Numerical Modeling of Financially Sustainable Urban Wastewater Systems

by

Atefeh Shadpour

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Masters of Science
in
Earth and Environmental Sciences

Waterloo, Ontario, Canada, 2016

©Atefeh Shadpour 2016

AUTHOR'S DECLARATION

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

A system dynamics model first developed using the software Stella 7.0.2, which explores the complex behavior of the financially sustainable management of water distribution infrastructure, was converted here into a system of coupled non-linear algebraic differential equations (DAEs). Each differential equation involved a time derivative on a primary variable specifying the temporal evolution of the system. In addition, algebraic (secondary) equations and variables specified the non-linearity inherent in the system as well as any controls on the primary variables constraining the physical evolution of the system relevant to the problem at hand. While Stella employed a Runge-Kutta numerical strategy, the numerical DAE method used a fully-explicit, fully-implicit and Crank-Nicolson Euler scheme combined with a fixed-point iteration to resolve the non-linearity. The Runge-Kutta and numerical DAE solutions deviate markedly when the non-linearity of the system becomes pronounced. I demonstrate point-wise stability of the numerical DAE solution as the timestep is refined. Furthermore, the refined numerical DAE solution does not exhibit any of the spurious oscillations inherent in the Runge-Kutta solution and is physically correct for the problem at hand.

Acknowledgements

I would like to express my deepest gratitude to my supervisors Dr. Andre Unger and my co-supervisor Dr. Mark Knight for their kind advice and exceptional mentorship throughout my program. This research truly benefited from their constant encouragement and support of this research. I am also specifically thankful for their patience throughout this work.

Dedication

To my mother who has always been there for me, and my father whose soul has been with me.

Table of Contents

List of Figures

List of Tables

Chapter 1: Literature Review

1.1 Urban Water Network

Clean and safe drinking water and proper collection of waste water are essential for a healthy community. In 2002, the United Nations (UN) declared right to water as a human right (Knight and World Health Organization 2003). In Canada the majority of urban water systems are owned and operated by local governments. In Ontario almost all water and wastewater systems are owned by the municipalities. Water is recognized as a public good. Public goods are characterized by their non-excludability and non-rivalrous nature (Lipsey 1999). During the 1990s, municipal governments in Ontario were transferred the responsibility for additional services from the province. Amongst the competing demands on the financial resources of municipalities, water and wastewater infrastructure often received inadequate attention of the decision makers due to the 'less visible' nature of these assets (Evengard 2011).

Water Opportunities and Water Conservation Act 2010, reiterates the requirement of financial sustainability plans for water and wastewater systems and in addition requires preparation of an asset management plan for physical infrastructure, a water conservation plan, and a risk assessment and mitigation plan (Water Opportunities and Water Conservation Act, 2010, S.O. 2010, c. 19 - Bill 72 n.d.). Mirza and Haider (Mirza and Haider 2003) estimated the total infrastructure deficit in Canada over 100 billion dollars. They also estimated that bringing the Canadian water distribution systems to an acceptable level of performance would require expenditure of more than 6 billion dollars. Citing an earlier study, they report the rehabilitation needs for sanitary and combined sewer in excess of 4 billion dollars.

Infrastructure is basic physical and organizational structures needed for the operation of a society or enterprise, or the services and facilities necessary for an economy to function. It can be generally defined as the set of interconnected structural elements that provide framework supporting an entire structure of development. Water infrastructure has varying definitions for different organizations.

Major components of an urban system are introduced below (Grigg 2003):

- Water supply system
    - Water source
    - Intake facilities

    - Water treatment plants
    - Water distribution networks
- Wastewater system
    - Collection and transmission plants
    - Wastewater treatment systems
    - Wastewater and sludge disposal systems

1.1.1 Water Network Sustainability

Most of the urban water supply networks are facing problems every day that need to be solved in an accurate way. In developing countries the networks are involved in several problems related to population, water scarcity, environmental pollution management, etc. Moreover, the networks will be affected by unexpected problems. The main problems are:

- Population growth
- Water scarcity
- Governmental issues

In terms of what advanced economies have suggested, the "Dublin Statement on Water and Sustainable Development" (The Dublin Statement on Water and Sustainable Development 1992), is a good example of the new trend to overcome these new problems. This statement has come up with some principles that are of great significance in the urban water supply networks. These are the followings:

1. Fresh water is a finite and vulnerable resource, essential to sustain life, development and the environment.

2. Water development and management should be based on a participatory approach, involving users, planners and policy-makers at all levels.

