Recognizing Even-Cycle and Even-Cut Matroids

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Even-cycle and even-cut matroids are classes of binary matroids that generalize respectively graphic and cographic matroids. We give algorithms to check membership for these classes of matroids. We assume that the matroids are 3-connected and are given by their $(0,1)$-matrix representations. We first give an algorithm to check membership for p-cographic matroids that is a subclass of even-cut matroids. We use this algorithm to construct algorithms for membership problems for even-cycle and even-cut matroids and the running time of these algorithms is polynomial in the size of the matrix representations. However, we will outline only how theoretical results can be used to develop polynomial time algorithms and omit the details of algorithms.

Acknowledgements

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Chapter 1

Introduction

1.1 Membership problems

A matroid is *graphic* if its circuits correspond to the circuits of some graph G (by a *cycle* in a graph we mean a subset of edges that induces a subgraph where all vertices have even degree, and by a *circuit* we mean an inclusion-wise minimal cycle, i.e. a subset of edges that induces a connected graph where every vertex has degree two). We say that G is a *representation* of that matroid and denote the matroid by $\text{cycle}(G)$. A matroid is *cographic* if it is the dual of a graphic matroid, or equivalently, if its circuits correspond to the inclusion-wise minimal cuts of some graph G. We say that G is a *representation* of that matroid and denote the matroid by $\text{cut}(G)$. In [15], Tutte gave a polynomial time algorithm to recognize if a binary matroid (described by its $(0, 1)$-matrix representation) is a graphic matroid (and hence to recognize if a binary matroid is a cographic matroid). The main contribution of this thesis is to prove analogous results for two classes of binary matroids arising from graph-like objects, namely, the class of even-cycle matroids and the class of even-cut matroids when the given matroids are 3-connected.

A *signed-graph* is a pair (G, Σ) where G is a graph and Σ is a subset of the edges of G, i.e. $\Sigma \subseteq E(G)$. We say that a set $B \subseteq E(G)$ is *even* (resp. *odd*) if $|B \cap \Sigma|$ is even (resp. odd). We will say that an edge e is *odd* when $e \in \Sigma$ and *even* otherwise. A matroid is an *even-cycle matroid* if its circuits correspond to the inclusion-wise minimal even cycles of some signed-graph (G, Σ). We say that (G, Σ) is a *signed-graph representation* (or simply *representation*) of that matroid and denote the matroid by $\text{ecycle}(G, \Sigma)$. Observe that when $\Sigma = \emptyset$, every cycle is even, hence $\text{ecycle}(G, \Sigma) = \text{cycle}(G)$. In particular, graphic matroids are even-cycle matroids. Even-cycle matroids are binary matroids, indeed, $\text{ecycle}(G, \Sigma)$ can be represented by the $(0, 1)$-matrix obtained by appending a row corresponding to the characteristic vector of

Σ to the vertex-edge incidence matrix of G. A *graft* is a pair (G, T) where G is a graph and T is a subset of the vertices of G of even cardinality. Vertices T are called *terminals*. A cut $\delta_G(U) := \{uv \in E(G) : u \in U, v \notin U\}$[1] is *even* (resp. *odd*) if $|U \cap T|$ is even (resp. odd). Note that as we have an even number of terminals, this is well-defined. A matroid is an *even-cut matroid* if its circuits corresponds to the inclusion-wise minimal even cuts of some graft (G, T). We say that (G, T) is a *graft representation* (or simply *representation*) of that matroid and denote the matroid by ecut(G, T). Observe that when $T = \emptyset$, every cut is even, hence ecut$(G, T) =$ cut(G). In particular, cographic matroids are even-cut matroids. Even-cut matroids are binary matroids, indeed, ecut(G, T) can be represented by the $(0, 1)$-matrix obtained by appending a row corresponding to the characteristic vector of a T*-join* (defined in 1.3.1) to the $(0, 1)$-matrix representing cut(G). Note, the term *representation* for an even-cycle (resp. even-cut) matroids will always refer to the signed-graph (resp. graft) representation. We will use the term *matrix representation* to indicate the $(0, 1)$-matrix representing the matroid over the two-element field.

Let $\mathcal{M}$ be a class of binary matroid. By the *membership* problem for $\mathcal{M}$ we mean the following: we are given a $(0, 1)$-matrix representation A of a binary matroid M, we return either YES if $M \in \mathcal{M}$ and NO if $M \notin \mathcal{M}$. An algorithm for testing membership problem is polynomial if its running time is polynomial in the size of A. The columns of A correspond to the elements $E(M)$ of M and the number of rows of A is given by the rank of M which is bounded by $|E(M)|$. Since A is a $(0, 1)$-matrix, saying that the algorithm runs in time polynomial in the size of A is equivalent to saying that the algorithm runs in time polynomial in the number of elements of M. The main results in the paper are,

(1) A polynomial time algorithm for testing membership in the class of even-cycle matroids when the given matroids are 3-connected;

(2) A polynomial time algorithm for testing membership in the class of even-cut matroids when the given matroids are 3-connected.

A class $\mathcal{M}$ of matroid is *minor closed* if for all $M \in \mathcal{M}$ every minor N of M is in $\mathcal{M}$. Even-cycle and even-cut matroids are minor closed classes of matroids. A binary matroid M is *minimally non-even-cycle* (resp. *minimally non-even-cut*) if it is not an even-cycle (resp. even-cut) matroid but every proper minor of M is an even-cycle (resp. even-cut) matroid. The following theorem is proven by the Matroid Minors Project [8].

Theorem 1.1.1. *Every minor-closed class of binary matroids are well-quasi-ordered.*

[1] We omit the sub-index G when there is no ambiguity.

It follows from Theorem 1.1.1 that minimally non-even-cycle (resp. non-even-cut) matroids have bounded sizes. The algorithm in (1) returns a compact certificate for membership. Namely, if the matroid, say M, being tested is an even-cycle matroid, the algorithm returns a signed-graph that is a representation of M. The algorithm in (2) returns a compact certificate for membership If the matroid, say M, being tested is an even-cut matroid, the algorithm returns a graft that is a representation of M.

In this thesis, we do not give all the details of the algorithms, but we focus on the theoretical underpinnings of the algorithms, namely the fact that we can keep track of the representations. We will outline only how these theoretical results can be used to develop polynomial time algorithms. We prove (1) and (2) are polynomial, but we do not work out the exact bound. Furthermore, the polynomial bound will depend on some constant efficient c that arises from the Matroid Minors Project and that has no explicit bound (see [8]). However, the algorithm does not use the value c for its computation.

1.2 The case of graphic matroids

Before identifying some of the challenges in designing a polynomial time algorithm for testing membership in even-cycle or even-cut matroid, we will sketch how the membership problem can be solved for the class of graphic matroids. The key property is the following result of Whitney in [17] that under some mild connectivity assumption, graphic matroids have a unique representation,

Theorem 1.2.1. *If M is a 3-connected graphic matroid there exists a unique graph G with cycle$(G) = M$.*[2]

We review the definition of k-connected matroids in Section 2.3.1. Given a matroid M and $I, J \subseteq E(M)$ where $I \cap J = \emptyset$, the matroid obtained by deleting elements I and contracting elements J is denoted by $M \setminus I/J$. Note, that the order in which the minor operations are conducted do not matter so the notation is well defined. Similarly, we denote by $G \setminus I/J$ the graph obtained from G by deleting edges I and contracting edges J. There is a one-to-one correspondence between minor operations in a graphic matroid and minor operations in its representation, namely,

Remark 1.2.2. *Let $I, J \subseteq E(G)$ where $I \cap J = \emptyset$. Then cycle$(G) \setminus I/J =$ cycle$(G \setminus I/J)$.*

[2]We will consider two graphs that only differ in vertex labels or isolated vertices to be identical.

Note, that the Remark 1.2.2 implies in particular that the class of graphic matroids is closed under minors. If M' is obtained from M by contracting (resp. deleting) some element e we say that M is obtained from M' by *uncontracting* (resp. *undeleting*) element e. We use the analogous definition for graphs as well. Suppose M is a graphic representation with representation G and let $M' = M \setminus I/J$ be a minor of M. Then by the previous remark, G arises from a representation of M' by undeleting a set of edges I and uncontracting a set of edges J.

We are now ready to sketch an algorithm for testing if a matroid M is graphic. We will limit ourselves to the case where M is 3-connected. In [13] Seymour proved that there exists a sequence of 3-connected matroids $M_1, M_2, \ldots, M_k$ where $M_1, M_2, \ldots, M_k$ are 3-connected, M_k is the graphic matroid of a wheel (possibly K_4) or the non-graphic matroid of a whirl and for every $i \in [k-1]$[3], M_{i+1} is isomorphic to a single element contraction or deletion of M_i. Moreover, this sequence of 3-connected matroids can be constructed in time polynomial in $E(M)$. If M_k is non-graphic, then so is M. We may assume M_k is graphic. Let G_k denote the wheel for which cycle$(G_k) = M_k$. Suppose that iteratively we constructed representations G_i for $i = r, \ldots, k$. If $r = 1$ then $M_1 = M$ is graphic. Otherwise M_{r-1} is obtained from M_r by undeleting (or uncontracting) some element e. If M_{r-1} is graphic then its representation G_{r-1} is obtained by undeleting (or uncontracting) some edge e from a representation of M_r. But by Theorem 1.2.1 the representation of M_r is unique, it is G_r. Thus it suffices to check if we can undelete (or uncontract) an edge of G_r to get a representation of M_{r-1}. We omit a description of this last step but refer the interested reader to [14] and [3] for a detailed explanation of how this can be carried.

We will see that the condition that M be 3-connected cannot be omitted in Theorem 1.2.1. Given a graph G and $X \subseteq E(G)$, we denote by $V_G(X)$ the set of vertices spanned by edges X, i.e. $V_G(X) = V(G[X])$. We denote by $\mathcal{B}_G(X)$ the set $V_G(X) \cap V_G(\bar{X})$[4]; i.e. $\mathcal{B}_G(X)$ is the set of vertices in the boundary of X and $\bar{X}$. Consider a graph G with $X \subseteq E(G)$ and $|X|, |\bar{X}| \geq 2$ where $\mathcal{B}_G(X)$ consists of two vertices u, v. Let G' be obtained by identifying vertex u and v of $G[X]$ with vertices v and u of $G[\bar{X}]$ (in that order). We say that G' is obtained from G by a *Whitney-flip on* X. We also call the operation that consists of identifying two components of a graph, or splitting two blocks of a graph a Whitney flip. It can be readily checked that if G and G' are related by a sequence of Whitney-flips, then they have the same cycles, i.e. they are representations of the same graphic matroid. In [17], Whitney proved the converse namely,

Theorem 1.2.3. *Any two representations of a graphic matroid are related by a sequence of Whitney flips.*

[3] $[k] = \{1, \ldots, k\}$.
[4] $\bar{S}$ denotes the complement of set S.

It is well known that if a graphic matroid M is 3-connected, then every representation G of M is 3-connected. Since there are no Whitney-flip possible in a 3-connected graph, Theorem 1.2.1 follows immediately from Theorem 1.2.3. Theorem 1.2.3 provides a very precise description of the set of all representations of a graphic matroid. We will see that the situation is markedly worse in the case of even-cycle and even-cut matroids.

1.3 What makes the problem difficult?

1.3.1 A first bad example

Unlike graphic matroids, which have a unique graph representation up to Whitney-flips, even-cycle and even-cut matroids may have multiple representations up to Whitney-flips, even though they are 3-connected. Before we see examples, we will introduce a natural ways to attain signed-graph (resp. graft) representations from one signed-graph (resp. graft) representation.

Two graphs G and G' are *equivalent* if G and G' are related by a sequence of Whitney-flips. Since $\text{cycle}(G) = \text{cycle}(G')$ it follows in particular, that for any $\Sigma \subseteq E(G)$ we also have $\text{ecycle}(G, \Sigma) = \text{ecycle}(G', \Sigma)$. The set $\Sigma' \subseteq E(G)$ is a *signature* of (G, Σ) if $\text{ecycle}(G, \Sigma) = \text{ecycle}(G, \Sigma')$. It can be readily checked that Σ' is a signature if and only if $\Sigma' = \Sigma \triangle \delta(U)$ for some $U \subseteq V(G)$. We say that (G, Σ') is obtained by *resigning* (G, Σ). Thus (G, Σ) and (G', Σ') are representations of the same even-cycle matroid if they are related by resigning and Whitney-flips. We will call such a pair of signed-graphs, *equivalent*. Note that this is indeed an equivalence relation (see [10]), thus we can partition the set of all representations of an even-cycle matroid into *equivalence classes*.

We say that a subset of edges J is a *T-join* of G where (G, T) is a graft, if the vertices of odd degree of $G[J]$ is given by T. A graft (G, T) is *equivalent* to a graft (G', T') if G' is equivalent to G and there exists a T-join of G that is a T'-join of G'. Note that as G and G' have the same cycles this implies readily that for every $J \subseteq E(G)$, J is a T-join of G if and only if J is a T'-join of G'. Clearly, if (G, T) and (G', T') are equivalent then $\text{ecut}(G, T) = \text{ecut}(G', T')$ since cocycles of $\text{ecut}(G, T)$ are precisely the cycles of G and the T-joins of (G, T). Note that this is indeed an equivalence class (see [6]), thus we can partition the set of all representations of an even-cut matroid into *equivalence classes*.

Now, we will introduce an example with multiple inequivalent representations. The matroid R_{10} is a 4-connected binary matroid with the following matrix representation in Figure 1.1 (see [7]). The matroid R_{10} is an even-cycle matroid and it has six inequivalent representations that are all isomorphic to the signed-graph $\big(K_5, E(K_5)\big)$(see [11]). We introduce two examples

$$
\begin{array}{cccccccccc}
a & b & c & d & e & f & g & h & i & j
\end{array}
$$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1
\end{pmatrix}
$$

Figure 1.1: Binary representation of R_{10}.

in Figure 1.2. The matroid R_{10} is also an even-cut matroid and it has ten inequivalent represen-

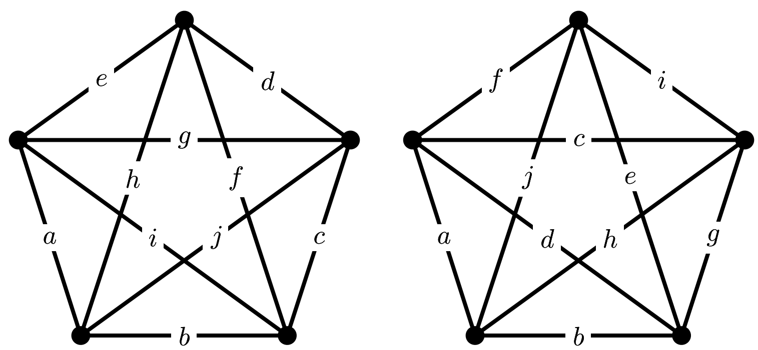

Figure 1.2: Inequivalent signed-graph representations of R_{10}. All edges are odd. The edge labelling is the same as in Figure 1.1.

tations that are all isomorphic to the graft given in Figure 1.3. We introduce two examples in Figure 1.3. Figure 1.2 and Figure 1.3 show that it is possible for a 4-connected even-cycle (or even-cut) matroid to have multiple inequivalent representations. Thus unlike the case of graphic matroid, a membership algorithm for either even-cycle of even-cut matroids would need to keep track of multiple representations. This would not be a problem as long as the number of pairwise inequivalent representations is bounded by a polynomial in the number of elements of matroids. Alas we will see that this is not case for the class of even-cycle matroid, nor is it the case for the class of even-cut matroids.

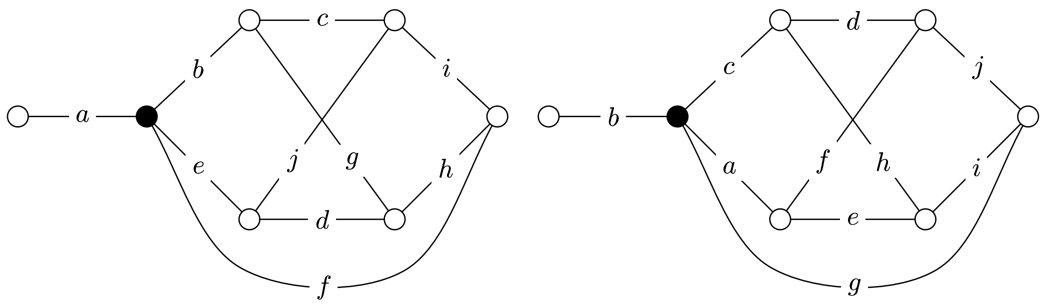

Figure 1.3: Inequivalent graft representations of R_{10}. White vertices are terminals. The edge labelling is the same as in Figure 1.1

1.3.2 An even-cycle matroid with an exponential number of inequivalent representations

In this section, we will introduce an even-cycle matroid that has an exponential number of pairwise inequivalent representations. A *mixed graph* is a graph with both edges and directed edges. Let $\vec{H}$ be a mixed graph with exactly two directed edges, say L and R, that are disjoint. Construct G from $\vec{H}$ by identifying the tail of L with the tail of R; by identifying the head of L with the head of R; and by removing L and R. Denote by s_1 (resp. s_2) the tail (resp. head) of L in $\vec{H}$. Let $\Sigma := \delta_{\vec{H}}(s_1) \triangle \delta_{\vec{H}}(s_2)$. Then (G, Σ) is a signed graph. Suppose we have a sets $X_0, X_1, \ldots, X_n$ where $n \geq 2$ with the following properties: (i) $X_0 = \{L\}$, $X_n = E(\vec{H}) \setminus \{R\}$; (ii) for all $i \in [n]$, $X_{i-1} \subset X_i$, $V_{\vec{H}}(X_{i-1}) \subset V_{\vec{H}}(X_i)$ and $|\mathcal{B}_{\vec{H}}(X_i)| = 2$. For all $i \in [n]$, let $P_i = X_i - X_{i-1}$. We illustrate the construction of (G, Σ) in Figure 1.4. Note that we can construct an example where $n = \Theta(|E(G)|)$ and $ecycle(G, \Sigma)$ is 3-connected. Now, construct a new mixed graph $\vec{H}'$ from $\vec{H}$ by repeatedly doing Whitney-flips on sets X_i for all $i \in I \subseteq [n]$ so that the ends of L and R remain disjoint. Let (G', Σ') be the signed-graph that arises from $\vec{H}'$ by the same way in the construction of (G, Σ). We say that (G, Σ) and (G', Σ) are related by *shuffling* $\{P_i | i \in I\}$. We illustrate this shuffling in Figure 1.5. It can be readily checked in [10] that $ecycle(G, \Sigma) = ecycle(G', \Sigma')$. It is now straightforward to see that we can have an exponential number (in the number of elements) of inequivalent signed-graph representations all related by shuffling subsets of edges. Hence, we can have an exponential number of pairwise inequivalent representations for an even-cycle matroid. In particular, it implies that a polynomial time recognition algorithm for even-cycle matroids cannot record the set of all pairwise inequivalent representations.

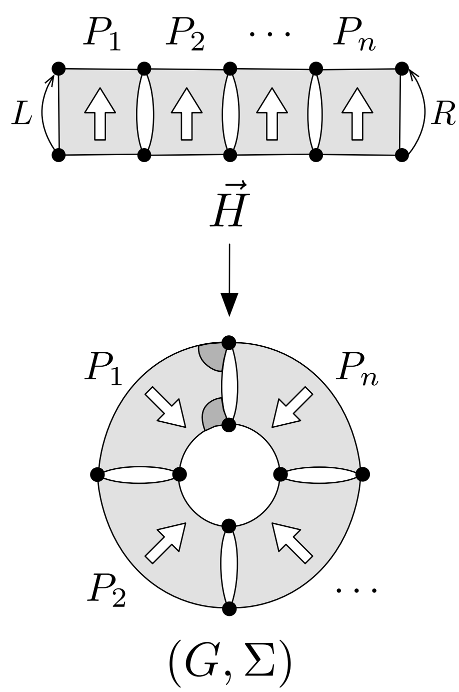

Figure 1.4: Construction of the example in Section 1.3.2. Shaded edges are odd.

1.3.3 An even-cut matroid with exponential number of inequivalent representations

In this section, we will introduce an even-cut matroid that has an exponential number of inequivalent representations. Consider a graft (G, T) where $|T| = 4$. Suppose that we have a partition $P_1, P_2, \ldots, P_n$ of $E(G)$ and for every $i \in [n]$, we have $\mathcal{B}_G(P_i) = T$. We illustrate the construction of (G, T) in Figure 1.6. Note that we can construct an example where $n = \Theta(|E(G)|)$ and $ecut(G, T)$ is 3-connected. For every $i \in [n]$ let $G_i = G[P_i]$. Denote by t_1, t_2, t_3, t_4 the terminal vertices T. For every $i \in I \subseteq [n]$ let G'_i be a graph constructed from G_i by relabelling the terminals in arbitrary one of three possible ways, (i) interchange the labels of t_1 and t_2 the labels of t_3 and t_4; (ii) interchange the labels of t_1 and t_3 the labels of t_2 and t_4; (iii) interchange the labels of t_1 and t_4 the labels of t_2 and t_3. Now let G' be obtained from G by identifying vertices t_1 (resp. t_2, t_3, t_4) in each of G_i for $i \in [n]$. Then (G', T) is a graft. We say that (G, T) and (G', T) are related by *shuffling* $\{P_i | i \in I\}$. We illustrate this shuffling in Figure 1.7. It can be readily checked in [6] that $ecut(G, T) = ecut(G', T)$. It is now straightforward to see

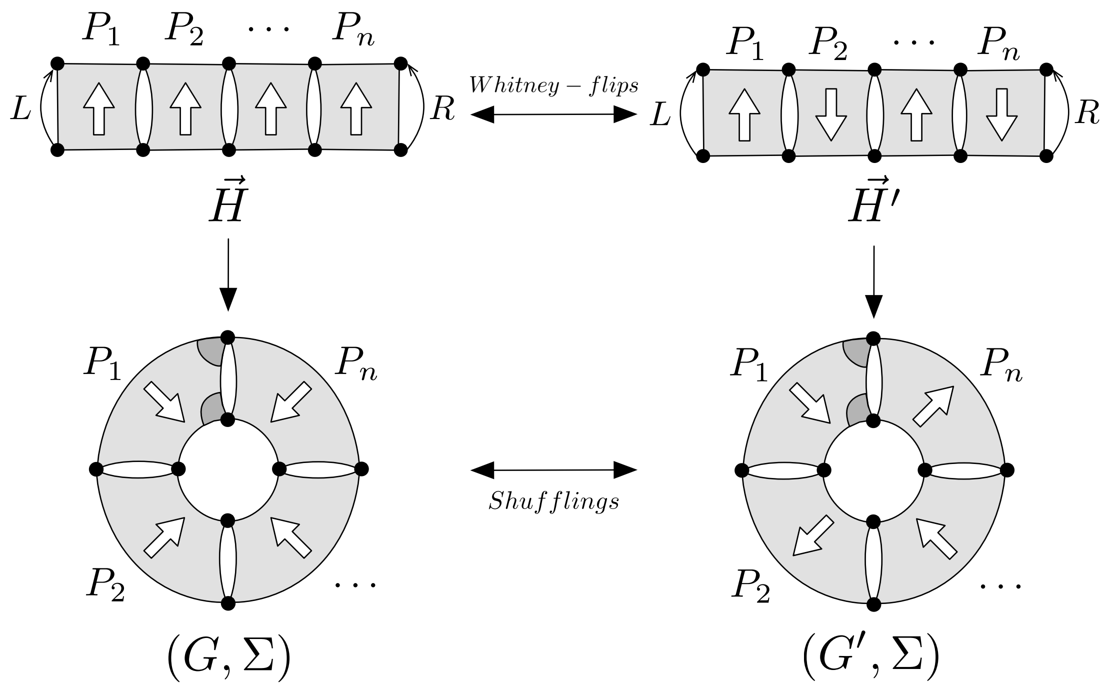

Figure 1.5: Shuffling in the example in Section 1.3.2. Shaded edges are odd.

