Exotic Group C*-algebras, Tensor Products, and Related Constructions

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Recently there has been a rejuvenated interest in exotic group C*-algebras, i.e., group C*-algebras which are "intermediate" to the full and reduced group C*-algebras. This resurgence began with the introduction of the class of group L^p-representations ($1 \leq p < \infty$) and their associated C*-algebras (a class of potentially exotic group C*-algebras) by Brown and Guentner. Unlike previous examples of exotic group C*-algebras, this class of examples is universally defined for all locally compact groups. In this thesis we compare this new class of exotic group C*-algebras to previously known examples of exotic group C*-algebras in several key examples and produces new examples of exotic group C*-algebras.

Similar to the definition of exotic group C*-algebras, an exotic C*-tensor product is a C*-tensor product which is intermediate to the minimal and maximal C*-tensor products. Borrowing from the theory of L^p-representations, we construct many exotic C*-tensor products for group C*-algebras.

We will also study the L^p-Fourier and Fourier-Stieltjes algebras of a locally compact group. These are ideals which of the Fourier-Stieltjes algebras containing the Fourier algebras and correspond to the class of L^p-representations. We study the structural properties of these algebras and classify the Fourier-Stieltjes spaces of $SL(2, \mathbb{R})$ which are ideals in the Fourier-Stieltjes algebra.

There are many different tensor products considered in the category of C*-algebras. In contrast, virtually the only tensor product ever considered for von Neumann algebras is the normal spatial tensor product. We propose a definition for what a generic tensor product in the category of von Neumann algebras should be and study properties of von Neumann algebras in relation to these tensor products.

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Dedication

This work is dedicated to the memories of Steven Bos and Cornelis Bos.

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Chapter 1

Introduction

1.1 Overview

Within operator algebras and abstract harmonic analysis, there are pairs of well studied objects where one is "bigger" than the other in some natural sense. Examples of interest to us include the reduced and full group C*-algebras, the minimal and maximal C*-tensor products, and the Fourier and Fourier-Stieltjes algebras. Roughly speaking, this thesis studies constructions which are "intermediate" to such pairs of objects in a natural way.

Let G be a locally compact group. The full and reduced group C*-algebras are each completions of the group algebra $L^1(G)$ with respect to particular C*-norms. The reduced group C*-algebra is the completion with respect to the norm arising from the left regular representation and the full group C*-algebra is the completion with respect to the largest possible C*-norm. It is known that these two C*-algebras agree if and only if G is amenable. We are most interested in the case when G is non-amenable and the full and reduced group C*-algebras differ. In this case we study particular constructions of C*-norms which are strictly between the full and reduced group C*-norms and the completions of $L^1(G)$ with respect to such a norm. Such a completion is said to be an *exotic* group C*-algebra.

Though it is generally agreed that there should be many exotic group C*-algebras for a non-amenable group, actually finding such C*-algebras is surprisingly difficult. One

method of constructing such group C*-algebras comes from the class of L^p-representations. For $1 \leq p < \infty$, a unitary representation $\pi : G \to B(\mathcal{H})$ is an L^p-representation as defined by Brown and Guentner (see [8]) if, roughly speaking, the map $G \ni s \mapsto \langle \pi(s)x, x \rangle$ is in $L^p(G)$ for many $x \in \mathcal{H}$. This class of representations and related constructions are a central object of study throughout this thesis and the associated L^p-C*-algebras, $C^*_{L^p}(G)$, are frequently examples of a continuum of exotic group C*-algebras for a particular group. Other constructions of exotic group C*-algebras are due to Bekka (see [3]) who showed that the class of finite dimensional representations produces exotic group C*-algebras for many residually finite groups, and Bekka, Kaniuth, Lau and Schlichting (see [5]) who considered locally compact groups when endowed with the discrete topology. Since the construction of the L^p-C*-algebras works for any locally compact group, it is natural to wonder if either of the other two constructions could produce the same C*-algebras. In Chapter 2, we show that this is not the case in several important examples and build some new examples of exotic group C*-algebras.

Given two C*-algebras $\mathcal{A}$ and $\mathcal{B}$, it is always possible to place a C*-norm on the algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$ but such a norm need not be unique. The smallest possible such norm is the minimal tensor norm $\|\cdot\|_{\min}$ and the largest is the maximal tensor norm $\|\cdot\|_{\max}$. Similar to the case of group C*-algebras, it is expected that there should be many different C*-norms on $\mathcal{A} \otimes \mathcal{B}$ when $\|\cdot\|_{\min} \neq \|\cdot\|_{\max}$ but finding such norms is difficult. Ozawa and Pisier were the first to exhibit an uncountable number of intermediate C*-tensor norms (see [37]). The main result of their 2014 paper is that $B(\mathcal{H}) \otimes B(\mathcal{H})$ admits a continuum of C*-norms. In Chapter 3, we show that many tensor products of group C*-algebras have this same property.

For a locally compact group G, the Fourier and Fourier-Stieltjes algebras of G are Banach algebras which arise in abstract harmonic analysis. The Fourier-Stieltjes algebra $B(G)$ can be viewed as a subalgebra of the continuous bounded function $C_b(G)$ equipped with a larger norm and the Fourier algebra $A(G)$ an ideal of $B(G)$ which is contained in $C_0(G)$. Despite always being commutative Banach algebras, the Fourier and Fourier-Stieltjes algebras each completely determine the underlying locally compact group G. The Fourier algebra $A(G)$ is naturally the predual of the group von Neumann algebra $VN(G)$ and the Fourier-Stieltjes $B(G)$ is the dual of the full group C*-algebra $C^*(G)$. In Chapter

4 we study analogues of the Fourier and Fourier-Stieltjes algebras which correspond to the L^p-representations of the group. The L^p-Fourier algebra $A_{L^p}(G)$ and L^p-Fourier-Stieltjes $B_{L^p}(G)$ are ideals of the Fourier-Stieltjes algebra $B(G)$ which contain the Fourier algebra $A(G)$. We study how these spaces reflect properties of the underlying group and study the structural properties of these algebras.

As alluded to above, studying the ways of tensoring two C*-algebras together remains an active area of research. In contrast, virtually the only tensor product of von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ ever considered is the normal spatial tensor product $\mathcal{M}\overline{\otimes}\mathcal{N}$. In Chapter 5, we propose a definition for what a generic tensor product in this category should be. We call these *weak* tensor products*. We study these weak* tensor products for many illustrative examples and give a complete characterization of the von Neumann algebras $\mathcal{M}$ which have the property that $\mathcal{M} \otimes \mathcal{N}$ has a unique weak* tensor product completion for every von Neumann algebra $\mathcal{N}$.

This thesis is essentially the compilation of the papers [53], [54], [55] and [56] which correspond to chapters 2, 3, 4 and 5, respectively. Each of these chapters contains its own introduction which elaborates further on its contents.

1.2 Background and notation

We assume a basic knowledge of harmonic analysis, Banach algebras, C*-algebras, and von Neumann algebras throughout this thesis. The books [9], [13] and [18] of Brown-Ozawa, Dixmier and Folland sufficiently cover these topics for the purpose of this thesis. Before proceeding to the body of this thesis, we first recall some of the finer points in these areas (parts of which are covered in the books listed above). We do not give individual references for each of the results listed in this section but instead provide at the beginning of each subsection a list of references in which all the results may be found. Additional background will be introduced as needed throughout this thesis.

1.2.1 Representations and group C*-algebras

In this subsection we review the connection between representations of locally compact groups and the associated group C*-algebras. All of the results mentioned in this section can be found in [13, Ch. 13] and [13, Ch. 18].

Let G be a locally compact group equipped with a fixed left Haar measure ds. The measure algebra $M(G)$ is an involutive Banach algebra with respect to convolution product

$$(\mu * \nu)(E) = \int \int 1_E(st)\, d\mu(s)\, d\nu(t)$$

for Borel subsets $E \subset G$ (where 1_E is the characteristic function of E) and an (isometric) involution is given by

$$d\mu^*(s) = \overline{d\mu(s^{-1})}.$$

With respect to this structure, the group algebra $L^1(G)$ is an involutive ideal of $M(G)$, and the restrictions of the convolution product and involution are given by

$$f * g(t) = \int f(s) g(s^{-1}t)\, ds$$

and

$$f^*(t) = \overline{f(t^{-1})}\Delta(t^{-1})$$

for almost every $t \in G$. Here Δ denotes the modular function of G.

Let $\pi : G \to B(\mathcal{H})$ be a (continuous unitary) representation of G. For each $\mu \in M(G)$, we define

$$\pi(\mu) = \int \pi(s)\, d\mu(s)$$

where this integral is defined in the weak sense, is that

$$\langle \pi(\mu)x, y \rangle = \int \langle \pi(s)x, y \rangle \, ds$$

for $x, y \in \mathcal{H}$. The following classical result establishes a bijective correspondence between the representations of G and the (non-degenerate) $*$-representations of $L^1(G)$.

Theorem 1.2.1. *The map $\mu \mapsto \pi(\mu)$ defines a $*$-representation of $M(G)$ whose restriction to $L^1(G)$ is non-degenerate. Moreover, every $*$-representation of $L^1(G)$ arises uniquely in this way.*

Recall that the left regular representation $\lambda : G \to B(L^2(G))$ is defined by

$$(\lambda(s)f)(t) = f(s^{-1}t)$$

for each $f \in L^2(G)$ and almost every $t \in G$. Then λ is a faithful representation of $L^1(G)$. The *reduced* and *full group C*-norms* of $L^1(G)$ are defined by

$$\|f\|_r = \|\lambda(f)\|$$

and

$$\|f\|_u = \sup\{\|\pi(f)\| : \pi \text{ is a represenation of G}\},$$

for $f \in L^1(G)$, respectively. The *reduced* and *full group C*-algebras* $C_r^*(G)$ and $C^*(G)$ are the completions of $L^1(G)$ with respect to $\|\cdot\|_r$ and $\|\cdot\|_u$.

Observe that by the one-to-one correspondence between representations of G and $*$-representations of $L^1(G)$, every representation π of G extends naturally to a $*$-representation of the full group C*-algebra $C^*(G)$ and vice versa. For a representation π of G, we let $N_\pi \subset C^*(G)$ denote the kernel of π when π is viewed as a $*$-representation of $C^*(G)$. If $\mathcal{S}$ is any collection of representations of G, we define $N_\mathcal{S} = \bigcap_{\pi \in \mathcal{S}} N_\pi$ and the C*-algebra C^*-algebra is defined to be $C^*(G)/N_\mathcal{S}$. Equivalently, we can define $C_\mathcal{S}^*$ to be the completion of $L^1(G)/(L^1(G) \cap N_\mathcal{S})$ with respect to the norm $\|\cdot\|_\mathcal{S}$ defined by

$$\|f\|_\mathcal{S} = \sup\{\|\pi(f)\| : \pi \in \mathcal{S}\}$$

for $f \in L^1(G)$. With this notation, we have that $C_r^*(G) = C_\lambda^*$ and $C^*(G) = C_\mathcal{S}^*$ where $\mathcal{S}$ is taken to be the collection of all representations of G.

A continuous function $\phi : G \to \mathbb{C}$ is *positive definite* if the matrix $\left[\phi(s_i^{-1}s_j)\right]$ is positive semidefinite for all choices of $s_1, \ldots, s_n \in G$. Equivalently, ϕ is positive definite if $\sum_{i,j=1}^n \overline{a_i} a_j \phi(s_i^{-1}s_j) \geq 0$ for all $a_1, \ldots, a_n \in \mathbb{C}$ and $s_1, \ldots, s_n \in G$. Note that positive definite functions ϕ are automatically bounded since

$$\begin{bmatrix} \phi(e) & \phi(s) \\ \phi(s^{-1}) & \phi(e) \end{bmatrix} \geq 0$$

implies that $|\phi(s)| \leq \phi(e)$.

Theorem 1.2.2. *The following are equivalent for a function $\phi \in C_b(G)$:*

1. *ϕ is positive definite,*
2. *ϕ defines a positive linear functional on $L^1(G)$, i.e.,*

$$\int \phi(s)(f^* * f)(s)\, ds \geq 0$$

   for every $f \in L^1(G)$,
3. *There exists a representation $\pi : G \to B(\mathcal{H})$ and $x \in \mathcal{H}$ such that*

$$\phi(s) = \langle \pi(s)x, x \rangle$$

   for every $s \in G$.

Observe that this result gives a one-to-one correspondence between positive definite functions ϕ and positive linear functionals on $L^1(G)$ (resp., $C^*(G)$) by associating ϕ with the functional $f \mapsto \int \phi(s)f(s)\, ds$ for $f \in L^1(G)$. We let $P_1(G)$ denote the set of positive definite functions where $\phi(e) = 1$.

Theorem 1.2.3 (Raikov's theorem)**.** *The following topologies coincide on $P_1(G)$:*

1. *Uniform convergence on compact subsets of G,*
2. *The weak* topology $\sigma(L^\infty(G), L^1(G))$, and*
3. *The weak* topology $\sigma(C^*(G)^*, C^*(G))$.*

Let $\mathcal{S}$ be a collection of representations of G. The collection $\mathcal{S}$ *weakly contains* a representation $\pi : G \to B(\mathcal{H})$ (write $\pi \prec \mathcal{S}$) if for any $x \in \mathcal{H}$, $\epsilon > 0$ and compact subset $K \subset G$, there exists (not necessarily distinct) representations $\pi_1 : G \to B(\mathcal{H}_1), \ldots, \pi_n : G \to B(\mathcal{H}_n)$ in $\mathcal{S}$ and $x_i \in \mathcal{H}_i$ such that

$$\left| \langle \pi(s)x, x \rangle - \sum_{i=1}^{n} \langle \pi_i(s)x_i, x_i \rangle \right| < \epsilon$$

for every $s \in K$. More generally, if $\mathcal{S}$ and $\mathcal{S}'$ are two collections of representations of G, we say that $\mathcal{S}$ is weakly contained in $\mathcal{S}'$ (write $\mathcal{S} \prec \mathcal{S}'$) if $\pi \prec \mathcal{S}'$ for every $\pi \in \mathcal{S}$. The following proposition is a consequence of Raikov's theorem.

Proposition 1.2.4. *Let $\mathcal{S}$ and $\mathcal{S}'$ be two collections of representations of G. Then $\mathcal{S}$ is weakly contained in $\mathcal{S}'$ if and only if $\|\cdot\|_{\mathcal{S}} \leq \|\cdot\|_{\mathcal{S}'}$ on $L^1(G)$.*

This proposition in particular implies that the reduced group C*-algebra $C_r^*(G)$ is canonically equal to the full group C*-algebra $C^*(G)$ if and only if every representation of G is weakly contained in the left regular representation λ. These are but two characterizations of a property called amenability.

A locally compact group G is *amenable* if $L^\infty(G)$ admits a left invariant mean, i.e., there exists a state $m \in L^\infty(G)^*$ such that $m({}_sf) = m(f)$ for every $s \in G$ and $f \in L^\infty(G)$, where ${}_sf(t) := f(st)$ for almost every $t \in G$. The amenability of groups admits "approximately $10^{10^{10}}$ different characterizations" (see [8, p. 48]). Below we list a couple characterizations of amenability which we will make use of throughout this thesis.

Theorem 1.2.5. *The following are equivalent for a locally compact group G.*

1. *G is amenable,*
2. *λ weakly contains every representation of G,*
3. *λ weakly contains the trivial representation 1_G of G.*

Weak containment is preserved under all the usual group operations.

Theorem 1.2.6. *Let G be a locally compact group.*

1. *If $\{\pi_i\}_i$ and $\{\sigma_i\}_i$ are representations of G such that $\pi_i \prec \sigma_i$ for every i, then $\bigoplus_i \pi_i \prec \bigoplus_i \sigma_i$,*
2. *If $\pi_1, \pi_2, \sigma_1, \sigma_2$ are representations of G such that $\pi_1 \prec \sigma_1$ and $\pi_2 \prec \sigma_2$, then $\pi_1 \otimes \pi_2 \prec \sigma_1 \otimes \sigma_2$,*
3. *If π, σ are representations of a closed subgroup H of G such that $\pi \prec \sigma$, then $\mathrm{Ind}_H^G \pi \prec \mathrm{Ind}_H^G \sigma$.*

Let $\mathcal{S}$ be a set of representations of G. For each representation π of G, $x_1, \ldots, x_n \in \mathcal{H}_\pi$, compact subset $K \subset G$, and $\epsilon > 0$ the set $W(\pi, x_1, \ldots, x_n, K, \epsilon)$ is defined to be the collection of representations σ in $\mathcal{S}$ such that for every $1 \leq i \leq n$, there exists $y_1, \ldots, y_k \in$

$\mathcal{H}_\sigma$ so that

$$\left| \langle \pi(s)x_i, x_i \rangle - \sum_{j=1}^{k} \langle \sigma(s)y_j, y_j \rangle \right| < \epsilon$$

for every $s \in K$. The topology on $\mathcal{S}$ generated by the basis of sets $W(\pi, x_1, \ldots, x_n, \epsilon)$ is called *Fell's topology*. This topology is intimately related to weak containment.

Proposition 1.2.7. *A net $\{\pi_i\}_i$ in $\mathcal{S}$ converges to π of G in Fell's topology if and only if $\pi \prec \oplus_j \pi_j$ for every subnet $\{\pi_j\}_j$ of $\{\pi_i\}_i$.*

Theorem 1.2.8. *Let $\mathcal{S}$ be a subset of $\widehat{G}$, the set of irreducible representations of G. An irreducible representation $\pi \in \widehat{G}$ is in the Fell closure $\overline{\mathcal{S}}$ if and only if π is weakly contained in $\mathcal{S}$.*

1.2.2 Quasi-containment of $*$-representations

All of the results listed here can be found in [49, Ch. III].

Let $\mathcal{A}$ be a C*-algebra and $\pi : \mathcal{A} \to B(\mathcal{H})$ a $*$-representation of $\mathcal{A}$. We let $\mathcal{M}_\pi := \pi(\mathcal{A})''$ denote the von Neumann algebra generated by $\pi(A)$. Observe that, since $\pi(\mathcal{A})$ is weak* dense in $\mathcal{M}_\pi$, the predual of $\mathcal{M}_\pi$ identifies isometrically with a subspace of $\mathcal{A}^*$. This subspace of $\mathcal{A}^*$ is denoted by V_π.

For a $*$-representation $\pi : \mathcal{A} \to B(\mathcal{H})$ and $x, y \in \mathcal{H}$, let $\pi_{x,y} : \mathcal{A} \to \mathbb{C}$ be defined by $\pi_{x,y}(a) = \langle \pi(a)x, y \rangle$. Observe that $\pi_{x,y}$ is clearly an element of V_π for each $x, y \in \mathcal{H}$.

Proposition 1.2.9. *Let $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a $*$-representation of a C*-algebra $\mathcal{A}$. Then*

$$V_\pi = \left\{ \sum_{i=1}^{\infty} \pi_{x_i,y_i} : x_i, y_i \in \mathcal{H} \text{ and } \sum_{i=1}^{\infty} \|x_i\|\|y_i\| < \infty \right\}.$$

Moreover if $u \in V_\pi$, then

$$\|u\| = \inf \left\{ \sum_{i=1}^{\infty} \|x_i\|\|y_i\| : u = \sum_{i=1}^{\infty} \pi_{x_i,y_i} \right\}$$

and this infimum is attained.

Observe that these spaces V_π are $\mathcal{A}$-bimodules with respect to the action $(a \cdot u \cdot b)(c) = u(bca)$ for $a, b, c \in \mathcal{A}$ and $u \in V_\pi$. This property distinguishes the V_π spaces amongst all closed subspaces of $\mathcal{A}^*$.

Theorem 1.2.10. *Let $\mathcal{A}$ be a C*-algebra and suppose that X is a closed subspace of $\mathcal{A}^*$ such that $a \cdot u \cdot b \in X$ for all $a, b \in \mathcal{A}$ and $u \in X$. Then $X = V_\pi$ for some $*$-representation π of $\mathcal{A}$.*

For $*$-representations π and σ of a C*-algebra $\mathcal{A}$, it is natural to wonder when V_π is contained in V_σ. This is answered by the concept of quasi-containment of representations.

Let π and σ be $*$-representations of a C*-algebra $\mathcal{A}$. The representation π is *quasi-contained* in σ if π is unitarily equivalent to an amplification of a subrepresentation of σ.

Theorem 1.2.11. *Let $\mathcal{A}$ be a C*-algebra, and π and σ $*$-representations of $\mathcal{A}$. The following are equivalent.*

1. *π is quasi-contained in σ,*
2. *The map $\sigma(a) \mapsto \pi(a)$ (for $a \in \mathcal{A}$) extends to a well-defined normal $*$-homomorphism from $\mathcal{M}_\sigma$ to $\mathcal{M}_\pi$,*
3. *$V_\pi \subset V_\sigma$.*

1.2.3 Fourier and Fourier-Stieltjes algebras

We now review the Fourier and Fourier-Stieltjes spaces of a locally compact group, placing a particular emphasis on the Fourier and Fourier-Stieltjes algebras. With the exception of the last theorem in this subsection, all the results listed here can be found in [16] and [1].

Let $\pi : G \to B(\mathcal{H})$ be a representation of a locally compact group G and $x, y \in \mathcal{H}$. Similar to the case for C*-algebras, we let $\pi_{x,y} : G \to \mathbb{C}$ be the function defined by $\pi_{x,y}(s) = \langle \pi(s)x, y \rangle$ for $s \in G$. The *Fourier-Stieltjes algebra* $B(G)$ is is the set of continuous bounded functions

$$\{\pi_{x,y} \mid \pi : G \to B(\mathcal{H}) \text{ is a representation of } G \text{ and } x, y \in \mathcal{H}\}.$$

The Fourier-Stieltjes algebra $B(G)$ naturally identifies with the dual of the full group C*-algebra $C^*(G)$ via the dual pairing

$$< u, f > := \int_G u(s) f(s)\, ds$$

for $u \in B(G)$ and $f \in L^1(G)$. The norm on $B(G)$ arising from this identification is given by

$$\|u\| = \inf\{\|x\|\|y\| : u = \pi_{x,y}\}.$$

When equipped with this norm, $B(G)$ is a commutative Banach algebra with respect to pointwise operations since

$$\pi_{x,y} + \sigma_{x',y'} = (\pi \oplus \sigma)_{(x,x'),(y,y')}$$

and

$$\pi_{x,y} \cdot \sigma_{x',y'} = (\pi \otimes \sigma)_{x\otimes x', y\otimes y'}.$$

Recall that when G is abelian, $C^*(G) = C_0(\widehat{G})$ and, hence, $B(G)$ is isometrically isomorphic to the measure algebra $M(\widehat{G})$ as a Banach space. In this case $B(G)$ is the set of all Fourier-Stieltjes transforms of measures in $M(\widehat{G})$ and the Fourier-Stieltjes transform is an isometric Banach algebra isomorphism between $M(\widehat{G})$ and $B(G)$.

Given a representation $\pi : G \to B(\mathcal{H})$ of G, the *Fourier space* A_π is defined to be the closed linear span of matrix coefficients $\pi_{x,y}$ for $x, y \in \mathcal{H}$ in $B(G)$. Then, by the identification between representations of G and $*$-representations of $C^*(G)$, A_π is the predual of the von Neumann algebra $VN_\pi := \pi(G)''$ $(= \pi(L^1(G))'' = \pi(M(G))'')$ and Proposition 1.2.9 also applies to these A_π spaces. Observe that the Fourier spaces A_π are invariant under both left and right-translation by G. Further, by using a weak* density argument, it can be shown to follow from Theorem 1.2.10 that every norm closed subspace of $B(G)$ which is invariant under both left and right translation by G is a Fourier space A_π for some representation π of G.

Proposition 1.2.12. *Let π be a representation of a locally compact group G. Then*

1. *A_π is a subalgebra of $B(G)$ if and only if $\pi \otimes \pi$ is quasi-contained in π,*

2. *A_π is an ideal of $B(G)$ if and only if $\pi \otimes \sigma$ is quasi-contained in π for every representation σ of G.*

More generally, if we are given a collection of representations $\mathcal{S}$, the *Fourier space* $A_\mathcal{S}$ is the closed linear span of all matrix coefficients for representations in $\mathcal{S}$. Observe that since $A_\mathcal{S}$ is invariant under left and right translation by G, $A_\mathcal{S} = A_\pi$ for some representation π of G. Such a representation π can be realized as taking a large enough direct sum of representations in $\mathcal{S}$. In the case when $\mathcal{S}$ is a set, it suffices to take π to be the direct sum of all representations in $\mathcal{S}$.

Given a collection of representations $\mathcal{S}$ of a locally compact group G, the *Fourier-Stieltjes space* $B_\mathcal{S}$ is the closure of $A_\mathcal{S}$ in the weak* topology $\sigma(B(G), C^*(G))$. This space $B_\mathcal{S}$ is exactly the subspace of $B(G) = C^*(G)^*$ which identifies as the dual of $C^*_\mathcal{S}(G)$ via

$$< u, f >= \int_G u(s) f(s)\, ds$$

for $u \in B_\mathcal{S}$ and $f \in L^1(G)$.