3. Women play a central part in the provision, management and safeguarding of water. Institutional arrangements should reflect the role of women in water provision and protection

4. Water has an economic value in all its competing uses and should be recognized as an economic good.

Recently there has been quite much attention to the fact that many publicly owned water utilities are not managed in a self-sustainable manner. In other words, their rates do not reflect the full costs of providing their services. It is argued that water systems are being subsidized which is not economically efficient. Utility rates should reflect both capital and maintenance costs of the provided services. Doing so, consumers will receive proper signals about the true cost of the services they receive and they will make informed choices about their consumption behavior. The rates charged to residential and commercial customers in Ontario are only one-third and one-fifth of the marginal costs of water supply and sewage treatment respectively (Renzetti 1999). Under the Sustainable Water and Sewage Act (Sustainable Water and Sewage Systems Act, Bill 175 2002), Ontario's public water utilities would be required to generate sufficient revenues to fully recover all of the long-term operating and capital costs. To assist the municipalities in preparing financial plans for ensuring the sustainability of their water and wastewater systems, Ontario's Ministry of Environment issued guidelines (Younis 2010).

In case of financial crisis, Ontario's municipalities will embark on implementing rates based on a full cost coverage, which implies significant increases to the prevailing rates.

1.1.2 Asset Management Models

At the beginning of the 20th century, there has been little attention to infrastructure rehabilitation. Lack of maintenance, aging processes, application of loadings and harsh environmental surroundings have deteriorated the infrastructure to the extent that they are in major need of repair. Replacement rates of deteriorating infrastructure are less than the wearing out of the public facilities (Choate and Walter 1981). Most of infrastructure systems in the urban areas are owned and operated by municipal governments. Traditionally infrastructures were owned and operated by Federal and Provincial governments. Municipalities are now faced with the problem of limited financial resources for rehabilitation and renewal of their infrastructure assets. In this context, asset management combines engineering knowledge with economic and financial practices.

US Federal Highway Administration (FHWA 1999) defines asset management as

> *"a systematic process of maintaining, upgrading and operating physical assets cost effectively. It combines engineering principles with sound business practices and economic theory, and it provides tools to facilitate a more organized, logical*

approach to decision-making. Thus asset management provides a framework for handling both short and long-range planning"

For wastewater utilities US Environmental Protection Agency (EPA 2002) has defined asset management as

"managing infrastructure capital assets to minimize the total cost of owning and operating them, while delivering the service levels customers desire. It is successfully practiced in urban centers and large regional sewer collection systems to improve operational, environmental and financial performance. Many of these large organizations base asset management planning on sophisticated information systems and extensive personnel resources"

Three important factors of asset management are condition assessment, deterioration modeling and selection of optimal rehabilitation strategies.

Fenner and Sweeting (Fenner and Sweeting 1999) presented a decision support system for rehabilitation of non-critical sewers. They used GIS techniques to analyze database containing asset information and pipe failure history. The authors suggest that given a rehabilitation budget, their framework can be used for identification of economically optimal group of pipe lengths with priority order for rehabilitation strategies.

Lalonde et al. (Lalonde and Bergeron 2003) used reliability theory for modeling asset failure risk and employed multi-criteria decision support tools in their research. They formulated the problem at three level analyses, segment level, network level and strategic management. They reported that the segment level problem could be resolved on the basis of bi-criteria methods.

Male et al. (Male, Walski and Slutsky 1990) used a net present value analysis to find the optimum replacement policy for a group of water mains.

