# Colouring Subspaces

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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**Abstract**

This thesis was originally motivated by considering vector space analogues of problems in extremal set theory, but our main results concern colouring a graph that is intimately related to these vector space analogues. The vertices of the $q$-Kneser graph are the $k$-dimensional subspaces of a vector space of dimension $v$ over $\mathbb{F}_q$, and two $k$-subspaces are adjacent if they have trivial intersection. The new results in this thesis involve colouring the $q$-Kneser graph when $k = 2$. There are two cases. When $k = 2$ and $v = 4$, the chromatic number is $q^2 + q$. If $k = 2$ and $v > 4$, the chromatic number is $\frac{q^{v-1}-1}{q-1}$. In both cases, we characterise the minimal colourings. We develop some theory for colouring the $q$-Kneser graph in general.

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# Contents

# Chapter 1

# Introduction

Many problems in extremal set theory ask for the maximum size of a family of sets that satisfy certain restrictions on the intersections of its members. By replacing the word "set" with "subspace," we can ask a similar question about a family of subspaces of a vector space. This thesis is motivated by these vector space analogues of problems in extremal set theory.

An interesting graph associated with extremal set theory is the Kneser graph. Its vertices are the $k$-subsets of a fixed set of size $v$ and two $k$-subsets are adjacent if they are disjoint. Classical problems in extremal set theory, such as the Erdos-Ko-Rado theorem, can be rewritten as questions about the size and structure of independent sets in the Kneser graph.

In 1955, Martin Kneser conjectured that when $v \geq 2k$ the chromatic number of the Kneser graph is $v - 2k + 2$. The problem remained open for twenty three years until Lovász found a proof that surprisingly uses algebraic topology.

Kneser's long standing conjecture motivates the problem of colouring the $q$-Kneser graph, the vector space analogue of the Kneser graph, which is constructed as follows. The vertices of the $q$-Kneser graph are the $k$-dimensional subspaces of a vector space of dimension $v$ over $\mathbb{F}_q$, and two $k$-subspaces are adjacent if they have trivial intersection. We need only consider the case $k \geq 2$ and $v \geq 2k$ since when $k = 1$, the $q$-Kneser graph is complete and when $v < 2k$, the $q$-Kneser graph is empty. The new results in this thesis involve colouring the $q$-Kneser graph when $k = 2$ and will appear in [3]. There are two cases. When $k = 2$ and $v = 4$, the chromatic number is $q^2 + q$. If $k = 2$ and $v > 4$, the chromatic number is $\frac{q^{v-1}-1}{q-1}$. In both cases, we characterise the minimal colourings. We develop some theory for colouring the $q$-Kneser graph in general.

## 1.1 The Kneser Graphs

The Kneser graph, $K_{v:k}$, has as its vertices the $k$-subsets of a fixed set of size $v$, and two $k$-subsets are adjacent if they are disjoint. The Kneser graph appears in many different combinatorial contexts, and we highlight three of its roles here.

First, the Kneser graph is related to extremal set theory; classical problems in extremal set theory, such as the Erdos-Ko-Rado theorem, can be rewritten as questions about the size and structure of independent sets in the Kneser graph. Second, the Kneser graphs play the same role in fractional graph colouring as the complete graphs do in graph colouring; understanding homomorphisms into the Kneser graph helps us obtain a lower bound on the chromatic number of any graph. Third, determining the chromatic number of the Kneser graph is in itself an interesting problem, which remained open for twenty three years until Lovasz found an ingenious proof using algebraic topology.

In extremal set theory, one common restriction on a family of sets is that it be *intersecting*, that is, any two members pairwise intersect. The Erdős-Ko-Rado and Hilton-Milner theorems, Theorem 1.1.1 and Theorem 1.1.2 respectively, are two important results in extremal set theory about intersecting families of $k$-subsets of a fixed $v$-set.

**1.1.1 Theorem (Erdős-Ko-Rado).** *Suppose $\mathcal{A}$ is an intersecting family of $k$-subsets of a $v$-set where $v \geq 2k$. Then*

$$|\mathcal{A}| \leq \binom{v-1}{k-1}. \tag{1.1.1}$$

*Moreover, when $v > 2k$, equality holds if and only if $\mathcal{A}$ consists of the $k$-subsets that contain a particular point.* □

An intersecting family is called *trivial* if all its members contain a fixed element. The Erdős-Ko-Rado theorem asserts that the bound (1.1.1) is tight only for trivial families. The Hilton-Milner theorem, Theorem 1.1.2, is concerned with the maximum size of non-trivial intersecting families.

**1.1.2 Theorem (Hilton-Milner).** *If $v > 2k$ then the maximum size of a non-trivial intersecting family of $k$-subsets of a $v$-set is*

$$\binom{v-1}{k-1} - \binom{v-k-1}{k-1} + 1.$$ □

Since two $k$-sets in the Kneser graph $K_{v:k}$ are adjacent if they are disjoint, an intersecting family of $k$-subsets of a $v$-set is an independent set in $K_{v:k}$. Thus, the Erdős-Ko-Rado theorem asserts that $\alpha(K_{v:k}) = \binom{v-1}{k-1}$ and gives the structure of maximal independent sets. Similarly, the Hilton-Milner theorem can be rephrased in terms of independent sets of Kneser graphs.

Next we consider the Kneser graph's relation to graph colouring. A colouring of a graph gives an upper bound on its chromatic number. Often, the difficulty in determining a graph's chromatic number is to demonstrate a lower bound. A graph's fractional chromatic number, $\chi^*(G)$, is a good lower bound on its chromatic number [7]. For the fractional chromatic number, the Kneser graphs have a role analogous to that of the complete graphs for the ordinary chromatic number.

**1.1.3 Theorem.** *For any graph $G$, we have*

$$\chi^*(G) = \min\{v/k : G \to K_{v:k}\}. \qquad \square$$

Consequently, studying graph homomorphisms into the Kneser graph (and hence studying the chromatic number of Kneser graphs) is useful for finding the chromatic number of any graph.

Determining the chromatic number of the Kneser graph, $K_{v:k}$, remained an open problem for twenty-three years until Lovász gave a proof using Borsuk's theorem from algebraic topology.

**1.1.4 Theorem.**

$$\chi(K_{v:k}) = v - 2k + 2. \qquad \square$$

Finding an optimal colouring of the Kneser graph, however, is easy. If $\alpha$ is a $k$-subset of $K_{v:k}$ and its largest element is greater than $2k$, define this element to be the colour of $\alpha$. Thus, the $k$-subsets not contained in $\{1, \dots, 2k\}$ can be coloured with $v-2k$ colours. The $k$-subsets not already coloured induce a copy of $K_{2k:k}$, which is bipartite, so the remaining $k$-subsets can be coloured with two colours. Thus we have coloured the Kneser graph $K_{v:k}$ with $v - 2k + 2$ colours, and obtained an upper bound on its chromatic number.

Giving a lower bound on the Kneser graph's chromatic number is not easy since many of the classical lower bounds fail. Consider the Kneser graph $K_{3k-1:k}$. It's chromatic number is large, $\chi(K_{3k-1:k}) = k + 1$, but it's clique number is small, $\omega(K_{3k-1:k}) = 2$, because it is triangle-free. Moreover, its independence number, $\alpha(K_{3k-1:k}) = \binom{3k-1}{k-1}$, is large and so the corresponding lower bound for the chromatic number

$$\frac{|V(K_{3k-1:k})|}{\alpha(K_{3k-1:k})} = \frac{\binom{3k-1}{k}}{\binom{3k-1}{k-1}} = \frac{3k-1}{k} < 3$$

becomes further and further away from the actual value $k+1$ as $k$ becomes large. The fractional chromatic number, which can be a very good lower bound for the chromatic number, is also much smaller than the chromatic number for the Kneser graphs. Since $K_{v:k}$ is vertex-transitive, its fractional chromatic number is $\chi^*(K_{v:k}) = v/k$, which is small in comparison to $v - 2k + 2$.

We see that the colouring problem for Kneser graphs is very interesting. Moreover, all known proofs of Theorem 1.1.4 use some form of Borsuk's theorem from algebraic topology, so there are no purely combinatorial proofs of Theorem 1.1.4.

## 1.2 The $q$-Kneser Graphs

The vertices of the $q$-Kneser graph $qK_{v:k}$ are the $k$-dimensional subspaces of a vector space of dimension $v$ over $\mathbb{F}_q$, and two $k$-subspaces are adjacent if they have trivial intersection. The $q$-Kneser graph is useful for studying the vector space analogues of the Erdős-Ko-Rado and Hilton-Milner theorems. Moreover, as we will see in the next chaper, the $q$-Kneser graph could potentially provide new insights into the Kneser graph. Our research in colouring the $q$-Kneser graphs is motivated by the Kneser graph's role in graph colouring as well as the difficulty of proving Kneser's conjecture. We hope to gain further intuition about colouring the Kneser graphs by colouring their vector space analogues, the $q$-Kneser graphs.

The vector space analogue of the Erdős-Ko-Rado problem is to determine the maximum size of an intersecting family of $k$-subspaces of $\mathbb{F}_q^v$. Since two $k$-subspaces are adjacent in $qK_{v:k}$ if they are disjoint, an intersecting family of $k$-subspaces of $\mathbb{F}_q^v$ is an independent set in $qK_{v:k}$. An intersecting family of $k$-spaces is trivial if all $k$-spaces contain a fixed one-dimensional vector space. Similarly, the vector space analogue of the Hilton-Milner theorem is to determine the maximum size of a non-trivial intersecting family of $k$-subspaces of $\mathbb{F}_q^v$. Understanding the size and structure of independent sets in the $q$-Kneser graphs is therefore useful for studying vector space analogues of problems in extremal set theory.

We now consider how the $q$-Kneser graphs could provide new information about the Kneser graphs. Formulas that enumerate combinatorial properties of the $q$-Kneser graphs, for example the number of vertices or their eigenvalues, involve the value $q$. Often, substituting $q = 1$ into these formulas yields the corresponding values for the Kneser graphs.

Finally, we summarize our results on colouring $q$-Kneser graphs. Godsil and Royle show in [3] that if $v > 2k$, then

$$\chi(qK_{v:k}) \leq \frac{q^{v-k+1} - 1}{q - 1}$$

and if $v = 2k$, then

$$\chi(qK_{2k:k}) \leq q^k + q^{k-1}.$$

We prove that the stated bounds are tight when $k = 2$, where we can also characterise the minimal colourings. When $v \geq 5$ these are essentially unique, but when $v = 4$ there are a number of colourings.

# Chapter 2

# Analogies between the Kneser and $q$-Kneser graphs

We define the $q$-binomial coefficient, a generalisation of the binomial coefficient. We show that the familiar binomial identities can be generalised to $q$-binomial identities. When $q$ is the order of a finite field $\mathbb{F}_q$, the $q$-binomial coefficients play the same role in the enumeration of subspaces of $\mathbb{F}_q^n$ that the binomial coefficients play in the enumeration of subsets. As a result, many properties of the Kneser and $q$-Kneser graphs have analogous expressions in terms of binomial coefficients and $q$-binomial coefficients respectively.

## 2.1 The Binomial and $q$-Binomial Coefficients

To explain the connections between the Kneser and $q$-Kneser graphs, we will define a generalisation of the binomial coefficient. The binomial coefficient $\binom{n}{k}$ is defined by

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}.$$

For a variable $q$ and an integer $n$, define

$$[n] := \frac{q^n - 1}{q - 1} = q^{n-1} + q^{n-2} + \ldots + q + 1.$$

The *$q$-factorial function* $[n]!$ is defined inductively by $[0]! = 1$ and

$$[n+1]! := [n+1][n]!.$$

We define the *$q$-binomial coefficient* $\begin{bmatrix} n \\ k \end{bmatrix}$ by

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]!}.$$

The $q$-binomial coefficients are a generalisation of the binomial coefficients because when $q = 1$, we have $[n] = n$, so $\begin{bmatrix} n \\ k \end{bmatrix} = \binom{n}{k}$. We now explore the relationship between the binomial coefficient and its $q$-analogue. For example, the binomial coefficient $\binom{n}{k}$ equals the number of $k$-subsets of a fixed $n$-set. We will show in Lemma 2.1.1 that when $q$ is the order of a finite field $\mathbb{F}_q$, the $q$-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$ equals the number of $k$-dimensional subspaces of the $n$-dimensional vector space $\mathbb{F}_q^n$.