Proposition 1.2.13. *Let $\mathcal{S}$ and $\mathcal{S}'$ be collections of representations of a locally compact group G. The following are equivalent:*

1. *$B_\mathcal{S} \subset B_{\mathcal{S}'}$,*
2. *$\mathcal{S}$ is weakly contained in $\mathcal{S}'$,*
3. *Every positive definite function u in $A_\mathcal{S}$ is the limit of a net $\{u_i\}$ of positive definite functions in $A_{\mathcal{S}'}$ with respect to uniform convergence on compact subsets of G,*
4. *Every matrix coefficient u in $A_\mathcal{S}$ is the limit of a bounded net $\{u_i\}$ of matrix coefficients in $A_{\mathcal{S}'}$ with respect to uniform convergence on compact subsets of G.*

Proposition 1.2.14. *Let π be a representation of a locally compact group G. Then*

1. *B_π is a subalgebra of $B(G)$ if and only if $\pi \otimes \pi$ is weakly-contained in π,*
2. *B_π is an ideal of $B(G)$ if and only if $\pi \otimes \sigma$ is weakly-contained in π for every representation σ of G.*

The Fourier and Fourier-Stieltjes spaces have nice restriction properties.

Proposition 1.2.15. *Let G be a locally compact group, H a closed subgroup of G, and π a representation of G. Then $A_\pi|_H = A_{\pi|_H}$. If H is an open subgroup of G, then $B_\pi|_H = B_{\pi|_H}$. Further, these restriction maps are quotient maps.*

Below we list some additional properties of these spaces which we will make use of in this thesis.

Proposition 1.2.16. *Let G be a locally compact group.*

1. *Suppose H is an open subgroup of G. Then we can extend every element $u \in B(H)$ to an element $\dot{u}$ in $B(G)$ by defining $\dot{u}(s) = 0$ for $s \in G \backslash H$. The map $u \mapsto \dot{u}$ is an isometry.*
2. *Let π and σ be representations of G. Then $A_{\pi\oplus\sigma} = A_\pi + A_\sigma = \{u+v : u \in A_\pi, v \in A_\sigma\}$ and $B_{\pi\oplus\sigma} = B_\pi + B_\sigma$.*

The *Fourier algebra* $A(G)$ is the Fourier space A_λ. Observe that since $\lambda \otimes \sigma$ is unitarily equivalent to an amplification of λ for every representation σ of G by Fell's absorption principle, $A(G)$ is an ideal of the Fourier algebra. The Fourier algebra is the predual of the group von Neumann algebra $VN(G) := VN_\lambda(G)$ and is a very important object of study in abstract harmonic analysis. When G is abelian, $A(G)$ is isometrically isomorphic as a Banach algebra to $L^1(\widehat{G})$ by applying the Fourier transform to functions in $L^1(\widehat{G})$. Below we list a few alternate characterizations and a couple of important theorems for the Fourier algebra.

Theorem 1.2.17. *Let G be a locally compact group.*

1. *$A(G)$ is the norm closure of $B(G) \cap C_c(G)$ in $B(G)$,*
2. *$A(G)$ is the norm closed linear span of $P(G) \cap C_c(G)$ in $B(G)$,*
3. *$A(G)$ is the norm closed linear span of $P(G) \cap L^2(G)$ in $B(G)$,*
4. *$A(G) = \{f * g^\vee : f, g \in L^2(G)\}$ and $\|u\| = \inf\{\|f\|_2\|g\|_2 : u = f * g^\vee\}$ for each $u \in A(G)$ (where $g^\vee(s) := g(s^{-1})$). Further, this infimum is attained.*

Theorem 1.2.18 (Herz's restriction theorem)**.** *Let G be a locally compact group and H a closed subgroup of G. Then $A(G)|_H = A(H)$.*

Theorem 1.2.19 (Leptin [32])**.** *Let G be a locally compact group. Then G is amenable if and only if $A(G)$ has a bounded approximate identity.*

Chapter 2

Constructions of exotic group C*-algebras

2.1 Introduction

An early class of exotic group C*-algebras is due to Bekka, Kaniuth, Lau, and Schlichting. Let G be a locally compact group and G_d be the group G endowed with the discrete topology. In their 1996 paper (see [5]), these authors give a characterization of when λ_G, the left regular representation of G viewed as a representation of G_d, is weakly contained in λ_{G_d}, the left regular representation of G_d. For a large class of groups G where λ_{G_d} does not weakly contain λ_G, the group C*-algebra $C^*_{\lambda_G}(G_d)$ lies strictly between the reduced and full group C*-algebras.

Recall that a group G is *residually finite* if homomorphisms $\phi_i : G \to G_i$ into finite groups G_i separate points in G. Similarly, a C*-algebra $\mathcal{A}$ is *residually finite dimensional* if $*$-representations $\pi_i : \mathcal{A} \to B(\mathcal{H}_i)$ on finite dimensional Hilbert spaces $\mathcal{H}_i$ separate points of $\mathcal{A}$. Bekka produced another class of exotic group C*-algebras by comparing these two concepts.

In 1999, Bekka demonstrated a very large class of arithmetic groups, which are automatically residually finite, whose full group C*-algebras are not residually finite dimen-

sional (see [3]). For such Γ, the group C*-algebra associated with the finite dimensional representations of Γ, $C^*_{\mathcal{F}}(\Gamma)$, is an exotic group C*-algebra when Γ is maximally almost periodic.

Recently, Brown and Guentner introduced the notion of ideal completions for discrete groups Γ (see [8]). This allows one to construct group C*-algebras of Γ associated to $\ell^p(\Gamma)$ (denoted $C^*_{\ell^p}(\Gamma)$) for $1 \leq p < \infty$. It turns out that the only interesting cases to consider are when $p \in (2, \infty)$ (see [8, Proposition 2.11]). Let $\mathbb{F}_d$ be a free group on $2 \leq d < \infty$ generators. In [8, Proposition 4.2], Brown and Guentner show that there exists a $p \in (2, \infty)$ so that $C^*_{\ell^p}(\Gamma)$ is an intermediate C*-algebra. Subsequently, Okayasu was able to adapt arguments due to Haagerup to show that each of these C*-algebras are distinct for $2 \leq p < \infty$ (see [36]), thus giving an infinite chain of intermediate C*-algebras associated to $\mathbb{F}_d$. It follows that the C*-algebras $C^*_{\ell^p}(\Gamma)$ are all distinct for any discrete group Γ containing a copy of the free group.

In this chapter, we aim to compare these existing constructions and introduce new constructions of exotic C*-algebras associated to a discrete group Γ. In section 2 we provide the necessary background on ideal completions and prove some supplementary results. Section 3 introduces an intuitive lattice structure which can be placed on the group C*-algebras of Γ. With the exception of examples given in section 2, all of our new constructions of C*-algebras arise by using this lattice structure. In sections 4 and 5 we focus our attention towards studying intermediate C*-algebras on specific groups. Section 4 studies $SL_n(S)$ where S is a dense subring of $\mathbb{R}$ while section 5 analyzes $SL_n(\mathbb{Z})$. Specific attention is paid in comparing the exotic group C*-algebras associated to ℓ^p with the constructions due to [5] and [3], respectively.

2.2 L^p-representations and associated C*-algebras

The theory of L^p-representations and their corresponding C*-algebras for discrete groups was recently developed by Brown and Guentner in [8]. This paper has inspired further work by a number of other authors (see [6, 27, 36, 53, 54]). Though Brown and Guentner defined L^p-representations in the context of discrete groups, their definitions and basic

results generalize immediately to our context of locally compact groups. Rather than making explicit note of this, we will simply state their results in the context of locally compact groups.

Let G be a locally compact group and D a linear subspace of $C_b(G)$. A representation $\pi : G \to B(\mathcal{H}_\pi)$ is said to be a *D-representation* if there exists a dense subspace $\mathcal{H}_0$ of $\mathcal{H}_\pi$ so that $\pi_{x,x} \in D$ for every $x \in \mathcal{H}_0$. The following facts are noted in [6] and [8], and are easily checked (see Proposition 1.2.12):

- The D-representations are closed under arbitrary direct sums.
- If D is a subalgebra of $C_b(G)$, then the tensor product of two D-representations remains a D-representation.
- If D is an ideal of $C_b(G)$, then the tensor product of a D-representation with any representation is a D-representation.

For our purposes, we will be most interested in studying the case when $D = L^p(G) \cap C_b(G)$ for $p \in [1, \infty)$. In this case, the left regular representation λ of G is an L^p-representation since the dense subspace $C_c(G)$ of $L^2(G)$ clearly satisfies the required condition.

To each linear subspace D of $C_b(G)$ define a C*-seminorm $\| \cdot \|_D : L^1(G) \to [0, \infty)$ by

$$\|f\|_D = \sup\{\|\pi(f)\| : \pi \text{ is a } D\text{-representation}\}.$$

The C*-algebra $C^*_D(G)$ is defined to be the "completion" of $L^1(G)$ with respect to this C*-seminorm. When $D = L^p$ $(= L^p(G) \cap C_b(G))$, we write $C^*_{L^p}(G) = C^*_D(G)$ and $\| \cdot \|_{L^p} = \| \cdot \|_D$. This process of building C*-algebras was originally applied in the case when D was an ideal of $\ell^\infty(\Gamma)$ for a discrete group Γ, and was called an *ideal completion* (see [8]). We note that in the case when $D = L^p$ for some $p \in [1, \infty)$, then $\| \cdot \|_{L^p}$ dominates the reduced C*-norm since λ is an L^p-representation. A fortiori $\| \cdot \|_{L^p}$ is a norm on $L^1(G)$ and the identity map on $L^1(G)$ extends to a quotient map from $C^*_{L^p}(G)$ onto $C^*_r(G)$.

In general, it is desirable that the space $D \subset C_b(G)$ used in this construction is translation invariant (under both left and right translation). Indeed, this guarantees that if u is a positive definite function on G which lies in D, then the GNS representation of u is a D-representation (see [8, Lemma 3.1]) and, hence, u extends to a positive linear functional on $C^*_D(G)$.

The subspaces D of $C_b(G)$ which have been most heavily studied in the context of D-representations are $C_0(G)$ and $L^p(G) \cap C_b(G)$. Brown and Guentner recognized both of these cases and developed much of the basic theory for the associated C*-algebras in their original paper. In the case when $D = L^p$, Brown and Guentner demonstrated that $C^*_{L^p}(G) = C^*_r(G)$ for every $p \in [1, 2]$ (see [8, Proposition 2.11]), and that if $C^*_{L^p}(G) = C^*(G)$ for some $p \in [1, \infty)$ then G is amenable (see [8, Proposition 2.12]).

Suppose G is a locally compact group, H is a closed subgroup, and μ a quasi-invariant measure on the homogeneous space G/H. We say the homogeneous space G/H is amenable if $L^\infty(G/H, \mu)$ admits a G-invariant mean (see [17]). This leads one to consider the question: can we give an analogous characterization of amenability of G/H as mentioned above? We partially address this question in the case when G is a discrete group.

For fixed a fixed subgroup H of a discrete group Γ, define

$$D_p = D_p(H) = \{f \in \ell^\infty(\Gamma) : f|_{sHt} \in \ell^p(sHt) \text{ for } s, t \in \Gamma\}.$$

Is it the case that $C^*_{D_p}(\Gamma) = C^*_{\ell^p}(\Gamma)$ if and only if Γ/H is amenable? In this case, taking H to be the trivial subgroup would recover the original result. Unfortunately, we do not know the answer to this question but we have attained some partial results including the reverse implication.

Proposition 2.2.1. *Let Γ be a discrete group and H a subgroup of Γ. If Γ/H is amenable, then $C^*_{D_p}(\Gamma) = C^*_{\ell^p}(\Gamma)$ canonically for every $p \in [1, \infty)$.*

Proof. For $s \in \Gamma$ and $f : \Gamma/H \to \mathbb{C}$, we let f_s denote the left translation of f by s. Since Γ/H is amenable, there exists a net $\{f_i\}$ of finitely supported functions in Γ/H with $\|f_i\|_2 = 1$ so that $\|(f_i)_s - f_i\|_2 \to 0$ for every $s \in \Gamma$ (see [17, p. 28]). Let σ be the induced representation $\mathrm{Ind}_H^\Gamma 1_H$. Then σ_{f_i, f_i} are positive definite functions converging pointwise to the trivial representation.

Fix a positive definite function $\varphi \in D_p$. Then, since σ_{f_i, f_i} is supported on only finitely many cosets sH for each i, we have that $\varphi\sigma_{f_i, f_i} \in \ell^p(\Gamma)$ and, hence, extends to a positive linear functional on $C^*_{\ell^p}(\Gamma)$ for every i. Since it is also the case that $\varphi\sigma_{f_i, f_i} \to \varphi$ pointwise, we conclude that φ extends to a positive linear functional on $C^*_{\ell^p}(\Gamma)$.

Now let φ be an arbitrary positive linear functional on $C^*_{D_p}(\Gamma)$. Then we can find a net $\{\varphi_i\}$ of sums of positive definite functions associated to D_p-representations converging pointwise to φ. By approximating each φ_i by positive definite functions in D_p, we may assume that $\{\varphi_i\} \subset D_p$. Then, since each φ_i extends to a positive linear functional on $C^*_{\ell^p}(\Gamma)$ and φ is the pointwise limit of these positive definite functions, we conclude that φ extends to a positive linear functional on $C^*_{\ell^p}(\Gamma)$. Hence, $\|x\|_{D_p} \leq \|x\|_{\ell^p}$ for every $x \in \mathbb{C}[\Gamma]$. As the reverse inequality is clear, we conclude that $C^*_{D_p}(\Gamma) = C^*_{\ell^p}(\Gamma)$ canonically. □

Proposition 2.2.2. *Let $p \in [1, \infty)$ and suppose $\Gamma = H \times K$. If $C^*_{\ell^p}(\Gamma) = C^*_{D_p(H)}(\Gamma)$, then K is amenable.*

Proof. Suppose that $C^*_{\ell^p}(\Gamma) = C^*_{D_p}(\Gamma)$ and let $\omega \in \ell^p(H)$ be a normalised positive definite function on H. Define $\varphi : \Gamma \to \mathbb{C}$ by $\varphi(h,k) = \omega(h)$. Then φ is a positive definite function which lies in $D_p(H)$ and, hence, extends to a positive linear functional on $C^*_{\ell^p}(\Gamma)$. So we can find a net $\{\varphi_i\}$ of positive definite functions in ℓ^p converging pointwise to φ.

Choose n large enough so that $p/n \leq 2$ and define $\psi = \varphi^n$, $\psi_i = \varphi_i^n$. Then $\{\psi_i\}$ is a net of ℓ^2-summable positive definite functions converging pointwise to ψ. Hence, ψ extends to a positive linear functional on $C^*_r(\Gamma)$. Thus, there is a net $\{f_i\} \subset \ell^2(\Gamma)$ with $\|f_i\|_2 = 1$ so that $\{\lambda_{f_i,f_i}\}$ converges to ψ pointwise.

Define $g_i : K \to \mathbb{C}$ by $g_i(k) = \|f_i|_{H\times\{k\}}\|_2$. Then $\|g_i\|_2 = 1$ for every i. Further,

$$\begin{aligned}
\left|\lambda_{f_i,f_i}(e,k)\right| &= \left| \sum_{(h,k') \in H\times K} f_i(h, k^{-1}k') f_i(h,k') \right| \\
&\leq \sum_{k' \in K} \left\| (f_i)_{(e,k^{-1})}|_{H\times\{k'\}} \cdot f_i|_{H\times\{k'\}} \right\|_1 \\
&\leq \sum_{k' \in K} \left\| f_i|_{H\times\{k^{-1}k'\}} \right\|_2 \left\| f_i|_{H\times\{k'\}} \right\|_2 \\
&= \sum_{k' \in K} g_i(k^{-1}k') g_i(k') \\
&= \lambda_{g_i,g_i}(k) \leq 1.
\end{aligned}$$

Consequently, $\{\lambda_{g_i,g_i}\}$ converges pointwise to the trivial representation since $\{\lambda_{f_i,f_i}(e,k)\}$ converges to $\psi(e,k) = 1$ for every $k \in K$. Hence, K is amenable. □

Brown and Guentner demonstrated that $C^*_{\ell^p}(\mathbb{F}_d) \neq C^*_r(\mathbb{F}_d)$ for some $p \in (2,\infty)$ where $\mathbb{F}_d$ is the free group on $2 \leq d < \infty$ generators (see [8, Proposition 4.14]). Subsequently, Okayasu showed that the C*-algebras $C^*_{\ell^p}(\mathbb{F}_d)$ are pairwise not canonically isomorphic for every $p \in [2,\infty)$ (this was also independently shown by both Higson and Ozawa). Making use of this fact, the following result immediately implies that an analogous result holds for every discrete group Γ containing a copy of $\mathbb{F}_2$.

Theorem 2.2.3. *Let H be an open subgroup of a locally compact group G and $\sigma : H \to \mathcal{H}$ an L^p-representation of H. Then $\pi := \mathrm{Ind}_H^G \sigma$ is an L^p-representation of G.*

Proof. Let $q : G \to G/H$ denote the canonical quotient map. Recall that the induced representation π is given by left translation on the completion $\mathcal{F}$ of the space

$$\mathcal{F}_0 = \{f : G \to \mathcal{H} \mid q(\operatorname{supp} f) \text{ is finite and } f(s\xi) = \sigma(\xi^{-1}) f(s) \text{ for all } s \in G, \xi \in H\}$$

with respect to the inner product

$$\langle f, g \rangle = \sum_{tH \in G/H} \langle f(t), g(t) \rangle_\sigma .$$

Let $\mathcal{H}_0$ be a dense linear subspace of $\mathcal{H}$ such that $\sigma_{x,y} \in \ell^p(H)$ for every $x, y \in \mathcal{H}_0$ (if $\pi_{x,x} \in \ell^p(H)$ for every $x \in \mathcal{H}_0$, then $\pi_{x,y} \in \ell^p(H)$ for all $x, y \in \mathcal{H}_0$ by the polarization identity). Fix a set of representatives $\{r_i\}_{i \in G/H}$ for G/H. Then the span of the functions $f \in \mathcal{F}_0$ such that $f(r_i)$ is nonzero for at most one i and $f(r_i) \in \mathcal{H}_0$ is dense in $\mathcal{F}$.

Fix f and g as above. Without loss of generality, we may assume that f and g are nonzero. Let i and j be the indices such that $f(r_i) \neq 0$, $g(r_j) \neq 0$. Then

$$\begin{aligned}
\sum_{s \in G} |\pi_{f,g}(s)|^p &= \sum_{s \in G} \left| \sum_{k \in G/H} \big\langle f(s^{-1} r_k), g(r_k) \big\rangle \right|^p \\
&= \sum_{s \in G} \left| \big\langle f(s^{-1} r_j), g(r_j) \big\rangle \right|^p \\
&= \sum_{\xi \in H} \left| \big\langle f(r_i \xi), g(r_j) \big\rangle \right|^p \\
&= \sum_{\xi \in H} \left| \big\langle \sigma(\xi^{-1}) f(r_i), g(r_j) \big\rangle \right|^p \\
&= \|\sigma_{f(r_i), g(r_j)}\|_p^p < \infty.
\end{aligned}$$

It follows that π is an L^p-representation. $\square$

Clearly this result implies that $\|\cdot\|_{L^p(G)}|_{L^1(H)} = \|\cdot\|_{L^p(H)}$ when H is an open subgroup of G and, so, indeed it follows that if Γ is a discrete group containing a copy of a noncommutative free group then $C^*_{\ell^p}(\Gamma)$ are pairwise not canonically isomorphic for every $p \in [2, \infty)$.

Much of the attention in this section has been focused towards the chain of L^p-C*-algebras. This raises the question, can we find exotic group C*-algebras which lie off this chain? We end this section by showing that the C*-algebras arising from the ideals D_p can satisfy this criteria.

Example 2.2.4. Fix $p \in [2, \infty)$ and let $\mathbb{F}_\infty$ be the free group on countably many generators $a_1, a_2, \ldots$ and view $\mathbb{F}_d$ as the subgroup of $\mathbb{F}_\infty$ generated by $a_1, \ldots, a_d$ for $d < \infty$. Take $H = \mathbb{F}_d$ for some fixed $2 \leq d < \infty$. Let $\varphi_\alpha : \mathbb{F}_\infty \to \mathbb{C}$ be the positive definite function defined by $\varphi_\alpha(s) = \alpha^{|s|}$ for each $\alpha \in (0, 1)$ (see [21, Lemma 1.2]). Then $\varphi_\alpha \in D_p = D_p(\mathbb{F}_d)$ for each $\alpha < (2d-1)^{-1/p}$ since

$$\sum_{s \in \mathbb{F}_d} \varphi_\alpha(t_1 s t_2) \leq \sum_{s \in \mathbb{F}_d} \alpha^{-|t_1|} \alpha^{-|t_2|} \varphi_\alpha(s)$$

for every $t_1, t_2 \in \mathbb{F}_\infty$ and $\sum_{s \in \mathbb{F}_d} \varphi_\alpha(s) < \infty$ if and only if $\alpha < (2d-1)^{-1/p}$. Hence, φ_α extends to a positive linear functional on $C^*_{D_p}(\mathbb{F}_\infty)$ for each $\alpha \leq (2d-1)^{-1/p}$. By [36, Corollary 3.5], we have that $\varphi_\alpha|_{\mathbb{F}_d}$ extends to a positive linear functional on $C^*_{\ell^p}(\mathbb{F}_d)$ if and only if $\alpha \leq (2d-1)^{-1/p}$. Therefore, this condition of $\alpha \leq (2d-1)^{-1/p}$ is necessary and sufficient for φ_α to extend to a positive linear functional on $C^*_{D_p}(\mathbb{F}_\infty)$.

Fix $\alpha \in (0, 1)$ and choose a positive integer d' large enough so that $(2d'-1)^{-1/p} < \alpha$. Then $\varphi_\alpha|_{\mathbb{F}_{d'}}$ does not extend to a positive linear functional on $C^*_{\ell^p}(\mathbb{F}_{d'})$. Hence, φ_α does not extend to a positive linear functional on $C^*_{\ell^p}(\mathbb{F}_\infty)$ for any $\alpha \in (0, 1)$. Therefore there is no canonical quotient map from $C^*_{\ell^q}(\mathbb{F}_\infty)$ to $C^*_{D_p}(\mathbb{F}_\infty)$ for any $p, q \in [2, \infty)$. Conversely, it is an easy consequence of Theorem 2.2.3 that there is no canonical quotient map from $C^*_{D_p}(\mathbb{F}_\infty)$ to $C^*_{\ell^q}(\mathbb{F}_\infty)$ for any $q > p \geq 2$.

2.3 The lattice of group C*-algebras

So far we have nonchalantly been talking about notions such as chains of group C*-algebras. We are able to do this because there is a natural partial ordering which can be placed on the group C*-algebras. In this section, we make this notion of a partial ordering specific and show that the group C*-algebras form a complete $\bigvee$-semilattice. This will allow us to build new exotic group C*-algebras.

Definition 2.3.1. Let G be a locally compact group. By a group C*-algebra associated to G, we will mean a C*-completion of $L^1(G)$. Place a partial ordering on the group C*-algebras by saying that $\mathcal{A} \preceq \mathcal{A}'$ if $\|f\|_{\mathcal{A}} \leq \|f\|_{\mathcal{A}'}$ for every $f \in L^1(G)$. Equivalently, we have that $\mathcal{A} \preceq \mathcal{A}'$ if and only if the identity map on $L^1(G)$ extends to a quotient from $\mathcal{A}'$ to $\mathcal{A}$.

We observe that with this definition, the group C*-algebras form a complete $\bigvee$-semilattice. Indeed, if $\{\mathcal{A}_i\}$ is a collection of group C*-algebras, then the completion of $L^1(G)$ with respect to the C*-norm $\|\cdot\|$ defined by $\|f\| = \sup_i \|f\|_{\mathcal{A}_i}$ for $f \in L^1(G)$ is the join $\bigvee_i \mathcal{A}_i$.

