Analysis of Time Dependent Aggregate Claims

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Estimation of aggregate claim amounts is a fundamental task in Actuarial science, based on which risk theory, ruin theory and reinsurance theory can be studied. Properties, including moments, Laplace transforms, and probability functions of aggregate claims have been extensively studied by many scholars under various models (see, e.g., Hogg and Klugman (1984)). The main classical model is the compound Poisson risk model, where the interclaim times are independent of the claim severities. Scholars started to explore this problem by considering more general counting processes, such as mixed Poisson processes (e.g., Willmot (1986)) and renewal processes (e.g., Andersen (1957)). Afterwards, the independence assumptions on multiple risk factors were gradually relaxed. Additionally, the observation times are further randomized to fit the reality better.

In this thesis, we propose to analyze the aggregate claims until both randomized and deterministic time horizons by incorporating inflation and payment (reporting) delays into the analysis. Dependence between the claim occurrence times (also interclaim times) and claim severities is further considered.

A comprehensive review on the study of the aggregate claims is given in Chapter 1. Chapter 2 introduces the relevant preliminary knowledge on the aggregate models and techniques used in this thesis.

Chapter 3 examines the Laplace transforms of the aggregate claims under a nonhomogeneous birth process, which covers Poisson, mixed Poisson and linear contagion model. Furthermore, the claim occurrence times influence the distribution of the claim severities. Under some assumptions on the counting process, the time-dependent aggregate claims are represented as a random sum of independent and identically distributed random variables.

The aggregate incurred but not reported (IBNR) claims are studied in Chapter 4 due to their essential role in reserving. A recursive formula is identified for the moments of the total discounted IBNR claims under a generalized renewal risk model where the interclaim times,

claim severities and random reporting lags have an arbitrary dependence structure. The probability mass function of the number of IBNR claims is obtained under certain assumptions on the marginal distributions of the interclaim times, claim severities and reporting lags. To address the influence of the economic environment, a Markovian arrival process is introduced in Chapter 5 to analyze the IBNR claim problem. A straightforward representation and a closed-form expression are identified for the moments of the total discounted IBNR claim amount and numbers respectively without adding much difficulty to the analysis.

Instead of a deterministic time horizon as considered in Chapters 3, 4 and 5, attention has also been paid to the analysis under a randomized observation time (see, e.g., Stanford et al. (2005) and Ramaswami et al. (2008)). Randomization in the time horizon usually leads to more tractable expressions for given quantities (e.g., Albrecher et al. (2011, 2013)). However, in the case of time-dependent aggregate claims, it only adds extra integration to the expressions of relevant quantities. In this thesis, instead of working with general random time horizons, we work with some specific random time horizons, i.e. two-sided exit time, in Chapter 6. The two-sided exit problem has been the subject matter of risk management analysis to better understand the dynamic of various insurance risk processes. In the two-sided exit setting, the discounted aggregate claims are investigated under a dependent renewal process (also known as dependent Sparre Andersen risk process). Utilizing Laplace transforms, we identify the fundamental solutions to a given integral equation, which will be shown to play a role similar to the scale matrix for spectrally-negative Markov-additive processes (e.g., Kyprianou and Palmowski (2008)). Explicit expressions and recursions are then identified for the two-sided probabilities and the moments of the aggregate claims respectively. Chapter 7 ends the thesis by some concluding remarks and directions for future research.

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Dedication

This is dedicated to my parents and my brother.

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Chapter 1

Background and Introduction

Insurance is an efficient way to transfer risks. Insurance companies manage risks shifted from individuals and get compensated by collecting premiums. The number of claims arising within a given time period from a specified block of insurance is referred to as claim frequency, which is usually modelled by a discrete random variable (rv). The claim severity, modelled by a nonnegative rv, gives the size of the individual claim. The premiums charged are dependent upon the frequency and severities of the claims occurred. Due to the fact that premiums are usually charged up front for the (non-life) contract, insurers are required by regulators to set aside adequate reserves to fulfil their promise to compensate the insured in future's claim causing events. Thus, accurate modelling of the total claim amount is vital for insurers in pricing, reserving, meeting solvency requirement and more generally, managing risks.

The aggregate loss is a mathematical representation of the total claims received by the insurer. Many models have been developed for the aggregate risks; the most classical ones are the individual and collective risk models (see Bühlman (1970) and Klugman et al. (2008, Chapter 9)). The individual risk model utilizes a sum of a fixed number of independent and identically distributed (iid) random variables (rv's) to quantify the aggregate loss. On the other hand, under the collective risk model, the aggregate loss is represented as a sum of a random number of iid rv's. The collective model is constructed based on the assumption that the claim severities are independent of the claim frequency. However, the assumption of independence is often viewed as too restrictive in real-world applications. The goal of this thesis is to study the aggregate claims by allowing dependence between claim severities and frequency, since adequate modelling of the dependence between different types of risks in an insurance company is vital. To consider the randomness in the claim severities, generic rv's are utilized in modelling. We use a counting process to model claim frequency, since it describes how the claim numbers develop over time. Dependence structure between them is introduced with consideration of both generality and tractability. The analysis in this thesis is very efficient and effective in capturing the properties of the aggregate claims, which are essential in risk theory, ruin theory and reinsurance theory.

Mathematically speaking, the goal of this thesis is to use the collective model and to quantify the aggregate claim model defined through the usual compound sum representation

$$S_t = \sum_{i=1}^{N_t} Y_i, \tag{1.1}$$

with the convention that $S_t = 0$ if $N_t = 0$, where the claim sizes $\{Y_i\}_{i=1}^{\infty}$ are assumed to form a sequence of positive rv's. The claim number process $\{N_t\}_{t\geq 0}$ is a counting process with claim occurrence times $\{T_i\}_{i=0}^{\infty}$ starting with $T_0 = 0$, and interclaim times $\tau_i = T_i - T_{i-1}$ for $i = 1, 2, \ldots$. This thesis first explores the distribution of the aggregate time-dependent claim amounts until a deterministic time horizon (i.e. t is a constant) by assuming an arbitrary dependence structure between Y_i and T_i (or τ_i). It allows for the incorporation of the time value of money and claims inflation, as well as payment delay into analysis.

From the deterministic time framework, many scholars contributed to the analysis of total time-dependent claim models. For instance, Willmot (1989) studied the total claim amounts through Laplace transform (LT) under inflationary condition in a mixed Poisson counting process. Jang (2004) considered a parallel problem under a shot noise counting process. Léveillé and Garrido (2001a, 2001b) derived a recursive formula for the moments of the discounted renewal sum of claim amounts. See also Woo and Cheung (2013) in the context of the dependent Sparre Andersen risk process. Kim and Kim (2007) and Ren (2008) also considered this problem in the framework of the Markovian claim arrival process. Using differential equations, Wang (2010) studied the moment generating function of a discounted compound renewal sum with phase-type interarrival times and general claim severities. The reader is also referred to Léveillé and Adékambi (2011, 2012) where the analysis of the joint distribution of the discounted compound renewal sums at different time points is considered.

The total incurred but not reported (IBNR) claim amount is defined as

$$S_{IBNR}(t) = \sum_{i=1}^{N_t} \mathbf{1}_{\{T_i+W_i>t\}} Y_i, \tag{1.2}$$

where W_i is the reporting lag associated with the ith loss Y_i. Thus, the IBNR problem is a particular application of the time-dependent aggregate claim model. In insurance contexts, IBNR claims arise from the natural lag between the occurrence and the report of a claim to the insurer. Indeed, insurers should make adequate provision for the total amount of claims incurred but not yet reported to the insurer. The IBNR claims are thus of central importance in claim reserving. In practice, the estimation of the IBNR claim amount is based on the "run-off triangle", which is a table recording the total reported claim amount by accident years and development years. Various deterministic (e.g., chain-ladder method (Taylor (1986)) and stochastic models (e.g., "macro-level" models (Wuthrich and Merz (2008)) are proposed to predict the IBNR reserving. A comprehensive review on the IBNR problem can also be found in Badescu et al. (2016) and references therein. However, the existing method mainly focuses on providing a point estimate for the total IBNR claim amount, which fails to account for the random variation in the value of variables that contribute to IBNR claims. The randomness of some important factors in this context, such as reporting lags, incurred claim severities and time value of money, is considered in this thesis. Guo et al. (2013) derived the distribution of the IBNR claim number for different distributional assumptions for the batch sizes, capitalizing on the self-decomposability property of the Poisson claim arrival process. Note that the IBNR problem has known connections with queueing theory. For instance, the IBNR claim number is equivalent to the number of busy servers in queues with infinite servers. More specifically, in this thesis, the IBNR claim number is analogous to the $GI^X/G/\infty$ queuing system with bulk arrivals (e.g., Liu et al., 1990). In light of this connection, we point out that Chaudhry and Templeton (1983) studied the probability generating function (pgf) of the number of customers in an $M^X/M/\infty$ queuing system. Later, Willmot and Drekic (2001, 2002, 2009) studied the transient distribution of the number of customers under various distributional assumptions for the reporting lags in a $M^X/G/\infty$ queue model. On the other hand, interpreting the reporting lag as an investigation time in the IBNR analysis leads to problems related to delays in claim settlement (e.g., Boogaert and Haezendonck (1989)). Therefore, the results obtained in this thesis are also applicable to address problems related to the reported but not settled claims.

Later on, more attention is paid to the model with randomized time horizon. From this standpoint, we further introduce an insurance surplus process $\{U_t\}_{t\geq 0}$ defined as

$$U_t = u + ct - S_t, \tag{1.3}$$

where $u = U_0 \geq 0$ is the initial surplus, c is a positive premium rate and S_t is the aggregate claim amount as defined in (1.1). Of particular interest in the risk analysis of the insurance surplus process $\{U_t\}_{t\geq 0}$ are the first passage times τ_b^+ and τ_0^- respectively defined as

$$\tau_b^+ = \inf\{t \geq 0 : U_t > b\}, \tag{1.4}$$

and

$$\tau_0^- = \inf\{t \geq 0 : U_t < 0\}. \tag{1.5}$$

Analysis of the total claim amount until a specific random time, including ruin time (i.e. τ_0^-) and the time surpassing certain levels (i.e. τ_b^+), has attracted extensive attention. For instance, Albrecher and Teugels (2006) provide exponential estimates for the infinite- (i.e. $\Pr(\tau_0^- < \infty | U_0 = u)$) and finite- (i.e. $\Pr(\tau_0^- < C | U_0 = u)$) time ruin probabilities by using copula to model the dependence between the interclaim times and claim severities. Ruin probabilities under the dependent Sparre Andersen risk model were further investigated by

Boudreault et al. (2006), Cossette et al. (2010) and Cheung et al. (2010). Cai et al. (2009) and Feng (2009a, 2009b) considered the mean of the total discounted operation costs under the compound Poisson risk model process and the phase-type renewal model, respectively. Recursive formulas were then derived by Cheung (2013) for the higher-order moments in a dependent Sparre Andersen risk model. See also Cheung and Landriault (2009) for the analysis of the maximum surplus level (i.e. $\max_{t\leq\tau_0^-} U_t$) before ruin in semi-Markov process. As an extension to the analysis of quantities related to τ_0^-, the expected accumulated discounted tax was investigated by Albrecher and Hipp (2007), which involves τ_b^+. However, most the papers mentioned above have focused on Gerber-Shiu discounted penalty functions. Less attention is paid to the analysis of the total discounted claim amount under the two-sided exit setting.

The two-sided exit probabilities are well studied under independence assumptions between the interclaim times and claim severities. We refer the reader to Kyprianou (2006, Chapter 8) and Kuznetsov et al. (2013) in the context of the Lévy insurance risk model. However, the assumption of independence is often viewed as too restrictive in real-world applications. Kyprianou and Palmowski (2008) further considered this problem under the Markov additive process, which can be viewed as a Markov regime switching Lévy insurance risk process. We further remark that occupation time problems (see Landriault et al. (2016)) and Parisian ruin problems (see Loeffen et al. (2013)) are both intimately connected to the two-sided exit problems. In this thesis, we analyze some quantities under a two-sided exit setting by allowing a (relatively) general dependence structure between the interclaim times and the claim severities.

To be specific, this thesis is constructed in the following way.

Chapter 2 introduces the mathematical quantities of interest and formally defines the terminology to be used. Relevant properties of some useful distributions and processes are reviewed.

Chapter 3 studies the distribution of the aggregate time-dependent claims in birth process claim count models. We derive an integral representation for the density of the claim values over the interval $(s,t]$ given that $N_s = k$ under a factorization assumption. Furthermore, the factorization assumption is extended to allow for a change point, which results in a piecewise factorization function. Thereafter, the mixed Erlang properties of the time-dependent sum are discussed.

In Chapter 4, the time-dependent renewal sum of IBNR claim amounts is investigated through LTs. Moments of the time-dependent renewal sum of IBNR claim amounts are obtained through defective renewal equation techniques. An explicit expression for the first moment is derived, and a recursive formula is identified for the higher-order moments. The joint distribution of the total discounted IBNR claim amount and the total incurred and reported (IR) claim amount at possibly different time points is then studied. Thus, the IBNR claims can be estimated by the known IR claims; this analysis is particularly relevant for reserving purposes. The self-decomposability of the IBNR claim number process is also considered when claim causing events arrive according to a compound Poisson process. Furthermore, properties of the IBNR claim number are analyzed under a Coxian distributional assumption for the interclaim times and exponentially distributed reporting lags.

Chapter 5 reconsiders the IBNR problem by assuming that claims occur according to a Markovian Arrival Process (MAP). The dynamic of such a process is assumed to change according to an external environment process. Thus, it allows the claim numbers and severities to fluctuate according to the state of the business environment. The Markovian arrival process is very general; it covers the Poisson process, a renewal process with phase-type interclaim times, and the Markov-modulated Poisson process. On the other hand, it allows for situation in which the interclaim times and/or claim severities are dependent. An explicit and simple expression for the first-order moment of the total discounted IBNR claim amount is derived and recursive formulas for its higher-order moments are obtained. Numerical applications are provided to examine the properties of the total IBNR claim number, which has closed-form expressions for its finite-order moments.

Chapter 6 assumes a dependent renewal model, where the pdf of the interclaim times and claim severities are expressed in the form of a summation of factorizations. In the two-sided exit setting, the discounted aggregate claim is investigated under a dependent renewal process. Utilizing LTs, we identify the fundamental solutions to a given integral equation, which will be shown to play a role similar to the scale matrix in the analysis of spectrally-negative Markov-additive processes (e.g., Kyprianou and Palmowski (2008)). Explicit expressions and recursions are then identified for the two-sided exit probabilities and the moments of the discounted aggregate claims incurred until the insurance surplus process first leaves the $[0, b]$ interval. A numerical example involving the Farlie-Gumbel-Morgenstern(FGM) copula is considered in the end.

Chapter 7 concludes the thesis and discusses future research directions. Note that most chapters of this thesis directly relate to scientific papers, and were written independently of one another. Due to the large amount of notation, efforts have been made to have consistent notation over the entire thesis to avoid ambiguity. Even though some inconsistencies may remain, no abuse of notation shall exist within a chapter.

Chapter 2

Preliminaries

This chapter summarizes the mathematical preliminaries relevant in this thesis. We adopt the conventions that the empty product and sum are 1 and 0 respectively throughout this thesis. Also, we assume that $\mathbb{N}^+ = \{1, 2, \ldots\}$ and $\mathbb{N} = \{0, 1, 2, \ldots\}$.

2.1 Quantities related to the aggregate risk model

The model of interest is the aggregate claim model as defined in (1.1). The distribution function (df) of Y_i is $F_i(\cdot) \in [0, 1]$ for $i \in \mathbb{N}^+$. Whenever the probability density function (pdf) of Y_i exists, we denote it as $f_i(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_i(x)$. Here, we assume Y_i is a nonnegative rv unless stated otherwise. The LT of Y_i is defined as

$$\tilde{f}_i(s) = \int_0^\infty e^{-sx} \mathrm{d}F_i(x),$$

for $s \in \mathbb{C}$ such that the integral exists. An equivalent definition to $\tilde{f}(s)$ being the Laplace transform of a nonnegative rv is that $\tilde{f}(s)$ is completely monotone (i.e. $(-1)^n \tilde{f}^{(n)}(s) \geq 0$ for $n \in \mathbb{N}$) and $\tilde{f}(0) = 1$ (Feller (1971, p. 439)). This characterization is very relevant in the inversion of Laplace transform. If $\{Y_i\}_{i=1}^\infty$ are iid with df F, we define its n-fold convolution as

$$F^{*n}(x) = \Pr(Y_1 + Y_2 + \ldots + Y_n \leq x),$$

for $n \in \mathbb{N}$ with the convention that $F^{*0}(x) = 1$ for $x \geq 0$. As for the claim count rv N_t, its probability mass function (pmf) and pgf are defined as $p_{n,t} = \Pr(N_t = n)$ for $n \in \mathbb{N}$ and $P_t(z) = \sum_{n=0}^{\infty} p_{n,t} z^n$ for some $z \in \mathbb{R}$ such that the summation converges, respectively. Also, in the case where there exists a random risk parameter Θ, we let $P_t(z|\theta)$ be the pgf of N_t given that $\Theta = \theta$. The marginal pgf of N_t is expressed as

$$P_t(z) = \int P_t(z|\theta) \mathrm{d}U(\theta),$$

where $U(\theta) = \Pr(\Theta \leq \theta)$ is the df of Θ. This defines a mixed counting process $\{N_t\}_{t \geq 0}$. The mixture distributions are motivated as a model to address heterogeneity within population, and thus improves the fitting power of the model to reality.

Usually at most one arrival is allowed at one time in the ordinary counting process. However, this assumption is violated in many real-world applications. For instance, buses arrive at a stop bringing multiple customers at once and people usually go to restaurants in groups instead of individually. The bulk arrival process (see, e.g., Chaudhry and Templeton (1983)) is utilized to model the counting process in which arrivals occur in groups. The size of an arriving group may be a random number or a fixed number. In the insurance context, this process is applicable in the situation where a single claim causing event might bring multiple claims. In this thesis, we allow the claim number to follow a counting process with random bulk arrivals.

2.2 Applications of time-dependent claim models

The time-dependent claim model refers to the situation where the claim severities depend on the time occurrences, interclaim times and other quantities. The two most popular applications of the time-dependent claim model in insurance practice are the inflation model and the payment (reporting) delay model.

The inflation model incorporates the time value of money and claim inflation into the analysis of claim values. We assume that a claim occurring at 0 is distributed as a common "baseline" rv Y with cdf $F(\cdot)$ and LT $\tilde{f}(\cdot)$. Considering a claim incurred at x, whose baseline rv is Y, its value at time x is $e^{\int_0^x \delta_{0,y}\mathrm{d}y}Y$, where $\delta_{0,y}$ is the instantaneous rate of claim inflation at time y. After taking the interest rate $\delta_{1,y}$ into account, the real value of this claim at time 0 carries a discount factor of $e^{-\int_0^x \delta_y \mathrm{d}y}$, where $\delta_y = \delta_{1,y} - \delta_{0,y}$ is the instantaneous effective interest rate net of inflation. Therefore, under the assumption that the interest rate is deterministic, the df of the time-0 value of a claim occurring at x is written as

$$\Pr\left(e^{-\int_0^x \delta_y \mathrm{d}y}Y \le y\right) = F\left(e^{\int_0^x \delta_y \mathrm{d}y}y\right),$$

and its LT is expressed as $\tilde{f}\left(e^{-\int_0^x \delta_y \mathrm{d}y}s\right)$. The amount of the discounted aggregate claims is then expressed as $\sum_{k=1}^{N_t} e^{-\int_0^{T_k} \delta_x \mathrm{d}x}Y_k$. Léveillé and Garrido (2001a, 2001b) and Léveillé and Adékambi (2012) studied its moments by assuming independence between interclaim times and claim severities in models under both deterministic and stochastic interest rate models. Woo and Cheung (2013) further analyzed the moments of the discounted aggregate claims, while relaxing the independence assumption, using moment generating functions and copula methods.

In the reporting (payment) delay model, we consider the process with bulk arrivals, i.e. a claim causing event generates a random number of independent claims. For each of the claims caused by the same claim causing event occurring at x, the claim severities are assumed to have a common LT $\tilde{f}_x(z)$ and to be independent of each other. Furthermore, a natural lag between the occurrence of a claim and the payment (or reporting) is taken into consideration. We assume this random delay is distributed as $K_x(\cdot)$ for a claim occurring at time $x > 0$ and all payment delays are independent of other payment delays. If payment delays are independent of all claim amounts, the LT for the amount of the claim occurring at time x that has not been paid up to time t is then given as $K_x(t-x) + \bar{K}_x(t-x)\tilde{f}_x(z)$. Therefore, the total value of all the claims occurring from one claim causing event at time x that has not been paid until t has LT

$$\tilde{f}_t(z|x) = B_x\left(K_x(t-x) + \bar{K}_x(t-x)\tilde{f}_x(z)\right),$$

where $B_x(z)$ is the pgf of the claims occurring at time x. The payment delay problem has a close mathematical relationship with the inflation model when claim causing events occur according to a Poisson process (see Klugman et al. (2013, Chapter 9)). The reporting delay model will be considered in depth under the general renewal process in Chapter 4 of this thesis. Useful (from the point of view of mathematics and computational feasibility) results related to the important quantities of this model will be derived.

2.3 Important processes and distribution classes

It is not easy to get closed-form expressions for quantities associated with the (discounted) aggregate claims. Here, we introduce some important distribution classes, which will facilitate the derivation of closed-form expressions for important quantities related to the aggregate claim.

2.3.1 Mixed Erlang distributions

The mixed Erlang distributions are frequently utilized to model the quantities associated with insurance claims. This class is dense, broad and computationally convenient. Klugman et al. (2013) summarized various contexts in which the use of mixed Erlang distributions is of interest.

Definition 1. *A rv X has a mixed Erlang distribution if its pdf is given as*

$$f(x) = \sum_{n=1}^{\infty} q_n \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}, \quad x > 0,$$

where $\lambda > 0$ and the mixing weights $\{q_n\}_{n=0}^{\infty}$ form a discrete counting distribution with pgf $Q(z) = \sum_{n=0}^{\infty} q_n z^n$.

Thus, the LT of X is easily obtained as

$$\mathbb{E}\left(e^{-sX}\right) = Q\left(\frac{\lambda}{\lambda+s}\right).$$

We further explore the distributional properties of mixed Erlang distribution. The tail distribution $\bar{F}(x) = \int_x^\infty f(y)\mathrm{d}y$ can be re-expressed as

$$\bar{F}(x) = e^{-\lambda x}\sum_{n=0}^{\infty}\bar{Q}_n\frac{(\lambda x)^n}{n!},$$

where $\bar{Q}_n = \sum_{i=n+1}^{\infty} q_i$. A special case of the mixed Erlang is the Erlang-r distribution, whose pdf is given by

$$e_{\lambda,r}(x) = \frac{\lambda^r e^{-\lambda x}}{(r-1)!},\ \ x>0,\ \lambda>0,\ r\in\mathbb{N}^+,$$

and its tail df can be written as

$$\int_x^\infty e_{\lambda,r}(y)\mathrm{d}y = e^{-\lambda x}\sum_{n=0}^{r-1}\frac{(\lambda x)^n}{n!}.$$

Moreover, it is of most importance to mention the mixed Erlang representation for exponential distributions. From Willmot and Woo (2007), exponential distributions can be expressed as a mixed Erlang distribution with the pmfs of a zero-truncated geometric distribution as the mixing weights, namely

$$\frac{\lambda_i}{\lambda_i+s} = Q_i\left(\frac{\lambda}{\lambda+s}\right),$$

where $\lambda_i < \lambda < \infty$ and $Q_i(z) = \frac{(\lambda_i/\lambda)z}{1-(1-\lambda_i/\lambda)z}$.

Thus, it is possible to express multiple mixed Erlang distributions with different scale parameters as mixed Erlang distributions with a common scale parameter. Therefore, the sum of independent mixed Erlang distributed rv's with various scale parameters also has a mixed Erlang distribution.

2.3.2 (Nonhomogeneous) Poisson processes

Poisson process is a classical model for claim frequency. It has broad applications in a variety of fields, including engineering, statistics and neuroscience. A Poisson process possesses many desirable properties, including thinning, superposition, and decomposition. The self-decomposability helps to bridge limiting distribution and its finite-time counterparts. Thus, it is of utmost importance in the analysis of a counting process through its limiting behaviour as illustrated in the later time. Here, we omit the detailed definitions of thinning and superposition; interested readers are encouraged to read Ross (2010, Chapter 5).

Before approaching the definition of Poisson process, we introduce some important definitions related to a counting process first.

Definition 2. *A stochastic process* $\{N_t\}_{t\geq 0}$ *is said to have stationary increment if the distribution of* $N_t - N_s$ *for* $t > s$ *depends only on the interval length, i.e.* $t - s$*.*

Definition 3. *A stochastic process* $\{N_t\}_{t\geq 0}$ *has independent increments if increments for any set of disjoint intervals are independent.*

Definition 4. *Conditional on* $N_t - N_0 = k$*, the successive jump times are distributed as the order statistics of* k *iid rv's with df on* $[0, t]$*, then we say the process has the order statistic property (see Feign (1979)).*

Poisson process can be defined through multiple equivalent definitions, see He (2014, Chapter 2) and Taylor and Karlin (1998, Chapter 5). We provide a definition next.

Definition 5. *The process* $\{N_t\}_{t\geq 0}$ *with* $N_0 = 0$ *is called a Poisson process if*

1. $\{N_t\}_{t\geq 0}$ *possesses the independent increment property and the stationary increment property; and*

2. $\Pr(N_t = 1) = \lambda t + o(t)$ *and* $\Pr(N_t \geq 2) = o(t)$*, where the intensity* $\lambda > 0$ *and* $o(t)/t \to 0$ *when* $t \to 0$*.*

The pmf of a Poisson process $\{N_t\}_{t\geq 0}$ with intensity λ is thus given by

$$\Pr(N_t = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \ n = 0, 1, 2, \ldots.$$

and its pgf is

$$P(z) = \mathbb{E}\left[z^{N_t}\right] = e^{\lambda t(z-1)}.$$

It is easily verified that $P(z)$ is discretely self-decomposable, i.e. $P(z) = P(1-\rho+\rho z)P_\rho(z)$ where $P_\rho(z)$ is itself a pgf for all ρ such that $0 < \rho < 1$. Also, it is noticed that, for $s < t$,

$$\Pr(N_s = k | N_t = k+n) = \binom{k+n}{n} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^n.$$

Thus, Poisson process has order statistics property, i.e. given that n claims occurring in $[0, t]$, the claim times are distributed as the order statistics of n iid rv's, which are uniformly distributed on $[0, t]$.

Moreover, the interclaim times of a Poisson process are exponentially distributed with mean of $1/\lambda$. Also, a Poisson process can be characterized by its interclaim times, i.e. a counting process with iid exponentially distributed interclaim times is a Poisson process. See Ross (2010, Chapter 5) for more detail.

In the case when the intensity rate changes over time but still is deterministic, denoted as $\lambda(t) > 0$, we have the nonhomogeneous Poisson process. Its pmf is written as

$$\Pr(N_t = n) = \frac{\left(\int_0^t \lambda(x)\mathrm{d}x\right)^n e^{-\int_0^t \lambda(x)\mathrm{d}x}}{n!}, \ \ n \in \mathbb{N}.$$

It is easily obtained that the nonhomogeneous Poisson process has independent but not stationary increments. The nonhomogeneous Poisson process also possesses the order statistic property, since

$$\Pr(N_s = k|N_t = k+n) = \binom{k+n}{n} \left(\frac{\int_0^s \lambda(x)\mathrm{d}x}{\int_0^t \lambda(x)\mathrm{d}x} \right)^k \left(1 - \frac{\int_0^s \lambda(x)\mathrm{d}x}{\int_0^t \lambda(x)\mathrm{d}x} \right)^n,$$

for $s \in [0, t]$, where the iid rv's have a pdf of $\lambda(x)/\int_0^t \lambda(y)\mathrm{d}y$ for $x \in [0, t]$.

2.3.3 Mixed Poisson processes

A mixed Poisson process can be viewed as a generalization of a Poisson process when the intensity of the Poisson process is regarded as a rv. This model accounts for heterogeneity within the population. Mathematically, for a mixed Poisson process $\{N_t\}_{t\geq 0}$, we have that $\{N_t|\Theta = \theta\}_{t\geq 0}$ is a Poisson process with rate θ. Its marginal pmf is given by

$$\Pr(N_t = n) = \int_0^\infty \frac{(\theta t)^n e^{-\theta t}}{n!} U(\mathrm{d}\theta),$$

where U is the df of Θ. U is called the mixing distribution and also called structure function. It represents the fluctuations in the risk levels. Various choices of mixing distributions lead to different models for claim frequency. For example, a mixed Poisson rv with Gamma distributed intensity follows a negative binomial distribution, see Klugman et al. (2013, Chapter 7) for more detail.

Let $\tilde{u}(s) = \int_0^\infty e^{-s\theta}\mathrm{d}U(\theta)$ be the LT of the mixing distribution, then, the pgf of N_t satisfies

$$P_t(z) = \tilde{u}[t(1-z)].$$

The moments are then easily obtainable via their relationship with the pgf. We compare the mean and variance of mixed Poisson process here, namely

$$\mathbb{E}[N_t] = \mathbb{E}[\Theta]t \leq Var[N_t] = \mathbb{E}[\Theta]t + Var[\Theta]t^2.$$

The variance is shown to exceed the mean. Thus, mixed Poisson processes have heavier tails than Poisson processes; it is then more proper to model long-tailed data. As demonstrated by McFadden (1965), mixed Poisson processes have stationary but not independent increments. The mixed Poisson process does have order statistic property and we will illustrate this later in Chapter 3.

Furthermore, the mixed Poisson process has an important characterization (Grandell (1997, p. 25-26)) given in the following theorem.

Theorem 1. *A pgf $P(z)$ satisfying $P(z) \neq 1, a.s.$ is a mixed Poisson pgf if and only if, as a function of z, $P(1 + (z-1)/\rho)$ is a pgf for all $\rho \in (0, 1)$.*

Theorem 1 is very relevant in insurance contexts in the sense that it guarantees both numbers of the ground-up loss and the claims with deductible are valid counting rv's. This characterization theorem also justifies the popularity of applying mixed Poisson processes in insurance practice.

2.3.4 (Delayed) Sparre Andersen risk processes

Consider a surplus process as defined in (1.3), where the pairs $\{(\tau_i, Y_i)\}_{i=1}^{\infty}$ are iid, and distributed as a generic pair (τ, Y). If τ and Y are independent of each other, the surplus process $\{U_t\}_{t\geq 0}$ as defined in (1.3) is called the Sparre Andersen risk model, where $\{N_t\}_{t\geq 0}$ is an ordinary renewal risk process. Relaxing the independence assumption between τ and Y yields the dependent Sparre-Andersen risk process. Thus, the (dependent) Sparre-Andersen risk process can be treated as a (dependent) renewal process. The Gerber-Shiu discounted penalty function (first proposed by Gerber and Shiu (1998)) has been well studied under the Sparre-Andersen risk model. Chapter 6 derived a recursive formula for the moments of the aggregate claims under the dependent Sparre Andersen risk model. The dependence assumed between the pair (τ_i, Y_i) helps to model the reality more accurately. For instance, in catastrophic events, data from practice has shown that a larger accident tends to occur as the interclaim time increases.

The first interclaim time in a renewal process or Sparre-Andersen risk process may sometimes be assumed to have a different distribution than the other interclaim times. This results in a delayed renewal risk process, where the distribution of the time to the first event is assumed to be different from that of the subsequent ones, which are assumed to be identically distributed. Its introduction is mainly motivated by the fact that the starting point 0 (also called observation time) is arbitrarily chosen, thus, the system might have been idling for some time. But instead exactly one claim occurs at time T_1, T_2, ..., thus the other interclaim times record the exact idling time of the system. The analysis under the renewal process can usually be extended to the delayed model without adding much difficulty.

2.4 Methodologies and techniques

In this section, we present some relevant mathematical tools and techniques used later in this thesis.

2.4.1 Lagrange polynomials

Suppose that $x_1, x_2, \ldots, x_k$ for $k \geq 2$ are distinct numbers and $h(x)$ is any polynomial of degree $k-1$ or less. Then, $h(x)$ can be expressed as

$$h(x) = \sum_{i=1}^{k} h(x_i) \left[\prod_{j=1, j\neq i}^{k} \frac{x - x_j}{x_i - x_j} \right].$$

By making use of the Lagrange polynomial expansions, we get the following lemma.

Lemma 2. *Consider an equation $h(x) = \sum_{i=0}^{k-1} c_i x^i + w(x)$ for $k \geq 2$, which has k distinct zeros $s_1, s_2, \ldots, s_k$, then*

$$\begin{aligned} h(x) &= w(x) - \sum_{i=1}^{k} w(s_i) \prod_{j=1, j\neq i}^{k} \frac{x - s_j}{s_i - s_j}, \\ &= w(x) + c_{k-1} \prod_{i=1, i\neq l}^{k} (x - s_i) - \sum_{i=1, i\neq l}^{k} \left[\prod_{j=1, j\neq i, l}^{k} \frac{x - s_j}{s_i - s_j} \right] w(s_i), \end{aligned}$$

for any $l \in \{1, 2, \ldots, k\}$ and $c_{k-1} = -\sum_{i=1}^{k} w(s_i) \prod_{j=1, j\neq i}^{k} \frac{1}{s_i - s_j}$.

2.4.2 Dickson-Hipp operator

Let s be a (possibly) complex number with a nonnegative real part, and define

$$\mathcal{T}_s h(x) = \int_x^\infty e^{-s(y-x)} h(y) \mathrm{d}y,$$

for a function h such that the integral exists. $\mathcal{T}_s$ is known as the Dickson-Hipp operator. The Dickson-Hipp operator is very relevant in ruin theory (see, e.g., Dickson and Hipp (2001)). For $x = 0$, it is equivalent to the Laplace transform operator, i.e. $\mathcal{T}_s h(0) = \tilde{h}(s)$. Furthermore, for any complex number $s_1, s_2, \ldots, s_k$, for $k \geq 2$, we have

$$\mathcal{T}_{s_1,s_2,\ldots,s_k} h(x) = \mathcal{T}_{s_1} \mathcal{T}_{s_2} \ldots \mathcal{T}_{s_k} h(x).$$

More specifically,

$$\mathcal{T}_{s_1,s_2} h(0) = \mathcal{T}_{s_2,s_1} h(0) = \frac{\tilde{h}(s_2) - \tilde{h}(s_1)}{s_1 - s_2},$$

for $s_1 \neq s_2$. By Li and Garrido (2004), it holds that, if $s_1, s_2, \ldots, s_k$ are distinct,

$$\mathcal{T}_{s_1,s_2,\ldots,s_k} h(x) = (-1)^{k-1} \sum_{i=1}^{k} \left[\prod_{j=1, j \neq i}^{k} \frac{1}{s_j - s_i} \right] \mathcal{T}_{s_i} h(x), \tag{2.1}$$

for $x \geq 0$.

