Character Polynomials and Lagrange Inversion

by

Amarpreet Rattan

A thesis
presented to the University of Waterloo
in fulfilment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, 2005

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

In this thesis, we investigate two expressions for symmetric group characters: Kerov's universal character polynomials and Stanley's character polynomials. We give a new explicit form for Kerov's polynomials, which exactly evaluate the characters of the symmetric group scaled by degree and a constant. We use this explicit expression to obtain specific information about Kerov polynomials, including partial answers to positivity questions. We then use the expression obtained for Kerov's polynomials to obtain results about Stanley's character polynomials.

Acknowledgements

I would like to thank Caroline Colijn. Our many great times together made my whole PhD. experience that much better. Without her love and support the making of this thesis would have been a very difficult experience.

I would like to thank various people for scientific support. I had many informative discussions with Chris Godsil and David Wagner, both of whom also gave useful comments as examiners. I would also like to thank my other examiners Philippe Biane, David Jackson and Andu Nica for their comments. As for technical support, I would like to thank Peter Colijn, James Muir and Simon Alexander.

A special thanks goes to John Irving, my mathematical travelling partner. Our many conference adventures will never be forgotten (hopefully for good reasons rather than bad). In addition, John and I had many fruitful conversations concerning the material in Chapter 4.

I would also like to thank my family: my parents, my brother Gurpreet, his wife Melissa Hagen and their new daughter Maia, for the many days and nights I spent in a welcoming, warm environment, typesetting this thesis.

Finally, I would especially like to thank my PhD. supervisor Ian Goulden. Always relaxed, encouraging and available, Ian invariably made me feel better about my work after one of our meetings. His excellent advice and great ideas helped me with many difficult mathematical concepts and his tireless reviews of theorems and drafts this thesis will always be greatly appreciated.

Contents

List of Figures

Index of Notation

Notation	Description	
H_λ	the product of hooks	13
h_λ	complete symmetric function indexed by λ	11
$H_\omega(z)$	moment generating series of continuous Young diagram ω	23
$h(u)$	hook length of a box u in a Young diagram	12
$H_{\mathbf{p};\,\mathbf{q}}(z)$	moment generating series for Stanley's character polynomial	73
J_n	Jucys-Murphy elements	29
K_λ	central element of $\mathbb{C}[\mathfrak{S}_n]$ indexed by λ	8
$\lambda \vdash d$	λ an integer partition of d	5
Λ	ring of symmetric functions	10
$\Lambda(n)$	ring of symmetric polynomials in n variables	10
Λ^*	ring of shift symmetric functions	64
$\Lambda^*(n)$	ring of shift symmetric polynomials in n variables	64
λ_μ	fixed permutation in C_μ	64
$[\lambda]$	the representation of $\mathfrak{S}_d$ associated with the partition $\lambda \vdash d$	7
$\ell(\lambda)$	number of parts in the partition λ	5
$\hat{m}_\lambda$	substitution of $1, \ldots, k-1$ into monomial symmetric function	42
m_λ	monomial symmetric function indexed by λ	10
$m_i(\lambda)$	number of parts of λ equal to i	9
$\mathcal{M}_k$	k^{th} moment of the Jucys-Murphy element	29
$(n)_k$	the falling factorial $n(n-1)\cdots(n-k+1)$	8
$[n]$	the set $\{1, 2, \ldots, n\}$	7
1^n	the partition of n with n parts equal to 1	7

Notation	Description	
$\phi_{\mathbf{p};\,\mathbf{q}}(z)$	generating series $\phi_{p\times q}(z)$ used in Lagrange inversion of Stanley's character polynomial	73
$\phi(x)$	generating series used in proof of main theorem of Chapter 3	58
$\Phi(x,u)$	generating series used in proof of main theorem of Chapter 3	58
$\Phi_i(x)$	generating series used in proof of main theorem of Chapter 3	58
$\mathbb{P}$	the set of positive integers	66
p_λ	power sum symmetric function indexed by λ	11
$p^\sharp_\mu$	p-sharp shift symmetric function indexed by μ	65
$P_\lambda(z)$	generating series in main theorem of Chapter 3	39
$p\times q$	the shape with p parts, all equal to q	64
$\mathbf{p}\times\mathbf{q}$	general partition used in Stanley's polynomials	67
$\mathrm{RTab}(\mu)$	reverse tableau of shape μ	66
$R_\omega(z)$	free cumulant generating series of a Young diagram ω	22
$R_i(\omega)$	free cumulant evaluated at ω	22
$\mathcal{R}_k$	k^{th} free cumulant of the Jucys-Murphy element	29
s_λ	Schur symmetric function indexed by λ	11
SSYT	semi-standard Young tableau	10
$\mathrm{sh}(\alpha)$	shape of a sequence α	9
$\Sigma_{k,2n}$	graded pieces of Kerov's polynomials	39
Σ_k	the k^{th} Kerov polynomial	22
$\mathrm{sign}(\lambda)$	sign of a partition λ	29
s^*_μ	Schur shift symmetric function indexed by μ	65
$s_\lambda(1^p)$	substituting 1 for the variables $x_1,\ldots,x_p$ and 0 for x_i, $i>p$, into the Schur function	13
SYT	standard Young tableau	10
$S(j,i)$	Stirling numbers of the second kind	43
$\mathfrak{S}_n$	the symmetric group on n letters	7

Notation	Description	
ϑ	substitution operator $R_i \mapsto u^i R_i$	59
$T(u)$	value assigned to the box u by the tableau T	66
$\widehat{\chi}_\omega$	the normalized character indexed by ω	8
$\widetilde{\chi}_\omega$	the central character indexed by ω	7
$\mathbf{x}^\alpha$	the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ when α is a sequence of length n	9
X^λ	the representation of $\mathfrak{S}_d$ associated with the partition $\lambda \vdash d$	7
z_λ	$1^{m_1(\lambda)} m_1(\lambda)! \cdots \lvert\lambda\rvert^{m_{\lvert\lambda\rvert}(\lambda)} m_{\lvert\lambda\rvert}(\lambda)!$	12
$\mathcal{Z}_\pi$	sum of a product of transpositions	30
$[z^{-1}]_\infty f(z)$	the coefficient of $1/z$ in $f(z)$ when $f(z)$ is expanded in powers of $1/z$	16
$[z^n]\ f(z)$	the coefficient of z^n in $f(z)$ when $f(z)$ is expanded in powers of z	14

Chapter 1

Introduction

Finding expressions for group characters is a very old task. In the case of the symmetric groups, much is known about their characters. In fact, there are well known combinatorial algorithms for computing the characters of the symmetric group, the Murnaghan-Nakayama rule (see Theorem 2.3.1) being such an example. As well as the Murnaghan-Nakayama rule, the ring of symmetric functions provides a calculus for computing symmetric group characters. Unfortunately, in the case of symmetric functions, one often needs to know the characters in order to use them effectively for computational purposes. Therefore, as a tool for computing characters, they are not as effective as one might hope. In the case of the Murnaghan-Nakayama rule, when the symmetric group is large, its use becomes quite cumbersome and other methods are needed to obtain information about symmetric group characters. In Kerov [18, 19], Kerov and Vershik [20] the authors recognize the shortcomings of such methods and approach the problem from a probabilistic point of view. Thus, instead of trying to compute symmetric group characters for large groups exactly, they try to obtain asymptotic information about characters.

The probability used is not classical probability; the authors use the theory of *free probability*, which has connections to both functional analysis and combinatorics. This use of free probability in the asymptotics of the symmetric group characters was studied by Biane [1, 2, 3], who obtained some remarkable asymptotic results. More specifically, Biane has an asymptotic expression for characters in terms of quantities called *free cumulants*, which are, very briefly and superficially, a sequence of functions $R_2, R_3, \ldots$ mapping Young diagrams to the complex numbers. Biane proves that $\widehat{\chi}_\omega(k\ 1^{n-k})$, which is the symmetric group character associated with an arbitrary partition ω and evaluated at a k-cycle, and scaled by degree and a constant, gets asymptotically close to $R_{k+1}(\omega)$. In fact, he proves a more general result, giving asymptotic results about $\widehat{\chi}_\omega(\sigma)$, where σ is an arbitrary shape (see Example 3.2.1).

The question arises of whether there is a useful expression in terms of $R_i(\omega)$ that *exactly* evaluates the character $\widehat{\chi}_\omega(k\ 1^{n-k})$. Such an expression would probably not give any additional information about the asymptotics of characters but, as we shall see, is interesting in its own right. Biane [1] (more correctly, Biane and Kerov, as Biane attributes this result to Kerov, who gave the result at a talk at an IHP conference) answered this question in the affirmative; the expressions for characters they found are known as *Kerov's universal character polynomials*. We shall see these expressions have some very remarkable algebraic and combinatorial properties.

The first property concerning these expressions for $\widehat{\chi}_\omega(k\ 1^{n-k})$ is that they are, indeed, polynomials (this is not obvious at the outset). The second, and by far more surprising, is that these polynomials are *independent of* ω (hence the adjective "universal"). At this point, an example is useful. The fifth Kerov polynomial is

$$\Sigma_5 = R_6 + 15\,R_4 + 5\,{R_2}^2 + 8\,R_2. \tag{1.1}$$

As mentioned above, the R_i can be thought of as functions from, in this example, partitions of $n \geq 5$ to complex numbers. Evaluating this at any such partition, say $\omega = (3\,2\,2)$, a partition of 7, we obtain

$$\begin{aligned}\widehat{\chi}_\omega(5\,1\,1) &= \Sigma_5(\omega)\\ &= R_6(\omega) + 15\,R_4(\omega) + 5\,R_2(\omega)^2 + 8\,R_2(\omega).\end{aligned}$$

Given that the $R_i(\omega)$ are readily obtained, we see that the above expression evaluates the character $\widehat{\chi}_\omega(5\,1\,1)$. For arbitrary ω, we emphasize, again, that the Kerov polynomial Σ_5 above is independent of ω and n.

The only expression known until now for Kerov's polynomials is an implicit one (due to Biane [1, Theorem 5.1]), which can be derived from a seemingly intractable formula of Frobenius. Other results have been obtained, for example, coefficients of some specific terms have been found, but otherwise these polynomials are somewhat of a mystery. They are the subject of the first half of this thesis. Here, we give a new explicit expression for Kerov's polynomials. The expression is obtained by using Biane's expression with Lagrange inversion, and considering the graded pieces of Kerov's polynomials. We use this explicit expression to obtain new results for Kerov's polynomials, in particular giving affirmative partial answers to some positivity conjectures; namely, it is conjectured by Biane and Kerov that the coefficients of Kerov's polynomials are all positive. Further, we use our explicit expression to reprove some results.

In the second half of this thesis, we discuss another polynomial expression for characters which was introduced by Stanley [28], and which we call *Stanley's character polynomials*. As an example, suppose that $p \times q$ is a partition of n with p parts, all equal to q. Then, for any partition μ of k, where $k \leq n$, Stanley proved that

$$\widehat{\chi}_{p\times q}(\mu\, 1^{n-k}) = (-1)^k \sum_{\substack{u,v \\ uv=\lambda_\mu}} p^{\ell(u)}(-q)^{\ell(v)}, \tag{1.2}$$

where λ_μ is any fixed permutation in the conjugacy class μ in the symmetric group on k letters, and $\ell(u)$ is the number of cycles in u. Stanley also obtained formulas for general shapes (that is, Stanley considers shapes more general than the rectangle $p \times q$) and the expressions are, predictably, more complex.

We shall see that there are connections between Kerov's character polynomials and Stanley's character polynomials. Both Kerov polynomials and Stanley's polynomials are based on the same formula of Frobenius. To make use of Frobenius' expression, we make heavy use of Lagrange's Inversion Theorem in the treatment of both Kerov's and Stanley's polynomials. Moreover, we shall see that we can use results concerning Kerov's polynomials and apply them to Stanley's polynomials. In particular, we are able to answer some positivity questions concerning Stanley's polynomials.

The thesis is organized as follows. In Chapter 2 we briefly review some fundamental concepts (representations, symmetric functions, Lagrange inversion) that many readers may already know, so we include no proofs (except for Section 2.5 where a suitable reference was not found). This chapter may, therefore, be omitted by those who feel comfortable with the background material. Chapter 3 deals exclusively with Kerov's character polynomials. In Section 3.1 and 3.2 we give the background and motivation for Kerov's polynomials, including a very brief discussion about the asymptotics of characters. Although this thesis offers no new results in this direction, it seems appropriate to provide some details about this motivating aspect of Kerov's polynomials. In Section 3.3.2, we start from some fairly basic expressions and begin to derive our explicit expression for Kerov's polynomials. We include the formula of Frobenius and the proof given by Macdonald [21, Section I.7, Exercise 6], and give an essentially self-contained derivation of Kerov's polynomials. In this way, we provide a complete expository account of the basic material leading up to Kerov's polynomials. Finally, in Section 3.4 we state the main result of this thesis, in Theorem 3.4.1, which gives an explicit expression for Kerov's polynomials. We also give some equivalent forms of the main theorem in Theorems 3.4.2 and 3.4.3, which are included since they help with later computations.

Our explicit expression for Kerov's polynomials is quite complicated, but we are able to show that a lot of useful information can be extracted from this expression in spite of its complexity, including some positivity results which are presented in Section 3.5. We postpone the proof of the main result until the end of Chapter 3, in Section 3.6.

In Chapter 4 we study Stanley's character polynomials. For Stanley's rectangular case, given in (1.2) above, we give a new proof in Section 4.1.2. The proof given here is slightly simpler than Stanley's proof and, more importantly, exploits in a new way an already known connection between *shift symmetric functions* and scaled characters $\widehat{\chi}_\omega(\mu)$. We then consider Stanley's character polynomials in the general case, and interpret them as a specialization of Kerov's polynomials. This enables us to use some of the results in Chapter 3 to obtain results about Stanley's polynomials. In particular, we are able to give some new positivity results.

We conclude the thesis with a theorem that gives a strong connection between the positivity of Kerov's polynomials and the positivity of Stanley's polynomials; that is, we show the former implies the latter. In particular, we introduce a *C-expansion* for Kerov's polynomials and it is immediate that the positivity of this C-expansion does imply positivity of Kerov's polynomials in the so called *R-expansion* given in (1.1). Furthermore, we show C-positivity of Kerov's polynomials does imply positivity for Stanley's polynomials. As we shall see, most of our results here concern the C-expansion, as they greatly simplify our expressions. Therefore, it is this author's belief that these C-expansions are the most likely to yield further information about Kerov's polynomials (and, consequently, Stanley's polynomials).

We make a final note about the results found in this thesis. In general, we label theorems, lemmas, proofs, *etc.* by the authors who gave them. When no label is given the results are new; however, in Chapter 3 most such results also appear in Goulden and Rattan [12].

Chapter 2

Fundamental Concepts

In this chapter we review the necessary terminology for the thesis. This chapter may be omitted by those who feel comfortable with the material. The notation in Sections 2.1 and 2.2, on representation theory and symmetric functions, is consistent with Macdonald [21] and Sagan [24], while the notation in Section 2.4 is consistent with Goulden and Jackson [8] and Stanley [27].

2.1 Partitions, Group Representations and the Symmetric Group

A *partition* is a weakly ordered list of positive integers $\lambda = \lambda_1 \lambda_2 \dots \lambda_k$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$. The integers $\lambda_1, \dots, \lambda_k$ are called the *parts* of the partition λ, and we denote the number of parts (often called the *length* of a partition) by $\ell(\lambda) = k$. If $\lambda_1 + \dots + \lambda_k = d$, then λ is a partition of d, and we write $\lambda \vdash d$. We denote by $\mathcal{P}$ the set of all partitions, including the single partition of 0 (which has no parts).

Let GL_d be the general linear group of dimension d (the set of all invertible $d \times d$ matrices) over the field $\mathbb{C}$. Given any group G, a *matrix representation of* G is a group homomorphism

$$X : G \longrightarrow GL_d,$$

or equivalently, X satisfies

1. $X(e) = I$, where e is the identity in G and I is the identity matrix in GL_d.
2. $X(gh) = X(g)X(h)$ for all $g, h \in G$.

The parameter d is called the *dimension* of the representation. We may also write $GL(V)$ for GL_d, where V is a d-dimensional vector space. Equivalently, we can use the language of modules to describe a representation. That is, a vector space V is a G-*module* if there is a multiplication $g \cdot v$ of elements in V by elements of G such that

1. $g \cdot v \in V$,
2. $g \cdot (cv + dw) = cg \cdot v + dg \cdot w$,
3. $(g \cdot h) \cdot v = g \cdot (h \cdot v)$,
4. $e \cdot v = v$, where e is the identity of G,

for all $g, h \in G$, $v, w \in V$ and scalars c, d.

If V is a G-module then W is a called a *submodule* of V if W is a subspace of V and W is a G-module. The module V is called *irreducible* if the only submodules of V are trivial subspaces. Furthermore, G-modules V and W are *equivalent* if there is a vector space isomorphism that commutes with the action of G on V and W, *i.e.*, if there exists an isomorphism $\theta : V \to W$ such that $\theta(g \cdot v) = g \cdot \theta(v)$.

For any representation X of G, the trace of the matrices $X(g)$ holds much of the information of the representation. Accordingly, define the *character* of a representation X to be the map $\chi : G \to \mathbb{C}$ given by $\chi(g) = \text{trace}(X(g))$. Characters are called irreducible, equivalent, *etc.*, if their associated representations have these properties. Also, the *degree* of a character is the dimension of the associated representation, which is clearly $\chi(e)$, where e is the identity element of the group.

The study of group characters can shed a lot of light on group representations. One can define an inner product on the space of group characters. In this space, a group character χ is irreducible if and only if the inner product of χ with itself is 1. Indeed, the character of a representation embodies much of the representation itself.

The group that we are most interested in is the symmetric group on n letters, denoted by $\mathfrak{S}_n$. We use either the standard cycle representation of a permutation (writing a permutation as the product of cycles), or write a permutation as a word.

Example 2.1.1. The simplest representation is the *trivial representation*. This is the representation

$$X : G \longrightarrow GL_1$$

such that $X(g) = [1]$ for all $g \in G$. □

Example 2.1.2. The *permutation representation* is obtained when a group G acts on a set S. We take the vector space $\mathbb{C}[S] = c_1 s_1 + c_2 s_2 + \cdots + c_n s_n$ where $c_i \in \mathbb{C}$ and

$S = \{s_1, s_2, \ldots, s_n\}$. Letting $v = c_1 s_1 + c_2 s_2 + \cdots + c_n s_n$, then $X(g)$ is defined as the matrix associated with the linear transformation $g \cdot v = c_1 g \cdot s_1 + c_2 g \cdot s_2 + \ldots + c_n g \cdot s_n$, where $g \cdot s_i$ is g acting on s_i, with respect to the basis $(s_1, s_2, \ldots, s_n)$. □

Example 2.1.3. The *left regular representation* is similar to the permutation representation and is one of the most important representations. In this case we take the *group algebra* $\mathbb{C}[G]$ (endowed with the obvious product), and an element $g \in G$ acts on a $v = c_1 g_1 + c_2 g_2 + \ldots + c_n g_n$ by $g \cdot v = c_1 g \cdot g_1 + c_2 g \cdot g_2 + \ldots + c_n g \cdot g_n$ where $g \cdot g_i$ is the usual multiplication in G. □

We now state some fundamental theorems of representation theory.

Theorem 2.1.4 (Maschke). *If V is a G-module then V is the direct sum of irreducible modules.*

Theorem 2.1.5. *The number of inequivalent irreducible representations of a group G is equal to the number of conjugacy classes.*

Theorem 2.1.6. *In the group algebra $\mathbb{C}[G]$, every irreducible representation appears with multiplicity equal to its dimension.*

Thus, we see that the irreducible representations play a fundamental role in the group algebra. Now define $[n] = \{1, 2, \ldots, n\}$. The *symmetric group on n letters*, denoted $\mathfrak{S}_n$, is the set of bijections from $[n]$ to itself. For the symmetric group $\mathfrak{S}_n$ the conjugacy classes can be naturally indexed by the partitions of n. Therefore, the number of inequivalent representations of $\mathfrak{S}_n$ is the number of partitions of n, and we write the conjugacy class associated with the partition λ as C_λ. Representations of $\mathfrak{S}_n$ are therefore indexed by partitions, and we write X^λ or $[\lambda]$ for the representation associated with the partition λ. Similarly, for characters we write χ^λ. Furthermore, since characters are class functions, we replace $\chi^\lambda(g)$ by $\chi^\lambda(\mu)$ when g belongs to the conjugacy class μ. We also use the notations χ^λ_μ and $\chi_\lambda(\mu)$ in place of $\chi^\lambda(\mu)$; each is the character associated with the partition λ, evaluated at the conjugacy class μ. We denote by 1^n the partition of n with n parts equal to 1 and, therefore, the conjugacy class C_{1^n} is the conjugacy class containing only the identity element. Thus, $\chi_\lambda(1^n)$ is the degree of the character χ_λ.

Various scalings of irreducible symmetric group characters have been considered in the recent literature. The *central character* is given by

$$\widetilde{\chi}_\omega(\lambda) = |\mathcal{C}_\lambda| \frac{\chi_\omega(\lambda)}{\chi_\omega(1^n)}.$$

For the symmetric group, we often denote the degree of χ_ω by f^ω. For results about the central character, see, for example, Corteel et al. [4], Frumkin et al. [6], Katriel [17]. Related to this scaling, for the conjugacy class $\mathcal{C}_{k\,1^{n-k}}$ only, is the *normalized character*, given by

$$\widehat{\chi}_\omega(k\,1^{n-k}) = (n)_k \frac{\chi_\omega(k\,1^{n-k})}{\chi_\omega(1^n)} = k\widetilde{\chi}_\omega(k\,1^{n-k}), \tag{2.1}$$

where $(n)_k = n(n-1)\cdots(n-k+1)$ is the falling factorial, with $(n)_0 = 1$ (we also allow n to be an indeterminate). This character evaluation is the central object of this thesis.

Example 2.1.7. The following are some operations on representations.

a. Induction: For any representation X of a subgroup H contained in a group G, the representation $X\uparrow_H^G$ is the representation of G *induced* by X to G.

b. Restriction: For any representation X of a group G containing a subgroup H, the representation $X\downarrow_H^G$ is the representation of H known as the *restriction* of X to H.

c. Kronecker product: For the representations X and Y of G, we denote by $X \otimes Y$ their *Kronecker product*.

d. Outer product: For the repsentations X and Y of G, we denote by $X \circ Y$ their *outer product*. □

We refer the reader to Sagan [24] for the definitions of these fundamental operations.

2.1.1 The Group Algebra of the Symmetric Group

In the symmetric group $\mathfrak{S}_n$, as we discussed above, conjugacy classes are indexed by partitions of n. Let the *cycle type* of a permutation σ be the partition whose parts are the lengths of the cycles in σ. In terms of cycle type of permutations, it is easy to describe the conjugacy classes of $\mathfrak{S}_n$; the *conjugacy class* C_λ consists of all members σ of $\mathfrak{S}_n$ with cycle type λ. The centre of $\mathbb{C}[\mathfrak{S}_n]$ is spanned by the elements

$$K_\alpha = \sum_{\sigma \in C_\alpha} \sigma.$$

The $(K_\alpha)_{\alpha\vdash n}$ form a linear basis for the centre of $\mathbb{C}[\mathfrak{S}_n]$. A natural task is to determine the structure constants of this basis are, *i.e.*, to determine the numbers $c^{\gamma}_{\alpha,\beta}$ such that

$$K_\alpha K_\beta = \sum_{\gamma} c^{\gamma}_{\alpha,\beta} K_\gamma. \tag{2.2}$$

This task, it turns out, is very difficult and has been heavily studied; see Corteel et al. [4], Goulden [7], Goulden and Jackson [9, 10], Goulden and Pepper [11], Goulden and Yong [13], Irving [15].

Since the group algebra is finite, its centre has a basis $\{ F_\alpha \mid \alpha \vdash n\}$ of orthogonal idempotents with

$$F_\alpha = \frac{f^\alpha}{n!} \sum_{\theta\vdash n} \chi_\alpha(\theta) K_\theta.$$

Furthermore, the previous equation can be inverted to obtain

$$K_\alpha = |C_\alpha| \sum_{\theta\vdash n} \frac{\chi_\theta(\alpha)}{f^\theta} F_\theta.$$

Finally, determining the product $K_\alpha K_\beta$ through the orthogonal idempotents we have

$$[K_\gamma]K_\alpha K_\beta = \frac{|C_\alpha||C_\beta|}{n!} \sum_{\theta\vdash n} \frac{1}{f^\theta} \chi_\theta(\gamma)\chi_\theta(\alpha)\chi_\theta(\beta). \tag{2.3}$$

2.2 Symmetric Functions

Letting $m_i(\lambda)$ to be the number of parts of a partition $\lambda \vdash n$ equal to i, we often rewrite $\lambda = 1^{m_1(\lambda)} 2^{m_2(\lambda)} \cdots n^{m_n(\lambda)}$. A sequence α of non-negative integers is said to have *shape* λ if its non-increasing rearrangement is λ, and we use $\mathrm{sh}(\alpha)$ to mean the shape of α. Let $\mathbf{x} = x_1, x_2, \ldots$ and for any sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$, we denote by $\mathbf{x}^\alpha$ the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots$. For the rest of this section, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \vdash n$.

A *tableau of shape* λ, *Young diagram of* λ or a *Young tableau of shape* λ is an array of *boxes* (i, j), where $1 \le i \le \ell$ and $1 \le j \le \lambda_i$. Visually, as with matrices, as i increases we move down the array and as j increases we move to the right (see Figure 2.1). This way of visualizing tableaux is often known as the "English convention". Some authors (most notably Francophones, hence we call the following the "French convention") prefer the use of coordinate geometry; a tableau is an array of boxes (i, j) where i increases left to right, j increases up, and $1 \le j \le \ell$ and $1 \le i \le \lambda_j$. As we will be most often using the English convention, we will specify the convention only when we decide to switch to the French one. A *standard Young* *tableau*, or an SYT, is a filling of the boxes of a tableau of shape λ with the numbers $1, 2, \ldots, n$, with rows and columns strictly increasing. A *semi-standard Young tableau*, or an SSYT, is a filling of the boxes of shape λ with positive integers such that rows are weakly increasing and columns strictly increasing. The following

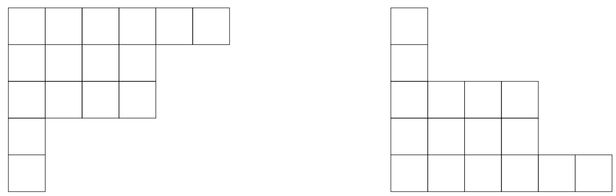

Figure 2.1: The tableau of shape $(6, 4, 4, 1, 1)$ drawn in the English convention (left) and French convention (right).

theorem connects SYT to characters of the symmetric group.

Theorem 2.2.1. *The number of SYT of shape λ is the degree f^λ of χ_λ.*

The algebra of symmetric functions is defined in the following way. Let $\Lambda(n)$ be the algebra of formal series symmetric in the n variables $x_1, x_2, \ldots, x_n$. Define a morphism from $\Lambda(n+1) \to \Lambda(n)$ by setting $x_{n+1} = 0$ in a symmetric function. Finally, let Λ, the algebra of symmetric functions, be the projective limit

$$\Lambda = \varprojlim \Lambda(n), \quad n \to \infty.$$

By definition, a function $f \in \Lambda$ is a sequence $f_1, f_2, \ldots$ where

1. $f_n \in \Lambda(n)$,
2. $f_{n+1}(x_1, \ldots, x_n, 0) = f_n(x_1, \ldots, x_n)$,
3. $\sup_n \deg f_n < \infty$.

Although this formally defines symmetric functions, informally a symmetric function $f(\mathbf{x})$ is a formal power series in a countable number of variables (which we assume to be $x_1, x_2, \ldots$) such that $(i\,j)f(\mathbf{x}) = f(\mathbf{x})$, where $(i\,j)f(\mathbf{x})$ is the series obtained by transposing the variables x_i and x_j in $f(\mathbf{x})$. The set of symmetric functions, with the operations addition and multiplication, form the *ring of symmetric functions*, which we denote by Λ. The ring of symmetric functions is a vector space; the following are some of its bases.

The *monomial symmetric functions* are the symmetric functions, indexed by partitions γ of n, defined by

$$m_\gamma = \sum_{\alpha : \operatorname{sh}(\alpha)=\gamma} \mathbf{x}^\alpha.$$

The set $\{ m_\gamma \mid \gamma \vdash n, n \geq 0\}$ of monomial symmetric functions forms a basis for Λ.

The *one-part elementary symmetric functions*, *one-part complete symmetric functions* and the *one-part power sum symmetric functions* are the symmetric functions, indexed with positive integers, given by

$$e_r = \sum_{1 \leq i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \ldots x_{i_r},$$

$$h_r = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \ldots x_{i_r},$$

and

$$p_r = \sum_{i \geq 1} x_i^r,$$

respectively, and we define e_0, h_0, and p_0 to equal 1. The sets $\{ e_r \mid r \geq 1\}$, $\{ h_r \mid r \geq 1\}$ and $\{ p_r \mid r \geq 1\}$ generate Λ. Furthermore, we define

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \ldots e_{\lambda_n},$$

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \ldots h_{\lambda_n},$$

and

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \ldots p_{\lambda_n},$$

as the *elementary symmetric functions*, *complete symmetric functions* and *power sum symmetric functions*, respectively. The sets $\{ e_\lambda \mid \lambda \vdash n, n \geq 0\}$, $\{ h_\lambda \mid \lambda \vdash n, n \geq 0\}$ and $\{ p_\lambda \mid \lambda \vdash n, n \geq 0\}$ are all bases for Λ.

The last symmetric functions we define here are the Schur functions; the Schur functions, s_λ, are defined combinatorially by

$$s_\lambda = \sum_{T \text{ an SSYT of shape } \lambda} x_T, \tag{2.4}$$

where x_T is the monomial $x_{i_1} x_{i_2} \cdots x_{i_n}$, and $i_1, i_2, \ldots, i_n$ are the numbers in the boxes of the SSYT T.

Alternatively, we can define the Schur functions in an algebraic way. For any $\sigma \in \mathfrak{S}_n$, define $\mathbf{x}^{\sigma\alpha}$ be the monomial $x_1^{\alpha_{\sigma(1)}} x_2^{\alpha_{\sigma(2)}} \cdots x_n^{\alpha_{\sigma(n)}}$. Let

$$a_\alpha = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \mathbf{x}^{\sigma\alpha}, \tag{2.5}$$

where $\varepsilon(\sigma)$ is the sign of the permutation σ. It is not difficult to see that a_α is zero unless all α_i are distinct and, in that case, we may assume that $\alpha_1 > \alpha_2 > \cdots > \alpha_n$. We define the *staircase sequence* to be $\delta = n-1\ n-2\ \ldots 0$, and write $\alpha = \lambda + \delta$ where λ is a partition with at most n parts. It is not hard to see that $a_{\lambda+\delta}$ is divisible by a_δ and that the quotient is symmetric in the n variables $x_1, x_2, \ldots, x_n$. We define $s_\lambda(x_1, x_2, \ldots, x_n)$ as

$$s_\lambda(x_1, x_2, \ldots, x_n) = \frac{a_{\lambda+\delta}}{a_\delta} \in \Lambda(n), \tag{2.6}$$

which are the Schur polynomials, and we obtain the Schur functions by extending these to the ring Λ.

There is a standard inner product $\langle \cdot, \cdot \rangle$ on Λ for which the Schur functions are an orthonormal basis, *i.e.*, $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$. Under this inner product, the power sums form an orthogonal basis; that is $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$, where

$$z_\lambda = 1^{m_1(\lambda)} m_1(\lambda)!\, 2^{m_2(\lambda)} m_2(\lambda)! \cdots |\lambda|^{m_{|\lambda|}(\lambda)} m_{|\lambda|}(\lambda)!.$$

The following theorem connects this inner product in symmetric functions to characters.

Theorem 2.2.2.

$$\chi_\lambda(\rho) = \langle s_\lambda, p_\rho \rangle.$$

2.2.1 Classical Results in Symmetric Function Theory

For any partition λ, the partition λ' is called the *conjugate* partition, and is the partition obtained by interchanging the rows and columns of the Young diagram of λ. The notation $u \in \lambda$ denotes the box u of λ. For any $u = (i, j) \in \lambda$ the *content* of u, denoted by $c(u)$, is the quantity $j - i$, and *hook length* of u, denoted by $h(u)$, is $\lambda_i + \lambda_j - i - j + 1$. Using the inner product at the end of the last section, we have the following two expressions.

Theorem 2.2.3. *Writing the Schur functions as a linear combination of the power sum symmetric functions, we have*

$$s_\lambda = \sum_{\rho \vdash n} z_\rho^{-1} \chi_\lambda(\rho) p_\rho,$$

where λ is a partition of n.

Theorem 2.2.4. *Writing the power sum symmetric functions as a linear combination of the Schur functions, we have*

$$p_\rho = \sum_{\lambda \vdash n} \chi_\lambda(\rho) s_\lambda,$$

where ρ is a partition of n.

From Theorem 2.2.4 and the algebraic definition of Schur functions given at the end of the previous section, we obtain the following theorem.

Theorem 2.2.5. *The character* $\chi_\lambda(\rho)$ *is* $[\mathbf{x}^{\lambda+\delta}]\, a_\delta p_\rho$.

We also require the following two results. We use the notation H_λ for $\prod_{u\in\lambda} h(u)$, where $\lambda \vdash n$.

Theorem 2.2.6. *For* $\lambda \vdash p$, *we have*

$$s_\lambda(\mathbf{1^p}) = \frac{\prod_{u\in\lambda}(p + c(u))}{H_\lambda},$$

where $\mathbf{1^p}$ *is the substitution* $x_i = 1$ *for* $1 \le i \le p$ *and* $x_i = 0$ *for* $i > p$.

The following is the famous hook formula of Frame, Robinson and Thrall (see [5]).

Theorem 2.2.7 (Frame, Robinson and Thrall). *For any partition* $\lambda \vdash n$ *we have*

$$f^\lambda = \frac{n!}{H_\lambda}.$$

The following theorem is a consequence of the previous two results.

Theorem 2.2.8.

$$\prod_{u\in\lambda}(x + c(u)) = \sum_{\beta\vdash n} \frac{|C_\beta|}{f^\lambda} \chi_\lambda(\beta) x^{\ell(\beta)}.$$

Theorem 2.2.8 follows from Theorems 2.2.6, 2.2.7 and 2.2.3, by noting that for any integer t, a substitution of $x_i = 1$ for all $1 \le i \le t$ into the equation in Theorem 2.2.6, yields the theorem for t. Noting that both sides of the equation are polynomials in t of degree n, gives the result with t replaced by the indeterminate x.

2.3 The Murnaghan-Nakayama Rule

In this section we state the *Murnaghan-Nakayama rule*, a combinatorial algorithm that computes symmetric group characters.

In a Young diagram λ with n boxes, a *border strip* is a connected set of boxes that contains no 2×2 subset of boxes. The *height* of a border strip B, $\mathrm{ht}(B)$, is one less than the number of columns occupied by B. Suppose that α is partition of n. A *border strip tableau* of shape λ and type α is an assignment of positive integers to the boxes of λ satisfying,

1. every row and column is weakly increasing,
2. the integer i appears α_i times,
3. the set of squares occupied by i forms a border strip B_i.

The *height* of a border strip tableau B of shape λ and type α with $B_1, B_2, \ldots B_k$ border strips, denoted by $\mathrm{ht}(B)$, is $\mathrm{ht}(B_1) + \mathrm{ht}(B_2) + \cdots + \mathrm{ht}(B_k)$.

Theorem 2.3.1 (Murnaghan-Nakayama Rule). *For any partitions λ and α of n, we have*

$$\chi_\lambda(\alpha) = \sum_T (-1)^{\mathrm{ht}(T)},$$

summed over all border strip tableaux of shape λ and type α.

2.4 Formal Power Series and Lagrange Inversion

For any ring K with a unit, let $K[[z]]$ and $K((z))$ denote the ring of formal power series and the ring of formal Laurent series in the indeterminate z. We need to deal with the compositional inverse of power series on many occasions, so knowing when they exist is pertinent. See Stanley [27, Proposition 5.4.1] for a proof of the following result.

Theorem 2.4.1. *A formal power series $f(z) = a_1 z + a_2 z^2 + \cdots \in K[[z]]$ has an inverse, denoted by $f(z)^{\langle -1 \rangle}$, if and only if a_1 is invertible in K, in which case the inverse of $f(z)$ is unique.*

Finally, given a formal power series the question of *how* to compute its inverse may arise. We require the following notation. Let $[z^n]\ f(z)$ be the *coefficient of z^n in the series $f(z)$*. We will use *Lagrange's Implicit Function Theorem* on a number of occasions; we state it in three forms, the second and third being clearly equivalent (see, *e.g.*, Goulden and Jackson [8, Section 1.2] or Stanley [27, Proposition 5.4.2] for a proof).

Theorem 2.4.2. *Suppose $\psi \in K[[z]]$ is a formal power series with invertible constant term. Then the functional equation $s = z\psi(s)$ has a unique formal power series solution $s = s(z)$. Moreover*

a. For a formal power series $F \in K[[x]]$, and $n \geq 0$, we have

$$[z^n]F(s)\frac{z}{s}\frac{ds}{dz} = [y^n]F(y)\psi(y)^n.$$

b. For a formal Laurent series $F \in K((x))$ and $n \neq 0$, we have

$$[z^n]F(s) = \frac{1}{n}[y^{n-1}]\left(\frac{d}{dy}F(y)\right)\psi(y)^n,$$

and if $n = 0$ we have

$$[z^0]\ F(s) = [y^0]\ F(y) + [y^{-1}]\ F'(y)\log\left(\frac{\phi(y)}{\phi(0)}\right).$$

c. Alternatively, suppose that $H(z)$ is a formal power series with no constant term and invertible linear coefficient and let $F \in K((x))$ be any Laurent series. Then, if $s = H(z)^{\langle -1\rangle}$ we have for $n \neq 0$

$$[z^n]F(s) = \frac{1}{n}[y^{n-1}]F'(y)\left(\frac{y}{H(y)}\right)^n.$$

Forms 2.4.2.b and 2.4.2.c of Lagrange's Theorem are equivalent from the observation that if $s = H(z)^{\langle -1\rangle}$ then $s = z\psi(s)$, where $\psi = z/H(z)$.

Theorem 2.4.2 is referred to as either Lagrange's Theorem or as Lagrange inversion. Throughout this thesis we use Lagrange's Theorem in all of the forms in Theorem 2.4.2, and we highlight which form we use when we feel it necessary.

2.5 Formal Residues

In this thesis, we shall on occasion need the residue theorem. In our application of the residue theorem, however, we shall be in the context of *formal* Laurent series.

We, thus, make sure that this is a valid application with the following two propositions. First, for any rational series $T(z)$, let $[z^{-1}]_\infty T(z)$ denote the coefficient of $1/z$ when $T(z)$ is expanded in powers of $1/z$ (so, we consider its formal Laurent series in $1/z$).

The next proposition expresses, essentially, that the residue is invariant under translation.

Proposition 2.5.1. *For any rational series $T(z)$, we have*

$$[z^{-1}]_\infty T(z) = [z^{-1}]_\infty T(z-c)$$

where c is any constant.

Proof. Using a partial fraction decomposition, for some $k, \alpha_1, \alpha_2, \ldots, \alpha_k, m_1, m_2, \ldots, m_k$ the rational series $T(z)$ is equal to

$$\begin{aligned} T(z) &= B_0(z) + \sum_{i=1}^{k} \frac{B_i(z)}{(z-\alpha_i)^{m_i}} \\ &= B_0(z) + \sum_{i=1}^{k} \frac{B_i(z)/z^{m_i}}{(1-\frac{\alpha_i}{z})^{m_i}}, \end{aligned}$$

where for $1 \leq i \leq k$ each $B_i(z)$ is a polynomial with $\deg B_i(z) < m_i$ and $B_0(z)$ is a polynomial. Then,

$$[z^{-1}]_\infty T(z) = \sum_{i=1}^{k} [z^{m_i-1}] B_i(z),$$

and since $\deg B_i(z) < m_i$, we have

$$[z^{m_i-1}]\, B_i(z) = [z^{m_i-1}]\, B_i(z-c),$$

and we obtain our result. □

Finally, we have the formal series version of the residue theorem. Note that the following only deals with the case where all poles are simple, which is all we use in this thesis.

Proposition 2.5.2. *For $D(z) = \prod_{i=1}^{k}(z-\alpha_i)$, with α_i all distinct, and a polynomial $N(z)$ we have*

$$[z^{-1}]_\infty \frac{N(z)}{D(z)} = \sum_{i=1}^{k} \frac{N(\alpha_i)}{D'(\alpha_i)}$$

Proof. Again, using partial fractions

$$\frac{N(z)}{D(z)} = B_0(z) + \sum_{i=1}^{k} \frac{B_i}{z - \alpha_i} \tag{2.7}$$

$$= B_0(z) + \sum_{i=1}^{k} \frac{B_i/z}{1 - \frac{\alpha_i}{z}}, \tag{2.8}$$

where B_i are constants and $B_0(z)$ is a polynomial. Multiplying (2.7) by $z - \alpha_j$ and evaluating the result at $z = \alpha_j$ we obtain

$$\frac{N(\alpha_j)}{\prod_{\substack{i=1 \\ i \neq j}}^{k} (\alpha_j - \alpha_i)} = B_j. \tag{2.9}$$

But

$$D'(\alpha_j) = \prod_{\substack{i=1 \\ i \neq j}}^{k} (\alpha_j - \alpha_i), \tag{2.10}$$

and comparing (2.9) and (2.10) to (2.8), the result follows. □

Chapter 3

Kerov's Character Polynomials

In this chapter we investigate the first type of character polynomial discussed in Chapter 1, Kerov's character polynomials. Briefly, Kerov's character polynomials are polynomials in variables $R_2, R_3, \ldots$, which are functions from Young diagrams to complex numbers, that exactly evaluate the normalized character given in (2.1). Recall from Chapter 1 that the fifth Kerov polynomial is

$$\Sigma_5 = R_6 + 15\,R_4 + 5\,{R_2}^2 + 8\,R_2.$$

For general k, the Kerov polynomial Σ_k is somewhat of a mystery. Notice that the term of "highest weight" in Σ_5 is R_6; it is known that the term of highest weight in Σ_k is R_{k+1}. But aside from a few other results, little is known about the coefficients of Kerov's polynomials. It is conjectured that the coefficient of each term is positive.

In Section 3.1, we introduce the basic material on Kerov polynomials. In Section 3.2, we describe the historical motivation behind Kerov's polynomials; this motivation comes from recent results by Biane concerning the *asymptotics* of characters. In Section 3.3 we cover preliminary results important to the rest of the thesis; in particular, we include the proof of Macdonald of Frobenius' expression for characters upon which the computation of Kerov's polynomials (and the polynomials in Chapter 4) relies. Finally, in Section 3.4, we state our main result in Theorem 3.4.1, followed by two variants of the main result. In Section 3.5, we give applications of the main result, including providing affirmative answers to some positivity conjectures. The proof of the main result is delayed until Section 3.6.

