)^T + \xi_{5i}\bar{D}_i(\bar{D}_i)^T + \xi_{6i}I)P_i + \frac{1}{\xi_{1i}}H_i^T H_i
$$

$$
+ (\frac{q_i}{\xi_{4i}} + \frac{q_i}{\xi_{5i}}\|\bar{H}_i\|^2 + \frac{\delta_i q_i}{\xi_{6i}})I + \gamma_i^{-2}P_i G_i G_i^T P_i - \gamma_i^{-2}P_i G_i G_i^T P_i]x + 2x^T P_i G_i w\Big\}\, dt
$$

$$
= V_i(x_0) + \int_{t_0}^{\infty}\Big\{x^T[(A_i + B_i K_i)^T P_i + P_i(A_i + B_i K_i) + P_i(\xi_{1i} D_i D_i^T + \xi_{4i}\bar{A}_i(\bar{A}_i)^T
$$

$$
+ \xi_{5i}\bar{D}_i(\bar{D}_i)^T + \xi_{6i}I)P_i + \frac{1}{\xi_{1i}}H_i^T H_i + (\frac{q_i}{\xi_{4i}} + \frac{q_i}{\xi_{5i}}\|\bar{H}_i\|^2 + \frac{\delta_i q_i}{\xi_{6i}})I + \gamma_i^{-2}P_i G_i G_i^T P_i
$$

$$
+ C_{ic}^T C_{ic}]x\Big\}\, dt - \int_{t_0}^{\infty}\gamma_i^2(w - \gamma_i^{-2}G_i^T P_i x)^T(w - \gamma_i^{-2}G_i^T P_i x)\, dt.
$$

The last term is strictly negative, using (4.20) with $\gamma_i^{-2} = \xi_{2i}$, we get $J_i \leq V_i(x_0)$ which leads to

$$
\|z\|_2^2 \leq \gamma^2\|w\|_2^2 + m_0,
$$

where $m_0 = \max_{i \in \mathcal{S}}\{V_i(x_0)\}$, and $\gamma = \max_{i \in \mathcal{S}}\{\gamma_i\}$. $\qquad\square$

Remark 4.2.2. *Corollary 4.2.1 provides sufficient conditions to ensure the robust global exponential ISS property. The algebraic Riccati-like equation in (4.20) is to guarantee the existence of the positive-definite matrix P_i (for all $i \in \mathcal{S}$), which implies that the*

solution trajectories of the subsystems are decreasing outside a certain neighbourhood of the disturbance $w(t)$. The role of the average dwell time condition is to organize the switching among the system modes. ξ_1, ξ_2 are tuning parameters to reduce the conservativeness of the Riccati-like equation.

Corollary 4.2.2 (Reliability)**.** *For any $i \in \mathcal{S}$, let the constant $\gamma_i > 0$ be given, and assume that Assumption A holds and there exist positive constants $\xi_{ji}, (j = 1, 2, 4, 5, 6)$, ϵ_i, α_i, $K_i = -\frac{1}{2}\epsilon_i B_{i\bar{\sigma}}^T P_i$, and a positive-definite matrix P_i such that the following algebraic Riccati-like equation holds*

$$A_i^T P_i + P_i A_i + P_i(\xi_{1i} D_i D_i^T + \xi_{2i} G_{ic} G_{ic}^T - \epsilon_i B_{i\bar{\Sigma}} B_{i\bar{\Sigma}}^T + \xi_{4i} \bar{A}_i (\bar{A}_i)^T + \xi_{5i} \bar{D}_i (\bar{D}_i)^T$$

$$+ \xi_{6i} I) P_i + (\frac{q_i}{\xi_{4i}} + \frac{q_i}{\xi_{5i}} \|\bar{H}_i\|^2 + \frac{\delta_i q_i}{\xi_{6i}}) I + \frac{1}{\xi_{1i}} H_i^T H_i + C_{ic}^T C_{ic} + \alpha_i P_i = 0, \tag{4.21}$$

where $\delta_i > 0$ such that $\|f_i(\psi)\|^2 \leq \delta_i \|\psi\|_r^2$. Assume further that $\|w_\sigma^F\|^2 \leq \xi_{2i} \alpha_i^* V_i(x)$ with $\alpha_i^* < \alpha_i$ and for all k, $r \leq t_k - t_{k-1} \leq \beta$ where $\beta > 0$, and the average dwell time condition holds. Then, system (4.19) is robustly globally exponentially ISS-H_∞.

Proof. Let $x(t) = x(t, t_0, \phi)$ be the solution of (4.19). $\forall i \in \mathcal{S}$, define $V_i(x) = x^T P_i x$ as a Lyapunov function candidate for the ith mode. Then, as shown earlier, condition (i) of Theorem 4.2.1 is satisfied. For condition (ii), the derivative of $V_i(x)$ along the trajectory

of (4.19) is given by

$$\dot{V}_i(x) \le x^T[A_i^T P_i + P_i A_i + P_i(\xi_{1i} D_i D_i^T + \xi_{2i} G_{ic} G_{ic}^T + \xi_{4i} \bar{A}_i (\bar{A}_i)^T + \xi_{5i} \bar{D}_i (\bar{D}_i)^T - \epsilon_i B_{i\bar{\Sigma}} B_{i\bar{\Sigma}}^T$$

$$+ \xi_{6i}I)P_i + \frac{1}{\xi_{1i}} H_i^T H_i + (\frac{q_i}{\xi_{4i}} + \frac{q_i}{\xi_{5i}} \|\bar{H}_i\|^2 + \frac{\delta_i q_i}{\xi_{6i}})I]x + \frac{1}{\xi_{2i}}(w_\sigma^F)^T w_\sigma^F$$

$$\le -\alpha_i V_i(x) + \frac{1}{\xi_{2i}}(w_\sigma^F)^T w_\sigma^F$$

$$\le -\lambda_i V_i(x) \le -\lambda V_i(x),$$

where $\lambda_i = \alpha_i - \alpha_i^*$, $\lambda = \min_{i \in \mathcal{S}}\{\lambda_i\}$ and we used Lemma 3.1.3, condition (4.21), and the

fact

$$B_{i\bar{\Sigma}} B_{i\bar{\Sigma}}^T \le B_{i\bar{\sigma}} B_{i\bar{\sigma}}^T.$$

Thus, we have

$$\|x\| \le b\|x_{t_0}\|_r e^{-\alpha(t-t_0)/2} + \bar{\gamma}(t), \quad t \ge t_0,$$

where $b = \mu\sqrt{e^\eta}$, and $\bar{\gamma}(t) = \sqrt{\Gamma(t)/c_1}$ is class $\mathcal{K}$ such that $\Gamma(s) = k(\mu^2)^{k-1} \frac{\|w_\sigma^F(s)\|^2}{\xi_2 \alpha^*}$ and

$\xi_2 \alpha^* = \min_{i \in \mathcal{S}}\{\xi_{2i} \alpha_i^*\}$. This completes the proof of globally exponentially ISS.

As for the upper bound $\|z\|$, one can follow the same steps in Corollary 4.2.1, where

$$J_i = \int_{t_0}^\infty (z^T z - \gamma_i^2 (w_\sigma^F)^T w_\sigma^F) dt.$$

$\square$

Example 4.2.3. *Consider system* (4.18) *with* $\mathcal{S} = \{1, 2\}$,

$$A_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0 & -6 \end{bmatrix}, B_1 = \begin{bmatrix} -3 & 1 \\ 0.1 & 0.2 \end{bmatrix}, C_1 = \begin{bmatrix} 2 & 0.1 \\ 0 & 2 \end{bmatrix}, F_1 = \begin{bmatrix} 0.1 & -2 \\ 0.1 & 0 \end{bmatrix},$$

$$\bar{A}_1 = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 1 \end{bmatrix}, D_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, H_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \bar{D}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \bar{H}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, f_1 = 0.1 \begin{bmatrix} \sin(x_1(t-1)) \\ \sin(x_2(t-1)) \end{bmatrix}, \mathcal{U}_1 = \sin(t),$$

$\epsilon_1 = 2$, $\xi_{11} = 0.2$, $\gamma_1 = 0.1$, $\alpha_1 = 2$, $\xi_{21} = \gamma_1^{-2}$, $\xi_{41} = 0.1$, $\xi_{51} = 0.3$, $\xi_{61} = 0.2$, $M_1 = 2$, $\beta = 3$, $\theta_1 = 0.05$, and $\delta_1 = 0.1$. As for the second mode,

$$A_2 = \begin{bmatrix} -9 & 0.2 \\ 0 & 0.1 \end{bmatrix}, B_2 = \begin{bmatrix} 0.1 & 0.5 \\ 0.1 & -1 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, F_2 = \begin{bmatrix} 0.1 & 0 \\ -3 & 0.1 \end{bmatrix},$$

$$\bar{A}_2 = \begin{bmatrix} 0.3 & 0.2 \\ 0 & 0.1 \end{bmatrix}, D_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, H_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \bar{D}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \bar{H}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}, f_2 = 0.01 \begin{bmatrix} \sin(x_1(t-1)) \\ \sin(x_2(t-1)) \end{bmatrix}, \mathcal{U}_2 = \sin(t),$$

$\epsilon_2 = 0.5$, $\xi_{12} = 0.3$, $\gamma_2 = 0.15$, $\alpha_2 = 2.5$, $\xi_{22} = \gamma_2^{-2}$, $\xi_{42} = 0.2$, $\xi_{52} = 0.09$, $\xi_{62} = 0.1$, $M_2 = 1.1$, $\theta_2 = 0.15$, and $\delta_2 = 0.01$. The disturbance $w^T(t) = 1.2[\sin(t) \ \sin(t)]$.

Case 1. *When all actuators are operational, we have*

$$P_1 = \begin{bmatrix} 0.7234 & -0.0157 \\ -0.0157 & 0.5559 \end{bmatrix}, P_2 = \begin{bmatrix} 11.6224 & -1.2007 \\ -1.2007 & 10.6159 \end{bmatrix},$$

with $c_{11} = \lambda_{\min}(P_1) = 9.8173$, $c_{12} = \lambda_{\max}(P_1) = 12.4211$, $c_{21} = \lambda_{\min}(P_2) = 26.6962$, $c_{22} = \lambda_{\max}(P_2) = 54.1990$, *so,* $c_1 = 9.8173$, $c_2 = 54.1990$, *and*

$$K_1 = \begin{bmatrix} 34.9874 & -4.6636 \\ -11.3823 & -0.9225 \end{bmatrix}, K_2 = \begin{bmatrix} -1.2381 & -0.5812 \\ -7.7135 & 7.3350 \end{bmatrix}.$$
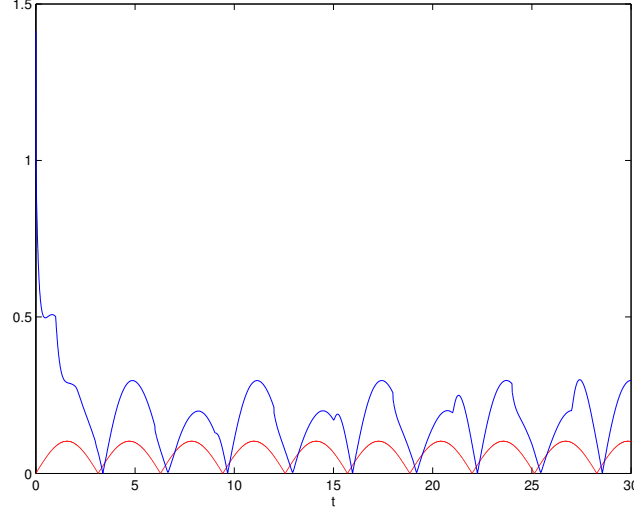
Figure 4.2.1: Input-to-state stabilization, $\phi(s) = 1 - s$, $s \in [-1, 0]$: Operational case.

Thus, the matrices $A_i + B_i K_i$ $(i = 1, 2)$ are Hurwitz and $\tau_a = \frac{\ln \mu}{\alpha^* - \nu} = 1.1783$, with $\nu = 0.5$, the upper bound of the disturbance magnitude is 0.1031, and the cheater bound $N_0 = 0.5853$.

Case 2. *When there is a failure in the first actuator, i.e., $B_{1\Sigma} = \{1\}$ and $B_{1\bar{\Sigma}} =$*

$\begin{bmatrix} 0 & 1 \\ 0 & 0.2 \end{bmatrix}$, *and $B_{2\Sigma} = \{2\}$ and $B_{2\bar{\Sigma}} = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0 \end{bmatrix}$, we have*

$$P_1 = \begin{bmatrix} 11.7139 & -3.1981 \\ -3.1981 & 11.5155 \end{bmatrix}, P_2 = \begin{bmatrix} 53.1251 & -4.8927 \\ -4.8927 & 27.3562 \end{bmatrix},$$

with $c_{11} = \lambda_{\min}(P_1) = 8.4151$, $c_{12} = \lambda_{\max}(P_1) = 14.8144$, $c_{21} = 26.4585$, $c_{22} = 54.0228$, so

97
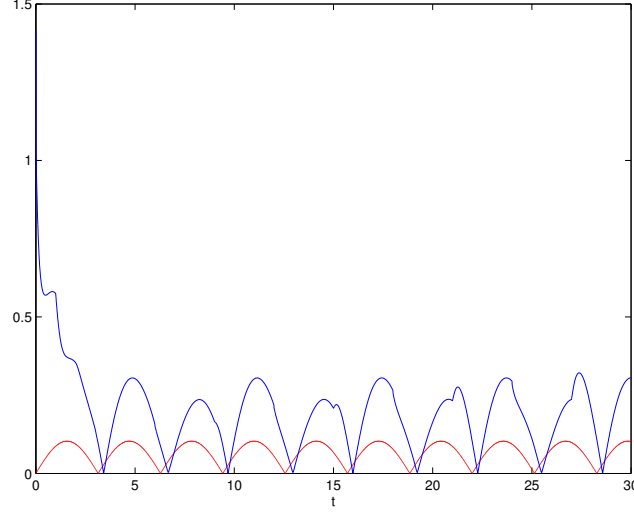
Figure 4.2.2: Input-to-state stabilization, $\phi(s) = 1 - s$, $s \in [-1, 0]$: Faulty actuators.

$c_1 = 8.4151$, $c_2 = 54.0228$, and the control gain matrices

$$K_1 = \begin{bmatrix} 35.4616 & -10.7459 \\ 0 & 0 \end{bmatrix}, K_2 = \begin{bmatrix} 0 & 0 \\ -7.8638 & 7.4506 \end{bmatrix}.$$

Thus, the matrices $A_i + B_i K_i$ ($i = 1, 2$) are Hurwitz and $\tau_a = 1.2823$, the upper bound of the disturbance magnitude is 0.1033, and the cheater bound $N_0 = 0.5378$.

Figure 4.2.1 and 4.2.2 show the simulation results of $\|x\|$ (top) and $\rho(s)$ (bottom) for both cases, where $\rho(s) = \max\{\rho_1(s), \rho_2(s)\}$ and $\rho_i(s) = s/\sqrt{c_2 \theta_i \xi_{2i}}$, $\tau_a = 3$. The figures show the input-to-state stability of the system where the state magnitude $\|x\|$ is bounded below by the system disturbance magnitude.

When we consider the system disturbance input

$$w(t) = \begin{bmatrix} e^{-0.2t}\sin(t) \\ e^{-0.2t}\sin(t) \end{bmatrix},$$

we get the same result, and this shows that the system state is decaying as well. The simulation results of $\|x\|$ (top) and $\gamma(\|w\|) = \sqrt{c_2/c_1}\rho(\|w\|)$ (bottom) are shown in Figures 4.2.3 and 4.2.4.
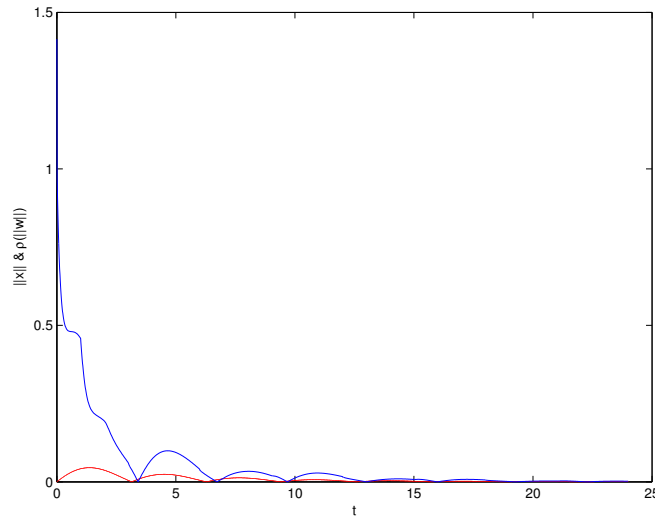
Figure 4.2.3: ISS with a decaying disturbance, $\phi(s) = 1 - s$, $s \in [-1,0]$: Operational case.
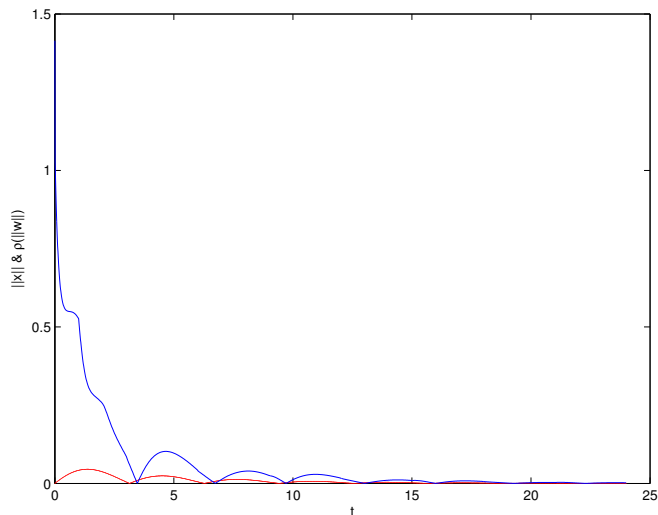
Figure 4.2.4: ISS with a decaying disturbance, $\phi(s) = 1 - s$, $s \in [-1, 0]$: Faulty actuators.

4.3 Conclusion

The system under investigation has been exponentially stabilized by state feedback robust reliable controllers. The Lyapunov-Razumikhin technique along with average dwell time approach by multiple Lyapunov functions has been utilized to fulfill our purpose, which implies that the results are delay independent. The output of the faulty actuators has been treated as a disturbing signal that has been augmented with the system disturbance.

Chapter 5

Impulsive Switched Systems with Time Delay

This chapter deals with the problem of exponentially input-to-state stabilization of impulsive switched systems with finite time delay. To analyze this qualitative property of each mode, we use the technique of multiple Lyapunov functions along with Razumikhin condition, and to achieve the ISS property of the switched system, we use the average dwell-time switching law. Some illustrative examples are presented to clarify the proposed theoretical results. The contents of this chapter forms the basis of [11].

5.1 Problem Formulation and Preliminaries

Consider the following impulsive switched system with time delay given by

$$
\begin{cases}
\dot{x} = f_{\varrho(t)}(x_t, w(t)), & t \neq t_k \\
\Delta x(t) = I_k(t, x_{t^-}), & t = t_k, \quad k \in \mathbb{N} \\
x_{t_0}(s) = \phi(s), & s \in [-r, 0], \quad r > 0,
\end{cases}
\tag{5.1}
$$

where $x \in \mathbb{R}^n$ is the system state, $f_{\varrho(t)}$, $I_{k_{\varrho(t)}} : \mathbb{R}_+ \times PC([-r, 0], \mathbb{R}^n) \to \mathbb{R}^n$, $\phi \in PC([-r, 0], \mathbb{R}^n)$, the impulsive times t_k satisfying $t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, with $\lim_{k \to \infty} t_k = \infty$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ where $x(t_k^+)$ and $x(t_k^-)$ are the right and left limits at t_k respectively, and $w \in \mathbb{R}^p$ is an input disturbance, which is assumed to be in $L_2[t_0, \infty)$. For $r > 0$, let $\mathcal{C}_r$ be the space of all continuous functions that are defined from $[-r, 0]$ to $\mathbb{R}^n$. For any $t \in \mathbb{R}_+$, let $x(t)$ be a function defined on $[t_0, \infty]$. Then, we define the functions x_t, $x_{t^-} \in PC([-r, 0], \mathbb{R}^n)$ are defined by $x_t(s) = x(t + s)$, $x_{t^-}(s) = x(t^- + s)$ for all $s \in [-r, 0]$, respectively, and the linear space $PC([-r, 0], \mathbb{R}^n)$ is equipped with the norm $||x_t||_r = \sup_{t-r \le \theta \le t} ||x(\theta)||$, where $r > 0$ is the time delay. ϱ is the switching rule which is a piecewise constant function defined by $\varrho : [t_0, \infty) \to \mathcal{S} = \{1, 2, \cdots, N\}$. So system (5.1) can be expressed as follows

$$
\begin{cases}
\dot{x} = f_i(x_t, w(t)), & i \in \mathcal{S} \\
\Delta x(t) = I_k(t, x_{t^-}), & t = t_k, \quad k \in \mathbb{N} \\
x_{t_0}(s) = \phi(s), & s \in [-r, 0], \quad r > 0,
\end{cases}
\tag{5.2}
$$

Definition 2. A function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is said to belong to class ν if

(i) $V \in C([t_{k-1}, t_k) \times \mathbb{R}^n, \mathbb{R}^n)$ and $V(t, x)$ is left continuous at each t_k;

(ii) $V(t, x)$ is continuously differentiable at all $x \in \mathbb{R}^n$, and for all $t \ge t_0$, $V(t, 0) \equiv 0$.

5.2 Main Results

In this section, we present our main results. The following theorem gives sufficient conditions of global exponential ISS property of the proposed system.

Theorem 5.2.1. *For any $i \in \mathcal{S}$, let K_i and a differentiable class $\mathcal{K}$ function γ be given. Assume that there exist positive constants c_1, c_2, r, β, d_{k_i}, and a class ν function V_i such that*

(i) *$c_1\|x\|^2 \leq V_i(x) \leq c_2\|x\|^2$ for all $t \geq t_0 - r$;*

(ii) *$\dot{V}_i(\psi(0)) < -\lambda V_i(\psi(0))$ whenever $V_i(\psi(s)) \leq qV_i(\psi(0))$, and $\gamma(\sup_{t_0 \leq \theta \leq t_k} |w(\theta)|) \leq V_i(\psi(0))$ for $\psi \in \mathcal{C}_r$, $s \in [-r, 0]$ and $t \in [t_{k-1}, t_k)$, where $q = \max\{\mu h, e^{\lambda r}\} > 1$ and $\mu = c_2/c_1$;*

(iii) *for all k, $r \leq t_k - t_{k-1} \leq \beta$ and the average dwell time condition holds, and $\beta > 0$;*

(iv) *for $s \in [-r, 0]$ and $h > 1$, $V_i(x(t+s)) \leq hV_j(x(t))$ for any $i, j \in \mathcal{S}$ and any $t \geq t_0$;*

(v) *$V_i(\psi(t_k^-) + I_{k_i}(\psi(t_k^-))) \leq (1 + d_{k_i})V_j(\psi(t_k^-))$ for any $i, j \in \mathcal{S}$ and any $t \geq t_0$ with $\sum_{k=1}^{\infty} d_{k_i} < \infty$, and $d_{0_i} = 0$.*

Then, system (5.2) *is globally exponentially ISS.*

Proof. Let $x(t, t_0, \phi)$ be any solution of system (5.2) with $x_{t_0} = \phi$ and $v_i(t) = V_i(x(t))$. First, using conditions (i) and (ii), we show that every mode is globally exponentially ISS.

For any $i \in \mathcal{S}$, and $k \in \mathbb{N}$, $t \in [t_{k-1}, t_k)$, we show that

$$v_i(t) \leq c_2 \prod_{j=0}^{k-1} (1 + d_{j_i}) ||x_{t_{k-1}}||_r^2 e^{-\lambda(t-t_{k-1})} + \gamma(\sup_{t_0 \leq s \leq t} ||w(s)||). \tag{5.3}$$

Let

$$Q_i(t) = \begin{cases} v_i(t) - c_2 \prod_{j=0}^{k-1} (1 + d_{j_i}) ||x_{t_{k-1}}||_r^2 e^{-\lambda(t-t_{k-1})} - \gamma(\sup_{t_0 \leq s \leq t} ||w(s)||), & t \in [t_{k-1}, t_k), \ k \in \mathbb{N} \\ v_i(t) - c_2 ||x_{t_0}||_r^2 e^{-\lambda(t-t_0)}, & t \in [t_0 - r, t_0). \end{cases}$$

We need to prove that $Q_i(t) \leq 0$ for all $t \geq t_0 - r$. It is clear that $Q_i(t) \leq 0$ for $t \in [t_0 - r, t_0]$.

From condition (i), we get

$$v_i(t) \leq c_2 ||x||^2 \tag{5.4}$$

$$\leq c_2 ||x_{t_0}||_r^2 \tag{5.5}$$

$$\leq c_2 ||x_{t_0}||_r^2 e^{-\lambda(t-t_0)}. \tag{5.6}$$

So, we have

$$Q_i(t) = v_i(t) - c_2 ||x_{t_0}||_r^2 e^{-\lambda(t-t_0)} \leq 0$$

Step 1, for $t \in [t_0, t_1)$, we need to show

$$Q_i(t) = v_i(t) - c_2 ||x_{t_0}||_r^2 e^{-\lambda(t-t_0)} - \gamma(\sup_{t_0 \leq \theta \leq t_1} ||w(\theta)||) \leq 0. \tag{5.7}$$

For any $i \in \mathcal{S}$, let $\alpha_i > 0$ be arbitrary, and we show $Q_i(t) \leq \alpha_i$ for $[t_0, t_1)$. If not, then there would exist some $t \in [t_0, t_1)$ so that $Q_i(t) > \alpha_i$. Let

$$t_i^* = \inf\{t \in [t_0, t_1) : Q_i(t) > \alpha_i, \ i \in \mathcal{S}\}.$$

104

$$Q_i(t_0) \leq v_i(t_0) - c_2\|x_{t_0}\|_r^2 \leq c_2(\|x(t_0)\|^2 - \|x_{t_0}\|_r^2) \leq 0$$

Since we have $Q_i(t) \leq 0 < \alpha_i$ for $t \in [t_0 - r, t_0]$, then $t_i^* \in (t_0, t_1)$. Also, since $Q_i(t)$ is continuous on $[t_0, t_1)$, then we have

$$Q_i(t_i^*) = \alpha_i \qquad \text{and } Q_i(t) \leq \alpha_i \text{ for } [t_0 - r, t_i^*].$$

Then, we have

$$v_i(t_i^*) = Q_i(t_i^*) + c_2\|x_{t_0}\|_r^2 e^{-\lambda(t_i^* - t_0)} + \gamma(\sup_{t_0 \leq \theta \leq t_i^*} \|w(\theta)\|) \tag{5.8}$$

and for $s \in [-r, 0]$, we have

$$v_i(t_i^* + s) = Q_i(t_i^* + s) + c_2\|x_{t_0}\|_r^2 e^{-\lambda(t_i^* + s - t_0)} + \gamma(\sup_{t_0 \leq \theta \leq t_i^* + s} \|w(\theta)\|)$$

$$\leq \alpha_i + c_2\|x_{t_0}\|_r^2 e^{-\lambda(t_i^* - t_0)} e^{\lambda r} + \gamma(\sup_{t_0 \leq \theta \leq t_i^*} \|w(\theta)\|)$$

$$\leq \left[\alpha_i + c_2\|x_{t_0}\|_r^2 e^{-\lambda(t_i^* - t_0)} + \gamma(\sup_{t_0 \leq \theta \leq t_i^*} \|w(\theta)\|)\right] e^{\lambda r}$$

$$= e^{\lambda r} v_i(t_i^*)$$

$$\leq q v_i(t^*), \tag{5.9}$$

where from (5.8), we use

$$\gamma(\sup_{t_0 \leq \theta \leq t_i^*} \|w(\theta)\|) \leq v_i(t_i^*).$$

Thus, from condition (ii), we have

$$\dot{v}_i(t_i^*) \leq -\lambda v_i(t_i^*)$$

105

which implies

$$\dot{Q}_i(t_i^*) = \dot{v}_i(t_i^*) + \lambda c_2 \|x_{t_0}\|_r^2 e^{-\lambda(t_i^*-t_0)} - \dot{\gamma}(\sup_{t_0 \leq \theta \leq t_i^*} \|w(\theta)\|)$$

$$\leq -\lambda v_i(t_i^*) + \lambda c_2 \|x_{t_0}\|_r^2 e^{-\lambda(t_i^*-t_0)} - \dot{\gamma}(\sup_{t_0 \leq \theta \leq t_i^*} \|w(\theta)\|)$$

$$\leq -\lambda\big[v_i(t_i^*) - c_2\|x_{t_0}\|_r^2 e^{-\lambda(t_i^*-t_0)} - \gamma(\sup_{t_0 \leq \theta \leq t_i^*} \|w(\theta)\|)\big]$$

$$= -\lambda\alpha_i. \tag{5.10}$$

Then, $Q_i(t)$ is decreasing at t_i^* which contradicts how t^* was defined. Thus, we get $Q_i(t) \leq \alpha_i$ for all $t \in [t_0, t_1)$. Let $\alpha_i \to 0^+$, then we have $Q_i(t) \leq 0$ for $t \in [t_0, t_1)$.

Step 2, for any $i \in \mathcal{S}$ assume $Q_i(t) \leq 0$ for all $t \in [t_{k-1}, t_k)$ for $k = 1, \cdots m$.

$$Q_i(t_m) = v_i(t_m) - c_2 \prod_{j=0}^{m}(1 + d_{j_i})\|x_{t_m}\|_r^2 - \gamma(\sup_{t_0 \leq \theta \leq t_{m+1}} \|w(\theta)\|)$$

$$\leq (1 + d_{m_i})\big[v_j(t_m^-) - c_2 \prod_{j=0}^{m-1}(1 + d_{j_i})\|x_{t_m}\|_r^2 e^{-\lambda(t_m^- - t_{m-1})}\big] - \gamma(\sup_{t_0 \leq \theta \leq t_{m+1}} \|w(\theta)\|)$$

$$= (1 + d_{m_i})Q_j(t_m^-) - \gamma(\sup_{t_0 \leq \theta \leq t_{m+1}} \|w(\theta)\|)$$

$$\leq 0 \ < \alpha_i.$$

Step 3, we will show that $Q_i(t) \leq 0$ for all $t \in [t_m, t_{m+1})$, i.e., we need to show that

$$v_i(t) \leq c_2 \prod_{j=0}^{m}(1 + d_{j_i})\|x_{t_m}\|_r^2 e^{-\lambda(t-t_m)} + \gamma(\sup_{t_0 \leq s \leq t} \|w(s)\|).$$

We need to prove that $Q_i(t) \leq \alpha_i$ for all $t \in [t_m, t_{m+1})$ and any $i \in \mathcal{S}$. If this were not true,

106

then there would exist some $t \in [t_m, t_{m+1})$ such that for any $i \in \mathcal{S}$ we have $Q_i(t) > \alpha_i$. Let

$$t_i^* = \inf\{t \in (t_m, t_{m+1}) : Q_i(t) > \alpha_i, \ i \in \mathcal{S}\}$$

by the continuity, we have $Q_i(t_i^*) = \alpha_i$ and $Q_i(t) \leq \alpha_i$ for all $t \in [t_m, t_i^*]$, i.e., $\dot{Q}_i(t_i^*) > 0$.

Thus, we have

$$v_i(t_i^*) = \alpha_i + c_2 \prod_{j=0}^{m}(1 + d_{j_i})\|x_{t_m}\|_r^2 e^{-\lambda(t_i^* - t_m)} + \gamma(\sup_{t_0 \leq \theta \leq t_i^*} \|w(\theta)\|). \qquad (5.11)$$

We want to show $v_i(t_i^* + s) \leq v_i(t_i^*)$ for $s \in [-r, 0]$.

Case 1. If $t_i^* + s \in [t_m, t_{m+1})$, then we have for each $i \in \mathcal{S}$

$$v_i(t_i^* + s) = Q_i(t_i^* + s) + c_2 \prod_{j=0}^{m}(1 + d_{j_i})\|x_{t_m}\|_r^2 e^{-\lambda(t_i^* + s - t_m)} + \gamma(\sup_{t_0 \leq \theta \leq t_i^* + s} \|w(\theta)\|)$$

$$\leq \alpha_i + c_2 \prod_{j=0}^{m}(1 + d_{j_i})\|x_{t_m}\|_r^2 e^{-\lambda(t_i^* - t_m)} e^{\lambda r} + \gamma(\sup_{t_0 \leq \theta \leq t_i^*} \|w(\theta)\|)$$

$$\leq \Big[\alpha_i + c_2 \prod_{j=0}^{m}(1 + d_{j_i})\|x_{t_m}\|_r^2 e^{-\lambda(t_i^* - t_m)} + \gamma(\sup_{t_0 \leq \theta \leq t_i^*} \|w(\theta)\|)\Big] e^{\lambda r}$$

$$= e^{\lambda r} v_i(t_i^*) \leq q v_i(t_i^*). \qquad (5.12)$$

Case 2. If $t_i^* + s \in [t_m - r, t_m)$. Then, since for any $i, j \in \mathcal{S}$ and for any $t \geq t_0 - r$,

$$v_i(t) \leq \mu v_j(t), \quad \mu = \frac{c_2}{c_1} \geq 1. \qquad (5.13)$$

Using the foregoing inequality and condition (iv), we get

$$v_i(t_i^* + s) \leq \mu v_j(t_i^* + s)$$

$$\leq \mu h v_i(t_i^*)$$

$$\leq q v_i(t_i^*), \tag{5.14}$$

where $q = \max\{\mu h, e^{\lambda r}\}$. Also, from (5.11), we have that

$$\gamma(\sup_{t_0 \leq \theta \leq t_i^*} \|w(\theta)\|) \leq v_i(t_i^*).$$

Thus, from condition (ii), we have

$$\dot{v}_i(t_i^*) \leq -\lambda v_i(t_i^*)$$

which implies

$$\dot{Q}_i(t_i^*) = \dot{v}_i(t_i^*) + \lambda c_2 \prod_{j=0}^{m}(1 + d_{j_i})\|x_{t_m}\|_r^2 e^{-\lambda(t_i^* - t_m)} - \dot{\gamma}(\sup_{t_0 \leq \theta \leq t_i^*} \|w(\theta)\|)$$

$$\leq -\lambda v_i(t_i^*) + \lambda c_2 \prod_{j=0}^{m}(1 + d_{j_i})\|x_{t_m}\|_r^2 e^{-\lambda(t_i^* - t_m)} - \dot{\gamma}(\sup_{t_0 \leq \theta \leq t_i^*} \|w(\theta)\|)$$

$$\leq -\lambda\left[v_i(t_i^*) - c_2 \prod_{j=0}^{m}(1 + d_{j_i})\|x_{t_m}\|_r^2 e^{-\lambda(t_i^* - t_m)} - \gamma(\sup_{t_0 \leq \theta \leq t_i^*} \|w(\theta)\|)\right]$$

$$= -\lambda \alpha_i. \tag{5.15}$$

Then, $Q_i(t)$ is decreasing at t_i^* which is a contradiction. Thus, we get $Q_i(t) \leq \alpha_i$ for all $t \in [t_m, t_{m+1})$. Let $\alpha_i \to 0^+$, then we have $Q_i(t) \leq 0$ for $t \in [t_m, t_{m+1})$. By induction, we

have $Q_i(t) \leq 0$ for all $t \geq t_0 - r$. Thus, we have for $t \in [t_{k-1}, t_k)$,

$$v_i(t) \leq c_2 \prod_{j=0}^{k-1}(1 + d_{j_i})||x_{t_{k-1}}||_r^2 e^{-\lambda(t-t_{k-1})} + \gamma(\sup_{t_0 \leq s \leq t} ||w(s)||), \qquad (5.16)$$

By condition (i), one can show

$$||x|| \leq \sqrt{\mu \prod_{j=0}^{k-1}(1 + d_{j_i})||x_{t_{k-1}}||_r e^{-\lambda(t-t_{k-1})/2}} + \sqrt{\frac{1}{c_1}\gamma(\sup_{t_0 \leq s \leq t} ||w(s)||)}. \qquad (5.17)$$

This proves the global exponential ISS for each subsystem. Second, we willshow that the whole switched impulsive system is globally exponentially ISS. Since condition (i) is assumed to hold for all $t \geq t_0 - r$, then we have from (5.16)

$$V_i(x(t)) \leq \mu \prod_{j=0}^{k-1}(1 + d_{j_i})V_i(x(t_{k-1} - r))e^{-\lambda(t-t_{k-1})} + \gamma(\sup_{t_0 \leq s \leq t} ||w(s)||), \qquad (5.18)$$

Activating modes i, j and l on the first, second and third intervals, respectively, we have for $t \in [t_0, t_1)$

$$V_i(x(t)) \leq V_i(x_{t_0})e^{-\lambda(t-t_0)} + \gamma(\sup_{t_0 \leq s \leq t_1} ||w(s)||),$$

109

and for $t \in [t_1, t_2)$

$$V_j(x(t)) \leq \mu \prod_{j=0}^{1} (1 + d_{j_i}) V_j(x_{t_1}) e^{-\lambda(t-t_1)} + \gamma(\sup_{t_0 \leq s \leq t_2} ||w(s)||)$$

$$\leq \mu^2 \prod_{j=0}^{1} (1 + d_{j_i}) V_i(x_{t_1}) e^{-\lambda(t-t_1)} + \gamma(\sup_{t_0 \leq s \leq t_2} ||w(s)||)$$

$$\leq \mu^2 \prod_{j=0}^{1} (1 + d_{j_i}) \left[\frac{c_2}{c_1} V_i(x_{t_0}) e^{-\lambda(t_1-r-t_0)} + \gamma(\sup_{t_0 \leq s \leq t_1-r} ||w(s)||)\right] e^{-\lambda(t-t_1)} + \gamma(\sup_{t_0 \leq s \leq t_2} ||w(s)||)$$

$$\leq \mu^3 \prod_{j=0}^{1} (1 + d_{j_i}) e^{\lambda r} V_i(x_{t_0}) e^{-\lambda(t_1-t_0)} e^{-\lambda(t-t_1)} + \mu^2 \prod_{j=0}^{1} (1 + d_{j_i}) \gamma(\sup_{t_0 \leq s \leq t_1} ||w(s)||) e^{-\lambda(t-t_1)}$$

$$+ \gamma(\sup_{t_0 \leq s \leq t_2} ||w(s)||)$$

$$\leq \mu^3 \prod_{j=0}^{1} (1 + d_{j_i}) e^{\lambda r} V_i(x_{t_0}) e^{-\lambda(t-t_0)} + (\mu^2 \prod_{j=0}^{1} (1 + d_{j_i}) + 1) \gamma(\sup_{t_0 \leq s \leq t_2} ||w(s)||)$$

and for $t \in [t_2, t_3)$

$$V_l(x(t)) \leq \mu \prod_{j=0}^{2} (1 + d_{j_i}) V_l(x_{t_2}) e^{-\lambda(t-t_2)} + \gamma(\sup_{t_0 \leq s \leq t_3} ||w(s)||)$$

$$\leq \mu^2 \prod_{j=0}^{2} (1 + d_{j_i}) V_j(x_{t_2}) e^{-\lambda(t-t_2)} + \gamma(\sup_{t_0 \leq s \leq t_3} ||w(s)||)$$

$$\leq \mu^2 \prod_{j=0}^{2}(1+d_{j_i})\Big[\mu^3 \prod_{j=0}^{1}(1+d_{j_i})e^{\lambda r}V_i(x_{t_0})e^{-\lambda(t_2-r-t_0)}$$

$$+ (\mu^2 \prod_{j=0}^{1}(1+d_{j_i})+1)\gamma(\sup_{t_0\leq s\leq t_2-r}||w(s)||)\Big]e^{-\lambda(t-t_2)} + \gamma(\sup_{t_0\leq s\leq t_3}||w(s)||)$$

$$\leq \mu^5(1+d_{2_i})\prod_{j=0}^{1}(1+d_{j_i})^2 e^{2\lambda r}V_i(x_{t_0})e^{-\lambda(t_2-t_0)}e^{-\lambda(t-t_2)}$$

$$+ \Big((\mu^2)^2(1+d_{2_i})\prod_{j=0}^{1}(1+d_{j_i})^2 + \mu^2 \prod_{j=0}^{2}(1+d_{j_i})\Big)\gamma(\sup_{t_0\leq s\leq t_2-r}||w(s)||)e^{-\lambda(t-t_2)}$$

$$+ \gamma(\sup_{t_0\leq s\leq t_3}||w(s)||)$$

$$\leq \mu^5(1+d_{2_i})\prod_{j=0}^{1}(1+d_{j_i})^2 e^{2\lambda r}V_i(x_{t_0})e^{-\lambda(t-t_0)}$$

$$+ \Big(\mu^4(1+d_{2_i})\prod_{j=0}^{1}(1+d_{j_i})^2 + \mu^2 \prod_{j=0}^{2}(1+d_{j_i})+1\Big)\gamma(\sup_{t_0\leq s\leq t_3}||w(s)||).$$

Generally, for $i \in \mathcal{S}$ and $t \in [t_{k-1}, t_k)$, we have

$$V_i(x(t)) \leq \mu^{2k-1}e^{(k-1)\lambda r}\prod_{j=0}^{1}(1+d_{j_i})\prod_{j=0}^{2}(1+d_{j_i})\cdots\prod_{j=0}^{k-1}(1+d_{j_i})e^{-\lambda(t-t_0)}V_i(x_{t_0})$$

$$+ k(\mu^2)^{k-1}\prod_{j=0}^{1}(1+d_{j_i})\prod_{j=0}^{2}(1+d_{j_i})\cdots\prod_{j=0}^{k-1}(1+d_{j_i})\gamma(\sup_{t_0\leq s\leq t_k}||w(s)||)$$

$$= \mu^k(\varrho)^{k-1} G e^{-\lambda(t-t_0)} V_i(x_{t_0}) + k(\mu^2)^{k-1} G\gamma(\sup_{t_0 \le s \le t_k} ||w(s)||)$$

$$= (\mu\varrho)^k \varrho^{-1} G e^{-\lambda(t-t_0)} V_i(x_{t_0}) + k(\mu^2)^{k-1} G\gamma(\sup_{t_0 \le s \le t_k} ||w(s)||)$$

$$\le e^{k\ln(\mu\varrho) - \ln(\varrho) + \ln G - \lambda(t-t_0)} V_i(x_{t_0}) + \Gamma(t)$$

where $\varrho = \mu e^{\lambda r}$ and $\Gamma(t) = k(\mu^2)^{k-1} G\gamma(\sup_{t_0 \le s \le t_k} ||w(s)||)$ is class $\mathcal{K}$ function with $G = \prod_{j=0}^{1}(1+d_{j_i}) \prod_{j=0}^{2}(1+d_{j_i}) \cdots \prod_{j=0}^{k-1}(1+d_{j_i})$. Using the average dwell-time condition with $N_0 = \frac{\eta}{\ln(\mu\varrho)}$, $\tau_a = \frac{\ln(\mu\varrho)}{\lambda-\nu}$, $(0 < \nu < \lambda)$, for some arbitrary positive constant η, we get

$$V_i(x(t)) \le e^{\eta + \ln G\mu - \nu(t-t_0)} V_i(x_{t_0}) + \Gamma(t)$$

$$\le D e^{-\nu(t-t_0)} ||x_{t_0}||_r^2 + \Gamma(t)$$

where $D = c_2 G\mu e^\eta$. This implies that

$$||x|| \le b||x_{t_0}||_r e^{-\nu(t-t_0)/2} + \bar{\gamma}(t), \quad t \ge t_0,$$

where $b = \mu\sqrt{Ge^\eta}$, and $\bar{\gamma}(t) = \sqrt{\Gamma(t)/c_1}$ is class $\mathcal{K}$. This completes the proof. $\quad\square$

As a special case, consider the following uncertain impulsive switched systems with time delay

$$\begin{cases} \dot{x} = (A_{\varrho(t)} + \Delta A_{\varrho(t)})x + (\bar{A}_{\varrho(t)} + \Delta\bar{A}_{\varrho(t)})x(t-r) + B_{\varrho(t)}u + G_{\varrho(t)}w \\ \quad + f_{\varrho(t)}(x(t-r)), \quad t \ne t_k \\ \Delta x(t) = I_{k_{\varrho(t)}}(x(t^-)) = C_{k_{\varrho(t)}}x(t^-), \quad t = t_k, \quad k = 1, 2, \cdots, \\ x_{t_0}(s) = \phi(s), \quad s \in [-r, 0], \quad r > 0, \end{cases} \quad (5.19)$$

where $x \in \mathbb{R}^n$ is the system state, the impulsive times t_k satisfying $t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, with $\lim_{k \to \infty} t_k = \infty$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ where $x(t_k^+)$(or $x(t_k^-)$) is the state just

112

after (or before) the impulse at t_k, $I_k : \mathbb{R}^n \to \mathbb{R}^n$ is the impulsive function, $u = Kx \in \mathbb{R}^l$ is the control input, and $w \in \mathbb{R}^p$ is an input disturbance, which is assumed to be in $L_2[t_0, \infty)$ For each $i \in \mathcal{S}$, A_i is a non Hurwitz matrix, $K_i \in \mathbb{R}^{l \times n}$ is the control gain matrix such that $u = K_i x$, where (A_i, B_i) is assumed to be stabilizable, $f_i(\cdot) \in \mathbb{R}^n$ is some nonlinearity, A_i, B_i, G_i are known real constant matrices with proper dimensions, and ΔA_i, $\Delta \bar{A}_i$ are piecewise continuous functions representing system parameter uncertainties with bounded norms.