2.4.3 Rouche's theorem

Solutions to Lundberg's generalized equation $\mathbb{E}\left[e^{-\delta W_1} e^{s(cW_1 - X_1)}\right] = 1$ for $\delta \geq 0$ are very relevant to the analysis of ruin-related quantities. Rouche's theorem is mainly utilized to verify the existence of the solutions to Lundberg's generalized equation with $\delta > 0$ in certain domain. As such, we recall Rouche's theorem here (see, e.g., Titchmarsh (1939)).

Theorem 3. *If $h(z)$ and $g(z)$ are analytic inside and on a closed contour $\boldsymbol{\mathcal{D}}$ and $|g(z)| < |h(z)|$ on $\boldsymbol{\mathcal{D}}$, then $h(z)$ and $g(z) + h(z)$ have the same number of zeros inside $\boldsymbol{\mathcal{D}}$.*

As an extension to Rouche's theorem, Klimenok (2001, Theorem 1) proposed a generalization which is particularly relevant in ruin cases where $\delta = 0$.

Theorem 4. *Let the function $g(z)$ and $h(z)$ be analytic in the open disk $|z| < 1$ and continuous on the boundary $|z| = 1$ and the following relations hold:*

$$|g(z)|_{|z|=1, z\neq 1} > |h(z)|_{|z|=1, z\neq 1},$$
$$g(1) = -h(1) \neq 0.$$

Let also the functions $g(z)$ and $h(z)$ have derivatives at the point $z = 1$ with the following inequality that holds:

$$\frac{g'(1) + h'(1)}{g(1)} > 0.$$

Then the numbers N_{g+h} and N_g of zeros of the functions $g(z) + h(z)$ and $g(z)$ in the domain $|z| < 1$ are related as follows:

$$N_{g+h} = N_g - 1.$$

2.5 Copula

Copula is a well known distribution-based aggregation method to specify the dependence structure between risk factors (e.g., Joe (1997), McNeil et al. (2005), and Nelsen (2006)). The copula method is utilized in this thesis also due to its easy computational implementation. Let $C : [0, 1] \times [0, 1] \to [0, 1]$ be a bivariate copula. Then, for any random vector (X, Y),

$$\Pr(X \leq x, Y \leq y) = C(\Pr(X \leq x), \Pr(Y \leq y)),$$

for $x, y \in \mathbb{R} \times \mathbb{R}$. Similarly, the survival copula relative to the joint survival function, i.e.

$$\Pr(X > x, Y > y) = \hat{C}(\Pr(X > x), \Pr(Y > y)),$$

is given by

$$\hat{C}(u,v) = u + v - 1 + C(1-u, 1-v),$$

for $(u,v) \in [0,1]^2$. Due to Sklar's theorem (Nelsen (2006)), the relationship between copula and distribution functions is uniquely defined for two continuous rv's X and Y.

Next, we introduce a special copula, called Bernstein copula. For a given copula C, the Bernstein copula (BC) is defined as

$$C_B(u,v) = \sum_{i=1}^{n} \sum_{j=1}^{\ell} C\left(\frac{i}{n}, \frac{j}{\ell}\right) B_n(i,u) B_\ell(j,v), \tag{2.2}$$

for $u, v \in [0,1]$, where the indexes n, ℓ are positive integers, and $B_n(i,p) = \binom{n}{i} p^i (1-p)^{n-i}$ for $p \in [0,1]$ and $i = 0, 1, \ldots, n$. Here, we adopt the convention that $B_n(k,p) = 0$ for $n < k$ or $k < 0$. We point out here that FGM copula is a special case of Bernstein copula with $n = \ell = 2$ and $C\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1+\theta}{4}$, for $\theta \in [-1,1]$ i.e.

$$C_{FGM}(u,v) = uv + \theta uv(1-u)(1-v). \tag{2.3}$$

It is easily obtainable that Bernstein density is given by

$$c_B(u,v) = \frac{\partial^2}{\partial u \partial v} C_B(u,v) = \sum_{i=0}^{n-1} \sum_{j=0}^{\ell-1} a_{n,\ell}\left(\frac{i}{n}, \frac{j}{\ell}\right) B_{n-1}(i,u) B_{\ell-1}(j,v), \tag{2.4}$$

where

$$a_{n,\ell}\left(\frac{i}{n}, \frac{j}{\ell}\right) = n\ell \left[C\left(\frac{i+1}{n}, \frac{j+1}{\ell}\right) - C\left(\frac{i}{n}, \frac{j+1}{\ell}\right) - C\left(\frac{i}{n}, \frac{j+1}{\ell}\right) + C\left(\frac{i}{n}, \frac{j}{\ell}\right) \right].$$

The motivation of introducing the Bernstein copula is also due to its denseness in the space of bounded continuous functions (see Nelsen (1998) for more detail). This means that for any continuous copula function in the 2 dimensional hypercube $[0,1]^2$, we can represent it as a Bernstein copula. The flexibility of Bernstein copula allows us to approximate an unknown underlying dependence structure in any realistic environment by fitting the Bernstein copula to the empirical data. See also Diers et al. (2012) for more discussions on the properties of the Bernstein copula. Chapter 6 will consider the analysis of the moments of the aggregate discounted claim costs under a two-sided exit setting by assuming the dependence between claim severities and interclaim times is a generalized Bernstein copula.

Chapter 3

Time-Dependent Claims in Birth Process Claim Count Models

In this chapter, we consider the case when $\{N_t\}_{t\geq 0}$ is a nonhomogeneous birth process, a model which is shown to be particularly suitable for use in a time-dependent claim context. We are interested in the behaviour of the process after a fixed time s, given the value of N_s, say k. Thus, the results hold for any counting process with the Markov property which behaves like a nonhomogeneous birth process thereafter.

Now, we are going to introduce the main processes under which we will work. First, we start with a Markovian counting process. A counting process is Markovian, if the manner in which the process behaves after a certain time, say s, is only related to the scenario at time s, without depending on the process history before s. Mathematically speaking, for $t > s$, the distribution of $N_t - N_s$ given N_s is the same regardless of the values $\{N_u\}_{0\leq u<s}$. For a Markovian counting process, of central importance to the analysis are the transition probabilities, for $n \in \mathbb{N}$, given by,

$$p_{k,k+n}(s,t) = \Pr(N_t - N_s = n | N_s = k),$$

and the pgf is denoted as

$$P_{k,s,t}(z) = \mathbb{E}[z^{N_t - N_s}|N_s = k] = \sum_{n=0}^{\infty} p_{k,k+n}(s,t) z^n. \tag{3.1}$$

Now we are going to approach the definition of a birth process.

Definition 6. *A Markovian counting process $\{N_t\}_{t\geq 0}$ is called a birth process if*

$$p_{k,k+1}(t, t+h) = \lambda_k(t)h + o(h),$$

and

$$p_{k,k+n}(t, t+h) = o(h)$$

for $n = 2, 3, \ldots$, where the functions $\{\lambda_0, \lambda_1, \ldots\}$ are called the transition intensity functions.

In birth process, the marginal probabilities are given (under the assumption that $N_0 = 0$) by $p_n(t) = \Pr(N_t = n) = p_{0,n}(0,t)$. It has been shown that birth process is defined in terms of the probability transition functions. The transition probabilities are then characterized by the transition intensities, which also have an alternative formulation, namely

$$p_{k,k}(s,t) = e^{-\int_s^t \lambda_k(y)\mathrm{d}y}, \tag{3.2}$$

for $k = 0, 1, \ldots$. For $n \geq 1$, $p_{k,k+n}(s,t)$ may be obtained recursively in n. The explicit formulas for the transition probabilities as a function of transition intensities can be obtained for some choices of $\lambda_m(t)$s, see Klugman et al. (2013, Chapter 7), Willmot (2010) and reference therein.

In what follows in this chapter, let the realizations of T_m, for $m \in \mathbb{N}^+$, be denoted by t_m. For convenience, we assume $T_0 = t_0 = 0$. Also, $h_{n,t}(t_{k+1}, t_{k+2}, \ldots, t_{k+n}|k, s)$ represents the density function associated with the event that there are exactly n claims in (s,t) at times $t_{k+1} < t_{k+2} < \ldots < t_{k+n}$ where $s < t_{k+1}$ and $t_{k+n} < t$, given that $N_s = k$. This density is of central importance in what follows, and is now given explicitly.

Lemma 5. *For $n \in \mathbb{N}^+$,*

$$h_{n,t}(t_{k+1}, t_{k+2}, \ldots, t_{k+n}|k, s) = e^{-\int_s^t \lambda_{k+n}(y)\mathrm{d}y} \prod_{m=1}^{n} \phi_{k+m}(t_{k+m}|s), \tag{3.3}$$

where

$$\phi_j(x|s) = \lambda_{j-1}(x) e^{\int_s^x \{\lambda_j(y) - \lambda_{j-1}(y)\}\mathrm{d}y}. \tag{3.4}$$

Proof. It is clear from (3.2) that for $m \in \mathbb{N}^+$, $\exp\{-\int_{t_{k+m-1}}^{t_*} \lambda_{k+m-1}(y)\mathrm{d}y\}$ may be interpreted as the probability that T_{k+m} exceeds t_*, given that $N_{t_{k+m-1}} = k + m - 1$. Thus, $\lambda_{k+m-1}(y)$ is the associated failure rate, and (assuming for the moment that $t_k = s$) the joint density of $T_{k+1}, T_{k+2}, \ldots, T_{k+n}|N_s = k$ may thus be expressed as

$$\prod_{m=1}^{n} \lambda_{k+m-1}(t_{k+m}) e^{-\int_{t_{k+m-1}}^{t_{k+m}} \lambda_{k+m-1}(y)\mathrm{d}y}.$$

In order to have exactly n claims in (s, t), there can be no more claims in (t_{k+n}, t) with probability $\exp\left\{-\int_{t_{k+n}}^{t} \lambda_{k+n}(y)\mathrm{d}y\right\}$, implying that

$$\begin{aligned}
&h_{n,t}(t_{k+1}, t_{k+2}, \ldots, t_{k+n}|k, s) \\
&= e^{-\int_{t_{k+n}}^{t} \lambda_{k+n}(y)\mathrm{d}y} \prod_{m=1}^{n} \lambda_{k+m-1}(t_{k+m}) e^{-\int_{t_{k+m-1}}^{t_{k+m}} \lambda_{k+m-1}(y)\mathrm{d}y} \\
&= e^{-\left[\int_s^t - \int_s^{t_{k+n}}\right]\lambda_{k+n}(y)\mathrm{d}y} \prod_{m=1}^{n} \lambda_{k+m-1}(t_{k+m}) e^{-\left[\int_s^{t_{k+m-1}} - \int_s^{t_{k+m}}\right]\lambda_{k+m-1}(y)\mathrm{d}y}.
\end{aligned} \tag{3.5}$$

Simple arrangement of (3.5) results in (3.3). □

An explicit expression for the probability transition function follows immediately from Lemma 5.

Lemma 6. *The transition probabilities may be expressed as*

$$p_{k,k+1}(s, t) = \int_s^t h_{1,t}(t_{k+1}|k, s)\mathrm{d}t_{k+1}, \tag{3.6}$$

and for n=2,3,...,

$$p_{k,k+n}(s,t) = \int_s^t \int_s^{t_{k+n}} \cdots \int_s^{t_{k+2}} h_{n,t}(t_{k+1},\ldots,t_{k+n}|k,s)\mathrm{d}t_{k+1}\mathrm{d}t_{k+2}\ldots\mathrm{d}t_{k+n}. \quad (3.7)$$

Proof. Integrating over all possible values of $t_{k+1}, t_{k+2}, \ldots, t_{k+n}$ results in (3.6) and (3.7). $\square$

We now turn to the problem of interest, namely the analysis of time-dependent claims. To this end, the sum total of claim values for claims incurred in (s,t) is denoted as

$$S_{s,t} = \sum_{i=k+1}^{N_t} X_i | N_s = k, \quad (3.8)$$

for $t \geq s$, where the claim severities $\{X_i\}_{i=k+1}^{\infty}$ depend on the particular quantity of interest to be analyzed. See Klugman et al. (2013, Section 9.1) for a discussion of this issue. We denote the conditional LT of $S_{s,t}$, given that there are exactly n claims in (s,t) at times $t_{k+1}, t_{k+2}, \ldots, t_{k+n}$ by $\tilde{f}_{n,t}(z|k,s,t_{k+1},\ldots,t_{k+n})$. If we further assume that the individual claim values are independent of all other claim values, with distribution depending on nothing more than possibly the incurral time, k, s and t, then we may write

$$\tilde{f}_{n,t}(z|k,s,t_{k+1},\ldots,t_{k+n}) = \prod_{m=1}^{n} \tilde{f}_t(z|k,s,t_{k+m}). \quad (3.9)$$

Note that in (3.9), $\tilde{f}_t(z|k,s,x)$ is the LT of the claim value associated with a claim incurral at $x \in (s,t)$. The independence assumption is not necessary, we are now in a position to state the general results for the aggregate claim values, conditional on $N_s = k$ without the assumption stated in (3.9).

Theorem 7. *Given that $N_s = k$, the aggregate claim values associated with claims incurred in (s,t) has LT*

$$\mathbb{E}[e^{-zS_{s,t}}|N_s = k] = p_{k,k}(s,t) + \sum_{n=1}^{\infty} p_{k,k+n}(s,t)\tilde{f}_{n,t}(z|k,s), \quad (3.10)$$

where

$$\tilde{f}_{1,t}(z|k,s) = \frac{\int_s^t h_{1,t}(t_{k+1}|k,s)\tilde{f}_{1,t}(z|k,s,t_{k+1})\mathrm{d}t_{k+1}}{\int_s^t h_{1,t}(t_{k+1}|k,s)\mathrm{d}t_{k+1}}, \tag{3.11}$$

and for n=2,3, ...,

$$\tilde{f}_{n,t}(z|k,s) = \frac{\int_s^t \int_s^{t_{k+n}} \ldots \int_s^{t_{k+2}} h_{n,t}(t_{k+1},\ldots,t_{k+n}|k,s)\tilde{f}_{n,t}(z|k,s,t_{k+1},\ldots,t_{k+n})\mathrm{d}t_{k+1}\ldots\mathrm{d}t_{k+n}}{\int_s^t \int_s^{t_{k+n}} \ldots \int_s^{t_{k+2}} h_{n,t}(t_{k+1},\ldots,t_{k+n}|k,s)\mathrm{d}t_{k+1}\ldots\mathrm{d}t_{k+n}}. \tag{3.12}$$

Proof. Obviously, $S_{s,t} = 0$ if $N_t - N_s = 0$, and otherwise (3.10) follows directly by conditioning on $N_t - N_s = n$, and the n claim times $T_{k+1},\ldots,T_{k+n}$, together with (3.6) and (3.7). □

Clearly, (3.11) and (3.12) imply that the LT $\tilde{f}_{n,t}(z|k,s,t_{k+1},\ldots,t_{k+n})$ may be represented as a mixture, with mixing weight proportional to $h_{n,t}(t_{k+1},\ldots,t_{k+n}|k,s)$. Also, it is useful to note that in the important special case when (3.9) holds, (3.3) implies that for any n, the integrand in (3.12) factors as a function of the integration variables $t_{k+1},t_{k+2},\ldots,t_{k+n}$. We also want to remark that the order of the claim incurral times won't influence the result if $h_{n,t}(t_{k+1},t_{k+2},\ldots,t_{k+n})$ is a symmetric function for $n \geq 1$. In this case, we can re-write (3.12) as

$$\tilde{f}_{n,t}(z|k,s) = \frac{\int_s^t \int_s^t \ldots \int_s^t h_{n,t}(t_{k+1},\ldots,t_{k+n}|k,s)\prod_{m=1}^n \tilde{f}_t(z|k,s,t_{k+m})\mathrm{d}t_{k+1}\ldots\mathrm{d}t_{k+n}}{\int_s^t \int_s^t \ldots \int_s^t h_{n,t}(t_{k+1},\ldots,t_{k+n}|k,s)\mathrm{d}t_{k+1}\ldots\mathrm{d}t_{k+n}}.$$

3.1 The birth process with factorization assumption

While the representation of Theorem 7 is extremely general, a very useful simplification results if (3.9) holds and $\lambda_j(x)$ for $j = k, k+1, \ldots$ is such that $\phi_{k+m}(x|s)$, defined in (3.4) for $m \in \mathbb{N}^+$, factors (for fixed k and s) as a function of m multiplied by a function of x. This is the case for (possibly) nonhomogeneous version of Poisson and mixed Poisson processes, and the contagion models, as is discussed later. This factorization assumption is motivated by Puri (1982), in the context of the evaluation of the marginal probabilities $\{p_n(t); n \in \mathbb{N}\}$.

In fact, the factorization essentially characterizes the so-called order statistic property (see Puri (1982)).

Theorem 8. *Suppose that (3.9) holds, and $\lambda_j(x)$ for $j = k, k+1, \ldots$ is such that*

$$\phi_{k+m}(x|s) = \alpha_{m,k,s} b(x|k,s), \tag{3.13}$$

for $m \in \mathbb{N}^+$. Then for $n \in \mathbb{N}^+$,

$$p_{k,k+n}(s,t) = e^{-\int_s^t \lambda_{k+n}(y)\mathrm{d}y} \frac{\left(\prod_{m=1}^n \alpha_{m,k,s}\right)}{n!} \left\{ \int_s^t b(x|k,s)\mathrm{d}x \right\}^n, \tag{3.14}$$

and (3.10) may be expressed in compound form as

$$\mathbb{E}[e^{-zS_{s,t}}|N_s = k] = P_{k,s,t}\left\{ \tilde{f}_t(z|k,s) \right\}, \tag{3.15}$$

where the pgf $P_{k,s,t}(z)$ is given by (3.1), and

$$\tilde{f}_t(z|k,s) = \frac{\int_s^t b(x|k,s)\tilde{f}_t(z|k,s,x)\mathrm{d}x}{\int_s^t b(x|k,s)\mathrm{d}x}, \tag{3.16}$$

is the LT of a mixed distribution.

Proof. We utilize the approach of Puri (1982). First note that (3.3) becomes

$$h_{n,t}(t_{k+1}, t_{k+2}, \ldots, t_{k+n}|k,s) = e^{-\int_s^t \lambda_{k+n}(y)\mathrm{d}y} \left\{ \prod_{m=1}^n \alpha_{m,k,s} \right\} \prod_{m=1}^n b(t_{k+m}|k,s), \tag{3.17}$$

for $n \in \mathbb{N}^+$. Also, combining (3.9) with (3.17) results in

$$\begin{aligned} & h_{n,t}(t_{k+1}, t_{k+2}, \ldots, t_{k+n}|k,s)\tilde{f}_{n,t}(z|k,s,t_{k+1},\ldots,t_{k+n}) \\ &= e^{-\int_s^t \lambda_{k+n}(y)\mathrm{d}y} \left\{ \prod_{m=1}^n \alpha_{m,k,s} \right\} \prod_{m=1}^n \left[b(t_{k+m}|k,s)\tilde{f}_t(z|k,s,t_{k+m}) \right], \end{aligned} \tag{3.18}$$

again for $n \in \mathbb{N}^+$. For $n = 1$, substitution of (3.17) into (3.6) yields (3.14) after changing the variable of integration from t_{k+1} to x. Similarly, (3.6) and (3.11) imply that

$$\begin{aligned} p_{k,k+1}(s,t)\tilde{f}_{1,t}(z|k,s) &= \int_s^t h_{1,t}(t_{k+1}|k,s)\tilde{f}_{1,t}(z|k,s,t_{k+1})\mathrm{d}t_{k+1} \\ &= e^{-\int_s^t \lambda_{k+1}(y)\mathrm{d}y}\alpha_{1,k,s}\int_s^t b(x|k,s)\tilde{f}_t(z|k,s,x)\mathrm{d}x \\ &= p_{k,k+1}(s,t)\tilde{f}_t(z|k,s), \end{aligned}$$

using (3.16) and (3.18) also. Note that for any integrable function $\gamma(x)$, it follows easily from $\gamma(t) = \frac{\partial}{\partial t}\int_s^t \gamma(x)\mathrm{d}x$ that

$$\int_s^t \int_s^{t_{k+n}} \dots \int_s^{t_{k+2}} \left\{\prod_{m=1}^n \gamma(t_{k+m})\right\} \mathrm{d}t_{k+1}\dots\mathrm{d}t_{k+n} = \frac{1}{n!}\left\{\int_s^t \gamma(x)\mathrm{d}x\right\}^n, \tag{3.19}$$

for $n = 2, 3, \dots$. Thus (3.14), which is essentially given by Puri (1982) for the marginal rather than the transitional properties, holds for $n = 2, 3, \dots$ by substituting (3.17) into (3.7) and using (3.19) with $\gamma(x) = b(x|k,s)$. Then (3.7) and (3.12) imply that

$$\begin{aligned} &p_{k,k+n}(s,t)\tilde{f}_{n,t}(z|k,s) \\ &= \int_s^t \int_s^{t_{k+n}} \dots \int_s^{t_{k+2}} h_{n,t}(t_{k+1},\dots,t_{k+n}|k,s)\tilde{f}_{n,t}(z|k,s,t_{k+1},\dots,t_{k+n})\mathrm{d}t_{k+1}\dots\mathrm{d}t_{k+n} \\ &= e^{-\int_s^t \lambda_{k+n}(y)\mathrm{d}y}\left(\prod_{m=1}^n \alpha_{m,k,s}\right)\int_s^t \int_s^{t_{k+n}} \dots \int_s^{t_{k+2}} \left\{\prod_{m=1}^n \left[b(t_{k+m}|k,s)\tilde{f}_t(z|k,s,t_{k+m})\right]\right\}\mathrm{d}\mathbf{t}, \end{aligned} \tag{3.20}$$

where $\mathbf{t} = [t_{k+1}, t_{k+2}, \dots, t_{k+m}]$, using (3.18) as well. Thus, we have

$$\begin{aligned} p_{k,k+n}(s,t)\tilde{f}_{n,t}(z|k,s) &= \frac{e^{-\int_s^t \lambda_{k+n}(y)\mathrm{d}y}}{n!}\left(\prod_{m=1}^n \alpha_{m,k,s}\right)\left\{\int_s^t b(x|k,s)\tilde{f}_t(z|k,s,x)\mathrm{d}x\right\}^n \\ &= p_{k,k+n}(s,t)\left\{\tilde{f}_t(z|k,s)\right\}^n, \end{aligned}$$

by (3.19) with $\gamma(x) = b(x|k,s)\tilde{f}_t(z|k,s,z)$, where the last line follows from (3.16). Therefore, (3.10) becomes

$$\mathbb{E}[e^{-zS_{s,t}}|N_s = k] = \sum_{n=0}^{\infty} p_{k,k+n}(s,t) \left\{ \tilde{f}_t(z|k,s) \right\}^n .$$

which immediately yields the representation (3.15). Finally, as $\lambda_m(x)$ is nonnegative for $m = k, k+1, \ldots$, it follows from (3.4) that $\phi_{k+m}(x|s)$ is also nonnegative for $m \in \mathbb{N}^+$, and thus it may be assumed without loss of generality that each of $\alpha_{m,k,s}$ and $b(x|k,s)$ is nonnegative, implying that (3.16) is a mixture LT of a distribution. □

The representation in Theorem 8 of the distribution of a random sum of conditionally independent, but not necessarily identically distributed rv's as a random sum of iid rv's is very convenient from the viewpoint of quantitative analysis. This is particularly true due to the fact that in many applications the transition probabilities are of a simple and well known form, as is discussed in further details in the next section.

Furthermore, the results of this section and Theorem 8 in particular make no assumptions about the behaviour of the process before time s except for the assumption about the Markov property and k claims have occurred. That is, the process $\{N_t\}_{t\geq 0}$ needs only be a nonhomogeneous birth process beyond a certain point.

Next, we provide some examples involving some common choices of the intensity function $\lambda_n(t)$.

Example 1. *A delayed nonhomogeneous Poisson process*
Suppose that $\lambda_n(x) = \lambda(x)$ for $s \leq x \leq t$ and $n = k, k+1, \ldots$. Then for $m \in \mathbb{N}^+$, (3.4) becomes $\phi_{k+m}(x|s) = \lambda(x)$, and the results of Theorem 8 may be applied with $\alpha_{m,k,s} = 1$ and $b(x|k,s) = \lambda(x)$. Then (3.14) becomes

$$p_{k,k+n}(s,t) = \frac{e^{-\int_s^t \lambda(x)\mathrm{d}x}\{\int_s^t \lambda(x)\mathrm{d}x\}^n}{n!}$$

a Poisson probability. As $p_{k,k}(s,t) = \exp\{-\int_s^t \lambda(y)\mathrm{d}y\}$, (3.15) is a compound Poisson LT *with secondary LT given by (3.16), namely*

$$\tilde{f}_t(z|k,s) = \frac{\int_s^t \lambda(x)\tilde{f}_t(z|k,s,x)\mathrm{d}x}{\int_s^t \lambda(x)\mathrm{d}x}.$$

Of course, homogeneous Poisson process is obtained when $\lambda(x)$ is a constant on (s,t). Besides, the ordinary nonhomogeneous Poisson process results if $\lambda_m(t) = \lambda(t)$ for all $m \in \mathbb{N}$ and $t \geq 0$.

Example 2. *Linear contagion*

In the linear contagion model, we assume that

$$\lambda_j(t) = (\alpha + \beta j)\lambda(t),$$

for $j = k, k+1, \ldots$. Thus (3.4) becomes, from $m \in \mathbb{N}^+$,

$$\phi_{m+k}(x|s) = \{\alpha + \beta(k+m-1)\}\,\lambda(x)e^{\beta\int_s^x \lambda(y)\mathrm{d}y},$$

implying that for $\beta \neq 0$, Theorem 8 applies with $\alpha_{m,k,s} = m+k-1+\frac{\alpha}{\beta}$, and $b(x|k,s) = \beta\lambda(x)e^{\beta\int_s^x \lambda(y)\mathrm{d}y}$. Then

$$\frac{\prod_{m=1}^n \alpha_{m,k,s}}{n!} = \frac{\Gamma(n+k+\frac{\alpha}{\beta})}{n!\Gamma(k+\frac{\alpha}{\beta})} = \binom{n+k+\frac{\alpha}{\beta}-1}{n},$$

and (3.14) becomes

$$p_{k,k+n}(s,t) = \binom{n+k+\frac{\alpha}{\beta}-1}{n}\left(1-e^{-\beta\int_s^t \lambda(x)\mathrm{d}x}\right)^n e^{-(\alpha+\beta k)\int_s^t \lambda(x)\mathrm{d}x}.$$

As in Klugman et al. (2013, p.112), $p_{k,k+n}(s,t)$ is of negative binomial form if $\beta > 0$, and is of binomial form if $\beta < 0$ with $-\alpha/\beta$ a positive integer. Also, the secondary distribution in the compound LT representation (3.15) itself has LT (3.16), namely

$$\tilde{f}(z|k,s) = \frac{\beta\int_s^t \lambda(x)e^{\beta\int_s^x \lambda(y)\mathrm{d}y}\tilde{f}_t(z|k,s,x)\mathrm{d}x}{e^{\beta\int_s^t \lambda(y)\mathrm{d}y}-1},$$

as is easily verified.

Again, we remark that the usual contagion model results when $\lambda_m(x) = (\alpha + \beta m)\lambda(x)$ *for all* $m \in \mathbb{N}^+$, *and* $x \geq 0$. *Furthermore, the homogeneous case results with* $\lambda(x) = 1$.

Example 3. *The nonhomogeneous mixed Poisson case*
Let $U(\theta)$, $\theta > 0$ *be the df of a nonnegative rv, and define* $\Gamma_m(t) = \int_0^\infty \theta^m e^{-\theta t} \mathrm{d}U(\theta)$ *for* $m = k, k+1, \ldots$. *Then consider the intensity function for* $t \geq s$ *defined by*

$$\lambda_m(t) = r(t) \frac{\Gamma_{m+1}\left\{R(s) + \int_s^t r(x)\mathrm{d}x\right\}}{\Gamma_m\left\{R(s) + \int_s^t r(x)\mathrm{d}x\right\}}, \; m = k, k+1, \ldots,$$

where $\{r(x); s \leq x \leq t\}$ *and* $R(s)$ *are nonnegative. For the motivation of this assumption, see Section 7.2 of Klugman et al. (2013). Then* $\Gamma_{m+1}(t) = -\Gamma'_m(t)$, *which implies that* $\lambda_m(t) = -\frac{\mathrm{d}}{\mathrm{d}t} \ln \Gamma_m \left\{R(s) + \int_s^t r(x)\mathrm{d}x\right\}$. *Therefore, for* $u, v \geq s$,

$$\begin{aligned}\int_u^v \lambda_m(y)\mathrm{d}y &= -\int_u^v \frac{\mathrm{d}}{\mathrm{d}y} \ln \Gamma_m \left\{R(s) + \int_s^y r(x)\mathrm{d}x\right\} \mathrm{d}y \\ &= \ln \frac{\Gamma_m\left\{R(s) + \int_s^v r(x)\mathrm{d}x\right\}}{\Gamma_m\left\{R(s) + \int_s^u r(x)\mathrm{d}x\right\}},\end{aligned}$$

resulting in

$$e^{-\int_u^v \lambda_m(y)\mathrm{d}y} = \frac{\Gamma_m\left\{R(s) + \int_s^v r(x)\mathrm{d}x\right\}}{\Gamma_m\left\{R(s) + \int_s^u r(x)\mathrm{d}x\right\}},$$

and

$$p_{k,k}(s,t) = \frac{\Gamma_k\left\{R(s) + \int_s^t r(x)\mathrm{d}x\right\}}{\Gamma_k\left\{R(s)\right\}}.$$

Also, for $m \in \mathbb{N}^+$

$$\begin{aligned}
\phi_{k+m}(x|s) &= \lambda_{k+m-1}(x) e^{-\int_x^s \lambda_{k+m}(y)\mathrm{d}y - \int_s^x \lambda_{k+m-1}(y)\mathrm{d}y} \\
&= r(x) \frac{\Gamma_{k+m}\left\{R(s) + \int_s^x r(y)\mathrm{d}y\right\}}{\Gamma_{k+m-1}\left\{R(s) + \int_s^x r(y)\mathrm{d}y\right\}} \frac{\Gamma_{k+m}\left\{R(s)\right\}}{\Gamma_{k+m}\left\{R(s) + \int_s^x r(y)\mathrm{d}y\right\}} \\
&\times \frac{\Gamma_{k+m-1}\left\{R(s) + \int_s^x r(y)\mathrm{d}y\right\}}{\Gamma_{k+m-1}\left\{R(s)\right\}} \\
&= r(x) \frac{\Gamma_{k+m}\left\{R(s)\right\}}{\Gamma_{k+m-1}\left\{R(s)\right\}}.
\end{aligned}$$

Therefore, Theorem 8 can be applied with

$$\alpha_{m,k,s} = \frac{\Gamma_{k+m}\left\{R(s)\right\}}{\Gamma_{k+m-1}\left\{R(s)\right\}},$$

and $b(x|k,s) = r(x)$. Then

$$p_{k,k+n}(s,t) = \frac{\Gamma_{k+n}\left\{R(s) + \int_s^t r(y)\mathrm{d}y\right\}}{\Gamma_{k+n}\left\{R(s)\right\}} \frac{\Gamma_{k+n}\left\{R(s)\right\}}{\Gamma_k\left\{R(s)\right\}} \frac{\left\{\int_s^t r(y)\mathrm{d}y\right\}^n}{n!},$$

i.e.

$$p_{k,k+n}(s,t) = \frac{\Gamma_{k+n}\left\{R(s) + \int_s^t r(y)\mathrm{d}y\right\}}{\Gamma_k\left\{R(s)\right\}} \frac{\left\{\int_s^t r(y)\mathrm{d}y\right\}^n}{n!},$$

a formula which evidently holds for all $n \in \mathbb{N}$. The secondary LT as defined in (3.16) becomes

$$\tilde{f}_t(z|k,s) = \frac{\int_s^t \lambda_x \tilde{f}_t(z|k,s,x)\mathrm{d}x}{\int_s^t r(x)\mathrm{d}x},$$

in this case.

The ordinary nonhomogeneous mixed Poisson process results if

$$\lambda_m(t) = r(t) \frac{\Gamma_m\left\{\int_0^t r(x)\mathrm{d}x\right\}}{\Gamma_m\left\{\int_0^t r(x)\mathrm{d}x\right\}},$$

for all $m \in \mathbb{N}$ and $t > 0$. The corresponding homogeneous process results with $r(x)$ being constant. This process may also be formulated as a conditional Poisson process rather than as a birth process. See Klugman et al. (2013, Section 7.2) or Grandell (1997) for further details.

In this process, we only consider the situation after time s, no assumption is needed for the process before time s. However, this process covers the case when it follows nonhomogeneous mixed Poisson process since time 0. To be specific, if $\{N_t, t \geq 0\}$ follows nonhomogeneous mixed Poisson process, with intensity as $\theta r(t)$, and Θ has cdf $U(\theta)$. Then we have $\{N_t - N_s | N_s = k\}$ also follow nonhomogeneous mixed Poisson process, with intensity as $\Theta^* r(t)$, and Θ^* has df $\Pr(\Theta^* \leq x) = \int_0^x \frac{(\theta \int_0^s r(x)\mathrm{d}x)^k e^{-\theta \int_0^s r(x)\mathrm{d}x}}{k! p_k(s)} U(\mathrm{d}\theta)$ for $x > 0$.