3.1 Background

In this chapter we shall see that the normalized character $\widehat{\chi}_\omega(k\,1^{n-k})$, given in (2.1), has a polynomial expression. The statement of this expression requires some notation involving partitions ω of n, which we develop now. We adapt the following description from Biane [2, 3]: consider the Young diagram of ω, in the French convention (see Section 2.2, Figure 2.1), and translate it, if necessary, so that the bottom left of the diagram is placed at the origin of an (x, y) plane. Finally, rotate the diagram counter-clockwise by 45°. Note that ω is uniquely determined by the

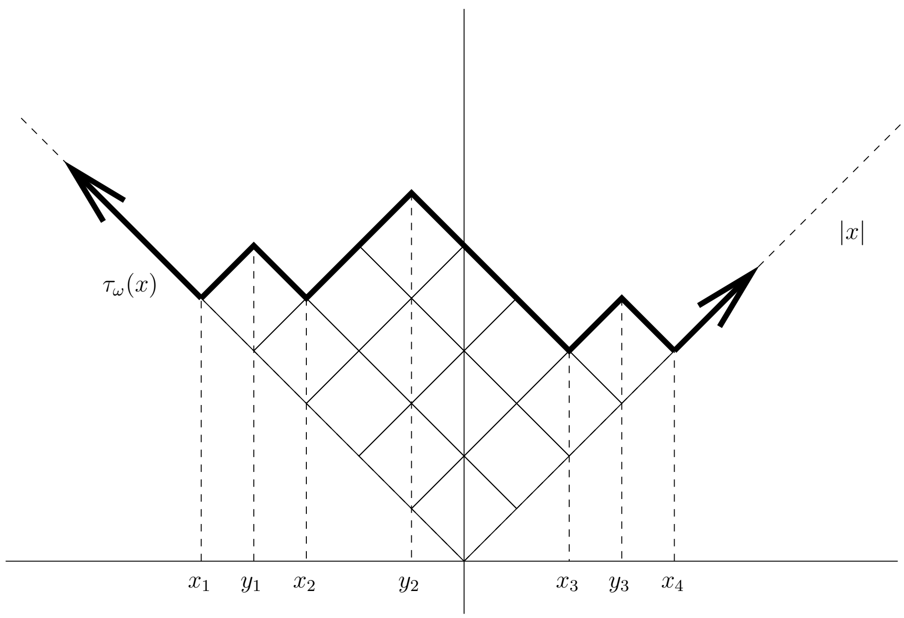

Figure 3.1: The partition (4 3 3 3 1) of 14, drawn in the French convention, and rotated by 45°.

curve $\tau_\omega(x)$ (see Figure 3.1). The value of $\tau_\omega(x)$ is equal to $|x|$ for large negative or positive values of x and it is clear that $\tau'_\omega(x) = \pm 1$, where differentiable. The interlacing sequence of points x_i and y_i in Figure 3.1 are the x-coordinates of the local minima and maxima, respectively, of the curve $\tau_\omega(x)$. We suitably scale the size of the boxes in Young diagrams so that the points x_i and y_i are integers. We call the sequence

$$x_1 < y_1 < x_2 < y_2 < \cdots < x_{m-1} < y_{m-1} < x_m$$

the *interlacing sequence of maxima and minima associated with* ω. Note that another way to look at this sequence of interlacing points is that they are the sequence of contents (see page 12) of the boxes immediately below the corners (after the above rotation has taken place). For example, in Figure 3.1, the box below the corner which is above the x-coordinate y_1 has content -4 (keeping in mind that the partition in Figure 3.1 was drawn in the *French* convention and then rotated), implying that $y_1 = -4$. We also call the local minima and maxima of the diagram the *inside* and *outside* corners, respectively, of the diagram. Setting

$$\sigma_\omega(x) = (\tau_\omega(x) - |x|)/2, \tag{3.1}$$

consider the function

$$H_\omega(z) = \frac{1}{z} \exp \int_{\mathbb{R}} \frac{1}{z-x}\, \sigma'_\omega(x)\, dx. \tag{3.2}$$

For now assume that an interlacing sequence of points x_i and y_i satisfy

$$x_1 < y_1 < x_2 < y_2 < \cdots < y_{t-1} < x_t < 0 < y_t < x_{t+1} < \cdots x_{n-1} < y_{n-1} < x_n,$$

so 0 lies between the t^{th} minimum and t^{th} maximum (other interlacing sequences are dealt with in a similar manner). Then (3.2) becomes

$$\begin{aligned}
&\frac{1}{z} \exp \Bigg(\sum_{i=1}^{t-1} \Bigg(\int_{x_i}^{y_i} \frac{1}{z-x}(1)\, dx + \int_{y_i}^{x_{i+1}} \frac{1}{z-x}(0)\, dx \Bigg) \\
&+ \int_{x_t}^{0} \frac{1}{z-x}(1)\, dx + \int_{0}^{y_t} \frac{1}{z-x}(0)\, dx \\
&+ \sum_{t}^{m-1} \Bigg(\int_{y_i}^{x_{i+1}} \frac{1}{z-x}(-1)\, dx + \int_{x_{i+1}}^{y_i} \frac{1}{z-x}(0)\, dx \Bigg) \Bigg) \\
&= \frac{1}{z} \exp \Bigg(\sum_{i=1}^{t-1} (\log(z-y_i) - \log(z-x_i)) - \sum_{i=t}^{m-1} (\log(z-x_{i+1}) - \log(z-y_i)) \Bigg) \\
&= \frac{1}{z} \exp \Bigg(\sum_{i=1}^{m-1} (\log(z-y_i) - \log(z-x_i)) + \log z - \log(z-x_m) \Bigg) \\
&= \frac{\prod_{i=1}^{m-1}(z-y_i)}{\prod_{i=1}^{m}(z-x_i)},
\end{aligned}$$

that is,

$$H_\omega(z) = \frac{\prod_{i=1}^{m-1}(z-y_i)}{\prod_{i=1}^{m}(z-x_i)}, \tag{3.3}$$

where m is the number of inside corners in the diagram ω. Note that $H_\omega(z)$ has a power series expansion in $1/z$ (see Section 2.5), *i.e.*

$$H_\omega(z) = z^{-1} + \sum_{k=1}^{\infty} M_k z^{-k-1}. \tag{3.4}$$

Now let $R_\omega(z) = 1 + R_i(\omega)z^i$, $i \geq 1$ be defined by

$$R_\omega(z) = zH_\omega^{\langle -1 \rangle}(z), \tag{3.5}$$

where $\langle -1 \rangle$ denotes compositional inverse.

Although the series $H_\omega(z)$ and $R_\omega(z)$ are derived purely from the shape of the tableau ω, they can be used to evaluate the normalized character $\widehat{\chi}_\omega(k\ 1^{n-k})$. In fact, one can express the normalized characters in terms of the $R_i(\omega)$. Furthermore, this expression is a polynomial in the $R_i(\omega)$ and this expression is *independent* of ω, as is expressed in the following theorem. We give Biane's proof of the theorem in Section 3.3.1.

Theorem 3.1.1 (Biane, Kerov). *For $k \geq 1$, there exist universal polynomials Σ_k, with integer coefficients, such that*

$$\widehat{\chi}_\omega(k\, 1^{n-k}) = \Sigma_k(R_2(\omega), R_3(\omega), \ldots, R_{k+1}(\omega)), \tag{3.6}$$

for all $\omega \vdash n$ with $n \geq k$.

These polynomials are the subject of this chapter. They first appear in the literature in Biane [1], and with proof in Biane [3, Theorem 1.1]. The author of those two papers, however, attributes Theorem 3.1.1 to Kerov, who described this result in a talk at an IHP conference in 2000. We have, therefore, associated both names with the theorem here. The polynomials Σ_k are known as *Kerov's character polynomials*. They are referred to as "universal polynomials" in Theorem 3.1.1 to emphasize that they are independent of ω and n, subject only to $n \geq k$. Thus, we now write Kerov's polynomials with $R_i(\omega)$ replaced by an indeterminate R_i, $i \geq 2$ for each i. In indeterminates R_i, the first six of Kerov's character polynomials, as listed in Biane [3], are given below:

$$\begin{aligned}
\Sigma_1 &= R_2 \\
\Sigma_2 &= R_3 \\
\Sigma_3 &= R_4 + R_2 \\
\Sigma_4 &= R_5 + 5R_3 \\
\Sigma_5 &= R_6 + 15R_4 + 5R_2^2 + 8R_2 \\
\Sigma_6 &= R_7 + 35R_5 + 35R_3R_2 + 84R_3
\end{aligned} \tag{3.7}$$

3.2 Motivation: Asymptotics of Characters and Free Probability

Although we largely consider Kerov's polynomials from a formal series aspect, we briefly look at their origins in studying the asymptotics of symmetric group characters.

Much is known about the characters of the symmetric group. The connections to the ring of symmetric functions provide a computational tool for computing the characters. There are also well known algorithms, such as the Murnaghan-Nakayama rule (see Theorem 2.3.1), to compute irreducible characters. When the group $\mathfrak{S}_n$ is large, however, these algorithms become cumbersome and somewhat ineffective. Thus, in order to answer questions about large symmetric group characters, we must move to a different approach.

The approach that has been recently explored is a probabilistic one and, more precisely, it appears that the theory of *free probability* provides the correct setting. Very briefly, free probability can be viewed as a highly non-commutative probability (that, in fact, does not reduce to classical probability in the commutative case), where the notion of independence is replaced by a notion of freeness. In the examples given later in this section, Biane [2] used the theory of free probability to obtain asymptotic results for characters. Futhermore, the presence of non-crossing partitions plays a role in both free probability and the asymptotics of the symmetric group, and this appears not to be a coincidence. In fact, this connection has been explored recently (see Śniady [25]).

The approach is as follows. Define a set of *generalized Young diagrams* to be the set of continuous real functions $\tau_\omega(x)$, as we did for the diagram in Figure 3.1. Note, that $\tau_\omega(x)$ has the properties

1. $|\tau_\omega(u_1) - \tau_\omega(u_2)| \leq |u_1 - u_2|$ for all $u_1, u_2 \in \mathbb{R}$,

2. $\tau_\omega(u) = |u|$ for all $u \in \mathbb{R}$, such that $|u|$ is sufficiently large.

It turns out there is a one-to-one correspondence between continuous Young diagrams ω and probability measures m_ω on $\mathbb{R}$ with compact support that satisfy

$$H_\omega(z) = \frac{1}{z} \exp \int_{\mathbb{R}} \frac{1}{z-x}\, \sigma'_\omega(x)\, dx = \int_{\mathbb{R}} \frac{1}{z-x}\, d(m_\omega),$$

where $\sigma'_\omega(x)$ is defined as in (3.2) (see Kerov [18, 19]). Thus, we can now think of Young diagrams as measures on the real line, and operations on those measures giving rise to other Young diagrams. The advantage of this is clear; we can now use the tools and techniques of analysis to study Young diagrams. The series

$$\int_{\mathbb{R}} \frac{1}{z-x}\, d(m_\omega)$$

is known as the *moment generating series* (since the coefficient of $(1/z)^{k+1}$ in this series is

$$\int_{\mathbb{R}} x^k d(m_\omega),$$

which is the k^{th} moment of the measure m_ω) or the *Cauchy transform* of the measure m_ω. From probability theory, the full set of moments (or the moment generating series) of the probability measure m_ω describes the measure uniquely, since m_ω has compact support.

In fact, the measure m_ω has a very concrete description in terms of ω. If the associated interlacing sequence of ω is $(x_i)_{1\le i\le m}$ and $(y_i)_{1\le i\le m-1}$ then the measure m_ω is

$$m_\omega = \sum_{i=1}^{m} \mu_k \delta_{x_k},$$

where δ_{x_k} is the usual delta function at x_k and

$$\mu_k = \frac{\prod_{i=1}^{m-1}(x_k - y_i)}{\prod_{\substack{i=1\\ i\neq k}}^{m}(x_k - x_i)}.$$

This gives the correct measure as

$$\begin{aligned}\int_{\mathbb{R}} \frac{1}{z-x}\, d(m_\omega) &= \int_{\mathbb{R}} \frac{1}{z-x} \sum_{k=1}^{m} \frac{\prod_{i=1}^{m-1}(x_k - y_i)}{\prod_{i\neq k}^{m}(x_k - x_i)} \delta_{x_k} \\ &= \sum_{k=1}^{m} \frac{1}{z-x_k} \frac{\prod_{i=1}^{m-1}(x_k - y_i)}{\prod_{\substack{i=1\\ i\neq k}}^{m}(x_k - x_i)},\end{aligned}$$

which is the partial fraction decomposition of the rational function on the right hand side of (3.3). The *free cumulant generating series* of the measure ω is defined as the inverse of the moment generating series, as in (3.5). Although free cumulants on the surface seem to simply complicate matters, some operations with measures are simpler in terms of the cumulants. For example, the moments of the (*free*) convolution of measures $\mu \boxplus \lambda$ have no simple expression in terms of the moments of the individual measures, which is a drawback, as we shall soon see. However, in terms of the free cumulants R_i we have the following very simple relationship:

$$R_i(\mu \boxplus \lambda) = R_i(\mu) + R_i(\lambda) \tag{3.8}$$

(in fact, (3.8) is often taken as the definition of free convolution). After defining these concepts and putting them in the context of free probability, one now has the tools of analysis and the theory of free probability at one's disposal to obtain asymptotic results about the symmetric group, as has been carried out in Biane [2].

Example 3.2.1 (Asymptotics of Characters). Prior to the use of free probability theory, Kerov and Vershik [20] gave asymptotic results concerning the representation of the infinite symmetric group. Their results, however, were mainly concerned with Young diagrams of order n, the shape of which has largest part approximately n. Most Young diagrams, however, do not have this property; in fact, it can be shown (see Biane [2]) that most Young diagrams of order n have largest part and number of parts approximately of order $\sqrt{n}$. We call a Young diagram *balanced* if it has this property. Now consider a sequence of permutations $\sigma_n \in \mathfrak{S}_n$, $n \geq 1$, where each σ_n is balanced and each σ_n has k_i cycles of length i. Setting $r = \sum_i ik_i$, we have

$$\lim_{n\to\infty} \frac{\chi_\omega(\sigma_n)}{\chi_\omega(1^n)} = \prod_{i\geq 2} R_{i+1}^{k_i}(\omega)n^{-r} + O(n^{-\frac{r+1}{2}}),$$

or, equivalently

$$\frac{\chi_\omega(\sigma_n)}{\chi_\omega(1^n)} \longrightarrow \prod_{i\geq 2} R_{i+1}^{k_i}(\omega)n^{-r}. \qquad \square$$

There are standard questions that can be asked and have been answered about small representations of the symmetric group. For example, the Kronecker product of two irreducible representations in Example 2.1.7.b is not itself irreducible, so a natural question to ask is what it is as the sum of irreducible characters. When considering large symmetric groups, however, one can give only statistical information about this sum.

Example 3.2.2 (Asymptotics of Restriction). Suppose that ω is a generalized Young diagram. For any real t such that $0 \leq t \leq 1$, there is a unique diagram ω_t whose free cumulants satisfy $R_n(\omega_t) = t^{n-1}R_n(\omega)$. Suppose ω_n is a sequence of generalized Young diagrams that, after a suitable rescaling, converges to the diagram ω as $n \longrightarrow \infty$, and p_n is a sequence such that $p_n/n \longrightarrow t$ as $n \longrightarrow \infty$. Then, the restriction of the representations $[\omega_n]$ to the group $\mathfrak{S}_{p_n}$ is "close" to the representation $[\omega_t]$. See Biane [2, 3] for details. $\square$

Example 3.2.3 (Asymptotics of Induction). We now consider the case of the outer product of representations. Recall that given two representations $[\lambda]$ and $[\mu]$ of $\mathfrak{S}_n$ and $\mathfrak{S}_m$, then the outer product $[\lambda] \circ [\mu]$ is the representation of $\mathfrak{S}_{n+m}$ induced by the Kronecker product $[\lambda] \otimes [\mu]$, a representation of $\mathfrak{S}_n \times \mathfrak{S}_m$. We note a few facts about the outer product. The structure constants of the outer product, that is the constants $g^{\gamma}_{\lambda,\mu}$ given by

$$[\lambda] \circ [\mu] = \sum_{\gamma} g^{\gamma}_{\lambda,\mu} [\gamma],$$

are known as the celebrated *Littlewood-Richardson coefficients*. They are most often seen as the structure constants in the product of Schur symmetric functions:

$$s_\lambda s_\mu = \sum_{\gamma} g^{\gamma}_{\lambda,\mu} s_\gamma.$$

Let p_n and q_n be sequences of integers asymptotic to $\sqrt{n}$, and λ_n and μ_n be diagrams with p_n and q_n boxes which, when scaled, converge to λ and μ, respectively. Then, the outer product $[\lambda_n] \circ [\mu_n]$, a representation of $\mathfrak{S}_{p_n+q_n}$ that is the induced representation of $[\lambda_n] \otimes [\mu_n]$ of $\mathfrak{S}_{p_n} \times \mathfrak{S}_{q_n}$, approaches the diagram $[\lambda] \boxplus [\mu]$, when properly scaled. As mentioned in (3.8), the free cumulants of $[\lambda] \boxplus [\mu]$ have a simple expression in terms of $[\lambda]$ and $[\mu]$. See Biane [2, 3] for details. □

The previous examples give motivation and a historical context for Kerov's polynomials. Although the asymptotics of characters were the original setting in which Kerov's polynomials first appear, we shall not be studying this aspect of Kerov's polynomials. Here, we will study Kerov's polynomials for their own sake, not only because they can facilitate the computation of characters, but also because Theorem 3.1.1 is certainly a surprising and significant result.

3.3 Preliminaries and Previous Results

Before we explain how to obtain the Kerov polynomials, we first give an example of how they can be used to compute the characters $\chi^\lambda(k\,1^{n-k})$. We do this by taking a Kerov polynomial from (3.7) and computing the $R_i(\lambda)$ from the series $H_\lambda(z)$.

Example 3.3.1. We use (3.4) and (3.5) to compute the character $\chi_{(4\,3\,3\,3\,1)}(5\,1^9)$, *i.e.* we compute the character for the shape in Figure 3.1 evaluated at a 5-cycle.

We will use Lagrange's Theorem 2.4.2 to compute the relevant $R_k(\omega)$ from (3.4) and (3.5). To use Lagrange's Theorem, we express $R_\omega(z)$ in terms of the *power series* $H_\omega(1/z)$. Clearly, from (3.5),

$$H_\omega(R_\omega(z)/z) = z,$$

so [1]

$$(H_\omega(1/z))^{\langle -1 \rangle} = \frac{z}{R_\omega(z)},$$

giving

$$R_\omega(z) = \frac{z}{(H_\omega(1/z))^{\langle -1 \rangle}}. \tag{3.9}$$

For the shape $(4\,3\,3\,3\,1)$ we have

$$\begin{aligned} H_\omega(1/z) &= \frac{z(1+4z)(1+z)(1-3z)}{(1+5z)(1+3z)(1-2z)(1-4z)} \\ &= z + 14z^3 - 14z^4 + 258z^5 - 502z^6 + \cdots \end{aligned}$$

and

$$\begin{aligned} R_k(\omega) &= [z^k]\, \frac{z}{(H_\omega(1/z))^{\langle -1 \rangle}} \\ &= -\frac{1}{k-1}\,[z]\,\left(\frac{1}{H_\omega(1/z)}\right)^{k-1}. \end{aligned}$$

One can easily compute that $R_2(\omega) = 14$, $R_4(\omega) = -134$ and $R_6(\omega) = 2358$. Using Kerov's polynomial for Σ_5 in (3.7) and specializing to the shape ω, we compute

$$\begin{aligned} \widehat{\chi}_\omega(5\,1^9) &= R_6 + 15R_4 + 5R_2^2 + 8R_2 \\ &= 2358 + 15(-134) + 5(14)^2 + 8(14) \\ &= 1440, \end{aligned}$$

which gives

$$\begin{aligned} \chi_\omega(5\,1^9) &= \frac{\chi_\omega(1^{14})}{(14)_5}\widehat{\chi}_\omega(5\,1^9) \\ &= \frac{21021}{240240}1440 \\ &= 126. \end{aligned}$$

□

All coefficients appearing in the list (3.7) are positive. It is conjectured that this holds in general: that for any $k \geq 1$, all nonzero coefficients in Σ_k are positive. In Biane [3], this conjecture, which we shall call the *R-positivity conjecture*, is attributed to Kerov. It has been verified for all k up to 15 by Biane [1], who computed Σ_k for

[1] To clarify this notation, $(H_\omega(1/z))^{\langle -1 \rangle}$ means "take the compositional inverse of the function $H_\omega(1/z)$" whereas $H_\omega^{\langle -1 \rangle}(1/z)$ means "take the compositional inverse of $H_\omega(z)$ and substitute $1/z$ for z in the result".

$k \le 15$, using an implicit formula for Σ_k (see (3.19) and (3.20) below or Biane [1, Theorem 5.1]) that he credits to Okounkov (private communication). Biane further comments that "It seems plausible that S. Kerov was aware of this (see especially the account of Kerov's central limit theorem in Ivanov and Olshanski [16])."

We are now in a position to find an explicit expression for Kerov's polynomials. Our treatment of the subject begins with a very brief summary of Biane's Theorem 3.1.1; we are less interested in the actual existence of Kerov polynomials and more interested in how to compute them. There is, however, one aspect of Biane's proof that we mention here, stated in Theorem 3.3.6.

3.3.1 The Existence of Kerov's Polynomials

We include this section for two reasons: to emphasize the combinatorics underlying Kerov polynomials and to give a proof of Theorem 3.3.6 below, as it is important in this chapter and the next. Our treatment of this material, however, is brief as we are primarily interested in determining Kerov's polynomials.

Let λ be a Young diagram with k boxes and let $n \ge k$. Suppose ϕ is an injective map from the cells of λ to the set $[n] = \{1, 2, \ldots, n\}$, and let σ_ϕ to be the permutation in $\mathfrak{S}_n$ whose cycles are given by the rows of $\phi(\lambda)$ (see Figure 3.2). Note, in λ parts of size 1 only contribute fixed point to σ_ϕ. Define Φ_λ be the collection of all such

10	15	5	19	12
1	13	4		
9	2	6		
7				

Figure 3.2: An example of an injection ϕ from cells of a diagram to $[19]$. The permutation σ_ϕ is $(10\,15\,5\,19\,12)(1\,13\,4)(9\,2\,6)$.

maps, and let $a_{\lambda;n}$ be the member of the group algebra of $\mathbb{C}[\mathfrak{S}_n]$ which is the formal sum of all the elements in Φ_λ; that is,

$$a_{\lambda;n} = \sum_{\phi \in \Phi_\lambda} \sigma_\phi.$$

We abbreviate $a_{(k);n}$ by $a_{k;n}$. It follows that $a_{1;n} = n.e$, where e is the identity in $\mathfrak{S}_n$. Furthermore, define the *sign* of a partition λ to be $(-1)^{|\lambda|-\ell(\lambda)}$, and denote it by $\text{sign}(\lambda)$ (hence, if a permutation σ is in the conjugacy class C_λ, then $\text{sign}(\lambda) = \varepsilon(\sigma)$). Finally, define the *weight* of a term $a_{i_1;n}a_{i_2;n}\cdots a_{i_p;n}$ in the group algebra $\mathbb{C}[\mathfrak{S}_n]$ as $i_1 + i_2 + \cdots + i_p - p$. The following theorem, although not stated exactly as below, is proved in Biane [1, Lemma 3.1]. We do not reproduce the proof here; the proof is by induction on the number of parts of a partition.

Lemma 3.3.2 (Biane). *There exist polynomials P_λ, with integer coefficients and independent of n, such that*

$$a_{\lambda;n} = P_\lambda(a_{1;n}, a_{2;n}, \ldots, a_{|\lambda|;n}).$$

Furthermore, each monomial in P_λ has weight congruent to $\text{sign}(\lambda) \pmod 2$.

Let J_n be the members of the group algebra $\mathbb{C}[\mathfrak{S}_{n+1}]$ given by

$$J_n = (1\ *) + (2\ *) + \cdots + (n\ *),$$

where the symbols on which $\mathfrak{S}_{n+1}$ acts are $1, 2, \ldots, n, *$ (we use the symbol "$*$" to distinguish it from the other symbols). The J_n are commonly known as *Jucys-Murphy elements*. There is a natural embedding of $\mathfrak{S}_n$ in $\mathfrak{S}_{n+1}$ (consisting of permutations with $*$ as a fixed point), and define an expectation E_n as the projection of $\mathbb{C}[\mathfrak{S}_{n+1}]$ onto $\mathbb{C}[\mathfrak{S}_n]$, given by $E_n(\sigma) = \sigma$ if $\sigma \in \mathfrak{S}_n$ and 0 otherwise. We can take the k^{th} moment of a Jucys-Murphy element with respect to this expectation, *i.e.* $\mathcal{M}_k = E_n(J_n^k)$. From this we can contruct the k^{th} free cumulant $\mathcal{R}_k$ of J_n^k by

$$\mathcal{R}_k = \sum_{\substack{l_1, l_2, \ldots l_r \\ \sum j l_j = k}} (-1)^{1+l_1+l_2+\cdots+l_r} \frac{k-2+\sum_i l_i}{l_1! l_2! \ldots l_r! (k-1)!} \mathcal{M}_1^{l_1} \mathcal{M}_2^{l_2} \cdots \mathcal{M}_r^{l_r} \qquad (3.10)$$

(note that this is obtained by using Lagrange inversion and finding the k^{th} coefficient on the right hand side of (3.5)). Let the *weight* of the monomial $\mathcal{R}_{j_1}^{i_1} \ldots \mathcal{R}_{j_t}^{i_t}$ be $\sum_{l=1}^t i_l j_l$. We apply the term "weight" as in the last context whenever it is appropriate; that is, the weights of the monomials $\mathcal{M}_{j_1}^{i_1} \ldots \mathcal{M}_{j_t}^{i_t}$ and $R_{j_1}^{i_1} \ldots R_{j_t}^{i_t}$ are also $\sum_{l=1}^t i_l j_l$. We also find it useful to refer to the *sign* of a monomial of R's (or M's) of weight k, which is $(-1)^k$.

The following lemma connects $\mathcal{R}_k$ and the free cumulants of the series (3.5) and is found in Biane [1, Lemma 4.1].

Lemma 3.3.3 (Biane).

$$\frac{\chi_\omega(\mathcal{R}_k)}{\chi_\omega(1^n)} = R_k(\omega).$$

We note that the proof of the previous lemma involves the computation of the eigenvalues of the Jucys-Murphy elements (*i.e.*, the images of the Jucys-Murphy element under the left regular representation), as computed in Okounkov and Vershik [23].

The following theorem has very important consequences and it is found in Biane [1, page 6]. Example 3.3.5 follows the proof of the theorem and amplifies some of the details omitted in the proof.

Theorem 3.3.4 (Biane). *For $k \geq 2$ and $n \geq k$, we have*

$$a_{k-1;n} = \mathcal{R}_k + \{\text{terms of } \mathcal{R}_j \text{ with } j < k\}. \tag{3.11}$$

Furthermore, the expression on the right hand side of (3.11) *only involves terms with sign* $(-1)^k$.

Proof (Biane). We see that $\mathcal{M}_k$, which equals the expectation $E_n(J_n^k)$, can be computed in the following way. Clearly,

$$J_n^k = \sum_{i_1,i_2,\ldots,i_k \in [n]} (i_1\ *)(i_2\ *)\cdots(i_k\ *). \tag{3.12}$$

By definition, a term $(i_1\ *)(i_2\ *)\cdots(i_k\ *)$ in this sum gives a non-trivial contribution to $\mathcal{M}_k$ if and only if $(i_1\ *)(i_2\ *)\cdots(i_k\ *)$ fixes $*$. Let us explore precisely when this happens by tracking successive partial products of transpositions. For convenience, set $\sigma = (i_1\ *)(i_2\ *)\cdots(i_k\ *)$.

If $i_1, i_2, \ldots, i_k$ are all distinct, then $*$ is not fixed by σ, since then $\sigma(*) = i_k$, implying σ does not contribute to $\mathcal{M}_k$. In fact, if σ fixes $*$ it is clear that one of $i_1, i_2, \ldots, i_{k-1}$ must equal i_k. Suppose that $i_{j_1} = i_k$. Then, we have $(i_{j_1}\ *)\cdots(i_k\ k)$ fixing $*$, and we are left to repeat the previous argument on $(i_1\ *)\cdots(i_{j_1-1}\ *)$; that is, if $*$ is fixed by $(i_1\ *)\cdots(i_{j_1-1}\ *)$, then for some j_2 we have $i_{j_2} = i_{j_1-1}$. In this manner we obtain a sequence $j_1, j_2, \ldots, j_t$, and σ fixes $*$ if and only if $j_t = 1$. The sign of all the permutations on the right hand side of (3.12) is $(-1)^k$.

Let π be the partition of $[k]$ such that l and m are in the same part if and only if $i_l = i_m$; we write $i_1, i_2, \ldots, i_k \sim \pi$ and say π is the partition *associated* with the sequence $i_1, \ldots, i_k$. The conjugacy class of σ in $\mathfrak{S}_{n+1}$ only depends on this partition and not on the actual sequence $i_1, i_2, \ldots, i_k$. Accordingly, let $\lambda(\pi)$ be the conjugacy class to which π gives rise, and set

$$\mathcal{Z}_\pi = \sum_{i_1,i_2,\ldots,i_k \sim \pi} (i_1\ *)(i_2\ *)\cdots(i_k\ *).$$

Furthermore, call the partitions π for which $\lambda(\pi)$ fixes $*$ *admissible*. Evidently,

$$\mathcal{M}_k = \sum_{\pi \text{ admissible}} \mathcal{Z}_\pi.$$

As usual, set $\ell(\pi)$ to be the number of parts in π. Then, we see that the number of sequences $i_1, i_2, \ldots, i_k$ associated with an admissible partition λ is $(n)_{\ell(\pi)}$ since, after linearly ordering the parts of π, the first part of π gives a set of integers $\{b_1, \ldots, b_r\}$ which means that $i_{b_1} = \cdots = i_{b_r}$, and the number of choices for this common integer is n. Similarly, the second part of π is some set $\{c_1, \ldots, c_t\}$ implying that $i_{c_1} = \cdots = i_{c_t}$, and there are $n-1$ choices for this common integer. The argument continues in this fashion. By symmetry, all terms in $a_{\lambda(\pi);n}$ appear the same number of times in the sum, and from the definition of $a_{\lambda(\pi);n}$ the number of terms in a $a_{\lambda(\pi);n}$ is $(n)_{|\lambda(\pi)|}$. Since $|\lambda(\pi)| \leq \ell(\pi)$, we arrive at the expression

$$\mathcal{Z}_\pi = \frac{(n)_{\ell(\pi)}}{(n)_{|\lambda(\pi)|}} a_{\lambda(\pi);n}.$$

The longest cycle for an admissible partition is $k-1$; this occurs if and only if the admissible partition is $\{1,k\}, \{2\}, \ldots, \{k-1\}$. For all other admissible partitions π we have $\ell(\pi) < k-1$. Thus, we see

$$\mathcal{M}_k = a_{k-1;n} + \sum_{\substack{\pi \text{ admissible} \\ \text{weight of } \pi < k-1}} \frac{(n)_{\ell(\pi)}}{(n)_{|\lambda(\pi)|}} a_{\lambda(\pi);n}.$$

From the comments earlier concerning the sign of the permutations in the sum, the right hand side of the last equation only contains terms of sign $(-1)^k$. Applying Lemma 3.3.2, we conclude that

$$\mathcal{M}_k = a_{k-1;n} + (\text{polynomial in } a_{j;n}\text{: where } j < k-1 \text{ and the sign of each term is } (-1)^k).$$

We invert this equation to obtain

$$a_{k-1;n} = \mathcal{M}_k + (\text{polynomial in } \mathcal{M}_j\text{: where } j < k \text{ and the sign of all terms is } (-1)^k).$$

Finally, from (3.10) we have

$$a_{k-1;n} = \mathcal{R}_k + (\text{polynomial in } \mathcal{R}_j\text{: where } j < k \text{ and the sign of all terms is } (-1)^k),$$

and the result follows. □

To illustrate the details in the proof of Theorem 3.3.4, we provide the following example.

Example 3.3.5. The product of the following $k = 8$ transpositions

$$(1\,*)(2\,*)(3\,*)(1\,*)(9\,*)(1\,*)(2\,*)(9\,*)$$

is $(1\,9)(2\,3)$. Here the sequence $i_1, \dots, i_k$ is $1, 2, 3, 1, 9, 1, 2, 9$. The sequence $j_1, \dots, j_t$ in the proof is $5, 1$ and, indeed, $*$ is a fixed point of the product of transpositions. The partition associated with the above sequence $1, 2, 3, 1, 9, 1, 2, 9$ is $\pi = \{1, 4, 6\}\{2, 7\}\{3\}\{5, 8\}$. The product

$$(5\,*)(2\,*)(4\,*)(5\,*)(1\,*)(5\,*)(2\,*)(1\,*)$$

also has π associated to it and, indeed, the product evaluates as $(1\,5)(2\,4)$, which has the same conjugacy class as $(1\,9)(2\,3)$. We, therefore, have confirmed that π is admissible in this case.

Note that the product

$$(2\,*)(2\,*)(3\,*)(1\,*)(9\,*)(1\,*)(2\,*)(9\,*)$$

is $(1\,9\,2\,3\,*)$, making the partition $\{1, 2, 7\}, \{3\}, \{4, 6\}, \{5, 8\}$ not admissible. The sequence $j_1, \dots, j_t$ in this case is 5, and $j_t \neq 1$. □

Proof of Theorem 3.1.1 (Biane). Applying Lemma 3.3.3 to Theorem 3.3.4, we obtain

$$\frac{\chi_\omega(a_{k:n})}{\chi_\omega(1^n)} = \frac{\chi_\omega(\mathcal{R}_{k+1})}{\chi_\omega(1^n)} + \left(\text{terms of } \frac{\chi_\omega(\mathcal{R}_j)}{\chi_\omega(1^n)} \text{ with } j \leq k \text{ and sign } (-1)^{k+1}\right),$$

and since the number of terms in $a_{k;n}$ is $(n)_k$ we have

$$(n)_k \frac{\chi_\omega(k\ 1^{n-k})}{\chi_\omega(1^n)} = R_{k+1}(\omega) + \left(\text{terms of } R_j(\omega) \text{ with } j \leq k \text{ and sign } (-1)^{k+1}\right),$$

allowing us to conclude that

$$\widehat{\chi}_\omega(k\ 1^{n-k}) = R_{k+1}(\omega) + \left(\text{terms of } R_j(\omega) \text{ with } j \leq k \text{ and sign } (-1)^{k+1}\right),$$

completing the proof. □

The proof of Theorem 3.1.1 also provides a proof of the following theorem.

Theorem 3.3.6 (Biane). *In the Kerov polynomial Σ_k only terms of sign $(-1)^{k+1}$ appear with non-zero coefficient; that is, only terms of weight i, where $i \equiv k + 1 (\text{mod } 2)$ appear with non-zero coefficient.*

3.3.2 Computation of Kerov's Polynomials and Frobenius' Expression for Characters

At this point we have given no indication of *how* to compute a Kerov polynomial. We have seen how one can use them to compute characters in Example 3.3.1 but, to this point, the Kerov polynomials given in (3.7), even the first two, are not obvious. We will now fully lay out the ground work that we use later to compute Kerov's polynomials.

From Theorem 2.2.5 we see that the character is the coefficient of $\mathbf{x}^\mu$ in $a_\delta p_\rho$. This, as we have seen in Chapter 2, is based on the basic character expansion of the Schur functions in terms of the power sum symmetric functions. Below, we include Frobenius' formula for the normalized character. This formula at first appears too complex to carry out the Lagrange inversion calculation, but we shall see an explicit formula for Kerov's polynomials can be determined from it.

Our first step is to find an expression for the degree of the character λ. We begin with a technical lemma.

Lemma 3.3.7. *For any $y_1, y_2, \ldots, y_n$ we have*

$$\det\left((y_i)_j\right)_{1\le i,j\le n} = \det\left(y_i^j\right)_{1\le i,j\le n}$$

Proof. It is well known that Stirling numbers of the second kind satisfy the equation

$$x^j = \sum_{r=0}^{j} S(j,r)\,(x)_r. \tag{3.13}$$

(see Stanley [26, Section 1.4]). Applying this to the n variables $y_1, y_2, \ldots, y_n$, we have

$$y_i^j = \sum_{r=0}^{j} S(j,r)\,(y_i)_r.$$

Since $S(i,j) = 0$ if $j > i$, the matrix $(S(i,j))_{1\le i,j\le n}$ is triangular. Moreover, $S(i,i) = 1$ for all i, making $\det(S(i,j)) = 1$. The result now follows. □

The determinant $\det\left(y_i^j\right)_{1\le i,j\le n}$ given in Lemma 3.3.7 is known as the *Vandermonde determinant* and is equal to $\prod_{i<j}(y_i - y_j)$. We use the notation $\Delta(y_1, y_2, \ldots, y_n)$ to denote the Vandermonde determinant. As in Section 2.2.1, let $\omega = \omega_1\omega_2\cdots\omega_\ell \vdash n$ be a partition of n with ℓ parts, and there are α_i parts of size i. For convenience, we define $\omega_{\ell+1}, \ldots, \omega_n = 0$ and consider $\omega = \omega_1 \ldots \omega_n$. Recall from Section 2.2 that the *staircase sequence* of length n is $\delta = n-1\ n-2\ \ldots 0$. Finally, set $\mu_i = \omega_i + \delta_i =$

$\omega_i + n - i$. With this notation we have the following expression for the degree of χ_ω. The proof is as presented in Macdonald [21, Section I.7, Exercise 6].

Lemma 3.3.8 (Frobenius). *The degree of the symmetric group character χ_ω is*

$$f^\omega = \frac{n!}{\mu!}\Delta(\mu_1, \mu_2, \ldots, \mu_n),$$

where $\mu! = \mu_1! \cdots \mu_n!$.

Proof (Macdonald). From Theorem 2.2.5 we see that $f^\omega = \chi_\omega(1^n)$ is

$$\begin{aligned}
[\mathbf{x}^\mu]a_\delta p_{(1^n)} &= \left(\sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)\mathbf{x}^{\sigma(\delta)}\right)\left(\sum_{i=1}^n x_i\right)^n \\
&= [\mathbf{x}^\mu]\left(\sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)\mathbf{x}^{\sigma(\delta)}\right)\left(\sum_{i=1}^n x_i\right)^n \\
&= \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)[\mathbf{x}^\mu]\mathbf{x}^{\sigma(\delta)}\left(\sum_{i=1}^n x_i\right)^n . \qquad (3.14)
\end{aligned}$$

But,

$$[\mathbf{x}^\mu]\mathbf{x}^{\sigma(\delta)}\left(\sum_{i=1}^n x_i\right)^n = \binom{n}{\mu_1 - n + \sigma(1), \mu_2 - n + \sigma(2), \ldots, \mu_n - n + \sigma(n)},$$

so,

$$[x^\mu]a_\delta p_\rho = \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)n!\frac{1}{\prod_{i=1}^n (\mu_i - n + \sigma(i))!}.$$

The last expression has a compact description; it is precisely the permutation characterization of the determinant:

$$\sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)n!\frac{1}{\prod_{i=1}^n (\mu_i - n + \sigma(i))!} = n!\det\left(\frac{1}{(\mu_i - n + j)!}\right)_{1\le i,j\le n}.$$

Proceeding, we see

$$\begin{aligned}
n!\det\left(\frac{1}{(\mu_i - n + j)!}\right) &= \frac{n!}{\mu!}\det\left((\mu_i)_{n-j}\right)_{1\le i,j\le n} \\
&= \frac{n!}{\mu!}\det\left(\mu_i^{n-j}\right)_{1\le i,j\le n} \\
&= \frac{n!}{\mu!}\Delta(\mu_1, \mu_2, \ldots, \mu_n),
\end{aligned}$$

where the second equality follows from Lemma 3.3.7. □

We now give the expression for characters due to Frobenius; this expression will eventually lead to our explicit expression for Kerov's polynomials, given below in Theorem 3.4.1.

Theorem 3.3.9 (Frobenius).

$$\widehat{\chi}_\omega(k\ 1^{n-k}) = -\frac{1}{k}[z^{-1}]_\infty (z)_k \frac{\theta(z-k)}{\theta(z)}, \tag{3.15}$$

where

$$\begin{aligned} \theta(z) &= \prod_{i=1}^{n}(z-\mu_i), \\ \mu_i &= \omega_i + n - i, \text{ for } 1 \le i \le n\ , \end{aligned} \tag{3.16}$$

Recall from Section 2.5, that $[z^{-1}]_\infty$ is the coefficient of $[z^{-1}]$ when the trailing series is expanded in powers of $\frac{1}{z}$. We point out that $\mu_i = n - i$ if $i \ge \ell + 1$. The following is a proof as presented in Macdonald [21, Section I.7, Exercise 7].

Proof (Macdonald). From Theorem 2.2.5 we have

$$\begin{aligned} \chi_\omega(k\ 1^{n-k}) &= [\mathbf{x}^\mu] a_\delta p_k p_{(1^{n-k})} \\ &= [\mathbf{x}^\mu] \left(\sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)\mathbf{x}^{\sigma(\delta)}\right) \left(\sum_{i=1}^{n} x_i^k\right) \left(\sum_{i=1}^{n} x_i\right)^{n-k} \\ &= \sum_{i=1}^{n} [\mathbf{x}^\mu] x_i^k \left(\sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)\mathbf{x}^{\sigma(\delta)}\right) \left(\sum_{i=1}^{n} x_i\right)^{n-k} \\ &= \sum_{i=1}^{n} \frac{(n-k)!\Delta(\mu_1,\ldots,\mu_i-k,\ldots,\mu_n)}{\mu_1!\cdots(\mu_i-k)!\cdots\mu_n!} \end{aligned}$$

where the last equation follows from (3.14). Thus, the normalized character $\widehat{\chi}_\omega(k\ 1^{n-k})$ is given by

$$\begin{aligned} \widehat{\chi}_\omega(k\ 1^{n-k}) &= \frac{n!}{(n-k)!}\frac{\chi_\omega(k\ 1^{n-k})}{f^\omega} \\ &= \sum_{i=1}^{n} \frac{\mu_i!}{(\mu_i-k)!} \prod_{j\neq i} \frac{\mu_i-\mu_j-k}{\mu_i-\mu_j} \\ &= -\frac{1}{k}\sum_{i=1}^{n} \frac{\mu_i!}{(\mu_i-k)!} \frac{\prod_j \mu_i-\mu_j-k}{\prod_{j\neq i}\mu_i-\mu_j} \\ &= -\frac{1}{k}\sum_{i=1}^{n} \mu_i(\mu_i-1)\cdots(\mu_i-k+1)\frac{\theta(\mu_i-k)}{\theta'(\mu_i)} \\ &= -\frac{1}{k}(z)_k\frac{\theta(z-k)}{\theta(z)}, \end{aligned}$$

where the last line follows from Proposition 2.5.2. This completes the proof. □

A brief version of the following lemma is given in Biane [1], and we amplify the details here.