For any $i \in \mathcal{S}$ the closed-loop system is

$$
\begin{cases}
\dot{x} = (A_i + \Delta A_i + B_i K_i)x + (\bar{A}_i + \Delta \bar{A}_i)x(t - r) + G_i w + f_i(x(t - r)), & t \neq t_k, \\
\Delta x(t) = I_{k_i}(x(t^-)) = C_{k_i} x(t^-), & t = t_k, \quad k = 1, 2, \cdots, \\
x_{t_0}(s) = \phi(s), & s \in [-r, 0], \ r > 0,
\end{cases}
\tag{5.20}
$$

Assume that the output of faulty actuators is an arbitrary energy-bounded signal which belongs to $L_2[t_0, \infty)$. To analyze the reliable stabilization with respect to actuator failures, for any $i \in \mathcal{S}$, we write the decomposition

$$
B_i = B_{i\sigma} + B_{i\bar{\sigma}}.
$$

Furthermore, the augmented disturbance input to the system becomes $w_\sigma^F = (w^T \ (u_\sigma^F)^T)^T$, where $u_\sigma^F \in \mathbb{R}^l$ is the failure vector whose elements corresponding to the set of faulty actuators σ, and F here stands for "failure". Since the control input u is applied to the system through the normal actuators, and the outputs of the faulty actuators are assumed

to be arbitrary signals, the closed-loop system becomes

$$
\begin{cases}
\dot{x} = (A_i + \Delta A_i + B_{i\bar{\sigma}}K_i)x + (\bar{A}_i + \Delta\bar{A}_i)x(t-r) + G_{ic}w_\sigma^F + f_i(x(t-r)), \\
\Delta x(t) = I_{k_i}(x(t^-)) = C_{k_i}x(t^-), \quad t = t_k, \quad k = 1, 2, \cdots, \\
x_{t_0}(s) = \phi(s), \quad s \in [-r, 0], \ r > 0,
\end{cases}
\tag{5.21}
$$

where $G_{ic} = (G_i \ \ B_{i\sigma})$. Then we have the following results

Corollary 5.2.1. *For any $i \in \mathcal{S}$, let K_i and $\gamma_i > 0$ be given. Assume that Assumption A holds and there exist positive constants ξ_{ji} ($j = 1, 2, 4, 5, 6$), a positive-definite matrix P_i such that*

$$
(A_i + B_iK_i)^T P_i + P_i(A_i + B_iK_i) + P_i(\xi_{1i}D_iD_i^T + \xi_{2i}G_iG_i^T + \xi_{4i}\bar{A}_i(\bar{A}_i)^T + \xi_{5i}\bar{D}_i(\bar{D}_i)^T
$$

$$
+ \xi_{6i}I)P_i + \frac{1}{\xi_{1i}}H_i^T H_i + (\frac{q_i}{\xi_{4i}} + \frac{q_i}{\xi_{5i}}\|\bar{H}_i\|^2 + \frac{\delta_i q_i}{\xi_{6i}})I + \alpha_i P_i = 0
\tag{5.22}
$$

where $\delta_i > 0$ such that

$$
\|f_i(\psi)\|^2 \le \delta_i\|\psi\|_r^2.
\tag{5.23}
$$

Assume further that $\|w\|^2 \le \xi_{2i}\alpha_i^* V_i(x)$ with $\alpha_i^* < \alpha_i$ and for all k, $r \le t_k - t_{k-1} \le \beta$ where $\beta > 0$, the average dwell time condition holds, and $V_i(x(t_k^-) + I_{k_i}(x(t_k^-))) \le d_{k_i}V_i(t_k^-)$, $0 < d_{k_i} < e^{-\lambda(t_{k+1}-t_k)} \le 1$, for all $k \in \mathbb{N}$. Then, system (5.20) is robustly globally exponentially ISS.

Proof. For all $t \in [t_0 - r, \infty)$, let $x(t) = x(t, t_0, \phi)$ be the solution of system (5.20). For any $i \in \mathcal{S}$, define $V_i(x) = x^T P_i x$ as a Lyapunov function candidate. We need to check if the conditions of Theorem 5.2.1 hold. It is clear that condition (i) holds as

$$
\lambda_{\min}(P_i)\|x\|^2 \le V_i(x) \le \lambda_{\max}(P_i)\|x\|^2
$$

114

and so

$$c_1\|x\|^2 \leq V_i(x) \leq c_2\|x\|^2$$

where $c_1 = \min_{i \in \mathcal{S}}\{\lambda_{\min}(P_i)\}$ and $c_2 = \max_{i \in \mathcal{S}}\{\lambda_{\max}(P_i)\}$.

For condition (ii), we have

$$\dot{V}(x) = \dot{x}^T P_i x + x^T P_i \dot{x}$$

$$= x^T[(A_i + B_i K_i)^T P_i + P_i(A_i + B_i K_i)]x + 2x^T P_i(\Delta A_i)x + 2x^T P_i G_i w$$

$$+ 2x^T P_i f_i((t-r)) + 2x^T P_i(\Delta \bar{A}_i)x(t-r) + 2x^T P_i \bar{A}_i x(t-r)$$

$$\leq x^T[(A_i + B_i K_i)^T P_i + P_i(A_i + B_i K_i) + P_i(\xi_{1i} D_i D_i^T + \xi_{2i} G_i G_i^T + \xi_{4i} \bar{A}_i(\bar{A}_i)^T$$

$$+ \xi_{5i}\bar{D}_i(\bar{D}_i)^T + \xi_{6i}I)P_i + \frac{1}{\xi_{1i}}H_i^T H_i + (\frac{q_i}{\xi_{4i}} + \frac{q_i}{\xi_{5i}}\|\bar{H}_i\|^2 + \frac{\delta_i q_i}{\xi_{6i}})I]x + \frac{1}{\xi_{2i}}w^T w$$

$$= -\alpha_i V_i(x) + \frac{1}{\xi_{2i}}w^T w$$

$$\leq -\lambda_i V_i(x) \leq -\lambda V_i(x),$$

where $\lambda_i = \alpha_i - \alpha_i^*$, $\lambda = \min_{i \in \mathcal{S}}\{\lambda_i\}$ and we used Lemma 3.1.3, and condition (5.22). Hence, condition (ii) holds. The rest of the proof is similar to the proof of Theorem 5.2.1. Thus, we have

$$\|x\| \leq b\|x_{t_0}\|_r e^{-\alpha(t-t_0)/2} + \bar{\gamma}(t), \quad t \geq t_0,$$

where $b = \mu\sqrt{Ge^\eta}$, and $\bar{\gamma}(t) = \sqrt{\Gamma(t)/c_1}$ is class $\mathcal{K}$ such that $\Gamma(s) = k(\mu^2)^{k-1}G\frac{\|w(s)\|^2}{\xi_2 \alpha^*}$ and $\xi_2\alpha^* = \min_{i \in \mathcal{S}}\{\xi_{2i}\alpha_i^*\}$. This completes the proof.

$\square$

115

Corollary 5.2.2 (Reliability). *For any $i \in \mathcal{S}$, let the constant $\gamma_i > 0$ be given, and assume that Assumption A holds and there exist positive constants $\xi_{ji}, (j = 1, 2, 4, 5, 6)$, ϵ_i, α_i, $K_i = -\frac{1}{2}\epsilon_i B_{i\bar{\sigma}}^T P_i$, and a positive-definite matrix P_i such that the following algebraic Riccati-like equation holds*

$$A_i^T P_i + P_i A_i + P_i(\xi_{1i} D_i D_i^T + \xi_{2i} G_{ic} G_{ic}^T - \epsilon_i B_{i\bar{\Sigma}} B_{i\bar{\Sigma}}^T + \xi_{4i} \bar{A}_i(\bar{A}_i)^T + \xi_{5i} \bar{D}_i(\bar{D}_i)^T$$

$$+ \xi_{6i} I)P_i + (\frac{q_i}{\xi_{4i}} + \frac{q_i}{\xi_{5i}}\|\bar{H}_i\|^2 + \frac{\delta_i q_i}{\xi_{6i}})I + \frac{1}{\xi_{1i}} H_i^T H_i + \alpha_i P_i = 0, \qquad (5.24)$$

where $\delta_i > 0$ such that $\|f_i(\psi)\|^2 \leq \delta_i \|\psi\|_r^2$. Assume further that $\|w_\sigma^F\|^2 \leq \xi_{2i} \alpha_i^* V_i(x)$ with $\alpha_i^* < \alpha_i$ and for all k, $r \leq t_k - t_{k-1} \leq \beta$ where $\beta > 0$, and the average dwell time condition holds. Then, system (5.21) is robustly globally exponentially ISS.

Proof. Let $x(t) = x(t, t_0, \phi)$ be the solution of (5.21). $\forall i \in \mathcal{S}$, define $V_i(x) = x^T P_i x$ as a Lyapunov function candidate for the ith mode. Then, as shown earlier, condition (i) of Theorem 5.2.1 is satisfied. For condition (ii), the derivative of $V_i(x)$ along the trajectory of (5.21) is given by

$$\dot{V}_i(x) \leq x^T[A_i^T P_i + P_i A_i + P_i(\xi_{1i} D_i D_i^T + \xi_{2i} G_{ic} G_{ic}^T + \xi_{4i} \bar{A}_i(\bar{A}_i)^T + \xi_{5i} \bar{D}_i(\bar{D}_i)^T - \epsilon_i B_{i\bar{\Sigma}} B_{i\bar{\Sigma}}^T$$

$$+ \xi_{6i} I)P_i + \frac{1}{\xi_{1i}} H_i^T H_i + (\frac{q_i}{\xi_{4i}} + \frac{q_i}{\xi_{5i}}\|\bar{H}_i\|^2 + \frac{\delta_i q_i}{\xi_{6i}})I]x + \frac{1}{\xi_{2i}}(w_\sigma^F)^T w_\sigma^F$$

$$\leq -\alpha_i V_i(x) + \frac{1}{\xi_{2i}}(w_\sigma^F)^T w_\sigma^F$$

$$\leq -\lambda_i V_i(x) \leq -\lambda V_i(x),$$

where $\lambda_i = \alpha_i - \alpha_i^*$, $\lambda = \min_{i \in \mathcal{S}}\{\lambda_i\}$ and we used Lemma 3.1.3, condition (5.24), and the fact

$$B_{i\bar{\Sigma}}B_{i\bar{\Sigma}}^T \leq B_{i\bar{\sigma}}B_{i\bar{\sigma}}^T.$$

Thus, we have

$$\|x\| \leq b\|x_{t_0}\|_r e^{-\alpha(t-t_0)/2} + \bar{\gamma}(t), \quad t \geq t_0,$$

where $b = \mu\sqrt{Ge^\eta}$, and $\bar{\gamma}(t) = \sqrt{\Gamma(t)/c_1}$ is class $\mathcal{K}$ such that $\Gamma(s) = k(\mu^2)^{k-1}G\frac{\|w_\sigma^F(s)\|^2}{\xi_2\alpha^*}$ and $\xi_2\alpha^* = \min_{i \in \mathcal{S}}\{\xi_{2i}\alpha_i^*\}$. This completes the proof. $\qquad\square$

Example 5.2.2. *Consider system* (5.20) *where* $\mathcal{S} = \{1, 2\}$,

$$A_1 = \begin{bmatrix} 0.3 & 0.1 \\ 0.2 & -6 \end{bmatrix}, B_1 = \begin{bmatrix} -3 & 0.5 \\ 1 & 0.2 \end{bmatrix}, C_1 = \begin{bmatrix} 3 & 0.3 \\ 0 & 2 \end{bmatrix}, F_1 = \begin{bmatrix} 0.1 & -2 \\ 1 & 1 \end{bmatrix},$$

$$\bar{A}_1 = \begin{bmatrix} 0.6 & 0.1 \\ 0.2 & 1 \end{bmatrix}, D_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, H_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \bar{D}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \bar{H}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, f_1 = 0.1\begin{bmatrix} \sin(x_1(t-1)) \\ \sin(x_2(t-1)) \end{bmatrix}, \mathcal{U}_1 = \sin(t),$$

$\epsilon_1 = 2.2$, $\xi_{11} = 0.1$, $\gamma_1 = 0.1$, $\alpha_1 = 2.5$, $\xi_{21} = \gamma_1^{-2}$, $\xi_{41} = 0.1$, $\xi_{51} = 0.03$, $\xi_{61} = 0.25$, $M_1 = 1.5$, *and* $\theta_1 = 0.05$ *with* $t_0 = 0$. *From* (5.23) *one may get* $\delta_1 = 0.1$.

As for the second mode, we take

$$A_2 = \begin{bmatrix} -9 & 0.2 \\ 0 & 0.1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 0.2 \\ 0.1 & -1 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, F_2 = \begin{bmatrix} 0.1 & 0.2 \\ -4 & 0.1 \end{bmatrix},$$

$$\overline{A}_2 = \begin{bmatrix} 3 & 0.2 \\ 2 & 0.1 \end{bmatrix}, D_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, H_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \overline{D}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \overline{H}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 2 \end{bmatrix}, f_2 = 0.01 \begin{bmatrix} \sin(x_1(t-1)) \\ \sin(x_{2t}(t-1)) \end{bmatrix}, \mathcal{U}_2 = \sin(t),$$

$\epsilon_2 = 0.5$, $\xi_{12} = 0.2$, $\gamma_2 = 0.15$, $\alpha_2 = 2$, $\xi_{22} = \gamma_2^{-2}$, $\xi_{42} = 0.2$, $\xi_{52} = 0.09$, $\xi_{62} = 0.15$, $M_2 = 1.1$ and $\theta_2 = 0.15$. From (5.23), one may get that $\delta_2 = 0.01$. Let the system input disturbance be defined by

$$w(t) = \begin{bmatrix} \sin(t) \\ \sin(t) \end{bmatrix}.$$

Case 1. *[All the actuators are operational]*
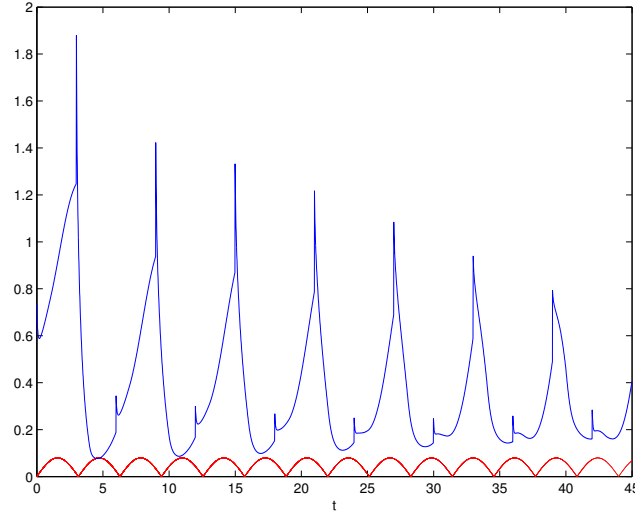
Figure 5.2.1: Input-to-state stabilization: Operational actuators.

118

When all the control actuators are operational, we have from Riccati-like equation,

$$P_1 = \begin{bmatrix} 0.6126 & -0.0560 \\ -0.0560 & 1.0827 \end{bmatrix}, P_2 = \begin{bmatrix} 1.4169 & -0.0746 \\ -0.0746 & 0.5356 \end{bmatrix},$$

with $c_1 = 0.5293$, $c_2 = 1.4232$, and the control gain matrices are

$$K_1 = \begin{bmatrix} 2.0832 & -1.3757 \\ -0.3246 & -0.2074 \end{bmatrix}, K_2 = \begin{bmatrix} -0.3524 & 0.0053 \\ -0.0895 & 0.1376 \end{bmatrix}.$$

Thus, the matrices

$$A_1 + B_1 K_1 = \begin{bmatrix} -6.1120 & 4.1235 \\ 2.2183 & -7.4172 \end{bmatrix}, \ and \ A_2 + B_2 K_2 = \begin{bmatrix} -9.3703 & 0.2328 \\ 0.0543 & -0.0371 \end{bmatrix}$$

are Hurwitz. The average dwell time is $\tau_a = \frac{\ln \mu}{\alpha^* - \nu} = 2.7835$ and $\delta = 3.0543$.

Figure 5.2.1 shows the simulation results of $\|x\|$ (top) and $\gamma(\|w\|) = \sqrt{c_2/c_1}\rho(\|w\|)$ (bottom), where $\rho(s) = \max\{\rho_1(s), \rho_2(s)\}$ and $\rho(s) = s/\sqrt{c_2\theta\xi_2}$, $\tau_a = 3$, and the initial function $\phi(s) = \cos(1 - s)$ for all $s \in [-1, 0]$.

***Case 2.** [Failure in the first actuator in the first mode and second actuator in the second mode] When there is a failure in the first actuator, i.e., $\Sigma_1 = \{1\}$ and $B_{1\overline{\Sigma}} = \begin{bmatrix} 0 & 0.5 \\ 0 & 0.2 \end{bmatrix},$*

and $\Sigma_2 = \{2\}$ and $B_{2\overline{\Sigma}} = \begin{bmatrix} 1 & 0 \\ 0.1 & 0 \end{bmatrix}$, we have from Riccati-like equation,

$$P_1 = \begin{bmatrix} 0.6025 & -0.0880 \\ -0.0880 & 1.0891 \end{bmatrix}, P_2 = \begin{bmatrix} 0.8166 & -0.0619 \\ -0.0619 & 0.5317 \end{bmatrix},$$

with $c_1 = 0.5188$, $c_2 = 1.1046$, and the control gain matrices

$$K_1 = \begin{bmatrix} 2.0849 & -1.4884 \\ 0 & 0 \end{bmatrix}, K_2 = \begin{bmatrix} 0 & 0 \\ -0.0563 & 0.1360 \end{bmatrix}.$$
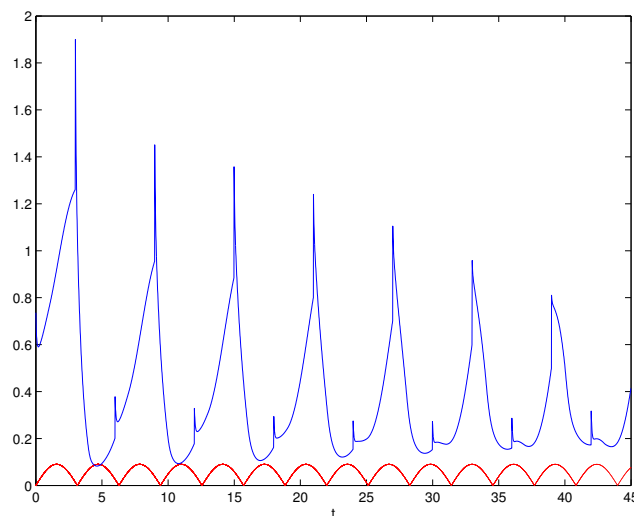
Figure 5.2.2: Input-to-state stabilization: Faulty actuators.

Thus, the matrices

$$A_1 + B_1 K_1 = \begin{bmatrix} -5.9547 & 4.5653 \\ 2.2849 & -7.4884 \end{bmatrix}, \text{ and } A_2 + B_2 K_2 = \begin{bmatrix} -9.0113 & 0.2272 \\ 0.0563 & -0.0360 \end{bmatrix}$$

are Hurwitz, and $\tau_a = 2.4678$.

Figure 5.2.2 shows the simulation results of $\|x\|$ (top) and $\gamma(\|w\|) = \sqrt{c_2/c_1}\rho(\|w\|)$ (bottom), where $\rho(s) = \max\{\rho_1(s), \rho_2(s)\}$ and $\rho(s) = s/\sqrt{c_2\theta\xi_2}$, $\tau_a = 3$, and the initial function $\phi(s) = \cos(1-s)$ for all $s \in [-1, 0]$.
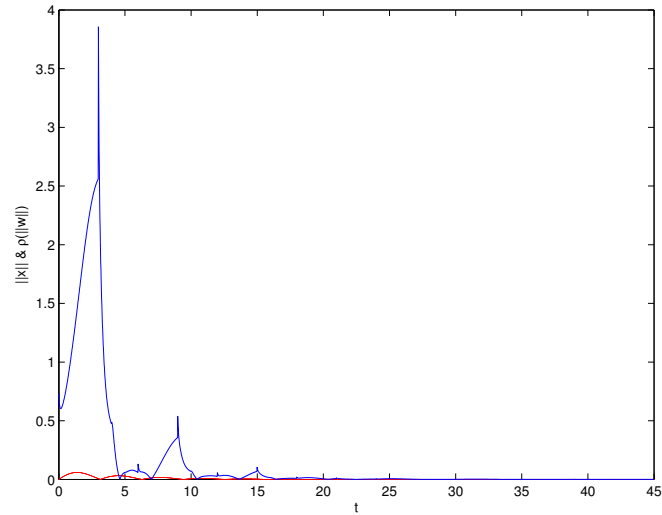
Figure 5.2.3: ISS with a decaying disturbance: Operational actuators.

When choosing a vanishing disturbance such as

$$w(t) = e^{-0.2t} \left[\begin{array}{c} \sin(t) \\ \sin(t) \end{array} \right],$$

the solution decays exponentially to zero in both cases as shown in Figures 5.2.3 and 5.2.4.
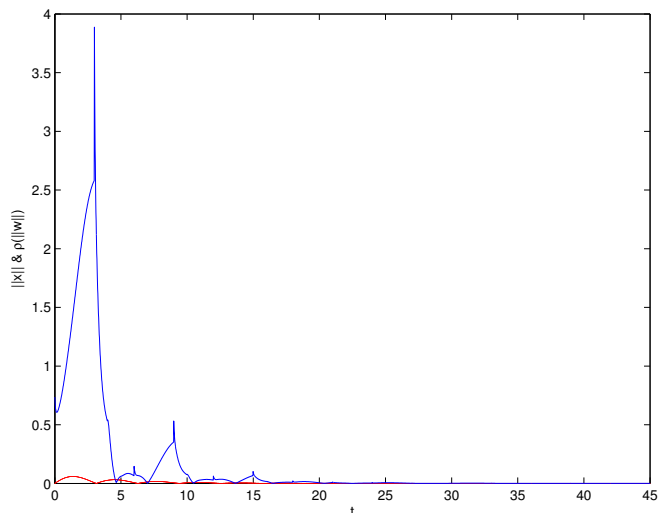
Figure 5.2.4: ISS with a decaying disturbance: Faulty actuators.

5.3 Conclusion

This chapter has discussed the problem of input-to-state stabilization via a robust and reliable H_∞ controller of a class of uncertain impulsive switched systems with time delay. The system under investigation has been input-to-state stabilized by the state feedback controller. The Lyapunov- Razumikhin technique along with average dwell time approach by multiple Lyapunov functions has been used to achieve the results. In addition, we have considered a time-varying parameter uncertainty in the system state, and an L_2 norm-bounded input disturbance. The output of the faulty actuators has been treated as a disturbing signal that has been augmented with the system disturbance. The results are delay independent, and robust with respect to any admissible uncertainty.

122

Part II

Stability and Stabilization of Uncertain Impulsive Large-Scale Systems (ILSSs)

This part discusses the stability and ISS of a class of uncertain impulsive large-scale deterministic and stochastic systems without and with time delay. In this part, we aim to design a robust reliable control that guarantees exponential stability and ISS not only when all the actuators are operational, but also when some of them experience failure.

The faulty output is treated as an outage (i.e., zero output in the case of exponential stability analysis) and as a non-zero disturbance that augmented with the system input disturbance (in the case of ISS analysis).

Thus, new sufficient conditions have been developed here to guarantee the exponential stability and input to state stabilization of the considered LSS in the presence of the state uncertainties, nonlinear lumped perturbation and input disturbance (in the case of ISS analysis) not only when all the actuators are operational, but also when some of them experience failure.

The methodology of scalar Lyapunov function (the linear combination of Lyapunov functions of the isolated subsystems) is used to analyze the stability and ISS for the interconnected system. Moreover, in the system with time delay, Lyapunov-Razumikhin technique is adopted.

Finally, some numerical examples with simulations are presented to clarify the theoretical results.

Chapter 6

Robust and Reliable Control of Uncertain ILSSs

The main objective of this chapter is to design a robust reliable control that guarantees global exponential stability of uncertain ILSS. The faulty actuator/sensor output is treated as an outage i.e., complete failure. Scalar Lyapunov function that is the linear combination of the Lyapunov functions of the corresponding isolated subsystems is used to analyze the stability of the LSS, and consequently, a Riccati-like equation is solved. For the ILSS to be exponentially stable, it is required that the degree of stability be greater than the interconnection. This type of relation is represented by the so-called test matrix. Furthermore, the state estimation to the large-scale system is also considered using the time-domain approach. Moreover, in this work, Luenberger observer is used to estimate the states. The material of this chapter forms the basis of [157].

6.1 Problem Formulation and Preliminaries

Consider the interconnected system

$$
\begin{cases}
\dot{w}^i = (A_i + \Delta A_i)w^i + B_i u_i + f_i(w^i) \\
\quad + g_i(w^1, w^2, \cdots, w^i, \cdots, w^l), \ t \neq t_k, \\
\Delta w^i(t) = I_k(w^i(t^-)) = C_{ik} w^i(t^-), \quad t = t_k, \quad k = 1, 2, \cdots, \\
w^i(t_0) = w_0^i,
\end{cases}
\tag{6.1}
$$

where $i = 1, 2, \cdots, l$, $w^i \in \mathbb{R}^{n_i}$ is the ith subsystem state, such that $\sum_{i=1}^{l} n_i = n$, $A_i \in \mathbb{R}^{n_i \times n_i}$ is not a Hurwitz matrix for each i, the impulsive times t_k satisfying $t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, with $\lim_{k \to \infty} t_k = \infty$, $\Delta w^i(t_k) = w^i(t_k^+) - w^i(t_k^-)$ where $w(t_k^+)$(or $w(t_k^-)$) is the state just after (or before) the impulse at t_k, and the function $I_k : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ is the impulsive function, $u_i = K_i w^i \in \mathbb{R}^q$ is the control input for the ith subsystem, where $K_i \in \mathbb{R}^{q \times n_i}$ is the control gain matrix, $f_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$, is some nonlinearity, $g_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_l} \to \mathbb{R}^n$ is the interconnection. The functions f_i and g_i satisfy Lipschitz condition. A_i, B_i, and C_{ik} are known real constant matrices with proper dimensions, and ΔA_i is a piecewise continuous function representing parameter uncertainty with bounded norm.

System (6.1) can be written in the following form

$$
\begin{cases}
\dot{x} = (A + \Delta A)x + Bu + F(x) + G(x), \quad t \neq t_k, \\
\Delta x(t) = I_k(x(t^-)) = C_k x(t^-), \quad t = t_k, \quad k = 1, 2, 3, \cdots, \\
x(t_0) = x_0,
\end{cases}
\tag{6.2}
$$

where

$$
x^T = (w^{1^T} \ w^{2^T} \ \cdots \ w^{l^T}),
$$

$$((A + \Delta A)x)^T = \left(\left((A^1 + \Delta A^1)w^1\right)^T \left((A^2 + \Delta A^2)w^2\right)^T \cdots \left((A^l + \Delta A^l)w^l\right)^T\right),$$

$$(Bu)^T = \left((B^1 u^1)^T \ (B^2 u^2)^T \ \cdots \ (B^l u^l)^T\right),$$

$$(F(x))^T = \left(f_1(w^1)^T \ f_2(w^2)^T \ \cdots \ f_l(w^l)^T\right),$$

$$(G(x))^T = \left(g^1(x)^T \ g^2(x)^T \ \cdots \ g^l(x)^T\right),$$

$$(C_k x)^T = \left((C_{1k} w^1)^T \ (C_{2k} w^2)^T \ \cdots \ (C_{lk} w^l)^T\right).$$

From (6.1), the corresponding isolated subsystems are

$$\begin{cases} \dot{w}^i = (A_i + \Delta A_i)w^i + B_i u_i + f_i(w^i), & t \neq t_k, \\ \Delta w^i(t) = C_{ik} w^i(t^-), & t = t_k, \quad k = 1, 2, \cdots, \\ w^i(t_0) = w_0^i, \end{cases} \tag{6.3}$$

where $i = 1, 2, \cdots, l$, and the corresponding closed-loop system is

$$\begin{cases} \dot{w}^i = (A_i + \Delta A_i + B_i K_i)w^i + f_i(w^i), & t \neq t_k, \\ \Delta w^i(t) = C_{ik} w^i(t^-), & t = t_k, \quad k = 1, 2, \cdots, \\ w^i(t_0) = w_0^i. \end{cases} \tag{6.4}$$

Then, the closed-loop systems for the faulty case becomes

$$\begin{cases} \dot{w}^i = (A_i + \Delta A_i + B_{i\bar{\sigma}} K_i)w^i + f_i(w^i), & t \neq t_k, \\ \Delta w^i(t) = C_{ik} w^i(t^-), & t = t_k, \quad k = 1, 2, \cdots, \\ w^i(t_0) = w_0^i. \end{cases} \tag{6.5}$$

The main objective of this chapter will be discussed in two sections, namely, the reliable control and the state estimation using Lunberger observer.

127

6.2 Reliable Control

In this section, we present four theorems. Theorem 6.2.1 and Theorem 6.2.4 discuss the robust controller for the impulsive isolated subsystems (6.3) to guarantee the global exponential stability for the operational and faulty actuator cases respectively. Theorem 6.2.7, and Theorem 6.2.9 deal with the interconnected system (6.2) without and with actuator failures respectively.

Theorem 6.2.1. *Let the controller gain K_i be given, and assume that Assumption A holds. Then, the trivial solution of system (6.4) is robustly globally exponentially stable if the following inequality holds*

$$\ln \alpha_{ik} - \nu_i(t_k - t_{k-1}) \leq 0, \quad k = 1, 2, \cdots, \tag{6.6}$$

where $\alpha_{ik} = \frac{\lambda_{\max}[(I+C_{ik})^T P_i(I+C_{ik})]}{\lambda_{\min}(P_i)}$, with P_i being a positive-definite matrix satisfying the Riccati-like equation

$$(A_i + B_i K_i)^T P_i + P_i(A_i + B_i K_i) + \xi_{1i} P_i D_i D_i^T P_i + \frac{1}{\xi_{1i}} H_i^T H_i + \xi_{3i} P_i^2$$

$$+ \frac{\delta_i I}{\xi_{3i}} - \sigma_i P_i = 0 \tag{6.7}$$

where ξ_{1i} and ξ_{3i} are any positive constants, $0 < \nu_i < -\sigma_i$, $\sigma_i < 0$, and δ_i is a positive constant such that

$$\|f_i(w^i)\|^2 \leq \delta_i \|w^i\|^2. \tag{6.8}$$

Proof. Let $w^i(t) = w^i(t, t_0, w_0^i)$ be the solution of system (6.4). For all $i = 1, 2, \cdots, l$, define $V^i(w^i) = w^{iT} P_i w^i$ as a Lyapunov function candidate for the ith subsystem. Then,

$$\dot{V}^i(w^i) = \dot{w^i}^T P_i w^i + (w^i)^T P_i \dot{w^i}$$

$$= [(A_i + \Delta A_i + B_i K_i) w^i + f_i(w^i)]^T P_i w^i + (w^i)^T P_i[(A_i + \Delta A_i + B_i K_i) w^i + f_i(w^i)]$$

$$= w^{iT} \Big[(A_i + B_i K_i)^T P_i + P_i(A_i + B_i K_i) \Big] w^i + 2w^{iT} P_i(\Delta A_i) w^i + 2w^{iT} P_i f_i(w^i)$$

$$\leq w^{iT} \Big[(A_i + B_i K_i)^T P_i + P_i(A_i + B_i K_i) + \xi_{1i} P_i D_i D_i^T P_i + \frac{1}{\xi_{1i}} H_i^T H_i + \xi_{3i} P_i^2$$

$$+ \frac{\delta_i I}{\xi_{3i}} \Big] w^i$$

$$= \sigma_i V^i(w^i),$$

where we used (6.8) and Lemma 3.1.3 in the second bottom line, and condition (6.7) in the last line. Then, for all $t \in (t_{k-1}, t_k]$, $k = 1, 2, \cdots$, one may have

$$V^i(w^i(t)) \leq V^i(w^i(t_{k-1}^+)) e^{\sigma_i(t-t_{k-1})}. \tag{6.9}$$

At $t = t_k^+$, we have

$$V^i(w^i(t_k^+)) \leq \lambda_{\max}(L_{ik}) w^{iT}(t_k) w^i(t_k)$$

$$\leq \alpha_{ik} V^i(w^i(t_k^-)), \tag{6.10}$$

where $\alpha_{ik} = \frac{\lambda_{\max}(L_{ik})}{\lambda_{\min}(P_i)}$, and $L_{ik} = [I + C_{ik}]^T P_i[I + C_{ik}]$.

From (6.9) and (6.10), we have for $t \in [t_0, t_1]$,

$$V^i(w^i(t)) \leq V^i(w_0^i) e^{\sigma_i(t-t_0)},$$

129

and for $t \in (t_1, t_2]$,

$$V^i(w^i(t_1^+)) \leq \alpha_{i1} V^i(w_0^i) e^{\sigma_i(t_1 - t_0)},$$

$$V^i(w^i(t)) \leq V^i(w^i(t_1^+)) e^{\sigma_i(t - t_1)},$$

which leads to

$$
\begin{aligned}
V^i(w^i(t)) &\leq \alpha_{i1} V^i(w_0^i) e^{\sigma_i(t_1 - t_0)} e^{\sigma_i(t - t_1)} \\
&= \alpha_{i1} V^i(w_0^i) e^{\sigma_i(t - t_0)}, \quad \text{for} \quad t \in [t_0, t_2].
\end{aligned}
$$

Generally, for $t \in (t_{k-1}, t_k]$, we have

$$
\begin{aligned}
V^i(w^i(t)) &\leq V^i(w_0^i) \, \alpha_{i1} \, \alpha_{i2} \, \cdots \, \alpha_{ik} \, e^{\sigma_i(t - t_0)} \\
&= V^i(w_0^i) \, \alpha_{i1} e^{-\nu_i(t_1 - t_0)} \, \cdots \, \alpha_{ik} e^{-\nu_i(t_k - t_{k-1})} e^{(\sigma_i + \nu_i)(t - t_0)} \\
&\leq V^i(w_0^i) e^{(\sigma_i + \nu_i)(t - t_0)}, \qquad t \geq t_0,
\end{aligned}
$$

where $0 < \nu_i < -\sigma_i$ and we used condition (6.6) to get the last inequality. The foregoing inequality implies that

$$\|w^i\| \leq \gamma_i \|w_0^i\| e^{(\sigma_i + \nu_i)(t - t_0)/2}, \quad t \geq t_0,$$

where $\gamma_i = \sqrt{\frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_i)}}$. This completes the proof of globally exponential stability of $w^i = 0$. $\square$

Remark 6.2.2. *Theorem 6.2.1 gives sufficient conditions to ensure robust global exponential stability for each isolated impulsive subsystem (6.4) by a state feedback controller. The*

time between impulses has to be bounded, and this condition is summarized in (6.6). *The nonlinearity is assumed to be bounded by some linear growth bound. Condition* (6.7) *guarantees that the Lyapunov function is decreasing along the trajectory of system* (6.4), *that is, the continuous system is stabilized by the feedback controller.*

Remark 6.2.3. *Condition* (6.8) *of Theorem* 6.2.1 *is assumed to hold globally, witch is a strong requirement on the function* $f_i(w^i)$. *If we just want local exponential stability, condition* (6.8) *may be relaxed to hold on a bounded region.*

The following theorem, on the other hand gives sufficient conditions to ensure robust global exponential stability for all the isolated impulsive subsystems when some control components (actuators) experience failure.

Theorem 6.2.4 (Reliability for isolated subsystems)**.** *The trivial solution of system* (6.5) *is robustly globally exponentially stable if Assumption A, and condition* (6.6) *hold with* P_i *being a positive-definite matrix satisfying the Riccati-like equation*

$$A_i^T P_i + P_i A_i + P_i(\xi_{1i} D_i D_i^T - \epsilon_{1i} B_{i\bar{\sigma}} B_{i\bar{\sigma}}^T + \xi_{3i} I)P_i + \frac{1}{\xi_{1i}}H_i^T H_i + \frac{\delta_i I}{\xi_{3i}} - \sigma_i P_i = 0, \quad (6.11)$$

where ξ_{1i}, ϵ_{1i} *and* ξ_{3i} *are positive constants such that the controller gain* $K_i = -\frac{1}{2}\epsilon_{1i}B_{i\bar{\sigma}}^T P_i$, $0 < \nu_i < -\sigma_i$, $\sigma_i < 0$, *and* δ_i *is a positive constant such that condition* (6.8) *holds.*

Proof. Let $w^i(t) = w^i(t, t_0, w_0^i)$ be the solution of system (6.5). As done in Theorem 6.2.1,

define $V^i(w^i) = w^{i^T} P_i w^i$. Then

$$\dot{V}^i(w^i) \leq w^{i^T} \Big[A_i^T P_i + P_i A_i + P_i(\xi_{1i} D_i D_i^T - \epsilon_{1i} B_{i\bar{\sigma}} B_{i\bar{\sigma}}^T + \xi_{3i} I) P_i$$

$$+ \frac{1}{\xi_{1i}} H_i^T H_i + \frac{\delta_i I}{\xi_{3i}} \Big] w^i$$

$$\leq w^{i^T} \Big[A_i^T P_i + P_i A_i + P_i(\xi_{1i} D_i D_i^T - \epsilon_{1i} B_{i\bar{\Sigma}} B_{i\bar{\Sigma}}^T + \xi_{3i} I) P_i$$

$$+ \frac{1}{\xi_{1i}} H_i^T H_i + \frac{\delta_i I}{\xi_{3i}} \Big] w^i$$

$$= \sigma_i V^i(w^i(t)),$$

where we used the fact $B_{i\bar{\Sigma}} B_{i\bar{\Sigma}}^T \leq B_{i\bar{\sigma}}(B_{i\bar{\sigma}})^T$ in the second last line and condition (6.11) in the last line. Following the same procedure as in the previous proof will show that the trivial solution of the closed-loop impulsive system (6.5) is robustly globally exponentially stable. $\qquad\square$

Having established the stabilizability of isolated subsystem in Theorems 6.2.1 and 6.2.4, we prove the same properties for the interconnected systems. The following definition is needed.