Wirahadikusumah et al. (Wirahadikusumah, et al. 1998) used probabilistic dynamic programming in conjunction with a Markov chain model to perform life cycle cost analysis.

Giustolisi (Giustolisi, Kapelan and Savic 2006) have used and evolutionary polynomial regression method to predict the burst rates of water mains. The study compared the reduction in burst rates after pipes' replacement against the cost of replacement. This is achieved by means of a genetic algorithm based multi-objective optimization procedure. Multi-objective genetic algorithm approach was adopted by Dandy and Engelhardt (Dandy and Engelhardt 2006) to develop trade-off curves between economic cost and reliability for replacement schedules of water pipes.

Rehan et al. (Rehan, et al. 2011) recognized the missing components of the water network management. As he mentions, these missing components are briefly

- Water and wastewater infrastructure are treated as separate isolated assets
- Budgetary constraints are either completely ignored or if considered, dynamic interaction with strategic rehabilitation/replacement action is not taken into account
- The socio-political environment of decision making processes is missing from the many of the suggested models.

He proposed an integrated asset management system for financially self-sustaining water and wastewater services which is helpful for municipalities to meet their regulatory obligations, using System Dynamics (SD) approach which is a feedback-based modeling paradigm having its origin in the work of Forrester (Forrester 1958). In this research, their model has been used as a prototype.

1.1.2.1 Description of Background Research

Rehan et al. (Rehan, et al. 2011) developed a management model for urban water and wastewater systems with interconnected components and complex system behavior. A detailed causal loop diagram for management of water and wastewater distribution networks is employed to identify feedback loops. The causal loop diagram is then developed into a system dynamics model comprising water and wastewater pipes, financial, and consumer sectors. Parameterization of the model was done using existing data sources. This was the first known causal loop diagram developed for a financially self-sustainable water utility. The model divides the entire water network to three sectors

- Physical infrastructure sector
- Finance sector

- Consumer sector

Policy levers are included in the model to facilitate formulation of different financing and rehabilitation strategies for the wastewater collection network. Financial and service performance indicators included in the model allow comparison of different financing and rehabilitation strategies.

They tried various simulation scenarios by changing three different policy levers

- Rehabilitation Rate (% of network replaced)
- Enforcing zero Funds Balance or not
- Price Elasticity of Demand (%/%)

In this work we replicate Scenario 3C which has the policy levers Rehabilitation Rate = 1.18, zero Funds Balance enforced, Price Elasticity of Demand = -0.35.

1.2 System Dynamics and DAE Modeling

System dynamics evolved in the 1950's (Forrester 1958) as a computation tool to quantify the behavior of complex transient systems. The mathematical vernacular associated with system dynamics involves breaking a problem down into stocks, flows, converters and connectors. Stocks are used to capture the transient behavior of select variables within the problem using discrete time derivatives. Flows, convertors and connectors are used to specify mathematical relationships needed to transmit information between the various stocks. Numerous off-the-shelf software packages are commercially available, and include: STELLA (isee systems n.d.), SIMILE (Simulistics n.d.), GoldSim (GoldSim n.d.), MATLAB/Simulink (MathWorks n.d.) and Mathematica (WOLFRAM n.d.). These software packages provide a graphical interface to apply the system dynamics vernacular to model a problem, and then allow the user to rapidly parameterize, execute and visualize simulations and test hypothesis. The ease by which these software packages have enabled the process of developing a prototype application has largely been responsible for the widespread use and acceptance of system dynamics in the field of civil engineering.

Sepplet and Richter (Sepplet and Richter 2005) and Rizzo et al. (Rizzo, et al. 2006) have called into question of the numerical accuracy of these various software packages. Specifically, they observed that for a given problem the various software packages may yield a different solution even when the same numerical algorithm to solve the set of equations is chosen. Rehan et al.