Note that it also makes sense to talk about the supremum and infimum of "completions" of $L^1(G)$ with respect to a C*-seminorm. If $\mathcal{S}$ is a collection of representations of G, then $C^*_{\mathcal{S}}(G)$ is defined to be the "completion" of $L^1(G)$ with respect to the C*-seminorm $\|x\|_{\mathcal{S}} := \sup_{\pi \in \mathcal{S}} \|\pi(x)\|$ (as in [16]). Moreover, every C*-seminorm arises in this way where $\mathcal{S}$ can be assumed to be a Fell closed subset of the irreducible representations $\widehat{G}$ (see [4, Proposition F.2.7]). Further, if $\mathcal{S}$ and $\mathcal{S}'$ are Fell closed subsets of $\widehat{G}$, then $C^*_{\mathcal{S}}(G) \preceq C^*_{\mathcal{S}'}(G)$ if and only if $\mathcal{S} \subset \mathcal{S}'$. If we place the same lattice structure on these "completions" as above, we get a complete lattice. Indeed, let $\{\mathcal{A}_i\}$ be a collection of such "completions" and write $\mathcal{A}_i = C^*_{\mathcal{S}_i}(\Gamma)$ for Fell closed subsets $\{\mathcal{S}_i\} \subset \widehat{\Gamma}$. Then $\bigwedge_i \mathcal{A}_i = C^*_{\cap \mathcal{S}_i}$. As the join arises as before, we conclude that we indeed get a complete lattice.

In the remainder of this section, we focus on using this lattice structure to produce a new class of examples of exotic group C*-algebras. Towards this goal, we give a characterization of when $C^*_D(G) = C^*(G)$ in terms of when the D-representations weakly contain

an *amenable representation*, i.e., a representation $\pi : \Gamma \to B(\mathcal{H})$ for which there exists a state μ on $B(\mathcal{H})$ such that $\mu(\pi(s)T\pi(s^{-1})) = \mu(T)$ for all $s \in \Gamma$ and $T \in B(\mathcal{H})$ (see [2]).

Proposition 2.3.2. *Let $D \triangleleft \ell^\infty(\Gamma)$ be an algebraic ideal. Then $C^*_D(\Gamma) = C^*(\Gamma)$ if and only if the D-representations weakly contain an amenable representation.*

Proof. If $C^*_D(\Gamma) = C^*(\Gamma)$, then the D-representations weakly contain all representations of Γ and, in particular, weakly contains the trivial representation which is evidently amenable.

Conversely, suppose that D weakly contains an amenable representation π. Then we can find a net $\{\pi_i\}$ of D-representations converging in Fell's topology to π. Note that $\overline{\pi}\otimes\pi_i$ is a D-representation for every i and $\overline{\pi}\otimes\pi_i$ converges to $\overline{\pi}\otimes\pi$ in the Fell topology. So the D-representations weakly contain $\overline{\pi}\otimes\pi$ and, hence, the trivial representation (see [2, Theorem 5.1]). A similar argument as used above now shows that the D-representations weakly contain every representation of Γ since $\pi \otimes 1 = \pi$ for every π. Hence, $C^*_D(\Gamma) = C^*(\Gamma)$. □

Remark 2.3.3. In particular, the previous proposition shows that if $C^*_D(\Gamma) \neq C^*(\Gamma)$, then the D-representations do not weakly contain any finite dimensional representations. Hence, if $C^*_D(\Gamma)$ is a group C*-algebra not coinciding with $C^*(\Gamma)$, then $C^*_D(\Gamma) \vee C^*_\pi(\Gamma)$ is a strictly larger C*-algebra than $C^*_D(\Gamma)$ for every finite dimensional representation and the only group C*-algebra $\mathcal{A}$ produced by an ideal completion for which $\mathcal{A} \succeq C^*_D(\Gamma)$ is $C^*(\Gamma)$.

Suppose Γ contains a copy of the free group and π is a finite dimensional representation of Γ. What does the group C*-algebra $C^*_D(\Gamma) \vee C^*_\pi(\Gamma)$ look like when we take $D = \ell^p$? Could it be the case that $C^*_{\ell^p}(\Gamma) \vee C^*_\pi(\Gamma)$ coincides with $C^*(\Gamma)$? Could $C^*_{\ell^p}(\Gamma) \vee C^*_\pi(\Gamma)$ dominate $C^*_{\ell^q}(\Gamma)$ for some $q > p \geq 2$? It turns out that neither of these cases can occur:

Proposition 2.3.4. *Suppose Γ contains a copy of the free group and $\mathcal{F}_0$ is a finite nonempty subset of the finite dimensional representations on Γ. Then $C^*_{\ell^q}(\Gamma) \not\preceq C^*_{\mathcal{F}_0}(\Gamma) \vee C^*_{\ell^p}(\Gamma)$ for any $q > p \geq 2$.*

Proof. Without loss of generality, we may assume that $\mathcal{F}_0$ is a subset of $\widehat{\Gamma}$. For each $p \geq q$, write $C^*_{\ell^p}(\Gamma) = C^*_{\mathcal{S}_p}(\Gamma)$ for some Fell closed subset $\mathcal{S}_p \subset \widehat{\Gamma}$. Then, since $\mathcal{S}_p$ is a proper subset of $\mathcal{S}_q$ for every $q > p$, $\mathcal{S}_q \backslash \mathcal{S}_p$ has infinite cardinality for $q > p$ (as there is an infinitude

of intermediate C*-algebras between $C^*_{\ell^p}(\Gamma)$ and $C^*_{\ell^p}(\Gamma)$). Now, write $C^*_{\ell^p}(\Gamma) \vee C^*_{\mathcal{F}_0}(\Gamma) = C^*_{\mathcal{S}_0}(\Gamma)$ for some Fell closed $S_0 \subset \widehat{\Gamma}$. Then since, $\mathcal{F}_0$ is a closed subset of $\widehat{\Gamma}$ in the Fell topology, $\mathcal{S}_0 \backslash \mathcal{S} = \mathcal{F}_0$ has finite cardinality. Hence, $C^*_{\ell^p}(\Gamma) \vee C^*_{\mathcal{F}_0}(\Gamma) \not\geq C^*_{\ell^q}(\Gamma)$ for $q > p$. $\square$

This gives us another class of examples of exotic group C*-algebras which lie off the chain $C^*_{\ell^p}(\Gamma)$. Note that we may always take $\mathcal{F}_0$ to be the singleton containing the trivial representation. Hence, for Γ containing a copy of the free group, this construction can always be used to produce exotic group C*-algebras differing from $C^*_{\ell^p}(\Gamma)$.

2.4 Exotic group C*-algebras of $SL_n(S)$

Let G be a locally compact group and G_d be the group G endowed with the discrete topology. Denote the left regular representation of G by λ_G. Then λ_G is a representation of G_d. In [5] Bekka, Kaniuth, Lau, and Schlichting show that the group C*-algebra $C^*_{\lambda_G}(G_d)$ is the reduced C*-algebra if and only if G admits an open subgroup H so that H_d is amenable. In particular, if G is a connected non-amenable group such as $SL_n(\mathbb{R})$ ($n \geq 2$), then $C^*_{\lambda_G}(G_d) \neq C^*_r(G_d)$. This inspires us to study the group C*-algebra $C^*_\delta(SL_n(S)) := C^*_{\lambda_{SL_n(\mathbb{R})}}(SL_n(S))$ where S is taken to be a dense (unital) subring of $\mathbb{R}$. We will demonstrate that $C^*_\delta(SL_n(S))$ is an exotic group C*-algebra and compare it to the C*-algebras $C^*_{\ell^p}(SL_n(S))$. Our first proposition shows that $C^*_{\ell^p}(SL_n(S)) \not\geq C^*_\delta(SL_n(S))$ for any $1 \leq p < \infty$.

Before proceeding to this proposition, we mention a result due to Breuillard and Gellander which we will make use of. In [7], these authors demonstrated that if Γ is a dense subgroup of a connected semi-simple real Lie group G, then Γ contains a copy of the free group on two generators which is dense in G. Moreover, their proof shows that these generators can be chosen arbitrarily close to the identity. (Compare the following statement to that of [5, Proposition 5]).

Proposition 2.4.1. *Let Γ be a dense subgroup of a connected semi-simple real Lie group G. If a representation π of the discrete group Γ is continuous in the ambient topology, then π is not weakly contained in the ℓ^p-representations for each $1 \leq p < \infty$.*

Proof. Let π be a continuous representation of Γ and $x \in \mathcal{H}_\pi$ be a unit vector. Then $\varphi := \pi_{x,x}$ is a continuous function on Γ with $\varphi(e) = 1$. We will demonstrate that φ does not extend to a state on $C^*_{\ell^p}(\Gamma)$ for any $1 \leq p < \infty$.

In the proof of [36, Theorem 3.4 (2)], Okayasu shows that a normalized positive definite function ψ on $\mathbb{F}_2$ extends to a state on $C^*_{\ell^p}(\mathbb{F}_2)$ if and only if $\|\psi\chi_k\|_p \leq k+1$ (where χ_k is the characteristic function of the set W_k of words with length k). Choose k large enough so that $(4 \cdot 3^{k-1})^{1/p} > k+1$. Next choose generators for a free subgroup of Γ close enough to the identity so that $|\varphi(s)| > \frac{k+1}{(4\cdot 3^{k-1})^{1/p}}$ for all $s \in W_k$. Then, on this copy of $\mathbb{F}_2$,

$$\|\varphi\chi_k\|_p > \left(|W_k| \left(\frac{k+1}{(4 \cdot 3^{k-1})^{1/p}}\right)^p\right)^{1/p} = k+1.$$

Hence, φ does not extend to a positive linear functional on $C^*_{\ell^p}(\mathbb{F}_2)$ and, so, we conclude that π is not weakly contained in the ℓ^p-representations. □

To observe that the previous proposition applies to our situation, we note that $SL_n(S)$ is a dense subgroup of $SL_n(\mathbb{R})$ since the subgroups

$$\begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{bmatrix}$$

generate $SL_n(\mathbb{R})$.

This proposition demonstrates that $C^*_\delta(SL_n(S))$ is a strictly larger group C*-algebra than $C^*_r(SL_n(S))$. Bekka, Kaniuth, Lau and Schlichting's result [5, Proposition 5] implies that this is the case when S is taken to be all of $\mathbb{R}$, but it was not a priori obvious that this would continue to hold for smaller rings S.

We are now led to ask similar questions as in the previous section. Could $C^*_\delta(SL_n(S))$ be the full group $C^*(SL_n(S))$? How does $C^*_\delta(SL_n(S)) \vee C^*_{\ell^p}(SL_n(S))$ compare to $C^*_{\ell^q}(SL_n(S)$? These questions are quite satisfactorily answered in the following proposition.

Proposition 2.4.2. *Suppose $q > p \geq 2$. Then $C^*_\delta(SL_n(S)) \vee C^*_{\ell^p}(SL_n(S)) \not\geq C^*_{\ell^q}(SL_n(S))$.*

Proof. Suppose that φ is a normalized positive definite function on $SL_n(S)$ which extends to a state on $C^*_\delta(SL_n(S))$. We will show that $\varphi|_{SL_n(\mathbb{Z})}$ extends to a state on $C^*_r(SL_n(\mathbb{Z}))$.

Since φ extends to a state on $C^*_\delta(SL_n(S))$. Then we can find a net $\{\varphi_i\}$ of sums of positive definite functions associated to $\lambda_{SL_n(\mathbb{R})}$ which converge pointwise to φ. By Herz's restriction theorem, $\varphi_i|_{SL_n(Z)}$ lies in $A(SL_n(\mathbb{Z}))$, the Fourier algebra of $SL_n(\mathbb{Z})$. Hence, as each $\varphi_i|_{SL_n(\mathbb{Z})}$ is positive definite and $\varphi_i|_{SL_n(\mathbb{Z})}$ converges pointwise to $\varphi|_{SL_n(\mathbb{Z})}$, we conclude that $\varphi|_{SL_n(\mathbb{Z})}$ extends to a positive linear functional on $C^*_r(SL_n(\mathbb{Z}))$.

This shows us that $\|x\|_\delta \leq \|x\|_{\ell^2}$ for every $x \in \mathbb{C}[SL_n(\mathbb{Z})]$ since $\|x\|^2_\delta = \|x^*x\|_\delta = \sup \varphi(x^*x)$ where the supremum is taken over states φ on $C^*_\delta(SL_n(S))$. Hence,

$$\|x\|_{C^*_\delta(SL_n(S))\vee C^*_{\ell^p}(SL_n(S))} = \|x\|_{\ell^p}$$

for $x \in \mathbb{C}[SL_n(\mathbb{Z})]$. As $\|\cdot\|_{\ell^q}$ is a larger norm on $\mathbb{C}[SL_n(\mathbb{Z})]$ than $\|\cdot\|_{\ell^p}$, we conclude that $C^*_\delta(SL_n(S)) \vee C^*_{\ell^p}(SL_n(S)) \not\geq C^*_{\ell^q}(SL_n(S))$. □

We note that this proposition adds to our list of examples of exotic group C*-algebras.

Remark 2.4.3. Suppose G is a non-amenable group containing a discrete copy of the free group. A similar analysis shows that $C^*_{\lambda_G}(G_d)$ is not the full group C*-algebra. This justifies our comment in the introduction of this chapter that $C^*_{\lambda_G}(G_d)$ leads to a large class of exotic group C*-algebras.

2.5 Exotic group C*-algebras of $SL_n(\mathbb{Z})$

Let $\mathcal{F}$ denote the set of finite dimensional representations on $SL_n(\mathbb{Z})$. Notice that the natural homomorphisms from $SL_n(\mathbb{Z})$ to $SL_n(\mathbb{Z}/N\mathbb{Z})$ for $N \geq 1$ separate the points of $SL_n(\mathbb{Z})$. So the set $\mathcal{F}$ of finite dimensional representations separate points in $\mathbb{C}[SL_n(\mathbb{Z})]$ and $\|\cdot\|_{\mathcal{F}} \geq \|\cdot\|_r$ (see [3, Proposition 1]). Since $SL_n(\mathbb{Z})$ is non-amenable, we have by Proposition 2.3.2 that the left regular representation does not weakly contain any finite dimensional representations. Hence, $C^*_{\mathcal{F}}(SL_n(\mathbb{Z}))$ is strictly larger than the reduced C*-algebra.

In [3], Bekka demonstrates that the universal C*-algebra $C^*(SL_n(\mathbb{Z}))$ is not residually finite dimensional for $n \geq 3$, hence showing that $C^*_{\mathcal{F}}(SL_n(\mathbb{Z}))$ is an exotic C*-algebra for $n \geq 3$. In fact, Bekka's proof shows something stronger.

Recall that the *congruence subgroup* $\Gamma(N)$ of $SL_n(\mathbb{Z})$ is the kernel of the canonical map $SL_n(\mathbb{Z}) \to SL_n(\mathbb{Z}/N\mathbb{Z})$. Let $\mathcal{F}_0$ denote the set of all finite dimensional representations of $SL_n(\mathbb{Z})$ which factor through a congruence subgroup, i.e., whose kernel contains $\Gamma(N)$ for some N. What Bekka actually showed was that $C^*_{\mathcal{F}_0}(SL_n(\mathbb{Z}))$ is not the full group C*-algebra for $n \geq 2$ and that $\mathcal{F}_0 = \mathcal{F}$ when $n \geq 3$.

Our questions about the exotic group C*-algebra $C^*_{\mathcal{F}_0}(SL_n(\mathbb{Z}))$ are again similar to those asked in the previous two sections. Our first is how does $C^*_{\mathcal{F}_0}(SL_n(\mathbb{Z}))$ compare to $C^*_{\ell^p}(SL_n(\mathbb{Z}))$? We provide a partial answer to this question below.

Proposition 2.5.1. *Let $n \geq 2$. There exists $p \in (2, \infty)$ so that $C^*_{\mathcal{F}_0}(SL_n(\mathbb{Z})) \ncong C^*_{\ell^p}(SL_n(\mathbb{Z}))$.*

Proof. Note that if π is a representation of $SL_n(\mathbb{Z})$ which factors through a congruence subgroup, then the restriction of π to $SL_2(\mathbb{Z})$ factors through a congruence subgroup of $SL_2(\mathbb{Z})$. Hence, it suffices to consider the case when $n = 2$.

Note that for each $\alpha \in (0, 1)$, the positive definite function $\varphi_\alpha : \mathbb{F}_2 \to \mathbb{C}$ defined by $\phi_\alpha(s) = \alpha^{-|s|}$ lies in $\ell^p(\mathbb{F}_2)$ for some p. Let π_α denote the GNS representation of φ_α. Then, since φ_α converges pointwise to the trivial representation as $\alpha \nearrow 1$, π_α converges to $1_{\mathbb{F}_2}$ in the Fell topology.

Let $\mathbb{F}_2 \subset SL_2(\mathbb{Z})$ be a finite index embedding of the free group in $SL_2(\mathbb{Z})$. Then $\mathrm{Ind}_{\mathbb{F}_2}^{SL_2(\mathbb{Z})}\pi_\alpha$ converges to $\mathrm{Ind}_{\mathbb{F}_2}^{SL_2(\mathbb{Z})}1_{\mathbb{F}_2}$. Note that $\mathrm{Ind}_{\mathbb{F}_2}^{SL_2(\mathbb{Z})}1_{\mathbb{F}_2}$ contains a copy of $1_{SL_2(\mathbb{Z})}$ as a subrepresentation since $\mathbb{F}_2$ is of finite index in $SL_2(\mathbb{Z})$. Hence, $\mathrm{Ind}_{\mathbb{F}_2}^{SL_2(\mathbb{Z})}\pi_\alpha \to 1_{SL_2(\mathbb{Z})}$ in the Fell topology.

Fix a finite generating set S for $SL_2(\mathbb{Z})$. In the proof of [3, Lemma 3] Bekka shows that the trivial representation is isolated among the set of restrictions $\pi|_{SL_2(\mathbb{Z})}$ where π is a representation which factors through a congruence subgroup. This is to say that there exists $\epsilon > 0$ so that if $\pi : SL_2(\mathbb{Z}) \to B(\mathcal{H})$ is a representation which factors through a congruence subgroup with the property that there exists a unit vector $x \in \mathcal{H}$ so that $\|\pi(s)x - x\| < \epsilon$ for every $s \in S$, then π contains the trivial representation as a subrepresentation.

Since $\mathrm{Ind}_{\mathbb{F}_2}^{SL_2(\mathbb{Z})}\pi_\alpha \to 1_{SL_2(\mathbb{Z})}$ in the Fell topology, we can find α and a unit vector x in the corresponding Hilbert space so that $\|\pi_\alpha(s)x - x\| < \epsilon$. By Theorem 2.2.3, $\mathrm{Ind}_{\mathbb{F}_2}^{SL_2(\mathbb{Z})}\pi_\alpha$ is an ℓ^p-representation for some p and, hence, does not weakly contain a copy of the trivial representation. Therefore π_α is not weakly contained in $\mathcal{F}_0$. □

Note that since $C^*_{\ell^q}(SL_n(\mathbb{Z})) \succeq C^*_{\ell^p}(SL_n(\mathbb{Z}))$ when $q > p$, the proposition provides the same conclusion for all $q > p$. This answer to our question is not as clean as that provided in the previous section and we are still left with questions. Does the conclusion of the proposition hold for any $p > 2$? If not, can we provide nontrivial estimates on the values of p which provide the conclusion of the proposition?

We conclude this chapter by showing that $C^*_{\mathcal{F}_0}(\Gamma) \vee C^*_{\ell^p}(\Gamma)$ forms a class of exotic group C*-algebras. This proposition provides a similar conclusion as the last:

Proposition 2.5.2. *Let $n \geq 2$. For every $p \in [1, \infty)$, there exists $q > p$ so that $C^*_{\mathcal{F}_0}(SL_n(\mathbb{Z})) \vee C^*_{\ell^p}(SL_n(\mathbb{Z})) \not\succeq C^*_{\ell^q}(\Gamma)$.*

Proof. Again it suffices to consider the case when $n = 2$.

Let S and ϵ be as in the previous proposition. Take $\pi_p : SL_n(\mathbb{Z}) \to B(\mathcal{H}_p)$ to be a faithful representation of $C^*_{\ell^p}(SL_2(\mathbb{Z}))$. Then, since the ℓ^p-representations do not weakly contain the trivial representation, there exists $\epsilon' > 0$ so that whenever $x \in \mathcal{H}_p$ is a unit vector, there exists $s \in S$ so that $\|\pi_p(s)x - x\| \geq \epsilon'$.

Suppose that $\sigma : SL_n(\mathbb{Z}) \to B(\mathcal{H}_\sigma)$ in $\mathcal{F}_0$ does not contain a copy of the trivial representation. Take $(x, y) \in \mathcal{H}_p \oplus \mathcal{H}_\sigma$ to be a unit vector. If $\|x\| \geq 1/\sqrt{2}$, there exists $s \in S$ so that $\|\pi_p(s)x - x\| \geq \epsilon'/\sqrt{2}$ which implies that $\|(\pi_p \oplus \sigma)(s)(x, y) - (x, y)\| \geq \epsilon'/\sqrt{2}$. Similarly, if $\|y\| \geq 1/\sqrt{2}$, $\|(\pi_p \otimes \sigma)(s)(x, y) - (x, y)\| \geq \epsilon/\sqrt{2}$. Hence, $\|(\pi_p \oplus \sigma)(s)(x, y) - (x, y)\| \geq \min\{\epsilon, \epsilon'\}/\sqrt{2}$.

Since we have that $\pi_q \to 1$ in the Fell topology as $q \to \infty$, we can find q so that there exists a unit vector $x \in \mathcal{H}_q$ with $\|\pi_q(s)x - x\| < \min\{\epsilon, \epsilon'\}/\sqrt{2}$ for every $s \in S$. As π_q does not weakly contain the trivial representation, we conclude that π_q is not weakly contained in $\{\pi_p\} \cup \mathcal{F}_0$. □

Chapter 3

C*-norms for tensor products of discrete group C*-algebras

3.1 Introduction

We begin this chapter by briefly reviewing the basics of C*-algebra tensor products. Those results mentioned without reference in this section can be found [9, Chapter 3].

Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. It is always possible to put a C*-norm on the algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$. Indeed, if $\mathcal{A} \subset B(\mathcal{H})$ and $\mathcal{B} \subset B(\mathcal{K})$, then it can be verified that the canonical map from $\mathcal{A} \otimes \mathcal{B}$ into $B(\mathcal{H} \otimes \mathcal{K})$ is injective. The C*-norm on $\mathcal{A} \otimes \mathcal{B}$ coming from this inclusion is called the *spatial* or *minimal C*-norm* and is denoted $\| \cdot \|_{\min}$. This norm is independent of the choices of embeddings $\mathcal{A} \subset B(\mathcal{H})$ and $\mathcal{B} \subset B(\mathcal{K})$, and can be shown to be the smallest among all C*-norms on $\mathcal{A} \otimes \mathcal{B}$. Consequently, every C*-norm on the tensor product $\mathcal{A} \otimes \mathcal{B}$ is a cross norm, i.e., a norm such that $\|a \otimes b\| = \|a\|\|b\|$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. The completion of $\mathcal{A} \otimes \mathcal{B}$ with respect to $\| \cdot \|_{\min}$ is denoted $\mathcal{A} \otimes_{\min} \mathcal{B}$.

Defined analogously to the full group C*-norm of $L^1(G)$, the *maximal C*-norm* of $\mathcal{A} \otimes B$ is defined by

$$\|x\|_{\max} = \sup\{\|\pi(x)\| : \pi \text{ is a } *\text{-representation of } \mathcal{A} \otimes \mathcal{B}\}$$

for $x \in \mathcal{A} \otimes \mathcal{B}$. This is the largest C*-norm which can be placed on $\mathcal{A} \otimes \mathcal{B}$. The completion of $\mathcal{A} \otimes \mathcal{B}$ with respect to $\|\cdot\|_{\max}$ is denoted by $\mathcal{A} \otimes_{\max} \mathcal{B}$.

Just as is the case for C*-norms on $L^1(G)$, $\mathcal{A} \otimes \mathcal{B}$ need not admit a unique C*-norm. However, if $\mathcal{A}$ is a C*-algebra with the property that $\mathcal{A} \otimes \mathcal{B}$ admits a unique C*-norm for every second C*-algebra $\mathcal{B}$, then $\mathcal{A}$ is said to be *nuclear*. In 1973 Lance showed that a discrete group Γ is amenable if and only if $C_r^*(\Gamma)$ is nuclear (see [31]). Since the quotient of a nuclear C*-algebra is nuclear, this also gives the characterization that a discrete group Γ is amenable if and only if $C^*(\Gamma)$ is nuclear. Lance's proof, however, does not provide us with a specific example of a C*-algebra $\mathcal{B}$ such that $C_r^*(\Gamma) \otimes \mathcal{B}$ does not have a unique norm when Γ is non-amenable. As a particular case of one result in this chapter, we show that the algebraic tensor products $C_r^*(\Gamma) \otimes C_r^*(\Gamma)$ and $C^*(\Gamma) \otimes C_r^*(\Gamma)$ do not admit a unique C*-norm when Γ is a non-amenable discrete group. We do not determine whether $C^*(\Gamma) \otimes C^*(\Gamma)$ admits a unique C*-norm for non-amenable Γ, but note that finding the solution to this problem for the case when $\Gamma = \mathbb{F}_\infty$ would solve the Connes embedding problem (see [29]).