Definition 6.2.5. *System (6.4) (or (6.5)) is said to possess property A (or B) if it satisfies the conditions in Theorem 6.2.1(or 6.2.4).*

Remark 6.2.6. *Property A implies that all the impulsive isolated subsystems are robustly globally exponentially stable in the normal actuators case, while Property B implies the same result is hold in the faulty case.*

Theorem 6.2.7. *Assume that system* (6.4) *possesses property A. Suppose further that, for any* $i, j = 1, 2, \cdots, l$, *there exist positive constants* b_{ij} *such that*

$$2w^{i^T} P_i g_i(w^1, w^2, \cdots, w^i, \cdots, w^l) \leq \|w^i\| \sum_{j=1}^{l} b_{ij} \|w^j\|, \tag{6.12}$$

and the test matrix $\mathcal{S} = [s_{ij}]_{l \times l}$ *is negative definite where*

$$s_{ij} = \begin{cases} \beta_i(\sigma_i^* + b_{ii}), & i = j \\ \frac{1}{2}(\beta_i b_{ij} + \beta_j b_{ji}), & i \neq j \end{cases}, \tag{6.13}$$

for some constant $\sigma_i^* = \sigma_i \lambda_{\max}(P_i) < 0$, *and positive constant* β_i. *Then, the trivial solution of system* (6.2) *is robustly globally exponentially stable if the following inequality holds*

$$\ln \alpha_k - \phi(t_k - t_{k-1}) \leq 0, \quad k = 1, 2, \cdots, \tag{6.14}$$

for $0 < \phi < \theta$ *where* $\theta = -\frac{\lambda_{\max}(\mathcal{S})}{\bar{\lambda} \beta^*}$ *with* $\bar{\lambda} = \min\{\lambda_{\max}(P_i) : i = 1, 2, \cdots, l\}$ *and* $\beta^* = \min\{\beta_i : i = 1, 2, \cdots, l\}$, $\alpha_k = \left[\max\{\lambda_{\max}[(I + C_{ik})^T P_i(I + C_{ik})] : i = 1, 2, \cdots, l\} \right]/\lambda^*$, *with* $\lambda^* = \min\{\lambda_{\min}(P_i) : i = 1, 2, \cdots, l\}$ *and* P_i *being a positive-definite matrix defined in Property A.*

Proof. Let $x(t) = x(t, t_0, x_0)$ be the solution of system (6.2). Define the composite Lyapunov function

$$V(x(t)) = \sum_{i=1}^{l} \beta_i V^i(w^i)$$

as a Lyapunov function candidate for interconnected system (6.2) where β_i is a positive constant, and $V^i(w^i)$ is a Lyapunov function for the ith isolated subsystem. Then, along the trajectory of (6.2), we have

$$\dot{V}(x) = \sum_{i=1}^{l} \beta_i \dot{V}^i(w^i)$$

$$\leq \sum_{i=1}^{l} \beta_i \{\sigma_i \|w^i\|^2 + 2w^{i^T} P_i g_i(w^1, w^2, \cdots, w^i, \cdots, w^l)\}$$

$$\leq \sum_{i=1}^{l} \beta_i \{\sigma_i \|w^i\|^2 + \|w^i\| \Sigma_{j=1}^{l} b_{ij} \|w^j\|\}$$

$$= z^T \mathcal{S} z,$$

where $z^T = (\|w^1\|, \|w^2\|, \cdots, \|w^i\|, \cdots, \|w^l\|)$ and $\mathcal{S}$ is a negative definite matrix with the maximum eigenvalue $\lambda_{\max}(\mathcal{S})$. Then, one can write

$$\dot{V}(x) \leq -\theta V(x),$$

where $\theta = -\frac{\lambda_{\max}(\mathcal{S})}{\bar{\lambda}\beta^*}$ with $\bar{\lambda} = \min\{\lambda_{\max}(P_i) : i = 1, 2, \cdots, l\}$ and $\beta^* = \min\{\beta_i : i = 1, 2, \cdots, l\}$. The last inequality implies that, for all $t \in (t_{k-1}, t_k]$,

$$V(x(t)) \leq V(x(t_{k-1}^+))e^{-\theta(t-t_{k-1})}, \tag{6.15}$$

and, at $t = t_k^+$,

134

$$V(x(t_k^+)) = \sum_{i=1}^{l} \beta_i {w^i}^T(t_k) \big[(I + C_{ik})^T P_i (I + C_{ik}) \big] w^i(t_k)$$

$$\leq L^{**} \sum_{i=1}^{l} \beta_i {w^i}^T(t_k) w^i(t_k)$$

$$\leq \frac{L^{**}}{\lambda^*} \sum_{i=1}^{l} \beta_i V^i(w^i)$$

$$= \alpha_k V(x(t)), \tag{6.16}$$

where $\alpha_k = \frac{L^{**}}{\lambda^*}$, $L^{**} = \max\{\lambda_{\max}(L_{ik}) : i = 1, \cdots, l\}$ with $L_{ik} = \big[(I + C_{ik})^T P_i (I + C_{ik}) \big]$ and $\lambda^* = \min\{\lambda_{\min}(P_i) : i = 1, \cdots, l\}$. From (6.15) and (6.16), we have for $t \in [t_0, t_1]$,

$$V(x(t)) \leq V(x_0) e^{-\theta(t-t_0)},$$

and for $t \in (t_1, t_2]$, we have

$$V(x(t_1^+)) \leq \alpha_1 V(x(t_1))$$

$$\leq \alpha_1 V(x_0) e^{-\theta(t_1-t_0)},$$

and

$$V(x(t)) \leq V(x(t_1^+)) e^{-\theta(t-t_1)}$$

$$\leq \alpha_1 V(x_0) e^{-\theta(t_1-t_0)} e^{-\theta(t-t_1)},$$

that is

$$V(x(t)) \leq \alpha_1 V(x_0) e^{-\theta(t-t_0)}, \quad t \in [t_0, t_2].$$

135

Therefore, for all $t \in (t_{k-1}, t_k]$,

$$V(x(t)) \leq V(x_0) \, \alpha_1 \, \alpha_2 \, \cdots \, \alpha_k \, e^{-\theta(t-t_0)}$$

$$\leq V(x_0) \, \alpha_1 e^{-\phi(t_1-t_0)} \, \alpha_2 e^{-\phi(t_2-t_1)} \, \cdots \, \alpha_k \, e^{-\phi(t_k-t_{k-1})} e^{-(\theta-\phi)(t-t_0)}$$

$$\leq V(x_0) \, e^{-(\theta-\phi)(t-t_0)}, \qquad t \geq t_0, \tag{6.17}$$

where $0 < \phi < \theta$. The forgoing inequality together with

$$C^* \|x\|^2 \leq V(x) \leq C^{**} \|x\|^2,$$

where $C^* = \lambda^* \beta^*$, and $C^{**} = \lambda^{**} \beta^{**}$ with $\lambda^{**} = \max\{\lambda_{\max}(P_i) \, : \, i = 1, \cdots, l\}$ and $\beta^{**} = \max\{\beta_i \, : \, i = 1, \cdots, l\}$, implies that

$$\|x\| \leq E \|x_0\| \, e^{-(\theta-\phi)(t-t_0)/2}, \qquad t \geq t_0,$$

where $E = \sqrt{\frac{C^{**}}{C^*}}$. That is, the trivial solution of the composite system (6.2) is robustly globally exponentially stable. $\qquad \square$

Remark 6.2.8. *Theorem 6.2.7 shows that the interconnected system can be robustly exponentially stabilized by the controllers of the isolated subsystems in the case where all the actuators are operational. Condition (6.12) estimates the interconnection, which is viewed as a perturbation, by an upper bound. The test matrix is needed to guarantee that the degree of stability is greater than the interconnection.*

The following theorem shows that the proposed reliable controllers are robust even in the presence of the interconnection effect. The proof is similar to that of Theorem 6.2.7, and thus, it is omitted here.

Theorem 6.2.9. *Assume that system* (6.5) *possesses property B. Suppose that for any* $i, j = 1, 2, \cdots, l$, *there exist positive constants* b_{ij} *such that the condition in* (6.12) *holds and the test matrix* $\mathcal{S} = [s_{ij}]_{l \times l}$, *defined in Theorem* 6.2.7, *is negative definite,* ϵ_{1i} *is a positive constant such that* $K_i = -\frac{1}{2}\epsilon_{1i}B_{i\bar{\sigma}}^T P_i$. *Then, the trivial solution of system* (6.2) *is robustly globally exponentially stable if* (6.14) *holds with* P_i *being a positive-definite matrix defined in Property B.*

Example 6.2.10. *Consider the composite system with* $l = 2$, *and the following information for the subsystems*

$$A^1 = \begin{bmatrix} 0 & 1 \\ -11 & 0 \end{bmatrix}, A^2 = \begin{bmatrix} 0 & 1 \\ -10 & 0 \end{bmatrix}, B^1 = \begin{bmatrix} -5 & 3 \\ -1 & 2 \end{bmatrix}, B^2 = \begin{bmatrix} 1 & -3 \\ 0.1 & -4 \end{bmatrix},$$

$$D^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, D^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, H^1 = \begin{bmatrix} 0 & 1 \end{bmatrix}, H^2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \mathcal{U}_1 = \mathcal{U}_2 = \sin(t),$$

$$f_1 = 0.5 \begin{bmatrix} 0 \\ \sin(w_2) \end{bmatrix}, f_2 = 1.5 \begin{bmatrix} 0 \\ \sin(w_4) \end{bmatrix}, C_{1k} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, C_{2k} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix},$$

for all $k = 1, 2, \cdots$, $\sigma_1 = -2$, $\sigma_2 = -2.5$, $\xi_{11} = 2$, $\xi_{12} = 0.5$, $\xi_{31} = 1$, $\xi_{32} = 1$, $\epsilon_{11} = 1$, $\epsilon_{12} = 0.7$, $\beta_1 = 1$, $\beta_2 = 2$, $b_{11} = 0.3$, $b_{22} = 1.5$, $b_{12} = 0.5$, $b_{21} = 0.3$ *and* $t_0 = 0$. *From* (6.8), *one may get* $\delta_1 = 0.25$ *and* $\delta_2 = 2.25$.

Case 1. *When all the control actuators are operational, we have from Riccati-like equation,*

$$P^1 = \begin{bmatrix} 0.5427 & -0.2419 \\ -0.2419 & 0.1955 \end{bmatrix}, P^2 = \begin{bmatrix} 2.9461 & -1.2229 \\ -1.2229 & 0.7834 \end{bmatrix},$$

with $\lambda_{\min}(P^1) = 0.0713$, $\lambda_{\max}(P^1) = 0.6669$, $\lambda_{\min}(P^2) = 0.2323$, $\lambda_{\max}(P^2) = 3.4971$, *so*

that $\lambda^* = 0.0713$, $\lambda^{**} = 3.4971$, *and the control gain matrices are*

$$K^1 = \begin{bmatrix} 1.2358 & -0.5071 \\ -0.5722 & 0.1674 \end{bmatrix}, K^2 = \begin{bmatrix} -0.9883 & 0.4006 \\ 1.3814 & -0.1873 \end{bmatrix}.$$

Thus, $A_i + B_i K_i$ *for* $i = 1, 2$, *are Hurwitz, and the time intervals* $t_k - t_{k-1} \geq 2.3328$ *for*

the first subsystem, and $t_k - t_{k-1} \geq 2.7421$ *for the second subsystem.*

The test matrix here is given by

$$\mathcal{S} = \begin{bmatrix} -1.0338 & 0.55 \\ 0.55 & -14.4855 \end{bmatrix},$$

which is negative definite matrix, and $t_k - t_{k-1} \geq 4.4142$ *for the interconnected system.*

(a) Isolated subsystem 1.

(b) Isolated subsystem 2.
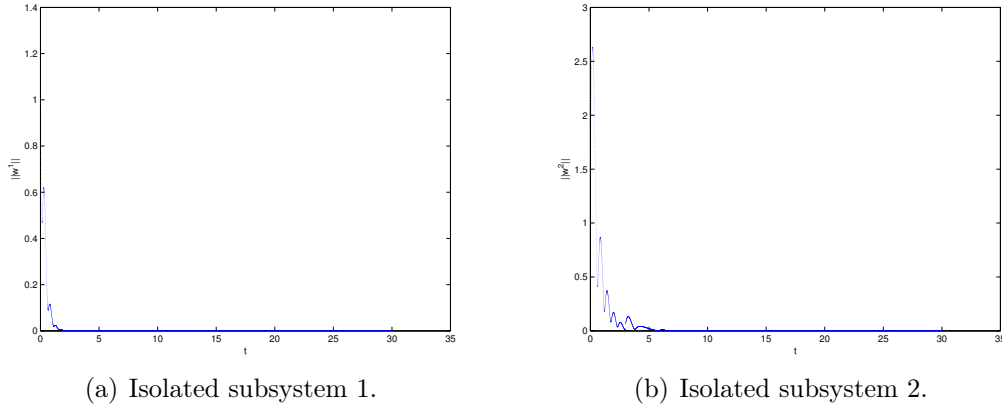
Figure 6.2.1: Operational actuators.
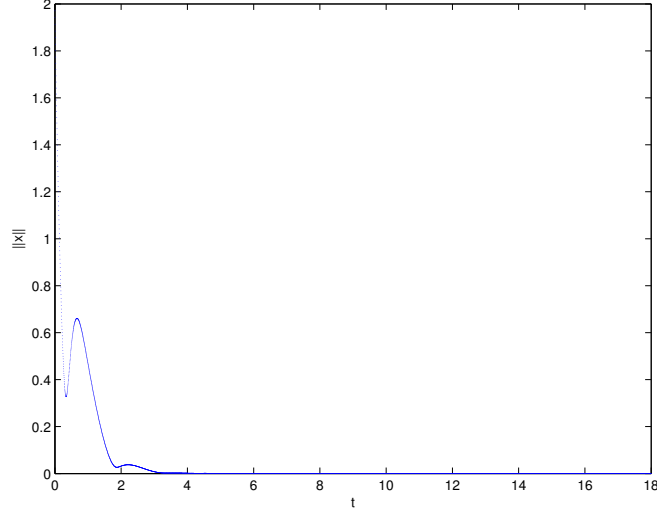
Figure 6.2.2: Interconnected system: Operational case.

Case 2. *When there is a failure in the second actuator in the first subsystem and first*

actuator in the second subsystem, i.e., $\Sigma^1 = \{2\}$ *and* $B^1_{\bar{\Sigma}} = \begin{bmatrix} -5 & 0 \\ -1 & 0 \end{bmatrix}$, *and* $\Sigma^2 = \{1\}$ *and*

$B^2_{\bar{\Sigma}} = \begin{bmatrix} 0 & -3 \\ 0 & -4 \end{bmatrix}$, *we have from Riccati-like equation,*

$$P^1 = \begin{bmatrix} 0.5806 & -0.2330 \\ -0.2330 & 0.2008 \end{bmatrix}, P^2 = \begin{bmatrix} 3.0616 & -1.2448 \\ -1.2448 & 0.7834 \end{bmatrix},$$

with $\lambda_{\min}(P^1) = 0.0901$, $\lambda_{\max}(P^1) = 0.6913$, $\lambda_{\min}(P^2) = 0.2351$, $\lambda_{\max}(P^2) = 3.6099$, *so*

$\lambda^* = 0.0901$, $\lambda^{**} = 3.6099$, *and the control gain matrices are*

$$K^1 = \begin{bmatrix} 1.3351 & -0.4820 \\ 0 & 0 \end{bmatrix}, K^2 = \begin{bmatrix} 0 & 0 \\ 1.4719 & -0.2103 \end{bmatrix}.$$

139

Thus, $A_i + B_{i\bar{\sigma}}K_i$ for $i = 1, 2,$ are Hurwitz, and the time intervals $t_k - t_{k-1} \geq 2.2286$ for the first subsystem and $t_k - t_{k-1} \geq 2.7519$ for the second subsystem.

(a) Isolated subsystem 1.

(b) Isolated subsystem 2.
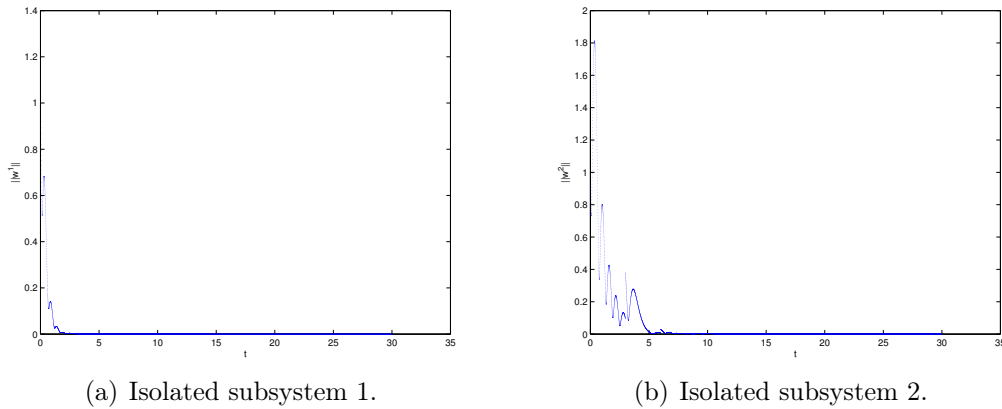
Figure 6.2.3: Faulty actuators.

Figures 6.2.1 and 6.2.3 show the isolated subsystems for both cases, while the inter-connected system is shown in Figures 6.2.2 and 6.2.4 for the operational and faulty cases respectively.

If we consider

$$f_1 = 0.5 \begin{bmatrix} w_1 \\ w_2^2 \end{bmatrix}, f_2 = 1.5 \begin{bmatrix} w_3 \\ w_4^2, \end{bmatrix},$$

one can show that condition (6.8) is satisfied only inside the region $\mathcal{D} = \{(w_1, w_2, w_3, w_4) \in \mathbb{R}^4 : w_1 \in \mathbb{R}, -2 \leq w_2 \leq 2, w_3 \in \mathbb{R}, -1.5 \leq w_4 \leq 1.5\}$. Thus, $x = 0$ is locally exponentially stable. The local stability and the instability of the trivial solution are shown in Figures 6.2.5(a), and 6.2.5(b), respectively.

140
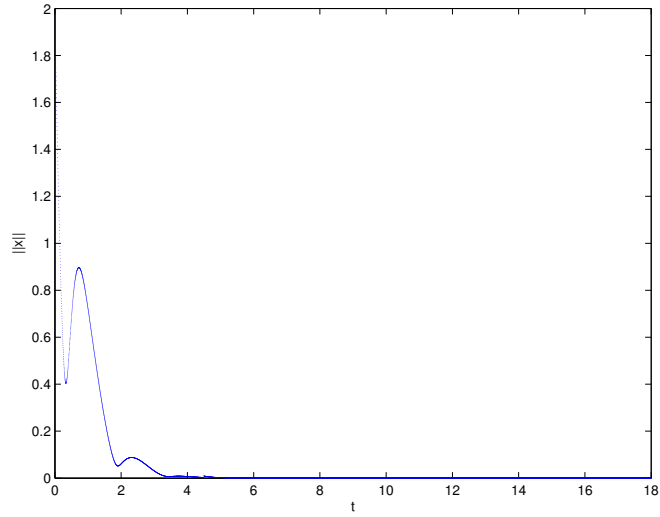
Figure 6.2.4: Interconnected system: Faulty case.
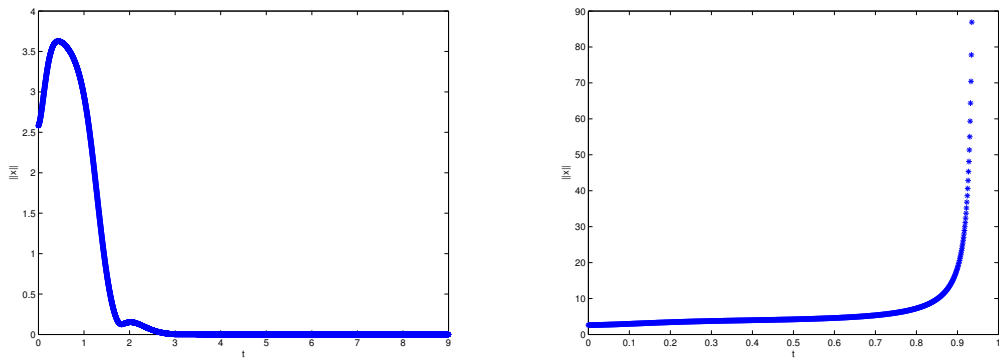
(a) $(w_{1_0}, w_{2_0}, w_{3_0}, w_{4_0}) = (1, 1.6, 0.4, 1.71)$.

(b) $(w_{1_0}, w_{2_0}, w_{3_0}, w_{4_0}) = (1, 1.6, 0.4, 1.751)$.

Figure 6.2.5: Normal case with $f_1 = 0.5[w_1 \ (w_2)^2]^T$ and $f_2 = 1.5[w_3 \ (w_4)^2]^T$.

141

6.3 State Estimation

To characterize the system state variables' evolution, it is helpful to have access to all these variables. However, this may not be the case due to complexity of output measurements or high cost. Therefore, it is necessary to design an observer to estimate the system output using the available information. This problem of state estimation has drawn much attention. See $[1, 47, 163, 165, 182]$ and many references therein. This section discusses the state estimation of the ILSS. Consider the isolated subsystem

$$
\begin{cases}
\dot{w}^i = (A_i + \Delta A_i)w^i + B_i u_i + f_i(w^i), \quad t \neq t_k, \\
\Delta w^i(t) = C_{ik} w^i(t^-), \quad t = t_k, \quad k = 1, 2, \cdots, \\
y^i(t) = C_i w^i(t), \\
w^i(t_0) = w_0^i,
\end{cases}
\tag{6.18}
$$

where $y^i(t) \in \mathbb{R}^{n_i}$ is the measured output vector. Define the Luenberger observer as follows

$$
\begin{cases}
\dot{\hat{w}}^i = (A_i + \Delta A_i)\hat{w}^i + B_i u_i + f_i(\hat{w}^i) + \mathcal{L}_i(y^i - C_i\hat{w}^i), \quad t \neq t_k, \\
\Delta \hat{w}^i(t) = C_{ik} \hat{w}^i(t^-), \quad t = t_k, \quad k = 1, 2, \cdots, \\
\hat{w}^i(t_0) = \hat{w}_0^i,
\end{cases}
\tag{6.19}
$$

where $\mathcal{L}_i \in \mathbb{R}^{n_i \times n_i}$ is the observer gain matrix. Define the state estimation error by $e^i = w^i - \hat{w}^i$. Then, the closed-loop error system becomes

$$
\begin{cases}
\dot{e}^i = (A_i + \Delta A_i - \mathcal{L}_i C_i)e^i + f_i(w^i) - f_i(\hat{w}^i), \quad t \neq t_k, \\
\Delta e^i(t) = C_{ik} e^i(t^-), \quad t = t_k, \quad k = 1, 2, \cdots, \\
e^i(t_0) = w_0^i - \hat{w}_0^i = e_0^i,
\end{cases}
\tag{6.20}
$$

Definition 6.3.1. *The pair (A, B) is said to be detectable if there exists a matrix F such that $A - FB$ is Hurwitz.*

We will use the same stability analysis followed in the last section to establish the observability problem of system (6.18).

Theorem 6.3.2. *Let the observer gain $\mathcal{L}_i$ be given, and assume that Assumption A holds, and (A_i, C_i) be detectable. Then, the trivial solution of the error system (6.20) is robustly globally exponentially stable if the following inequality holds*

$$\ln \alpha_{ik} - \nu_i(t_k - t_{k-1}) \leq 0, \quad k = 1, 2, \cdots, \tag{6.21}$$

where $\alpha_{ik} = \frac{\lambda_{\max}[(I+C_{ik})^T P_i(I+C_{ik})]}{\lambda_{\min}(P_i)}$, with P_i being a positive-definite matrix satisfying the Riccati-like equation

$$(A_i - \mathcal{L}_i C_i)^T P_i + P_i(A_i - \mathcal{L}_i C_i) + \xi_{1i} P_i D_i D_i^T P_i + \frac{1}{\xi_{1i}} H_i^T H_i + a_i I$$

$$- \sigma_i P_i = 0, \tag{6.22}$$

where ξ_{1i} is any positive constants, $0 < \nu_i < -\sigma_i$, $\sigma_i < 0$, $a_i > 0$ such that

$$2 e^{i^T} P_i[f_i(w^i) - f_i(\hat{w}^i)] \leq a_i \|e^i\|^2. \tag{6.23}$$

Proof. Let $e^i(t) = e^i(t, t_0, e_0^i)$ be the solution of the error system (6.20). For all $i = 1, 2, \cdots, l$, define $V^i(e^i) = e^{i^T} P_i e^i$ as a Lyapunov function candidate for the ith subsystem. Then,

$$\dot{V}^i(e^i) = e^{i^T}[(A_i - \mathcal{L}_i C_i)^T P_i + P_i(A_i - \mathcal{L}_i C_i)]e^i + 2 e^{i^T} P_i \Delta A_i e^i + 2 e^{i^T} P_i f_i e^i$$

$$\leq e^{i^T}[(A_i - \mathcal{L}_i C_i)^T P_i + P_i(A_i - \mathcal{L}_i C_i) + \xi_{1i} P_i D_i D_i^T P_i + \frac{1}{\xi_{1i}} H_i^T H_i$$

$$+ a_i I]e^i = \sigma_i V^i(e^i),$$

143

where we used (6.23) and Lemma 3.1.3 in the second bottom line, and (6.22) in the last line. The last inequality implies that, for $t \in (t_{k-1}, t_k]$, $k = 1, 2, \cdots$,

$$V^i(e^i(t)) \leq V^i(e^i(t_{k-1}^+))e^{\sigma_i(t-t_{k-1})}, \tag{6.24}$$

and at $t = t_k$,

$$V^i(e^i(t_k^+)) \leq \alpha_{ik} V^i(e^i(t_k^-)), \tag{6.25}$$

where $\alpha_{ik} = \frac{\lambda_{\max}(L_{ik})}{\lambda_{\min}(P_i)}$, and $L_{ik} = [I + C_{ik}]^T P_i [I + C_{ik}]$.

From (6.21), (6.24), and (6.25), we have for $t \geq t_0$,

$$V^i(e^i(t)) \leq V^i(e_0^i)e^{(\sigma_i + \nu_i)(t-t_0)},$$

where $0 < \nu_i < -\sigma_i$. The last inequality implies that

$$\|e^i\| \leq \gamma_i \|e_0^i\| e^{-(\xi_i - \nu_i)(t-t_0)/2}, \quad t \geq t_0,$$

where $\gamma_i = \sqrt{\frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_i)}}$. Then, the trivial solution is globally exponentially stable which completes the proof. $\square$

The following theorem gives sufficient conditions to ensure robust global exponential stability for all the isolated impulsive subsystems when some control components experience failure.

As done in the reliable stabilization, for $i = 1, 2, \cdots, l$, consider the decomposition of the observer matrix $C_i = C_{i\Omega} + C_{i\bar{\Omega}}$, where $C_{i\Omega}$, $C_{i\bar{\Omega}}$ are the observer matrices associated with

144

Ω, $\bar{\Omega}$ respectively, and $C_{i\Omega}$, $C_{i\bar{\Omega}}$ are generated by zeroing out the columns corresponding to $\bar{\Omega}$ and Ω, respectively. For a fixed $i \in \{1, 2, \cdots, l\}$, let $\omega \subseteq \Omega$ corresponds to some of the sensors that experience failure, and assume that the output of faulty sensors is zero. Then, the decomposition becomes $C_i = C_{i\omega} + C_{i\bar{\omega}}$, where $C_{i\omega}$ and $C_{i\bar{\omega}}$ have the same definition of $C_{i\Omega}$ and $C_{i\bar{\Omega}}$, respectively. The closed-loop impulsive error system for the faulty case becomes

$$
\begin{cases}
\dot{e}^i = (A_i + \Delta A_i - \mathcal{L}_i C_{i\bar{\omega}})e^i + f_i(w^i) - f_i(\hat{w}^i), & t \neq t_k \\
\Delta e^i(t) = C_{ik} e^i(t^-), & t = t_k, \quad k = 1, 2, \cdots \\
e^i(t_0) = w_0^i - \hat{w}_0^i = e_0^i,
\end{cases}
\tag{6.26}
$$

Theorem 6.3.3. *The trivial solution of system* (6.26) *is robustly globally exponentially stable if Assumption A holds,* $(A_i, C_{i\bar{\omega}})$ *is detectable, and condition* (6.21) *holds with* P_i *being a positive-definite matrix satisfying the Riccati-like equation*

$$
A_i^T P_i + P_i A_i + P_i[\xi_{1i} D_i D_i^T - \epsilon_{1i} C_{i\bar{\Omega}} C_{i\bar{\Omega}}^T]P_i + \frac{1}{\xi_{1i}} H_i^T H_i + a_i I - \sigma_i P_i = 0, \tag{6.27}
$$

where ξ_{1i}, ϵ_{1i} *are positive constants such that the observer gain* $\mathcal{L}_i = \frac{1}{2}\epsilon_{1i} C_{i\bar{\omega}}^T P_i$, $0 < \nu_i < -\sigma_i$, $\sigma_i < 0$, *the matrices* $P_i, C_{i\bar{\omega}}$ *are commutative, and* $a_i > 0$ *such that* (6.23) *holds.*

Proof. For all $i = 1, 2, \cdots, l$, let $e^i(t) = e^i(t, t_0, e_0^i)$ be the solution of system (6.26). As done in the previous proof, define $V^i(e^i) = (e^i)^T P_i e^i$ as a Lyapunov function candidate for

the ith subsystem. Then, one may have

$$\dot{V}^i(e^i) \leq e^{iT}\Big[(A_i - \mathcal{L}_iC_i)^T P_i + P_i(A_i - \mathcal{L}_iC_i) + \xi_{1i}P_iD_iD_i^T P_i$$

$$+ \frac{1}{\xi_{1i}}H_i^T H_i + a_i I\Big]e^i$$

$$\leq e^{iT}\Big[A_i^T P_i + P_iA_i + P_i(\xi_{1i}D_iD_i^T + \epsilon_{1i}C_{i\bar{\omega}}C_{i\bar{\omega}}^T)P_i + \frac{1}{\xi_{1i}}H_i^T H_i$$

$$+ a_i I\Big]e^i$$

$$\leq e^{iT}\Big[A_i^T P_i + P_iA_i + P_i(\xi_{1i}D_iD_i^T - \epsilon_{1i}C_{i\bar{\Omega}}C_{i\bar{\Omega}}^T)P_i + \frac{1}{\xi_{1i}}H_i^T H_i$$

$$+ a_i I\Big]e^i$$

$$= \sigma_i V^i(e^i(t)),$$

where we used the fact $C_{i\bar{\Omega}}(C_{i\bar{\Omega}})^T \leq C_{i\bar{\omega}}(C_{i\bar{\omega}})^T$, in the second last line and condition (6.27) in the last line. Following the same procedure as in the previous proof will show that the trivial solution of the closed-loop impulsive error system (6.26) is robustly globally exponentially stable. □

Definition 6.3.4. *System (6.20)(or (6.26)) is said to possess property C (or D) if it satisfies the conditions in Theorem 6.3.2 (or 6.3.3).*

Remark 6.3.5. *Property C implies that all the impulsive error isolated subsystems are robustly globally exponentially stable in the normal actuators case, while Property D implies the same result is held in the faulty case.*

Considering the interconnection g_i in system (6.18) results in the interconnected system

$$
\begin{cases}
\dot{w}^i = (A_i + \Delta A_i)w^i + B_i u_i + f_i(w^i) + g_i(w^1, w^2, \cdots, w^l), & t \neq t_k, \\
\Delta w^i(t) = C_{ik} w^i(t^-), & t = t_k, \quad k = 1, 2, \cdots, \\
y^i(t) = C_i w^i(t), \\
w^i(t_0) = w_0^i,
\end{cases}
\tag{6.28}
$$

Similarly, we define the response system as follows

$$
\begin{cases}
\dot{\hat{w}}^i = (A_i + \Delta A_i)\hat{w}^i + B_i u_i + f_i(\hat{w}^i) + g_i(\hat{w}^1, \hat{w}^2, \cdots, \hat{w}^l) \\
\qquad + \mathcal{L}_i(y^i - C_i\hat{w}^i), & t \neq t_k, \\
\Delta \hat{w}^i(t) = C_{ik} \hat{w}^i(t^-), & t = t_k, \quad k = 1, 2, \cdots, \\
\hat{w}^i(t_0) = \hat{w}_0^i,
\end{cases}
\tag{6.29}
$$

Then the closed-loop error system becomes

$$
\begin{cases}
\dot{e}^i = (A_i + \Delta A_i - \mathcal{L}_i C_i)e^i + f_i(w^i) - f_i(\hat{w}^i) + g_i(w^1, w^2, \cdots, w^l) \\
\qquad - g_i(\hat{w}^1, \hat{w}^2, \cdots, \hat{w}^l), & t \neq t_k, \\
\Delta e^i(t) = C_{ik} e^i(t^-), & t = t_k, \quad k = 1, 2, \cdots \\
e^i(t_0) = w_0^i - \hat{w}_0^i = e_0^i,
\end{cases}
\tag{6.30}
$$

System (6.30) can be written in the following form

$$
\begin{cases}
\dot{e}_c = (A + \Delta A - \mathcal{L}C)e_c + F(x) - F(\hat{x}) + G(x) - G(\hat{x}), & t \neq t_k, \\
\Delta e_c(t) = I_k(e_c(t^-)) = C_k e_c(t^-), & t = t_k, \quad k = 1, 2, 3, \cdots, \\
e_c(t_0) = e_{c0},
\end{cases}
\tag{6.31}
$$

such that

$$
x^T = (w^{1^T} \ w^{2^T} \ \cdots \ w^{l^T}),
$$

$$
\hat{x}^T = \left((\hat{w}^1)^T \ (\hat{w}^2)^T \ \cdots \ (\hat{w}^l)^T \right),
$$

147

$$e_c^T = (e^{1^T} \; e^{2^T} \; \cdots \; e^{l^T}),$$

$$((A + \Delta A - \mathcal{L}C)e_c)^T = \left[\left[(A_1 + \Delta A_1 - \mathcal{L}_1 C_1)e^1 \right]^T \; \left[(A_2 + \Delta A_2 - \mathcal{L}_2 C_2)e^2 \right]^T \right.$$

$$\left. \cdots \left[(A_l + \Delta A_l - \mathcal{L}_l C_l)e^l \right]^T \right],$$

$$(F(x))^T = \left(f_1^T(w^1) \; f_2^T(w^2) \; \cdots \; f_l^T(w^l) \right),$$

$$(F(\hat{x}))^T = \left(f_1^T(\hat{w}^1) \; f_2^T(\hat{w}^2) \; \cdots \; f_l^T(\hat{w}^l) \right),$$

$$(G(x))^T = \left(g_1^T(x) \; g_2^T(x) \; \cdots \; g_l^T(x) \right),$$

$$(G(\hat{x}))^T = \left(g_1^T(\hat{x}) \; g_2^T(\hat{x}) \; \cdots \; g_l^T(\hat{x}) \right),$$

$$(C_k e_c)^T = \left((C_{1k}e^1)^T \; (C_{2k}e^2)^T \; \cdots \; (C_{lk}e^l)^T \right).$$

Theorem 6.3.6. *Assume that system* (6.20) *possesses property C, and the observer gain* $\mathcal{L}$ *is given. Suppose further that for any* $i, j = 1, 2, \cdots, l$, *there exists a positive constant* b_{ij} *such that*

$$2e^{i^T} P_i [g_i(w^1, w^2, \cdots, w^l) - g_i(\hat{w}^1, \hat{w}^2, \cdots, \hat{w}^l)] \leq \|e^i\| \Sigma_{j=1}^l b_{ij} \|e^j\|, \tag{6.32}$$

and the test matrix $\mathcal{S} = [s_{ij}]_{l \times l}$ *is negative definite where*

$$s_{ij} = \begin{cases} \beta_i(\sigma_i^* + b_{ii}), & i = j \\ \frac{1}{2}(\beta_i b_{ij} + \beta_j b_{ji}), & i \neq j \end{cases}, \tag{6.33}$$

148

for some constant $\sigma_i^* = \sigma_i \lambda_{\max}(P_i) < 0$, and a positive constant β_i. Then, the trivial solution of system (6.31) is robustly globally exponentially stable if the following inequality holds

$$\ln \alpha_k - \phi(t_k - t_{k-1}) \leq 0, \quad k = 1, 2, \cdots, \tag{6.34}$$

for $0 < \phi < \theta$ where $\theta = -\frac{\lambda_{\max}(\mathcal{S})}{\bar{\lambda}\beta^*}$ with $\bar{\lambda} = \min\{\lambda_{\max}(P_i) : i = 1, 2, \cdots, l\}$ and $\beta^* = \min\{\beta_i : i = 1, 2, \cdots, l\}, \alpha_k = \left[\max\{\lambda_{\max}[(I + C_{ik})^T P_i(I + C_{ik})] : i = 1, 2, \cdots, l\}\right]/\lambda^*$, with $\lambda^* = \min\{\lambda_{\min}(P_i) : i = 1, 2, \cdots, l\}$ and P_i being a positive-definite matrix defined in Property C.

Proof. Let $e_c(t) = e_c(t, t_0, e_{c0})$ be the solution of system (6.31). Define the composite Lyapunov function

$$V(e_c(t)) = \sum_{i=1}^{l} \beta_i V^i(e^i)$$

with $V^i(e^i)$ being the Lyapunov function for the ith isolated subsystem and $\beta_i > 0$. Then, one may get after using property C and (6.32),

$$\dot{V}(e_c) \leq \sum_{i=1}^{l} \beta_i \left\{ \sigma_i \|e^i\|^2 + 2e^{iT} P_i[g_i(w^1, \cdots, w^l) - g_i(\hat{w}^1, \cdots, \hat{w}^l)] \right\}$$

$$\leq \sum_{i=1}^{l} \beta_i \left\{ \sigma_i \|e^i\|^2 + \|e^i\| \Sigma_{j=1}^{l} b_{ij} \|e^j\| \right\}$$

$$= z^T \mathcal{S} z,$$

where $z^T = (\|e^1\|, \|e^2\|, \cdots, \|e^l\|)$. Then,

$$\dot{V}(e_c) \leq -\theta V(e_c),$$

where $\theta = -\frac{\lambda_{\max}(\mathcal{S})}{\bar{\lambda}\beta^*}$ with $\bar{\lambda} = \min\{\lambda_{\max}(P_i) : i = 1, 2, \cdots, l\}$ and $\beta^* = \min\{\beta_i : i = 1, 2, \cdots, l\}$. The rest of the proof is similar to that of Theorem 6.2.7, thus, it is omitted here. $\square$

The following theorem shows that the proposed reliable sensors are robust in the presence of the interconnection effect. One can prove this result as done in the previous theorem.

Theorem 6.3.7. *Assume that system* (6.26) *possesses property D. Suppose further that for any $i, j = 1, 2, \cdots, l$, there exist positive constants b_{ij} such that the condition in* (6.32) *holds, and the test matrix $\mathcal{S} = [s_{ij}]_{l \times l}$, defined in theorem 6.3.6, is negative definite, ϵ_{1i} is a positive constant such that $\mathcal{L}_i = \frac{1}{2}\epsilon_{1i}C_{i\bar{\omega}}{}^T P_i$, where P_i and $C_{i\bar{\omega}}$ are commutative. Then, the trivial solution of system* (6.31) *is robustly globally exponentially stable if* (6.34) *holds with P_i being a positive-definite matrix defined in Property D.*

Example 6.3.8. *Consider the composite system with $l = 2$, where*

$$A^1 = \begin{bmatrix} -4 & 0 \\ 0 & 4 \end{bmatrix}, A^2 = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}, C^1 = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}, C^2 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix},$$

$$D^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, D^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, H^1 = \begin{bmatrix} 0 & 1 \end{bmatrix}, H^2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \mathcal{U}_1 = \mathcal{U}_2 = \sin(t),$$

150

$$f_1 = 0.5 \begin{bmatrix} 0 \\ \sin(w^2) \end{bmatrix}, f_2 = 1.5 \begin{bmatrix} 0 \\ \sin(w^4) \end{bmatrix}, C_{1k} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, C_{2k} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix},$$

for all $k = 1, 2, \cdots,$ $\sigma_1 = -2,$ $\sigma_2 = -2.5,$ $\xi_{11} = 2,$ $\xi_{12} = 0.5,$ $\epsilon_{11} = 1,$ $\epsilon_{12} = 0.7,$ $\beta_1 = 1,$ $\beta_2 = 2,$ $b_{11} = 1,$ $b_{22} = 1.5,$ $b_{12} = 0.5,$ $b_{21} = 0.3.$ *From* (6.8), *one may get* $\delta_1 = 0.25$ *and* $\delta_2 = 2.25.$

Case 1. *When all the control sensors are operational, we have from Riccati-like equation,*

$$P^1 = \begin{bmatrix} 0.0416 & 0 \\ 0 & 4.5182 \end{bmatrix}, P^2 = \begin{bmatrix} 2.2800 & 0 \\ 0 & 0.2512 \end{bmatrix},$$

with $\lambda_{\min}(P^1) = 0.0416,$ $\lambda_{\max}(P^1) = 4.5182,$ $\lambda_{\min}(P^2) = 0.2512,$ $\lambda_{\max}(P^2) = 2.2800,$ *so,* $\lambda^* = 0.0416,$ $\lambda^{**} = 4.5182,$ *and the observer gain matrices are*

$$\mathcal{L}^1 = \begin{bmatrix} 0.0312 & 0 \\ 0 & 3.3887 \end{bmatrix}, \mathcal{L}^2 = \begin{bmatrix} 2.3940 & 0 \\ 0 & 0.2638 \end{bmatrix}.$$

Thus, $A_i - \mathcal{L}_i C_i$ *for* $i = 1, 2,$ *are Hurwitz, and the time intervals* $t_k - t_{k-1} \geq 3.6238$ *for the first subsystem, and* $t_k - t_{k-1} \geq 2.4891$ *for the second subsystem.*

Figure 6.3.1 shows the isolated subsystems while Figure 6.3.2 shows the interconnected error system in the operational case.
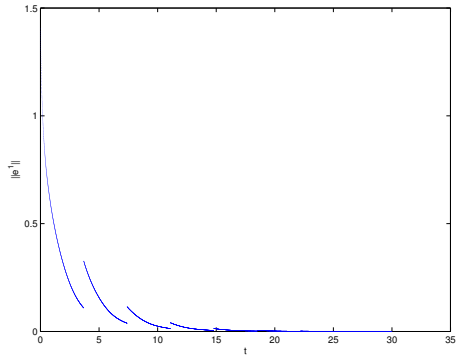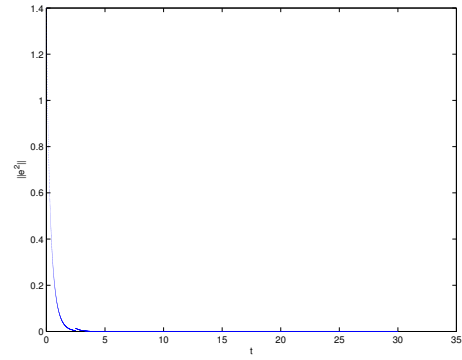
(a) Isolated subsystem 1.   (b) Isolated subsystem 2.