Lemma 3.3.10 (Biane). *The rational function $H_\omega(z)$ for $\omega \vdash n$ given in* (3.3) *is related to the function θ above by*

$$\frac{1}{H_\omega(z-n)} = \frac{z\theta(z-1)}{\theta(z)}.$$

Proof (Biane). Let ω and μ be as defined in (3.16). Then,

$$\begin{aligned}\frac{z\theta(z-1)}{\theta(z)} &= z\prod_{i=1}^{n}\frac{(z-1-\mu_i)}{(z-\mu_i)}\\ &= z\prod_{i=1}^{\ell}\frac{(z-n-(\omega_i-i+1))}{(z-n-(\omega_i-i))}\prod_{i=\ell+1}^{n}\frac{(z-n+i-1)}{(z-n+i))}\\ &= (z-n+\ell)\prod_{i=1}^{\ell}\frac{(z-n-(\omega_i-i+1))}{(z-n-(\omega_i-i))}. \end{aligned} \tag{3.17}$$

Note that for a block $t, t+1, \ldots, r$, where $\omega_t = \omega_{t+1} = \cdots = \omega_r$, we have

$$\prod_{i=t}^{r}\frac{(z-n-(\omega_i-i+1))}{(z-n-(\omega_i-i))} = \frac{(z-n-(\omega_t-t+1))}{(z-n-(\omega_r-r))}. \tag{3.18}$$

Thus, for a block in ω where all the parts are equal, only two terms of the product

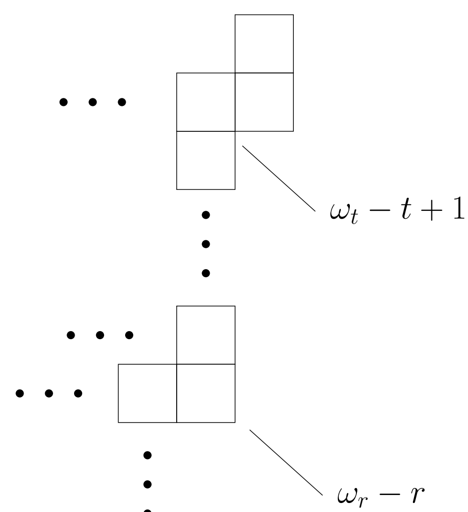

Figure 3.3: Only the corners of a diagrams survive as non-trivial terms.

on the left hand side of (3.17) survive; the first box (in the numerator) and the last box (in the denominator). Note that both of these values are the contents of the boxes immediately up and left from a corner (see Figure 3.3). Recall that the corners of type $\omega_t - t + 1$ and $\omega_t - t$ are *inside* and *outside* corners, respectively. The factor $z - n + \ell$ outside the product in (3.17) is the last corner. Furthermore, if we were to draw the diagram as in Figure 3.1, we would see that the values of the inside corners $\omega_t - t + 1$ correspond to the local minima and, likewise, the outside corners to the local maxima. Thus, the numerator of (3.17) is a product of the form

$$\prod_{i=1}^{\ell}(z - n - x_i),$$

where the x_i are the values of the local minima when the diagram of ω is rotated as in Figure 3.1, and similarly for the denominator. Therefore, we see that (3.17) is simply the reciprocal of the rational function given in (3.3). □

From (3.15) we obtain

$$\begin{aligned}
\widehat{\chi}_\omega(k\,1^{n-k}) &= -\frac{1}{k}[z^{-1}]_\infty(z)\cdot(z-1)\cdots(z-k+1)\frac{\theta(z-k)}{\theta(z)} \\
&= -\frac{1}{k}[z^{-1}]_\infty\, z\frac{\theta(z-1)}{\theta(z)}(z-1)\frac{\theta(z-2)}{\theta(z-1)}\cdots(z-(k-1))\frac{\theta(z-k)}{\theta(z-(k-1))} \\
&= -\frac{1}{k}[z^{-1}]_\infty\,\frac{1}{H_\omega(z-n)}\frac{1}{H_\omega(z-n-1)}\cdots\frac{1}{H_\omega(z-n-(k-1))}.
\end{aligned}$$

Applying Proposition 2.5.1 to the previous expression, we have

$$\widehat{\chi}_\omega(k\,1^{n-k}) = -\frac{1}{k}[z^{-1}]_\infty\,\frac{1}{H_\omega(z)}\frac{1}{H_\omega(z-1)}\cdots\frac{1}{H_\omega(z-(k-1))}. \tag{3.19}$$

However, we saw in (3.9) that

$$R_{i+1}(\omega) = -\frac{1}{i}[z^{-1}]_\infty\left(\frac{1}{H_\omega(z)}\right)^i. \tag{3.20}$$

The last two equations hold for any ω, so substituting $G(z) = \frac{1}{H(z)}$ and replacing the coefficients in $G(z)$ by indeterminates, and noting by Theorem 3.1.1 that Kerov's polynomials are universal, we obtain the following implicit formula for Kerov's polynomials, which can be found, with essentially the above proof, in Biane [1, Theorem 5.1].

Theorem 3.3.11 (Biane). *Let G be the power series*

$$G(z) = \sum_{j\geq 1} g_j z^j.$$

Define S_k to be

$$S_k = -\frac{1}{k}[z^{-1}]_\infty \prod_{i=0}^{k-1} G(z-j),$$

and

$$R_{i+1} = -\frac{1}{i}[z^{-1}]_\infty \, G(z)^i.$$

Then, when S_k is expressed in terms of the R_i, it gives Kerov's polynomials.

We shall, however, mainly be using the following slight modification of Theorem 3.3.11, as it is more convenient in terms of our notation. This corollary can be found in Stanley [29] (without proof).

Corollary 3.3.12 (Stanley). *Let $R(x) = 1 + \sum_{i \geq 2} R_i z^i$ and*

$$F(z) = \frac{z}{R(z)}, \qquad G(z) = \frac{1}{F^{\langle -1 \rangle}(z^{-1})}. \tag{3.21}$$

Then, for $k \geq 1$,

$$\Sigma_k = -\frac{1}{k}[z^{-1}]_\infty \prod_{j=0}^{k-1} G(z-j).$$

Proof. By Lagrange inversion (Theorem 2.4.2.c) and Theorem 3.3.11, we have

$$\begin{aligned} R_i &= [z^i]\, R(z) \\ &= [z^{i-1}]\, \frac{1}{F(z)} \\ &= -\frac{1}{i-1}[z^i] \left(\frac{z}{F^{\langle -1 \rangle}(z)} \right)^{i-1} \\ &= -\frac{1}{i-1}[z] \left(\frac{1}{F^{\langle -1 \rangle}(z)} \right)^{i-1}. \end{aligned} \tag{3.22}$$

But

$$G(z) = \frac{1}{F^{\langle -1 \rangle}(1/z)},$$

which implies that

$$G(1/z) = \frac{1}{F^{\langle -1 \rangle}(z)}.$$

Thus, from Theorem 3.3.11 and (3.22) we have

$$R_i = -\frac{1}{i-1}[z^{-1}]_\infty \, G(z)^{i-1},$$

and

$$\Sigma_k = -\frac{1}{k}[z^{-1}]_\infty \prod_{j=0}^{k-1} G(z-j). \qquad \square$$

3.4 The Main Result

We now obtain an explicit formula for Kerov's polynomials; this is the main result of this chapter, stated as Theorem 3.4.1. Variants of the main theorem are given also, as Theorems 3.4.2 and 3.4.3. The main result is obtained by considering the graded pieces of Kerov's polynomials, as follows. For $n \geq 0$, we define

$$\Sigma_{k,2n} = [u^{k+1-2n}]\Sigma_k(R_2u^2, \dots, R_{k+1}u^{k+1}), \tag{3.23}$$

the sum of all terms of weight $k+1-2n$ in Σ_k (from Theorem 3.3.6 all other terms are 0). In order to state the main result, we introduce the generating series $C(z) = \sum_{m\geq 0} C_m z^m$, given by

$$C(z) = \frac{1}{1 - \sum_{i\geq 2}(i-1)R_i z^i}. \tag{3.24}$$

The initial terms of $C(z)$ are $C_0 = 1$, $C_1 = 0$, and the general terms C_m are given by

$$C_m = \sum_{\substack{j_2,j_3,\dots\geq 0 \\ 2j_2+3j_3+\dots=m}} (j_2+j_3+\dots)! \prod_{i\geq 2} \frac{((i-1)R_i)^{j_i}}{j_i!}, \quad m \geq 2. \tag{3.25}$$

Note, that as a sum of monomials in the R_i, the weight of C_m is m; thus, we define the *weight* of the monomial $C_{j_1}^{i_1} \dots C_{j_t}^{i_t}$ to be $\sum_{l=1}^t i_l j_l$. We emphasize that weights of monomials R's and C's are compatible.

As in Section 2.2, for $\lambda \vdash n$ we denote the monomial symmetric function with exponents given by the parts of λ, in indeterminates $x_1, x_2, \dots$, by m_λ. Here, we consider the particular evaluation of the monomial symmetric function at $x_i = i$, for $i = 1, \dots, k-1$, and $x_i = 0$, for $i \geq k$, and write this as $\hat{m}_\lambda$. Let $D = z\frac{d}{dz}$, and I be the identity operator, and define $P_m(z)$ by

$$P_m(z) = -\frac{1}{m!}C(z)(D+(m-2)I)C(z)\dots(D+I)C(z)DC(z), \quad m \geq 1. \tag{3.26}$$

For example, we have

$$P_1(z) = -C(z), \quad P_2(z) = -\frac{1}{2}C(z)DC(z), \tag{3.27}$$

$$P_3(z) = -\frac{1}{6}\left(C(z)^2DC(z) + C(z)(DC(z))^2 + C(z)^2D^2C(z)\right).$$

Finally, for a partition λ, we write $P_\lambda(z) = \prod_{j=1}^{l(\lambda)} P_{\lambda_j}(z)$. We now state the main result.

Theorem 3.4.1 (Main Theorem). *For $n \geq 1$, $k \geq 2n-1$,*

$$\Sigma_{k,2n} = -\frac{1}{k}[z^{k+1-2n}] \sum_{\lambda \vdash 2n} \hat{m}_\lambda \frac{P_\lambda(z)}{C(z)}.$$

We postpone the proof of Theorem 3.4.1 until Section 3.6. There is a slight modification of this result, given below, in which the term corresponding to the partition with one part is given a simpler (but equivalent) evaluation.

Theorem 3.4.2. *For $n \geq 1$, $k \geq 2n-1$,*

$$\Sigma_{k,2n} = -\frac{1}{k}[z^{k+1-2n}] \left(\frac{k-1}{2n} \hat{m}_{2n} P_{2n-1}(z) + \sum_{\substack{\lambda \vdash 2n \\ l(\lambda) \geq 2}} \hat{m}_\lambda \frac{P_\lambda(z)}{C(z)} \right).$$

The following result gives a generating series form of the main result.

Theorem 3.4.3. *For $n \geq 1$, $k \geq 2n-1$,*

$$\Sigma_{k,2n} = -\frac{1}{k}[u^{2n} z^{k+1}] \frac{1}{C(z)} \prod_{j=1}^{k-1} (1 + \sum_{i \geq 1} j^i P_i(z) u^i z^i),$$

and

$$\Sigma_k = -\frac{1}{k}[z^{k+1}] \frac{1}{C(z)} \prod_{j=1}^{k-1} (1 + \sum_{i \geq 1} j^i P_i(z) z^i).$$

Note that, for each $n \geq 1$, these results give $\Sigma_{k,2n}$ as the coefficient of z^{k+1-2n} in a polynomial in $C(z)$ and

$$D^i C(z) = \sum_{m \geq 2} m^i C_m z^m, \quad i \geq 1.$$

Thus $\Sigma_{k,2n}$ is written as a polynomial in the C_m's, with coefficients that are rational in k, so the results here give *C-expansions* for $\Sigma_{k,2n}$, for $n \geq 1$.

We also postpone the proofs of Theorems 3.4.2 and 3.4.3 until Section 3.6. In the meantime we give some applications of the main theorems.

Using the above results, with the help of Maple, we have determined the C-expansions and the R-expansions of Σ_k (see Appendices A and B where we have listed the first 20 R-expansions and 22 C-expansions, respectively, of Kerov's polynomials. We listed only the first 20 R-expansions as for higher k the expansions are a number of pages long). Note that it easily follows from the main theorems that

$\Sigma_{k,0} = R_{k+1}$ (see Theorem 3.5.3 below). The R-expansions are in complete agreement with those reported in Biane [1] for $k \leq 11$. The C-expansions are given below for $k \leq 10$:

$$\begin{aligned}
\Sigma_1 - R_2 &= 0 \\
\Sigma_2 - R_3 &= 0 \\
\Sigma_3 - R_4 &= C_2 \\
\Sigma_4 - R_5 &= \tfrac{5}{2}C_3 \\
\Sigma_5 - R_6 &= 5\,C_4 + 8\,C_2 \\
\Sigma_6 - R_7 &= \tfrac{35}{4}\,C_5 + 42\,C_3 \\
\Sigma_7 - R_8 &= 14\,C_6 + \tfrac{469}{3}\,C_4 + \tfrac{203}{3}\,{C_2}^2 + 180\,C_2 \\
\Sigma_8 - R_9 &= 21\,C_7 + \tfrac{1869}{4}\,C_5 + \tfrac{819}{2}\,C_3C_2 + 1522\,C_3 \\
\Sigma_9 - R_{10} &= 30\,C_8 + 1197\,C_6 + \tfrac{963}{2}\,{C_3}^2 + 1122\,C_4C_2 + 81\,{C_2}^3 + \tfrac{26060}{3}\,C_4 + \tfrac{17680}{3}\,{C_2}^2 \\
&+ 8064\,C_2 \\
\Sigma_{10} - R_{11} &= \tfrac{165}{4}\,C_9 + \tfrac{5467}{2}\,C_7 + \tfrac{4433}{2}\,C_4C_3 + \tfrac{1133}{2}\,C_3{C_2}^2 + \tfrac{11033}{4}\,C_5C_2 + 38225\,C_5 \\
&+ 52580\,C_3C_2 + 96624\,C_3
\end{aligned}$$

Note the form of the data presented above. We have

$$\Sigma_k - \Sigma_{k,0} = \sum_{k \geq 1} \Sigma_{k,2n},$$

where $\Sigma_{k,0} = R_{k+1}$ remains on the left hand side, and we can recover the individual $\Sigma_{k,2n}$ on the right hand side: recall that the weight of the monomial $C_{m_1} \dots C_{m_i}$ is $m_1 + \dots + m_i$ and, therefore, from (3.23) and (3.25), $\Sigma_{k,2n}$ is the sum of all terms of weight $k + 1 - 2n$.

In the above C-expansions for $k \leq 10$, all nonzero coefficients are positive rationals, with apparently small denominators. In fact, we have computed all the data for $k = 25$ (though not included $k = 23, 24$ and 25 in Appendix B as each polynomial is a number of pages long). We do not have a precise conjecture about the denominators, but conjecture that the positivity holds for all k.

Conjecture 3.4.4. *For $n \geq 1$, $k \geq 2n - 1$, $\Sigma_{k,2n}$ is C-positive.*

This C-positivity conjecture implies the R-positivity conjecture, from (3.25) (so, the data in Appendix B also confirm the R-positivity conjecture for $k \leq 25$). Theorem 3.5.4 gives an immediate proof that Conjecture 3.4.4 holds for $n = 1$ and all k. In Corollary 3.5.10, we are able to prove that Conjecture 3.4.4 holds for $n = 2$ and all k. We are not able to prove the conjecture for any larger value of n, though Theorem 3.5.14 below, together with (3.24), proves that the linear terms are C-positive for all n. We shall see that the introduction of the indeterminates C_k and the generating series $C(z)$ simplify expressions a great deal. Moreover, we shall see how this introduction leads to new results about Stanley's polynomials in the next chapter.

The conjecture does not hold for $n = 0$, as described below. We have $\Sigma_{k,0} = R_{k+1}$, and it is straightforward to determine the C-expansion for the R_i's: from (3.24), we obtain

$$\begin{aligned} 1 - \sum_{i\geq 2}(i-1)R_i z^i &= \frac{1}{C(z)} \\ &= \sum_{j_2,j_3,\ldots\geq 0}(j_2+j_3+\ldots)!\prod_{m\geq 2}\frac{(-C_m z^m)^{j_m}}{j_m!}, \end{aligned}$$

so we conclude that

$$R_i = \frac{1}{i-1}\sum_{\substack{j_2,j_3,\ldots\geq 0\\ 2j_2+3j_3+\ldots=i}}(-1)^{1+j_2+j_3+\cdots}(j_2+j_3+\ldots)!\prod_{m\geq 2}\frac{C_m^{j_m}}{j_m!}, \qquad i\geq 2.$$

Thus, terms of negative sign appear in the C-expansion of R_i, for $i \geq 4$. This is the reason that we have presented the data for k up to 10 with R_{k+1} subtracted on the left hand side. This is also the reason that the R-positivity conjecture does not imply the C-positivity conjecture, so R-positivity and C-positivity are not equivalent.

3.5 Special Cases of the Main Result

We now give some special cases of the main result.

3.5.1 Monomial Symmetric Functions: A Computational Tool

To make the expression for $\Sigma_{k,2n}$ that arises from Theorem 3.4.1 (or Theorem 3.4.2) explicit, we need to evaluate the $\hat{m}_\lambda$, which are monomial symmetric functions in $1, 2, \ldots, k-1$. For general results about symmetric functions, see Macdonald [21].

Proposition 3.5.1. *For indeterminates a_i, $i \geq 1$, let $A(x) = 1 + \sum_{i\geq 1} a_i x^i$, and $a_\lambda = \prod_{j=1}^{l(\lambda)} a_{\lambda_j}$, where $\lambda = \lambda_1 \ldots \lambda_{l(\lambda)}$ is a partition. Then*

$$\sum_{\lambda\in\mathcal{P}} \hat{m}_\lambda a_\lambda = \exp \sum_{j\geq 1} \hat{m}_j \sum_{i\geq 1} \frac{(-1)^{i-1}}{i}[x^j](A(x)-1)^i.$$

Proof. We have

$$\begin{aligned}\sum_{\lambda\in\mathcal{P}} m_\lambda a_\lambda &= \prod_{n\geq 1} A(x_n)\\ &= \exp\sum_{n\geq 1}\log(A(x_n))\\ &= \exp\sum_{n\geq 1}\sum_{i\geq 1}\frac{(-1)^{i-1}}{i}(A(x_n)-1)^i,\end{aligned}$$

and the result follows. □

Proposition 3.5.1 gives an expression for $\hat{m}_\lambda$ as a polynomial in $\hat{m}_i$, $i \geq 1$, by equating coefficients of a_λ. To evaluate the $\hat{m}_i$, $i \geq 1$, we apply the following result.

Proposition 3.5.2. *For $j \geq 1$,*

$$\hat{m}_j = \sum_{i=1}^{j} S(j,i)i!\binom{k}{i+1},$$

where $S(j,i)$, the Stirling numbers of the second kind, *are given by*

$$\sum_{j\geq 0}\sum_{i=0}^{j} S(j,i)u^i\frac{x^j}{j!} = \exp u(e^x-1).$$

Proof. Using (3.13), we have

$$x^j = \sum_{i=0}^{n} S(j,i)i!\binom{x}{i}.$$

Summing both sides from $x=1$ to $x=k-1$ and using the identity

$$\sum_{j=1}^{k-1}\binom{j}{i} = \binom{k}{i+1},$$

we obtain the result. □

As special cases of this result, we have the following, well-known sums of integer powers.

$$\hat{m}_1 = \tfrac{1}{2}(k-1)k, \quad \hat{m}_2 = \tfrac{1}{6}(k-1)k(2k-1), \quad \hat{m}_3 = \tfrac{1}{4}(k-1)^2k^2, \tag{3.28}$$

$$\hat{m}_4 = \tfrac{1}{30}(k-1)k(2k-1)(3k^2-3k-1).$$

3.5.2 The Cases $n = 0, 1, 2$

In this section, we apply the main theorem to obtain specific results about the coefficients of terms in Kerov's polynomials. Note, from Biane's Theorem 3.3.4 it follows that the largest term in Σ_k is R_{k+1}. This follows easily from Theorem 3.4.1, which we consider now.

Proposition 3.5.3. *The term of highest degree in Σ_k is R_{k+1}, and it is the only term of weight $k+1$.*

Proof. From Theorem 3.4.1 the term of highest weight is $\Sigma_{k,0}$, which is

$$-\frac{1}{k}[z^{k+1-2n}]\frac{1}{C(z)} = R_{k+1},$$

giving the result. □

Next we consider the case $n = 1$ of Theorem 3.4.2. An expression for this case was conjectured by Biane [1, Conjecture 6.4]; specifically Biane conjectured that the terms of weight $k-1$ in Kerov's polynomials are given by

$$\frac{(k+1)k(k-1)}{24} \sum_{\substack{j_2,j_3,\ldots\geq 0 \\ 2j_2+3j_3+\ldots=k-1}} (j_2+j_3+\ldots)! \prod_{i\geq 2} \frac{((i-1)R_i)^{j_i}}{j_i!}, \quad k-1 \geq 2. \tag{3.29}$$

which is $\frac{1}{4}\binom{k+1}{3}$ times C_{k-1} given in (3.25). This was later proven in Śniady [25], by a combinatorial method, but more along the lines of the work done by Biane; it appears that the combinatorial proof given by Śniady is inspired by the free probability approach developed by Biane. The proof we give below is far more direct than Śniady's proof. We state the result as a theorem now.

Theorem 3.5.4. *In Σ_k for $k \geq 1$, the terms of weight $k-1$ are given by*

$$\Sigma_{k,2} = \tfrac{1}{24}(k-1)k(k+1)C_{k-1}.$$

In particular, $\Sigma_{k,2}$ is C-positive.

Proof. From Theorem 3.4.2, with $n = 1$, we obtain

$$\begin{aligned} \Sigma_{k,2} &= -\frac{1}{k}[z^{k-1}]\left(-\tfrac{1}{2}(k-1)\hat{m}_2 C(z) + \hat{m}_{11}C(z)\right) \\ &= \frac{1}{k}\left(\tfrac{1}{2}(k-1)\hat{m}_2 - \hat{m}_{11}\right)[z^{k-1}]C(z). \end{aligned}$$

But from Proposition 3.5.1, we obtain

$$\hat{m}_{11} = \tfrac{1}{2}(\hat{m}_1^2 - \hat{m}_2),$$

and the result follows from (3.28), by routine manipulation. □

From Theorem 3.5.4 we obtain easily the following corollaries.

Corollary 3.5.5. *In the R-expansion of Kerov's polynomial Σ_k, the terms of weight $k-1$ have positive coefficients.*

Proof. This follows directly from Theorem 3.5.4 and (3.25). □

Corollary 3.5.6. *The sum of the coefficients of the R's of terms of weight $k-1$ in Σ_k is $\frac{1}{4}\binom{k+1}{3}2^{k-3}$.*

Proof. By Theorem 3.5.4 the coefficient of $[z^{k-1}]$ in $C(z)$ is the collection of terms of degree $k-1$ in Σ_k. Of course, setting $R_i = 1$ for all i will yield the result. From (3.24) we see

$$C(z) = \frac{1}{-t^2 \frac{d}{dt}\frac{R(z)}{t}}.$$

Setting $R_i = 1$ for all i in $R(z)$, we obtain

$$\frac{d}{dz}\frac{R(z)}{z} = -\frac{1}{z^2} + \frac{1}{(1-z)^2} = \frac{(2z-1)}{z^2(1-z)^2},$$

from which it follows that

$$C(z) = \frac{(1-z)^2}{1-2z} = 1 + \frac{z^2}{1-2z}.$$

Taking the coefficient $[t^{k-1}]$ in the last expression and multiplying by $\frac{1}{4}\binom{k+1}{3}$ yields the result. □

Next we consider the case $n = 2$ of Theorem 3.4.2, to obtain an explicit C-expansion for $\Sigma_{k,4}$.

Theorem 3.5.7. *In Σ_k for $k \geq 3$, the terms of weight $k-3$ are given by*

$$\Sigma_{k,4} = \alpha(k) \sum_{\substack{i,j,m\geq 0 \\ i+j+m=k-3}} C_i C_j C_m + \beta(k) \sum_{\substack{i,j,m\geq 0 \\ i+j+m=k-3}} i^2 C_i C_j C_m,$$

where

$$\begin{aligned} \alpha(k) &= -\tfrac{1}{17280}(k-3)(k-1)^2 k(k+1)(k^2-4k-6), \\ \beta(k) &= \tfrac{1}{2880}(k-1)k(k+1)(2k^2-3). \end{aligned}$$

Proof. From Theorem 3.4.2, with $n = 2$, letting $b = \frac{1}{6}(\hat{m}_{31} - \frac{1}{4}(k-1)\hat{m}_4)$, we obtain

$$\begin{aligned}
\Sigma_{k,4} &= -\frac{1}{k}[t^{k-3}]\left(b\left(C(z)^2DC(z) + C(z)(DC(z))^2 + C(z)^2D^2C(z)\right)\right.\\
&+ \left.\tfrac{1}{4}\hat{m}_{22}C(z)\ (DC(z))^2 - \tfrac{1}{2}\hat{m}_{211}C(z)^2DC(z) + \hat{m}_{1111}C(z)^3\right)\\
&= -\frac{1}{k}[t^{k-3}]\left(\hat{m}_{1111}C(z)^3 + (b - \tfrac{1}{2}\hat{m}_{211})C(z)^2DC(z)\right.\\
&+ \left.bC(z)^2D^2C(z) + (b + \tfrac{1}{4}\hat{m}_{22})C(z)(DC(z))^2\right)\\
&= -\frac{1}{k}[t^{k-3}]\left(\hat{m}_{1111}C(z)^3 + (b - \tfrac{1}{2}\hat{m}_{211})\tfrac{1}{3}DC(z)^3\right.\\
&+ \left.bC(z)^2D^2C(z) + (b + \tfrac{1}{4}\hat{m}_{22})\left(\tfrac{1}{6}D^2C(z)^3 - \tfrac{1}{2}C(z)^2D^2C(z)\right)\right)\\
&= -\frac{1}{k}\left(\hat{m}_{1111} + \tfrac{1}{3}(k-3)(b - \tfrac{1}{2}\hat{m}_{211}) + \tfrac{1}{6}(k-3)^2(b + \tfrac{1}{4}\hat{m}_{22})\right)[t^{k-3}]C(z)^3\\
&- \frac{1}{k}\left(\tfrac{1}{2}b - \tfrac{1}{8}\hat{m}_{22}\right)[t^{k-3}]C(z)^2D^2C(z).
\end{aligned}$$

But from Proposition 3.5.1, we obtain

$$\begin{aligned}
\hat{m}_{31} &= \hat{m}_3\hat{m}_1 - \hat{m}_4,\\
\hat{m}_{22} &= \tfrac{1}{2}(\hat{m}_2^2 - \hat{m}_4),\\
\hat{m}_{211} &= \tfrac{1}{2}(\hat{m}_2\hat{m}_1^2 - 2\hat{m}_3\hat{m}_1 - \hat{m}_2^2 + 2\hat{m}_4),\\
\hat{m}_{1111} &= \tfrac{1}{24}(\hat{m}_1^4 - 6\hat{m}_2\hat{m}_1^2 + 8\hat{m}_3\hat{m}_1 + 3\hat{m}_2^2 - 6\hat{m}_4),
\end{aligned}$$

so from (3.28), by routine manipulation, we obtain

$$\Sigma_{k,4} = \alpha(k)[z^{k-3}]C(z)^3 + \beta(k)[z^{k-3}]C(z)^2D^2C(z), \tag{3.30}$$

where $\alpha(k)$ and $\beta(k)$ are given above. The result follows. □

For monomials in $R_2, R_3, \ldots$ that are pure powers of a single R_m, we have the following form of the above result.

Corollary 3.5.8. *For $m \geq 2$, $i \geq 1$,*

$$\begin{aligned}
[R_m^i]\Sigma_{mi+3,4} &= \tfrac{1}{34560}(m-1)^i mi(i+1)(i+2)(mi+2)(mi+3)(mi+4)\\
&\times\ (m^3i^3 + 2m^2(m+4)i^2 + 4m(3m+5)i + 15m + 18).
\end{aligned}$$

Proof. From Theorem 3.5.7, we obtain

$$[R_m^i]\Sigma_{mi+3,4} = \alpha(mi+3)[R_m^iz^{mi}]C(z)^3 + \beta(mi+3)[R_m^iz^{mi}]C(z)^2D^2C(z).$$

Now, setting $R_j = 0$ for $j \neq m$, we obtain $C(z) = (1-(m-1)R_m z^m)^{-1}$, so

$$[R_m^i z^{mi}]C(z)^3 = (m-1)^i \binom{i+2}{2}.$$

Also, we have

$$\begin{aligned}
D^2C(z) &= Dm(m-1)R_m z^m (1-(m-1)R_m z^m)^{-2} \\
&= Dm\left((1-(m-1)R_m z^m)^{-2} - (1-(m-1)R_m z^m)^{-1}\right) \\
&= m^2(m-1)\left(2R_m z^m (1-(m-1)R_m z^m)^{-3}\right. \\
&\qquad \left. -R_m z^m (1-(m-1)R_m z^m)^{-2}\right),
\end{aligned}$$

so

$$[R_m^i z^{mi}]C(z)^2 D^2 C(z) = (m-1)^i m^2 \left(2\binom{i+3}{4} - \binom{i+2}{3}\right).$$

The result follows by routine manipulation. □

The following conjecture of Stanley, communicated by Biane (private communication), is an immediate consequence of Corollary 3.5.8.

Corollary 3.5.9 (Conjectured by Stanley). *For $i \geq 1$,*

$$[R_2^i]\Sigma_{2i+3,4} = \tfrac{1}{540} i(i+1)^3(i+2)^3(i+3)(2i+3).$$

Proof. We set $m = 2$ in Corollary 3.5.8. Then the factor that is cubic in i becomes

$$8i^3 + 48i^2 + 88i + 48 = 8(i+1)(i+2)(i+3),$$

and the result follows. □

As the final result of this section, we are able to use the explicit C-expansion given in Theorem 3.5.7, to prove the C-positivity of $\Sigma_{k,4}$.

Corollary 3.5.10. *$\Sigma_{k,4}$ is C-positive for all $k \geq 3$.*

Proof. Consider $0 \leq i \leq j \leq m$, with $i + j + m = k - 3$, and let $\gamma = |\mathrm{Aut}(i,j,m)|$. Thus when $k = 12$, for example, $\gamma = 1$ for $(i,j,m) = (2,3,4)$ or $(0,2,7)$, $\gamma = 2$ for $(i,j,m) = (2,2,5)$ or $(1,4,4)$, and $\gamma = 6$ for $(i,j,m) = (3,3,3)$. Then, from Theorem 3.5.7, we obtain

$$[C_i C_j C_m]\Sigma_{k,4} = \frac{6}{\gamma}\alpha(k) + \frac{2}{\gamma}(i^2 + j^2 + m^2)\beta(k). \tag{3.31}$$

Now, the minimum value of $x^2 + y^2 + z^2$ over the reals, subject to $x + y + z = c$, for any fixed real c, is achieved at $x = y = z = c/3$, so in the above expression we have $i^2 + j^2 + m^2 \geq \frac{1}{3}(k-3)^2$. But $\beta(k) > 0$ for $k \geq 3$, so we obtain

$$\begin{aligned}
[C_iC_jC_m]\Sigma_{k,4} &\geq \frac{2}{\gamma}\left(3\alpha(k) + \tfrac{1}{3}(k-3)^2\beta(k)\right) \\
&= \tfrac{1}{8640\gamma}(k-3)(k-1)k(k+1)\left(-3(k-1)(k^2-4k-6)\right. \\
&\qquad\qquad\qquad\qquad\qquad \left.+2(k-3)(2k^2-3)\right) \\
&= \tfrac{1}{8640\gamma}(k-3)(k-1)k^3(k+1)(k+3) \\
&\geq 0,
\end{aligned}$$

for $k \geq 3$, giving the result. □

Corollary 3.5.11 (Conjectured by Biane and Kerov). *$\Sigma_{k,4}$ is R-positive for all $k \geq 3$.*

Proof. Follows from Corollary 3.5.10 and (3.25). □

3.5.3 The Case $n = 3$

In this section, we give a compact expression for $\Sigma_{k,6}$ in terms of our C's. We are not able, however, to use this expression to show positivity. This illustrates that Theorem 3.4.1 alone does not, unfortunately, fully explain Kerov's polynomials. We do, however, hope that the work done in the chapter serves as a good basis for future work. To simplify our notation in the following theorem and proof, we will replace $C(z)$ with C and $P_\lambda(z)$ with P_λ.

Theorem 3.5.12. *In Σ_k for $k \geq 5$, the terms of weight $k-5$ are given by*

$$\begin{aligned}
\Sigma_{k,6} = -\frac{1}{k}\,[z^{k-5}]\ \Big(&-\frac{1}{362880}\,k\,(k-1)\,\Big(1918\,k^7 - 21041\,k^6 + 74635\,k^5 - 102143\,k^4 \\
&+31879\,k^3 + 26860\,k^2 - 4416\,k - 3780\Big)\,C^2(DC)^3 \\
&- \frac{1}{725760}\,k\,(k-1)\,\Big(30\,k^7 + 213\,k^6 - 2009\,k^5 + 4193\,k^4 - 2254\,k^3 - 847\,k^2 \\
&+292\,k + 60\Big)\,C^4(D^3C) \\
&+ \frac{1}{120960}\,k\,(3\,k-5)\,(k-1)\,\Big(111\,k^5 - 392\,k^4 + 277\,k^3 + 132\,k^2 - 42\,k - 24\Big)\,C(DC)^4 \\
&- \frac{1}{362880}\,k\,(k-1)\,\Big(507\,k^6 - 2589\,k^5 + 4159\,k^4 - 1511\,k^3 - 1154\,k^2 + 232\,k \\
&+180\Big)\,C^3(D^2C)^2 \\
&+ \frac{1}{725760}\,k\,(k-1)\,\Big(249\,k^6 - 1299\,k^5 + 2096\,k^4 - 739\,k^3 - 592\,k^2 + 101\,k \\
&+90\Big)\,C^4(D^4C) \\
&+ \frac{1}{241920}\,k\,(k-1)\,\Big(630\,k^8 - 10052\,k^7 + 59791\,k^6 - 161489\,k^5 + 190331\,k^4 - 51967\,k^3 \\
&-46584\,k^2 + 7036\,k + 6080\Big)\,C^3(DC)^2 \\
&+ \frac{1}{725760}\,k\,(k-1)\,\Big(15\,k^8 + 162\,k^7 - 2407\,k^6 + 8424\,k^5 - 10357\,k^4 + 1907\,k^3 + 3159\,k^2 \\
&-313\,k - 390\Big)\,C^4(D^2C) \\
&- \frac{1}{483840}\,k\,(k-1)\,\Big(210\,k^9 - 4305\,k^8 + 35392\,k^7 - 147530\,k^6 \\
&+322402\,k^5 - 332609\,k^4 + 80524\,k^3 + 74812\,k^2 - 10560\,k - 9120\Big)\,C^4(DC) \\
&+ \frac{1}{2903040}\,k\,(k-1)\,(k-2)\,(k-3)\,(k-4)\,(k-5)\,(k-6)\,\Big(63\,k^5 - 315\,k^4 + 315\,k^3 \\
&+91\,k^2 - 42\,k - 16\Big)\,C^5\Big)\,.
\end{aligned}$$

Proof. From Theorem 3.4.2 we have

$$\Sigma_{k,6} = -\frac{1}{k}\,[z^{k-5}]\ \left(\frac{k-1}{6}\hat{m}_6 P_5 + \sum_{\substack{\lambda\vdash 6\\ \ell(\lambda)\geq 2}} \hat{m}_\lambda \frac{P_\lambda}{C}\right).$$

In (3.27) we have already computed P_1, P_2 and P_3. We now find P_4 and P_5. Recall that the differential operator D does not commute with C. Therefore, we use brackets to indicate when an operation has taken place; that is, $D(CDC)$ indicates that the preceding D is still to operate on CDC, whereas $(DC)(DC) = (DC)^2$. We have

$$\begin{aligned}
P_4 &= -\frac{1}{24}C(D+2I)C(D+I)CDC \\
&= -\frac{1}{24}C(D+2I)(-6P_3) \\
&= -\frac{1}{24}\left(CD\left(C^2DC + C(DC)^2 + C^2D^2C\right) + 2C^3DC + 2C^2(DC)^2 + 2C^3(D^2C)\right) \\
&= -\frac{1}{24}\left(4C^2(DC)^2 + 3C^3D^2C + C(DC)^3 + 4C^2(DC)(D^2C) + C^3D^3C + 2C^3DC\right),
\end{aligned}$$

and

$$\begin{aligned}
P_5 =& -\frac{1}{120}C(D+3I)(-24P_4) \\
=& -\frac{1}{120}\Big(8C^2(DC)^3 + 8C^3(DC)(D^2C) + 9C^3(DC)(D^2C) + 3C^4D^3C + C(DC)^4 \\
&+ 3C^2(DC)^2(D^2C) + 8C^2(DC)^2D^2C + 4C^3(D^2C)^2 + 4C^3(DC)(D^3C) \\
&+ 3C^3(DC)(D^3C) + C^4D^4C + 6C^3(DC)^2 + 2C^4(D^2C) + 12C^3(DC)^2 \\
&+ 9C^4D^2C + 3C^2(DC)^3 + 12C^3(DC)(D^2C) + 3C^4(D^3C) + 6C^4DC\Big) \\
=& -\frac{1}{120}\Big(11C^2(DC)^3 + 29C^3(DC)(D^2C) + 6C^4D^3C + C(DC)^4 + 11C^2(DC)^2(D^2C) \\
&+ 4C^3(D^2C)^2 + 7C^3(DC)D^3C + C^4D^4C + 18C^3(DC)^2 + 11C^4D^2C \\
&+ 6C^4DC\Big).
\end{aligned}$$

This gives

$$\begin{aligned}
\Sigma_{k,6} =& -\frac{1}{k}[z^{k-5}]\left(\left(\frac{k-1}{6}\hat{m}_6 - \hat{m}_{51}\right)P_5 - \frac{1}{2}\hat{m}_{42}(DC)P_4 + \hat{m}_{33}\frac{P_3^2}{C} + \hat{m}_{411}CP_4\right. \\
&- \hat{m}_{3111}C^2P_3 - \hat{m}_{321}P_3P_2 - \frac{1}{8}\hat{m}_{222}C^2(DC)^3 + \hat{m}_{2211}CP_2^2 + \hat{m}_{21111}C^3P_2 \\
&\left. + \hat{m}_{111111}C^5\right) \\
=& -\frac{1}{k}[z^{k-5}]\left(\left(\frac{k-1}{6}\hat{m}_6 - \hat{m}_{51}\right)P_5 + \left(\hat{m}_{411}C - \frac{1}{2}\hat{m}_{42}(DC)\right)P_4\right. \\
&+ \left(\hat{m}_{33}\frac{P_3}{C} - \hat{m}_{3111}C^2 + \frac{1}{2}\hat{m}_{321}CDC\right)P_3 - \frac{1}{8}\hat{m}_{222}C^2(DC)^3 \\
&\left. + \frac{1}{4}\hat{m}_{2211}C^3(DC)^2 - \frac{1}{2}\hat{m}_{21111}C^4(DC) + \hat{m}_{111111}C^5\right).
\end{aligned}$$

In order to deal with the last expression, we divide it up into two parts: the terms involving P_5 and P_4 in the first part, and the remaining terms in the second part.

Setting $d = -\frac{1}{120}\left(\frac{k-1}{6}\hat{m}_6 - \hat{m}_{51}\right)$, the first part becomes

$$
\begin{aligned}
&dP_5 + \left(\hat{m}_{411}C - \frac{\hat{m}_{42}}{2}DC\right)P_4 \\
&\quad = \left(11d - \frac{\hat{m}_{411}}{24} + \frac{\hat{m}_{42}}{12}\right)C^2(DC)^3 + \left(29d - \frac{\hat{m}_{411}}{6} + \frac{\hat{m}_{42}}{16}\right)C^3(DC)D^2C \\
&\quad + \left(6d - \frac{\hat{m}_{411}}{24}\right)C^4D^3C + \left(d + \frac{\hat{m}_{42}}{48}\right)C(DC)^4 + \left(11d + \frac{\hat{m}_{42}}{12}\right)C^2(DC)^2D^2C \\
&\quad + 4dC^3(D^2C)^2 + \left(7d + \frac{\hat{m}_{42}}{48}\right)C^3(DC)D^3C + dC^4D^4C \\
&+ \left(18d - \frac{\hat{m}_{411}}{6} + \frac{\hat{m}_{42}}{24}\right)C^3(DC)^2 + \left(11d - \frac{\hat{m}_{411}}{8}\right)C^4D^2C + \left(6d - \frac{\hat{m}_{411}}{12}\right)C^4DC.
\end{aligned}
$$

Simplifying the second part we have

$$
\begin{aligned}
&-\frac{\hat{m}_{33}}{6}\left(\left(CDC + (DC)^2 + CD^2C\right) - \hat{m}_{3111}C^2 + \frac{\hat{m}_{321}}{2}CDC\right) \\
&\qquad \cdot \left(-\frac{1}{6}\left(C^2DC + C(DC)^2 + C^2D^2C\right)\right) \\
&\qquad = \left(\frac{\hat{m}_{33}}{36} + \frac{\hat{m}_{3111}}{6} - \frac{\hat{m}_{321}}{12}\right)C^3(DC)^2 \\
&\qquad + \left(\frac{\hat{m}_{33}}{18} - \frac{\hat{m}_{321}}{12}\right)\left(C^2(DC)^3 + C^3(DC)(D^2C)\right) \\
&+ \frac{\hat{m}_{33}}{36}\left(C(DC)^4 + 2C^2(DC)^2(D^2C) + C^3(D^2C)^2\right) + \frac{\hat{m}_{3111}}{6}\left(C^4DC + C^4D^2C\right).
\end{aligned} \tag{3.32}
$$

If we set

$$
a = \frac{\hat{m}_{33}}{36} + \frac{\hat{m}_{3111}}{6} - \frac{\hat{m}_{321}}{12}, \qquad b = \frac{\hat{m}_{33}}{18} - \frac{\hat{m}_{321}}{12},
$$

then the expression in (3.32) becomes

$$
\begin{aligned}
&\left(a + \frac{\hat{m}_{2211}}{4}\right)C^3(DC)^2 + \left(\frac{\hat{m}_{3111}}{6} - \frac{\hat{m}_{21111}}{2}\right)C^4DC \\
&\qquad + \left(b - \frac{\hat{m}_{222}}{8}\right)C^2(DC)^3 + bC^3(DC)(D^2C) \\
&+ \frac{\hat{m}_{33}}{36}\left(C(DC)^4 + 2C^2(DC)^2(D^2C) + C^3(D^2C)^2\right) + \frac{\hat{m}_{3111}}{36}\left(C^4D^2C\right) + \hat{m}_{111111}C^5.
\end{aligned}
$$

Therefore, we have

$$\begin{aligned}\Sigma_{k,6} = -\frac{1}{k}[z^{k-5}]\Bigg(&\left(b-\frac{1}{8}\hat{m}_{222}+11d-\frac{\hat{m}_{411}}{24}+\frac{\hat{m}_{42}}{12}\right)C^2(DC)^3\\
&+\left(b+29d-\frac{\hat{m}_{411}}{6}+\frac{\hat{m}_{42}}{18}\right)C^3(DC)(D^2C)+\left(6d-\frac{\hat{m}_{411}}{24}\right)C^4D^3C\\
&+\left(\frac{\hat{m}_{33}}{36}+d+\frac{\hat{m}_{42}}{48}\right)C(DC)^4+\left(\frac{\hat{m}_{33}}{18}+11d+\frac{\hat{m}_{42}}{12}\right)C^2(DC)^2D^2C\\
&+\left(\frac{\hat{m}_{33}}{36}+4d\right)C^3(D^2C)^2+\left(7d+\frac{\hat{m}_{42}}{48}\right)C^3(DC)D^3C+dC^4D^4C\\
&+\left(a+\frac{\hat{m}_{2211}}{4}+18d-\frac{\hat{m}_{411}}{6}+\frac{\hat{m}_{42}}{24}\right)C^3(DC)^2\\
&+\left(\frac{\hat{m}_{3111}}{6}+11d-\frac{\hat{m}_{411}}{8}\right)C^4D^2C\\
&+\left(\frac{\hat{m}_{3111}}{6}-\frac{\hat{m}_{21111}}{2}+6d-\frac{\hat{m}_{411}}{12}\right)C^4DC+\hat{m}_{111111}C^5\Bigg).\end{aligned}$$

To simplify the above expression further, we apply the rule $[z^{k-5}]\,Df = (k-5)\,[z^{k-5}]\,f$. Using the product rule for differentiation, we apply this rule to the following terms; the aim is to reduce the number of distinct terms involving the series C in the last expression.