Recently Ozawa and Pisier demonstrated pairs of C*-algebras $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \otimes \mathcal{B}$ admits $2^{\aleph_0}$ distinct C*-norms (see [37]), including the case when $\mathcal{A} = \mathcal{B} = \mathcal{B}(\mathcal{H})$. Although their paper mainly focuses on von Neumann algebras, Ozawa and Pisier also show that $C_r^*(\mathbb{F}_d) \otimes C_r^*(\mathbb{F}_d)$ admits $2^{\aleph_0}$ distinct C*-norms where $\mathbb{F}_d$ is the noncommutative free group on $d \geq 2$ generators. We generalize this result by showing that if π is a representation which weakly contains λ and $1 \leq p < \infty$, then $C_\pi^*(\Gamma_1) \otimes C_{\ell^p}^*(\Gamma_2)$ admits $2^{\aleph_0}$ distinct C*-norms for every pair of discrete groups Γ_1 and Γ_2 containing copies of noncommutative free groups. As a special case of this result, we get that $C_r^*(\Gamma_1) \otimes C_r^*(\Gamma_2)$ and $C^*(\Gamma_1) \otimes C_r^*(\Gamma_2)$ each admit $2^{\aleph_0}$ C*-norms when Γ_1 and Γ_2 contain a copy of $\mathbb{F}_2$. In contrast to the approach of Ozawa and Pisier which takes advantage of the C*-simplicity of $C_r^*(\mathbb{F}_d)$, we make use of Fourier-Stieltjes spaces and the theory of L^p-representations to establish our results.

3.2 C*-norms of tensor products

The first step towards our results in this chapter is the following observation of a one-to-one correspondence between certain Fourier-Stieltjes spaces and tensor product norms for group C*-algebras.

Proposition 3.2.1. *Let G_1 and G_2 be locally compact groups and π_1, π_2 be representations of G_1 and G_2. We can identify the C*-norms on $C^*_{\pi_1}(G_1) \otimes C^*_{\pi_2}(G_2)$ with the Fourier-Stieltjes spaces B_σ of $G_1 \times G_2$ which satisfy $B_\sigma|_{G_1} = B_{\pi_1}$, $B_\sigma|_{G_2} = B_{\pi_2}$, and $B_\sigma \supset B_{\pi_1 \times \pi_2}$. For $f_1, \ldots, f_n \in L^1(G_1)$ and $g_1, \ldots, g_n \in L^1(G_2)$, the norm of $\sum_{i=1}^n \pi_1(f_i) \otimes \pi_2(g_i)$ associated to B_σ is given by*

$$\left\| \sum_{i=1}^n \pi_1(f_i) \otimes \pi_2(g_i) \right\| = \left\| \sum_{i=1}^n \sigma(f_i \times g_i) \right\|.$$

Proof. Note that since $L^1(G_1 \times G_2) = L^1(G_1) \widehat{\otimes} L^1(G_2)$ (where $\widehat{\otimes}$ denotes the projective Banach space tensor product), we may consider $*$-representations of $C^*_{\pi_1}(G_1) \otimes C^*_{\pi_2}(G_2)$ as being representations of $G_1 \times G_2$. It can be checked that this gives a one-to-one correspondence between $*$-representations of $C^*_{\pi_1}(G_1) \otimes C^*_{\pi_2}(G_2)$ and representations σ of $G_1 \times G_2$ such that $B_\sigma|_{G_1} \subset B_{\pi_1}$ and $B_\sigma|_{G_2} \subset B_{\pi_2}$. Moreover, this immediately gives that if σ is a representation of $G_1 \times G_2$ corresponding to a $*$-representation $\widetilde{\sigma}$ of $C^*_{\pi_1}(G_1) \otimes C^*_{\pi_2}(G_2)$ and $f_1, \ldots, f_n \in L^1(G_1)$, $g_1, \ldots, g_n \in L^1(G_2)$, then

$$\left\| \widetilde{\sigma}\Big(\sum_{i=1}^n \pi_1(f_i) \otimes \pi_2(g_i) \Big) \right\| = \left\| \sum_{i=1}^n \sigma(f_i \times g_i) \right\|.$$

Finally, a $*$-representation of $C^*_{\pi_1}(G_1) \otimes C^*_{\pi_2}(G_2)$ corresponding to a representation σ of $G_1 \times G_2$ separates points of $C^*_{\pi_1}(G_1) \otimes C^*_{\pi_2}(G_2)$ if and only if $\|\sigma(\cdot)\| \geq \|\pi_1 \times \pi_2(\cdot)\|$ on $L^1(G_1 \times G_2)$ if and only if $B_\sigma \supset B_{\pi_1 \times \pi_2}$. □

The key idea behind producing different C*-norms on tensor products of group C*-algebras in this section will be to construct Fourier-Stieltjes spaces satisfying the conditions of the above proposition which differ on the diagonal subgroup Δ of $\Gamma \times \Gamma$. We begin with the following lemma.

Lemma 3.2.2. *Suppose Γ is a discrete group and $\mathcal{S}$ is a subset of $B(\Gamma\times\Gamma)$ supported on the diagonal subgroup Δ of $\Gamma\times\Gamma$. Let $A_{\mathcal{S}}$ denote the norm closed translation invariant subspace of $B(\Gamma\times\Gamma)$ generated by $\mathcal{S}$. Then $A_{\mathcal{S}}|_{\Gamma\times\{e\}} \subset A(\Gamma\times\{e\})$ and $A_{\mathcal{S}}|_{\{e\}\times\Gamma} \subset A(\{e\}\times\Gamma)$.*

Proof. Let $\varphi \in \mathcal{S}$ and fix $s = (s_1, s_2)$, $t = (t_1, t_2)$ in $\Gamma\times\Gamma$. Then $\varphi(s(x_1, e)t)$ is nonzero only if $s_1x_1t_1 = s_2t_2$, i.e., only if $x_1 = s_1^{-1}s_2t_2t_1^{-1}$. Therefore, the translated element $x \mapsto \varphi(sxt)$ in $B(\Gamma\times\Gamma)$ has finite support when restricted to $\Gamma\times\{e\}$ and, thus, its restriction is an element of $A(\Gamma\times\{e\})$. So $A_{\mathcal{S}}|_{\Gamma\times\{e\}} \subset A(\Gamma\times\{e\})$. Similarly, $A_{\mathcal{S}}|_{\{e\}\times\Gamma} \subset A(\{e\}\times\Gamma)$. □

We are now prepared to prove our first main theorem of this chapter.

Theorem 3.2.3. *Let Γ be a non-amenable discrete group, π a representation of Γ weakly containing the left regular representation, and $1 \le p < \infty$. Then $C^*_\pi(\Gamma) \otimes C^*_{\ell^p}(\Gamma)$ does not admit a unique C*-norm.*

Proof. By Proposition 3.2.1, it suffices to construct two distinct weak*-closed translation invariant subspaces B_σ of $B(\Gamma\times\Gamma)$ with the prescribed conditions.

Choose a representation π' of Γ which extends to a faithful $*$-representation of $C^*_{\ell^p}(\Gamma)$. By taking π' to be a large enough direct sum of ℓ^p-representations, we may assume that π' is itself an ℓ^p-representation of Γ. We first consider the space $B_{\pi\times\pi'}$ associated to the minimal tensor product $C^*_\pi(\Gamma) \otimes_{\min} C^*_{\ell^p}(\Gamma)$. Notice that on the diagonal subgroup Δ of $\Gamma\times\Gamma$, the space $A_{\pi\times\pi'}$ restricts to $A_{\pi\otimes\pi'}$. Since $\pi\otimes\pi'$ is an ℓ^p-representation and $\Gamma \cong \Delta$ is non-amenable, the space $B_{\pi\times\lambda}|_\Delta = B_{\pi\otimes\pi'}(\Delta)$ is not all of $B(\Delta)$.

Let $\mathcal{S}$ be the set of all elements of $B(\Gamma\times\Gamma)$ supported on Δ. Then the weak*-closure B_σ of $A_{\mathcal{S}} + A_{\pi\times\pi'}$ satisfies the conditions of Proposition 3.2.1 as $B_\sigma|_{\Gamma\times\{e\}} = B_\pi + \overline{A_{\mathcal{S}}|_{\Gamma\times\{e\}}}^{w^*} = B_\pi$ since $A_{\mathcal{S}}|_{\Gamma\times\{e\}} = A(\Gamma\times\{e\})$ and π weakly contains λ. Similarly, $B_\sigma|_{\{e\}\times\Gamma} = B_{\pi'}$. As $B_\sigma|_\Delta$ is the entire Fourier-Stieltjes algebra $B(\Delta)$, we conclude that $C^*_\pi(\Gamma) \otimes C^*_{\ell^p}(\Gamma)$ does not admit a unique C*-norm. □

In the case when Γ contains a noncommutative free group, we can greatly improve this result by taking further advantage of the theory of L^p-representations.

Before proving this second theorem, we make note of some particular Fourier spaces. For a discrete group Γ, we will let $A_{\ell^p}(\Gamma)$ denote the Fourier space $A_{\mathcal{S}}$ where $\mathcal{S}$ is taken to be the collection of ℓ^p-representations of Γ. Then by Theorem 2.2.3, these spaces have the property that $A_{\ell^p}(\Gamma)|_H = A_{\ell^p}(H)$ for every subgroup H of Γ.

Theorem 3.2.4. *Let Γ_1 and Γ_2 be discrete groups containing copies of noncommutative free groups. If π is a representation of Γ_1 weakly containing a copy of the left regular representation, then $C^*_\pi(\Gamma_1) \otimes C^*_{\ell^p}(\Gamma_2)$ admits $2^{\aleph_0}$ distinct C*-norms for every $p \in [2, \infty)$.*

Proof. Choose a faithful ℓ^p-representation π' for $C^*_{\ell^p}(\Gamma_2)$ and identify a copy of $\mathbb{F}_2$ inside each of Γ_1 and Γ_2. Denote the diagonal subgroup of $\mathbb{F}_2 \times \mathbb{F}_2 \leq \Gamma_1 \times \Gamma_2$ by Δ. Then, since $(\pi \times \sigma)|_\Delta = \pi|_{\mathbb{F}_2} \otimes \pi'|_{\mathbb{F}_2}$ is an ℓ^p-representation of Δ, the restriction $A_{\pi\times\pi'}|_\Delta$ is contained in $A_{\ell^p}(\Delta)$.

For each $q > p$, let $\mathcal{S}_q$ be the set of all functions in $A_{\ell^q}(\Gamma_1 \times \Gamma_2)$ supported on Δ. Then $A_{\mathcal{S}_q}|_\Delta = A_{\ell^q}(\Delta)$. Let B_{σ_q} be the weak*-closure of $A_{\mathcal{S}_q} + A_{\pi\times\pi'}$. Then, by similar reasoning as in the proof of the previous theorem, B_{σ_q} satisfies the conditions of Proposition 3.2.1 and $B_{\sigma_q}|_\Delta = B_{\ell^q}(\Delta)$. Since the C*-norms $\|\cdot\|_{\ell^p}$ on $\ell^1(\Delta)$ are distinct for every $q > p$, we conclude that B_{σ_q} are distinct for every $q > p$. □

Remark 3.2.5. Let G be a locally compact group containing an open normal compact subgroup K. This happens, for instance, when G is a totally disconnected SIN group. Let $q : G \to G/K$ be the canonical quotient map. If m_K is the normalized Haar measure on K, then $\varphi \mapsto m_K * \varphi$ is a contraction on $B(G)$ (see [16, Proposition 2.18]) mapping $A(G)$ onto $A(G/K) \circ q$ since $A(G) = \overline{B(G) \cap C_c(G)}$ and $\varphi \in C_c(G)$ if and only if $q(\operatorname{supp}\varphi)$ is finite. It follows that $m_K * B_{\lambda_G} = B_{\lambda_{G/K}} \circ q$.

Suppose G is non-amenable. Then the analogue of Theorem 3.2.3 is true for G, i.e., if π is any representation of G weakly containing λ_G, then $C^*_\pi(G) \otimes C^*_{L^p}(G)$ does not admit a unique C*-norm. Indeed, denote the diagonal subgroup of $G/K \times G/K$ by Δ. Then, making the appropriate changes to the proof of Theorem 3.2.3 and taking $\mathcal{S}$ to be the set of all elements in $B(G/K \times G/K) \circ q$ supported on $q^{-1}(\Delta)$ produces a second B_σ satisfying the conditions of Proposition 3.2.1. Suppose G_1 and G_2 are two locally compact groups containing open normal compact subgroups K_1 and K_2 so that G/K_1 and G/K_2 contain

noncommutative free subgroups. Then a similar trick as above shows that $C^*_\pi(G_1) \otimes C^*_{L^p}(G_2)$ admits $2^{\aleph_0}$ distinct C*-norms for every representation π of G_1 weakly containing λ_{G_1}.

Chapter 4

L^p-Fourier and Fourier-Stieltjes algebras

4.1 Introduction

Let G be a locally compact group and D an algebraic ideal of $C_b(G)$. Motivated by the important role played by the Fourier-Stieltjes algebra $B(G)$ and the Fourier algebra $A(G)$ in studying the structure of a locally compact group, Brannan and Ruan (see [6]) defined and developed the basic theory of D-Fourier-Stieltjes algebra $B_D(G)$ and D-Fourier algebra $A_D(G)$, which naturally identify with the dual of the C*-algebra $C^*_D(G)$ and predual of the analogously defined von Neumann algebra $VN_D(G)$, respectively. In this chapter we study these algebras in the more concrete setting when $D = L^p$ $(= L^p(G) \cap C_b(G))$ for $1 \leq p < \infty$.

The L^p-Fourier and L^p-Fourier-Stieltjes algebras are ideals of the Fourier-Stieltjes algebra containing the Fourier algebra. Similar to the case of the C*-algebra $C^*_{L^p}(G)$, the L^p-Fourier algebra coincides with the Fourier algebra $A(G)$ and the L^p-Fourier-Stieltjes algebra with the reduced Fourier-Stieltjes algebra $B_r(G) := B_\lambda$ when $p \in [1, 2]$. This is generally not the case for $p > 2$. In fact we demonstrate rich classes of groups G so that $A_{L^p}(G)$ is distinct for every $p \in (2, \infty)$ and $B_{L^p}(G)$ is distinct for each $p \in (2, \infty)$.

As an application of the theory developed throughout this chapter, we characterize the Fourier-Stieltjes ideals of $SL_2(\mathbb{R})$ in terms of L^p-Fourier-Stieltjes algebras.

Just as is the case for the classical Fourier algebra, we show that each L^p-Fourier algebra is a complete invariant for the underlying locally compact group. However, we also show that even for abelian groups, when $p > 2$ the L^p-Fourier algebra can lack many nice properties that hold for $A(G)$. In particular, we show that when G is a noncompact abelian group, $A_{L^p}(G)$ is not even square dense for each $p \in (2, \infty)$ and, hence, lacks any reasonable notion of amenability. As such the analogues of Ruan's (see [42]) and Leptin's (see Theorem 1.2.19) characterizations of amenability fail for the L^p-Fourier algebras. Though the analogues of these characterizations of amenability fail for the L^p-Fourier algebra, we show that the analogue of Losert's characterization of amenability in terms of multipliers (see [33]) holds for $A_{L^p}(G)$ and Runde-Spronk's characterization of amenability in terms of operator Connes amenability (see [45]) holds for $B_{L^p}(G)$.

This chapter is organized as follows. In section 2 we recall the definition of the L^p-Fourier and Fourier-Stieltjes algebras and note some basic properties. Section 3 is devoted to studying the L^p-Fourier algebras for abelian groups. In section 4 we study the structural properties of the L^p-Fourier and Fourier-Stieltjes algebras. As a final application of this theory, we characterize the Fourier-Stieltjes ideals of $SL_2(\mathbb{R})$ in section 5.

4.2 Definition and basic properties of L^p-Fourier and Fourier-Stieltjes algebras

In [6] Brannan and Ruan define the D-Fourier algebra $A_D(G)$ and D-Fourier-Stieltjes algebra $B_D(G)$ when D is a subalgebra of $C_b(G)$. When $D = C_0(G)$, the D-Fourier algebra $A_D(G)$ is already well studied and is known in the literature as the Rajchman algebra. In contrast, very little has been done in regards to the L^p-Fourier and L^p-Fourier-Stieltjes algebras (though we did briefly make use of it in the previous chapter). We aim to help fill this gap with this chapter. We begin by recalling the definitions of these spaces and prove some of their basic properties.

Let D be a linear subspace of $C_b(G)$. The D-Fourier space is defined to be

$$A_D(G) = A_D := \{\pi_{x,y} : \pi \text{ a } D\text{-representation}, x, y \in \mathcal{H}_\pi\}.$$

Similarly, the D-Fourier-Stieltjes space $B_D(G) = B_D$ is defined to be the closure of A_D with respect to the weak*-topology $\sigma(B(G), C^*(G))$. When the subspace D of $C_b(G)$ is a subalgebra (resp., ideal) of $C_b(G)$, then Brannan and Ruan noted that $A_D(G)$ and $B_D(G)$ are subalgebras (resp., ideals) of $B(G)$ (see [6, Proposition 3.11]). In these cases, we may call $A_D(G)$ and $B_D(G)$ the D-Fourier algebra and D-Fourier-Stieltjes algebra, respectively. Note that since L^p is an ideal in $C_b(G)$, $A_{L^p}(G)$ and $B_{L^p}(G)$ are ideals in $B(G)$.

Let D be a subspace of $C_b(G)$. As an immediate consequence of the next proposition, we get that $A_D = A_\mathcal{S}$ and $B_D = B_\mathcal{S}$ when $\mathcal{S}$ is taken to be the collection of D-representations of G.

Proposition 4.2.1. *Let D be a subspace of $C_b(G)$. Then A_D is a closed translation invariant subspace of $B(G)$. Moreover,*

$$\|u\|_{B(G)} = \inf\{\|x\|\|y\| : u = \pi_{x,y} \text{ and } \pi \text{ is a } D\text{-representation}\}$$

and this infimum is attained for some D-representation π and $x, y \in \mathcal{H}_\pi$.

Proof. For every $u \in A_D$, choose a D-representation $\pi_u : G \to B(\mathcal{H}_u)$ so that $u = (\pi_u)_{x,y}$ for some $x, y \in \mathcal{H}_u$. Then $\pi := \bigoplus_{u \in A_D} \pi_u$, being a direct sum of D-representations, is also a D-representation. Moreover $A_\pi \supset A_D$ since every element $u \in A_D$ is a coefficient function of π.

Now let $u \in A_\pi$. Then we can find sequences $\{x_n\}, \{y_n\}$ in $\mathcal{H}_\pi$ so that $u = \sum_{n=1}^\infty \pi_{x_n,y_n}$ and $\|u\| = \sum_{n=1}^\infty \|x_n\|\|y_n\|$. Let $\tilde{\pi} : G \to B(\mathcal{H}_\pi^{\bigoplus \infty})$ be the infinite amplification $\infty \cdot \pi$. Then, since $\tilde{\pi}$ is a D-representation and $u = \tilde{\pi}_{(x_n),(y_n)}$, we arrive at the desired conclusions. □

Since we have shown that A_D is a Fourier space, it is clear that B_D is the Fourier-Stieltjes space which is dual to the C*-algebra $C_D^*(G)$.

Let $P(G)$ denote the set of positive definite functions on G. Then A_D has a very nice description in terms of the linear span of positive definite functions when D is a translation invariant subspace of $C_b(G)$.

Proposition 4.2.2. *Suppose that D is a translation invariant subspace of $C_b(G)$. Then A_D is the closed linear span of $P(G) \cap D$ in $B(G)$.*

Proof. Let $u \in P(G) \cap D$. Then, since the GNS representation of u is a D-representation, u is clearly in A_D. As A_D is a closed subspace of $B(G)$, we conclude that A_D contains the closed linear span of $P(G) \cap D$.

Now let $u \in A_D$. Then we can write $u = \pi_{x,y}$ for some D-representation π of G and $x, y \in \mathcal{H}_\pi$. Let $\mathcal{H}_0$ be a dense subspace of $\mathcal{H}_\pi$ so that $\pi_{z,z} \in D$ for every $z \in \mathcal{H}_0$ and choose sequences $\{x_n\}$, $\{y_n\}$ in $\mathcal{H}_0$ converging in norm to x and y, respectively. Then

$$\pi_{x_n,y_n} = \sum_{k=0}^{3} i^k \pi_{x_n+i^k y_n, x_n+i^k y_n}$$

converges to $u = \pi_{x,y}$ in norm. Hence, A_D is the closed linear span of $P(G) \cap D$. □

For the remainder of this section, we will focus specifically on L^p-Fourier and Fourier-Stieltjes algebras. We begin by identifying cases when these spaces are familiar subspaces of $B(G)$.

Proposition 4.2.3. *Let G be a locally compact group.*

(i) $A_{L^p}(G) = A(G)$ *for every* $p \in [1, 2]$.
(ii) If G is compact, then $A_{L^p}(G) = B(G)$ *for every* $p \in [1, \infty)$.
(iii) If G is amenable, then $B_{L^p}(G) = B(G)$ *for every* $p \in [1, \infty)$.
(iv) If $B_{L^p}(G) = B(G)$ *for some* $p \in [1, \infty)$, *then G is amenable.*

Proof. (i) Recall that the Fourier algebra $A(G)$ is both the closed linear span of $P(G) \cap C_c(G)$ and of $P(G) \cap L^2(G)$ (see [16, Proposition 3.4]). Since $P(G) \cap C_c(G) \subset P(G) \cap L^p(G) \subset P(G) \cap L^2(G)$ for every $p \in [1, 2]$, we arrive at the desired conclusion.

(ii) Let π be a representation of G and $x \in \mathcal{H}_\pi$. Then $\pi_{x,x}$ is bounded in uniform norm by $\|x\|^2$. Since $x \in \mathcal{H}_\pi$ was arbitrary, we conclude that every representation π of G is an L^p-representation and, hence, that $A_{L^p}(G) = B(G)$.

(iii) Since G is amenable, $C_r^*(G) = C^*(G)$. Hence, the reduced Fourier-Stieltjes algebra $B_r(B) = B(G)$. Since $A_{L^p}(G) \supset A(G)$, we conclude that $B_{L^p}(G)$ must also be all of $B(G)$.

(iv) If $B_{L^p}(G) = B(G)$, then $C^*_{L^p}(G) = C^*(G)$ and, hence, G is amenable. □

Proposition 4.2.4. *Let G be a locally compact group and $p, q, r \in [1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then $uv \in A_{L^r}(G)$ for every $u \in A_{L^p}(G)$ and $v \in A_{L^q}(G)$. Similarly, $uv \in B_{L^r}(G)$ for all $u \in B_{L^p}(G)$ and $v \in B_{L^q}(G)$.*

Proof. Let $u \in A_{L^p}(G)$ and $v \in A_{L^q}(G)$. By Proposition 4.2.2, we can approximate u and v well in norm by linear combinations $a_1u_1 + \ldots + a_nu_n$ and $b_1v_1 + \ldots + b_mv_m$ of positive definite elements in $L^p(G)$ and $L^q(G)$, respectively. Then the product

$$\sum_{i,j} a_i b_j u_i v_j$$

is a linear combination of elements in $P(G) \cap L^r(G)$ approximating uv well in norm. Hence, $uv \in A_{L^r}(G)$ by Proposition 4.2.2.

Note that since multiplication in $B(G)$ is separately weak*-weak* continuous, it follows that $uv \in B_{L^r}(G)$ for all $u \in B_{L^p}(G)$ and $v \in B_{L^q}(G)$. □

Proposition 4.2.5. *Suppose H is an open subgroup of a locally compact group G and $1 \leq p < \infty$. Then $A_{L^p}(H) = A_{L^p}(G)|_H$ and $B_{L^p}(H) = B_{L^p}(G)|_H$.*

Proof. The equality $A_{L^p}(G)|_H = A_{L^p}(H)$ follows from the definition of L^p-Fourier spaces since $\pi|_H$ is an L^p-representation for every L^p-representation π of G, and $\mathrm{ind}_H^G \sigma$ is an L^p-representation for every L^p-representation σ of H.

We now proceed to prove the second part of the statement. Notice that since H is an open subgroup of G, $L^1(H)$ embeds naturally into $L^1(G)$. Let $u \in C^*_{L^p}(H)^* = B_{L^p}(H)$. Then, by the Hahn-Banach theorem, there is an element $\widetilde{u} \in C^*_{L^p}(G)^* = B_{L^p}(G)$ extending u as a linear functional. Then for $f \in L^1(H) \subset L^1(G)$,

$$\int_H \widetilde{u}(s) f(s)\, ds = \int_G \widetilde{u}(s) f(s)\, ds = < \widetilde{u}, f > = < u, f > = \int_H u(s) f(s)\, ds.$$

So $u = \widetilde{u}|_H$ almost everywhere. Since u and $\widetilde{u}$ are each continuous functions, this implies that $u = \widetilde{u}|_H$ and, hence, that $B_{L^p}(H) \subset B_{L^p}(G)|_H$. A similar but simpler argument shows that $B_{L^p}(H) \supset B_{L^p}(G)|_H$. □

The above proposition can fail, even for A_{L^p}, when H is a non-open closed subgroup of G. This is demonstrated later in Remark 4.4.4.