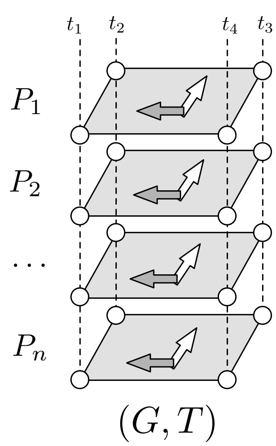

Figure 1.6: Construction of the example in Section 1.3.3. White vertices are terminals. Dotted edges mean identifying vertices.

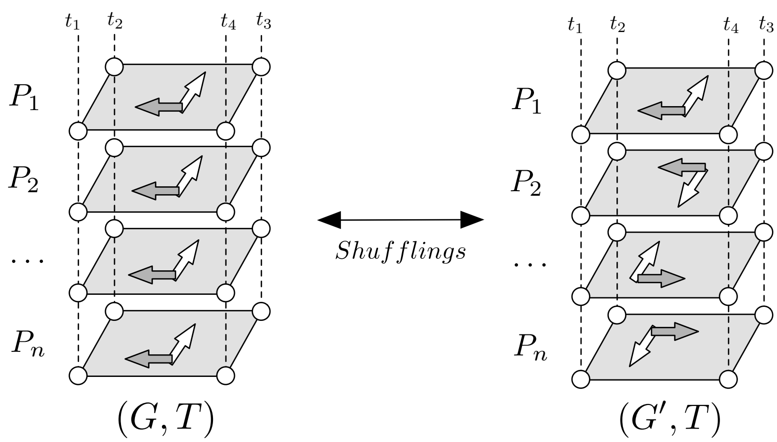

Figure 1.7: Shuffling of the example in Section 1.3.3. White vertices are terminals.

that we can have an exponential number (in the number of elements) of inequivalent graft representations all related by shuffling subsets of edges. Hence, we can have an exponential number of pairwise inequivalent representations for an even-cut matroid. In particular, it implies that a polynomial time recognition algorithm for even-cut matroids cannot record the set of all pairwise inequivalent representations.

1.4 P-graphic and p-cographic matroids

In Section 1.3.2 and Section 1.3.3, we showed that there exist examples that have an exponential number of inequivalent representations. In this section, we will introduce a subclass of even-cycle matroids, *p-graphic* matroids, such that even-cycle matroids that are not p-graphic are nicely behaved. We also introduce a subclass of even-cut matroids, *p-cographic* matroids, such that even-cut matroids that are not p-cographic are nicely behaved.

Before we proceed, we require a number of definitions from [11]. Denote by $\mathrm{loop}(H)$ the set of loops of a graph H. Consider a graph H and a vertex v and $\alpha \subseteq \delta_H(v) \cup \mathrm{loop}(H)$. We say that G is obtained from H by *splitting* v into v_1, v_2 according to α if $V(G) = V(H) - \{v\} \cup \{v_1, v_2\}$ and for every $e = (u, w) \in E(H)$:

(1) if $e \notin \delta_H(v) \cup \mathrm{loop}(H)$, then $e = (u, w) \in E(G)$;

(2) if $e \in \text{loop}(H) - \alpha$, then $e = (u, w) \in E(G)$[5];

(3) if $e \in \text{loop}(H) \cap \alpha$, then $e = (v_1, v_2) \in E(G)$;

(4) if $e \in \delta_H(v) \cap \alpha$ and $w = v$ then $e = (u, v_1) \in E(G)$;

(5) if $e \in \delta_H(v) - \alpha$ and $w = v$ then $e = (u, v_2) \in E(G)$.

Note that if you identify v_1 and v_2 in H you get the graph G back. Consider a signed graph (G, Σ) we say that $u \in V(G)$ is a *blocking vertex* if every odd circuit of (G, Σ) uses u. Equivalently, u is a blocking vertex if there exists a signature Γ of (G, Σ) where $\Gamma \subseteq \delta_G(u) \cup \text{loop}(G)$ (see [11]).

Remark 1.4.1. *If an even-cycle matroid has a representation with a blocking vertex then it is graphic.*

Proof. Let (G, Σ) be the representation. We may assume that $\Sigma \subseteq \delta(u) \cup \text{loop}(G)$ for some vertex u. Let G' be obtained from G by splitting u according to Σ. Then clearly $\text{cycle}(G') = \text{ecycle}(G, \Sigma)$. □

Consider a signed graph (G, Σ) we say that $u, v \in V(G)$ is a *blocking pair* if every odd circuit of (G, Σ) uses at least one of u, v. Equivalently, u, v is a blocking pair if there exists a signature Γ of (G, Σ) where $\Gamma \subseteq \delta_G(u) \cup \delta_G(v) \cup \text{loop}(G)$ (see [11]). We say that an even-cycle matroid M is *pinch-graphic* (or p-graphic for short) if it has a representation that has a blocking pair. We say that (G, Σ) is a *blocking pair* representation or *BP-representation* of M. The name arises from the fact that if we identify a blocking pair u, v of a representation (G, Σ) where $\Sigma \subseteq \delta(u) \cup \delta(v) \cup \text{loop}(G)$ then we obtain a blocking vertex which is, by Remark 1.4.1, a representation of a graphic matroid. In other words, pinch-graphic matroids are one 'pinch' away from being graphic. Observe that examples in Section 1.3.2 have by construction a blocking pair.

Consider a signed graph (G, Σ) and $I, J \subseteq E(G)$ where $I \cap J \neq \emptyset$. If I contains an odd circuit, then let $\Gamma = \emptyset$ otherwise there exists a signature Γ of Σ where $\Gamma \cap I = \emptyset$. Then $(G, \Sigma)/I \setminus J$ denotes the signed graph $(G/I \setminus J, \Gamma)$. Note that minors of signed graph are only defined up to resigning. Analogously to Remark 1.2.2 we have

Remark 1.4.2. *Let $I, J \subseteq E(G)$ where $I \cap J = \emptyset$. Then ecycle$(G, \Sigma)/I \setminus J =$ ecycle$((G, \Sigma)/I \setminus J)$.*

Similarly, for even-cut matroids we have an analogous Remark 1.4.3.

[5] If e is a loop on v in H, then it can be arbitrarily chosen to be a loop at v_1 or at v_2 in G.

Remark 1.4.3. *If an even-cut matroid has a representation with exactly two terminals then it is cographic.*

Proof. Let (G, T) be a representation where $T = \{t_1, t_2\}$. Let G' be obtained from G by identifying t_1 and t_2. Then clearly cut$(G') =$ ecut(G, T). $\square$

We say that an even-cut matroid M is *pinch-cographic* (or p-cographic for short) if it has a representation (G, T) where $|T| \leq 4$. We say that (G, T) is a T_4*-representation* of M. The name arises from the fact that if we identify say $t_1, t_2 \in T$ in that representation, we obtain a representation with only at most two terminals which is, by Remark 1.4.3, a representation of a cographic matroid. In other words, p-cographic matroids are one 'pinch' away from being cographic. Observe that examples in Section 1.3.3 have by construction four terminals.

Consider a graft (G, T) and $I, J \subseteq E(G)$ where $I \cap J \neq \emptyset$. Let $G' = G/I \setminus J$. If J contains an odd cut then $(G, T)/I \setminus J$ denotes $(G', \emptyset)$. Otherwise then there exists a T-join L of G that is disjoint from J and $(G, T)/I \setminus J$ denotes (G', T') where T' is the set of vertices of odd degree of $G'[L - I]$. Analogously to Remark 1.4.2 we have

Remark 1.4.4. *Let $I, J \subseteq E(G)$ where $I \cap J = \emptyset$. Then ecut$(G, T)/I \setminus J =$ ecut$((G, T) \setminus I/J)$.*

1.5 Thesis Overview

In the rest of the thesis, we will introduce two steps for solving the membership problems for even-cycle and even-cut matroids. The main idea starts from the following nice properties in Theorem 1.5.1 and Theorem 1.5.2.

Theorem 1.5.1. *There exists a constant c such that every 3-connected even-cycle matroid that is not p-graphic has at most c inequivalent representations.*

Theorem 1.5.2. *There exists a constant c such that every 3-connected even-cut matroid that is not p-cographic has at most c inequivalent representations.*

We postpone the proof of Theorem 1.5.1 and Theorem 1.5.2 until Section 2.4. According to these theorems, to keep track of all representations up to equivalence classes is a good strategy as the number of equivalence classes is bounded by a constant. In Chapter 2, by relying on these nice properties, we assume that we have polynomial time algorithms that recognize p-graphic and p-cographic matroids when the given matroids are 3-connected. Then we will solve the weakened membership problems for even-cycle and even-cut matroids using these algorithms, namely,

(1) Assuming the existence of a polynomial time algorithm recognizing p-graphic matroids, we show how to design a polynomial time algorithm for testing membership for the class of even-cycle matroids when the given matroids are 3-connected;

(2) Assuming the existence of a polynomial-time algorithm recognizing p-cographic matroids, we show how to design a polynomial time algorithm for testing membership for the class of even-cut matroids when the given matroids are 3-connected.

To complete our algorithms, we need algorithms to recognize p-graphic matroids and p-cographic matroids. However, membership problems for p-graphic and p-cographic are equivalent since they are dual to each other.

Proposition 1.5.3. *A matroid M is p-graphic if and only if dual of M is p-cographic.*

We postpone the proof of Proposition 1.5.3 until Section 3.1. In Chapter 3, we will solve the membership problem for p-cographic matroids, namely,

(3) We show how to design a polynomial time algorithm for testing membership for the class of p-cographic matroids when the given matroids are 3-connected.

Chapter 2

Recognizing even-cycle and even-cut matroids

The goal of this chapter is to solve the membership problems for even-cycle and even-cut matroids when the given matroids are 3-connected. In this chapter, we assume that there exist algorithms for recognizing p-graphic and p-cographic matroids, respectively. These recognition algorithms will be explained in Chapter 3. We also assume that these algorithms return a compact certificate for membership or non-membership. We will use these algorithms to solve membership problem for even-cycle and even-cut matroids. In Section 2.1 and 2.2, we give brief outlines of algorithms for recognizing even-cycle and even-cut matroids. In Section 2.3, we will introduce concepts of matroid connectivity and some ingredients to prove the main results that are used in the algorithms. The proofs for these results will be in Section 2.4.

2.1 An algorithm for recognizing even-cycle matroids

To solve the membership problem for even-cycle matroids, we will use a similar strategy for the membership problem for graphic matroids as the one mentioned in Section 1.2. In this section, we give a brief overview of these algorithms as this will also be helpful in outlining the membership algorithm for even-cycle and even-cut matroids. We start with the following observation,

Proposition 2.1.1. *Let $\mathcal{M}$ be a minor closed class of binary matroids and suppose that we have a polynomial time algorithm to solve the membership problem for $\mathcal{M}$ when the given matroids are 3-connected. Then given any 3-connected binary matroid $M \notin \mathcal{M}$ described by its $(0,1)$-matrix* *representation, we can find in polynomial time a 3-connected minor N of M that is minimally not in $\mathcal{M}$.*

Proof. Pick $e \in E(M)$. We check if $M/e \notin \mathcal{M}$ and M/e is 3-connected. If it is, we define M' as M/e. Otherwise we check if $M \setminus e \notin \mathcal{M}$ and $M \setminus e$ is 3-connected. If it is, we define M' as $M \setminus e$. We repeat for every element $e \in M$. If M' is not defined, then return $N = M$. Otherwise, we repeat the above procedure with M'. We recursively find minors of M, until we find a 3-connected minor N of M that is minimally not in $\mathcal{M}$. □

Observe that if a signed graph (G, Σ) has a blocking pair, say u, v, then for every minor (G', Σ') the vertices u', v' corresponding to respectively u, v will form a blocking pair. This proves in particular, that the class of p-graphic matroids is minor closed.

Let M be an even-cycle matroid and let $N = M/I \setminus J$ be a minor of M. Suppose that (G, Σ) is a representation of M. It follows from Remark 1.4.2 that $(H, \Gamma) := (G, \Sigma)/I \setminus J$ is a representation of N. We say that representation (H, Γ) of N *extends* to the representation (G, Σ) of M. Remark 1.4.2 implies that every representation of M is obtained by extending some representation of N. It is possible, however, that a single representation of N extends to several inequivalent representations of M. If N is a minor of a matroid M then M is a *major* of N. The following results are proved in [10],

Proposition 2.1.2. *Let M be a matroid and let $N = M \setminus e$ for some element e. Suppose that N is an even-cycle matroid. Let $\mathcal{F}$ be an equivalence class of representations of N. Let $\mathcal{F}'$ be the set of all extensions of $\mathcal{F}$ to M. Then $\mathcal{F}'$ is contained in at most one equivalence class of the representations of M.*

Proposition 2.1.3. *Let M be a matroid without loops and coloops, and let $N = M/e$ for some element e. Suppose that N is an even-cycle matroid that is not graphic. Let $\mathcal{F}$ be an equivalence class of representations of N. Let $\mathcal{F}'$ be the set of all extensions of $\mathcal{F}$ to M. Then $\mathcal{F}'$ is contained in at most two equivalence classes of the representations of M.*

Recall that an equivalence class for an even-cycle matroid is given by the set of all signed graphs that are equivalent to a fixed signed graph that is a representation of that matroid. Partition the set of all signed-graph representations into equivalence classes. Let $\mathcal{E}(M)$ be obtained by selecting one signed-graph from each equivalence classe, i.e. $\mathcal{E}(M)$ is a maximal collection of representations of M that are pairwise inequivalent. The membership algorithm will keep track of $\mathcal{E}(M)$ for even-cycle matroids that are not p-graphic.

Now let us outline how having a polynomial algorithm for membership in the class of p-graphic yields a polynomial algorithm for membership in the class of even-cycle matroids. Note that we assume for such an algorithm that if the answer is YES then the algorithm returns a representation that has a blocking pair. Consider a 3-connected binary matroid M given by a $(0, 1)$-matrix representation. We check if M is p-graphic; if it is, then it is also an even-cycle matroid. Moreover, the membership algorithm returns a representation (G, Σ). In fact, (G, Σ) has a blocking pair. We can then return (G, Σ) as a certificate and stop. Otherwise, because of Proposition 2.1.1 we can find a minor N of M that is minimal with respect to (i) N is non p-graphic and (ii) N is 3-connected. However, unlike the membership problem for graphic matroids in Section 1.2, we cannot use Theorem 1.1.1 to find a small minor of M, since 3-connectivity is not minor-closed. Instead, we have the following similar result.

Theorem 2.1.4. *There exists a constant c such that if a binary matroid N is minimal with respect to the following properties,*

(1) N is non-p-graphic, and

(2) N is 3-connected,

then $|E(N)| < c$.

We will postpone the proof of Theorem 2.1.4 until Section 2.4. It follows from Theorem 2.1.4 that the size of N is a constant independent of M. Then, we check if N an even-cycle matroid (it has constant size so this can be done in constant time). If it is not, then M is not an even-cycle matroid either and we can stop. Otherwise, we can use brute force to find $\mathcal{E}(N)$. As M and N are 3-connected, [13] implies that we can construct a sequence of 3-connected matroids $M_1, \ldots, M_k$ where $M = M_1$, $M_k = N$, and for every $i \in [k-1]$, M_{i+1} is obtained from M_i by deleting or contracting an element, say e_i of M_i. Inductively, we constructed $\mathcal{E}(M_{i+1})$. Using Proposition 2.1.2 and Proposition 2.1.3, we can construct $\mathcal{E}(M_i)$ from $\mathcal{E}(M_{i+1})$ in time polynomial in the size of M_i and the cardinality of $\mathcal{E}(M_i)$. However, the cardinality of $\mathcal{E}(M_i)$ is bounded by a constant because of Theorem 1.5.1. It follows that $\mathcal{E}(M_i)$ can be constructed in polynomial time in the size of M_i. If $\mathcal{E}(M_i) = \emptyset$ then M_i is not an even-cycle matroid and neither is M. Otherwise, if $i = 1$ then $M_i = M$ is an even-cycle matroid with representations $\mathcal{E}(M_i)$, and if $i > 1$ then we recursively construct $\mathcal{E}(M_{i-1})$.

2.2 An algorithm for recognizing even-cut matroids

We can proceed in a similar way to obtain a polynomial membership algorithm for even-cut matroid using a polynomial recognition algorithm for p-cographic matroids. Observe that if a graft (G, T) has at most four terminals then so does every minor (G', T') of (G, T). This proves in particular, that the class of p-cographic matroids is minor closed.

Let M be an even-cut matroid and let $N = M/I \setminus J$ be a minor of M. Suppose that (G, T) is a representation of M. It follows from Remark 1.4.4 that $(G', T') := (G, T) \setminus I/J$ is a representation of N. We say that representation (G', T') of N *extends* to the representation (G, T) of M. Remark 1.4.4 implies that every representation of M is obtained by extending some representation of N. It is possible, however, that a single representation of N extends to several inequivalent representations of M. Recall that an equivalence class for an even-cut matroid is given by the set of all grafts that are equivalent to a fixed graft that is a representation of that matroid. The following result appears in [6],

Proposition 2.2.1. *Let M be a matroid and let $N = M \setminus e$ for some element e. Suppose that N is an even-cut matroid. Let $\mathcal{F}$ be an equivalence class of representations of N. Let $\mathcal{F}'$ be the set of all extensions of $\mathcal{F}$ to M. Then $\mathcal{F}'$ is contained in at most one equivalence class of the representations of M.*

Proposition 2.2.2. *Let M be a matroid without loops and coloops, and let $N = M/e$ for some element e. Suppose that N is an even-cut matroid that is not cographic. Let $\mathcal{F}$ be an equivalence class of representations of N. Let $\mathcal{F}'$ be the set of all extensions of $\mathcal{F}$ to M. Then $\mathcal{F}'$ is contained in at most two equivalence classes of the representations of M.*

Let M be an even-cut matroid. Partition the set of all graft representations into equivalence classes. Let $\mathcal{E}(M)$ be obtained by selecting one graft from each equivalence class, i.e. $\mathcal{E}(M)$ is a maximal collection of representations of M that are pairwise inequivalent. The membership algorithm will keep track of $\mathcal{E}(M)$ for even-cut matroids that are not p-cographic.

Now let us outline how having a polynomial algorithm for membership in the class of p-cographic yields a polynomial algorithm for membership in the class of even-cut matroids. Note we assume for such an algorithm that if the answer is YES then the algorithm returns a representation that has 4 terminals. Consider a 3-connected binary matroid M given by a $(0, 1)$-matrix representation. We check if M is p-cographic; if it is, then it is also an even-cut matroid. Moreover, the membership algorithm returns a representation (G, T). In fact, $|T| = 4$. We can then return (G, T) as a certificate and stop. Otherwise, because of Proposition 2.1.1 we can find a minor N of M that is minimal with respect to (i) N is non p-cographic and (ii) N is 3-connected. The following theorem follows the Theorem 2.1.4 since p-cographic matroids are dual of p-graphic matroids by Proposition 1.5.3,

Theorem 2.2.3. *There exists a constant c such that if a binary matroid N is minimal with respect to the following properties,* *(1)* N *is non-p-cographic, and*

(2) N *is 3-connected,*

then $|E(N)| < c$*.*

It follows from theorem 2.2.3 that the size of N is a constant independent of M. Then, we check if N an even-cut matroid (it has constant size so this can be done in constant time). If it is not, then M is not an even-cut matroid either and we can stop. Otherwise, we can use brute force to find $\mathcal{E}(N)$. As M and N are 3-connected, [13] implies that we can construct a sequence of 3-connected matroids $M_1, \ldots, M_k$ where $M = M_1$, $M_k = N$, and for every $i \in [k-1]$, M_{i+1} is obtained from M_i by deleting or contracting an element, say e_i of M_i. Inductively, we constructed $\mathcal{E}(M_{i+1})$. Using Proposition 2.2.1 and Proposition 2.2.2, we can construct $\mathcal{E}(M_i)$ from $\mathcal{E}(M_{i+1})$ in time polynomial in the size of M_i and the cardinality of $\mathcal{E}(M_i)$. However, the cardinality of $\mathcal{E}(M_i)$ is bounded by a constant because of Theorem 1.5.2. It follows that $\mathcal{E}(M_i)$ can be constructed in polynomial time in the size of M_i. If $\mathcal{E}(M_i) = \emptyset$ then M_i is not an even-cut matroid and neither is M. Otherwise, if $i = 1$ then $M_i = M$ is an even-cut matroid with representations $\mathcal{E}(M_i)$, and if $i > 1$ then we recursively construct $\mathcal{E}(M_{i-1})$.

2.3 Matroid connectivity

2.3.1 The connectivity function

Tutte [16] introduced connectivity and separations in matroids. In this section, we review these definitions and present their applications to even-cycle and even-cut matroids. Let M be a matroid with a rank function r. For $X \subseteq E(M)$, the connectivity function is defined as $\lambda_M(X) = r(X) + r(\bar{X}) - r(M)$. A partition $(X, \bar{X})$ of $E(M)$ is k-*separating* if $\lambda_M(X) \leq k-1$. It is *exactly* k-separating when equality holds. A partition $(X, \bar{X})$ is a k-separation if it is exactly k-separating and $|X|, |\bar{X}| \geq k$. M is k-connected if it has no r-separations for any $r < k$. We simply say that M is connected if M is 2-connected. Let G be a connected graph and let $X \subseteq E(G)$. The partition $(X, \bar{X})$ is a k-*separation* of G if $|X|, |\bar{X}| \geq k$, $|\mathcal{B}_G(X)| = k$ and both $G[X]$ and $G[\bar{X}]$ are connected. Note that with this definition two parallel edges of G form a 2-separation of G. A connected graph G is k-*connected* if it has no r-separations for any $r < k$.

A signed-graph is *bipartite* if all its cycles are even. The number of components of a graph G is denoted by $\kappa(G)$. The next propositions from [10] describes the connectivity function for even-cycle matroids.