**2.1.1 Lemma.** *When $q$ is the order of a finite field $\mathbb{F}_q$, the $q$-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$ equals the number of $k$-dimensional subspaces of the $n$-dimensional vector space $\mathbb{F}_q^n$.*

*Proof.* There are $(q^n-1)(q^n-q)...(q^n-q^{k-1})$ ways to choose $k$-tuple independent vectors from $\mathbb{F}_q^n$. Since a given $k$-space has $(q^k-1)(q^k-q)...(q^k-q^{k-1})$ distinct ordered bases, the number of $k$-dimensional subspaces of $\mathbb{F}_q^n$ is

$$\begin{aligned} &\frac{(q^n-1)(q^n-q)\dots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\dots(q^k-q^{k-1})} \\ &= \frac{(q^n-1)(q^{n-1}-1)\dots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\dots(q-1)} \\ &= \frac{[n][n-1]\dots[n-k+1]}{[k]!} \\ &= \frac{[n][n-1]\dots[n-k+1]}{[k]!}\frac{[n-k]!}{[n-k]!} \\ &= \frac{[n]!}{[k]![n-k]!} = \begin{bmatrix} n \\ k \end{bmatrix}. \end{aligned}$$

Thus, when $q$ is the order of a finite field $\mathbb{F}_q$, the $q$-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$ equals the number of $k$-dimensional subspaces of the $n$-dimensional vector space $\mathbb{F}_q^n$. □

One interpretation of the binomial coefficient $\binom{n}{k}$ is as the coefficient of $x^k y^{n-k}$ in the expansion of $(x+y)^n$:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \tag{2.1.1}$$

It is well known that the $q$-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$ can be viewed as the coefficient of $x^k y^{n-k}$ in a noncommutative binomial formula; a proof is given in [10].

**2.1.2 Theorem.** *Let $x$ and $y$ be elements satisfying the commutation relation $yx = qxy$, where $q$ is a number commuting with both $x$ and $y$, then*

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k}. \tag{2.1.2}$$

□

When $q = 1$, the variables $x$ and $y$ commute so (2.1.2) reduces to (2.1.1). As an example of elements satisfying such a commutation relation for $q \neq 1$, consider the linear operators, $\mu_x$ and $\sigma_q$, on the space of polynomials whose actions on a polynomial $f(x)$ are

$$\mu_x[f(x)] = xf(x),\ \sigma_q[f(x)] = f(qx).$$

For any $f(x)$ we have

$$\sigma_q \mu_x[f(x)] = \sigma_q[xf(x)] = qxf(qx) = q\mu_x\sigma_q[f(x)].$$

Therefore

$$\sigma_q \mu_x[f(x)] = q\mu_x\sigma_q.$$

A familiar identity involving binomial coefficients is *Pascal's Identity*,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k},\ 1 \leq k \leq n-1. \tag{2.1.3}$$

Often, naively changing binomial coefficients in an identity to $q$-binomial coefficients yields a $q$-identity: for example,

$$\binom{n}{k} = \binom{n}{n-k},\ \begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix}$$

for $0 \leq k \leq n$. In this case, however, changing binomial coefficients to $q$-binomial coefficients does not give a $q$-Pascal identity. When $q \neq 1$, we have

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 1 + q \neq 2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

As we will show in Theorem 2.1.3, there are interestingly *two* $q$-Pascal identities. We provide a proof in the case where $q$ is the order of a finite field $\mathbb{F}_q$. Again, when $q = 1$, both $q$-Pascal identities (2.1.4) and (2.1.5) reduce to Pascal's identity (2.1.3).

**2.1.3 Theorem.** *Let $q$ be the order of a finite field $\mathbb{F}_q$. There are two $q$-Pascal identities:*

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix} \tag{2.1.4}$$

*and*

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + q^k\begin{bmatrix} n-1 \\ k \end{bmatrix} \tag{2.1.5}$$

*where $1 \leq k \leq n-1$.*

*Proof.* Let $H$ be an $(n-1)$-subspace of $\mathbb{F}_q^n$. To show the first identity, (2.1.4), we partition the $k$-subspaces of $\mathbb{F}_q^n$ into $k$-subspaces that are contained in $H$ and $k$-subspaces that aren't contained in $H$. By Lemma 2.1.1, since $H$ is isomorphic to $\mathbb{F}_q^{n-1}$, there are $\left[\begin{smallmatrix} n-1 \\ k \end{smallmatrix}\right]$ $k$-spaces contained in $H$. If a $k$-subspace does not lie in $H$, then it intersects $H$ in a $(k-1)$-space. By Lemma 2.1.1, there are $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ $(k-1)$-subspaces in $H$, each of which is contained in

$$\begin{bmatrix} n-(k-1) \\ 1 \end{bmatrix} = \begin{bmatrix} n-k+1 \\ 1 \end{bmatrix} = \frac{q^{n-k+1}-1}{q-1}$$

$k$-subspaces of $V$ and

$$\begin{bmatrix} (n-1)-(k-1) \\ 1 \end{bmatrix} = \begin{bmatrix} n-k \\ 1 \end{bmatrix} = \frac{q^{n-k}-1}{q-1}$$

$k$-subspaces of $H$. Therefore, each $(k-1)$-subspace of $H$ is contained in $q^{n-k}$ $k$-subspaces of $V$ that are not contained in $H$. We consequently have (2.1.4):

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix}.$$

By the symmetry property of the coefficients, $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix}$, we have

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix} = q^{n-(n-k)} \begin{bmatrix} n-1 \\ n-k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ n-k \end{bmatrix} = q^{k} \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix},$$

which gives the second identity, (2.1.5). □

We can use induction and Pascal's identity to show that the binomial coefficient $\binom{n}{k}$ is an integer. Similarly, we can use induction and Theorem 2.1.3 to show that the $q$-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$ is a polynomial in $q$. We have the following combinatorial interpretation of the coefficients of the $q$-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$ as a polynomial in $q$; a proof is given in [12].

**2.1.4 Theorem.** *Let*

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{\ell=0}^{k(n-k)} a_\ell q^\ell.$$

*Then the coefficient $a_\ell$ is the number of partitions of $\ell$ whose Ferrers diagrams fit in a box of size $k \times n-k$.* □

When we set $q = 1$ in the statement of Theorem 2.1.4, we obtain as a corollary that there are $\binom{n}{k}$ partitions whose Ferrers diagrams fit in a box of size $k \times n-k$.

## 2.2 From $q$-Kneser to Kneser

As we saw in the last section, we can recover binomial identities from their corresponding $q$-binomial identities by setting $q = 1$. In this section, we ask to what extent does setting $q = 1$ in results about the $q$-Kneser graphs yield information about the Kneser graphs? It is easy to see that the Kneser graph $K_{v:k}$ has $\binom{v}{k}$ vertices and is regular with valency $\binom{v-k}{k}$. In Lemma 2.2.1, we show that the $q$-Kneser graph, $qK_{v:k}$, has $\begin{bmatrix} v \\ k \end{bmatrix}$ vertices and is regular with valency $q^{k^2} \begin{bmatrix} v-k \\ k \end{bmatrix}$.

**2.2.1 Lemma.** *The $q$-Kneser graph $qK_{v:k}$ has $\begin{bmatrix} v \\ k \end{bmatrix}$ vertices and is regular with valency $q^{k^2}\begin{bmatrix} v-k \\ k \end{bmatrix}$.*

*Proof.* The vertices of the $q$-Kneser graph $qK_{v:k}$ are the $k$-dimensional subspaces of $\mathbb{F}_q^v$. By Lemma 2.1.1, the $q$-Kneser graph has $\begin{bmatrix} v \\ k \end{bmatrix}$ vertices.

Now we show that the $q$-Kneser graph is regular and we determine its valency. Let $\alpha$ be a vertex of $qK_{v:k}$; it is a k-dimensional subspace and contains $q^k$ elements of $\mathbb{F}_q^v$. There are $(q^v - q^k)(q^v - q^{k+1})...(q^v - q^{2k-1})$ ways to choose $k$-tuple independent vectors in $\mathbb{F}_q^v$ that are not in $\alpha$. Since a given $k$-space has $(q^k - 1)(q^k - q)...(q^k - q^{k-1})$ distinct ordered bases, there are

$$\frac{(q^v - q^k)(q^v - q^{k+1})...(q^v - q^{2k-1})}{(q^k - 1)(q^k - q)...(q^k - q^{k-1})} = q^{k^2}\begin{bmatrix} v-k \\ k \end{bmatrix}$$

$k$-spaces in $\mathbb{F}_q^v$ that have trivial intersection with $\alpha$. Consequently, the $q$-Kneser graph $qK_{v:k}$ is regular with valency $q^{k^2}\begin{bmatrix} v-k \\ k \end{bmatrix}$. □

When $q = 1$, the expressions for the number of vertices and valency of the $q$-Kneser graph $qK_{v:k}$ reduce to those of the Kneser graph $K_{v:k}$, respectively $\binom{v}{k}$ and $\binom{v-k}{k}$. Many other parameters of the $q$-Kneser graphs, for example the eigenvalues of the adjacency matrix and their multiplicities, are given by expressions that involve $q$-binomial coefficients and which reduce to those of the Kneser graph when we set $q = 1$. Theorem 2.2.2 appears in [4] and a proof of Theorem 2.2.3 can be found in [7].

**2.2.2 Theorem.** *The eigenvalues of the $q$-Kneser graph $qK_{v:k}$ are the integers*

$$\lambda_i = (-1)^i q^{k(k-i)}\begin{bmatrix} v-r-i \\ r-i \end{bmatrix},\ 0 \le i \le k$$

*occuring with multiplicities*

$$m_0 = 1,\ m_i = \begin{bmatrix} v \\ i \end{bmatrix} - \begin{bmatrix} v \\ i-1 \end{bmatrix}.$$ □

**2.2.3 Theorem.** *The eigenvalues of the Kneser graph $K_{v:k}$ are the integers*

$$\lambda_i = (-1)^i \binom{v-r-i}{r-i},\ 0 \le i \le k$$

*occuring with multiplicities*

$$m_0 = 1,\ m_i = \binom{v}{i} - \binom{v}{i-1}.$$ □

Substituting $q = 1$ in the expression for $\chi(qK_{v:k})$ does not, however, give $\chi(K_{v:k})$. For example, we show in this paper that $\chi(qK_{v:2}) = [v-1]$ when $v > 4$. Consequently, the $q$-Kneser graph $qK_{5:2}$ has chromatic number $\chi(qK_{5:2}) = [4]$, but the Kneser graph, $K_{5:2}$, also known as the Petersen graph, has chromatic number $\chi(K_{5:2}) = 3$. Therefore, the relationship between the chromatic numbers of the Kneser graph and the $q$-Kneser graph is more complex than setting $q = 1$.

## 2.3 Independent Sets

Another area where there are analogies between the Kneser and $q$-Kneser graphs is the size and structure of their maximal independent sets. An independent set in the Kneser graph $K_{v:k}$ is a set of $k$-subsets in $\{1,\ldots,v\}$ that pairwise intersect nontrivially. Similarly, an independent set in the $q$-Kneser graph $qK_{v:k}$ is a set of $k$-dimensional spaces in $\mathbb{F}_q^v$ that pairwise intersect nontrivially.