We finish this section by establishing the first class of examples of groups G where $A_{L^p}(G)$ and $B_{L^p}(G)$ are distinct subspaces of $B(G)$ for $2 < p < \infty$.

Proposition 4.2.6. *Let Γ be a discrete group containing a copy of a noncommutative free group. Then $B_{\ell^p}(\Gamma)$ is distinct for every $p \in [2, \infty)$. Hence, $A_{\ell^p}(\Gamma)$ is also distinct for every $p \in [2, \infty)$.*

Proof. The first statement is immediate from previous comments since $C^*_{\ell^p}(\Gamma)$ is distinct for each $p \in [2, \infty)$. The second statement follows from the first since $B_{\ell^p}(\Gamma)$ is the weak*-closure of $A_{\ell^p}(\Gamma)$. □

4.3 L^p-Fourier algebras of abelian groups

In this section we show that the algebras $A_{L^p}(G)$ are distinct for every $p \in [2, \infty)$ when G is a noncompact locally compact abelian group. We will later see that this phenomena does not generalize to the setting of general noncompact locally compact groups. Before entering into proofs, we provide a brief summary of the tools of which we use and refer the reader to Graham and McGehee's book [20] for more details.

Let Γ be a discrete abelian group. A subset Θ of Γ is said to be *dissociate* if every element $\omega \in \Gamma$ can be written in at most one way as a product

$$\omega = \prod_{j=1}^{n} \theta_j^{\epsilon_j}$$

where $\theta_1, \ldots, \theta_n \in \Theta$ are distinct elements, $\epsilon_j = \pm 1$ if $\theta_j^2 \neq 1$, and $\epsilon_j = 1$ if $\theta_j^2 = 1$. As an example, if $\Gamma = \mathbb{Z}$ then the set $\{3^j : j \geq 1\}$ is dissociate. As in the case of the integers, every infinite discrete abelian group admits an infinite dissociate set.

Let G be a compact abelian group with normalized Haar measure and $\Gamma = \widehat{G}$ be the dual group of G. If γ is a group element of Γ such that $\gamma^2 \neq 1$ and $a(\gamma)$ is a constant with

$|a(\gamma)| \leq 1/2$, then the trigonometric polynomial

$$q_\gamma := 1 + a(\gamma)\gamma + \overline{a(\gamma)}\overline{\gamma}$$

is a positive function on G with $\|q_\gamma\|_1 = 1$. Similarly, if $\gamma \in \Gamma\backslash\{1\}$ has the property that $\gamma^2 = 1$ and $0 \leq a(\gamma) < 1$, then $q_\gamma := 1 + a(\gamma)\gamma$ is a positive function which integrates over G to 1. We will consider weak* limits of products of polynomials of this type.

Let $\Theta \subset \Gamma$ be a dissociate set. To each $\theta \in \Theta$ assign a value $a(\theta) \in \mathbb{C}$ with the imposed restrictions from above. For each finite subset $\Phi \subset \Theta$ define $P_\Phi = \prod_{\theta\in\Phi} q_\theta$. This being a product of positive functions is a positive function on G with Fourier transform

$$\widehat{P_\Phi}(\gamma) = \begin{cases} \prod_{\theta\in\Phi} a(\theta)^{(\epsilon_\theta)}, & \gamma = \prod_{\theta\in\Phi} \theta^{\epsilon_\theta} \\ 0, & \text{otherwise} \end{cases}$$

where ϵ_θ range over $\{-1, 0, 1\}$ and

$$a(\theta)^{(\epsilon_\theta)} = \begin{cases} 1, & \epsilon_\theta = 0 \\ a(\theta), & \epsilon_\theta = 1 \\ \overline{a(\theta)}, & \epsilon_\theta = -1. \end{cases}$$

It follows that as $\Phi \nearrow \Theta$, P_Φ converges weak* to a measure μ on G where

$$\widehat{\mu}(\gamma) = \begin{cases} \prod_{\theta\in\Theta} a(\theta)^{(\epsilon_\theta)}, & \gamma = \prod_{\theta\in\Theta} \theta^{\epsilon_\theta},\ \epsilon_\theta = 0 \text{ for all but finitely many } \theta \\ 0, & \text{otherwise} \end{cases}$$

The measure μ is said to be based on Θ and a. This method of constructing measures is called the Riesz product construction and the set of all such constructions is denoted $R(G)$.

In 1959 Zygmund (see [57]) proved that a measure $\mu \in R(\mathbb{T})$ based on Θ and a is an element of $L^1(\mathbb{T})$ if and only if $a \in \ell^2(\Theta)$. This result was extended to all compact abelian groups G by Hewitt and Zuckerman in 1966 (see [24]). If Γ is the dual of a compact abelian group G and $\mu \in R(G)$ is based on Θ and a, then $\widehat{\mu} \in A(\Gamma) = A_{\ell^2}(\Gamma)$ if and only if $a \in \ell^2(\Theta)$. We will demonstrate that the analogue of this theorem holds when 2 is replaced with p for $2 \leq p < \infty$. Towards this goal, we begin by proving an elementary lemma which is surely known but we include for the convenience of the reader and because we lack a suitable reference.

Lemma 4.3.1. *Suppose that $0 < \alpha < 1$ and $\{x_n\}$ is a bounded sequence but $\{x_n\} \notin \ell^p$. Then there exists a bounded sequence $\{y_n\}$ so that $\{x_n y_n\} \in \ell^p$ but $\{x_n y_n^\alpha\} \notin \ell^p$.*

Proof. Clearly it suffices to consider the case when $p = 1$. We first focus our attention to the case when $\{x_n\} \in c_0$. Then we can choose mutually disjoint subsets $I_1, I_2, \ldots$ of $\mathbb{N}$ so that $\sum_{n \in I_k} |x_n| = 1$ for each k. Define

$$y_n = \begin{cases} k^{-\frac{1}{\alpha}} & \text{if } n \in I_k \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\sum_{n \in \mathbb{N}} |x_n y_n| = \sum_k \sum_{n \in I_k} |x_n y_n| = \sum_k \sum_{n \in I_k} |x_n| k^{-\frac{1}{\alpha}} = \sum_k k^{-\frac{1}{\alpha}} < \infty,$$

but

$$\sum_{n \in \mathbb{N}} |x_n y_n^\alpha| = \sum_k \sum_{n \in I_k} |x_n y_n^\alpha| = \sum_k \sum_{n \in I_k} |x_n| k^{-1} = \sum_k k^{-1} = \infty.$$

Now assume that $\limsup |x_n| > 0$. Then we can find $\delta > 0$ and a subsequence $\{x_{n_k}\}$ so that $|x_{n_k}| \geq \delta$ for every k. Defining

$$y_n = \begin{cases} k^{-\frac{1}{\alpha}} & \text{if } n = n_k \\ 0 & \text{otherwise.} \end{cases}$$

gives the desired result.

□

We are now prepared to prove the main result of this section in the case of discrete groups.

Theorem 4.3.2. *Let G be a compact abelian group with dual group Γ and $\mu \in R(G)$ be based on Θ and a. Then $\widehat{\mu} \in A_{\ell^p}(\Gamma)$ if and only if $a \in \ell^p(\Theta)$.*

Proof. First we suppose that $\sum_{\theta \in \Theta} \left|a(\theta)\right|^p < \infty$ and let

$$\Omega(\Theta) = \{\theta_1 \cdots \theta_n \mid \theta_1^{\epsilon_1}, \ldots, \theta_n^{\epsilon_n} \in \Theta \text{ distinct}, \epsilon_1, \ldots, \epsilon_n = \pm 1, n \geq 0\}.$$

Then

$$\begin{aligned}
\|\widehat{\mu}\|_p^p &= \sum_{\omega\in\Omega(\Theta)} \left|\widehat{\mu}(\omega)\right|^p \\
&\leq 1+\sum_{n=1}^{\infty}\sum_{\substack{\Phi\subset\Theta\\|\Phi|=n}} 2^n \prod_{\theta\in\Phi}|a(\theta)|^p \\
&\leq 1+\sum_{n=1}^{\infty}\frac{2^n}{n!}\left(\sum_{\theta\in\Theta}\left|a(\theta)\right|^p\right)^n \\
&= \exp\left\{2\sum_{\theta\in\Theta}\left|a(\theta)\right|^p\right\} < \infty.
\end{aligned}$$

Hence, $\widehat{\mu}\in A_{\ell^p}(\Gamma)$.

Now suppose that $\sum_{\theta\in\Theta}\left|a(\theta)\right|^p=\infty$ but $\widehat{\mu}\in A_{\ell^p}(\Gamma)$. Hewitt and Zuckerman showed this is not possible for $p=2$, so we will assume without loss of generality that $p>2$. Choose a sequence $\{b(\theta)\}\in\ell^\infty(\Theta)$ with $\|b\|_\infty\leq 1$ so that $\{a(\theta)b(\theta)\}\in\ell^p(\Theta)$ but $\{a(\theta)b(\theta)^\alpha\}\notin\ell^p(\Theta)$ for $\alpha=\frac{p-2}{2p}$. Let $\nu\in R(G)$ be based on Θ and $c:=\{(a(\theta)b(\theta))^{\frac{p-2}{2}}\}$. Define $q=\frac{2p}{p-2}$ (this is chosen so that $1/p+1/q=1/2$). Then

$$\sum_{\theta\in\Theta}\left|c(\theta)\right|^q=\sum_{\theta\in\Theta}\left|a(\theta)b(\theta)\right|^p<\infty$$

implies that $\widehat{\nu}\in\ell^q(\Theta)$. So $\widehat{\mu}\cdot\widehat{\nu}\in A_{\ell^2}(\Gamma)=A(\Gamma)$ by Proposition 4.2.4. Observe that $\mu*\nu$ is the element in $R(G)$ generated by Θ and $a\cdot c$. So $a\cdot c\in\ell^2(\Theta)$. But, by our assumption on b,

$$\sum_{\theta\in\Theta}\left|a(\theta)c(\theta)\right|^2=\sum_{\theta\in\Theta}\left|a(\theta)b(\theta)^{\frac{p-2}{2p}}\right|^p=\infty,$$

a contradiction. Therefore, $\widehat{\mu}\in A_{\ell^p}(\Gamma)$ if and only if $a\in\ell^p(\Theta)$. □

Corollary 4.3.3. *Let Γ be an infinite discrete abelian group. The subspaces $A_{\ell^p}(\Gamma)$ of $B(\Gamma)$ are distinct for every $p\in[2,\infty)$.*

Our next step is show that $A_{L^p}(G)$ is distinct for each $2\leq p<\infty$ for another class of locally compact abelian groups G.

Remark 4.3.4. Suppose Γ is a lattice in a locally compact abelian group G, i.e. a discrete subgroup of G such that the homogeneous space G/Γ admits a finite G-invariant measure. Further, suppose that $v \in A(G)$ is a normalized positive definite function with the property that $\operatorname{supp} v \cap \Gamma = \{e\}$ and $(s + \operatorname{supp} v) \cap \Gamma$ is finite for every $s \in G$. Then the map $J = J_v$ from $B(\Gamma)$ into $B(G)$ defined by

$$Ju(s) = \sum_{\xi \in \Gamma} u(\xi) v(s - \xi)$$

is a well defined isometry with the following properties (see [20, Theorem A.7.1]):

(i) $Ju \in P(G)$ if and only if $u \in P(\Gamma)$;
(ii) $Ju \in A(G)$ if and only if $u \in A(\Gamma)$.

Lemma 4.3.5. *Let $G = \mathbb{R}^n \times K$ for some compact abelian group K and $n \geq 1$. Choose a normalized $v \in A(G) \cap P(G)$ so that $\operatorname{supp} v \subset [-1/3, 1/3]^n \times K$ and suppose $\mu \in R(\mathbb{Z}^n)$ is based on Θ and a. Then $J_v \widehat{\mu} \in A_{L^p}(\mathbb{R}^n)$ if and only if $a \in \ell^p(\Theta)$.*

Proof. We leave it as an exercise to the reader to check that if $\operatorname{supp} v \subset [-1/3, 1/3]^n \times K$, then $\operatorname{supp} v \cap \mathbb{Z}^n = \{e\}$ and $(s + \operatorname{supp} v) \cap \mathbb{Z}^n$ is finite for every $s \in G$.

Let $u \in P(\Gamma) \cap \ell^p(\Gamma)$. For $(x_1, \ldots, x_n, k) \in \mathbb{R}^n \times K$, write $x_i = m_i + y_i$ for some $m_i \in \mathbb{Z}$ and $y_i \in [-1/2, 1/2]$ $(1 \leq i \leq n)$. Then

$$\begin{aligned} J_v u(x_1, \ldots, x_n, k) &= \sum_{(\ell_1, \ldots, \ell_n) \in \mathbb{Z}^n} u(\ell_1, \ldots, \ell_n) v(m_1 + y_1 - \ell_1, \ldots, m_n + y_1 - \ell_n, k) \\ &= u(m_1, \ldots, m_n) v(y_1, \ldots, y_n, k). \end{aligned}$$

For each $(m_1, \ldots, m_n) \in \mathbb{Z}^n$, define $M_{m_1, \ldots, m_n} = [m_1 - 1/2, m_1 + 1/2] \times \cdots \times [m_n - 1/2, m_n + 1/2] \times K$. Then

$$\begin{aligned} & \int_G |J_v u|^p \\ = & \sum_{(m_1, \ldots, m_n) \in \mathbb{Z}^n} \int_{M_{m_1, \ldots, m_n}} |J_v u|^p \\ = & \sum_{(m_1, \ldots, m_n) \in \mathbb{Z}^n} \int_{[-1/2, 1/2]^n \times K} |u(m_1, \ldots, m_n) v(y_1, \ldots, y_n, k)|^p \, d(y_1, \ldots, y_n, k) \\ = & \|u\|_p^p \int_{[-1/2, 1/2]^n \times K} |v(y_1, \ldots, y_n, k)|^p \, d(y_1, \ldots, y_n, k) < \infty. \end{aligned}$$

Hence, $J_v u \in L^p(G)$. As J is an isometry mapping $P(\Gamma)$ into $P(G)$ and $A_{\ell^p}(\Gamma)$ is the closed linear span of $P(\Gamma) \cap \ell^p(\Gamma)$, it follows that J_v maps $A_{\ell^p}(\Gamma)$ into $A_{L^p}(G)$.

Let $\mu \in R(\mathbb{Z}^n)$ be based on Θ and a, and suppose that $a \notin \ell^p(\Theta)$. Let c be chosen as in the proof of Theorem 4.3.2 and $\nu \in R(\mathbb{Z}^n)$ be based on Θ and c. Then $\widehat{\nu} \in A_{\ell^q}(\Gamma)$ and, hence, $J_v\widehat{\nu} \in A_{L^q}(G)$ where q satisfies $1/p + 1/q = 1/2$. For $m_1, \ldots, m_n \in \mathbb{Z}$, $y_1, \ldots, y_n \in [-1/2, 1/2]$ and $k \in K$,

$$\begin{aligned} & J_v\widehat{\mu}(m_1 + y_1, \ldots, m_n + y_n, k) J_v\widehat{\nu}(m_1 + y_1, \ldots, m_n + y_n, k) \cdot \\ = \; & \widehat{\mu}(m_1, \ldots, m_n)\widehat{\nu}(m_1, \ldots, m_n) v(y_1, \ldots, y_n, k)^2 \\ = \; & J_{v^2}\widehat{\mu * \nu}. \end{aligned}$$

Since v^2 is a positive definite function with support contained in $[-1/3, 1/3]^n \times K$, $J_v\widehat{\mu * \nu} \in A(G)$ if and only if $\widehat{\mu * \nu} \in A(G)$. But $\mu * \nu$ is the element of $R(G)$ based on Θ and $a \cdot c$ and, as in the proof of Theorem 4.3.2, $a \cdot c \notin \ell^2(\Theta)$. So $\widehat{\mu * \nu} \notin A(\Gamma)$ and, hence, $J_v\widehat{\mu} \cdot J_v\widehat{\nu}$ is not in $A(G)$. It follows that $J_v\widehat{\mu} \notin A_{L^p}(G)$. □

Corollary 4.3.6. *$A_{L^p}(G)$ is distinct for every $p \in [2, \infty)$ when $G = \mathbb{R}^n \times K$ where K is some compact abelian group and $n \geq 1$.*

Proof. It suffices to check that there is a nonzero positive definite function v whose support is contained in $[-1/3, 1/3]^n \times K$. Observe that

$$\omega(x) := \chi_{[-1/6,1/6]} * \chi_{[-1/6,1/6]}(x) = \begin{cases} 1 - 3|x|, & |x| \leq 1/3 \\ 0, & \text{otherwise} \end{cases}$$

is a positive definite function on $\mathbb{R}$ with support contained in $[-1/3, 1/3]$. Taking $v = \omega \times \cdots \times \omega \times 1_K$ clearly does the trick. □

We now prove one last lemma before we show that $A_{L^p}(G)$ is distinct for each $p \in [2, \infty)$ when G is any noncompact locally compact abelian group.

Lemma 4.3.7. *Suppose K is a compact subgroup of a locally compact group G. Then*

$$A_{L^p}(G : K) := \{u \in A_{L^p}(G) : u(sk) = u(s) \text{ for all } s \in G, k \in K\}$$

is isometrically isomorphic to $A_{L^p}(G/K)$.

Proof. Let m_K denote the normalized Haar measure for K and note that m_K is a central idempotent measure. Denote the universal representation of G by ϖ and define $p_K = \varpi(m_K)$. Observe that if π is a representation of G, then $p_K\pi$ is constant on cosets of K and, hence, defines a representation $\pi_K : G/K \to \mathcal{U}(\mathcal{H}_\pi)$ by $\pi_K(sK) = p_K\pi(s)$ for $s \in G$.

Suppose π is an L^p-representation of G and $\mathcal{H}_0$ is a dense subspace of $\mathcal{H}_\pi$ so that $\pi_{x,x} \in L^p(G)$ for all $x \in \mathcal{H}_0$. Let $q : G \to G/K$ be the canonical quotient map. Then

$$(\pi_K)_{p_K x, p_K x} \circ q = \langle p_K\pi(\cdot)x, x\rangle = m_K * \pi_{x,x} \in L^p(G)$$

for all $x \in \mathcal{H}_0$. Since $m_K * L^p(G) \cong L^p(G/K)$, it follows that π_K is an L^p-representation of G/K.

Conversely, suppose that $\widetilde{\pi}$ is an L^p-representation of G/K. Then Weyl's integral formula implies that $\widetilde{\pi} \circ q$ is an L^p-representation of G. Furthermore, $m_K * (\widetilde{\pi} \circ q)_{x,y} = (\widetilde{\pi} \circ q)_{x,y}$ for all $x, y \in \mathcal{H}_{\widetilde{\pi}}$. □

We thank Nico Spronk for pointing out this previous lemma, which has allowed for cleaner arguments throughout this section.

Theorem 4.3.8. *Let G be a noncompact locally compact abelian group. Then $A_{L^p}(G)$ is distinct for every $p \in [2, \infty)$.*

Proof. By the structure theorem for locally compact abelian groups, G has an open subgroup of the form $\mathbb{R}^n \times K$ where $n \geq 0$ and K is compact. If $n > 0$, then the result follows from Lemma 4.3.5. Otherwise, it follows from Lemma 4.3.7 that $A_{L^p}(\mathbb{R}^n \times K)$ is distinct for every $p \in [2, \infty)$ and, hence, $A_{L^p}(G)$ is distinct for every $p \in [2, \infty)$ by Proposition 4.2.5. □

Recall that a SIN group (standing for small invariant neighbourhoods) is a locally compact group G which contains a neighbourhood base for the identity consisting of compact sets which are invariant under inner automorphisms. We finish this section by showing that this same phenomenon which occurs for abelian groups also occurs in SIN groups with noncompact connected components.

Theorem 4.3.9. *Let G be a SIN group with noncompact connected component. Then $A_{L^p}(G)$ is distinct for every $p \in [2, \infty)$.*

Proof. By the structure theorem for SIN groups, G contains an open subgroup of finite index which is of the form $\mathbb{R}^n \times K =: H$ for some $n \geq 0$ and compact group K (see [38]). Then, since the (noncompact) connected component of the identity is contained in H, it is necessarily the case that $n \geq 1$. So it suffices to check this for groups G of the form $\mathbb{R}^n \times K$ for some $n \geq 1$. As this follows from Lemma 4.3.7, we conclude that $A_{L^p}(G)$ is distinct for all $p \in [2, \infty)$. □

4.4 The structure of L^p-Fourier(-Stieltjes) algebras

In this section investigate the structural properties of the L^p-Fourier and Fourier-Stieltjes algebras with an emphasis on the former. Similar to the Fourier algebra, we find that the L^p-Fourier algebras completely determine the group. However, armed with our knowledge of these spaces in the cases when G is either an abelian locally compact group or a discrete group containing a copy of a noncommutative free group, we observe that many nice properties which hold for Fourier algebras fail for L^p-Fourier algebras. We begin this section by determining the spectrum of the L^p-Fourier algebras.

Proposition 4.4.1. *Let G be a locally compact group. Then the spectrum of $A_{L^p}(G)$ is G, where we identify elements of G with their point evaluations.*

Proof. Clearly we have that $G \subset \sigma(A_{L^p}(G))$, so it suffices to check that $\sigma(A_{L^p}(G)) \subset G$. Let $\chi \in \sigma(A_{L^p}(G))$ and choose an integer n so that $p/n \leq 2$. Then, since u^n is in $A(G)$ for every $u \in A_{L^p}(G)$, there exists $s \in G$ so that $< \chi, u^n > = u(s)^n$ for all $u \in A_{L^p}(G)$ (see [16, Théoréme 3.34]). As $< \chi, u >< \chi, u^n > = < \chi, u^{n+1} > = u(s)^{n+1}$, it follows that χ is evaluation at s. Hence, we conclude that $\sigma(A_{L^p}(G)) = G$. □

One of the most coveted properties of the Fourier algebra $A(G)$ is that it completely determines the underlying locally compact group G. We now show that the analogue of

this theorem holds for $A_{L^p}(G)$. The proof is similar to that given by Martin Walter and we refer the reader to his original paper [51] for most of the details.

Theorem 4.4.2. *Let G_1 and G_2 be locally compact groups and suppose $A_{L^p}(G_1)$ is isometrically isomorphic to $A_{L^q}(G_2)$ as Banach algebras for some $p, q \in [2, \infty)$. Then G_1 is homeomorphically isomorphic to G_2.*

Proof. Most of this proof is identical to that given by Walter and a careful read of his paper reveals that the only detail that is left to be verified is that the identification of G with $\sigma(A_{L^p}(G))$ is homeomorphic when $\sigma(A_{L^p}(G))$ is equipped with the weak*-topology.

Let $VN_{L^p}(G)$ be the canonical choice of von Neumann algebra which is dual to $A_{L^p}(G)$. Then the canonical embedding of G into $VN_{L^p}(G)$ is continuous in the weak*-topology. Denote this map by σ. Then, since $A(G)$ is contained in $A_{L^p}(G)$, the map

$$\sigma(G) \ni \sigma(s) \mapsto \lambda(s) \in \lambda(G)$$

is weak*-weak* continuous. Finally, Eymard showed that the map $\lambda(s) \mapsto s$ from $\lambda(G)$ to G is continuous (see [16, Théoréme 3.34]). Hence, we conclude the identification of G with $\sigma(A_{L^p}(G))$ is a homeomorphic one. □

The Fourier algebra admits many beautiful properties and it is natural to wonder whether analogues of these continue to hold for the L^p-Fourier algebra. In many cases, such as with Walter's theorem, analogues do exist, but we will now see that this is not always the case.

So far we have found several classes of noncompact groups G so that $A_{L^p}(G)$ is distinct for every $p \in [2, \infty)$. The following example shows in a strong way that this need not happen in general.

Example 4.4.3. Let G be the $ax+b$ group. In 1974 Khalil demonstrated that the Fourier algebra $A(G)$ coincides with its Rajchman algebra $B_0(G) := B(G) \cap C_0(G)$ (see [28]). Since elements in $B(G)$ are uniformly continuous, if $u \in B(G)$ is L^p-integrable then $u \in C_0(G)$. As $A_{L^p}(G)$ is the closed linear span of $P(G) \cap L^p(G)$ and the norm on $B(G)$ dominates the uniform norm, it follows that $A_{L^p}(G) \subset B_0(G)$. Therefore $A_{L^p}(G) = A(G)$ for every $1 \leq p < \infty$.