Figure 6.3.1: Operational sensors.
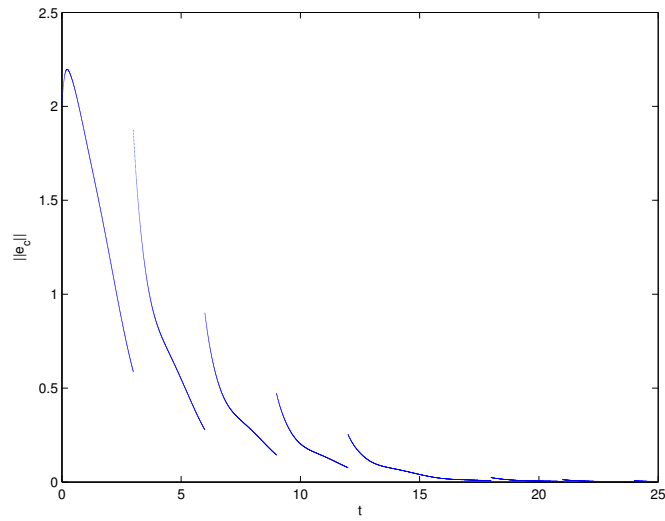
Figure 6.3.2: Interconnected system: Operational case.

Case 2. *When there is a failure in the first sensor in the first subsystem and second sensor in the second subsystem, i.e., $\Omega^1 = \{1\}$ and $C_{\bar{\Omega}}^1 = \begin{bmatrix} 0 & 0 \\ 0 & 1.5 \end{bmatrix}$, and $\Omega^2 = \{2\}$ and $C_{\bar{\Omega}}^2 = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$, we have from Riccati-like equation,*

$$P^1 = \begin{bmatrix} 0.0411 & 0 \\ 0 & 4.5182 \end{bmatrix}, P^2 = \begin{bmatrix} 2.2800 & 0 \\ 0 & 0.2942 \end{bmatrix},$$

with $\lambda_{\min}(P^1) = 0.0411$, $\lambda_{\max}(P^1) = 4.5182$, $\lambda_{\min}(P^2) = 0.2942$, $\lambda_{\max}(P^2) = 2.2800$, so $\lambda^* = 0.0411$, $\lambda^{**} = 4.5182$, and the observer gain matrices

$$\mathcal{L}^1 = \begin{bmatrix} 0 & 0 \\ 0 & 3.3887 \end{bmatrix}, \mathcal{L}^2 = \begin{bmatrix} 2.3940 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, $A_i + \mathcal{L}_i C_{i\bar{\omega}}$ for $i = 1, 2$, are Hurwitz, and the time intervals $t_k - t_{k-1} \geq 3.6300$, and $t_k - t_{k-1} \geq 2.4101$ for the first and second subsystems respectively. For Case 1 and Case 2, the test matrix is

$$\mathcal{S} = \begin{bmatrix} -8.364 & 0.55 \\ 0.55 & -8.4 \end{bmatrix}.$$

Figure 6.3.3 shows the isolated subsystems while Figure 6.3.4 shows the interconnected error system, $\|e_c\|$ for the faulty sensors case.
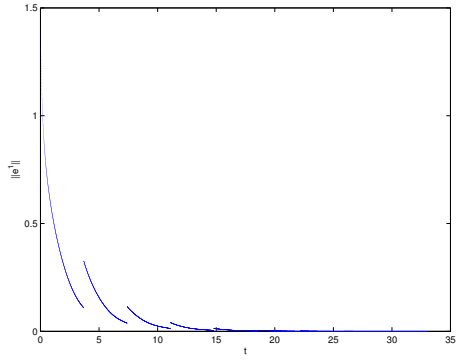
(a) Isolated subsystem 1.
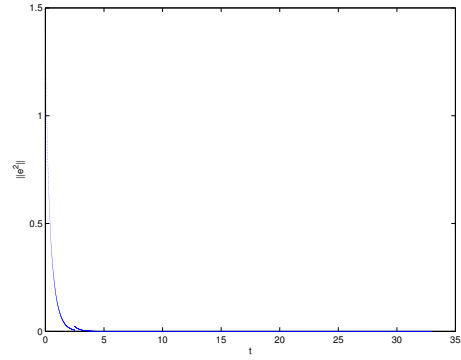
(b) Isolated subsystem 2.

Figure 6.3.3: Faulty sensors.
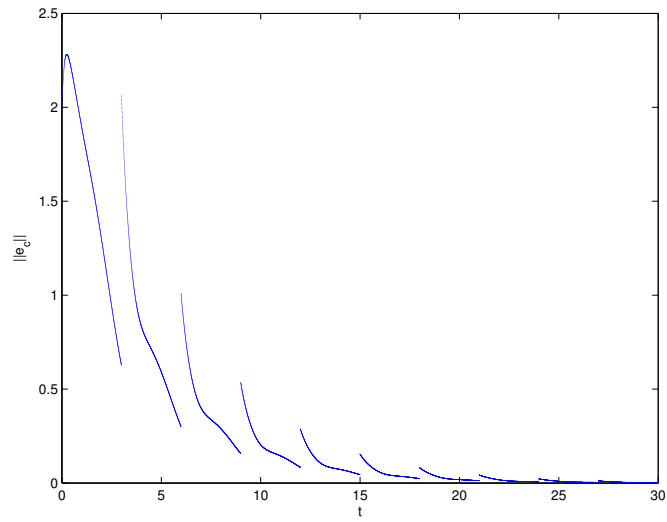
Figure 6.3.4: Interconnected system: Faulty case.

154

6.4 Conclusion

This chapter has addressed the problem of designing a robust reliable controller that guarantees the global exponential stability of uncertain ILSS with fixed impulses. We have analyzed the stability for such a complex system by decomposing the system into lower order, isolated subsystems, and the interconnection was treated as a system perturbation. The isolated subsystems were assumed to be globally exponentially stabilized by the state feedback controllers and the interconnection was estimated by an upper bound that is smaller than the stability degree of the isolated subsystems in order to guarantee the stability of the interconnected system. The scalar Lyapunov functions have been utilized to fulfil our purpose, and this approach has led to solving a Riccati-like equation. In addition, the output of the faulty actuators has been treated as an outage. As an application to this result, the problem of state estimation has been considered, where scalar Lyapunov functions (or time-domain) approach has been used. To the best of author's knowledge, this approach has not been used before where the frequency-domain approach has been used instead in most of the available results [125, 163, 165, 182]. To illustrate the theoretical results, two examples have been discussed with simulations.

Chapter 7

ISS and Stabilization of LSSIS with Time Delay

This chapter addresses the input-to-state stabilization of nonlinear large-scale stochastic impulsive systems (LSSIS) with time delay. Scalar Lyapunov function is utilized to analyze ISS. Furthermore, the impulsive stabilization is discussed for LSSIS with time delay. Lyapunov-Razumikhin approach is used to accomplish our goal. The materials of this chapter form the basis of [12].

7.1 Problem Formulation and Preliminaries

Consider the following LSSIS with time delay

$$
\begin{cases}
dw^i = f_i(t, w_t^i + g_i(t, w_t^1, w_t^2, \cdots, w_t^i, \cdots, w_t^l) + G_i w_i)dt + \sigma_{ii}(t, w_t^i)dW_i(t), \ t \neq t_k, \\
\Delta w^i(t) = I_{ik}(t^-, w_{t-}^i), \quad t = t_k, \quad k \in \mathbb{N}, \\
w_{t_0}^i(s) = \phi_i(s), \quad s \in [-r, 0)
\end{cases}
\tag{7.1}
$$

where $i = 1, 2, \cdots, l$, $w^i \in \mathbb{R}^{n_i}$ is the ith subsystem state, such that $\sum_{i=1}^{l} n_i = n$, the impulsive times t_k satisfying $t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, with $\lim_{k\to\infty} t_k = \infty$, $\Delta w^i(t_k) = w^i(t_k^+) - w^i(t_k^-)$ where $w(t_k^+)$ (or $w(t_k^-)$) is the state just after (or before) the impulse at t_k, $I_k : \mathbb{R}_+ \times \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ is the impulsive function, $f_i : \mathbb{R}_+ \times \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$, is a nonlinear function, $g^i : \mathbb{R}_+ \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_l} \to \mathbb{R}^n$ is the interconnection and $w_i \in \mathbb{R}^{p_i}$ is an input disturbance to the ith isolated subsystem, which is assumed to be in $L_2[t_0, \infty)$ and $G_i \in \mathbb{R}^{n_i \times p_i}$ where $\sum_{i=1}^{l} p_i = p$. The functions f_i and g^i satisfy Lipschitz condition. $\sigma : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$, where $\sigma(t, x_t) = (\sigma_{ij}(t, w_t^i))$, $W_i : \mathbb{R}_+ \to \mathbb{R}^{n_i}$.

System (7.1) can be written in the following form

$$\begin{cases} dx = F(t, x_t + Gw)dt + \sigma(t, x_t)dW, & t \neq t_k, \quad x \in \mathbb{R}^n \\ \Delta x(t) = I_k(t^-, x_{t-}), \quad t = t_k, \quad k \in \mathbb{N}, \\ x_{t_0}(s) = \Phi(s), \quad s \in [-r, 0] \end{cases} \tag{7.2}$$

where

$$x^T = (w^{1T} \ w^{2T} \ \cdots \ w^{lT}),$$

$$f^T(t, x_t) = \left(f_1^T(t, w_t^1) \ f_2(t, w_t^2)^T \ \cdots \ f_l(t, w_t^l)^T \right),$$

$$g^T(t, x_t) = \left(g_1^T(t, x_t) \ g_2^T(t, x_t) \ \cdots \ g_l^T(t, x_t) \right),$$

$$F(t, x_t) = f(t, x_t) + g(t, x_t),$$

$$I_k^T(t, x_t) = \left(I_{1k}^T(t, w_t^1) \ I_{2k}(t, w_t^2)^T \ \cdots \ I_{lk}(t, w_t^l)^T \right),$$

$$W^T = (W_1{}^T \ W_2{}^T \ \cdots \ W_l{}^T), \quad \Phi(s)^T = (\phi_1(s) \ \phi_2(s) \ \cdots, \phi_l(s)),$$

$$G^T = (G_1{}^T \ G_2{}^T \ \cdots \ G_l{}^T), \quad w^T = (w_1{}^T \ w_2{}^T \ \cdots \ w_l{}^T),$$

From (7.1), the corresponding isolated subsystems are

$$\begin{cases} dw^i = f_i(t, w_t^i + G_i w_i)dt + \sigma_{ii}(t, w_t^i)dW_i(t), \ t \neq t_k, \\ \Delta w^i(t) = I_{ik}(t^-, w_{t^-}^i), \quad t = t_k, \quad k \in \mathbb{N}, \\ w_{t_0}^i(s) = \phi_i(s), \quad s \in [-r, 0), \end{cases} \tag{7.3}$$

where $i = 1, 2, \cdots, l$.

Definition 7.1.1. *The trivial solution of system* (7.2) *is said to be robustly globally input-to-state stable in the mean square if there exist positive constants* λ, $\bar{\lambda}$ *such that*

$$\mathbb{E}\big[\|x(t)\|^2\big] \leq \bar{\lambda} \mathbb{E}\big[\|\Phi\|_r^2\big]e^{-\lambda(t-t_0)} + \rho(\sup_{t_0 \leq \tau \leq t} \|w(\tau)\|), \qquad \forall t \geq t_0,$$

for any solution $x(t) = x(t, t_0, \Phi)$ *of* (7.2), $\Phi = (\phi_1 \ \phi_2 \ \cdots \phi_l)^T \in \mathbb{R}^n$, *and* $t_0 \in \mathbb{R}^+$.

7.2 Input-to-State Stabilization via Reliable Control

In this section, we present the main objective of this chapter.

Theorem 7.2.1. *Assume there exist positive constants* a_i, b_i, λ, $\alpha > r$, $\sigma_i < 0$ *and a positive-definite and decrescent function* $V^i(t, \psi^i(0))$ *for all* $(t, \psi(0)) \in [t_0 - r, \infty) \times PC([-r, 0], \mathbb{R}^n)$. *Then, the trivial solution,* $w^i \equiv 0$, *of system* (7.3) *is ISS in the mean square if the following conditions hold*

(i) $c_{1i}\|\psi^i(0)\|^2 \leq V^i(t, \psi^i(0)) \leq c_{2i}\|\psi^i(0)\|^2;$

158

(ii) for all $k \in \mathbb{N}$, $t \neq t_k$ and $\psi^i \in PC([-r, 0], \mathbb{R}^n)$, we have $\mathbb{E}\big[\mathcal{L}_i V^i(t, \psi^i)\big] \leq \sigma_i \mathbb{E}\big[\|\psi^i(0)\|^2\big]$

    provided that $\mathbb{E}\big[V^i(t+s, \psi^i)\big] \leq q\mathbb{E}\big[V^i(t, \psi^i(0))\big]$ where $q \geq e^{2\lambda\alpha}$ with $\lambda = \max_{i \in \mathcal{S}}\{-\sigma_i\}$,

    $s \in [-r, 0]$, and $\gamma(\sup_{t_{k-1} \leq s \leq t_k} \|w_i(s)\|) \leq \mathbb{E}\big[V^i(t, \psi^i(0))\big]$ for $\psi^i \in \mathcal{C}_r$, and $t \in$

    $[t_{k-1}, t_k)$;

(iii) for all $t = t_k$, $k \in \mathbb{N}$,

$$\mathbb{E}\big[V^i\big(t_k, \psi^i(0) + I_{ik}(t_k^-, \psi^i(t_k^-))\big)\big] \leq d_{ik}\mathbb{E}\big[V^i(t_k^-, \psi^i(0))\big] \tag{7.4}$$

    where $\psi^i(0^-) = \psi^i(0)$, and $d_{ik} > 0$;

(iv) for all $k \in \mathbb{N}$, $r \leq t_{k+1} - t_k \leq \alpha$, and $\ln(d_{ik}) + \lambda\alpha < -\lambda(t_{k+1} - t_k)$.

Proof. To prove the assertion of the theorem, we have the following claim

    Claim. For any $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}$, conditions $(i) - (iv)$ imply that

$$\mathbb{E}\big[V^i(t, w^i(t))\big] \leq M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t-t_{k-1})} + \gamma(\sup_{t_{k-1} \leq s \leq t} \|w_i(s)\|) \tag{7.5}$$

where $\lambda > 0$ and $M > 1$.

Proof of the claim. Choose $M > 1$ such that

$$c_2\mathbb{E}\big[\|\phi_i\|_r^2\big] < M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_1-t_0)} + \overline{\gamma}(t_1) \leq qc_2\mathbb{E}\big[\|\phi_i\|_r^2\big], \tag{7.6}$$

where $\overline{\gamma}(t) = \gamma(\sup_{t_0 \leq s \leq t}(\|w_i(s)\|))$ and $c_2 = max_{i=1,\cdots,l}\{c_{2i}\}$. Using the mathematical induction method, we prove the claim for all $k \in \mathbb{N}$. Let $w^i = w^i(t, t_0, \phi_i)$ be the solution

of the isolated subsystem (7.3).

Step 1, for $k = 1$, i.e. $t \in [t_0, t_1)$, we show that

$$\mathbb{E}\big[V^i(t, w^i(t))\big] \leq M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_1-t_0)} + \overline{\gamma}(t_1). \tag{7.7}$$

From (7.6), we have for $t \in [t_0 - r, t_0]$

$$\mathbb{E}\big[V^i(t, w^i(t))\big] \leq c_2\mathbb{E}\big[\|w^i\|^2\big]$$

$$\leq c_2\mathbb{E}\big[\|\phi_i\|_r^2\big]$$

$$< M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_1-t_0)} + \overline{\gamma}(t_1). \tag{7.8}$$

If (7.7) were not true, then for $s \in [-r, 0]$, there would exist $\bar{t} \in (t_0, t_1)$ such that

$$\mathbb{E}\big[V^i(\bar{t}, w^i(\bar{t}))\big] > M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_1-t_0)}$$

$$> c_2\mathbb{E}\big[\|\phi_i\|_r^2\big]$$

$$\geq \mathbb{E}\big[V^i(t_0 + s, w^i(t_0 + s))\big]. \tag{7.9}$$

From the continuity, there exists $t^* \in (t_0, \bar{t})$ such that

$$\mathbb{E}\big[V^i(t^*, w^i(t^*))\big] = M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_1-t_0)} + \overline{\gamma}(t_1) \tag{7.10}$$

and for all $t \in [t_0 - r, t^*]$, we have

$$\mathbb{E}\big[V^i(t, w^i(t))\big] \leq M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_1-t_0)} + \overline{\gamma}(t_1). \tag{7.11}$$

Also, there exists $t^{**} \in [t_0, t^*)$ such that

$$\mathbb{E}\big[V^i(t^{**}, w^i(t^{**}))\big] = c_2\mathbb{E}\big[\|\phi_i\|_r^2\big] \tag{7.12}$$

and for $t \in [t^{**}, t^*]$

$$\mathbb{E}\big[V^i(t, w^i(t))\big] \geq c_2 \mathbb{E}\big[\|\phi_i\|_r^2\big]. \tag{7.13}$$

Hence, from (7.11), (7.6), and (7.13), for all $t \in [t^{**}, t^*]$, and $s \in [-r, 0]$, we have

$$\mathbb{E}\big[V^i(t + s, w^i(t + s))\big] \leq c_2 \mathbb{E}\big[\|\phi_i\|_r^2\big]$$

$$< M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_1 - t_0)} + \overline{\gamma}(t_1)$$

$$\leq qc_2\mathbb{E}\big[\|\phi_i\|_r^2\big]$$

$$\leq q\mathbb{E}\big[V^i(t, w^i(t))\big].$$

Therefore, we have $\mathbb{E}\big[\mathcal{L}_i V^i(t, w_t^i)\big] \leq 0$ for $t \in [t^{**}, t^*]$. By Itô's formula over $[t^{**}, t^*]$, we have

$$\mathbb{E}\big[V^i(t^*, w^i(t^*))\big] = \mathbb{E}\big[V^i(t^{**}, w^i(t^{**}))\big] + \underbrace{\int_{t^{**}}^{t^*} \mathbb{E}\big[\mathcal{L}_i V^i(s, w_s^i)\big]ds}_{\leq 0}$$

which implies

$$\mathbb{E}\big[V^i(t^{**}, w^i(t^{**}))\big] \geq \mathbb{E}\big[V^i(t^*, w^i(t^*))\big]. \tag{7.14}$$

By (7.10), (7.12) and (7.14), we have

$$c_2\mathbb{E}\big[\|\phi_i\|_r^2\big] \geq M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_1 - t_0)} + \overline{\gamma}(t_1),$$

which is a contradiction, and so (7.5) is true when $k = 1$.

Step 2, assume (7.5) is true for $k = 1, 2, \cdots, m$, that is

$$\mathbb{E}\big[V^i(t, w^i(t))\big] \leq M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_k - t_0)} + \gamma(\sup_{t_0 \leq s \leq t_k} \|w_i(s)\|), \quad t \in [t_{k-1}, t_k) \tag{7.15}$$

161

Step 3, we show (7.15) is true for $k = m + 1$, i.e.,

$$\mathbb{E}\big[V^i(t, w^i(t))\big] \leq M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_{m+1}-t_0)} + \gamma(\sup_{t_0 \leq s \leq t_{m+1}} \|w_i(s)\|), \quad t \in [t_m, t_{m+1}) \quad (7.16)$$

If (7.16) is not true, we define

$$\bar{t} = \inf\left\{t \in [t_m, t_{m+1}) : \mathbb{E}\big[V^i(t, w^i(t))\big] > M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_{m+1}-t_0)} + \gamma(\sup_{t_0 \leq s \leq t_{m+1}} \|w_i(s)\|)\right\}.$$

By the continuity of $\mathbb{E}\big[V^i(t, w^i(t))\big]$, there exists $\tilde{t} \in [t_m, \bar{t})$, such that

$$\mathbb{E}\big[V^i(\tilde{t}, w^i(\tilde{t}))\big] = M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_{m+1}-t_0)} + \gamma(\sup_{t_0 \leq s \leq t_{m+1}} \|w_i(s)\|) \quad (7.17)$$

and

$$\mathbb{E}\big[V^i(t, w^i(t))\big] \leq M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_{m+1}-t_0)} + \gamma(\sup_{t_0 \leq s \leq t_{m+1}} \|w_i(s)\|), \quad t \in [t_m, \tilde{t}) \quad (7.18)$$

Since, at $t = t_m^+$, we have

$$\mathbb{E}\big[V^i(t_m, w^i(t_m))\big] \leq d_{im}\mathbb{E}\big[V^i(t_m^-, w^i(t_m^-))\big]$$

$$< d_{im}\left\{M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_m-t_0)} + \overline{\gamma}(t_m)\right\}$$

$$\leq e^{-\lambda\alpha}e^{-\lambda(t_{m+1}-t_m)}\left\{M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_m-t_0)} + \overline{\gamma}(t_m)\right\}$$

$$= e^{-\lambda\alpha}\left\{M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_{m+1}-t_0)} + \overline{\gamma}(t_m)e^{-\lambda(t_{m+1}-t_m)}\right\}$$

$$\leq M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_{m+1}-t_0)} + \overline{\gamma}(t_m)e^{-\lambda(t_{m+1}-t_m)}$$

162

$$\leq M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_{m+1}-t_0)} + \overline{\gamma}(t_m)$$

$$\leq M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_{m+1}-t_0)} + \overline{\gamma}(t_{m+1})$$

$$< \mathbb{E}\big[V^i(\overline{t}, w^i(\overline{t}))\big] \tag{7.19}$$

i.e., $\mathbb{E}\big[V^i(t_m, w^i(t_m))\big] < e^{-\lambda\alpha}\Big\{ M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_{m+1}-t_0)} + \overline{\gamma}(t_{m+1})\Big\} < \mathbb{E}\big[V^i(\overline{t}, w^i(\overline{t}))\big]$, so

that there exists $t^* \in (t_m, \overline{t})$ such that

$$\mathbb{E}\big[V^i(t^*, w^i(t^*))\big] = e^{-\lambda\alpha}\Big\{ M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_{m+1}-t_0)} + \overline{\gamma}(t_{m+1})\Big\} \text{ and } \mathbb{E}\big[\mathcal{L}_i^+ V^i(t^*, w^i(t^*))\big] > 0.$$
$$\tag{7.20}$$

We know $t^* + s \in [t_{m-1}, \overline{t})$ for $s \in [-r, 0]$.

By (7.15) and (7.20), we have

$$\mathbb{E}\big[V^i(t^* + s, w^i(t^* + s))\big] \leq M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_m - t_0)} + \overline{\gamma}(t_m)$$

$$= M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_{m+1}-t_0)}e^{\lambda(t_{m+1}-t_m)} + \overline{\gamma}(t_{m+1})$$

$$\leq e^{\lambda\alpha} M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_{m+1}-t_0)} + \overline{\gamma}(t_{m+1})$$

$$\leq e^{\lambda\alpha}\Big\{ M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_{m+1}-t_0)} + \overline{\gamma}(t_{m+1})\Big\}$$

$$\leq e^{2\lambda\alpha}\mathbb{E}\big[V^i(t^*, w^i(t^*))\big]$$

$$\leq q\mathbb{E}\big[V^i(t^*, w^i(t^*))\big]$$

where $q \geq e^{2\lambda\alpha} > 1$ and $s \in [-r, 0]$. Thus, from (ii), we have $\mathbb{E}\big[\mathcal{L}_i^+ V^i(t^*, w^i(t^*))\big] < 0$ which

contradicts (7.20). Thus, (7.5) must be true for $k = m + 1$. Hence, by the mathematical

induction, (7.5) is true for $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}$. $\qquad\square$

Remark 7.2.2. *Theorem 7.2.1 gives sufficient conditions to ensure ISS for each isolated stochastic impulsive subsystem (7.3). The time between impulses has to be bounded, and this condition is summarized in (iv). Condition (iii) guarantees that the Lyapunov function is decreasing along the solution trajectories of the system. We should remark that the pth moment ISS can be proved with slight modifications in the proof and theorem statement by replacing each $\| \cdot \|^2$ by $\| \cdot \|^p$.*

Having established the stabilizability of isolated subsystem in Theorems 7.2.1, we prove the same properties for the interconnected systems.

Theorem 7.2.3. *Assume that the composite system, system (7.2), satisfies the following conditions:*

  (i) every isolated subsystem satisfies the conditions in Theorem 7.2.1.

  (ii) for any $i, j = 1, 2, \cdots, l$, there exist positive constants b_{ij} such that

$$g_i^T(t, \psi) V_{\psi^i(0)}^i(t, \psi^i(0)) \leq \|\psi^i(0)\| \sum_{j=1}^{l} q b_{ij} \|\psi^j(0)\|, \tag{7.21}$$

  where q is defined in Theorem 7.2.1.

  (iii) the test matrix $\mathcal{S} = [s_{ij}]_{l \times l}$ is negative definite where

$$s_{ij} = \begin{cases} \beta_i(\sigma_i + q b_{ii}), & i = j \\ \frac{q}{2}(\beta_i b_{ij} + \beta_j b_{ji}), & i \neq j \end{cases}, \tag{7.22}$$

  for some constant $\sigma_i < 0$, and positive constant β_i.

164

Then, the trivial solution of system (7.2) *is ISS in the mean square.*

Proof. Let $x(t) = x(t, t_0, \Phi)$ be the solution of system (7.2). Define the composite Lyapunov function $V(t, x(t)) = \sum_{i=1}^{l} \beta_i V^i(t, w^i)$ as a Lyapunov function candidate for interconnected system (7.2) where β_i is a positive constant, and $V^i(t, w^i)$ is a Lyapunov function for the ith isolated subsystem. From (i) in theorem 7.2.1, for any i, there exist $c_{1i}, c_{2i} > 0$ such that

$$c_{1i}\|w^i\|^2 \leq V^i(t, w^i) \leq c_{2i}\|w^i\|^2 \leq c_{2i}\|w_t^i\|_r^2$$

which implies

$$\sum_{i=1}^{l} \beta_i c_{1i}\|w^i\|^2 \leq \underbrace{\sum_{i=1}^{l} \beta_i V^i(t, w^i)}_{V(t,x(t))} \leq \sum_{i=1}^{l} \beta_i c_{2i}\|w_t^i\|_r^2$$

Clearly, $V(t, x(t))$ is positive-definite and decrescent function. Therefore, there exist c_1, $c_2 > 0$ such that

$$c_1\|x(t)\|^2 \leq V(t, x(t)) \leq c_2\|x\|^2 \leq c_2\|x_t\|_r^2$$

Since, $\sigma_{ij}(t, w^j) \equiv 0$ for all $i \neq j$, the infinitesimal diffusion operator becomes

$$\mathcal{L}V^i(t, x) = \mathcal{L}_i V^i(t, w^i) + g_i^T(t, x_t)V_{w^i}^i(t, w^i)$$

Thus, we have

$$\mathbb{E}\big[\mathcal{L}V(t, x)\big] = \sum_{i=1}^{l} \beta_i \mathbb{E}\big[\mathcal{L}V^i(t, x)\big]$$

165

$$= \sum_{i=1}^{l} \beta_i \mathbb{E}\big[\mathcal{L}_i V^i(t, w^i) + g_i^T(t, x_t) V_{w^i}^i(t, w^i)\big]$$

$$\leq \sum_{i=1}^{l} \beta_i \Big\{ \sigma_i \mathbb{E}\big[\|w^i\|^2\big] + \mathbb{E}\big[\|w^i\| \sum_{j=1}^{l} q b_{ij} \|w^j\|\big]\Big\}$$

$$= z^T \mathcal{S} z,$$

where $z^T = \Big(\mathbb{E}\big[\|w^1\|\big], \mathbb{E}\big[\|w^2\|\big], \cdots, \mathbb{E}\big[\|w^i\|\big], \cdots, \mathbb{E}\big[\|w^l\|\big]\Big)$ and $\mathcal{S}$ is a negative definite

matrix with the maximum eigenvalue $\lambda_{\max}(\mathcal{S})$. Then, one can write

$$\mathbb{E}\big[\mathcal{L}V(t, x)\big] \leq \lambda_{\max}(\mathcal{S}) \sum_{i=1}^{l} \mathbb{E}\big[\|w^i\|^2\big] \leq 0,$$

whenever $\mathbb{E}\big[V(t, x_t))\big] \leq q\mathbb{E}\big[V(t, x)\big]$. Moreover, for each $\psi^i \in \mathcal{C}_r$ and $t \in [t_{k-1}, t_k)$, we

have

$$\gamma\big(\sup_{t_{k-1} \leq s \leq t_k} \|w_i(s)\|\big) \leq \mathbb{E}\big[V^i(t, \psi^i(0))\big]$$

We know that

$$\|w\| \leq \sum_{i=1}^{l} \|w_i\|$$

Then,

$$\sup_{t_{k-1} \leq s \leq t_k} \|w\| \leq \sum_{i=1}^{l} \sup_{t_{k-1} \leq s \leq t_k} \|w_i\|$$

So,

$$\gamma\big(\sup_{t_{k-1} \leq s \leq t_k} \|w\|\big) \leq \sum_{i=1}^{l} \gamma\big(\sup_{t_{k-1} \leq s \leq t_k} \|w_i\|\big)$$

166

which leads to

$$\gamma(\sup_{t_{k-1}\leq s\leq t_k}\|w\|)\leq\sum_{i=1}^{l}\gamma(\sup_{t_{k-1}\leq s\leq t_k}\|w_i\|)\leq\sum_{i=1}^{l}\mathbb{E}\big[V^i(t,w^i)\big]$$

Then, we have

$$\gamma(\sup_{t_{k-1}\leq s\leq t_k}\|w\|)\leq\beta_i\gamma(\sup_{t_{k-1}\leq s\leq t_k}\|w\|)\leq\sum_{i=1}^{l}\beta_i\mathbb{E}\big[V^i(t,w^i)\big]$$

At $t=t_k^+$, we have

$$\mathbb{E}\big[V(t_k^+,x(t_k^+))\big]=\sum_{i=1}^{l}\beta_i\mathbb{E}\big[V^i(t_k^+,w^i(t_k^+))\big]$$

$$\leq\sum_{i=1}^{l}\beta_i d_k\mathbb{E}\big[V^i(t_k^-,w^i(t_k^-))\big]$$

$$=d_k\mathbb{E}\big[V(t_k^-,x(t_k^-))\big]. \tag{7.23}$$

Thus, the conditions of Theorem 7.2.1 are all satisfied and so $x\equiv 0$ is ISS in the mean square. $\qquad\square$

Remark 7.2.4. *Theorem 7.2.3 shows that the unperturbed interconnected system is exponentially stable when the isolated subsystems are stable. Condition (7.66) estimates the interconnection, which is viewed as a perturbation, by an upper bound. The test matrix is needed to guarantee that the degree of stability of the isolated subsystems is greater than the interconnection.*

167

The following theorem shows that the interconnected system is ISS in the presence of the stochastic perturbation.

Theorem 7.2.5. *Assume that system* (7.2) *satisfies conditions (i) and (ii) of Theorem* 7.2.3 *and the following conditions hold*

(iii) for any $i, j = 1, 2, \cdots, l$, there exist positive constants e_i such that

$$\mathbb{E}\big[(y^i)^T V^i_{\psi^i(0)\psi^i(0)}(t, \psi^i(0))(y^i)\big] \leq q e_i \mathbb{E}\big[\|y^i(0)\|^2\big], \qquad (7.24)$$

where $y^i = \sigma(t, \psi^j)$, the ith row of the matrix σ.

(iv) for any $\sigma(t, \psi^j)$, $i, j = 1, 2, \cdots, l$, there exists $d_{ij} > 0$ such that

$$\mathbb{E}\big[\|\sigma_{ij}(t, \psi^j)\|^2\big] \leq d_{ij}\mathbb{E}\big[\|\psi^i(0)\|^2\big]$$

(v) the test matrix $\mathcal{S} = [s_{ij}]_{l \times l}$ is negative definite where

$$s_{ij} = \begin{cases} \beta_i(\sigma_i + q b_{ii}) + \frac{1}{2}\sum_{k=1,\, k\neq i} q\beta_k e_k d_{ki}, & i = j \\ \frac{q}{2}(\beta_i b_{ij} + \beta_j b_{ji}), & i \neq j \end{cases}, \qquad (7.25)$$

for some constant $\sigma_i < 0$, and positive constant β_i.

Then, the trivial solution of system (7.2) *is ISS in the mean square.*

Proof. Let $x(t) = x(t, t_0, \Phi)$ be the solution of system (7.2). Define the composite Lyapunov function as in Theorem 7.2.3. The infinitesimal diffusion operator becomes

$$\mathcal{L}V^i(t, x) = \mathcal{L}_i V^i(t, w^i) + g_i^T(t, x_t)V^i_{w^i}(t, w^i) + \frac{1}{2}\sum_{i=1}^{l} \operatorname{tr}\big[\sigma_{ij}^T(t, w_t^j)V^i_{w^i w^i}(t, w^i)\sigma_{ij}(t, w_t^j)\big]$$

Thus, we have

$$\mathbb{E}\big[\mathcal{L}V(t,x)\big] = \sum_{i=1}^{l} \beta_i \mathbb{E}\Big[\mathcal{L}_i V^i(t,w^i) + g_i^T(t,x_t)V_{w^i}^i(t,w^i)$$

$$+ \frac{1}{2}\sum_{i=1}^{l} \mathrm{tr}\,\big[\sigma_{ij}^T(t,w_t^j)V_{w^iw^i}^i(t,w^i)\sigma_{ij}(t,w_t^j)\big]\Big]$$

$$\leq \sum_{i=1}^{l} \beta_i\Big\{\sigma_i \mathbb{E}\big[\|w^i\|^2\big] + \mathbb{E}\big[\|w^i\|\sum_{j=1}^{l} qb_{ij}\|w^j\|\big]$$

$$+ \frac{1}{2}\sum_{j=1,i\neq j}^{l} qe_i \mathbb{E}\big[\|\sigma_{ij}(t,w_t^j)\|^2\big]$$

$$\leq \sum_{i=1}^{l} \beta_i\Big\{\sigma_i \mathbb{E}\big[\|w^i\|^2\big] + \mathbb{E}\big[\|w^i\|\sum_{j=1}^{l} qb_{ij}\|w^j\|\big]$$

$$+ \frac{1}{2}\sum_{j=1,i\neq j}^{l} qe_i d_{ij}\mathbb{E}\big[\|w^j\|^2\big]\Big\}$$

$$= z^T \mathcal{S}z.$$

The rest of the proof is similar to the previous one and thus omitted here.　　□

Consider the following interconnected system

$$d\dot{w}^i = \Big[(A_i + \Delta A_i)w^i + (\bar{A}_i + \Delta\bar{A}_i)w_t^i + B_iu_i + f_i(w_t^i)$$
$$+ g_i(w_t^1, w_t^2, \cdots, w_t^l) + G_iw_i\Big]dt + \sigma_{ii}(t,w_t^i)dW_i(t), \quad t \neq t_k,$$
$$\Delta w^i(t) = I_{ik}(w^i(t^-)) = C_{ik}w^i(t^-), \quad t = t_k, \quad k \in \mathbb{N},$$
$$w_{t_0}^i(s) = \phi_i(s), \qquad s \in [-r, 0], \tag{7.26}$$

where $i = 1, 2, \cdots, l$, $w^i \in \mathbb{R}^{n_i}$ is the ith subsystem state, such that $\Sigma_{i=1}^l n_i = n$, $A_i \in \mathbb{R}^{n_i \times n_i}$ is a non-Hurwitz matrix for each i, the impulsive times t_k satisfying $t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, with $\lim_{k \to \infty} t_k = \infty$, $\Delta w^i(t_k) = w^i(t_k^+) - w^i(t_k^-)$ where $w(t_k^+)$(or $w(t_k^-)$) is the state just after (or before) the impulse at t_k, and $I_k : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ is the impulsive function, $u_i = K_i w^i \in \mathbb{R}^q$ is the control input for the ith subsystem, where $K_i \in \mathbb{R}^{q \times n_i}$ is the control gain matrix, $f_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ is some nonlinearity, $g_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_l} \to \mathbb{R}^n$ is the interconnection. f_i and g^i satisfy Lipschitz condition. A_i, B_i, and C_{ik} are known real constant matrices with proper dimensions, and ΔA_i is a piecewise continuous function representing parameter uncertainty with bounded norm. $w_i \in \mathbb{R}^{p_i}$ is an input disturbance to the ith isolated subsystem, which is assumed to be in $L_2[t_0, \infty)$ and $G_i \in \mathbb{R}^{n_i \times p_i}$ where $\sum_{i=1}^l p_i = p$.