Proposition 2.3.1. *Let (G, Σ) be a non-bipartite signed-graph where G is connected and let (X_1, X_2) be a partition of $E(G)$. For $i = 1, 2$ let $p_i = 0$ if $(G[X_i], \Sigma \cap X_i)$ is bipartite and let $p_i = 1$ otherwise. Then*

$$\lambda_{\mathit{ecycle}(G,\Sigma)}(X_i) = |\mathcal{B}_G(X_1)| + p_1 + p_2 - \kappa(G[X_1]) - \kappa(G[X_2]).$$

We will use the following application of this proposition,

Proposition 2.3.2. *Suppose that ecycle(G, Σ) is 3-connected. Let S denote the set of all odd loops. Then*

(1) (G, Σ) *has no even loop,*

(2) $|S| \leq 1$ *and*

(3) $G \setminus S$ *is 2-connected.*

Moreover, if G has a 2-separation $(X, \bar{X})$, then $(G[X], \Sigma \cap X)$ and $(G[\bar{X}], \Sigma \cap \bar{X})$ are both non-bipartite.

The next propositions [6] describes the connectivity function for even-cut matroids.

Proposition 2.3.3. *Let (G, T) be a graft where $T \neq \emptyset$ and G is connected and let X_1, X_2 be a partition of $E(G)$. For $i = 1, 2$ let $p_i = 0$ if $(G, T)/\bar{X}_i$ has no odd cut and let $p_i = 1$ otherwise. Then*

$$\lambda_{\mathit{ecut}(G,T)}(X_i) = |\mathcal{B}_G(X_1)| + p_1 + p_2 - \kappa(G[X_1]) - \kappa(G[X_2]).$$

Given a separation X of G, we define the *interior* of X in G to be $\mathcal{I}_G(X) = V_G(X) - \mathcal{B}_G(X)$. We will use the following application of this proposition,

Proposition 2.3.4. *Suppose that ecut(G, T) is 3-connected. Let S denote the set of all odd bridges. Then*

(1) (G, T) *has no even bridge,*

(2) $|S| \leq 1$ *and*

(3) G/S *is 2-connected.*

Moreover, if G has a 2-separation $(X, \bar{X})$ then $T \cap \mathcal{I}_{G/\bar{X}}(X)$ and $T \cap \mathcal{I}_{G/X}(\bar{X})$ are both non-empty.

2.3.2 2-Sums in p-graphic matroids

Let M_1 and M_2 be binary matroids that have exactly one element, say e, in common. The *2-sum* $M = M_1 \oplus_2 M_2$ of M_1 and M_2 is defined as the matroid with elements $E(M_1) \triangle E(M_2)$ and circuits that are either circuits of M_i avoiding e for $i \in \{1, 2\}$ or; are of the form $C_1 \triangle C_2$ where for $i \in \{1, 2\}$, C_i is a circuit of M_i using e. We will use the following results from [9].

Proposition 2.3.5. *Let M be a binary matroid and let $X \subseteq E(M)$. Then $(X, \bar{X})$ is a 2-separation of M if and only if there exists M_1, M_2 with $E(M_1) - E(M_2) = X$ and $M = M_1 \oplus_2 M_2$. Moreover, if M is 2-connected then M_1 and M_2 are minors of M.*

Next we present two natural ways of constructing 2-sums of even-cycle matroids in terms of representations. Let (G_1, Σ) be a signed graph and let G_2 be a graph. Suppose that $E(G_1) \cap E(G_2) = \{e\}$ where e is not a loop of G_1 or G_2 and suppose also that $e \notin \Sigma$. Let G be obtained from G_1 and G_2 by identifying e and then deleting e (we can get two possible graphs in that way and these graphs are related by a single Whitney-flip). Then it can be readily checked that,

$$\text{ecycle}(G, \Sigma) = \text{ecycle}(G_1, \Sigma) \oplus_2 \text{cycle}(G_2). \tag{2.1}$$

We then say that (G, Σ) is obtained from (G_1, Σ) and G_2 by *summing on an edge*.

Let (G_1, Σ_1) and (G_2, Σ_2) be non-bipartite signed graphs. Suppose that $E(G_1) \cap E(G_2) = \{e\}$ where e is an odd loop of both (G_1, Σ_1) and (G_2, Σ_2). Let G be obtained by taking the disjoint union of G_1 and G_2 and deleting the loop e. Let $\Sigma = \Sigma_1 \triangle \Sigma_2$. Then it can be readily checked that,

$$\text{ecycle}(G, \Sigma) = \text{ecycle}(G_1, \Sigma_1) \oplus_2 \text{ecycle}(G_2, \Sigma_2). \tag{2.2}$$

We then say that (G, Σ) is obtained from (G_1, Σ_1) and (G_2, Σ_2) by *summing on a loop*.

By Proposition 2.3.5, to understand 2-sums in p-graphic matroids, we need to understand 2-separations in p-graphic matroids. Before we proceed, we require a definition. Let G be a graph and let (E_1, E_2) be a partition of $E(G)$. Then we define *auxiliary graph* $\tilde{G}$ of G according to E_1, E_2 as follows:

(1) each component of $G[E_1], G[E_2]$ is a vertex of $\tilde{G}$.

(2) each vertex of G that is incident to two components of $G[E_1], G[E_2]$ is an edge between corresponding vertices in $\tilde{G}$.

Note that every auxiliary graphs are bipartite.

Proposition 2.3.6. *Let M be an even-cycle matroid that is not graphic where (X_1, X_2) is a 2-separation of M. Let (G', Σ) be a representation of M. Then there exists a graph G equivalent to G' and its auxiliary graph $\tilde{G}$ according to (X_1, X_2) where for $i \in \{1, 2\}$, $G_i = G[X_i]$ and $\Sigma_i = \Sigma \cap X_i$, such that either*

(1) both (G_1, Σ_1) and (G_2, Σ_2) are non-bipartite and $\tilde{G}$ is a 1-path graph,

(2) exactly one of (G_1, Σ_1), (G_2, Σ_2) is non-bipartite and $\tilde{G}$ is a 2-circuit graph,

(3) both (G_1, Σ_1) and (G_2, Σ_2) are bipartite and $\tilde{G}$ is a union of two 2-circuit graphs sharing one vertex in common.

Moreover, in (3), let G_1 be the corresponding graph to a vertex of degree 4 in the auxiliary graph and let G_2 be the union of two components C_1 and C_2 respectively corresponding to two vertices of degree 2 in the auxiliary graph. Let $\mathcal{B}_G(E(C_1)) = \{a_1, a_2\}$ and let $\mathcal{B}_G(E(C_2)) = \{b_1, b_2\}$. Then $(\delta(a_1) \cup \delta(b_1)) \cap X_1$ is a signature of Σ.

Proof. Let G be a graph equivalent to G' such that $\tilde{G}$ has the minimum number of vertices. Since G is connected, $\tilde{G}$ is connected. Let (A_1, A_2) be a bipartition of $\tilde{G}$ where for $i \in \{1, 2\}$, A_i are components of G_i.

Claim 1. *There exists a cut-edge in $\tilde{G}$ if and only if (1) occurs.*

Proof. Suppose that there exists a cut-edge $e = uv$ of $\tilde{G}$ where $u \in A_1$ and $v \in A_2$. Then deleting e divides $\tilde{G}$ into two components C_1 containing u and C_2 containing v. By the way of contradiction, let us assume $|V(\tilde{G})| \geq 3$. We may assume that C_1 contains at least two vertices. Then there exists a vertex $w \in A_2$ in C_1. Let H be a graph obtained from G by splitting the vertex corresponding to e according to u and v, and identifying one vertex of the component corresponding to w and one vertex of the component corresponding to v. Then H is a graph equivalent to G where $\tilde{H}$ has less number of vertices than $\tilde{G}$, giving a contradiction. Thus, u, v are the only vertices and $\tilde{G}$ is a 1-path graph. Since $\lambda_M(E(M_1)) = 1$, by Proposition 2.3.1, both (G_1, Σ_1), (G_2, Σ_2) are non-bipartite. The opposite direction is trivial. $\Diamond$

Claim 2. *There is no path (v_1, v_2, v_3, v_4) of length 3 in $\tilde{G}$ such that $deg(v_2) = deg(v_3) = 2$. Also, there is no 3-circuit (v_1, v_2, v_3) in $\tilde{G}$ such that $deg(v_2) = deg(v_3) = 2$.*

Proof. Suppose that there exists a path (v_1, v_2, v_3, v_4) in $\tilde{G}$. Let $X = E(v_2) \cup E(v_3)$ in G. Since $(X, \bar{X})$ is a 2-separation of G, we can perform a Whitney-flip on X to obtain an auxiliary graph with less number of vertices, giving a contradiction. We can use the same argument for the second result. $\Diamond$

Let $k = |\mathcal{B}_G(E(G_1))|$. For $i \in \{1,2\}$, let $p_i = 0$ if (G_i, Σ_i) is bipartite and let $p_i = 1$ otherwise. By Proposition 2.3.1, $k = \kappa(G_1)+\kappa(G_2)+1-p_1-p_2$. Thus, $|E(\tilde{G})| = k = |V(\tilde{G})|+1-p_1-p_2$. There are the following cases.

Case 1. Both (G_1, Σ_1) and (G_2, Σ_2) are non-bipartite.
Since $p_1 = p_2 = 1$, we have $|E(\tilde{G})| = |V(\tilde{G})| - 1$. Thus, $\tilde{G}$ is a tree. By Claim 1, (1) occurs.

Case 2. Exactly one of (G_1, Σ_1) and (G_2, Σ_2) is bipartite.
We may assume (G_1, Σ_1) is bipartite. Since $p_1 = 0$ and $p_2 = 1$, $|E(\tilde{G})| = |V(\tilde{G})|$. By Claim 1, there is no vertex of degree 1. Thus, $\tilde{G}$ is a circuit. By Claim 2, $\tilde{G}$ is a 2-circuit and (2) occurs.

Case 3. Both (G_1, Σ_1) and (G_2, Σ_2) are bipartite.
Since $p_1 = p_2 = 0$, $|E(\tilde{G})| = |V(\tilde{G})| + 1$. By Claim 1, there is no vertex of degree 1. Thus, in $\tilde{G}$, either every vertex has degree 2 except two vertices of degree 3, or every vertex has degree 2 except one vertex of degree 4.

Claim 3. *Every vertex has degree 2 except one vertex of degree 4 in $\tilde{G}$.*

Proof. Suppose that every vertex has degree 2 except two vertices of degree 3 in $\tilde{G}$. Let u, v be vertices of degree 3. Then, there exists a path P_1 between u and v, and consider $H = \tilde{G} \backslash P_1$. Since every vertex in H except isolated vertices has degree 2, H is a circuit or a union of two disjoint circuits. If H is a union of two disjoint cycles, then each edge of P_1 is a cut-edge contradicting Claim 1. Thus, H is a cycle containing both u, v, so $\tilde{G}$ is a union of three internally disjoint u, v-paths P_1, P_2, P_3. By Claim 2, length of P_1, P_2, P_3 is at most 2. Suppose that u, v are contained in the different partition of (A_1, A_2). Then, P_1, P_2, P_3 are 1-paths and for $i \in \{1,2\}$, G_i is connected. Let $\mathcal{B}_G(X_1) = \{v_1, v_2, v_3\}$. Since (G_2, Σ_2) is bipartite, we may assume $\Sigma = \Sigma_1$. Since (G_1, Σ_1) is bipartite, Σ_1 is a cut of G_1. If this cut does not separate v_1, v_2, v_3 in G_1, then (G, Σ) is bipartite, giving a contradiction. If this cut separate v_1, v_2, v_3 (say, v_1 and v_2, v_3), then v_1 is a blocking vertex, giving a contradiction by Remark 1.4.1. Thus, u, v are contained in the same partition of (A_1, A_2). Thus, P_1, P_2, P_3 are 2-paths. We can use the similar argument to show that (G, Σ) is graphic, giving a contradiction. $\Diamond$

By Claim 3, every vertex has degree 2 except one vertex u of degree 4 in $\tilde{G}$. $\tilde{G} \backslash u$ is a union of disjoint two paths. Thus, $\tilde{G}$ is a union of two circuits sharing one vertex in common. By Claim 2, these circuits are 2-circuits and (3) occurs. We can use the similar argument in Claim 3 to prove the last part. $\square$

Now we can prove the following Proposition 2.3.7 and Proposition 2.3.8 about a 2-sum in p-graphic matroids.

Proposition 2.3.7. *Let M be a p-graphic matroid that is not graphic. Suppose there exist matroids M_1 and M_2 such that $E(M_1) \cap E(M_2) = \{e\}$ and $M = M_1 \oplus_2 M_2$. Then for some $i \in \{1, 2\}$, M_i is p-graphic and non-graphic but M_{3-i} is graphic.*

Proof. Proposition 2.3.5 implies that M_1, M_2 are minor of M. It follows that M_1, M_2 are both p-graphic. Moreover, if M_1, M_2 are both graphic then so is M, a contradiction. It suffices to show that for some $i \in \{1, 2\}$, M_i is graphic. Let (G', Σ) be a BP-representation of M and for $i \in \{1, 2\}$, let $X_i = E(M_i) - \{e\}$. Apply Proposition 2.3.6 to M and (G', Σ) with the 2-separation (X_1, X_2) and let G be the result graph. For $i \in \{1, 2\}$, let $G_i = G[X_i]$ and let $\Sigma_i = \Sigma \cap X_i$.

Case 1. The outcome (1) occurs.
Then, (G, Σ) is obtained by summing on a loop from (G_1, Σ_1) and (G_2, Σ_2) where for $i \in \{1, 2\}$, $M_i = \text{ecycle}(G_i, \Sigma_i)$. Suppose that none of (G_1, Σ_1) and (G_2, Σ_2) is graphic. By Remark 1.4.1, for $i \in \{1, 2\}$, (G_i, Σ_i) has at least two vertices to intersect all odd circuits. Thus, (G, Σ) require at least three vertices to intersect all odd circuits (as (G_1, Σ_1) and (G_2, Σ_2) share at most one vertex). But this contradicts the fact that (G, Σ) is a BP-representation.

Case 2. The outcome (2) occurs.
Then, (G, Σ) is obtained by summing on an edge from (G_1, Σ_1) and (G_2, Σ_2) where for $i \in \{1, 2\}$, $M_i = \text{ecycle}(G_i, \Sigma_i)$. Suppose that (G_1, Σ_1) is bipartite. Then, M_1 is graphic.

Case 3. The outcome (3) occurs.
Let G_1 be the corresponding graph to a vertex of degree 4 in the auxiliary graph. Let G_2 be the union of two components C_1 and C_2 respectively corresponding to two vertices of degree 2 in the auxiliary graph. Let $\mathcal{B}_G(E(C_1)) = \{a_1, a_2\}$ and let $\mathcal{B}_G(E(C_2)) = \{b_1, b_2\}$. Since $(\delta(a_1) \cup \delta(b_1)) \cap X_1$ is a signature of Σ, an edge set $C \subseteq E(M_2)$ is a circuit of M_2 if and only if C is either a circuit of G_2 or the union of $\{e\}$ and an inclusion-wise minimal $\{a_1, a_2, b_1, b_2\}$-join of G_2. Now construct a graph H from G_2 by (i) identifying a_1 and b_1, and (ii) adding an edge $e = (a_2, b_2)$. Then, $\text{cycle}(H) = M_2$. Thus, M_2 is graphic. □

Proposition 2.3.8. *Let M_1 be a p-graphic matroid and let M_2 be a graphic matroid. Suppose that $E(M_1) \cap E(M_2) = \{e\}$. Then $M = M_1 \oplus_2 M_2$ is p-graphic. Moreover, if M_1 is not graphic, then so is M.*

Before we can proceed to the proof of Proposition 2.3.8, we require some preliminaries. Consider a signed graph (G, Σ) and vertices $v_1, v_2 \in V(G)$ where $\Sigma \subseteq \delta_G(v_1) \cup \delta_G(v_2) \cup \text{loop}(G)$. We can construct a signed graph (G', Σ) from (G, Σ) by replacing the incidences of every odd edge e as follows:

- if $e = v_1v_2$ in G, then e becomes a loop in G' incident to v_1;
- if e is a loop in G, then $e = v_1v_2$ in G';
- if $e = xv_i$, for $i \in \{1, 2\}$ and $x \neq v_1, v_2$, then $e = xv_{3-i}$ in G'.

In this case, we say that (G', Σ) is obtained from (G, Σ) by a *Lovász-flip*. In [11] it is shown that $\text{ecycle}(G', \Sigma) = \text{ecycle}(G, \Sigma)$ i.e. they are both representations of the same p-graphic matroid. An immediate corollary is the following observation,

Remark 2.3.9. *Let M be a connected p-graphic matroid and let e be some element of M. Then there exists a BP-representation of M where e is not a loop.*

Proof of Proposition 2.3.8. Let e be the unique element in $E(M_1) \cap E(M_2)$. Let (G_1, Σ) be a BP-representation of M_1. Because of Remark 2.3.9 we may assume that e is not an odd loop of (G_1, Σ_1). By resigning we may also assume that $e \notin \Sigma_1$. Let G_2 be representation of M_2. Then by summing (G_1, Σ) and G_2 on edge e we obtain a BP-representation (G, Σ) of M. The last part directly follows from Proposition 2.3.5. □

2.4 Bounding the number of representations

The following results respectively appears in [10] and [6].

Theorem 2.4.1. *Let N be a 3-connected even-cycle matroid that is not p-graphic. Let M be a 3-connected major of N. For every equivalence class $\mathcal{F}$ of N, the set of extensions of $\mathcal{F}$ to M is the union of at most two equivalence classes.*

Theorem 2.4.2. *Let N be a 3-connected even-cut matroid that is not p-cographic. Let M be a 3-connected major of N. For every equivalence class $\mathcal{F}$ of N, the set of extensions of $\mathcal{F}$ to M is the union of at most two equivalence classes.*

Now, we are ready to prove Theorem 1.5.1 and Theorem 1.5.2.

Proof of Theorem 1.5.1. Let c be the constant from Theorem 2.1.4. Let M be a 3-connected even-cycle matroid that is not p-graphic. Let N be a minimal matroid with respect to (1) N is non-p-graphic and (2)N is 3-connected. By the Theorem 2.1.4, $|E(N)| < c$. Let c' be the maximum number of inequivalent representations for a matroid on at most c elements. Then $|\mathcal{S}(N)| \leq c'$. It follows from Theorem 2.4.1 that $|\mathcal{S}(M)| \leq 2c'$. □

Proof of Theorem 1.5.2. Let c be the constant from Theorem 2.2.3. Let M be a 3-connected even-cut matroid that is not p-cographic. Let N be a minimal matroid with respect to (1) N is non-p-cographic and (2)N is 3-connected. By the Theorem 2.2.3, $|E(N)| < c$. Let c' be the maximum number of inequivalent representations for a matroid on at most c elements. Then $|\mathcal{S}(N)| \leq c'$. It follows from Theorem 2.4.2 that $|\mathcal{S}(M)| \leq 2c'$. □

Now it remains that to prove Theorem 2.1.4. Before we prove Theorem 2.1.4, we need following lemmas.

Lemma 2.4.3. *Let N be a binary matroid that is minimally non-p-graphic. Then N is connected. Moreover, if N is not 3-connected, then N has at most one 2-separation.*

Proof. Suppose N is not connected. Then there exists a 1-separation (X_1, X_2) of N. Since N is a minimally non-p-graphic matroid, $N|_{X_1}$ and $N|_{X_2}$ are p-graphic. For $i \in \{1, 2\}$, let (G_i, Σ_i) be BP-representations for $N|_{X_i}$. By Remark 2.3.1, since (X_1, X_2) is a 1-separation of N, at least one of (G_i, Σ_i) is bipartite. Thus, N is p-graphic, giving a contradiction.

For the second part, we assume that N has at least two 2-separations. By [1], there exists N_1, N_2, N_3 such that $N = N_1 \oplus_2 N_2 \oplus_2 N_3$ where $E(N_1) \cap E(N_2) = \{e\}$, $E(N_2) \cap E(N_3) = \{f\}$ and $E(N_1) \cap E(N_3) = \emptyset$. Since N is minimally non-p-graphic, $N_1 \oplus_2 N_2$ and $N_2 \oplus_2 N_3$ are p-graphic. By Proposition 2.3.8, N_1 and N_3 are not graphic, otherwise N is p-graphic. Since N_2 is connected, there exists a circuit C containing e and f in N_2. Then $N \backslash (N_2 - C)/(C - \{e, f\})$ is 2-sum of N_1 and N_3 where e and f are regarded as the same edge. It contradicts Proposition 2.3.7 as N_1 and N_3 are both not graphic. □

The following lemmas are proven respectively in [4] and [2].

Lemma 2.4.4. *Let M be a matroid and let N be a minor of M. Suppose that N has a unique exact 2-separation (X, Y) and that M is minimal major of N that bridges (X, Y). Then M is 3-connected.*

Lemma 2.4.5. *If (X, Y) is an exact 2-separation in a matroid N and M is a minimal matroid that bridges the 2-separation (X, Y) in N, then $|E(M)| < |E(N)| + 5$.*

We now prove Theorem 2.1.4.

Proof of Theorem 2.1.4. Let N be a binary matroid that is minimal with respect to (1) N is non-p-graphic and (2) N is 3-connected. Let N' be a minor of N that is minimally non-p-graphic. Let c be the maximum size of an excluded minor for p-graphic. By Theorem 1.1.1, c is a constant.

Note that $|E(N')| \leq c$. By Lemma 2.4.3, either N' is 3-connected, or N' is connected and has a unique 2-separation. If N' is 3-connected, then $N = N'$. Thus, we may assume that N' is connected and has unique 2-separation (X, Y). Let N'' be a minor of N and a major of N' that minimally bridges (X, Y). It follows from Lemma 2.4.4 that N'' is 3-connected. Thus, $N = N''$. By Lemma 2.4.5, $|E(N)| < |E(N')| + 5 \leq c + 5$. □

Chapter 3

Recognizing p-graphic and p-cographic matroids

The goal of this chapter is solving membership problems for p-graphic and p-cographic matroids when the given matroids are 3-connected. By Proposition 1.5.3, it suffices to have an algorithm for recognizing p-cographic matroids. In Section 3.1, we introduce folding/unfolding operations between a BP-representation and a T_4 representation, and prove Proposition 1.5.3. In Section 3.2, we give a brief outline of an algorithm for recognizing p-cographic matroids. Unlike other algorithms mentioned in the previous sections, this algorithm keep track of two types of classes: equivalence classes and anemone classes. Both classes and related results will be considered respectively in Section 3.3 and Section 3.4.

3.1 From p-graphic to p-cographic

Before we proceed to the proof for Proposition 1.5.3, we need a number of definitions from [11]. Consider a signed graph (G, Σ) where $\Sigma \subseteq \delta_G(s) \cup \delta_G(t)$ and suppose that there are no loops and edges between s and t. The graft (H, T) obtained from (G, Σ) by *unfolding* on s, t is defined as follows:

(1) split s into s_1, s_2 according to $\Sigma \cap \delta_G(s)$;

(2) split t into t_1, t_2 according to $\Sigma \cap \delta_G(t)$;

(3) set $T = \{s_1, s_2, t_1, t_2\}$.