The Erdős-Ko-Rado theorem states that for $v > 2k$, the independence number of the Kneser graph, $\alpha(K_{v:k})$, is $\binom{v-1}{k-1}$ and that an independent set with size $\binom{v-1}{k-1}$ consists of the $k$-subsets of $\{1,\ldots,v\}$ that contain a particular point. Chris Godsil and Mike Newman give a proof of the Erdős-Ko-Rado theorem using linear algebra in [8]. For $v = 2k$, the bound $\alpha(K_{2k:k}) = \binom{2k-1}{k-1}$ is correct, but an independent set of size $\binom{2k-1}{k-1}$ is not necessarily the set of $k$-subsets of $\{1,\ldots,2k\}$ that contain a particular point because $K_{2k:k}$ is isomorphic to $\binom{2k-1}{k-1}$ vertex-disjoint copies of $K_2$.

The maximal independent sets of the $q$-Kneser graph have a similar structure to those of the Kneser graph. The $q$-analogue of the maximal independent set in $K_{v:k}$ is the set of $k$-spaces in $\mathbb{F}_q^v$ that contain a particular one-dimensional subspace, and is called a *point pencil*. A special case of the Erdős-Ko-Rado theorem for vector spaces [5] is the following theorem.

**2.3.1 Theorem.** *For $v > 2k$, $\alpha(qK_{v:k}) = \begin{bmatrix} v-1 \\ k-1 \end{bmatrix}$ and an independent set with size $\begin{bmatrix} v-1 \\ k-1 \end{bmatrix}$ is a point pencil.* □

Mike Newman's and Chris Godsil's proof of the Erdős-Ko-Rado theorem in [8] can be generalised to the $q$-Kneser graphs. The case $v = 2k$ again requires special attention. For $v = 2k$, there are two non-isomorphic classes of maximal independent sets, the point pencils and the set of $k$-spaces contained in a $(2k-1)$-dimensional space, both of which have size $\begin{bmatrix} 2k-1 \\ k-1 \end{bmatrix}$. Mike Newman and Chris Godsil can prove this, and as Chris Godsil points out in [3], there is some confusion about where a proof of this claim appears in the literature.

# Chapter 3

# Projective Prerequisites

We will use concepts from projective geometry to determine the chromatic number of the $q$-Kneser graph when $k = 2$. In this chapter, we define the *projective geometry* of dimension $v - 1$ over the finite field $\mathbb{F}_q$, denoted by $PG(v-1, q)$, and discuss some of its properties, most important of which is duality. We will also prove a special case of the Bose-Burton theorem.

## 3.1 Definitions

The objects of the projective geometry, $PG(v-1, q)$, are:

- *points*, which are the one-dimensional spaces of $\mathbb{F}_q^v$.
- *lines*, which are the two-dimensional spaces of $\mathbb{F}_q^v$.
- *planes*, which are the three-dimensional spaces of $\mathbb{F}_q^v$.
- *i-flats*, which are the $(i + 1)$-dimensional spaces of $\mathbb{F}_q^n$ for $0 \leq i < v$.
- *hyperplanes*, which are the $(v - 1)$-dimensional spaces of $\mathbb{F}_q^v$.

The incidence relation between the objects of the projective geometry, $PG(v-1, q)$, is defined by containment of the corresponding subspaces. The incidence relation is symmetric; we say that a point is incident with a line (the point is on the line) or that a line is incident with a point (the line passes through the point) if the corresponding one-space is contained in the corresponding two-space. It is easy to check that any line in $PG(v-1, q)$ contains at least three points, and that two distinct points lie on a unique line.

The introduction of the projective geometry, $PG(v - 1, q)$, allows us to visualise the $q$-Kneser graphs. For example, the vertices of $qK_{v:2}$ are the lines of $PG(v-1, q)$. An independent set in $qK_{v:2}$ is a set of 2-spaces in $\mathbb{F}_q^v$ that pairwise have non-trivial intersection, and corresponds to a set of lines in $PG(v - 1, q)$ any two of which intersect. The term point pencil suggests its pictorial representation as the set of lines through a point. We can visualise the other canonical independent set of $qK_{v:2}$, the set of 2-spaces in a 3-space, as the lines on a projective plane.

## 3.2 Duality

We now explore the concept of duality in $PG(v-1,q)$. The table below lists geometric properties of $PG(v-1,q)$ that we will use to determine the chromatic number of $qK_{v:2}$. Counting arguments similar to the one in Lemma 2.1.1 give the values in the table.

| Property | $PG(3,q)$ | $PG(v-1,q)$ |
|---|---|---|
| Number of points | $q^3+q^2+q+1$ | $[v]$ |
| Number of lines | $(q^2+1)(q^2+q+1)$ | $\begin{bmatrix} v \\ 2 \end{bmatrix}$ |
| Number of planes/hyperplanes | $q^3+q^2+q+1$ | $[v]$ |
| Number of points on a line | $q+1$ | $q+1$ |
| Number of planes on a line | $q+1$ | $[v-2]$ |
| Number of lines through a point | $q^2+q+1$ | $[v-1]$ |
| Number of lines on a plane | $q^2+q+1$ | $q^2+q+1$ |
| Number of points on a plane/hyperplane | $q^2+q+1$ | $[v-1]$ |
| Number of planes/hyperplanes on a point | $q^2+q+1$ | $[v-1]$ |

It is not a coincidence that the number of points equals the number of hyperplanes in $PG(v-1,q)$. Moreover, if we swap the words "point" and "plane" in any of the statements above, the resulting statement is valid in $PG(3,q)$. The following statements, for example, are both valid in $PG(3,q)$.

The number of points on a line is $q+1$.
The number of planes on a line is $q+1$.

This is a consequence of duality. For concreteness, we will explain duality for the projective geometry, $PG(3,q)$.

The usual dot product is an inner product in $\mathbb{F}_q^4$, and we say two vectors are *orthogonal* if their inner product is zero. For a subspace $S$, define $S^\perp$, the *orthogonal complement* of $S$, to be the set of vectors that are orthogonal to every element of $S$. Observe that we have a bijection between the points and the planes of $PG(3,q)$ that sends a point $x$ to the plane $x^\perp$. Similarly, we have a bijection between the lines of $PG(3,q)$ that sends the line $L$ to the line $L^\perp$. Let $x$ be a point of $PG(3,q)$ on a line $L$, which in turn is on a plane $P$. Then the point $P^\perp$ is on the line $L^\perp$, which in turn is on the plane $x^\perp$. Consequently, if we swap the words "point" and "plane" in any statement about $PG(3,q)$ in this thesis, the resulting statement is valid in $PG(3,q)$. We say that point and plane are dual in $PG(3,q)$ and that line is self-dual.

## 3.3 The Bose-Burton Theorem

Consider a finite projective geometry $PG(v-1,q)$. The Bose-Burton theorem asks: Given an integer $0 \leq s \leq v-1$, what is the cardinality of the smallest set of points $T$ with the property that every subspace of dimension $s$ is incident with at least one element of $T$? Bose and Burton solved this question in [2]. We will need the following special case of the Bose-Burton theorem.

**3.3.1 Theorem.** *Suppose $C$ is a set of points of $PG(v-1,q)$ such that every line is incident with a point from $C$. We have $|C| \geq [v-1]$ and, moreover, if $|C| = [v-1]$ then $C$ is the set of points on a hyperplane.*

*Proof.* Let $x$ be a point not in $C$. Since the point pencil on $x$ has size $[v-1]$, we must have $|C| \geq [v-1]$.

We will show by induction that if $S$ is a subspace of dimension $k < v-1$ in $C$, then $C$ contains a subspace of dimension $k+1$ which contains $S$. Since $|C| = [v-1]$, we see that $C$ is the set of points on a hyperplane. First, assume that $S$ is a one-dimensional subspace, and let $T$ be a one-dimensional subspace in $C$ distinct from $S$. Suppose, for contradiction, that a point $y$ on the line $S \vee T$ is not in $C$. Since both $S$ and $T$ lie on the line $S \vee T$, at least one line through $y$ is not incident with a point from $C$ because the size of the point pencil on $y$ is $[v-1] = |C|$. This contradicts our definition of $C$ so $S \vee T$ is contained in $C$.

Suppose we have shown the statement for all $k < d < v-1$, and let $S$ be a $d$-dimensional subspace in $C$. Let $T$ be a one-dimensional space not in $S$ so $S \vee T$ is a $(d+1)$-dimensional subspace. Suppose, for contradiction, that a point $y$ in $S \vee T$ is not contained in $C$. We have that $y \notin S$ so all lines through $y$ on $S \vee T$ are incident with the points in $S$. Now $T$ lies on a line through $y$ so one line through $y$ on $S \vee T$ contains two points from $C$. Hence, at least one line on $y$ is not incident with a point from $C$ because the size of the point pencil on $y$ is $[v-1] = |C|$. This contradicts our definition of $C$ so $S \vee T$ is contained in $C$. □

# Chapter 4

# Colouring $qK_{v:2}$

We determine the chromatic number of the $q$-Kneser graphs with $k = 2$. Godsil and Royle proved that if $v \geq 2k$ then $\chi(qK_{v:k}) \leq [v-k+1]$ and if $v = 2k$ then $\chi(qK_{v:k}) \leq q^k + q^{k-1}$. One new result of this thesis is that these upper bounds are tight when $k = 2$. In addition, we characterise the minimal colourings. This chapter forms the basis for the forthcoming article [3].

## 4.1 Covers of the $q$-Kneser graphs

We say that a family of independent sets covers a graph if every vertex in the graph lies in some independent set. Recall that a point pencil is a maximal independent set of the $q$-Kneser graph $qK_{v:k}$, and that the independent set consisting of the $k$-spaces in a $(2k-1)$-space is maximal when $v = 2k$. In this chapter, we say a *cover* of the $q$-Kneser graph $qK_{v:k}$ is a set of points and $(2k-1)$-spaces in $PG(v-1, q)$ such that every $k$-space is either on a point or contained in a $(2k-1)$-space. A cover of the $q$-Kneser graph $qK_{v:k}$ by $n$ points and $(2k-1)$-spaces gives rise to a colouring of $qK_{v:k}$ with $n$ colours. Lemma 4.1.1, which is due to Chris Godsil and Gordon Royle, describes two covers of the $q$-Kneser graph that give upper bounds on its chromatic number.

**4.1.1 Lemma.** *If $v \geq 2k$ then*

$$\chi(qK_{v:k}) \leq [v-k+1].$$

*If $v = 2k$ then*

$$\chi(qK_{v:k}) \leq q^k + q^{k-1}.$$

*Proof.* A subspace of dimension $v-k+1$ in $\mathbb{F}_q^v$ has non-trivial intersection with each $k$-space. It follows that the pencils on the points of such a subspace give a colouring with $[v-k+1]$ colours.

If $v = 2k$, choose a subspace $U$ of dimension $k+1$, and a subspace $T$ of dimension $k$ in $U$. We claim the point-pencils of the points in $U \backslash T$, together with the $(2k-1)$-spaces that contain $T$ but not $U$ cover every $k$-space in $\mathbb{F}_q^v$. By considering dimension, any $k$-space $S$ has nontrivial intersection with $U$, and thus any $k$-space not covered by a point in $U\backslash T$ must contain a point of $T$. Hence $\dim(S \vee T) \leq 2k-1$ and so $S$ is contained in a $(2k-1)$-subspace that contains $T$. If $U \subseteq S \vee T$, then

$$U = U \cap (S \vee T) = (U \cap S) \vee T = T$$

by the modular identity, which is a contradiction. Consequently, $U$ is not contained in $S \vee T$ so there is a $(2k-1)$-space that contains $S \vee T$ but not $U$. Hence the $(2k-1)$-spaces that meet $U$ in $T$ cover those $k$-spaces that do not contain a point of $U\backslash T$.