(Rehan, et al. 2011) present simulation results using STELLA 7.0.2 for which some variables exhibit what appear to be spurious oscillations; otherwise, local maxima or minima which may indicate non-monotone behavior of the solution. Rehan et al. (Rehan, et al. 2011) do not discuss these oscillations and their impact on their proposed financially sustainable strategy for managing a water distribution network. These same spurious oscillations appear in Son and Rojas work (Son and Rojas 2011) and in the case of their work, are interpreted as productivity dynamics within the context of a construction management system. While these software packages specify their respective numerical algorithm for solving the set of equations arising from the system dynamics problem, the implementation of the algorithm is proprietary and not open to inspection. Clearly there is a need within the civil engineering community to be able to accept solutions obtained using system dynamics software packages on a prototype basis, and then be able to transition the prototype to a numerical algorithm for which convergence of the variables to a unique and physically correct solution is assured. Liability for damages arising from a professional engineer accepting simulation results without due diligence are just one example. This work centers around the hypothesis that the set of equations developed using system dynamics vernacular can be translated directly into a series of coupled and potentially non-linear differential equations as well as a series of algebraic equations, commonly termed "differential algebraic equations" or DAEs ((Petzold and Ascher 1998); (Brenan, Campbell and Petzold 1989); (Petzold and Ascher 1998)). The differential equations capture the transient behavior of the primary variables, while the algebraic equations represent secondary variables and are used to directly specify the non-linearity inherent in the system as well as any controls on the primary variables. DAEs have been used in solving problems of various research areas such as; solving control problems (Pytlak and Zawadzki 2014), solving non-linear mechanics problems (Liu, Elastoplastic models and oscillators solved by a Lie-group differential algebraic equations method 2014), solving breakage population balance equations in computer science and informatics (Narni 2013), solving mass transfer equations (Liu, On-line detecting heat source of a nonlinear heat conduction equation by a differential algebraic equation method 2014). I test the utility of fully-implicit, fully-explicit and Crank-Nicolson Euler schemes to discretize the time derivatives inherent in the differential equations, with a fixed-point iteration to resolve the non-linearity arising from the coupled differential and algebraic equations. Point-wise stability of this numerical DAE scheme dictates that as the timestep used to approximate the time derivative is reduced, the solution will converge to a single answer (Petzold and Ascher 1998). The rate of convergence is dictated by the order of the error arising from the method used to discretize the time derivative. The premise here is that this answer will be absent from any spurious oscillations and be physically correct for the engineering problem at hand. Specifically, I revisit the work of Rehan et al. (Rehan, et al. 2011) to demonstrate that this hypothesis is correct.

Chapter 2: A Numerical DAE Approach for Urban Wastewater System

2.1 A Numerical DAE Approach

The objective of this work is to specifically focus on Scenario 3C of Rehan et al. (Rehan, et al. 2011) as a prototype problem to highlight the proposed approach for converting a system dynamics problem into a series of coupled non-linear ordinary differential equations. Rehan et al. [(Rehan, et al. 2011) - Appendix A] present the entirety of the equations constituting the system dynamics problem in the STELLA 7.0.2 notation of “stocks”, “flows”, “connectors” and “convertors”. This same system is presented graphically on Figures 2 and 3 of Rehan et al. (Rehan, et al. 2011). The premise in this section is that the stocks in system dynamics can instead be called primary variables within a set of DAEs, with the flows, connectors and convertors constituting the remainder of a given DAE as well as secondary equations and variables. The clearest indicator for a stock to primary variable representation is that both involve time derivatives. To begin, let $\boldsymbol{X}$ denote the vector of primary variables identified from a given system dynamics problem and $\boldsymbol{Y}(\boldsymbol{X})$ be the vector of secondary variables, some of which may be non-linearly dependent on the primary variables. The general form of a semi-explicit index-1 DAE system is given by Equation 2-1.