1.
$$D(C^3(DC)D^2C) = 3C^2(DC)^2D^2C + C^3(D^2C)^2 + C^3(DC)D^3C,$$

implying

$$C^2(DC)^2(D^2C) = \frac{1}{3}D(C^3(DC)D^2C) - \frac{1}{3}C^3(D^2C)^2 - \frac{1}{3}C^3(DC)D^3C.$$

2.
$$D(C^4D^3C) = 4C^3(DC)D^3C + C^4D^4C,$$

implying

$$C^3(DC)D^3C = \frac{1}{4}D(C^4D^3C) - \frac{1}{4}C^4D^4C.$$

3.
$$D(C^4D^2C) = 4C^3(DC)D^2C + C^4D^3C,$$

implying

$$C^3(DC)D^2C = \frac{1}{4}D(C^4D^2C) - \frac{1}{4}C^4D^3C.$$

Thus, for example using 2. above, we eliminate the term $C^2(DC)^2D^2C$ by

$$\begin{aligned}
[z^{k-5}]\, C^2(DC)^2(D^2C) &= [z^{k-5}]\, \left(\frac{1}{4}D(C^4D^3C) - \frac{1}{4}C^4D^4C\right) \\
&= [z^{k-5}]\, \frac{1}{4}D(C^4D^3C) - [z^{k-5}]\, \frac{1}{4}C^4D^4C \\
&= [z^{k-5}]\, \frac{1}{4}(k-5)C^4D^3C - [z^{k-5}]\, \frac{1}{4}C^4D^4C \\
&= [z^{k-5}]\, \left(\frac{1}{4}(k-5)C^4D^3C - \frac{1}{4}C^4D^4C\right).
\end{aligned}$$

Doing this in turn for the expressions in 1., 2. and 3., and substituting the original values for the parameters a and b into $\Sigma_{k,6}$, we obtain after simplifying,

$$\begin{aligned}
\Sigma_{k,6} = -\frac{1}{k}\,[z^{k-5}]\, \Bigg(&\left(\frac{\hat{m}_{33}}{18} - \frac{\hat{m}_{321}}{12} - \frac{\hat{m}_{222}}{8} + 11d - \frac{\hat{m}_{411}}{24} + \frac{\hat{m}_{42}}{12}\right) C^2(DC)^3 \\
&+ \left(\left(-\frac{5}{4} - \frac{(k-5)}{12}\right) d + \left(-\frac{5}{576}(k-5) - \frac{1}{64}\right)\hat{m}_{42} + \left(-\frac{1}{108}(k-5)\right.\right. \\
&\left.\left. -\frac{1}{72}\right)\hat{m}_{33} + \frac{\hat{m}_{321}}{48}\right) C^4D^3C + \left(d + \frac{\hat{m}_{33}}{36} + \frac{\hat{m}_{42}}{48}\right) C(DC)^4 \\
&+ \left(\frac{d}{3} + \frac{\hat{m}_{33}}{108} - \frac{\hat{m}_{42}}{36}\right) C^3(D^2C)^2 + \left(\frac{d}{6} + \frac{\hat{m}_{42}}{576} + \frac{\hat{m}_{33}}{216}\right) C^4D^4C \\
&+ \left(\frac{\hat{m}_{33}}{36} + \frac{\hat{m}_{3111}}{6} - \frac{\hat{m}_{321}}{12} + \frac{\hat{m}_{2211}}{4} + 18d - \frac{\hat{m}_{411}}{6} + \frac{\hat{m}_{42}}{24}\right) C^3(DC)^2 \\
&+ \left(\frac{\hat{m}_{3111}}{6} + \left(\frac{11(k-5)^2}{12} + \frac{29(k-5)}{4} + 11\right) d - \left(\frac{(k-5)}{24} + \frac{1}{8}\right)\hat{m}_{411}\right. \\
&\left. + \left(\frac{(k-5)^2}{216} + \frac{k-5}{72}\right)\hat{m}_{33} - \frac{k-5}{48}\hat{m}_{321} + \left(\frac{(k-5)^2}{144} + \frac{k-5}{64}\right)\hat{m}_{42}\right) C^4D^2C \\
&+ \left(\frac{\hat{m}_{3111}}{6} - \frac{\hat{m}_{21111}}{2} + 6d - \frac{\hat{m}_{411}}{12}\right) C^4DC + \hat{m}_{111111}C^5 \Bigg).
\end{aligned} \tag{3.33}$$

Using Propositions 3.5.1 and 3.5.2 we have

$$\begin{aligned}
\hat{m}_{111111} =& \frac{1}{2903040}\, k\,(k-1)\,(k-2)\,(k-3)\,(k-4)\,(k-5)\,(k-6) \\
&\cdot \left(63\,k^5 - 315\,k^4 + 315\,k^3 + 91\,k^2 - 42\,k - 16\right) \\
\hat{m}_{21111} =& \frac{1}{241920}\, k\,(k-1)\,(k-2)\,(k-3)\,(k-4)\,(k-5) \\
&\cdot \left(210\,k^5 - 945\,k^4 + 868\,k^3 + 273\,k^2 - 118\,k - 48\right) \\
\hat{m}_{3111} =& \frac{1}{20160}\, k\,(k-1)\,(k-2)\,(k-3)\,(k-4) \\
&\cdot \left(105\,k^5 - 399\,k^4 + 315\,k^3 + 123\,k^2 - 44\,k - 20\right) \\
\hat{m}_{222} =& \frac{1}{45360}\, k\,(k-1)\,(k-2)\,(k-3)\,(2\,k-1)\,(2\,k-3)\,(2\,k-5)\,\left(35\,k^2 + 21\,k + 4\right) \\
\hat{m}_{51} =& \frac{1}{168}\, k\,(k-1)\,(k-2)\,\left(14\,k^5 - 38\,k^4 + 19\,k^3 + 14\,k^2 - 3\,k - 2\right) \\
\hat{m}_{321} =& \frac{1}{2520}\, k\,(k-1)\,(k-2)\,(k-3) \\
&\cdot \left(105\,k^5 - 378\,k^4 + 279\,k^3 + 113\,k^2 - 39\,k - 20\right) \\
\hat{m}_{2211} =& \frac{1}{60480}\, k\,(k-1)\,(k-2)\,(k-3)\,(k-4) \\
&\cdot \left(420\,k^5 - 1736\,k^4 + 1477\,k^3 + 494\,k^2 - 205\,k - 90\right) \\
\hat{m}_{33} =& \frac{1}{672}\, k\,(k-1)\,(k-2)\,\left(21\,k^5 - 69\,k^4 + 45\,k^3 + 21\,k^2 - 6\,k - 4\right) \\
\hat{m}_{42} =& \frac{1}{1260}\, k\,(k-1)\,(k-2)\,(2\,k-1)\,(2\,k-3)\,\left(21\,k^3 - 24\,k^2 - 22\,k - 5\right) \\
\hat{m}_{6} =& \frac{1}{42}\, k\,(2\,k-1)\,(k-1)\,\left(3\,k^4 - 6\,k^3 + 3\,k + 1\right) \\
\hat{m}_{411} =& \frac{1}{5040}\, k\,(k-1)\,(k-2)\,(k-3) \\
&\cdot \left(126\,k^5 - 399\,k^4 + 258\,k^3 + 134\,k^2 - 39\,k - 20\right).
\end{aligned}$$

Substituting these monomial symmetric functions into (3.33) and simplifying gives the desired result. □

We see from the above proof that $P_i(z)$ becomes substantially more difficult to compute as we increase i.

We end this section with an observation that may seem trivial in light of Theorems 3.5.7 and 3.5.12 (or simply Theorem 3.4.1); we shall, however, find it useful in the next chapter.

Theorem 3.5.13. *For $k \geq 1$,*

$$\Sigma_{k,2n} = \sum_{\substack{i_1,i_2,\ldots,i_{2n-1}\geq 0 \\ i_1+i_2+\cdots+i_{2n-1}=k+1-2n}} \gamma_{i_1,i_2,\ldots,i_{2n-1}} C_{i_1} \cdot C_{i_2} \cdots C_{i_{2n-1}}$$

where the C_t are given in (3.24) *and the γ's are rational. In particular, $\Sigma_{k,2n}$ is C-positive (and, consequently, R-positive) if all $\gamma_{i_1,i_2,\ldots,i_{2n-1}}$ are positive.*

3.5.4 The Linear Terms

Previously, for $n \geq 1$, only one explicit result was known; the following result for the linear coefficients is due to Biane [1] and Stanley [29]. What follows is an original proof based on the results of the last section.

Theorem 3.5.14 (Biane, Stanley). *For $n \geq 1$, $k \geq 2n-1$, the coefficient of R_{k+1-2n} in $\Sigma_{k,2n}$ is equal to the number of k-cycles c in $\mathfrak{S}_k$ such that $(1\ldots k)c$ has $k-2n$ cycles.*

Proof. For $i \geq 1$, let $A^{(i)}(z)$ consist of the terms in $P_i(z)$ that are linear in the C_m's. Also, let $L_{n,k} = [R_{k+1-2n}]\Sigma_{k,2n}$. We apply Theorem 3.4.3 to determine $L_{n,k}$. From (3.24), we have

$$\begin{aligned}
L_{n,k} &= \left[\frac{C_{k+1-2n}}{k-2n}\right]\Sigma_{k,2n} = \left[\frac{C_{k+1-2n}}{k-2n}\right]\Sigma_k \\
&= -\frac{1}{k}\left[\frac{C_{k+1-2n}}{k-2n}z^{k+1}\right]\frac{1}{C(z)}\prod_{j=1}^{k-1}(1-jz+\sum_{i\geq 1} j^i A^{(i)}(z)z^i) \\
&= -\frac{1}{k}\left[\frac{C_{k+1-2n}}{k-2n}z^{k+1}\right]\frac{1}{C(z)}\left(\prod_{j=1}^{k-1}(1+\sum_{i\geq 1}\frac{j^i A^{(i)}(z)z^i}{1-jz})\right)\prod_{a=1}^{k-1}(1-az) \\
&= -\frac{1}{k}\left[\frac{C_{k+1-2n}}{k-2n}z^{k+1}\right]\left(1-C(z)+\sum_{j=1}^{k-1}\sum_{i\geq 1}\frac{j^i A^{(i)}(z)z^i}{1-jz}\right)\prod_{a=1}^{k-1}(1-az).
\end{aligned}$$

But, for $i \geq 1$,

$$A^{(i)}(z) = -\frac{1}{i!}(D+(i-2)I)\ldots(D+I)DC(z) = -\sum_{m\geq 2}\binom{-(m-1)}{i}(-1)^i\frac{C_m}{m-1}z^m.$$

Now let $\frac{C_m}{m-1} = x^{m-1}$, $m \geq 2$, which gives

$$\begin{aligned}
\sum_{i\geq 1} j^i A^{(i)}(z)z^i &= -\sum_{m\geq 2}\left((1-jz)^{-(m-1)}-1\right)x^{m-1}z^m \\
&= -\frac{z}{1-\frac{xz}{1-jz}}+\frac{z}{1-xz},
\end{aligned}$$

and

$$1 - C(z) = -\sum_{m\geq 2}(m-1)x^{m-1}z^m = -\frac{z}{(1-xz)^2} + \frac{z}{1-xz}.$$

Thus we obtain

$$L_{n,k} = \frac{1}{k}[x^{k-2n}z^{k+1}]\left(\frac{z}{(1-xz)^2} - \frac{z}{1-xz} + \sum_{j=1}^{k-1}\left(\frac{z}{1-(j+x)z} - \frac{z}{(1-jz)(1-xz)}\right)\right) \cdot \prod_{a=1}^{k-1}(1-az).$$

We now finish the proof using the method of Biane [1, Theorem 6.1]: Replace z by z^{-1}, and multiply by z^k, to obtain

$$L_{n,k} = \frac{1}{k}[x^{k-2n}][z^{-1}]_\infty(z)_k\left(\frac{z}{(z-x)^2} - \frac{1}{z-x} + \sum_{j=1}^{k-1}\left(\frac{1}{z-j-x} - \frac{z}{(z-j)(z-x)}\right)\right).$$

Using Proposition 2.5.1, in the previous equation we may substitute $z+c$ for z, where c is independent of z. Thus, substituting $z+j+x$ for z in the first term of the summation over j, and substituting $z+x$ for z in all other terms, we obtain

$$L_{n,k} = \frac{1}{k}[x^{k-2n}]\left([z](z+x)(z+x)_k - (x)_k + \sum_{j=1}^{k-1}\left((x+j)_k - \frac{x(x)_k}{x-j}\right)\right) \tag{3.34}$$

$$= \frac{1}{k}[x^{k-2n}]\sum_{j=0}^{k-1}(x+j)_k. \tag{3.35}$$

The rest of the proof is found in Biane [1]; there, however, the proof is very brief, so we include a more complete version here.

For any partition $\lambda \vdash k$ consider the generating series

$$Q_\lambda(x) = \prod_{u\in\lambda}(x+c(u)) \tag{3.36}$$

as well as the series

$$T_k(x,y) = \frac{1}{k!}\sum_{\lambda\vdash k} f^\lambda \chi_\lambda(c_k)Q_\lambda(x)Q_\lambda(y).$$

where c_k is the k-cycle. By Theorem 2.2.8 series $Q_\lambda(x)$ satisfies

$$Q_\lambda(x) = \sum_{\sigma\in\mathfrak{S}_k}\frac{\chi_\lambda(\sigma)}{f^\lambda}x^{\ell(\sigma)}$$

$$= \sum_{\beta\vdash k}\frac{|C_\beta|}{f^\lambda}\chi_\lambda(\beta)x^{\ell(\beta)},$$

where, from Section 2.2.1, the set C_β is the conjugacy class in $\mathfrak{S}_n$ of elements associated with the partition β and f^λ is the degree of χ_λ. Thus, we have

$$\begin{aligned} T_k(x,y) &= \frac{1}{k!}\sum_{\lambda\vdash k}\sum_{\alpha\vdash k}\sum_{\beta\vdash k} f^\lambda \chi_\lambda(c_k)\frac{|C_\alpha|}{f^\lambda}\chi_\lambda(\alpha)\frac{|C_\beta|}{f^\lambda}\chi_\lambda(\beta) \\ &= \sum_{\alpha,\beta\vdash k} x^{\ell(\alpha)}y^{\ell(\beta)}\frac{|C_\alpha||C_\beta|}{k!}\sum_{\lambda\vdash k}\frac{1}{f^\lambda}\chi_\lambda(c_k)\chi_\lambda(\alpha)\chi_\lambda(\beta). \end{aligned} \tag{3.37}$$

But, from (2.3), Section 2.1.1, we have

$$c^{\gamma}_{\alpha,\beta} = [K_\gamma]K_\alpha K_\beta = \frac{|C_\alpha||C_\beta|}{k!}\sum_{\lambda\vdash k}\frac{1}{f^\lambda}\chi_\lambda(\gamma)\chi_\lambda(\alpha)\chi_\lambda(\beta).$$

Therefore, it follows from (3.37) that

$$T_k(x,y) = \sum_{\alpha,\beta\vdash k} c^{c_k}_{\alpha,\beta} x^{\ell(\alpha)}y^{\ell(\beta)}.$$

which gives

$$T_k(x,y) = \sum_{\sigma\in\mathfrak{S}_n} x^{\ell(\sigma^{-1}c_k)}y^{\ell(\sigma)}. \tag{3.38}$$

To prove the final result, we find the coefficient of $x^{k-2n}y$ of the left hand side and the right hand side of (3.38).

We first find the coefficient of the left hand side of (3.38). Note that $\chi_\lambda(c_k) = 0$ unless $\lambda = (k-i\ 1^i)$; that is, $\chi_\lambda(c_k)$ is 0 unless λ is a hook. When λ is the hook $(k-i\ 1^i)$, the character $\chi_\lambda(c_k) = (-1)^i$ (this is a direct consequence of the Murnaghan-Nakayama rule, Theorem 2.3.1). Thus, from (3.36), when λ is the hook $(k-i\ 1^i)$ we have

$$\begin{aligned} Q_\lambda(x) &= \prod_{u\in\lambda}(x+c(u)) \\ &= (x-1)\cdots(x-i)\cdot x\cdot(x+1)\cdots(x+k-i-1) \\ &= (x+k-i-1)_k. \end{aligned}$$

The degree $f^{(k-i\ 1^i)}$ of the hook λ can be computed as follows. By Theorem 2.2.1, the degree $f^{(k-i\ 1^i)}$ is the number of SYT of the hook $(k-i\ 1^i)$, which is clearly $\binom{k-1}{i}$.

Thus, we obtain

$$\begin{aligned}
[y]\,\frac{1}{k!}Q_\lambda(y) &= [y]\,\frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}\frac{\chi_\lambda(\sigma)}{f^\lambda}y^{\ell(\sigma)}\\
&= \sum_{\substack{\sigma\in\mathfrak{S}_k\\ \ell(\sigma)=1}}\frac{\chi_\lambda(\sigma)}{f^\lambda}\\
&= \frac{1}{k!}\frac{(-1)^i}{\binom{k-1}{i}}(k-1)!\\
&= \frac{1}{k}\frac{(-1)^i}{\binom{k-1}{i}}.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
[x^{k-2n}y]\,T_k(x,y) &= [x^{k-2n}y]\,\frac{1}{k!}\sum_{\lambda\vdash k}f^\lambda\chi_\lambda(c_k)Q_\lambda(x)Q_\lambda(y)\\
&= [x^{k-2n}]\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}(x+k-j-1)_k\frac{1}{k}\frac{(-1)^j}{\binom{k-1}{j}}\\
&= [x^{k-2i}]\frac{1}{k}\sum_{j=0}^{k-1}(x+j)_k.
\end{aligned}$$

But, from (3.35), this is the coefficient of the linear term $[R_{k+1-2i}]\Sigma_k$. Now, the right hand side of (3.38) is

$$\begin{aligned}
[x^{k-2n}y]\sum_{\sigma\in\mathfrak{S}}x^{\ell(\sigma^{-1}c_k)}y^{\ell(\sigma)} &= [x^{k-2n}]\sum_{\substack{\sigma\in\mathfrak{S}\\ \ell(\sigma)=1}}x^{\ell(\sigma^{-1}c_k)}\\
&= \sum_{\substack{\sigma\in\mathfrak{S}_k\\ \ell(\sigma)=1,\ \ell(\sigma^{-1}c_k)=k-2n}}1,
\end{aligned}$$

completing the proof. □

3.6 Lagrange Inversion and the Proof of the Main Result

As a first step, we translate Corollary 3.3.12 into formal power series, using the notation

$$\phi(x)=xG(x^{-1}),\qquad \Phi(x,u)=\sum_{i\geq 0}\Phi_i(x)u^i=(1-ux)\phi(x(1-ux)^{-1}),\qquad (3.39)$$

where $G(x)$ is defined in (3.21).

Proposition 3.6.1. *The following two equations hold.*

1. *For $k \geq 1$,*

$$\Sigma_k = -\frac{1}{k}[x^{k+1}] \prod_{j=0}^{k-1} \Phi(x, j). \tag{3.40}$$

2. *For $k, n \geq 1$,*

$$\Sigma_{k,2n} = -\frac{1}{k}[u^{2n} x^{k+1}] \prod_{j=0}^{k-1} \Phi(x, ju). \tag{3.41}$$

Proof. For (3.40), we first replace x by x^{-1} in Corollary 3.3.12, to obtain

$$\Sigma_k = -\frac{1}{k}[x^{k+1}] \prod_{j=0}^{k-1} xG(x^{-1}(1-jx)),$$

and the result follows immediately.

For (3.41), we let ϑ be the substitution operator $R_i \mapsto u^i R_i$, $i \geq 2$. Then, from (3.23), we have

$$\Sigma_{k,2n} = [u^{k+1-2n}]\vartheta\Sigma_k. \tag{3.42}$$

Now, from (3.21), we have

$$\vartheta F(x) = \frac{x}{\vartheta R(x)} = \frac{x}{R(ux)} = \frac{1}{u}F(ux).$$

Applying ϑ to both sides of the equation $F(F^{\langle -1 \rangle}(x)) = x$ we obtain

$$\begin{aligned} x &= \vartheta F(\vartheta F^{\langle -1 \rangle}(x)) \\ &= \frac{1}{u}F(u\vartheta F^{\langle -1 \rangle}(x)) \end{aligned}$$

implying

$$\vartheta F^{\langle -1 \rangle}(x) = \frac{1}{u}F^{\langle -1 \rangle}(ux).$$

Thus, combining this with (3.21) and (3.39), we obtain

$$\vartheta\phi(x) = x\vartheta G(x^{-1}) = \frac{x}{\vartheta F^{\langle -1 \rangle}(x)} = \frac{ux}{F^{\langle -1 \rangle}(ux)} = \phi(ux),$$

and then

$$\vartheta\Phi(x, j) = (1-jx)\phi(ux(1-jx)^{-1}) = \Phi(ux, ju^{-1}).$$

Combining this with (3.42) and (3.40) gives

$$\Sigma_{k,2n} = -\frac{1}{k}[u^{k+1-2n}x^{k+1}]\prod_{j=0}^{k-1}\Phi(ux, ju^{-1})$$

and (3.41) now follows, by substituting first $x = xu^{-1}$, and then $u = u^{-1}$. □

Next, we give an expression for the coefficients Φ_i, $i \geq 0$, defined in (3.39).

Proposition 3.6.2. *For $i \geq 0$,*

$$\Phi_i(x) = \frac{x}{i!}\left(x^2\frac{d}{dx}\right)^i \frac{\phi(x)}{x}. \tag{3.43}$$

Note that for $i = 0$, this specializes to $\Phi_0(x) = \phi(x)$.

Proof. From (3.21) and (3.39), we have

$$\phi(x) = 1 + \sum_{j\geq 2}\phi_j x^j,$$

where ϕ_j, $j \geq 2$ are polynomials in the R_i's. For $i = 0$, we have $\Phi_0(x) = \Phi(x, 0) = \phi(x)$. For $i \geq 1$, we have

$$\begin{aligned}
\Phi_i(x) &= [u^i]\Phi(x,u) = [u^i]\left(1 - ux + \sum_{j\geq 2}\phi_j x^j(1-ux)^{1-j}\right) \\
&= -\binom{1}{i}x + \sum_{j\geq 2}\phi_j\binom{j+i-2}{i}x^{j+i} \\
&= \frac{x}{i!}\left(x^2\frac{d}{dx}\right)^i\left(\frac{1}{x} + \sum_{j\geq 2}\phi_j x^{j-1}\right),
\end{aligned}$$

and the result follows. □

We consider the functional equation

$$w = z\phi(w), \tag{3.44}$$

where ϕ is given by (3.39). Then from (3.21) and (3.39), we have

$$w = zwG(w^{-1}) = \frac{zw}{F^{\langle -1\rangle}(w)},$$

so $F^{\langle -1\rangle}(w) = z$, and from (3.21) we deduce that

$$z = wR(z). \tag{3.45}$$

We now relate the series $C(z)$ and differential operator D of Section 2 to the variable w.

Proposition 3.6.3.

$$\frac{Dw}{w} = \frac{1}{R(z)C(z)} \tag{3.46}$$

$$w^2 \frac{d}{dw} = zC(z)D \tag{3.47}$$

Proof. From (3.24) and (3.21), we obtain

$$C(z) = \frac{1}{-zD\frac{R(z)}{z}}. \tag{3.48}$$

But

$$\frac{Dw}{w} = -wD\frac{1}{w} = -\frac{z}{R(z)}D\frac{R(z)}{z},$$

from (3.45), and result (3.46) follows.

Now, (3.46) gives the operator identity

$$w\frac{d}{dw} = R(z)C(z)D,$$

and multiplying by w and using (3.45), we obtain result (3.47). □

Proof of Theorem 3.4.1. For a partition λ, let $\Phi_\lambda(x) = \prod_{j=1}^{l(\lambda)} \Phi_{\lambda_j}(x)$. Then from (3.41) and (3.43), we have

$$\begin{aligned}
\Sigma_{k,2n} &= -\frac{1}{k}[x^{k+1}] \sum_{\lambda \vdash 2n} \hat{m}_\lambda \Phi_\lambda(x)\phi(x)^{k-l(\lambda)} \\
&= -\frac{1}{k}[x^{k+1}] \sum_{\lambda \vdash 2n} \hat{m}_\lambda \frac{\Phi_\lambda(x)}{\phi(x)^{l(\lambda)+1}}\phi(x)^{k+1} \\
&= -\frac{1}{k}[z^{k+1}] \sum_{\lambda \vdash 2n} \hat{m}_\lambda \frac{1}{R(z)C(z)} \frac{\Phi_\lambda(w)}{\phi(w)^{l(\lambda)+1}},
\end{aligned}$$

where the last equality follows from Theorem 2.4.2.a and (3.46). But, from (3.43), (3.44) and (3.47), for $i \geq 1$ we have

$$\begin{aligned}
\frac{\Phi_i(w)}{\phi(w)} &= \frac{1}{i!}\frac{w}{\phi(w)}(w^2\frac{d}{dw})^i \frac{\phi(w)}{w} \\
&= \frac{z}{i!}(zC(z)D)^{i-1} zC(z)D\frac{1}{z} \\
&= -\frac{z}{i!}(zC(z)D)^{i-1}C(z).
\end{aligned}$$

Finally, we prove by induction on $i \geq 1$ that

$$-\frac{1}{i!}(zC(z)D)^{i-1}C(z) = z^{i-1}P_i(z),$$

where $P_i(z)$ is defined in Section 2. The result is clearly true for $i = 1$. For the induction step, we have

$$\begin{aligned} -\frac{1}{(i+1)!}(zC(z)D)^iC(z) &= \frac{1}{i+1}zC(z)Dz^{i-1}P_i(z) \\ &= \frac{1}{i+1}\left(z^iC(z)D + (i-1)z^iC(z)I\right)P_i(z) \\ &= z^iP_{i+1}(z), \end{aligned}$$

as required. Together, these results give

$$\frac{\Phi_i(w)}{\phi(w)} = z^iP_i(z),$$

so

$$\frac{\Phi_\lambda(w)}{\phi(w)^{l(\lambda)+1}} = z^{2n}\frac{P_\lambda(z)}{\phi(w)},$$

since $\lambda \vdash 2n$, and the result follows from (3.44) and (3.45). □

Proof of Theorem 3.4.2. In the proof of Theorem 3.4.1, the term in $\Sigma_{k,2n}$ corresponding to the partition with the single part $2n$ can be treated in the following modified way. We obtain

$$\begin{aligned} -\frac{1}{k}[x^{k+1}]\hat{m}_{2n}\Phi_{2n}(x)\phi(x)^{k-1} &= -\frac{1}{k}[x^{k-2}]\hat{m}_{2n}x^{-3}\Phi_{2n}(x)\phi(x)^{k-1} \\ &= -\frac{1}{k}[x^{k-2}]\hat{m}_{2n}x^{-3}\frac{x}{(2n)!}x^2\frac{d}{dx}\left(x^2\frac{d}{dx}\right)^{2n-1} \\ &\qquad \cdot \frac{\phi(x)}{x}\phi(x)^{k-1} \\ &= -\frac{k-1}{k}[z^{k-1}]\hat{m}_{2n}\frac{1}{(2n)!}\left(w^2\frac{d}{dw}\right)^{2n-1}\frac{\phi(w)}{w}, \end{aligned}$$

from Theorem 2.4.2.b, and the result follows as in the above proof of Theorem 3.4.1. □

Chapter 4

Stanley's Character Polynomials

In this chapter we explore expressions for the normalized characters in terms of polynomials introduced by Stanley [28]. We shall see that there are some connections between the polynomials in this chapter and Kerov's polynomials. In particular, we show that there are positivity conjectures for Stanley's polynomials whose proofs follow from the positivity results we have thus far obtained for Kerov's polynomials; we end the chapter by showing a strong connection between positivity of Kerov's polynomials and positivity of Stanley's polynomials in general.

In Section 4.1, we give the "rectangular shape" version of Stanley's polynomials. The main theorem in this case was introduced in Chapter 1 in (1.2) and is also found below in Theorem 4.1.1. This particular expression for the rectangular character connects it to permutation factorizations. We devote all of Section 4.1 to this rectangular case. The new proof of this result promised in Chapter 1 is at the end of this section in Section 4.1.2. As mentioned earlier, we make use of *shift symmetric functions*, and we give a brief account of these in Section 4.1.1. Finally, Sections 4.2 and 4.3 deal with the general case of non-rectangular shapes. In particular, in the general case Stanley conjectures a certain kind of positivity (here we have called this $\mathbf{p}, \mathbf{q}$-positivity) for a particular form of his polynomials. We are able to prove that the terms of highest degree in Stanley's polynomials are $\mathbf{p}, \mathbf{q}$-positive and, furthermore, using results from Chapter 3, we are able to prove that $\mathbf{p}, \mathbf{q}$-positivity holds for the terms of second and third highest degrees, all of which are new results. As in the case of Kerov's polynomials we are, unfortunately, unable to show positivity in general. As mentioned above, however, we are able to show a strong connection between positivity of Kerov's polynomials (specifically C-positivity) and $\mathbf{p}, \mathbf{q}$-positivity for Stanley's polynomials.

4.1 Stanley's Polynomials for Rectangular Shapes

As in Chapter 3, in this chapter we shall discuss expressions for the normalized characters $\widehat{\chi}_\omega$. We begin with a specific two variable case of Stanley's results - as they have a particularly simple form - and discuss the general form later.

We begin with the character $\widehat{\chi}_\omega$ when ω has the rectangular shape of p parts, all equal to q. We denote this shape by $p \times q$. The following theorem can be found in Stanley [28].

Theorem 4.1.1 (Stanley). *Suppose that $p \times q \vdash n$ and $\mu \vdash k$ for $k \leq n$. Let λ_μ be any fixed permutation in the conjugacy class indexed by μ in $\mathfrak{S}_k$. Then,*

$$\widehat{\chi}_{p\times q}(\mu\ 1^{n-k}) = (-1)^k \sum_{\substack{u,v \\ uv=\lambda_\mu}} p^{\ell(u)}(-q)^{\ell(v)}.$$

This result can be written in terms of the connection coefficients of the symmetric group, given in (2.2); Theorem 4.1.1 then becomes

$$\widehat{\chi}_\omega(\mu\ 1^{n-k}) = (-1)^k \sum_{u,v\vdash k} c^{\mu}_{u,v} p^{\ell(u)}(-q)^{\ell(v)}.$$

Stanley's proof of this involves a combination of results; results about certain tableaux, the Murnaghan-Nakayama rule, Theorem 2.3.1, and the following symmetric function identity

$$\sum_{\omega\vdash k} H_\omega\ s_\omega(x)s_\omega(y)s_\omega(z) = \sum_{\omega\vdash k} p_\omega(x)p_\omega(y)p_\omega(z),$$

which appears in Hanlon et al. [14]. Here, we present an original proof with the aim of making the result more transparent and, in addition, of showing more connections between what are known as *shift symmetric functions* and the normalized character $\widehat{\chi}_\omega$ (we shall see that there is already a known relationship between these objects). Sections 4.1.1 gives the necessary background for this proof.

4.1.1 A Brief Account of Shift Symmetric Functions

In Section 2.2, on page 10, we have given the formal definition of a symmetric function $f \in \Lambda$ as the limit of functions $f_1, f_2, \ldots$ where $f_i \in \Lambda(i)$. In a similar manner, we can define the *shift symmetric algebra* $\Lambda^*(n)$ as the set of series in n variables that are *shift symmetric*; that is, the algebra $\Lambda^*(n)$ is the set of series f in n

variables $x_1, x_2, \ldots, x_n$ such that f is symmetric in the new variables

$$x_i' = x_i - i.$$

Finally, define the algebra Λ^* of *shift symmetric functions* as the limit

$$\Lambda^* = \varprojlim \Lambda^*(n).$$

Just as the ordinary Schur polynomials $s_\lambda(x_1, x_2, \ldots, x_n)$ can be defined as

$$s_\lambda(x_1, x_2, \ldots, x_n) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{1\le i,j\le n}}{\det\left(x_i^{n-j}\right)_{1\le i,j\le n}},$$

we can analogously define shift Schur polynomials $s_\lambda^*(x_1, x_2, \ldots, x_n)$ by

$$s_\lambda^*(x_1, x_2, \ldots, x_n) = \frac{\det\left((x_i+n-i)_{\lambda_j+n-j}\right)_{1\le i,j\le n}}{\det\left((x_i)_{n-j}\right)_{1\le i,j\le n}}.$$

Finally, the *shift Schur functions*, denoted by $s_\lambda^* \in \Lambda^*$ are defined as the limit of the sequence $(s_\lambda^*(x_1, x_2, \ldots, x_n))_{n\ge 1}$. Furthermore, recall from Theorem 2.2.4, that the power sum symmetric functions p_μ can be written as a linear combination of Schur functions by

$$p_\mu = \sum_{\rho\vdash k} \chi_\rho(\mu) s_\rho,$$

where μ is a partition of k. Analogous to this we *define* the *p-sharp* shift symmetric functions $p_\mu^\sharp$ by

$$p_\mu^\sharp = \sum_{\rho\vdash k} \chi_\rho(\mu) s_\rho^*$$

(see Okounkov and Olshanski [22, Section 1] for more details).

The following result connects shift symmetric functions and the normalized characters $\widehat{\chi}$, and can be found in Okounkov and Olshanski [22, (15.21)].

Theorem 4.1.2 (Okounkov, Olshanski). *Suppose that $\mu \vdash k$ and $\lambda \vdash n$, with $k \le n$. Then*

$$p_\mu^\sharp(\lambda) = \widehat{\chi}_\lambda(\mu\, 1^{n-k}).$$

The following theorem gives a combinatorial interpretation to shift Schur functions; it is also found in Okounkov and Olshanski [22, Theorem 11.1]. For any shape μ, a *reverse tableau of shape* μ is a function T : boxes of $\mu \mapsto \mathbb{P}$, where $\mathbb{P}$ is the set of positive integers, such that T is weakly decreasing along the rows of μ and strongly decreasing along the columns of μ. We denote by $\mathrm{RTab}(\mu)$ the set of reverse tableau of shape μ.

Theorem 4.1.3 (Okounkov, Olshanski). *For $\lambda \in \mathcal{P}$,*

$$s_\lambda^* = \sum_{T \in \mathrm{RTab}(\mu)} \prod_{u \in \mu} (x_{T(u)} - c(u)),$$

where $T(u)$ is the value assigned to the box u by the tableau T and, again, $c(u)$ is the content of the box u.

4.1.2 Proof of Theorem 4.1.1

We are now ready to give a proof of Theorem 4.1.1.

Proof of Theorem 4.1.1. As a first step to this proof, for a partition $\lambda \vdash k$ we evaluate $s_\lambda^*(x_1, x_2, \ldots, x_p)$ with $x_i = q$ for $1 \leq i \leq p$; that is, we compute the evaluation $s_\lambda^*(p \times q)$. Using Theorem 4.1.3 we obtain

$$\begin{aligned} s_\lambda^*(p \times q) &= \sum_{T \in \mathrm{RTab}(\lambda)} \prod_{u \in \lambda} (x_{T(u)} - c(u)) \Big|_{(x_1,\ldots,x_p)=(q,\ldots,q)} \\ &= \sum_{T \in \mathrm{RTab}(\lambda)} \prod_{u \in \lambda} (q - c(u)) \\ &= (-1)^k \prod_{u \in \lambda} (-q + c(u)) \sum_{T \in \mathrm{RTab}(\lambda)} 1 \Big|_{(x_1,\ldots,x_p)=(q,\ldots,q)} . \end{aligned} \tag{4.1}$$

The number of $\mathrm{RTab}(\lambda)$ is clearly the number of SSYT of shape λ filled with only numbers $1, 2, \ldots, p$, which is $s_\lambda(\mathbf{1^p})$ from (2.4). Thus, from (4.1) above and Theorem 2.2.6 in Section 2.2, we have

$$\begin{aligned} s_\lambda^*(p \times q) &= (-1)^k \prod_{u \in \lambda} (-q + c(u))\, s_\lambda(\mathbf{1^p}) \\ &= \frac{(-1)^k}{H_\lambda} \prod_{u \in \lambda} (-q + c(u))(p + c(u)). \end{aligned}$$

Therefore, from Theorem 4.1.2 and Theorem 2.2.8 we have

$$\begin{aligned}
\widehat{\chi}_{p\times q}(\mu\, 1^{n-k}) &= \sum_{\lambda\vdash k}\chi_\lambda(\mu)s^*_\lambda(p\times q)\\
&= (-1)^k\sum_{\lambda\vdash k}\frac{\chi_\lambda(\mu)}{H_\lambda}\prod_{u\in\lambda}(-q+c(u))(p+c(u))\\
&= (-1)^k\sum_{\alpha,\beta,\lambda\vdash k}\frac{\chi_\lambda(\mu)}{H_\lambda}\frac{|C_\alpha|}{f^\lambda}\chi_\lambda(\alpha)p^{\ell(\alpha)}\frac{|C_\beta|}{f^\lambda}\chi_\lambda(\beta)(-q)^{\ell(\beta)}\\
&= (-1)^k\sum_{\alpha,\beta,\vdash k}p^{\ell(\alpha)}(-q)^{\ell(\beta)}\frac{|C_\alpha||C_\beta|}{k!}\sum_{\lambda\vdash k}\frac{1}{f^\lambda}\chi_\lambda(\alpha)\chi_\lambda(\beta)\chi_\lambda(\mu)\\
&= (-1)^k\sum_{\alpha,\beta\vdash k}p^{\ell(\alpha)}(-q)^{\ell(\beta)}c^\mu_{\alpha,\beta},
\end{aligned}$$

where the third equality follows from Theorem 2.2.7 in Section 2.2, and the last equality follows from (2.3). This completes the proof. □

4.2 Generalizations to Non-Rectangular Shapes

In the previous section we gave a polynomial form for the normalized character $\widehat{\chi}_\omega(\mu\, 1^{n-k})$ when the shape ω is a rectangle. Naturally, there is an analogous question for arbitrary shapes σ. To consider that question, let σ be the shape with p_i parts of size q_i, for i from 1 to m and where q_1 is the size of the largest part (see Figure 4.1). Thus, $p_1, p_2, \ldots, p_m$ are positive integers and $q_1 > q_2 > \cdots > q_m$. We denote the partition σ with the notation $\mathbf{p}\times\mathbf{q}$. Define the function F_k in indeterminates $p_1, \ldots, p_m, q_1, \ldots, q_m$ by

$$F_k(p_1, p_2, \ldots, p_m;\ q_1, q_2, \ldots, q_m) = \widehat{\chi}_{\mathbf{p}\times\mathbf{q}}(k\, 1^{n-k}) \tag{4.2}$$

We often denote $(p_1, \ldots, p_m)$ by $\mathbf{p}$ and $(q_1, \ldots, q_m)$ by $\mathbf{q}$, giving us the notation $F_k(\mathbf{p};\ \mathbf{q})$ for $F_k(p_1, p_2, \ldots, p_m;\ q_1, q_2, \ldots, q_m)$. The following theorem with proof appears in Stanley [28, Proposition 1].

Theorem 4.2.1 (Stanley). *$F_k(p_1, p_2, \ldots, p_m;\ q_1, q_2, \ldots, q_m)$ is a polynomial in the p's and q's such that $F_k(1, 1, \ldots, 1;\ -1, -1, \ldots, -1) = (k+m-1)_k$.*

In light of this theorem, we call the polynomials in (4.2) *Stanley's character polynomials*. Note that the rectangular case of Theorem 4.1.1 is the case $m = 1$ in (4.2). We emphasize that Stanley's proof below also uses Frobenius' Theorem 3.3.9, just as the proof of Theorem 3.4.1.

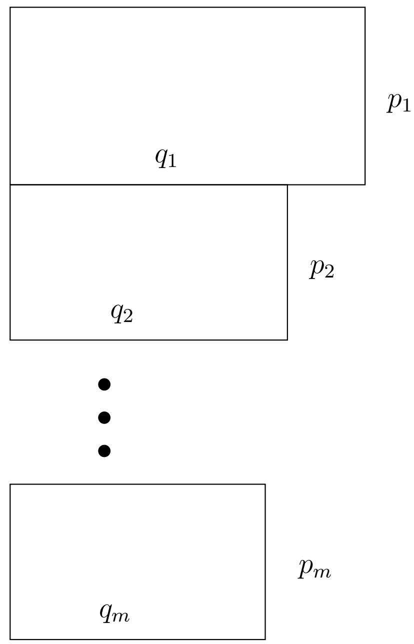

Figure 4.1: The shape $\mathbf{p} \times \mathbf{q}$.

Proof (Stanley). Using Frobenius' formula (3.15) with $\lambda = \mathbf{p} \times \mathbf{q}$ and μ and θ defined as in (3.16), we obtain

$$\begin{aligned} F_k(\mathbf{p};\ \mathbf{q}) &= -\frac{1}{k}[z^{-1}]_\infty\ (z)_k \frac{\theta(z-k)}{\theta(z)} \\ &= -\frac{1}{k}[z^{-1}]_\infty \frac{(z)_k \displaystyle\prod_{i=1}^{m} (z-(q_i+p_i+p_{i+1}+\cdots+p_m))_k}{\displaystyle\prod_{i=1}^{m} (z-(q_i+p_{i+1}+p_{i+2}+\cdots+p_m))_k}, \end{aligned} \tag{4.3}$$

where the last equation is obtained by cancelling common factors (similar to the proof of Lemma 3.3.10 where the only surviving factors were corners). Since

$$\frac{1}{z-a} = \frac{1}{z} + \frac{a}{z^2} + \frac{a^2}{z^3} \cdots,$$

it follows that $F_k(\mathbf{p};\ \mathbf{q})$ is a polynomial in the p's and q's. To show that it has integer coefficients we can, equivalently, show that

$$[z^{-1}]_\infty \frac{(z)_k \displaystyle\prod_{i=1}^{m} (z-(q_i+p_i+p_{i+1}+\cdots+p_m))_k}{\displaystyle\prod_{i=1}^{m} (z-(q_i+p_{i+1}+p_{i+2}+\cdots+p_m))_k}$$

is divisible by k. Note that it is clear that

$$(z)_k \frac{\theta(z-k)-\theta(z)}{\theta(z)}$$

is divisible by k, implying that

$$(z)_k \frac{\theta(z-k)}{\theta(z)} \equiv (z)_k (\mathrm{mod}\ k).$$

Finally, we have

$$[z^{-1}]_\infty (z)_k = 0,$$

proving that $F_k(\mathbf{p};\ \mathbf{q})$ has integer coefficients. For the rest of the theorem, we have

$$F_k(1,1,\ldots,1;\ -1,-1,\ldots,-1) = -\frac{1}{k}[z^{-1}]_\infty \frac{(z-k+1)(z-m+1)_k}{z+1}.$$

From Proposition 2.5.2, we have

$$\begin{aligned} -\frac{1}{k}[z^{-1}]_\infty \frac{(z-k+1)(z-m+1)_k}{z+1} &= (-m)_k \\ &= (-1)^k(k+m-1)_k. \end{aligned} \qquad \square$$

Stanley also generalizes $F_k(\mathbf{p};\ \mathbf{q})$ to

$$F_\mu(\mathbf{p};\ \mathbf{q}) = \widehat{\chi}_{\mathbf{p}\times\mathbf{q}}(\mu\ 1^{n-k}),$$

where μ is a partition of k. Stanley states that $F_\mu(\mathbf{p};\ \mathbf{q})$ is, by the Murnaghan-Nakayama rule, Theorem 2.3.1, a polynomial with integer coefficients. Finally, in [28, Conjecture 1], Stanley gives a positivity conjecture for the series $F_\mu(\mathbf{p};\ \mathbf{q})$. For convenience, we use the notation $-\mathbf{q} = (-q_1, -q_2, \ldots, -q_m)$, and $F_\mu(\mathbf{p};\ -\mathbf{q})$ is the series $F_\mu(\mathbf{p};\ \mathbf{q})$ with q_i replaced by $-q_i$.