In the following, we take a further look at a special case of (3.13), given by

$$\phi_m(x|s) = \alpha_{m,s} b(x|s), \tag{3.21}$$

for $m = k+1, k+2, \ldots$. It follows directly that, for $x \in [s,t]$ and $i = 1, 2, \ldots, n \in \mathbb{N}^+$,

$$\begin{aligned}
&\Pr(T_{i+k} \leq x | N_t = n+k, N_s = k) \\
&= \sum_{j=i}^{n} \Pr(N_x = j+k | N_t = n+k, N_s = k) \\
&= \sum_{j=i}^{n} \frac{\Pr(N_t = n+k | N_s = k, N_x = j+k) \Pr(N_x = j+k | N_s = k)}{\Pr(N_t = n+k | N_s = k)} \\
&= \sum_{j=i}^{n} \frac{\Pr(N_t = n+k | N_x = j+k) \Pr(N_x = j+k | N_s = k)}{\Pr(N_t = n+k | N_s = k)},
\end{aligned}$$

where the last line holds due to Markovian property of the process. By (3.14), we have

$$\begin{aligned}
&\Pr(T_{i+k} \leq x | N_t = n+k, N_s = k) \\
&= \sum_{j=i}^{n} \binom{n}{j} \frac{\left(\int_s^x b(y|s)\mathrm{d}y\right)^j \left(\int_x^t b(y|s)\mathrm{d}y\right)^{n-j}}{\left(\int_s^t b(y|s)\mathrm{d}y\right)^n} \left\{ e^{\int_s^x [\lambda_{k+n}(y) - \lambda_{k+j}(y)]\mathrm{d}y} \prod_{m=j+1}^{n} \frac{\alpha_{m+k,x} \int_x^t b(y|x)\mathrm{d}y}{\alpha_{m+k,s} \int_x^t b(y|s)\mathrm{d}y} \right\},
\end{aligned}$$

where the last term can be re-written, by (3.4), as

$$
\begin{aligned}
&e^{\int_s^x [\lambda_{k+n}(y)-\lambda_{k+j}(y)]\mathrm{d}y} \prod_{m=j+1}^{n} \frac{\alpha_{m+k,x} \int_x^t b(y|x)\mathrm{d}y}{\alpha_{m+k,s} \int_x^t b(y|s)\mathrm{d}y} \\
&= \prod_{m=j+1}^{n} e^{\int_s^x [\lambda_{k+m}(y)-\lambda_{k+m-1}(y)]\mathrm{d}y} \frac{\int_x^t \lambda_{k+m-1}(y) e^{\int_x^y [\lambda_{k+m}(z)-\lambda_{k+m-1}(z)]\mathrm{d}z}\mathrm{d}y}{\int_x^t \lambda_{k+m-1}(y) e^{\int_s^y [\lambda_{k+m}(z)-\lambda_{k+m-1}(z)]\mathrm{d}z}\mathrm{d}y} = 1.
\end{aligned}
$$

Hence, it follows that

$$
\Pr(T_{i+k} \leq x | N_t = n+k, N_s = k) = \sum_{j=i}^{n} \binom{n}{j} \frac{\left(\int_s^x b(y|s)\mathrm{d}y\right)^j \left(\int_x^t b(y|s)\mathrm{d}y\right)^{n-j}}{\left(\int_s^t b(y|s)\mathrm{d}y\right)^n}.
$$

Therefore, conditional on $N_t = n+k$, $N_s = k$, the $(k+i)$th claim occurrence time is distributed as the ith order statistic of n iid rv's with df $\int_s^x b(y|s)\mathrm{d}y / \int_s^t b(y|s)\mathrm{d}y$ for $i = 1, 2, \ldots, n$ and $x \in [s, t]$. Then, the resulting process $\{N_t\}_{t \geq s}$ conditional on $N_s = k$ has the order statistic property.

3.2 A birth process model with "two stages"

In this section, we indicate how the representation in Theorem 8 is modified if the factorization (3.13) changes form but (3.9) holds. We have the following result.

Corollary 9. *Suppose* (3.9) *holds, and* $\lambda_m(x)$ *for* $m = k, k+1, \ldots$ *is such that*

$$
\phi_{k+m}(x|s) = \begin{cases} \alpha_{m,k,s} b_1(x|k,s) & m \leq n_0, \\ \alpha_{m,k,s} b_2(x|k,s) & m > n_0. \end{cases} \tag{3.22}
$$

Then, the LT of the total claim size may be expressed as

$$\mathbb{E}[e^{-zS_{s,t}}|N_s = k] = \sum_{n=0}^{n_0} p_{k,k+n}(s,t)\,[\tilde{g}_{1,t}(z|k,s)]^n$$
$$+ \sum_{n=n_0+1}^{\infty} p_{k,k+n}(s,t) \int_s^t c_n(x|k,s,t)\,[\tilde{g}_{1,x}(z|k,s)]^{n_0}\,[\tilde{g}_{2,x}(z|k,s)]^{n-n_0-1}\,\tilde{f}_t(z|k,s,x)\mathrm{d}x, \tag{3.23}$$

where

$p_{k,k+n}(s,t)$

$$= \begin{cases} e^{-\int_s^t \lambda_k(y)\mathrm{d}y} & n = 0, \\ e^{-\int_s^t \lambda_{k+n}(y)\mathrm{d}y} \prod_{m=1}^n \alpha_{m,k,s} \frac{(\int_s^t b_1(x|k,s)\mathrm{d}x)^n}{n!} & 0 < n \le n_0, \\ e^{-\int_s^t \lambda_{k+n}(y)\mathrm{d}y} \prod_{m=1}^n \alpha_{m,k,s} \int_s^t \frac{(\int_s^x b_1(y|k,s)\mathrm{d}y)^{n_0} (\int_x^t b_2(y|k,s)\mathrm{d}y)^{n-n_0-1}}{n_0!(n-n_0-1)!} b_2(x|k,s)\mathrm{d}x & n > n_0, \end{cases}$$

and

$$\tilde{g}_{1,x}(z|k,s) = \frac{\int_s^x b_1(y|k,s)\tilde{f}_t(z|k,s,y)\mathrm{d}y}{\int_s^x b_1(y|k,s)\mathrm{d}y}, \quad s < x \le t, \tag{3.24}$$

$$\tilde{g}_{2,x}(z|k,s) = \frac{\int_x^t b_2(y|k,s)\tilde{f}_t(z|k,s,y)\mathrm{d}y}{\int_x^t b_2(y|k,s)\mathrm{d}y}, \quad s < x < t, \tag{3.25}$$

$$c_n(x|k,s,t) = \frac{(\int_s^x b_1(y|k,s)\mathrm{d}y)^{n_0}(\int_x^t b_2(y|k,s)\mathrm{d}y)^{n-n_0-1} b_2(x|k,s)}{\int_s^t \left[(\int_s^x b_1(y|k,s)\mathrm{d}y)^{n_0}(\int_x^t b_2(y|k,s)\mathrm{d}y)^{n-n_0-1}\right] b_2(x|k,s)\mathrm{d}x}. \tag{3.26}$$

Proof. Recall that $h_{n,t}(t_{k+1}, t_{k+2}, \ldots, t_{k+n}|k,s) = e^{-\int_s^t \lambda_{k+n}(y)\mathrm{d}y} \prod_{m=1}^n \phi_{k+m}(t_{k+m})$. For $n \le n_0$, the process is unaffected by the change of factorization and Theorem 8 applies. Thus, we focus on the case when $n > n_0$. Indeed, substituting the two-stage factorization $\phi_{k+m}(x|s)$ into (3.3) results in

$$h_{n,t}(t_{k+1}, t_{k+2}, \ldots, t_{k+n}|k,s) = e^{-\int_s^t \lambda_{k+n}(y)\mathrm{d}y} \prod_{m=1}^n \alpha_{m,k,s} \left\{ \prod_{m=1}^{n_0} b_1(t_{k+m}|k,s) \right\} \prod_{m=n_0+1}^{n} b_2(t_{k+m}|k,s). \tag{3.27}$$

In turn, by using (3.27), the integration of (3.18) becomes

$$\int_s^t \int_s^{t_{k+n}} \cdots \int_s^{t_{k+2}} h_{n,t}(t_{k+1}, t_{k+2}, \ldots, t_{k+n}|k, s) \prod_{m=1}^{n} \tilde{f}_t(z|k, s, t_{k+m}) \mathrm{d}t_{k+m}$$
$$= \int_s^t \int_s^{t_{k+n}} \ldots \int_s^{t_{k+n_0+2}} \left(\int_s^{t_{k+n_0+1}} \ldots \int_s^{t_{k+2}} \prod_{m=1}^{n_0} b_1(t_{k+m}|k, s) \tilde{f}_t(z|k, s, t_{k+m}) \mathrm{d}t_{k+m} \right)$$
$$\times \left(\prod_{m=n_0+1}^{n} b_2(t_{k+m}|k, s) \tilde{f}_t(z|k, s, t_{k+m}) \mathrm{d}t_{k+m} \right) e^{-\int_s^t \lambda_{k+n}(y)\mathrm{d}y} \prod_{m=1}^{n} \alpha_{m,k,s} \tag{3.28}$$

Rearranging the integration field in (3.28) results in

$$e^{-\int_s^t \lambda_{k+n}(y)\mathrm{d}y}$$
$$\times \prod_{m=1}^{n} \alpha_{m,k,s} \int_s^t \int_{t_{k+n_0+1}}^{t} \int_{t_{k+n_0+1}}^{t_{k+n}} \cdots \int_{t_{k+n_0+1}}^{t_{k+n_0+3}} \left[\prod_{m=n_0+2}^{n} b_2(t_{k+m}|k, s) \tilde{f}_t(z|k, s, t_{k+m}) \mathrm{d}t_{k+m} \right]$$
$$\times \frac{\left(\int_s^{t_{k+n_0+1}} b_1(x|k, s) \tilde{f}_t(z|k, s, x) \mathrm{d}x \right)^{n_0}}{n_0!} b_2(t_{k+n_0+1}|k, s) \tilde{f}_t(z|k, s, t_{k+n_0+1}) \mathrm{d}t_{k+n_0+1}. \tag{3.29}$$

Using the similar logic as that in proof of Theorem 8, (3.29) can be simplified to

$$\int_s^t \int_s^{t_{k+n}} \cdots \int_s^{t_{k+2}} h_{n,t}(t_{k+1}, t_{k+2}, \ldots, t_{k+n}|k, s) \prod_{m=1}^{n} \tilde{f}_t(z|k, s, t_{k+m}) \mathrm{d}t_{k+m}$$
$$= \int_s^t \frac{\left(\int_s^x b_1(y|k, s) \tilde{f}_t(z|k, s, y) \mathrm{d}y \right)^{n_0} \left(\int_x^t b_2(y|k, s) \tilde{f}_t(z|k, s, y) \mathrm{d}y \right)^{n-n_0-1}}{n_0!(n - n_0 - 1)!} b_2(x|k, s) \tilde{f}_t(z|k, s, x) \mathrm{d}x$$
$$\times e^{-\int_s^t \lambda_{k+n}(y)\mathrm{d}y} \prod_{m=1}^{n} \alpha_{m,k,s}.$$

Thus, from (3.10), the LT of $S_{s,t}$ is given by

$$\begin{aligned}
&\mathbb{E}[e^{-zS_{s,t}}|N_s = k] \\
&= e^{-\int_s^t \lambda_k(y)\mathrm{d}y} + \sum_{n=1}^{n_0} e^{-\int_s^t \lambda_{k+n}(y)\mathrm{d}y} \left(\prod_{m=1}^{n} \alpha_{m,k,s}\right) \frac{(\int_s^t b_1(x|k,s)\tilde{f}_t(z|k,s,x)\mathrm{d}x)^n}{n!} \\
&+ \sum_{n=n_0+1}^{\infty} \left[\int_s^t \frac{(\int_s^x b_1(y|k,s)\tilde{f}_t(z|k,s,y)\mathrm{d}y)^{n_0}(\int_x^t b_2(y|k,s)\tilde{f}_t(z|k,s,y)\mathrm{d}y)^{n-n_0-1}}{n_0!(n-n_0-1)!}\right. \\
&\left.\times b_2(x|k,s)\tilde{f}_t(z|k,s,x)\mathrm{d}x\right] e^{-\int_s^t \lambda_{k+n}(y)\mathrm{d}y} \prod_{m=1}^{n} \alpha_{m,k,s}. \qquad (3.30)
\end{aligned}$$

The pgf of the number of claims during (s,t) given there are k claim occurrences up to time s is easily obtained by replacing the LT of the claim severity by z, namely

$$\begin{aligned}
P_{k,s,t}(z) &= e^{-\int_s^t \lambda_k(y)\mathrm{d}y} + \sum_{n=1}^{n_0} e^{-\int_s^t \lambda_{k+n}(y)\mathrm{d}y} \prod_{m=1}^{n} \alpha_{m,k,s} \frac{(\int_s^t b_1(x|k,s)\mathrm{d}x)^n}{n!} z^n \\
&+ \sum_{n=n_0+1}^{\infty} e^{-\int_s^t \lambda_{k+n}(y)\mathrm{d}y} \prod_{m=1}^{n} \alpha_{m,k,s} \int_s^t \frac{(\int_s^x b_1(y|k,s)\mathrm{d}y)^{n_0}(\int_x^t b_2(y|k,s)\mathrm{d}y)^{n-n_0-1}}{n_0!(n-n_0-1)!} b_2(x|k,s)\mathrm{d}x z^n.
\end{aligned}$$

Thus coefficient of z^n immediately leads to the pmf in Corollary 9. After substituting $p_{k,k+n}(s,t)$ into (3.30), (3.23) is easily verified. $\square$

As a special case when $b_1(x|k,s) = b_2(x|k,s) = b(x|k,s)$, replacing $\int_s^x b(y|k,s)\tilde{f}_t(z|k,s,y)\mathrm{d}y$ by $A(x)\int_s^t b(y|k,s)\tilde{f}_t(z|k,s,y)\mathrm{d}y$, with $A(s) = 0$ and $A(t) = 1$ results in

$$\begin{aligned}
&\int_s^t \left[\frac{(\int_s^x b_1(y|k,s)\tilde{f}_t(z|k,s,y)\mathrm{d}y)^{n_0}}{n_0!} \frac{(\int_x^t b_2(y|k,s)\tilde{f}_t(z|k,s,y)\mathrm{d}y)^{n-n_0-1}}{(n-n_0-1)!}\right] b_2(x|k,s)\tilde{f}_t(z|k,s,x)\mathrm{d}x \\
&= \left[\int_s^t b(y|k,s)\tilde{f}_t(z|k,s,y)\mathrm{d}y\right]^n \int_s^t \frac{A(x)^{n_0}(1-A(x))^{n-n_0-1}}{n_0!(n-n_0-1)!}\mathrm{d}A(x).
\end{aligned}$$

The second integral part in the last line is the pdf of a Beta$(n_0+1, n-n_0)$ distribution. Since the domain of Beta distribution is $(0,1)$, the second integral has a value of 1. Thus (3.29) is simplified to be equivalent to (3.20). Therefore, the result of Theorem 8 may be recovered from Corollary 9. Clearly, more than two stages may be handled in a similar manner, but the details are tedious and omitted here.

3.3 The mixed Erlang properties

It is generally very difficult to pursue explicit expressions for the quantities associated with the aggregate claims. As illustrated in Section 2.3.1, the use of mixed Erlang distributions provides a better approach to address the problem. Given $N_s = k$ and $T_{k+i} = x$, we assume the corresponding individual claim X_{k+i} in (3.8) is mixed Erlang distributed, whose LT is given as

$$\tilde{f}_t(z|k,s,x) = Q_x\left(\frac{\beta_{x,s,t}}{z+\beta_{x,s,t}}\middle| k,s,t\right) \quad \text{for } s<x<t, \tag{3.31}$$

with the pgf $Q_x(z|k,s,t) = \sum_{n=0}^{\infty} q_n(x|k,s,t)z^n$. We also assume that $\beta_{s,t} = \sup_{s\le x\le t}\beta_{x,s,t} < \infty$ and set $\alpha_{x,s,t} = \beta_{x,s,t}/\beta_{s,t}$. Thus,

$$Q_x\left(\frac{\beta_{x,s,t}}{z+\beta_{x,s,t}}\middle| k,s,t\right) = Q_x\left(\frac{\alpha_{x,s,t}\frac{\beta_{s,t}}{z+\beta_{s,t}}}{1-(1-\alpha_{x,s,t})\frac{\beta_{s,t}}{z+\beta_{s,t}}}\middle| k,s,t\right) = R_x\left(\frac{\beta_{s,t}}{z+\beta_{s,t}}\middle| k,s,t\right),$$

with

$$R_x(z|k,s,t) = \sum_{j=0}^{\infty} r_j(x|k,s,t)z^j = Q_x\left(\frac{\alpha_{x,s,t}z}{1-(1-\alpha_{x,s,t})z}\middle| k,s,t\right).$$

We find that $R_x(z|k,s,t)$ is a compound distribution with a zero-truncated geometric distribution being its secondary distribution. After some easy mathematical manipulations, we have that $r_0(x|k,s,t) = q_0(x|k,s,t)$, and for $j \in \mathbb{N}^+$,

$$r_j(x|k,s,t) = \sum_{i=1}^{j}\binom{j-1}{i-1}q_i(x|k,s,t)(\alpha_{x,s,t})^i(1-\alpha_{x,s,t})^{j-i}.$$

See also Klugman et al. (2013, p.162). We further assume that (3.9) holds, then from (3.10), (3.11) and (3.12), our total claim amount $S_{s,t}$ in (3.8) is also mixed Erlang distributed, expressed as

$$\mathbb{E}[e^{-zS_{s,t}}|N_s = k] = Q_{k,s,t}\left(\frac{\beta_{s,t}}{z+\beta_{s,t}}\right), \tag{3.32}$$

where

$$Q_{k,s,t}(z) = p_{k,k}(s,t) + \sum_{n=1}^{\infty} p_{k,k+n}(s,t) Q_n^*(z|k,s,t),$$

with $Q_n^*(z|k,s,t)$ defined as

$$Q_n^*(z|k,s,t) = \frac{\int_s^t \int_s^{t_{k+n}} \dots \int_s^{t_{k+2}} h_{n,t}(t_{k+1},\dots,t_{k+n}|k,s) \left(\prod_{m=1}^n R_{t_{k+m}}(z|k,s,t)\right) \mathrm{d}t_{k+1}\dots \mathrm{d}t_{k+n}}{\int_s^t \int_s^{t_{k+n}} \dots \int_s^{t_{k+2}} h_{n,t}(t_{k+1},\dots,t_{k+n}|k,s) \mathrm{d}t_{k+1}\dots \mathrm{d}t_{k+n}}. \tag{3.33}$$

Now, we apply the mixed Erlang properties in two-stage models, i.e. by (3.22), to specify the mixing weights.

Proposition 10. *If the individual time-dependent claim value is mixed Erlang distributed as defined in* (3.31)*, then total claim amount $S_{s,t}$ defined by* (3.8) *in the two-stage models is also mixed Erlang distributed, i.e.*

$$\mathbb{E}[e^{-zS_{s,t}}|N_s = k] = Q_{k,s,t}\left(\frac{\beta_{s,t}}{z+\beta_{s,t}}\right) = \sum_{n=0}^{\infty} q_n(k,s,t) \left[\frac{\beta_{s,t}}{z+\beta_{s,t}}\right]^n,$$

whose mixing weights $q_n(k,s,t)$ are given as

$$\begin{aligned} &\sum_{m=n_0+1}^{\infty} p_{k,k+m} \int_s^t c_m(x|k,s,t) \left(q_1^{*n_0}(x|k,s,t) * q_2^{*(m-n_0-1)}(x|k,s,t) * r(x|k,s,t)\right)_n \mathrm{d}x \\ &+ \sum_{m=1}^{n_0} p_{k,k+m} (q_1^{*m})_n(t|k,s,t), \end{aligned} \tag{3.34}$$

for $n \in \mathbb{N}^+$, and

$$\begin{aligned} q_0(k,s,t) = &\sum_{n=n_0+1}^{\infty} p_{k,k+n} \int_s^t c_n(x|k,s,t) q_{1,0}(x|k,s,t)^{n_0} q_{2,0}(x|k,s,t)^{n-n_0-1} r_0(x|k,s,t) \mathrm{d}x \\ &+ \sum_{n=0}^{n_0} p_{k,k+n} q_{1,0}(t|k,s,t)^n, \end{aligned} \tag{3.35}$$

where, for $s < x \le t$ and $j = 0, 1, \ldots,$

$$q_{1,j}(x|k,s,t) = \frac{\int_s^x b_1(y|k,s) r_j(y|k,s,t)\mathrm{d}y}{\int_s^x b_1(y|k,s)\mathrm{d}y},$$
$$q_{2,j}(x|k,s,t) = \frac{\int_x^t b_2(y|k,s) r_j(y|k,s,t)\mathrm{d}y}{\int_x^t b_2(y|k,s)\mathrm{d}y}.$$

Proof. As shown in (3.33), we need to identify $Q_n^*(z|k,s,t)$ in order to find the mixing weight of the mixed Erlang distribution. From (3.23), (3.24), (3.25) and (3.26), we have

$$Q_n^*(z|k,s,t) = \begin{cases} Q_{t,1}^n(z|k,s,t), & n \le n_0 \\ \int_s^t c_n(x|k,s,t) Q_{x,1}(z|k,s,t)^{n_0} Q_{x,2}(z|k,s,t)^{n-n_0-1} R_x(z|k,s,t)\mathrm{d}x, & n \ge n_0+1, \end{cases} \tag{3.36}$$

where

$$Q_{x,1}(z|k,s,t) = \sum_{j=0}^{\infty} q_{1,j}(x|k,s,t) z^j = \frac{\int_s^x b_1(y|k,s) R_y(z|k,s,t)\mathrm{d}y}{\int_s^x b_1(y|k,s)\mathrm{d}y},$$
$$Q_{x,2}(z|k,s,t) = \sum_{j=0}^{\infty} q_{2,j}(x|k,s,t) z^j = \frac{\int_x^t b_2(y|k,s) R_y(z|k,s,t)\mathrm{d}y}{\int_x^t b_2(y|k,s)\mathrm{d}y},$$

and $c_n(x|k,s,t)$ defined by (3.26). Expanding (3.36) and equating the coefficient of z^n yield (3.34) and (3.35) directly. □

In the case where (3.13) holds, i.e. $\phi_{k+m}(x|s) = \alpha_{m,k,s} b(x|k,s)$ for $m \ge 1$, (3.32) can be simplified to

$$\mathbb{E}[e^{-zS_{s,t}}|N_s = k] = P_{k,s,t}\left[Q\left(\left.\frac{\beta_{s,t}}{z+\beta_{s,t}}\right| k,s,t\right)\right],$$

where $Q(z|k,s,t) = \frac{\int_s^t b(x|k,s) R_x(z|k,s,t)\mathrm{d}x}{\int_s^t b(x|k,s)\mathrm{d}x}$, by (3.15) and (3.16).

Thus, we can express the aggregate claim $S_{s,t}$ as a mixed Erlang distributed rv with the mixing weight distributed as a compound distribution of a compound distribution. The mixed Erlang properties obtained can be applied to address the inflation model and payment delay model, details are omitted here.

Chapter 4

IBNR Claims In Renewal Models

Chapter 3 analyzed the aggregate claims under the models where claim severities and claim occurrence times are dependent. Starting from this chapter, we work under a renewal risk model with dependence between interclaim times and claim severities, also known as a dependent Sparre Andersen risk model.

IBNR claims are of central importance in claim reserving. Indeed, insurers should make adequate provision for the total amount of claims incurred but not yet reported to the insurer at fixed point in time. This chapter considers the randomness of some important factors in this context, such as reporting lags and incurred claim sizes. Furthermore, the independence assumption made in Section 9.3 of Klugman et al. (2013) is overly strong, the claim amounts and delays are known to be highly correlated in insurance practice, and thus the model is not appropriate. To address this issue, a dependency structure among interclaim times, claim severities and delays are incorporated into the analysis of the IBNR claim problem in this chapter.

LTs are utilized to characterize the total discounted IBNR claim amounts, and the IBNR claim number is further examined using pgfs.

Definition 7. *The claim causing event process $\{N_t\}_{t\geq 0}$ is an ordinary renewal process defined through the interarrival times $\{\tau_k\}_{k\geq 1}$, which form a sequence of iid rv's having common df*

$F_\tau(\cdot) = 1 - \bar{F}_\tau(\cdot)$ *and LT* $\tilde{f}_\tau(s)$.

Renewal-type risk models consider processes with independent interarrival times (see, e.g., Andersen, 1957). Moreover, we further introduce a series of definitions with their underlying assumptions.

1. The batch size C_x corresponds to the number of independent claims generated by the claim causing event at time x. The batch sizes $\{C_x\}_{x\geq 0}$ have a common pgf $B(z) = \mathbb{E}[z^{C_x}]$. Batch sizes at different time points are mutually independent (i.e. C_x and C_y are independent for $x \neq y$). They are also assumed to be independent of any other rv's in the risk model.

2. Let $W_{i,k}$ be the reporting lag of the ith claim in the kth claim causing event. The rv's $\{W_{i,k}\}_{i,k\in\mathbb{N}^+}$ are assumed to be iid with df $K(\cdot)$.

3. Let $X_{i,k}$ be the deflated (or baseline) severity of the ith claim in the kth claim causing event. The nonnegative rv's $\{X_{i,k}\}_{i,k\in\mathbb{N}^+}$ form a sequence of iid rv's with df $P(\cdot)$ and LT $\tilde{p}(s) = \mathbb{E}\left[e^{-sX_{i,k}}\right] = \int_0^\infty e^{-sx}P(\mathrm{d}x)$ for $s \geq 0$.

We assume that all claim severities and reporting lags in different claim causing events are independent (i.e. $X_{j,n}$ and $W_{i,k}$ are independent rv's if $i \neq j$ or $k \neq n$). Also, the random vectors $(\tau_k, W_{i,k}, X_{i,k})_{k\in\mathbb{N}^+}$ are mutually independent with common joint df J, which we conventionally express as

$$J(t, w, x) = F(t)K_{W|\tau}(w|t)P_{X|\tau,W}(x|t, w), \tag{4.1}$$

for $t, w, x \geq 0$, where $P_{X|\tau,W}$ is the df of $X_{i,k}|(\tau_k, W_{i,k})$, and $K_{W|\tau}$ is the df of $W_{i,k}|\tau_k$. Furthermore, we let $\tilde{p}_{X|\tau,W}$ be the LT of $X_{i,k}|(\tau_k, W_{i,k})$, and $\mu_n(t, w) = \mathbb{E}[X^n_{i,k}|\tau_k = t, W_{i,k} = w]$ for $n \in \mathbb{N}$. Note that the claim arrival dynamic described above is a generalization of the one governing the dependent Sparre Andersen model (e.g., Cheung et al. (2010)).

To introduce the total discounted IBNR claim amount of interest here, first, we assume that the time-0 value of a (deflated) claim of amount y occurring at x and reported at time $x+w$

is $e^{-\delta x}l(w)y$, where δ is the constant force of interest (net of inflation) and l is a non-negative function of the reporting lag. Thus, the total discounted IBNR claim amount $Z(t)$ is defined as

$$Z(t) = \sum_{k=1}^{N_t} \sum_{i=1}^{C_{T_k}} e^{-\delta T_k} l(W_{i,k}) \mathbf{1}_{\{W_{i,k}+T_k>t\}} X_{i,k}, \tag{4.2}$$

for $\delta \geq 0$ with LT $L_\gamma(t) = \mathbb{E}\left[e^{-\gamma Z(t)}\right]$, which is known to exist on (at least) $\gamma \geq 0$. A possible candidate for l is $l(w) = e^{-\epsilon w}$ for $w, \epsilon \geq 0$, which corresponds to the situation where both the force of interest and inflation rate are assumed constant. Other relevant applications for the function l (e.g., l may be censored at a given reporting lag threshold) can be found in Huang et al. (2015). Also, note that the renewal sum of discounted claim amounts studied by e.g., Léveillé and Garrido (2001a, 2001b) and Léveillé and Adékambi (2011), is a special case of (4.2) with infinitely long reporting lags (i.e. $W_{i,k} = \infty$ a.s. for $i, k \in \mathbb{N}^+$), $l(\cdot) = 1$, and $B(z) = z$.

The remainder of this chapter is structured as follows. In Section 4.1, the generic model for the total discounted IBNR claim amounts is formally introduced and analyzed under a generic dependence structure between the interclaim times, claim severities and reporting lags. An expression for the LT of the total discounted IBNR claim amount is derived, and recursive formulas for the moments are then obtained using their defective renewal equation representation. Also, we later analyze the joint distribution of the IBNR claim amounts and the incurred and reported claim amounts at different time points. Section 4.2 pays special attention to the IBNR claim number problem when the claim arrival process follows a compound Poisson process. Section 4.3 studies the IBNR claim number when batch arrivals are of size 1. The tractability of the model when reporting lags are distributed as a mixture of exponentials is later shown.

4.1 Total discounted IBNR claim amount

In this section, the total discounted IBNR claim amount $Z(t)$ is analyzed through LTs.

4.1.1 Moments of IBNR claims

Following similar arguments as in Léveillé and Garrido (2001b), we condition on the characteristics of the first claim causing event and make use of the representation (4.1) for the joint df J. It follows that

$$L_\gamma(t) = \bar{F}(t) + \int_0^t L_{\gamma e^{-\delta x}}(t-x) B\left(1 + \int_{t-x}^\infty \left[\tilde{p}_{X|\tau,W}(\gamma l(w) e^{-\delta x}|x,w) - 1\right] K_{W|\tau}(\mathrm{d}w|x)\right) F(\mathrm{d}x). \tag{4.3}$$

Assuming that the moments of $Z(t)$ exist, differentiating n times (4.3) wrt γ and evaluating the resulting equation at $\gamma = 0$, one obtains

$$\mathbb{E}[Z^n(t)] = \int_0^t e^{-n\delta x} \mathbb{E}[Z^n(t-x)] F(\mathrm{d}x) + v_n(t), \tag{4.4}$$

for $n \geq 1$, where

$$v_n(t) = \sum_{m=1}^n \binom{n}{m} \int_0^t e^{-n\delta x} \mathbb{E}[Z^{n-m}(t-x)] \sum_{k=1}^m B^{(k)}(1) B_{m,k}\left(\xi_1(x,t), \ldots, \xi_{m-k+1}(x,t)\right) F(\mathrm{d}x), \tag{4.5}$$

$B^{(k)}(\cdot)$ is the kth-order derivative of $B(\cdot)$ and

$$\xi_i(x,t) = \int_{t-x}^\infty l(w)^i \mu_i(x,w) K_{W|\tau}(dw|x), \quad 0 \leq x \leq t, \ i = 1, 2, \ldots, n.$$

Also, $B_{m,k}$ holds for Bell's polynomial (e.g., Bell (1927) and Johnson (2002)) defined as

$$B_{m,k}(x_1, x_2, \ldots, x_{m-k+1}) = \sum \frac{m!}{j_1! j_2! \ldots j_{m-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{m-k+1}}{(m-k+1)!}\right)^{j_{m-k+1}}, \tag{4.6}$$

for $m = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$. Note that the sum in (4.6) is taken over all sequences $j_1, j_2, j_3, \ldots, j_{m-k+1}$ of non-negative integers such that

$$j_1 + j_2 + \ldots + j_{m-k+1} = k; \quad j_1 + 2j_2 + 3j_3 + \ldots + (m-k+1) j_{m-k+1} = m.$$

For $n \in \mathbb{N}^+$, the moments $\{\mathbb{E}[Z^n(t)]\}_{t \geq 0}$ satisfy the defective (proper) renewal equation (4.4) when $\delta > 0$ ($\delta = 0$). Results pertaining to renewal equations are discussed in great length in, e.g., Feller (1971) and Resnick (2002). In Theorem 11, a recursive formula for the moments of $Z(t)$ is obtained, which involves the renewal function $\{H(t)\}_{t\geq 0}$ of the renewal process $\{N_t\}_{t\geq 0}$ satisfying

$$\int_0^\infty e^{-sx} H(\mathrm{d}x) = \frac{\tilde{f}(s)}{1-\tilde{f}(s)}, \qquad s > 0. \tag{4.7}$$

Theorem 11. *For $n \in \mathbb{N}^+$, the nth moment of $Z(t)$ is given recursively as*

$$\mathbb{E}[Z^n(t)] = v_n(t) + \int_0^t e^{-n\delta x} v_n(t-x) H(\mathrm{d}x), \tag{4.8}$$

where $\mathbb{E}[Z^0(t)] = 1$ for $t > 0$.

Proof. Letting $M_n(t) = \mathbb{E}[Z^n(t)]$, Equation (4.4) is a renewal equation of the form

$$M_n(t) = \tilde{f}(n\delta) \int_0^t M_n(t-x) F_{n\delta}(\mathrm{d}x) + v_n(t), \tag{4.9}$$

for $n \in \mathbb{N}^+$, where $F_{n\delta}(\mathrm{d}x) = e^{-n\delta x} F(\mathrm{d}x)/\tilde{f}(n\delta)$ is the Esscher transform of the df F (see, e.g., Gerber and Shiu (1994)). Taking the LT on both sides of (4.9) leads to

$$\tilde{M}_n(s) = \frac{\tilde{v}_n(s)}{1-\tilde{f}(n\delta+s)}, \tag{4.10}$$

where $\tilde{M}_n(s) = \int_0^\infty e^{-sx} M_n(x)\mathrm{d}x$ and $\tilde{v}_n(s) = \int_0^\infty e^{-st} v_n(t)\mathrm{d}t$ for $s > 0$. It is immediate from (4.7) that (4.10) can be re-expressed as

$$\tilde{M}_n(s) = \tilde{v}_n(s) \int_0^\infty e^{-sx} e^{-n\delta x} H(\mathrm{d}x) + \tilde{v}_n(s). \tag{4.11}$$

Inversion of (4.11) results in (4.8). □

In Theorem 11, the nth moment of the total discounted IBNR claim amount is expressed in terms of v_n which in turn is characterized by the lower-order moments of $Z(t)$. Given that v_n itself is of an integral form, the double integral representation (4.8) can be simplified when interarrival times are independent of the claim severities and reporting lags, i.e.

$P_{X|\tau,W}(x|t,w) = P_{X|W}(x|w)$ and $K_{W|\tau}(w|t) = K(w)$ for all $x, w, t \geq 0$. The joint distribution (4.1) then becomes

$$J(t,w,x) = F(t)K(w)P_{X|W}(x|w), \tag{4.12}$$

for $t, w, x \geq 0$.