System (7.26) can be written in the following form

$$
\begin{cases}
d\dot{x} = \left[(A + \Delta A)x + (\bar{A} + \Delta\bar{A})x_t + Bu + f(x_t) + g(x_t) + Gw\right]dt + \sigma(t, x_t)dW, & t \neq t_k, \\
\Delta x(t) = I_k(x(t^-)) = C_k x(t^-), & t = t_k, \quad k \in \mathbb{N}, \\
x_{t_0}(s) = \Phi(s), & s \in [-r, 0],
\end{cases}
\tag{7.27}
$$

where

$$
x^T = (w^{1T} \ w^{2T} \ \cdots \ w^{lT}),
$$

$$
((A + \Delta A)x)^T = \left(\left((A_1 + \Delta A_1)w^1\right)^T \ \left((A_2 + \Delta A_2)w^2\right)^T \ \cdots \ \left((A_l + \Delta A_l)w^l\right)^T\right),
$$

$$
(Bu)^T = \left((B_1 u^1)^T \ (B_2 u^2)^T \ \cdots \ (B_l u^l)^T\right),
$$

$$
(F(x))^T = \left(f_1(w^1)^T \ f_2(w^2)^T \ \cdots \ f_l(w^l)^T\right),
$$

170

$$(G(x))^T = \left(g_1(x)^T \ g_2(x)^T \ \cdots \ g_l(x)^T \right),$$

$$(C_k x)^T = \left((C_{1k}w^1)^T \ (C_{2k}w^2)^T \ \cdots \ (C_{lk}w^l)^T \right).$$

$$G^T = (G_1{}^T \ G_2{}^T \ \cdots \ G_l{}^T), \quad w^T = (w_1{}^T \ w_2{}^T \ \cdots \ w_l{}^T),$$

From (7.26), the corresponding isolated subsystems are

$$\begin{cases} d\dot{w}^i = \left[(A_i + \Delta A_i)w^i + (\bar{A}_i + \Delta \bar{A}_i)w_t^i + B_i u_i + f_i(w_t^i) + G_i w_i \right] dt \\ \qquad + \sigma_{ii}(t, w_t^i) dW_i(t), \quad t \neq t_k, \\ \Delta w^i(t) = I_{ik}(w^i(t^-)) = C_{ik}w^i(t^-), \quad t = t_k, \quad k \in \mathbb{N}, \\ w_{t_0}^i(s) = \phi_i(s), \qquad s \in [-r, 0], \end{cases} \tag{7.28}$$

where $i = 1, 2, \cdots, l$, and the corresponding closed-loop system is

$$\begin{cases} d\dot{w}^i = \left[(A_i + \Delta A_i + B_i K_i)w^i + (\bar{A}_i + \Delta \bar{A}_i)w_t^i + f_i(w_t^i) + G_i w_i \right] dt \\ \qquad + \sigma_{ii}(t, w_t^i) dW_i(t), \quad t \neq t_k, \\ \Delta w^i(t) = I_{ik}(w^i(t^-)) = C_{ik}w^i(t^-), \quad t = t_k, \quad k \in \mathbb{N}, \\ w_{t_0}^i(s) = \phi_i(s), \qquad s \in [-r, 0], \end{cases} \tag{7.29}$$

To analyze the reliable stabilization with respect to actuator failures, for $i = 1, 2, \cdots, l$, consider the decomposition of the control matrix $B_i = B_{i\sigma} + B_{i\bar{\sigma}}$ Then, the closed-loop systems for the faulty case becomes

$$\begin{cases} d\dot{w}^i = \left[(A_i + \Delta A_i + B_{\bar{\sigma}}^i K_i)w^i + (\bar{A}_i + \Delta \bar{A}_i)w_t^i + f_i(w_t^i) + G_i w_i \right] dt \\ \qquad + \sigma_{ii}(t, w_t^i) dW_i(t), \quad t \neq t_k, \\ \Delta w^i(t) = I_{ik}(w^i(t^-)) = C_{ik}w^i(t^-), \quad t = t_k, \quad k \in \mathbb{N}, \\ w_{t_0}^i(s) = \phi_i(s), \qquad s \in [-r, 0], \end{cases} \tag{7.30}$$

Corollary 7.2.1. *Let the controller gain K^i be given, and assume that Assumption A holds. Assume further that there exist positive constants λ, $\alpha > r$, $\sigma_i < 0$ and a positive-definite*

171

and decrescent function $V^i(t, \psi^i(0))$ for all $(t, \psi(0)) \in [t_0 - r, \infty) \times PC([-r, 0], \mathbb{R}^n)$. Then, the trivial solution, $w^i \equiv 0$, of system (7.29) is ISS in the mean square if the following conditions hold

(i) $\lambda_{\min}(P_i)\|\psi^i(0)\|^2 \leq \mathbb{E}\big[V^i(t, \psi^i(0))\big] \leq \lambda_{\max}(P_i)\|\psi^i(0)\|^2$;

(ii) for all $k \in \mathbb{N}$, $t \neq t_k$ and $\psi^i \in PC([-r, 0], \mathbb{R}^n)$, we have $\mathbb{E}\big[\mathcal{L}_i V^i(t, \psi^i)\big] \leq \sigma_i \mathbb{E}\big[\|\psi^i(0)\|^2\big]$ provided that $\mathbb{E}\big[V^i(t+s, \psi^i)\big] \leq q\mathbb{E}\big[V^i(t, \psi^i(0))\big]$, where $q \geq e^{2\lambda\alpha}$ with $\lambda = \max_{i \in \mathcal{S}}\{-\sigma_i\}$, $s \in [-r, 0]$, and $\gamma(\sup_{t_{k-1} \leq s \leq t_k} \|w_i(s)\|) \leq \mathbb{E}\big[V^i(t, \psi^i(0))\big]$ for $\psi^i \in \mathcal{C}_r$, and $t \in [t_{k-1}, t_k)$;

(iii) for all $t = t_k$, $k \in \mathbb{N}$,

$$\mathbb{E}\big[V^i\big(t_k, \psi^i(0) + I_{ik}(t_k^-, \psi^i(t_k^-)))\|\big] \leq d_{ik}\mathbb{E}\big[V^i(t_k^-, \psi^i(0))\big] \qquad (7.31)$$

where $\psi^i(0^-) = \psi^i(0)$, and $d_{ik} > 0$;

(iv) for all $k \in \mathbb{N}$, $r \leq t_{k+1} - t_k \leq \alpha$, and $\ln(d_{ik}) + \lambda\alpha < -\lambda(t_{k+1} - t_k)$, where $d_{ik} = \frac{\lambda_{\max}[(I+C_{ik})^T P^i(I+C_{ik})]}{\lambda_{\min}(P^i)}$, with P^i being a positive-definite matrix satisfying

$$(A_i + B_i K_i)^T P_i + P_i(A_i + B_i K_i) + \big(\frac{q_i}{\xi_{4i}} + \frac{q_i}{\xi_{5i}}\|\bar{H}_i\|^2 + \frac{q_i\delta_i}{\xi_{6i}}\big)I + \frac{1}{\xi_{1i}}H_i^T H_i + \gamma_i q_i P_i$$

$$+ P_i\big[\xi_{1i}D_i D_i^T + \xi_{2i}G_i G_i^T + \xi_{4i}\bar{A}_i(\bar{A}_i)^T + \xi_{5i}\bar{D}_i \bar{D}_i^T + \xi_{6i}I\big]P_i - \alpha_i P_i = 0 \qquad (7.32)$$

where $\xi_{ji}, j = 1, \cdots 4$, are any positive constants, $0 < \nu_i < -\alpha_i$, $\alpha_i < 0$, γ_i and δ_i are positive constants such that

$$\mathrm{tr}[\sigma_{ii}^T(t, w_t^i)P_i\sigma_{ii}(t, w_t^i)] \leq 2\gamma_i q_i \psi^{i^T}(0)P_i\psi^i(0) \qquad (7.33)$$

172

and

$$\|f_i(w^i)\|^2 \le \delta_i \|w^i\|^2. \tag{7.34}$$

Proof. Let $w^i = w^i(t, t_0, \phi_i)$ be the solution of the isolated subsystem (7.29), and $V^i(t, w^i(t)) = w^i P_i w^i$ be a Lyapunov function candidate. Then,

$$\mathcal{L}V^i(t, w^i) = w^{i^T}\big[(A_i + B_i K_i)^T P_i + P_i(A_i + B_i K_i)\big]w^i + 2w^{i^T}P_i\Delta A_i w^i + 2w^{i^T}P_i\bar{A}_i w_t^i$$

$$+ 2w^{i^T}P_i\Delta\bar{A}_i w_t^i + 2w^{i^T}P_i f_i(w_t^i) + 2w^{i^T}P_i G_i w^i + \frac{1}{2}\operatorname{tr}[\sigma_{ii}^T(t, w_t^i)P_i\sigma_{ii}(t, w_t^i)]$$

Claim. For any $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}$, conditions $(i) - (iv)$ imply that

$$\mathbb{E}\big[V^i(t, w^i(t))\big] \le M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t-t_0)} + \gamma\big(\sup_{t_{k-1}\le s\le t_k}\|w_i(s)\|\big) \tag{7.35}$$

where $\lambda > 0$ and $M > 1$.

Proof of the claim. Choose $M > 1$ such that

$$c_2\mathbb{E}\big[\|\phi_i\|_r^2\big] < M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_1-t_0)} + \overline{\gamma}(t_1) \le qc_2\mathbb{E}\big[\|\phi_i\|_r^2\big], \tag{7.36}$$

where $\overline{\gamma}(t) = \gamma(\sup_{t_0\le s\le t}(\|w_i(s)\|))$ and $c_2 = max_{i=1,\cdots,l}\{\lambda_{\max}(P_i)\}$. Using the mathematical induction method, one can follow the same proof of Theorem 7.2.1 to prove the claim

173

for all $k \in \mathbb{N}$. Thus, by the claim, Lemma 3.1.3, and condition (iv), we have

$$\mathcal{L}V^i(t, w^i) \leq w^{iT}\Big\{(A_i + B_iK_i)^T P_i + P_i(A_i + B_iK_i) + (\frac{q_i}{\xi_{4i}} + \frac{q_i}{\xi_{5i}}\|\bar{H}_i\|^2 + \frac{q_i\delta_i}{\xi_{6i}})I$$

$$+ P_i\big[\xi_{1i}D_iD_i^T + \xi_{2i}G_iG_i^T + \xi_{4i}\bar{A}_i(\bar{A}_i)^T + \xi_{5i}\bar{D}_i\bar{D}_i^T + \xi_{6i}I\big]P_i + \frac{1}{\xi_{1i}}H_i^T H_i$$

$$+ \gamma_i q_i P_i\Big\}w^i + \frac{1}{\xi_{2i}}w_i^T w_i$$

$$= \alpha_i V^i(t, w^i) + \frac{1}{\xi_{2i}}w_i^T w_i$$

Applying Itô's formula, and take the expectation gives

$$\mathcal{L}\mathbb{E}[V^i(t, w^i)] \leq \alpha_i\mathbb{E}[V^i(t, w^i)] - \theta_i\mathbb{E}[V^i(t, w^i)] + \theta_i\mathbb{E}[V^i(t, w^i)] + \frac{1}{\xi_{2i}}w_i^T w_i, \quad t \in (t_{k-1}, t_k),$$

for all $i = 1, \cdots l$ and all $t \neq t_k$.

Then, we have for each subinterval $t \in (t_{k-1}, t_k)$,

$$\mathcal{L}\mathbb{E}[V^i(t, w^i)] \leq \overline{\alpha}_i\mathbb{E}[V^i(t, w^i)] - \theta_i\mathbb{E}[V^i(t, w^i)] + \frac{1}{\xi_{2i}}w_i^T w_i,$$

where $\overline{\alpha}_i = \alpha_i + \theta_i$, and $0 < \theta_i < -\alpha_i$. The forgoing inequality implies that

$$\mathcal{L}\mathbb{E}[V^i(t, w^i)] \leq \overline{\alpha}_i\mathbb{E}[V^i(t, w^i)], \quad \text{for all} \quad t \in (t_{k-1}, t_k),$$

provided that

$$-\theta_i\mathbb{E}[V^i(t, w^i)] + \frac{1}{\xi_{2i}}w_i^T w_i < 0$$

that is

$$\mathbb{E}[V^i(t, w^i)] > \frac{1}{\theta_i\xi_{2i}}w_i^T w_i$$

174

This implies, by (i)

$$\mathbb{E}\big[\|w^i\|^2\big] > \frac{\|w_i\|^2}{\theta_i \xi_{2i} c_2} := \rho_i(\|w_i\|),$$

where $c_2 = \max\{\lambda_{\max}(P_i) : i = 1, \cdots, l\}$.

At $t = t_k$, we have

$$\mathbb{E}[V^i(t, w^i(t_k^+))] \leq d_{ik}\mathbb{E}\big[V^i(t_k^-, w^i(t_k^-))\big]$$

$$\leq d_{ik}M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t-t_0)} + \gamma(\sup_{t_{k-1}\leq s\leq t_k}\|w_i(s)\|)$$

$$\leq e^{-\lambda(\alpha+t_{k+1}-t_k)}M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t-t_0)} + \gamma(\sup_{t_{k-1}\leq s\leq t_k}\|w_i(s)\|)$$

$$\leq M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t-t_0)} + \gamma(\sup_{t_{k-1}\leq s\leq t_k}\|w_i(s)\|)$$

Which implies

$$\mathbb{E}\big[\|w^i\|^2\big] \leq \hat{M}\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t-t_0)} + \frac{1}{\lambda_{\min}(P_i)}\gamma(\sup_{t_{k-1}\leq s\leq t_k}\|w_i(s)\|)$$

where $\hat{M} = \frac{M}{\lambda_{\min}(P_i)}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 7.2.2 (Reliability for isolated subsystems). *The trivial solution of system* (7.30) *is robustly ISS in the mean square if all the conditions of Corollary* 7.2.1 *hold with replacing* (7.32) *with*

$$A_i^T P_i + P_i A_i + P_i\big[\xi_{1i}D_i D_i^T + \xi_{2i}G_{ci}G_{ci}^T - \epsilon_i B_{i\bar{\Sigma}}B_{i\bar{\Sigma}}^T + \xi_{4i}\bar{A}_i(\bar{A}_i)^T + \xi_{5i}\bar{D}_i\bar{D}_i^T + \xi_{6i}I\big]P_i$$

$$+ (\frac{q_i}{\xi_{4i}} + \frac{q_i}{\xi_{5i}}\|\bar{H}_i\|^2 + \frac{q_i\delta_i}{\xi_{6i}})I + \frac{1}{\xi_{1i}}H_i^T H_i + \gamma_i q_i P_i - \alpha_i P_i = 0 \qquad (7.37)$$

such that the control gain matrix $k_i = -\frac{1}{2}\epsilon_i B_{i\bar{\sigma}}^T P_i$, *where* $\epsilon_i > 0$.

Proof. Let $w^i = w^i(t, t_0, \phi_i)$ be the solution of the isolated subsystem (7.30), and $V^i(t, w^i(t)) = w^i P_i w^i$ be a Lyapunov function candidate. Then,

$$\mathcal{L}V^i(t, w^i) \leq w^{iT} \Big\{ A_i^T P_i + P_i A_i + (\frac{q_i}{\xi_{4i}} + \frac{q_i}{\xi_{5i}} \|\bar{H}_i\|^2 + \frac{q_i \delta_i}{\xi_{6i}}) I + \frac{1}{\xi_{1i}} H_i^T H_i + \gamma_i q_i P_i$$

$$+ P_i \big[\xi_{1i} D_i D_i^T + \xi_{2i} G_{ci} G_{ci}^T - \epsilon_i B_{i\bar{\sigma}} B_{i\bar{\sigma}}^T + \xi_{4i} \bar{A}_i (\bar{A}_i)^T + \xi_{5i} \bar{D}_i \bar{D}_i^T + \xi_{6i} I \big] P_i \Big\} w^i$$

$$\leq w^{iT} \Big\{ A_i^T P_i + P_i A_i + (\frac{q_i}{\xi_{4i}} + \frac{q_i}{\xi_{5i}} \|\bar{H}_i\|^2 + \frac{q_i \delta_i}{\xi_{6i}}) I + \frac{1}{\xi_{1i}} H_i^T H_i + \gamma_i q_i P_i$$

$$+ P_i \big[\xi_{1i} D_i D_i^T + \xi_{2i} G_{ci} G_{ci}^T - \epsilon_i B_{i\bar{\Sigma}} B_{i\bar{\Sigma}}^T + \xi_{4i} \bar{A}_i (\bar{A}_i)^T + \xi_{5i} \bar{D}_i \bar{D}_i^T + \xi_{6i} I \big] P_i \Big\} w^i$$

$$= \alpha_i V^i(t, w^i) + \frac{1}{\xi_{2i}} (w_\sigma^F)^T w_\sigma^F$$

where we used the claim in Corollary 7.2.1, Lemma 3.1.3, condition (7.37), and the fact $B_{i\bar{\Sigma}} B_{i\bar{\Sigma}}^T \leq B_{i\bar{\sigma}} B_{i\bar{\sigma}}^T$. The rest of the proof is similar to the proof of Corollary 7.2.1 and thus omitted here. $\square$

Definition 7.2.6. *System* (7.29) *(or* (7.30)*) is said to possess property A (or B) if it satisfies the conditions in Corollary 7.2.1(or 7.2.2).*

Remark 7.2.7. *Property A implies that all the stochastic impulsive isolated subsystems are robustly ISS in the mean square in the normal actuators case, while Property B implies the same result is hold in the faulty case.*

Corollary 7.2.3 (Interconnected system (Normal Case))**.** *Assume that the composite system, system* (7.27)*, satisfies the following conditions:*

(i) System (7.29) possess property A;

(ii) for any $i, j = 1, 2, \cdots, l$, there exist positive constants b_{ij} such that

$$g_i^T(t, \psi) V_{\psi^i(0)}^i(t, \psi^i(0)) \leq \|\psi^i(0)\| \sum_{j=1}^{l} q_i b_{ij} \|\psi^j(0)\|, \qquad (7.38)$$

where q_i is defined in Corollary 7.2.1;

(iii) for any $i, j = 1, 2, \cdots, l$, there exist positive constants e_i such that

$$\mathbb{E}\big[(y^i)^T V_{\psi^i(0)\psi^i(0)}^i(t, \psi^i(0))(y^i)\big] \leq q e_i \mathbb{E}\big[\|y^i(0)\|^2\big], \qquad (7.39)$$

where $y^i = \sigma(t, \psi^j)$, the ith row of the matrix σ.

(iv) for any $\sigma(t, \psi^j)$, $i, j = 1, 2, \cdots, l$, there exists $d_{ij} > 0$ such that

$$\mathbb{E}\big[\|\sigma_{ij}(t, \psi^j)\|^2\big] \leq d_{ij} \mathbb{E}\big[\|\psi^i(0)\|^2\big];$$

(v) the test matrix $\mathcal{S} = [s_{ij}]_{l \times l}$ is negative definite where

$$s_{ij} = \begin{cases} \beta_i(\sigma_i + q b_{ii}) + \frac{1}{2} \sum_{k=1, \, k \neq i} q \beta_k e_k d_{ki}, & i = j \\ \frac{q}{2}(\beta_i b_{ij} + \beta_j b_{ji}), & i \neq j \end{cases}, \qquad (7.40)$$

for some constant $\sigma_i = \alpha_i \lambda_{\max}(P^i) < 0$, and positive constant β_i.

Then, the trivial solution of system (7.27) is ISS in the mean square.

Corollary 7.2.4 (Interconnected system (Faulty Case)). *Assume that system* (7.30) *possesses property B. Suppose further that the conditions (ii)-(iv) of Corollary 7.2.3 hold, and ϵ_i is a positive constant such that $K^i = -\frac{1}{2}\epsilon_i B_{\bar{\sigma}}^{i}{}^{T} P^i$ where P_i is a positive-definite matrix defined in Property B. Then, the trivial solution of system* (7.27) *is ISS in the mean square.*

The proof of Corollary 7.2.3(7.2.4) follows directly from the proof of Theorem 7.2.5.

7.3   Stabilization via Impulses

In this section, we state and prove a result on the exponential stabilization of the large-scale stochastic system by Impulsive controller

Theorem 7.3.1. *Assume there exist positive constants λ, $\beta, c_{1i}, c_{2i}, \bar{c}_i$ and $V^i(t, \psi^i(0))$ for all $(t, \psi^i(0)) \in [t_0 - r, \infty) \times \mathbb{R}^n$ with $\psi^i(0) \in PC([-r, 0], \mathbb{R}^n)$ such that*

(i) $c_{1i}\|\psi^i(0)\|^2 \leq V^i(t, \psi^i(0)) \leq c_{2i}\|\psi^i(0)\|^2$;

(ii) *for all $k \in \mathbb{N}$, $t \neq t_k$ and $\psi^i \in PC([-r, 0], \mathbb{R}^n)$, we have $\mathbb{E}\big[\mathcal{L}_i V^i(t, \psi)\big] \leq \bar{c}_i \mathbb{E}\big[V^i(t, \psi^i(0))\big]$ provided that $\mathbb{E}^i\big[V^i(t + s, \psi^i)\big] \leq q\mathbb{E}\big[V^i(t, \psi^i(0))\big]$, where $q \geq \gamma e^{\lambda r} > 1$, $s \in [-r, 0]$, with $\gamma \geq 1$;*

(iii) *for all $t = t_k$, $k \in \mathbb{N}$,*

$$\mathbb{E}\big[V^i(t_k, \psi^i(0) + I_{ik}(t_k^-, \psi^i(t_k^-)))\|\big] \leq d_{ik}\mathbb{E}\big[V^i(t_k^-, \psi^i(0))\big] \qquad (7.41)$$

where $0 < d_{ik} \leq 1$;

(iv) for all $k \in \mathbb{N}$, $\ln(d_{ik}) + (\lambda + \alpha)(t_{k+1} - t_k) \leq 0$, and $\gamma \geq e^{(\lambda+c)(t_1-t_0)}$.

Then, the trivial solution, $w^i \equiv 0$, of system (7.3) is exponentially stabilizable in the mean square.

Proof. To prove the assertion of this theorem, we have the following claim.

Claim. For any $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}$, conditions $(i) - (iv)$ imply that

$$\mathbb{E}\big[V^i(t, w^i(t))\big] \leq M\mathbb{E}\big[\|\phi_i\|_r^2\big] e^{-\lambda(t-t_0)} \tag{7.42}$$

where $\lambda > 0$ and $M > 1$.

Proof of the claim. Choose $M > 1$ such that

$$0 < c_2\mathbb{E}\big[\|\phi_i\|_r\big] e^{(\lambda+c)(t_1-t_0)} \leq M\mathbb{E}\big[\|\phi_i\|_r^2\big] \leq c_2\gamma e^{\lambda r}\mathbb{E}\big[\|\phi_i\|_r^2\big], \tag{7.43}$$

From (i), we have

$$\mathbb{E}\big[V^i(t, w^i)\big] \leq c_2\mathbb{E}\big[\|w^i\|^2\big]$$

$$\leq c_2\mathbb{E}\big[\|\phi_i\|_r^2\big]$$

$$< M\mathbb{E}\big[\|\phi_i\|_r^2\big] e^{-(\lambda+c)(t_1-t_0)}$$

$$< M\mathbb{E}\big[\|\phi_i\|_r^2\big] e^{-\lambda(t_1-t_0)}$$

That is, for all $t \in (t_0 - r, t_0]$,

$$\mathbb{E}\big[V^i(t, w^i)\big] \leq M\mathbb{E}\big[\|\phi_i\|_r^2\big] e^{-\lambda(t_1-t_0)} \tag{7.44}$$

We want to prove

$$\mathbb{E}\big[V^i(t, w^i)\big] \le M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t-t_0)}, \qquad t \in [t_{k-1}, t_k) \tag{7.45}$$

Using the mathematical induction method, we prove the claim for all $k \in \mathbb{N}$. Let $w^i = w^i(t, t_0, \phi)$ be the solution of the isolated subsystem (7.3) with $w_{t_0}^i = \phi_i(s)$.

Step 1, for $k = 1$, i.e. $t \in [t_0, t_1)$, we show that

$$\mathbb{E}\big[V^i(t, w^i(t))\big] \le M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_1-t_0)} \tag{7.46}$$

If (7.46) were not true, then for $s \in [-r, 0]$, there would exist $\bar{t} \in [t_0, t_1)$ such that

$$\mathbb{E}\big[V^i(\bar{t}, w^i(\bar{t}))\big] > M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_1-t_0)} \tag{7.47}$$

Define

$$t^* = \inf\{t \in [t_0, t_1) : \mathbb{E}\big[V^i(t, w^i(t))\big] > M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_1-t_0)}\} \tag{7.48}$$

From the continuity of $\mathbb{E}\big[V^i(t, w^i(t))\big]$ over (t_0, t_1), then $t^* \in (t_0, t_1)$ and

$$\mathbb{E}\big[V^i(t^*, w^i(t^*))\big] = M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_1-t_0)} \tag{7.49}$$

and for all $t \in [t_0 - r, t^*]$, we have

$$\mathbb{E}\big[V^i(t, w^i(t))\big] \le M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_1-t_0)} \tag{7.50}$$

Define

$$t^{**} = \sup\{t \in [t_0 - r, t^*) : \mathbb{E}\big[V^i(t, w^i(t))\big] \le c_2\mathbb{E}\big[\|\phi_i\|_r^2\big]\} \tag{7.51}$$

Then, $t^{**} \in [t_0, t^*)$ and

$$\mathbb{E}\big[V^i(t^{**}, w^i(t^{**}))\big] = c_2\mathbb{E}\big[\|\phi_i\|_r^2\big] \tag{7.52}$$

180

and for $t \in (t^{**}, t^*]$

$$\mathbb{E}\big[V^i(t, w^i(t))\big] > c_2 \mathbb{E}\big[\|\phi_i\|_r^2\big] \tag{7.53}$$

In fact, (7.52), and (7.53) imply that

$$\mathbb{E}\big[V^i(t^{**}, w^i(t^{**}))\big] < \mathbb{E}\big[V^i(t, w^i(t))\big] \tag{7.54}$$

Now, for all $t \in [t^{**}, t^*]$, and $s \in [-r, 0]$ and $t + s \in [t^{**}, t^*]$, we have

$$\mathbb{E}\big[V^i(t+s, w^i(t+s))\big] \leq M\mathbb{E}\big[\|\phi_i\|_r^2\big] e^{-\lambda(t_1 - t_0)}$$

$$\leq c_2 \gamma e^{\lambda r} \mathbb{E}\big[\|\phi_i\|_r^2\big] e^{-\lambda(t_1 - t_0)}$$

$$< c_2 \gamma e^{\lambda r} \mathbb{E}\big[\|\phi_i\|_r^2\big]$$

$$= \gamma e^{\lambda r} \mathbb{E}\big[V^i(t^{**}, w^i(t^{**}))\big]$$

$$\leq q\mathbb{E}\big[V^i(t^{**}, w^i(t^{**}))\big]$$

$$\leq q\mathbb{E}\big[V^i(t, w^i(t))\big]$$

Therefore, we have $\mathbb{E}\big[V^i(t+s, w^i(t+s))\big] \leq q\mathbb{E}\big[V^i(t, w^i(t))\big]$. Thus, from (ii), we have

$$\mathbb{E}\big[\mathcal{L}_i V^i(t, w_t^i)\big] \leq c\mathbb{E}\big[V^i(t, w^i(t))\big], \qquad t \in [t^{**}, t^*]. \tag{7.55}$$

By Itô's formula over $[t^{**}, t^*]$ and the forgoing inequality, one may get

$$\mathbb{E}\big[V^i(t^*, w^i(t^*))\big] = \mathbb{E}\big[V^i(t^{**}, w^i(t^{**}))\big] + \int_{t^{**}}^{t^*} \mathbb{E}\big[\mathcal{L}_i V^i(s, w_s^i)\big] ds$$

$$\leq \mathbb{E}\big[V^i(t^{**}, w^i(t^{**}))\big] + c\int_{t^{**}}^{t^*} \mathbb{E}\big[V^i(s, w^i(s))\big] ds$$

Then, by Gronwall inequality, we have

181

$$\mathbb{E}\big[V^i(t^*, w^i(t^*))\big] \leq \mathbb{E}\big[V^i(t^{**}, w^i(t^{**}))\big]e^{c(t^*-t^{**})}$$

$$= c_2\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{c(t^*-t^{**})}$$

$$< c_2\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{c(t_1-t_0)}$$

$$\leq M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_1-t_0)}$$

$$= \mathbb{E}\big[V^i(t^*, w^i(t^*))\big] \tag{7.56}$$

which is a contradiction, and so (7.45) is true when $k = 1$. Step 2, assume (7.45) is true for $k = 1, 2, \cdots, m$, that is

$$\mathbb{E}\big[V^i(t, w^i(t))\big] \leq M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_k-t_0)}, \quad t \in [t_{k-1}, t_k) \tag{7.57}$$

Step 3, we show (7.57) is true for $k = m + 1$, i.e.,

$$\mathbb{E}\big[V^i(t, w^i(t))\big] \leq M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_{m+1}-t_0)}, \quad t \in [t_m, t_{m+1}). \tag{7.58}$$

If (7.58) is not true, we define

$$t^* = \inf\left\{t \in [t_m, t_{m+1}) : \mathbb{E}\big[V^i(t, w^i(t))\big] > M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_{m+1}-t_0)}\right\}$$

Since, at $t = t_m^+$, we have

$$\mathbb{E}\big[V^i(t_m, w^i(t_m))\big] \leq d_m\mathbb{E}\big[V^i(t_m^-, w^i(t_m^-))\big]$$

$$< d_m M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_m-t_0)}$$

$$\leq e^{-(\lambda+c)(t_{m+1}-t_m)}M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_m-t_0)}$$

$$< M\mathbb{E}\big[\|\phi_i\|_r^2\big]e^{-\lambda(t_{m+1}-t_0)} \tag{7.59}$$

182

Thus, at $t = t_m$, we have $\mathbb{E}[V^i(t_m, w^i(t_m))] < M\mathbb{E}[\|\phi_i\|_r^2]e^{-\lambda(t_{m+1}-t_0)}$. We know that $t^* \in (t_m, t_{m+1})$ and

$$\mathbb{E}[V^i(t^*, w^i(t^*))] = M\mathbb{E}[\|\phi_i\|_r^2]e^{-\lambda(t_{m+1}-t_0)} \tag{7.60}$$

and

$$\mathbb{E}[V^i(t, w^i(t))] < M\mathbb{E}[\|\phi_i\|_r^2]e^{-\lambda(t_{m+1}-t_0)}, \quad t \in [t_m, t^*) \tag{7.61}$$

Define

$$t^{**} = \sup\{t \in [t_0 - r, t^*) : \mathbb{E}[V^i(t, w^i(t))] \leq d_m M\mathbb{E}[\|\phi_i\|_r^2]e^{-\lambda(t_m-t_0)}\} \tag{7.62}$$

Then, $t^{**} \in [t_0, t^*)$ and

$$\mathbb{E}[V^i(t^{**}, w^i(t^{**}))] = d_m M\mathbb{E}[\|\phi_i\|_r^2]e^{-\lambda(t_m-t_0)} \tag{7.63}$$

and for $t \in (t^{**}, t^*]$

$$\mathbb{E}[V^i(t, w^i(t))] > d_m M\mathbb{E}[\|\phi_i\|_r^2]e^{-\lambda(t_m-t_0)} \tag{7.64}$$

By (7.63) and (7.64), we have

$$\mathbb{E}[V^i(t, w^i(t))] > \mathbb{E}[V^i(t^{**}, w^i(t^{**}))] \tag{7.65}$$

For $t \in [t^{**}, t^*]$ and some $s \in (-r, 0]$, we have two cases. Either $t + s \geq t_m$, or $t + s < t_m$.

Case 1. When $t + s \geq t_m$, we have

$$\mathbb{E}\big[V^i(t+s, w^i(t+s))\big] \leq M\mathbb{E}\big[\|\phi_i\|_r^2\big] e^{-\lambda(t_{m+1}-t_0)}$$

$$\leq M\mathbb{E}\big[\|\phi_i\|_r^2\big] e^{-\lambda(t+s-t_0)}$$

$$\leq M\mathbb{E}\big[\|\phi_i\|_r^2\big] e^{-\lambda(t-t_0)} e^{\lambda r}$$

$$\leq \gamma d_m M\mathbb{E}\big[\|\phi_i\|_r^2\big] e^{-\lambda(t_m-t_0)} e^{\lambda r}$$

$$\leq \gamma e^{\lambda r}\mathbb{E}\big[V^i(t^{**}, w^i(t^{**}))\big] \leq q\mathbb{E}\big[V^i(t^{**}, w^i(t^{**}))\big]$$

$$< q\mathbb{E}\big[V^i(t, w^i(t))\big]$$

Case 2. When $t + s < t_m$, we assume that $t + s < [t_{l-1}, t_l]$, for some $l \in \mathbb{N}$, and $l < m$, then we have

$$\mathbb{E}\big[V^i(t+s, w^i(t+s))\big] \leq M\mathbb{E}\big[\|\phi_i\|_r^2\big] e^{-\lambda(t_l-t_0)}$$

$$\leq M\mathbb{E}\big[\|\phi_i\|_r^2\big] e^{-\lambda(t+s-t_0)}$$

$$< q\mathbb{E}\big[V^i(t, w^i(t))\big]$$

In both cases we have for $t \in (t^{**}, t^*)$, $s \in (-r, 0]$, we have $\mathbb{E}\big[V^i(t+s, w^i(t+s))\big] < q\mathbb{E}\big[V^i(t, w^i(t))\big]$. Thus, we have

$$\mathbb{E}\big[\mathcal{L}_i V^i(t, w_t^i)\big] \leq c\mathbb{E}\big[V^i(t, w^i(t))\big]$$

As done in Step 1, we use Itô's formula and Grownwall inequality to get a contradiction. Thus, (7.45) must be true for $k = m + 1$. Hence, by the mathematical induction, (7.45) is true for $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}$.

From (i) and (7.45), one can get

$$\mathbb{E}\big[\|w^i\|^2\big] \le \bar{\lambda}\mathbb{E}\big[|\phi_i|_r^2\big]e^{-\lambda(t-t_0)}, \qquad t \in [t_{k-1}, t_k), \; k \in \mathbb{N}$$

where $\bar{\lambda} = \frac{M}{c_1}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Theorem 7.3.2. *Assume that the composite system, system* (7.2) *with* $\sigma_{ij}(t, w^j) \equiv 0$ *for all* $i \ne j,$, *satisfies the following conditions:*

(i) *every isolated subsystem is impulsively stabilized, i.e., every isolated subsystem satis-fies the conditions in Theorem* 7.3.1*;*

(ii) *for any* $i, j = 1, 2, \cdots, l$, *there exist positive constants* b_{ij} *such that*

$$g_i^T(t, \psi)V_{\psi^i(0)}^i(t, \psi^i(0)) \le \|\psi^i(0)\| \sum_{j=1}^{l} qb_{ij}\|\psi^j(0)\|, \tag{7.66}$$

where q *is defined in Theorem* 7.2.1*;*

(iii) *the test matrix* $\mathcal{S} = [s_{ij}]_{l \times l}$ *is positive-definite where*

$$s_{ij} = \begin{cases} \beta_i(\bar{c}_i + qb_{ii}), & i = j \\ \frac{q}{2}(\beta_i b_{ij} + \beta_j b_{ji}), & i \ne j \end{cases}, \tag{7.67}$$

for some positive constant β_i.

Then, the trivial solution of system (7.2) *is exponentially stable in the mean square.*

185

Proof. Let $x(t) = x(t, t_0, \Phi)$ be the solution of system (7.2). Define the composite Lyapunov function $V(t, x(t)) = \sum_{i=1}^{l} \beta_i V^i(t, w^i)$ as a Lyapunov function candidate for interconnected system (7.2) where β_i is a positive constant, and $V^i(t, w^i)$ is a Lyapunov function for the ith isolated subsystem. From (i) in Theorem 7.2.1, for any i, one can show that there exist c_1, $c_2 > 0$ such that

$$c_1 \|x(t)\|^2 \leq V(t, x(t)) \leq c_2 \|x\|^2 \leq c_2 \|x_t\|_r^2$$

Since, $\sigma_{ij}(t, w^j) \equiv 0$ for all $i \neq j$, the infinitesimal diffusion operator becomes

$$\mathcal{L}V^i(t, w^i) = \mathcal{L}_i V^i(t, w^i) + g_i^T(t, x_t) V_{w^i}^i(t, w^i)$$

Thus, we have

$$\mathbb{E}\big[\mathcal{L}V(t, x)\big] = \sum_{i=1}^{l} \beta_i \mathbb{E}\big[\mathcal{L}V^i(t, x)\big]$$

$$= \sum_{i=1}^{l} \beta_i \mathbb{E}\big[\mathcal{L}_i V^i(t, w^i) + g_i^T(t, x_t) V_{w^i}^i(t, w^i)\big]$$

$$\leq \sum_{i=1}^{l} \beta_i \Big\{ \bar{c}_i \mathbb{E}\big[\|w^i\|^2\big] + \mathbb{E}\big[\|w^i\| \sum_{j=1}^{l} q b_{ij} \|w^j\|\big] \Big\}$$

$$= z^T \mathcal{S} z,$$

where $z^T = \Big(\mathbb{E}\big[\|w^1\|\big], \mathbb{E}\big[\|w^2\|\big], \cdots, \mathbb{E}\big[\|w^i\|\big], \cdots, \mathbb{E}\big[\|w^l\|\big] \Big)$ and $\mathcal{S}$ is a positive-definite matrix with the maximum eigenvalue $C = \lambda_{\max}(\mathcal{S})$. Then, one can write

$$\mathbb{E}\big[\mathcal{L}V(t, x)\big] \leq C \sum_{i=1}^{l} \mathbb{E}\big[\|w^i\|^2\big],$$

186

whenever $\mathbb{E}[V(t, x_t))] \leq q\mathbb{E}[V(t, x)]$.

At $t = t_k^+$, we have

$$\mathbb{E}[V(t_k^+, x(t_k^+))] = \sum_{i=1}^{l} \beta_i \mathbb{E}[V^i(t_k^+, w^i(t_k^+))]$$

$$\leq \sum_{i=1}^{l} \beta_i d_k \mathbb{E}[V^i(t_k^-, w^i(t_k^-))]$$

$$= d_k \mathbb{E}[V(t_k^-, x(t_k^-))]. \tag{7.68}$$

Thus, the conditions of Theorem 7.3.1 are all satisfied and so $x \equiv 0$ is exponentially stable in the mean square.

Theorem 7.3.3. *Assume that system* (7.2) *satisfies conditions (i) and (ii) of Theorem 7.3.2 and the following conditions hold*

(iii) for any $i, j = 1, 2, \cdots, l$, there exist positive constants e_i such that

$$\mathbb{E}[(y^i)^T V^i_{\psi^i(0)\psi^i(0)}(t, \psi^i(0))(y^i)] \leq qe_i\mathbb{E}[\|y^i(0)\|^2], \tag{7.69}$$

where $y^i = \sigma(t, \psi^j)$, the ith row of the matrix σ;

(iv) for any $\sigma(t, \psi^j)$, $i, j = 1, 2, \cdots, l$, there exists $d_{ij} > 0$ such that

$$\mathbb{E}[\|\sigma_{ij}(t, \psi^j)\|^2] \leq d_{ij}\mathbb{E}[\|\psi^i(0)\|^2];$$

(v) the test matrix $\mathcal{S} = [s_{ij}]_{l \times l}$ is positive definite where

$$s_{ij} = \begin{cases} \beta_i(\bar{c}_i + qb_{ii}) + \frac{1}{2}\sum_{k=1, k\neq i} q\beta_k e_k d_{ki}, & i = j \\ \frac{q}{2}(\beta_i b_{ij} + \beta_j b_{ji}), & i \neq j \end{cases}, \tag{7.70}$$

187

for some positive constant β_i.

Then, the trivial solution of system (7.2) *is exponentially stable in the mean square.*

Proof. Let $x(t) = x(t, t_0, \Phi)$ be the solution of system (7.2). Define the composite Lyapunov function as in Theorem 7.3.2. The infinitesimal diffusion operator becomes

$$\mathcal{L}V^i(t, x) = \mathcal{L}_i V^i(t, w^i) + g_i^T(t, x_t) V_{w^i}^i(t, w^i) + \frac{1}{2} \sum_{i=1}^{l} \text{tr}\left[\sigma_{ij}^T(t, w_t^j) V_{w^i w^i}^i(t, w^i) \sigma_{ij}(t, w_t^j)\right]$$

Thus, we have

$$\mathbb{E}\left[\mathcal{L}V(t, x)\right] = \sum_{i=1}^{l} \beta_i \mathbb{E}\left[\mathcal{L}_i V^i(t, w^i) + g_i^T(t, x_t) V_{w^i}^i(t, w^i)\right.$$

$$+ \frac{1}{2} \sum_{i=1}^{l} \text{tr}\left[\sigma_{ij}^T(t, w_t^j) V_{w^i w^i}^i(t, w^i) \sigma_{ij}(t, w_t^j)\right]\right]$$

$$\leq \sum_{i=1}^{l} \beta_i \left\{ \bar{c}_i \mathbb{E}\left[\|w^i\|^2\right] + \mathbb{E}\left[\|w^i\| \sum_{j=1}^{l} q b_{ij}\|w^j\|\right]\right.$$

$$+ \frac{1}{2} \sum_{j=1, i \neq j}^{l} q e_i \mathbb{E}\left[\|\sigma_{ij}(t, w_t^j)\|^2\right]$$

$$\leq \sum_{i=1}^{l} \beta_i \left\{ \bar{c}_i \mathbb{E}\left[\|w^i\|^2\right] + \mathbb{E}\left[\|w^i\| \sum_{j=1}^{l} q b_{ij}\|w^j\|\right]\right.$$

$$\left. + \frac{1}{2} \sum_{j=1, i \neq j}^{l} q e_i d_{ij} \mathbb{E}\left[\|w^j\|^2\right]\right\}$$

$$= z^T \mathcal{S} z.$$

188

where $z^T = \Big(\mathbb{E}\big[\|w^1\|\big], \mathbb{E}\big[\|w^2\|\big], \cdots, \mathbb{E}\big[\|w^i\|\big], \cdots, \mathbb{E}\big[\|w^l\|\big] \Big)$ and $\mathcal{S}$ is a positive-definite matrix with the maximum eigenvalue $C = \lambda_{\max}(\mathcal{S})$. The rest of the proof is similar to the previous one and thus omitted here.

7.4   Conclusion

Throughout this chapter, we have addressed LSSISs. The focus has been on developing some sufficient conditions to guarantee ISS and stabilization by reliable controller and impulsive effects. To prove the qualitative properties, we have considered the decomposition approach followed in Chapter 6 and used the Lyapunov-Razumikhin technique.

Part III

Reliable Control Stabilization for Singularly Perturbed Systems

In this part, we address the switched control singularly perturbed systems (SCSPS) without and with time delay where the controllers are subject to faulty actuators. The continuous states are viewed as an interconnected system with two-time scale (slow and fast) subsystems. Moreover, due to dominant behaviour of the reduced systems, the stabilization of the full order systems is achieved through the controller of the slow reduced order subsystem. This in turn results in lessening some unnecessary sufficient conditions imposed on the fast subsystem. The stability analysis is obtained by multiple Lyapunov function method after decomposing the system into isolated, lower order, slow and fast subsystems, and the interconnection between them.

It has been observed that if the degree of stability of each isolated mode is greater than the interconnection between them, the underlying interconnected mode of the switched system is exponentially stable. Moreover, if switching among the system modes follows the average dwell-time rule, then the SCSPS is also exponentially stable. Finally, numerical examples and simulations are provided to justify the proposed theoretical results.

Chapter 8

Switched Control Singularly Perturbed Systems

In the present chapter, we aim to study the stability property of switched singularly perturbed systems via reliable controller for two cases, namely when all the actuators are operational and when some of them experience failures. The faulty actuator output is treated as an outage. The reduced system, which depends on the slow (dominant) system, is used to design the stabilizing reliable controller. The Lyapunov function and average dwell time condition argument are used to establish the exponential stability criteria. As said, we have adopted the decomposition approach. The relationship between the stability degrees of the isolated subsystems and the interconnection strength is usually formulated by the so-called M-matrix. An illustrative example is provided to clarify the validity of our results. The material of this chapter forms the basis of [13].