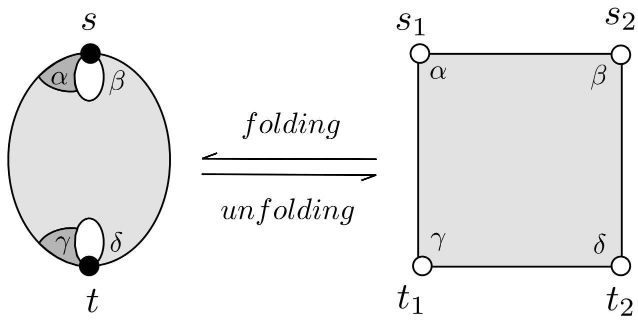

Figure 3.1: Folding/unfolding between a BP-representation and a T_4-representation. Shaded edges are odd and white vertices are terminals.

We illustrate the construction in the following picture, Let us extend this definition to the case where we have loops or edges e with ends s, t of G as follows:

(a) If e is an even loop in G it remains an even loop at the same vertex in H; If e is an even loop at s (resp. t) in G, it remains an even loop at s_1 or s_2 (resp. t_1 or t_2) in H.

(b) If e is an odd loop in G at s, we have $e = (s_1, s_2)$.

(c) If e is an odd loop in G at t, we have $e = (t_1, t_2)$.

(d) If $e = (s, t)$ is even we have a choice $e = (s_1, t_1)$ or $e = (s_2, t_2)$ in H.

(e) If $e = (s, t)$ is odd we have a choice $e = (s_1, t_2)$ or $e = (s_2, t_1)$ in H.

Note that, for (a)-(e), the edge e behaves similarly to a path Q of length two with respect to the previous definition. For instance for (d), we think of e as a path Q with two odd edges or two even edges. For (e), we think of e as a path Q where the edge incident to s is odd and the edge incident to t is even or vice-versa. We say that (G, Σ) is obtained from (H, T) by *folding* with *pairing* s_1, s_2 *and* t_1, t_2.

The following characterization of the cocycles of even-cycle and even-cut matroids appears in [11]. We follow closely the proof given in that paper.

Proposition 3.1.1.

(1) The cocycles of ecycle(G, Σ) *are precisely cuts of* G *and signatures of* (G, Σ)*;*

(2) The cocycles of ecut(G,T) are precisely cycles of G and T-joins of G.

Proof of Proposition 1.5.3. Let M be a p-graphic matroid. By definition there exists a representation (G,Σ) with vertices s,t such that $\Sigma \subseteq \delta_G(s) \cup \delta_G(t) \cup \text{loop}(G)$. Let $(H,T) = (H,\{s_1,s_2,t_1,t_2\})$ be a T_4-representation obtained by unfolding (G,Σ) on s,t where s_1,s_2 correspond to s and t_1,t_2 correspond to t. Let $N = \text{ecut}(H,T)$. We need to show that the cycles of M are equal to the cocycles of N. Because of Proposition 3.1.1 it suffices to show that the set of all even-cycles of (G,Σ) is equal to the union of the set of all cycles and all T-joins of H. Suppose that C is an even cycle of (H,Γ). For every $v \in V(H)-\{s,t\}$, $|\delta_H(v)\cap C| = |\delta_G(v)\cap C|$, which is even. For $i=1,2$ define $d(s,i) = |C\cap\delta_G(s_i)|$ and $d(t,i) = |C\cap\delta_G(t_i)|$. Since C is a cycle $d(s,1), d(s,2)$ have the same parity and so do $d(t,1), d(t,2)$. Note that $\alpha = \delta_G(s_1)$, $\beta = \delta_G(t_1)$ and $\Gamma = \alpha\Delta\beta$. Thus, as $|C\cap\Gamma|$ is even, $d(s,1)$ and $d(t,1)$ have the same parity. Thus $d(s,1)$, $d(s,2)$, $d(t,1)$, $d(t,2)$ are either all even or all odd. In the former case C is a cycle of G, in the later case it is a T-join of G. The converse is similar. □

3.2 An algorithm for recognizing p-cographic matroids

In this section, we give a brief version of an algorithms to solve the membership problem for p-graphic(resp. p-cographic) matroids when the given matroids is 3-connected and is not graphic(resp. cographic). By Proposition 1.5.3, it is enough to solve the membership problem for p-cographic matroids. We will use the same strategy for membership problem for graphic matroids that is mentioned in Section 1.2. In section 1.3.3, the example shows that we may have exponentially many equivalence classes for a p-cographic matroid. To solve this, we will use an additional class of representations, called an *anemone class*. These classes will be used to consider all representations obtained by shuffling petals in Section 1.3.3 as one class of representations. The formal definition for anemone class will be introduced in Section 3.3. Recall that an equivalence class for an p-graphic matroid is given by the set of all signed graphs that are equivalent to a fixed signed graph that is a representation of that matroid. We only consider T_4-representations, since p-graphic matroids are minor-closed. Let $\mathcal{A}(M)$ be obtained by selecting one T_4-representation from each anemone classes and let $\mathcal{S}(M)$ be obtained by selecting one T_4-representation from each equivalence classes that have at least one T_4-representation that is not contained in any anemone classes. The membership algorithm will keep track of $\mathcal{A}(M)\cup\mathcal{S}(M)$ for p-cographic matroids that are not cographic. Then as an analogue of Theorem 1.5.2, we have the following result,

Theorem 3.2.1. *Let M be a 3-connected p-cographic matroid that is not cographic. Then the* *set of all T_4-representations of M is contained in a polynomial number of equivalence representations and a polynomial number of anemone classes.*

Now let us outline a polynomial time algorithm for the membership problem for p-cographic matroids. Consider a 3-connected binary matroid M given by a $(0,1)$-matrix representation. We check if M is cographic by using the membership algorithm for cographic matroids. If M is cographic, then it is also a p-cographic matroid. Moreover, the membership algorithm returns a representation (G,T) where $T=\emptyset$. We can then return $(G,\emptyset)$ as a certificate and stop. Otherwise, because of Proposition 2.1.1 we can find a minor N of M that is minimally non-cographic. Note that every minimally non-cographic matroids is 3-connected. By Theorem 1.1.1, N has a bounded size. As M and N are 3-connected, by[13], this implies that we can construct a sequence of 3-connected matroids $M_1,\dots,M_k$ where $M=M_1$, $M_k=N$, and for every $i\in[k-1]$, M_{i+1} is obtained from M_i by deleting or contracting an element, say e_i of M_i. Iteratively, we constructed $\mathcal{S}(M_{i+1})\cup\mathcal{A}(M_{i+1})$. Then, we can construct $\mathcal{S}(M_i)\cup\mathcal{A}(M_i)$ from $\mathcal{S}(M_{i+1})\cup\mathcal{A}(M_{i+1})$ in polynomial time in the size of M_i and the cardinality of $\mathcal{S}(M_i)\cup\mathcal{A}(M_i)$. However, the cardinality of $\mathcal{S}(M_i)\cup\mathcal{A}(M_i)$ is bounded by a polynomial number because of Theorem 3.2.1. It follows that $\mathcal{S}(M_i)\cup\mathcal{A}(M_i)$ can be constructed in polynomial time in the size of M_i. If $\mathcal{S}(M_i)\cup\mathcal{A}(M_i)=\emptyset$ then M_i is not a p-cographic matroid and neither is M. Otherwise, if $i=1$ then $M_i=M$ is a p-cographic matroid with representations $\mathcal{S}(M_i)\cup\mathcal{A}(M_i)$, and if $i>1$ then we iteratively construct $\mathcal{S}(M_{i-1})\cup\mathcal{A}(M_{i-1})$.

3.3 Anemone classes

3.3.1 Flowers

Let us review the definition of flowers from [12]. Given a matroid M with rank function r, the *connectivity function* λ_M returns for every $X\subseteq E(M)$ the integer, $\lambda_M(X):=r(X)+r(\bar{X})-r(E(M))$. A *flower* Φ of a 3-connected matroid M is an ordered partition $(P_1,\dots,P_n)$ of $E(M)$ where $|P_i|\geq 2$ and $\lambda_M(P_i)=\lambda_M(P_i\cup P_{i+1})=2$ for all $i\in[n]$ where all indices are taken modulo n. The sets $P_1,\dots,P_n$ are the *petals* of the flower.

By a *cyclic* subset of $[n]$ we mean a proper subset of the form $\{i,i+1,\dots,j-1,j\}$ with $i\leq j$ or the complement of such a set. The flower Φ is a *daisy* if for every cyclic subset I of $[n]$, the set $\cup_{i\in I}P_i$ is 3-separating (i.e. the union of consecutive petals is 3-separating) and no other union of petals is 3-separating. The flower Φ is an *anemone* if for every non-empty proper subset I of $[n]$, the set $\cup_{i\in I}P_i$ is 3-separating (i.e. the union of an arbitrary set of petals is 3-separating). It is shown in [12] that for a 3-connected matroid, every flower is either a daisy or an anemone.

Definition 3.3.1. *Let (G,T) be a graft with $|T| = 4$. Denote by E_0 the set of edges of G that have both ends in T. Let $P_1, \ldots, P_n$ be a partition of $E(G) - E_0$. The pair $\Phi = (E_0, \{P_1, \ldots, P_n\})$ where $n \geq 4$ is an* ordinary anemone *of (G,T) if for every $i \in [n]$:*

(1) $G[P_i]$ is connected,

(2) $\mathcal{B}_G(P_i) \subseteq T$ and $|\mathcal{B}_G(P_i)| \geq 3$.

We say that $P_1, \ldots, P_n$ are the petals *of the ordinary anemone and edges in E_0 are* loose.

Definition 3.3.2. *Let $\Phi = (E_0, \{P_1, \ldots, P_k\})$ be an ordinary anemone of (G,T) where $T = \{t_1, t_2, t_3, t_4\}$. We say that (G,T) and (G',T) are related by* rearranging *loose edges if G can be obtained from G' by repeatedly replacing edge $e = t_p t_q$ by $e = t_r t_s$ where $\{p,q,r,s\} = [4]$. For every $i \in [n]$ let G'_i be the graph obtained from $G[P_i]$ by relabeling the terminals in one of three possible ways,*

(1) interchange the labels of t_1 and t_2 the labels of t_3 and t_4;

(2) interchange the labels of t_1 and t_3 the labels of t_2 and t_4;

(3) interchange the labels of t_1 and t_4 the labels of t_2 and t_3. [1]

Now let G' be obtained by identifying vertices t_1 (resp. t_2, t_3, t_4) in each of G'_i for $i = 1, \ldots, k$. We then say that (G,T) and (G',T') are related by rearranging *petals.*

In the previous definition, it can be readily checked that if Φ is an ordinary anemone for (G,T) and (G',T') is obtained from (G,T) by rearranging petals and loose edges, then Φ is an ordinary anemone for (G',T') as well. Moreover, it can be checked that $C \subseteq E(G)$ is a cycle or a T-join of (G,T) if and only if C is a cycle or a T'-join of (G',T'). Thus, (G,T) and (G',T') are representations of the same p-cographic matroid. This leads to the following definition, given Φ an ordinary anemone of (G,T) the set of all grafts obtained from (G,T) by rearranging the loose edges or petals is the *anemone class* generated by (G,T) and Φ. Note no two members of an anemone class are related by Whitney-flips in general, i.e. are equivalent.

We saw in the introduction that ordinary anemones may give rise to p-cographic matroids that have an exponential number of pairwise inequivalent T_4-representations. In this section we gain a better understanding of these types of anemone.

[1] proceed as if $T \subseteq V_G(P_i)$ for all $i \in [k]$ by thinking of vertices in $T \setminus V_G(P_i)$ as isolated vertices of G.

3.3.2 Representations of ordinary anemones

We will show that for a p-cographic matroid, an ordinary anemone for one representation will correspond, for another representation, to either an ordinary anemone or to a *skewed anemone* that we define next.

Definition 3.3.3. *Let (G,T) be a graft with $|T|=4$. Let v_1,v_2,v_3 be distinct vertices of G and let $X=\{v_1,v_2,v_3\}$. Denote by E_0 as follows:*

(a) if $X\subseteq T$, then let E_0 be the set of all edges that have both ends in T,

(b) if $|T\cap X|=2$, then let E_0 be the set of all edges that have both ends in X or that have both ends in $T-X$.

(c) if $|T\cap X|\leq 1$, then let E_0 be the set of all edges that have both ends in X.

The pair $\Phi=(E_0,\{P_1,\ldots,P_n\})$ where $E_0,P_1,\ldots,P_n$ is a partition of $E(G)$ and where $n\geq 4$ is a skewed anemone *for (G,T) if*

(1) $T\subseteq V_G(P_i)$ for some $i\in[n]$.

Moreover, for every $i\in[n]$:

(2) $G[P_i]$ is connected, and

(3) $\mathcal{B}_G(P_i)=X$.

$P_1,\ldots,P_n$ are the petals *of the skewed anemone and edges in E_0 are* loose.

We illustrate the construction in Figure 3.2, Let E be a set and let $\mathcal{S}=\{S_1,\ldots,S_p\}$ and $\mathcal{R}=\{R_1,\ldots,R_q\}$ be two partitions of E. If for every $i\in[q]$, $R_i\subseteq S_j$ for some $j\in[p]$, then $\mathcal{R}$ is a *refinement* of $\mathcal{S}$. If $\mathcal{R}$ is a refinement of $\mathcal{S}$ then $\mathcal{S}$ is a *coarsening* of $\mathcal{R}$. An ordinary (resp. skewed) anemone Φ of a graft (G,T) is *maximal* if no refinement Φ' of Φ where $|\Phi'|>|\Phi|$ is an ordinary (resp. skewed) anemone.

We are now ready to state the main result of this section,

Proposition 3.3.4. *Let M be a 3-connected p-cographic matroid with T_4-representations (G,T) and (G',T'). Then Φ is an ordinary or skewed anemone of (G',T') if Φ is an ordinary anemone of (G,T). Moreover, if Φ is a maximal in (G,T) if and only if it is maximal in (G',T').*

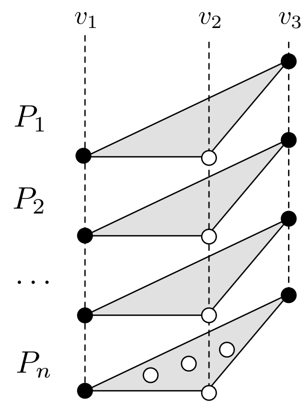

Figure 3.2: Construction of a skewed anemone. White vertices are terminals. Dotted edges mean identifying vertices.

We next define a set of matroid properties that captures the behaviour of ordinary and skewed anemones with exactly three petals. What we need is essentially the notion of a copaddle, a type of matroid anemone; however, it will be convenient in our case to modify the definition in [12] to treat separately the elements of M that are in the co-span of each petal P_i. This leads to the following definition.

Definition 3.3.5. *Consider a binary matroid M with a partition $(E_0, P_1, P_2, \dots, P_n)$ of $E(M)$. Let $N = M/E_0$. Then $(E_0, \{P_1, \dots, P_n\})$ is a* copaddle *of M if for every $i \in [n]$,*

(1) $e \in E_0$ if and only if e is a coloop of $M \setminus P_i$,

(2) $\lambda_N(P_i) = 2$,

and for all distinct $i, j \in [n]$,

(3) $\sqcap_N(P_i, P_j) = 0$,

(4) $\sqcap_{N^*}(P_i, P_j) = 2$,

where $P_1, \dots, P_n$ are the *petals* of the copaddle. Observe that this is related to the common notion of anemone as we can move the elements in E_0 to any petal P_i, that is,

Remark 3.3.6. *Let M be a 3-connected binary matroid. If $(E_0, \{P_1, \dots, P_n\})$ is a copaddle of M, then $(E_0 \cup P_1, \dots, P_n)$ is an anemone of M.*

The proof of Proposition 3.3.4 is covered in the next three sections. In Section 3.3.2 we study 1- and 3-separations in even-cut matroids. In Section 3.3.2 we show that ordinary and skewed anemone give rise to copaddles. As a corollary we prove that ordinary and skewed anemones correspond matroid anemone. In Section 3.3.2 we show that copaddles correspond to ordinary or skewed anemone. We complete the proof of Proposition 3.3.4 in Section 3.3.2.

1- and 3-separations in grafts.

Consider a graft (G, T) and $X \subset E(G)$ where $X \neq \emptyset$. We say that X *induces a* 1*-separation of Type I* in (G, T) if $G[X], G[\bar{X}]$ are connected; $|\mathcal{B}_G(X)| = 1$; and $T \subseteq V_G(X)$ or $T \subseteq V_G(\bar{X})$. We say that X *induces a* 1*-separation of Type II* in (G, T) if $G[X], G[\bar{X}]$ are connected; $|\mathcal{B}_G(X)| = 2$; and $T = \mathcal{B}_G(X)$.

We will need the following result about 1-separations,

Proposition 3.3.7. *Let $M = ecut(G, T)$ where G is connected. Consider $X \subset E(G)$ where $X \neq \emptyset$ and $G[X], G[\bar{X}]$ are connected. Then $\lambda_M(X) = 0$ if and only if X induces a* 1*-separation of Type I or II.*

Proof. Follows immediately from Proposition 2.3.3. □

Consider a graft (G, T) and $X \subset E(G)$ where $X \neq \emptyset$. We say that X *induces a* 3*-separation of Type I* in (G, T) if $G[X], G[\bar{X}]$ are connected; $|\mathcal{B}_G(X)| = 2$; and $\mathcal{I}_G(X) \cap T \neq \emptyset$, and $\mathcal{I}_G(\bar{X}) \cap T \neq \emptyset$. We say that X *induces a* 3*-separation of Type II* in (G, T) if $G[X], G[\bar{X}]$ are connected; $|\mathcal{B}_G(X)| = 3$; $T \nsubseteq \mathcal{B}_G(X)$; and $T \subseteq V_G(X)$ or $T \subseteq V_G(\bar{X})$. We say that X *induces a* 3*-separation of Type III* in (G, T) if $G[X], G[\bar{X}]$ are connected; $|\mathcal{B}_G(X)| = 4$; and $T \subseteq \mathcal{B}_G(X)$ where $T \neq \emptyset$.

We will need the following result about 3-separations,

Proposition 3.3.8. *Let $M = ecut(G, T)$ where G is connected. Consider $X \subset E(G)$ where $X \neq \emptyset$ and $G[X], G[\bar{X}]$ are connected. Then $\lambda_M(X) = 2$ if and only if X induces a* 3*-separation of Type I, II or III.*

Proof. Follows immediately from Proposition 2.3.3. □

Proposition 3.3.9. *Let (G, T) be a graft with a partition P_1, P_2, P_3 of $E(G)$. Suppose that $G[P_i]$ and $G \setminus P_i$ is connected for all $i \in [3]$. Let $M = ecut(G, T)$ and let $\{i, j, k\} = [3]$. Then*

(1) $\lambda_M(P_i) = 2$ if and only if P_i induces a 3*-separation of Type I, II, or III in (G, T);*

(2) $\sqcap_M(P_i, P_j) = 0$ *if and only if* P_i *induces a* 1*-separation of Type I or II of* $(G,T)/P_k$*;*

(3) $\sqcap_{M^*}(P_i, P_j) = 2$ *if and only if* P_i *induces a* 3*-separation of Type I, II, or III in* $(G,T) \setminus P_k$*.*

Proof. **(1)** Follows directly from Proposition 3.3.8. **(2)** Remark 1.4.4 implies that $(G', T') = (G,T)/P_k$ is a representation of $M \backslash P_k$. Thus $\sqcap_M(P_i, P_j) = \lambda_{\text{ecut}(G',T')}(P_i)$. By Proposition 3.3.7, $\lambda_{\text{ecut}(G',T')}(P_i) = 0$ if and only if P_i induces a 1-separation of Type I or II of (G', T'). **(3)** As the function λ is invariant under duals, $\lambda_{(M/P_k)^*}(P_i) = \lambda_{M/P_k}(P_i)$. Remark 1.4.4 implies that $(G', T') := (G,T) \setminus P_k$ is a representation of M/P_k. Thus $\sqcap_{M^*}(P_i, P_j) = \lambda_{\text{ecut}(G',T')}(P_i)$. By Proposition 3.3.8, $\lambda_{\text{ecut}(G',T')}(P_i) = 2$ if and only if P_i induces a 3-separation of Type I, II, or III of (G', T'). □

Proposition 3.3.10. *Let* (G,T) *be a representation of a matroid* M *with disjoint subsets* P_1, P_2 *of* $E(G)$*. Suppose that* $G[P_i]$ *is connected for* $i = 1, 2$*. Let* $M = ecut(G,T)$*. If* $\sqcap_{M^*}(P_1, P_2) = 2$ *then* $G[P_1 \cup P_2]$ *is connected.*

Proof. Let $P_3 = E(G) - (P_1 \cup P_2)$. As in the proof of Proposition 3.3.9(c) we show that $\lambda_{M^*}(P_1, P_2) = \lambda_{\text{ecut}(G',T')}(P_i)$ where $(G', T') = (G,T) \setminus P_3$. Suppose for a contradiction $G[P_1 \cup P_2]$ is not connected. As $G[P_1], G[P_2]$ are connected, $V_G(P_1) \cap V_G(P_2) = \emptyset$. But then Proposition 2.3.3 implies that $\lambda_{\text{ecut}(G',T')}(P_1) \leq 1$, a contradiction. □

From ordinary or skewed anemones to copaddles

Proposition 3.3.11. *Let* M *be a 3-connected p-cographic matroid* M *with a* T_4*-representation* (G,T)*. If* Φ *is an ordinary anemone of* (G,T)*, then* Φ *is a copaddle of* M*.*

Proof. We have $\Phi = (E_0, \{P_1, \ldots, P_n\})$ for some partition $E_0, P_1, \ldots, P_n$ of $E(M)$. We need to prove properties (1)-(4) of Definition 3.3.5. By (1) and (2) in Definition 3.3.1, $G[P_i]$ is connected for $i \in [n]$. **(1)** $e \in E_0$ if and only if e has both ends in T. Let $i \in [n]$. Remark 1.4.4 implies that $(G,T)/P_i$ is a representation of $M \setminus P_i$. By (2) in Definition 3.3.1, either e is a loop of G/P_i or e is an edge of $(G,T)/P_i$ where the unique terminals are the ends of e. In either cases e is a coloop of $M \setminus P_i$. **(2)** Let $i, j \in [n]$. Remark 1.4.4 implies that $(H,T) := (G,T) \setminus E_0$ is a representation of $N = M/E_0$. By Definition 3.3.1(1), $G[P_i]$ is connected, and by Definition 3.3.1(1) and (2), $G[E(M) - P_i]$ is connected. By Definition 3.3.1(2), $\mathcal{B}_H(P_i) \subseteq T$ where $\alpha := |\mathcal{B}_H(P_i) \cap T| \in \{3, 4\}$. If $\alpha = 3$ then P_i induces a 3-separation of Type II in (H,T). If $\alpha = 4$ then P_i induces a 3-separation of Type III in (H,T). In both cases, Proposition 3.3.9(1) implies that $\lambda_N(P_i) = 2$ as required. **(3)** Let $\alpha := |\mathcal{B}_G(E(M) - P_i) \cap T|$. By Definition 3.3.1(2), $\alpha \in \{3, 4\}$. If $\alpha = 3$ then P_i induces a 1-separation of Type II in $(H,T)/(E(M) - P_i)$. If $\alpha = 4$ then P_i induces a

1-separation of Type I in $(H,T)/(E(M)-P_i)$. In both cases, Proposition 3.3.9(2) implies that $\sqcap_N(P_i,P_j)=0$ as required. **(4)** Let $(H',T')=(H,T)\setminus(E(M)-P_i)$. By Definition 3.3.1(2), $\alpha:=|\mathcal{B}_{G'}(P_i)|\in\{2,3,4\}$. If $\alpha=2$ then P_i induces a 3-separation of Type I in (G',T'). If $\alpha=3$ then P_i induces a 3-separation of Type II in (G',T'). If $\alpha=4$ then P_i induces a 3-separation of Type III in (G',T'). In all cases, Proposition 3.3.9(3) implies that $\sqcap_{N^*}(P_i,P_j)=2$ as required. □

We leave the proof of the following result as an exercise as it is very similar to that of Proposition 3.3.11.