Corollary 12. *If* (4.12) *holds, the nth moment of $Z(t)$ can be expressed as*

$$\mathbb{E}[Z^n(t)] = \sum_{m=1}^{n} \binom{n}{m} \int_0^t e^{-n\delta x} \mathbb{E}[Z^{n-m}(t-x)] \Xi_m(t-x) H(\mathrm{d}x), \tag{4.13}$$

where

$$\Xi_m(x) = \sum_{k=1}^{m} B^{(k)}(1) B_{m,k}\left(\xi_1(x), \xi_2(x), \ldots, \xi_{m-k+1}(x)\right),$$

with $\xi_i(x) = \int_x^\infty l(w)^i \mu_i(w) K(dw)$, *and* $\mathbb{E}[X_{j,k}^i | \tau_k = t, W_{j,k} = w] =: \mu_i(w)$ *for all* $j, k \in \mathbb{N}^+$.

Proof. Under (4.12), (4.5) can be rewritten as

$$v_n(t) = \sum_{m=1}^{n} \binom{n}{m} \int_0^t e^{-n\delta x} \mathbb{E}[Z^{n-m}(t-x)] \Xi_m(t-x) F(\mathrm{d}x). \tag{4.14}$$

From (4.7), it is clear that $H(\mathrm{d}x) = F(\mathrm{d}x) + \int_0^x F(\mathrm{d}x - y) H(\mathrm{d}y)$, which allows to rewrite (4.14) as

$$\begin{aligned} v_n(t) = &\sum_{m=1}^{n} \binom{n}{m} \int_0^t e^{-n\delta x} \mathbb{E}[Z^{n-m}(t-x)] \Xi_m(t-x) H(\mathrm{d}x) \\ &- \sum_{m=1}^{n} \binom{n}{m} \int_0^t e^{-n\delta x} \mathbb{E}[Z^{n-m}(t-x)] \Xi_m(t-x) \int_0^x F(\mathrm{d}x - y) H(\mathrm{d}y). \end{aligned} \tag{4.15}$$

By interchanging the order of integration and later using (4.14), the last term of (4.15) becomes

$$\begin{aligned}
&\sum_{m=1}^{n}\binom{n}{m}\int_0^t e^{-n\delta x}\mathbb{E}[Z^{n-m}(t-x)]\Xi_m(t-x)\int_0^x F(\mathrm{d}x-y)H(\mathrm{d}y)\\
&=\sum_{m=1}^{n}\binom{n}{m}\int_0^t\int_y^t e^{-n\delta x}\mathbb{E}[Z^{n-m}(t-x)]\Xi_m(t-x)F(\mathrm{d}x-y)H(\mathrm{d}y)\\
&=\sum_{m=1}^{n}\binom{n}{m}\int_0^t\int_0^{t-y} e^{-n\delta x-n\delta y}\mathbb{E}[Z^{n-m}(t-y-x)]\Xi_m(t-y-x)F(\mathrm{d}x)H(\mathrm{d}y)\\
&=\int_0^t e^{-n\delta x}\upsilon_n(t-x)H(\mathrm{d}x). \qquad (4.16)
\end{aligned}$$

Substituting (4.16) into (4.15) yields

$$\upsilon_n(t)=\sum_{m=1}^{n}\binom{n}{m}\int_0^t e^{-n\delta x}\mathbb{E}[Z^{n-m}(t-x)]\Xi_m(t-x)H(\mathrm{d}x)-\int_0^t e^{-n\delta x}\upsilon_n(t-x)H(\mathrm{d}x). \qquad (4.17)$$

Equation (4.13) is obtained by further substituting (4.17) into (4.8). □

Furthermore, if all batch sizes are of size 1 (i.e., $B(z)=z$) and $l(w)=e^{-\epsilon w}$ for $\epsilon\geq 0$, (4.13) becomes

$$\mathbb{E}\left[Z^n(t)\right]=\sum_{m=1}^{n}\binom{n}{m}\int_0^t e^{-n\delta x}\mathbb{E}[Z^{n-m}(t-x)]\left[\int_{t-x}^{\infty}e^{-m\epsilon w}\mu_m(w)K(\mathrm{d}w)\right]H(\mathrm{d}x), \qquad (4.18)$$

for $n\in\mathbb{N}^+$, which is a generalization of Equation (2.2) of Léveillé and Garrido (2001b).

4.1.2 Joint moments of IR and IBNR claims

In this section, we consider the joint moments of the total discounted IR claim amount by time t and the total discounted IBNR claim amount at time $t+\Delta$ ($\Delta\geq 0$). For claim reserving purposes, the knowledge of the IR claims up to time t in conjuncture with the joint moments can help in the prediction of the IBNR claim amount at a future time $t+\Delta$.

The time-0 value of the total IR claim amount up to time t is defined as

$$Z_{ir}(t) = \sum_{k=1}^{N_t} \sum_{i=1}^{C_{T_k}} e^{-\delta T_k} l(W_{i,k}) \mathbf{1}_{\{W_{i,k}+T_k \leq t\}} X_{i,k}.$$

For $u, v \geq 0$, let $L_{u,v}(t, \Delta) = \mathbb{E}\left[e^{-uZ_{ir}(t)-vZ(t+\Delta)}\right]$ be the joint LT of $(Z_{ir}(t), Z(t+\Delta))$.

By conditioning on $\tau_1 = x$ and making use of the law of total probability on the following three events:

- No claim occurs up to time $t+\Delta$ (i.e. $x > t+\Delta$), then $Z_{ir}(t) = Z(t+\Delta) = 0$;
- The first claim occurs at time $x \in (t, t+\Delta]$, then $Z_{ir}(t) = 0$ and $L_{u,v}$ reduces to a univariate LT related to the IBNR component only;
- The first claim occurs by or at time t (i.e. $x \leq t$). We further condition on the characteristics of the first claim causing event and use the regenerating property of $\{N_t\}_{t \geq 0}$ at $\tau_1 = x$;

the joint LT $L_{u,v}$ can be expressed as

$$L_{u,v}(t,\Delta) = \bar{F}(t+\Delta) + \int_t^{t+\Delta} L_{ve^{-\delta x}}(t+\Delta-x) B\left(\mathbb{E}\left[e^{-ve^{-\delta x} l(W_{1,1}) X_{1,1} \mathbf{1}_{\{W_{1,1} > t+\Delta-x\}}} \middle| \tau_1 = x\right]\right) F(\mathrm{d}x)$$
$$+ \int_0^t L_{ue^{-\delta x}, ve^{-\delta x}}(t-x, \Delta) B\left(\mathbb{E}\left[e^{-e^{-\delta x} l(W_{1,1}) X_{1,1}\left[u\mathbf{1}_{\{W_{1,1} \leq t-x\}} + v\mathbf{1}_{\{W_{1,1} > t+\Delta-x\}}\right]} \middle| \tau_1 = x\right]\right) F(\mathrm{d}x).$$

By appropriately differentiating the joint LT $L_{u,v}(t,\Delta)$ and using a similar methodology to that of Section 4.1.1, the joint moments are given by

$$\mathbb{E}[Z_{ir}^m(t) Z^n(t+\Delta)] = \int_0^t e^{-(m+n)\delta x} \mathbb{E}[Z_{ir}^m(t-x) Z^n(t+\Delta-x)] F(\mathrm{d}x) + v_{m,n,\Delta}(t), \quad (4.19)$$

for $m, n \in \mathbb{N}^+$, where

$$v_{m,n,\Delta}(t) = \sum_{\substack{j=0 \\ i+j>0}}^{n} \sum_{i=0}^{m} \binom{m}{i} \binom{n}{j} \int_0^t e^{-(m+n)\delta x} \mathbb{E}[Z_{ir}^{m-i}(t-x) Z^{n-j}(t+\Delta-x)] B_{i,j}^*(x; t, t+\Delta) F(\mathrm{d}x),$$

with, for $0 \leq t_1 \leq t_2$,

$$B_{i,j}^*(\cdot;t_1,t_2)=\sum_{k=1\wedge i}^{i}\sum_{\ell=1\wedge j}^{j} B^{(k+\ell)}(1)B_{i,k}\left(\eta_1(\cdot,t_1),\ldots,\eta_{i-k+1}(\cdot,t_1)\right)B_{j,\ell}\left(\xi_1(\cdot,t_2),\ldots,\xi_{j-\ell+1}(\cdot,t_2)\right),$$

where $B_{0,0}(\cdot) = 1$, $a \wedge b = \min\{a,b\}$, and $\eta_i(x,t_1) = \int_0^{t_1-x} l(w)^i \mu_i(x,w) K_{W|\tau}(\mathrm{d}w|x)$ for $0 \leq x \leq t_1$ and $i = 1,2,\ldots,m$.

Note that (4.19) also holds for $n = 0$, which yields an expression for the mth moment of $Z_{ir}(t)$. Also, for $\delta > (=)\ 0$, Equation (4.19) is a defective (proper) renewal equation of the same form as Equation (4.9). From the proof of Theorem 11, the following result is immediate.

Theorem 13. *For $m \in \mathbb{N}^+$ and $n \in \mathbb{N}$, the joint moments of $Z_{ir}(t)$ and $Z(t+\Delta)$ are given by*

$$\mathbb{E}[Z_{ir}^m(t)Z^n(t+\Delta)] = v_{m,n,\Delta}(t) + \int_0^t e^{-(m+n)\delta x} v_{m,n,\Delta}(t-x)H(\mathrm{d}x).$$

In light of Corollary 12, we provide a similar result in the joint moment setting.

Corollary 14. *If (4.12) holds, we have, for $m \in \mathbb{N}^+$ and $n \in \mathbb{N}$,*

$$\begin{aligned}&\mathbb{E}[Z_{ir}^m(t)Z^n(t+\Delta)]\\ &=\sum_{\substack{j=0\\ i+j>0}}^{n}\sum_{i=0}^{m}\mathbf{1}_{i+j>0}\binom{m}{i}\binom{n}{j}\int_0^t e^{-(n+m)\delta x}\mathbb{E}[Z_{ir}^{m-i}(t-x)Z^{n-j}(t+\Delta-x)]B_{i,j}^*(x;t,t+\Delta)H(\mathrm{d}x).\end{aligned}$$

By further assuming that all batch sizes are of size 1 and $l(w) = e^{-\epsilon w}$, the joint moment becomes

$$\begin{aligned}
&\mathbb{E}[Z_{ir}^m(t)Z^n(t+\Delta)] \\
&= \sum_{i=1}^{m}\binom{m}{i}\int_0^t e^{-(m+n)\delta x}\mathbb{E}[Z_{ir}^{m-i}(t-x)Z^n(t+\Delta-x)]\int_0^{t-x} e^{-i\epsilon w}\mu_i(w)K(\mathrm{d}w)H(\mathrm{d}x) \\
&+ \sum_{j=1}^{n}\binom{n}{j}\int_0^t e^{-(m+n)\delta x}\mathbb{E}[Z_{ir}^{m}(t-x)Z^{n-j}(t+\Delta-x)]\int_{t+\Delta-x}^{\infty} e^{-j\epsilon w}\mu_j(w)K(\mathrm{d}w)H(\mathrm{d}x),
\end{aligned} \tag{4.20}$$

for $m \in \mathbb{N}^+$ and $n \in \mathbb{N}$.

The above moment representations are in integral form, and their evaluation clearly requires distributional assumptions. In particular, the flexible mixed Erlang distributional class for the df F and K allows these quantities to be expressed in terms of Erlang densities (see, e.g., Landriault and Shi (2014)), from which explicit evaluation of these integrals is straightforward but tedious. A particular example is considered next.

Example 4. *We assume that $K(w) = 1 - e^{-\theta w}$ for $\theta, w \geq 0$ and $F(x) = 1 - (1 + \lambda x)e^{-\lambda x}$ for $\lambda, x \geq 0$. The renewal function H (defined in (4.7)) is then given by $H(t) = \left(2\lambda t - 1 + e^{-2\lambda t}\right)/4$ for $t \geq 0$. From (4.18), one deduces that*

$$\mathbb{E}[Z(t)] = \frac{\mu_1\lambda\theta}{2(\epsilon+\theta)}\left(\frac{e^{-(\epsilon+\theta)t} - e^{-\delta t}}{\delta - \epsilon - \theta} - \frac{e^{-(\epsilon+\theta)t} - e^{-(\delta+2\lambda)t}}{\delta + 2\lambda - \epsilon - \theta}\right),$$

when none of the denominators are 0. Also, the second moment is given by

$$\begin{aligned}
\mathbb{E}[Z^2(t)] &= \frac{\mu_1\lambda\theta}{\epsilon+\theta}\int_0^t e^{-2\delta x}\mathbb{E}[Z(t-x)]e^{-(\epsilon+\theta)(t-x)}(1-e^{-2\lambda x})\mathrm{d}x \\
&+ \frac{\mu_2\lambda\theta}{2(2\epsilon+\theta)}\int_0^t e^{-2\delta x}e^{-(2\epsilon+\theta)(t-x)}(1-e^{-2\lambda x})\mathrm{d}x,
\end{aligned}$$

whose closed form expression, although tedious, can be found explicitly. A similar reasoning leads to closed-form expressions for the moments of $Z_{ir}(t)$, and the joint moments of $Z_{ir}(t)$ and $Z(t+\Delta)$ using Equation (4.20). We omit the details here.

Numerical values for the correlation coefficient of $Z_{ir}(t)$ and $Z(t+\Delta)$ are given in Table 4.1. We assume that $\lambda = 3$, $\theta = 0.5$, $\mu_1 = 1$, $\mu_2 = 1.1$, $\delta = 0.05$ and $\epsilon = 0.06$.

Table 4.1: Correlation coefficients between $Z_{ir}(t)$ and $Z(t+\Delta)$

$t\backslash\Delta$	0	0.25	0.5	0.75	1	2
1	-0.1371	-0.1162	-0.0963	-0.0802	-0.0673	-0.0355
2	-0.2088	-0.1794	-0.1533	-0.1312	-0.1127	-0.0633
5	-0.2079	-0.1829	-0.1604	-0.1405	-0.1231	-0.0730
10	-0.1426	-0.1261	-0.1111	-0.0978	-0.0861	-0.0516
20	-0.0740	-0.0655	-0.0577	-0.0508	-0.0448	-0.0269
50	-0.0157	-0.0139	-0.0123	-0.0109	-0.0096	-0.0059
100	-0.0013	-0.0012	-0.0011	-0.0009	-0.0008	-0.0005

As expected, we observe that the values of the correlation coefficient are negative indicating a negative relation between the total discounted IR claim amount and the future discounted IBNR claim amounts. The influence of the total amount of IR claims on the future IBNR claims decays as the prediction horizon Δ increases. Also, for a given prediction horizon Δ, we see that the values of the correlation coefficient first decrease and later increase to approach an asymptotic value of 0, as expected.

4.1.3 Alternative representation for the LT of the total discounted sum under certain assumption

Under the general model, the representation (4.3) for the LT of $Z(t)$ seems rather difficult to work with. However, in what follows, we show that an alternative and more compelling representation can be obtained when (4.12) holds.

Proposition 15. *If the interclaim times are independent of the claim severities and reporting* *lags, i.e. under (4.12), the LT of $Z(t)$ satisfies*

$$L_\gamma(t) = 1 - \int_0^t L_{\gamma e^{-\delta x}}(t-x)\left\{1 - B\left(\tilde{\sigma}(\gamma e^{-\delta x}, t-x)\right)\right\} H(\mathrm{d}x), \tag{4.21}$$

where, for $a, y \geq 0$,

$$\tilde{\sigma}(a,y) = K(y) + \int_y^\infty \tilde{f}_{X|W}(al(w)|w)K(\mathrm{d}w). \tag{4.22}$$

Proof. Equation (4.3) becomes

$$L_\gamma(t) = \bar{F}(t) + \int_0^t L_{\gamma e^{-\delta x}}(t-x)B\left(\tilde{\sigma}(\gamma e^{-\delta x}, t-x)\right) F(\mathrm{d}x), \tag{4.23}$$

with $\tilde{\sigma}(a,y)$ as defined in (4.22). Using the representation that $H(\mathrm{d}x) = F(\mathrm{d}x) + \int_0^x F(\mathrm{d}x - y)H(\mathrm{d}y)$, (4.23) can be re-expressed as

$$\begin{aligned} L_\gamma(t) = \bar{F}(t) &+ \int_0^t L_{\gamma e^{-\delta x}}(t-x)B\left(\tilde{\sigma}(\gamma e^{-\delta x}, t-x)\right) H(\mathrm{d}x) \\ &- \int_0^t L_{\gamma e^{-\delta x}}(t-x)B\left(\tilde{\sigma}(\gamma e^{-\delta x}, t-x)\right) \int_0^x F(\mathrm{d}x-y)H(\mathrm{d}y). \end{aligned} \tag{4.24}$$

Similarly to (4.16), the last term of (4.24) is equivalent to

$$\int_0^t \left[L_{\gamma e^{-\delta y}}(t-y) - \bar{F}(t-y)\right] H(\mathrm{d}y)$$

which implies that

$$\begin{aligned} L_\gamma(t) = \bar{F}(t) &+ \int_0^t L_{\gamma e^{-\delta x}}(t-x)B\left(\tilde{\sigma}(\gamma e^{-\delta x}, t-x)\right) H(\mathrm{d}x) \\ &- \int_0^t \left(L_{\gamma e^{-\delta y}}(t-y) - \bar{F}(t-y)\right) H(\mathrm{d}y). \end{aligned} \tag{4.25}$$

Equation (4.21) is found by substituting $H(x) = F(x) + \int_0^x F(x-y)H(\mathrm{d}y)$ into (4.25). $\square$

Moreover, by repeatedly substituting (4.21) within itself, $L_\gamma(t)$ can be expressed as

$$L_\gamma(t) = \sum_{k=0}^{\infty} H_k(t;\gamma), \tag{4.26}$$

where

$$H_k(t;\gamma) = \int_0^t \int_0^{t-x_1} \cdots \int_0^{t-\sum_{i=1}^{k-1} x_i} \prod_{i=1}^{k} \left\{ \left[B\left(\tilde{\sigma}(\gamma e^{-\delta \sum_{j=1}^{i} x_j}, t - \sum_{j=1}^{i} x_j) \right) - 1 \right] H(\mathrm{d}x_i) \right\},$$

for $k \geq 1$ with initial value $H_0(t;\gamma) = 1$. Furthermore, $H_k(t;\gamma)$ can also be computed recursively via

$$H_k(t;\gamma) = \int_0^t H_{k-1}(t-x;\gamma e^{-\delta x}) \left\{ B\left(\tilde{\sigma}(\gamma e^{-\delta x}, t-x) \right) - 1 \right\} H(\mathrm{d}x), \tag{4.27}$$

for $k \geq 1$.

Note that Equations (4.26) and (4.27) generalize their counterparts for the renewal sums of the discounted claims in Theorem 2.1 of Léveillé et al. (2010).

4.2 IBNR claim number for a Poisson process with bulk arrivals

We now examine in more detail distributional properties of the IBNR claim number

$$U(t) = \sum_{k=1}^{N_t} \sum_{i=1}^{C_{T_k}} \mathbf{1}_{\{W_{i,k}+T_k>t\}},$$

with pmf $p_n(t) = \mathbb{P}(U(t) = n)$ $(n \in \mathbb{N})$, df $P_n(t) = 1 - \bar{P}_n(t) = \sum_{j=0}^{n} p_j(t)$ $(n \in \mathbb{N})$, and pgf $P(z;t) = \mathbb{E}\left[z^{U(t)}\right]$ for $|z| \leq 1$. In what follows, we assume in this section that $K_{W|\tau}(w|t) = K(w)$ for all $w, t \geq 0$ in (4.1) (with, additionally, no consideration given to claim severities). In other words, we assume that interarrival times (times between claim causing events) are independent of reporting lags.

When the number of claim causing event follows a Poisson process with rate $\lambda > 0$, and the reporting lags have df $K(x) = 1 - e^{-\theta x}$ for $x > 0$, the IBNR claim number $U(t)$ is known to have pgf

$$P(z;t) = \sum_{n=0}^{\infty} p_n(t) z^n = \exp\left(\lambda \int_0^t \left\{B\left(1 + e^{-\theta x}(z-1)\right) - 1\right\} \mathrm{d}x\right), \tag{4.28}$$

(e.g., Klugman et al. (2013, Chapter 9)). When the batch size pgf is of the combination or generalized mixture form

$$B(z) = \sum_{i=1}^{\infty} w_i B_i(z), \tag{4.29}$$

where $B_i(z)$ is a proper pgf and $\sum_{i=1}^{\infty} w_i = 1$, (4.28) can be expressed as

$$P(z;t) = \prod_{i=1}^{\infty} P_i(z;t), \tag{4.30}$$

with

$$P_i(z;t) = \exp\left(\lambda w_i \int_0^t \left\{B_i[1 + e^{-\theta x}(z-1)] - 1\right\} \mathrm{d}x\right). \tag{4.31}$$

In the mixture case (i.e. $w_i \geq 0$ for $i \in \mathbb{N}^+$), (4.30) corresponds to the pgf of a sum of independent compound Poisson rv's. Note also that $B_i\left(1 + e^{-\theta x}(z-1)\right)$ is the pgf of a compound distribution with primary pgf $B_i(z)$ and a Bernoulli secondary distribution with mean $e^{-\theta x}$.

4.2.1 Recursion method

We propose to analyze the IBNR claim number $U(t)$ when the batch size pgf is of the form (4.29) with

$$B_i(z) = \alpha_i + (1 - \alpha_i) \frac{\phi_i\left(\beta_i^{-1}(1-z)\right) - \phi_i(\beta_i^{-1})}{1 - \phi_i(\beta_i^{-1})}, \tag{4.32}$$

for $i \in \mathbb{N}^+$, $\beta_i > 0$ and $0 \leq \alpha_i \leq 1$, complementing in the process results in Guo et al. (2013). The class of pgfs (4.32) was proposed by Klugman et al. (2013, Chapter 8) and covers all distributions in the $(a, b, 1)$ class. In what follows, we further assume that $\phi_i(x)$ is the LT of a nonnegative rv, i.e. $\phi_i(s) = \int_0^\infty e^{-sx} F_i(\mathrm{d}x)$. Thus, $\phi_i(\beta_i^{-1}(1-z))$ is a mixed Poisson pgf, i.e.

$$\phi_i(\beta_i^{-1}(1-z)) = \sum_{j=0}^{\infty} r_{i,j}(\beta_i^{-1}) z^j,$$

where for $x > 0$, $i \in \mathbb{N}^+$ and $j \in \mathbb{N}$,

$$r_{i,j}(x) = \int_0^\infty \frac{(xy)^j e^{-xy}}{j!} F_i(\mathrm{d}y). \tag{4.33}$$

Proposition 16. *Assume the claim arrival process follows a compound Poisson process with arrival rate $\lambda > 0$ and secondary distribution with pgf (4.29), where $B_i(z)$ is as defined in (4.32). When the reporting lag df $K(x) = 1 - e^{-\theta x}$ for $x, \theta \geq 0$, the pmf of $U(t)$ can be obtained recursively as*

$$p_n(t) = \frac{\lambda}{n\theta} \sum_{j=1}^{n} p_{n-j}(t) \sum_{i=1}^{\infty} \frac{w_i(1-\alpha_i)}{1-\phi_i(\beta_i^{-1})} \sum_{k=0}^{j-1} \left[r_{i,k}(\beta_i^{-1} e^{-\theta t}) - r_{i,k}(\beta_i^{-1}) \right], \tag{4.34}$$

for $n \in \mathbb{N}^+$ where

$$p_0(t) = P(0;t) = \prod_{i=1}^{\infty} \exp\left(\lambda w_i \frac{1-\alpha_i}{1-\phi_i(\beta_i^{-1})} \int_0^t \left[\phi_i(\beta_i^{-1} e^{-\theta x}) - 1 \right] \mathrm{d}x \right).$$

Proof. Indeed, substituting (4.32) into (4.28) results in

$$P(z;t) = \exp\left(\sum_{i=1}^{\infty} \lambda w_i \frac{1-\alpha_i}{1-\phi_i(\beta_i^{-1})} \int_0^t \left[\phi_i(\beta_i^{-1} e^{-\theta x}(1-z)) - 1 \right] \mathrm{d}x \right). \tag{4.35}$$

Using (4.33) and (4.35), it is easily verified that

$$\ln P(z;t) = \sum_{i=1}^{\infty} \lambda w_i \frac{1-\alpha_i}{1-\phi_i(\beta_i^{-1})} \int_0^t \left[\sum_{j=0}^{\infty} r_{i,j}(\beta_i^{-1} e^{-\theta x}) z^j - 1 \right] \mathrm{d}x. \tag{4.36}$$

Taking derivative on both sides of (4.36) wrt z results in

$$\begin{aligned}\frac{\partial P(z;t)}{\partial z} &= P(z;t)\sum_{j=1}^{\infty}\left[\sum_{i=1}^{\infty}\frac{\lambda w_i(1-\alpha_i)}{1-\phi_i(\beta_i^{-1})}\int_0^t r_{i,j}(\beta_i^{-1}e^{-\theta x})\mathrm{d}x\right] jz^{j-1}\\ &= P(z;t)\sum_{j=1}^{\infty}\left[\sum_{i=1}^{\infty}\frac{\lambda w_i(1-\alpha_i)}{1-\phi_i(\beta_i^{-1})}\int_0^t\int_0^{\infty}\frac{\beta_i^{-j}e^{-j\theta x}y^j e^{-\beta_i^{-1}e^{-\theta x}y}}{(j-1)!}F_i(\mathrm{d}y)\mathrm{d}x\right] z^{j-1}.\end{aligned} \tag{4.37}$$

Interchanging the order of integration followed by changing the variable of integration x to $x^* = e^{-\theta x}$ in (4.37), Equation (4.34) is obtained by equating the coefficients of z^{n-1} on both sides of (4.37). □

Example 5. *When $\alpha_i = 0$ and $\phi_i(x) = (1+x)^{-n_i}$ for $n_i \in \mathbb{N}^+$ in (4.32), Equation (4.29) becomes*

$$B(z) = \sum_{i=1}^{\infty} w_i \frac{\left(\frac{\beta_i}{1+\beta_i-z}\right)^{n_i} - \left(\frac{\beta_i}{1+\beta_i}\right)^{n_i}}{1-\left(\frac{\beta_i}{1+\beta_i}\right)^{n_i}}, \tag{4.38}$$

which is the pgf of a zero-truncated negative binomial distribution. Given that $\phi_i(x)$ is an Erlang LT, it follows that $\phi_i(x) = \int_0^\infty e^{-xy}\frac{y^{n_i-1}}{(n_i-1)!}e^{-y}\mathrm{d}y$. Therefore, the pmf of the IBNR claim number is directly obtainable by substituting

$$r_{i,k}(x) = \binom{n_i+k-1}{k} x^k (x+1)^{-n_i-k},$$

for $i \in \mathbb{N}^+$ and $k \in \mathbb{N}$, into (4.34) to obtain

$$p_n(t) = \frac{\lambda}{n\theta}\sum_{j=1}^{n} p_{n-j}(t)\sum_{i=1}^{\infty}\frac{w_i\beta_i^{n_i}}{1-\left(\frac{\beta_i}{1+\beta_i}\right)^{n_i}}\sum_{k=0}^{j-1}\binom{n_i+k-1}{k}\left[\frac{e^{-k\theta t}}{(e^{-\theta t}+\beta_i)^{n_i+k}} - \frac{1}{(1+\beta_i)^{n_i+k}}\right],$$

for $n \in \mathbb{N}^+$, with

$$p_0(t) = \prod_{i=1}^{\infty}\exp\left(\frac{\lambda w_i}{1-\left(\frac{\beta_i}{1+\beta_i}\right)^{n_i}}\int_0^t\left[\left(\frac{\beta_i}{e^{-\theta x}+\beta_i}\right)^{n_i} - 1\right]\mathrm{d}x\right).$$

Furthermore, assuming that the batch size follows a combination of zero truncated geometrics (i.e. $B(z)$ is as defined in (4.38) with $n_i = 1$ for all i), the pmf of $U(t)$ satisfies

$$p_n(t) = \frac{\lambda}{n\theta}\sum_{j=1}^{n} p_{n-j}(t) \sum_{i=1}^{\infty} w_i(1+\beta_i)\left[\left(\frac{1}{1+\beta_i}\right)^j - \left(\frac{e^{-\theta t}}{e^{-\theta t}+\beta_i}\right)^j\right], \tag{4.39}$$

for $n \in \mathbb{N}^+$, with

$$p_0(t) = P(0;t) = \prod_{i=1}^{\infty}\left(\frac{e^{-\theta t}+\beta_i}{1+\beta_i}\right)^{\frac{\lambda w_i(1+\beta_i)}{\theta}}.$$

4.2.2 Self-decomposition of limiting distributions

A recursive formula for the pmf of the IBNR claim number is obtained in (4.34). We now aim to derive an explicit form for the pgf of the IBNR claim number by making use of the self-decomposability property of the Poisson process (e.g., Steutel and Van Harn (1979)). We first recall an important theorem on self-decomposability from Klugman et al. (2013, Theorem 9.9).

Theorem 17. *For $\lambda_*, \theta_* > 0$ and $B_*(z)$ a pgf, consider the pgf*

$$P_*(z;t) = \exp\left(\lambda_* \int_0^t \left\{B_*\left(1+e^{-\theta_* x}(z-1)\right) - 1\right\} \mathrm{d}x\right).$$

Then, its limiting pgf

$$P_*(z;\infty) \equiv \lim_{t\to\infty} P_*(z;t) = \exp\left(\frac{\lambda_*}{\theta_*}\int_1^z \frac{1-B(y)}{1-y}\mathrm{d}y\right)$$

is discrete self-decomposable, and thus,

$$P_*(z;\infty) = P_*(z;t)P_*(1+e^{-\theta_* t}(z-1);\infty).$$

If we further assume that $B_*(z)$ is a zero-truncated mixed Poisson pgf with mixing distribution LT $\tilde{f}_0$, namely

$$B_*(z) = \frac{\tilde{f}_0(1-z) - \tilde{f}_0(1)}{1-\tilde{f}_0(1)}, \quad |z| \le 1,$$

then

$$P_*(z;\infty) = \lim_{t\to\infty} P_*(z;t) = \exp\left\{-\frac{\lambda_*}{\theta_*(1-\tilde{f}_0(1))}\int_0^{1-z}\frac{1-\tilde{f}_0(y)}{y}\mathrm{d}y\right\}. \tag{4.40}$$

Note that Klugman et al. (2013, Chapter 9) considered models with certain distributional assumptions for the batch size pgf. In what follows, we complement these results and derive the limiting pgf of the IBNR claim number when the batch size is modelled as a combination of zero-truncated negative binomials defined by (4.38). Thus, we define $B_i(z)$ as

$$B_i(z) = \frac{\left(\frac{\beta_i}{1+\beta_i-z}\right)^{n_i} - \left(\frac{\beta_i}{1+\beta_i}\right)^{n_i}}{1-\left(\frac{\beta_i}{1+\beta_i}\right)^{n_i}},$$

which is a zero-truncated mixed Poisson pgf with mixing LT

$$\tilde{f}_i(s) = \left(\frac{\beta_i}{\beta_i+s}\right)^{n_i}. \tag{4.41}$$

Assuming $\beta^* := \max_{i\ge1}\beta_i < \infty$, the mathematical convenience of rewriting (4.41) using a common scale parameter for convolution or compounding purposes was illustrated by Willmot and Woo (2007). Hence, the LT (4.41) can alternatively be rewritten as $\tilde{f}_i(s) = Q_i\left(\frac{\beta^*}{\beta^*+s}\right)$ where $Q_i(z) = \left[\frac{(\beta_i/\beta^*)z}{1-(1-\beta_i/\beta^*)z}\right]^{n_i}$. Note that

$$\frac{1-Q(z)}{1-z} = \sum_{n=1}^{\infty}\bar{Q}_{i,n}z^n, \tag{4.42}$$

where

$$\bar{Q}_{i,n} = \begin{cases} 1, & n < n_i, \\ 1-\sum_{j=n_i}^{n}\binom{j-1}{n_i-1}\left(\frac{\beta_i}{\beta^*}\right)^{n_i}\left(1-\frac{\beta_i}{\beta^*}\right)^{j-n_i}, & n \ge n_i. \end{cases}$$

By making use of Theorem 17, an explicit form for the limiting pgf of the IBNR claim number is obtained in the following proposition.

Proposition 18. *When the claim number process follows a compound Poisson process with arrival rate $\lambda > 0$ and secondary distribution with pgf (4.38), and the reporting lags have df $K(t) = 1 - e^{-\theta t}$ for $t > 0$, the IBNR claim number has a limiting pgf of the form*

$$P(z;\infty) = \left(\frac{\beta^*}{\beta^* + 1 - z}\right)^{\sum_{i=1}^{\infty} w_i \frac{\varepsilon_i}{\theta}} \exp\left\{\lambda^* \left(Q^*(z) - 1\right)\right\}, \tag{4.43}$$

where $\varepsilon_i = \frac{\lambda(1+\beta_i)^{n_i}}{(1+\beta_i)^{n_i} - \beta_i^{n_i}}$, $\lambda^* = \sum_{i=1}^{\infty} w_i \frac{\varepsilon_i}{\theta} \sum_{n=1}^{\infty} \frac{\bar{Q}_{i,n}}{n}$, *and*

$$Q^*(z) = \sum_{n=1}^{\infty} \frac{1}{\lambda^*} \sum_{i=1}^{\infty} \frac{w_i \varepsilon_i \bar{Q}_{i,n}}{n\theta} \left(\frac{\beta^*}{\beta^* + 1 - z}\right)^n.$$

Proof. Using (4.40), Equation (4.31) as $t \to \infty$ can be written as

$$P_i(z;\infty) = \exp\left\{-w_i \frac{\varepsilon_i}{\theta} \int_0^{1-z} \frac{1 - \tilde{f}_i(y)}{y} \mathrm{d}y\right\}.$$

By (4.42), the limiting pgf $P_i(z;\infty)$ can be re-expressed as

$$\begin{aligned} P_i(z;\infty) &= \exp\left\{-w_i \frac{\varepsilon_i}{\theta} \int_0^{1-z} \frac{1 - Q_i(\frac{\beta^*}{\beta^*+y})}{1 - \frac{\beta^*}{\beta^*+y}} \frac{1}{\beta^* + y} \mathrm{d}y\right\} \\ &= \exp\left\{-w_i \frac{\varepsilon_i}{\theta} \int_0^{1-z} \frac{1}{\beta^*} \sum_{n=1}^{\infty} \bar{Q}_{i,n-1} \left(\frac{\beta^*}{\beta^* + y}\right)^n \mathrm{d}y\right\}. \end{aligned} \tag{4.44}$$

Evaluating the integral in (4.44) yields

$$P_i(z;\infty) = \left(\frac{\beta^*}{\beta^* + 1 - z}\right)^{w_i \bar{Q}_{i,0} \frac{\varepsilon_i}{\theta}} \exp\left\{w_i \frac{\varepsilon_i}{\theta} \sum_{n=1}^{\infty} \frac{\bar{Q}_{i,n}}{n} \left[\left(\frac{\beta^*}{\beta^* + 1 - z}\right)^n - 1\right]\right\}. \tag{4.45}$$

Finally, (4.43) can be easily obtained by substituting (4.45) into (4.30) when $t \to \infty$. □

Remark 1. When both $\sum_{i=1}^{\infty} w_i\ \varepsilon_i/\theta$ and λ^* are positive, (4.43) is the pgf of an independent sum of a negative binomial and a compound Poisson rv's. Therefore, the limiting pgf (4.43) is numerically tractable. The condition that $\sum_{i=1}^{\infty} w_i\ \varepsilon_i/\theta$ and λ^* are positive is guaranteed when the batch size is a mixture of zero-truncated geometrics. However, it generally does not hold true for all combinations of zero-truncated geometrics. A counter-example consists in setting $\beta_1 = 1, \beta_2 = 100$ and $w_1 = 1.1, w_2 = -0.1, w_i = 0$ $(i = 3, 4, \ldots)$. The identification of the limiting distribution of the IBNR claim number is non-trivial in this case.