Conjecture 4.2.2 (Stanley). *For any partition $\mu \vdash k$ with $k \leq n$, the polynomial $(-1)^k F_\mu(\mathbf{p};\ -\mathbf{q})$ has non-negative integer coefficients summing to $(k+m-1)_k$.*

We refer to this property (all coefficients of all terms in the p's and q's being positive) as $\mathbf{p},\mathbf{q}$-*positivity*. Although this is conjectured for all partitions $\mu \vdash k$, it is not yet even proven when μ has a single part; *i.e.* it is not proven that $(-1)^k F_k(\mathbf{p};\ -\mathbf{q})$ has non-negative coefficients. The expressions $(-1)^k F_k(\mathbf{p};\ -\mathbf{q})$ for $k = 1,2,3,4$ and

$m = 2$ are given in (4.4). These data also appear in Stanley [28, page 8].

$$\begin{aligned}
-F_1(a,p;\ -b,-q) &= ab + pq, \\
F_2(a,p;\ -b,-q) &= a^2b + ab^2 + 2\,apq + p^2q + pq^2, \\
-F_3(a,p;\ -b,-q) &= a^3b + 3\,a^2b^2 + 3\,a^2pq + ab^3 + 3\,abpq + 3\,ap^2q \\
&\quad + 3\,apq^2 + p^3q + 3\,p^2q^2 + pq^3 + ab + pq, \\
F_4(a,p;\ -b,-q) &= a^4b + 6\,a^3b^2 + 4\,a^3pq + 6\,a^2b^3 + 12\,a^2bpq \\
&\quad + 6\,a^2p^2q + 6\,a^2pq^2 + ab^4 + 4\,ab^2pq + 4\,abp^2q \\
&\quad + 4\,abpq^2 + 4\,ap^3q + 14\,ap^2q^2 + 4\,apq^3 + p^4q \\
&\quad + 6\,p^3q^2 + 6\,p^2q^3 + pq^4 + 5\,a^2b + 5\,ab^2 + 10\,apq + 5\,p^2q \\
&\quad + 5\,pq^2.
\end{aligned} \tag{4.4}$$

Finally, Stanley mentions that the terms of highest degree in $F_k(\mathbf{p};\ \mathbf{q})$, *i.e.* the terms of degree $k+1$, have a particularly nice expression. Keeping Stanley's notation, let $G_k(\mathbf{p};\ \mathbf{q})$ be the terms of highest degree in $F_k(\mathbf{p};\ \mathbf{q})$. We have the following expression for the generating series of $G_k(\mathbf{p};\ \mathbf{q})$, which we call $G_{\mathbf{p};\,\mathbf{q}}(z)$. This theorem appears, with proof, in [28, Proposition 2].

Theorem 4.2.3 (Stanley). *The generating series for* $G_k(\mathbf{p};\ \mathbf{q})$ *is*

$$G_{\mathbf{p};\,\mathbf{q}}(z) = 1 + \sum_{i\geq 1} G_{i-1}(\mathbf{p};\ \mathbf{q})z^i = \frac{z}{\left(\dfrac{z \displaystyle\prod_{i=1}^{m} \left(1 - \left(q_i + \sum_{j=i+1}^{m} p_j\right) z\right)}{\displaystyle\prod_{i=1}^{m} \left(1 - \left(q_i + \sum_{j=i}^{m} p_j\right) z\right)} \right)^{\langle -1 \rangle}}. \tag{4.5}$$

Proof (Stanley). From (4.3) we have

$$G_{k-1}(\mathbf{p};\ \mathbf{q}) = -\frac{1}{k}[z^{-1}]_\infty \frac{z^k \displaystyle\prod_{i=1}^{m} \left(z - \left(q_i + \sum_{j=i}^{m} p_j\right)\right)^k}{\displaystyle\prod_{i=1}^{m} \left(z - \left(q_i + \sum_{j=i+1}^{m} p_j\right)\right)^k}.$$

Call the quantity after the "$[z^{-1}]_\infty$" operator in the last equation $L(z)^k$. Setting $M(z) = zL(1/z)$ note that $M(0) = 1$. Then, using Lagrange Theorem 2.4.2, the last equation becomes

$$-\frac{1}{k}[z]\,M(z)^k = [z^{k+1}]\,\frac{z}{\left(\dfrac{z}{M(z)}\right)^{\langle -1 \rangle}},$$

giving the desired result. □

Of course, $\mathbf{p}, \mathbf{q}$-positivity of $(-1)^k F_k(\mathbf{p};\ -\mathbf{q})$ would imply that $(-1)^k G_k(\mathbf{p};\ -\mathbf{q})$ is also $\mathbf{p}, \mathbf{q}$-positive. Stanley does not prove $\mathbf{p}, \mathbf{q}$-positivity for the latter series in [28] but states that Elizalde has proven this in a private communication to him. In fact, Elizalde shows (according to Stanley)

$$(-1)^k G_k(\mathbf{p};\ \mathbf{q}) = \frac{1}{k} \sum_{i_1+\cdots+i_m+j_1+\cdots+j_m=k+1} \binom{k}{i_1} \left(\!\binom{i_1}{j_1}\!\right)$$
$$\prod_{s=2}^{m} \left(\sum_{r=0}^{\min(i_s,j_s)} \binom{k}{r} \left(\!\binom{r}{j_s - r}\!\right) \binom{k - r - i_i - \cdots - i_{s-1} - j_1 - \cdots - j_{s-1}}{i_s - r} \right)$$
$$\cdot\, p_1^{i_1} \cdots p_m^{i_m} q_1^{i_1} \cdots q_m^{i_m},$$

where $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$. However, as far as this author can see, no proof exists in the literature.

In the next sections we give partial answers to the positivity questions concerning $(-1)^k F(\mathbf{p};\ -\mathbf{q})$. As alluded to at the beginning of this chapter, we use Kerov's polynomials to answer these questions.

4.3 Applying Kerov Polynomials to Stanley's Polynomials

Note that both (3.6) and (4.2) give expressions for the normalized character $\widehat{\chi}_\omega$, the former directly and the second through the series H in (3.4). Since they hold for any shapes $\mathbf{p} \times \mathbf{q}$, we can conclude that they give the same expression for $\widehat{\chi}_\omega$. Thus, we will use (3.4) and (3.6) to obtain results about Stanley's polynomials. More specifically, using (3.4) we obtain the R_i in Kerov's polynomials for a general shape ω. It turns out that the generating series $R_\omega(z)$ is *almost* the same as the generating series $G_{\mathbf{p};\,\mathbf{q}}(z)$ in (4.5) (we qualify our use of "almost" later). We prove this in Section 4.3.2. We shall, in addition, see that $R_\omega(z)$ has a much nicer form than $G_{\mathbf{p};\,\mathbf{q}}(z)$, and this nicer form allows us to show the positivity of the top terms of $(-1)^k F_k(\mathbf{p};\ -\mathbf{q})$. The main result needed to show the positivity of the top terms of $(-1)^k F_k(\mathbf{p};\ -\mathbf{q})$ is given in Theorem 4.3.3 of Section 4.3.2, the main theorem of this chapter. In Section 4.3.3 we use Theorem 4.3.3 and the results from Chapter 3 to prove the positivity of the terms of degree $k-1$ and $k-3$ in $F_k(\mathbf{p};\ \mathbf{q})$. Finally, we end the chapter by showing in Theorem 4.3.10 that C-positivity for Kerov's polynomials implies $\mathbf{p}, \mathbf{q}$-positivity for Stanley's polynomials.

4.3.1 The Series H for the Shape $\mathbf{p} \times \mathbf{q}$

We now compute what the series H must be for the shape $\mathbf{p} \times \mathbf{q}$. For the shape $\mathbf{p} \times \mathbf{q}$, it is not difficult to see that its interlacing sequence of maxima and minima is

$$x_1 = q_1, \quad y_1 = q_1 - p_1, \quad x_2 = q_2 - p_1, \quad y_2 = q_2 - p_1 - p_2, \quad x_3 = q_3 - p_1 - p_2,$$

$$y_3 = q_3 - p_1 - p_2 - p_3, \quad \ldots, \quad x_m = q_m - \sum_{i=1}^{m-1} p_i, \;\; y_m = q_m - \sum_{i=1}^{m} p_i, \;\; x_{m+1} = -\sum_{i=1}^{m} p_i.$$

Using the notation developed in Chapter 3, and from (3.3) in Example 3.3.1, we have

$$\begin{aligned} H_{\mathbf{p}\times\mathbf{q}}(1/z) &= \frac{z\,(1-(q_1-p_1)\,z)\,(1-(q_2-(p_1+p_2))\,z)\cdots\left(1-\left(q_m-\sum_{i=1}^{m}p_i\right)z\right)}{(1-q_1z)\,(1-(q_2-p_1)\,z)\cdots\left(1-\left(q_m-\sum_{i=1}^{m-1}p_i\right)z\right)\left(1+\sum_{i=1}^{m}p_i\right)} \\ &= \frac{z\prod_{i=1}^{m}\left(1-\left(q_i-\sum_{j=1}^{i}p_j\right)z\right)}{\left(1+\sum_{j=1}^{m}p_jz\right)\prod_{i=1}^{m}\left(1-\left(q_i-\sum_{j=1}^{i-1}p_j\right)z\right)}, \end{aligned} \tag{4.6}$$

and we obtain

$$R_{\mathbf{p}\times\mathbf{q}}(z) = \frac{z}{\left(\dfrac{z\prod_{i=1}^{m}\left(1-\left(q_i-\sum_{j=1}^{i}p_j\right)z\right)}{\left(1+\sum_{j=1}^{m}p_jz\right)\prod_{i=1}^{m}\left(1-\left(q_i-\sum_{j=1}^{i-1}p_j\right)z\right)}\right)^{\langle -1\rangle}}. \tag{4.7}$$

Alternatively, from (3.9), it follows from the comment immediately following Theorem 2.4.2 that if

$$\begin{aligned} \phi_{\mathbf{p}\times\mathbf{q}}(z) &= \frac{z}{H_{\mathbf{p}\times\mathbf{q}}(1/z)} \\ &= \frac{\left(1+\sum_{j=1}^{m}p_jz\right)\prod_{i=1}^{m}\left(1-\left(q_i-\sum_{j=1}^{i-1}p_j\right)z\right)}{\prod_{i=1}^{m}\left(1-\left(q_i-\sum_{j=1}^{i}p_j\right)z\right)}, \end{aligned} \tag{4.8}$$

then

$$\frac{z}{R_{\mathbf{p}\times\mathbf{q}}(z)} = z\,\phi_{\mathbf{p}\times\mathbf{q}}\left(\frac{z}{R_{\mathbf{p}\times\mathbf{q}}(z)}\right). \tag{4.9}$$

Applying Theorem 2.4.2.b, we obtain for $k \geq 2$

$$\begin{aligned} R_k(\mathbf{p} \times \mathbf{q}) &= [z^{k-1}] \frac{R_{\mathbf{p}\times\mathbf{q}}(z)}{z} \\ &= \frac{1}{k-1}[y^{k-2}] - \frac{1}{y^2}\phi^{k-1}_{\mathbf{p}\times\mathbf{q}}(y) \\ &= -\frac{1}{k-1}[y^k]\phi^{k-1}_{\mathbf{p}\times\mathbf{q}}(y). \end{aligned} \tag{4.10}$$

Remark 1. The ϕ in the previous equations is the same as the ϕ in (3.39), with G replaced by a series determined by the partition $\mathbf{p} \times \mathbf{q}$ rather than a general series. Indeed by (3.21) we see that $F(z) = (H(1/z))^{\langle -1 \rangle}$, implying that

$$\begin{aligned} \phi(z) &= zG(\tfrac{1}{z}) \\ &= \frac{z}{F^{\langle -1 \rangle}(z)} \\ &= \frac{z}{(H(\frac{1}{z}))^{\langle -1 \rangle}}. \end{aligned}$$

Also, note that (4.9) is essentially (3.44) and (3.45). □

Of course, substituting $R_i(\mathbf{p} \times \mathbf{q})$ for R_i in Kerov's polynomials will give us the normalized character $\widehat{\chi}_{\mathbf{p}\times\mathbf{q}}(k\ 1^{n-k})$. In fact, in Appendix C we have done just that (using Maple) to produce the polynomials $(-1)^k F_k(a, p; -b, -q)$ for k from 1 to 10. Note that the data agree with Stanley's data given in (4.4). We, therefore, can now use Kerov's polynomials to better understand Stanley's character polynomials. It is clear from (4.6) and (4.8) that $R_i(\mathbf{p} \times \mathbf{q})$ is a homogenous polynomial of degree i in the p's and q's. Therefore, since Kerov's polynomial Σ_k is graded with terms of weight $k + 1 (\mathrm{mod}\ 2)$ (Theorem 3.3.6) in the R_i's, we see that Stanley's character polynomials are also graded with terms of degree $k + 1 (\mathrm{mod}\ 2)$. We state this now as a proposition, for easy reference later.

Proposition 4.3.1. *Terms of degree i in $F_k(\mathbf{p};\ \mathbf{q})$ are obtained from the terms of weight i in Kerov's polynomials Σ_k with the R_i's evaluated at the shape $\mathbf{p} \times \mathbf{q}$.*

To further reinforce the idea that we are dealing with polynomials, and to make convenient variable substitutions, we depart from the notation of Chapter 3. We shall replace $R_i(\mathbf{p} \times \mathbf{q})$ with $R_i(\mathbf{p};\ \mathbf{q})$ and $R_{\mathbf{p}\times\mathbf{q}}(z)$ with $R_{\mathbf{p};\,\mathbf{q}}(z)$ to emphasize that these objects are polynomials in p's and q's. We do this analogously with $H_{\mathbf{p}\times\mathbf{q}}(z)$ and $\phi_{\mathbf{p}\times\mathbf{q}}(z)$; that is, the series $\phi_{\mathbf{p};\,\mathbf{q}}(z)$ will denote the series in (4.8) and $H_{\mathbf{p};\,\mathbf{q}}(z)$

will denote the series in (4.6). We shall deal with the terms of different weights separately, starting with the terms of highest degree, namely the terms of degree $k+1$.

4.3.2 Terms of Degree $k+1$

The expression for the top terms in Stanley's polynomials is given implicitly in (4.5). From (3.9) and Proposition 4.3.1, we can obtain a similar formula for the top terms; that is, the top term in $F_k(\mathbf{p};\ \mathbf{q})$ is $R_{k+1}(\mathbf{p};\ \mathbf{q})$ and, therefore, the generating series for the top terms is

$$R_{\mathbf{p};\,\mathbf{q}}(z) = \frac{z}{\left(\dfrac{z\prod_{i=1}^{m}\left(1-\left(q_i-\sum_{j=1}^{i}p_j\right)z\right)}{\left(1+\sum_{j=1}^{m}p_jz\right)\prod_{i=1}^{m}\left(1-\left(q_i-\sum_{j=1}^{i-1}p_j\right)z\right)}\right)^{\langle -1\rangle}}. \tag{4.11}$$

Evidently, the two generating series $R_{\mathbf{p};\,\mathbf{q}}(z)$ and $G_{\mathbf{p};\,\mathbf{q}}(z)$ should be equal; after all they both generate the top terms of $F_k(\mathbf{p};\ \mathbf{q})$, although it is not obvious from (4.5) and (4.11) that this is the case. It turns out that $R_{\mathbf{p};\,\mathbf{q}}(z)$ and $G_{\mathbf{p};\,\mathbf{q}}(z)$ are *almost* the same; we state this more precisely in the next proposition.

Proposition 4.3.2. *The generating series $R_{\mathbf{p};\,\mathbf{q}}(z)$ and $G_{\mathbf{p};\,\mathbf{q}}(z)$ are identical except for the linear terms; more precisely*

$$R_{\mathbf{p};\,\mathbf{q}}(z) = G_{\mathbf{p};\,\mathbf{q}}(z) - \sum_{i=1}^{m} p_i z.$$

Proof. From Theorem 2.4.2, it suffices to show that $R_{\mathbf{p};\,\mathbf{q}}(z) + \sum_{i=1}^{m} p_i z$ satisfies the same equation as $G_{\mathbf{p};\,\mathbf{q}}(z)$. In this proof, we denote $R_{\mathbf{p};\,\mathbf{q}}(z)$ and $G_{\mathbf{p};\,\mathbf{q}}(z)$ by R and G, respectively. From (4.11) we have

$$\frac{z}{R} = \left(\frac{z\prod_{i=1}^{m}\left(1-\left(q_i-\sum_{j=1}^{i}p_j\right)z\right)}{\left(1+\sum_{j=1}^{m}p_jz\right)\prod_{i=1}^{m}\left(1-\left(q_i-\sum_{j=1}^{i-1}p_j\right)z\right)}\right)^{\langle -1\rangle}.$$

By the definition of compositional inverse we have, from the last expression,

$$
\begin{aligned}
z &= \frac{z\prod_{i=1}^{m}\left(R-\left(q_i-\sum_{j=1}^{i}p_j\right)z\right)}{\left(R+\sum_{j=1}^{m}p_jz\right)\prod_{i=1}^{m}\left(R-\left(q_i-\sum_{j=1}^{i-1}p_j\right)z\right)} \\
&= \frac{z\prod_{i=1}^{m}\left(\left(R+\sum_{j=1}^{m}p_jz\right)-\left(q_i+\sum_{j=i+1}^{m}p_j\right)z\right)}{\left(R+\sum_{j=1}^{m}p_jz\right)\prod_{i=1}^{m}\left(\left(R+\sum_{j=1}^{m}p_jz\right)-\left(q_i+\sum_{j=i}^{m}p_j\right)z\right)} \\
&= \frac{\frac{z}{\left(R+\sum_{j=1}^{m}p_jz\right)}\prod_{i=1}^{m}\left(1-\left(q_i+\sum_{j=i+1}^{m}p_j\right)\frac{z}{\left(R+\sum_{j=1}^{m}p_jz\right)}\right)}{\prod_{i=1}^{m}\left(1-\left(q_i+\sum_{j=i}^{m}p_j\right)\frac{z}{\left(R+\sum_{j=1}^{m}p_jz\right)}\right)}.
\end{aligned}
$$

Again, from the definition of compositional inverse, we conclude that

$$
\frac{z}{\left(R+\sum_{j=1}^{m}p_jz\right)} = \left(\frac{z\prod_{i=1}^{m}\left(1-\left(q_i+\sum_{j=i+1}^{m}p_j\right)z\right)}{\prod_{i=1}^{m}\left(1-\left(q_i+\sum_{j=i}^{m}p_j\right)z\right)}\right)^{\langle -1\rangle}.
$$

Comparing this expression with (4.5), the result follows. □

Remark 2. Using Theorem 2.4.2.b we can directly compute the linear terms. From the comment following Theorem 2.4.2, note that since $G_{\mathbf{p};\,\mathbf{q}}(z)$ satisfies (4.5), it must also satisfy

$$
\frac{z}{G_{\mathbf{p};\,\mathbf{q}}(z)} = z\psi_{\mathbf{p};\,\mathbf{q}}\left(\frac{z}{G_{\mathbf{p};\,\mathbf{q}}(z)}\right),
$$

where

$$
\psi_{\mathbf{p};\,\mathbf{q}}(z) = \frac{\prod_{i=1}^{m}\left(1-\left(q_i+\sum_{j=i}^{m}p_j\right)z\right)}{\prod_{i=1}^{m}\left(1-\left(q_i+\sum_{j=i+1}^{m}p_j\right)z\right)}.
$$

Thus, using Lagrange inversion Theorem 2.4.2.b, we have

$$
\begin{aligned}
[z]\, G_{\mathbf{p};\,\mathbf{q}}(z) &= [z^0]\, \frac{1}{\left(\dfrac{z \prod_{i=1}^{m} \left(1 - \left(q_i + \sum_{j=i+1}^{m} p_j\right) z\right)}{\prod_{i=1}^{m} \left(1 - \left(q_i + \sum_{j=i}^{m} p_j\right) z\right)} \right)^{\langle -1 \rangle}} \\
&= [y^0] \frac{-1}{y} + [y^{-1}] \frac{-1}{y^2} \log \left(\frac{\prod_{i=1}^{m} \left(1 - \left(q_i + \sum_{j=i}^{m} p_j\right) z\right)}{\prod_{i=1}^{m} \left(1 - \left(q_i + \sum_{j=i+1}^{m} p_j\right) z\right)} \right) \\
&= -[y] \sum_{i=1}^{m} \left(-\log \left(\left(1 - \left(q_i + \sum_{j=i}^{m} p_j\right) y\right)^{-1} \right) \right. \\
&\qquad \left. + \log \left(\left(1 - \left(q_i + \sum_{j=i+1}^{m} p_j\right) z\right)^{-1} \right) \right) \\
&= \sum_{i=1}^{m} \left(\left(q_i + \sum_{j=i}^{m} p_j\right) - \left(q_i + \sum_{j=i+1}^{m} p_j\right) \right) \\
&= \sum_{i=1}^{m} p_i.
\end{aligned}
$$

Similarly, for $R_{\mathbf{p};\,\mathbf{q}}(z)$ we use (4.9) and Theorem 2.4.2.b to obtain

$$
\begin{aligned}
[z]\, R_{\mathbf{p};\,\mathbf{q}}(z) &= [z^0]\, \frac{1}{\left(\dfrac{z \prod_{i=1}^{m} \left(1 - \left(q_i - \sum_{j=1}^{i} p_j\right) z\right)}{\left(1 + \sum_{j=1}^{m} p_j z\right) \prod_{i=1}^{m} \left(1 - \left(q_i - \sum_{j=1}^{i-1} p_j\right) z\right)} \right)^{\langle -1 \rangle}} \\
&= [y^0]\, \frac{1}{y} + [y^{-1}] - \frac{1}{y^2} \log \left(\frac{\left(1 + \sum_{j=1}^{m} p_j y\right) \prod_{i=1}^{m} \left(1 - \left(q_i - \sum_{j=1}^{i-1} p_j\right) y\right)}{y \prod_{i=1}^{m} \left(1 - \left(q_i - \sum_{j=1}^{i} p_j\right) y\right)} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= -[y]\left(\sum_{i=1}^{m} -\log\left(\left(1+\sum_{j=1}^{m} p_j y\right)^{-1}\right) - \log\left(\left(1-\left(q_i - \sum_{j=1}^{i-1} p_j\right) y\right)^{-1}\right)\right) \\
&\qquad\qquad\qquad\qquad + \log\left(\left(1-\left(q_i - \sum_{j=1}^{i} p_j\right) y\right)^{-1}\right) \\
&= -\sum_{j=1}^{m} p_j + \sum_{i=1}^{m}\left(q_i - \sum_{j=1}^{i-1} p_j - \left(q_i - \sum_{j=1}^{i} p_j\right)\right) \\
&= -\sum_{j=1}^{m} p_j + \sum_{i=1}^{m} p_i \\
&= 0.
\end{aligned}
$$

The last equation comes as no surprise, as is clear from the combinatorial origins in Section 3.3.1. Note also, from the data in Appendix A, that the term R_1 never appears. □

Through Lagrange Inversion, we see that the R_i are written in terms of the series $\phi_{\mathbf{p};\,\mathbf{q}}$ given in (4.10). We use the notation $\phi_{\mathbf{p};-\mathbf{q}}$, $R_k(\mathbf{p};-\mathbf{q})$ and $G_k(\mathbf{p};-\mathbf{q})$ to denote that we are substituting $-q_i$ for q_i for all i in these series. We have the following compact expression for the series $\phi_{\mathbf{p};-\mathbf{q}}(-z)$.

Theorem 4.3.3. *For $p_1, p_2, \ldots, p_m$ and $q_1, q_2, \ldots, q_m$, we have*

$$\phi_{\mathbf{p};-\mathbf{q}}(-z) = \prod_{i=1}^{n}\left(1+\frac{p_i q_i z}{(1-r_{i-1}z)\,(1-(q_i+r_i)z)}\right).$$

where $r_i = \sum_{j=1}^{i} p_j$.

Proof. We have, from (4.8),

$$\phi_{\mathbf{p};-\mathbf{q}}(-z) = \frac{(1-r_m z)\prod_{i=1}^{m}(1-(q_i+r_{i-1})\,z)}{\prod_{i=1}^{m}(1-(q_i+r_i)\,z)}.$$

Now set $A_n(z) = 1 - r_n z, F_0 = 1$ and

$$F_n(z) = A_n(z)\frac{\prod_{i=1}^{n}(1-(q_i+r_{i-1})\,z)}{\prod_{i=1}^{n}(1-(q_i+r_i)\,z)}. \tag{4.12}$$

Note that $\phi_{\mathbf{p};-\mathbf{q}}(-z) = F_m(z)$. Then,

$$\begin{aligned}
F_n(z) &= \frac{F_{n-1}(z)}{A_{n-1}(z)} \frac{1-(q_n+r_{n-1})\,z}{1-(q_n+r_n)\,z} A_n(z) \\
&= \frac{F_{n-1}(z)}{A_{n-1}(z)} \frac{A_{n-1}(z)\left(1-\dfrac{q_n z}{A_{n-1}(z)}\right)}{A_{n-1}(z)\left(1-\dfrac{(q_n+p_n)\,z}{A_{n-1}(z)}\right)} A_{n-1}(z)\left(1-\frac{p_n z}{A_{n-1}(z)}\right) \\
&= F_{n-1}(z)\frac{1-\dfrac{(q_n+p_n)z}{A_{n-1}(z)}+\dfrac{p_n q_n z}{A_{n-1}^2(z)}}{1-\dfrac{(q_n+p_n)z}{A_{n-1}(z)}} \\
&= F_{n-1}(z)\left(1+\frac{p_n q_n z}{A_{n-1}^2(z)\left(1-\frac{(q_n+p_n)z}{A_{n-1}(z)}\right)}\right) \\
&= F_{n-1}(z)\left(1+\frac{p_n q_n z}{A_{n-1}(z)\,(1-(q_n+r_n)z)}\right) \\
&= F_{n-1}(z)\left(1+\frac{p_n q_n z}{(1-r_{n-1}z)\,(1-(q_n+r_n)z)}\right).
\end{aligned} \tag{4.13}$$

Therefore, from (4.13) we have

$$\begin{aligned}
\phi_{\mathbf{p};-\mathbf{q}}(-z) &= F_m(z) \\
&= \frac{F_m(z)}{F_0(z)} \\
&= \frac{F_m(z)}{F_{m-1}(z)} \cdot \frac{F_{m-1}(z)}{F_{m-2}(z)} \cdots \frac{F_1(z)}{F_0(z)} \\
&= \prod_{i=1}^{m}\left(1+\frac{p_i q_i z}{(1-r_{i-1}z)\,(1-(q_i+r_i)z)}\right).
\end{aligned}$$

□

Corollary 4.3.4. *$\phi_{\mathbf{p};-\mathbf{q}}(-z)$ is $\mathbf{p},\mathbf{q}$-positive.*

Proof. Each multiplicand in Theorem 4.3.3 is $\mathbf{p},\mathbf{q}$-positive, making the product $\mathbf{p},\mathbf{q}$-positive. □

Corollary 4.3.5. *For all $k \geq 1$, the series in p's and q's $(-1)^k R_{k+1}(\mathbf{p};-\mathbf{q})$ and $(-1)^k G_k(\mathbf{p};-\mathbf{q})$ are $\mathbf{p},\mathbf{q}$-positive. That is, the terms of highest degree in $(-1)^k F_k(\mathbf{p};\ -\mathbf{q})$ all have positive coefficients.*

Proof. The series $(-1)^k G_k(\mathbf{p};-\mathbf{q})$ is by definition the terms of highest degree in $(-1)^k F_k(\mathbf{p};\ -\mathbf{q})$, and by Proposition 4.3.2, $(-1)^k G_k(\mathbf{p};-\mathbf{q}) = (-1)^k R_{k+1}(\mathbf{p};-\mathbf{q})$

are equal for all $k \geq 1$. Thus, it suffices to show that $(-1)^k R_{k+1}(\mathbf{p}; -\mathbf{q})$ is $\mathbf{p}, \mathbf{q}$-positive for all $k \geq 1$.

By (4.10) we have

$$\begin{aligned}
(-1)^k R_{k+1}(\mathbf{p}; -\mathbf{q}) &= (-1)^k \left(-\frac{1}{k}[y^{k+1}]\ \phi^k_{\mathbf{p};-\mathbf{q}}(y) \right) \\
&= \frac{1}{k}[(-y)^{k+1}]\phi^k_{\mathbf{p};-\mathbf{q}}(y) \\
&= \frac{1}{k}[y^{k+1}]\phi^k_{\mathbf{p};-\mathbf{q}}(-y),
\end{aligned}$$

and the result follows. □

4.3.3 Terms of Degree $k-1$, $k-3$ and a General Connection Between Kerov's Polynomials and Stanley's Polynomials

In this section we deal with terms of degree $k-1$ and $k-3$ in Stanley's polynomials. We note that in Stanley [28] there are no results concerning terms not of highest degree; Stanley comments only on the series $G_{\mathbf{p};-\mathbf{q}}(z)$, the terms of highest degree in $k+1$. Moreover, we note the complication that in $(-1)^k\Sigma_k$ there are negative terms when one evaluates the R_i in terms of the shape $\mathbf{p};\ \mathbf{q}$ and substitutes $-q$'s for all the q's. More precisely, consider, for example, Σ_5 given in Appendix A. We see from the comments at the beginning of Section 4.3 that

$$\begin{aligned}
(-1)^5 F_5(\mathbf{p}; -\mathbf{q}) &= (-1)^5 \Sigma_5(\mathbf{p};\ \mathbf{q})|_{\mathbf{q}\to-\mathbf{q}} \\
&= (-1)^5 \left(R_6(\mathbf{p}; -\mathbf{q}) + 15R_4(\mathbf{p}; -\mathbf{q}) + 5R_2(\mathbf{p}; -\mathbf{q})^2 \right. \\
&\qquad \left. + 8R_2(\mathbf{p}; -\mathbf{q}) \right) \\
&= (-1)^5 R_6(\mathbf{p}; -\mathbf{q}) + 15(-1)^3 R_4(\mathbf{p}; -\mathbf{q}) \\
&\qquad - 5((-1)R_2(\mathbf{p}; -\mathbf{q}))^2 + 8(-1)R_2(\mathbf{p}; -\mathbf{q}).
\end{aligned}$$

Note that all terms are $\mathbf{p}, \mathbf{q}$-positive except for the term $-5((-1)R_2(\mathbf{p}; -\mathbf{q}))^2$. Thus, $\mathbf{p}, \mathbf{q}$-positivity would not immediately follow from positivity of Kerov's polynomials. For the terms of degree $k-1$ and $k-3$, however, we can use the results given in Chapter 3. We begin with the following theorem.

Theorem 4.3.6. *For $k \geq 3$, the terms of degree $k-1$ in $F_k(\mathbf{p};\ \mathbf{q})$ are given by*

$$-\frac{k(k+1)}{24}[y^{k-3}]\phi''_{\mathbf{p};-\mathbf{q}}(y)\phi^{k-1}_{\mathbf{p};-\mathbf{q}}(y).$$

Proof. From Proposition 4.3.1 and Theorem 3.5.4, the terms of degree $k-1$ in $F_k(\mathbf{p};\ \mathbf{q})$ are given by

$$\frac{1}{4}\binom{k+1}{3}C_{k-1}(\mathbf{p};\ \mathbf{q}).$$

From (3.44), (3.45) and (3.48) we obtain the system of equations

$$z = wR(z), \qquad w = z\phi_{\mathbf{p};-\mathbf{q}}(w), \qquad C(z) = \frac{1}{-z^2\frac{d}{dz}\frac{1}{w}},$$

where $\phi_{\mathbf{p};-\mathbf{q}}(z)$ is given in (4.8). Thus,

$$z\frac{d}{dz}w = \frac{w}{1 - z\phi'_{\mathbf{p};-\mathbf{q}}(w)},$$

from which we obtain

$$\begin{aligned} C(z) &= \frac{1}{-z^2\frac{d}{dz}\frac{1}{w}} \\ &= \frac{1}{\frac{z^2}{w^2}\frac{d}{dz}w} \\ &= \frac{w}{z}(1 - z\phi'_{\mathbf{p};-\mathbf{q}}(w)) \\ &= \phi_{\mathbf{p};-\mathbf{q}}(w) - w\phi'_{\mathbf{p};-\mathbf{q}}(w). \end{aligned}$$

Therefore, for all $k \geq 2$, we have by Lagrange Theorem 2.4.2.b that

$$\begin{aligned} [z^{k-1}]\ C(z) &= [z^{k-1}]\ \phi_{\mathbf{p};-\mathbf{q}}(w) - [z^{k-1}]\ w\phi'_{\mathbf{p};-\mathbf{q}}(w) \\ &= \frac{1}{k-1}[y^{k-2}]\ \phi'_{\mathbf{p};-\mathbf{q}}(y)\phi^{k-1}_{\mathbf{p};-\mathbf{q}}(y) \\ &\qquad - \frac{1}{k-1}[y^{k-2}]\ \left(\phi'_{\mathbf{p};-\mathbf{q}}(y) + y\phi''_{\mathbf{p};-\mathbf{q}}(y)\right)\phi^{k-1}_{\mathbf{p};-\mathbf{q}}(y) \\ &= -\frac{1}{k-1}[y^{k-3}]\phi''_{\mathbf{p};-\mathbf{q}}(y)\phi^{k-1}_{\mathbf{p};-\mathbf{q}}(y), \end{aligned}$$

and the result follows. □

Example 4.3.7. Define $S_k(a, p; -b, -q)$ to be the terms of degree $k-1$ in $F_k(a, p; -b, -q)$. In the following equations, we give the polynomials $S_k(a, p; -b, -q)$, for k from 2 to 6, using Theorem 4.3.6 and Maple.

$$\begin{aligned}
-S_3(a,p;-b,-q) &= ab + pq \\
S_4(a,p;-b,-q) &= 5\,a^2b + 5\,ab^2 + 10\,apq + 5\,p^2q + 5\,pq^2 \\
-S_5(a,p;-b,-q) &= 15\,a^3b + 40\,a^2b^2 + 45\,a^2pq + 15\,ab^3 + 35\,abpq + 45\,ap^2q \\
&\quad + 45\,apq^2 + 15\,p^3q + 40\,p^2q^2 + 15\,pq^3 \\
S_6(a,p;-b,-q) &= 35\,a^4b + 175\,a^3b^2 + 140\,a^3pq + 175\,a^2b^3 + 315\,a^2bpq \\
&\quad + 210\,a^2p^2q + 210\,a^2pq^2 + 35\,ab^4 + 105\,ab^2pq + 105\,abp^2q \\
&\quad + 105\,abpq^2 + 140\,ap^3q + 420\,ap^2q^2 + 140\,apq^3 + 35\,p^4q \\
&\quad + 175\,p^3q^2 + 175\,p^2q^3 + 35\,pq^4.
\end{aligned}$$

One can compare these polynomials to those given in Appendix C. □

From Theorem 4.3.6 we obtain the following positivity result.

Corollary 4.3.8. *For $k \geq 3$, the terms of degree $k-1$ in $(-1)^k F_k(\mathbf{p};-\mathbf{q})$ are $\mathbf{p},\mathbf{q}$-positive.*

Proof. The terms of degree $k-1$ in $F_k(\mathbf{p};\,\mathbf{q})$ are given in Theorem 4.3.6. Therefore, the terms of degree $k-1$ in $(-1)^k F_k(\mathbf{p};\,-\mathbf{q})$ are

$$\begin{aligned}
(-1)^k \frac{1}{4}\binom{k+1}{3} C_{k-1}(\mathbf{p};-\mathbf{q}) &= -\frac{1}{4}\binom{k+1}{3} [z^{k-1}]\; C_{\mathbf{p};-\mathbf{q}}(-z) \\
&= \frac{k(k+1)}{24}[y^{k-1}](-y)^2 \frac{d^2}{d(-y)^2}\left(\phi_{\mathbf{p};-\mathbf{q}}(-y)\right) \\
&\qquad \cdot \phi^{k-1}_{\mathbf{p};-\mathbf{q}}(-y) \\
&= \frac{k(k+1)}{24}[y^{k-1}]y^2\left(\frac{d^2}{dy^2}(-1)^2\right)\left(\phi_{\mathbf{p};-\mathbf{q}}(-y)\right) \\
&\qquad \cdot \phi^{k-1}_{\mathbf{p};-\mathbf{q}}(-y) \\
&= \frac{k(k+1)}{24}[y^{k-1}]y^2 \frac{d^2}{dy^2}\left(\phi_{\mathbf{p};-\mathbf{q}}(-y)\right) \phi^{k-1}_{\mathbf{p};-\mathbf{q}}(-y).
\end{aligned}$$

From Theorem 4.3.4 both $\phi_{\mathbf{p};-\mathbf{q}}(-y)$ and, of course then, $\frac{d^2}{dy^2}\phi_{\mathbf{p};-\mathbf{q}}(-y)$ are $\mathbf{p},\mathbf{q}$-positive, proving the result. □

In addition, we can give a positivity result for the terms of third highest degree in $(-1)^k F(\mathbf{p};-\mathbf{q})$.

Corollary 4.3.9. *For $k \geq 5$, the terms of degree $k-3$ in $(-1)^k F(\mathbf{p};-\mathbf{q})$ are $\mathbf{p},\mathbf{q}$-positive.*

Note that the $k \geq 5$ restriction is there simply because the terms are otherwise 0.

Proof. From Theorem 3.5.7, the terms of third highest degree are

$$\Sigma_{k,4} = \sum_{\substack{i,j,m \geq 0 \\ i+j+m=k-3}} \tau_{i,j,m} C_i C_j C_m,$$

where $\tau_{i,j,m} \geq 0$ and is given in (3.31). Therefore,

$$\begin{aligned}(-1)^k \Sigma_{k,4} &= (-1)^k \sum_{\substack{i,j,m \geq 0 \\ i+j+m=k-3}} \tau_{i,j,m} C_i(\mathbf{p};\ \mathbf{q}) C_j(\mathbf{p};\ \mathbf{q}) C_m(\mathbf{p};\ \mathbf{q}) \\ &= \sum_{\substack{i,j,m \geq 0 \\ i+j+m=k-3}} \tau_{i,j,m} \left((-1)^{i+1} C_i(\mathbf{p};\ \mathbf{q})\right) \left((-1)^{j+1} C_j(\mathbf{p};\ \mathbf{q})\right) \\ &\qquad \cdot \left((-1)^{m+1} C_m(\mathbf{p};\ \mathbf{q})\right). \qquad (4.14)\end{aligned}$$

Substituting $-\mathbf{q}$ for $\mathbf{q}$ in the (4.14), we see from the proof of Corollary 4.3.8, that each $(-1)^{t+1} C_{t+1}(\mathbf{p}; -\mathbf{q})$ is $\mathbf{p}, \mathbf{q}$-positive. Thus, after the subtitution of $-\mathbf{q}$ for $\mathbf{q}$, each summand in (4.14) is the product of $\mathbf{p}, \mathbf{q}$-positive terms, making the last expression in (4.14) $\mathbf{p}, \mathbf{q}$-positive, completing the proof. □

Finally, the following theorem gives a general connection between Kerov's polynomials and Stanley's polynomials. The proof we give is essentially the proof of Corollary 4.3.9 (in particular, Corollary 4.3.9 would follow as a consequence).