Now we count the number of point pencils and $(2k-1)$-spaces in our cover. The number of point pencils on the points of $U\backslash T$ is $[k+1]-[k]$. The number of $(2k-1)$-spaces that contain $T$ but not $U$ is $[k]-[k-1]$ so we have a total of

$$([k+1]-[k]) + ([k]-[k-1]) = [k+1]-[k-1] = q^k + q^{k-1}$$

point pencils and $(2k-1)$-spaces in our cover, which gives rise to a colouring of $qK_{v:k}$ with $q^k + q^{k-1}$ colours. □

## 4.2 Covers of $qK_{v:2}$

Restricting to the $q$-Kneser graphs with $k=2$, we have that a cover of $qK_{v:2}$ is a set of points and planes such that every line is either on one of the the points or lies in one of the planes. We have seen in Lemma 4.1.1 that covers of $qK_{v:2}$ can be used to give upper bounds on the chromatic number of $qK_{v:2}$. A property that is unique to $qK_{v:2}$ is that covers of $qK_{v:2}$ can be used to obtain a *lower bound* on the chromatic number of $qK_{v:2}$. As Lemma 4.2.1 shows, every independent set in $qK_{v:2}$ lies in a point pencil or a plane. Consequently, a colouring of $qK_{v:2}$ with $n$ colours gives rise to a cover of $qK_{v:2}$ using at most $n$ points and planes. If we can show that a cover of $qK_{v:2}$ uses at least $[v-1]$ points when $v>4$ and at least $q^2+q$ points and planes when $v=4$, then $[v-1]$ and $q^2+q$ are lower bounds for the chromatic number of $qK_{v:2}$ when $v>4$ and when $v=4$ respectively. We will do this in Section 4.3 for $qK_{4:2}$ and in Section 4.5 for $qK_{v:2}$ where $v>4$, and thus determine the chromatic number of $qK_{v:2}$ by Lemma 4.1.1.

**4.2.1 Lemma.** *Every independent set in $qK_{v:2}$ lies in a point pencil or a plane.*

*Proof.* Let $S$ be an independent set in $qK_{v:2}$. Suppose that the lines in $S$ are not coplanar. We show that $S$ lies in a point pencil. Let $\ell_1$ and $\ell_2$ be two lines in $S$ that intersect in a one-dimensional subspace $a$. Let $P$ be the plane determined by $\ell_1$ and $\ell_2$ and let $\ell_3$ be a line in $S$ not on $P$. Since $S$ is independent, $\ell_3$ intersects $\ell_1$ in a one-dimensional space $x$ and intersects $\ell_2$ in a one-dimensional space $y$. We must have $x=y$ otherwise $\ell_3$ would lie on $P$. Consequently, $\ell_1$ and $\ell_2$ intersect in $x$ and, as two points determine a line, we have $x=a$. Now, any line, $\ell_4$ in $S$ on the plane $P$ must intersect $\ell_3$ since $S$ is independent. Thus, $\ell_4$ intersects $\ell_3$ in $a$ so $S$ lies in the point pencil on $a$. □

In Lemma 4.1.1, we showed that the set of points on a hyperplane is one cover for $qK_{v:2}$ that uses $[v-1]$ points. We will see in Section 4.5 that all minimal covers of $qK_{v:2}$ for $v > 4$ look like this. Lemma 4.2.2 describes several covers of $qK_{4:2}$ that use $q^2+q$ points and planes.

**4.2.2 Lemma.** *Choose a plane $U$ and a point $a$ in $U$. Let $m_1,\dots,m_s$ be $s$ lines in $U$ through $a$ where $0 < s < q+1$. The points in $U$ not on these lines and the planes distinct from $U$ on these lines form a cover of $qK_{4:2}$ with $q^2+q$ points and planes.*

*Proof.* Let $\ell$ be a line. We have $\ell$ intersects $U$ in a point, $x$, or $\ell$ lies on $U$. First suppose $\ell$ intersects $U$ in a point $x$. If $x$ is not on one of the $s$ lines, then $\ell$ is covered by a point. Therefore, assume that $x$ lies on one of the $s$ lines, $m_i$, through $a$. Then $\ell$ and $m_i$ determine a plane $P$ on $m_i$ distinct from $U$, and $\ell$ lies on $P$.

Now, suppose $\ell$ lies on $U$. If $\ell$ is one of the $s$ lines, $m_i$, then $\ell$ is covered by a plane on it. Otherwise, since $\ell$ contains $q+1$ points and $s < q+1$, $\ell$ intersects a point, $y$, in $U$ not on one of the $s$ lines; in this case, $\ell$ is covered by a point.

We have shown that the points in $U$ not on the $s$ lines and the planes distinct from $U$ on the $s$ lines form a cover of $qK_{4:2}$. There are

$$(q^2+q+1)-1-sq = q(q+1-s)$$

points in our cover and $sq$ planes. This gives a total of $q^2+q$ points and planes. □

We call a cover of the type described in Lemma 4.2.2 a *standard cover*. The cover of $qK_{4:2}$ given in Lemma 4.1.1 is an example of a standard cover where $s=1$. The standard covers economise the number of points and planes used because they minimise the number of lines on multiple points from the cover or contained in multiple planes from the cover. For example, in a standard cover, no plane from the cover contains a point from the cover so there are no lines in $PG(3,q)$ covered by both a point and a plane. Corollary 4.4.4 shows that standard covers contain $q$ collinear points: if the line $\ell$ contains $\alpha_1,\dots,\alpha_q$ from the cover, then $\ell$ is the only line in $PG(3,q)$ on more than one point from $\alpha_1,\dots,\alpha_q$. Standard covers similarly contain $q$ collinear planes. We will see in Section 4.4 that every minimal cover of $qK_{4:2}$ is a standard cover.

## 4.3 The chromatic number of $qK_{4:2}$

In this section we show that $\chi(qK_{4:2}) = q^2+q$ by showing that a minimal cover of $qK_{4:2}$ must contain at least $q^2+q$ points and planes. Recall that there are two maximal independent sets in $qK_{4:2}$: the point pencils and the set of lines in a plane, both of which have size $q^2+q+1$. We also have that point and plane are dual in $PG(3,q)$ and that line is self-dual.

One cover of $qK_{4:2}$ that uses $q^2+q+1$ points is the set of points on a plane. We have seen that if we use both points and planes, we can obtain smaller covers using only $q^2+q$ points and planes. Lemma 4.3.1, which is due to Chris Godsil and Gordon Royle, shows that a minimal cover of $qK_{4:2}$ must use sufficient quantities of both points and planes.

**4.3.1 Lemma.** *Suppose we have a cover of $qK_{4:2}$ formed from $r$ points and $s$ planes. If $r+s \leq q^2+q$, then $r, s \geq q$.*

*Proof.* Suppose $r \leq q-1$, and let $a$ be a point not used in the cover. There are $q^2+q+1$ lines on $a$, and so at least $q^2+2$ of these lines are not covered by one of our $r$ points. Hence they must be covered by one of the $s$ planes. The first plane on $a$ covers $q+1$ lines through $a$, and each additional plane on $a$ covers at most $q$ further lines. Hence if there are $t$ of our $s$ planes on $a$, then

$$(q+1)+(t-1)q \geq q^2+2$$

and therefore

$$t-1 \geq q-1+\frac{1}{q}.$$

This implies that $t \geq q+1$ since $t$ is an integer.

Now count pairs $(a, H)$ where $a$ is point not in our cover and $H$ is a plane in the cover that contains $a$. We find that

$$s(q^2+q+1) \geq (q+1)[(q^3+q^2+q+1)-(q-1)]$$

and hence

$$s \geq \frac{(q^3+q^2+q-(q-2))(q+1)}{q^2+q+1} = q(q+1) - \frac{(q+1)(q-2)}{q^2+q+1}.$$

Since $s$ is an integer, this implies that $s \geq q^2+q$ and consequently $r+s \geq q^2+q+1$. Consequently, if $r+s \leq q^2+q$ then $r \geq q$ and, by duality, $s \geq q$ as well. □

Theorem 4.3.2 shows that a minimal cover of $qK_{4:2}$ uses exactly $q^2+q$ points and planes, and so proves that $\chi(qK_{4:2}) = q^2+q$. We saw in Lemma 4.3.1 that the number of points and the number of planes in a minimal cover of $qK_{4:2}$ are both at least $q$. Theorem 4.3.2 gives a stronger condition, namely, that the number of points and the number of planes in a minimal cover of $qK_{4:2}$ are divisible by $q$.

**4.3.2 Theorem.** *Suppose $C$ is a cover of $qK_{4:2}$ with $r+s \leq q^2+q$ points and planes. Then $C$ contains exactly $q^2+q$ point pencils and planes, and, moreover, $q \mid r$.*

*Proof.* Let $C$ be a cover of $qK_{4:2}$ with $r+s \leq q^2+q$ point pencils and planes. Write $r = kq+x$ where $0 \leq x < q$. By Lemma 4.3.1, $s \geq q$ which implies that $r \leq q^2$ and $k \leq q$. Suppose, for contradiction, that $s < (q^2+q)-kq$.

Let $P$ be a plane that is not in the cover. Then $P$ must contain some points from the cover; otherwise, we'd need $q^2+q+1$ planes in the cover to cover the lines on $P$. We want to know how many points from the cover $P$ must contain. Suppose $P$ contains at most $k$ points from the cover. Since $k \leq q$, any $k$ points on $P$ cover at most $kq+1$ lines, so at least $(q^2+q+1)-(kq+1) = (q^2+q)-kq$ lines on $P$ are not covered by one of these $k$ points. As $s < (q^2+q)-kq$, the remaining $(q^2+q)-kq$ lines on $P$ cannot all be covered by planes. Therefore, $P$ must contain at least $k+1$ points from the cover.

Each point of $PG(3,q)$ lies in exactly $q^2+q+1$ lines; since $r \leq q^2$, any point not in the cover lies on a line that is disjoint from the points in the cover. Let $\ell$ be a line that does not contain points from the cover. Since $r = kq+x$ where $x < q$, at least two of the $q+1$ planes on $\ell$, $P_1$ and $P_2$, will each contain fewer than $k+1$ points from the cover. By the preceding paragraph, $P_1$ and $P_2$ must be in the cover. To summarize, if $\ell$ is a line containing no points from the cover, then two planes on $\ell$ are in the cover.

The plane $P_1$ on $\ell$ contains at most $k$ points from the cover so at least $(q^2+q)-kq$ lines on $P_1$ don't contain a point from the cover. By the preceding paragraph, we need at least $(q^2+q)-kq$ additional planes from the cover on these lines. But $s < (q^2+q)-kq$ so we have a contradiction.

Consequently, $s \geq (q^2+q)-kq$. The number of point pencils and planes in $C$ is

$$r+s \geq (kq+x)+((q^2+q)-kq) = (q^2+q)+x.$$

As $C$ satisfies $r+s \leq q^2+q$, we must have $x=0$. Therefore, $C$ contains exactly $r+s = q^2+q$ point pencils and planes, and, moreover, $q|r$. $\square$

## 4.4 Characterising minimal colourings of $qK_{4:2}$

We have shown that the minimal covers of $qK_{4:2}$ have size $q^2+q$, and seen that the standard covers defined in Lemma 4.2.2 are minimal covers. In this section, we show conversely that every minimal cover of $qK_{4:2}$ is a standard cover. Thus, a minimal colouring of $qK_{4:2}$ induces a standard cover of $qK_{4:2}$; conversely, a standard cover of $qK_{4:2}$ gives rise to a minimal colouring of $qK_{4:2}$. We have, consequently, characterised the minimal colourings of $qK_{4:2}$.

Let $C$ be a minimal cover of $qK_{4:2}$ and consider a plane $P$ that is not in $C$. The lines on $P$ are either covered by points in the cover that lie on $P$ or planes in the cover that intersect $P$. Lemma 4.4.1 gives a lower bound on the number of points from the minimal cover, $C$, that $P$ must contain.