Numerical values of the pmf and mean of the IBNR claim number $U(t)$ are given when $\lambda = 1$, $\theta = 2$ and the batch size has pgf (4.38) with $n_1 = \ldots = n_4 = 1$, $(\beta_1, \beta_2, \beta_3, \beta_4) = (1, 0.5, 3, 5)$, and $(w_1, w_2, w_3, w_4) = (0.1, 0.2, 0.4, 0.3)$.

Table 4.2: Pmf and mean of IBNR claim number with mixture of zero-truncated geometric batch sizes

Items		t=1	t=2	t=3	t=4	t=5	t=6	$t = \infty$
$p_n(t)$	n=0	0.5943	0.5417	0.5346	0.5336	0.5335	0.5335	0.5335
	n=1	0.2365	0.2626	0.2662	0.2667	0.2667	0.2667	0.2667
	n=2	0.0935	0.1087	0.1108	0.1111	0.1111	0.1111	0.1111
	n=3	0.0390	0.0454	0.0463	0.0464	0.0464	0.0464	0.0464
	n=4	0.0176	0.0202	0.0206	0.0206	0.0206	0.0206	0.0206
	n=5	0.0086	0.0097	0.0099	0.0099	0.0099	0.0099	0.0099
$\mathbb{E}[U(t)]$		0.7321	0.8312	0.8446	0.8464	0.8466	0.8467	0.8467

In Table 4.2, the values of the pmf of the IBNR claim number for a finite t are obtained from (4.39) while the asymptotic values (as $t \to \infty$) are calculated using (4.43). As expected, we observe that the average number of IBNR claims initially increases in t. Also, the distribution of the IBNR claim number appears to be stable for large enough t (i.e. ≥ 6 in this setting).

4.3 IBNR claim number for a renewal process with single arrivals

In this section, we focus on the distribution of the IBNR claim number under the assumption that batch arrivals are all of size 1. Two sources of randomness affecting the IBNR claim number will be studied: reporting lags and interarrival times. Section 4.3.1 will study the problem through the specification of the interarrival time distribution, while Section 4.3.2 will consider this problem under distributional assumptions on the reporting lags. Henceforth, we assume the existence of the renewal density $h(x)$ (with $h(x)\mathrm{d}x = H(\mathrm{d}x)$) for $x > 0$.

Lemma 19. *When all batch arrivals are of size* 1*, the IBNR claim number* $U(t)$ *has pgf*

$$P(z;t) = 1 + \int_0^t \left[(z-1)\bar{K}(t-x)\right] P(z;t-x)h(x)\mathrm{d}x. \tag{4.46}$$

Proof. Conditioning on the first claim occurrence time, we have

$$P(z;t) = \bar{F}(t) + \int_0^t \left[K(t-x) + \bar{K}(t-x)z\right] P(z;t-x)F(\mathrm{d}x). \tag{4.47}$$

Using (4.7), (4.47) can be re-expressed as

$$\begin{aligned} P(z;t) = \bar{F}(t) &+ \int_0^t \left[K(t-x) + \bar{K}(t-x)z\right] P(z;t-x)H(\mathrm{d}x) \\ &- \int_0^t \left[K(t-x) + \bar{K}(t-x)z\right] P(z;t-x) \int_0^x F(\mathrm{d}x-y)H(\mathrm{d}y). \end{aligned} \tag{4.48}$$

Similarly to (4.16), the last term of (4.48) can be shown to be equivalent to

$$\int_0^t \left[P(z;t-y) - \bar{F}(t-y)\right] H(\mathrm{d}y),$$

which implies that

$$\begin{aligned} P(z;t) = \bar{F}(t) &+ \int_0^t \left[K(t-x) + \bar{K}(t-x)z\right] P(z;t-x)H(\mathrm{d}x) \\ &- \int_0^t \left(P(z;t-y) - \bar{F}(t-y)\right) H(\mathrm{d}y). \end{aligned} \tag{4.49}$$

Equation (4.46) is found by substituting $H(x) = F(x) + \int_0^x F(x-y)H(\mathrm{d}y)$ (which is immediate from (4.7)) into (4.49). $\square$

Letting $\bar{P}(z;t) = (1 - P(z;t))/(1-z)$ be the pgf of the tail distribution of the IBNR claim number, it is immediate from (4.46) that

$$\bar{P}(z;t) = \int_0^t \bar{K}(t-v)P(z;t-v)h(v)\mathrm{d}v.$$

This implies that

$$\tilde{\bar{P}}(z;s) := \int_0^\infty e^{-st}\bar{P}(z;t)\mathrm{d}t = \tilde{\nu}(z;s)\tilde{h}(s), \tag{4.50}$$

for $s > 0$, where $\tilde{\nu}(z;s) = \int_0^\infty e^{-st}\nu(z;t)\mathrm{d}t = \int_0^\infty e^{-st}\bar{K}(t)P(z;t)\mathrm{d}t$. Applying the Final Value Theorem on the transform relationship (4.50), it follows that

$$\bar{P}(z;\infty) = \frac{1}{\mu_\tau}\int_0^\infty \bar{K}(x)P(z;x)\mathrm{d}x, \tag{4.51}$$

given that $\lim_{s\to 0} s\tilde{h}(s) = 1/\mu_\tau$ where $\mu_\tau = \int_0^\infty \bar{F}(x)\mathrm{d}x$.

4.3.1 Model with Coxian-n distributed interarrival times

In this sub-section, we assume that the interarrival times $\{\tau_k\}_{k\geq 1}$ are from the K_n-family of distributions with LT

$$\tilde{f}(s) = \frac{\alpha(s)}{\prod_{i=1}^n (s+\lambda_i)}, \tag{4.52}$$

where $\lambda_i > 0$ for $i = 1, 2, \ldots, n$, and $\alpha(s) = \sum_{i=0}^{n-1} \alpha_i s^i$ is a polynomial of degree $n-1$ (or less) in s with $\alpha_0 = \prod_{i=1}^n \lambda_i$. It is clear from the definition of the renewal density that

$$\tilde{h}(s) = \int_0^\infty e^{-sx}h(x)\mathrm{d}x = \frac{\tilde{f}(s)}{1-\tilde{f}(s)} = \frac{\alpha(s)}{sr(s)}, \tag{4.53}$$

where

$$r(s) = \sum_{i=0}^{n-1} r_i s^i = \frac{1}{s}\left[\prod_{i=1}^{n}(s+\lambda_i) - \alpha(s)\right],$$

with $r_{n-1} = 1$.

Theorem 20. *When interarrival times have LT (4.52), the pgf of $U(t)$ satisfies the nth order ordinary differential equation (ODE)*

$$\sum_{i=1}^{n} r_{i-1}\left\{\frac{\partial^i}{\partial t^i}P(z;t)\right\} = (z-1)\sum_{i=0}^{n-1}\alpha_i \frac{\partial^i}{\partial t^i}[\bar{K}(t)P(z;t)], \quad t \geq 0. \tag{4.54}$$

Proof. By substituting (4.53) into (4.50), it follows that

$$\tilde{\tilde{P}}(z;s) = \tilde{\nu}(z;s)\frac{\alpha(s)}{sr(s)},$$

or equivalently

$$r(s)\left(1 - s\tilde{P}(z;s)\right) = (1-z)\alpha(s)\tilde{\nu}(z;s). \tag{4.55}$$

Dividing both sides of (4.55) by s^{n+1}, one finds that

$$\sum_{i=0}^{n-1} r_i\left[\frac{1}{s^{n-i+1}} - \tilde{P}(z;s)\frac{1}{s^{n-i}}\right] = (1-z)\sum_{i=0}^{n-1}\alpha_i\tilde{\nu}(z;s)\frac{1}{s^{n-i+1}}. \tag{4.56}$$

Inverting the LT in (4.56) yields

$$\sum_{i=0}^{n-1} r_i\left[\frac{t^{n-i}}{(n-i)!} - \int_0^t P(z;x)\frac{(t-x)^{n-i-1}}{(n-i-1)!}\mathrm{d}x\right] = (1-z)\sum_{i=0}^{n-1}\alpha_i\int_0^t \nu(z;x)\frac{(t-x)^{n-i}}{(n-i)!}\mathrm{d}x. \tag{4.57}$$

Finally, taking the $(n+1)$th order derivative of (4.57) wrt t results in (4.54). □

The general solution to (4.54), which is of a linear form, can be found for some selections of

$\bar{K}(t)$. From the boundary conditions

$$P(z;0) = 1, \tag{4.58}$$

$$\frac{\partial^k}{\partial t^k} P(z;t)|_{t=0} = 0, \text{ for } k = 1, 2, \ldots, n-1, \tag{4.59}$$

and $P(z;\infty)$ given by (4.51), numerical solutions to the nth order ODE (4.54) are readily available.

Example 6. *Consider a renewal claim arrival process with $\tilde{f}(s) = (\lambda/(\lambda + s))^2$. Using (4.54), the pgf of the IBNR claim number $U(t)$ satisfies the ODE*

$$\frac{\partial^2}{\partial t^2} P(z;t) + 2\lambda \frac{\partial}{\partial t} P(z;t) + (1-z)\lambda^2 \bar{K}(t) P(z;t) = 0. \tag{4.60}$$

By further assuming that the reporting lags are diatomic rv's with survival function

$$\bar{K}(t) = \begin{cases} a, & 0 \le t < b, \\ 0, & t \ge b, \end{cases}$$

where $b > 0$ and $0 < a \le 1$, the ODE (4.60) becomes

$$\frac{\partial^2}{\partial t^2} P(z;t) + 2\lambda \frac{\partial}{\partial t} P(z;t) + a(1-z)\lambda^2 P(z;t) = 0, \tag{4.61}$$

for $t < b$, and for $t \ge b$,

$$\frac{\partial^2}{\partial t^2} P(z;t) + 2\lambda \frac{\partial}{\partial t} P(z;t) = 0. \tag{4.62}$$

The solution to (4.61) is easily found to be

$$P(z;t) = \frac{e^{-\lambda t}}{2} \left[e^{\lambda t \sqrt{1-a+az}} + e^{-\lambda t \sqrt{1-a+az}} \right] + \frac{e^{-\lambda t}}{2\sqrt{1-a+az}} \left[e^{\lambda t \sqrt{1-a+az}} - e^{-\lambda t \sqrt{1-a+az}} \right], \tag{4.63}$$

for $0 \le t < b$, with the help of the boundary conditions (4.58) and (4.59). Next, we rewrite

(4.63) *into a compound distribution form, i.e.*

$$P(z;t) = Q(1-a+az;t), \tag{4.64}$$

where

$$\begin{aligned}
Q(z;t) &= \frac{e^{-\lambda t}}{2}\left[e^{\lambda t\sqrt{z}} + e^{-\lambda t\sqrt{z}}\right] + \frac{e^{-\lambda t}}{2\sqrt{z}}\left[e^{\lambda t\sqrt{z}} - e^{-\lambda t\sqrt{z}}\right] \\
&= \frac{e^{-\lambda t}}{2}\sum_{m=0}^{\infty}\left[\frac{(\lambda t\sqrt{z})^m}{m!} + \frac{(-\lambda t\sqrt{z})^m}{m!}\right] + \frac{e^{-\lambda t}}{2\sqrt{z}}\sum_{m=0}^{\infty}\left[\frac{(\lambda t\sqrt{z})^m}{m!} - \frac{(-\lambda t\sqrt{z})^m}{m!}\right] \\
&= \sum_{m=0}^{\infty} e^{-\lambda t}\left[\frac{(\lambda t)^{2m}}{(2m)!} + \frac{(\lambda t)^{2m+1}}{(2m+1)!}\right] z^m,
\end{aligned}$$

is a valid pgf. From (4.64)*, one immediately obtains*

$$p_n(t) = \sum_{m=n}^{\infty} e^{-\lambda t}\left[\frac{(\lambda t)^{2m}}{(2m)!} + \frac{(\lambda t)^{2m+1}}{(2m+1)!}\right]\binom{m}{n} a^n (1-a)^{m-n},\ n\in\mathbb{N},\ t<b.$$

Also, for $t \geq b$*, the solution to* (4.62)*, under boundary condition* (4.51)*, is given by*

$$P(z;t) = (1-e^{-2\lambda(t-b)})P(z;\infty) + e^{-2\lambda(t-b)}P(z;b-), \tag{4.65}$$

where $P(z;b-)$ *holds for* (4.63) *with* $t=b$*. By inverting* (4.65)*, one finds*

$$p_n(t) = (1-e^{-2\lambda(t-b)})p_n(\infty) + e^{-2\lambda(t-b)}p_n(b-), \tag{4.66}$$

for $n\in\mathbb{N}$*. Note that from* (4.51)*, we have*

$$\bar{P}(z;\infty) = \frac{a}{\mu_\tau}\int_0^b P(z;x)\mathrm{d}x. \tag{4.67}$$

Equating the coefficients of z^n for $n \in \mathbb{N}$ on both sides of (4.67) yields

$$\begin{aligned}\sum_{j=n+1}^{\infty} p_j(\infty) &= \frac{a\lambda}{2}\int_0^b p_n(x)\mathrm{d}x \\ &= \frac{1}{2}\sum_{m=n}^{\infty}\binom{m}{n}a^{n+1}(1-a)^{m-n}\int_0^b e^{-\lambda x}\left[\frac{\lambda^{2m+1}x^{2m}}{(2m)!}+\frac{\lambda^{2m+2}x^{2m+1}}{(2m+1)!}\right]\mathrm{d}x \\ &= \sum_{m=n}^{\infty}\binom{m}{n}a^{n+1}(1-a)^{m-n}\left(1-\sum_{j=0}^{2m}\frac{e^{-\lambda b}(\lambda b)^j}{j!}-\frac{e^{-\lambda b}(\lambda b)^{2m+1}}{2(2m+1)!}\right),\end{aligned}$$

which easily leads to the evaluation of the pmf $p_n(\infty)$ in (4.66).

4.3.2 (Mixture of) Exponential(s) reporting lags

In this section, we analyze the IBNR claim number by assuming an exponential distribution with mean $1/\theta$ for the reporting lags. Hence, for $s > 0$ and $|z| \leq 1$, by (4.50),

$$\begin{aligned}\tilde{\tilde{P}}(z;s) &= \tilde{P}(z;s+\theta)\tilde{h}(s) \\ &= \left[\frac{1}{s+\theta}+(z-1)\tilde{\tilde{P}}(z;s+\theta)\right]\tilde{h}(s).\end{aligned}$$

By induction, $\tilde{\tilde{P}}(z;s)$ can be expressed in terms of the LT $\tilde{h}(s)$ of the renewal density as

$$\tilde{\tilde{P}}(z;s) = \sum_{n=1}^{k}(z-1)^{n-1}\frac{1}{s+n\theta}\prod_{j=0}^{n-1}\tilde{h}(s+j\theta)+(z-1)^k\left[\prod_{j=0}^{k-1}\tilde{h}(s+j\theta)\right]\tilde{\tilde{P}}(z;s+k\theta), \quad (4.68)$$

for $k \in \mathbb{N}^+$. By letting $k \to \infty$, the second term on the right hand side of (4.68) vanishes and thus,

$$\begin{aligned}\tilde{\tilde{P}}(z;s) &= \sum_{n=1}^{\infty}(z-1)^{n-1}\frac{1}{s+n\theta}\prod_{j=0}^{n-1}\tilde{h}(s+j\theta) \\ &= \tilde{h}(s)\left[\frac{1}{s+\theta}+\sum_{n=2}^{\infty}\frac{\prod_{j=1}^{n-1}\tilde{h}(j\theta)}{n\theta}\tilde{a}_n(s)(z-1)^{n-1}\right], \quad (4.69)\end{aligned}$$

where, for $n = 2, 3, \ldots$,

$$\tilde{a}_n(s) = \frac{n\theta}{s + n\theta} \prod_{j=1}^{n-1} \frac{\tilde{h}(s + j\theta)}{\tilde{h}(j\theta)}. \tag{4.70}$$

Note that $\tilde{h}(s + j\theta)/\tilde{h}(j\theta)$ for $j \in \mathbb{N}^+$ is the LT of the Esscher transform of the renewal density. As such, $\tilde{a}_n(s)$ is the LT of the independent sum of an exponential rv with mean $1/(n\theta)$ and a sequence of rv's with LT $\tilde{h}(s + j\theta)/\tilde{h}(j\theta)$ for $j = 1, \ldots, n-1$.

Inverting (4.69) in s yields

$$\bar{P}(z; t) = \int_0^t e^{-\theta(t-y)} h(y) \mathrm{d}y + \sum_{n=2}^{\infty} \left[\frac{\prod_{j=1}^{n-1} \tilde{h}(j\theta)}{n\theta} \int_0^t a_n(t-y) h(y) \mathrm{d}y \right] (z-1)^{n-1}, \tag{4.71}$$

where $\tilde{a}_n(s) = \int_0^\infty e^{-st} a_n(t) \mathrm{d}t$. Thus, from Willmot et al. (2005, Equation (5.6)), it follows that

$$\bar{P}(z; t) = \sum_{n=1}^{\infty} \frac{\mathbb{E}[U^{(n)}(t)]}{n!} (z-1)^{n-1}, \tag{4.72}$$

where $U^{(n)}(t) = U(t)(U(t)-1)\ldots(U(t)-n+1)$ for $n \in \mathbb{N}^+$. By comparing (4.71) and (4.72), one deduces that

$$\mathbb{E}[U^{(n)}(t)] = \begin{cases} \int_0^t e^{-\theta(t-y)} h(y) \mathrm{d}y, & n = 1, \\ \frac{(n-1)! \prod_{j=1}^{n-1} \tilde{h}(j\theta)}{\theta} \int_0^t a_n(t-y) h(y) \mathrm{d}y, & n = 2, 3, \ldots. \end{cases} \tag{4.73}$$

Also, the tail pmf of the IBNR claim number can be obtained by expanding $\bar{P}(z; t)$ and equating the coefficients of z^n on both sides of (4.71), namely

$$\bar{P}_k(t) = \begin{cases} \sum_{n=k+1}^{\infty} \binom{n-1}{k} \frac{(-1)^{n-k-1} \prod_{j=1}^{n-1} \tilde{h}(j\theta)}{n\theta} \int_0^t a_n(t-y) h(y) \mathrm{d}y, & k \in \mathbb{N}^+, \\ \int_0^t e^{-\theta(t-y)} h(y) \mathrm{d}y + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} \prod_{j=1}^{n-1} \tilde{h}(j\theta)}{n\theta} \int_0^t a_n(t-y) h(y) \mathrm{d}y, & k = 0. \end{cases} \tag{4.74}$$

Next, we focus on specifying $a_n(t)$ for $n \geq 2$ under the assumption that the interarrival times are mixed Erlang distributed with LT

$$\tilde{f}(s) = G\left(\frac{\beta}{\beta + s}\right), \tag{4.75}$$

where $G(z) = \sum_{n=1}^{\infty} g_n z^n$. Mixed Erlang distributions are known to be a large class of distribution functions which are mathematically tractable and dense in the set of all continuous distributions on $[0, \infty)$ (see, e.g., Willmot and Woo (2007)). Under (4.75), the LT of the renewal density is given by

$$\tilde{h}(s) = \frac{G(\frac{\beta}{\beta+s})}{1 - G(\frac{\beta}{\beta+s})}.$$

For $j = 1, 2, \ldots, n-1$, we express

$$\frac{\tilde{h}(s + j\theta)}{\tilde{h}(j\theta)} = G_j\left(\frac{\beta + j\theta}{\beta + j\theta + s}\right), \tag{4.76}$$

where

$$G_j(z) = \frac{G(\frac{\beta}{\beta+j\theta}z)}{1 - G(\frac{\beta}{\beta+j\theta}z)} \frac{1 - G(\frac{\beta}{\beta+j\theta})}{G(\frac{\beta}{\beta+j\theta})}.$$

We point out that $G_j(1) = 1$ for $j = 1, 2, \ldots, n-1$. Given that $G(z)$ is a valid pgf, $G_j(z)$ is absolutely monotone on $(0, 1)$. From Feller (1971, p.223), $G_j(z)$ is a pgf and (4.76) is a mixed Erlang LT. Using Equation (2.1) of Willmot and Woo (2007), we propose to convert the LT $\tilde{h}(s + j\theta)/\tilde{h}(j\theta)$ for $j = 1, \ldots, n-1$, to a mixed Erlang LT with a common scale parameter. Indeed, given that

$$\frac{\tilde{h}(s + j\theta)}{\tilde{h}(j\theta)} = G_{j,n-1}\left(\frac{\beta + (n-1)\theta}{\beta + (n-1)\theta + s}\right),$$

for $j = 1, \ldots, n-1$ and $n = 2, 3, \ldots$, where

$$G_{j,n-1}(z) = G_j \left(\frac{\frac{\beta+j\theta}{\beta+(n-1)\theta} z}{1 - (1 - \frac{\beta+j\theta}{\beta+(n-1)\theta}) z} \right),$$

we have

$$\prod_{j=1}^{n-1} \frac{\tilde{h}(s+j\theta)}{\tilde{h}(j\theta)} = \prod_{j=1}^{n-1} G_{j,n-1} \left(\frac{\beta+(n-1)\theta}{\beta+(n-1)\theta+s} \right) = G^*_{n-1} \left(\frac{\beta+(n-1)\theta}{\beta+(n-1)\theta+s} \right), \tag{4.77}$$

where $G^*_{n-1}(z) = \prod_{j=1}^{n-1} G_{j,n-1}(z) = \sum_{i=1}^{\infty} g^*_{i,n-1} z^i$ for $n = 2, 3, \ldots$. Thus, (4.77) is also a LT of a mixed Erlang distribution. Substituting (4.77) into (4.70) followed by a LT inversion, one finds

$$a_n(t) = \int_0^t n\theta e^{-n\theta(t-y)} \sum_{i=1}^{\infty} g^*_{i,n-1} \frac{[\beta+(n-1)\theta]^i \, y^{i-1}}{(i-1)!} e^{-[\beta+(n-1)\theta]y} \mathrm{d}y. \tag{4.78}$$

Evaluating the integral on the right hand side of (4.78) yields

$$a_n(t) = n\theta e^{-n\theta t} \sum_{i=1}^{\infty} g^*_{i,n-1} [\beta+(n-1)\theta]^i \sum_{m=i}^{\infty} \frac{(\beta-\theta)^{m-i} t^m e^{-(\beta-\theta)t}}{m!}. \tag{4.79}$$

The moments and tail pmf of the IBNR claim number can be obtained by substituting (4.79) into (4.73) and (4.74), respectively.

Remark 2. If the interarrival time LT is further given by (4.52) for a finite n, the LT $\tilde{a}_n(s)$ defined in (4.70) can be explicitly inverted using partial fraction expansions. Using (4.74), this would lead to an explicit expression for the tail $\bar{P}_k(t)$ for $k \in \mathbb{N}$. The calculations are straightforward but tedious, and thus the details are omitted.

In the following, we take a closer look at the process with a specific choice of interclaim times. The LT $\tilde{a}_n(s)$ as defined in (4.70) can be explicitly inverted into a more attractive form.

4.3.2.1 Specific choice of interclaim times

Here, we assume that the renewal density has LT

$$\tilde{h}(s) = \frac{c_0}{s} \prod_{i=1}^{l} \frac{1}{s + c_i}, \tag{4.80}$$

with $l \geq 0$. This model is still very general and it covers Erlang distributed interclaim times. The distribution of IBNR claim number can be expressed explicitly in this model as shown thereafter. Note that the LT (4.80) is a special case of the more general Coxian class of LT (4.52). For simplicity, we choose to work with (4.80); however, a similar but more involved methodology will lead to equivalent results in the more general Coxian class as illustrated in the later time.

In the case where $l = 0$, (4.80) represents the renewal density for exponentially distributed interclaim times. The claim number thus follows a Poisson process; the pgf of the survival distribution of IBNR claim number is easily obtained as

$$\bar{P}(z;t) = \frac{1 - \exp\left(\frac{c_0(z-1)(1-e^{-\mu t})}{\mu}\right)}{1-z}, \; c_0 > 0,$$

from (4.28). In the following, we'll work in the scenarios where $l \geq 1$.

Substituting (4.80) into (4.69) results in

$$\tilde{\bar{P}}(z;s) = \sum_{n=1}^{\infty} c_0^n (z-1)^{n-1} \left(\prod_{j=0}^{n} \frac{1}{s + j\mu} \right) \left[\prod_{i=1}^{l} \prod_{j=0}^{n-1} \frac{1}{s + c_i + j\mu} \right]. \tag{4.81}$$

Also we know that, for $i = 1, 2, \ldots, l$,

$$\prod_{j=0}^{n-1} \frac{1}{s + c_i + j\mu} = \int_0^{\infty} e^{-sv} \frac{(1-e^{-\mu v})^{n-1} e^{-c_i v}}{\mu^{n-1}(n-1)!} \mathrm{d}v,$$

which can be proved via integration by parts or by induction on n. Thus, (4.81) can be inverted to a convolution of $l+1$ functions and the pgf of the survival function is obtained as

$$\bar{P}(z;t) = \sum_{n=1}^{\infty} c_0^n (z-1)^{n-1} \vartheta_{0,n} * \vartheta_{1,n} * \cdots * \vartheta_{l,n}(t),$$

where

$$\vartheta_{i,n}(v) = \begin{cases} \frac{(1-e^{-\mu v})^n}{\mu^n n!}, & i = 0, \\ \\ \frac{(1-e^{-\mu v})^{n-1} e^{-c_i v}}{\mu^{n-1}(n-1)!}, & i = 1, 2, \ldots, l. \end{cases}$$

The survival distribution is then obtained accordingly

$$\bar{P}_k(t) = \sum_{n=k+1}^{\infty} (-1)^{n-1-k} c_0^n \binom{n-1}{k} \vartheta_{0,n} * \vartheta_{1,n} * \cdots * \vartheta_{l,n}(t), \ k \geq 0. \tag{4.82}$$

If it holds that,

$$\begin{aligned} &c_i + (k-j)\mu \neq 0, \text{ for } i \in \{1, 2, \ldots, l\}, \text{ and } j, k \in \{1, 2, \ldots, n\} \\ &c_i + j\mu \neq c_m + k\mu, \text{ if } i \neq m \in \{1, 2, \ldots, l\} \text{ or } j \neq k \in \{0, 1, \ldots, n-1\}, \end{aligned} \tag{4.83}$$

an alternative simpler expression for (4.82) is available. More specifically, by using partial fraction expansions, we have

$$\left(\prod_{j=0}^{n} \frac{1}{s+j\mu}\right) \left[\prod_{i=1}^{l} \prod_{j=0}^{n-1} \frac{1}{s+c_i+j\mu}\right] = \frac{v_{n,0,0}}{s} + \sum_{j=1}^{n} \frac{v_{n,0,j}}{s+j\mu} + \sum_{i=1}^{l} \sum_{j=0}^{n-1} \frac{v_{n,i,j}}{s+c_i+j\mu},$$

where, for $i = 1, 2, \ldots, l$ and $j = 0, 1, \ldots, n$,

$$v_{n,0,j} = \frac{1}{\mu^n} \left(\prod_{\substack{k=0 \\ k \neq j}}^{n} \frac{1}{k-j}\right) \left[\prod_{i=1}^{l} \prod_{k=0}^{n-1} \frac{1}{c_i + (k-j)\mu}\right],$$

$$v_{n,i,j} = \left(\prod_{k=0}^{n} \frac{1}{-c_i + (k-j)\mu}\right) \left[\prod_{m=1}^{l} \prod_{\substack{k=0 \\ (m,k) \neq (i,j)}}^{n-1} \frac{1}{c_m - c_i + (k-j)\mu}\right], \ j \neq n.$$

Thus, (4.82) can be easily re-expressed as

$$\bar{P}_k(t) = \sum_{n=k+1}^{\infty} (-1)^{n-1-k} c_0^n \binom{n-1}{k} \left[\sum_{j=0}^{n} v_{n,0,j} e^{-j\mu t} + \sum_{i=1}^{l} \sum_{j=0}^{n-1} v_{n,i,j} e^{-(c_i + j\mu)t} \right]. \tag{4.84}$$

Therefore, if the interclaim times are distributed as a Coxian distribution, the LT of the renewal density can be expressed as a summation of the form (4.80) by partial fraction expansions. The survival function of the IBNR claim number can thus be expressed as a summation of convolutions similar to those defined in (4.82) or exponential terms as defined in (4.84). To conclude, under the exponential reporting lag assumption, the pmf of the IBNR claim number is tractable when the interarrival times are Coxian distributed.

Example 7. *We assume the interclaim times follow an Erlang-2 with mean $2/\lambda$, then*

$$\tilde{h}(s) = \frac{\lambda}{s} \frac{\lambda}{s + 2\lambda}.$$

which is a special case of (4.80) with $c_0 = \lambda^2$, $c_1 = 2\lambda$ and $l = 1$. Since it is unknown whether (4.83) holds, we use (4.82) to obtain the survival function of the IBNR claim number as

$$\bar{P}_k(t) = \sum_{n=k+1}^{\infty} (-1)^{n-1-k} \lambda^{2n} \binom{n-1}{k} h_n(t),$$

where

$$h_n(t) = \int_0^t \frac{\left[1 - e^{-\mu(t-v)}\right]^n}{\mu^n n!} \frac{(1 - e^{-\mu v})^{n-1} e^{-2\lambda v}}{\mu^{n-1}(n-1)!} \mathrm{d}v.$$

The pmf and mean of the IBNR claim number are easily computed, see Table 4.3 for example, where $\lambda = 1, \mu = 1$.

Table 4.3: Pmf and mean of IBNR claim number with exponential reporting lags

Items		$t=1$	$t=2$	$t=5$	$t=10$	$t=15$	$t=20$
$p_n(t)$	n=0	0.8094	0.6638	0.5819	0.5792	0.5805	0.5817
	n=1	0.1818	0.3008	0.3500	0.3503	0.3491	0.3479
	n=2	0.0087	0.0337	0.0625	0.0645	0.0645	0.0645
	n=3	0.0002	0.0016	0.0053	0.0057	0.0057	0.0057
E[U(t)]		0.1996	0.3733	0.4920	0.4975	0.4963	0.4950

The IBNR claim number is concentrated around 0 *early on. Over time (i.e. as* t *increases), the distribution of the IBNR claim number seems to stabilize, and the mean IBNR claim number first increases and then decreases after it reach its peak.*

4.3.2.2 Models with reporting lag distributed as mixture of exponentials

In this section, we generalize the distributional assumption for the reporting lags from exponential to mixture of exponentials. For illustrating purpose, we first consider

$$\bar{K}(x) = \rho e^{-b_1 x} + (1-\rho)e^{-b_2 x}, \quad b_1, b_2 > 0, \ 0 \le \rho \le 1. \tag{4.85}$$

Substituting (4.85) into (4.50) and taking LT gives

$$\tilde{\tilde{P}}(z;s) = \left(\frac{\rho}{s+b_1} + \frac{1-\rho}{s+b_2}\right)\tilde{h}(s) + \rho\tilde{h}(s)(z-1)\tilde{\tilde{P}}(z;s+b_1) + (1-\rho)\tilde{h}(s)(z-1)\tilde{\tilde{P}}(z;s+b_2).$$

Using a similar methodology as for the exponential reporting lags of Section 4.3.2, one obtains

$$\begin{aligned}\tilde{\tilde{P}}(z;s) = \left(\frac{\rho}{s+b_1} + \frac{1-\rho}{s+b_2}\right)\tilde{h}(s) + \sum_{n=1}^{\infty}(z-1)^n \Bigg\{ \sum_{\{x_i=0,1\}_{1\le i\le n}} \rho^{\sum_{i=1}^n x_i}(1-\rho)^{n-\sum_{i=1}^n x_i} \\ \times \left[\frac{\rho\prod_{i=0}^n \tilde{h}(y_i)}{s+b_1+y_n}\prod_{i=0}^n \frac{\tilde{h}(s+y_i)}{\tilde{h}(y_i)} + \frac{(1-\rho)\prod_{i=0}^n \tilde{h}(y_i)}{s+b_2+y_n}\prod_{i=0}^n \frac{\tilde{h}(s+y_i)}{\tilde{h}(y_i)}\right]\Bigg\},\end{aligned} \tag{4.86}$$

where $y_i = \sum_{j=1}^i [b_1 x_j + b_2(1-x_j)]$ for $i \ge 1$ and $y_0 = 0$. The term in the square bracket of

(4.86) is easily invertible, as shown in Section 4.3.2.