8.1 Problem Formulation and Preliminaries

Consider the following system

$$\dot{x} = A_{11_{\varrho(t)}} x + A_{12_{\varrho(t)}} z + B_{1_{\varrho(t)}} u, \tag{8.1a}$$

$$\epsilon_{\varrho(t)} \dot{z} = A_{21_{\varrho(t)}} x + A_{22_{\varrho(t)}} z + B_{2_{\varrho(t)}} u, \tag{8.1b}$$

$$x(t_0) = x_0, \quad z(t_0) = z_0, \tag{8.1c}$$

where $x \in \mathbb{R}^m$, $z \in \mathbb{R}^n$ are the system slow and fast states respectively, $u \in \mathbb{R}^l$ is the control input of the form $u = Kx$ for some control gain $K \in \mathbb{R}^{l \times m}$, $\varrho : [t_0, \infty) \to \mathcal{S} = \{1, 2, \cdots, N\}$ is a piecewise constant function known as the switching signal (or law). For each $i \in \mathcal{S}$, $A_{11_i} \in \mathbb{R}^{m \times m}$, $A_{12_i} \in \mathbb{R}^{m \times n}$, $A_{21_i} \in \mathbb{R}^{n \times m}$, $A_{22_i} \in \mathbb{R}^{n \times n}$, are known real constant matrices with A_{22_i} is a nonsingular Hurwitz matrix, $B_1 \in \mathbb{R}^{m \times l}$, $B_2 \in \mathbb{R}^{n \times l}$, and $0 < \epsilon_i \ll 1$. Setting $\epsilon_i = 0$ implies that $z = h_i(x) = -A_{22_i}^{-1}[A_{21_i} x + B_{2_i} u]$. Plug z into (8.1a) gives *the slow reduced subsystem* $\dot{x}_s = A_{0_i} x_s + B_{0_i} u$ where $A_{0_i} = A_{11_i} - A_{12_i} A_{22_i}^{-1} A_{21_i}$, and $B_{0_i} = B_{1_i} - A_{12_i} A_{22_i}^{-1} B_{2_i}$. Choose $u = Kx_s$ such that (A_{0_i}, B_{0_i}) is stabilizable.

For simplicity of notation, we use x in lieu of x_s to refer to the slow reduced system.

Definition 8.1.1. *The trivial solution of system* (8.1) *is said to be globally exponentially stable (g.e.s.) if there exist positive constants L, and λ such that*

$$\|x(t)\| + \|z(t)\| \leq L(\|x(t_0)\| + \|z(t_0)\|) e^{-\lambda(t-t_0)}, \quad t \geq t_0 \in \mathbb{R}_+,$$

for all $x(t)$ and $z(t)$, the solutions of system (8.1), *and any $x_0 \in \mathbb{R}^m$, $z_0 \in \mathbb{R}^n$.*

8.2   The Main Results

In this section, we present our main results.

8.2.1   Normal Case

For any $i \in \mathcal{S}$, the closed-loop system becomes

$$\begin{cases} \dot{x} = (A_{11_i} + B_{1_i}K_i)x + A_{12_i}z, \\ \epsilon_i \dot{z} = (A_{21_i} + B_{2_i}K_i)x + A_{22_i}z, \\ x(t_0) = x_0, \quad z(t_0) = z_0. \end{cases} \tag{8.2}$$

Theorem 8.2.1. *The trivial solution of system* (8.1) *is globally exponentially stable if the average dwell-time condition holds, and the following assumptions hold*

(i) $Re[\lambda(A_{22_i})] < 0, \quad$ *and* (A_{0_i}, B_{0_i}) *is stabilizable;*

(ii) there exist positive constants $a_{ji}, \ j = 1, \cdots, 6$ *such that*

$$2x^T P_{1_i} A_{12_i} h_i(x) \le a_{1i} x^T x, \tag{8.3}$$

$$2x^T P_{1_i} A_{12_i}(z - h_i(x)) \le a_{2i} x^T x + a_{3i}(z - h_i(x))^T(z - h_i(x)), \tag{8.4}$$

$$2(z - h_i(x))^T P_{2_i} \mathcal{R}_{1_i} x \le a_{4i} x^T x + a_{5i}(z - h_i(x))^T(z - h_i(x)), \tag{8.5}$$

$$(z - h_i(x))^T \mathcal{R}_{2_i}(z - h_i(x)) \le a_{6i}(z - h_i(x))^T(z - h_i(x)) \tag{8.6}$$

where $h_i(x) = -A_{22_i}^{-1}(A_{21_i} + B_{2_i}K_i)x$, P_{2_i} *is the solution of the Lyapunov equation* $A_{22_i}^T P_{2_i} + P_{2_i}A_{22_i} = -I_{i_n}$, *where* I_{i_n} *is an identity matrix,* $\mathcal{R}_{1_i} = A_{22_i}^{-1}(A_{21_i} + B_{2_i}K_i)[A_{11_i} + B_{1_i}K_i - A_{12_i}A_{22_i}^{-1}(A_{21_i} + B_{2_i}K_i)]$, *and* $\mathcal{R}_{2_i} = 2P_{2_i}A_{22_i}^{-1}(A_{21_i} + B_{2_i}K_i)A_{12_i}$;

194

(iii) there exist a positive constant ϵ_i such that $-\bar{A}_i$ is an M-matrix where

$$\bar{A}_i = \begin{bmatrix} \frac{\lambda_{\max}(N_i)}{\lambda_{\max}(P_{1_i})} & \frac{a_{3i}}{\lambda_{\min}(P_{2_i})} \\ \frac{a_{4i}}{\lambda_{\min}(P_{1_i})} & \frac{a_{6i}}{\lambda_{\min}(P_{2_i})} - \frac{(1-a_{5i}\epsilon_i)}{\epsilon_i\lambda_{\max}(P_{2_i})} \end{bmatrix},$$

where $N_i = -Q_i + (a_{1i} + a_{2i})I + M^T P_i + P_i M^T$ such that $M = A_{12_i} A_{22_i}^{-1}(A_{21_i} + B_{2_i} K_i)$

and $(A_{0_i} + B_{0_i} K_i)^T P_{1_i} + P_{1_i}(A_{0_i} + B_{0_i} K_i) = -Q_i$ for a given K_i.

Proof. Let $V_i(x) = x^T P_{1_i} x$ and $W_i((z - h_i(x))(t)) = (z - h_i(x))^T P_{2_i}(z - h_i(x))$ be Lyapunov

function candidates for the slow and the fast subsystem, respectively. Then,

$$\dot{V}_i(x) = \dot{x}^T P_{1_i} x + x^T P_{1_i} \dot{x}$$

$$= \left[(A_{11_i} + B_{1_i} K_i)x + A_{12_i} z\right]^T P_{1_i} x + x^T P_{1_i}[(A_{11_i} + B_{1_i} K_i)x + A_{12_i} z]$$

$$= x^T[(A_{11_i} + B_{1_i} K_i)^T P_{1_i} + P_{1_i}(A_{11_i} + B_{1_i} K_i)]x + 2x^T P_{1_i} A_{12_i} z$$

$$= -x^T Q_i x + 2x^T P_{1_i} A_{12_i}(z - h_i(x)) + 2x^T P_{1_i} A_{12_i} h_i(x)$$

$$\leq x^T(-Q_i + a_{1i}I + a_{2i}I)x + a_{3i}(z - h_i(x))^T(z - h_i(x))$$

$$\leq \frac{\lambda_{\max}(N_i)}{\lambda_{\max}(P_{1_i})} V_i(x) + \frac{a_{3i}}{\lambda_{\min}(P_{2_i})} W_i((z - h_i(x))(t)), \tag{8.7}$$

where $N_i = -Q_i + (a_{1i} + a_{2i})I + M^T P_i + P_i M^T$ such that $M = A_{12_i} A_{22_i}^{-1}(A_{21_i} + B_{2_i} K_i)$ is

negative definite. We also have

$$\dot{W}_i((z - h_i(x))(t)) = (\dot{z} - \dot{h}_i(x))^T P_{2_i}(z - h_i(x)) + (z - h_i(x))^T P_{2_i}(\dot{z} - \dot{h}_i(x))$$

$$= [\frac{1}{\epsilon}((A_{21} + B_2 K)x + A_{22}z) - \dot{h}(x)]^T P(z - h(x))$$

$$+ (z - h(x))^T P[\frac{1}{\epsilon}((A_{21} + B_2 K)x + A_{22}z) - \dot{h}(x)]$$

$$= [\frac{1}{\epsilon_i} A_{22_i}(z - h_i(x)) - \dot{h}_i(x)]^T P_{2_i}(z - h_i(x))$$

$$+ (z - h_i(x))^T P_{2_i}[\frac{1}{\epsilon_i} A_{22_i}(z - h_i(x)) - \dot{h}_i(x)]$$

$$= \frac{1}{\epsilon_i}(z - h_i(x))^T [A_{22_i}^T P_{2_i} + P_{2_i} A_{22_i}](z - h_i(x)) - 2(z - h_i(x))^T P_{2_i} \dot{h}_i(x)$$

$$= -\frac{1}{\epsilon_i}(z - h_i(x))^T(z - h_i(x)) - 2(z - h_i(x))^T P_{2_i} \dot{h}_i(x)$$

$$\leq (a_{5i} - \frac{1}{\epsilon_i})(z - h_i(x))^T(z - h_i(x)) + (z - h_i(x))^T \mathcal{R}_{2_i}(z - h_i(x))$$

$$+ a_{4i} x^T x$$

$$\leq \frac{a_{4i}}{\lambda_{\min}(P_{1_i})} V_i(x) + \left[\frac{a_{6i}}{\lambda_{\min}(P_{2_i})} - \frac{(1 - a_{5i}\epsilon_i)}{\epsilon_i \lambda_{\max}(P_{2_i})}\right] W_i((z - h_i(x))(t)).$$

$$(8.8)$$

where $\mathcal{R}_{2_i} = 2P_{2_i} A_{22_i}^{-1}(A_{21_i} + B_{2_i} K_i) A_{12_i}$. Combining (8.7) and (8.8), we get

$$\begin{bmatrix} \dot{V}_i(x) \\ \dot{W}_i\big((z - h_i(x))(t)\big) \end{bmatrix} \leq \begin{bmatrix} \frac{\lambda_{\max}(N_i)}{\lambda_{\max}(P_{1_i})} & \frac{a_{3i}}{\lambda_{\min}(P_{2_i})} \\ \frac{a_{4i}}{\lambda_{\min}(P_{1_i})} & \frac{a_{6i}}{\lambda_{\min}(P_{2_i})} - \frac{(1 - a_{5i}\epsilon_i)}{\epsilon_i \lambda_{\max}(P_{2_i})} \end{bmatrix} \begin{bmatrix} V_i(x) \\ W_i\big((z - h_i(x))(t)\big) \end{bmatrix}.$$

196

Then, we have

$$\bar{A}_i = \left[\begin{array}{cc} \frac{\lambda_{\max}(N_i)}{\lambda_{\max}(P_{1_i})} & \frac{a_{3i}}{\lambda_{\min}(P_{2_i})} \\ \frac{a_{4i}}{\lambda_{\min}(P_{1_i})} & \frac{a_{6i}}{\lambda_{\min}(P_{2_i})} - \frac{(1-a_{5i}\epsilon_i)}{\epsilon_i\lambda_{\max}(P_{2_i})} \end{array} \right].$$

Then there exists $\eta_i = -\lambda_{\max}(\bar{A}_i) > 0$ such that for $t \in [t_{k-1}, t_k)$,

$$V_i(x) \le \Big(V_i(x(t_{k-1})) + W_i\big((z - h_i(x))(t_{k-1})\big) \Big) e^{-\eta_i(t-t_{k-1})},$$

and

$$W_i\big((z - h_i(x))(t)\big) \le \Big(V_i(x(t_{k-1})) + W_i\big((z - h_i(x))(t_{k-1})\big) \Big) e^{-\eta_i(t-t_{k-1})},$$

For any $i, j \in \mathcal{S}, \ M > 1$, we have

$$V_j(x(t)) \le \mu_1 V_i(x(t)),$$

$$W_j\big((z - h_j(x))(t)\big) \le \mu_2 W_i\big((z - h_i(x))(t)\big).$$

Let $\mu = \max\{\mu_1, \mu_2\}$, then we have

$$V_j(x(t)) \le \mu V_i(x(t)),$$

$$W_j\big((z - h_j(x))(t)\big) \le \mu W_i\big((z - h_i(x))(t)\big).$$

Starting with V_i, we have for $t \in [t_0, t_1)$

$$V_1(x(t)) \le \Big[V_1(x(t_0)) + W_1\big((z - h_1(x))(t_0)\big) \Big] e^{-\eta_1(t-t_0)}$$

For $t \in [t_1, t_2)$, we have

$$V_2(x(t)) \le \Big[V_2(x(t_1)) + W_2\big((z - h_2(x))(t_1)\big) \Big] e^{-\eta_2(t-t_1)} \tag{8.9}$$

197

we know that for $t = t_1$, we have

$$V_2(x(t_1)) \leq \mu \Big[V_1(x(t_0)) + W_1\big((z - h_1(x))(t_0)\big) \Big] e^{-\eta_1(t_1 - t_0)}$$

and similarly,

$$W_2\big((z - h_2(x))(t_1)\big) \leq \mu \Big[V_1(x(t_0)) + W_1\big((z - h_1(x))(t_0)\big) \Big] e^{-\eta_1(t_1 - t_0)}.$$

Then for $t \in [t_0, t_2)$, (8.9) becomes

$$V_2(x(t)) \leq 2\mu \Big[V_1(x(t_0)) + W_1\big((z - h_1(x))(t_0)\big) \Big] e^{-\eta_1(t_1 - t_0)} e^{-\eta_2(t - t_1)}.$$

Then, for all $t \geq t_0$, we have

$$V_i(x(t)) \leq 2\mu e^{-\eta_1(t_1 - t_0)} \cdot 2\mu e^{-\eta_2(t_2 - t_1)} \cdots 2\mu e^{-\eta_{k-1}(t_{k-1} - t_{k-2})}$$

$$\times \Big[V_1(x(t_0)) + W_1\big((z - h_1(x))(t_0)\big) \Big] e^{-\eta_k(t - t_{k-1})}.$$

Let $\eta = \min\{\eta_j : j = 1, 2, \cdots, k\}$. Then

$$V_i(x(t)) \leq (2\mu)^{k-1} \Big[V_1(x(t_0)) + W_1\big((z - h_1(x))(t_0)\big) \Big] e^{-\eta(t - t_0)}$$

$$\leq \Big[V_1(x(t_0)) + W_1\big((z - h_1(x))(t_0)\big) \Big] e^{(k-1)\ln\rho - \eta(t - t_0)},$$

where $\rho = 2\mu$. Applying the average dwell-time condition with $N_0 = \frac{\gamma}{\ln\rho}$, γ is an arbitrary

constant, $\tau_a = \frac{\ln\rho}{(\eta - \eta^*)}$ with $\eta > \eta^*$ leads to

$$V_i(x(t)) \leq \Big[V_1(x(t_0)) + W_1\big((z - h_1(x))(t_0)\big) \Big] e^{\rho - \eta(t - t_0)}$$

and

$$W_i\big((z - h_i(x))(t)\big) \leq \Big[V_1(x(t_0)) + W_1\big((z - h_1(x))(t_0)\big)\Big]e^{\rho - \eta(t - t_0)}$$

This implies that there exists $L > 0$ such that

$$\|x(t)\| + \|z(t)\| \leq L(\|x(t_0)\| + \|z(t_0)\|)e^{-\eta^*(t - t_0)/2}.$$

$\square$

8.2.2   Faulty Case

To analyze the reliable stabilization with respect to actuator failures, for any $i \in \mathcal{S}$, consider the decomposition of the control matrix $B_i = B_{i\sigma} + B_{i\bar{\sigma}}$. Since the control input u is applied to the system through the normal actuators, the closed-loop system becomes

$$\dot{x} = (A_{11_i} + B_{1\bar{\sigma}_i}K_{i\bar{\sigma}})x + A_{12_i}z, \tag{8.10a}$$

$$\epsilon_i \dot{z} = (A_{21_i} + B_{2\bar{\sigma}_i}K_{i\bar{\sigma}})x + A_{22_i}z, \tag{8.10b}$$

$$x(t_0) = x_0, \quad z(t_0) = z_0. \tag{8.10c}$$

where $K_{i\bar{\sigma}} = -\frac{1}{2}\beta_i B_{0i\bar{\sigma}}^T P_{i\bar{\sigma}}$, with $B_{0i\bar{\sigma}} = B_{1\bar{\sigma}_i} - A_{12_i}A_{22_i}^{-1}B_{2\bar{\sigma}_i}$, and $P_{i\bar{\sigma}}$ is a positive definite matrix such that $(A_{0_i} + B_{0i\bar{\sigma}}K_{i\bar{\sigma}})^T P_{i\bar{\sigma}} + P_{i\bar{\sigma}}(A_{0_i} + B_{0i\bar{\sigma}}K_{i\bar{\sigma}}) = -I$. Setting $\epsilon_i = 0$, one may get $z = h_{i\bar{\sigma}}(x) = -A_{22_i}^{-1}(A_{21_i} + B_{2\bar{\sigma}_i}K_{i\bar{\sigma}})x$. In the following theorem, we assume that $\bar{\sigma}_i = \bar{\Sigma}_i$.

Theorem 8.2.2. *The trivial solution of system* (8.10) *is globally exponentially stable if the average dwell-time condition and the following assumptions hold for any $i \in \mathcal{S}$*

(i) $Re[\lambda(A_{22_i})] < 0$, and $A_{11_i}^T P_{1_i} + P_{1_i} A_{11_i} + \beta_i P_{1_i}\left(A_{12_i} A_{22_i}^{-1} B_{2\bar{\Sigma}_i} B_{1\bar{\Sigma}_i}^T - B_{1\bar{\Sigma}_i} B_{1\bar{\Sigma}_i}^T\right) P_{1_i} + \alpha_i I = 0$;

(ii) there exist positive constants a_{ji}, $j = 1, \cdots, 6$ such that

$$2x^T P_{1_i} A_{12_i} h_{i\bar{\Sigma}}(x) \le a_{1i} x^T x, \tag{8.11}$$

$$2x^T P_{1_i} A_{12_i}(z - h_{i\bar{\Sigma}}(x)) \le a_{2i} x^T x + a_{3i}(z - h_{i\bar{\Sigma}}(x))^T(z - h_{i\bar{\Sigma}}(x)), \tag{8.12}$$

$$2(z - h_{i\bar{\Sigma}}(x))^T P_{2_i} \mathcal{R}_{1_{i\bar{\Sigma}}} x \le a_{4i} x^T x + a_{5i}(z - h_{i\bar{\Sigma}}(x))^T(z - h_{i\bar{\Sigma}}(x)), \tag{8.13}$$

$$(z - h_{i\bar{\Sigma}}(x))^T \mathcal{R}_{2\bar{\Sigma}_i}(z - h_{i\bar{\Sigma}}(x)) \le a_{6i}(z - h_{i\bar{\Sigma}}(x))^T(z - h_{i\bar{\Sigma}}(x)), \tag{8.14}$$

where $h_{i\bar{\Sigma}}(x) = -A_{22_i}^{-1}(A_{21_i} + B_{2\bar{\Sigma}_i} K_{i\bar{\Sigma}})x$, P_{2_i} is the solution of $A_{22_i}^T P_{2_i} + P_{2_i} A_{22_i} = -I_{i_n}$, $\mathcal{R}_{1_{i\bar{\Sigma}}} = A_{22_i}^{-1}(A_{21_i} + B_{2\bar{\Sigma}_i} K_{i\bar{\Sigma}})[A_{11_i} + B_{1_i} K_i - A_{12_i} A_{22_i}^{-1}(A_{21_i} + B_{2\bar{\Sigma}_i} K_{i\bar{\Sigma}})]$ where $K_{i\bar{\Sigma}} = -\frac{1}{2}\beta_i B_{0i\bar{\Sigma}}^T P_{i\bar{\Sigma}}$, and $\mathcal{R}_{2\bar{\Sigma}_i} = 2P_{2_i} A_{22_i}^{-1}(A_{21_i} + \frac{1}{2}\beta_i B_{2\bar{\Sigma}_i} B_{2\bar{\Sigma}_i}^T(A_{12_i} A_{22_i}^{-1})^T P_{1_i})A_{12_i} - \frac{1}{2}\beta_i B_{2\bar{\Sigma}_i} B_{1\bar{\Sigma}_i}^T P_{1_i}$;

(iii) there exist a positive constant ϵ_i such that $-\bar{A}_{i\bar{\Sigma}}$ is an M-matrix where

$$\bar{A}_{i\bar{\Sigma}} = \begin{bmatrix} \frac{\lambda_{\max}(N_{i\bar{\Sigma}})}{\lambda_{\max}(P_{1_i})} & \frac{a_{3i}}{\lambda_{\min}(P_{2_i})} \\ \frac{a_{4i}}{\lambda_{\min}(P_{1_i})} & \frac{\epsilon_i(a_{5i}+a_{6i})-1}{\epsilon_i \lambda_{\max}(P_{2_i})} \end{bmatrix}.$$

Proof. Let $V_i(x) = x^T P_{1_i} x$ and $W_i\big((z - h_{\bar{\Sigma}_i}(x))(t)\big) = (z - h_{\bar{\Sigma}_i}(x))^T P_{2_i}(z - h_{\bar{\Sigma}_i}(x))$ be

Lyapunov function candidates. Then, we have

$$\dot{V}_i(x) = \dot{x}^T P_{1_i} x + x^T P_{1_i} \dot{x}$$

$$= \left[(A_{11_i} + B_{1\bar{\Sigma}_i} K_{i\bar{\Sigma}})x + A_{12_i} z\right]^T P_{1_i} x + x^T P_{1_i} \left[(A_{11_i} + B_{1\bar{\Sigma}_i} K_{i\bar{\Sigma}})x + A_{12_i} z\right]$$

$$= x^T \left[(A_{11_i} - \frac{1}{2}\beta_i B_{1\bar{\Sigma}_i} B_{0i\bar{\Sigma}}^T P_{i\bar{\Sigma}})^T P_{1_i} + P_{1_i}(A_{11_i} - \frac{1}{2}\beta_i B_{1\bar{\Sigma}_i} B_{0i\bar{\Sigma}}^T P_{i\bar{\Sigma}})\right] x$$

$$+ 2x^T P_{1_i} A_{12_i}(z - h_{i\bar{\Sigma}}(x)) + 2x^T P_{1_i} A_{12_i} h_{i\bar{\Sigma}}(x)$$

$$= x^T \left[A_{11_i}^T P_{1_i} + P_{1_i} A_{11_i} + \beta_i P_{1_i}\left(A_{12_i} A_{22_i}^{-1} B_{2\bar{\Sigma}_i} B_{1\bar{\Sigma}_i}^T - B_{1\bar{\Sigma}_i} B_{1\bar{\Sigma}_i}^T\right) P_{1_i}\right] x$$

$$+ 2x^T P_{1_i} A_{12_i}(z - h_{i\bar{\Sigma}}(x)) + 2x^T P_{1_i} A_{12_i} h_{i\bar{\Sigma}}(x)$$

$$\leq x^T(-\alpha_i + a_{1i} + a_{2i}) I x + a_{3i}(z - h_{i\bar{\Sigma}}(x))^T (z - h_{i\bar{\Sigma}}(x))$$

$$\leq \frac{-\alpha_i + a_{1i} + a_{2i}}{\lambda_{\max}(P_{1_i})} V_i(x) + \frac{a_{3i}}{\lambda_{\min}(P_{2_i})} W_i\big((z - h_{\bar{\Sigma}_i}(x))(t)\big) \tag{8.15}$$

We also have

$$\dot{W}_i\big((z - h_{\bar{\Sigma}_i}(x))(t)\big) = (\dot{z} - \dot{h}_{i\bar{\Sigma}}(x))^T P_{2_i}(z - h_{i\bar{\Sigma}}(x)) + (z - h_{i\bar{\Sigma}}(x))^T P_{2_i}(\dot{z} - \dot{h}_{i\bar{\Sigma}}(x))$$

$$= \left[\frac{1}{\epsilon_i}((A_{21_i} + B_{2\bar{\Sigma}_i} K_{i\bar{\Sigma}})x + A_{22_i} z) - \dot{h}_{i\bar{\Sigma}}(x)\right]^T P_{2_i}(z - h_{i\bar{\Sigma}}(x))$$

$$+ (z - h_{i\bar{\Sigma}}(x))^T P_{2_i}\left[\frac{1}{\epsilon_i}((A_{21_i} + B_{2\bar{\Sigma}_i} K_{i\bar{\Sigma}})x + A_{22_i} z) - \dot{h}_{i\bar{\Sigma}}(x)\right]$$

$$= -\frac{1}{\epsilon_i}(z - h_{i\bar{\Sigma}}(x))^T (z - h_{i\bar{\Sigma}}(x)) - 2(z - h_{i\bar{\Sigma}}(x))^T P_{2_i} \dot{h}_{i\bar{\Sigma}}(x)$$

201

$$\leq (a_{5i} - \frac{1}{\epsilon_i})(z - h_{i\bar{\Sigma}}(x))^T(z - h_{i\bar{\Sigma}}(x)) + a_{4i}x^Tx$$

$$+ (z - h_{i\bar{\Sigma}}(x))^T \mathcal{R}_{2\bar{\Sigma}_i}(z - h_{i\bar{\Sigma}}(x))$$

$$\leq \frac{a_{4i}}{\lambda_{\min}(P_{1_i})}V_i(x) + \left[\frac{\epsilon_i(a_{5i} + a_{6i}) - 1}{\epsilon_i\lambda_{\max}(P_{2_i})}\right]W_i((z - h_{i\bar{\Sigma}}(x))(t)), \quad\quad (8.16)$$

where $\mathcal{R}_{2\bar{\Sigma}_i} = 2P_{2_i}A_{22_i}^{-1}(A_{21_i} - \frac{1}{2}\beta_i B_{2\bar{\Sigma}_i}B_{1\bar{\Sigma}_i}^T P_{1_i} + \frac{1}{2}\beta_i B_{2\bar{\Sigma}_i}B_{2\bar{\Sigma}_i}^T(A_{12_i}A_{22_i}^{-1})^T P_{1_i})A_{12_i}$.

Combining (8.15) and (8.16), we get the M-matrix $-\bar{A}_{i\bar{\Sigma}}$ with

$$\bar{A}_{i\bar{\Sigma}} = \begin{bmatrix} \frac{\lambda_{\max}(N_{i\bar{\Sigma}})}{\lambda_{\max}(P_{1_i})} & \frac{a_{3i}}{\lambda_{\min}(P_{2_i})} \\ \frac{a_{4i}}{\lambda_{\min}(P_{1_i})} & \frac{\epsilon_i(a_{5i} + a_{6i}) - 1}{\epsilon_i\lambda_{\max}(P_{2_i})} \end{bmatrix}.$$

Proceeding as done in the proof of Theorem 8.2.1, we get the desired result. □

Example 8.2.3. *Consider system* (8.1) *with* $\mathcal{S} = \{1, 2\}$,

$$A_{11_1} = \begin{bmatrix} -5 & 0 \\ 0 & -10 \end{bmatrix}, A_{12_1} = \begin{bmatrix} 0.1 & 2 \\ 0.1 & 0 \end{bmatrix}, A_{21_1} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, A_{22_1} = \begin{bmatrix} 1 & -2 \\ 3 & -2 \end{bmatrix},$$

$$A11_2 = \begin{bmatrix} -3 & 1 \\ 0 & -6 \end{bmatrix}, A_{12_2} = \begin{bmatrix} 1 & 0 \\ 0.1 & 0.3 \end{bmatrix}, A_{21_2} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}, A_{22_2} = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix},$$

$$B_{1_1} = \begin{bmatrix} -5 & 0.5 \\ 0.1 & 0.15 \end{bmatrix}, B_{2_1} = \begin{bmatrix} 3 & -1 \\ 1 & 4 \end{bmatrix}, B_{1_2} = \begin{bmatrix} 4 & 5 \\ 0.5 & 1 \end{bmatrix}, B_{2_2} = \begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix},$$

$\epsilon_1 = 0.01$, $\beta_1 = 0.5$, $a_{11} = 0.1$, $a_{21} = 0.15$, $a_{31} = 0.02$, $a_{41} = 0.01$, $a_{51} = 70$, $Q_1 = -4I$,

$\epsilon_2 = 0.02$, $\beta_2 = 0.25$, $a_{12} = 0.3$, $a_{22} = 0.2$, $a_{32} = 0.2$, $a_{42} = 0.02$, $a_{52} = 30$, *and* $Q_2 = -I$.
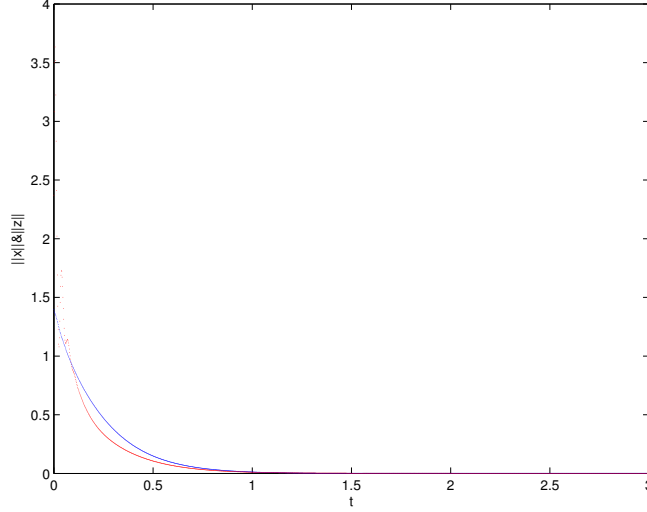
Figure 8.2.1: Singularly perturbed switched system: Operational actuators.

Case 1. *When all actuators are operational, we have*

$$P_{1_1} = \begin{bmatrix} 0.1025 & 0.0274 \\ 0.0274 & 0.0615 \end{bmatrix}, P_{1_2} = \begin{bmatrix} 0.1697 & 0.0616 \\ 0.0616 & 0.1322 \end{bmatrix},$$

$$P_{2_1} = \begin{bmatrix} 1.5 & 1 \\ 1 & 1.75 \end{bmatrix}, P_{2_1} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 1 \end{bmatrix},$$

and

$$K_1 = \begin{bmatrix} 0.0217 & 0.0031 \\ 0.0840 & 0.0238 \end{bmatrix}, K_2 = \begin{bmatrix} -0.1638 & -0.0869 \\ -0.1449 & -0.0842 \end{bmatrix}.$$

Thus, the matrices $A_{0_i} + B_{0_i} K_i$ ($i = 1, 2$) are Hurwitz and $\tau_a = \frac{\ln \mu}{\alpha^* - \nu} = 1.8330$.

Case 2. *When there are failures in the first actuator of B_{1_i}, and the second actuator of B_{2_i} for both modes, i.e.,*

$$B_{1\bar{\Sigma}_1} = \begin{bmatrix} 0 & 0.5 \\ 0 & 0.15 \end{bmatrix}, B_{2\bar{\Sigma}_1} = \begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix}, B_{1\bar{\Sigma}_2} = \begin{bmatrix} 0 & 5 \\ 0 & 1 \end{bmatrix}, B_{2\bar{\Sigma}_2} = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix},$$

we have

$$P_{1_1} = \begin{bmatrix} 0.0993 & 0.0257 \\ 0.0257 & 0.0606 \end{bmatrix}, P_{1_2} = \begin{bmatrix} 0.2828 & 0.1570 \\ 0.1570 & 0.2145 \end{bmatrix},$$

P_{2_1} and P_{2_2} are the same as for the normal case, and

$$K_1 = \begin{bmatrix} -0.1024 & -0.0278 \\ -0.0134 & -0.0055 \end{bmatrix}, K_2 = \begin{bmatrix} -0.1355 & -0.0991 \\ -0.1964 & -0.1249 \end{bmatrix}.$$
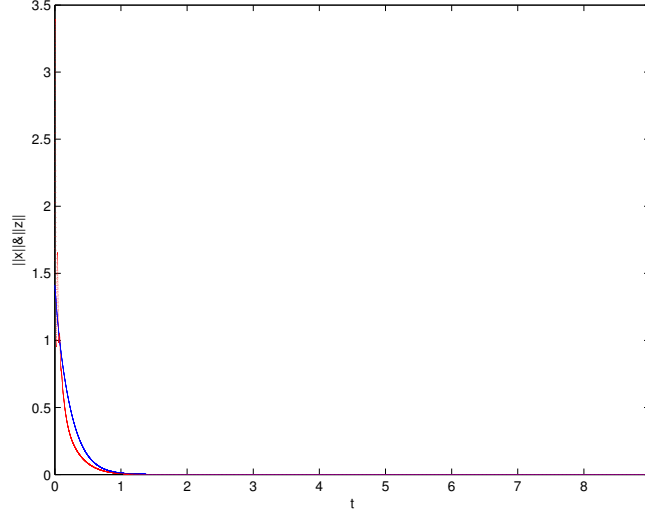
Figure 8.2.2: Singularly perturbed switched system: Faulty actuators.

Thus, the matrices $A_{0_i} + B_{0_i} K_i$ (i = 1, 2) are Hurwitz and $\tau_a = \frac{\ln \mu}{\alpha^* - \nu} = 4.1498$.

Figures 8.2.1 and 8.2.2 show the simulation results of $\|x\|$ (top) and $\|z\|$ (bottom) for the normal and the faulty cases respectively.

8.3 Conclusion

This chapter has established new sufficient conditions that guaranteed the global exponential stability of SCSPS. The output of the faulty actuators has been treated as an outage. We have shown that, using the average dwell-time condition with multiple Lyapunov functions, the full order switched system has been exponentially stabilized by using the state feedback control law $u = K_i x$ A numerical example has been introduced to clarify the proposed results.

Chapter 9

Switched Control SPS with Time Delay

In the present chapter, we extend the results of chapter 8 for switched singularly perturbed systems with time delay via reliable controller when all the actuators are operational as well as when some of them experience failures. An illustrative example is provided to illustrate our results. The contents of this chapter form the basis of [14].

9.1   Problem Formulation and Preliminaries

Consider the following system

$$\dot{x} = A_{11_i}x + \widetilde{A}_{11_i}x(t - r_1) + A_{12_i}z + \widetilde{A}_{12_i}z(t - r_1) + B_{1_i}u, \tag{9.1a}$$

$$\epsilon_i \dot{z} = A_{21_i}x + \widetilde{A}_{21_i}x(t - r_1) + A_{22_i}z + B_{2_i}u, \tag{9.1b}$$

$$x_{t_0}(s) = \phi_1(s), \quad z_{t_0}(s) = \phi_2(s), \quad s \in [-r, 0], \quad r = \max\{r_1, r_2, r_3\}, \tag{9.1c}$$

where $x \in \mathbb{R}^m$, $z \in \mathbb{R}^n$ are the system slow and fast states respectively, $u \in \mathbb{R}^l$ is the control input, and A_{jk_i}, $\widetilde{A}_{jk_i}$, $j, k \in \{1, 2\}$, $i \in \mathcal{S} = \{1, 2, \cdots, N\}$ are known real constant

matrices with A_{22_i} is a nonsingular Hurwitz matrix, and $0 < \epsilon_i \ll 1$. For $r = \max\{r_1, r_2, r_3\}$ where $r_j = jr_1 > 0$ for all $j = \{1, 2, 3\}$, let $\mathcal{C}_r$ be the space of all continuous functions that are defined from $[-r, 0]$ to $\mathbb{R}^n$. For any $t \in \mathbb{R}_+$, let $x(t)$ be a function defined on $[t_0, \infty)$. Then, we define $x_t : [-r, 0] \rightarrow \mathbb{R}^n$ by $x_t(s) = x(t + s)$ for all $s \in [-r, 0]$, and its norm by $\|x_t\|_r = \sup_{t-r \le \theta \le t} \|x(\theta)\|$, where $r > 0$ is the time delay. $\phi_1(t)$, $\phi_2(t) \in \mathcal{C}_r$. $K \in \mathbb{R}^{l \times n}$ is the control gain matrix such that $u = Kx$, where (A_{11_i}, B_{1_i}) is assumed to be stabilizable.

Setting $\epsilon_i = 0$ turns (9.1b) into the algebraic equation

$$z = h_i(x) = -A_{22_i}^{-1}[(A_{21_i} + B_{2_i}K_i)x + \widetilde{A}_{21_i}x(t - r_1)], \tag{9.2}$$

and $z_t = z(t - r_1) = -A_{22_i}^{-1}[(A_{21_i} + B_{2_i}K_i)x(t - r_1) + \widetilde{A}_{21_i}x(t - r_2)]$. Plug z and z_t into (9.1a) gives the slow reduced subsystem

$$\dot{x}_s = (A_{0_i} + B_{0_i}K_i)x_s(t) + C_{0_i}x_s(t - r_1) + D_{0_i}x_s(t - r_2), \tag{9.3}$$

where $A_{0_i} = A_{11_i} - A_{12_i}A_{22_i}^{-1}A_{21_i}$, $B_{0_i} = B_{1_i} - A_{12_i}A_{22_i}^{-1}B_{2_i}$, $C_{0_i} = \widetilde{A}_{11_i} - A_{12_i}A_{22_i}^{-1}\widetilde{A}_{21_i} - \widetilde{A}_{12_i}A_{22_i}^{-1}(A_{21_i} + B_{2_i}K_i)$, and $D_{0_i} = -\widetilde{A}_{12_i}A_{22_i}^{-1}\widetilde{A}_{21_i}$. Choose K_i such that $A_{0_i} + B_{0_i}K_i$ is Hurwitz.

Then, the closed-loop system becomes

$$\begin{cases} \dot{x} = (A_{11_i} + B_{1_i}K_i)x + \widetilde{A}_{11_i}x(t - r_1) + A_{12_i}z + \widetilde{A}_{12_i}z(t - r_1), \\ \epsilon_i \dot{z} = (A_{21_i} + B_{2_i}K_i)x + \widetilde{A}_{21_i}x(t - r_1) + A_{22_i}z, \\ x_{t_0}(s) = \phi_1(s), \quad z_{t_0}(s) = \phi_2(s), \quad s \in [-r, 0], \quad r = \max\{r_1, r_2, r_3\}, \end{cases} \tag{9.4}$$

Definition 9.1.1. *[105] The trivial solution of system* (9.4) *is said to be exponentially*

207

stable if there exist positive constants L, and λ such that

$$\|x(t)\| + \|z(t)\| \leq L(\|x_{t_0}\|_r + \|z_{t_0}\|_r)e^{-\lambda(t-t_0)}, \quad t \geq t_0$$

for all $x(t)$ and $z(t)$, the solutions of system (9.4).

Lemma 9.1.2. *[59] Consider the following differential inequality*

$$\dot{y} \leq -\alpha y(t) + \beta \sup_{t-r \leq \theta \leq t} y(\theta), \quad t \in [t_0, \infty), \quad t_0 \geq 0$$

where α, and β are positive constants such that $\alpha > \beta > 0$. Then, there exists a positive constant η such that $y(t) \leq \|y_0\|_r e^{-\eta(t-t_0)}, \quad t \geq t_0$, where η is a unique positive solution of $g(\eta) = -\eta + \alpha - \beta e^{\eta r} = 0$.

9.2 The Main Results

In this section we introduce our main theorems and proofs.