Theorem 4.3.10. *If Kerov's polynomials Σ_k are C-positive then Stanley's polynomials $(-1)^k F_k(\mathbf{p};\ -\mathbf{q})$ are $\mathbf{p}, \mathbf{q}$-positive.*

Proof. From Proposition 4.3.1 the terms of degree i in Stanley's polynomials are obtained from the terms of weight i in Kerov's polynomials. From Theorem 3.5.13 the terms of degree $k+1-2n$ in Stanley's polynomials are obtained from

$$\sum_{\substack{i_1,\ldots,i_{2n-1} \geq 0 \\ i_1+\cdots+i_{2n-1}=k+1-2n}} \gamma_{i_1,\ldots,i_{2n-1}} C_{i_1}(\mathbf{p};\ \mathbf{q}) \cdots C_{i_{2n-1}}(\mathbf{p};\ \mathbf{q}).$$

Thus, the terms of degree $k+1-2n$ in $(-1)^k F_k(\mathbf{p};\ -\mathbf{q})$ are given by

$$\sum_{\substack{i_1,\ldots,i_{2n-1} \geq 0 \\ i_1+\cdots+i_{2n-1}=k+1-2n}} \gamma_{i_1,\ldots,i_{2n-1}} \left((-1)^{i_1-1} C_{i_1}(\mathbf{p};\ -\mathbf{q})\right) \cdots \left((-1)^{i_{2n-1}-1} C_{i_{2n-1}}(\mathbf{p};\ -\mathbf{q})\right).$$

From the proof of Corollary 4.3.8, each $(-1)^{j-1} C_j(\mathbf{p};\ -\mathbf{q})$ is $\mathbf{p}, \mathbf{q}$-positive, and the result follows. □

Appendix A

The R-expansions of Kerov's Character Polynomials for $k \leq 20$

$\Sigma_2 = R_3$

$\Sigma_3 = R_4 + R_2$

$\Sigma_4 = R_5 + 5\,R_3$

$\Sigma_5 = R_6 + 15\,R_4 + 5\,{R_2}^2 + 8\,R_2$

$\Sigma_6 = R_7 + 35\,R_5 + 35\,R_3R_2 + 84\,R_3$

$\Sigma_7 = R_8 + 70\,R_6 + 84\,R_4R_2 + 469\,R_4 + 56\,{R_3}^2 + 14\,{R_2}^3 + 224\,{R_2}^2 + 180\,R_2$

$\Sigma_8 = R_9 + 126\,R_7 + 168\,R_5R_2 + 1869\,R_5 + 252\,R_4R_3 + 126\,R_3{R_2}^2 + 2688\,R_3R_2 + 3044\,R_3$

$\Sigma_9 = R_{10} + 210\,R_8 + 300\,R_6R_2 + 5985\,R_6 + 480\,R_5R_3 + 270\,{R_4}^2 + 270\,R_4{R_2}^2 + 10548\,R_4R_2 + 26060\,R_4 + 360\,{R_3}^2R_2 + 6714\,{R_3}^2 + 30\,{R_2}^4 + 2400\,{R_2}^3 + 14580\,{R_2}^2 + 8064\,R_2$

$\Sigma_{10} = R_{11} + 330\,R_9 + 495\,R_7R_2 + 16401\,R_7 + 825\,R_6R_3 + 990\,R_5R_4 + 495\,R_5{R_2}^2 + 32901\,R_5R_2 + 152900\,R_5 + 1485\,R_4R_3R_2 + 46101\,R_4R_3 + 330\,{R_3}^3 + 330\,R_3R_2^3 + 33000\,R_3{R_2}^2 + 258060\,R_3R_2 + 193248\,R_3$

$\Sigma_{11} = R_{12} + 495\,R_{10} + 770\,R_8R_2 + 39963\,R_8 + 1320\,R_7R_3 + 1650\,R_6R_4 + 825\,R_6{R_2}^2 + 87890\,R_6R_2 + 696905\,R_6 + 880\,{R_5}^2 + 2640\,R_5R_3R_2 + 130108\,R_5R_3 + 1485\,{R_4}^2R_2 + 71214\,{R_4}^2 + 1980\,R_4R_3^2 + 660\,R_4{R_2}^3 + 105545\,R_4{R_2}^2 + 1459700\,R_4R_2 + 2286636\,R_4 + 1320\,{R_3}^2{R_2}^2 + 136345\,{R_3}^2R_2 + 902440\,{R_3}^2 + 55\,{R_2}^5 + 15400\,{R_2}^4 + 386980\,{R_2}^3 + 1401444\,{R_2}^2 + 604800\,R_2$

$\Sigma_{12} = R_{13} + 715\,R_{11} + 1144\,R_9R_2 + 88803\,R_9 + 2002\,R_8R_3 + 2574\,R_7R_4 + 1287\,R_7{R_2}^2 + 209352\,R_7R_2 + 2641925\,R_7 + 2860\,R_6R_5 + 4290\,R_6R_3R_2 + 321750\,R_6R_3 + 5148\,R_5R_4R_2 + 369798\,R_5R_4 + 3432\,R_5{R_3}^2 + 1144\,R_5{R_2}^3 + 280995\,R_5{R_2}^2 + 6390956\,R_5R_2 + 18128396\,R_5 + 3861\,{R_4}^2R_3 + 5148\,R_4R_3{R_2}^2 + 802230\,R_4R_3R_2 + 8581144\,R_4R_3 + 2288\,{R_3}^3R_2 + 173745\,{R_3}^3 + 715\,R_3{R_2}^4 + 240240\,R_3{R_2}^3 + 7379372\,R_3{R_2}^2 + 33549516\,R_3R_2 + 19056960\,R_3$

$\Sigma_{13} = R_{14} + 1001\,R_{12} + 1638\,R_{10}R_2 + 183183\,R_{10} + 2912\,R_9R_3 + 3822\,R_8R_4 + 1911\,R_8{R_2}^2 + 456092\,R_8R_2 + 8691683\,R_8 + 4368\,R_7R_5 + 6552\,R_7R_3R_2 + 720902\,R_7R_3 + 2275\,R_6^2 + 8190\,R_6R_4R_2 + 855400\,R_6R_4 + 5460\,R_6{R_3}^2 + 1820\,R_6{R_2}^3 + 662025\,R_6{R_2}^2 + 23377562\,R_6R_2 + 109425316\,R_6 + 4368\,{R_5}^2R_2 + 448084\,{R_5}^2 + 13104\,R_5R_4R_3 + 8736\,R_5R_3{R_2}^2 + 1996540\,R_5R_3R_2 + 32944938\,R_5R_3 + 2457\,{R_4}^3 + 4914\,R_4^2{R_2}^2 + 1100190\,{R_4}^2R_2 + 17749927\,{R_4}^2 + 13104\,R_4{R_3}^2R_2 + 1436435\,R_4{R_3}^2 + 1365\,R_4{R_2}^4 + 672490\,R_4{R_2}^3 + 32392724\,R_4{R_2}^2 + 253452836\,R_4R_2 + 292271616\,R_4 + 1456\,{R_3}^4 + 3640\,{R_3}^2{R_2}^3 + 1314495\,{R_3}^2{R_2}^2 + 40788384\,{R_3}^2R_2 + 153561772\,{R_3}^2 + 91\,{R_2}^6 + 71344\,{R_2}^5 + 5475652\,{R_2}^4 + 73713276\,{R_2}^3 + 190217664\,{R_2}^2 + 68428800\,R_2$

$\Sigma_{14} = R_{15} + 1365\,R_{13} + 2275\,R_{11}R_2 + 355355\,R_{11} + 4095\,R_{10}R_3 + 5460\,R_9R_4 + 2730\,R_9{R_2}^2 + 924742\,R_9R_2 + 25537655\,R_9 + 6370\,R_8R_5 + 9555\,R_8R_3R_2 + 1494402\,R_8R_3 + 6825\,R_7R_6 + 12285\,R_7R_4R_2 + 1815177\,R_7R_4 + 8190\,R_7{R_3}^2 + 2730\,R_7{R_2}^3 + 1424150\,R_7{R_2}^2 + 74586655\,R_7R_2 + 539651112\,R_7 + 13650\,R_6R_5R_2 + 1957865\,R_6R_5 + 20475\,R_6R_4R_3 + 13650\,R_6R_3{R_2}^2 + 4452175\,R_6R_3R_2 + 108780815\,R_6R_3 + 10920\,{R_5}^2R_3 + 12285\,R_5{R_4}^2 + 16380\,R_5R_4{R_2}^2 + 5165615\,R_5R_4R_2 + 121953975\,R_5R_4 + 21840\,R_5{R_3}^2R_2 + 3384745\,R_5{R_3}^2 + 2275\,R_5{R_2}^4 + 1603875\,R_5{R_2}^3 + 116367160\,R_5{R_2}^2 + 1457761032\,R_5R_2 + 2961802480\,R_5 + 24570\,{R_4}^2R_3R_2 + 3741465\,{R_4}^2R_3 + 10920\,R_4{R_3}^3 + 13650\,R_4R_3{R_2}^3 + 6951945\,R_4R_3{R_2}^2 + 319646600\,R_4R_3R_2 + 1900585960\,R_4R_3 + 9100\,{R_3}^3{R_2}^2 + 3030755\,{R_3}^3R_2 + 67649400\,{R_3}^3 + 1365\,R_3{R_2}^5 + 1248520\,R_3R_2^4 + 113233120\,R_3{R_2}^3 + 1831663288\,R_3{R_2}^2 + 5823745200\,R_3R_2 + 2699672832\,R_3$

$$\Sigma_{15} = R_{16} + 1820\,R_{14} + 3080\,R_{12}R_2 + 654654\,R_{12} + 5600\,R_{11}R_3 + 7560\,R_{10}R_4 + 3780\,R_{10}R_2^2 + 1767024\,R_{10}R_2 + 68396900\,R_{10} + 8960\,R_9R_5 + 13440\,R_9R_3R_2 + 2907744\,R_9R_3 + 9800\,R_8R_6 + 17640\,R_8R_4R_2 + 3597384\,R_8R_4 + 11760\,R_8R_3^2 + 3920\,R_8R_2^3 + 2851800\,R_8R_2^2 + 213459960\,R_8R_2 + 2273360089\,R_8 + 5040\,R_7^2 + 20160\,R_7R_5R_2 + 3961104\,R_7R_5 + 30240\,R_7R_4R_3 + 20160\,R_7R_3R_2^2 + 9151800\,R_7R_3R_2 + 319751360\,R_7R_3 + 10500\,R_6^2R_2 + 2037000\,R_6^2 + 33600\,R_6R_5R_3 + 18900\,R_6R_4^2 + 25200\,R_6R_4R_2^2 + 10970400\,R_6R_4R_2 + 368042400\,R_6R_4 + 33600\,R_6R_3^2R_2 + 7209300\,R_6R_3^2 + 3500\,R_6R_2^4 + 3448200\,R_6R_2^3 + 363356700\,R_6R_2^2 + 6893328064\,R_6R_2 + 22556777880\,R_6 + 20160\,R_5^2R_4 + 13440\,R_5^2R_2^2 + 5767440\,R_5^2R_2 + 190928800\,R_5^2 + 80640\,R_5R_4R_3R_2 + 16801680\,R_5R_4R_3 + 17920\,R_5R_3^3 + 22400\,R_5R_3R_2^3 + 15800400\,R_5R_3R_2^2 + 1047424280\,R_5R_3R_2 + 9387340928\,R_5R_3 + 15120\,R_4^3R_2 + 3103380\,R_4^3 + 30240\,R_4^2R_3^2 + 12600\,R_4^2R_2^3 + 8746920\,R_4^2R_2^2 + 568600060\,R_4^2R_2 + 5006452864\,R_4^2 + 50400\,R_4R_3^2R_2^2 + 22949640\,R_4R_3^2R_2 + 727222860\,R_4R_3^2 + 2520\,R_4R_2^5 + 3182550\,R_4R_2^4 + 418373760\,R_4R_2^3 + 10422033664\,R_4R_2^2 + 55848839760\,R_4R_2 + 51381813456\,R_4 + 11200\,R_3^4R_2 + 2508240\,R_3^4 + 8400\,R_3^2R_2^4 + 8340780\,R_3^2R_2^3 + 800181760\,R_3^2R_2^2 + 12869508064\,R_3^2R_2 + 33336787680\,R_3^2 + 140\,R_2^7 + 263424\,R_2^6 + 51093280\,R_2^5 + 1933747200\,R_2^4 + 17295397560\,R_2^3 + 34907328000\,R_2^2 + 10897286400\,R_2$$

$$\Sigma_{16} = R_{17} + 2380\,R_{15} + 4080\,R_{13}R_2 + 1154062\,R_{13} + 7480\,R_{12}R_3 + 10200\,R_{11}R_4 + 5100\,R_{11}R_2^2 + 3212320\,R_{11}R_2 + 169537940\,R_{11} + 12240\,R_{10}R_5 + 18360\,R_{10}R_3R_2 + 5366832\,R_{10}R_3 + 13600\,R_9R_6 + 24480\,R_9R_4R_2 + 6740432\,R_9R_4 + 16320\,R_9R_3^2 + 5440\,R_9R_2^3 + 5386280\,R_9R_2^2 + 558874320\,R_9R_2 + 8433097673\,R_9 + 14280\,R_8R_7 + 28560\,R_8R_5R_2 + 7540792\,R_8R_5 + 42840\,R_8R_4R_3 + 28560\,R_8R_3R_2^2 + 17640560\,R_8R_3R_2 + 855690920\,R_8R_3 + 30600\,R_7R_6R_2 + 7906360\,R_7R_6 + 48960\,R_7R_5R_3 + 27540\,R_7R_4^2 + 36720\,R_7R_4R_2^2 + 21642360\,R_7R_4R_2 + 1004725160\,R_7R_4 + 48960\,R_7R_3^2R_2 + 14255180\,R_7R_3^2 + 5100\,R_7R_2^4 + 6868000\,R_7R_2^3 + 1018466260\,R_7R_2^2 + 28015362432\,R_7R_2 + 138687993080\,R_7 + 25500\,R_6^2R_3 + 61200\,R_6R_5R_4 + 40800\,R_6R_5R_2^2 + 23470200\,R_6R_5R_2 + 1066471880\,R_6R_5 + 122400\,R_6R_4R_3R_2 + 34340000\,R_6R_4R_3 + 27200\,R_6R_3^3 + 34000\,R_6R_3R_2^3 + 32609400\,R_6R_3R_2^2 + 3033562280\,R_6R_3R_2 + 39375602368\,R_6R_3 + 10880\,R_5^3 + 65280\,R_5^2R_3R_2 + 18083920\,R_5^2R_3 + 73440\,R_5R_4^2R_2 + 20084820\,R_5R_4^2 + 97920\,R_5R_4R_3^2 + 40800\,R_5R_4R_2^3 + 38092920\,R_5R_4R_2^2 + 3436658120\,R_5R_4R_2 + 43442253696\,R_5R_4 + 81600\,R_5R_3^2R_2^2 + 50098320\,R_5R_3^2R_2 + 2210817700\,R_5R_3^2 + 4080\,R_5R_2^5 + 7003150\,R_5R_2^4 + 1302945280\,R_5R_2^3 + 47980320192\,R_5R_2^2 + 403288092320\,R_5R_2 + 640277565264\,R_5 + 36720\,R_4^3R_3 +$$

$91800\,{R_4}^2R_3{R_2}^2 + 55581840\,{R_4}^2R_3R_2 + 2409452500\,{R_4}^2R_3 + 81600\,R_4{R_3}^3R_2 + 24356920\,R_4{R_3}^3 + 30600\,R_4R_3{R_2}^4 + 40807480\,R_4R_3{R_2}^3 + 5459072640\,R_4R_3{R_2}^2 + 128040338880\,R_4R_3R_2 + 514531785200\,R_4R_3 + 5440\,R_3^5 + 27200\,{R_3}^3{R_2}^3 + 26801180\,{R_3}^3{R_2}^2 + 2333847040\,{R_3}^3R_2 + 26603024736\,{R_3}^3 + 2380\,R_3{R_2}^6 + 5117952\,R_3{R_2}^5 + 1145198880\,R_3{R_2}^4 + 50584764800\,R_3{R_2}^3 + 536391335160\,R_3{R_2}^2 + 1314589943808\,R_3R_2 + 520105017600\,R_3$

$\Sigma_{17} = R_{18} + 3060\,R_{16} + 5304\,R_{14}R_2 + 1958502\,R_{14} + 9792\,R_{13}R_3 + 13464\,R_{12}R_4 + 6732\,R_{12}{R_2}^2 + 5596536\,R_{12}R_2 + 393481660\,R_{12} + 16320\,R_{11}R_5 + 24480\,R_{11}R_3R_2 + 9471720\,R_{11}R_3 + 18360\,R_{10}R_6 + 33048\,R_{10}R_4R_2 + 12047832\,R_{10}R_4 + 22032\,R_{10}{R_3}^2 + 7344\,R_{10}{R_2}^3 + 9687960\,R_{10}{R_2}^2 + 1358203032\,R_{10}R_2 + 28157550993\,R_{10} + 19584\,R_9R_7 + 39168\,R_9R_5R_2 + 13653312\,R_9R_5 + 58752\,R_9R_4R_3 + 39168\,R_9R_3{R_2}^2 + 32256480\,R_9R_3R_2 + 2118341712\,R_9R_3 + 9996\,{R_8}^2 + 42840\,R_8R_6R_2 + 14522760\,R_8R_6 + 68544\,R_8R_5R_3 + 38556\,R_8{R_4}^2 + 51408\,R_8R_4{R_2}^2 + 40283880\,R_8R_4R_2 + 2527831320\,R_8R_4 + 68544\,R_8{R_3}^2R_2 + 26582220\,R_8{R_3}^2 + 7140\,R_8{R_2}^4 + 12880560\,R_8{R_2}^3 + 2616205956\,R_8{R_2}^2 + 100776536520\,R_8R_2 + 720447491400\,R_8 + 22032\,{R_7}^2R_2 + 7398468\,{R_7}^2 + 73440\,R_7R_6R_3 + 88128\,R_7R_5R_4 + 58752\,R_7R_5{R_2}^2 + 44631120\,R_7R_5R_2 + 2727348216\,R_7R_5 + 176256\,R_7R_4R_3R_2 + 65539080\,R_7R_4R_3 + 39168\,R_7{R_3}^3 + 48960\,R_7R_3{R_2}^3 + 62723880\,R_7R_3{R_2}^2 + 7988140608\,R_7R_3R_2 + 145251275016\,R_7R_3 + 45900\,{R_6}^2R_4 + 30600\,{R_6}^2{R_2}^2 + 23001000\,{R_6}^2R_2 + 1393439340\,{R_6}^2 + 48960\,R_6{R_5}^2 + 195840\,R_6R_5R_3R_2 + 71257200\,R_6R_5R_3 + 110160\,R_6{R_4}^2R_2 + 39642300\,R_6{R_4}^2 + 146880\,R_6R_4{R_3}^2 + 61200\,R_6R_4{R_2}^3 + 75765600\,R_6R_4{R_2}^2 + 9299362848\,R_6R_4R_2 + 163914689928\,R_6R_4 + 122400\,R_6{R_3}^2{R_2}^2 + 99847800\,R_6{R_3}^2R_2 + 6011731692\,R_6{R_3}^2 + 6120\,R_6{R_2}^5 + 14047950\,R_6{R_2}^4 + 3611048608\,R_6{R_2}^3 + 189612052992\,R_6{R_2}^2 + 2367891394224\,R_6R_2 + 5943136639504\,R_6 + 117504\,{R_5}^2R_4R_2 + 41815920\,{R_5}^2R_4 + 78336\,{R_5}^2{R_3}^2 + 32640\,{R_5}^2{R_2}^3 + 39939120\,{R_5}^2{R_2}^2 + 4842223968\,{R_5}^2R_2 + 84530220708\,{R_5}^2 + 176256\,R_5{R_4}^2R_3 + 293760\,R_5R_4R_3{R_2}^2 + 234004320\,R_5R_4R_3R_2 + 13694468184\,R_5R_4R_3 + 130560\,R_5{R_3}^3R_2 + 51375360\,R_5{R_3}^3 + 48960\,R_5R_3{R_2}^4 + 86622480\,R_5R_3{R_2}^3 + 15888265824\,R_5R_3{R_2}^2 + 528470969472\,R_5R_3R_2 + 3143822584896\,R_5R_3 + 16524\,{R_4}^4 + 55080\,{R_4}^3{R_2}^2 + 43347960\,{R_4}^3R_2 + 2496204180\,{R_4}^3 + 220320\,{R_4}^2{R_3}^2R_2 + 85640220\,{R_4}^2{R_3}^2 + 27540\,{R_4}^2{R_2}^4 + 48109320\,{R_4}^2{R_2}^3 + 8676349296\,{R_4}^2{R_2}^2 + 283879059264\,{R_4}^2R_2 + 1664828612280\,{R_4}^2 + 48960\,R_4{R_3}^4 + 146880\,R_4{R_3}^2{R_2}^3 + 189973980\,R_4{R_3}^2{R_2}^2 + 22369864800\,R_4{R_3}^2R_2 + 356862570912\,R_4{R_3}^2 + 4284\,R_4{R_2}^6 + 12172272\,R_4{R_2}^5 + 3734409312\,R_4{R_2}^4 + 236162036160\,R_4{R_2}^3 + 3798814728504\,R_4{R_2}^2 +$

$15435445558464\, R_4R_2 + 11905898330880\, R_4 + 48960\, R_3^4R_2^2 + 41669040\, R_3^4R_2 + 2402180104\, R_3^4 + 17136\, R_3^2R_2^5 + 40026840\, R_3^2R_2^4 + 9608537088\, R_3^2R_2^3 + 443605042080\, R_3^2R_2^2 + 4620143919648\, R_3^2R_2 + 9107976760416\, R_3^2 + 204\, R_2^8 + 822528\, R_2^7 + 353456928\, R_2^6 + 31642208256\, R_2^5 + 752018450616\, R_2^4 + 5003578123776\, R_2^3 + 8355017145600\, R_2^2 + 2324754432000\, R_2$

$\Sigma_{18} = R_{19} + 3876\, R_{17} + 6783\, R_{15}R_2 + 3215142\, R_{15} + 12597\, R_{14}R_3 + 17442\, R_{13}R_4 + 8721\, R_{13}R_2^2 + 9398331\, R_{13}R_2 + 862928092\, R_{13} + 21318\, R_{12}R_5 + 31977\, R_{12}R_3R_2 + 16084431\, R_{12}R_3 + 24225\, R_{11}R_6 + 43605\, R_{11}R_4R_2 + 20683305\, R_{11}R_4 + 29070\, R_{11}R_3^2 + 9690\, R_{11}R_2^3 + 16715250\, R_{11}R_2^2 + 3098068389\, R_{11}R_2 + 86027797713\, R_{11} + 26163\, R_{10}R_7 + 52326\, R_{10}R_5R_2 + 23694957\, R_{10}R_5 + 78489\, R_{10}R_4R_3 + 52326\, R_{10}R_3R_2^2 + 56424870\, R_{10}R_3R_2 + 4909530555\, R_{10}R_3 + 27132\, R_9R_8 + 58140\, R_9R_6R_2 + 25494390\, R_9R_6 + 93024\, R_9R_5R_3 + 52326\, R_9R_4^2 + 69768\, R_9R_4R_2^2 + 71483130\, R_9R_4R_2 + 5939392476\, R_9R_4 + 93024\, R_9R_3^2R_2 + 47238750\, R_9R_3^2 + 9690\, R_9R_2^4 + 22994370\, R_9R_2^3 + 6249977325\, R_9R_2^2 + 327662916825\, R_9R_2 + 3262699892088\, R_9 + 61047\, R_8R_7R_2 + 26331606\, R_8R_7 + 101745\, R_8R_6R_3 + 122094\, R_8R_5R_4 + 81396\, R_8R_5R_2^2 + 80480295\, R_8R_5R_2 + 6490978284\, R_8R_5 + 244188\, R_8R_4R_3R_2 + 118532925\, R_8R_4R_3 + 54264\, R_8R_3^3 + 67830\, R_8R_3R_2^3 + 114157890\, R_8R_3R_2^2 + 19468614081\, R_8R_3R_2 + 482319595929\, R_8R_3 + 52326\, R_7^2R_3 + 130815\, R_7R_6R_4 + 87210\, R_7R_6R_2^2 + 84666375\, R_7R_6R_2 + 6730584852\, R_7R_6 + 69768\, R_7R_5^2 + 279072\, R_7R_5R_3R_2 + 131716170\, R_7R_5R_3 + 156978\, R_7R_4^2R_2 + 73387215\, R_7R_4^2 + 209304\, R_7R_4R_3^2 + 87210\, R_7R_4R_2^3 + 141149385\, R_7R_4R_2^2 + 23119387473\, R_7R_4R_2 + 553802501121\, R_7R_4 + 174420\, R_7R_3^2R_2^2 + 186324165\, R_7R_3^2R_2 + 15006465012\, R_7R_3^2 + 8721\, R_7R_2^5 + 26351955\, R_7R_2^4 + 9151327732\, R_7R_2^3 + 664563711816\, R_7R_2^2 + 11825317043640\, R_7R_2 + 44160980070544\, R_7 + 72675\, R_6^2R_5 + 145350\, R_6^2R_3R_2 + 67951125\, R_6^2R_3 + 348840\, R_6R_5R_4R_2 + 159957675\, R_6R_5R_4 + 232560\, R_6R_5R_3^2 + 96900\, R_6R_5R_2^3 + 153707625\, R_6R_5R_2^2 + 24676446441\, R_6R_5R_2 + 582100836033\, R_6R_5 + 261630\, R_6R_4^2R_3 + 436050\, R_6R_4R_3R_2^2 + 451747800\, R_6R_4R_3R_2 + 35145645504\, R_6R_4R_3 + 193800\, R_6R_3^3R_2 + 99346725\, R_6R_3^3 + 72675\, R_6R_3R_2^4 + 168387975\, R_6R_3R_2^3 + 41576328732\, R_6R_3R_2^2 + 1908984783384\, R_6R_3R_2 + 16166926380888\, R_6R_3 + 62016\, R_5^3R_2 + 28159140\, R_5^3 + 279072\, R_5^2R_4R_3 + 232560\, R_5^2R_3R_2^2 + 238432140\, R_5^2R_3R_2 + 18336328860\, R_5^2R_3 + 104652\, R_5R_4^3 + 261630\, R_5R_4^2R_2^2 + 265423635\, R_5R_4^2R_2 + 20116645428\, R_5R_4^2 + 697680\, R_5R_4R_3^2R_2 + 350148150\, R_5R_4R_3^2 + 87210\, R_5R_4R_2^4 + 197690535\, R_5R_4R_2^3 + 47504097084\, R_5R_4R_2^2 + 2127087877032\, R_5R_4R_2 +$

$17639100251208\, R_5R_4 + 77520\, R_5R_3{}^4 + 232560\, R_5R_3{}^2R_2{}^3 + 391006035\, R_5R_3{}^2R_2{}^2 + 61506255612\, R_5R_3{}^2R_2 + 1346569377120\, R_5R_3{}^2 + 6783\, R_5R_2{}^6 + 25192062\, R_5R_2{}^5 + 10441525392\, R_5R_2{}^4 + 922360363320\, R_5R_2{}^3 + 21593547979560\, R_5R_2{}^2 + 135419270647824\, R_5R_2 + 177317274898944\, R_5 + 261630\, R_4{}^3R_3R_2 + 129899295\, R_4{}^3R_3 + 174420\, R_4{}^2R_3{}^3 + 261630\, R_4{}^2R_3R_2{}^3 + 434959875\, R_4{}^2R_3R_2{}^2 + 67373930460\, R_4{}^2R_3R_2 + 1452375874728\, R_4{}^2R_3 + 348840\, R_4R_3{}^3R_2{}^2 + 382255965\, R_4R_3{}^3R_2 + 29061223444\, R_4R_3{}^3 + 61047\, R_4R_3R_2{}^5 + 184565430\, R_4R_3R_2{}^4 + 59068658592\, R_4R_3R_2{}^3 + 3760094847960\, R_4R_3R_2{}^2 + 56357253645120\, R_4R_3R_2 + 169914189023568\, R_4R_3 + 46512\, R_3{}^5R_2 + 25192062\, R_3{}^5 + 67830\, R_3{}^3R_2{}^4 + 162113700\, R_3{}^3R_2{}^3 + 38155561056\, R_3{}^3R_2{}^2 + 1580651249760\, R_3{}^3R_2 + 11537475926976\, R_3{}^3 + 3876\, R_3R_2{}^7 + 17581536\, R_3R_2{}^6 + 8561138256\, R_3R_2{}^5 + 875902898640\, R_3R_2{}^4 + 24050910615048\, R_3R_2{}^3 + 187683394106304\, R_3R_2{}^2 + 376118453760768\, R_3R_2 + 130859579289600\, R_3$

$\Sigma_{19} = R_{20} + 4845\, R_{18} + 8550\, R_{16}R_2 + 5126010\, R_{16} + 15960\, R_{15}R_3 + 22230\, R_{14}R_4 + 11115\, R_{14}R_2{}^2 + 15283866\, R_{14}R_2 + 1801329010\, R_{14} + 27360\, R_{13}R_5 + 41040\, R_{13}R_3R_2 + 26413116\, R_{13}R_3 + 31350\, R_{12}R_6 + 56430\, R_{12}R_4R_2 + 34289376\, R_{12}R_4 + 37620\, R_{12}R_3{}^2 + 12540\, R_{12}R_2{}^3 + 27823125\, R_{12}R_2{}^2 + 6690798510\, R_{12}R_2 + 243582356589\, R_{12} + 34200\, R_{11}R_7 + 68400\, R_{11}R_5R_2 + 39650340\, R_{11}R_5 + 102600\, R_{11}R_4R_3 + 68400\, R_{11}R_3R_2{}^2 + 95027550\, R_{11}R_3R_2 + 10751609040\, R_{11}R_3 + 35910\, R_{10}R_8 + 76950\, R_{10}R_6R_2 + 43069770\, R_{10}R_6 + 123120\, R_{10}R_5R_3 + 69255\, R_{10}R_4{}^2 + 92340\, R_{10}R_4R_2{}^2 + 121832370\, R_{10}R_4R_2 + 13162873770\, R_{10}R_4 + 123120\, R_{10}R_3{}^2R_2 + 80606835\, R_{10}R_3{}^2 + 12825\, R_{10}R_2{}^4 + 39381300\, R_{10}R_2{}^3 + 14038852905\, R_{10}R_2{}^2 + 978155763966\, R_{10}R_2 + 13178203976145\, R_{10} + 18240\, R_9{}^2 + 82080\, R_9R_7R_2 + 44957496\, R_9R_7 + 136800\, R_9R_6R_3 + 164160\, R_9R_5R_4 + 109440\, R_9R_5R_2{}^2 + 138929520\, R_9R_5R_2 + 14540098080\, R_9R_5 + 328320\, R_9R_4R_3R_2 + 205115640\, R_9R_4R_3 + 72960\, R_9R_3{}^3 + 91200\, R_9R_3R_2{}^3 + 198558360\, R_9R_3R_2{}^2 + 44469403860\, R_9R_3R_2 + 1466199316140\, R_9R_3 + 41895\, R_8{}^2R_2 + 22779708\, R_8{}^2 + 143640\, R_8R_7R_3 + 179550\, R_8R_6R_4 + 119700\, R_8R_6R_2{}^2 + 148368150\, R_8R_6R_2 + 15244146120\, R_8R_6 + 95760\, R_8R_5{}^2 + 383040\, R_8R_5R_3R_2 + 231651420\, R_8R_5R_3 + 215460\, R_8R_4{}^2R_2 + 129228120\, R_8R_4{}^2 + 287280\, R_8R_4R_3{}^2 + 119700\, R_8R_4R_2{}^3 + 249849810\, R_8R_4R_2{}^2 + 53646016830\, R_8R_4R_2 + 1707778339500\, R_8R_4 + 239400\, R_8R_3{}^2R_2{}^2 + 330264270\, R_8R_3{}^2R_2 + 34937664930\, R_8R_3{}^2 + 11970\, R_8R_2{}^5 + 46908435\, R_8R_2{}^4 + 21566912920\, R_8R_2{}^3 + 2110776824691\, R_8R_2{}^2 + 51770353887060\, R_8R_2 + 275057386118488\, R_8 + 92340\, R_7{}^2R_4 + 61560\, R_7{}^2R_2{}^2 + 75688875\, R_7{}^2R_2 + 7730011680\, R_7{}^2 + 205200\, R_7R_6R_5 + 410400\, R_7R_6R_3R_2 + 244099650\, R_7R_6R_3 +$

$492480\, R_7R_5R_4R_2 + 288001620\, R_7R_5R_4 + 328320\, R_7R_5{R_3}^2 + 136800\, R_7R_5{R_2}^3 +$
$278165700\, R_7R_5{R_2}^2 + 58249583070\, R_7R_5R_2 + 1819579841292\, R_7R_5 +$
$369360\, R_7{R_4}^2R_3 + 615600\, R_7R_4R_3{R_2}^2 + 819743220\, R_7R_4R_3R_2 +$
$83469926190\, R_7R_4R_3 + 273600\, R_7{R_3}^3R_2 + 180525840\, R_7{R_3}^3 + 102600\, R_7R_3{R_2}^4 +$
$307313790\, R_7R_3{R_2}^3 + 100324623840\, R_7R_3{R_2}^2 + 6207119862102\, R_7R_3R_2 +$
$72472927653840\, R_7R_3 + 35625\, {R_6}^3 + 256500\, {R_6}^2R_4R_2 + 148720125\, {R_6}^2R_4 +$
$171000\, {R_6}^2{R_3}^2 + 71250\, {R_6}^2{R_2}^3 + 143597250\, {R_6}^2{R_2}^2 + 29822367225\, {R_6}^2R_2 +$
$926094355230\, {R_6}^2 + 273600\, R_6{R_5}^2R_2 + 157268700\, R_6{R_5}^2 + 820800\, R_6R_5R_4R_3 +$
$684000\, R_6R_5R_3{R_2}^2 + 894432600\, R_6R_5R_3R_2 + 89352654510\, R_6R_5R_3 +$
$153900\, R_6{R_4}^3 + 384750\, R_6{R_4}^2{R_2}^2 + 498507750\, R_6{R_4}^2R_2 + 49148644440\, R_6{R_4}^2 +$
$1026000\, R_6R_4{R_3}^2R_2 + 658529550\, R_6R_4{R_3}^2 + 128250\, R_6R_4{R_2}^4 + 373384200\, R_6R_4{R_2}^3 +$
$117903765840\, R_6R_4{R_2}^2 + 7080614444394\, R_6R_4R_2 + 80661647984100\, R_6R_4 +$
$114000\, R_6{R_3}^4 + 342000\, R_6{R_3}^2{R_2}^3 + 739596375\, R_6{R_3}^2{R_2}^2 + 153218346120\, R_6{R_3}^2R_2 +$
$4509698600727\, R_6{R_3}^2 + 9975\, R_6{R_2}^6 + 47872020\, R_6{R_2}^5 + 26369945910\, R_6{R_2}^4 +$
$3175405116900\, R_6{R_2}^3 + 104413147057500\, R_6{R_2}^2 + 958343893560768\, R_6R_2 +$
$1954656775501200\, R_6 + 145920\, {R_5}^3R_3 + 246240\, {R_5}^2{R_4}^2 + 410400\, {R_5}^2R_4{R_2}^2 +$
$526823640\, {R_5}^2R_4R_2 + 51380833980\, {R_5}^2R_4 + 547200\, {R_5}^2{R_3}^2R_2 + 347937120\, {R_5}^2{R_3}^2 +$
$68400\, {R_5}^2{R_2}^4 + 197225700\, {R_5}^2{R_2}^3 + 61587533280\, {R_5}^2{R_2}^2 + 3663655405623\, {R_5}^2R_2 +$
$41432280110400\, {R_5}^2 + 1231200\, R_5{R_4}^2R_3R_2 + 775481580\, R_5{R_4}^2R_3 +$
$547200\, R_5R_4{R_3}^3 + 820800\, R_5R_4R_3{R_2}^3 + 1740605580\, R_5R_4R_3{R_2}^2 +$
$351642690000\, R_5R_4R_3R_2 + 10105007432094\, R_5R_4R_3 + 547200\, R_5{R_3}^3{R_2}^2 +$
$765952320\, R_5{R_3}^3R_2 + 76114633840\, R_5{R_3}^3 + 95760\, R_5R_3{R_2}^5 + 371500920\, R_5R_3{R_2}^4 +$
$156825338040\, R_5R_3{R_2}^3 + 13527648124560\, R_5R_3{R_2}^2 + 283569436423800\, R_5R_3R_2 +$
$1247637548296416\, R_5R_3 + 115425\, {R_4}^4R_2 + 72009810\, {R_4}^4 + 307800\, {R_4}^3{R_3}^2 +$
$153900\, {R_4}^3{R_2}^3 + 323136135\, {R_4}^3{R_2}^2 + 64374037560\, {R_4}^3R_2 + 1823325743637\, {R_4}^3 +$
$923400\, {R_4}^2{R_3}^2{R_2}^2 + 1279634895\, {R_4}^2{R_3}^2R_2 + 125380658760\, {R_4}^2{R_3}^2 +$
$53865\, {R_4}^2{R_2}^5 + 206817660\, {R_4}^2{R_2}^4 + 86041245780\, {R_4}^2{R_2}^3 + 7310914509420\, {R_4}^2{R_2}^2 +$
$151162866081900\, {R_4}^2R_2 + 657260785021416\, {R_4}^2 + 410400\, R_4{R_3}^4R_2 +$
$281494500\, R_4{R_3}^4 + 359100\, R_4{R_3}^2{R_2}^4 + 1091552280\, R_4{R_3}^2{R_2}^3 +$
$334724507460\, R_4{R_3}^2{R_2}^2 + 18557168880240\, R_4{R_3}^2R_2 + 187336894109520\, R_4{R_3}^2 +$
$6840\, R_4{R_2}^7 + 39642246\, R_4{R_2}^6 + 25394402880\, R_4{R_2}^5 + 3531762669060\, R_4{R_2}^4 +$
$137312829420660\, R_4{R_2}^3 + 1605438838388808\, R_4{R_2}^2 + 5263620826167360\, R_4R_2 +$
$3518998580742912\, R_4 + 18240\, {R_3}^6 + 159600\, {R_3}^4{R_2}^3 + 360025680\, {R_3}^4{R_2}^2 +$
$72322105590\, {R_3}^4R_2 + 1961702574480\, {R_3}^4 + 31920\, {R_3}^2{R_2}^6 + 156847698\, {R_3}^2{R_2}^5 +$
$82209481200\, {R_3}^2{R_2}^4 + 8939828459460\, {R_3}^2{R_2}^3 + 254146381414020\, {R_3}^2{R_2}^2 +$
$1928968296107184\, {R_3}^2R_2 + 3077385808793760\, {R_3}^2 + 285\, {R_2}^9 +$

$2257200\,R_2^{8} + 1949355540\,R_2^{7} + 366283126092\,R_2^{6} + 19653117610140\,R_2^{5} + 333416706139944\,R_2^{4} + 1766923640720640\,R_2^{3} + 2533181737248000\,R_2^{2} + 640237370572800\,R_2$

$\Sigma_{20} = R_{21} + 5985\,R_{19} + 10640\,R_{17}R_2 + 7963242\,R_{17} + 19950\,R_{16}R_3 + 27930\,R_{15}R_4 + 13965\,R_{15}R_2^{2} + 24161312\,R_{15}R_2 + 3600529450\,R_{15} + 34580\,R_{14}R_5 + 51870\,R_{14}R_3R_2 + 42114982\,R_{14}R_3 + 39900\,R_{13}R_6 + 71820\,R_{13}R_4R_2 + 55133022\,R_{13}R_4 + 47880\,R_{13}R_3^{2} + 15960\,R_{13}R_2^{3} + 44884175\,R_{13}R_2^{2} + 13777132940\,R_{13}R_2 + 645643728093\,R_{13} + 43890\,R_{12}R_7 + 87780\,R_{12}R_5R_2 + 64275442\,R_{12}R_5 + 131670\,R_{12}R_4R_3 + 87780\,R_{12}R_3R_2^{2} + 154858550\,R_{12}R_3R_2 + 22412869490\,R_{12}R_3 + 46550\,R_{11}R_8 + 99750\,R_{11}R_6R_2 + 70390250\,R_{11}R_6 + 159600\,R_{11}R_5R_3 + 89775\,R_{11}R_4^{2} + 119700\,R_{11}R_4R_2^{2} + 200570650\,R_{11}R_4R_2 + 27730238750\,R_{11}R_4 + 159600\,R_{11}R_3^{2}R_2 + 132830425\,R_{11}R_3^{2} + 16625\,R_{11}R_2^{4} + 65090200\,R_{11}R_2^{3} + 29904022575\,R_{11}R_2^{2} + 2713702951204\,R_{11}R_2 + 48296666026245\,R_{11} + 47880\,R_{10}R_9 + 107730\,R_{10}R_7R_2 + 74113452\,R_{10}R_7 + 179550\,R_{10}R_6R_3 + 215460\,R_{10}R_5R_4 + 143640\,R_{10}R_5R_2^{2} + 231144690\,R_{10}R_5R_2 + 30916566300\,R_{10}R_5 + 430920\,R_{10}R_4R_3R_2 + 341946990\,R_{10}R_4R_3 + 95760\,R_{10}R_3^{3} + 119700\,R_{10}R_3R_2^{3} + 332406900\,R_{10}R_3R_2^{2} + 96095650770\,R_{10}R_3R_2 + 4132917328806\,R_{10}R_3 + 111720\,R_9R_8R_2 + 75869052\,R_9R_8 + 191520\,R_9R_7R_3 + 239400\,R_9R_6R_4 + 159600\,R_9R_6R_2^{2} + 249760700\,R_9R_6R_2 + 32704206000\,R_9R_6 + 127680\,R_9R_5^{2} + 510720\,R_9R_5R_3R_2 + 391137040\,R_9R_5R_3 + 287280\,R_9R_4^{2}R_2 + 218424570\,R_9R_4^{2} + 383040\,R_9R_4R_3^{2} + 159600\,R_9R_4R_2^{3} + 424129020\,R_9R_4R_2^{2} + 117454289680\,R_9R_4R_2 + 4873746557946\,R_9R_4 + 319200\,R_9R_3^{2}R_2^{2} + 561265320\,R_9R_3^{2}R_2 + 76707501290\,R_9R_3^{2} + 15960\,R_9R_2^{5} + 79996175\,R_9R_2^{4} + 47823086640\,R_9R_2^{3} + 6170892626493\,R_9R_2^{2} + 203052301928980\,R_9R_2 + 1482541157911384\,R_9 + 97755\,R_8^{2}R_3 + 251370\,R_8R_7R_4 + 167580\,R_8R_7R_2^{2} + 258538700\,R_8R_7R_2 + 33506088840\,R_8R_7 + 279300\,R_8R_6R_5 + 558600\,R_8R_6R_3R_2 + 418531050\,R_8R_6R_3 + 670320\,R_8R_5R_4R_2 + 494817190\,R_8R_5R_4 + 446880\,R_8R_5R_3^{2} + 186200\,R_8R_5R_2^{3} + 479977050\,R_8R_5R_2^{2} + 129220584220\,R_8R_5R_2 + 5249012612186\,R_8R_5 + 502740\,R_8R_4^{2}R_3 + 837900\,R_8R_4R_3R_2^{2} + 1417670940\,R_8R_4R_3R_2 + 186131320130\,R_8R_4R_3 + 372400\,R_8R_3^{3}R_2 + 312564630\,R_8R_3^{3} + 139650\,R_8R_3R_2^{4} + 534002980\,R_8R_3R_2^{3} + 226662265480\,R_8R_3R_2^{2} + 18501103588926\,R_8R_3R_2 + 290018736202160\,R_8R_3 + 143640\,R_7^{2}R_5 + 287280\,R_7^{2}R_3R_2 + 213654525\,R_7^{2}R_3 + 149625\,R_7R_6^{2} + 718200\,R_7R_6R_4R_2 + 522211200\,R_7R_6R_4 + 478800\,R_7R_6R_3^{2} + 199500\,R_7R_6R_2^{3} + 506311050\,R_7R_6R_2^{2} + 134425148200\,R_7R_6R_2 + 5407355314242\,R_7R_6 + 383040\,R_7R_5^{2}R_2 + 276392620\,R_7R_5^{2} + 1149120\,R_7R_5R_4R_3 + 957600\,R_7R_5R_3R_2^{2} + 1582035000\,R_7R_5R_3R_2 + 202786960670\,R_7R_5R_3 + 215460\,R_7R_4^{3} + 538650\,R_7R_4^{2}R_2^{2} +$

$882739620\,R_7R_4{}^2R_2 + 111810118140\,R_7R_4{}^2 + 1436400\,R_7R_4R_3{}^2R_2 + 1167446070\,R_7R_4R_3{}^2 + 179550\,R_7R_4R_2{}^4 + 664315050\,R_7R_4R_2{}^3 + 271766087320\,R_7R_4R_2{}^2 + 21475567775466\,R_7R_4R_2 + 327783015866040\,R_7R_4 + 159600\,R_7R_3{}^4 + 478800\,R_7R_3{}^2R_2{}^3 + 1317499995\,R_7R_3{}^2R_2{}^2 + 354258608480\,R_7R_3{}^2R_2 + 13748469525693\,R_7R_3{}^2 + 13965\,R_7R_2{}^6 + 85607312\,R_7R_2{}^5 + 61659245550\,R_7R_2{}^4 + 9898982639700\,R_7R_2{}^3 + 443808528768220\,R_7R_2{}^2 + 5724588717327484\,R_7R_2 + 17142759609274320\,R_7 + 399000\,R_6{}^2R_5R_2 + 285700625\,R_6{}^2R_5 + 598500\,R_6{}^2R_4R_3 + 498750\,R_6{}^2R_3R_2{}^2 + 817351500\,R_6{}^2R_3R_2 + 103937388625\,R_6{}^2R_3 + 638400\,R_6R_5{}^2R_3 + 718200\,R_6R_5R_4{}^2 + 1197000\,R_6R_5R_4R_2{}^2 + 1929843300\,R_6R_5R_4R_2 + 240065731500\,R_6R_5R_4 + 1596000\,R_6R_5R_3{}^2R_2 + 1275962100\,R_6R_5R_3{}^2 + 199500\,R_6R_5R_2{}^4 + 725761050\,R_6R_5R_2{}^3 + 291463568200\,R_6R_5R_2{}^2 + 22684702093242\,R_6R_5R_2 + 342284494563900\,R_6R_5 + 1795500\,R_6R_4{}^2R_3R_2 + 1423532250\,R_6R_4{}^2R_3 + 798000\,R_6R_4R_3{}^3 + 1197000\,R_6R_4R_3R_2{}^3 + 3210274200\,R_6R_4R_3R_2{}^2 + 836718853600\,R_6R_4R_3R_2 + 31554503733942\,R_6R_4R_3 + 798000\,R_6R_3{}^3R_2{}^2 + 1414421750\,R_6R_3{}^3R_2 + 181673930200\,R_6R_3{}^3 + 139650\,R_6R_3R_2{}^5 + 688660700\,R_6R_3R_2{}^4 + 378586122600\,R_6R_3R_2{}^3 + 43431665508000\,R_6R_3R_2{}^2 + 1239640039043560\,R_6R_3R_2 + 7659265765864468\,R_6R_3 + 255360\,R_5{}^3R_4 + 212800\,R_5{}^3R_2{}^2 + 340256560\,R_5{}^3R_2 + 41901955900\,R_5{}^3 + 1915200\,R_5{}^2R_4R_3R_2 + 1505714280\,R_5{}^2R_4R_3 + 425600\,R_5{}^2R_3{}^3 + 638400\,R_5{}^2R_3R_2{}^3 + 1697306100\,R_5{}^2R_3R_2{}^2 + 437776713920\,R_5{}^2R_3R_2 + 16357803585333\,R_5{}^2R_3 + 718200\,R_5R_4{}^3R_2 + 559872810\,R_5R_4{}^3 + 1436400\,R_5R_4{}^2R_3{}^2 + 718200\,R_5R_4{}^2R_2{}^3 + 1892774205\,R_5R_4{}^2R_2{}^2 + 482039954480\,R_5R_4{}^2R_2 + 17770606709373\,R_5R_4{}^2 + 2872800\,R_5R_4R_3{}^2R_2{}^2 + 5002877460\,R_5R_4R_3{}^2R_2 + 627792712680\,R_5R_4R_3{}^2 + 167580\,R_5R_4R_2{}^5 + 811552700\,R_5R_4R_2{}^4 + 435577112040\,R_5R_4R_2{}^3 + 48811009650480\,R_5R_4R_2{}^2 + 1364825110628980\,R_5R_4R_2 + 8288663588798152\,R_5R_4 + 638400\,R_5R_3{}^4R_2 + 550928560\,R_5R_3{}^4 + 558600\,R_5R_3{}^2R_2{}^4 + 2144353680\,R_5R_3{}^2R_2{}^3 + 850028506740\,R_5R_3{}^2R_2{}^2 + 62291109499080\,R_5R_3{}^2R_2 + 852305979460320\,R_5R_3{}^2 + 10640\,R_5R_2{}^7 + 78187242\,R_5R_2{}^6 + 65274085800\,R_5R_2{}^5 + 12152587839700\,R_5R_2{}^4 + 652209543128220\,R_5R_2{}^3 + 10957473664583484\,R_5R_2{}^2 + 54731025150293520\,R_5R_2 + 61290148786433280\,R_5 + 269325\,R_4{}^4R_3 + 1077300\,R_4{}^3R_3R_2{}^2 + 1859383890\,R_4{}^3R_3R_2 + 230350983240\,R_4{}^3R_3 + 1436400\,R_4{}^2R_3{}^3R_2 + 1228459155\,R_4{}^2R_3{}^3 + 628425\,R_4{}^2R_3R_2{}^4 + 2390137680\,R_4{}^2R_3R_2{}^3 + 934884554940\,R_4{}^2R_3R_2{}^2 + 67549101390540\,R_4{}^2R_3R_2 + 912161144802600\,R_4{}^2R_3 + 191520\,R_4R_3{}^5 + 1117200\,R_4R_3{}^3R_2{}^3 + 3157169960\,R_4R_3{}^3R_2{}^2 + 810515337240\,R_4R_3{}^3R_2 + 28715565324600\,R_4R_3{}^3 + 111720\,R_4R_3R_2{}^6 + 690329052\,R_4R_3R_2{}^5 +$