To obtain a simpler representation for (4.86), we introduce the rv's $Y_i = \sum_{j=1}^{i}(b_1 I_j + b_2[1 - I_j])$ for $i \geq 1$ and $Y_0 = 0$, where I_j $(j \geq 1)$ are identically distributed and independent Bernoulli rv's with mean ρ. Thus Y_i is distributed as a linearly transformed binomial$(i,\, \rho)$ for $i \geq 1$. The term in the brace of (4.86) can be re-written as

$$\mathbf{E}_{I_i,\, 1 \leq i \leq n}\left[\frac{\rho \prod_{i=0}^{n} \tilde{h}(Y_i)}{s + b_1 + Y_n} \prod_{i=0}^{n} \frac{\tilde{h}(s + Y_i)}{\tilde{h}(Y_i)} + \frac{(1-\rho) \prod_{i=0}^{n} \tilde{h}(Y_i)}{s + b_2 + Y_n} \prod_{i=0}^{n} \frac{\tilde{h}(s + Y_i)}{\tilde{h}(Y_i)} | I_i, 1 \leq i \leq n\right].$$

Equation (4.86) can therefore be re-written as

$$\tilde{\tilde{P}}(z; s) = \sum_{n=0}^{\infty} (z-1)^n \mathbf{E}_{\{Y_i; i=0,1,\ldots,n\}}\left[\frac{\rho}{s + b_1 + Y_n} \prod_{i=0}^{n} \tilde{h}(s + Y_i) + \frac{1-\rho}{s + b_2 + Y_n} \prod_{i=0}^{n} \tilde{h}(s + Y_i)\right]. \tag{4.87}$$

Similarly to $\tilde{a}_n(s)$, term by term inversion of (4.87) follows under distributional assumptions for the df F as in Section 4.3.2. The generalization to more general mixtures of exponentials follows along the same lines.

Chapter 5

IBNR Claims under a MAP Model

This chapter will study the total discounted IBNR claim amount and the number of IBNR claims under a Markovian setting. The model considered in this chapter also allows for claim severities and reporting lags to depend on an underlying continuous-time Markov chain (CTMC) which can be used to model changes in economic environments, for instance.

As a claim counting process, the MAP model has been extensively studied in the actuarial literature. For instance, Asmussen (1989) utilized a Markov process to estimate ruin probabilities. See also Bäuerle (1996), Jasiulewicz (2001), and Cheung and Landriault (2010) for more ruin-related problems under a MAP claim counting process. Ahn et al. (2007) considered a MAP model under a dividend barrier problem. See also Li and Lu (2007) and Badescu et al. (2007) for other relevant work on this topic. Moreover, the aggregate claims under the framework of the MAP model were analyzed by Kim and Kim (2007), where the first two moments of the discounted aggregate claims were obtained under a Markov modulated Poisson risk model. Ren (2008) further extended Kim and Kim's results to a general MAP model. In this chapter, recursive formulas are derived for all finite-order moments of the total discounted IBNR claim amount and the pmf of the IBNR claim number in the MAP model with dependence between the interclaim times and claim severities.

In the MAP risk model, there exists a homogeneous CTMC, say $\mathbf{J} = \{J(t); t \geq 0\}$, with finite state space $E = \{1, 2, 3, \ldots, m\}$ which describes the evolution of an (unobservable) environment process. We consider two types of transitions in a MAP risk model:

1. transitions of the CTMC $\mathbf{J}$ from state i to state j without a claim occurrence (type-1 transitions); or

2. transitions of the CTMC $\mathbf{J}$ from state i to state j with an accompanying claim (type-2 transitions).

In what follows, we refer to either type of transition as a system change. The type-1 transition is governed by the matrix $\mathbf{D} = (D_{ij})_{i,j \in E}$. Its (i, j)th element corresponds to the instantaneous rate of transition from state i to state j ($j \neq i$) in E without an accompanying claim. The type-2 transition is governed by the matrix $\mathbf{T} = (T_{ij})_{i,j \in E}$, for which its (i, j)th element corresponds to the instantaneous rate of transition from state i to state j in E with an accompanying claim. The diagonal elements of $\mathbf{D}$ are assumed to be negative such that the sum of the elements on each row of the matrix $\mathbf{Q} = \mathbf{D} + \mathbf{T}$ is zero. Under the above assumptions, the counting process $\{N_t\}_{t \geq 0}$ is said to be MAP($\mathbf{D}, \mathbf{T}$).

The MAP counting process is very general. It covers many important counting processes. For example, when $\mathbf{D} = \lambda > 0$ and $\mathbf{T} = -\lambda$, it reduces to a Poisson process with intensity λ; with $\mathbf{D} = \mathbf{B}$ and $\mathbf{T} = \mathbf{b}^\top \boldsymbol{\alpha}$, where $\boldsymbol{\alpha}$ is a vector and $\mathbf{b} = -\mathbf{B}\mathbf{e}$ with $\mathbf{e}$ being a vector of 1's, it reduces to a renewal process with the interclaim times following a phase-type distribution with representation $(\boldsymbol{\alpha}, \mathbf{B})$; with $\mathbf{D} = \mathbf{Q} - diag(\lambda_i)$ and $\mathbf{T} = diag(\lambda_i)$, where $diag(\lambda_i)$ denotes a diagonal matrix with λ_i on the diagonal, it reduces to a Markov modulated Poisson process with rate λ_i for $i \in E$ and infinitesimal generator $\mathbf{Q} = (q_{ij})_{i,j \in E}$. Interested readers are referred to Neuts (1981), and Latouche and Ramaswami (1999) for a more general introduction on MAPs.

We first introduce the risk model of interest through a series of definitions together with their underlying assumptions.

1. Let s_n denote the time of the nth system change and $\tau_n = s_n - s_{n-1}$ for $n \in \mathbb{N}^+$ with $s_0 = 0$. Given that $J(s_{n-1}) = j$, τ_n has pdf f_j, df $F_j = 1 - \bar{F}_j$ and LT $\tilde{f}_j$.

2. Let X_n be the claim size accompanying the nth system change. We assume that $X_n = 0$ if the nth system change does not involve a claim while X_n has pdf g_{jk}, df G_{jk} and LT $\tilde{g}_{jk}$ if the nth system change involves a type-2 transition from state j to state k.

3. Let W_n be the reporting lag for the claim accompanying the nth system change. We assume that $W_n = 0$ if the nth system change does not involve a claim while W_n has df $K_{jk} = 1 - \bar{K}_{jk}$ if the nth system change involves a type-2 transition from state j to state k.

Given the underlying states $\{J(t)\}_{t\geq 0}$, we further assume that the random vectors $(\tau_n, W_n, X_n)_{n\in\mathbb{N}^+}$ are mutually independent. For notational convenience, we express the joint distribution of (τ_n, W_n, X_n) (conditional on $J(s_{n-1}) = j$ and $J(s_n) = k$) as

$$\Pr(\tau_n \leq t, W_n \leq w, X_n \leq x | J(s_{n-1}) = j, J(s_n) = k) = F_j(t) K_{jk|\tau}(w|t) G_{jk|\tau,W}(x|t,w), \quad (5.1)$$

for $t, w, x \geq 0$ and $j, k \in E$, where given $J(s_{n-1}) = j, J(s_n) = k$, $G_{jk|\tau,W}$ is the df of $X_n|(\tau_n, W_n)$ and $K_{jk|\tau}$ is the df of the reporting lag $W_n|\tau_n$. Furthermore, we let $\tilde{g}_{jk|\tau,W}$ be the LT of $X_n|(\tau_n, W_n)$, and $\mu_{jk}^{(i)}(t,w) = E[X_n^i|\tau_n = t, W_n = w, J(s_{n-1}) = j, J(s_n) = k]$ for $i \in \mathbb{N}$.

We will examine the following two special cases of (5.1):

- Special Case A. Conditional on $J(s_{n-1}) = j$ and $J(s_n) = k$, the interclaim time τ_n is independent of the claim severity X_n and reporting lag W_n, i.e. $G_{jk|\tau,W}(x|t,w) = G_{jk|W}(x|w)$ and $K_{jk|\tau}(w|t) = K_{jk}(w)$. Thus,

$$\Pr(\tau_n \leq t, W_n \leq w, X_n \leq x | J(s_{n-1}) = j, J(s_n) = k) = F_j(t) K_{jk}(w) G_{jk|W}(x|w),$$

  for $t, w, x \geq 0$ and $j, k \in E$. Therefore, $\mu_{jk}^{(i)}(w) = \mathbb{E}[X_n^i|W_n = w, J(s_{n-1}) = j, J(s_n) = k]$ for $i \in \mathbb{N}$.

- Special Case B. Conditional on $J(s_{n-1}) = j$ and $J(s_n) = k$, the interclaim time τ_n, claim severity X_n and reporting lag W_n are all independent, i.e.

$$\Pr\left(\tau_n \leq t, W_n \leq w, X_n \leq x | J(s_{n-1}) = j, J(s_n) = k\right) = F_j(t)K_{jk}(w)G_{jk}(x),$$

for $t, w, x \geq 0$ and $j, k \in E$. Accordingly, $\mathbb{E}[X_n^i | J(s_{n-1}) = j, J(s_n) = k] =: \mu_{jk}^{(i)}$ for $i \in \mathbb{N}$.

The primary quantity of interest in this chapter is the total discounted IBNR claim amount $Z(t)$ defined as

$$Z(t) = \sum_{n=1}^{N_t} e^{-\delta s_n} l(W_n) \mathbf{1}_{\{W_n + s_n > t\}} X_n,$$

whose LT is denoted as

$$L_{ij}(\gamma, t) = \mathbb{E}[e^{-\gamma Z(t)} \mathbf{1}_{\{J(t)=j\}} | J(0) = i],$$

for $i, j \in E$. Let $\mathbf{L}(\gamma, t) = \left(L_{ij}(\gamma, t)\right)_{i,j \in E}$. Similarly, let

$$U(t) = \sum_{n=1}^{N_t} \mathbf{1}_{\{W_n + s_n > t\}} \mathbf{1}_{\{X_n > 0\}},$$

be the IBNR claim number at time t, whose pgf is denoted as $\mathbf{P}(z; t)$ with (i, j)th element

$$P_{ij}(z; t) = \mathbb{E}[z^{U(t)} \mathbf{1}_{\{J(t)=j\}} | J(0) = i].$$

The pgf $\mathbf{P}(z; t)$ will be the subject matter of Section 5.2.

5.1 Total discounted IBNR claim amount

In this section, we work under the MAP$(\mathbf{D}, \mathbf{T})$ with the dependence structure defined in (5.1), where $F_j(x) = 1 - e^{D_{jj} x}$ for $x \geq 0$ and $j \in E$. Also the probability that the system change is a transition of $\mathbf{J}$ from state j to state k with (without) a claim is given by $-T_{jk}/D_{jj}$ ($-D_{jk}/D_{jj}$ for $k \neq j$) for $j, k \in E$.

Under these assumptions, we derive a renewal equation for the LT of the total discounted IBNR claim amount. Conditioning on the time and characteristic of the first system change, it follows that

$$L_{ij}(\gamma,t) = e^{D_{ii}t}\mathbf{1}_{\{i=j\}} + \sum_{k=1,k\neq i}^{m} D_{ik}\int_0^t L_{kj}(\gamma e^{-\delta x}, t-x)e^{D_{ii}x}\mathrm{d}x$$
$$+\sum_{k=1}^{m}T_{ik}\int_0^t L_{kj}(\gamma e^{-\delta x}, t-x)\left[K_{ik|\tau}(t-x|x)+\int_{t-x}^{\infty}\tilde{g}_{ik|W,\tau}(\gamma e^{-\delta x}l(w)|w,x)K_{ik|\tau}(\mathrm{d}w|x)\right]e^{D_{ii}x}\mathrm{d}x. \tag{5.2}$$

Each of the three terms on the right-hand side of (5.2) represents

- the state remains in state i and no claim occurs up to time t,
- the first system change is a type-1 transition at time $x \leq t$ from state i to state k $(i \neq k)$,
- the first system change is a type-2 transition at time $x \leq t$ from state i to state k.

From (5.2), it is immediate that $\mathbf{L}(\gamma,0) = \mathbf{I}$ and $\mathbf{L}(0,t) = e^{\mathbf{Q}t}$.

5.1.1 Moments of the total discounted IBNR claim amount

Next, we provide a closed-form expression for the expected discounted IBNR claim amount. We later derive a recursive formula for its higher-order moments. A few definitions are first introduced to ease the subsequent presentation. For $n \in \mathbb{N}^+$, $x \geq 0$, and $i,j \in E$,

- Let $\mathbf{E}^{(n)}(x)$ be a $m \times m$ matrix whose (i,j)th entry is

$$E_{ij}^{(n)}(x) = \mathbb{E}[Z^n(x)\mathbf{1}_{\{J(x)=j\}}|J(0)=i],$$

- Let $\boldsymbol{\mu}^{(n)}(x)$ (or $\boldsymbol{\mu}^{(n)}$) be the $m \times m$ matrix whose (i,j)th element is $\mu_{ij}^{(n)}(x)$ (or $\mu_{ij}^{(n)}$),
- Let $\mathbf{B}^{(n)}(x)$ be a $m \times m$ matrix whose (i,j)th entry is $\int_x^\infty l^n(w)\mu_{ij}^{(n)}(w)K_{ij}(\mathrm{d}w)$.

Here, for notation convenience, we assume that $\mathbf{E}^{(0)}(x) = e^{\mathbf{Q}x}$ for $x \geq 0$. The first-order moment is obtained by differentiating the LT (5.2) with respect to γ and subsequently evaluating it at $\gamma = 0$. This results in

$$
\begin{aligned}
E_{ij}^{(1)}(t) = &\sum_{k=1, k\neq i}^{m} D_{ik} \int_0^t e^{-\delta x} E_{kj}^{(1)}(t-x) e^{D_{ii}x} \mathrm{d}x + \sum_{k=1}^{m} T_{ik} \int_0^t e^{-\delta x} E_{kj}^{(1)}(t-x) e^{D_{ii}x} \mathrm{d}x \\
&+ \sum_{k=1}^{m} T_{ik} \int_0^t e^{-\delta x} E_{kj}^{(0)}(t-x) \left[\int_{t-x}^{\infty} l(w) \mu_{ik}^{(1)}(w,x) K_{ik|\tau}(\mathrm{d}w|x) \right] e^{D_{ii}x} \mathrm{d}x.
\end{aligned}
$$

Along the same lines, the n-th derivative of (5.2) with respect to γ (which is further evaluated at $\gamma = 0$) yields

$$
\mathbf{E}^{(n)}(t) = \int_0^t e^{-n\delta x} e^{\mathbf{\Delta}x} [\mathbf{Q} - \mathbf{\Delta}] \mathbf{E}^{(n)}(t-x) \mathrm{d}x + \mathbf{M}^{(n)}(t), \tag{5.3}
$$

for $n \in \mathbb{N}^+$, where $\mathbf{\Delta} = diag(D_{ii})$ and $\mathbf{M}^{(n)}(t)$ is an $m \times m$ matrix whose (i,j)th entry is,

$$
M_{ij}^{(n)}(t) = \sum_{d=1}^{n} \binom{n}{d} \sum_{k=1}^{m} T_{ik} \int_0^t e^{-n\delta x} E_{kj}^{(n-d)}(t-x) \left[\int_{t-x}^{\infty} l^d(w) \mu_{ik}^{(d)}(w,x) K_{ik|\tau}(\mathrm{d}w|x) \right] e^{D_{ii}x} \mathrm{d}x,
$$

for $n \in \mathbb{N}^+$, where $M_{ij}^{(n)}(0) = 0$. Taking LT on both sides of (5.3) yields

$$
\begin{aligned}
\tilde{\mathbf{E}}^{(n)}(s) &= \left\{ \mathbf{I} - [s\mathbf{I} + n\delta\mathbf{I} - \mathbf{\Delta}]^{-1} [\mathbf{Q} - \mathbf{\Delta}] \right\}^{-1} \tilde{\mathbf{M}}^{(n)}(s) \\
&= \{ s\mathbf{I} + n\delta\mathbf{I} - \mathbf{Q} \}^{-1} [s\mathbf{I} + n\delta\mathbf{I} - \mathbf{\Delta}] \tilde{\mathbf{M}}^{(n)}(s),
\end{aligned} \tag{5.4}
$$

for $s \geq 0$. Then, the inversion of the LT (5.4) results in

$$
\begin{aligned}
\mathbf{E}^{(n)}(t) &= \int_0^t e^{-n\delta(t-x)} e^{\mathbf{Q}(t-x)} [n\delta\mathbf{I} - \mathbf{\Delta}] \mathbf{M}^{(n)}(x) \mathrm{d}x + \int_0^t e^{-n\delta(t-x)} e^{\mathbf{Q}(t-x)} \frac{\mathrm{d}\mathbf{M}^{(n)}(x)}{\mathrm{d}x} \mathrm{d}x \\
&= \int_0^t e^{-n\delta(t-x)} e^{\mathbf{Q}(t-x)} \left\{ [n\delta\mathbf{I} - \mathbf{\Delta}] \mathbf{M}^{(n)}(x) + \frac{\mathrm{d}\mathbf{M}^{(n)}(x)}{\mathrm{d}x} \right\} \mathrm{d}x,
\end{aligned}
$$

for $n \in \mathbb{N}^+$. In the following, we show how the moments can be obtained recursively.

Theorem 21. *When the counting process $\{N_t\}_{t\geq 0}$ is a MAP($\mathbf{D}$,$\mathbf{T}$) and an arbitrary dependence among the interclaim times, claim severities and reporting lags is allowed, the moments* *of $Z(t)$ can be written in a recursion form, i.e.*

$$\mathbf{E}^{(n)}(t) = \int_0^t e^{-n\delta(t-x)} e^{\mathbf{Q}(t-x)} \left\{ [n\delta\mathbf{I} - \mathbf{\Delta}]\mathbf{M}^{(n)}(x) + \frac{\mathrm{d}\mathbf{M}^{(n)}(x)}{\mathrm{d}x} \right\} \mathrm{d}x,$$

for $n \in \mathbb{N}^+$.

More specifically, under Special Case A, it follows

$$\mathbf{E}^{(1)}(t) = \int_0^t e^{-\delta(t-x)} \mathbf{Q}\mathbf{E}^{(1)}(x)\mathrm{d}x + \int_0^t e^{-\delta(t-x)} \left[\mathbf{T} \circ \mathbf{B}^{(1)}(x)\right] e^{\mathbf{Q}x}\mathrm{d}x, \tag{5.5}$$

and for $n \geq 2$,

$$\mathbf{E}^{(n)}(t) = \int_0^t e^{-n\delta(t-x)} \mathbf{Q}\mathbf{E}^{(n)}(x)\mathrm{d}x + \sum_{i=1}^{n} \binom{n}{i} \int_0^t e^{-n\delta(t-x)} \left[\mathbf{T} \circ \mathbf{B}^{(i)}(x)\right] \mathbf{E}^{(n-i)}(x)\mathrm{d}x. \tag{5.6}$$

Considering the boundary conditions $\mathbf{E}^{(n)}(0) = \mathbf{0}$ for $n \in \mathbb{N}^+$ (where $\mathbf{0}$ is a $m \times m$ matrix of 0), the integral equations (5.5) and (5.6) can easily be solved as

$$\mathbf{E}^{(1)}(t) = \int_0^t e^{-\delta(t-x)} e^{\mathbf{Q}(t-x)} \left[\mathbf{T} \circ \mathbf{B}^{(1)}(x)\right] e^{\mathbf{Q}x}\mathrm{d}x, \tag{5.7}$$

and

$$\mathbf{E}^{(n)}(t) = \sum_{i=1}^{n} \binom{n}{i} \int_0^t e^{-n\delta(t-x)} e^{\mathbf{Q}(t-x)} \left[\mathbf{T} \circ \mathbf{B}^{(i)}(x)\right] \mathbf{E}^{(n-i)}(x)\mathrm{d}x. \tag{5.8}$$

Thus, the first-order moment can be obtained explicitly from (5.7) and the higher-order ones can be obtained recursively from (5.8).

Example 8. *In this example, we provide further detail on the derivation of a closed-form expression for the first order moment when the underlying CTMC $\mathbf{J}$ has two states, i.e. $m = 2$. The interclaim times are distributed as a sum of two independent, exponentially distributed rv's with rates η and v. Therefore, it follows*

$$\mathbf{Q} = \begin{pmatrix} -\eta & \eta \\ v & -v \end{pmatrix}, \mathbf{T} = \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix}.$$

We further assume that $\mathbf{B}^{(1)}(x)$ *is of an exponential form for computational convenience, i.e.*

$$\mathbf{B}^{(1)}(x) = \begin{pmatrix} b_{11}e^{-\beta_{11}x} & b_{12}e^{-\beta_{12}x} \\ b_{21}e^{-\beta_{21}x} & b_{22}e^{-\beta_{22}x} \end{pmatrix},$$

where $b_{ij}, \beta_{ij} \geq 0$ *for* $i, j \in \{1, 2\}$*. This assumption covers the case when* $l(w)$ *is a compound discount factor, the reporting lags are exponentially distributed (*k_{ij} *is an exponential df) and the expected individual losses depend on the reporting lags through exponential functions. Next, let* $\boldsymbol{m}_Z^{(1)}(t) = \mathbf{E}^{(1)}(t)\mathbf{e}$*, whose ith element is* $\mathbb{E}[Z(t)|J_0 = i]$ *for* $i = 1, 2$*. From (5.7), it follows that*

$$\boldsymbol{m}_Z^{(1)}(t) = \int_0^t e^{-\delta(t-x)} e^{\mathbf{Q}(t-x)} \left[\mathbf{T} \circ \mathbf{B}^{(1)}(x)\right] \mathbf{e}\mathrm{d}x.$$

By applying matrix decomposition on $\mathbf{Q}$*, it holds that*

$$e^{\mathbf{Q}x} = \begin{pmatrix} \frac{v}{\eta+v} + \frac{\eta}{\eta+v}e^{-(\eta+v)x} & \frac{\eta}{\eta+v}\left[1 - e^{-(\eta+v)x}\right] \\ \frac{v}{\eta+v}\left[1 - e^{-(\eta+v)x}\right] & \frac{\eta}{\eta+v} + \frac{v}{\eta+v}e^{-(\eta+v)x} \end{pmatrix}.$$

Then, it is easily obtained that

$$\boldsymbol{m}_Z^{(1)}(t) = \begin{pmatrix} \frac{\eta v}{\eta+v}(c_1(t) + c_2(t)) \\ \frac{v}{\eta+v}(\eta c_1(t) - v c_2(t)) \end{pmatrix},$$

where

$$c_1(t) = \frac{b_{11}(e^{-\delta t} - e^{-\beta_{11}t})}{\beta_{11} - \delta} + \frac{b_{12}(e^{-\delta t} - e^{-\beta_{12}t})}{\beta_{12} - \delta},$$

and

$$c_2(t) = \frac{b_{11}(e^{-\beta_{11}t} - e^{-(\eta+v+\delta)t})}{\eta + v + \delta - \beta_{11}} + \frac{b_{12}(e^{-\beta_{12}t} - e^{-(\eta+v+\delta)t})}{\eta + v + \delta - \beta_{12}},$$

provided that all the denominators are not zero. Thus, a closed expression for the first moment of the total discounted IBNR claim amount is obtained under this setting.

5.1.2 Laplace transform under Special Case A

In this section, the focus is to find an alternative representation for the LTs of the total discounted IBNR claims. Under Special Case A, it holds that

$$\begin{aligned} L_{ij}(\gamma,t) = e^{D_{ii}t}\mathbf{1}_{\{i=j\}} &+ \sum_{k=1,k\neq i}^{m} D_{ik}\int_0^t L_{kj}(\gamma e^{-\delta x},t-x)e^{D_{ii}x}\mathrm{d}x \\ &+ \sum_{k=1}^{m} T_{ik}\int_0^t L_{kj}(\gamma e^{-\delta x},t-x)A_{ik}(\gamma e^{-\delta x},t-x)e^{D_{ii}x}\mathrm{d}x, \end{aligned} \tag{5.9}$$

where $A_{ik}(a,y) = K_{ik}(y) + \int_y^\infty \tilde{g}_{ik|W}(al(w)|w)K_{ik}(\mathrm{d}w)$ for $i,k \in E$ and $a,y \geq 0$. Thus, $A_{ik}(0,y) = 1$ for all $y \geq 0$.

Theorem 22. *When $\{N_t\}_{t\geq 0}$ is a MAP $(\mathbf{D},\mathbf{T})$ and under Special Case A, the LT of $Z(t)$ follows*

$$\mathbf{L}(\gamma,t) = \mathbf{I} + \int_0^t \left\{\mathbf{D} + \left[\mathbf{T}\circ\mathbf{A}(\gamma e^{-\delta x},t-x)\right]\right\}\mathbf{L}(\gamma e^{-\delta x},t-x)\mathrm{d}x, \tag{5.10}$$

where $\circ$ is Hadamard product symbol, $\mathbf{A}(\cdot,\cdot)$ is an $m\times m$ matrix with $A_{ij}(\cdot,\cdot)$ as its (i,j)th element, and $\mathbf{I}$ is a $m\times m$ identity matrix.

Proof. By making use of the relationship between $F_i(x)$ and its renewal equation $m_i(x)$, namely

$$m_i(\mathrm{d}x) = F_i(\mathrm{d}x) + \int_0^x F_i(\mathrm{d}x-y)m_i(\mathrm{d}y), \tag{5.11}$$

(5.9) can be rewritten as

$$
\begin{aligned}
L_{ij}(\gamma,t) &= e^{D_{ii}t}\mathbf{1}_{\{i=j\}} + \sum_{k=1,k\neq i}^{m} \frac{D_{ik}}{-D_{ii}} \int_0^t L_{kj}(\gamma e^{-\delta x}, t-x) m_i(\mathrm{d}x) \\
&\quad + \sum_{k=1}^{m} \frac{T_{ik}}{-D_{ii}} \int_0^t L_{kj}(\gamma e^{-\delta x}, t-x) A_{ik}(\gamma e^{-\delta x}, t-x) m_i(\mathrm{d}x) \\
&\quad - \sum_{k=1,k\neq i}^{m} \frac{D_{ik}}{-D_{ii}} \int_0^t L_{kj}(\gamma e^{-\delta x}, t-x) \int_0^x F_i(\mathrm{d}x-y) m_i(\mathrm{d}y) \\
&\quad - \sum_{k=1}^{m} \frac{T_{ik}}{-D_{ii}} \int_0^t L_{kj}(\gamma e^{-\delta x}, t-x) A_{ik}(\gamma e^{-\delta x}, t-x) \int_0^x F_i(\mathrm{d}x-y) m_i(\mathrm{d}y). \qquad (5.12)
\end{aligned}
$$

By interchanging the order of integration in the last two terms of (5.12) and later using (5.11), it becomes

$$
\begin{aligned}
L_{ij}(\gamma,t) &= \mathbf{1}_{\{i=j\}} - \int_0^t L_{ij}(\gamma e^{-\delta x}, t-x) m_i(\mathrm{d}x) + \sum_{k=1,k\neq i}^{m} \frac{D_{ik}}{-D_{ii}} \int_0^t L_{kj}(\gamma e^{-\delta x}, t-x) m_i(\mathrm{d}x) \\
&\quad + \sum_{k=1}^{m} \frac{T_{ik}}{-D_{ii}} \int_0^t L_{kj}(\gamma e^{-\delta x}, t-x) A_{ik}(\gamma e^{-\delta x}, t-x) m_i(\mathrm{d}x). \qquad (5.13)
\end{aligned}
$$

Substituting $m_i(dx) = -D_{ii}dx$ into (5.13) yields

$$
\begin{aligned}
&L_{ij}(\gamma,t) \\
&= \mathbf{1}_{\{i=j\}} + \sum_{k=1}^{m} D_{ik} \int_0^t L_{kj}(\gamma e^{-\delta x}, t-x)\mathrm{d}x + \sum_{k=1}^{m} T_{ik} \int_0^t L_{kj}(\gamma e^{-\delta x}, t-x) A_{ik}(\gamma e^{-\delta x}, t-x)\mathrm{d}x,
\end{aligned}
$$

from which (5.10) is obtained directly. □

5.2 IBNR claim number

Next, we examine in more detail some distributional properties of the IBNR claim number.

Corollary 23. *When $\{N_t\}_{t\geq 0}$ is a MAP $(\mathbf{D},\mathbf{T})$ and under Special Case A, the pgf of $U(t)$*

is given as

$$\mathbf{P}(z;t) = \mathbf{I} + \int_0^t \left\{\mathbf{D} + \mathbf{T} \circ \left[\mathbf{K}(x) + z\bar{\mathbf{K}}(x)\right]\right\} \mathbf{P}(z;x)\mathrm{d}x, \tag{5.14}$$

where $\mathbf{K}(x)$ and $\bar{\mathbf{K}}(x)$ are $m \times m$ matrices with (i,j)th elements $K_{ij}(x)$ and $\bar{K}_{ij}(x)$, respectively.

Proof. It is directly obtained from (5.10) by letting $\gamma = -\ln(z)$, $\delta = 0$ and all claim sizes are of size 1 almost surely. $\square$

It is obvious that $\mathbf{P}(1;t) = e^{\mathbf{Q}t}$ and $\mathbf{P}(z;0) = \mathbf{I}$. From Magnus (1954), the solution to (5.14) is in the form of Magnus expansion, i.e. $P(z;t) = e^{\sum_{k=1}^{\infty} \Omega_k(z;t)}$, where

$$\begin{aligned}
\Omega_1(z;t) &= \int_0^t A(z;t_1)\mathrm{d}t_1, \\
\Omega_2(z;t) &= \frac{1}{2}\int_0^t \int_0^{t_1} [A(z;t_1), A(z;t_2)]\mathrm{d}t_2\mathrm{d}t_1, \\
\Omega_3(z;t) &= \frac{1}{6}\int_0^t \int_0^{t_1} \int_0^{t_2} \left([A(z;t_1),[A(z;t_2),A(z;t_3)]] + [A(z;t_3),[A(z;t_2,A(z;t_1)]]\right)\mathrm{d}t_3\mathrm{d}t_2\mathrm{d}t_1, \\
&\vdots
\end{aligned}$$

where $[\mathbf{A},\mathbf{B}] = \mathbf{AB} - \mathbf{BA}$ for any matrix $\mathbf{A}$ and $\mathbf{B}$ and $A(z;x) = \mathbf{D} + \mathbf{T} \circ \left[\mathbf{K}(x) + z\bar{\mathbf{K}}(x)\right]$ for $x \in [0,t]$. The calculation of the pmf from Magnus expansion is in general a computationally intensive task. Next, we provide a recursive formula for the pmf of $U(t)$, which is a $m \times m$ matrix $\mathbf{P}_n(t)$ whose (i,j) element is $\Pr(U(t) = n, J(t) = j | J(0) = i)$. By taking derivatives of (5.14) with respect to z and evaluating at $z = 0$, one finds that $\{\mathbf{P}_n(t)\}_{t\geq 0}$ satisfies the recursion

$$\mathbf{P}_n(t) = \int_0^t [\mathbf{T} \circ \bar{\mathbf{K}}(x)]\mathbf{P}_{n-1}(x)\mathrm{d}x + \int_0^t \left\{\mathbf{D} + \mathbf{T} \circ \mathbf{K}(x)\right\} \mathbf{P}_n(x)\mathrm{d}x. \tag{5.15}$$

From (5.15), the pmf of the IBNR claim number is expressed in an integral form, which relies on both the pmf of the same level at previous times and those of lower levels. Also, explicit expressions for the moments of $U(t)$ can be obtained. First, we define $U^{(n)}(t) =$

$U(t)(U(t)-1)\dots(U(t)-n+1)$ for $n \in \mathbb{N}^+$ and let

$$E_{ij}^{*(n)}(t) = \mathbb{E}[U^{(n)}(t)\mathbf{1}_{\{J(t)=j\}}|J(0)=i],$$

which can be obtained by taking the nth order derivative of (5.14) with respect to z and evaluating at $z=1$. Furthermore, let $\mathbf{m}^{(n)}(t)$ be a vector, whose ith element is $m_i^{(n)}(t) = \mathbb{E}[U^{(n)}(t)|J(0)=i]$ for $i \in E$. Then, explicit expressions for all the moments can be obtained.

Corollary 24. *When $\{N_t\}_{t\geq 0}$ is a MAP $(\mathbf{D}, \mathbf{T})$ and under Special Case A, the factorial moments of $U(t)$ are given as*

$$\boldsymbol{m}^{(n)}(t) = n! \int_0^t \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-1}} \prod_{i=1}^{n} [e^{\mathbf{Q}(t_{i-1}-t_i)}\mathbf{T} \circ \bar{\mathbf{K}}(t_i)] \mathrm{d}t_n \mathrm{d}t_{n-1} \cdots \mathrm{d}t_1 \mathbf{e}, \tag{5.16}$$

for $n \in \mathbb{N}^+$, and $t_0 = t$.