Remark 4.4.4. Recall that Herz's restriction theorem states that if G is a locally compact group and H is a closed subgroup of G, then $A(G)|_H = A(H)$. Our previous example also shows that the analogue of this theorem does not hold for A_{L^p} when $p > 2$. Indeed, $\mathbb{R}$ is a closed subgroup of the $ax + b$ group G, but $A_{L^p}(G)|_{\mathbb{R}} = A(G)|_{\mathbb{R}} = A(\mathbb{R}) \neq A_{L^p}(\mathbb{R})$ for $p > 2$ by Theorem 4.3.8.

Before proceeding to discuss the amenability of these Banach algebras, we pause to recall some definitions and properties. See [44] for a reference.

Recall that if E is a Banach-$\mathcal{A}$-bimodule, then E^* is also. A Banach algebra $\mathcal{A}$ is *amenable* if every bounded derivation $\delta : A \to E^*$ is inner for all Banach-$\mathcal{A}$-bimodules E. This definition is motivated by the fact that Johnson showed the group algebra $L^1(G)$ of a locally compact group G has this property if and only if G is amenable. The analogue of this result fails when $L^1(G)$ is replaced with the Fourier algebra $A(G)$. Indeed, Forrest and Runde showed that the Fourier algebra $A(G)$ is amenable if and only if G is virtually abelian, i.e., contains an open abelian subgroup of finite index (see [19]). However, the analogue of Johnson's result does hold for the Fourier algebra when the operator space structure of $A(G)$ is taken into account. For a reference on operator spaces, see [39].

A completely contractive Banach algebra $\mathcal{A}$ is *operator amenable* if every completely bounded derivation $\delta : A \to E$ into an operator-$\mathcal{A}$-bimodule E is inner. In 1995 Ruan showed that a locally compact group G is amenable if and only if the Fourier algebra $A(G)$ is operator amenable (see [41]).

A weaker condition than amenability is that of weak amenability. A (completely contractive) Banach algebra $\mathcal{A}$ is (*operator*) *weakly amenable* if every (completely) bounded derivation $\delta : \mathcal{A} \to \mathcal{A}^*$ is inner. In [47], Spronk showed that $A(G)$ is always operator weakly amenable (see also [46]). What will be important to us in the next example is the fact that (operator) weakly amenable Banach algebras are square dense, i.e., the span of products ab for $a, b \in \mathcal{A}$ is dense in $\mathcal{A}$.

Remark 4.4.5. Let G be a noncompact abelian group and $p > 2$. Then $uv \in A_{L^{p/2}}(G)$ for all $u, v \in A_{L^p}(G)$ implies that $A_{L^p}(G) \cdot A_{L^p}(G) \subset A_{L^{p/2}}(G)$. By Theorem 4.3.8 we know that $A_{L^{p/2}}(G)$ is strictly contained in $A_{L^p}(G)$. So $A_{L^p}(G)$ is not square dense and, hence,

the analogues of both Leptin's and Ruan's characterizations of amenability in terms of the Fourier algebra fail horribly for $A_{L^p}(G)$.

As a consequence of this observation, we find that $A_{L^p}(G)$ is never an amenable Banach algebra when G is noncompact and $p > 2$.

Proposition 4.4.6. *Let G be a locally compact group and $p > 2$. If $A_{L^p}(G) \neq A(G)$, then $A_{L^p}(G)$ is not (operator) weakly amenable.*

Proof. Without loss of generality, we may assume that $A_{L^{p/2}}(G) \neq A_{L^p}(G)$. Indeed, if not we define

$$\widetilde{p} = \inf\{q \in [2, \infty) : A_{L^q}(G) = A_{L^p}(G)\}$$

and replace p with $\widetilde{p} + \epsilon$ for some $0 < \epsilon < \min\{1, p - \widetilde{p}\}$. Then the space $A_{L^p}(G)$ has not changed and $A_{L^p}(G) \neq A_{L^{p/2}}(G)$ since $\widetilde{p} > 1$ implies $p/2 < (1 + \widetilde{p})/2 < \widetilde{p}$. So indeed we may assume that $A_{L^p}(G) \neq A_{L^{p/2}}(G)$. Then a similar argument as in the previous remark shows that $A_{L^p}(G)$ is not square dense and, therefore, is not (operator) weakly amenable. □

Corollary 4.4.7. *Let G be a noncompact locally compact group and $p > 2$. Then $A_{L^p}(G)$ is a non-amenable Banach algebra.*

Proof. By the above proposition, we may assume without loss of generality that $A_{L^p}(G) = A(G)$. Then G does not contain an open abelian subgroup of finite index by Proposition 4.2.5 and Theorem 4.3.8 since such a subgroup is necessarily noncompact. In particular, this implies that G is not almost abelian. Hence, $A_{L^p}(G) = A(G)$ is non-amenable by Forrest and Runde's result mentioned above. □

Recall that a linear functional D on a Banach algebra $\mathcal{A}$ is a *point derivation* if there exists some multiplicative linear functional χ on $\mathcal{A}$ so that $D(ab) = \chi(a)D(b) + D(a)\chi(b)$ for all $a, b \in \mathcal{A}$. The existence of nonzero point derivations is an obstruction to the (operator) weak amenability of $\mathcal{A}$. Since the Fourier algebra is always operator weakly amenable, the Fourier algebra does not admit any nonzero point derivations. As a corollary to Proposition 4.4.1, we show that the L^p-Fourier algebras admit no nonzero point derivations either. This corollary was pointed out to us by Nico Spronk.

Corollary 4.4.8. *Let G be a locally compact group and $p \in [1, \infty)$. Then $A_{L^p}(G)$ does not admit any nonzero point derivations.*

Proof. Suppose that $A_{L^p}(G)$ admits a nonzero point derivation D and choose a multiplicative linear functional χ on $A_{L^p}(G)$ so that $D(uv) = D(u)\chi(v) + D(v)\chi(u)$ for all $u, v \in A_{L^p}(G)$. By the above proposition, χ is the point evaluation functional at some point $s \in G$. Choose $u \in A_{L^p}(G)$ and $v \in A(G)$ so that $D(u) \neq 0$ and $v(s) \neq 0$. Then

$$D(uv) = D(u)\chi(v) + \chi(u)D(v) = v(s)D(u) \neq 0$$

since $v \in A(G)$ implies that $D(v) = 0$. But since $A(G)$ is an ideal in $B(G)$ and $A(G)$ admits no nonzero point derivations, we must have that $D(uv) = 0$. This contradicts the above calculation and, therefore, we conclude that $A_{L^p}(G)$ admits no point derivations. □

Let G_1 and G_2 be locally compact groups. The Effros-Ruan tensor product formula (see [15]) implies that $A(G_1)\widehat{\otimes}A(G_2) = A(G_1 \times G_2)$ where $\widehat{\otimes}$ denotes the operator projective tensor product and $u \otimes v \in A(G_1)\widehat{\otimes}A(G_2)$ is identified with $u \times v \in A(G_1 \times G_2)$. The next example shows that the analogue of this formula fails for A_{L^p}. Before this, we observe that the algebraic tensor product $A_{L^p}(G_1) \otimes A_{L^p}(G_2)$ embeds in $A_{L^p}(G_1 \times G_2)$ via the above identification.

Proposition 4.4.9. *Let G_1 and G_2 be locally compact groups and $p > 2$. Then $u \times v \in A_{L^p}(G_1 \times G_2)$ for all $u \in A_{L^p}(G_1)$ and $v \in A_{L^p}(G_2)$.*

Proof. First suppose that u and v are positive definite functions which are L^p-integrable. Then $u \times v$ is a positive definite function on $G_1 \times G_2$ and

$$\int_{G_1 \times G_2} |u \times v|^p = \int_{G_1} \int_{G_2} |u(s)v(t)|^p \, ds \, dt = \|u\|_p \|v\|_p < \infty.$$

Similar arguments as used previously in this chapter now show that $u \times v \in A_{L^p}(G_1 \times G_2)$ for all $u \in A_{L^p}(G_1)$ and $v \in A_{L^p}(G_2)$. □

Example 4.4.10. Let Γ_1 and Γ_2 be discrete groups containing copies of nonabelian free groups and $p > 2$. Then $A_{\ell^p}(\Gamma_1) \otimes A_{\ell^p}(\Gamma_2)$ is not norm dense in $A_{\ell^p}(\Gamma_1 \times \Gamma_2)$. Indeed,

identify copies of $\mathbb{F}_2$ in both Γ_1 and Γ_2 and let Δ be the diagonal subgroup of $\mathbb{F}_2 \times \mathbb{F}_2 \subset \Gamma_1 \times \Gamma_2$. Then $u \times v|_\Delta \in A_{\ell^{p/2}}(\Delta)$ for all $u \in A_{\ell^p}(\Gamma_1)$ and $v \in A_{\ell^p}(\Gamma_2)$ by Proposition 4.2.4 and Proposition 4.2.5. But $A_{\ell^p}(\Gamma_1 \times \Gamma_2)|_\Delta = A_{\ell^p}(\Delta)$. As $A_{\ell^{p/2}}(\Delta)$ is a proper subspace of $A_{\ell^p}(\Delta)$, we conclude that $A_{\ell^p}(\Gamma_1) \otimes A_{\ell^p}(\Gamma_2)$ is not norm dense in $A_{\ell^p}(\Gamma_1) \otimes A_{\ell^p}(\Gamma_2)$.

The observations made in this previous example have applications to constructing exotic C*-norms on $C^*_{\ell^p}(\Gamma_1) \otimes C^*_{\ell^p}(\Gamma_2)$ which are different from those constructed in the previous chapter.

Proposition 4.4.11. *Let Γ_1 and Γ_2 be discrete groups containing copies of noncommutative free groups and $p > 2$. Then $C^*_{\ell^p}(\Gamma_1 \times \Gamma_2)$ gives rise to a C*-norm on the algebraic tensor product $C^*_{\ell^p}(\Gamma_1) \otimes C^*_{\ell^p}(\Gamma_2)$ in the natural way. This norm is distinct from the minimal and maximal tensor product norms.*

Proof. Similar to the constructions in Chapter 3, we will again appeal to Proposition 3.2.1.

Let π_1 and π_2 be faithful ℓ^p-representations for $C^*_{\ell^p}(\Gamma_1)$ and $C^*_{\ell^p}(\Gamma_2)$, respectively. It follows from Proposition 4.4.9 that $B_{\pi_1 \times \pi_2} \subset B_{\ell^p}(\Gamma_1 \times \Gamma_2)$ and, by Proposition 4.2.5, $B_{\ell^p}(\Gamma_1 \times \Gamma_2)|_{\Gamma_1} = B_{\ell^p}(\Gamma_1) = B_{\pi_1}$ and $B_{\ell^p}(\Gamma_1 \times \Gamma_2)|_{\Gamma_2} = B_{\ell^p}(\Gamma_2) = B_{\pi_2}$. So $C^*_{\ell^p}(\Gamma_1 \times \Gamma_2)$ indeed induces a C*-norm on $C^*_{\ell^p}(\Gamma_1) \otimes C^*_{\ell^p}(\Gamma_2)$ in the natural way. From the observations in the previous example, we have that $B_{\ell^p}(\Gamma_1 \times \Gamma_2) \neq B_{\pi_1 \times \pi_2}$ and, hence, that the norm coming from $C^*_{\ell^p}(\Gamma_1 \times \Gamma_2)$ is not the spatial tensor product norm. □

Remark 4.4.12. For Γ_1 and Γ_2 as above, fix a copy of $\mathbb{F}_2$ in both Γ_1 and Γ_2 and let Δ denote the diagonal subgroup of $\mathbb{F}_2 \times \mathbb{F}_2 \leq \Gamma_1 \times \Gamma_2$. In the proof of Theorem 3.2.4, we constructed Fourier Stieltjes spaces B_{σ_q} for $q > p$ such that $B_{\sigma_q}|_\Delta = B_{\ell^q}(\Delta)$. Since $B_{\ell^p}(\Gamma_1 \times \Gamma_2)|_\Delta = B_{\ell^p}(\Delta)$, this indeed gives a different C*-norm that those constructed in the previous section.

In a previous example we observed that characterizations of amenability in terms of the Fourier algebra can fail when $A(G)$ is replaced with $A_{L^p}(G)$. We finish this section by identifying some characterizations of amenability which do translate over.

Let $\mathcal{A}$ be a Banach algebra. A linear operator $T : \mathcal{A} \to \mathcal{A}$ is said to be a multiplier of $\mathcal{A}$ if $T(ab) = aT(b) = T(a)b$ for all $a, b \in \mathcal{A}$. In the context of Fourier algebras $A(G)$,

every multiplier every multiplier is bounded and is realizable as multiplication by some function on G (see [33]). Losert characterized the amenability of a locally compact group G in terms of multipliers by showing that G is amenable if and only if $M(A(G))$, the set of multipliers of $A(G)$, is exactly $B(G)$ if and only if the norm on $A(G)$ is equivalent to the norm it attains as a multiplier on itself (see [33, Theorem 1]). We show that the analogue of this theorem holds for L^p-Fourier algebras.

Theorem 4.4.13. *The following are equivalent for a locally compact group G and $1 \leq p < \infty$.*

(i) *G is amenable.*
(ii) *$M(A_{L^p}(G)) = B(G)$.*
(iii) *$\|\cdot\|_{B(G)}$ is equivalent to $\|\cdot\|_{M(A_{L^p}(G))}$ on $B(G)$.*
(iv) *$\|\cdot\|_{B(G)}$ is equivalent to $\|\cdot\|_{M(A_{L^p}(G))}$ on $A(G)$.*
(v) *$\|\cdot\|_{B(G)}$ is equivalent to $\|\cdot\|_{M(A(G))}$ on $A(G)$.*

Proof. (i) $\Rightarrow$ (ii): It is an application of the closed graph theorem that every element of $M(A_{L^p}(G))$ is bounded and given by a multiplication operator. Suppose that $v \in C_b(G)$ is a multiplier of $A_{L^p}(G)$. Since G is amenable, $A(G)$ admits a bounded pointwise approximate identity $\{u_\alpha\}$. Then $\{u_\alpha v\}$ is a bounded sequence converging pointwise to v and, hence, $v \in B(G)$ (see [16, Corollaire 2.25]).

(ii) $\Rightarrow$ (iii): This is a standard application of the open mapping theorem.

(iii) $\Rightarrow$ (iv): This is clear.

(iv) $\Rightarrow$ (v): Suppose that there exists $c > 0$ so that

$$\sup\{\|uv\|_{B(G)} : v \in A_{L^p}(G), \|v\|_{B(G)} \leq 1\} > c\|u\|_{B(G)}$$

for every $u \in A(G)$ and choose n sufficiently large so that $p/n < 2$. Fix $u \in A_{L^p}(G)$ and choose a unit vector v_1 in $A_{L^p}(G)$ so that $\|uv_1\|_{B(G)} > c\|u\|_{B(G)}$. Next choose $v_2 \in A_{L^p}(G)$ so that

$$\|(uv_1)v_2\|_{B(G)} > c\|uv_1\|_{B(G)} > c^2\|u\|_{B(G)}.$$

Repeat this process until we arrive at n unit vectors $v_1, \ldots, v_n \in A_{L^p}(G)$ and define $v = v_1 \cdots v_n$. Then $v \in A(G)$ has norm at most 1 and $\|uv\|_{B(G)} > c^n\|u\|_{B(G)}$. Hence, $\|\cdot\|_{B(G)}$ is equivalent to $\|\cdot\|_{M(A(G))}$ on $A(G)$.

(v) $\Rightarrow$ (i): As mentioned above, this was shown by Losert. $\square$

We now prove a characterization of amenability in terms of the L^p-Fourier-Stieltjes algebra in terms of operator Connes amenability, which is a weak* version of operator amenability.

A pairing of a Banach algebra $\mathcal{A}$ with a predual $\mathcal{A}_*$ is said to be a *dual Banach algebra* if $\mathcal{A}_*$ is a closed sub-$\mathcal{A}$-module of $\mathcal{A}^*$. For our purposes, we will consider the dual Banach algebra $B_{L^p}(G)$ with predual $C^*_{L^p}(G)$. A dual Banach algebra $\mathcal{A}$ is said to be (*operator*) *Connes amenable* if every weak*-weak* continuous derivation $\delta : \mathcal{A} \to E^*$ is inner for all operator-$\mathcal{A}$-bimodule E. In [45] Runde and Spronk showed that a locally compact group G is amenable if and only if its reduced Fourier-Stieltjes algebra $B_r(G)$ is operator Connes amenable. We generalize this result to L^p-Fourier-Stieltjes algebras in the following result.

Proposition 4.4.14. *Let G be a locally compact group and $p \in [1, \infty)$. Then G is amenable if and only if $B_{L^p}(G)$ is operator Connes amenable.*

Proof. First suppose that G is amenable. Then $B_{L^p}(G) = B_r(G) = B(G)$ is operator Connes amenable (see [45, Theorem 4.4]).

Next suppose that $B_{L^p}(G)$ is operator Connes amenable. Then, as in the proof of [45, Theorem 4.4], $B_{L^p}(G)$ has an identity. So $B_{L^p}(G) = B(G)$ and, hence, G is amenable by Proposition 4.2.3. $\square$

4.5 Fourier-Stieltjes ideals of $SL_2(\mathbb{R})$

In this section, we study the L^p-Fourier-Stieltjes algebras for $SL_2(\mathbb{R})$ and characterize the Fourier-Stieltjes ideals of $SL_2(\mathbb{R})$. The representation theory of $SL_2(\mathbb{R})$ is very well understood, and this knowledge is used intimately throughout this section.

The irreducible representations of $SL_2(\mathbb{R})$ fall into the following five categories:

$$\begin{aligned}
\text{Trivial representation} &: \tau, \\
\text{Discrete series} &: \{T_n : n \in \mathbb{Z}, |n| \geq 2\}, \\
\text{Mock discrete series} &: T_{-1}, T_1, \\
\text{Principal series} &: \{\pi_{it,\epsilon} : t \in \mathbb{R}, \epsilon = \pm 1\}, \\
\text{Complementary series} &: \{\pi_r : -1 < r < 0\}.
\end{aligned}$$

There is no standard for the notation and parametrizations of these representations, so, for convenience, we will follow that used by Repka in [41] – a paper which we will refer to again. The Fell topology on these representations is also completely understood. Rather than detailing this topology we refer the reader to [18, Figure 7.3] for a nice description.

Kunze and Stein studied the integrability properties of the coefficients of irreducible representations of $SL_2(\mathbb{R})$ and demonstrated the remarkable fact that for every nontrivial irreducible representation π of $SL_2(\mathbb{R})$, there exists a $p \in [2, \infty)$ so that $\pi_{x,x} \in L^p$ for every $x \in \mathcal{H}_\pi$. In fact, they showed the following for an irreducible representation π of $SL_2(\mathbb{R})$ (see [30, Theorem 10]):

- π is an element of the discrete series if and only if every coefficient function of π is L^2-integrable,
- π is an element of the mock discrete series or the continuous principal series if and only if every coefficient function of π is $L^{2+\epsilon}$-integrable for every $\epsilon > 0$, but not every coefficient function is L^2-integrable,
- π is an element of the complementary series with parameter $r \in (-1, 0)$ if and only if every coefficient function of r is $L^{2/(1+r)+\epsilon}$-integrable for every $\epsilon > 0$, but not every coefficient function is $L^{2/(1+r)}$-integrable.

A fortiori, every nontrivial irreducible representation of $SL_2(\mathbb{R})$ is an L^p-representation for some $p \in [2, \infty)$. We use this and a result of Repka to show that the spaces $B_{L^p}(SL_2(\mathbb{R}))$ are distinct for every $p \in [2, \infty)$.

Lemma 4.5.1. *Let G be the group $SL_2(\mathbb{R})$. Then*

(i) The discrete series, mock discrete series, and principal series are weakly contained in the L^p-representations for every $p \in [2, \infty)$,

(ii) The complementary series representations π_r is weakly contained in the L^p-representations (for $p \in [2, \infty)$) if and only if $r \in [2/p - 1, 0)$.

Proof. Let π be a representation of $SL_2(\mathbb{R})$. Then [41, Theorem 9.1] immediately implies that if π is an L^p-representation for some $p > 2$, then the direct integral decomposition of π does not include the representations π_r for $-1 < r < 2/p - 1$ (apart from on a null set). Hence, π does not weakly contain π_r for any $-1 < r < 2/p - 1$ since the set $\{\pi_r : -1 < r < 2/p - 1\}$ is open in the Fell topology.

Note that by the results of Kunze and Stein mentioned above, π_r is an L^p-representation for every $2/p - 1 < r < 0$. Hence, the L^p-representations weakly contain π_r if and only if $2/p - 1 \leq r < 0$.

To complete our proof we must know that the mock discrete series and principal series are weakly contained in the left regular representation. However this is given by the Cowling-Haagerup-Howe theorem (see [12, Theorem 1]) since they are each $L^{2+\epsilon}$-representations for every $\epsilon > 0$. □

Corollary 4.5.2. *Let $G = SL_2(\mathbb{R})$. Then the Fourier-Stieltjes spaces $B_{L^p}(G)$ are distinct for every $p \in [2, \infty)$. Equivalently, the C^*-algebras $C^*_{L^p}(G)$ are distinct for every $p \in [2, \infty)$.*

We now proceed to prove the main result of this section: a characterization of the Fourier-Stieltjes ideals of $SL_2(\mathbb{R})$.

Theorem 4.5.3. *Let I be a nontrivial Fourier-Stieltjes ideal of $SL_2(\mathbb{R})$. Then $I = B(G)$ or $I = B_{L^p}(G)$ for some $p \in [2, \infty)$.*

Proof. Write $I = B_\pi$ for some representation π of G. Then, since $\pi \otimes \lambda$ is unitarily equivalent to an amplification of λ by Fell's absorption principle, it is an easy exercise to see that $B_\pi \supset B_r(G) = B_{L^2}(G)$.

Consider the case when the trivial representation τ is weakly contained in π. Then I contains the unit and, hence, is all of $B(G)$.

Next consider the case when π does not contain the complementary representation π_r for any $r \in (-1, 0)$. Then, by Lemma 4.5.1, B_π is a subset of $B_{L^2}(G)$. Since we already know the reverse inclusion, we conclude that $I = B_{L^2}(G)$.

Finally, we consider the case when π weakly contains some element of the complementary series. Let

$$r = \inf\{r' \in (-1, 0) : \pi_{r'} \text{ is weakly contained in } \pi\}.$$

Then $r > -1$ since π_r converges to the trivial representation τ in the Fell topology as $r \to -1$. Also notice that π weakly contains π_r since $\pi_{r'} \to \pi_r$ in the Fell topology as $r' \to r$. In [39] Pukánszky showed that if $r_1, r_2 \in (-1, 0)$ with $r_1 + r_2 < -1$, then $\pi_{r_1+r_2+1}$ is a subrepresentation of $\pi_{r_1} \otimes \pi_{r_2}$ (see also [41, Theorem 5.9]). Since $r + r' < -1$ for $-1 < r' < -r - 1$ and π weakly contains $\pi_r \otimes \pi_{r'}$ for every $-1 < r < 0$, it follows that π weakly contains $\pi_{r'}$ for each $r \leq r' < r$. Therefore, by Lemma 4.5.1, we conclude that $I = B_{L^p}(G)$ where $p = 2/(1 + r)$. □

It is natural to wonder for which other groups are the Fourier-Stieltjes ideals characterizable as above. Unfortunately this characterization does not hold for arbitrary locally compact groups G.

Example 4.5.4. Consider the free group $\mathbb{F}_\infty$ on countably many generators $a_1, a_2, \ldots$ and let $\mathbb{F}_d$ denote the subgroup generated by $a_1, \ldots, a_d$ for some $2 \leq d < \infty$. For each $p \in [1, \infty)$, define

$$D_p = \{f \in \ell^\infty(\mathbb{F}_\infty) : f|_{s\mathbb{F}_d t} \in \ell^p(s\mathbb{F}_d t) \text{ for all } s, t \in \mathbb{F}_\infty\}.$$

Then D_p is an ideal of $\ell^\infty(\mathbb{F}_\infty)$ which implies that B_{D_p} is an ideal of $B(G)$. Moreover, we showed that $C^*_{D_p}(\mathbb{F}_\infty) \neq C^*_{\ell^q}(\mathbb{F}_\infty)$ for any $1 \leq q < \infty$ and that $C^*_{D_p}(\mathbb{F}_\infty)$ is distinct for each $p \in [2, \infty)$ in Example 2.2.4. Hence, $\mathbb{F}_\infty$ has a continuum of Fourier-Stieltjes ideals which are not of the form $B_{\ell^p}(\mathbb{F}_\infty)$ for some $p \in [2, \infty)$.