$466045965000\,R_4R_3{R_2}^4 + 67021753121400\,R_4R_3{R_2}^3 + 2600964850573320\,R_4R_3{R_2}^2 + 28098738300314436\,R_4R_3R_2 + 67764383615834640\,R_4R_3 + 223440\,{R_3}^5{R_2}^2 + 416998624\,{R_3}^5R_2 + 52689230790\,{R_3}^5 + 148960\,{R_3}^3{R_2}^5 + 759612210\,{R_3}^3{R_2}^4 + 403580839200\,{R_3}^3{R_2}^3 + 42639983920380\,{R_3}^3{R_2}^2 + 1078026577321540\,{R_3}^3R_2 + 5683381632400984\,{R_3}^3 + 5985\,R_3{R_2}^8 + 52668000\,R_3{R_2}^7 + 50829096780\,R_3{R_2}^6 + 10743281472924\,R_3{R_2}^5 + 653621753210580\,R_3{R_2}^4 + 12704844357384984\,R_3{R_2}^3 + 78267651477160320\,R_3{R_2}^2 + 133371684885600000\,R_3R_2 + 41680704936960000\,R_3$

Appendix B

The C-expansions of Kerov's Character Polynomials for $k \leq 22$

$\Sigma_1 - R_2 = 0$

$\Sigma_2 - R_3 = 0$

$\Sigma_3 - R_4 = C_2$

$\Sigma_4 - R_5 = 5/2\, C_3$

$\Sigma_5 - R_6 = 5\, C_4 + 8\, C_2$

$\Sigma_6 - R_7 = \frac{35}{4}\, C_5 + 42\, C_3$

$\Sigma_7 - R_8 = 14\, C_6 + \frac{469}{3}\, C_4 + \frac{203}{3}\, {C_2}^2 + 180\, C_2$

$\Sigma_8 - R_9 = 21\, C_7 + \frac{1869}{4}\, C_5 + \frac{819}{2}\, C_2 C_3 + 1522\, C_3$

$\Sigma_9 - R_{10} = 30\, C_8 + 1197\, C_6 + \frac{963}{2}\, {C_3}^2 + 1122\, C_2 C_4 + 81\, {C_2}^3 + \frac{26060}{3}\, C_4 + \frac{17680}{3}\, {C_2}^2 + 8064\, C_2$

$\Sigma_{10} - R_{11} = \frac{165}{4}\, C_9 + \frac{5467}{2}\, C_7 + \frac{4433}{2}\, C_3 C_4 + \frac{1133}{2}\, C_3 {C_2}^2 + \frac{11033}{4}\, C_2 C_5 + 38225\, C_5 + 52580\, C_2 C_3 + 96624\, C_3$

$\Sigma_{11} - R_{12} = 55\, C_{10} + 5709\, C_8 + 139381\, C_6 + 762212\, C_4 + 604800\, C_2 + 639232\, {C_2}^2 +$

$\frac{623414}{3}\,C_2C_4 + 6160\,C_2C_6 + 86229\,{C_3}^2 + \frac{9691}{2}\,C_3C_5 + \frac{119383}{3}\,{C_2}^3 + \frac{6611}{3}\,{C_4}^2 + \frac{3982}{3}\,C_4{C_2}^2 + \frac{4433}{4}\,C_2{C_3}^2$

$\Sigma_{12} - R_{13} = \frac{143}{2}\,C_{11} + \frac{88803}{8}\,C_9 + \frac{2641925}{6}\,C_7 + 4532099\,C_5 + 9528480\,C_3 + 7710560\,C_2C_3 + \frac{2151292}{3}\,C_2C_5 + \frac{50765}{4}\,C_2C_7 + 549549\,C_3C_4 + \frac{39897}{4}\,C_3C_6 + \frac{1287}{2}\,{C_3}^3 + \frac{2309879}{6}\,C_3{C_2}^2 + \frac{18161}{4}\,C_4C_2C_3 + \frac{34463}{4}\,C_5C_4 + \frac{23595}{8}\,C_5{C_2}^2$

$\Sigma_{13} - R_{14} = 91\,C_{12} + \frac{61061}{3}\,C_{10} + 1241669\,C_8 + \frac{109425316}{5}\,C_6 + 97423872\,C_4 + 68428800\,C_2 + 92793792\,{C_2}^2 + \frac{610712284}{15}\,C_2C_4 + \frac{10960872}{5}\,C_2C_6 + \frac{73346}{3}\,C_2C_8 + \frac{82526899}{5}\,{C_3}^2 + \frac{6539117}{4}\,C_3C_5 + \frac{116207}{6}\,C_3C_7 + \frac{166710908}{15}\,{C_2}^3 + \frac{3786068}{45}\,{C_2}^4 + \frac{55237}{6}\,C_3C_2C_5 + \frac{6574906}{9}\,{C_4}^2 + \frac{55220789}{45}\,C_4{C_2}^2 + \frac{10103561}{10}\,C_2{C_3}^2 + \frac{43043}{12}\,{C_3}^2C_4 + 4095\,C_2{C_4}^2 + \frac{48958}{3}\,C_4C_6 + \frac{91819}{12}\,{C_5}^2 + \frac{18382}{3}\,C_6{C_2}^2$

$\Sigma_{14} - R_{15} = \frac{455}{4}\,C_{13} + \frac{71071}{2}\,C_{11} + \frac{25537655}{8}\,C_9 + 89941852\,C_7 + 740450620\,C_5 + 1349836416\,C_3 + 1430971360\,C_2C_3 + 184556554\,C_2C_5 + \frac{72560345}{12}\,C_2C_7 + \frac{178087}{4}\,C_2C_9 + \frac{410641868}{3}\,C_3C_4 + \frac{17974671}{4}\,C_3C_6 + 35672\,C_3C_8 + \frac{6162403}{8}\,{C_3}^3 + \frac{420037072}{3}\,C_3{C_2}^2 + \frac{21934523}{4}\,C_4C_2C_3 + \frac{7556835}{2}\,C_5C_4 + \frac{87440665}{24}\,C_5{C_2}^2 + \frac{2660372}{3}\,C_3{C_2}^3 + \frac{180271}{12}\,C_4C_2C_5 + \frac{35945}{2}\,C_6C_2C_3 + \frac{18109}{3}\,C_3{C_4}^2 + \frac{178633}{6}\,C_4C_7 + \frac{108381}{16}\,C_5{C_3}^2 + \frac{107289}{4}\,C_5C_6 + \frac{71617}{6}\,C_7{C_2}^2$

$\Sigma_{15} - R_{16} = 140\,C_{14} + 59514\,C_{12} + \frac{68396900}{9}\,C_{10} + 324765727\,C_8 + 4511355576\,C_6 + 17127271152\,C_4 + 10897286400\,C_2 + 17780056848\,{C_2}^2 + 9593568768\,C_2C_4 + \frac{3645670794}{5}\,C_2C_6 + \frac{137654720}{9}\,C_2C_8 + 77308\,C_2C_{10} + 3822841344\,{C_3}^2 + 523886162\,C_3C_5 + \frac{103019720}{9}\,C_3C_7 + 62706\,C_3C_9 + 3190473216\,{C_2}^3 + \frac{3145878221}{45}\,{C_2}^4 + 14513315\,C_3C_2C_5 + \frac{2083561321}{9}\,{C_4}^2 + \frac{26029444313}{45}\,C_4{C_2}^2 + \frac{2330866761}{5}\,C_2{C_3}^2 + \frac{16718455}{3}\,{C_3}^2C_4 + \frac{9021695}{3}\,{C_3}^2{C_2}^2 + \frac{57691360}{9}\,C_2{C_4}^2 + \frac{84031640}{9}\,C_4C_6 + \frac{22071320}{9}\,C_4{C_2}^3 + \frac{39000550}{9}\,{C_5}^2 + \frac{29836760}{3}\,C_6{C_2}^2 + 20986\,C_5C_3C_4 + 33502\,C_7C_2C_3 + 27244\,C_6C_2C_4 + 53700\,{C_2}^5 + 3150\,{C_4}^3 + 12579\,C_2{C_5}^2 + 46018\,C_5C_7 + 52276\,C_8C_4 + 21966\,C_8{C_2}^2 + 12579\,C_6{C_3}^2 + 21966\,{C_6}^2$

$\Sigma_{16} - R_{17} = 170\,C_{15} + \frac{577031}{6}\,C_{13} + 16953794\,C_{11} + \frac{8433097673}{8}\,C_9 + \frac{69343996540}{3}\,C_7 + 160069391316\,C_5 + 260052508800\,C_3 + 337156189272\,C_2C_3 + \frac{163778076160}{3}\,C_2C_5 + \frac{10243810615}{4}\,C_2C_7 + 35951702\,C_2C_9 + \frac{386665}{3}\,C_2C_{11} + \frac{118577899520}{3}\,C_3C_4 + \frac{36585716371}{20}\,C_3C_6 + 27213192\,C_3C_8 + \frac{317441}{3}\,C_3C_{10} + \frac{150420961280}{3}\,C_3{C_2}^2 + \frac{109769}{3}\,C_6C_3C_4 + \frac{64218590023}{20}\,C_4C_2C_3 + \frac{17678202573}{40}\,{C_3}^3 + \frac{6047653559}{4}\,C_5C_4 + \frac{17590797787}{8}\,C_5{C_2}^2 + \frac{9312340563}{10}\,C_3{C_2}^3 + \frac{35888836}{3}\,C_3{C_4}^2 + \frac{53889337}{4}\,{C_3}^2C_5 + \frac{8152163}{2}\,C_2{C_3}^3 + \frac{5536730}{9}\,C_3{C_2}^4 + \frac{197194288}{9}\,C_4C_7 + 19416006\,C_5C_6 + \frac{59236262}{9}\,C_5{C_2}^3 + \frac{225622028}{9}\,C_7{C_2}^2 + \frac{131473648}{9}\,C_3C_4{C_2}^2 + 36471664\,C_3C_2C_6 + \frac{270981394}{9}\,C_4C_2C_5 +$

$\frac{67864}{3}\, C_7C_3^2 + \frac{30821}{2}\, C_5C_4^2 + \frac{50558}{3}\, C_3C_5^2 + \frac{230911}{3}\, C_5C_8 + \frac{265523}{3}\, C_9C_4 + \frac{230911}{6}\, C_9C_2^2 + \frac{127075}{3}\, C_5C_2C_6 + \frac{178993}{3}\, C_8C_2C_3 + 48127\, C_7C_2C_4 + \frac{213605}{3}\, C_7C_6$

$\Sigma_{17} - R_{18} = 204\, C_{16} + 150654\, C_{14} + 35771060\, C_{12} + 3128616777\, C_{10} + 102921070200\, C_8 + \frac{5943136639504}{5}\, C_6 + 3968632776960\, C_4 + 2324754432000\, C_2 + 4386384368640\, C_2^2 + \frac{13839469318432}{5}\, C_2C_4 + \frac{1338680692224}{5}\, C_2C_6 + \frac{56975901642}{7}\, C_2C_8 + 79369328\, C_2C_{10} + 207468\, C_2C_{12} + \frac{5441834311016}{5}\, C_3^2 + 187135682712\, C_3C_5 + 5847039364\, C_3C_7 + 60854237\, C_3C_9 + 172278\, C_3C_{11} + \frac{5235284660944}{5}\, C_2^3 + 82059886720\, C_4^2 + \frac{1289581576192}{5}\, C_4C_2^2 + \frac{1021326327216}{5}\, C_2C_3^2 + 62798\, C_7C_3C_4 + \frac{74873222791}{7}\, C_5C_2C_3 + \frac{206925200064}{5}\, C_2^4 + \frac{6663318827}{35}\, C_2^5 + \frac{23550440}{3}\, C_4^3 + \frac{3825085}{2}\, C_3^4 + \frac{23352062206}{5}\, C_6C_4 + \frac{265556157497}{35}\, C_6C_2^2 + \frac{8618088069}{4}\, C_5^2 + \frac{19941030379}{5}\, C_4C_3^2 + \frac{163660978681}{35}\, C_2C_4^2 + \frac{113751303108}{35}\, C_4C_2^3 + \frac{136983821263}{35}\, C_3^2C_2^2 + 52411629\, C_4C_3C_5 + \frac{345324264}{5}\, C_4C_2C_6 + 26949114\, C_4C_2C_3^2 + 35657653\, C_5C_3C_2^2 + 86410116\, C_7C_2C_3 + \frac{46721984}{3}\, C_4^2C_2^2 + 48830800\, C_4C_8 + \frac{4599860}{3}\, C_4C_2^4 + 31761712\, C_2C_5^2 + 42097389\, C_5C_7 + 58861072\, C_8C_2^2 + \frac{158159517}{5}\, C_6C_3^2 + \frac{99832568}{5}\, C_6^2 + \frac{83179368}{5}\, C_6C_2^3 + 2500275\, C_3^2C_2^3 + 33354\, C_2C_6^2 + 144908\, C_{10}C_4 + 64634\, C_{10}C_2^2 + 125358\, C_9C_5 + 39219\, C_8C_3^2 + 113628\, C_8C_6 + 54859\, C_7^2 + 25534\, C_6C_4^2 + 101898\, C_9C_2C_3 + 82348\, C_8C_2C_4 + 70618\, C_7C_2C_5 + 54978\, C_6C_3C_5 + 23579\, C_5^2C_4$

$\Sigma_{18} - R_{19} = \frac{969}{4}\, C_{17} + 229653\, C_{15} + \frac{215732023}{3}\, C_{13} + \frac{86027797713}{10}\, C_{11} + 407837486511\, C_9 + \frac{22080490035272}{3}\, C_7 + 44329318724736\, C_5 + 65429789644800\, C_3 + 99400589430912\, C_2C_3 + \frac{57403472915324}{3}\, C_2C_5 + 1155211200918\, C_2C_7 + \frac{950092202421}{40}\, C_2C_9 + \frac{4979564707}{30}\, C_2C_{11} + \frac{1295553}{4}\, C_2C_{13} + \frac{40796114441240}{3}\, C_3C_4 + \frac{4005088325334}{5}\, C_3C_6 + \frac{1207208811663}{70}\, C_3C_8 + \frac{773582093}{6}\, C_3C_{10} + \frac{543609}{2}\, C_3C_{12} + 19893520260584\, C_3C_2^2 + 113827138\, C_6C_3C_4 + \frac{8624464303302}{5}\, C_4C_2C_3 + \frac{1166646696471}{5}\, C_3^3 + \frac{183141}{16}\, C_5^3 + 654250047912\, C_5C_4 + 1210202085609\, C_5C_2^2 + \frac{3368284829784}{5}\, C_3C_2^3 + \frac{1103642081327}{105}\, C_3C_4^2 + \frac{334910356587}{28}\, C_3^2C_5 + \frac{1812379517331}{280}\, C_2C_3^3 + \frac{188430360467}{70}\, C_3C_2^4 + \frac{406837382257}{30}\, C_4C_7 + \frac{237989645181}{20}\, C_5C_6 + \frac{1309518988019}{120}\, C_5C_2^3 + \frac{1433185387549}{60}\, C_7C_2^2 + \frac{9910640096179}{420}\, C_3C_4C_2^2 + \frac{4688116263351}{140}\, C_3C_2C_6 + \frac{326574932023}{12}\, C_4C_2C_5 + \frac{711451751}{10}\, C_7C_3^2 + \frac{143273110}{3}\, C_5C_4^2 + \frac{418794371}{8}\, C_3C_5^2 + \frac{263997913}{3}\, C_5C_8 + \frac{152800641}{40}\, C_5C_2^4 + \frac{621920027}{6}\, C_9C_4 + \frac{5195632973}{40}\, C_9C_2^2 + \frac{1365663703}{20}\, C_5C_4C_2^2 + \frac{2666514549}{20}\, C_5C_2C_6 + \frac{609610497}{10}\, C_5C_2C_3^2 + \frac{1933209587}{10}\, C_8C_2C_3 + \frac{4578172607}{30}\, C_7C_2C_4 + \frac{1684006043}{20}\, C_6C_3C_2^2 + \frac{1607214731}{30}\, C_4^2C_2C_3 + \frac{670508917}{60}\, C_4C_3C_2^3 + \frac{1207972196}{15}\, C_7C_6 + \frac{2363793781}{60}\, C_7C_2^3 + \frac{93700685}{6}\, C_4C_3^3 + \frac{559985423}{120}\, C_3^3C_2^2 + \frac{1049427}{16}\, C_3^2C_9 + 167637\, C_3C_2C_{10} + \frac{84303}{2}\, C_3C_6^2 + \frac{460275}{2}\, C_4C_{11} + \frac{84303}{2}\, C_4^2C_7 + \frac{795549}{4}\, C_5C_{10} + 167637\, C_7C_8 + \frac{712215}{4}\, C_9C_6 + 104652\, C_{11}C_2^2 + \frac{210273}{2}\, C_3C_4C_8 + \frac{716091}{8}\, C_3C_5C_7 +$

$\frac{545547}{4} C_4C_2C_9 + \frac{295545}{4} C_4C_5C_6 + \frac{462213}{4} C_5C_2C_8 + \frac{210273}{2} C_7C_2C_6$

$\Sigma_{19} - R_{20} = 285\, C_{18} + 341734\, C_{16} + 492214\, C_{14}C_2 + 138563770\, C_{14} + \frac{834157}{2} C_{13}C_3 + 355604\, C_{12}C_4 + 164141\, C_{12}{C_2}^2 + 331126870\, C_{12}C_2 + 22143850599\, C_{12} + \frac{615581}{2} C_{11}C_5 + \frac{533615}{2} C_{11}C_3C_2 + 260452912\, C_{11}C_3 + 273638\, C_{10}C_6 + 218994\, C_{10}C_4C_2 + \frac{631158910}{3} C_{10}C_4 + \frac{424327}{4} C_{10}{C_3}^2 + \frac{814623575}{3} C_{10}{C_2}^2 + 64396272576\, C_{10}C_2 + \frac{4392734658715}{3} C_{10} + \frac{506293}{2} C_9C_7 + \frac{369683}{2} C_9C_5C_2 + 177250525\, C_9C_5 + \frac{342361}{2} C_9C_4C_3 + \frac{1641178029}{4} C_9C_3C_2 + \frac{189399024243}{4} C_9C_3 + 123158\, {C_8}^2 + 164350\, C_8C_6C_2 + 158419492\, C_8C_6 + \frac{287717}{2} C_8C_5C_3 + 68514\, C_8{C_4}^2 + \frac{969972838}{3} C_8C_4C_2 + \frac{259245538114}{7} C_8C_4 + \frac{305514243}{2} C_8{C_3}^2 + \frac{263259326}{3} C_8{C_2}^3 + 69280372562\, C_8{C_2}^2 + \frac{93812776439170}{21} C_8C_2 + \frac{275057386118488}{7} C_8 + \frac{315039}{4} {C_7}^2C_2 + \frac{228476330}{3} {C_7}^2 + \frac{260395}{2} C_7C_6C_3 + \frac{233073}{2} C_7C_5C_4 + \frac{3291521335}{12} C_7C_5C_2 + \frac{63056251045}{2} C_7C_5 + \frac{481825807}{2} C_7C_4C_3 + \frac{1137609401}{6} C_7C_3{C_2}^2 + 97848576596\, C_7C_3C_2 + \frac{9332762596030}{3} C_7C_3 + 54853\, {C_6}^2C_4 + 129237525\, {C_6}^2C_2 + \frac{74499618051}{5} {C_6}^2 + \frac{205751}{4} C_6{C_5}^2 + \frac{840751539}{4} C_6C_5C_3 + 97308690\, C_6{C_4}^2 + \frac{445674640}{3} C_6C_4{C_2}^2 + \frac{2685041310908}{35} C_6C_4C_2 + \frac{7346860279390}{3} C_6C_4 + 136000347\, C_6{C_3}^2C_2 + \frac{691080377949}{20} C_6{C_3}^2 + \frac{27375485}{3} C_6{C_2}^4 + \frac{170852955694}{5} C_6{C_2}^3 + \frac{35744738324805}{7} C_6{C_2}^2 + \frac{3957833393740496}{35} C_6C_2 + 390931355100240\, C_6 + \frac{1078264535}{12} {C_5}^2C_4 + \frac{270804625}{4} {C_5}^2{C_2}^2 + \frac{561510198935}{16} {C_5}^2C_2 + \frac{3375817861985}{3} {C_5}^2 + \frac{1329304657}{6} C_5C_4C_3C_2 + \frac{1577882763237}{28} C_5C_4C_3 + \frac{132707419}{4} C_5{C_3}^3 + \frac{307885139}{12} C_5C_3{C_2}^3 + \frac{284189873839}{4} C_5C_3{C_2}^2 + \frac{49022595196035}{7} C_5C_3C_2 + \frac{541568082522388}{7} C_5C_3 + 32902110\, {C_4}^3C_2 + \frac{175384768130}{21} {C_4}^3 + \frac{87899187}{2} {C_4}^2{C_3}^2 + 11108825\, {C_4}^2{C_2}^3 + \frac{1076797387186}{35} {C_4}^2{C_2}^2 + \frac{21265702002485}{7} {C_4}^2C_2 + \frac{708436340027840}{21} {C_4}^2 + \frac{172789591}{6} C_4{C_3}^2{C_2}^2 + \frac{733224560821}{14} C_4{C_3}^2C_2 + 2547878026595\, C_4{C_3}^2 + \frac{58064535211}{7} C_4{C_2}^4 + \frac{60408198352160}{21} C_4{C_2}^3 + \frac{12981413205554944}{105} C_4{C_2}^2 + 972677565188640\, C_4C_2 + 1172999526914304\, C_4 + \frac{32651861}{8} {C_3}^4C_2 + \frac{18294308688}{5} {C_3}^4 + \frac{132896134353}{10} {C_3}^2{C_2}^3 + \frac{23840385374335}{7} {C_3}^2{C_2}^2 + \frac{3379097580196164}{35} {C_3}^2C_2 + 378415097098200\, {C_3}^2 + \frac{3440157689}{15} {C_2}^6 + \frac{5286009473005}{21} {C_2}^5 + \frac{2485798266001168}{105} {C_2}^4 + 403314720431760\, {C_2}^3 + 1360182210333696\, {C_2}^2 + 640237370572800\, C_2$

$\Sigma_{20} - R_{21} = \frac{665}{2} C_{19} + \frac{3981621}{8} C_{17} + \frac{2921611}{4} C_{15}C_2 + 257180675\, C_{15} + \frac{2497607}{4} C_{14}C_3 + \frac{6432811}{12} C_{13}C_4 + \frac{6008807}{24} C_{13}{C_2}^2 + \frac{1901199185}{3} C_{13}C_2 + \frac{215214576031}{4} C_{13} + \frac{1861601}{4} C_{12}C_5 + \frac{1649599}{4} C_{12}C_3C_2 + 504405445\, C_{12}C_3 + \frac{1649599}{4} C_{11}C_6 + \frac{4100789}{12} C_{11}C_4C_2 + \frac{1229939825}{3} C_{11}C_4 + \frac{332899}{2} C_{11}{C_3}^2 + \frac{1082828335}{2} C_{11}{C_2}^2 + \frac{1637630071049}{10} C_{11}C_2 + \frac{9659333205249}{2} C_{11} + \frac{4524793}{12} C_{10}C_7 + \frac{3464783}{12} C_{10}C_5C_2 + \frac{1033296475}{3} C_{10}C_5 + \frac{3252781}{12} C_{10}C_4C_3 + 830421885\, C_{10}C_3C_2 + \frac{243998460503}{2} C_{10}C_3 + \frac{1437597}{4} C_9C_8 + \frac{1013593}{4} C_9C_6C_2 + 303243800\, C_9C_6 + 226898\, C_9C_5C_3 +$

$$\frac{2616775}{24} C_9C_4^2 + 656931080\, C_9C_4C_2 + \frac{381861940929}{4} C_9C_4 + \frac{5016203205}{16} C_9C_3^2 + \frac{555002540}{3} C_9C_2^3 + \frac{7478365873853}{40} C_9C_2^2 + \frac{31444409071747}{2} C_9C_2 + 185317644738923\, C_9 + \frac{2828777}{12} C_8C_7C_2 + 283402670\, C_8C_7 + \frac{801591}{4} C_8C_6C_3 + \frac{2192771}{12} C_8C_5C_4 + 548801035\, C_8C_5C_2 + 79857448134\, C_8C_5 + 493037935\, C_8C_4C_3 + \frac{1218511135}{3} C_8C_3C_2^2 + \frac{2674461641817}{10} C_8C_3C_2 + 11056290809191\, C_8C_3 + \frac{574693}{6} C_7^2C_3 + \frac{1980769}{12} C_7C_6C_4 + 496995160\, C_7C_6C_2 + \frac{726378891259}{10} C_7C_6 + \frac{234346}{3} C_7C_5^2 + \frac{3337310005}{8} C_7C_5C_3 + 195653165\, C_7C_4^2 + \frac{2830314860}{9} C_7C_4C_2^2 + \frac{1032662358322}{5} C_7C_4C_2 + \frac{25652503028593}{3} C_7C_4 + \frac{2346874585}{8} C_7C_3^2C_2 + \frac{3792224610069}{40} C_7C_3^2 + \frac{371539775}{18} C_7C_2^4 + \frac{1986144353137}{20} C_7C_2^3 + \frac{116903069433311}{6} C_7C_2^2 + \frac{1750388490230204}{3} C_7C_2 + 2857126601545720\, C_7 + \frac{589589}{8} C_6^2C_5 + \frac{392625405}{2} C_6^2C_3 + 342699675\, C_6C_5C_4 + \frac{812679305}{3} C_6C_5C_2^2 + 178896551886\, C_6C_5C_2 + 7454891522946\, C_6C_5 + 460381970\, C_6C_4C_3C_2 + \frac{743948589328}{5} C_6C_4C_3 + \frac{564216115}{8} C_6C_3^3 + \frac{173445395}{3} C_6C_3C_2^3 + \frac{821116861005}{4} C_6C_3C_2^2 + 26519230552588\, C_6C_3C_2 + \frac{1976456435543004}{5} C_6C_3 + \frac{2549050925}{48} C_5^3 + \frac{5037310115}{24} C_5^2C_3C_2 + \frac{2177801711965}{32} C_5^2C_3 + \frac{1712753195}{9} C_5C_4^2C_2 + \frac{122857227055}{2} C_5C_4^2 + \frac{1383608405}{8} C_5C_4C_3^2 + \frac{415741945}{9} C_5C_4C_2^3 + \frac{1642774719023}{10} C_5C_4C_2^2 + \frac{63904696399901}{3} C_5C_4C_2 + \frac{960260028766000}{3} C_5C_4 + \frac{2969662855}{48} C_5C_3^2C_2^2 + \frac{5774518913501}{40} C_5C_3^2C_2 + \frac{18425301534137}{2} C_5C_3^2 + \frac{1015502915473}{40} C_5C_2^4 + \frac{23482580555049}{2} C_5C_2^3 + \frac{2089209436560898}{3} C_5C_2^2 + 7968503084481940\, C_5C_2 + 15322537196608320\, C_5 + \frac{463872365}{9} C_4^3C_3 + \frac{971272685}{18} C_4^2C_3C_2^2 + 125790280406\, C_4^2C_3C_2 + \frac{48171959117441}{6} C_4^2C_3 + \frac{754636585}{24} C_4C_3^3C_2 + \frac{180637049432}{5} C_4C_3^3 + \frac{1448594574357}{20} C_4C_3C_2^3 + \frac{74387110579337}{3} C_4C_3C_2^2 + \frac{4867677973687652}{5} C_4C_3C_2 + 5579810732881000\, C_4C_3 + 1356125\, C_3^5 + \frac{1189408281397}{40} C_3^3C_2^2 + 6689263658631\, C_3^3C_2 + \frac{649068861012996}{5} C_3^3 + \frac{34429198881}{10} C_3C_2^5 + \frac{25645850568511}{6} C_3C_2^4 + \frac{6884348761637992}{15} C_3C_2^3 + 9045629032098120\, C_3C_2^2 + 36040768049583360\, C_3C_2 + 20840352468480000\, C_3$$

$$\Sigma_{21} - R_{22} = 385\, C_{20} + 711018\, C_{18} + 1061060\, C_{16}C_2 + \frac{1385037566}{3} C_{16} + \frac{1828827}{2} C_{15}C_3 + 790328\, C_{14}C_4 + 372603\, C_{14}C_2^2 + \frac{3507397388}{3} C_{14}C_2 + 124242757177\, C_{14} + \frac{1377607}{2} C_{13}C_5 + \frac{1242241}{2} C_{13}C_3C_2 + \frac{11290334077}{12} C_{13}C_3 + 609840\, C_{12}C_6 + 519596\, C_{12}C_4C_2 + \frac{2310015928}{3} C_{12}C_4 + \frac{1016631}{4} C_{12}C_3^2 + \frac{3106284313}{3} C_{12}C_2^2 + 393708729722\, C_{12}C_2 + 14797534469445\, C_{12} + \frac{1106875}{2} C_{11}C_7 + \frac{881265}{2} C_{11}C_5C_2 + \frac{7764215261}{12} C_{11}C_5 + \frac{836143}{2} C_{11}C_4C_3 + \frac{4832746391}{3} C_{11}C_3C_2 + \frac{2970985784459}{10} C_{11}C_3 + 519596\, C_{10}C_8 + 384230\, C_{10}C_6C_2 + 564473668\, C_{10}C_6 + \frac{700777}{2} C_{10}C_5C_3 + 169554\, C_{10}C_4^2 + \frac{11545968082}{9} C_{10}C_4C_2 + \frac{700017932306}{3} C_{10}C_4 + 616977559\, C_{10}C_3^2 + \frac{3343330936}{9} C_{10}C_2^3 + 473553871714\, C_{10}C_2^2 + 50992044863900\, C_{10}C_2 + \frac{7077838516218976}{9} C_{10} + \frac{1016631}{4} C_9^2 + \frac{700777}{2} C_9C_7C_2 + \frac{6204823757}{12} C_9C_7 + \frac{610533}{2} C_9C_6C_3 + \frac{565411}{2} C_9C_5C_4 + 1064489668\, C_9C_5C_2 +$$

$$\frac{1546817550201}{8} C_9C_5 + \frac{3894044143}{4} C_9C_4C_3 + \frac{9957847757}{12} C_9C_3{C_2}^2 + \frac{6873909137557}{10} C_9C_3C_2 +$$
$$\frac{290357000949015}{8} C_9C_3 + 169554\, {C_8}^2C_2 + \frac{752430844}{3} {C_8}^2 + \frac{565411}{2} C_8C_7C_3 +$$
$$248864\, C_8C_6C_4 + \frac{2819450656}{3} C_8C_6C_2 + \frac{857135391882}{5} C_8C_6 + \frac{475167}{4} C_8{C_5}^2 +$$
$$\frac{9725872211}{12} C_8C_5C_3 + \frac{3457984244}{9} C_8{C_4}^2 + \frac{1927290178}{3} C_8C_4{C_2}^2 + \frac{7931856944858}{15} C_8C_4C_2 +$$
$$27875380678440\, C_8C_4 + \frac{1822930318}{3} C_8{C_3}^2C_2 + \frac{4925609142531}{20} C_8{C_3}^2 + 44213565\, C_8{C_2}^4 +$$
$$\frac{4035812228756}{15} C_8{C_2}^3 + 67926023784005\, C_8{C_2}^2 + \frac{168767042934507028}{63} C_8C_2 +$$
$$17897083115235280\, C_8 + \frac{475167}{4} {C_7}^2C_4 + \frac{8093880553}{18} {C_7}^2C_2 + \frac{493343217521}{6} {C_7}^2 +$$
$$\frac{430045}{2} C_7C_6C_5 + \frac{1464321639}{2} C_7C_6C_3 + \frac{23463486343}{36} C_7C_5C_4 + \frac{6436323949}{12} C_7C_5{C_2}^2 +$$
$$\frac{5323441926803}{12} C_7C_5C_2 + \frac{94136371878045}{4} C_7C_5 + \frac{8462394083}{9} C_7C_4C_3C_2 +$$
$$\frac{3800430811947}{10} C_7C_4C_3 + \frac{439766261}{3} C_7{C_3}^3 + \frac{2255671385}{18} C_7C_3{C_2}^3 + \frac{16831897327861}{30} C_7C_3{C_2}^2 +$$
$$\frac{372061576548095}{4} C_7C_3C_2 + \frac{16383277117811428}{9} C_7C_3 + 33957\, {C_6}^3 + 306965901\, {C_6}^2C_4 +$$
$$\frac{753096784}{3} {C_6}^2{C_2}^2 + 208408794209\, {C_6}^2C_2 + 11097032286312\, {C_6}^2 +$$
$$\frac{3453497399}{12} C_6{C_5}^2 + \frac{1618240393}{2} C_6C_5C_3C_2 + 328684950594\, C_6C_5C_3 +$$
$$\frac{3358074544}{9} C_6{C_4}^2C_2 + \frac{2258403863408}{15} C_6{C_4}^2 + 345404576\, C_6C_4{C_3}^2 + \frac{868617970}{9} C_6C_4{C_2}^3 +$$
$$\frac{6484803160066}{15} C_6C_4{C_2}^2 + 71888517206462\, C_6C_4C_2 + \frac{63800029849136636}{45} C_6C_4 +$$
$$\frac{398040610}{3} C_6{C_3}^2{C_2}^2 + \frac{3898308311051}{10} C_6{C_3}^2C_2 + \frac{127551908595069}{4} C_6{C_3}^2 +$$
$$\frac{221728417295}{3} C_6{C_2}^4 + 44348315138844\, C_6{C_2}^3 + \frac{122595298723281868}{35} C_6{C_2}^2 +$$
$$55471381061750896\, C_6C_2 + \frac{786428482773580416}{5} C_6 + \frac{2051053763}{6} {C_5}^2C_4C_2 +$$
$$\frac{6634956102005}{48} {C_5}^2C_4 + \frac{472517177}{3} {C_5}^2{C_3}^2 + \frac{174532083}{4} {C_5}^2{C_2}^3 + \frac{1569614727989}{8} {C_5}^2{C_2}^2 +$$
$$32764304190625\, {C_5}^2C_2 + \frac{11693830821540175}{18} {C_5}^2 + \frac{10304016217}{36} C_5{C_4}^2C_3 +$$
$$\frac{2559003667}{12} C_5C_4C_3{C_2}^2 + \frac{9401745383492}{15} C_5C_4C_3C_2 + \frac{205448157647235}{4} C_5C_4C_3 +$$
$$\frac{766893259}{12} C_5{C_3}^3C_2 + \frac{7402164521367}{80} C_5{C_3}^3 + \frac{12113379545267}{60} C_5C_3{C_2}^3 +$$
$$\frac{718170365491155}{8} C_5C_3{C_2}^2 + \frac{98686652869909687}{21} C_5C_3C_2 + 37307675379381880\, C_5C_3 +$$
$$\frac{193738402}{9} {C_4}^4 + \frac{283322105}{9} {C_4}^3{C_2}^2 + \frac{4151946322832}{45} {C_4}^3C_2 + \frac{22636800335545}{3} {C_4}^3 +$$
$$\frac{757072745}{9} {C_4}^2{C_3}^2C_2 + \frac{729420366367}{6} {C_4}^2{C_3}^2 + \frac{3901474135328}{45} {C_4}^2{C_2}^3 +$$
$$38582154471008\, {C_4}^2{C_2}^2 + \frac{638400239419113116}{315} {C_4}^2C_2 + 16197890261019280\, {C_4}^2 +$$
$$\frac{296958695}{24} C_4{C_3}^4 + \frac{1105554319562}{5} C_4{C_3}^2{C_2}^2 + \frac{258512022828477}{4} C_4{C_3}^2C_2 +$$
$$\frac{8378690015197106}{5} C_4{C_3}^2 + \frac{433344704408}{45} C_4{C_2}^5 + 15872049129137\, C_4{C_2}^4 +$$
$$\frac{245541358459097756}{105} C_4{C_2}^3 + 66363251422136768\, C_4{C_2}^2 + \frac{2058579138243817728}{5} C_4C_2 +$$
$$430926266757888000\, C_4 + \frac{1233151775933}{40} {C_3}^4C_2 + \frac{35543774185101}{8} {C_3}^4 +$$
$$\frac{289141078303}{15} {C_3}^2{C_2}^4 + 25014224340111\, {C_3}^2{C_2}^3 + \frac{286563107279919371}{105} {C_3}^2{C_2}^2 +$$
$$51228739999477524\, {C_3}^2C_2 + \frac{793897768725836544}{5} {C_3}^2 + 100061577\, {C_2}^7 +$$
$$\frac{2379030797816}{3} {C_2}^6 + \frac{81518838944024812}{315} {C_2}^5 + 14369083146112816\, {C_2}^4 +$$
$$\frac{914744603149001856}{5} {C_2}^3 + 519085894846464000\, {C_2}^2 + 221172909834240000\, C_2$$

$$\Sigma_{22} - R_{23} = \frac{1771}{4} C_{21} + \frac{2994761}{3} C_{19} + \frac{18147437}{12} C_{17}C_2 + \frac{6434962655}{8} C_{17} +$$

$\frac{3938704}{3} C_{16}C_3 + \frac{6851999}{6} C_{15}C_4 + \frac{1627549}{3} C_{15}C_2^2 + \frac{8349313973}{4} C_{15}C_2 + \frac{548419232603}{2} C_{15} +$
$\frac{11994983}{12} C_{14}C_5 + \frac{5484787}{6} C_{14}C_3C_2 + \frac{6788107645}{4} C_{14}C_3 + \frac{10627771}{12} C_{13}C_6 +$
$\frac{3086853}{4} C_{13}C_4C_2 + \frac{25177214293}{18} C_{13}C_4 + \frac{18179315}{48} C_{13}C_3^2 + \frac{137405194355}{72} C_{13}C_2^2 +$
$\frac{2701936792445}{3} C_{13}C_2 + \frac{169969248772505}{4} C_{13} + \frac{4801181}{6} C_{12}C_7 + \frac{7893347}{12} C_{12}C_5C_2 +$
$\frac{2355490973}{2} C_{12}C_5 + \frac{1887886}{3} C_{12}C_4C_3 + \frac{12034804441}{4} C_{12}C_3C_2 + 688051965205\, C_{12}C_3 +$
$\frac{2229689}{3} C_{11}C_8 + \frac{3433969}{6} C_{11}C_6C_2 + \frac{20445945289}{20} C_{11}C_6 + \frac{12710467}{24} C_{11}C_5C_3 +$
$\frac{515361}{2} C_{11}C_4^2 + \frac{434726998849}{180} C_{11}C_4C_2 + \frac{16300332146153}{30} C_{11}C_4 + \frac{46739816321}{40} C_{11}C_3^2 +$
$\frac{64307781109}{90} C_{11}C_2^3 + \frac{5671586795842}{5} C_{11}C_2^2 + \frac{769863883063401}{5} C_{11}C_2 + \frac{15232017986753678}{5} C_{11} +$
$\frac{8576953}{12} C_{10}C_9 + 515361\, C_{10}C_7C_2 + \frac{33248400053}{36} C_{10}C_7 + \frac{2750363}{6} C_{10}C_6C_3 +$
$\frac{1719641}{4} C_{10}C_5C_4 + \frac{72031191551}{36} C_{10}C_5C_2 + \frac{5379487307293}{12} C_{10}C_5 + \frac{22268670127}{12} C_{10}C_4C_3 +$
$\frac{29220797419}{18} C_{10}C_3C_2^2 + 1671151488809\, C_{10}C_3C_2 + \frac{999053349038507}{9} C_{10}C_3 +$
$\frac{5842529}{12} C_9C_8C_2 + \frac{1751343671}{2} C_9C_8 + \frac{9976043}{24} C_9C_7C_3 + \frac{4475317}{12} C_9C_6C_4 +$
$\frac{34787640921}{20} C_9C_6C_2 + \frac{15634691064399}{40} C_9C_6 + \frac{8608831}{48} C_9C_5^2 + 1534016418\, C_9C_5C_3 +$
$\frac{8802190045}{12} C_9C_4^2 + \frac{45507459899}{36} C_9C_4C_2^2 + \frac{154610056429477}{120} C_9C_4C_2 +$
$\frac{20444196353349455}{24} C_9C_4 + \frac{2415987827}{2} C_9C_3^2C_2 + \frac{97091729946079}{160} C_9C_3^2 + \frac{810575062}{9} C_9C_2^4 +$
$\frac{82105272758641}{120} C_9C_2^3 + \frac{26237434776737807}{120} C_9C_2^2 + \frac{111189221091312573}{10} C_9C_2 +$
$98479238251514890\, C_9 + \frac{602140}{3} C_8^2C_3 + \frac{688919}{2} C_8C_7C_4 + \frac{58069844987}{36} C_8C_7C_2 +$
$\frac{727690062165}{2} C_8C_7 + \frac{3791711}{12} C_8C_6C_5 + \frac{26996311111}{20} C_8C_6C_3 + \frac{22006963891}{18} C_8C_5C_4 +$
$\frac{12471356953}{12} C_8C_5C_2^2 + \frac{4248139382003}{4} C_8C_5C_2 + \frac{282229564075195}{4} C_8C_5 +$
$\frac{33600536585}{18} C_8C_4C_3C_2 + \frac{13978441985002}{15} C_8C_4C_3 + \frac{1179464495}{4} C_8C_3^3 + \frac{4676569139}{18} C_8C_3C_2^3 +$
$\frac{43431948068531}{30} C_8C_3C_2^2 + \frac{13633873212160447}{45} C_8C_3C_2 + \frac{266803847737746638}{35} C_8C_3 +$
$\frac{2413873}{16} C_7^2C_5 + \frac{15478628803}{24} C_7^2C_3 + \frac{862477}{6} C_7C_6^2 + \frac{22096184341}{20} C_7C_6C_4 +$
$\frac{83815466537}{90} C_7C_6C_2^2 + \frac{57420251310433}{60} C_7C_6C_2 + \frac{319674568301494}{5} C_7C_6 +$
$\frac{18802575187}{36} C_7C_5^2 + \frac{13992905278}{9} C_7C_5C_3C_2 + \frac{18712369927163}{24} C_7C_5C_3 +$
$\frac{2910217255}{4} C_7C_4^2C_2 + \frac{3618817848833}{10} C_7C_4^2 + \frac{2734823993}{4} C_7C_4C_3^2 + 197746824\, C_7C_4C_2^3 +$
$\frac{65949511551649}{60} C_7C_4C_2^2 + \frac{8285980050025069}{36} C_7C_4C_2 + \frac{262058373765775246}{45} C_7C_4 +$
$\frac{2494321687}{9} C_7C_3^2C_2^2 + \frac{242585790753053}{240} C_7C_3^2C_2 + \frac{18744932208844051}{180} C_7C_3^2 +$
$\frac{2426036919955}{12} C_7C_2^4 + \frac{27775400613303217}{180} C_7C_2^3 + \frac{712079738884590461}{45} C_7C_2^2 +$
$\frac{1005560670496102028}{3} C_7C_2 + 1331809866212570976\, C_7 + \frac{19734348887}{40} C_6^2C_5 +$
$\frac{7268658881}{10} C_6^2C_3C_2 + \frac{7316321799193}{20} C_6^2C_3 + \frac{113232374629}{90} C_6C_5C_4C_2 +$
$\frac{2512360105541}{4} C_6C_5C_4 + \frac{5876712083}{10} C_6C_5C_3^2 + \frac{15136740289}{90} C_6C_5C_2^3 +$
$\frac{112840406348527}{120} C_6C_5C_2^2 + \frac{990888330974698}{5} C_6C_5C_2 + \frac{25271712462896097}{5} C_6C_5 +$
$\frac{32564606833}{60} C_6C_4^2C_3 + \frac{19261611556}{45} C_6C_4C_3C_2^2 + \frac{46816597345177}{30} C_6C_4C_3C_2 +$
$\frac{5796176016604757}{36} C_6C_4C_3 + \frac{1313302001}{10} C_6C_3^3C_2 + \frac{18863584826713}{80} C_6C_3^3 +$
$\frac{10973011573491}{20} C_6C_3C_2^3 + \frac{18594547219829807}{60} C_6C_3C_2^2 + \frac{147234598534568479}{7} C_6C_3C_2 +$
$\frac{1114403568091052932}{5} C_6C_3 + \frac{6973211047}{36} C_5^3C_2 + \frac{18573163276415}{192} C_5^3 + \frac{8950075123}{18} C_5^2C_4C_3 +$

$$\frac{4649043751}{24} C_5{}^2C_3C_2{}^2 + \frac{8501853238241}{12} C_5{}^2C_3C_2 + \frac{5280723464413745}{72} C_5{}^2C_3 + \frac{909498821}{6} C_5C_4{}^3 + \frac{4198138549}{24} C_5C_4{}^2C_2{}^2 + \frac{9547228266818}{15} C_5C_4{}^2C_2 + \frac{2361746942897485}{36} C_5C_4{}^2 + \frac{2864253986}{9} C_5C_4C_3{}^2C_2 + \frac{22840224408749}{40} C_5C_4C_3{}^2 + \frac{17318474049041}{40} C_5C_4C_2{}^3 + \frac{88285082317708487}{360} C_5C_4C_2{}^2 + \frac{753639792348084103}{45} C_5C_4C_2 + \frac{537883396502498200}{3} C_5C_4 + \frac{767683719}{32} C_5C_3{}^4 + \frac{91328503577993}{160} C_5C_3{}^2C_2{}^2 + \frac{3182725168324144}{15} C_5C_3{}^2C_2 + \frac{250016402650880382}{35} C_5C_3{}^2 + \frac{3289481283671}{120} C_5C_2{}^5 + \frac{6991507370698399}{120} C_5C_2{}^4 + \frac{1019960205968778943}{90} C_5C_2{}^3 + \frac{1317449568399997186}{3} C_5C_2{}^2 + 3920438943718640112\, C_5C_2 + 6471032463145324800\, C_5 + \frac{377501047}{4} C_4{}^3C_3C_2 + \frac{7594583700176}{45} C_4{}^3C_3 + \frac{3047785499}{72} C_4{}^2C_3{}^3 + \frac{9873413253231}{20} C_4{}^2C_3C_2{}^2 + \frac{33014547844759393}{180} C_4{}^2C_3C_2 + \frac{649449859252043894}{105} C_4{}^2C_3 + \frac{13594491798239}{48} C_4C_3{}^3C_2 + \frac{414991944017273}{8} C_4C_3{}^3 + \frac{5818863142433}{60} C_4C_3C_2{}^4 + \frac{29241250643311313}{180} C_4C_3C_2{}^3 + \frac{1478410979219565743}{63} C_4C_3C_2{}^2 + \frac{3021521521513835956}{5} C_4C_3C_2 + 2712067327341076320\, C_4C_3 + \frac{384720588967}{32} C_3{}^5 + \frac{6365425901263}{120} C_3{}^3C_2{}^3 + \frac{7875539720705189}{120} C_3{}^3C_2{}^2 + \frac{436808216691304139}{70} C_3{}^3C_2 + \frac{398098512218656788}{5} C_3{}^3 + \frac{28670278021}{18} C_3C_2{}^6 + \frac{42268370443088}{3} C_3C_2{}^5 + \frac{1627219738615076276}{315} C_3C_2{}^4 + \frac{4877286472561078576}{15} C_3C_2{}^3 + 4754022059945728416\, C_3C_2{}^2 + 15808640764412390400\, C_3C_2 + 8198570710149120000\, C_3$$