Proof. By utilizing the relationship between $\mathbf{m}^{(n)}$ and $\mathbf{E}^{(n)}$, one has

$$\begin{aligned} \mathbf{m}^{(1)}(t) &= \mathbf{E}^{*(1)}(t)\mathbf{e} \\ &= \int_0^t \mathbf{Q}\mathbf{m}^{(1)}(x)\mathrm{d}x + \int_0^t \mathbf{T} \circ \bar{\mathbf{K}}(x) e^{\mathbf{Q}x} \mathrm{d}x \mathbf{e} \\ &= \int_0^t \mathbf{Q}\mathbf{m}^{(1)}(x)\mathrm{d}x + \int_0^t \mathbf{T} \circ \bar{\mathbf{K}}(x) \mathrm{d}x \mathbf{e}, \end{aligned} \tag{5.17}$$

where the last line holds due to the fact that the $\mathbf{Q}$'s row sums are equal to 0. Taking LT on both sides of (5.17) yields

$$\begin{aligned} \tilde{\mathbf{m}}^{(1)}(s) &= \mathbf{Q}s^{-1}\tilde{\mathbf{m}}^{(1)}(s) + s^{-1}\int_0^\infty e^{-sx}\mathbf{T} \circ \bar{\mathbf{K}}(x)\mathrm{d}x\mathbf{e} \\ &= [s\mathbf{I} - \mathbf{Q}]^{-1} \int_0^\infty e^{-sx}\mathbf{T} \circ \bar{\mathbf{K}}(x)\mathrm{d}x\mathbf{e}. \end{aligned} \tag{5.18}$$

Inverting the LT (5.18) gives

$$\mathbf{m}^{(1)}(t) = \int_0^t e^{\mathbf{Q}(t-x)}[\mathbf{T} \circ \bar{\mathbf{K}}(x)]\mathrm{d}x\mathbf{e}.$$

Repeating the above procedure yields

$$\mathbf{m}^{(n)}(t) = n \int_0^t e^{\mathbf{Q}(t-x)}[\mathbf{T} \circ \bar{\mathbf{K}}(x)]\mathbf{m}^{(n-1)}(x)\mathrm{d}x,$$

from which (5.16) holds directly by repeated substitutions. □

As a special case of the MAP model, the Markov modulated Poisson process (MMPP) is obtained when $\mathbf{D} = \mathbf{Q} - diag(\lambda_i)$ and $\mathbf{T} = diag(\lambda_i)$. As a result, only type-1 (type-2) transitions from state i to j are possible when $i \neq j$ $(i = j)$. Next, we consider a numerical example under the MMPP model where

$$\mathbf{Q} = \begin{pmatrix} -9 & 2 & 3 & 4 \\ 16/3 & -28/3 & 8/3 & 4/3 \\ 4/3 & 4/3 & -14/3 & 2 \\ 25/3 & 5/3 & 5/3 & -35/3 \end{pmatrix},$$

and $\mathbf{T} = diag(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. Also the survival function of the reporting lag $\bar{\mathbf{K}}$ is a diagonal matrix with $\bar{K}_{11}(x) = e^{-\beta_1 x}$, $\bar{K}_{22}(x) = (1+\beta_2 x)e^{-\beta_2 x}$, $\bar{K}_{33}(x) = \left(\frac{1}{x+1}\right)^{\beta_3}$ and $\bar{K}_{44}(x) = e^{-\beta_4 x}$, where $(\beta_1, \beta_2, \beta_3, \beta_4) = (3, 1, 2, 1)$. We further assume that $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (1, 4, 5, 3)$. The moments of the IBNR claim number are illustrated in Table 5.1.

Table 5.1: Mean and variance of the IBNR claim number in the MMPP model

Quantity	States \t	1	1.25	1.5	1.75	2	2.25	2.5
$\mathbb{E}[U(t)\|J(0)]$	J(0)=1	1.773	2.039	2.258	2.439	2.589	2.714	2.819
	J(0)=2	2.020	2.261	2.453	2.608	2.734	2.837	2.923
	J(0)=3	1.914	2.149	2.345	2.510	2.649	2.765	2.864
	J(0)=4	1.826	2.081	2.291	2.464	2.608	2.728	2.830
$Var[U(t)\|J(0)]$	J(0)=1	2.005	2.305	2.550	2.751	2.916	3.052	3.165
	J(0)=2	2.272	2.547	2.762	2.933	3.071	3.182	3.274
	J(0)=3	2.126	2.399	2.625	2.813	2.969	3.098	3.206
	J(0)=4	2.053	2.344	2.580	2.774	2.933	3.065	3.174

From Table 5.1, the initial state of the CTMC $\mathbf{J}$ influences the expectation and also the variance of the IBNR claim number within certain periods. For instance, state 2 is a “bad” environment for insurance companies, since the IBNR claim number is large and uncertain. Thus, they need to set more reserves to prepare for the future liabilities.

Chapter 6

Total Discounted Claim Costs under A Two-sided Exit Setting

In this chapter, the aggregate claims until a randomized observation time is analyzed. We work with the surplus process as defined in (1.3) with the surpassing times τ_b^+ and τ_0^- as in (1.4) and (1.5) respectively. Among many quantities of interest involving τ_b^+ and τ_0^- in probability analysis, we specially mention the two-sided exit probabilities

$$m_\alpha(u;b) = \Pr(\tau_b^+ < \tau_0^- \wedge e_\alpha | U_0 = u) = \mathbb{E}\left[e^{-\alpha\tau_b^+} \mathbf{1}_{\{\tau_b^+ < \tau_0^-\}} \middle| U_0 = u \right], \tag{6.1}$$

and

$$M_\alpha(u;b) = \Pr(\tau_0^- < \tau_b^+ \wedge e_\alpha | U_0 = u) = \mathbb{E}\left[e^{-\alpha\tau_0^-} \mathbf{1}_{\{\tau_0^- < \tau_b^+\}} \middle| U_0 = u \right], \tag{6.2}$$

where e_α is an exponential rv with mean $1/\alpha$, independent of $\{U_t\}_{t\geq 0}$.

In what follows, we propose to enhance the risk analysis on the two-sided exit problem for the insurance surplus process (1.3) by further examining two specific quantities pertaining to the excursions of $\{U_t\}_{t\geq 0}$ in $[0, \tau_b^+ \wedge \tau_0^-]$, namely

$$\phi_{m,\alpha,\delta}(u;b) = \mathbb{E}\left[e^{-\alpha\tau_b^+} \left(\sum_{k=1}^{N_{\tau_b^+}} e^{-\delta T_k} h(X_k) \right)^m 1_{\{\tau_b^+ < \tau_0^-\}} \middle| U_0 = u \right], \tag{6.3}$$

and

$$\Phi_{m,\alpha,\delta}(u;b) = \mathbb{E}\left[e^{-\alpha\tau_0^-} \left(\sum_{k=1}^{N_{\tau_0^-}} e^{-\delta T_k} h(X_k) \right)^m 1_{\{\tau_0^- < \tau_b^+\}} \middle| U_0 = u \right], \tag{6.4}$$

for a nonnegative integer m, where $\delta \geq 0$ and $h(\cdot)$ is a so-called cost function. By definition, $\phi_{m,\alpha,\delta}(u;b) = \Phi_{m,\alpha,\delta}(u;b) = 0$ for $u \geq b$ and $m \geq 1$. Furthermore, we have $\phi_{0,\alpha,\delta}(u;b) = m_\alpha(u;b)$ and $\Phi_{0,\alpha,\delta}(u;b) = M_\alpha(u;b)$. Note that the expected total discounted claim costs until ruin (i.e., a special case of (6.4) with $m = 1$ and $b \to \infty$) was studied by Cai et al. (2009) and Feng (2009). See also Cheung (2013) for the higher-order moments of the total discounted claim costs until ruin in a Sparre Andersen risk process.

The two-sided exit probabilities (6.1) and (6.2) have been the subject matter of various risk analysis. We refer the reader to Kyprianou (2006, Chapter 8) and Kuznetsov et al. (2013) in the context of the Lévy insurance risk model, and Kyprianou and Palmowski (2008) in its Markov additive generalization. In this chapter, we consider the insurance surplus process (1.3) in the framework of the dependent Sparre Andersen risk model (see, e.g., Cheung et al. (2010)). More specifically, we assume that the pairs $\{(W_i, X_i)\}_{i=1}^{\infty}$ form a sequence of iid random vectors distributed as a generic random vector (W, X). Similarly to Willmot and Woo (2012), the joint pdf $f_{W,X}$ of (W, X) is assumed to be of the form

$$f_{W,X}(t,x) = \sum_{i=1}^{n} \lambda_i e^{-\lambda_i t} g_i(x), \tag{6.5}$$

for $t, x \geq 0$, where $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n$ for $n \geq 1$ and $\int_0^\infty |g_i(x)| \mathrm{d}x < \infty$ for $i = 1, 2, \ldots, n$. It is immediate from (6.5) that the marginal survival function of W is given by $\bar{F}_W(t) = \int_t^\infty f_W(x)\mathrm{d}x = \sum_{i=1}^n \eta_i e^{-\lambda_i t}$ for $t \geq 0$, where $\eta_i = \int_0^\infty g_i(x)\mathrm{d}x$. For convenience, we also define $\tilde{g}_i(s)$ as $\tilde{g}_i(s) = \int_0^\infty e^{-sx} g_i(x)\mathrm{d}x$ for all $s \in \mathbb{R}$ such that the integral exists.

Remark that, if all $g_i(x)'s$ are valid pdfs, (6.5) defines a negative correlation between the interclaim times and claim severities. In general, a positive dependence is also allowed under (6.5). A simple example is obtained when $n = 2$, $\lambda_1 = 1$, $\lambda_2 = 2$, $g_1 = 1.5e^{-x} - e^{-2x}$, and $g_2 = -0.5e^{-x} + e^{-2x}$.

One important example for the joint pdf (6.5) is the Bernstein copula defined in (2.2), where W is an exponential rv with mean $1/\lambda$ and X has a marginal pdf $g(\cdot)$; the joint df of (W, X) is then given by

$$f_{W,X}(t,x) = \lambda e^{-\lambda t} \sum_{i=0}^{n-1} \sum_{j=0}^{\ell-1} a_{n,\ell}\left(\frac{i}{n}, \frac{j}{n}\right) B_{n-1}(i, 1 - e^{-\lambda t}) B_{\ell-1}(j, G(x)) g(x),$$

which is a special case of (6.5) with $\lambda_i = i\lambda$ and

$$g_i(x) = \frac{1}{i} \sum_{k=n-i}^{n-1} \frac{(n-1)!}{(n-1-k)!(n-i)!(k-n+i)!} \sum_{j=0}^{\ell-1} a_{n,\ell}\left(\frac{k}{n}, \frac{j}{n}\right) B_{\ell-1}(j, G(x)) g(x),$$

for $i = 1, 2, \ldots, n$.

Considering the denseness of the Bernstein copula in the space of bounded continuous functions (see Nelsen (1998) for more detail), it follows that the dependence defined in (6.5) allows a very general model to be studied under the insurance surplus process (1.3). We refer the reader to Willmot and Woo (2012) for other examples for the joint density (6.5).

The one-sided exit problem involving τ_0^- have been extensively examined in the (dependent) Sparre Andersen risk model (e.g., Li and Garrido (2004, 2005), Gerber and Shiu (2005), Boudreault et al. (2006) and Cossette et al. (2010)). See also Cheung et al. (2010) for some extensions. Less attention has been paid to the analysis of two-sided exit problems in the context of this class of risk processes. In this chapter, we first investigate the two-sided exit probability $m_\alpha(u; b)(M_\alpha(u; b))$, and the moments of the total discounted claim costs $\phi_{m,\alpha,\delta}(u;b)$ ($\Phi_{m,\alpha,\delta}(u;b)$) in the dependent renewal risk model (1.3) whose dependence is introduced through the joint density (6.5). As expected, the solutions to Lundberg's generalized equation $\mathbb{E}\left[e^{-\alpha W_1}e^{s(cW_1-X_1)}\right] = 1$ for $\alpha \geq 0$ are very relevant in this context, whose closed-form representation under (6.5) is given by

$$\sum_{i=1}^{n} \frac{\lambda_i}{\lambda_i+\alpha-cs}\tilde{g}_i(s) = 1. \tag{6.6}$$

The rest of the chapter is structured as follows. In Section 6.1, we identify n fundamental solutions to a given integral equation which will be shown to play a crucial role in the subsequent analysis. In Section 6.2, we show that the two-sided exit quantities $\phi_{m,\alpha,\delta}(u;b)$ and $\Phi_{m,\alpha,\delta}(u;b)$ satisfy an nth order integro-differential equation (IDE). It is later shown that the n fundamental solutions derived in Section 6.1 are a group of independent solutions to the corresponding homogeneous IDEs. From the general theory on IDEs, $m_\alpha(u;b)$ and $M_\alpha(u;b)$ are expressed in terms of the n fundamental solutions together with a particular solution to the associated IDE. A recursive formula is then provided in Section 6.2 to calculate the moments under the two-sided exit setting. Section 6.3 provides explicit expressions for the two-sided exit probabilities under the Farlie-Gumbel-Morgenstern (FGM) copula and exponentially distributed claim sizes.

6.1 Solutions to integral equations

By conditioning on the time and the amount of the first claim and using the regenerative property of the aggregate claim process at claim instants, (6.3) can be expressed as

$$\phi_{m,\alpha,\delta}(u;b) = \int_0^{\frac{b-u}{c}} e^{-(\alpha+m\delta)t}\int_0^{u+ct} \phi_{m,\alpha,\delta}(u+ct-x;b)f_{W,X}(t,x)\mathrm{d}x\mathrm{d}t + v_{m,\alpha,\delta}(u;b), \tag{6.7}$$

for $m \in \mathbb{N}$ and $0 \leq u \leq b$, where

$$v_{m,\alpha,\delta}(u;b) = \begin{cases} \sum_{j=0}^{m-1}\int_0^{\frac{b-u}{c}} e^{-(\alpha+m\delta)t}\int_0^{u+ct} h(x)^{m-j}\phi_{j,\alpha,\delta}(u+ct-x;b)f_{W,X}(t,x)\mathrm{d}x\mathrm{d}t, & m \in \mathbb{N}^+, \\ e^{-\alpha\frac{b-u}{c}}\bar{F}_W\left(\frac{b-u}{c}\right), & m = 0, \end{cases}$$

Similarly for $\Phi_{m,\alpha,\delta}(u;b)$ defined in (6.4), we have

$$\Phi_{m,\alpha,\delta}(u;b) = \int_0^{\frac{b-u}{c}} e^{-(\alpha+m\delta)t} \int_0^{u+ct} \Phi_{m,\alpha,\delta}(u+ct-x;b) f_{W,X}(t,x)\mathrm{d}x\mathrm{d}t + V_{m,\alpha,\delta}(u;b), \quad (6.8)$$

for $m \in \mathbb{N}$ and $0 \leq u \leq b$, where

$$\begin{aligned} V_{m,\alpha,\delta}(u;b) = &\sum_{j=0}^{m-1} \int_0^{\frac{b-u}{c}} e^{-(\alpha+m\delta)t} \int_0^{u+ct} h(x)^{m-j} \Phi_{j,\alpha,\delta}(u+ct-x;b) f_{W,X}(t,x)\mathrm{d}x\mathrm{d}t \\ &+ \int_0^{\frac{b-u}{c}} e^{-(\alpha+m\delta)t} \int_{u+ct}^{\infty} h(x)^m f_{W,X}(t,x)\mathrm{d}x\mathrm{d}t. \end{aligned}$$

We point out that $\phi_{m,\alpha,\delta}$ and $\Phi_{m,\alpha,\delta}$ satisfy integral equations (6.7) and (6.8) respectively, which are of an identical form. As a cornerstone to the analysis of the solution to (6.7) and (6.8) is the integral equation

$$w_\alpha(u) = \frac{1}{c} \int_u^\infty e^{-\alpha \frac{y-u}{c}} \int_0^y w_\alpha(y-x) f_{W,X}\left(\frac{y-u}{c}, x\right) \mathrm{d}x\mathrm{d}y, \quad (6.9)$$

for $u \geq 0$. A set of independent solutions to (6.9) plays a similar role in the analysis of the dependent renewal risk model with joint pdf (6.5) as the scale function and scale matrix are to the analysis of the Lévy insurance risk process and the spectrally negative Markov-additive process, respectively.

Next, we focus on solving for a group of independent solutions to (6.9) when $f_{W,X}$ is as given in (6.5). Thus, (6.9) becomes

$$w_\alpha(u) = \sum_{i=1}^n \frac{\lambda_i}{c} \int_u^\infty e^{-(\lambda_i+\alpha)\frac{y-u}{c}} \int_0^y w_\alpha(y-x) g_i(x)\mathrm{d}x\mathrm{d}y. \quad (6.10)$$

Taking LT on both sides of (6.10) with respect to u yields

$$\tilde{w}_\alpha(s) = \sum_{i=1}^n \frac{\lambda_i}{c} \frac{\tilde{w}_\alpha\left(\frac{\lambda_i+\alpha}{c}\right)\tilde{g}_i\left(\frac{\lambda_i+\alpha}{c}\right) - \tilde{w}_\alpha(s)\tilde{g}_i(s)}{s - \frac{\lambda_i+\alpha}{c}},$$

which can be re-organized as

$$\tilde{w}_\alpha(s) = \frac{Q_{n-1}(s)}{\prod_{j=1}^{n}\left(s - \frac{\lambda_j+\alpha}{c}\right) + \sum_{i=1}^{n}\frac{\lambda_i}{c}\left\{\prod_{j=1,j\neq i}^{n}\left(s-\frac{\lambda_j+\alpha}{c}\right)\right\}\tilde{g}_i(s)}, \tag{6.11}$$

where $Q_{n-1}(s)$ is a $(n-1)$th order polynomial in s given by

$$Q_{n-1}(s) = \sum_{i=1}^{n}\frac{\lambda_i}{c}\tilde{g}_i\left(\frac{\lambda_i+\alpha}{c}\right)\tilde{w}_\alpha\left(\frac{\lambda_i+\alpha}{c}\right)\left\{\prod_{j=1,j\neq i}^{n}\left(s-\frac{\lambda_j+\alpha}{c}\right)\right\}.$$

We define by $\{m_{i,\alpha}(u)\}_{u\geq 0}$ for $i = 1, 2, \ldots, n$ a set of n fundamental solutions to (6.10), whose LT can be written as

$$\tilde{m}_{i,\alpha}(s) = \frac{s^{i-1}}{\prod_{j=1}^{n}\left(s - \frac{\lambda_j+\alpha}{c}\right) + \sum_{i=1}^{n}\frac{\lambda_i}{c}\left\{\prod_{j=1,j\neq i}^{n}\left(s-\frac{\lambda_j+\alpha}{c}\right)\right\}\tilde{g}_i(s)}. \tag{6.12}$$

Thus, all functions which satisfy the integral equation (6.10) can be expressed as a linear combination of the fundamental solutions $\{m_{i,\alpha}(u)\}_{u\geq 0}$ for $i = 1, 2, \ldots, n$. By construction, the n fundamental solutions $\{m_{i,\alpha}(u)\}_{u\geq 0}$ $(i = 1, 2, \ldots, n)$ to (6.10) defined through the LT (6.12) are independent (i.e. no solution can be expressed as a linear combination of the other $(n-1)$ solutions). Note that, by the initial value theorem, we have

$$m_{i,\alpha}(0) = \lim_{s\to\infty}\frac{s^i}{\prod_{j=1}^{n}\left(s - \frac{\lambda_j+\alpha}{c}\right) + \sum_{k=1}^{n}\frac{\lambda_k}{c}\left\{\prod_{j=1,j\neq k}^{n}\left(s-\frac{\lambda_j+\alpha}{c}\right)\right\}\tilde{g}_k(s)} = 0,$$

for $i = 1, 2, \ldots, n-1$ and $m_{n,\alpha}(0) = 1$. Thus, together with (6.12), it is immediate that

$$m_{i+1,\alpha}(x) = m_{i,\alpha}^{(1)}(x) = m_{i-1,\alpha}^{(2)}(x) = \ldots = m_{1,\alpha}^{(i)}(x),$$

for $i = 1, 2, \ldots, n-1$.

Furthermore, it is obvious that the denominator in (6.11) has the same zeros as Lundberg's generalized equation (6.6). According to Landriault et al. (2014c), for $\alpha > 0$, by Rouche's theorem, there are exactly n zeros (which we denote $\rho_{1,\alpha}, \rho_{2,\alpha}, \ldots, \rho_{n,\alpha}$) with a positive real part to Lundberg's generalized equation (6.6). By Cossette et al. (2010, Proposition 4.2) and Klimenok (2001, Theorem 1), for $\alpha = 0$ and under the safety loading condition $\mathbb{E}[cW_1 - X_1] > 0$, (6.6) has a zero $\rho_{1,\alpha} = 0$ and exactly $n-1$ zeros (say, $\rho_{2,\alpha}, \rho_{3,\alpha}, \ldots, \rho_{n,\alpha}$) with positive real parts. Henceforth, we assume that $\rho_{i,\alpha} \neq \rho_{j,\alpha}$ for $i, j = 1, 2, \ldots, n$ with $i \neq j$.

Note that an alternative and convenient representation for the LT associated with the fundamental solution $m_{i,\alpha}$ can be found by tying up (6.12) to the known ladder height LT in the insurance surplus process (1.3). Indeed from Equation 50 of Cheung et al. (2010) and Willmot and Woo (2012), we have

$$\tilde{m}_{i,\alpha}(s) = \frac{s^{i-1} \prod_{j=1}^{n} (s - \rho_{j,\alpha})^{-1}}{1 - \varphi_\alpha \tilde{r}_\alpha(s)}, \tag{6.13}$$

for $i = 1, 2, \ldots, n$, where

$$\varphi_\alpha = \frac{\prod_{j=1}^{n}(-\rho_{j,\alpha}) - \prod_{j=1}^{n}\left(-\frac{\lambda_j+\alpha}{c}\right) - \sum_{i=1}^{n} \frac{\lambda_i}{c} \left\{\prod_{j=1, j\neq i}^{n} \left(-\frac{\lambda_j+\alpha}{c}\right)\right\} \tilde{g}_i(0)}{\prod_{j=1}^{n}(-\rho_{j,\alpha})}, \tag{6.14}$$

and $\tilde{r}_\alpha(s)$ is the LT of the ladder height density $\{r_\alpha(x)\}_{x\geq 0}$ given by

$$r_\alpha(x) = \frac{1}{\varphi_\alpha} \sum_{i=1}^{n} \frac{\lambda_i}{c} \sum_{k=1}^{n} \left\{ \frac{\prod_{j=1, j\neq i}^{n} \left(\rho_{k,\alpha} - \frac{\lambda_j+\alpha}{c}\right)}{\prod_{m=1, m\neq k}^{n} \left(\rho_{k,\alpha} - \rho_{m,\alpha}\right)} \right\} \int_x^\infty e^{-\rho_{k,\alpha}(y-x)} g_i(y) \mathrm{d}y. \tag{6.15}$$

From (6.13), a defective renewal equation is easily obtainable for $\{m_{i,\alpha}(x)\}_{x\geq 0}$.

Lemma 25. *The integral equation (6.10) has n independent solutions given by*

$$m_{i,\alpha}(x) = \sum_{j=1}^{n} \xi_{i,j} e^{\rho_{j,\alpha} x} + \varphi_\alpha \int_0^x m_{i,\alpha}(x-y) r_\alpha(y) \mathrm{d}y, \tag{6.16}$$

for $i = 1, 2, \ldots, n$ and $x \geq 0$, where $\xi_{i,j} = \rho_{j,\alpha}^{i-1} \prod_{k=1, k\neq j}^{n} \left(\rho_{k,\alpha} - \rho_{j,\alpha}\right)^{-1}$ for $i, j = 1, 2, \ldots, n$, and $r_\alpha(y)$ is as defined in (6.15).

Also, note that under some distributional assumptions for $g_i(x)$ (for instance, if g_i has a

Coxian density function), the LT (6.12) can be inverted directly through partial fraction expansions.

6.2 Probabilities and moments

In this section, we focus on deriving a nth order IDE for $\phi_{m,\alpha,\delta}(u;b)$ and $\Phi_{m,\alpha,\delta}(u;b)$. By showing that $\{m_{i,\alpha}(x)\}_{i=1}^{n}$ are a set of independent solutions to the IDE, we can express $\phi_{m,\alpha,\delta}(u;b)$ and $\Phi_{m,\alpha,\delta}(u;b)$ for $m \geq 0$ as a function of their lower-order moments.

6.2.1 A homogeneous IDE and the exit probability $m_\alpha(u;b)$

From (6.7) with the joint pdf (6.5), the two-sided exit probability (6.1) can be expressed as

$$m_\alpha(u;b) = \sum_{i=1}^{n} \frac{\lambda_i}{c} \int_u^b e^{-(\lambda_i+\alpha)\frac{y-u}{c}} \int_0^y m_\alpha(y-x;b) g_i(x) \mathrm{d}x \mathrm{d}y + \sum_{j=1}^{n} \eta_j e^{-(\lambda_j+\alpha)\frac{b-u}{c}}, \qquad (6.17)$$

for $0 \leq u \leq b$. Applying the nth order derivative operator $\prod_{i=1}^{n}\left(\mathcal{D} - \frac{\lambda_i+\alpha}{c}\mathcal{I}\right)$ on both sides of (6.17) yields

$$\prod_{j=1}^{n}\left(\mathcal{D} - \frac{\lambda_j+\alpha}{c}\mathcal{I}\right) m_\alpha(u;b) = -\sum_{k=1}^{n} \frac{\lambda_k}{c} \prod_{j=1, j\neq k}^{n}\left(\mathcal{D} - \frac{\lambda_j+\alpha}{c}\mathcal{I}\right) \int_0^u m_\alpha(u-x;b) g_k(x) \mathrm{d}x, \qquad (6.18)$$

for $0 \leq u \leq b$ where $\mathcal{D}$ and $\mathcal{I}$ are respectively the identity and the differentiation operators with respect to u, respectively. By comparing (6.10) and (6.17), it is easily shown that the fundamental solutions $\{m_{i,\alpha}(x)\}_{x\geq 0}$ also satisfy the IDE (6.18). Thus, $\{m_{i,\alpha}(\cdot)\}_{i=1}^{n}$ provide a set of independent solutions to (6.18). According to the general theory on IDEs, the solution to (6.17) can be expressed as a linear combination of $\{m_{i,\alpha}(x)\}_{i=1}^{n}$, namely

$$m_\alpha(u;b) = \beta_{1,b} m_{1,\alpha}(u) + \beta_{2,b} m_{2,\alpha}(u) + \ldots + \beta_{n,b} m_{n,\alpha}(u), \qquad (6.19)$$

for $0 \leq u \leq b$, where the coefficients $\{\beta_{i,b}\}_{i=1}^{n}$ can be found through boundary conditions of $m_\alpha(u;b)$ at b. The result is stated in the following theorem.

Theorem 26. *The two-sided exit probability $m_\alpha(u;b)$ can be expressed as*

$$m_\alpha(u;b) = Z_{1,\alpha}(u;b) + \sum_{i=1}^{n-1} \nu_i Z_{i+1,\alpha}(u;b), \tag{6.20}$$

for $u \leq b$, where for $i = 1, 2, \ldots, n$,

$$\nu_i = \sum_{k=i+1}^{n} \eta_k \left(\prod_{l=1}^{i} \frac{\lambda_k - \lambda_l}{c} \right),$$

$$Z_{i,\alpha}(u;b) = \begin{pmatrix} m_{1,\alpha}(u) \\ m_{2,\alpha}(u) \\ \vdots \\ m_{n,\alpha}(u) \end{pmatrix}^{\intercal} \begin{pmatrix} m_{1,\alpha}(b) & m_{2,\alpha}(b) & \ldots & m_{n,\alpha}(b) \\ \chi_{1,1,\alpha}(b) & \chi_{2,1,\alpha}(b) & \ldots & \chi_{n,1,\alpha}(b) \\ \vdots & \vdots & \ddots & \vdots \\ \chi_{1,n-1,\alpha}(b) & \chi_{2,n-1,\alpha}(b) & \ldots & \chi_{n,n-1,\alpha}(b) \end{pmatrix}^{-1} \mathbf{e}_i,$$

provided the associated matrix is invertible, and $\mathbf{e}_i$ is an n-dimensional column vector with 1 at the ith position and 0 otherwise.

Proof. From the representation (6.19), the coefficients $\{\beta_{i,b}\}_{i=1}^n$ of the two-sided exit probability $m_\alpha(u;b)$ are derived through a set of equations related to its boundary conditions at b (i.e., $m_\alpha^{(i)}(b;b)$ for $i = 0, 1, 2, \ldots, n-1$). To derive these boundary conditions, we apply the derivative operator $\prod_{j=1}^{i} \left(\mathcal{D} - \frac{\lambda_j + \alpha}{c} \mathcal{I} \right)$ on both sides of (6.17) to obtain

$$\begin{aligned}
&\prod_{j=1}^{i} \left(\mathcal{D} - \frac{\lambda_j + \alpha}{c} \mathcal{I} \right) m_\alpha(u;b) \qquad\qquad (6.21) \\
&= -\sum_{k=1}^{i} \frac{\lambda_k}{c} \prod_{j=1, j\neq k}^{i} \left(\mathcal{D} - \frac{\lambda_j + \alpha}{c} \mathcal{I} \right) \int_0^u m_\alpha(u-x;b) g_k(x) \mathrm{d}x + \sum_{j=i+1}^{n} \eta_j \left(\prod_{l=1}^{i} \frac{\lambda_j - \lambda_l}{c} \right) e^{-(\lambda_j+\alpha)\frac{b-u}{c}} \\
&- \sum_{k=i+1}^{n} \frac{\lambda_k}{c} \sum_{l=1}^{i} \left(\prod_{q=1}^{i-1} \frac{\lambda_k - \lambda_q}{c} \right) \prod_{j=l+1}^{i} \left(\mathcal{D} - \frac{\lambda_j + \alpha}{c} \mathcal{I} \right) \int_0^u m_\alpha(u-x;b) g_k(x) \mathrm{d}x \\
&+ \sum_{k=i+1}^{n} \frac{\lambda_k}{c} \left(\prod_{q=1}^{i} \frac{\lambda_k - \lambda_q}{c} \right) \int_u^b e^{-(\lambda_k+\alpha)\frac{y-u}{c}} \int_0^y m_\alpha(y-x;b) g_k(x) \mathrm{d}x \mathrm{d}y,
\end{aligned}$$

for $i = 1, 2, \ldots, n-1$. By substituting (6.19) into (6.21), we have

$$\begin{aligned} \beta_{1,b} m_{1,\alpha}(b) + \beta_{2,b} m_{2,\alpha}(b) + \ldots + \beta_{n,b} m_{n,\alpha}(b) &= 1, \\ \beta_{1,b} \chi_{1,1,\alpha}(b) + \beta_{2,b} \chi_{2,1,\alpha}(b) + \ldots + \beta_{n,b} \chi_{n,1,\alpha}(b) &= \textstyle\sum_{k=2}^{n} \eta_k \left(\frac{\lambda_k - \lambda_1}{c} \right), \\ &\vdots \\ \beta_{1,b} \chi_{1,n-1,\alpha}(b) + \beta_{2,b} \chi_{2,n-1,\alpha}(b) + \ldots + \beta_{n,b} \chi_{n,n-1,\alpha}(b) &= \eta_n \left(\textstyle\prod_{l=1}^{n-1} \frac{\lambda_n - \lambda_l}{c} \right), \end{aligned} \tag{6.22}$$

where for $i = 1, 2, \ldots, n-1$, and $j = 1, 2, \ldots, n$,

$$\begin{aligned} \chi_{j,i,\alpha}(b) = &\prod_{l=1}^{i} \left(\mathcal{D} - \frac{\lambda_l + \alpha}{c} \mathcal{I} \right) m_{j,\alpha}(b) + \sum_{k=1}^{i} \frac{\lambda_k}{c} \prod_{l=1, l \neq k}^{i} \left(\mathcal{D} - \frac{\lambda_l + \alpha}{c} \mathcal{I} \right) \int_0^b m_{j,\alpha}(b-x) g_k(x) \mathrm{d}x \\ &+ \sum_{k=i+1}^{n} \frac{\lambda_k}{c} \sum_{l=1}^{i} \left(\prod_{q=1}^{i-1} \frac{\lambda_k - \lambda_q}{c} \right) \prod_{m=l+1}^{i} \left(\mathcal{D} - \frac{\lambda_m + \alpha}{c} \mathcal{I} \right) \int_0^b m_{j,\alpha}(b-x) g_k(x) \mathrm{d}x. \end{aligned}$$

Thus, (6.20) is obtained by some simple matrix operations on the equations in (6.22). □

6.2.2 Inhomogeneous IDEs and moments

In this section, we propose to solve the integral equation

$$W_\alpha(u; b) = \frac{1}{c} \int_u^b e^{-\alpha \frac{y-u}{c}} \int_0^y W_\alpha(y-x; b) f_{W,X} \left(\frac{y-u}{c}, x \right) \mathrm{d}x \mathrm{d}y + V_b(u), \tag{6.23}$$

for $0 \leq u \leq b$, where the joint pdf $f_{W,X}$ is of the form (6.5). It is further assumed that $V_b(u)$ is n-time differentiable with respect to u on $[0, b]$. From (6.7) and (6.8), it is clear that the interest in Equation (6.23) resides in the fact that both $\phi_{m,\alpha,\delta}(u; b)$ and $\Phi_{m,\alpha,\delta}(u; b)$ satisfy an integral equation of this form. More specifically, we have $\phi_{m,\alpha,\delta}(u; b) = W_{\alpha+m\delta}(u; b)$ with $V_b(u) = v_{m,\alpha,\delta}(u; b)$ for $m \in \mathbb{N}^+$ and $0 \leq u \leq b$. Similarly, for $m \in \mathbb{N}$ and $0 \leq u \leq b$, $\Phi_{m,\alpha,\delta}(u; b) = W_{\alpha+m\delta}(u; b)$ with $V_b(u) = V_{m,\alpha,\delta}(u; b)$.