$$\frac{\partial \boldsymbol{X}}{\partial t} = f(\boldsymbol{X}, \boldsymbol{Y}, t) \qquad \textbf{2-1}$$

$$\boldsymbol{0} = \boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{t})$$

where $\partial \boldsymbol{X}/\partial t \equiv \boldsymbol{X}'$, and where $(\partial g/\partial \boldsymbol{X}')^{-1}$ exists and is bounded in a neighborhood of the solution. In other words, it is possible to differentiate the algebraic equations $g(\boldsymbol{X}, \boldsymbol{Y}, t)$ once with respect to $\boldsymbol{X}$; hence, index-1. The notion of semi-explicit DAE (and explicitness in general) implies that the algebraic equations are separated from the differential equations as presented in Equation 2-1 above. In contrast, an implicit DAE follows from application of the implicit function theorem which states that there exists a function $\tilde{g}$ such that $\boldsymbol{Y} = \tilde{g}(\boldsymbol{X}, t)$. Thus, the DAE is equivalent to the ordinary differential equation (ODE): $\partial \boldsymbol{X}/\partial t = f(\boldsymbol{X}, \tilde{g}(\boldsymbol{X}, t), t)$. Here I make the distinction that in the case of the problem which follows, system dynamics leads to a semi-explicit DAE. The function $\tilde{g}$ does not exist because $\boldsymbol{Y}$ is non-linearly dependant on $\boldsymbol{X}$.

Brenan et al. (Brenan, Campbell and Petzold 1989) review the application of Runge-Kutta and Backward Differentiating Formulations (i.e. Euler methods) for the numerical solution of DAEs. For semi-explicit index-1 DAEs, they demonstrate that the use of an Euler method of order $k \leq 7$ evaluated with a constant time step size Δt converges with an accuracy of order $\mathcal{O}\left(\Delta t^{k}\right)$ after $k+1$ fixed-point iterations provided each fixed point iteration is solved to an accuracy of order $\mathcal{O}\left(\Delta t^{k+1}\right)$. In other words, as the timestep Δt is progressively reduced along with the numerical error in approximating the time derivative, the solution should converge to a single solution over the entire simulation period and thereby exhibit point-wise stability. In fact, Petzold and Ascher (Petzold and Ascher 1998) state that application of Euler methods to semi-explicit index-1 DAEs retain all properties (i.e. order, stability, convergence) relative to analogous ODEs. For semi-explicit index-1 DAEs, Bendtsen and Thomsen (Bendtsen and Thomsen 1999) demonstrate that Runge-Kutta methods have poor stability properties. In addition, the semi-explicit property whereby the differential and algebraic equations cannot be combined and are non-linearly dependent on one-another creates a situation where the internal Runge-Kutta solutions within the timestep Δt must also be coupled with algebraic solutions at the same stage. In summary, application of Runge-Kutta algorithms to semi-explicit DAEs is not a straight-forward extension of their application to analogous ODEs.

I begin this work by introducing the following numerical (Euler) discretization of Equation 2-1 as:

$$\frac{\boldsymbol{X}^{n+1}-\boldsymbol{X}^{n}}{\Delta t}=\theta f(\boldsymbol{X}^{n+1},\boldsymbol{Y}^{n+1})+(1-\theta)f(\boldsymbol{X}^{n},\boldsymbol{Y}^{n})+\mathcal{O}\left(\Delta t^{k}\right) \tag{2-2}$$

$$0=g(\boldsymbol{X}^{n+1},\boldsymbol{Y}^{n+1},t)$$

where; $\theta=1.0$, 0.5 and 0.0 yields a shift from a fully-implicit, Crank-Nicolson, to fully-explicit temporal discretization, yielding a discretization error of order $k=1$ (i.e. first-order accurate), 2 (i.e. second-order accurate) and 1, respectively. Note that "fully-implicit" and "fully-explicit" now refer to the actual Euler method used to discretize the time derivative, and not the form of the DAE. Superscripts $n+1$ and n denote the current and past time solutions separated over an increment of time Δt. For the fully-implicit and Crank-Nicolson discretization, the set of equations involving the primary variables $\boldsymbol{X}$ leads to a system of equations of the form $[a]\{\boldsymbol{X}\}=\{b\}$ which is solved via Gaussian elimination. The non-linear dependence expressed by $\boldsymbol{Y}(\boldsymbol{X})$ is resolved by invoking a fixed-point iteration as illustrated on Figure 3.1. Specifically, the simultaneous solution of the primary variables is followed by the sequential solution of the secondary equations. For a given timestep, the fixed-point iteration involves looping around the coupled set of equations involving the primary and secondary variables until a convergence tolerance is achieved.