9.2.1 Normal Case

Theorem 9.2.1. *The trivial solution of system (9.4) is globally exponentially stable if the following assumptions hold for each $i \in \mathcal{S}$*

(i) $Re[\lambda(A_{22_i})] < 0, \quad and \quad (A_{0_i} + B_{0_i} K_i)^T P_{1_i} + P_{1_i}(A_{0_i} + B_{0_i} K_i) = -Q_i;$

(ii) there exist positive constants ν, a_{ji}, β_{ji}, $j = 1, \cdots, 7$ such that

$$2x^T P_{1_i} \left[A_{12_i} h_i(x) + \widetilde{A}_{12_i} h_i(x(t - r_1)) \right] \leq a_{1i} \|x\|^2 + a_{2i} \|x(t - r_1)\|_{r_1}^2$$

$$+ a_{3i} \|x(t - r_2)\|_{r_2}^2, \tag{9.5}$$

$$2x^T P_{1_i} \left[\widetilde{A}_{11_i} x(t - r_1) + A_{12_i} \big((z - h_i(x))(t)\big) + \widetilde{A}_{12_i} \big((z - h_i(x))(t - r_1)\big) \right]$$

$$\leq a_{4i} \|x\|^2 + a_{5i} \|z - h_i(x)\|^2 + a_{6i} \|x(t - r_1)\|_{r_1}^2 + a_{7i} \|(z - h_i(x))(t - r_1)\|_{r_1}^2, \tag{9.6}$$

$$(z - h_i(x))^T \mathcal{R}_{1_i} (z - h_i(x)) \leq \nu \|z - h_i(x)\|^2, \tag{9.7}$$

$$-2(z - h_i(x))^T P_{2_i} \left[\alpha_1 x + \alpha_2 x(t - r_1) + \alpha_3 x(t - r_2) + \alpha_4 x(t - r_3) \right.$$

$$\left. + \alpha_5 \big((z - h_i(x))(t - r_1)\big) + \alpha_6 \big((z - h_i(x))(t - r_2)\big) \right]$$

$$\leq \beta_{1i} \|x\|^2 + \beta_{2i} \|z - h_i(x)\|^2 + \beta_{3i} \|x(t - r_1)\|_{r_1}^2 + \beta_{4i} \|x(t - r_2)\|_{r_2}^2$$

$$+ \beta_{5i} \|x(t - r_3)\|_{r_3}^2 + \beta_{6i} \|(z - h_i(x))(t - r_1)\|_{r_1}^2 + \beta_{7i} \|(z - h_i(x))(t - r_2)\|_{r_2}^2, \tag{9.8}$$

where $h_i(x) = -A_{22_i}^{-1} \left[(A_{21_i} + B_{2_i} K_i)x + \widetilde{A}_{21_i} x(t - r_1) \right]$, P_{2_i} is the solution of the

Lyapunov equation $A_{22_i}^T P_{2_i} + P_{2_i} A_{22_i} = -I_{i_n}$, where I_{i_n} is an identity matrix, $\alpha_1 = $

$-A_{22_i}^{-1}(A_{21_i} + B_{2_i} K_i) \left[A_{11_i} + B_{1_i} K_i - A_{12_i} A_{22_i}^{-1}(A_{21_i} + B_{2_i} K_i) \right]$, $\alpha_2 = A_{22_i}^{-1} \left[(A_{21_i} + \right.$

$B_{2_i} K_i) \left[A_{12_i} A_{22_i}^{-1} \widetilde{A}_{21_i} + \widetilde{A}_{12_i} A_{22_i}^{-1}(A_{21_i} + B_{2_i} K_i) - \widetilde{A}_{11_i} \right] - \widetilde{A}_{21_i} \left[A_{12_i} A_{22_i}^{-1}(A_{21_i} + B_{2_i} K_i) - \right.$

$(A_{11_i} + B_{1_i} K_i) \right]$, $\alpha_3 = A_{22_i}^{-1} \left[[(A_{21_i} + B_{2_i} K_i) \widetilde{A}_{12_i} + \widetilde{A}_{21_i} A_{12_i}] A_{22_i}^{-1} A_{21_i} + \widetilde{A}_{21_i} [\widetilde{A}_{12_i} A_{22_i}^{-1}(A_{21_i} + \right.$

$B_{2_i} K_i) - \widetilde{A}_{11_i}]$, $\alpha_4 = A_{22_i}^{-1} \widetilde{A}_{21_i} \widetilde{A}_{12_i} A_{22_i}^{-1} \widetilde{A}_{21_i}$, $\alpha_5 = -A_{22_i}^{-1} \left[(A_{21_i} + B_{2_i} K_i) \widetilde{A}_{12_i} + \widetilde{A}_{21_i} A_{12_i} \right]$

and $\alpha_6 = -A_{22_i}^{-1} \widetilde{A}_{21_i} \widetilde{A}_{12_i}$;

209

(iii) $\gamma_{1i} = \frac{\lambda_{\max}(N_i)}{\lambda_{\max}(P_{1_i})} + \frac{\beta_{1i}}{\lambda_{\min}(P_{1_i})} < 0$, $\gamma_{2i} = \frac{a_{5i}}{\lambda_{\min}(P_{2_i})} + \frac{\epsilon_i(\nu+\beta_{2i})-1}{\epsilon_i\lambda_{\max}(P_{2_i})} < 0$, and $-\gamma_i > \delta_i$, where

$$\gamma_i = \max\left\{\gamma_{1i}, \gamma_{2i}\right\}, \ \delta_{1i} = \max\left\{\frac{a_{2i}+a_{6i}+\beta_{3i}}{\lambda_{\min}(P_{1_i})}, \frac{a_{7i}+\beta_{6i}}{\lambda_{\min}(P_{2_i})}\right\}, \ \delta_{2i} = \max\left\{\frac{a_{3i}+\beta_{4i}}{\lambda_{\min}(P_{1_i})}, \frac{\beta_{7i}}{\lambda_{\min}(P_{2_i})}\right\},$$

and $\delta_i = \delta_{1i} + \delta_{2i} + \frac{\beta_{5i}}{\lambda_{\min}(P_{1_i})}$;

(iv) for each $i \in \mathcal{S}$, the average dwell-time condition holds.

Proof. Let $V_i(x) = x^T P_{1_i} x$ and $W_i((z-h_i(x))(t)) = (z-h_i(x))^T P_{2_i}(z-h_i(x))$ be Lyapunov function candidates for the slow and the fast subsystem, respectively. Then, we have

$$\dot{V}_i(x) = \dot{x}^T P_{1_i} x + x^T P_{1_i}\dot{x}$$

$$= \left[(A_{11_i} + B_{1_i}K_i)x + \widetilde{A}_{11_i}x(t-r_1) + A_{12_i}z + \widetilde{A}_{12_i}z(t-r_1)\right]^T P_{1_i}x$$

$$\quad + x^T P_{1_i}\left[(A_{11_i} + B_{1_i}K_i)x + \widetilde{A}_{11_i}x(t-r_1) + A_{12_i}z + \widetilde{A}_{12_i}z(t-r_1)\right]$$

$$= -x^T Q_i x + 2x^T P_{1_i}\left[\widetilde{A}_{11_i}x(t-r_1) + A_{12_i}z + \widetilde{A}_{12_i}z(t-r_1)\right]$$

$$\leq x^T(-Q_i + (a_{1i} + a_{4i})I)x + a_{5i}\|z - h_i(x)\|^2 + (a_{2i} + a_{6i})\|x(t-r_1)\|_{r_1}^2$$

$$\quad + a_{7i}\|(z - h_i(x))(t-r_1)\|_{r_1}^2 + a_{3i}\|x(t-r_2)\|_{r_2}^2$$

$$\leq \frac{\lambda_{\max}(N_i)}{\lambda_{\max}(P_{1_i})}V_i(x) + \frac{a_{5i}}{\lambda_{\min}(P_{2_i})}W_i\big((z-h_i(x))(t)\big) + \frac{a_{2i}+a_{6i}}{\lambda_{\min}(P_{1_i})}\|V_i(x(t-r_1))\|_{r_1}$$

$$\quad + \frac{a_{7i}}{\lambda_{\min}(P_{2_i})}\|W_i\big((z-h_i(x))(t-r_1)\big)\|_{r_1} + \frac{a_{3i}}{\lambda_{\min}(P_{1_i})}\|V_i(x(t-r_2))\|_{r_2}, \quad (9.9)$$

where $N_i = -Q_i + (a_{1i} + a_{4i})I$ is negative definite. We also have

$$\dot{W}_i((z - h_i(x))(t)) = (\dot{z} - \dot{h}_i(x))^T P_{2_i}(z - h_i(x)) + (z - h_i(x))^T P_{2_i}(\dot{z} - \dot{h}_i(x))$$

$$= [\frac{1}{\epsilon_i} A_{22_i}(z - h_i(x)) - \dot{h}_i(x)]^T P_{2_i}(z - h_i(x))$$

$$+ (z - h_i(x))^T P_{2_i}[\frac{1}{\epsilon_i} A_{22_i}(z - h_i(x)) - \dot{h}_i(x)]$$

$$= -\frac{1}{\epsilon_i}(z - h_i(x))^T(z - h_i(x)) - 2(z - h_i(x))^T P_{2_i}\dot{h}_i(x)$$

$$\leq (\beta_{2i} - \frac{1}{\epsilon_i})(z - h_i(x))^T(z - h_i(x)) + (z - h_i(x))^T \mathcal{R}_{1_i}(z - h_i(x))$$

$$+ \beta_{1i}\|x\|^2 + \beta_{3i}\|x(t - r_1)\|_{r_1}^2 + \beta_{4i}\|x(t - r_2)\|_{r_2}^2 + \beta_{5i}\|x(t - r_3)\|_{r_3}^2$$

$$+ \beta_{6i}\|(z - h_i(x))(t - r_1)\|_{r_1}^2 + \beta_{7i}\|(z - h_i(x))(t - r_2)\|_{r_2}^2$$

$$\leq \frac{\beta_{1i}}{\lambda_{\min}(P_{1_i})}V_i(x) + \left[\frac{\epsilon_i(\nu + \beta_{2i}) - 1}{\epsilon_i\lambda_{\max}(P_{2_i})}\right]W_i((z - h_i(x))(t))$$

$$+ \frac{\beta_{3i}}{\lambda_{\min}(P_{1_i})}\|V_i(x(t - r_1))\|_{r_1} + \frac{\beta_{4i}}{\lambda_{\min}(P_{1_i})}\|V_i(x(t - r_2))\|_{r_2}$$

$$+ \frac{\beta_{5i}}{\lambda_{\min}(P_{1_i})}\|V_i(x(t - r_3))\|_{r_3} + \frac{\beta_{6i}}{\lambda_{\min}(P_{2_i})}\|W_i((z - h_i(x))(t - r_1))\|_{r_1}$$

$$+ \frac{\beta_{7i}}{\lambda_{\min}(P_{2_i})}\|W_i((z - h_i(x))(t - r_2))\|_{r_2} \tag{9.10}$$

where $\mathcal{R}_{1_i} = 2P_{2_i}A_{22_i}^{-1}(A_{21_i} + B_{2_i}K_i)A_{12_i}$.

Adding (9.9) and (9.21), we get

$$\dot{U}_i(x) = \dot{V}_i(x) + \dot{W}_i((z - h_i(x))(t))$$

$$\leq \left(\frac{\lambda_{\max}(N_i)}{\lambda_{\max}(P_{1_i})} + \frac{\beta_{1i}}{\lambda_{\min}(P_{1_i})}\right)V_i(x) + \left(\frac{a_{2i} + a_{6i} + \beta_{3i}}{\lambda_{\min}(P_{1_i})}\right)\|V_i(x(t - r_1))\|_{r_1}$$

$$+ \frac{a_{3i} + \beta_{4i}}{\lambda_{\min}(P_{1_i})}\|V_i(x(t - r_2))\|_{r_2} + \frac{\beta_{5i}}{\lambda_{\min}(P_{1_i})}\|V_i(x(t - r_3))\|_{r_3}$$

$$+ \left(\frac{a_{5i}}{\lambda_{\min}(P_{2_i})} + \frac{\epsilon_i(\nu + \beta_{2i}) - 1}{\epsilon_i\lambda_{\max}(P_{2_i})}\right)W_i((z - h_i(x))(t))$$

$$+ \frac{a_{7i} + \beta_{6i}}{\lambda_{\min}(P_{2_i})}\|W_i((z - h_i(x))(t - r_1))\|_{r_1} + \frac{\beta_{7i}}{\lambda_{\min}(P_{2_i})}\|W_i((z - h_i(x))(t - r_2))\|_{r_2}$$

$$\leq \gamma_i\left(V_i(x) + W_i((z - h_i(x))(t))\right) + \delta_{1_i}\left(\|V_i(x(t - r_1))\|_{r_1} + \|W_i((z - h_i(x))(t - r_1))\|_{r_1}\right)$$

$$+ \delta_{2i}\left(\|V_i(x(t - r_2))\|_{r_2} + \|W_i((z - h_i(x))(t - r_2))\|_{r_2}\right)$$

$$+ \frac{\beta_{5i}}{\lambda_{\min}(P_{1_i})}\left(\|V_i(x(t - r_3))\|_{r_3} + \|W_i((z - h_i(x))(t - r_3))\|_{r_3}\right)$$

$$\leq \gamma_i U_i(x(t)) + \delta_i \sup_{t - r < \theta < t} U_i(\theta) \tag{9.11}$$

where $\gamma_i = \max\left\{\frac{\lambda_{\max}(N_i)}{\lambda_{\max}(P_{1_i})} + \frac{\beta_{1i}}{\lambda_{\min}(P_{1_i})}, \frac{a_{5i}}{\lambda_{\min}(P_{2_i})} + \frac{\epsilon_i(\nu + \beta_{2i}) - 1}{\epsilon_i\lambda_{\max}(P_{2_i})}\right\}$, $\delta_{1i} = \max\left\{\frac{a_{2i} + a_{6i} + \beta_{3i}}{\lambda_{\min}(P_{1_i})}, \frac{a_{7i} + \beta_{6i}}{\lambda_{\min}(P_{2_i})}\right\}$,

$\delta_{2i} = \max\left\{\frac{a_{3i} + \beta_{4i}}{\lambda_{\min}(P_{1_i})}, \frac{\beta_{7i}}{\lambda_{\min}(P_{2_i})}\right\}$, and $\delta_i = \delta_{1i} + \delta_{2i} + \frac{\beta_{5i}}{\lambda_{\min}(P_{1_i})}$.

Then, by Lemma 9.1.2, there exists $\eta_i > 0$ such that for $t \in [t_{k-1}, t_k)$,

$$U_i(t) \leq \|U_i(x(t_{k-1} - r))\|_r e^{-\eta_i(t-t_{k-1})}$$

$$= \|V_i(x(t_{k-1} - r)) + W_i\big((z - h_i(x))(t_{k-1} - r)\big)\|_r e^{-\eta_i(t-t_{k-1})}$$

$$\leq \Big(\|V_i(x(t_{k-1} - r))\|_r + \|W_i\big((z - h_i(x))(t_{k-1} - r)\big)\|_r\Big) e^{-\eta_i(t-t_{k-1})}$$

which leads to

$$V_i(x) \leq V_i(x(t)) + W_i((z - h_i(x))(t))$$

$$\leq \Big(\|V_i(x(t_{k-1} - r))\|_r + \|W_i\big((z - h_i(x))(t_{k-1} - r)\big)\|_r\Big) e^{-\eta_i(t-t_{k-1})}, \qquad (9.12)$$

Similarly,

$$W_i\big((z - h_i(x))(t)\big) \leq \Big(\|V_i(x(t_{k-1} - r))\|_r + \|W_i\big((z - h_i(x))(t_{k-1} - r)\big)\|_r\Big) e^{-\eta_i(t-t_{k-1})},$$

For any $i, j \in \mathcal{S}$, we have

$$V_j(x(t)) \leq \mu_1 V_i(x(t)),$$

$$W_j\big((z - h_j(x))(t)\big) \leq \mu_2 W_i\big((z - h_i(x))(t)\big).$$

Let $\mu = \max\{\mu_1, \mu_2\} \geq 1$, then we have

$$V_j(x(t)) \leq \mu V_i(x(t)),$$

$$W_j\big((z - h_j(x))(t)\big) \leq \mu W_i\big((z - h_i(x))(t)\big).$$

Starting with V_1, we have for $t \in [t_0, t_1)$

$$V_1(x(t)) \leq \Big[\|V_1(x(t_0 - r))\|_r + \|W_1\big((z - h_1(x))(t_0 - r)\big)\|_r\Big] e^{-\eta_1(t-t_0)}$$

For $t \in [t_1, t_2)$, we have

$$V_2(x(t)) \leq \Big[\|V_2(x(t_1 - r))\|_r + \|W_2\big((z - h_2(x))(t_1 - r)\big)\|_r\Big] e^{-\eta_2(t - t_1)} \qquad (9.13)$$

we know that for $t = t_1$, we have

$$V_2(x(t_1)) \leq \mu\Big[\|V_1(x(t_0 - r))\|_r + \|W_1\big((z - h_1(x))(t_0 - r)\big)\|_r\Big] e^{-\eta_1(t_1 - t_0)}.$$

Similarly,

$$W_2\big((z - h_2(x))(t_1)\big) \leq \mu\Big[\|V_1(x(t_0 - r))\|_r + \|W_1\big((z - h_1(x))(t_0 - r)\big)\|_r\Big] e^{-\eta_1(t_1 - t_0)}$$

So that

$$\|V_2(x(t_1 - r))\|_r \leq \mu\Big[\|V_1(x(t_0 - r))\|_r + \|W_1\big((z - h_1(x))(t_0 - r)\big)\|_r\Big] e^{-\eta_1(t_1 - t_0 - r)}$$

and

$$\|W_2\big((z - h_2(x))(t_1 - r)\big)\|_r \leq \mu\Big[\|V_1(x(t_0 - r))\|_r + \|W_1\big((z - h_1(x))(t_0 - r)\big)\|_r\Big] e^{-\eta_1(t_1 - t_0 - r)}.$$

Then, for $t \in [t_0, t_2)$, (9.13) becomes

$$V_2(x(t)) \leq 2\mu\Big[\|V_1(x(t_0 - r))\|_r + \|W_1\big((z - h_1(x))(t_0 - r)\big)\|_r\Big] e^{-\eta_1(t_1 - t_0)} e^{\eta_1 r} e^{-\eta_2(t - t_1)}$$

Generally, we have for $t \in [t_{k-1}, t_k)$,

$$V_i(x(t)) \leq \prod_{l=1}^{i-1} 2\mu e^{-\eta_l(t_l - t_{l-1})} e^{\eta_l r} \times \Big[\|V_1(x(t_0 - r))\|_r + \|W_1\big((z - h_1(x))(t_0 - r)\big)\|_r\Big] e^{-\eta_i(t - t_{i-1})}$$

214

Let $\eta = \min\{\eta_j : j = 1, 2, \cdots, i\}$, and $\bar{\eta} = \max\{\eta_j : j = 1, 2, \cdots, i-1\}$. Then

$$V_i(x(t)) \leq (2\mu e^{\bar{\eta}r})^{i-1}\Big[\|V_1(x(t_0 - r))\|_r + \|W_1\big((z - h_1(x))(t_0 - r)\big)\|_r\Big]e^{-\eta(t-t_0)}$$

$$= \big(\|V_1(x(t_0 - r))\|_r + \|W_1\big((z - h_1(x))(t_0 - r)\big)\|_r\big)e^{(i-1)\ln\rho - \eta(t-t_0)}$$

where $\rho = 2\mu e^{\bar{\eta}r}$. Applying the average dwell-time condition with $N_0 = \frac{\gamma}{\ln\rho}$, γ is an arbitrary constant, $\tau_a = \frac{\ln\rho}{(\eta - \eta^*)}$ with $\eta > \eta^*$ leads to

$$V_i(x(t)) \leq \big(\|V_1(x(t_0 - r))\|_r + \|W_1\big((z - h_1(x))(t_0 - r)\big)\|_r\big)e^{\gamma - \eta^*(t-t_0)}$$

and similarly,

$$W_i\big((z - h_i(x))(t)\big) \leq \big(\|V_1(x(t_0 - r))\|_r + \|W_1\big((z - h_1(x))(t_0 - r)\big)\|_r\big)e^{\gamma - \eta^*(t-t_0)}$$

This implies that there exists $L > 0$ such that

$$\|x(t)\| + \|z(t)\| \leq L(\|x(t_0 - r)\|_r + \|z(t_0 - r)\|_r)e^{-\eta^*(t-t_0)/2},$$

which completes the proof. $\qquad\square$

Remark 9.2.2. *Every subsystem in* (9.4) *is treated as an interconnected system. The adequate approach to analyze the stability of this type of systems is to decompose it into lower order subsystems ignoring the interconnection, study the stability of each mode. Then, use this information with the interconnection to draw a conclusion about the stability property. Condition* (ii) *means that the perturbation part (the interconnection) is assumed to be bounded. Condition* (iii) *is needed to guarantee the exponential stability property, which needs the stability degree to be larger than the interconnection.*

9.2.2 Faulty Case

For any $i \in \{1, 2\}$, consider the decomposition of the control matrix $B_i = B_{i\sigma} + B_{i\bar{\sigma}}$. Then, the closed-loop system becomes

$$\dot{x} = (A_{11_i} + B_{1\bar{\sigma}_i} K_{i\bar{\sigma}})x + \widetilde{A}_{11_i} x(t - r_1) + A_{12_i} z + \widetilde{A}_{12_i} z(t - r_1), \tag{9.14a}$$

$$\epsilon_i \dot{z} = (A_{21_i} + B_{2\bar{\sigma}_i} K_{i\bar{\sigma}})x + \widetilde{A}_{21_i} x(t - r_1) + A_{22_i} z, \tag{9.14b}$$

$$x_{t_0}(s) = \phi_1(s), \quad z_{t_0}(s) = \phi_2(s), \quad s \in [-r, 0], \quad r > 0, \tag{9.14c}$$

where $K_{i\bar{\sigma}} = -\frac{1}{2}\beta_i B_{0i\bar{\sigma}}^T P_{i\bar{\sigma}}$, with $B_{0i\bar{\sigma}} = B_{1\bar{\sigma}_i} - A_{12_i} A_{22_i}^{-1} B_{2\bar{\sigma}_i}$, and $P_{i\bar{\sigma}}$ is a positive definite matrix such that $(A_{0_i} + B_{0i\bar{\sigma}} K_{i\bar{\sigma}})^T P_{i\bar{\sigma}} + P_{i\bar{\sigma}}(A_{0_i} + B_{0i\bar{\sigma}} K_{i\bar{\sigma}}) = -Q_{i\bar{\sigma}}$. Setting $\epsilon_i = 0$ turns (9.14b) into an algebraic equation which has the following solution

$$z = h_{i\bar{\sigma}}(x) = -A_{22_i}^{-1}[(A_{21_i} + B_{2\bar{\sigma}_i} K_{i\bar{\sigma}})x + \widetilde{A}_{21_i} x(t - r_1)] \tag{9.15}$$

and $z_t = z(t - r_1) = -A_{22_i}^{-1}[(A_{21_i} + B_{2\bar{\sigma}_i} K_{i\bar{\sigma}})x(t - r_1) + \widetilde{A}_{21_i} x(t - r_2)]$.

In the following theorem, we assume that all susceptible actuators have experience failures, i.e. $\bar{\sigma} = \bar{\Sigma}$.

Theorem 9.2.3. *The trivial solution of system* (9.14) *is globally exponentially stable if the following assumptions hold for each $i \in \mathcal{S}$*

(i) $Re[\lambda(A_{22_i})] < 0$, and

$$A_{11_i}^T P_{1_i} + P_{1_i} A_{11_i} + \frac{1}{2}\beta_i P_{1_i}\left(A_{12_i} A_{22_i}^{-1} B_{2\bar{\Sigma}_i} B_{1\bar{\Sigma}_i}^T + (A_{12_i} A_{22_i}^{-1} B_{2\bar{\Sigma}_i} B_{1\bar{\Sigma}_i}^T)^T - 2B_{1\bar{\Sigma}_i} B_{1\bar{\Sigma}_i}^T\right)P_{1_i}$$
$$+ \alpha_i I = 0$$

(ii) there exist positive constants a_{ji}, β_{ji}, $j = 1, \cdots, 7$ such that

$$2x^T P_{1_i} \left[A_{12_i} h_{i\bar{\Sigma}}(x) + \widetilde{A}_{12_i} h_{i\bar{\Sigma}}(x(t - r_1)) \right] \leq a_{1i} \|x\|^2 + a_{2i} \|x(t - r_1)\|_{r_1}^2$$

$$+ a_{3i} \|x(t - r_2)\|_{r_2}^2, \tag{9.16}$$

$$2x^T P_{1_i} \left[\widetilde{A}_{11_i} x(t - r_1) + A_{12_i} \big((z - h_{i\bar{\Sigma}}(x))(t) \big) + \widetilde{A}_{12_i} \big((z - h_{i\bar{\Sigma}}(x))(t - r_1) \big) \right]$$

$$\leq a_{4i} \|x\|^2 + a_{5i} \|z - h_{i\bar{\Sigma}}(x)\|^2 + a_{6i} \|x(t - r_1)\|_{r_1}^2 + a_{7i} \|(z - h_{i\bar{\Sigma}}(x))(t - r_1)\|_{r_1}^2, \tag{9.17}$$

$$-2(z - h_{i\bar{\Sigma}}(x))^T P_{2_i} \dot{h}_{i\bar{\Sigma}}(x) \leq \beta_{1i} \|x\|^2 + \beta_{2i} \|z - h_{i\bar{\Sigma}}(x)\|^2 + \beta_{3i} \|x(t - r_1)\|_{r_1}^2$$

$$+ \beta_{4i} \|x(t - r_2)\|_{r_2}^2 + \beta_{5i} \|x(t - r_3)\|_{r_3}^2$$

$$+ \beta_{6i} \|(z - h_{i\bar{\Sigma}}(x))(t - r_1)\|_{r_1}^2$$

$$+ \beta_{7i} \|(z - h_{i\bar{\Sigma}}(x))(t - r_2)\|_{r_2}^2, \tag{9.18}$$

where $h_{i\bar{\Sigma}}(x) = -A_{22_i}^{-1} \left[(A_{21_i} + B_{2\bar{\Sigma}_i} K_{i\bar{\Sigma}})x + \widetilde{A}_{21_i} x(t - r) \right]$ with $K_{i\bar{\Sigma}} = -\frac{1}{2} \beta_i B_{0i\bar{\Sigma}} P_{i\bar{\Sigma}}$,

and P_{2_i} is the solution of the Lyapunov equation $A_{22_i}^T P_{2_i} + P_{2_i} A_{22_i} = -I_{i_n}$, where I_{i_n}

is an identity matrix.

(iii) $\gamma_{1i} = \frac{\lambda_{\max}(N_{i\bar{\Sigma}})}{\lambda_{\max}(P_{1_i})} + \frac{\beta_{1i}}{\lambda_{\min}(P_{1_i})} < 0$, $\gamma_{2i} = \frac{a_{5i}}{\lambda_{\min}(P_{2_i})} + \frac{\epsilon_i \beta_{2i} - 1}{\epsilon_i \lambda_{\max}(P_{2_i})} < 0$, and $-\gamma_i > \delta_i$, where

$\gamma_i = \max \left\{ \gamma_{1i}, \gamma_{2i} \right\}$, $\delta_{1i} = \max \left\{ \frac{a_{2i} + a_{6i} + \beta_{3i}}{\lambda_{\min}(P_{1_i})}, \frac{a_{7i} + \beta_{6i}}{\lambda_{\min}(P_{2_i})} \right\}$, $\delta_{2i} = \max \left\{ \frac{a_{3i} + \beta_{4i}}{\lambda_{\min}(P_{1_i})}, \frac{\beta_{7i}}{\lambda_{\min}(P_{2_i})} \right\}$,

and $\delta_i = \delta_{1i} + \delta_{2i} + \frac{\beta_{5i}}{\lambda_{\min}(P_{1_i})}$.

(iv) for each $i \in \mathcal{S}$, the average dwell-time condition holds.

217

Proof. Let $V_i(x) = x^T P_{1_i} x$ and $W_i\big((z - h_{\bar{\Sigma}_i}(x))(t)\big) = (z - h_{\bar{\Sigma}_i}(x))^T P_{2_i}(z - h_{\bar{\Sigma}_i}(x))$ be Lyapunov function candidates for the slow and the fast subsystem, respectively. Then, we have

$$\dot{V}_i(x) = x^T\Big[(A_{11_i} + B_{1\bar{\Sigma}_i}K_{i\bar{\Sigma}})^T P_{1_i} + P_{1_i}(A_{11_i} + B_{1\bar{\Sigma}_i}K_{i\bar{\Sigma}})\Big]x + 2x^T P_{1_i}A_{12_i}(z - h_{i\bar{\Sigma}}(x))$$

$$+ 2x^T P_{1_i}A_{12_i}h_{i\bar{\Sigma}}(x) + 2x^T P_{1_i}\widetilde{A}_{11_i}x(t - r_1) + 2x^T P_{1_i}\widetilde{A}_{12_i}(z - h_{i\bar{\Sigma}}(x))(t - r_1)$$

$$+ 2x^T P_{1_i}\widetilde{A}_{12_i}h_{i\bar{\Sigma}}(x(t - r_1))$$

$$= x^T\Big[A_{11_i}^T P_{1_i} + P_{1_i}A_{11_i} + \frac{1}{2}\beta_i P_{1_i}\big(A_{12_i}A_{22_i}^{-1}B_{2\bar{\Sigma}_i}B_{1\bar{\Sigma}_i}^T + (A_{12_i}A_{22_i}^{-1}B_{2\bar{\Sigma}_i}B_{1\bar{\Sigma}_i}^T)^T$$

$$- 2B_{1\bar{\Sigma}_i}B_{1\bar{\Sigma}_i}^T\big)P_{1_i}\Big]x + 2x^T P_{1_i}\Big[\widetilde{A}_{11_i}x(t - r_1) + A_{12_i}(z - h_{i\bar{\Sigma}}(x))$$

$$+ \widetilde{A}_{12_i}(z - h_{i\bar{\Sigma}}(x))(t - r_1) + \widetilde{A}_{12_i}h_{i\bar{\Sigma}}(x(t - r_1)) + A_{12_i}h_{i\bar{\Sigma}}(x)\Big]$$

$$\leq \frac{\lambda_{\max}(N_{i\bar{\Sigma}})}{\lambda_{\max}(P_{1_i})}V_i(x) + \frac{a_{5i}}{\lambda_{\min}(P_{2_i})}W_i((z - h_{i\bar{\Sigma}}(x))(t)) + \frac{a_{2i} + a_{6i}}{\lambda_{\min}(P_{1_i})}\|V_i(x(t - r_1))\|_{r_1}$$

$$+ \frac{a_{7i}}{\lambda_{\min}(P_{2_i})}\|W_i((z - h_{i\bar{\Sigma}}(x))(t - r_1))\|_{r_1} + \frac{a_{3i}}{\lambda_{\min}(P_{1_i})}\|V_i(x(t - r_2))\|_{r_2}, \qquad (9.19)$$

218

where $N_{i\bar{\Sigma}} = (-\alpha_i + a_{1i} + a_{2i})I$ is negative definite.

$$\dot{W}_i\big((z - h_{\bar{\Sigma}_i}(x))(t)\big) = \left[\frac{1}{\epsilon_i}\big((A_{21_i} + B_{2\bar{\Sigma}_i}K_{i\bar{\Sigma}})x + \widetilde{A}_{21_i}x(t - r_1) + A_{22_i}z\big) - \dot{h}_{i\bar{\Sigma}}(x)\right]^T P_{2_i}(z - h_{i\bar{\Sigma}}(x))$$

$$+ (z - h_{i\bar{\Sigma}}(x))^T P_{2_i}\left[\frac{1}{\epsilon_i}\big((A_{21_i} + B_{2\bar{\Sigma}_i}K_{i\bar{\Sigma}})x + \widetilde{A}_{21_i}x(t - r_1) + A_{22_i}z\big) - \dot{h}_{i\bar{\Sigma}}(x)\right]$$

$$= \left[\frac{1}{\epsilon_i}A_{22_i}(z - h_{i\bar{\Sigma}}(x)) - \dot{h}_{i\bar{\Sigma}}(x)\right]^T P_{2_i}(z - h_{i\bar{\Sigma}}(x))$$

$$+ (z - h_{i\bar{\Sigma}}(x))^T P_{2_i}\left[\frac{1}{\epsilon_i}A_{22_i}(z - h_{i\bar{\Sigma}}(x)) - \dot{h}_{i\bar{\Sigma}}(x)\right]$$

$$= -\frac{1}{\epsilon_i}(z - h_{i\bar{\Sigma}}(x))^T(z - h_{i\bar{\Sigma}}(x)) - 2(z - h_{i\bar{\Sigma}}(x))^T P_{2_i}\dot{h}_{i\bar{\Sigma}}(x)$$

$$\leq (\beta_{2i} - \frac{1}{\epsilon_i})\|z - h_{i\bar{\Sigma}}(x)\|^2 + \beta_{1i}\|x\|^2 + \beta_{3i}\|x(t - r_1)\|_{r_1}^2$$

$$+ \beta_{4i}\|x(t - r_2)\|_{r_2}^2 + \beta_{5i}\|x(t - r_3)\|_{r_3}^2 + \beta_{6i}\|(z - h_{i\bar{\Sigma}}(x))(t - r_1)\|_{r_1}^2$$

$$+ \beta_{7i}\|(z - h_{i\bar{\Sigma}}(x))(t - r_2)\|_{r_2}^2$$

$$\leq \frac{\beta_{1i}}{\lambda_{\min}(P_{1_i})}V_i(x) - \frac{(1 - \beta_{2i}\epsilon_i)}{\epsilon_i\lambda_{\max}(P_{2_i})}W_i\big((z - h_i(x))(t)\big)$$

$$+ \frac{\beta_{3i}}{\lambda_{\min}(P_{1_i})}\|V_i(x(t - r_1))\|_{r_1} + \frac{\beta_{4i}}{\lambda_{\min}(P_{1_i})}\|V_i(x(t - r_2))\|_{r_2}$$

$$+ \frac{\beta_{5i}}{\lambda_{\min}(P_{1_i})}\|V_i(x(t - r_3))\|_{r_3} + \frac{\beta_{6i}}{\lambda_{\min}(P_{2_i})}\|W_i\big((z - h_{i\bar{\Sigma}}(x))(t - r_1)\big)\|_{r_1}$$

$$+ \frac{\beta_{7i}}{\lambda_{\min}(P_{2_i})}\|W_i\big((z - h_{i\bar{\Sigma}}(x))(t - r_2)\big)\|_{r_2}, \tag{9.20}$$

Adding (9.19) and (9.20), we get

$$\dot{U}_i(x) \leq \Big(\frac{\lambda_{\max}(N_{i\bar{\Sigma}})}{\lambda_{\max}(P_{1_i})} + \frac{\beta_{1i}}{\lambda_{\min}(P_{1_i})}\Big)V_i(x) + \Big(\frac{a_{2i} + a_{6i} + \beta_{3i}}{\lambda_{\min}(P_{1_i})}\Big)\|V_i(x(t - r_1))\|_{r_1}$$

$$+ \frac{a_{3i} + \beta_{4i}}{\lambda_{\min}(P_{1_i})}\|V_i(x(t - r_2))\|_{r_2} + \frac{\beta_{5i}}{\lambda_{\min}(P_{1_i})}\|V_i(x(t - r_3))\|_{r_3}$$

$$+ \Big(\frac{a_{5i}}{\lambda_{\min}(P_{2_i})} + \frac{\epsilon_i\beta_{2i} - 1}{\epsilon_i\lambda_{\max}(P_{2_i})}\Big)W_i((z - h_i(x))(t))$$

$$+ \frac{a_{7i} + \beta_{6i}}{\lambda_{\min}(P_{2_i})}\|W_i((z - h_i(x))(t - r_1))\|_{r_1} + \frac{\beta_{7i}}{\lambda_{\min}(P_{2_i})}\|W_i((z - h_i(x))(t - r_2))\|_{r_2}$$

$$\leq \gamma_i\Big(V_i(x) + W_i((z - h_i(x))(t))\Big) + \delta_{1_i}\Big(\|V_i(x(t - r_1))\|_{r_1} + \|W_i((z - h_i(x))(t - r_1))\|_{r_1}\Big)$$

$$+ \delta_{2i}\Big(\|V_i(x(t - r_2))\|_{r_2} + \|W_i((z - h_i(x))(t - r_2))\|_{r_2}\Big)$$

$$+ \frac{\beta_{5i}}{\lambda_{\min}(P_{1_i})}\Big(\|V_i(x(t - r_3))\|_{r_3} + \|W_i((z - h_i(x))(t - r_3))\|_{r_3}\Big)$$

$$\leq \gamma_i U_i(x(t)) + \delta_i \sup_{t - r < \theta < t} U_i(\theta) \tag{9.21}$$

where $\gamma_i = \max\Big\{\frac{\lambda_{\max}(N_{i\bar{\Sigma}})}{\lambda_{\max}(P_{1_i})} + \frac{\beta_{1i}}{\lambda_{\min}(P_{1_i})}, \frac{a_{5i}}{\lambda_{\min}(P_{2_i})} + \frac{\epsilon_i\beta_{2i} - 1}{\epsilon_i\lambda_{\max}(P_{2_i})}\Big\}$, $\delta_{1i} = \max\Big\{\frac{a_{2i} + a_{6i} + \beta_{3i}}{\lambda_{\min}(P_{1_i})}, \frac{a_{7i} + \beta_{6i}}{\lambda_{\min}(P_{2_i})}\Big\}$,

$\delta_{2i} = \max\Big\{\frac{a_{3i} + \beta_{4i}}{\lambda_{\min}(P_{1_i})}, \frac{\beta_{7i}}{\lambda_{\min}(P_{2_i})}\Big\}$, and $\delta_i = \delta_{1i} + \delta_{2i} + \frac{\beta_{5i}}{\lambda_{\min}(P_{1_i})}$.

Proceeding as done in the proof of Theorem 9.2.1, we get the desired result.

Example 9.2.4. *Consider system* (9.1) *with* $\mathcal{S} = \{1, 2\}$,

$$A_{11_1} = \begin{bmatrix} -3 & 0 \\ 1 & -7 \end{bmatrix}, A_{12_1} = \begin{bmatrix} 1 & 2 \\ 0.1 & 0 \end{bmatrix}, A_{21_1} = \begin{bmatrix} 2 & 0.5 \\ 1 & 1 \end{bmatrix}, A_{22_1} = \begin{bmatrix} 1.5 & -2 \\ 3 & -3 \end{bmatrix},$$

$$\widetilde{A}_{11_1} = \begin{bmatrix} -1 & -0.4 \\ 0.2 & 0.8 \end{bmatrix}, \widetilde{A}_{12_1} = \begin{bmatrix} -1 & 0.1 \\ -0.5 & -0.2 \end{bmatrix}, \widetilde{A}_{21_1} = \begin{bmatrix} -1 & 0.5 \\ -0.5 & -1 \end{bmatrix},$$

$$A_{11_2} = \begin{bmatrix} -5 & 1 \\ 0 & -3 \end{bmatrix}, A_{12_2} = \begin{bmatrix} -2 & 0 \\ 0.1 & 3 \end{bmatrix}, A_{21_2} = \begin{bmatrix} 1.5 & 3 \\ 2 & 1 \end{bmatrix}, A_{22_2} = \begin{bmatrix} -3 & 2 \\ 1.5 & -2 \end{bmatrix},$$

$$\widetilde{A}_{11_2} = \begin{bmatrix} 1 & 0.3 \\ 0.2 & -1 \end{bmatrix}, \widetilde{A}_{12_2} = \begin{bmatrix} 0.01 & -0.2 \\ -1 & 0.5 \end{bmatrix}, \widetilde{A}_{21_2} = \begin{bmatrix} -2 & 0.5 \\ 0.5 & 0.5 \end{bmatrix},$$

$$B_{1_1} = \begin{bmatrix} -6 & 0.5 \\ 01 & 0.15 \end{bmatrix}, B_{2_1} = \begin{bmatrix} 3 & -2 \\ 1.5 & 4 \end{bmatrix}, B_{1_2} = \begin{bmatrix} 3 & 4 \\ 1 & 1.5 \end{bmatrix}, B_{2_2} = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix},$$

$\epsilon_1 = 0.01$, $\beta_1 = 0.5$, $a_{11} = 0.1$, $a_{21} = 0.15$, $a_{31} = 0.02$, $a_{41} = 0.15$, $a_{51} = 2$, $a_{61} =$

0.15, $a_{71} = 0.05$, $\beta_{11} = 0.1$, $\beta_{21} = 1$, $\beta_{31} = 0.01$, $\beta_{41} = 0.02$, $\beta_{51} = 0.015$, $\beta_{61} =$

0.2, $\beta_{71} = 0.1$, $Q_1 = -4I$. $\epsilon_2 = 0.02$, $\beta_2 = 0.25$, $a_{12} = 0.3$, $a_{22} = 0.02$, $a_{32} = 0.015$, $a_{42} =$

0.2, $a_{52} = 0.3$, $a_{62} = 0.01$, $a_{72} = 0.4$, $\beta_{12} = 0.1$, $\beta_{22} = 0.5$, $\beta_{32} = 0.02$, $\beta_{42} = 0.02$, $\beta_{52} =$

0.25, $\beta_{62} = 0.1$, $\beta_{72} = 0.01$, $Q_2 = -I$, and the initial conditions $\phi_1 = \sin(t)$, $\phi_2 =$

$\cos(t + 1)$.