Appendix C

Stanley's Character Polynomials $(-1)^k F_k(a,p,-b,-q)$ for $k \le 10$

$-F_1(a,p,-b,-q) = ab + pq$

$F_2(a,p,-b,-q) = a^2b + ab^2 + 2\,apq + p^2q + pq^2$

$-F_3(a,p,-b,-q) = a^3b + 3\,a^2b^2 + 3\,a^2pq + ab^3 + 3\,abpq + 3\,ap^2q + 3\,apq^2 + p^3q + 3\,p^2q^2 + pq^3 + ab + pq$

$F_4(a,p,-b,-q) = a^4b + 6\,a^3b^2 + 4\,a^3pq + 6\,a^2b^3 + 12\,a^2bpq + 6\,a^2p^2q + 6\,a^2pq^2 + ab^4 + 4\,ab^2pq + 4\,abp^2q + 4\,abpq^2 + 4\,ap^3q + 14\,ap^2q^2 + 4\,apq^3 + p^4q + 6\,p^3q^2 + 6\,p^2q^3 + pq^4 + 5\,a^2b + 5\,ab^2 + 10\,apq + 5\,p^2q + 5\,pq^2$

$-F_5(a,p,-b,-q) = a^5b + 10\,a^4b^2 + 5\,a^4pq + 20\,a^3b^3 + 30\,a^3bpq + 10\,a^3p^2q + 10\,a^3pq^2 + 10\,a^2b^4 + 30\,a^2b^2pq + 20\,a^2bp^2q + 20\,a^2bpq^2 + 10\,a^2p^3q + 40\,a^2p^2q^2 + 10\,a^2pq^3 + ab^5 + 5\,ab^3pq + 5\,ab^2p^2q + 5\,ab^2pq^2 + 5\,abp^3q + 20\,abp^2q^2 + 5\,abpq^3 + 5\,ap^4q + 35\,ap^3q^2 + 35\,ap^2q^3 + 5\,apq^4 + p^5q + 10\,p^4q^2 + 20\,p^3q^3 + 10\,p^2q^4 + pq^5 + 15\,a^3b + 40\,a^2b^2 + 45\,a^2pq + 15\,ab^3 + 35\,abpq + 45\,ap^2q + 45\,apq^2 + 15\,p^3q + 40\,p^2q^2 + 15\,pq^3 + 8\,ab + 8\,pq$

$F_6(a,p,-b,-q) = a^6b + 15\,a^5b^2 + 6\,a^5pq + 50\,a^4b^3 + 60\,a^4bpq + 15\,a^4p^2q + 15\,a^4pq^2 + 50\,a^3b^4 + 120\,a^3b^2pq + 60\,a^3bp^2q + 60\,a^3bpq^2 + 20\,a^3p^3q + 90\,a^3p^2q^2 + 20\,a^3pq^3 + 15\,a^2b^5 + 60\,a^2b^3pq + 45\,a^2b^2p^2q + 45\,a^2b^2pq^2 + 30\,a^2bp^3q + 135\,a^2bp^2q^2 + 30\,a^2bpq^3 + 15\,a^2p^4q + 120\,a^2p^3q^2 + 120\,a^2p^2q^3 + 15\,a^2pq^4 + ab^6 + 6\,ab^4pq + 6\,ab^3p^2q + 6\,ab^3pq^2 + 6\,ab^2p^3q + 27\,ab^2p^2q^2 + 6\,ab^2pq^3 + 6\,abp^4q + 48\,abp^3q^2 +$

$48\,abp^2q^3 + 6\,abpq^4 + 6\,ap^5q + 69\,ap^4q^2 + 146\,ap^3q^3 + 69\,ap^2q^4 + 6\,apq^5 + p^6q + 15\,p^5q^2 + 50\,p^4q^3 + 50\,p^3q^4 + 15\,p^2q^5 + pq^6 + 35\,a^4b + 175\,a^3b^2 + 140\,a^3pq + 175\,a^2b^3 + 315\,a^2bpq + 210\,a^2p^2q + 210\,a^2pq^2 + 35\,ab^4 + 105\,ab^2pq + 105\,abp^2q + 105\,abpq^2 + 140\,ap^3q + 420\,ap^2q^2 + 140\,apq^3 + 35\,p^4q + 175\,p^3q^2 + 175\,p^2q^3 + 35\,pq^4 + 84\,a^2b + 84\,ab^2 + 168\,apq + 84\,p^2q + 84\,pq^2$

$-F_7(a,p,-b,-q) = a^7b + 21\,a^6b^2 + 7\,a^6pq + 105\,a^5b^3 + 105\,a^5bpq + 21\,a^5p^2q + 21\,a^5pq^2 + 175\,a^4b^4 + 350\,a^4b^2pq + 140\,a^4bp^2q + 140\,a^4bpq^2 + 35\,a^4p^3q + 175\,a^4p^2q^2 + 35\,a^4pq^3 + 105\,a^3b^5 + 350\,a^3b^3pq + 210\,a^3b^2p^2q + 210\,a^3b^2pq^2 + 105\,a^3bp^3q + 525\,a^3bp^2q^2 + 105\,a^3bpq^3 + 35\,a^3p^4q + 315\,a^3p^3q^2 + 315\,a^3p^2q^3 + 35\,a^3pq^4 + 21\,a^2b^6 + 105\,a^2b^4pq + 84\,a^2b^3p^2q + 84\,a^2b^3pq^2 + 63\,a^2b^2p^3q + 315\,a^2b^2p^2q^2 + 63\,a^2b^2pq^3 + 42\,a^2bp^4q + 378\,a^2bp^3q^2 + 378\,a^2bp^2q^3 + 42\,a^2bpq^4 + 21\,a^2p^5q + 273\,a^2p^4q^2 + 609\,a^2p^3q^3 + 273\,a^2p^2q^4 + 21\,a^2pq^5 + ab^7 + 7\,ab^5pq + 7\,ab^4p^2q + 7\,ab^4pq^2 + 7\,ab^3p^3q + 35\,ab^3p^2q^2 + 7\,ab^3pq^3 + 7\,ab^2p^4q + 63\,ab^2p^3q^2 + 63\,ab^2p^2q^3 + 7\,ab^2pq^4 + 7\,abp^5q + 91\,abp^4q^2 + 203\,abp^3q^3 + 91\,abp^2q^4 + 7\,abpq^5 + 7\,ap^6q + 119\,ap^5q^2 + 427\,ap^4q^3 + 427\,ap^3q^4 + 119\,ap^2q^5 + 7\,apq^6 + p^7q + 21\,p^6q^2 + 105\,p^5q^3 + 175\,p^4q^4 + 105\,p^3q^5 + 21\,p^2q^6 + pq^7 + 70\,a^5b + 560\,a^4b^2 + 350\,a^4pq + 1050\,a^3b^3 + 1540\,a^3bpq + 700\,a^3p^2q + 700\,a^3pq^2 + 560\,a^2b^4 + 1414\,a^2b^2pq + 1036\,a^2bp^2q + 1036\,a^2bpq^2 + 700\,a^2p^3q + 2324\,a^2p^2q^2 + 700\,a^2pq^3 + 70\,ab^5 + 266\,ab^3pq + 238\,ab^2p^2q + 238\,ab^2pq^2 + 266\,abp^3q + 938\,abp^2q^2 + 266\,abpq^3 + 350\,ap^4q + 1974\,ap^3q^2 + 1974\,ap^2q^3 + 350\,apq^4 + 70\,p^5q + 560\,p^4q^2 + 1050\,p^3q^3 + 560\,p^2q^4 + 70\,pq^5 + 469\,a^3b + 1183\,a^2b^2 + 1407\,a^2pq + 469\,ab^3 + 959\,abpq + 1407\,ap^2q + 1407\,apq^2 + 469\,p^3q + 1183\,p^2q^2 + 469\,pq^3 + 180\,ab + 180\,pq$

$F_8(a,p,-b,-q) = a^8b + 28\,a^7b^2 + 8\,a^7pq + 196\,a^6b^3 + 168\,a^6bpq + 28\,a^6p^2q + 28\,a^6pq^2 + 490\,a^5b^4 + 840\,a^5b^2pq + 280\,a^5bp^2q + 280\,a^5bpq^2 + 56\,a^5p^3q + 308\,a^5p^2q^2 + 56\,a^5pq^3 + 490\,a^4b^5 + 1400\,a^4b^3pq + 700\,a^4b^2p^2q + 700\,a^4b^2pq^2 + 280\,a^4bp^3q + 1540\,a^4bp^2q^2 + 280\,a^4bpq^3 + 70\,a^4p^4q + 700\,a^4p^3q^2 + 700\,a^4p^2q^3 + 70\,a^4pq^4 + 196\,a^3b^6 + 840\,a^3b^4pq + 560\,a^3b^3p^2q + 560\,a^3b^3pq^2 + 336\,a^3b^2p^3q + 1848\,a^3b^2p^2q^2 + 336\,a^3b^2pq^3 + 168\,a^3bp^4q + 1680\,a^3bp^3q^2 + 1680\,a^3bp^2q^3 + 168\,a^3bpq^4 + 56\,a^3p^5q + 812\,a^3p^4q^2 + 1904\,a^3p^3q^3 + 812\,a^3p^2q^4 + 56\,a^3pq^5 + 28\,a^2b^7 + 168\,a^2b^5pq + 140\,a^2b^4p^2q + 140\,a^2b^4pq^2 + 112\,a^2b^3p^3q + 616\,a^2b^3p^2q^2 + 112\,a^2b^3pq^3 + 84\,a^2b^2p^4q + 840\,a^2b^2p^3q^2 + 840\,a^2b^2p^2q^3 + 84\,a^2b^2pq^4 + 56\,a^2bp^5q + 812\,a^2bp^4q^2 + 1904\,a^2bp^3q^3 + 812\,a^2bp^2q^4 + 56\,a^2bpq^5 + 28\,a^2p^6q + 532\,a^2p^5q^2 + 2044\,a^2p^4q^3 + 2044\,a^2p^3q^4 + 532\,a^2p^2q^5 + 28\,a^2pq^6 + ab^8 + 8\,ab^6pq + 8\,ab^5p^2q + 8\,ab^5pq^2 + 8\,ab^4p^3q + 44\,ab^4p^2q^2 + 8\,ab^4pq^3 + 8\,ab^3p^4q + 80\,ab^3p^3q^2 + 80\,ab^3p^2q^3 + 8\,ab^3pq^4 + 8\,ab^2p^5q + 116\,ab^2p^4q^2 +$

$272\,ab^2p^3q^3 + 116\,ab^2p^2q^4 + 8\,ab^2pq^5 + 8\,abp^6q + 152\,abp^5q^2 + 584\,abp^4q^3 +$
$584\,abp^3q^4 + 152\,abp^2q^5 + 8\,abpq^6 + 8\,ap^7q + 188\,ap^6q^2 + 1016\,ap^5q^3 + 1742\,ap^4q^4 +$
$1016\,ap^3q^5 + 188\,ap^2q^6 + 8\,apq^7 + p^8q + 28\,p^7q^2 + 196\,p^6q^3 + 490\,p^5q^4 + 490\,p^4q^5 +$
$196\,p^3q^6 + 28\,p^2q^7 + pq^8 + 126\,a^6b + 1470\,a^5b^2 + 756\,a^5pq + 4410\,a^4b^3 + 5460\,a^4bpq +$
$1890\,a^4p^2q + 1890\,a^4pq^2 + 4410\,a^3b^4 + 9576\,a^3b^2pq + 5544\,a^3bp^2q + 5544\,a^3bpq^2 +$
$2520\,a^3p^3q + 9156\,a^3p^2q^2 + 2520\,a^3pq^3 + 1470\,a^2b^5 + 4872\,a^2b^3pq + 3612\,a^2b^2p^2q +$
$3612\,a^2b^2pq^2 + 2856\,a^2bp^3q + 11004\,a^2bp^2q^2 + 2856\,a^2bpq^3 + 1890\,a^2p^4q +$
$11844\,a^2p^3q^2 + 11844\,a^2p^2q^3 + 1890\,a^2pq^4 + 126\,ab^6 + 588\,ab^4pq + 504\,ab^3p^2q +$
$504\,ab^3pq^2 + 504\,ab^2p^3q + 2100\,ab^2p^2q^2 + 504\,ab^2pq^3 + 588\,abp^4q + 3864\,abp^3q^2 +$
$3864\,abp^2q^3 + 588\,abpq^4 + 756\,ap^5q + 6762\,ap^4q^2 + 13272\,ap^3q^3 + 6762\,ap^2q^4 +$
$756\,apq^5 + 126\,p^6q + 1470\,p^5q^2 + 4410\,p^4q^3 + 4410\,p^3q^4 + 1470\,p^2q^5 + 126\,pq^6 +$
$1869\,a^4b + 8526\,a^3b^2 + 7476\,a^3pq + 8526\,a^2b^3 + 14364\,a^2bpq + 11214\,a^2p^2q +$
$11214\,a^2pq^2 + 1869\,ab^4 + 4788\,ab^2pq + 4788\,abp^2q + 4788\,abpq^2 + 7476\,ap^3q +$
$20790\,ap^2q^2 + 7476\,apq^3 + 1869\,p^4q + 8526\,p^3q^2 + 8526\,p^2q^3 + 1869\,pq^4 + 3044\,a^2b +$
$3044\,ab^2 + 6088\,apq + 3044\,p^2q + 3044\,pq^2$

$-F_9(a,p,-b,-q) \quad = \quad a^9b + 36\,a^8b^2 + 9\,a^8pq + 336\,a^7b^3 + 252\,a^7bpq +$
$36\,a^7p^2q + 36\,a^7pq^2 + 1176\,a^6b^4 + 1764\,a^6b^2pq + 504\,a^6bp^2q + 504\,a^6bpq^2 +$
$84\,a^6p^3q + 504\,a^6p^2q^2 + 84\,a^6pq^3 + 1764\,a^5b^5 + 4410\,a^5b^3pq + 1890\,a^5b^2p^2q +$
$1890\,a^5b^2pq^2 + 630\,a^5bp^3q + 3780\,a^5bp^2q^2 + 630\,a^5bpq^3 + 126\,a^5p^4q + 1386\,a^5p^3q^2 +$
$1386\,a^5p^2q^3 + 126\,a^5pq^4 + 1176\,a^4b^6 + 4410\,a^4b^4pq + 2520\,a^4b^3p^2q + 2520\,a^4b^3pq^2 +$
$1260\,a^4b^2p^3q + 7560\,a^4b^2p^2q^2 + 1260\,a^4b^2pq^3 + 504\,a^4bp^4q + 5544\,a^4bp^3q^2 +$
$5544\,a^4bp^2q^3 + 504\,a^4bpq^4 + 126\,a^4p^5q + 2016\,a^4p^4q^2 + 4956\,a^4p^3q^3 + 2016\,a^4p^2q^4 +$
$126\,a^4pq^5 + 336\,a^3b^7 + 1764\,a^3b^5pq + 1260\,a^3b^4p^2q + 1260\,a^3b^4pq^2 + 840\,a^3b^3p^3q +$
$5040\,a^3b^3p^2q^2 + 840\,a^3b^3pq^3 + 504\,a^3b^2p^4q + 5544\,a^3b^2p^3q^2 + 5544\,a^3b^2p^2q^3 +$
$504\,a^3b^2pq^4 + 252\,a^3bp^5q + 4032\,a^3bp^4q^2 + 9912\,a^3bp^3q^3 + 4032\,a^3bp^2q^4 +$
$252\,a^3bpq^5 + 84\,a^3p^6q + 1764\,a^3p^5q^2 + 7224\,a^3p^4q^3 + 7224\,a^3p^3q^4 + 1764\,a^3p^2q^5 +$
$84\,a^3pq^6 + 36\,a^2b^8 + 252\,a^2b^6pq + 216\,a^2b^5p^2q + 216\,a^2b^5pq^2 + 180\,a^2b^4p^3q +$
$1080\,a^2b^4p^2q^2 + 180\,a^2b^4pq^3 + 144\,a^2b^3p^4q + 1584\,a^2b^3p^3q^2 + 1584\,a^2b^3p^2q^3 +$
$144\,a^2b^3pq^4 + 108\,a^2b^2p^5q + 1728\,a^2b^2p^4q^2 + 4248\,a^2b^2p^3q^3 + 1728\,a^2b^2p^2q^4 +$
$108\,a^2b^2pq^5 + 72\,a^2bp^6q + 1512\,a^2bp^5q^2 + 6192\,a^2bp^4q^3 + 6192\,a^2bp^3q^4 +$
$1512\,a^2bp^2q^5 + 72\,a^2bpq^6 + 36\,a^2p^7q + 936\,a^2p^6q^2 + 5436\,a^2p^5q^3 + 9576\,a^2p^4q^4 +$
$5436\,a^2p^3q^5 + 936\,a^2p^2q^6 + 36\,a^2pq^7 + ab^9 + 9\,ab^7pq + 9\,ab^6p^2q + 9\,ab^6pq^2 +$
$9\,ab^5p^3q + 54\,ab^5p^2q^2 + 9\,ab^5pq^3 + 9\,ab^4p^4q + 99\,ab^4p^3q^2 + 99\,ab^4p^2q^3 + 9\,ab^4pq^4 +$
$9\,ab^3p^5q + 144\,ab^3p^4q^2 + 354\,ab^3p^3q^3 + 144\,ab^3p^2q^4 + 9\,ab^3pq^5 + 9\,ab^2p^6q +$
$189\,ab^2p^5q^2 + 774\,ab^2p^4q^3 + 774\,ab^2p^3q^4 + 189\,ab^2p^2q^5 + 9\,ab^2pq^6 + 9\,abp^7q +$

$234\,abp^6q^2 + 1359\,abp^5q^3 + 2394\,abp^4q^4 + 1359\,abp^3q^5 + 234\,abp^2q^6 + 9\,abpq^7 + 9\,ap^8q + 279\,ap^7q^2 + 2109\,ap^6q^3 + 5499\,ap^5q^4 + 5499\,ap^4q^5 + 2109\,ap^3q^6 + 279\,ap^2q^7 + 9\,apq^8 + p^9q + 36\,p^8q^2 + 336\,p^7q^3 + 1176\,p^6q^4 + 1764\,p^5q^5 + 1176\,p^4q^6 + 336\,p^3q^7 + 36\,p^2q^8 + pq^9 + 210\,a^7b + 3360\,a^6b^2 + 1470\,a^6pq + 14700\,a^5b^3 + 15750\,a^5bpq + 4410\,a^5p^2q + 4410\,a^5pq^2 + 23520\,a^4b^4 + 44730\,a^4b^2pq + 21420\,a^4bp^2q + 21420\,a^4bpq^2 + 7350\,a^4p^3q + 28980\,a^4p^2q^2 + 7350\,a^4pq^3 + 14700\,a^3b^5 + 43050\,a^3b^3pq + 27090\,a^3b^2p^2q + 27090\,a^3b^2pq^2 + 16590\,a^3bp^3q + 69300\,a^3bp^2q^2 + 16590\,a^3bpq^3 + 7350\,a^3p^4q + 50610\,a^3p^3q^2 + 50610\,a^3p^2q^3 + 7350\,a^3pq^4 + 3360\,a^2b^6 + 13950\,a^2b^4pq + 10440\,a^2b^3p^2q + 10440\,a^2b^3pq^2 + 8460\,a^2b^2p^3q + 37800\,a^2b^2p^2q^2 + 8460\,a^2b^2pq^3 + 6840\,a^2bp^4q + 49320\,a^2bp^3q^2 + 49320\,a^2bp^2q^3 + 6840\,a^2bpq^4 + 4410\,a^2p^5q + 43560\,a^2p^4q^2 + 89220\,a^2p^3q^3 + 43560\,a^2p^2q^4 + 4410\,a^2pq^5 + 210\,ab^7 + 1170\,ab^5pq + 990\,ab^4p^2q + 990\,ab^4pq^2 + 930\,ab^3p^3q + 4500\,ab^3p^2q^2 + 930\,ab^3pq^3 + 990\,ab^2p^4q + 7650\,ab^2p^3q^2 + 7650\,ab^2p^2q^3 + 990\,ab^2pq^4 + 1170\,abp^5q + 11970\,abp^4q^2 + 24960\,abp^3q^3 + 11970\,abp^2q^4 + 1170\,abpq^5 + 1470\,ap^6q + 18990\,ap^5q^2 + 60540\,ap^4q^3 + 60540\,ap^3q^4 + 18990\,ap^2q^5 + 1470\,apq^6 + 210\,p^7q + 3360\,p^6q^2 + 14700\,p^5q^3 + 23520\,p^4q^4 + 14700\,p^3q^5 + 3360\,p^2q^6 + 210\,pq^7 + 5985\,a^5b + 42588\,a^4b^2 + 29925\,a^4pq + 77028\,a^3b^3 + 110502\,a^3bpq + 59850\,a^3p^2q + 59850\,a^3pq^2 + 42588\,a^2b^4 + 96606\,a^2b^2pq + 74628\,a^2bp^2q + 74628\,a^2bpq^2 + 59850\,a^2p^3q + 180900\,a^2p^2q^2 + 59850\,a^2pq^3 + 5985\,ab^5 + 19377\,ab^3pq + 16497\,ab^2p^2q + 16497\,ab^2pq^2 + 19377\,abp^3q + 63612\,abp^2q^2 + 19377\,abpq^3 + 29925\,ap^4q + 150975\,ap^3q^2 + 150975\,ap^2q^3 + 29925\,apq^4 + 5985\,p^5q + 42588\,p^4q^2 + 77028\,p^3q^3 + 42588\,p^2q^4 + 5985\,pq^5 + 26060\,a^3b + 63600\,a^2b^2 + 78180\,a^2pq + 26060\,ab^3 + 49020\,abpq + 78180\,ap^2q + 78180\,apq^2 + 26060\,p^3q + 63600\,p^2q^2 + 26060\,pq^3 + 8064\,ab + 8064\,pq$

$F_{10}(a,p,-b,-q) = a^{10}b + 45\,a^9b^2 + 10\,a^9pq + 540\,a^8b^3 + 360\,a^8bpq + 45\,a^8p^2q + 45\,a^8pq^2 + 2520\,a^7b^4 + 3360\,a^7b^2pq + 840\,a^7bp^2q + 840\,a^7bpq^2 + 120\,a^7p^3q + 780\,a^7p^2q^2 + 120\,a^7pq^3 + 5292\,a^6b^5 + 11760\,a^6b^3pq + 4410\,a^6b^2p^2q + 4410\,a^6b^2pq^2 + 1260\,a^6bp^3q + 8190\,a^6bp^2q^2 + 1260\,a^6bpq^3 + 210\,a^6p^4q + 2520\,a^6p^3q^2 + 2520\,a^6p^2q^3 + 210\,a^6pq^4 + 5292\,a^5b^6 + 17640\,a^5b^4pq + 8820\,a^5b^3p^2q + 8820\,a^5b^3pq^2 + 3780\,a^5b^2p^3q + 24570\,a^5b^2p^2q^2 + 3780\,a^5b^2pq^3 + 1260\,a^5bp^4q + 15120\,a^5bp^3q^2 + 15120\,a^5bp^2q^3 + 1260\,a^5bpq^4 + 252\,a^5p^5q + 4410\,a^5p^4q^2 + 11340\,a^5p^3q^3 + 4410\,a^5p^2q^4 + 252\,a^5pq^5 + 2520\,a^4b^7 + 11760\,a^4b^5pq + 7350\,a^4b^4p^2q + 7350\,a^4b^4pq^2 + 4200\,a^4b^3p^3q + 27300\,a^4b^3p^2q^2 + 4200\,a^4b^3pq^3 + 2100\,a^4b^2p^4q + 25200\,a^4b^2p^3q^2 + 25200\,a^4b^2p^2q^3 + 2100\,a^4b^2pq^4 + 840\,a^4bp^5q + 14700\,a^4bp^4q^2 + 37800\,a^4bp^3q^3 + 14700\,a^4bp^2q^4 + 840\,a^4bpq^5 + 210\,a^4p^6q + 4830\,a^4p^5q^2 +$

$21000\,a^4p^4q^3 + 21000\,a^4p^3q^4 + 4830\,a^4p^2q^5 + 210\,a^4pq^6 + 540\,a^3b^8 + 3360\,a^3b^6pq + 2520\,a^3b^5p^2q + 2520\,a^3b^5pq^2 + 1800\,a^3b^4p^3q + 11700\,a^3b^4p^2q^2 + 1800\,a^3b^4pq^3 + 1200\,a^3b^3p^4q + 14400\,a^3b^3p^3q^2 + 14400\,a^3b^3p^2q^3 + 1200\,a^3b^3pq^4 + 720\,a^3b^2p^5q + 12600\,a^3b^2p^4q^2 + 32400\,a^3b^2p^3q^3 + 12600\,a^3b^2p^2q^4 + 720\,a^3b^2pq^5 + 360\,a^3bp^6q + 8280\,a^3bp^5q^2 + 36000\,a^3bp^4q^3 + 36000\,a^3bp^3q^4 + 8280\,a^3bp^2q^5 + 360\,a^3bpq^6 + 120\,a^3p^7q + 3420\,a^3p^6q^2 + 21240\,a^3p^5q^3 + 38400\,a^3p^4q^4 + 21240\,a^3p^3q^5 + 3420\,a^3p^2q^6 + 120\,a^3pq^7 + 45\,a^2b^9 + 360\,a^2b^7pq + 315\,a^2b^6p^2q + 315\,a^2b^6pq^2 + 270\,a^2b^5p^3q + 1755\,a^2b^5p^2q^2 + 270\,a^2b^5pq^3 + 225\,a^2b^4p^4q + 2700\,a^2b^4p^3q^2 + 2700\,a^2b^4p^2q^3 + 225\,a^2b^4pq^4 + 180\,a^2b^3p^5q + 3150\,a^2b^3p^4q^2 + 8100\,a^2b^3p^3q^3 + 3150\,a^2b^3p^2q^4 + 180\,a^2b^3pq^5 + 135\,a^2b^2p^6q + 3105\,a^2b^2p^5q^2 + 13500\,a^2b^2p^4q^3 + 13500\,a^2b^2p^3q^4 + 3105\,a^2b^2p^2q^5 + 135\,a^2b^2pq^6 + 90\,a^2bp^7q + 2565\,a^2bp^6q^2 + 15930\,a^2bp^5q^3 + 28800\,a^2bp^4q^4 + 15930\,a^2bp^3q^5 + 2565\,a^2bp^2q^6 + 90\,a^2bpq^7 + 45\,a^2p^8q + 1530\,a^2p^7q^2 + 12420\,a^2p^6q^3 + 33705\,a^2p^5q^4 + 33705\,a^2p^4q^5 + 12420\,a^2p^3q^6 + 1530\,a^2p^2q^7 + 45\,a^2pq^8 + ab^{10} + 10\,ab^8pq + 10\,ab^7p^2q + 10\,ab^7pq^2 + 10\,ab^6p^3q + 65\,ab^6p^2q^2 + 10\,ab^6pq^3 + 10\,ab^5p^4q + 120\,ab^5p^3q^2 + 120\,ab^5p^2q^3 + 10\,ab^5pq^4 + 10\,ab^4p^5q + 175\,ab^4p^4q^2 + 450\,ab^4p^3q^3 + 175\,ab^4p^2q^4 + 10\,ab^4pq^5 + 10\,ab^3p^6q + 230\,ab^3p^5q^2 + 1000\,ab^3p^4q^3 + 1000\,ab^3p^3q^4 + 230\,ab^3p^2q^5 + 10\,ab^3pq^6 + 10\,ab^2p^7q + 285\,ab^2p^6q^2 + 1770\,ab^2p^5q^3 + 3200\,ab^2p^4q^4 + 1770\,ab^2p^3q^5 + 285\,ab^2p^2q^6 + 10\,ab^2pq^7 + 10\,abp^8q + 340\,abp^7q^2 + 2760\,abp^6q^3 + 7490\,abp^5q^4 + 7490\,abp^4q^5 + 2760\,abp^3q^6 + 340\,abp^2q^7 + 10\,abpq^8 + 10\,ap^9q + 395\,ap^8q^2 + 3970\,ap^7q^3 + 14585\,ap^6q^4 + 22252\,ap^5q^5 + 14585\,ap^4q^6 + 3970\,ap^3q^7 + 395\,ap^2q^8 + 10\,apq^9 + p^{10}q + 45\,p^9q^2 + 540\,p^8q^3 + 2520\,p^7q^4 + 5292\,p^6q^5 + 5292\,p^5q^6 + 2520\,p^4q^7 + 540\,p^3q^8 + 45\,p^2q^9 + pq^{10} + 330\,a^8b + 6930\,a^7b^2 + 2640\,a^7pq + 41580\,a^6b^3 + 39270\,a^6bpq + 9240\,a^6p^2q + 9240\,a^6pq^2 + 97020\,a^5b^4 + 164010\,a^5b^2pq + 66990\,a^5bp^2q + 66990\,a^5bpq^2 + 18480\,a^5p^3q + 78540\,a^5p^2q^2 + 18480\,a^5pq^3 + 97020\,a^4b^5 + 254100\,a^4b^3pq + 138600\,a^4b^2p^2q + 138600\,a^4b^2pq^2 + 69300\,a^4bp^3q + 311850\,a^4bp^2q^2 + 69300\,a^4bpq^3 + 23100\,a^4p^4q + 173250\,a^4p^3q^2 + 173250\,a^4p^2q^3 + 23100\,a^4pq^4 + 41580\,a^3b^6 + 155100\,a^3b^4pq + 102300\,a^3b^3p^2q + 102300\,a^3b^3pq^2 + 69300\,a^3b^2p^3q + 331650\,a^3b^2p^2q^2 + 69300\,a^3b^2pq^3 + 42900\,a^3bp^4q + 336600\,a^3bp^3q^2 + 336600\,a^3bp^2q^3 + 42900\,a^3bpq^4 + 18480\,a^3p^5q + 199650\,a^3p^4q^2 + 425700\,a^3p^3q^3 + 199650\,a^3p^2q^4 + 18480\,a^3pq^5 + 6930\,a^2b^7 + 34815\,a^2b^5pq + 26400\,a^2b^4p^2q + 26400\,a^2b^4pq^2 + 21450\,a^2b^3p^3q + 109725\,a^2b^3p^2q^2 + 21450\,a^2b^3pq^3 + 18150\,a^2b^2p^4q + 150975\,a^2b^2p^3q^2 + 150975\,a^2b^2p^2q^3 + 18150\,a^2b^2pq^4 + 14685\,a^2bp^5q + 164175\,a^2bp^4q^2 + 356400\,a^2bp^3q^3 + 164175\,a^2bp^2q^4 + 14685\,a^2bpq^5 + 9240\,a^2p^6q + 130845\,a^2p^5q^2 + 441375\,a^2p^4q^3 + 441375\,a^2p^3q^4 + 130845\,a^2p^2q^5 + 9240\,a^2pq^6 + 330\,ab^8 + 2145\,ab^6pq + 1815\,ab^5p^2q + 1815\,ab^5pq^2 + 1650\,ab^4p^3q + 9075\,ab^4p^2q^2 +$

$1650\,ab^4pq^3 + 1650\,ab^3p^4q + 14850\,ab^3p^3q^2 + 14850\,ab^3p^2q^3 + 1650\,ab^3pq^4 + 1815\,ab^2p^5q + 21450\,ab^2p^4q^2 + 47850\,ab^2p^3q^3 + 21450\,ab^2p^2q^4 + 1815\,ab^2pq^5 + 2145\,abp^6q + 31185\,abp^5q^2 + 107250\,abp^4q^3 + 107250\,abp^3q^4 + 31185\,abp^2q^5 + 2145\,abpq^6 + 2640\,ap^7q + 46365\,ap^6q^2 + 216480\,ap^5q^3 + 354750\,ap^4q^4 + 216480\,ap^3q^5 + 46365\,ap^2q^6 + 2640\,apq^7 + 330\,p^8q + 6930\,p^7q^2 + 41580\,p^6q^3 + 97020\,p^5q^4 + 97020\,p^4q^5 + 41580\,p^3q^6 + 6930\,p^2q^7 + 330\,pq^8 + 16401\,a^6b + 167013\,a^5b^2 + 98406\,a^5pq + 471240\,a^4b^3 + 589050\,a^4bpq + 246015\,a^4p^2q + 246015\,a^4pq^2 + 471240\,a^3b^4 + 954690\,a^3b^2pq + 602250\,a^3bp^2q + 602250\,a^3bpq^2 + 328020\,a^3p^3q + 1067880\,a^3p^2q^2 + 328020\,a^3pq^3 + 167013\,a^2b^5 + 490545\,a^2b^3pq + 362835\,a^2b^2p^2q + 362835\,a^2b^2pq^2 + 314325\,a^2bp^3q + 1108800\,a^2bp^2q^2 + 314325\,a^2bpq^3 + 246015\,a^2p^4q + 1355805\,a^2p^3q^2 + 1355805\,a^2p^2q^3 + 246015\,a^2pq^4 + 16401\,ab^6 + 65505\,ab^4pq + 52305\,ab^3p^2q + 52305\,ab^3pq^2 + 52305\,ab^2p^3q + 205920\,ab^2p^2q^2 + 52305\,ab^2pq^3 + 65505\,abp^4q + 385935\,abp^3q^2 + 385935\,abp^2q^3 + 65505\,abpq^4 + 98406\,ap^5q + 769560\,ap^4q^2 + 1446720\,ap^3q^3 + 769560\,ap^2q^4 + 98406\,apq^5 + 16401\,p^6q + 167013\,p^5q^2 + 471240\,p^4q^3 + 471240\,p^3q^4 + 167013\,p^2q^5 + 16401\,pq^6 + 152900\,a^4b + 659340\,a^3b^2 + 611600\,a^3pq + 659340\,a^2b^3 + 1060620\,a^2bpq + 917400\,a^2p^2q + 917400\,a^2pq^2 + 152900\,ab^4 + 353540\,ab^2pq + 353540\,abp^2q + 353540\,abpq^2 + 611600\,ap^3q + 1624480\,ap^2q^2 + 611600\,apq^3 + 152900\,p^4q + 659340\,p^3q^2 + 659340\,p^2q^3 + 152900\,pq^4 + 193248\,a^2b + 193248\,ab^2 + 386496\,apq + 193248\,p^2q + 193248\,pq^2$

Bibliography

[1] P. Biane. Characters of symmetric groups and free cumulants. *Asymptotic Combinatorics with Applications to Mathematical Physics, A. Vershik (Ed.), Springer Lecture Notes in Mathematics*, 1815:185–200, 2003.

[2] P. Biane. Representations of the symmetric groups and free probability. *Advances in Mathematics*, 138:126–181, 1998.

[3] P. Biane. Free cumulants and representations of large symmetric groups. *Proceedings of the XIIIth International Congress of Mathematical Physics, London, Int. Press*, pages 321–326, 2000.

[4] Sylvie Corteel, Alain Goupil, and Gilles Schaeffer. Content evaluation and class symmetric functions. *Advances in Mathematics*, 188(2):315–336, 2004.

[5] J. S. Frame, G. de B. Robinson, and R. M. Thrall. The hook of graphs of $\mathfrak{S}_n$. *Canadian Journal of Mathematics*, 6:316–324, 1954.

[6] A. Frumkin, G. James, and Y. Roichman. On trees and characters. *Journal of Algebraic Combinatorics*, 17(3):323–334, 2003.

[7] I. P. Goulden. A differential operator for symmetric functions and the combinatorics of multiplying transpositions. *Transactions of the American Mathematical Society*, 344(1):421–440, 1994.

[8] I. P. Goulden and D. M. Jackson. *Combinatorial Enumeration*. John Wiley and Sons, Dover Reprint, 2004.

[9] I. P. Goulden and D. M. Jackson. Symmetric functions and Macdonald's result for top connexion coefficients in the symmetric group. *Journal of Algebra*, 166 (2):364–378, 1994.

[10] I.P. Goulden and D.M. Jackson. Transitive factorizations into transpositions and holomorphic mapings on the sphere. *Proceedings of the American Mathematics Society*, 125:51–60, 1997.

[11] I.P. Goulden and S. Pepper. Labelled trees and factorizations of a cycle into transpositions. *Discrete Mathematics*, 113:263–268, 1993.

[12] I.P. Goulden and A. Rattan. An explicit form for Kerov's character polynomials. preprint, 2005, math.CO/0505317.

[13] I.P. Goulden and A. Yong. Tree-like properties of cycle factorizations. *Journal of Combinatorial Theory Series A*, 98:106–117, 2002.

[14] P. Hanlon, R. Stanley, and J. Stembridge. Some combinatorial aspects of the spectra of normal distributed random matrices. *Contemporary Mathematica*, 158:151–174, 1992.

[15] J. C. Irving. *Combinatorial Constructions for Transitive Factorizations in the Symmetric Group*. PhD thesis, University of Waterloo, 2004.

[16] V. Ivanov and G. Olshanski. Kerov's central limit theorem for the Plancherel measure on Young diagrams. *Symmetric functions 2001: Surveys of developments and perspectives, S. Fomin (Ed.), NATO Science series II. Mathematics, Physics and Chemistry*, 74:93–151, 2002.

[17] J. Katriel. Explicit expression for the central characters of the symmetric group. *Discrete Applied Mathematics*, 67:149–156, 1996.

[18] S Kerov. Transition probabilities of continual Young diagrams and the Markov moment problem. *Functional Analysis and Applications*, 27:104–117, 1993.

[19] S Kerov. The differential model of growth of Young diagrams. *Proceedings of St. Petersburg Mathematical Society*, 4:167–194, 1996.

[20] S. Kerov and A. Vershik. The asymptotic character theory of the symmetric group. *Functional Analysis and Applications*, 15:246–255, 1981.

[21] I. G. Macdonald. *Symmetric Functions and Hall Polynomials*. Oxford University Press, Oxford, second edition, 1995.

[22] A. Okounkov and G. Olshanski. Shifted Schur functions. *St. Petersburg Mathematics Journal*, 2:239–300, 1998. English Version.

[23] A. Okounkov and A. Vershik. A new approach to representation theory of the symmetric group. *Selecta Mathematica, N.S.*, 2(4):581–605, 1996.

[24] B. E. Sagan. *The Symmetric Group: Representations, Combinatorial Algorithms and Symmetric Functions*. Wadsworth and Brook/Cole, Pacific Grove, California, 1991.

[25] P. Śniady. Asymptotics of characters of symmetric groups and free probability. preprint, 2003.

[26] R. P. Stanley. *Enumerative Combinatorics*, volume 1. Cambridge University Press, Cambridge, 1996.

[27] R. P. Stanley. *Enumerative Combinatorics*, volume 2. Cambridge University Press, 1998.

[28] R. P. Stanley. Irreducible symmetric group characters of rectangular shape. *Séminaire Lothar. Combin.*, 50:B50d, 11pp, 2003.

[29] R.P. Stanley. Kerov's character polynomial and irreducible symmetric group characters of rectangular shape. Transparencies from a talk at CMS meeting, Quebec City, 2002.

Index