Proposition 3.3.12. *Let M be a p-cographic matroid M with a T_4-representation (G,T). If Φ is a skewed anemone of (G,T), then Φ is a copaddle of M.*

This together with Remark 3.3.6 implies the following remark.

Remark 3.3.13. *Let M be a 3-connected p-cographic matroid with a T_4-representation (G,T). If $(E_0,\{P_1,\ldots,P_n\})$ is an ordinary or a skewed anemone of (G,T) then $(E_0\cup P_1,P_2,\ldots,P_n)$ is an anemone of M.*

From copaddles to ordinary or skewed anemones

Proposition 3.3.14. *Let M be a 3-connected p-cographic matroid with a T_4-representation (G,T). Consider a partition $E_0,P_1,\ldots,P_n$ of $E(M)$ where $G[P_i]$ is connected for every $i\in[n]$. Suppose that for every coarsening Q_1,Q_2,Q_3 of $P_1,\ldots,P_n$, $(E_0,\{Q_1,Q_2,Q_3\})$ is a copaddle of M. Then $\Phi=(E_0,\{P_1,\ldots,P_n\})$ is an ordinary or a skewed anemone of (G,T).*

We will proceed by induction on the number of petals. Let us first consider the base case where we have exactly three petals.

Proposition 3.3.15. *Let M be a 3-connected p-cographic matroid that is not cographic. Let (G,T) be a T_4-representation of M. If $\Phi=(E_0,\{Q_1,Q_2,Q_3\})$ is a copaddle of M and $G[Q_i]$ is connected for all $i\in[3]$ then Φ is an ordinary or a skewed anemone of (G,T).*

Proof. Let $H=G\setminus E_0$. Remark 1.4.4 implies that (H,T) is a representation of $N=M/E_0$.

Claim 1.

(a) If $\mathcal{B}_G(Q_i)\subseteq T$ for all $i\in[3]$, then Φ is an ordinary anemone.

(b) If $T \subseteq V_G(Q_i)$ for some $i \in [3]$ and there exists $a, b, c \in V(G)$ such that for all $i \in [3]$, $\mathcal{B}_G(Q_i) \subseteq \{a, b, c\}$, then Φ is a skewed anemone.

Proof. **(a)** Let $e \in E_0$ and let $i \in [3]$. Since e is a coloop of $M \setminus Q_i$ it is either a loop of G/Q_i or an edge of $(G,T)/Q_i$ where the only terminals are then end of e. Since this holds for all $i \in [3]$ and since by hypothesis $\mathcal{B}_G(Q_i) \subseteq T$ the ends of e must be in T. We need to show that each property (1) and (2) of Definition 3.3.1 holds. (1) holds by hypothesis, (2) by hypothesis $\mathcal{B}_G(Q_i) \subseteq T$. By Definition 3.3.5(4), $\lambda_M(Q_i) = 2$. By Proposition 3.3.9(1) Q_i induces a 3-separation of Type II, or III in (H,T). It follows that $|\mathcal{B}_H(Q_3) \cap T| \in \{3, 4\}$ as required. **(b)** Similarly as in (a) we prove that edges in E_0 have both ends in $\{a, b, c\}$. We need to show that each property (1)-(3) of Definition 3.3.3 holds. (1) by hypothesis, (2) by Claim 1. The proof for (3) is similar to (2) of part (a). $\Diamond$

Consider first the case where for some $i \in [3]$, $T \cap \mathcal{I}_H(Q_i) \neq \emptyset$. We may assume $i = 3$. Let $(H', T') = (H,T) \setminus Q_3$. As $T \cap \mathcal{I}_H(Q_3) \neq \emptyset$, $T' = \emptyset$. By Definition 3.3.5(4), $\sqcap_{M^*}(Q_1, Q_2) = 2$. Equivalently, by Proposition 3.3.9(3) Q_1 induces a 3-separation of Type I, II, or III in (H', T'). But as $T' = \emptyset$ it is must be of Type II. Denote by a, b, c the vertices in $\mathcal{B}_{H'}(Q_1)$. Thus $V_H(Q_1) \cap V_H(Q_2) = \{a, b, c\}$. By Definition 3.3.5(2), $\lambda_M(Q_1) = 2$. By Proposition 3.3.9(1) Q_1 induces a 3-separation of Type I, II, or III in (H,T). Since $\mathcal{B}_H(Q_1) \subseteq \{a, b, c\}$ and since $T \cap \mathcal{I}_H(Q_3) \setminus V(Q_1) \neq \emptyset$, it must be of Type II. Thus $\mathcal{B}_H(Q_1) = \{a, b, c\}$ and $T \cap \mathcal{I}_H(Q_1) = \emptyset$. Similarly, $\mathcal{B}_H(Q_2) = \{a, b, c\}$ and $T \cap \mathcal{I}_H(Q_2) = \emptyset$. Thus $\mathcal{B}_H(Q_3) \subseteq \{a, b, c\}$ and $T \subseteq V_H(Q_3)$. It then follows from the Claim that Φ is a skewed anemone of (G,T).

Thus we will assume that,

$$\mathcal{I}_H(Q_i) \cap T = \emptyset \qquad \text{for all } i \in [3]. \tag{3.1}$$

Let $(H', T') := (H,T)/Q_3$. By Definition 3.3.5(3), $\sqcap_N(Q_1, Q_2) = 0$. By Proposition 3.3.9(2), Case 1 or Case 2 occurs.

Case 1. Q_1 (resp. Q_2) induces a 1-separation of (H', T') of Type I.

It follows that $\mathcal{B}_H(Q_i) \subseteq \mathcal{B}_H(Q_3)$ for $i = 1, 2$. Because of (3.1), $T' = \emptyset$ thus $T \subseteq V_H(Q_3)$. But then (3.1) implies in fact that $T \subseteq \mathcal{B}_H(Q_3)$. Proposition 3.3.10 implies that $H[Q_1 \cup Q_2]$ is connected. By Definition 3.3.5(2), $\lambda_M(Q_3) = 2$. By Proposition 3.3.9(a) Q_3 induces a 3-separation of Type I, II, or III in (H,T). As $T \subseteq \mathcal{B}_H(Q_3)$ it must be of Type III. Since M is a p-cographic matroid that is not cographic, $|T| = 4$ and $\mathcal{B}_H(Q_3) = T$. Then $\mathcal{B}_H(Q_i) \subseteq \mathcal{B}_H(Q_3) = T$ for $i = 1, 2$. By Definition 3.3.5(2), $\lambda_M(Q_i) = 2$ for $i = 1, 2$. By Proposition 3.3.9(1) Q_i induces a 3-separation of Type I, II, or III in (H,T). As $\mathcal{I}_H(Q_i) \cap T = \emptyset$, it must be of Type II or III. In all cases it follows from the Claim that Φ is an ordinary anemone for (H,T).

Case 2. Q_1 (resp. Q_2) induces a 1-separation of (H', T') of Type II.

Denote by t_1, t_2, t_3, t_4 the vertices in T. Then we have, say, t_1 which is the unique vertex in $V_H(Q_1) \cap V_H(Q_2) - V_H(Q_3)$, and $t_2, t_3, t_4 \in \mathcal{B}_H(Q_3)$. Because of t_1, $H[Q_1 \cup Q_2]$ is connected. By Definition 3.3.5(2), $\lambda_M(Q_3) = 2$. By Proposition 3.3.9(1) Q_3 induces a 3-separation of Type I, II, or III in (H, T). Because of t_1 it is not of Type III. Because of t_2, t_3, t_4 it is not of Type I, thus it is of Type II, i.e. $\mathcal{B}_H(Q_3) = \{t_2, t_3, t_4\}$. By Definition 3.3.5(2), $\lambda_M(Q_i) = 2$ for $i = 1, 2$. By Proposition 3.3.9(1) Q_i induces a 3-separation of Type I, II, or III in (H, T). As $\mathcal{I}_H(Q_i) \cap T = \emptyset$, it must be of Type II or III. In all cases it follows from the Claim that Φ is an ordinary anemone for (H, T). □

We now generalize the previous result to the case where we have an arbitrary number of petals.

Proof of Proposition 3.3.14. We have $\Phi = (E_0, \{P_1, \ldots, P_n\})$. Let us proceed by induction on n. We may assume $n \geq 4$ for otherwise the result follows from Proposition 3.3.15. Define $\Phi^+(i,j)$ as the partition $\Phi - \{P_i, P_j\} \cup \{P_i \cup P_j\}$ and define $\Phi^-(i,j)$ as the partition $\cup_{\ell \in [k]-\{i,j\}} P_\ell, P_i, P_j$. Observe that if $\{Q_1, Q_2, Q_3\}$ is a coarsening of $\Phi^+(i,j)$ (resp. $\Phi^-(i,j)$) then it is also a coarsening of Φ. Moreover, if the petals $G[P_i]$ are connected in G then so are the petals in any coarsening. Thus the hypothesis of the theorem also hold for $\Phi^+(i,j)$ and $\Phi^-(i,j)$.

For all distinct $i, j \in [k]$ it follows by induction that $\Phi^+(i,j)$ is an ordinary or skewed anemone of (G, T) and that $\Phi^-(i,j)$ is an ordinary or skewed anemone of (G, T). Consider first the case where $\mathcal{I}_G(P_i) \cap T = \emptyset$ for all $i \in [k]$. Pick distinct $i, j \in [k]$. Then $\Phi^+(i,j)$ and $\Phi^+(i,j)$ are both ordinary anemones of (G, T). It follows that for all $P \in \Phi^+(i,j) \cup \Phi^-(i,j)$, $\mathcal{B}_G(P) \subseteq T$ and $|\mathcal{B}_G(P) \cap T| \in \{3, 4\}$. In particular, this holds for all $P = P_i$ with $i = 1, \ldots, k$, hence Φ is an ordinary anemone. Consider the case where $\mathcal{I}_G(P_i) \cap T \neq \emptyset$ for some $i \in [k]$. Pick $j \in [k] - \{i\}$. Then $\Phi^+(i,j)$ and $\Phi^+(i,j)$ are both skewed anemones of (G, T). It follows that there exists $a, b, c \in V(G)$ such that for all $P \in \Phi^+(i,j) \cup \Phi^-(i,j)$, $\mathcal{B}_G(P) = \{a, b, c\}$. In particular, this holds for all $P = P_i$ with $i = 1, \ldots, k$, hence Φ is a skewed anemone. □

The proof of Proposition 3.3.4

Let M be a matroid and let $X, Y \subseteq E(M)$. We say that X and Y are *crossing* if none of $X \cap Y$, $X \cap \bar{Y}$, $\bar{X} \cap Y$ and $\bar{X} \cap \bar{Y}$ is an empty set.

Lemma 3.3.16. *Let M be a 3-connected p-cographic matroid with T_4-representations (G, T) and (G', T'). Let $\Phi = (E_0, \{P_1, \ldots, P_n\})$ be an ordinary anemone of (G, T). Then for all $i \in [n]$, $G'[E(M) - P_i]$ is connected.*

Proof. Suppose not. Then there exists $X \subseteq E(M) - P_i$ such that $(X, \bar{X})$ is a 1- or 2-separation of $M' = M \backslash P_i$. Thus, $\lambda_{M'}(X) \leq 1$. Suppose that for every $i \in [n]$, X and P_i are not crossing. Then, by taking complement of X if necessary, we may assume $X \subseteq P_j$ for some $j \in [n] - \{i\}$. Then $\lambda_M(X) = \lambda_{M'}(X) \leq 1$, a contradiction. Thus, X and P_j are crossing for some $j \in [n] - \{i\}$. Then by submodularity, $\lambda_{M'}(X \cap P_j) + \lambda_{M'}(X \cup P_j) \leq \lambda_{M'}(X) + \lambda_{M'}(P_j) \leq 3$. Thus, $\lambda_{M'}(X \cap P_j) \leq 1$ or $\lambda_{M'}(X \cup P_j) \leq 1$. Since M is 3-connected, $X \cap P_j$ or $\bar{X} \cap \bar{P}_j$ is an edge. By the similar argument, $X \cap \bar{P}_j$ or $\bar{X} \cap P_j$ is an edge. If $\bar{X} \cap P_j$ is an edge, then $|P_j| = 2$. This contradicts that $|\mathcal{B}_G(P_j)| \geq 3$. Thus, $X \cap \bar{P}_j$ is an edge. Thus, for some $k \in [n] - \{i, j\}$, $X \cap P_k \neq \emptyset$ and $\mathcal{I}_G(X) \cap T \neq \emptyset$. Thus, $\lambda_{M'}(X) = 2$, giving a contradiction. □

We are now ready for the proof of the main result of this section.

Proof of Proposition 3.3.4. Let $\Phi = (E_0, \{P_1, \dots, P_n\})$ be an ordinary anemone of (G, T).

Claim 1. *$G'[P_i]$ is connected for all $i \in [n]$.*

Proof. Suppose there exists $i \in [n]$ such that $G'[P_i]$ is not connected. Let $C_1, \dots, C_k$ be components of $G'[P_i]$. By Proposition 2.3.3, $\lambda(P_i) = |\mathcal{B}_{G'}(P_i)| + p_1 + p_2 - k - 1$ where p_1, p_2 are corresponding constant in Proposition 2.3.3. Since M is 3-connected, $|\mathcal{B}_{G'}(C_j)| \leq 2$ for all $j = 1, \dots, k$. Suppose $k \leq 4$. Then, $\lambda(P_i) \leq |\mathcal{B}_{G'}(P_i)| - k - 1 \leq 3$ contradicting $\lambda(P_i) = 2$. If $k = 3$, then $|\mathcal{B}_{G'}(C_j)| = 2$ for all $j = 1, 2, 3$ and $p_1 = p_2 = 0$. Since M is 3-connected $\mathcal{I}_{G'}(C_j) \neq \emptyset$, so $p_1 = 1$, giving a contradiction. Thus, $k = 2$. If $|\mathcal{B}_{G'}(P_i)| = 5$, then $|\mathcal{B}_{G'}(C_j)| = 2$ for some $j \in [2]$ and $p_1 = p_2 = 0$, but by the similar argument, $p_1 = 1$, giving a contradiction. Thus, $|\mathcal{B}_{G'}(P_i)| = 4$, $|\mathcal{B}_{G'}(C_j)| = 2$, $p_1 = 1$ and $p_2 = 0$. Since M is 3-connected, $\mathcal{I}_{G'}(C_j) \neq \emptyset$ for $j = 1, 2$ and $\mathcal{I}_{G'}(\bar{P}_i) = \emptyset$. Thus, P_i is the unique petal such that $G'[P_i]$ is not connected. Then we can use the same argument in the proof of Proposition 3.3.15 to show that there exists $a, b, c \in V(G')$ such that for all $j \in [n]$ but i, $\mathcal{B}_{G'}(P_j) = \{a, b, c\}$. By Proposition 3.3.16, $\mathcal{B}_{G'}(C_j) \cap \{a, b, c\} \neq \emptyset$ for $j = 1, 2$. We may assume $|\mathcal{B}_{G'}(C_1)| = 1$. Thus, there exists $m \in [n]$ such that $\mathcal{B}_{G'}(C_j) \subseteq V_{G'}(P_m)$. Then $\lambda(P_i, P_m) \leq 1$, giving a contradiction. $\Diamond$

Let Q_1, Q_2, Q_3 be a coarsening of $P_1, \dots, P_n$ and let $\Phi' = (E_0, \{Q_1, Q_2, Q_3\})$. If Φ' is an ordinary anemone of (G, T) then by Proposition 3.3.11 Φ' is a copaddle. If Φ' is a skewed anemone of (G, T) then by Proposition 3.3.12 Φ' is a copaddle. It follows from Claim 2 and Proposition 3.3.14 that Φ is an ordinary or a skewed anemone of (G', T'). Suppose for a contradiction, Φ is maximal in (G, T) but that Φ is not maximal in (G', T'). Then there is a refinement $\hat{\Phi}$ of Φ that is an ordinary or skewed anemone of (G', T'). Applying the first part of the theorem it follows that $\hat{\Phi}$ is an ordinary or skewed anemone of (G, T). Observe that Φ is an ordinary (resp. skewed) anemone of (G,T) if and only if $\hat{\Phi}$ is an ordinary (resp. skewed) anemone of (G,T), contradicting the fact that Φ is maximal. $\square$

3.3.3 Ordinary anemones related by arranging petals and loose edges

If Φ is an ordinary anemone for two T_4-representations of a p-cographic matroid, then we can describe exactly the relationship between these representations. The analogous statement where we replace ordinary anemones by skewed ones is not true.

Proposition 3.3.17. *Let M be a 3-connected p-cographic matroid that is not cographic. Let (G,T) and (G',T') be T_4-representations of M. If $\Phi = (E_0, \{P_1,\dots,P_n\})$ is a maximal ordinary anemone of both (G,T) and (G',T'), then (G,T), (G',T') are related by rearranging petals and loose edges.*

Proof. Let $i \in [n]$. Let H be obtained from $G[P_i]$ by identifying vertices in $\mathcal{B}_G(P_i)$ and let H' be obtained from $G'[P_i]$ by identifying vertices in $\mathcal{B}_{G'}(P_i)$.

Claim 1. *H and H' are equivalent.*

Proof. Let $D = E(M) - P_i$. By Definition 3.3.1(2), $\alpha := |\mathcal{B}_G(P_i)| \in \{3,4\}$. Consider first the case where $\alpha = 4$. Then $H = G/D$, and $T \subseteq V_G(D)$. In particular, $(G,T)/D = (H,\emptyset)$. It follows that

$$\text{ecut}(G,T) \setminus D = \text{ecut}((G,T)/D) = \text{ecut}(H,\emptyset) = \text{cut}(H),$$

Consider now the case where $\alpha = 3$, say $\mathcal{B}_G(G) = \{t_1,t_2,t_3\}$ where $T = \{t_1,t_2,t_3,t_4\}$. Denote by $\hat{H}$ the graph obtained $G[P_i]$ by identifying t_1,t_2,t_3 to a single vertex, say $\hat{t}$. Then $\hat{H} = G/D$. Note that H is obtained from $\hat{H}$ by identifying $\hat{t}$ and t_4. It follows that,

$$\text{ecut}(G,T) \setminus D = \text{ecut}((G,T)/D) = \text{ecut}((\hat{H},\{\hat{t},t_4\}) = \text{cut}(H),$$

where the first equation holds by Remark 1.4.4. Thus in both cases $\text{cut}(H) = \text{ecut}(G,T) \setminus D$. Similarly, for G' we show that $\text{cut}(H') = \text{ecut}(G',T') \setminus D$. It follows that $\text{cut}(H) = \text{cut}(H')$, or equivalently, $\text{cycle}(H) = \text{cycle}(H')$. By Theorem 1.2.3 this implies that H and H' are related by Whitney-flips. Thus, H and H' are equivalent. $\Diamond$

Claim 2. *H and H' are isomorphic.*

Proof. By Claim 1, H and H' are equivalent. Suppose H and H' are not isomorphic. Suppose there is a 2-separation $(X, P_i - X)$ of H that is also a 2-separation of G. By Proposition 2.3.3, $\lambda_M(X) = 1$. This contradicts 3-connectivity of M. Thus, every 2-separation of H has the identified vertex as its one of boundary vertices and it is a 3-,4- or 5-separation in G. Thus, no two 2-separations in the sequence of Whitney-flips from H to H' are crossing. Let $(X, P_i - X)$ be one of 2-separations that is performed from H to H'. Then, $(X, P_i - X)$ is also a 2-separation of H'. By rearranging petals if necessary, we may assume there exist two petals P_j, P_k where i, j, k are all distinct and $\mathcal{B}_G(P_j) \cap \mathcal{B}_G(P_k) = T$. Then we can find a path Q_1 in $G[P_j]$ and a path Q_2 in $G[P_k]$, such that $Q_1 \cup Q_2$ is a T-join of (G, T). Let $Q = Q_1 \cup Q_2$ and let $R = \bar{P}_i - Q$. In fact, there are at least two choices for the pair (Q_1, Q_2), thus we may assume that in $G/Q \setminus R$, $(X, P_i - X)$ is a 3-separation. Since Q is a cocircuit of ecut(G, T), Q is either a T'-join of (G', T') or a circuit of G'. Consider first the case where Q is a T'-join of (G', T'). Let $F = G/Q \setminus R$ and let $F' = G'/Q \setminus R$. Then,

$$\text{cut}(F) = \text{ecut}(F, \emptyset) = \text{ecut}(G, T)/Q \setminus R = \text{ecut}(G', T')/Q \setminus R = \text{ecut}(F', \emptyset) = \text{cut}(F')$$

By Theorem 1.2.3, F and F' are related by Whitney-flips, giving a contradiction. Consider now the case where Q is a circuit of G'. Note that $V_{G'}(Q) \cap T \geq 2$, otherwise Q is a disjoint union of two circuits in ecut$(G, T)/P_i$, but Q is a circuit in ecut$(G', T')/P_i$, giving a contradiction. Let $F = G/Q \setminus R$ and let $F' = G'/Q \setminus R$. Note that in ecut$(G', T')/Q \setminus R$, there are at most two terminals. By identifying possible two terminals, we may assume that ecut$(G', T')/Q \setminus R$ has no terminals. Then,

$$\text{cut}(F) = \text{ecut}(F, \emptyset) = \text{ecut}(G, T)/Q \setminus R = \text{ecut}(G', T')/Q \setminus R = \text{ecut}(F', \emptyset) = \text{cut}(F')$$

By Theorem 1.2.3, F and F' are related by Whitney-flips, giving a contradiction. $\Diamond$

Claim 3. *Let $i \in [n]$ then $G[P_i]$ and $G'[P_i]$ are isomorphic.*

Proof. Suppose for a contradiction that $G[P_i]$ and $G'[P_i]$ are not isomorphic. It follows from Claim 1 that we must have edges $e_1, e_2 \in P_i$ such that in G, e_j is incident to t_j for $j = 1, 2$ where t_1, t_2 are distinct elements of T, and in G', e_1, e_2 are incident to the same vertex $t' \in T'$. Since Φ is maximal, $G[P_i] \setminus \mathcal{B}(P_i)$ is connected. It follows that there exists a path Q in $G[P_i] \setminus \mathcal{B}(P_i)$ such that $Q \cup \{e_1, e_2\}$ forms a t_1t_2-path in $G[P_i]$. But then Claim 1 implies that $Q \cup \{e_1, e_2\}$ forms a circuit in $G'[P_i]$. By Proposition 3.1.1 it implies that $Q \cup \{e_1, e_2\}$ is a cocircuit of M. But, then, Proposition 3.1.1 implies that $Q \cup \{e_1, e_2\}$ is either a circuit of G or a T-join of G, a contradiction as $Q \cup \{e_1, e_2\}$ is a path of G and $|T| = 4$. $\Diamond$

By a *tripod* of $G[P_i]$ (resp. $G'[P_i]$) we mean a triple (Q_1, Q_2, Q_3) where Q_1, Q_2, Q_3 are three edge disjoint paths in G (resp. G') that share exactly one vertex $v_0 \in \mathcal{I}_G(P_i)$ as an initial vertex and where the other end of Q_j, for $j = 1, 2, 3$ is a vertex in T.