**4.4.1 Lemma.** *Let $C$ be a cover of $qK_{4:2}$ with $r = kq$ points and $s = q(q+1-k)$ planes. If $P$ is a plane not in $C$, then it contains at least $k$ points from $C$; if equality holds then the $k$ points are collinear.*

*Proof.* Since $P$ is not a plane in $C$, $P$ must contain some points from $C$; otherwise, we'd need $q^2+q+1$ planes to cover the lines on $P$. The $s = (q+1-k)q$ planes in $C$ cover at most $(q+1-k)q$ lines on $P$ so at least $kq+1$ lines on $P$ remain. If $t < q+1$, then $t$ points in a plane cover at most $tq+1$ lines, with equality if and only if they are collinear. Consequently $P$ contains at least $k$ points from $C$, and if it contains exactly $k$, they are collinear. $\square$

We would like to obtain an upper bound on the number of lines covered by points from a minimal cover, $C$. Lemma 4.4.2 gives such an upper bound when there is a plane containing $q + 1$ points from the cover. (As we shall see, such a plane exists whenever the cover contains more than $q$ points.)

**4.4.2 Lemma.** *Let $C$ be a cover of $qK_{4:2}$ with $r = kq$ points and $s = q(q+1-k)$ planes. Suppose $P$ is a plane that contains at least $q + 1$ points from the cover. Then the points of $C$ cover at most $kq^3$ lines not in $P$.*

*Proof.* A point of $C$ in $P$ covers $q^2$ lines not in $P$. If $x$ is a point in $C$ not on $P$, at least $q+1$ lines incident with $x$ are incident with a point in $C$ that lies on $P$. Consequently $x$ covers at most $q^2$ lines that are not incident with a point of $C$ on $P$. Therefore the $kq$ points of $C$ cover at most $kq^3$ lines not on $P$. $\square$

In a standard cover, no plane from the cover contains a point from the cover. We will see in Lemma 4.4.7 that any minimal cover also has this economical property. As a first step, Lemma 4.4.3 shows that every plane in $PG(3, q)$ contains a line that is disjoint from the points in a minimal cover.

**4.4.3 Lemma.** *If $C$ is a cover of $qK_{4:2}$ with $r = kq$ point pencils and $s = q(q + 1 - k)$ planes, then each plane contains at least one line disjoint from the points in $C$.*

*Proof.* Let $P$ be a plane and assume for contradiction that each line on $P$ is incident with a point from $C$. Since all planes from $C$ meet $P$ in a line, each plane in $C$ contains at least one point from $C$. Therefore on each plane of $C$ there are at least $q + 1$ lines that are incident with a point from $C$. Each plane in $C$, consequently, covers at most $q^2$ of the lines that do not contain points from $C$. As there are

$$(q^2 + 1)(q^2 + q + 1) = q^4 + q^3 + 2q^2 + q + 1$$

lines in total, the number of lines covered by the points in $C$ is at least

$$(q^4+q^3+2q^2+q+1)-sq^2 = q^4+q^3+2q^2+q+1-(q+1-k)q^3 = kq^3+2q^2+q+1.$$

We know that $q^2 + q + 1$ of these lines lie in $P$, the remaining lines, of which there are at least $kq^3 + q^2$, must intersect $P$ in a point.

Since every line in $P$ is incident with a point from $C$, there are at least $q + 1$ points from $C$ on $P$. By Lemma 4.4.2, the $kq$ points of $C$ cover at most $kq^3$ lines not on $P$, a contradiction. $\square$

An immediate corollary of Lemma 4.4.3 is Corollary 4.4.4 which shows that a minimal cover cannot contain $q + 1$ collinear points. We saw that one feature of standard covers is that they contain $q$ collinear points: if the line $\ell$ contains

$\alpha_1, \ldots, \alpha_q$ from the cover, then $\ell$ is the only line in $PG(3, q)$ on more than one point from $\alpha_1, \ldots, \alpha_q$. Consequently, one might think that we could construct a smaller cover by having $q+1$ collinear points in the cover. As Corollary 4.4.4 shows, this is not the case. The reason is that we do not want planes from the cover to contain points from the cover.

**4.4.4 Corollary.** *Let $C$ be a minimal cover of $qK_{4:2}$. Then $C$ does not contain $q+1$ collinear points. (Dually, $C$ does not contain $q+1$ collinear planes.)*

*Proof.* Suppose, for contradiction, that there is a line $\ell$ consisting of $q+1$ points from $C$. Let $P$ be any plane on $\ell$. Since any line on $P$ intersects $\ell$ in a point, every line on $P$ is incident with a point from $C$, which is a contradiction by Lemma 4.4.3. □

From Lemma 4.3.1, we know that a minimal cover of $qK_{4:2}$ must contain at least $q$ points and at least $q$ planes. Lemma 4.4.5 shows that minimal covers of $qK_{4:2}$ with exactly $q$ points or exactly $q$ planes are standard.

**4.4.5 Lemma.** *Let $C$ be a cover of $qK_{4:2}$ with $q^2$ points and $q$ planes. Then $C$ is standard. (Dually, if $C$ is a cover with $q$ points and $q^2$ planes, then $C$ is standard.)*

*Proof.* Assume $C$ contains $q^2$ points and $q$ planes. We will show first that there is a plane containing at least $q+1$ points from $C$.

Let $\ell_1$ be a line not incident with a point in $C$ By Corollary 4.4.4 there is a plane $H$ on $\ell_1$ that is not in $C$ and by Lemma 4.4.1, there are at least $q$ points from $C$ on $H$. If there are exactly $q$ points then they lie on a line $\ell_2$, and any plane containing $\ell_2$ and a point in $C$ not on $\ell_2$ contains $q+1$ points from $C$. Otherwise, the plane $H$ contains at least $q+1$ points from $C$.

We next show that no plane of $C$ contains a point of $C$, and that the $q$ planes of $C$ lie on a common line.

Let $P$ be a plane that contains at least $q+1$ points from $C$. By Lemma 4.4.2, the $q^2$ points in $C$ cover at most $q^4$ lines not in $P$. By Lemma 4.4.3, there is a line on $P$ that contains no point of $C$ and so at most $q^2+q$ lines on $P$ are incident with points of $C$. Hence the number of lines incident with the $q^2$ points in $C$ is at most $q^4+q^2+q$.

Since any two planes have a line in common, the $q$ planes in $C$ cover at most $q(q^2+q)+1 = q^3+q^2+1$ lines. The total number of lines is

$$q^4 + q^3 + 2q^2 + q + 1 = (q^4 + q^2 + q) + (q^3 + q^2 + 1),$$

so the $q^2$ points in $C$ must cover exactly $q^4+q^2+q$ lines and the $q$ planes must cover exactly $q^3+q^2+1$. We also see that the set of lines covered by the points of $C$ is disjoint from the set of lines covered by the planes, and consequently no point of $C$ can lie in a plane of $C$.

Further, since the $q$ planes cover exactly $q^3+q^2+1$ lines, the $q$ planes must lie on a line $\ell$.

Let $Q$ be the unique plane on $\ell$ not in the cover. Then the $q^2$ points of our cover must lie on $Q$, and hence the points of the cover are the points of $Q \backslash \ell$. □

In a standard cover with $kq$ points and $q(q+1-k)$ planes, there is a distinguished plane that contains all $kq$ points from the cover. Any plane that is not in the cover and not the distinguished plane contains exactly $k$ points from the cover. On the distinguished plane, any line not incident with a point from the cover lies on $q$ planes from the cover. As a step towards proving that every minimal cover is a standard cover, Lemma 4.4.6 shows that if $P$ is a plane containing at least $k+1$ points from the cover, then any line not incident with a point from the cover lies on two planes from the cover. We show in Theorem 4.4.8 that a plane $P$ containing at least $k+1$ points from a minimal cover is the distinguished plane in a standard cover.

**4.4.6 Lemma.** *Let $C$ be a cover of $qK_{4:2}$ with $r = kq$ points and $s = q(q+1-k)$ planes and let $P$ be a plane not in $C$ that contains at least $k+1$ points from $C$. Then any line on $P$ not incident with a point from $C$ lies on a least two planes from $C$.*

*Proof.* Let $\ell$ be a line on $P$ that is not incident with a point from the cover. Let

$$H_1, \ldots, H_{q+1}$$

denote the $q+1$ planes on $\ell$, where $H_1 = P$. Since $\ell$ contains no point of the cover, these planes partition the $kq$ points of the cover. By Lemma 4.4.1, each plane not in $C$ contains at least $k$ points from $C$. Since $P$ contains $k+1$ points from the cover, it follows that at least two of the planes on $\ell$ must lie in the cover. □

We have seen in Lemma 4.4.5 that a minimal cover with exactly $q$ points or exactly $q$ planes has the property that no plane in the cover contains a point from the cover. Lemma 4.4.7 extends this result to minimal covers with greater than $q$ points and greater than $q$ planes. Lemma 4.4.7 assumes that there is a plane $P$ containing $q+1$ points from the cover and that there is a point $y$ on $q+1$ planes from the cover, but as we shall see in Theorem 4.4.8, this is always the case for minimal covers with greater than $q$ points and greater than $q$ planes.

**4.4.7 Lemma.** *Let $C$ be a cover of $qK_{4:2}$ with $r = kq$ points and $s = q(q+1-k)$ planes. Suppose there is a plane $P$ that contains at least $q+1$ points from the cover and a point $y$ that lies on at least $q+1$ planes. Then no plane in the cover contains a point from the cover.*

*Proof.* Let $P$ be a plane that contains at least $q+1$ points from $C$. By Lemma 4.4.2, our $r = kq$ points cover at most $kq^3$ lines not in $P$. Since there is a line in $P$ that contains no points from $C$, our $r$ points cover at most $kq^3 + q^2 + q$ lines. Dually, the number of lines covered by the $s$ planes in $C$ is at most

$$sq^2 + q^2 + q = (q+1-k)q^3 + q^2 + q.$$

Suppose, for contradiction, that some plane in $C$ contains a point from $C$. Then there are $q+1$ lines that are covered by both by a point in $C$ and a plane from $C$. Hence the number of lines covered by the points and planes of $C$ is at most

$$(kq^3 + q^2 + q) + ((q + 1 - k)q^3 + q^2 + q) - (q + 1) = q^4 + q^3 + 2q^2 + q - 1$$

Since there are $q^4 + q^3 + 2q^2 + q + 1$ lines altogether, this provides our contradiction. $\square$

We have seen in Lemma 4.4.5 that minimal covers with exactly $q$ points or exactly $q$ planes are standard. Theorem 4.4.8 extends this result to all minimal covers.

**4.4.8 Theorem.** *A cover of $qK_{4:2}$ with $q^2 + q$ points and planes is standard.*

*Proof.* Let $C$ be a cover of $qK_{4:2}$ with $q^2 + q$ point pencils and planes. We may assume that there are $r = kq$ point pencils and $s = (q + 1 - k)q$ planes. By Lemma 4.4.5 and duality, we may assume that $2 \leq k \leq q - 1$. We will prove the statement via Lemma 4.4.7 by showing that there is a plane $P$ containing $q + 1$ points from the cover, and that there is a point $y$ on $q + 1$ planes from the cover.

As a first step, we show that there is a line that contains no point from $C$ and lies on exactly one plane from $C$. Let $m$ be a line that contains no point from $C$. There are $q + 1$ planes on $m$ and $kq$ points in the cover, so there is a plane $H$ on $m$ that contains fewer than $k$ points. By Lemma 4.4.1 we see that $H$ lies in the cover. At most $(k - 1)q + 1$ lines on $H$ are incident with points of $C$ and therefore there are at least $q(q + 2 - k)$ lines in $H$ not incident with a point from $C$. Since there are only $q(q + 1 - k)$ planes in $C$, there is a line $\ell$ in $H$ which is not contained in a second plane from $C$.