Case 1. *[All the actuators are operational] For the first mode, the closed loop system is*

given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \epsilon \dot{z}_1 \\ \epsilon \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -2.0074 & -0.0059 & 1 & 2 \\ 0.9425 & -7.0131 & 0.1 & 0 \\ 0.6941 & 0.6085 & 1.5 & -2 \\ 2.6602 & 0.7528 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} -1 & -0.4 & -1 & 0.1 \\ 0.2 & 0.8 & -0.5 & -0.2 \\ -1 & 0.5 & 0 & 0 \\ -0.5 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t - r_1) \\ x_2(t - r_1) \\ z_1(t - r_1) \\ z_2(t - r_1) \end{bmatrix}.$$

For the second mode, the closed loop system is given by

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \epsilon \dot{z}_1 \\ \epsilon \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -5.7150 & -0.2098 & -2 & 0 \\ -0.2563 & -3.4302 & 0.1 & 3 \\ 1.6338 & 3.1093 & -3 & 2 \\ 1.6898 & 0.4890 & 1.5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 & 0.3 & 0.01 & -0.2 \\ 0.2 & -1 & -1 & 0.5 \\ -2 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-r_1) \\ x_2(t-r_1) \\ z_1(t-r_1) \\ z_2(t-r_1) \end{bmatrix}.
$$

Case 2. *[Faulty actuators] For the first mode, assume we have faulty in the second actuator of B_{1_1}, and the first actuator of B_{2_1}, and for the second mode, assume we have faulty in the first actuator of both B_{1_2} and B_{2_2} i.e.,*

$$
B_{1\bar{\Sigma}_1} = \begin{bmatrix} -6 & 0 \\ 1 & 0 \end{bmatrix}, B_{2\bar{\Sigma}_1} = \begin{bmatrix} 0 & -2 \\ 0 & 4 \end{bmatrix}, B_{1\bar{\Sigma}_2} = \begin{bmatrix} 0 & 4 \\ 0 & 1.5 \end{bmatrix}, B_{2\bar{\Sigma}_2} = \begin{bmatrix} 0 & -3 \\ 0 & 2 \end{bmatrix},
$$

The closed loop system is given by

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \epsilon \dot{z}_1 \\ \epsilon \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -3.6758 & 0.1850 & 1 & 2 \\ 1.1126 & -7.0308 & 0.1 & 0 \\ 1.0713 & 0.5768 & 1.5 & -2 \\ 2.8574 & 0.8463 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} -1 & -0.4 & -1 & 0.1 \\ 0.2 & 0.8 & -0.5 & -0.2 \\ -1 & 0.5 & 0 & 0 \\ -0.5 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-r_1) \\ x_2(t-r_1) \\ z_1(t-r_1) \\ z_2(t-r_1) \end{bmatrix}.
$$

For the second mode, the closed loop system is given by

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \epsilon \dot{z}_1 \\ \epsilon \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -5.6224 & 0.0651 & 1 & 2 \\ -0.2334 & -3.3506 & 0.1 & 0 \\ 1.9668 & 3.7012 & 1.5 & -2 \\ 1.6888 & 0.5325 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} -1 & -0.4 & -1 & 0.1 \\ 0.2 & 0.8 & -0.5 & -0.2 \\ -1 & 0.5 & 0 & 0 \\ -0.5 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-r_1) \\ x_2(t-r_1) \\ z_1(t-r_1) \\ z_2(t-r_1) \end{bmatrix}.
$$

Figure 9.2.1 shows the simulation results of $\|x\|$ (blue) and $\|z\|$ (red) when all the actuators are operational. Figure 9.2.2 shows the simulation results of $\|x\|$ (in blue) and $\|z\|$ (in red) when failure occurs.
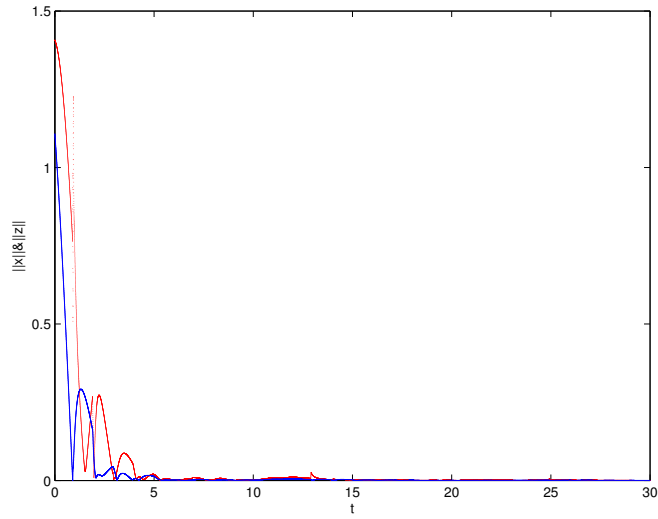
222

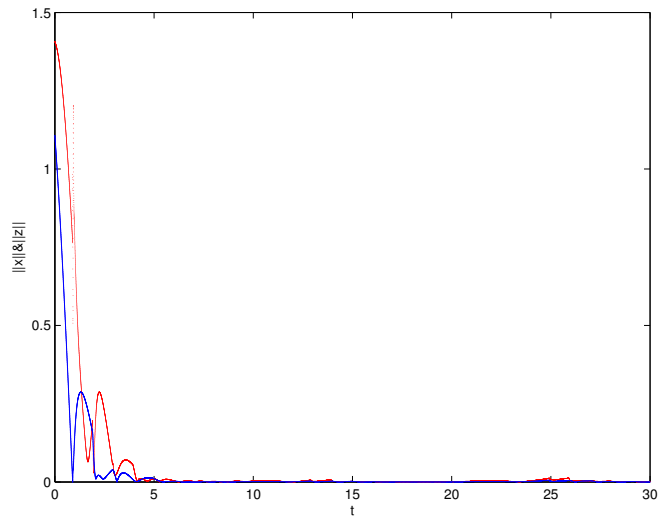Figure 9.2.1: Singularly perturbed switched system with time delay: Operational actuators.

Figure 9.2.2: Singularly perturbed switched system with time delay: Faulty actuators.

223

9.3   Conclusion

This chapter has addressed the global exponential stability problem of switched singularly perturbed systems with time delay via reliable controller of the individual slow subsystems under the average dwell-time signal law. The output of the faulty actuators has been treated as an outage. Halanay inequality has been employed to achieve the desired results. We have shown that, using the average dwell-time with multiple Lyapunov functions, the switched system is exponentially stabilizable, when the slow subsystem is exponentially stabilized by a reliable feedback controller.

Chapter 10

Conclusions and Future Works

Throughout this thesis, the focus has been on studying some qualitative properties of hybrid systems including switched systems, impulsive systems, and impulsive switched systems. As stated in the early chapters, hybrid systems are very important in describing many dynamical systems in engineering and sciences. These systems become more realistic if a part of the system state history and some environment random processes are taken into account, which results in the so-called *stochastic hybrid systems with time delay*.

Having introduced the definitions of hybrid systems and their usefulness, and provided the background and preliminaries in Chapters 1 and 2, the contributions of this thesis and some possible future works are summarized in this chapter.

In Part I, we have established some results on input-to-state stability and stabilization (ISS) of switched, impulsive systems with and without time delay. The ISS analysis has been achieved by multiple Laypunov functions and, to organize the switching among the system modes, we have used the average dwell time switching law. As for the delay systems,

Lyapunov-Razumikhin approach has been considered. So that, as a future work, one may use Laypunov functional or La Salle's Theorems. Moreover, throughout this part the stabilization has been established by using state feedback control law with possible faulty actuators. This may suggest considering systems with faulty sensors.

Part II deals with the stability and stabilization of large-scale systems with/without random noise and with/without time delay. These systems have been decomposed into smaller, low order subsystems. In fact, one may apply this approach to study some network systems that are stabilized by decentralized controllers with possible failures in pre-specified sets of actuators or sensors.

In Part III, the focus has been on developing some results regarding stability and stabilization of singularly perturbed systems with/without time delay. The reliable stabilization has been achieved through the slow subsystem due to their dominant behaviour. One may study these systems with impulsive effects, or consider some external input disturbance and establish the ISS properties. Furthermore, if these systems are subject to some random noises, then one may conduct some researches on the stability and stabilization in certain stochastic senses.

References

[1] A. Abid, M. Ben Hamed, Z. Ben Mabrouk, and L. Sbita. Multiple Lunberger observer for an induction motor represented by decoupled multiple model. *2013 10th International Multi-Conference on Systems, Signals and Devices, SSD 2013*, pages 1–6, 2013.

[2] M.U. Akhmet. On the general problem of stability for impulsive differential equations. *Journal of Mathematical Analysis and Applications*, 288(1):182–196, 2003.

[3] R. Alur, C. Courcoubetis, N. Halbwachs, T.a. Henzinger, P.-H. Ho, X. Nicollin, A. Olivero, J. Sifakis, and S. Yovine. The algorithmic analysis of hybrid systems. *Theoretical Computer Science*, 138(1):3–34, 1995.

[4] R. Alur, T.A. Henzinger, and E.D. Sontag. *Hybrid Systems III: Verification and Control.* Springer, 1996.

[5] M.S. Alwan. Stability of Hybrid Singularly Perturbed Systems with Time Delay. Master's thesis, University of Waterloo, 2006.

[6] M.S. Alwan. *Qualitative Properties of Stochastic Hybrid Systems and Applications*. PhD thesis, University of Waterloo, 2011.

[7] M.S. Alwan and X.Z. Liu. On stability of linear and weakly nonlinear switched systems with time delay. *Mathematical and Computer Modelling*, 48(7-8):1150–1157, 2008.

[8] M.S. Alwan and X.Z. Liu. Stability of singularly perturbed switched systems with time delay and impulsive effects. *Nonlinear Analysis, Theory, Methods and Applications*, 71(9):4297–4308, 2009.

[9] M.S. Alwan, X.Z. Liu, and B. Ingalls. Exponential stability of singularly perturbed switched systems with time delay. *Nonlinear Analysis: Hybrid Systems*, 2(3):913–921, 2008.

[10] M.S. Alwan, X.Z. Liu, and T.G. Sugati. h_∞ control and input-to-state stabilization for hybrid systems. *(Submitted)*.

[11] M.S. Alwan, X.Z. Liu, and T.G. Sugati. Input-to-state stabilization for impulsive switched systems with time delay. *(In preparation)*.

[12] M.S. Alwan, X.Z. Liu, and T.G. Sugati. Stability and stabilization for time-delayed stochastic large-scale systems with impulsive effects. *(In preparation)*.

[13] M.S. Alwan, X.Z. Liu, and T.G. Sugati. Switched singularly perturbed systems with reliable controllers. *Proceedings of the AMMCS-CAIMS2015 Conference*.

[14] M.S. Alwan, X.Z. Liu, and T.G. Sugati. Switched singularly perturbed systems with reliable controllers and time delay. *(In preparation).*

[15] M.S. Alwan, X.Z. Liu, and W.-C. Xie. Existence, continuation, and uniqueness problems of stochastic impulsive systems with time delay. *Journal of the Franklin Institute*, 347(7):1317–1333, September 2010.

[16] M.S. Alwan, X.Z. Liu, and W.-C. Xie. Robust reliable control for uncertain stochastic impulsive systems with time delay. *proceedings of the international conference on dynamical systems and applications*, 6:1–8, 2011.

[17] M.S. Alwan, X.Z. Liu, and W.-C. Xie. On input-to-state stability of nonlinear stochastic hybrid systems. *Dynamics of Continuous, Discrete and Impulsive Systems (Series A: Mathematical Analysis)*, 19:513–533, 2012.

[18] M.S. Alwan, X.Z. Liu, and W.-C. Xie. Comparison principle and stability of differential equations with piecewise constant arguments. *Journal of the Franklin Institute*, 350(2):211–230, 2013.

[19] M.S. Alwan, X.Z. Liu, and W.-C. Xie. On design of robust reliable control and input-to-state stabilization of uncertain stochastic systems with state delay. *Communications in Nonlinear Science and Numerical Simulation*, 18(4):1047–1056, April 2013.

[20] M.S. Alwan, X.Z. Liu, and W.-C. Xie. Stochastic switching reliable control for a class of stochastic hybrid systems with state delay. *Journal of Nonlinear Systems and Applications*, pages 86–91, 2013.

[21] M.S. Alwan, X.Z. Liu, and W.-C. Xie. Input-to-state stability of large-scale stochastic impulsive systems with time-delay and application to control systems. *Interdisciplinary Topics in Applied Mathematics, Modeling and Computational Science*, 117:21–27, 2015.

[22] M.S. Andersen, S.K. Pakazad, A. Hansson, and A. Rantzer. Robust stability analysis of sparsely interconnected uncertain systems. *IEEE Transactions on Automatic Control*, 59(8):2151–2156, 2014.

[23] J. Anderson, Y.-C. Chang, and A. Papachristodoulou. Model decomposition and reduction tools for large-scale networks in systems biology. *Automatica*, 47(6):1165–1174, 2011.

[24] P. Antsaklis. *Hybrid Systems IV: Digital Signal Processing*. Springer, v.4 edition, 1997.

[25] P.J. Antsaklis, W. Kohn, M. Lemmon, A. Nerode, and S. Sastry. *Hybrid Systems V: Lecture Notes in Computer Science*. Springer Berlin Heidelberg, 2003.

[26] P.J. Antsaklis, W. Kohn, A. Nerode, and S. Sastry. *Hybrid Systems II*, volume 999. Springer Science & Business Media, 1995.

[27] D.D. Bainov and P.S. Simeonov. *Systems with Impulse Effect Stability, Theory and Applications.* Mathematics and its Applications. Ellis Horwood Ltd., Chichester, England, 1989.

[28] D.D. Bainov and P.S. Simeonov. *Impulsive Differential Equations: Asymptotic Properties of the Solutions*, volume 28 of *Series on Advances in Mathematics for Applied Sciences.* World Scientific, Singapore, December 1995.

[29] L. Bakule. Decentralized control: An overview. *Annual Reviews in Control*, 32(1):87–98, 2008.

[30] G.H. Ballinger. *Qualitative Theory of Impulsive Delay Differential Equations.* PhD thesis, University of Waterloo, 1999.

[31] G.H. Ballinger and X.Z. Liu. Practical stability of impulsive delay differential equations and applications to control problems. *In optimization Methods and Applications*, 52:3–21, 2001.

[32] R.E. Bellman and K.L. Cooke. *Differential-Difference Equations.* Rand Corporation, 1963.

[33] M.S. Branicky. Stability of switched and hybrid systems. *Proceedings of 1994 33rd IEEE Conference on Decision and Control*, 4:3498–3503, 1994.

[34] M.S. Branicky. Stability of hybrid systems: State of the art. In *Decision and Control, 1997., Proceedings of the 36th IEEE Conference on*, volume 1, pages 120–125. IEEE, 1997.

[35] M.S. Branicky. Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):475–482, 1998.

[36] E. Braverman and R. Mamdani. Continuous versus pulse harvesting for population models in constant and variable environment. *Journal of Mathematical Biology*, 57(3):413–434, 2008.

[37] F. Breitenecker. Development of simulation software - from simple ODE modelling to structural dynamic systems. *Proceedings of the 22th European Conference on Modelling and Simulation (ECMS 2008)*, 5(Cd):20–37, 2008.

[38] T.A. Burton. *Stability and periodic solutions of ordinary and functional differential equations.* Dover, 2005.

[39] M. Cao and A.S. Morse. Dwell-time switching. *Systems and Control Letters*, 59(1):57–65, 2010.

[40] D. Chatterjee and D. Liberzon. Stability analysis of deterministic and stochastic switched systems via a comparison principle and multiple lyapunov functions. *SIAM Journal on Control and Optimization*, 45(1):174–206, 2006.

[41] W.-H. Chen, G. Yuan, and W.X. Zheng. Robust stability of singularly perturbed impulsive systems under nonlinear perturbation. *IEEE Transactions on Automatic Control*, 58(1):168–174, 2013.

[42] W.-H. Chen and W.X. Zheng. Input-to-state stability and integral input-to-state stability of nonlinear impulsive systems with delays. *Automatica*, 45(6):1481–1488, 2009.

[43] W.-H. Chen and W.X. Zheng. Input-to-state stability for networked control systems via an improved impulsive system approach. *Automatica*, 47(4):789–796, 2011.

[44] D. Cheng and Y. Zhu. Stabilization of switched systems via common Lyapunov function. *2006 6th World Congress on Intelligent Control and Automation*, (1):183–187, 2006.

[45] P. Cheng, F. Deng, and F. Yao. Exponential stability analysis of impulsive stochastic functional differential systems with delayed impulses. *Communications in Nonlinear Science and Numerical Simulation*, 19(6):2104–2114, 2014.

[46] X.-M. Cheng, W.-H. Gui, and Z.-J. Gan. Robust reliable control for a class of time-varying uncertain impulsive systems. *Journal of Central South University of Technology*, 12(1):199–202, 2005.

[47] Y. Cheng, B. Jiang, Y. Fu, and Z. Gao. Robust observer based reliable control for satellite attitude control systems with sensor faults. *International Journal of Innovative Computing, Information and Control*, 7(7):4149–4160, 2011.

[48] S. Dashkovskiy, M. Kosmykov, A. Mironchenko, and L. Naujok. Stability of interconnected impulsive systems with and without time delays, using Lyapunov methods. *Nonlinear Analysis: Hybrid Systems*, 6(3):899–915, 2012.

[49] B. De Schutter, W.P.M.H. Heemels, J. Lunze, and C. Prieur. Survey of modeling, analysis, and control of hybrid systems. *Chapter 2 in Handbook of Hybrid Systems Control âĂŞ Theory, Tools, Applications*, 19:31–55, 2009.

[50] J. Dong and G.-H. Yang. Reliable state feedback control of t–s fuzzy systems with sensor faults. *Fuzzy Systems, IEEE Transactions on*, 23(2):421–433, 2015.

[51] R.D. Driver. *Ordinary and Delay Differential Equations*, volume 20 Applied. Springer-Verlag, 1977.

[52] D. Du, Shi P. Jiang, B., and Z. Mao. Observer-based reliable control for discrete time systems: An average dwell time approach. *Asian Journal of Control*, 14(6):1625–1633, 2012.

[53] L. Erbe and W. Krawcewicz. Existence of solutions to boundary value problems for impulsive second order differential inclusions. *Rocky Mountain Journal of Mathematics*, 22(2):519–539, 1992.

[54] R. Goebel, R.G. Sanfelice, and A. Teel. Hybrid Dynamical Systems. *IEEE Control Systems*, 29(2):28–93, 2009.

[55] K. Gu and S.-L. Niculescu. Survey on recent results in the stability and control of time-delay systems. *Journal of Dynamic Systems, Measurement, and Control*, 125(2):158, 2003.

[56] Z. Gu, J. Liu, C. Peng, and E. Tian. Reliable control for interval time-varying delay systems subjected to actuator saturation and stochastic failure. *Optimal Control Applications and Methods*, 33:739–750, 2012.

[57] Z.-H. Guan, G. Chen, X. Yu, and Y. Qin. Robust decentralized stabilization for a class of large-scale time-delay uncertain impulsive dynamical systems. *Automatica*, 38(12):2075–2084, 2002.

[58] Z.-H. Guan, D. J. Hill, and X. Shen. On hybrid impulsive and switching systems and application to nonlinear control. *IEEE Transactions on Automatic Control*, 50(7):1058–1062, 2005.

[59] A. Halanay. *Differential equations: stability, oscillations, time lags*, volume 23 Mathema. Academic Press, 1966.

[60] J.K. Hale. *Functional differential equations*. Springer, 1971.

[61] J.K. Hale and S.M.V. Lunel. *Introduction to functional differential equations*, volume 99 Applied. Springer, 1993.

[62] P. Hartman. Some examples in the theory of singular boundary value problems. *American Journal of Mathematics*, 74(1):107–126, 1952.

[63] J.P. Hespanha. Lecture Notes on Hybrid Control and Switched Systems, 2005.

[64] J.P. Hespanha and A.S. Morse. Stability of switched systems with average dwell-time. *Proceedings of the 38th IEEE Conference on Decision and Control*, 3:2655–2660, 1999.

[65] Y.Y. Hou. Reliable stabilization of networked control systems with actuator faults. *Nonlinear Dynamics*, 71(3):447–455, 2013.

[66] B. Hu, A.N. Michel, and P.J. Antsaklis. Stability analysis for a class of nonlinear switched systems. *proceedings of the 38th IEEE conference on Decision and Control*, pages 4374–4379, 1999.

[67] H. Hu, B. Jiang, and H. Yang. Robust H_∞ reliable control for a class of uncertain switched nonlinear systems. *12th International Conference on Control, Automation, Robotics and Vision*, 2012:5068–5073, 2012.

[68] H. Hu, B. Jiang, and H. Yang. Robust H_∞ reliable control for uncertain switched systems with circular disk pole constraints. *Journal of the Franklin Institute*, 350(4):802–817, 2013.

[69] H. Hu, B. Jiang, and H. Yang. A new approach to robust reliable H_∞ control for uncertain nonlinear systems. *International Journal of Systems Science*, pages 1–8, 2014.

[70] C. Hua, L. Zhang, and X. Guan. Decentralized output feedback controller design for nonlinear interconnected systems with unknown control direction and time-varying delays. *International Journal of Adaptive Control and Signal Processing*, 28:1160–1173, 2014.

[71] J. Huang and J. Liu. Continuous and discrete halanay delayed inequalities and their applications in stability of neural networks. In *Advances in Neural Networks–ISNN*, pages 262–271. Springer, 2009.

[72] Y.-S. Huang, M. Wu, Y. He, L.-L. Yu, and Q.-X. Zhu. Decentralized adaptive fuzzy control of large-scale nonaffine nonlinear systems by state and output feedback. *Nonlinear Dynamics*, 69(4):1665–1677, 2012.

[73] A.F. Ivanov and M.A. Mammadov. Global asymptotic stability in a class of nonlinear differential delay equations. *Discrete and Continuous Dynamical Systems*, pages 727–736, 2011.

[74] M. Jankovic. Extension of control Lyapunov functions to time-delay systems. *Proceedings of the 39th IEEE Conference on Decision and Control*, 5:4403–4408, 2000.

[75] Y. Jiang and Z.-P. Jiang. Robust adaptive dynamic programming for large-scale systems with an application to multimachine power systems. *IEEE Transactions on Circuits and Systems II: Express Briefs*, 59(10):693–697, 2012.

[76] K.-I. Kang, K.-S. Park, and J-T. Lim. Exponential stability of singularly perturbed switched systems with time delay and uncertainties. *International Journal of Systems Science*, 46(1):170–178, 2015.

[77] A. Khadra, X. Liu, and X. Shen. Application of impulsive synchronization to communication security. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 50(3):341–351, 2003.

[78] A. Khadra, X.Z. Liu, and X. Shen. Impulsively synchronizing chaotic systems with delay and applications to secure communication. *Automatica*, 41(9):1491–1502, 2005.

[79] H.K. Khalil. *Nonlinear Systems*. Prentice Hall, third edition, 2002.

[80] P.P. Khargonekar, I.R. Petersen, and M.A. Rotea. H_∞ optimal control with state-feedback. *IEEE Transactions on Automatic Control*, 33(8):786–788, 1988.

[81] P. Kokotovic, H.K. Khalil, and J. O'Reilly. *Singular Perturbation Methods in Control: Analysis and Design*. SIAM, 1986.

[82] P.V. Kokotovic, R.E. O'malley, and P. Sannuti. Singular perturbations and order reduction in control theory-An overview. *Automatica*, 12(4):123–132, 1976.

[83] F. Kozin. A survey of stability of stochastic systems. *Automatica*, 5(1):95–112, 1969.

[84] Y. Kuang. *Delay Differential Equations with Applications in Population Dynamics*, volume 191. Academic press, INC. Mathematics in science and engineering, 1993.

[85] V. Lakshmikantham, D.D. Bainov, and P.S. Simeonov. *Theory of Impulsive Differential Equations.*, volume 6 of *Modern Applied Mathematics*. World Scientific, Singapore, 1989.

[86] V. Lakshmikantham, S. Leela, and L. Moreau. *Differential and Integral Inequalities; Theory and Applications PART A: Ordinary Differential Equations*. Mathematics in Science and Engineering. Elsevier Science, New York, 1969.

[87] S.-H. Lee and J.-T. Lim. Stability analysis of switched systems with impulse effects. *Proceedings of the 1999 IEEE International Symposium on Intelligent Control Intelligent Systems and Semiotics*, pages 79–83, 1999.

[88] C. Li, L. Chen, and K. Aihara. Impulsive control of stochastic systems with applications in chaos control, chaos synchronization, and neural networks. *Chaos: An interdisciplinary Journal of Nonlinear Science*, 18(2):023132–(1–11), 2008.

[89] C. Li, F. Ma, and G. Feng. Hybrid impulsive and switching time-delay systems. *IET Control Theory & Applications*, 3(11):1487, 2009.

[90] X. Li, J. Shen, H. Akca, and R. Rakkiyappan. LMI-based stability for singularly perturbed nonlinear impulsive differential systems with delays of small parameter. *Applied Mathematics and Computation*, 250:798–804, 2015.

[91] Z Li, Y Soh, and C Wen. *Switched and impulsive systems. Analysis, design and applications.* 2005.

[92] Y.C. Liao. Switching and impulsive control of a reflected diffusion. *Applied Mathematics & Optimization*, 11(1):153–159, 1984.

[93] D. Liberzon. *Switching in Systems and Control.* Systems & Control: Foundations & Applications. Birkhäuser Boston, 2003.

[94] D. Liberzon and A.S. Morse. Basic problems in stability and design of switched systems. *IEEE Control Systems*, 19(5):59–70, 1999.

[95] H. Lin and P.J. Antsaklis. Stability and stabilizability of switched linear systems: A survey of recent results. *IEEE Transactions on Automatic Control*, 54(2):308–322, 2009.

[96] B. Liu and D.J. Hill. Input-to-state stability for discrete time-delay systems via the Razumikhin technique. *Systems and Control Letters*, 58(8):567–575, 2009.

[97] B. Liu, K.L. Teo, and X.Z. Liu. Robust exponential stabilization for large-scale uncertain impulsive systems with coupling time-delays. *Nonlinear Analysis, Theory, Methods and Applications*, 68(5):1169–1183, 2008.

[98] J. Liu, X. Liu, and W.-C. Xie. Input-to-state stability of impulsive and switching hybrid systems with time-delay. *Automatica*, 47(5):899–908, 2011.

[99] X.Z. Liu. Impulsive stabilization of nonlinear systems. *Journal, IMA Journal of Mathematical Control and Information*, 10:11–19, 1993.

[100] X.Z. Liu. Stability results for impulsive differential systems with applications to population growth models. *Dynamics and Stability of Systems: An international Journal*, 9(2):163–174, January 1994.

[101] X.Z. Liu. Impulsive stabilization and applications to population growth models. *The Rocky Mountain Journal of Mathematics*, 25(1):381–395, 1995.

[102] X.Z. Liu and X. Liao. Comparison method and robust stability of Large-scale dynamic systems. *Dynamics of continuous, discrete and impulsive systems, series A: Mathematical Analysis*, 11:413–430, 2004.

[103] X.Z. Liu and K. Rohlf. Impulsive control of a Lotka-Volterra system. *IMA Journal of Mathematical Control and Information*, 15:269–284, 1998.

[104] X.Z. Liu, X. Shen, and Y. Zhang. Exponential stability of singularly perturbed systems with time delay. *Applicable Analysis*, 82(2):117–130, 2003.

[105] X.Z. Liu, X. Shen, and Y. Zhang. Exponential stability of singularly perturbed systems with time delay. *Applicable Analysis*, 82(2):117–130, 2003.

[106] X.Z. Liu, X. Shen, and Y. Zhang. A comparison principle and stability for large-scale impulsive delay differential systems. *The ANZIAM Journal*, 47(02):203, 2005.

[107] X.Z. Liu and P. Stechlinski. Pulse and constant control schemes for epidemic models with seasonality. *Nonlinear Analysis: Real World Applications*, 12(2):931–946, 2011.

[108] X.Z. Liu and Q. Wang. Impulsive stabilization of high-order Hopfield-type neural networks with time-varying delays. *IEEE Transactions on Neural Networks*, 19(1):71–79, 2008.

[109] Z. Liu, Y. Wang, and G. Zhao. Exponential stability for positive switched linear systems with delays and impulses. *Proceedings of the 33rd CCC*, pages 2469–2474, 2014.

[110] F. Long and S. Fei. Neural networks stabilization and disturbance attenuation for nonlinear switched impulsive systems. *Neurocomputing*, 71(7-9):1741–1747, 2008.

[111] F. Long, S. Fei, Z. Fu, and S. Zheng. H^∞ neural networks control for uncertain nonlinear switched impulsive systems. *ICONIP 2006, LNCS 4234*, (III):664–673, 2006.

[112] J. Lu and Z. Wu. Robust reliable H_∞ control for uncertain switched linear systems with disturbances. *2nd International Conference (ICIECS), IEEE Xplore 2010*, pages 1–4, December 2010.

[113] J. Lygeros. Lecture Notes on Hybrid Systems. *Course Notes*, page 82, 2004.

[114] Y. Ma and J. Sun. Stability criteria for impulsive systems on time scales. *Journal of Computational and Applied Mathematics*, 213(2):400–407, 2008.

[115] Y. Ma, N. Zhang, and X. Zhong. Decentralized robust control for uncertain nonlinear descriptor large-scale composite systems with input saturation. *International Journal of Innovative Computing, Information and Control*, 9(10):3991–4000, 2013.

[116] M.S. Mahmoud. Reliable decentralized control of interconnected discrete delay systems. *Automatica*, 48:986–990, 2012.

[117] M.A. Mahmud, H.R. Pota, and M. J. Hossain. Full-order nonlinear observer-based excitation controller design for interconnected power systems via exact linearization approach. *International Journal of Electrical Power and Energy Systems*, 41(1):54–62, 2012.

[118] I. Malloci, J. Daafouz, and C. Iung. Stabilization of continuous-time singularly perturbed switched systems. *Proceedings of the 48th IEEE Conference on Decision and Control and 28th CCC*, 2(2):6371–6376, 2009.

[119] X. Mao. Razumikhin-type theorems on exponential stability of stochastic functional differential equations. *Stochastic Processes and their Applications*, 65(2):233–250, 1996.

[120] X. Mao. Stability of stochastic differential equations with Markovian switching. *Stochastic Processes and their Applications*, 79:45–67, 1999.

[121] X. Mao. *Stochastic Differential Equations and Applications*. Horwood Publishing Limited, 2 edition, 2008.

[122] Z.-Z. Mao and X.-S. Xiao. Decentralized adaptive tracking control of nonaffine nonlinear large-scale systems with time delays. *Information Sciences*, 181(23):5291–5303, 2011.

[123] K. Mathiyalagan, R. Sakthivel, and Ju.H. Park. Robust reliable control for neutral-type nonlinear systems with time-varying delays. *Reports on Mathematical Physics*, 74(2):181–203, 2014.

[124] A.S. Matveev and A.V. Savkin. *Qualitative Theory of Hybrid Dynamical Systems*. Springer Science & Business Media, 2012.

[125] E.A. Medina and D.A. Lawrence. State estimation for linear impulsive systems. *Proceedings of the American Control Conference*, (4):1183–1188, 2009.

[126] S. Mehraeen, S. Jagannathan, and M.L. Crow. Decentralized dynamic surface control of large-scale interconnected systems in strict-feedback form using neural networks with asymptotic stabilization. *IEEE Transactions on Neural Networks*, 22(11):1709–1722, 2011.

[127] A.N. Michel and B. Hu. Stability analysis of interconnected hybrid dynamical systems. *Proc. IFAC symposium on large scale system*, 1998.

[128] A.N. Michel and R.K. Miller. *Qualitative Analysis of Large Scale Dynamical Systems*, volume 134 of *Mathematics in Science and Engineering*. Elsevier Science, 1977.

[129] M. Morari and L. Thiele. *Hybrid Systems: Computational and Control.* 2005.

[130] A.S. Morse. Control Using Logic-Based Switching. *Lecture Notes in Control and Information Sciences 222*, 1997.

[131] S. Morse. Supervisory control of families of linear set-point controllers, part 1: exact matching. *IEEE Transactions on Automatic Control*, 41(10):1413–1431, 1996.

[132] K. Narendra and J. Balakrishnan. A Common Lyapunov function for stable LTI systems with commuting A-Matrices. *IEEE Transactions on Automatic Control*, 39(12):2469–2471, 1994.

[133] C. Ning, Y. He, M. Wu, and J. She. Improved razumikhin-type theorem for input-to-state stability of nonlinear time-delay systems. *IEEE Transactions on Automatic Control*, 59(7):1983–1988, 2014.

[134] S.G. Pandit and S.G. Deo. *Lecture Notes in Mathematics: Differential Systems Involving Impulses.* Springer-Verlag, 1982.

[135] K-S. Park and J.-T. Lim. Exponential stability of singularly perturbed systems. *International Journal of Innovative Computing, Information and Control (ICIC International)*, 9(2):865–874, 2013.

[136] B.M. Randelovic, L.V. Stefanovic, and B.M. Dankovic. Numerical solution of impulsive differential equations. *Facta Univ. Ser. Math. Inform.*, 15:101–111, 2000.

[137] A. Saberi and H.K. Khalil. Quadratic-type Lyapunov functions for singularly perturbed systems. *IEEE Transactions on Automatic Control*, 29(6):542–550, 1984.

[138] V.R. Saksena, J. O'Reilly, and P.V. Kokotovic. Singular perturbations and time-scale methods in control theory: Survey 1976âĂŞ1983. *Automatica*, 20(3):273–293, 1984.

[139] R. Sakthivel, S. Santra, K. Mathiyalagan, and A. Arunkumar. Reliable robust control design for uncertain mechanical systems. *Journal of Dynamic Systems, Measurement, and Control*, 137(2):021003, 2014.

[140] Jr. Sandell, N., P. Varaiya, M. Athans, and M. Safonov. Survey of decentralized control methods for large scale systems. *IEEE Transactions on Automatic Control*, 23(2):108–128, 1978.

[141] Nils R. Sandell. Robust stability of systems with application to singular perturbations. *Automatica*, 15(4):467–470, 1979.

[142] S. Selvi, R Sakthivel, and K. Mathiyalagan. Delay-dependent robust reliable control for uncertain switched neutral systems. *Complexity*, 2015.

[143] C.-J. Seo and B.K. Kim. Robust and reliable H_∞ control for linear systems with parameter uncertainty and actuator failure. *Automatica*, 32(3):3–5, 1996.

[144] I. Shames, A.M.H. Teixeira, H. Sandberg, and K.H. Johansson. Distributed fault detection for interconnected second-order systems. *Automatica*, 47(12):2757–2764, 2011.

[145] Z. H. Shao and M. E. Sawan. Robust stability of singularly perturbed systems. *International Journal of Control*, 58(6):1469–1476, 1993.

[146] B. Shen, Z. Wang, D. Ding, and H. Shu. H_∞ state estimation for complex networks with uncertain inner coupling and incomplete measurements. *IEEE transactions on neural networks and learning systems*, 24(12):2027–37, 2013.

[147] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King. Stability criteria for switched and hybrid systems. *SIAM Review*, 49(4):545–592, January 2007.

[148] D.D. Siljak. *Large-Scale Dynamic Systems: Stability and Structure*. Dover Civil and Mechanical Engineering Series. Dover Publications, 2007.

[149] A. Sinha. *Linear Systems: Optimal and Robust Control*. CRC Press, 2007.

[150] H. Smith. *An Introduction to Delay Differential Equations with Applications to the Life Sciences*. Texts in Applied Mathematics. Springer New York, 2010.

[151] E.D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, 34(4):435–443, 1989.

[152] E.D. Sontag. The ISS philosophy as a unifying framework for stability-like behavior. *Nonlinear Control in the year 2000, Springer, London*, 2(0):443–467, 2001.

[153] E.D. Sontag and Y. Wang. On characterizations of the input-to-state stability property. *Systems and Control Letters*, 24(5):351–359, 1995.

[154] E.D. Sontag and Y. Wang. New characterizations of input-to-state stability. *IEEE Transactions on Automatic Control*, 41(9):1283–1294, 1996.

[155] P. Stechlinski. A study of infectious disease models with switching. Master's thesis, University of Waterloo, 2009.

[156] T.G. Sugati, M.S. Alwan, and X.Z. Liu. H_∞ control and input-to-state stabilization for hybrid systems with time delay. *Communication faculty of sciences University of Ankara. Series A1: Mathematics and Statistics*, 64(1):29–38, 2015.

[157] T.G. Sugati, M.S. Alwan, and X.Z. Liu. Robust reliable control for uncertain impulsive large-scale systems. *DCDIS, Series B: Applications and Algorithms*, 22:381–400, 2015.

[158] X.M. Sun, D. Wu, G.P. Liu, and W. Wang. Input-to-state stability for networked predictive control with random delays in both feedback and forward channels. *IEEE Transactions on Industrial Electronics*, 61(7):3519–3526, 2014.

[159] Y. Tang, C. Prieur, A. Girard, Y. Tang, C. Prieur, and A. Girard. Lyapunov stability of a singularly perturbed system of two conservation laws. *CPDE2013- 1st IFAC Workshop on Control of Systems Modeled by Partial Differential Equations*, page 6, 2014.

[160] A.R. Teel. Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem. *IEEE Transactions on Automatic Control*, 43(7):960–964, 1998.

[161] A.R. Teel, L. Moreau, and D. Nesic. A note on the robustness of input-to-state stability. *Proceedings of the 40th IEEE Conference on Decision and Control*, 1:875–880, 2001.

[162] A. van der Schaft and H. Schumacher. *An Introduction to Hybrid Dynamical Systems*, volume 251. Springer-Verlag London, 1 edition, 2000.

[163] R.J. Veillette. Reliable State Feedback and Reliable Observers. *Proceedings of the 31st conference on decision and control*, pages 2898–2903, 1992.

[164] R.J. Veillette. Reliable Linear-quadratic State-feedback Control. *Automatica*, 31(1):137–143, 1995.

[165] R.J. Veillette, J.B. Medanic, and W.R. Perkins. Design of reliable control systems. *IEEE Transactions on Automatic Control*, 37(3):290–304, 1992.

[166] L. Vu and D. Liberzon. Common Lyapunov functions for families of commuting nonlinear systems. *Systems & Control Letters*, 54(5):405–416, 2005.

[167] Q. Wang and X.Z. Liu. Impulsive stabilization of delay differential systems via the Lyapunov-Razumikhin method. *Applied Mathematics Letters*, 20(8):839–845, 2007.

[168] Q. Wang and C. Wei. Decentralized robust adaptive output feedback control of stochastic nonlinear interconnected systems with dynamic interactions. *Automatica*, 54:124–134, 2015.

[169] Q. Wang and Q. Zhu. Razumikhin-type stability criteria for differential equations with delayed impulses. *Electronic Journal of Qualitative Theory of Differential Equations*, (14):1–18, 2013.

[170] W. Wang. A generalized Halanay inequality for stability of nonlinear neutral functional differential equations. *Journal of Inequalities and Applications*, 2010.

[171] Y.E. Wang, X.M. Sun, and B. Wu. Lyapunov-Krasovskii functionals for input-to-state stability of switched non-linear systems with time-varying input delay. *IET Control Theory & Applications*, 9:1717–1722, 2015.

[172] M. Wicks, P. Peleties, and R. DeCarlo. Switched controller synthesis for the quadratic stabilisation of a pair of unstable linear systems. *European Journal of Control*, 4(2):140–147, January 1998.

[173] W.K. Wong, E. Tian, D. Yue, and R.W.M. Au. Robust reliable control for systems with random actuator fault and probabilistic nonlinearity with new characters. *International Journal of Robust and Nonlinear Control*, 23:2013–2027, 2013.

[174] M. Xiang and Z. Xiang. Reliable control of positive switched systems with time-varying delays. *Advances in Difference Equations*, 2013(1):1–15, 2013.

[175] G. Xie and L. Wang. Stability and stabilization of switched impulsive systems. *American Control Conference*, (4):4406–4411, 2006.

[176] L. Xie and C.D. de Souza. Robust H_∞ control for linear systems with norm-bounded time-varying uncertainty. *IEEE Transactions on Automatic Control*, 37(8):1188–1191, 1992.

[177] L. Xie and C.E. de Souza. Robust H_∞ control for linear systems with norm-bounded time-varying uncertainty. *Proceedings of the 29th Conference on Decision and Control*, pages 1034–1035, 1990.

[178] L. Xie and C.E. de Souza. Robust H_∞ control for linear time-invariant systems with norm-bounded uncertainty in the input matrix. *Systems & Control Letters*, 14(5):389–396, June 1990.

[179] L. Xie and C.E. de Souza. Robust H_∞ control for class of uncertain linear time-variant systems. *IEE Proceedings-D*, 138(5):479–483, 1991.

[180] H. Xu, X.Z. Liu, and K. L. Teo. Robust H_∞ stabilisation with definite attenuance of an uncertain impulsive switched system. *The ANZIAM Journal*, 46(04):471, 2005.

[181] C. Yang and W. Zhu. Stability analysis of impulsive switched systems with time delays. *Mathematical and Computer Modelling*, 50(7-8):1188–1194, 2009.

[182] G.-H. Yang, J.L. Wang, and Y.C. Soh. Reliable H_∞ controller design for linear systems. *Automatica*, 37:717–725, 2001.

[183] T. Yang. *Impulsive control theory*, volume 272. Springer Science \& Business Media, 2001.

[184] Y. Yang and Y. Wang. Robust stability of singularly perturbed systems with delays. *52nd IEEE Conference on Decision and Control*, pages 3975–3980, 2013.

[185] Z. Yang. Impulsive decentralized stabilization of stochastic large-scale systems. *Proceedings of the 26th Chinese Control Conference, CCC*, (3):794–797, 2007.

[186] Z. Yang, Y. Hong, and Z. P. Jiang. Input-to-state stability of hybrid switched systems with impulsive effects. *Proceedings of the 7th World Congress on Intelligent Control and Automation (WCICA)*, pages 7167–7171, 2008.

[187] S.J. Yoo and J.B. Park. Decentralized adaptive output-feedback control for a class of nonlinear large-scale systems with unknown time-varying delayed interactions. *Information Sciences*, 186(1):222–238, 2012.

[188] G. Zhai, B. Hu, K. Yasuda, and A.N. Michel. Piecewise Lyapunov functions for switched systems with average dwell time. *Control*, 2(3):192–197, 2000.

[189] G. Zhai, B. Hu, K. Yasuda, and A.N. Michel. Stability analysis of switched systems with stable and unstable subsystems: an average dwell time approach. *International Journal of Systems Science*, 32(8), 2001.

[190] Y. Zhang, D.S. Naidu, C. Cai, and Y. Zou. Singular perturbations and time scales in control theory and applications : an overview 2002-2012. *International Journal of Informaton and Systems Sciences*, 9(1):233–278, 2014.

[191] P. Zhao, W. Feng, and Y. Kang. Stochastic input-to-state stability of switched stochastic nonlinear systems. *Automatica*, 48(10):2569–2576, 2012.

[192] X. Zhao, X. Liu, S. Yin, and H. Li. Improved results on stability of continuous-time switched positive linear systems. *Automatica*, 50(2):614–621, 2014.

[193] X. Zhao, S. Yin, H. Li, and B. Niu. Switching stabilization for a class of slowly switched systems. *IEEE Transactions on Automatic Control*, 60(99):1, 2015.

[194] X. Zhao, L. Zhang, P. Shi, and M. Liu. Stability and stabilization of switched linear systems with mode-dependent average dwell time. *IEEE Transactions on Automatic Control*, 57(7):1809–1815, 2012.

[195] Y. Zhao, A. Xiao, and L. Li. Input-to-state stability of singularly perturbed control systems with delays. *Journal of Applied Mathematics*, 2013, 2013.

[196] W. Zhu. Stability analysis of switched impulsive systems with time delays. *Nonlinear Analysis: Hybrid Systems*, 4(3):608–617, 2010.

[197] X.-L. Zhu, H. Yang, Y. Wang, and Y.-L. Wang. New stability criterion for linear switched systems with time-varying delay. *International Journal of Robust and Nonlinear Control*, 24:214–227, 2014.