Claim 4. *Let $i \in [n]$. Then there exists a tripod (Q_1, Q_2, Q_3) of $G[P_i]$ (resp. $G'[P_i]$) and distinct elements $t_1, t_2, t_3 \in T$ where t_j is the end of Q_j for $j = 1, 2, 3$.*

Proof. Since Φ is maximal, $G[P_i] \setminus \mathcal{B}(P_i)$ is connected. It follows that there exists in G: a t_1, t_2-path $L \subseteq P_i$ that is internally disjoint from T, and a path L' with end in $V_G(L) - T$ and t_3 that is internally disjoint from L. Then $L \cup L'$ contain the edges of the required tripod. Similarly, we can find the tripod in $G'[P_i]$. $\diamondsuit$

Since M is not cographic, there exists $j \in [n]$, say $j = 1$, such that $\mathcal{B}_G(P_j) = T$ (for otherwise we can rearrange the petals to get another representation of M where a terminal becomes an isolated vertex, which implies that M is cographic). Denote by t_1, t_2, t_3, t_4 the elements in T and T'. Recall from Claim 2, that $G[P_i]$ and $G'[P_i]$ are isomorphic for all $i \in [n]$. We may assume that $G[P_1]$ are $G[P_2]$ are identical, in particular that vertex t_j of $G[P_1]$ corresponds to vertex t_j of $G'[P_1]$ for all $j = 1, \ldots, 4$. It can be readily checked that edges of E_0 in G and G' are related by rearranging edges. We need to show that $G \setminus E_0$ are $G' \setminus E_0$ are related by rearranging petals. Let $i \in \{2, \ldots, n\}$. It suffices to show that $G'[P_i]$ is obtained from $G[P_i]$ by either:

i. keeping same labels for vertices of $G[P_i]$ and $G'[P_i]$; or

ii. interchanging the labels of t_1 and t_2 & the labels of t_3 and t_4; or

iii. interchanging the labels of t_1 and t_3 & the labels of t_2 and t_4; or

iv. interchanging the labels of t_1 and t_4 & the labels of t_2 and t_3. [2]

By Claim 3 there exists a tripod (Q_1, Q_2, Q_3) of $G[P_1]$ where t_j is the end of Q_j for $j = 1, 2, 3$. We may assume (after possibly relabeling the vertices of T in both G and G' that $\mathcal{B}_G(P_i) \supseteq \{t_1, t_2, t_3\}$. By Claim 3 there exists a tripod (R_1, R_2, R_3) of $G[P_i]$ where t_j is the end of R_j for $j = 1, 2, 3$. By Claim 2 (Q_1, Q_2, Q_3) is a tripod of $G'[P_1]$ and (R_1, R_2, R_3) is a tripod of $G'[P_i]$. Denote by $C(i, j)$ the set $Q_i \cup Q_j \cup R_i \cup R_j$. Exactly one of the following cases 1-4 occurs.

Case 1: R_1 has end t_1 in G'. $C(1, 2)$ is a circuit of G. By Proposition 3.1.1 it is either a circuit of a T'-join of G', but it cannot be the latter, hence it is a circuit. It follows that R_2 has end t_2 in G'. By considering $C(1, 3)$ we prove similarly that R_3 has end t_3 in G'. Hence, (i) occurs.

Case 2: R_2 has end t_1 in G'. $C(1, 2)$ is a circuit of G. By Proposition 3.1.1 it is either a circuit of a T'-join of G', but it cannot be the latter, hence it is a circuit. It follows that R_2 has end t_1 in G'. $C(1, 3)$ is a circuit of G. By Proposition 3.1.1 it is either a circuit of a T'-join of G'. But it cannot be the former, hence it is a T'-join. It follows that R_3 has end t_4 in G'. Hence, (ii) occurs.

Case 3: R_3 has end t_1 in G'. Similarly, as in the previous case, we have that R_3 has end t_1 in G', R_1 has end t_3 in G' and R_2 has end t_4 in G'. Hence, outcome (iii) occurs.

[2] If $T \not\subseteq \mathcal{B}_G(P_i)$ then for the purpose of relabeling we think of the vertex in $T - \mathcal{B}_G(P_i)$ as part of $G[P_i]$.

Case 4: none of R_1, R_2, R_3 has end t_1 in G'. $C(2,3)$ is a circuit of G. By Proposition 3.1.1 it is either a circuit of a T'-join of G', but it cannot be the latter, hence it is a circuit. It follows that R_2, R_3 have ends t_2, t_3 or t_3, t_2 respectively in G'. Then r_1 has end t_4 in G'. $C(1,2)$ is a circuit of G. But it cannot be the former, hence it is a T'-join. Hence, R_3 has end t_2 in G' and R_2 has end t_3 in G'. Hence, outcome (iv) occurs. □

3.3.4 Bounding the number of representations with anemones

The key result in this section is the following,

Proposition 3.3.18. *Let M be a 3-connected p-cographic matroid. Then there exist anemone classes $\mathcal{F}_1, \dots, F_k$ where $k \in O\left(rank(M)^3\right)$ such that if (G,T) is a T_4-representation of M and there exists Φ that is a maximal ordinary anemone for (G,T), then $(G,T) \in \mathcal{F}_i$ for some $i \in [k]$.*

Before we proceed with the proof we wish to show that k cannot be chosen as a constant in the previous result. Consider a wheel W_n with vertices $v_0, v_1, \dots, v_n$ and for $i = 1, \dots, n$, edges v_0v_i and v_iv_{i+1} (where $n+1 = 1$). Construct a graph G from W_n by applying the following construction for each $i \in [n]$: (a) add to W_n, 2-connected graphs $G[P_i]$ and $G[P_i']$ where $\mathcal{B}_G(P_i) = \mathcal{B}_G(P_i') = \{v_0, v_i\}$, and (b) let Σ such that $\Sigma \cap \{v_iv_{i+1} | i \in [n]\} = v_1v_2$ and such that for all $i \in [n]$, $\Sigma \cap (P_i \cup P_i') \subseteq \delta_G(v_0)$ and there exists an odd circuit of (G,Σ) contained in P_i (resp. P_i') that is using v_0 and avoiding v_i. Then for every $i \in [n]$ we can resign (G,Σ) to get a signature $\Gamma_i \subseteq \delta_G(v_0) \cup \delta_G(v_i)$. Denote by (H_i, T_i) the graft obtained by unfolding (G, Γ_i) on v_0, v_i. Then $\Phi = (\emptyset, G \setminus (P_i \cup P_i'), P_i, P_i')$ is an ordinary anemone for (H_i, T_i). Moreover, by Proposition 1.5.3 (H_i, T_i) for all $i \in [n]$ are representations of the same even-cut matroid. We illustrate the construction in Figure 3.3,

Let G be a connected graph and let $S \subseteq V(G)$. By a *proper S-bridge* we mean the connected graph obtained from a component of $G \setminus S$ by adding all the edges of G that have one end in the component and one end in S. An *S-bridge* is either an edge with both ends in S or a proper S-bridge. We have the following nice relation between proper bridges and petals of maximal flowers,

Remark 3.3.19. *Let $\Phi = (E_0, \{P_1, \dots, P_n\})$ be a maximal ordinary anemone for (G,T). Then the petals $P_1, \dots, P_n$ are exactly the proper T-bridges of G.*

Remark 3.3.20. *Let $\Phi = (E_0, \{P_1, \dots, P_n\})$ be a maximal skewed anemone for (G,T). Then the petals $P_1, \dots, P_n$ are exactly the proper $\mathcal{B}_G(P_1)$-bridges of G.*

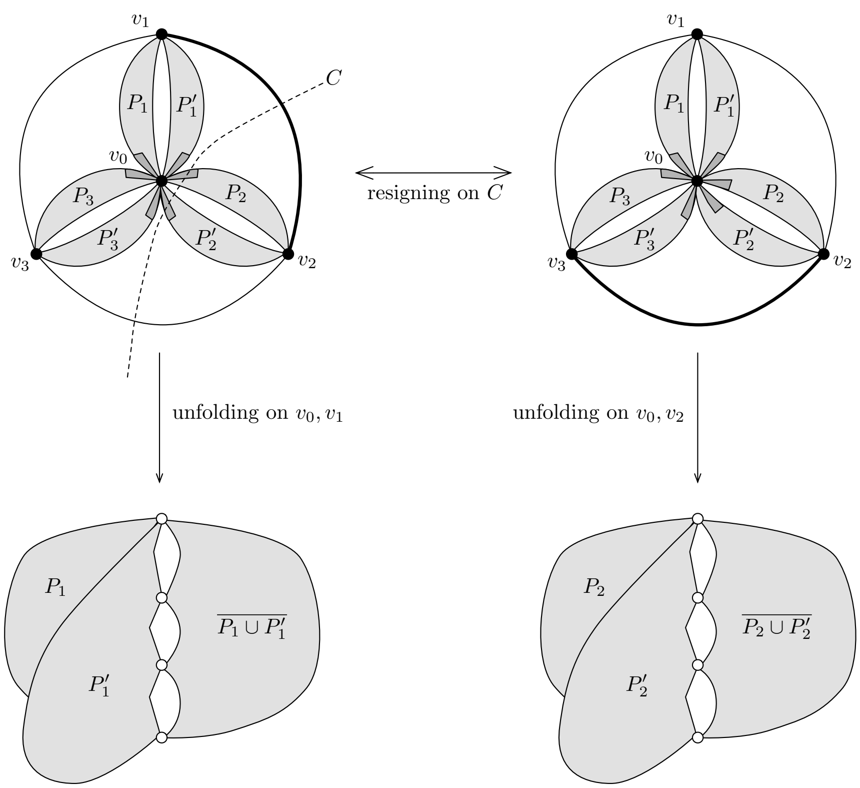

Figure 3.3: Linearly many ordinary anemones. Bold edges and shaded edges are odd. White vertices are terminal.

We are now ready for the proof of the main result in this section.

Proof of Theorem 3.3.18. Let $(G_1, T_1), \ldots, (G_k, T_k)$ be a maximal set of T_4-representations with the following properties:

- for each $i \in [k]$, there exists a maximal ordinary anemone Φ_i for (G_i, T_i),
- for every distinct $i, j \in [k]$, (G_i, T_i) and (G_j, T_j) are in distinct anemone equivalence classes.

We need to show that $k \in \mathcal{O}(|V|^3)$.

Claim 1. $\Phi_1, \ldots, \Phi_k$ *are all distinct*

Proof. Suppose that for distinct $i, j \in [k]$, $\Phi_i = \Phi_j$. Then Φ_i is a maximal ordinary anemone for both (G_i, T_i) and (G_j, T_j). By Proposition 3.3.17, (G_i, T_i) and (G_j, T_j) are related by rearranging petals and edges. Hence, (G_i, T_i) and (G_j, T_j) are in the same anemone class, a contradiction. $\Diamond$

Let $i \in [k]$. By Proposition 3.3.4 Φ_i is either: (i) a maximal ordinary anemone of (G_1, T_1) or (ii) a maximal skewed anemone of (G_1, T_1). By Remark 3.3.19 it follows that there is a unique ordinary anemone as in (i). By Remark 3.3.20 it follows that there are at most $\binom{|V|}{3}$ skewed anemones as in (ii) because it is the number of choices for the three-vertex boundary. It follows that $k \in \mathcal{O}(|V|^3)$ as required. $\square$

3.3.5 Extending anemones

Suppose that we have a p-cographic matroid M, and that N is obtained by deleting or contracting a single element of M. The proof of Theorem 3.2.1 (as well as the membership algorithm for p-cographic matroids) relies on the fact that we can describe the representations of M from the representations of N. In particular, we need to show that the number of equivalence classes and anemone classes of N and M only differ by a polynomial in $|E(N)|$. Two key steps are propositions 3.3.25 and 3.3.27 in this section.

First let us count the maximum number of grafts in an anemone class.

Remark 3.3.21. *Let Φ be an anemone for some graft (G, T). If Φ has k petals, then the anemone class obtained generated by Φ and (G, T) has cardinality at most 8×3^k.*

Proof. There are at most 2^3 ways of rearranging loose edges, and each petal can be rearranged in three possible ways (see Definition 3.3.2). $\square$

Before we proceed further we need some additional results and definitions.

Remark 3.3.22. *Let M be a matroid and $N = M \backslash e$ for some $e \in E(M)$. Then M is determined by the cycles of N as well as a unique cycle of M using e.*

Proof. Let C be the unique cycle of M using e. For any cycle C_1 of M using e, $C_2 = C \triangle C_1$ is a cycle of M avoiding e. Thus, $C_1 = C \triangle C_2$. $\square$

Proposition 3.3.23. *Let M be an p-cographic matroid and let $eE(M)$. Let $N = M/e$ and suppose that both M and N are 3-connected. Consider a set $\Gamma \subseteq E(N)$ where $\Gamma \cup \{e\}$ is a cocycle of M. Suppose that (G, T) is representation of N. Then (G', T') is a representation of M that extends N if and only if (G', T') satisfies one of the following,* *i.* Γ *is a* $\{v_1, v_2\}$*-join and* G' *is obtained from* G *by adding edge* $e = (v_1, v_2)$*, or*

ii. For every T*-join* J *of* (G, T)*,* $\Gamma \triangle J$ *is a* $\{v_1, v_2\}$*-join and* G' *is obtained from* G *by adding edge* $e = (v_1, v_2)$*.*

Proof. Suppose (G', T') is a representation of M that extends N. By Remark 1.4.4, $(G', T') \backslash e = (G, T)$. Thus $T' = T$. Since $\Gamma \cup \{e\}$ is a cocycle of M, $\Gamma \cup \{e\}$ is either a cycle of G' or a T-join of (G', T). For the former case, (i) holds. For the latter case, let J be a T-join of (G, T). Since J is also a T-join of (G', T), $(\Gamma \cup \{e\}) \triangle J$ is a cycle of G'. Thus, (ii) holds. The converse follows from Remark 3.3.22 (by duality). □

Proposition 3.3.24. *Let* M *be an p-cographic matroid and let* $e \in E(M)$*. Let* $N = M \setminus e$ *and suppose that both* M *and* N *are* 3*-connected. Consider a set* $\Gamma \subseteq E(N)$ *where* $\Gamma \cup \{e\}$ *is a cycle of* M*. Suppose that* (G, T) *is representation of* N*. Then* (G', T') *is a representation of* M *that extends* (G, T) *where* $|T'| = |T|$ *if and only if* (G', T') *can be constructed in the following way,*

i. find $\Gamma' = \Gamma \triangle D$ *where* D *is an even cut of* (G, T) *and* $\Gamma' \subseteq \delta_G(v)$ *for some* $v \in V(G)$*,*

ii. G' *is obtained from* G *by splitting* v *according to* Γ' *into* v_1, v_2 *and adding edge* $e = (v_1, v_2)$*,*

iii. if $v \notin T$ *then* $T' = T$*,*

iv. if $v \in T$ *then* $T' = T - \{v\} \cup \{v_{3-i}\}$ *for some* $i \in [2]$ *where* v_i *is incident to the edges of* Γ' *in* G'*.*

Moreover, if e *is not a loop of* M*, then* $\Gamma' \neq \emptyset$*. If there is no parallel edge of* e *in* M*, then* $|\Gamma'| > 1$ *and* $|\delta(v) \triangle \Gamma'| > 1$*.*

Proof. Suppose (G', T') is a representation of M that extends N. By Remark 1.4.4, $(G', T')/e = (G, T)$. Let $e = (v_1, v_2)$ and let v be a vertex obtained by contracting e. Since $|T'| = |T|$, not all of v_1, v_2 are terminals. If both v_1, v_2 are not terminals, then $T' = T$ and $v \notin T$. If exactly one of v_1, v_2 is a terminal, then $T' = T - \{v\} \cup \{v_i\}$ for some $i \in [2]$ and $v \in T$. We may assume that v_1 is not a terminal. Then $D = \delta_{G'}(v_1) \triangle (\Gamma \cup \{e\})$ is an even cut of (G', T') avoiding e. Thus, D is an even cut of (G, T). Note that v_1 is incident to all edges of Γ' in G'. The converse follows from Remark 3.3.22. The last part follows from Proposition 2.3.3. □

Let M be a p-cographic matroid and let (G, T) be a T_4-representation. We say that (G, T) is of *anemone type* if there exists Φ that is an anemone for (G, T).

Proposition 3.3.25. *Let M be a p-cographic matroid, let $e \in E(M)$ and let $N = M/e$ where M and N are 3-connected. Let (G_o, T) be a T_4-representation of N and let $\Phi = (E_0, \{P_1, \dots, P_k\})$ be an ordinary anemone for (G_o, T). Let $\mathcal{F}_N$ be the anemone class generated by Φ and (G_o, T). Let $\mathcal{F}_M$ be the set of T_4-representations of M that extend some representation in $\mathcal{F}_N$. Then either,*

i. $\mathcal{F}_M$ is contained in at most 216 equivalence classes, or

ii. $\mathcal{F}_M$ is contained in a unique anemone class.

Moreover, suppose that we are given a set $\Gamma \subseteq E(N)$ where $\Gamma \cup \{e\}$ is a cocycle of M. Then together with (G_o, T) we can construct in time polynomial in $|E(G)|$ the equivalence classes in (i) or the anemone class in (ii).

Proof. In this thesis, we will omit the proof for the algorithmic part. Let $\Gamma \subseteq E(N)$ where $\Gamma \cup \{e\}$ is a cocycle of M. For $i \in [k]$, let $H_i = G[P_i]$ and let $\Gamma_i = \Gamma \cap P_i$. Let $(G, T) \in \mathcal{F}_N$. Suppose that (G, T) extends to some representation in $\mathcal{F}_M$. It follows from Proposition 3.3.23 that, after possibly replacing Γ by $\Gamma \triangle J$ where J is a T-join of (G, T), we have $V_G(\Gamma) = \{v_1, v_2\}$ for some $\{v_1, v_2\} \subseteq V(G)$ and $e = (v_1, v_2)$. Consider first the case where $v_1, v_2 \in V(H_i)$ for some $i \in [k]$. Then for an arbitrary (G', T) in the anemone class $\mathcal{F}_N$, (G', T) is obtained from (G, T) by rearranging petals and loose edges. It means that every representation in $\mathcal{F}_N$ extends to a representation of M. Moreover, it is easy to check that the resulting representations remain related by rearranging petals and edges. Hence, outcome (ii) occurs.

Thus, we may assume that for distinct $i, j \in [k]$, $v_1 \in \mathcal{I}_G(P_i)$ and $v_2 \in \mathcal{I}_G(P_j)$. Let (G', T) be an arbitrary representation in $\mathcal{F}_N$.

Claim 1. *(G', T) extends to M if and only if P_i and P_j are rearranged from (G, T) in the same way.*

Proof. Suppose that P_i and P_j are rearranged in a different way. Since $V_G(\Gamma) = \{v_1, v_2\}$, $V_{G'}(\Gamma) = \{v_1, v_2, t_1, t_2\}$ for some distinct terminals $t_1, t_2 \in T$. This contradicts Proposition 3.3.23. ◊

It follows that all representations $(G, T) \in \mathcal{F}_N$ that extend to some representation of M are exactly the representations for which P_i and P_j are rearranged like they are one petal. Moreover, either we have at least 4 petals, in which case outcome (ii) occurs, or there are at most 3 petals in which case outcome (i) occurs as $8 \times 3^3 = 216$. □

We leave the proof of the following remark as an exercise (it is equivalent to the problem of checking if a graph is bipartite).

Remark 3.3.26. *Let (G, Γ) be a signed graph and let $R \subseteq V(G)$. Then we can check in time polynomial in $|E(G)|$ if there exists a cut $\delta(U)$ such that all edges in $\Gamma \triangle \delta(U)$ have at least one endpoint in R.*

We say that an ordinary anemone $\Phi = (E_0, \{P_1, \ldots, P_k\})$ is *degenerate* if it is also a skewed anemone.

Proposition 3.3.27. *Let M be a p-cographic matroid, let $e \in E(M)$ and let $N = M \backslash e$ where M and N are 3-connected. Let (G_o, T) be a T_4-representation of N and let $\Phi = (E_0, \{P_1, \ldots, P_k\})$ be a maximal ordinary anemone for (G_o, T). Let $\mathcal{F}_N$ be the anemone class generated by Φ and (G_o, T). Let $\mathcal{F}_M$ be the set of T_4-representations of M that extend some representation (G, T) in $\mathcal{F}_N$ such that Φ is not degenerate in (G, T). Then either,*

i. $\mathcal{F}_M$ is contained in at most 216 equivalence classes, or

ii. $\mathcal{F}_M$ is contained in a unique anemone class.

Moreover, suppose that we are given a set $\Gamma \subseteq E(N)$ where $\Gamma \cup \{e\}$ is a cycle of M. Then together with (G_o, T) we can construct in time polynomial in $|E(G)|$ the equivalence classes in (i) or find a representative for the anemone class in (ii).