Chapter 5

Weak* tensor products for von Neumann algebras

5.1 Introduction

There are many C*-tensor products which are studied in the category of C*-algebras. In contrast, virtually the only tensor product in the category of von Neumann algebras ever studied is the normal spatial tensor product. Further, within the literature there is not a description of what a tensor product in this category should be. We propose that in the category of von Neumann algebras, a generic tensor product of von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ should be a von Neumann algebra $\mathcal{S}$ which contains a weak* dense copy of the algebraic tensor product $\mathcal{M} \otimes \mathcal{N}$ such that $\mathcal{M}$ and $\mathcal{N}$ are identifiable as von Neumann algebras with the copies of $\mathcal{M} \otimes 1$ and $1 \otimes \mathcal{N}$ in $\mathcal{S}$, respectively. We call such a von Neumann algebra $\mathcal{S}$ a *weak* tensor product* of $\mathcal{M}$ and $\mathcal{N}$. In section 2 of this chapter, we define this concept more rigorously and investigate weak* tensor products of factors.

Similar to the maximal tensor product for C*-algebras, there is a "largest" weak* tensor product completion of $\mathcal{M} \otimes \mathcal{N}$ for two von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$. We briefly draw some connections between this "largest" weak*-tensor product and related structures in Section 3. Surprisingly, in general there is no "smallest" weak* tensor product despite the

normal spatial tensor product being defined analogously to the minimal tensor product of C*-algebras.

Tensor products play an invaluable role within the field of C*-algebras and many properties of C*-algebras (such as nuclearity (see [39, Definition 11.4]), exactness (see [39, Chapter 17]), and the WEP (see [39, Proposition 15.3])) are either defined or have a characterization in terms of tensor products. Perhaps weak* tensor products could play a similar role within von Neumann algebras.

Recall that a C*-algebra $\mathcal{A}$ is nuclear if and only if the algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$ has a unique C*-completion for every second C*-algebra $\mathcal{B}$. In section 4, we completely characterize the analogous property for weak* tensor products. The class of nuclear C*-algebras is large and contains many interesting examples. In contrast, a von Neumann algebra $\mathcal{M}$ has the property that $\mathcal{M} \otimes \mathcal{N}$ has a unique weak* tensor product completion for every von Neumann algebra $\mathcal{N}$ if and only if $\mathcal{M}$ is the direct product of type I factors. In particular, this implies that even abelian von Neumann algebras need not admit this property.

In the final section, we apply the theory developed throughout to studying weak* tensor product completions of $L^\infty(\mathbb{R}) \bar{\otimes} L^\infty(\mathbb{R})$. In particular, we construct $2^\mathfrak{c}$ nonequivalent weak* tensor product completions of $L^\infty(\mathbb{R}) \bar{\otimes} L^\infty(\mathbb{R})$ and note that a generic weak* tensor product completion of $L^\infty(\mathbb{R}) \otimes L^\infty(\mathbb{R})$ need not have a separable predual, despite $L^\infty(\mathbb{R})$ having a separable predual.

5.2 Definitions and the case of factors

In this section we make precise the notion of weak* tensor products and examine particular examples involving factors. In particular, we show that if $\mathcal{M}$ is a factor of type II or III, then $\mathcal{M} \otimes \mathcal{M}'$ does not admit a unique weak* tensor product completion. The case when $\mathcal{M} = B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ is completely different, and we show that $B(\mathcal{H})$ has the property that $B(\mathcal{H}) \otimes \mathcal{N}$ has a unique weak* tensor product completion for each von Neumann algebra $\mathcal{N}$.

Definition 5.2.1. Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras and suppose that $\alpha : \mathcal{M} \otimes \mathcal{N} \to B(\mathcal{H})$ is an injective $*$-representation on some Hilbert space $\mathcal{H}$ such that $\alpha(\mathcal{M} \otimes 1)$ and $\alpha(1 \otimes \mathcal{N})$ are von Neumann subalgebras of $B(\mathcal{H})$. The *weak* tensor product* $\mathcal{M} \overline{\otimes}_\alpha \mathcal{N}$ is defined to be the weak* closure of $\alpha(\mathcal{M} \otimes \mathcal{N})$ in $B(\mathcal{H})$. Such a weak* tensor product $\mathcal{M} \overline{\otimes}_\alpha \mathcal{N}$ is also called a *weak* tensor product completion* of $\mathcal{M} \otimes \mathcal{N}$.

We recover the definition of the *normal spatial tensor product* $\mathcal{M} \overline{\otimes} \mathcal{N}$ of von Neumann algebras $\mathcal{M} \subset B(\mathcal{H})$ and $\mathcal{N} \subset B(\mathcal{H})$ by letting α be the canonical inclusion ι of $\mathcal{M} \otimes \mathcal{N}$ inside $B(\mathcal{H} \otimes \mathcal{K})$. The following example gives another construction of a weak* tensor product for factors.

Example 5.2.2. Let $\mathcal{M} \subset B(\mathcal{H})$ be a factor. It is an early result of Murray and von Neumann that the multiplication map $m : a \otimes b \mapsto ab$ from $\mathcal{M} \otimes \mathcal{M}' \to B(\mathcal{H})$ is injective (see [35]). Observe that $a \in B(\mathcal{H})$ commutes with $\mathcal{M} \cdot \mathcal{M}'$ if and only if $a \in \mathbb{C}1$ as both $\mathcal{M}$ and $\mathcal{M}'$ are subsets of $\mathcal{M} \cdot \mathcal{M}'$. Hence, $\mathcal{M} \overline{\otimes}_m \mathcal{M}' = B(\mathcal{H})$.

Recall that if $\mathcal{M}$ is a factor, then $\mathcal{M} \overline{\otimes} \mathcal{M}'$ is a factor of the same type. Hence, if $\mathcal{M}$ is a factor of type II or III and m is as in Example 5.2.2, then $\mathcal{M} \overline{\otimes}_m \mathcal{M}'$ and $\mathcal{M} \overline{\otimes} \mathcal{M}'$ are not $*$-isomorphic. On the other hand, if $\mathcal{M} = B(\mathcal{H})$ is a type I factor then $\mathcal{M} \overline{\otimes}_m \mathcal{M}'$ is trivially canonically $*$-isomorphic to $\mathcal{M} \overline{\otimes} \mathcal{M}'$ since $\mathcal{M}' = \mathbb{C}1$. This motivates us to define a means of comparison for weak* tensor products.

Definition 5.2.3. Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. Two weak* tensor products $\mathcal{M} \overline{\otimes}_\alpha \mathcal{N}$ and $\mathcal{M} \overline{\otimes}_\beta \mathcal{N}$ of $\mathcal{M}$ and $\mathcal{N}$ are *equivalent* if the map $\alpha(a \otimes b) \mapsto \beta(a \otimes b)$ for $a \in \mathcal{M}$, $b \in \mathcal{N}$ extends to a (normal) $*$-isomorphism of $\mathcal{M} \overline{\otimes}_\alpha \mathcal{N}$ onto $\mathcal{M} \overline{\otimes}_\beta \mathcal{N}$.

Let $\mathcal{A}$ be a $*$-algebra and $\pi : \mathcal{A} \to B(\mathcal{H})$, $\sigma : \mathcal{A} \to B(\mathcal{K})$ be two $*$-representations. Recall that the map $\sigma(a) \mapsto \pi(a)$ extends to a normal $*$-homomorphism from $\pi(\mathcal{A})''$ to $\sigma(\mathcal{A})''$ if and only if π is quasi-contained in σ, i.e., if and only if π is unitarily equivalent to a subrepresentation of some amplification of σ. In particular, this immediately gives that two weak* tensor products $\mathcal{M} \overline{\otimes}_\alpha \mathcal{N}$ and $\mathcal{M} \overline{\otimes}_\beta \mathcal{N}$ are equivalent if and only if α and β are quasi-equivalent (i.e., each is quasi-contained in the other) and, further, that the identity map on $\mathcal{M} \otimes \mathcal{N}$ extends to a normal $*$-homomorphism from $\mathcal{M} \overline{\otimes}_\alpha \mathcal{N}$ to $\mathcal{M} \overline{\otimes}_\beta \mathcal{N}$ if and

only if β is quasi-equivalent to a subrepresentation of α. This observation allows us to see that there is a universal weak* tensor product.

Proposition 5.2.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. There exists a unique (up to equivalence) weak* tensor product $\mathcal{M}\overline{\otimes}_\alpha\mathcal{N}$ of $\mathcal{M}$ and $\mathcal{N}$ with the property that if $\mathcal{M}\overline{\otimes}_\beta\mathcal{N}$ is any other weak* tensor product, then the identity map on $\mathcal{M}\otimes\mathcal{N}$ extends to a normal $*$-homomorphism from $\mathcal{M}\overline{\otimes}_\alpha\mathcal{N}$ onto $\mathcal{M}\overline{\otimes}_\beta\mathcal{N}$.*

Proof. The uniqueness of such a weak* tensor product is clear, so we focus on showing existence. Since the collection of all quasi-equivalence classes of representations of $\mathcal{M}\otimes\mathcal{N}$ forms a set, the collection of equivalence classes of weak* tensor products of $\mathcal{M}$ and $\mathcal{N}$ must form a set. Let $\{\mathcal{M}\overline{\otimes}_{\alpha_i}\mathcal{N} : i\in I\}$ be the set formed by choosing one representative from each equivalence class, and define $\alpha = \bigoplus_{i\in I}\alpha_i$. Then, since α_i is a subrepresentation of α for every $i\in I$, it is clear that $\mathcal{M}\overline{\otimes}_\alpha\mathcal{N}$ has the desired universal property among weak* tensor products of $\mathcal{M}$ and $\mathcal{N}$. □

Definition 5.2.5. Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. The *maximal* (or *universal*) *weak* tensor product* of $\mathcal{M}$ and $\mathcal{N}$ is the weak* tensor product $\mathcal{M}\overline{\otimes}_\alpha\mathcal{N}$ of $\mathcal{M}$ and $\mathcal{N}$ with universal property described in Proposition 5.2.4. This weak* tensor product is denoted $\mathcal{M}\overline{\otimes}_{w^*\text{-max}}\mathcal{N}$.

It is readily verified that this weak* tensor product is both commutative and associative. It is also easily checked that this weak* tensor product has the projectivity property since if $\mathcal{J}$ is any weak* closed ideal of a von Neumann algebra $\mathcal{M}$, then $\mathcal{M} = \mathcal{N}\oplus\mathcal{J}$ where $\mathcal{N} = \mathcal{M}/\mathcal{J}$.

We find it interesting to observe that if $\mathcal{M}\subset B(\mathcal{H})$ is a factor of type II or III and $m:\mathcal{M}\otimes\mathcal{M}'\to B(\mathcal{H})$ is the multiplication map, then $\mathcal{M}\overline{\otimes}_{w^*\text{-max}}\mathcal{M}'$ is not a factor since both $B(\mathcal{H}) = \mathcal{M}\overline{\otimes}_m\mathcal{M}'$ and $\mathcal{M}\overline{\otimes}\mathcal{M}'$ are normal quotients of $\mathcal{M}\overline{\otimes}_{w^*\text{-max}}\mathcal{M}'$.

We find it useful to think of the normal spatial and maximal weak* tensor products as being analogues of the minimal and maximal tensor products of C*-algebras. Recall that the minimal tensor product $\mathcal{A}\otimes_{\min}\mathcal{B}$ of C*-algebras $\mathcal{A}$ and $\mathcal{B}$ has the property that if $\mathcal{A}\otimes_\alpha\mathcal{B}$ is any other C*-completion of $\mathcal{A}\otimes\mathcal{B}$, then the identity map on $\mathcal{A}\otimes\mathcal{B}$ extends to a $*$-homomorphism from $\mathcal{A}\otimes_\alpha\mathcal{B}\to\mathcal{A}\otimes_{\min}\mathcal{B}$.

Remark 5.2.6. The examples we have already studied show that the analogue of this property does not hold for the normal spatial tensor product among the class of weak* tensor products. Indeed, let $\mathcal{M} \subset B(\mathcal{H})$ be a factor of type II or III and $m : \mathcal{M} \otimes \mathcal{M}' \to B(H)$ the multiplication operator. Then $\mathcal{M}\overline{\otimes}_m\mathcal{M}' = B(\mathcal{H})$ is not a normal quotient of $\mathcal{M}\overline{\otimes}\mathcal{M}'$ since $B(\mathcal{H})$ is a type I factor but $\mathcal{M}\overline{\otimes}\mathcal{M}'$ is a factor not of type I. However, it is interesting to note that both $\mathcal{M}\overline{\otimes}\mathcal{M}'$ and $\mathcal{M}\overline{\otimes}_m\mathcal{M}'$, being factors, are each minimal among the weak* tensor products of $\mathcal{M}$ and $\mathcal{M}'$.

We finish this section with a brief discussion of an analogue of nuclearity for C*-algebras phrased in terms of weak* tensor products. Typically the injectivity of von Neumann algebras is thought of as being the proper analogue of nuclearity of C*-algebras. We will later see that our following definition is a much stronger condition than injectivity and does not even include the class of abelian von Neumann algebras.

Definition 5.2.7. A von Neumann algebra $\mathcal{M}$ has the *weak* tensor uniqueness property* (or *WTU property*) if for any von Neumann algebra $\mathcal{N}$, any two weak* tensor products $\mathcal{M}\overline{\otimes}_\alpha\mathcal{N}$ and $\mathcal{M}\overline{\otimes}_\beta\mathcal{N}$ of $\mathcal{M}$ and $\mathcal{N}$ are equivalent.

Proposition 5.2.8. *Let $\mathcal{H}$ be a Hilbert space. Then $B(\mathcal{H})$ has the WTU property.*

Proof. Let $\mathcal{N}$ be a von Neumann algebra and suppose that $B(\mathcal{H})\overline{\otimes}_\alpha\mathcal{N} \subset B(\mathcal{K})$ is a weak* tensor product of $B(\mathcal{H})$ and $\mathcal{N}$. Then $\alpha|_{B(\mathcal{H})}$ is unitarily equivalent to an amplification map of $B(\mathcal{H})$. So we may assume that $\mathcal{K} = \mathcal{H} \otimes \ell^2(I)$ for some index set I and

$$\alpha(a \otimes 1) = a \otimes 1 \in B(\mathcal{H})\overline{\otimes}B(\ell^2(I)) = B(\mathcal{K})$$

for every $a \in B(\mathcal{H})$. Since $\alpha(1\otimes\mathcal{N})$ commutes with $\alpha(B(\mathcal{H})\otimes 1) = B(\mathcal{H})\otimes 1$, we have that $\alpha(1 \otimes \mathcal{N}) \subset 1 \otimes B(\ell^2(I))$. It follows that $B(\mathcal{H})\overline{\otimes}_\alpha\mathcal{N}$ is equivalent to the normal spatial tensor product $B(\mathcal{H})\overline{\otimes}\mathcal{N}$. □

5.3 Remarks on the maximal weak* tensor product

Before continuing onto the main results of this chapter, we pause to record a couple of connections between the maximal weak* tensor product and related constructions. We begin by establishing a connection with the maximal tensor product of C*-algebras.

Proposition 5.3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. Then the identity map on $\mathcal{A}\otimes\mathcal{B}$ extends to a normal $*$-isomorphism $(\mathcal{A}\otimes_{\max}\mathcal{B})^{**}\cong\mathcal{A}^{**}\overline{\otimes}_{w^*\text{-max}}\mathcal{B}^{**}$.*

Proof. We first assume that $\mathcal{A}$ and $\mathcal{B}$ are unital.

Let α be a $*$-representation of $\mathcal{A}^{**}\otimes\mathcal{B}^{**}$ such that $\mathcal{A}^{**}\overline{\otimes}_{\alpha}\mathcal{B}^{**}=\mathcal{A}^{**}\overline{\otimes}_{w^*\text{-max}}\mathcal{B}$. Then, since $a\otimes b\mapsto\alpha(a\otimes b)$ extends to a $*$-representation from $\mathcal{A}\otimes_{\max}\mathcal{B}\to\mathcal{A}^{**}\overline{\otimes}_{w^*\text{-max}}\mathcal{B}^{**}$, we have that the identity map on $\mathcal{A}\otimes\mathcal{B}$ extends to a normal $*$-homomorphism from $(\mathcal{A}\otimes_{\max}\mathcal{B})^{**}\to\mathcal{A}^{**}\overline{\otimes}_{w^*\text{-max}}\mathcal{B}^{**}$.

Now let $\pi_u:\mathcal{A}\otimes\mathcal{B}\to B(H_u)$ be the universal representation of $\mathcal{A}\otimes_{\max}\mathcal{B}$. Then $\pi_u|_{\mathcal{A}}$ extends to a normal $*$-homomorphism $\mathcal{A}^{**}\to B(\mathcal{H}_u)$ and $\pi_u|_{\mathcal{B}}$ extends to a normal $*$-homomorphism $\mathcal{B}^{**}\to B(\mathcal{H}_u)$. As the ranges of $\pi_u|_{\mathcal{A}}$ and $\pi_u|_{\mathcal{B}}$ commute, the identity map on $\mathcal{A}\otimes\mathcal{B}$ extends to a normal $*$-homomorphism $\mathcal{A}^{**}\overline{\otimes}_{w^*\text{-max}}\mathcal{B}^{**}\to(\mathcal{A}\otimes_{\max}\mathcal{B})^{**}$. Hence, the identity map on $\mathcal{A}\otimes\mathcal{B}$ extends to a normal $*$-isomorphism $(\mathcal{A}\otimes_{\max}\mathcal{B})^{**}\cong\mathcal{A}^{**}\overline{\otimes}_{w^*\text{-max}}\mathcal{B}^{**}$.

In the case that $\mathcal{A}$ and $\mathcal{B}$ are not necessarily unital, then the canonical embedding of $\mathcal{A}^{**}\otimes\mathcal{B}^{**}$ into $\mathcal{A}_I^{**}\otimes\mathcal{B}_I^{**}$ (where $\mathcal{A}_I$ and $\mathcal{B}_I$ are the unitizations of $\mathcal{A}$ and $\mathcal{B}$) is readily verified to extend to a binormal embedding of $\mathcal{A}^{**}\overline{\otimes}_{w^*\text{-max}}\mathcal{B}^{**}$ into $\mathcal{A}_I^{**}\overline{\otimes}_{w^*\text{-max}}\mathcal{B}_I^{**}$ since every $*$-representation of $\mathcal{A}\otimes\mathcal{B}$ canonically extends to a $*$-representation of $\mathcal{A}_I\otimes\mathcal{B}_I$. Therefore the identity map on $\mathcal{A}\otimes\mathcal{B}$ extends to a normal $*$-isomorphism $(\mathcal{A}\otimes_{\max}\mathcal{B})^{**}\cong\mathcal{A}^{**}\overline{\otimes}_{w^*\text{-max}}\mathcal{B}^{**}$ for any C*-algebras $\mathcal{A}$ and $\mathcal{B}$. □

Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. The binormal C*-norm $\|\cdot\|_{\text{bin}}$ of $\mathcal{M}\otimes\mathcal{N}$ is defined by

$$\|x\|_{\text{bin}}=\sup\{\varphi(x^*x)^{1/2}:\varphi\in S(\mathcal{M}\otimes\mathcal{N}),\,(a,b)\mapsto\varphi(a\otimes b)\text{ is separately weak* continuous}\}$$

where $S(\mathcal{M}\otimes\mathcal{N})$ is the set of states on $\mathcal{M}\otimes\mathcal{N}$, i.e. the set of linear maps $\varphi:\mathcal{M}\otimes\mathcal{N}\to\mathbb{C}$ such that $\varphi(1)=1$ and $\varphi(x^*x)\geq 0$ for every $x\in\mathcal{M}\otimes\mathcal{N}$. This C*-tensor norm was defined and studied by Effros and Lance in [14]. The main result of their paper on this norm is that a von Neumann algebra $\mathcal{M}$ has the property that the binormal C*-norm $\|\cdot\|_{\text{bin}}$ agrees with the minimal C*-norm $\|\cdot\|_{\min}$ on $\mathcal{M}\otimes\mathcal{N}$ for all choices of von Neumann algebras $\mathcal{N}$ if and only if $\mathcal{M}$ is semidiscrete (see [14, Theorem 4.1]). We will next observe observe that the norm on $\mathcal{M}\otimes\mathcal{N}$ arising from the inclusion in $\mathcal{M}\overline{\otimes}_{w^*\text{-max}}\mathcal{N}$ is exactly the binormal C*-norm.

Proposition 5.3.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras and $\|\cdot\|_{w^*\text{-max}}$ denote the norm on $\mathcal{M}\otimes\mathcal{N}$ arising from the inclusion into $\mathcal{M}\overline{\otimes}_{w^*\text{-max}}\mathcal{N}$. Then $\|\cdot\|_{w^*\text{-max}} = \|\cdot\|_{\text{bin}}$.*

Proof. Let $\varphi \in S(\mathcal{M}\otimes\mathcal{N})$ and suppose that $\varphi|_{\mathcal{M}}$ and $\varphi|_{\mathcal{N}}$ are weak* continuous. Denote the GNS representation of φ by $\pi_\varphi : \mathcal{M}\otimes\mathcal{N} \to B(\mathcal{H}_\varphi)$. We claim that the restrictions $\pi_\varphi|_{\mathcal{M}}$ and $\pi_\varphi|_{\mathcal{N}}$ define normal maps on $\mathcal{M}$ and $\mathcal{N}$, respectively. Indeed, let $x = \sum_{j=1}^n a_j \otimes b_j$ and $y = \sum_{k=1}^m a'_k \otimes b'_k$ be elements of $\mathcal{M}\otimes\mathcal{N}$. Denoting the images of x and y in H_φ by $\Lambda(x)$ and $\Lambda(y)$, respectively, we then have that the map

$$\mathcal{M}\times\mathcal{N} \ni (a,b) \mapsto \langle \pi_\varphi(a\otimes b)\Lambda(x), \Lambda(y)\rangle = \sum_{j,k} \varphi(((a'_k)^* a a_j) \otimes ((b'_k)^* b b_j))$$

is separately weak* continuous. Since $\Lambda(\mathcal{M}\otimes\mathcal{N})$ is dense in $\mathcal{H}_\varphi$, it follows that $\pi_\varphi|_{\mathcal{M}}$ is WOT-WOT continuous on the unit ball of $\mathcal{M}$. Therefore $\pi_\varphi|_{\mathcal{M}}$ is normal and, similarly, $\pi_\varphi|_{\mathcal{N}}$ is also normal. So $\|\cdot\|_{\text{bin}} \leq \|\cdot\|_{w^*\text{-max}}$.

Now let $\alpha : \mathcal{M}\otimes\mathcal{N} \to B(\mathcal{H})$ be a $*$-representation which satisfies the conditions required for a weak* tensor product. It is clear that $(a,b) \mapsto \langle \alpha(a\otimes b)x, y\rangle$ is separately weak* continuous for all $x, y \in \mathcal{H}$ and, so, $\|\cdot\|_{w^*\text{-max}} \leq \|\cdot\|_{\text{bin}}$. □

5.4 Characterization of the weak* tensor uniqueness property

In this section we study the WTU property and give a complete characterization of the von Neumann algebras with this property. We have already seen in Section 5.2 that a factor $\mathcal{M}$ has the WTU property if and only if $\mathcal{M}$ is of type I. We will show in this section that a von Neumann algebra $\mathcal{M}$ has the WTU property if and only if $\mathcal{M}$ is a direct product of type I factors.

Lemma 5.4.1. *Let $\{\mathcal{M}_i : i \in I\}$ be a set of von Neumann algebras. Then $\mathcal{M} := \prod_{i\in I}\mathcal{M}_i$ has the WTU property if and only if $\mathcal{M}_i$ has the WTU property for each $i \in I$.*

Proof. We first suppose that there exists an index $j \in I$ so that $\mathcal{M}_j$ fails to have the WTU property. Then there exist two inequivalent weak* tensor products $\mathcal{M}_j\overline{\otimes}_\alpha\mathcal{N}$ and $\mathcal{M}_j\overline{\otimes}_\beta\mathcal{N}$

for some von Neumann algebra $\mathcal{N}$. Let $\mathcal{M}_i\overline{\otimes}_{\alpha_i}\mathcal{N}$ and $\mathcal{M}_i\overline{\otimes}_{\beta_i}\mathcal{N}$ for $i \in I$ be arbitrary weak* tensor products such that $\alpha_j = \alpha$ and $\beta_j = \beta$. Then $\mathcal{M}\overline{\otimes}_{\bigoplus_i \alpha_i}\mathcal{N}$ and $\mathcal{M}\overline{\otimes}_{\bigoplus \beta_i}\mathcal{N}$ are inequivalent weak* tensor products since $\bigoplus_i \alpha_i$ is not quasi-equivalent to $\bigoplus_i \beta_i$.