Next we show that there is a plane that contains at least $q + 2$ points from $C$.

Let $H_1, \ldots, H_q$ denote the planes on $\ell$ other than $H$. These $q$ planes do not belong to $C$ and therefore by Lemma 4.4.1, there are at least $k$ points from $C$ on each of them. Since these planes partition the points of $C$ into $q$ classes, each plane contains exactly $k$ points from $C$ and, by Lemma 4.4.1, each set of $k$ points lies on a line. Denote the line on $H_i$ by $m_i$. The $k$ points on $H_i$ cover exactly $kq + 1$ lines on $H_i$; the remaining $(q + 1 - k)q$ lines on $H_i$ are covered by planes of $C$. Since there are exactly $(q + 1 - k)q$ planes in $C$, each line of $H_i$ that is not covered by a point of $C$ is contained in exactly one plane from $C$. Note that $H$ contains no points of $C$.

The plane $H_1$ contains $\ell$ and therefore $m_1$ intersects $\ell$ in a point $x$. The lines other than $m_1$ on $x$ in $H_1$ are covered by planes of $C$, and so there are $q$ planes from $C$ on $x$. For $i = 2, \ldots, q$ these planes intersect $H_i$ in $q$ distinct lines through $x$, and these lines do not contain points of $C$. Therefore each of the lines $m_2, \ldots, m_q$ intersects $\ell$ in $x$.

Let $P$ be the plane determined by $m_1$ and $m_2$. The lines on $P$ incident with $x$ are $m_1$, $m_2$, the intersection of $P$ with $H$ and the intersection of $P$ with $H_3, \ldots, H_q$. The planes in $C$ intersect $H_1$ in lines that contain no points of $C$, but $P \cap H_1 = m_1$ which does contain points from $C$. Therefore $P$ is not in $C$.

Since $P$ contains $2k$ points from $C$, by Lemma 4.4.6 any line on $P$ not incident with a point from $C$ lies on at least two planes from $C$. As there are $q$ planes from $C$ on $x$, at most $q/2$ lines on $P$ incident with $x$ do not contain points from $C$. Consequently at least $(q+2)/2$ lines on $P$ incident with $x$ contain points from $C$. Referring to our listing above of the lines on $x$ in $P$, we see that $m_1$ and $m_2$ contain $k$ points from $C$. As $H$ contains no points of $C$, the line $P \cap H$ is disjoint from $C$. If $P \cap H_i$ contains a point from $C$ then $P \cap H_i$ is $m_i$, because this is the only line on $x$ in $H_i$ that contains points from $C$. Then $P \cap H_i$ contains $k$ points from $C$. Since $k \geq 2$, it follows that the number of points from $C$ on $P$ is at least

$$k\frac{q+2}{2} \geq q+2.$$

So we have shown that there is a plane that contains at least $q+2$ points from $C$; the dual of our argument shows that there is a point $y$ on at least $q+2$ planes from $C$.

By Lemma 4.4.7, no plane in $C$ contains a point from $C$. The $(q+1-k)q$ planes in $C$ each meet $P$ in a line, and so by Lemma **??** there are at least $q+1-k$ lines in $P$ that contain no point of $C$. Thus there are at most $q^2+k$ lines in $P$ that do contain points of $C$. Each point of $C$ in $P$ covers $q^2$ lines not in $P$. Since there are at least $q+2$ points of $C$ in $P$, each point of $C$ not in $P$ covers at most

$$(q^2+q+1)-(q+2) = q^2-1$$

lines not covered by points of $C$ in $P$. So the number of lines covered by the points of $C$ is at most

$$kq^3+q^2+k,$$

and if equality holds, all points of $C$ lie in $P$ and there are exactly $q+1-k$ lines in $P$ disjoint from $C$, each of which lies in $q$ planes from $C$.

Dually, the number of lines covered by the $(q+1-k)q$ planes of $C$ is at most

$$(q+1-k)q^3+q^2+(q+1-k)$$

and, if equality holds, these planes have a common point $y$ and there are exactly $k$ lines incident with $y$ that do not lie on a plane from $C$. Since the total number of lines is

$$q^4+q^3+2q^2+q+1 = (kq^3+q^2+k)+((q+1-k)q^3+q^2+q+1-k),$$

our last two inequalities must be tight. Therefore all points of $C$ lie in $P$ and all the planes of $C$ contain $y$. Hence $C$ is a standard cover. □

## 4.5 The chromatic number of $qK_{v:2}$ for $v > 4$

In this section, we show $\chi(qK_{v:2}) = [v-1]$ when $v > 4$ and characterise the minimal colourings. Theorem 4.5.1, which is joint work with Chris Godsil, shows that in a cover of $qK_{v:2}$ with at most $[v-1]$ points and planes the number of points is at least $[v-1]$. By Lemma 4.1.1, $\chi(qK_{v:2}) = [v-1]$. We work projectively in the space $PG(v-1,q)$.

**4.5.1 Theorem.** *Suppose we have a cover of $qK_{v:2}$ by $r$ points and $s$ planes such that $r+s \le [v-1]$. Then $r \ge [v-1]$.*

*Proof.* Suppose, for contradiction, that $r < [v-1]$, and define $\delta := [v-1] - r$. Let $C$ denote the set of points in the cover so $|C| = r$.

We determine a lower bound on the number of lines that do not contain a point of $C$, by counting the flags $(x, \ell_x)$ where $x$ is a point not in $C$ and $\ell_x$ is a line on $x$ disjoint from $C$. Each point $x$ lies on $[v-1]$ lines, and therefore there are at least $[v-1]-r = \delta$ lines through $x$ that are disjoint from $C$. As there are $[v]-r$ points not in $C$, the number of flags is at least

$$([v]-r)\delta = ([v]-[v-1]+[v-1]-r)\delta = (q^{v-1}+\delta)\delta.$$

Since each line disjoint from $C$ lies in exactly $q+1$ flags, it follows that the number of lines disjoint from $C$ is at least

$$\frac{(q^{v-1}+\delta)\delta}{v+1}.$$

Each of the lines disjoint from $C$ must be contained in one of the $s$ planes, and a plane contains exactly $q^2+q+1$ lines. Therefore

$$s \ge \frac{(q^{v-1}+\delta)\delta}{(q+1)(q^2+q+1)}. \tag{4.5.1}$$

Since $r+s \le [v-1]$, we have $s \le \delta$ so

$$\frac{(q^{v-1}+\delta)\delta}{(q+1)(q^2+q+1)} \le \delta,$$

from which we have

$$\delta \le (q+1)(q^2+q+1) - q^{v-1}. \tag{4.5.2}$$

Observe that

$$q^4 - (q+1)(q^2+q+1) = q^4 - q^3 - 2q^2 - 2q - 1 = q(q(q^2-q-2)-2)-1$$

and therefore

$$q^{v-1} - (q+1)(q^2+q+1) = (q^{v-1}-q^4) + q(q(q^2-q-2)-2)-1.$$

If $q > 2$ then $q^2-q-2 > 0$ and so the right side is positive. If $q = 2$ then the right side is equal to

$$(2^{v-1}-16)-5,$$

which is positive if $v > 5$. Consequently, we conclude that the right hand side of (2) is negative in these cases, which is a contradiction. Therefore, $r = [v-1]$ if $v > 5$ or if $v = 5$ and $q > 2$.

Finally we consider the case where $v = 5$ and $q = 2$. Let $x$ be a point not in $C$. Since $r < [4] = 15$, by assumption, at least one of the 15 lines on $x$ must be covered by one of the $s$ planes in the cover. Consequently, $x$ must lie on one of the $s$ planes in the cover. Since, $r + s \leq [4] = 15$, we have $r \leq 15 - s$ so at least

$$31 - (15 - s) = 16 + s \tag{4.5.3}$$

points don't lie in $C$, and must lie on one of the $s$ planes in the cover. Since planes contain 7 points, we must have $7s \geq 16 + s$ so $s \geq 3$.

Suppose $s > 3$. Then $r \leq 11$, so at least four of the 15 lines on $x$ must lie on the $s$ planes. Since a plane on $x$ covers three lines on $x$, we must have that $x$ lies on at least two planes in the cover. Consequently, $7s/2 \geq 16 + s$, which implies that $s \geq 7$. Since $s \leq \delta \leq 5$ by (2), this is a contradiction.

Therefore, $s = 3$ so at least 19 points must lie on the three planes in the cover by (4.5.3). However in $2K_{5:2}$ distinct planes intersect, so three planes cannot contain 19 distinct points. We have the desired contradiction, so $r = [4] = 15$ when $v = 5$ and $q = 2$. □

A minimal cover of $qK_{v:2}$ has exactly $[v-1]$ points. Theorem 3.3.1, which is a special case of the Bose-Burton theorem, shows that these $[v-1]$ points must lie on a hyperplane so this is the only type of minimal colouring of $qK_{v:2}$.

# Chapter 5

# Colouring $qK_{v:k}$

In this chapter, we develop some theory for colouring the $q$-Kneser graph $qK_{v:k}$ in general. We first survey some of the work done on the Kneser conjecture before Lovász's result, most notably Garey's and Johnson's determination of the chromatic number of $K_{v:3}$ [6]. We then try to extend the ideas in [6] to colouring the $q$-Kneser graph, $qK_{v:3}$. Finally, we note the usefulness of graph homomorphisms for colouring the Kneser graphs and their $q$-analogues.

## 5.1 The Kneser conjecture before Lovász

Before Lovász's result, only the chromatic numbers of the Kneser graphs $K_{v:2}$ and $K_{v:3}$ had been determined. We will show how these were computed in Lemma 5.1.1 and Lemma 5.1.2, respectively.

**5.1.1 Lemma.** *For $v \geq 4$, the chromatic number of the Kneser graph, $K_{v:2}$, is $\chi(K_{v:2}) = v - 2$.*

*Proof.* We will prove the statement by induction on $v$. The statement is true for $v = 4$ because $K_{4:2}$ is bipartite.

Suppose, for contradiction, that $K_{v:2}$ has a colouring $C$ with $v-3$ colours. If a colour class $C_i$ contains more than three vertices, then it lies in a point pencil. Recolour $K_{v:2}$ so $C_i$ is a point pencil and then remove all vertices in $C_i$ to obtain a colouring of $K_{v-1:2}$ with $v - 4 < (v - 1) - 2$ colours. This contradicts the induction hypothesis, so all colour classes in $C$ contain at most three vertices. Consequently, we must have that

$$3(v - 3) \geq |K_{v:2}| = \frac{v(v - 1)}{2}.$$

This implies that $v^2 - 7v + 18 \leq 0$, which is a contradiction because $v^2 - 7v + 18$ has no real zeros. Thus, the chromatic number of $K_{v:2}$ is $\chi(K_{v:2}) = v - 2$. $\square$

An important step in Lemma 5.1.1 is the assertion that if a colour class contains more than three vertices then it lies in a point pencil. Garey and Johnson show in [6] that if an independent set in $K_{v:3}$ contains greater than $3v-8$ vertices, then it lies in a point pencil. Consequently, we are able to compute the chromatic number of $K_{v:3}$ in a similar manner to that of $K_{v:2}$.

**5.1.2 Lemma.** *The chromatic number of the Kneser graph, $K_{v:3}$, is $\chi(K_{v:3}) = v-4$.*

*Proof.* We prove the statement by induction on $v$. The statement is true for $v=6$ because $K_{6:3}$ is bipartite.