Proof. In this thesis, we will omit the proof for the algorithmic part. Let $(G, T), (G', T) \in \mathcal{F}_N$ such that Φ is non-degenerate ordinary anemone in (G, T) and (G', T), respectively. By Proposition 3.3.24, we may assume that there exist $v \in V(G)$ and $\Gamma \subseteq E(G)$ such that $\Gamma \subseteq \delta_G(v)$ and that (G, T) extends by splitting v according to Γ. Similarly, we may assume that there exists $v' \in V(G')$ and $\Gamma' \subseteq E(G')$ such that $\Gamma' \subseteq \delta_{G'}(v')$ and that (G', T) extends by splitting v' according to Γ'. Then, for some $U \subseteq V(G)$, $\Gamma \triangle \Gamma' = \delta_G(U)$. For $i \in [k]$, let $\Gamma_i = \Gamma \cap P_i$ and let $\Gamma'_i = \Gamma' \cap P_i$. Then, $\Gamma_i \triangle \Gamma'_i = \delta_{G[P_i]}(U_i)$ for some $U_i \subseteq V(P_i)$.

Claim 1. *If $\Gamma_i \neq \Gamma'_i$, then there exists a vertex $u \in V_G(P_i)$ such that $\Gamma_i = \Gamma'_i \triangle \delta_{G[P_i]}(u)$.*

Proof. Since $\Gamma_i \neq \Gamma'_i$, we may assume that $\Gamma_i \neq \emptyset$. Then, $v \in V_G(P_i)$. Suppose $\Gamma'_i = \emptyset$. Since $\Gamma_i \triangle \Gamma'_i$ is a cut of $G[P_i]$, so Γ_i is a cut of $G[P_i]$. Thus, $\Gamma_i = \delta_{G[P_i]}(v)$. Thus, $\Gamma_i = \Gamma'_i \triangle \delta_{G[P_i]}(v)$. Now, suppose $\Gamma'_i \neq \emptyset$. Then there exists a vertex $w \in V_G(P_i)$ such that $\Gamma'_i \subseteq \delta_{G[P_i]}(w)$. If $v = w$, then $\Gamma_i \triangle \Gamma'_i \subseteq \delta_{G[P_i]}(v)$ is a cut of $V_G(P_i)$. Since both Γ_i and Γ'_i are not empty, $\Gamma'_i = \Gamma_i \triangle \delta_{G[P_i]}(v)$. Thus, we may assume that $v \neq w$. Then $\{v, w\}$ forms a two-vertex cut-set of $G[P_i]$. Let $(X, \bar{X})$ be a 2-separation of $G[P_i]$. If $\mathcal{I}_{G[P_i]}(X) \cap T = \emptyset$, then $\lambda_M(X) = 1$ by Proposition 2.3.3, contradicting 3-connectivity of M. Thus, $\mathcal{I}_{G[P_i]}(X) \cap T \neq \emptyset$, and similarly, $\mathcal{I}_{G[P_i]}(\bar{X}) \cap T \neq \emptyset$. Let $T = \{t_1, t_2, t_3, t_4\}$. There are following three cases up to exchanging X and $\bar{X}$.

Case 1. $\mathcal{B}_{G[P_i]}(X) \cap T = \emptyset$.

Without loss of generality, we may assume that $t_1, t_2 \in \mathcal{I}_{G[P_i]}(X)$ and that $t_3 \in \mathcal{I}_{G[P_i]}(\bar{X})$. Since $v, w \notin T$, $\Gamma = \Gamma_i$ and $\Gamma' = \Gamma'_i$. Since $\Gamma \triangle \Gamma'$ is a cut of G, for every $j \neq i$, $\mathcal{B}_G(P_j) = \{t_1, t_2, t_4\}$. Thus, Φ is a degenerate ordinary anemone in (G, T), a contradiction.

Case 2. $|\mathcal{B}_{G[P_i]}(X) \cap T| = 1$.

Without loss of generality, we may assume the $\mathcal{B}_{G[P_i]}(X) \cap T = \{t_1\}$, $t_2 \in \mathcal{I}_{G[P_i]}(X)$ and $t_3 \in \mathcal{I}_{G[P_i]}(\bar{X})$. Since $\Gamma \triangle \Gamma'$ is a cut of G, either for every $j \neq i$, $\mathcal{B}_G(P_j) = \{t_1, t_2, t_4\}$ or for every $j \neq i$, $\mathcal{B}_G(P_j) = \{t_1, t_3, t_4\}$. In both cases, Φ is a degenerate ordinary anemone in (G, T), a contradiction.

Case 3. $|\mathcal{B}_{G[P_i]}(X) \cap T| = 2$. Without loss of generality, we may assume the $\mathcal{B}_{G[P_i]}(X) \cap T = \{t_1, t_2\}$, $t_3 \in \mathcal{I}_{G[P_i]}(X)$ and $t_4 \in \mathcal{I}_{G[P_i]}(\bar{X})$. Then, X and $\bar{X}$ are both petals, contradicting maximality of Φ. $\Diamond$

Suppose $v \notin T$. Let P_i be the petal such that $v \in \mathcal{I}_G(P_i)$. By 3-connectivity of M and Proposition 3.3.24, $|\Gamma| > 1$ and $|\delta(v) \triangle \Gamma| > 1$. By Claim 1, $v = v'$ and either $\Gamma = \Gamma'$ or $\Gamma = \Gamma' \triangle \delta_G(v)$, in which case outcome (ii) occurs. Now suppose $v \in T$. Note that $D := \Gamma \triangle \Gamma'$ is an even cut of (G, T) and every even cut of (G, T) can be generated by cuts of the form $\delta_G(\mathcal{I}_G(P_i))$ and a cuts of the form $\delta(t_a) \triangle \delta(t_b)$ where a, b are distinct and $a, b \in [4]$. By Claim 1, for $i \in [k]$, $D \cap P_i$ is either the empty set or is of the form $\delta_{G[P_i]}(t_j)$ for some $t_j \in T$. Thus, $D \cap P_i = \emptyset$ for all $i \in [k]$. Let I be the set of all $i \in [k]$ such that $\Gamma_i \neq \emptyset$. Then, either $|[k] - I| \leq 4$, in which case outcome (i) occurs, or $|[k] - I| \geq 5$, in which case outcome (ii) occurs. $\square$

3.4 Bounding the number of representations

3.4.1 Equivalence classes and anemone classes

In this section, we will prove Theorem 3.2.1, which we restate here:

Theorem 3.2.1. *Let M be a 3-connected p-cographic matroid that is not cographic. Then the set of all T_4-representations of M is contained in the union of a polynomial number of equivalence classes and a polynomial number of anemone classes.*

Let M be a 3-connected p-cographic matroid that is not cographic. For a equivalence class $\mathcal{F}$ of M, we say that $\mathcal{F}$ is *anemone type* if for every T_4-representation $(G,T) \in \mathcal{F}$, (G,T) has a non-degenerate ordinary anemone. The Proposition 3.3.18 shows that the number of anemone classes for M is bounded by a polynomial function of $E(M)$. Thus, it suffices to bound the number of equivalence classes that are not anemone types by a polynomial function of $|E(M)|$. First, in Section 3.4.2, we will prove Lemma 3.4.1 that there are polynomially many equivalence classes containing a representation with a degenerate ordinary anemone.

Lemma 3.4.1. *Let M be a 3-connected p-cographic matroid that is not cographic. Then the number of equivalence classes for M that contains a representation with a degenerate ordinary anemone is bounded by polynomial function of $|E(M)|$.*

Fortunately, by Proposition 2.2.1, column extension of an equivalence class does not increase the number of equivalence classes. In fact, if we have possible anemone classes after column extension, then we may reduce the number of equivalence classes. However, by Proposition 2.2.2, row extension of equivalence classes may double the number of equivalence classes so that eventually, we have exponentially many equivalence classes as we saw in Section 1.3.3. Let $\mathcal{F}$ be a equivalence class of N and let M be a 3-connected p-cographic row major of N. We say that $\mathcal{F}$ is *row stable* for M, if the set of all extensions of $\mathcal{F}$ to M is contained in at most one equivalence class. Otherwise, we say that $\mathcal{F}$ is *row unstable* for M. In the Section 3.4.5, we will prove the following lemma stating that the number of row unstable equivalence classes $\mathcal{F}$ such that every T_4-representation in $\mathcal{F}$ has no ordinary anemone, is bounded by polynomial function of $|E(M)|$. Thus, row extension only increase the number of equivalence classes that are not anemone type by a polynomial function of $|E(M)|$.

Lemma 3.4.2. *Let N be a 3-connected p-cographic matroid that is not cographic. Let M be a 3-connected p-cographic major of N. Let $\mathcal{F}_1, \dots, \mathcal{F}_k$ be equivalence classes for N that are row unstable for M. Suppose that every T_4-representation in $\mathcal{F}_1 \cup \dots \cup \mathcal{F}_k$ has no ordinary anemone. Then $k = O(|E(M)^{14}|)$.*

Proof of Theorem 3.2.1. Let N be a minimally non-cographic minor of M. Note that N is 3-connected and p-cographic. By Seymour [13], there exists a sequence of 3-connected matroids $N_1, \dots, N_k$, where $N = N_1$, $M = N_k$ and N_i is a column or row minor of N_{i+1} for all $i \in [k-1]$. Since N is non-cographic and M is p-cographic, N_i is p-cographic and non-cographic for all $i \in [k]$. By Theorem 1.1.1, N has a constant size. Thus, the set of all T_4-representations of M is contained in a constant number equivalence classes. By the Proposition 3.3.18, there is a polynomial function p_1 such that for each $i \in [k]$, the number of anemone classes for N_i is bounded by $p_1(|E(M)|)$. Thus, it suffices to prove that there exists a polynomial function p_2

such that for $i \in [k-1]$ the number of equivalence classes that are not anemone type from N_i to N_{i+1} is increased by $p_2(|E(M)|)$. Note that $N = N_1$ has the constant size, so there is a bounded number of equivalence classes for N.

Consider the case that N_i is a column minor of N_{i+1}. By Proposition 2.2.1, an extension from each equivalence class for N_i does not increase the number of equivalence classes for N_{i+1}. By Proposition 3.3.25, an extension from each anemone classes for N_i increases the number of equivalence classes for N_{i+1} by at most 216. By Proposition 3.3.18, there are at most $p_1(|E(M)|)$ anemone classes.

Now consider the case that N_i is a row minor of N_{i+1}. By Proposition 2.2.2, an extension from each equivalence class for N_i increases the number of equivalence classes for N_{i+1} by at most 1. By Lemma 3.4.1 and Lemma 3.4.2, there exists a polynomial function p_3 such that the number of such extensions is bounded by $p_3(|E(M)|)$. By Proposition 3.3.27, an extension from each anemone classes for N_i increases the number of equivalence classes for N_{i+1} by at most 216. By Proposition 3.3.18, there are at most $p_1(|E(M)|)$ anemone classes. Thus, $p_2 = 216 \times p_1 + p_3$ is a polynomial function that we want. $\square$

3.4.2 Equivalence classes that are not anemone type

Let M be a 3-connected p-cographic matroid that is not cographic. Let (H, T) be a T_4-representation for M that contains a non-degenerate ordinary anemone $\Phi = (E_0, \{P_1, \ldots, P_k\})$. Note that there exists three distinct petals P_i, P_j, P_l for $i, j, l \in [k]$ such that

(1) $\mathcal{B}_H(P_i) = T$, and

(2) $\mathcal{B}_H(P_j) \cup \mathcal{B}_H(P_l) = T$.

The following lemma implies that equivalence classes which contain a representation with a non-degenerate ordinary anemone are anemone type.

Lemma 3.4.3. *Let M be a 3-connected p-cographic matroid that is not cographic. Let (H, T) be a T_4-representation for M that contains a non-degenerate ordinary anemone $\Phi = (E_0$, $\{P_1$, …, $P_k\})$. Suppose $\mathcal{F}$ is an equivalence class containing (H, T). Then $\mathcal{F} = \{(H, T)\}$.*

Proof. Let $T = \{t_1, t_2, t_1, t_2\}$ and let $(X, \bar{X})$ be a 2-separation of H where $\mathcal{B}_H(X) = \{u, v\}$. If $X \subseteq P_i$ for some $i \in [k]$, then by Proposition 2.3.3, $\lambda_M(X) = 1$ contradicting 3-connectivity of M. Thus, X and P_i are crossing for some $i \in [k]$. We may assume that $u \in \mathcal{I}_H(P_i)$ and $v \in \mathcal{I}_H(\bar{P}_i)$. Let $N = cycle(H)$. Then $\lambda_N(X) = 1$ and $\lambda_N(P_i) \in \{2, 3\}$. By submodularity of connectivity function, $\lambda_N(X \cap P_i) + \lambda_N(X \cup P_i) \leq \lambda_N(X) + \lambda_N(P_i) \leq 4$. Suppose that $\lambda_N(X \cap P_i) = \lambda_N(X \cup P_i) = 2$. Then, $|\mathcal{B}_H(X \cap P_i) \cap \mathcal{B}_H(X \cap \bar{P}_i)| = 2$. and $|\mathcal{B}_H(\bar{X} \cap P_i) \cap \mathcal{B}_H(\bar{X} \cap \bar{P}_i)| = 2$. Since for $j \in [k] - \{i\}$, $H[P_j] \backslash \mathcal{B}_H(P_j)$ is connected, $v \in \mathcal{I}_H(P_j)$ contradicting that every pair of petals can have a common vertex only in T. Thus, we may assume $\lambda_N(X \cap P_i) = 1$. By the similar argument, $\lambda_N(\bar{X} \cap P_i) = 1$ or $\lambda_N(X \cap \bar{P}_i) = 1$. If $\lambda_N(\bar{X} \cap P_i) = 1$, then $|\mathcal{B}_H(P_i)| = 2$, giving a contradiction. Thus, $\lambda_N(X \cap \bar{P}_i) = 1$. Since M is 3-connected, $X \cap P_i$ and $X \cap \bar{P}_i$ share a vertex (say t_1) in T. We may assume that $X \cap \bar{P}_i$ is an edge of P_j where $j \in [k]$ and $i \neq j$. Then for $l \in [k] - \{i, j\}$, $\mathcal{B}_H(P_l) = \{t_2, t_3, t_4\}$. Thus, X is the unique 2-separation of H. However, Whitney-flip on X increases the number of terminal vertices to size of 6, giving a contradiction. □

Proof of Theorem 3.4.1. Let $\mathcal{F}$ be a equivalence class that contains a T_4-representation with a degenerate ordinary anemone Φ. By Lemma 3.4.3, for every $(H, T) \in \mathcal{F}$, Φ is not a non-degenerate ordinary anemone in (H, T). Thus, Φ has a petal P_i where $|\mathcal{B}_H(P_i)| = 4$ and there exists $t_1, t_2, t_3 \in T$ such that for a petal $P_j \neq P_i$ of Φ, $\mathcal{B}_H(P_j) = \{t_1, t_2, t_3\}$. Thus, for each anemone class $\mathcal{F}'$, there are at most 4 representations that has a degenerate ordinary anemone. By Proposition 3.3.18, there exists a polynomial function p such that the number of anemone classes for M is bounded by $p(|E(M)|)$. Thus, the number of equivalence classes that contains a T_4-representation with a degenerate ordinary anemone is bounded by $4 \times p(|E(M)|)$. □

3.4.3 Homologous representations

In [6], the following lemma explains how row unstable equivalence classes extend.

Lemma 3.4.4. *Let M be an even cut matroid that is not cographic and let $\mathcal{F}$ be an equivalence class of M. Let N be a row major of M with no loops or coloops. Suppose that the set $\mathcal{F}'$ of extensions of $\mathcal{F}$ to N is the union of two equivalence classes $\mathcal{F}_1$ and $\mathcal{F}_2$. Then for any $(H_1, T_1) \in \mathcal{F}_1$ and $(H_2, T_2) \in \mathcal{F}_2$, (H_1, T_1) and (H_2, T_2) are basic siblings or nested siblings.*

A basic sibling and a nested sibling are defined in [11]. As corollary, if M and N are p-cographic, then we have the following result from [5].

Corollary 3.4.5. *Let M be a 3-connected p-cographic matroid that is not cographic. Let $\mathcal{F}$ be a row unstable equivalence class of M. Then there exists a representation (H, T) in $\mathcal{F}$ and a partition $P = (\{u_1, v_1\}, \{u_2, v_2\})$ of T, such that, for $i \in \{1, 2\}$, a representation obtained by adding an edge (u_i, v_i) to (H, T) is contained in $\mathcal{F}_i$.*

In this case, we call (H, T) a *representative* of $\mathcal{F}$ with a *pairing* P and we call (H, T, P) an *extended* T_4*-representation* for M.

For $i = 1, 2$, let (H_i, T_i) be a representative of a row unstable equivalence class $\mathcal{F}_i$ with a pairing P_i and let (G_i, Σ_i) be a signed graph obtained by folding (H_i, T_i) according to P_i. Then we say that (H_1, T_1, P_1) and (H_2, T_2, P_2) are *homologous* if (G_1, Σ_1) and (G_2, Σ_2) are equivalent. The following lemma explains the relation between row unstable equivalence classes.

Lemma 3.4.6. *Let N be a 3-connected p-cographic matroid which is not cographic and let $\mathcal{F}_1, \ldots, \mathcal{F}_k$ be equivalence classes for N that is row unstable for a row major M of N. For $i \in [k]$, Let (H_i, T_i) be a representative of $\mathcal{F}_i$ with a pairing P_i. Then (H_1, T_1, P_1), $\cdots$, $(H_k$, T_k, $P_k)$ are homologous.*

Proof. Let $e \in E(M)$ where $N = M/e$. For $i \in \{1, 2\}$, let $(H_i', T_i') = (H_i, T_i)/e$. By Corollary 3.4.5, ends of e are vertices of T_1, so $|T_1'| = 2$. Thus, G_1 can be obtained by identifying two vertices of $|T_1'|$ from (H_1', T_1') and $cut(G_1) = ecut(H_1', T_1')$. Similarly, $cut(G_2) = ecut(H_2', T_2')$. Since $ecut(H_1, T_1) = ecut(H_2, T_2)$, $ecut(H_1', T_1') = ecut(H_2', T_2')$. Thus, $cut(G_1) = cut(G_2)$. Thus, G_1 and G_2 are equivalent, and (G_1, Σ_1), (G_2, Σ_2) are also equivalent. □

3.4.4 Whitney flip sequences and the gap function

Let (H_1, T_1, P_1) and (H_2, T_2, P_2) be homologous extended T_4-representations for M. For $i = 1, 2$, let (G_i, Σ_i) be a signed-graph obtained by folding (H_i, T_i) according to P_i and let $\{u_i, v_i\}$ be blocking pair obtained by identifying vertices of T_i. Then, $\Sigma_i \subseteq \delta(u_i) \cup \delta(v_i)$. $(G_i, \Sigma_i, \{u_i, v_i\})$ is an *extended BP-representation* for M if (G_i, Σ_i) is a BP-representation for M and $\Sigma_i \subseteq \delta(u_i) \cup \delta(v_i)$. In this case, $R_i = (G_i, \Sigma_i, \{u_i, v_i\})$ is obtained by folding (H_i, T_i, P_i). We say that R_1, R_2 are *equivalent* if $(G_1, \Sigma_1), (G_2, \Sigma_2)$ are equivalent. Since R_1, R_2 are equivalent, we can find a sequence of Whitney-flips from G_1 to G_2.

Let $G_1, \ldots, G_{k+1}$ be graphs and let $X_1, \ldots, X_k$ be a 2-separation of $G_1, \ldots, G_k$, respectively. Suppose that for $i = 1, \ldots, k$, G_{i+1} is a graph obtained by performing a Whitney-flip on X_i in G_i. Then, we say that $\mathcal{S} = (X_1, \ldots, X_k)$ is a *w-sequence* of G_1 and denote $G_{k+1} = W_{flip}[G, \mathcal{S}]$. For $X, Y \in \mathcal{S}$, X, Y are *crossing* if none of $X \cap Y$, $X \cap \bar{Y}$, $\bar{X} \cap Y$, $\bar{X} \cap Y$ is an empty set. Otherwise, X, Y are *non-crossing.* We say that $\mathcal{S}$ is *non-crossing* if for every pairs $X, Y \in \mathcal{S}$, X, Y are non-crossing. Note that if $\mathcal{S}$ is non-crossing, then we can rearrange the order of $\mathcal{S}$ arbitrarily.

Let $(G_1, \Sigma_1, \{u_1, v_1\})$ and $(G_2, \Sigma_2, \{u_2, v_2\})$ be extended BP-representations for a matroid M. We say that $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a *nice* Whitney-flip sequence from $(G_1, \Sigma_1, \{u_1, v_1\})$ to $(G_2, \Sigma_2, \{u_2, v_2\})$ if $\mathcal{S}_2$ is non-crossing w-sequence and there exist graphs H_1 and H_2, such that,

(1) for all $X \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$, X is a minimal 2-separation(that is, X induces connected graph after deleting boundary vertices of X).

(2) $H_1 = W_{flip}[G_1, \mathcal{S}_1]$, where $\{u_1, v_1\} \cap \mathcal{B}_{G_1}(X) = \emptyset$ for all $X \in \mathcal{S}_1$,

(3) $H_2 = W_{flip}[G_2, \mathcal{S}_3]$, where $\{u_2, v_2\} \cap \mathcal{B}_{G_2}(X) = \emptyset$ for all $X \in \mathcal{S}_3$, and

(4) $H_2 = W_{flip}[H_1, \mathcal{S}_2]$, where $\{u_1, v_1\} \cap \mathcal{B}_{H_1}(X) \neq \emptyset$ and $\{u_2, v_2\} \cap \mathcal{B}_{H_2}(X) \neq \emptyset$ for all $X \in \mathcal{S}_2$.

By using the following lemma from [11], there exists a nice w-sequence $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ from $(G_1, \Sigma_1, \{u_1, v_1\})$ to $(G_2, \Sigma_2, \{u_2, v_2\})$.

Lemma 3.4.7. *Let G_1, G_2 be 2-connected equivalent graphs and let $Z \subseteq V(G_1)$, where $|Z| \leq 2$. Then, there exist a w-sequence $\mathcal{S}_1$ of G_1 and a graph H with a non-crossing w-sequence $\mathcal{S}_2$ such that:*

(1) $H = W_{flip}[G_1, \mathcal{S}_1]$, where $Z \cup \mathcal{B}_{G_1}(X) = \emptyset$ for all $X \in \mathcal{S}_1$ and

(2) $G_2 = W_{flip}[H, \mathcal{S}_2]$, where $Z \cup \mathcal{B}_{G_1}(X) \neq \emptyset$ for all $X \in \mathcal{S}_2$.

To use this lemma, we set $Z = \{u_1, v_1\}$. We may assume that each Whitney-flip in $\mathcal{S}_1 \cup \mathcal{S}_2$ is minimal. Let $\mathcal{S}'_2$ be a set of all $X \in \mathcal{S}_2$ such that $\{u_2, v_2\} \cap \mathcal{B}_{G_2}(X) \neq \emptyset$ and let $\mathcal{S}_3 = \mathcal{S}_2 - \mathcal{S}'_2$. We can rearrange $\mathcal{S}_2$ into $(\mathcal{S}'_2, \mathcal{S}_4)$, because $\mathcal{S}_2$ is non-crossing. Then $(\mathcal{S}_1, \mathcal{S}'_2, \mathcal{S}_3)$ is a nice w-sequence.