Similarly as in the homogeneous case, we apply the operator $\prod_{j=1}^{n} \left(\mathcal{D} - \frac{\lambda_j + \alpha}{c} \mathcal{I} \right)$ on both sides of (6.23) to obtain

$$\prod_{j=1}^{n}\left(\mathcal{D}-\frac{\lambda_j+\alpha}{c}\mathcal{I}\right)W_\alpha(u;b)=-\sum_{k=1}^{n}\frac{\lambda_k}{c}\prod_{j=1,j\neq k}^{n}\left(\mathcal{D}-\frac{\lambda_j+\alpha}{c}\mathcal{I}\right)\int_0^u W_\alpha(u-x;b)g_k(x)\mathrm{d}x+\kappa_b(u), \tag{6.24}$$

for $u \leq b$, where $\kappa_b(u) = \prod_{j=1}^{n}\left(\mathcal{D}-\frac{\lambda_j+\alpha}{c}\mathcal{I}\right)V_b(u)$. We remark that $\kappa_b(u)$ is independent of b for the two-sided exit probabilities $m_\alpha(u;b)$ and $M_\alpha(u;b)$. However, it is not true in general for $\phi_{m,\alpha,\delta}(u;b)$ and $\Phi_{m,\alpha,\delta}(u;b)$ when $m \geq 1$. Thus, the results in Cheung (2013) to deal with the discounted aggregate claims until ruin cannot be directly applied here. A particular solution to (6.23) can be found by extending the domain of definition of the IDE (6.23) to $u \geq 0$ and looking for a solution to the resulting IDE:

$$\prod_{j=1}^{n}\left(\mathcal{D}-\frac{\lambda_j+\alpha}{c}\mathcal{I}\right)W_{\alpha,b}(u)=-\sum_{k=1}^{n}\frac{\lambda_k}{c}\prod_{j=1,j\neq k}^{n}\left(\mathcal{D}-\frac{\lambda_j+\alpha}{c}\mathcal{I}\right)\int_0^u W_{\alpha,b}(u-x)g_k(x)\mathrm{d}x+\kappa_b(u), \tag{6.25}$$

for $u \geq 0$, where we extend the definition of $\kappa_b(u)$ to $u > b$. As illustrated later, the solution to (6.25) on $[0,b]$ is independent of the extended definition of $\kappa_b(u)$ on $u > b$. Thus, we assume that $\kappa_b(u)$ for $u > b$ is arbitrarily defined, but such that $\tilde{\kappa}_b(s) = \int_0^\infty e^{-sx}\kappa_b(x)\mathrm{d}x$ exists for $s > 0$. A particular solution, which we denote by $\{M_{\alpha,b}(u)\}_{u\geq 0}$, to the IDE (6.25) is found by taking LT on both sides of (6.25) and letting the boundary conditions $W_{\alpha,b}^{(i)}(0) = 0$ for $i = 0, 1, \ldots, n-1$. It follows that

$$\tilde{M}_{\alpha,b}(s) = \frac{\tilde{\kappa}_b(s)}{\prod_{j=1}^{n}\left(s-\frac{\lambda_j+\alpha}{c}\right)+\sum_{i=1}^{n}\frac{\lambda_i}{c}\left\{\prod_{j=1,j\neq i}^{n}\left(s-\frac{\lambda_j+\alpha}{c}\right)\right\}\tilde{g}_i(s)}.$$

which, using (6.13), can be rewritten as

$$\tilde{M}_{\alpha,b}(s) = \frac{1}{1-\varphi_\alpha\tilde{r}_\alpha(s)}\frac{\tilde{\kappa}_b(s)}{\prod_{i=1}^{n}(s-\rho_{i,\alpha})}. \tag{6.26}$$

By inversion of (6.26), we conclude

$$M_{\alpha,b}(x) = \sum_{i=1}^{n} \xi_i \int_0^x e^{\rho_{i,\alpha}(x-y)} \kappa_b(y)\mathrm{d}y + \varphi_\alpha \int_0^x M_{\alpha,b}(x-y) r_\alpha(y)\mathrm{d}y, \tag{6.27}$$

for $x \geq 0$, where φ_α and $r_\alpha(y)$ are as defined in (6.14) and (6.15) respectively, and $\xi_i = \prod_{j=1, j\neq i}^{n} (\rho_{i,\alpha} - \rho_{j,\alpha})^{-1}$ for $i = 1, 2, \ldots, n$. We further remark that the specification of $M_{\alpha,b}(u)$ on $u > b$ is not unique due to the arbitrary extension of $\kappa_b(u)$ on $u > b$.

Thus, $W_\alpha(u; b) - M_{\alpha,b}(u)$ on $u \in [0, b]$ satisfies the IDE (6.18). By utilizing the properties of inhomogeneous IDEs, the solution to (6.23) is given by

$$W_\alpha(u; b) = M_{\alpha,b}(u) + \beta^*_{1,b} m_{1,\alpha}(u) + \beta^*_{2,b} m_{2,\alpha}(u) + \ldots + \beta^*_{n,b} m_{n,\alpha}(u), \tag{6.28}$$

for $0 \leq u \leq b$, where $\{m_{i,\alpha}(x)\}_{i=1}^{n}$ are given by (6.16) for $x \geq 0$.

Note that Equation (6.28) holds for $0 \leq u \leq b$, and thus makes use of the particular solution $M_{\alpha,b}(u)$ (defined in (6.27)) on $[0, b]$ only, which is independent of the extended definition of $\kappa_b(u)$ on $u \geq b$.

Theorem 27. *The integral equation* (6.23) *can be represented as*

$$W_\alpha(u; b) = M_{\alpha,b}(u) + \left[V_b(b) - M_{\alpha,b}(b)\right] Z_{1,\alpha}(u; b) - \sum_{i=1}^{n-1} \gamma_{i,\alpha}(b) Z_{i+1,\alpha}(u; b), \tag{6.29}$$

for $0 \leq u \leq b$, *where*

$$\begin{aligned}
\gamma_{i,\alpha}(b) = &\prod_{j=1}^{i} \left(\mathcal{D} - \frac{\lambda_j + \alpha}{c}\mathcal{I}\right) M_{\alpha,b}(b) + \sum_{k=1}^{i} \frac{\lambda_k}{c} \prod_{j=1, j\neq k}^{i} \left(\mathcal{D} - \frac{\lambda_j + \alpha}{c}\mathcal{I}\right) \int_0^b M_{\alpha,b}(b-x) g_k(x)\mathrm{d}x \\
&+ \sum_{k=i+1}^{n} \frac{\lambda_k}{c} \sum_{l=1}^{i} \left(\prod_{q=1}^{i-1} \frac{\lambda_k - \lambda_q}{c}\right) \prod_{j=l+1}^{i} \left(\mathcal{D} - \frac{\lambda_j + \alpha}{c}\mathcal{I}\right) \int_0^b M_{\alpha,b}(b-x) g_k(x)\mathrm{d}x \\
&- \prod_{j=1}^{i} \left(\mathcal{D} - \frac{\lambda_j + \alpha}{c}\mathcal{I}\right) V_b(b),
\end{aligned} \tag{6.30}$$

for $i = 1, 2, \ldots, n-1$.

Proof. Similarly as for the two-sided exit probability $m_\alpha(u;b)$, for $i = 1, 2, \ldots, n-1$, we apply the derivative operator $\prod_{j=1}^{i}\left(\mathcal{D} - \frac{\lambda_j+\alpha}{c}\mathcal{I}\right)$ on both sides of (6.23) to obtain the boundary conditions $\{W_\alpha^{(i)}(b;b)\}_{i=0}^{n-1}$, namely

$$\prod_{j=1}^{i}\left(\mathcal{D} - \frac{\lambda_j+\alpha}{c}\mathcal{I}\right) W_\alpha(u;b) \tag{6.31}$$
$$= -\sum_{k=1}^{i}\frac{\lambda_k}{c}\prod_{j=1,j\neq k}^{i}\left(\mathcal{D} - \frac{\lambda_j+\alpha}{c}\mathcal{I}\right)\int_0^u W_\alpha(u-x;b)g_k(x)\mathrm{d}x + \prod_{j=1}^{i}\left(\mathcal{D} - \frac{\lambda_j+\alpha}{c}\mathcal{I}\right)V_b(u)$$
$$- \sum_{k=i+1}^{n}\frac{\lambda_k}{c}\sum_{l=1}^{i}\left(\prod_{q=1}^{i-1}\frac{\lambda_k-\lambda_q}{c}\right)\prod_{j=l+1}^{i}\left(\mathcal{D} - \frac{\lambda_j+\alpha}{c}\mathcal{I}\right)\int_0^u W_\alpha(u-x;b)g_k(x)\mathrm{d}x$$
$$+ \sum_{k=i+1}^{n}\frac{\lambda_k}{c}\left(\prod_{q=1}^{i}\frac{\lambda_k-\lambda_q}{c}\right)\int_u^b e^{-(\lambda_k+\alpha)\frac{y-u}{c}}\int_0^y W_\alpha(y-x;b)g_k(x)\mathrm{d}x\mathrm{d}y,$$

for $i = 1, 2, \ldots, n-1$ and it is also easily known that $W_\alpha(b;b) = V_b(b)$ from (6.23). The coefficients $\beta^*_{1,b}, \beta^*_{2,b}, \ldots, \beta^*_{n,b}$ in (6.28) can be determined by (6.31), i.e.

$$W_\alpha^{(i)}(b;b) = M_{\alpha,b}^{(i)}(b) + \beta^*_{1,b}m_{1,\alpha}^{(i)}(b) + \beta^*_{2,b}m_{2,\alpha}^{(i)}(b) + \ldots + \beta^*_{n,b}m_{n,\alpha}^{(i)}(b),$$

for $i = 0, 1, \ldots, n-1$. $\square$

Thus, the moments of the discounted aggregate claims in the two-sided exit setting are easily obtainable from (6.29).

Corollary 28. *For the insurance risk process* (1.3) *with joint pdf* (6.5), *the moments of the total discounted claims* (6.3) *and* (6.4) *are respectively given by*

$$\phi_{m,\alpha,\delta}(u;b) = w_{m,\alpha,\delta,b}(u) - w_{m,\alpha,\delta,b}(b)Z_{1,\alpha+m\delta}(u;b) - \sum_{i=1}^{n-1}\vartheta_{i,m}(b)Z_{i+1,\alpha+m\delta}(u;b), \tag{6.32}$$

for $m \geq 1$ *and*

$$\Phi_{m,\alpha,\delta}(u;b) = W_{m,\alpha,\delta,b}(u) - W_{m,\alpha,\delta,b}(b)Z_{1,\alpha+m\delta}(u;b) - \sum_{i=1}^{n-1}\Theta_{i,m}(b)Z_{i+1,\alpha+m\delta}(u;b), \tag{6.33}$$

for $m \geq 0$ and $0 \leq u \leq b$, where $w_{m,\alpha,\delta,b}(u)$ ($W_{m,\alpha,\delta,b}(u)$) is the particular solution $M_{\alpha+m\delta,b}(u)$ defined in (6.27) with $\kappa_b(u) = \prod_{j=1}^{n}\left(\mathcal{D} - \frac{\lambda_j+\alpha}{c}\mathcal{I}\right) v_{m,\alpha,\delta}(u;b)$ ($\kappa_b(u) = \prod_{j=1}^{n}\left(\mathcal{D} - \frac{\lambda_j+\alpha}{c}\mathcal{I}\right) V_{m,\alpha,\delta}(u;b)$) and $\vartheta_{i,m}(b)$ ($\Theta_{i,m}(b)$) is given by $\gamma_{i,\alpha+m\delta}(b)$ defined in (6.30) with $M_{\alpha+m\delta,b}(u)$ replaced by $w_{m,\alpha,\delta,b}(u)$ ($W_{m,\alpha,\delta,b}(u)$) and $V_b(u)$ by $v_{m,\alpha,\delta}(u;b)$ ($V_{m,\alpha,\delta}(u;b)$).

Note that the terms $w_{m,\alpha,\delta,b}$ and $W_{m,\alpha,\delta,b}$ defined in (6.32) and (6.33) respectively only depend on the lower order moments, therefore, the total discounted claim costs as defined in (6.7) and (6.8) are obtainable recursively.

To illustrate the relationship between the two-sided exit probabilities, for $m = 0$, we re-write (6.33) as

$$M_\alpha(u;b) = W_{0,\alpha,\delta,b}(u) - W_{0,\alpha,\delta,b}(b)m_\alpha(u;b) - \sum_{i=1}^{n-1}(\Theta_{i,0}(b) - W_{0,\alpha,\delta,b}(b)\nu_i)Z_{i+1,\alpha}(u;b). \quad (6.34)$$

Under the independence case (i.e. $n = 1$), (6.34) recovers the classical result for spectrally negative Lévy processes (see Kyprianou (2006, Chapter 8)).

6.3 An example under FGM copula

In this section, we consider the FGM copula

$$C_{FGM}(u,v) = uv + \theta uv(1-u)(1-v), \quad (6.35)$$

which is a special case of the Bernstein copula with $n = \ell = 2$ and $C\left(\frac{1}{2},\frac{1}{2}\right) = \frac{1+\theta}{4}$ for $\theta \in [-1,1]$. We further assume that W is exponentially distributed with mean $1/\lambda$ while X has pdf g and df $G(\cdot) = 1 - \bar{G}(\cdot)$. Hence, the joint pdf of (W,X) satisfies (6.5) with $\lambda_1 = \lambda$, $\lambda_2 = 2\lambda$, $g_1(x) = [1-\theta+2\theta G(x)]g(x)$ and $g_2(x) = \theta[1-2G(x)]g(x)$. In what follows, we explicitly exclude the independent case (i.e. $\theta = 0$) as the resulting insurance surplus process is spectrally negative Lévy. Let $0 \leq \rho_{1,\alpha} < \rho_{2,\alpha}$ be the two nonnegative distinct solutions of (6.6). Then, the two fundamental solutions $m_{1,\alpha}$ and $m_{2,\alpha}$ defined through the LT (6.12) are given by

$$m_{i,\alpha}(x) = \frac{\rho_{1,\alpha}^{i-1} e^{\rho_{1,\alpha}x} - \rho_{2,\alpha}^{i-1} e^{\rho_{2,\alpha}x}}{\rho_{2,\alpha} - \rho_{1,\alpha}} + \int_0^x m_{i,\alpha}(x-y)\varphi_\alpha r_\alpha(y)\mathrm{d}y,$$

for $x \geq 0$ and $i = 1, 2$, where

$$\begin{aligned}\varphi_\alpha r_\alpha(x) = &\frac{\lambda}{c}\int_x^\infty \left[\frac{\rho_{1,\alpha} - \frac{2\lambda+\alpha}{c}}{\rho_{1,\alpha} - \rho_{2,\alpha}} e^{-\rho_{1,\alpha}(y-x)} + \frac{\rho_{2,\alpha} - \frac{2\lambda+\alpha}{c}}{\rho_{2,\alpha} - \rho_{1,\alpha}} e^{-\rho_{2,\alpha}(y-x)}\right] g(y)[1+\theta(1-2G(y))]\mathrm{d}y \\ &+ \left(\frac{\lambda}{c}\right)^2 \int_x^\infty \left[\frac{e^{-\rho_{1,\alpha}(y-x)} - e^{-\rho_{2,\alpha}(y-x)}}{\rho_{1,\alpha} - \rho_{2,\alpha}}\right] 2\theta g(y)[1-2G(y)]\mathrm{d}y.\end{aligned}$$

Furthermore, from Theorem 26, we have

$$Z_{1,\alpha}(u;b) = \frac{\chi_{2,\alpha}(b)m_{1,\alpha}(u) - \chi_{1,\alpha}(b)m_{2,\alpha}(u)}{\chi_{2,\alpha}(b)m_{1,\alpha}(b) - \chi_{1,\alpha}(b)m_{2,\alpha}(b)},$$

and

$$Z_{2,\alpha}(u;b) = \frac{m_{2,\alpha}(b)m_{1,\alpha}(u) - m_{1,\alpha}(b)m_{2,\alpha}(u)}{\chi_{2,\alpha}(b)m_{1,\alpha}(b) - \chi_{1,\alpha}(b)m_{2,\alpha}(b)},$$

where

$$\chi_{i,\alpha}(b) = m'_{i,\alpha}(b) - \frac{\lambda+\alpha}{c} m_{i,\alpha}(b) + \frac{\lambda}{c}\int_0^b m_{i,\alpha}(b-x)g(x)\left[1+\theta(1-2G(x))\right]\mathrm{d}x.$$

6.3.1 The two-sided exit probabilities

From (6.20) and $\eta_2 = 0$, the two-sided exit probability $m_\alpha(u;b)$ is given by

$$m_\alpha(u;b) = Z_{1,\alpha}(u;b).$$

Also, the other two-sided exit probability $M_\alpha(u;b)$ satisfying IDE (6.24) with

$$\begin{aligned}\kappa_b(x) &= \frac{\lambda_1}{c}\left[g_1(x) + \frac{\lambda_2+\alpha}{c}\bar{G}_1(x)\right] + \frac{\lambda_2}{c}\left[g_2(x) + \frac{\lambda_1+\alpha}{c}\bar{G}_2(x)\right] \\ &= \frac{\lambda}{c}g(x) + \frac{2\lambda^2+\lambda\alpha}{c^2}\bar{G}(x) + \theta\left[\frac{\lambda}{c}g(x)[1-2G(x)] - \frac{\lambda\alpha}{c^2}G(x)\bar{G}(x)\right].\end{aligned}$$

Thus, a particular solution to (6.25) is given by

$$M_\alpha(x) = \int_0^x \frac{e^{\rho_{1,\alpha}(x-y)} - e^{\rho_{2,\alpha}(x-y)}}{\rho_{1,\alpha} - \rho_{2,\alpha}} \kappa_b(y)\mathrm{d}y + \int_0^x M_\alpha(x-y)\varphi_\alpha r_\alpha(y)\mathrm{d}y,$$

for $x \geq 0$. Finally, the probability $M_\alpha(u;b)$ can be represented as

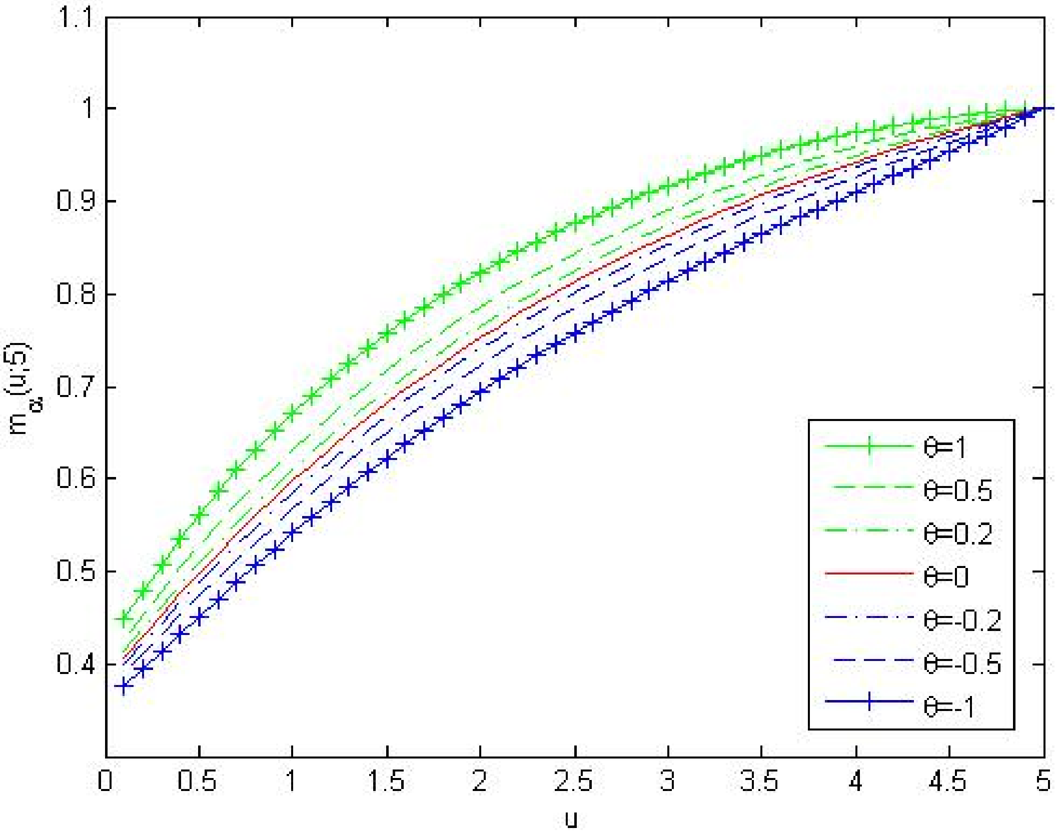
$$M_\alpha(u;b) = M_\alpha(u) - M_\alpha(b)m_\alpha(u;b) - \gamma_\alpha(b)Z_{2,\alpha}(u;b),$$

for $u \leq b$, where

$$\gamma_\alpha(b) = M_\alpha'(b) - \frac{\lambda+\alpha}{c}M_\alpha(b) + \frac{\lambda}{c}\left[\int_0^b M_\alpha(b-x)g(x)[1+\theta(1-2G(x))]\,\mathrm{d}x + \bar{G}(b)(1-\theta G(b))\right].$$

Example 9. *Under the FGM copula (6.35), we consider the two-sided exit probability $m_0(u;b) = 1 - M_0(u;b)$. We provide a numerical example for the model where the claim severity X also follows an exponential distribution with mean 1, the premium rate $c = 3$ and $\lambda = 2$. For $b = 5$, we draw the curves of $m_\alpha(u;b)$ for different dependence parameters: $\theta = -1$, -0.5, -0.2, 0, 0.2, 0.5, 1.*

Figure 6.1: The probability of reaching level 5 before ruin under different θ



From Figure 6.1, we can see that the probability that the surplus process reaches level 5 *before dropping below* 0 *increases with the dependence parameter* θ*.*

6.3.2 Mean of the discounted aggregate claims

From (6.32), we have

$$\phi_{1,\alpha,\delta}(u;b) = w_{1,\alpha,\delta,b}(u) - w_{1,\alpha,\delta,b}(b)m_{\alpha+\delta}(u;b) - \vartheta_{1,1}(b)Z_{2,\alpha+\delta}(u;b),$$

for $0 \leq u \leq b$, where

$$w_{1,\alpha,\delta,b}(u) = \int_0^u \frac{e^{\rho_{1,\alpha}(u-y)} - e^{\rho_{2,\alpha}(u-y)}}{\rho_{1,\alpha} - \rho_{2,\alpha}}\kappa_{1,b}(y)\mathrm{d}y + \int_0^u w_{1,\alpha,\delta,b}(u-y)\varphi_{\alpha+\delta}r_{\alpha+\delta}(y)\mathrm{d}y, \quad (6.36)$$

with

$$\kappa_{1,b}(u) = -\frac{\lambda}{c}\left\{m_\alpha(0;b)[1+\theta(1-2G(u))]ug(u) + \int_0^u m'_\alpha(u-x;b)[1+\theta(1-2G(x))]xg(x)\mathrm{d}x\right\}$$
$$+ \int_0^u m_\alpha(u-x;b)\left[\frac{2\lambda^2+\lambda(\alpha+\delta)}{c^2} + \theta\frac{\lambda(\alpha+\delta)}{c^2}(1-2G(x))\right]xg(x)\mathrm{d}x,$$

and

$$\vartheta_{1,1}(b) = w'_{1,\alpha,\delta,b}(b) - \frac{\lambda+\alpha+\delta}{c}w_{1,\alpha,\delta,b}(b) + \frac{\lambda}{c}\int_0^b w_{1,\alpha,\delta,b}(b-x)\left[1+\theta-2\theta G(x)\right]g(x)\mathrm{d}x$$
$$+ \frac{\lambda}{c}\int_0^b m_\alpha(b-x;b)\left[1+\theta-2\theta G(x)\right]xg(x)\mathrm{d}x.$$

Similarly, from (6.33), the other expectation is written as

$$\Phi_{1,\alpha,\delta}(u;b) = W_{1,\alpha,\delta,b}(u) - W_{1,\alpha,\delta,b}(b)m_{\alpha+\delta}(u;b) - \varTheta_{1,1}(b)Z_{2,\alpha+\delta}(u;b),$$

for $0 \le u \le b$, where $W_{1,\alpha,\delta,b}(u)$ is the same as $w_{1,\alpha,\delta,b}(u)$ as defined in (6.36) with $\kappa_{1,b}(u)$ replaced by

$$\kappa^*_{1,b}(u) = -\frac{\lambda}{c}\left\{[M_\alpha(0;b)-1]\left[1+\theta(1-2G(u))\right]ug(u) + \int_0^u M'_\alpha(u-x;b)[1+\theta(1-2G(x))]xg(x)\mathrm{d}x\right\}$$
$$+ \left(\int_0^u M_\alpha(u-x;b) + \int_u^\infty\right)\left[\frac{2\lambda^2+\lambda(\alpha+\delta)}{c^2} + \theta\frac{\lambda(\alpha+\delta)}{c^2}(1-2G(x))\right]xg(x)\mathrm{d}x,$$

and $\varTheta_{1,1}(b)$ is given by

$$\varTheta_{1,1}(b) = W'_{1,\alpha,\delta,b}(b) - \frac{\lambda+\alpha+\delta}{c}W_{1,\alpha,\delta,b}(b) + \frac{\lambda}{c}\int_0^b W_{1,\alpha,\delta,b}(b-x)g(x)\left[1+\theta-2\theta G(x)\right]\mathrm{d}x$$
$$+ \frac{\lambda}{c}\left(\int_0^b m_\alpha(b-x;b) + \int_b^\infty\right)\left[1+\theta-2\theta G(x)\right]xg(x)\mathrm{d}x.$$

Example 10. *We provide a numerical example with the same setting as for the two-exit probabilities except that we let $\lambda = 1$, $\beta = 1.5$, $\alpha = 0.1$ and $\delta = 0.05$. Figures 6.2 and 6.3 display the expected discounted aggregate claims $\phi_{1,\alpha,\delta}(u;b)$ and $\Phi_{1,\alpha,\delta}(u;b)$ when the dependence parameter θ is 0.5 and -0.5 respectively.*

Figure 6.2: The expectation of the discounted aggregate claims with $\theta = 0.5$

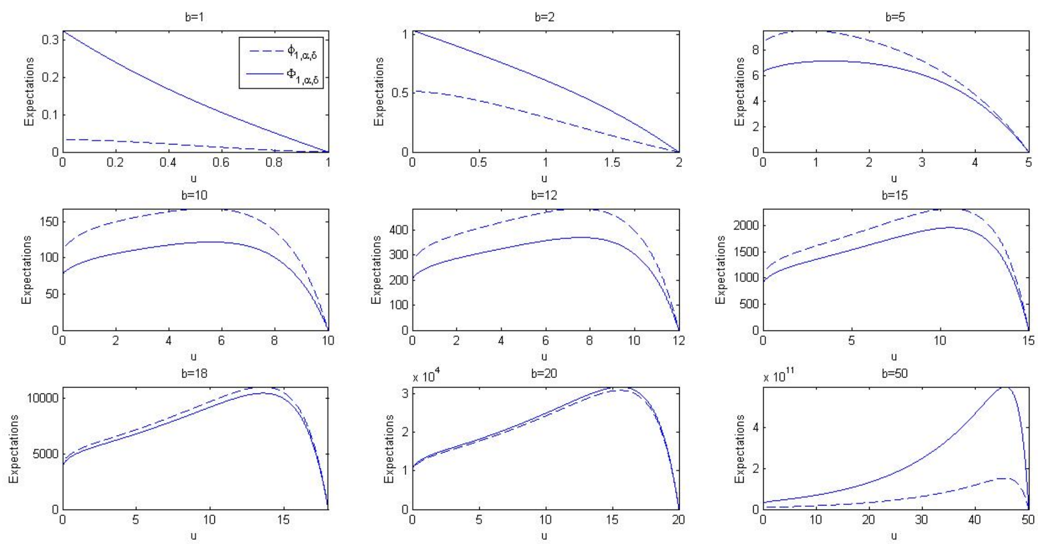
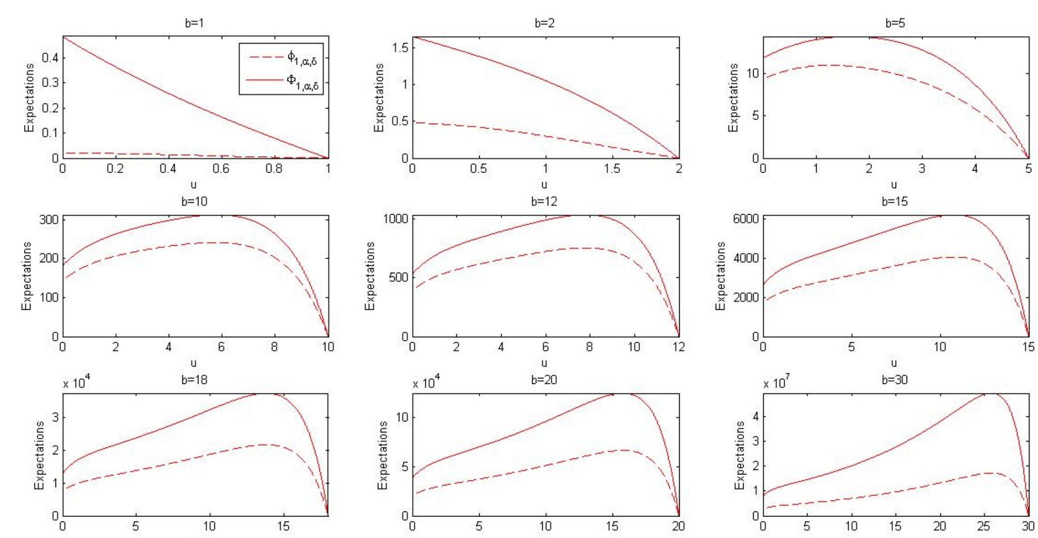


Figure 6.3: The expectation of the discounted aggregate claims with $\theta = -0.5$



We point out that the case $\theta = (-)0.5$ *corresponds to a positive (negative) dependence between the interclaim time and the resulting claim size. From Figures 6.2 and 6.3, two notable observations will be made:*

1. *We notice that, for small b, both $\phi_{1,\alpha,\delta}(u;b)$ and $\Phi_{1,\alpha,\delta}(u;b)$ decrease in u. For larger b, this monotonicity is lost as middle-range values of initial capital u leads on average to larger discounted claim amount until the first $[0,b]$ exit.*

2. *We observe that the ordering of the solid and dashed lines are often reversed. This can be partially explained from Figure 6.1 as, all else being equal, a smaller dependence parameter θ leads to a larger exit probability $M(u;b)$ from level 0, which then translates into a larger value $\Phi_{1,\alpha,\delta}(u;b)$ (compared to $\phi_{1,\alpha,\delta}(u;b)$).*

Chapter 7

Concluding Remarks and Future Work

The time-dependent aggregate claims have been examined in depth in this thesis. In Chapter 3, an integral representation for the transition probabilities of the birth process is derived, which is then applied to the analysis of the time-dependent claims. The present derivation is purely analytic, involving only elementary calculus. The sum of time-dependent and not necessarily identically distributed rv's is represented as that of iid rv's. Finally, due to the conditional (as opposed to marginal) nature of the results, the analysis holds for any Markov counting process which behaves as a nonhomogeneous birth process beyond a certain point.

As a special case of the time-dependent aggregate claim model, the IBNR claim problem is considered in Chapter 4. The formulas derived in this thesis for the LT of the total discounted IBNR claim amount extend those obtained by Léveillé and Garrido (2001a, 2001b) for the discounted renewal sums. The moments and joint moments derived for the discounted sum recover the results by Léveillé and Adékambi (2010, 2012) and Wang (2010) as special cases. The nth order ODE for the pgf of the IBNR claim number in the renewal process with Coxian interclaim times can be solved numerically with constraints to the boundary conditions. In addition, the closed form for the pmf of the IBNR claim number derived for models with exponential reporting lags and mixed Erlang interarrival times are very meaningful from the point of view of both mathematical and computational applications. Results obtained in the renewal model can be easily extended to the delayed case given that only the first interarrival time is impacted (see, e.g., Cox (1962) and Ross (1996)). An underlying environmental process is further incorporated into the analysis in Chapter 5 by considering a MAP claim arrival process.

Chapter 6 considers the aggregate claims until a randomized time horizon (also called observation time) instead of a deterministic one as in the previous chapters. However, a generic randomization of the time horizon does not ease the analysis of the discounted aggregate claims. Thus, the focus is on examining the moments and probabilities of the aggregate claims under some specific randomized time horizons, i.e. τ_b^+ and τ_0^-. Analysis involving one passage time τ_0^- (or τ_b^+) has been extensively studied in ruin theory (dividend payment and taxation problems). Therefore, the model involving two passage times provides contributions to the study of the randomized observation problem.

Based on the research topics studied in this thesis, I propose to generalize the aggregate claim study in the following directions.

First, I propose to continue the analysis of the two-sided exit problem with dependence in spectrally-negative Markov-additive process (spectrally negative Lévy processes) as an extension to Chapter 6. Kyprianou and Palmowski (2008) discussed the properties of the spectrally negative Markov additive process. See also Ivanovs and Palmowski (2012). Later on, more attention has been paid to analyzing ruin-related problems in the Markov-additive process (e.g. Asmussen and Albrecher (2010) and reference therein). Introducing this process helps to incorporate the influence of the financial markets on an insurer's surplus process (e.g., Garrido and Morales (2006)). See also Yang and Zhang (2001). Despite the popularity of these processes, relatively few papers have examined the properties of the aggregate claims under a two-sided exit setting in the Markov-additive process. Some earlier attempts involve considering this problem in a MAP risk model (e.g., Cheung and Landriault (2010)). This project is aimed at examining probabilities and moments of the aggregate claims until some randomized time determined by a spectrally-negative Markov-additive process.

Second, I plan to examine the properties of aggregate claims under the Cox process. Cox processes, also called doubly stochastic Poisson process, allows for a random arrival rate over time, and it would be expected to fit the reality better. Assuming a generic stochastic process for the intensity seems very limited (at least for now) in deriving useful properties of the aggregate claims. Over the years, many scholars have contributed to deriving properties of this process by specifying its intensity function and by application of this process in reality. For instance, Basu and Dassios (2002) calculated the stop-loss expectation of the counting process by assuming the intensity follows a lognormal process. Bouzas et al. (2002), on the other hand, derived the pmf of this process and provided two methods for forecasting the relevant parameters using a Gaussian process as the intensity. See also Bouzas et al. (2006), who considered a periodic intensity. By making use of the infinitesimal generator, Dassios and Jang (2003, 2005, 2008) addressed this problem by assuming a shot noise intensity. See also Badescu et al. (2016), where the intensity follows a hidden Markov model.

As a stepping stone, we will first consider a shot noise process due to its simplicity. The shot noise process is a Cox process with intensity defined as

$$\lambda_t = \lambda_0 e^{-\int_0^t \delta(x)\mathrm{d}x} + \sum_{i=1}^{M_t} Y_i e^{-\int_{\tau_i}^t \delta(x)\mathrm{d}x}, \tag{7.1}$$

where λ_0 is the initial value of λ, Y_i is the jump size of catastrophe whose distribution depends on its occurrence time, τ_i is the time at which catastrophe i occurs, where $\tau_i < t < \infty$, $\delta(x)$ is the instantaneous exponential decay rate at time x, and $\{M_t\}_{t\geq 0}$ is a (nonhomogeneous) Poisson process.

Note that, from (7.1), the intensity λ_t decays exponentially as time passes since the last occurrence of a catastrophe event. The decrease continues until another catastrophe occurs which will bring in a positive jump in the intensity of the shot noise process. Therefore, λ_t can be treated as a discounted aggregate claim amount under the Poisson claim count processes. Using the techniques for the time-dependent aggregate claims and useful properties of Poisson process, interesting results about the Cox process are likely to be found. Further generalization includes allowing $\{M_t\}_{t\geq 0}$ to be a renewal process.

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