Next, for each index j let $e_j : \prod_{i\in I} \mathcal{M}_i \to \mathcal{M}_j$ be the restriction to the jth component and suppose that $\mathcal{M}_i$ has the WTU property for every $i \in I$. Let $\mathcal{N}$ be an arbitrary von Neumann algebra and $\prod_{i\in I}(\mathcal{M}_i)\overline{\otimes}_\alpha\mathcal{N}$ a weak* tensor product of $\prod_i \mathcal{M}_i$ and $\mathcal{N}$. Then $e_j \otimes 1 \in (\mathcal{M}\overline{\otimes}_\alpha\mathcal{N})'$ since $e_j \otimes 1$ commutes with $a \otimes b$ for all $a \in \mathcal{M}$, $b \in \mathcal{N}$ and $\mathcal{M} \otimes \mathcal{N}$ is weak* dense in $\mathcal{M}\overline{\otimes}_\alpha\mathcal{N}$. Since $\sum_{i\in I} e_i \otimes 1 = 1$, it follows that

$$\mathcal{M}\overline{\otimes}_\alpha\mathcal{N} = \prod_{i\in I}(e_i \otimes 1)\mathcal{M}\overline{\otimes}_\alpha\mathcal{N} = \prod_{i\in I} \mathcal{M}_i\overline{\otimes}_{\alpha_i}\mathcal{N}$$

where α_i denotes $\alpha|_{\mathcal{M}_i\otimes\mathcal{N}}$. But $\mathcal{M}_i\overline{\otimes}_{\alpha_i}\mathcal{N} = \mathcal{M}\overline{\otimes}\mathcal{N}$ for each $i \in I$ by the WTU property. Therefore $\mathcal{M}\overline{\otimes}_\alpha\mathcal{N} = \prod_{i\in I} \mathcal{M}_i\overline{\otimes}\mathcal{N}$ and, hence, we conclude that $\mathcal{M}$ has the WTU property. □

Choi and Effros showed in [11, Lemma 2.1] that a von Neumann algebra $\mathcal{M} \subset B(\mathcal{H})$ is injective if and only if the multiplication map $m : \mathcal{M} \otimes \mathcal{M}' \to B(\mathcal{H})$ is continuous with respect to the minimal tensor product norm. It is interesting to see in the following theorem that an analogous characterization of the WTU property also holds.

Theorem 5.4.2. *The following are equivalent for a von Neumann algebra $\mathcal{M} \subset B(\mathcal{H})$.*

(i) $\mathcal{M}$ has the WTU property;
(ii) The weak tensor products $\mathcal{M}\overline{\otimes}\mathcal{M}'$ and $\mathcal{M}\overline{\otimes}_{w^*\text{-max}}\mathcal{M}'$ are equivalent;*
(iii) The multiplication map $m : a \otimes b \mapsto ab$ extends to a normal $*$-homomorphism from $\mathcal{M}\overline{\otimes}\mathcal{M}'$ to $B(\mathcal{H})$;
(iv) $\mathcal{M}$ is of the form $\prod_{i\in I} B(\mathcal{H}_i)$ for some choices of Hilbert spaces $\mathcal{H}_i$.

Proof. (i) ⇒ (ii): Trivial.

(ii) ⇒ (iii): Suppose that the multiplication map $m : \mathcal{M} \otimes \mathcal{M}' \to B(\mathcal{H})$ does not extend to a normal $*$-homomorphism $\widetilde{m} : \mathcal{M}\overline{\otimes}\mathcal{M}' \to B(\mathcal{H})$. Then m is not quasi-contained in the spatial embedding $\iota : \mathcal{M} \otimes \mathcal{M}' \to B(\mathcal{H} \otimes \mathcal{H})$. Defining $\alpha = \iota \oplus m$, we have that $\mathcal{M}\overline{\otimes}_\alpha\mathcal{M}'$ is a nonequivalent weak* tensor product to $\mathcal{M}\overline{\otimes}\mathcal{M}'$ with the property that the identity map

on $\mathcal{M} \otimes \mathcal{M}'$ extends to a normal $*$-homomorphism from $\mathcal{M}\overline{\otimes}_\alpha\mathcal{M}'$ to $\mathcal{M}\overline{\otimes}\mathcal{M}'$. It follows that $\mathcal{M}\overline{\otimes}\mathcal{M}'$ and $\mathcal{M}\overline{\otimes}_{w^*\text{-max}}\mathcal{M}'$ are inequivalent weak* tensor products.

(iii) $\Rightarrow$ (iv): Suppose that m extends to a normal $*$-homomorphism $\widetilde{m} : \mathcal{M}\overline{\otimes}\mathcal{M}' \to B(\mathcal{H})$. We claim that $Z(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$ must then be of the form $\ell^\infty(S)$ for some set S. Indeed, suppose otherwise. Then, without loss of generality, we may assume that $Z(\mathcal{M}) = L^\infty(X, \mu)$ for some locally compact space X equipped with a necessarily non-discrete positive Radon measure μ. Since μ is non-discrete, there exists a Borel subset E of X such that $\mu(E) > \sum_{x\in E} \mu(\{x\})$. By inner regularity of μ, we can then find a compact subset K_0 of X so that $\mu(K_0) > \sum_{x\in K_0} \mu(\{x\})$. Let $\{U_1, \ldots, U_n\}$ be a finite covering for K_0 consisting of of precompact open sets and define K be the closure of $\bigcup_{i=1}^n U_i$. Observe that K also has the property that $\mu(K) > \sum_{x\in K} \mu(\{x\})$ since μ is a positive measure, $K \supset K_0$, and $\mu(K) < \infty$. Further, $C(K)$ injects into $L^\infty(X)$ in the natural way since K is the closure of an open subset of X. We will show that μ being non-discrete leads to a contradiction of the normality of $\widetilde{m}$.

Let $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ be functions in $C(K) \subset L^\infty(X)$. Then

$$m(f_1 \otimes g_1 + \ldots + f_n \otimes g_n)(x) = (f_1 \otimes g_1 + \ldots + f_n \otimes g_n)(x, x)$$

for $x \in X$. By norm continuity, it follows that $\widetilde{m}(f)(x) = f(x, x)$ for almost every $f \in C(K \times K) = C(K) \otimes_{\min} C(K) \subset L^\infty(X)\overline{\otimes}L^\infty(X)$.

Choose a sequence of descending relatively open subsets $U_1 \supset U_2 \supset \ldots$ of $K \times K$ so that U_n contains $\Delta := \{(x, x) : x \in K\}$ for every n and $(\mu \times \mu)(U_n) \to (\mu \times \mu)(\Delta)$ as $n \to \infty$. By Urysohn's lemma, there exists functions $f_1, f_2, \ldots \in C(K \times K) \subset L^\infty(X)\overline{\otimes}L^\infty(X)$ mapping into $[0, 1]$ such that $f_n(x, x) = 1$ for every $x \in K$ and $f_n(x, y) = 0$ for $(x, y) \notin U_n$. Then, by Lebesgue's dominated convergence theorem,

$$\int f_n(x, y)g(x, y)\, d\mu(x)d\mu(y) \to \int_\Delta g(x, y)\, d\mu(x)d\mu(y)$$

as $n \to \infty$ for every $g \in L^1(X \times X)$. Hence, $f_n \to 1_\Delta$ in the weak* topology. Then, since $\widetilde{m}$ is normal and $\widetilde{m}(f_n) = 1_K$ for every n, we have that $\widetilde{m}(1_\Delta) = 1_K$.

Next, define a net of elements in $L^\infty(X \times X)$ indexed under inclusion by the finite

subsets F of K by $h_F = \sum_{x\in F} 1_{\{(x,x)\}} = \sum_{x\in F} 1_{\{x\}} \otimes 1_{\{x\}}$. Then

$$\begin{aligned}\int h_F(x,y)g(x,y)\,d\mu(x)d\mu(y) &= \sum_{x\in F} g(x,x)\mu(x)^2 \\ &\to \sum_{x\in K} g(x,x)\mu(x)^2 \\ &= \int_\Delta g(x,y)\,d\mu(x)d\mu(y)\end{aligned}$$

in the limit as $F \to K$ for every $g \in L^1(X\times X)$. Hence, h_F converges weak* to 1_Δ.

Observe that

$$\int_K m(h_F)(x)\,d\mu(x) = \sum_{x\in F}\mu(\{x\}) \to \sum_{x\in K}\mu(\{x\})$$

as $F \to K$. Recall that $\mu(K) > \sum_{x\in K}\mu(\{x\})$. It follows that $\widetilde{m}(h_F)$ does not converge to 1_K. This contradicts the normality of $\widetilde{m}$ and, so, we conclude that μ must be a discrete measure. Hence, $Z(\mathcal{M})$ is of the form $\ell^\infty(S)$ for some set S.

Since $Z(\mathcal{M})$ is of the form $\ell^\infty(S)$, there exists a set $\{e_i : i \in S\}$ of minimal central projections in $\mathcal{M}$ such that $\sum_{i\in S} e_i = 1$. Then $\mathcal{M} = \prod_{i\in S} e_i\mathcal{M}$ where each term $e_i\mathcal{M} \subset B(e_i\mathcal{H})$ is a factor by the minimality of e_i. Therefore the desired result follows from Lemma 5.4.1 since a factor has the WTU property if and only if it is of type I.

(iv) $\Rightarrow$ (i): This is clear from Proposition 5.2.8 and Lemma 5.4.1. □

5.5 Weak* tensor product completions of $L^\infty(\mathbb{R})\otimes L^\infty(\mathbb{R})$

This section is dedicated to studying weak* tensor product completions of $L^\infty(\mathbb{R})\otimes L^\infty(\mathbb{R})$. The main result of this section is the construction of $2^\mathfrak{c}$ distinct such weak* tensor product completions. As a consequence of our constructions, we also find that the weak* maximal tensor product $L^\infty(\mathbb{R})\overline{\otimes}_{w^*\text{-max}}L^\infty(\mathbb{R})$ does not have a separable predual despite $L^\infty(\mathbb{R})$ having separable predual $L^1(\mathbb{R})$.

The approach that we take in constructing weak* tensor products is to use the fact that $L^\infty(\mathbb{R})$ contains a weak* dense copy of the continuous bounded functions $C_b(\mathbb{R})$ (we choose

to use $C_b(\mathbb{R})$ over $C_0(\mathbb{R})$ for convenience since $C_b(\mathbb{R})$ is unital) and, hence, any weak* tensor product completion $L^\infty(\mathbb{R})\overline{\otimes}_\alpha L^\infty(\mathbb{R})$ of $L^\infty(\mathbb{R})\otimes L^\infty(\mathbb{R})$ must contain a weak* dense copy of $C_b(\mathbb{R})\otimes_{\min} C_b(\mathbb{R})$. In particular, this implies that if $\pi : C_b(\mathbb{R})\otimes_{\min} C_b(\mathbb{R}) \to B(\mathcal{H})$ is a $*$-representation such that the restrictions $\pi|_{C_b(\mathbb{R})\otimes 1}$ and $\pi|_{1\otimes C_b(\mathbb{R})}$ are quasi-contained in the canonical representation $\sigma : C_b(\mathbb{R}) \to L^\infty(\mathbb{R})$, then π extends uniquely to a normal $*$-representation $\widetilde{\pi} : L^\infty(\mathbb{R})\overline{\otimes}_{w^*\text{-max}} L^\infty(\mathbb{R}) \to B(\mathcal{H})$.

We will identify $C_b(\mathbb{R})\otimes_{\min} C_b(\mathbb{R})$ as being a C*-subalgebra of $C_b(\mathbb{R}\times\mathbb{R})$. For every $x\in\mathbb{R}$, define an (unbounded) measure μ_x on $\mathbb{R}\times\mathbb{R}$ by

$$\int f\, d\mu_x = \int_{\mathbb{R}} f(y, x+y)\, dy$$

for $f\in C_c(\mathbb{R}\times\mathbb{R})$. Further, we let

$$\pi_x : C_b(\mathbb{R})\otimes_{\min} C_b(\mathbb{R}) \to L^\infty(\mathbb{R}\times\mathbb{R}, \mu_x)$$

be the natural inclusion. These $*$-representations π_x will be our building blocks in constructing many weak* tensor product completions of $L^\infty(\mathbb{R})\otimes L^\infty(\mathbb{R})$.

Lemma 5.5.1. *For every $x\in\mathbb{R}$, the $*$-representations $\pi_x|_{C_b(\mathbb{R})\otimes 1}$ and $\pi_x|_{1\otimes C_b(\mathbb{R})}$ are unitarily equivalent to the canonical representation $\sigma : C_b(\mathbb{R}) \to L^\infty(\mathbb{R})$.*

Proof. Define a map from $U : L^2(\mu_x) \to L^2(\mathbb{R})$ by $U(g)(y) = g(y, x+y)$. Then U is clearly a well defined surjective isometry. Further, we observe that if $f\in C_b(\mathbb{R})$ and $\xi\in L^2(\mu_x)$, then

$$U\big(\pi_x(f\otimes 1)\xi\big)(y) = f(y)\xi(y, x+y) = \big(\sigma(f)U(\xi)\big)(y)$$

for almost every $y\in\mathbb{R}$. So we have shown that $\pi_x|_{C_b(\mathbb{R})\otimes 1}$ is unitarily equivalent to σ. A similar argument shows that the same holds for $\pi_x|_{1\otimes C_b(\mathbb{R})}$. □

The author wishes to thank Nico Spronk for suggesting the following lemma.

Let X be a measure space. We say that a measure μ on X is absolutely continuous with respect to a family of measures $\{\nu_i : i\in I\}$ on X if $\nu_i(E) = 0$ for every $i\in I$ implies that $\mu(E) = 0$ whenever $E\subset X$ is a measurable subset.

Lemma 5.5.2. *Let X be a locally compact space, and μ and $\{\nu_i : i \in I\}$ be Radon measures on X. Denote the canonical inclusions of $C_0(X)$ in $L^\infty(\mu)$ and $L^\infty(\nu_i)$ by π and σ_i, respectively. Then π is quasi-contained in $\bigoplus_{i\in I}\sigma_i$ if and only if μ is absolutely continuous with respect to $\{\nu_i : i \in I\}$.*

Proof. If μ is absolutely continuous with respect to $\{\nu_i : i \in I\}$, then the identity map on the bounded measurable functions $\mathcal{M}_b(X)$ of X clearly extends to a surjective $*$-homomorphism from $\big(\bigoplus_{i\in I}\sigma_i\big)(\mathcal{M}_b(X))$ to $\pi(\mathcal{M}_b(X))$. Since $\pi(\mathcal{M}_b(X)) = L^\infty(\mu) = \pi(C_b(X))''$ and $\big(\bigoplus_{i\in I}\sigma_i\big)(\mathcal{M}_b(X)) = \big(\bigoplus_{i\in I}\sigma_i\big)(C_b(X))''$, this implies that π is quasi-contained in $\bigoplus_{i\in I}\sigma_i$.

Next we suppose towards a contradiction that π is quasi-contained in $\bigoplus_{i\in I}\sigma_i$, but μ is not absolutely continuous with respect to $\{\nu_i : i \in I\}$. Since π is quasi-contained in $\bigoplus_{i\in I}\sigma_i$, there exists a cardinal ω such that π is unitarily equivalent to a subrepresentation of the amplification $\omega\cdot\bigoplus_{i\in I}\sigma_i$. Note that since μ is not absolutely continuous with respect to $\{\nu_i : i \in I\}$, there exists a compact set K such that $\mu(K) > 0$ but $\nu_i(K) = 0$ for every $i \in I$ by inner regularity. So the function $\varphi{:}C_b(X) \to \mathbb{C}$ defined by

$$\varphi(f) = \int_K f\, d\mu = \langle \pi(f)\xi, \xi\rangle\,,$$

where $\xi \in L^2(\mu)$ is the characteristic function 1_K, is a vector state of φ. This implies that φ must also be a vector state of $\omega\cdot\bigoplus_{i\in I}\sigma_i$.

Observe that the Hilbert space on which $\omega\cdot\bigoplus_{i\in I}\sigma_i$ acts is

$$\bigoplus_{i\in I} L^2(\nu_i)^{\oplus\omega}.$$

So, assuming that φ is a vector state of $\omega\cdot\bigoplus_{i\in I}\sigma_i$, then we can approximate φ arbitrarily well in norm by maps $\psi{:}C_b(X) \to \mathbb{C}$ of the form

$$\psi(f) = \sum_{j=1}^{m}\sum_{k=1}^{n_j}\big\langle \sigma_{i_j}(f)\xi_{jk}, \xi_{jk}\big\rangle = \sum_{j=1}^{m}\sum_{k=1}^{n_j}\int f|\xi_{jk}|^2\, d\nu_{i_j}$$

for choices of $m, n_j \in \mathbb{N}$, $i_j \in I$, and $\xi_{jk} \in L^2(\mu_{i_j})$. We will show that this is not possible.

By outer regularity, we can choose a decreasing sequence of open subsets $U_1 \supset U_2 \supset \cdots$ of X containing K such that $\nu_{i_j}(U_p) \to 0$ as $p \to \infty$ for all $j = 1, \ldots, m$. By Urysohn's lemma, there exists functions $f_p : X \to [0,1]$ such that $f_p(x) = 1$ for $x \in K$ and $f_p(x) = 0$ for $x \notin U_p$. So, by Lebesgue's dominated convergence theorem,

$$\int f_p |\xi_{jk}|^2 \, d\mu_{i_j} \to 0$$

as $p \to \infty$ and all $j = 1, \ldots, m$ and $k = 1, \ldots, n_j$. Hence, $\psi(f_p) \to 0$ as $p \to \infty$. Since $\varphi(f_p) = \mu(K)$ for every p, this implies that $\|\psi - \varphi\| \geq \mu(K) > 0$. This contradicts that we should be able to approximate φ arbitrarily well with functionals with the form of ψ. So we conclude that π is not quasi-contained in $\bigoplus_{i \in I} \sigma_i$. □

Corollary 5.5.3. *For each $x \in \mathbb{R}$, the representation π_x is not quasi-contained in*

$$\bigoplus_{x' \neq x} \pi_{x'} \oplus \sigma,$$

where $\sigma : C_b(\mathbb{R}) \otimes_{\min} C_b(\mathbb{R}) \to L^\infty(\mathbb{R}) \overline{\otimes} L^\infty(\mathbb{R})$ denotes the canonical inclusion.

Before proving the main theorem of this section, we pause to note a further corollary of the proof of Lemma 5.5.2.

Corollary 5.5.4. *The von Neumann algebra $L^\infty(\mathbb{R}) \overline{\otimes}_{w^*\text{-max}} L^\infty(\mathbb{R})$ does not have a separable predual.*

Proof. For each $x \in \mathbb{R}$, let $\varphi_x : C_b(\mathbb{R}) \otimes_{\min} C_b(\mathbb{R}) \to \mathbb{C}$ be the function defined by

$$\varphi_x(f) = \int_{[0,1]} f(y, x+y) \, dy = \langle \pi_x(f)\xi, \xi \rangle$$

where $\xi \in L^2(\mu_x)$ is the characteristic function $1_{[0,1] \times [x, 1+x]}$. Then, as φ_x is a vector state of π_x, it follows that φ_x is in the predual of $L^\infty(\mathbb{R}) \overline{\otimes}_{w^*\text{-max}} L^\infty(\mathbb{R})$. So the predual of $L^\infty(\mathbb{R}) \overline{\otimes}_{w^*\text{-max}} L^\infty(\mathbb{R})$ cannot be separable since, by an argument similar to the proof of Lemma 5.5.2, $\|\varphi_x - \varphi_y\| \geq 1$ for all distinct $x, y \in \mathbb{R}$. □

Theorem 5.5.5. *$L^\infty(\mathbb{R}) \otimes L^\infty(\mathbb{R})$ admits $2^{\mathfrak{c}}$ nonequivalent weak* tensor products.*

Proof. Let $\sigma : C_b(\mathbb{R}) \otimes_{\min} C_b(\mathbb{R}) \to L^\infty(\mathbb{R})\overline{\otimes}L^\infty(\mathbb{R})$ be the canonical inclusion. For each subset $S \subset \mathbb{R}$, define $\alpha_S = \bigoplus_{x\in S} \pi_x \oplus \sigma$. Then, by Lemma 5.5.1, the restrictions $\alpha_S|_{C_b(\mathbb{R})\otimes 1}$ and $\alpha_S|_{1\otimes C_b(\mathbb{R})}$ are each quasi-equivalent to the canonical inclusion $C_b(\mathbb{R}) \to L^\infty(\mathbb{R})$. Further, since α_S contains σ as a subrepresentation, we conclude that each α_S extends to a $*$-representation $\widetilde{\alpha}_S$ of $L^\infty(\mathbb{R}) \otimes L^\infty(\mathbb{R})$ which satisfies the conditions required to construct a weak* tensor product. It is clear from Corollary 5.5.3 that these weak* tensor products $L^\infty(\mathbb{R})\overline{\otimes}_{\widetilde{\alpha}_S}L^\infty(\mathbb{R})$ are pairwise nonequivalent for $S \subset \mathbb{R}$. Hence, $L^\infty(\mathbb{R}) \otimes L^\infty(\mathbb{R})$ admits $2^{\mathfrak{c}}$ nonequivalent weak* tensor products. □

Remark 5.5.6. Let $\mathcal{M}$ be any infinite dimensional abelian von Neumann algebra with separable predual. Recall that then $\mathcal{M} = \ell^\infty(\mathbb{N})$, $\mathcal{M} = L^\infty(\mathbb{R})$ or $\mathcal{M} = L^\infty(\mathbb{R}) \oplus \ell^\infty(S)$ where $S = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ or $S = \mathbb{N}$. In particular, it follows that if $\mathcal{M}$ does not have the WTU property (or, equivalently, $\mathcal{M} \neq \ell^\infty(\mathbb{N})$), then $\mathcal{M} \otimes \mathcal{M}$ admits $2^{\mathfrak{c}}$ distinct weak* tensor product completions.

Chapter 6

Further directions

We finish off this thesis with a brief mention of some open problems and a discussion of future directions of research.

Recall that Okayasu showed that the C*-algebras $C^*_{\ell^p}(\mathbb{F}_d)$ are not canonically $*$-isomorphic for each $2 \leq p < \infty$. However, it remains an open problem whether any such pair could be non-canonically $*$-isomorphic when $2 < p < \infty$.

Open Problem 1. *Let $2 \leq d \leq \infty$. Does there exist distinct $p, q \in (2, \infty)$ so that $C^*_{\ell^p}(\mathbb{F}_d)$ is $*$-isomorphic to $C^*_{\ell^q}(\mathbb{F}_d)$?*

We note that if $2 < p < \infty$, then $C^*_{\ell^p}(\mathbb{F}_d)$ is $*$-isomorphic to neither $C^*_r(\mathbb{F}_d)$ nor $C^*(\mathbb{F}_d)$ since $C^*_r(\mathbb{F}_d)$ is simple and $C^*(\mathbb{F}_d)$ admits finite dimensional representations. This problem has also been considered independently by Brannan, Brown, Buss-Echterhoff-Willet (see [10, Remark 2.8]), and possibly others. In the case of $SL_2(\mathbb{R})$, Buss, Echterhoff, and Willett (see [10, Remark 2.8]) recently used work of the author (see [55]) and Miličić (see [34]) to show that $C^*_{L^p}(SL_2(\mathbb{R}))$ is $*$-isomorphic to $C^*_{L^q}(SL_2(\mathbb{R}))$ for every $p, q \in (2, \infty)$.

Much of this thesis is spent constructing exotic group C*-norms and exotic C*-tensor norms. We now know that these exist for a large class of examples, but it remains open whether we are always able to find such a C*-norm in every nontrivial case.

Open Problem 2. *Let G be a non-amenable locally compact group. Does $L^1(G)$ admit an exotic group C*-norm?*

Open Problem 3. *Let A and B be C*-algebras such that $A \otimes B$ does not admit a unique C*-norm. Does there exist an exotic C*-norm on $A \otimes B$?*

The theory of weak* tensor products developed in Chapter 5 is still a very young field and there remains much work to be done. Within the theory of C*-algebras, many important properties have characterizations in terms of C*-tensor products. It would be very interesting to see whether the same is true for weak* tensor products within the category of von Neumann algebras. It would also be interesting to exhibit von Neumann algebras M and N lacking the WTU property so that $M \otimes N$ admits a unique weak* tensor product completion, or to show that no such pair exists.

Another further direction of study to consider is that of exotic quantum group C*-algebras. A problem which remains open in this area is to give an example of an exotic quantum group C*-algebra for a discrete quantum group which is not of Kac type. Brannan and Ruan extended the concept of L^p-representations to discrete quantum groups of Kac type and proved that Okayasu's result holds for free orthogonal and unitary quantum groups (see [6]). In joint work with Ruan, the author demonstrated that exotic group C*-algebras typically admits very poor local and approximation properties (see [43]). Jointly with Brannan and Ruan, we are currently writing a paper where we show that the same is true for L^p-quantum group C*-algebras of free orthogonal and unitary quantum groups.

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