Let $R_1 = (G_1, \Sigma_1, \{u_1, v_1\}), R_2 = (G_2, \Sigma_2, \{u_2, v_2\})$ be extended BP-representations for a matroid M. By using a nice w-sequence, we can define a function $gap(R_1, R_2)$ by the minimum $|\mathcal{S}_2|$ among all nice w-sequences $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ from R_1 to R_2.

3.4.5 The proof of Lemma 3.4.2

Extended BP-representations with large gap

In this section, we will prove following Lemma 3.4.8 stating that if we have two homologous extended T_4-representations for a matroid with large gap, then at least one of them has an ordinary anemone.

Lemma 3.4.8. *Let M be a 3-connected p-cographic matroid which is not cographic. Let (H_1, T_1, P_1), (H_2, T_2, P_2) are homologous extended T_4-representation for M. For $i = 1, 2$, let $R_i = (G_i, \Sigma_i, \{u_i, v_i\})$ be an extended BP-representation obtained by folding (H_i, T_i, P_i). If $gap(R_1, R_2) > 10$, then at least one of (H_1, T_1) and (H_2, T_2) has an ordinary anemone.*

A w-sequence $\mathcal{S} = (X_1, \ldots, X_k)$ of a graph G is a *w-star* of G with a *center* z if $\mathcal{S}$ is non-crossing and there exist distinct $z, v_1, \ldots, v_k \in V(G)$ such that for $i \in [k]$, $\mathcal{B}_G(X_i) = \{z, v_i\}$. To prove Lemma 3.4.8, we need following Lemma 3.4.9 from [11] and Remark 3.4.10. We omit the proof of Remark 3.4.10.

Lemma 3.4.9. *Let G_1, G_2 be equivalent graphs with $G_2 = W_{flip}[G_1, \mathcal{S}]$ for some non-crossing w-sequence $\mathcal{S}$. Suppose that there exist vertices $z_1 \in V(G_1)$ and $z_2 \in V(G_2)$ such that $z_1 \in \mathcal{B}_{G_1}(X)$ and $z_2 \in \mathcal{B}_{G_2}(X)$ for every $X \in \mathcal{S}$. Then $G_2 = W_{flip}[G_1, \mathcal{S}']$ for some $\mathcal{S}'$ which is a w-star of G_1 with center z_1 and a w-star of G_2 with center z_2.*

Remark 3.4.10. *Let X_1, X_2, X_3 be non-crossing 2-separations of a graph G. Suppose $z \in \mathcal{B}_G(X_1) \cap \mathcal{B}_G(X_2)$ and $z \notin \mathcal{B}_G(X_3)$. Then, $X_3 \cap (X_1 \cup X_2) = \emptyset$ or $\bar{X}_3 \cap (X_1 \cup X_2) = \emptyset$.*

Now, we need to define some terminology for signed graphs. Let G be a connected graph and let u, v be distinct vertices of G. Let E_0 be a set of edges of G that have both ends in $\{u, v\}$ and let $P_1, \ldots, P_k$ be a partition of $E(G) - E_0$. We say that $\Phi = (E_0, \{P_1, \ldots, P_k\})$ where $k \geq 3$ is a *graphic anemone* of G with a *spine* $\{u, v\}$, if

(1) for each $i \in [n]$, $\mathcal{B}_G(P_i) = \{u, v\}$,

(2) for each $i \in [n]$, $\mathcal{I}_G(P_i) \neq \emptyset$, and

(3) for each $i \in [n]$, $G[P_i] \backslash \{u, v\}$ is connected.

The sets $P_1, P_2, \ldots, P_n$ are called the *petals* of Φ.

Lemma 3.4.11. *Let M be a 3-connected p-cographic matroid that is not cographic. Let (H, T, P) be an extended T_4-representation for M and let $(G, \Sigma, \{u, v\})$ be an extended BP-representation obtained by folding (H, T, P). Suppose that $\Phi = (E_0, \{P_1, \ldots, P_k\})$ is a graphic anemone of G with a spine $\{u, v\}$. Then Φ is an ordinary anemone of (H, T).*

Proof. Since (H, T) is a graft obtained by unfolding (G, Σ), $|T| = 4$. Suppose that we split u into u_1, u_2 and split v into v_1, v_2 in H. Then, for every edge $e \in E_0$, the ends of e are in $\{u_1, u_2, v_1, v_2\}$. Since $G[P_i] \setminus \{u, v\}$ is connected and M is 3-connected, $H[P_i]$ is connected. Since $\mathcal{B}_G(P_i) = \{u, v\}$, $\mathcal{B}_H(P_i) \subseteq T$. By 3-connectivity of M, $|\mathcal{B}_H(P_i)| \geq 3$. □

We now prove Lemma 3.4.8.

Proof of Lemma 3.4.8. Suppose (H_1, T_1) and (H_2, T_2) have no ordinary anemone. Let $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ be a nice w-sequence from R_1 to R_2. Let $G'_1 = W_{flip}[G_1, \mathcal{S}_1]$ and Let $G'_2 = W_{flip}[G_2, \mathcal{S}_2]$. Note that for every 2-separation $X \in \mathcal{S}_2$, $\mathcal{B}_{G'_1}(X) \cap \{u_1, v_1\} \neq \emptyset$ and $\mathcal{B}_{G'_2}(X) \cap \{u_2, v_2\} \neq \emptyset$. Since, (H_1, T_1) and (H_2, T_2) have no ordinary anemone, there are at most two 2-separations in $\mathcal{S}_2$ that have both u_1, v_1 as its boundary vertices in G'_1. Thus, we may assume there is no such 2-separations and $|\mathcal{S}_2| \geq 11$. Similarly, we may assume there is no 2-separations in $\mathcal{S}_2$ that have both u_2, v_2 as its boundary vertices in G'_2 and $|\mathcal{S}_2| \geq 13$. Let $(X_u u, X_u v, X_v u, X_v v)$ be a partition of $\mathcal{S}_2$ such that, for $s, t \in \{u, v\}$, X_{st} be 2-separation in $\mathcal{S}_2$ incident to s_1 in (G'_1, Σ_1) and incident to t_2 in (G'_2, Σ_2). Since $|\mathcal{S}_2| \geq 13$, without loss of generality, $|X_{uu}| \geq 4$. By Lemma 3.4.9, X_{uu} is a w-star of $W_{flip}[(G'_1, \Sigma_1), X_{vu} \cup X_{vv}]$ with center u_1 and a w-star of $W_{flip}[(G'_2, \Sigma_2), X_{uv}]$ with a center u_2. Let A be a set of vertices in the boundary of X_{uu} which is not u_1.

Case 1. $|A| \geq 3$.
Let $a_1, a_2, a_3 \in A$ and let X_i be a 2-separation where $\mathcal{B}_{G'_1}(X_i)$ for $i = 1, 2, 3$. By Remark 3.4.10, $G_1[X_1 \cup X_2 \cup X_3]$ and $G'_1[X_1 \cup X_2 \cup X_3]$ are isomorphic since every 2-separations in $\mathcal{S}_1$ does not contain u_1, v_1 at the boundary. Since M is 3-connected and each X_i is minimal, $G'_1[X_i]$ contains an odd circuit C_i of G'. Since u_1, v_1 is a blocking pair of G_1, C_i contains u_1 and avoids a_i in G'_1. Thus, C_1, C_2, C_3 are pairwise disjoint in G'_2. By Remark 3.4.10, C_1, C_2, C_3 are pairwise disjoint in G_2 since every 2-separations in $\mathcal{S}_3$ does not contain u_2, v_2 at the boundary. Thus, G_2 has no blocking pair, giving a contradiction.

Case 1. $|A| \leq 2$.
There exists $a \in A$ such that there are at least three 2-separations X_1, X_2, X_3 in $\mathcal{S}_2$ with $\{u_1, a\}$ as boundary vertices. By Remark 3.4.10, $G_1[X_1 \cup X_2 \cup X_3]$ and $G'_1[X_1 \cup X_2 \cup X_3]$ are isomorphic since every 2-separations in $\mathcal{S}_1$ does not contain u_1, v_1 at the boundary. Since M is 3-connected and each X_i is minimal, $G'_1[X_i]$ contains an odd circuit C_i of G'. Since u_1, v_1 is a blocking pair of G_1, C_i contains u_1 and avoids a in G'_1. Thus, all of C_1, C_2, C_3 have a common vertex and avoid u_2 in G'_2. By Remark 3.4.10, C_1, C_2, C_3 have a common vertex w and avoid u_2 in G_2 since every 2-separations in $\mathcal{S}_3$ does not contain u_2, v_2 at the boundary. Since $\{u_2, v_2\}$ is a blocking pair of G_2, $w = v_2$. Thus, (H_2, T_2) has an ordinary anemone, giving a contradiction. □

Extended BP-representations with small gap

In this section, we will prove following Lemma 3.4.12 stating that if we have large number of extended T_4-representations with pairwise small gap, then at least one of them has an ordinary anemone.

Lemma 3.4.12. *Let M be a 3-connected p-cographic matroid which is not cographic. For $i \in [k]$, let (H_i, T_i, P_i) be an extended T_4-representation for M. Suppose that for $i, j \in [k]$, (H_i, T_i, P_i), (H_j, T_j, P_j) are homologous. For $i \in [k]$, let $R_i = (G_i, \Sigma_i, \{u_i, v_i\})$ be an extended BP-representation obtained by folding (H_i, T_i, P_i). Suppose that for $i, j \in [k]$, $gap(G_i, G_j) \leq 10$. Then either (H_i, T_i) has an ordinary anemone for some $i \in [k]$ or $k = O(|E(M)|^{14})$.*

To prove Lemma 3.4.12, we need following lemmas. We postpone the proof for Lemma 3.4.13 until the end of this section.

Lemma 3.4.13. *Let $(G_1, \Sigma_1, \{u_1, v_1\}), \cdots, (G_k, \Sigma_k, \{u_k, v_k\})$ be extended BP-representations that are equivalent. Suppose for any $i, j \in [k]$, $gap(G_i, G_j) \leq 10$. Let $M = ecycle(G_1, \Sigma_1)$. Then either $G_i = G_j$ for some $i \neq j$ or $k = O(|E(M)|^{12})$.*

Lemma 3.4.14. *Let M be a 3-connected p-cographic matroid which is not cographic. Let (H_1, T_1, P_1),..., (H_k, T_k, P_k) be extended T_4-representations that are pairwise homologous and let $(G_i, \Sigma_i, \{u_i, v_i\})$ be an extended BP-representation obtained by folding (H_i, T_i, P_i). Suppose there exists a graph G such that for every $i \in [k]$, $G = G_i$. Then either (H_i, T_i) has an ordinary anemone for some $i \in [k]$ or $k = O(|E(M)|^2)$.*

Proof. Suppose that $k \neq O(|E(M)|^2)$. Let m be the maximum number of extended BP-representations with the same blocking pair $\{u, v\}$ and let $I \subseteq [k]$ be the set of indices of these extended BP-representations. Note that there are at most $|V(G)|^2$ choices for a blocking pair $\{u, v\}$ in G. Thus, $m \neq O(1)$. Let E_0 be a set of edges of G that have both ends in u, v. Let $C_1, \ldots, C_l$ be components of $G \setminus \{u, v\}$. For $a \in [l]$, let $P_a = G[V(C_a) \cup \{u, v\}]$. By 3-connectivity of M, for every $a \in [l]$, $\mathcal{B}_G(P_a) = \{u, v\}$. For $i, j \in I$, since (G, Σ_i) and (G, Σ_j) are equivalent, $D := \Sigma_i \triangle \Sigma_j$ is a cut of G. Note that $\Sigma_i \cup \Sigma_j \subseteq \delta(u) \cup \delta(v)$. Thus, D is a cut of G generated by $\delta(u), \delta(v)$ and $\delta(P_1) \ldots, \delta(P_l)$. Thus $l \geq 3$ and $(E_0, \{P_1, \ldots, P_l\})$ is a graphic anemone. By Lemma 3.4.11, for $i \in I$, (H_i, T_i) has an ordinary anemone. □

Before we prove Lemma 3.4.13, we need following lemmas.

Lemma 3.4.15. *Let $(H_1, T_1, P_1), (H_2, T_2, P_2)$ be extended T_4-representations that are homologous and for $i = 1, 2$, let $R_i = (G_i, \Sigma_i, \{u_i, v_i\})$ be extended BP-representations obtained by folding (H_i, T_i, P_i). Then there exists (H_2', T_2', P_2') and its folding $R_2' = (G_2', \Sigma_2', \{u_2', v_2'\})$ such that*

(1) (H_2', T_2', P_2') and (H_2, T_2, P_2) are equivalent.

(2) (H_2', T_2', P_2') *and* (H_1, T_1, P_1) *are homologous.*

(3) there exists a nice w-sequence $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ *from* R_1 *to* R_2' *such that* $|\mathcal{S}_2| = gap(R_1, R_2) = gap(R_1, R_2')$ *and* $|\mathcal{S}_3| = 0$.

Proof. Let $(\mathcal{S}_1', \mathcal{S}_2', \mathcal{S}_3')$ be a nice w-sequence from R_1 to R_2 where $|\mathcal{S}_2'| = gap(R_1, R_2)$ and let $\mathcal{S}_3' = (X_1, \dots, X_k)$ and let $\bar{\mathcal{S}}_3'$ be a w-sequence of G_2 obtained by reverse the order of $\mathcal{S}_3$. Since $X_i \cap \{u_2, v_2\} = \emptyset$ for all $i \in [k]$, $\bar{\mathcal{S}}_3'$ is a w-sequence of H_2. Let $(H_2', T_2') = W_{flip}[(H_2, T_2), \bar{\mathcal{S}}_3']$. Since $X_i \cap \{u_2, v_2\} = \emptyset$, $\mathcal{I}_{H_2}(X_i) \cap T_2$ has even size. Thus, $T_2' = T_2$ and (H_2', T_2, P_2), (H_2, T_2, P_2) are equivalent. Let R_2' be a extended BP-representation obtained by folding (H_2', T_2, P_2). Then, $(\mathcal{S}_1', \mathcal{S}_2', \emptyset)$ is a nice w-sequence from R_1 to R_2'. Thus, (H_2', T_2', P_2') and (H_1, T_1, P_1) are homologous. For the last part, $|\mathcal{S}_2'| = gap(R_1, R_2')$, otherwise $gap(R_1, R_2)$ can be smaller. □

We may assume that (H_2', T_2, P_2) is chosen with minimum $|\mathcal{S}_1|$. In this case, we call (H_2', T_2, P_2) a *alternative* of (H_2, T_2, P_2) for (H_1, T_1, P_1).

Lemma 3.4.16. *Let* M *be a 3-connected p-cographic matroid that is not cographic. Let* (H_1, T_1, P_1), (H_2', T_2, P_2) *be extended* T_4*-representations for* M *that are homologous and let* (H_2, T_2, P_2) *be an alternative of* (H_2', T_2, P_2) *for* (H_1, T_1, P_1) *For* $i \in [2]$*, let* $R_i = (G_i, \Sigma_i, \{u_i, v_i\})$ *be extended BP-representations obtained by folding* (H_i, T_i, P_i)*. Then* $\mathcal{S}_1$ *is non-crossing. Moreover, there exists a set* $\mathcal{S}$ *only dependent of* (H_1, T_1, P_1) *such that* $|\mathcal{S}| \le 16$ *and* $\mathcal{S}_1 \in \mathcal{S}$ *for any choice of* (H_2', T_2, P_2).

Proof. Let $\mathcal{X} = \{X_1, \dots, X_k\}$ be the set of all minimal 2-separations of G_1 avoiding u_1, v_1. Note that $\mathcal{S}_1 \subseteq \mathcal{X}$.

Claim 1. $\mathcal{X}$ is non-crossing.

Proof. Suppose $\mathcal{X}$ is crossing. Let $(X_i, \bar{X}_i), (X_j, \bar{X}_j)$ be the pair of 2-separations in $\mathcal{X}$. Since X_i, X_j are minimal, all of $X_i \cap X_j$, $\bar{X}_i \cap X_j$, $X_i \cap \bar{X}_j$, $\bar{X}_i \cap \bar{X}_j$ are either a 2-separation in G_1 or a set of one edge. By 3-connectivity, at least three of them contains at least two edges and each of them contains an odd cycle. Since they cannot contain u_1 or v_1 in their boundary, there is no blocking pair. Contradiction. ♢

The first part of the lemma follows from Claim 1 since $\mathcal{S}_1 \subseteq \mathcal{X}$.

Claim 2. We may assume $X_1 \subseteq \cdots \subseteq X_k$.

Proof. By 3-connectivity of M, for $i \in [k]$, each of X_i and $\bar{X}_i$ contains one vertex of $\{u_1, v_1\}$ in its interior. We may assume $u_1 \in \mathcal{I}_{G_1}(X_i)$ for all $i \in [k]$. For distinct $i, j \in [k]$, X_i, X_j are non-crossing, so $X_i \subseteq X_j$ or $X_j \subseteq X_i$. Since $\mathcal{X}$ is non-crossing, we can rearrange $\mathcal{X}$ such that $X_1 \subseteq \cdots \subseteq X_k$. $\diamondsuit$

If $|\mathcal{X}| \leq 2$, then there are only four possibilities for $\mathcal{S}_1$. Thus, we may assume that $|\mathcal{X}| \geq 3$.

Claim 3. There is no $X \in \mathcal{S}_2$ such that X, X_2 are crossing.

Proof. Suppose there exists $X \in \mathcal{S}_2$ such that X, X_2 are crossing. We may assume $u_1 \in \mathcal{B}_{G_1}(X)$. Since $u_1 \in \mathcal{I}_{G_1}(X_1)$ and $X_1 \subseteq X_2$, X_1, X are crossing. Since X, X_1, X_2 are minimal, all of $X \cap X_1$, $\bar{X} \cap X_1$, $X \cap (X_2 - X_1)$, $\bar{X} \cap (X_2 - X_1)$, $X \cap \bar{X}_2$, $\bar{X} \cap \bar{X}_2$ are either a 2-separation of G_1, an empty set or a set of one edge. Since X crosses with X_1 and X_2 respectively, $X \cap X_1$, $\bar{X} \cap X_1$, $X \cap \bar{X}_2$, $\bar{X} \cap \bar{X}_2$ are not empty and at least one of $X \cap (X_2 - X_1)$, $\bar{X} \cap (X_2 - X_1)$ is not empty, otherwise $X_1 = X_2$. By 3-connectivity of M, at least 4 of them contain at least two edges and each of them contains an odd cycle of (G_1, Σ_1). Thus, $\mathcal{B}_{G_1}(X) = \{u_1, v_1\}$. Thus, X is crossing with each element of $\mathcal{S}_2$. By the similar argument above, at least 6 of $X \cap X_1$, $\bar{X} \cap X_1$, $X \cap (X_2 - X_1)$, $\bar{X} \cap (X_2 - X_1)$, $X \cap (X_3 - X_2)$, $\bar{X} \cap (X_3 - X_2)$, $X \cap \bar{X}_3$, $\bar{X} \cap \bar{X}_3$ contain an odd cycle of (G_1, Σ_1) and then there is no blocking pair. Contradiction. $\diamondsuit$

It follows from Claim 3 that for each $X \in \mathcal{S}_2$, X crosses with none of $X_2, \ldots, X_{k-1}$. Thus for $i = 2, \ldots, k-1$, X_i is a 2-separation of G_2 . By the construction of alternative representation, X_i is not a 2-separation of H_2. Thus, when we unfold (G_2, Σ_2) to (H_2, T_2), at least one vertex of $\mathcal{B}_{G_2}(X_i)$ is splitted, namely, $\mathcal{B}_{G_2}(X_i)$ contains at least one of u_2, v_2. We may assume that $u_2 \in \mathcal{B}_{G_2}(X_i)$ for some $2 \leq i \leq k-1$.

Claim 4. For $2 \leq j \leq k-1$, $j \neq i$, $u_2 \in \mathcal{B}_{G_2}(X_j)$.

Proof. Suppose $v_2 \in \mathcal{B}_{G_2}(X_j)$. We may assume $j < i$. Since X_i is not a 2-separation after unfolding and M is 3-connected, there exists an odd circuit C containing u_2 and avoiding v_2 such that $C \subseteq X_i$. Since C avoids v_2, $C \subseteq (X_i - X_j)$. Then C avoids u_1, v_1 in G_1. Contradiction. $\diamondsuit$

Since for all $2 \leq i \leq k-1$, $\mathcal{B}_{G_2}(X_i)$ share a vertex in common, there are fixed choices of $X_3, \ldots, X_{k-2}$ for $\mathcal{S}_1$. Thus,there are only constantly many $\mathcal{S}_1$ for all (H_2', T_2, P_2). $\square$

Now, we are ready to prove Lemma 3.4.13 and Lemma 3.4.12.

Proof of Lemma 3.4.13. Let us fix $(G_1, \Sigma_1, \{u_1, v_1\}$ and for $i = 2, \ldots, k$, let $(G'_i, \Sigma'_i, \{u'_i, v'_i\}$ be alternatives of $(G_i, \Sigma_i, \{u_i, v_i\}$. Let $(\mathcal{S}'_1, \mathcal{S}_2, \emptyset)$ be a nice sequence from G_1 to G'_i. By Lemma 3.4.16, $\mathcal{S}_1$ is non-crossing Whitney flips and $|\mathcal{S}_1| \leq 2$. Thus, there are $O(|E(M)|^2)$ choices for $\mathcal{S}'_1$. Let $K_i = W_{flip}[G'_i, \mathcal{S}'_1]$. Then $\mathcal{S}_2$ is non-crossing Whitney flips of K_i and $|\mathcal{S}_2| \leq 9$. Thus, there are $O(|E(M)|^{10})$ choices for $\mathcal{S}_2$. Thus, $k = O(|E(M)|^{12})$. □

Proof of Lemma 3.4.12. Suppose that $k \neq O(|E(M)|^{14})$. Let m be the maximum number of extended BP-representations with the same graph G and let I be the set of indices of these extended BP-representations. By Lemma 3.4.13, $m \neq O(|E(M)^2)$. By Lemma 3.4.14, there exists $i \in [k]$ such that (H_i, T_i) has an ordinary anemone. □

The proof of Lemma 3.4.2

We now prove Theorem 3.4.2.

Proof of Theorem 3.4.2. For $i \in [k]$, let $R_i = (H_i, T_i, P_i)$ be an extended T_4-representation for N that is a representative for $\mathcal{F}_i$. By Lemma 3.4.6, $R_1, \ldots, R_k$ are homologous. For $i \in [k]$, let $S_i = (G_i, \Sigma_i, \{u_i, v_i\})$ be an extended BP-representation obtained by folding R_i. Let m be the maximum gap among $S_1, \ldots, S_k$. Let $i, j \in [k]$ be the indices such that $gap(S_i, S_j) = m$. If $m > 10$, then by Lemma 3.4.8, then at least one of (H_i, T_i) and (H_j, T_j) has an ordinary anemone, giving a contradiction. If $m \leq 10$, then by Lemma 3.4.12, $k = O(|E(M)|^{14})$. □

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