Suppose, for contradiction, that $K_{v:3}$ has a colouring $C$ with $v-5$ colours. As in Lemma 5.1.1, if $C$ contains a colour class with greater than $3v-8$ vertices, then we could obtain a colouring of $K_{v-1:3}$ with $v-6 < (v-1)-4$ colours, which would contradict the induction hypothesis. Consequently, all colour classes of $C$ contain fewer than $3v-8$ vertices, so we must have

$$(v-5)(3(v-8)) \geq |K_{v:3}| = \frac{v(v-1)(v-2)}{6}.$$

This implies that $v^3-6v^2+25v-40 \leq 0$, which is never true for $v \geq 6$. Therefore, the chromatic number of the Kneser graph, $K_{v:3}$, is $\chi(K_{v:3}) = v-4$. □

In Lemma 5.1.1, we asserted that an independent set in $K_{v:2}$ of size greater than three lies in a point pencil and in Lemma 5.1.2 we claimed that an independent set of $K_{v:3}$ with size greater than $3v-8$ lies in a point pencil. Both of these statements are special cases of Theorem 5.1.3, which is due to Hilton and Milner [9].

**5.1.3 Theorem (Hilton-Milner).** *The maximum size of an independent set in $K_{v:k}$ that does not lie in a point pencil is*

$$1 + \binom{v-1}{k-1} + \binom{v-k-1}{k-1}. \qquad \square$$

Unfortunately, we cannot use the Hilton-Milner bound and an argument similar to those of Lemma 5.1.1 and Lemma 5.1.2 to prove the Kneser conjecture because

$$(v-2k+1)\left(1 + \binom{v-1}{k-1} + \binom{v-k-1}{k-1}\right) > \binom{v}{k}$$

for $v=10$ and $k=4$ so we cannot reach a contradiction.

## 5.2 Covering Planes

The proof of Theorem 4.5.1 is similar to the proofs of Lemma 5.1.1 and Lemma 5.1.2 because it uses the fact that the $q^2+q+1$ lines in a plane are the largest independent set in $qK_{v:2}$ that do not lie in a point pencil.

In this section we describe an approach for colouring $qK_{v:3}$ for $v > 6$ that is similar to Theorem 4.5.1, which is due to Chris Godsil. To make the strategy into a proof, we would need a reasonable upper bound on the size of a maximal independent set in $qK_{v:3}$ that does not lie in a point pencil. For $k = 3$, a $q$-analogue of the Hilton-Milner bound, Theorem 5.1.3, would provide such an upper bound, but, to our knowledge, such a result is not yet known. In the next chapter, we will discuss our conjectures for a $q$-analogue of the Hilton-Milner bound for $k = 3$ and their implications for the approach described below.

Define a *blob* to be a maximal independent set in $qK_{v:3}$ that does not lie in a point pencil. Suppose we have a cover of $qK_{v:3}$ with at most $[v-2]$ point pencils and blobs. As in Theorem 4.5.1, we wish to show the number of points in our cover is at least $[v-2]$. Let $C$ denote the set of centres of the point pencils and let $r = |C|$. Let $s$ equal the number of blobs in the cover. Define $\delta := [v-2] - r$, and suppose for a contradiction that $\delta > 0$.

We first derive a lower bound on the number of planes disjoint from $C$. The number of points not in $C$ is

$$[v] - r = [v] - ([v-2] + \delta) = q^{v-1} + q^{v-2} + \delta.$$

Let $x$ be a point not in $C$. There are $[v-1]$ lines on $x$. Since a point in $C$ cannot lie on two lines on $x$, the number of lines on $x$ disjoint from $C$ is at least

$$[v-1] - r = [v-1] - [v-2] + \delta = q^{v-2} + \delta.$$

Let $\ell$ be a line disjoint from $C$. There are $[v-2]$ planes on $\ell$. Since a point in $C$ cannot lie on two planes on $\ell$, there are at least $[v-2] - r = \delta$ planes on $\ell$ that are disjoint from $C$.

It follows that the number of flags $(x, \ell, P)$ consisting of a point $x$ not in $C$, a line $\ell$ on $x$ disjoint from $C$, and a plane $P$ on $\ell$ disjoint from $C$ is at least

$$(q^{v-1} + q^{v-2} + \delta)(q^{v-2} + \delta)\delta.$$

Since each plane disjoint from $C$ lies in exactly $(q^2+q+1)(q+1)$ of these flags, the number of planes disjoint from $C$ is at least

$$\frac{(q^{v-1} + q^{v-2} + \delta)(q^{v-2} + \delta)\delta}{(q^2+q+1)(q+1)}.$$

Let $\beta$ denote the maximum size of a blob. Then the number of blobs in our cover must be at least

$$\delta \geq s \geq \frac{(q^{v-1} + q^{v-2} + \delta)(q^{v-2} + \delta)\delta}{\beta(q^2+q+1)(q+1)}.$$

Since $\delta \geq 1$ by assumption, we obtain the following lower bound on $\beta$:

$$\beta \geq \frac{(q^{v-1} + q^{v-2} + 1)(q^{v-2} + 1)}{(q^2+q+1)(q+1)}.$$

We conjecture that this lower bound is too large and so we can arrive at a contradiction. A proof of this conjecture would show that the number of points in the cover is exactly $[v-2]$, so by Lemma 4.1.1 we would have $\chi(qK_{v:3}) = [v-2]$.

## 5.3 Homomorphisms

We show using graph homomorphisms that to colour all the $q$-Kneser graphs, $qK_{v:k}$, we only need to colour the $q$-Kneser graphs $qK_{2k:k}$ and $qK_{2k+1:k}$ for each $k$.

A homomorphism from a graph $G$ to a graph $H$ is a function $f$ from $V(G)$ to $V(H)$ such that if $u$ and $v$ are adjacent vertices in $G$, then $f(u)$ and $f(v)$ are adjacent in $H$. If there is a homomorphism $f : G \to H$, then $\chi(G) \leq \chi(H)$, so homomorphisms are useful for colouring problems.

We have the following homomorphisms between Kneser graphs. The Kneser graph, $K_{v:k}$, is an induced subgraph of $K_{v+1:k}$, and this embedding is called the *extension map*. For a positive integer $t$, the Kneser graph, $K_{v:k}$, is an induced subgraph of $K_{tv:tk}$, and this embedding is called the *multiplication map*. Stahl found a homomorphism from $K_{v:k}$ to $K_{v-2:k-1}$. By induction, this implies there is a homomorphism from $K_{v:k}$ to $K_{v-2k+2:1}$, which is the complete graph on $v - 2k + 2$ vertices. Consequently, $\chi(K_{v:k}) \leq v - 2k + 2$ so we call Stahl's map the *colouring map*. In [11], Stahl makes a conjecture about necessary and sufficient conditions for the existence of a homomorphism from $K_{v:k}$ to $K_{w:\ell}$. A reformulation of this conjecture, due to Chris Godsil, is that there is a homomorphism from $K_{v:k}$ to $K_{w:\ell}$ if and only if there is a homomorphism from $K_{v:k}$ to $K_{w:\ell}$ that is a composition of extension, multiplication and colouring maps.

The following homomorphisms between the $q$-Kneser graphs are known [3]:

(a) The extension map, embedding $qK_{v:k}$ in $qK_{v+1:k}$.

(b) The field of order $q$ is a subfield of the field of order $q^r$ so we have a subfield map $qK_{v:k} \to q^r K_{v:k}$.

(c) A $k$-space in a $v$-dimensional vector space over $\mathbb{F}_{q^r}$ can be viewed as a subspace of dimension $rk$ in a space of dimension $rv$ over $\mathbb{F}_q$. This leads to a $q$-analogue of the multiplication map, embedding $q^r K_{v:k}$ as an induced subgraph of $qK_{rv:rk}$.

(d) Each $k$-subspace is the row space of a unique $k \times v$ matrix in reduced row echelon form. The subspace spanned by the last $k - 1$ rows of this matrix is a $(k - 1)$-subspace of a $(v - 1)$-dimensional space. Hence we have a homomorphism from $qK_{v:k}$ to $qK_{v-1:k-1}$.

(e) Finally $qK_{v:k}$ is an induced subgraph of $K_{[v]:[k]}$ because we can view $\mathbb{F}_q^v$ as a set of $[v]$ vectors, and a $k$-dimensional subspace of $\mathbb{F}_q^v$ is a subset of $[k]$ vectors from $\mathbb{F}_q^v$.

We do not have a $q$-analogue of Stahl's colouring map because

$$\chi(qK_{3:1}) = q^2 + q + 1 < q^3 + q^2 + q + 1 = \chi(qK_{5:2})$$

so there is no homomorphism from $qK_{5:2}$ to $qK_{3:1}$.

The fourth homomorphism above shows that we have the following homomorphisms

$$qK_{2k+1:k} \to qK_{2k:k-1} \to \ldots \to qK_{k+3:2}.$$

If we could show for each $k$ that $\chi(qK_{2k+1:k}) = [k+2]$, then by Lemma 4.1.1 we would be able to colour all $q$-Kneser graphs excluding those of the form $qK_{2k:k}$. Therefore, colouring the $q$-Kneser graphs in general reduces to colouring the $q$-Kneser graphs $qK_{2k:k}$ and $qK_{2k+1:k}$ for each $k$. We restate this observation:

**5.3.1 Lemma.** *If $\chi(qK_{2k:k}) = q^k + q^{k-1}$ and $\chi(qK_{2k+1:k}) = [k+2]$ for all positive integers $k$, then for $v \geq 2k$ we have $\chi(qK_{v:k}) = [v-k+1]$.*

# Chapter 6

# Hilton-Milner and Kruskal-Katona

We discuss two classical results in extremal set theory, the Hilton-Milner and Kruskal-Katona theorems, and suggest possible vector space analogues of these results.

## 6.1 The Hilton-Milner theorem

The Hilton-Milner theorem [9] was originally stated in terms of intersecting families of sets. For our purposes, we use the following reformulation of the Hilton-Milner problem: What is the size of the largest independent set in the Kneser graph that does not lie in a point pencil? We give two important examples of such independent sets and then give the full statement of the Hilton-Milner theorem, Theorem 6.1.1.

Example 1: Let $V = \{1, ..., v\}$, let $\alpha$ be a $k$-set of $V$, and suppose $1 \notin \alpha$. Define $F_1$ to be the set of all $k$-sets of $V$ that contain 1 and intersect $\alpha$, together with the set $\alpha$. It is easy to see that $F_1$ is an independent set of $K_{v:k}$ that is not contained in a point pencil, and that

$$|F_1| = \binom{v-1}{k-1} - \binom{v-k-1}{k-1} + 1.$$

Example 2: Let $V = \{1, ...v\}$ and define $F_2$ to be the set of all $k$-sets of $V$ that contain at least two elements from $\{1, 2, 3\}$. It is easy to see that $F_2$ is an independent set in $K_{v:k}$. For $k = 2$ we have $F_1 = F_2$, and for k=3 we have $|F_1| = |F_2|$. When $v > 2k$ and $k \geq 4$ we have $|F_1| > |F_2|$.

**6.1.1 Theorem (Hilton-Milner).** *For $v > 2k$, the maximum size of an independent set in the Kneser graph, $K_{v:k}$, that does not lie in a point pencil is*

$$1 + \binom{v-1}{k-1} + \binom{v-k-1}{k-1}.$$

*Moreover, equality is possible only for $F_1$ or $F_2$; the latter occurs only for $k \leq 3$.* □

The restriction $v > 2k$ is necessary because $K_{2k:k}$ is isomorphic to $\binom{2k-1}{k-1}$ vertex-disjoint copies of $K_2$. As a corollary of the Hilton-Milner theorem, we have:

**6.1.2 Corollary.** *The point pencils are the only independent sets that attain equality in the Erdős-Ko-Rado theorem.* □

## 6.2 $q$-Hilton-Milner conjectures

In this section we present our conjectures for a $q$-analogue of the Hilton-Milner theorem for $k = 3$, and their implications for the covering planes approach described in Section 5.2. We first give the $q$-analogues of the independent sets defined in the examples of the previous section.