Key outcomes of the numerical DAE approach are: first, to clearly identify the source of non-linear dependence and hence the key primary variable(s) $\boldsymbol{X}$ limiting convergence; second, to identify those secondary variables $\boldsymbol{Y}$ that serve as controllers on the range of $\boldsymbol{X}$ obtained as part of the physically acceptable solution. These latter two concepts are illustrated hereafter in the context of Scenario 3C of Rehan et al. (Rehan, et al. 2011).

2.1.1 Physical Infrastructure Network

Details regarding the formulation and parameterization of the physical infrastructure sector are described in Section 4.1 of Rehan et al. (Rehan, et al. 2011) and are not repeated here for brevity. However, to clarify our intent to focus on the combined water and wastewater distribution network as the infrastructure sector, I rename $ConditionGroup$ stocks from Rehan et al. (Rehan, et al. 2011) here as $PipeCondition$. This section presents that component of the system dynamics model shown in Figure 3 of Rehan et al. (Rehan, et al. 2011) belonging to the physical infrastructure sector, with the presentation involving primary and secondary variables.

2.1.1.1 Primary Variables and Equations

Degradation of pipes is captured by the transient accumulation and subsequent depletion of pipe lengths in five different age groups over the course of the simulation period. This processes is captured by five primary variables denoted as: $PipeCondition_{0-20}^{n+1}$, $PipeCondition_{20-40}^{n+1}$, $PipeCondition_{40-60}^{n+1}$, $PipeCondition_{60-80}^{n+1}$ and $PipeCondition_{80-100}^{n+1}$ $[km]$. The subscript indicates the age span for each group, under the assumption that service life of any given pipe is 100 years. These five primary variables lead to the following five differential equations:

$$\frac{PipeCondition_{0-20}^{n+1} - PipeCondition_{0-20}^{n}}{\Delta t} = \theta(RehabLength^{n+1} - Deterioration_{20-40}^{n+1}) + (1-\theta)(RehabLength^{n} - Deterioration_{20-40}^{n}) \quad \textbf{2-3}$$

$$\frac{PipeCondition_{20-40}^{n+1} - PipeCondition_{20-40}^{n}}{\Delta t} = \theta(Deterioration_{20-40}^{n+1} - Deterioration_{40-60}^{n+1}) + (1-\theta)(Deterioration_{20-40}^{n} - Deterioration_{40-60}^{n}) \quad \text{2-4}$$

$$\frac{PipeCondition_{40-60}^{n+1} - PipeCondition_{40-60}^{n}}{\Delta t} = \theta(Deterioration_{40-60}^{n+1} - Deterioration_{60-80}^{n+1}) + (1-\theta)(Deterioration_{40-60}^{n} - Deterioration_{60-80}^{n}) \quad \text{2-5}$$

$$\frac{PipeCondition_{60-80}^{n+1} - PipeCondition_{60-80}^{n}}{\Delta t} = \theta(Deterioration_{60-80}^{n+1} - Deterioration_{80-100}^{n+1}) + (1-\theta)(Deterioration_{60-80}^{n} - Deterioration_{80-100}^{n}) \quad \text{2-6}$$

$$\frac{PipeCondition_{80-100}^{n+1} - PipeCondition_{80-100}^{n}}{\Delta t} = \theta(Deterioration_{80-100}^{n+1} - RehabLength^{n+1}) + (1-\theta)(Deterioration_{80-100}^{n} - RehabLength^{n}) \quad \text{2-7}$$

Initial conditions for these five primary variables are provided on Table 2.1.