Example 1: Let $P$ be a plane in $\mathbb{F}_q^v$, and let $p$ be a point not on $P$. The naive analogue of $F_1$ is the set of all planes that contain $p$ and meet $P$, together with the plane $P$. This independent set, however, is not maximal. Let $P' \neq P$ be a plane in the 4-space $p \vee P$ that does not contain $p$. We have $P'$ intersects every plane in the independent set non-trivially so $P'$ can be added to the set. Consequently, define $qF_1$ to be the union of the set of planes that contain $p$ and meet $P$ and the set of planes in the 4-space $p \vee P$. It is not hard to show that $qF_1$ is a maximal independent set and that

$$|qF_1| = \begin{bmatrix} v-1 \\ 2 \end{bmatrix} - q^6 \begin{bmatrix} v-4 \\ 2 \end{bmatrix} + q^3.$$

Example 2: Let $P$ be a plane in $\mathbb{F}_q^v$. Define $qF_2$ to be the set of planes that intersect $P$ in a line, together with the plane $P$. We have that $qF_2$ is a maximal independent set and that

$$|qF_2| = \begin{bmatrix} v-1 \\ 2 \end{bmatrix} - q^6 \begin{bmatrix} v-4 \\ 2 \end{bmatrix} + q^3 = |qF_1|.$$

Conjecture: For $v > 6$, the maximum size of an independent set, $F$, in the $q$-Kneser graph, $qK_{v:k}$, that does not lie in a point pencil is

$$|F| = \begin{bmatrix} v-1 \\ 2 \end{bmatrix} - q^6 \begin{bmatrix} v-4 \\ 2 \end{bmatrix} + q^3.$$

Moreover, equality is possible only for $F = qF_1$ or $F = qF_2$.

When $q = 1$, we have $|qF_1| = |F_1| = 3v - 8 = |F_2| = |qF_2|$ so our conjecture for the q-analogue of the Hilton-Milner theorem reduces to the Hilton-Milner theorem in the case $k = 3$. Furthermore, if our conjecture is correct, then we have the desired contradiction in the covering planes argument of Section 5.2 since

$$\beta \geq \frac{(q^{v-1} + q^{v-2} + 1)(q^{v-2} + 1)}{(q^2 + q + 1)(q + 1)} \geq |qF_1|.$$

## 6.3 The Kruskal-Katona Theorem

The Kruskal-Katona theorem is an important theorem in extremal set theory that implies the Erdős-Ko-Rado theorem. Although the Kruskal-Katona theorem is not directly related to our research on colouring $q$-Kneser graphs, we include it here as part of a broader approach of considering vector space analogues of problems in extremal set theory.

Let $X = \{1, 2, \ldots, n\}$, and let $X^k$ denote the set of $k$-subsets of $X$. Suppose $\mathcal{A} \subseteq X^k$ is a family of $k$-subsets of $X$ where $1 \leq k \leq n-1$. Define the *shadow* $\partial\mathcal{A}$ of $\mathcal{A}$ to be

$$\partial\mathcal{A} := \{B \in X^{k-1} | B \subseteq A \in \mathcal{A}\}$$

The Kruskal-Katona theorem addresses two questions. First, given $n$, $k$, and $|\mathcal{A}|$, what is the minimum size of the shadow of $\mathcal{A}$? Second, what is the structure of families $\mathcal{A}$ whose shadows have the minimum size? We provide an example.

EXAMPLE: The Kruskal-Katona problem for $n = 8$, $k = 2$ and $|\mathcal{A}| = 7$ can be stated as follows: Choose seven edges of the complete graph $K_8$ so that they are incident with the fewest number of vertices possible. One choice for the seven edges is

$$\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 5\}.$$

The strategy for choosing the edges is to keep the largest vertex as small as possible. Given $|\mathcal{A}|$, let $v$ be the least integer such that $|\mathcal{A}| \leq \binom{v}{2}$. The minimum size for the shadow of $\mathcal{A}$ when $k = 2$ is $|\partial\mathcal{A}| = v$.

One simple lower bound on $|\partial\mathcal{A}|$ is given by the LYM inequality, Lemma 6.3.1.

**6.3.1 Lemma.**

$$|\partial\mathcal{A}| \geq \frac{|\mathcal{A}|}{\binom{n}{k}} \binom{n}{k-1}$$

*Proof.* We count ordered pairs $(A, B)$ where $A \in \mathcal{A}$, $B \in \partial\mathcal{A}$, and $B \subset A$. Every set $B \in \partial\mathcal{A}$ is contained in $n - k + 1$ elements of $X^k$, and thus in at most $n - k + 1$ elements of $\mathcal{A}$. Every set $A \in \mathcal{A}$ contains $k$ elements of $\partial\mathcal{A}$. Consequently,

$$|\partial\mathcal{A}|(n - k + 1) \geq \{(A, B) | A \in \mathcal{A}, B \in \partial\mathcal{A}, B \subseteq A\} = |\mathcal{A}|k$$

and so

$$|\partial\mathcal{A}| \binom{n}{k} \geq \frac{|\mathcal{A}|k}{n - k + 1} \binom{n}{k} = |\mathcal{A}| \binom{n}{k-1}.$$

Dividing both sides by $\binom{n}{k}$ gives the desired inequality. □

### 6.3.1 The Colex Order

Recall that in the example above we chose edges in a way that kept the largest vertex as small as possible. This strategy can be generalized to build families of $k$-sets whose shadows have the minimum size. To formalize this, we define the *colex order* on $X^k$. Write $A, B \in X^k$ as $A = \{a_1, a_1, \dots, a_k\}$, $B = \{b_1, b_2, \dots, b_k\}$ where $a_1 < a_2, \dots, < a_k$ and $b_1 < b_2, \dots, < b_k$. We say $A < B$ in the colex order if $A \neq B$ and for $s = \max\{t | a_t \neq b_t\}$, we have $a_s < b_s$. The colex order defines a total order on $X^k$. Observe that in the example, we picked the edges in colex order. The Kruskal-Katona theorem asserts that the shadow of $\mathcal{A} \subseteq X^k$ is at least as large as the shadow of the first $|\mathcal{A}|$ sets in the colex order on $X^k$.

We now show how to find the size of the shadow of the first $m$ sets in the colex order on $X^k$. Proofs of the following theorems appear in [1]. Theorem 6.3.3 is due to Kruskal and Katona.

**6.3.2 Theorem.** *For any positive integers $m \geq 1$ and $k \geq 1$, there is a unique solution to the equation*

$$m = \binom{m_k}{k} + \binom{m_k - 1}{k-1} + \dots + \binom{m_h}{h}$$

*with integers $m_k > m_{k-1}, \cdots, > m_h \geq h \geq 1$. Furthermore, the size of the shadow of the first $m$ sets in the colex order on $X^k$ is*

$$\partial^{(k)}(m) := \sum_{j=h}^{k} \binom{m_j}{j-1}. \qquad \square$$

**6.3.3 Theorem (Kruskal-Katona).** *Let $k$ be a positive integer and let $\mathcal{A} \subseteq X^k$. Then*

$$|\partial \mathcal{A}| \geq \partial^{(k)}(|\mathcal{A}|) \tag{6.3.1}$$

*i.e. the shadow of $\mathcal{A}$ is at least as large as the shadow of the first $|\mathcal{A}|$ sets in the colex order on $X^k$. If $|\mathcal{A}| = \binom{m_k}{k}$ for some $m_k \geq k$ then equality holds in (6.3.1) if and only if $\mathcal{A} \simeq \{1, 2, \dots, m_k\}^k$.* $\square$

### 6.3.2 Kruskal-Katona implies EKR

We now give Daykin's proof of the Erdős-Ko-Rado theorem from the Kruskal-Katona theorem.

**6.3.4 Theorem.** *Suppose $\mathcal{A}$ is an intersecting family of $k$-subsets of an $n$-set, $X$, where $n > 2k$. Then*

$$|\mathcal{A}| \leq \binom{n-1}{k-1}$$

*Moreover, equality holds if and only if $\mathcal{A}$ consists of the $k$-subsets of $X$ that contain a particular point.*

*Proof.* Let $\mathcal{B} = \{X - A | A \in \mathcal{A}\} \subseteq X^{n-k}$. Since $\mathcal{A}$ is an intersecting family, for $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we have $A \not\subset B$. Let $\partial^t$ denote the operation of taking shadows $t$ times. Since $n > 2k$, $\partial^{n-2k}$ is defined and we have $\partial^{n-2k}\mathcal{B} \subseteq X^k$ and $\partial^{n-2k}\mathcal{B} \cap \mathcal{A} = \emptyset$. Hence,

$$|\partial^{n-2k}\mathcal{B}| + |\mathcal{A}| \leq \binom{n}{k}. \tag{6.3.2}$$

Since $|\mathcal{A}| = |\mathcal{B}|$, if $|\mathcal{B}| \leq \binom{n-1}{k-1}$, then $|\mathcal{A}| \leq \binom{n-1}{k-1}$, and the Erdős-Ko-Rado bound holds. If $|\mathcal{B}| \geq \binom{n-1}{k-1}$, then by Theorem 6.3.3

$$|\partial\mathcal{B}| \geq \binom{n-1}{n-k-1}, |\partial^2\mathcal{B}| \geq \binom{n-1}{n-k-2}, \ldots, |\partial^{n-2k}\mathcal{B}| \geq \binom{n-1}{k}. \tag{6.3.3}$$

By (6.3.2), this implies

$$|\mathcal{A}| \leq \binom{n}{k} - \binom{n-1}{k} = \binom{n-1}{k-1},$$

so the Erdős-Ko-Rado bound holds.

Now suppose $|\mathcal{A}| = \binom{n-1}{k-1}$. We have $|\mathcal{B}| = \binom{n-1}{k-1} = \binom{n-1}{n-k}$. We show that all the inequalities in (6.3.3) are tight. By (6.3.2), $|\partial^{n-2k}\mathcal{B}| = \binom{n-1}{k}$. Theorem 6.3.2 gives unique integers $m_{k+1} > m_k, \cdots, > m_h \geq h \geq 1$ such that

$$|\partial^{n-2k-1}\mathcal{B}| = \sum_{j=h}^{k+1} \binom{m_j}{j}.$$

Moreover,

$$|\partial^{n-2k}\mathcal{B}| = \sum_{j=h}^{k+1} \binom{m_j}{j-1}.$$

Since $|\partial^{n-2k}\mathcal{B}| = \binom{n-1}{k}$, we must have $m_{k+1} = n-1$ and $h = k+1$. Consequently, $|\partial^{n-2k-1}\mathcal{B}| = \binom{n-1}{k+1}$. By induction, $|\partial\mathcal{B}| = \binom{n-1}{n-k-1} = \binom{n-1}{n-k}$, so by Theorem 6.3.3, $\mathcal{B} \simeq \{1, 2, \ldots, n-1\}^{n-k}$. Consequently,

$$\mathcal{A} \simeq \{S \in \{1, 2, \ldots, n\}^k | n \in S\}.$$

so the Erdős-Ko-Rado theorem is proved. □

We now formulate a $q$-analogue of the Kruskal-Katona problem. Let $\mathbb{V} = \mathbb{F}_q^n$ and let $\mathbb{V}^k$ denote the set of $k$-subspaces of $\mathbb{V}$. Let $\mathcal{A} \subset \mathbb{V}^k$ be a family of $k$-subspaces of $\mathbb{V}$. Define the $q$-shadow of $\mathcal{A}$, $\partial_q\mathcal{A}$ to be

$$\partial_q\mathcal{A} = \{B \in \mathbb{V}^{k-1} | B \subseteq A \in \mathcal{A}\}$$

Given $n$, $k$, and $|\mathcal{A}|$, what is the minimum size of the $q$-shadow of $\mathcal{A}$? Moreover, what is the structure of families $\mathcal{A}$ whose shadows have the minimum size? To our knowledge, no $q$-analogue of the Kruskal-Katona theorem is known. It would be interesting if there is a $q$-analogue of the Kruskal-Katona theorem, and if Daykin's argument could be generalized to prove $q$-EKR, Theorem 2.3.1.

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