2.1.1.2 Secondary Variables and Equations

The ageing process of the pipes is facilitated thought the set of secondary variables denoted as: $Deterioration_{0-20}^{n+1}$, $Deterioration_{20-40}^{n+1}$, $Deterioration_{40-60}^{n+1}$, and $Deterioration_{60-80}^{n+1}$ $[km/yr]$. These secondary variables are quantified as:

$$Deterioration_{20-40}^{n+1} = \frac{PipeCondition_{0-20}^{n+1}}{20} \quad \text{2-8}$$

$$Deterioration_{40-60}^{n+1} = \frac{PipeCondition_{20-40}^{n+1}}{20} \quad \text{2-9}$$

$$Deterioration_{60-80}^{n+1} = \frac{PipeCondition_{40-60}^{n+1}}{20} \qquad \textbf{2-10}$$

$$Deterioration_{80-100}^{n+1} = \frac{PipeCondition_{60-80}^{n+1}}{20} \qquad \textbf{2-11}$$

The rate at which capital works is conducted, which involve replacing deteriorated pipes in age group $PipeCondition_{80-100}^{n+1}$ with newly emplaced pipes in age group $PipeCondition_{0-20}^{n+1}$, is controlled by the secondary variable $RehabLength^{n+1}$ $[km/yr]$. In the context of Scenario 3C of Rehan et al. (Rehan, et al. 2011), it is a constant given by:

$$RehabLength^{n+1} = TotalLengthofPipes \times \frac{RehabFraction}{100} \qquad \textbf{2-12}$$

where the constants $TotalLengthofPipes$ and $RehabFraction$ are provided on Table 2.1. $RehabFraction$ is manually adjusted to ensure that no more than 5% of the network has pipes in $PipeCondition_{80-100}^{n}$ for the entire 100-year simulation period. As such, $RehabFraction$ is a control isolated to the infrastructure sector and independent of the remainder of the system.

The average condition of the pipe network exerts an important influence on operation expenditures which are a secondary variable in the finance sector. As such, the average condition of the entire network is denoted as $AverageCondition^{n+1}$ $[-]$ and is a secondary variable calculated as:

$$\begin{aligned} AverageCondition^{n+1} &= (PipeCondition_{0-20}^{n+1} + PipeCondition_{20-40}^{n+1} \\ &+ PipeCondition_{40-60}^{n+1} + PipeCondition_{60-80}^{n+1} \\ &+ PipeCondition_{80-100}^{n+1}) \,/\, TotalLengthofPipes \quad . \end{aligned} \qquad \textbf{2-13}$$

Table 2.2 listed the relationship between $AverageCondition^{n+1}$ and $ConditionFactor^{n+1}$ $[-]$ that is used to adjust operation expenditures based on the age of the network.

Table 2.1: **Initial conditions and constants**

Physical infrastructure sector	
Initial conditions	
$PipeCondition_{0-20}^{0}$	$140\ [km]$

$PipeCondition_{20-40}^{0}$	280 $[km]$
$PipeCondition_{40-60}^{0}$	140 $[km]$
$PipeCondition_{60-80}^{0}$	105 $[km]$
$PipeCondition_{80-100}^{0}$	35 $[km]$
Constants	
$TotalLengthofPipes$	700 $[km]$
$RehabFraction$	1.18 $[\%\ of\ TotalLengthofPipes/yr]$
Finance sector	
Initial conditions	
$UserFee^{0}$	3.75 $[\$/m^3]$
$FundBalance^{0}$	0 $[\$]$
Constants	
$MaxFundBalance$	0 $[\$]$
$MinFundBalance$	0 $[\$]$
$MaxFeeHike$	10 $[\%]$
$UnitPriceOpEx$	50 $[\$/m/year]$
$UnitPriceCapEx$	1000 $[\$/m]$
Consumer sector	
Initial conditions	
$WaterDemand^{0}$	300 $[lpcd]^{\dagger}$
Constants	
$ElasticityofDemand$	-0.35 $[-]$
$MinimumDemand$	200 $[lpcd]^{\dagger}$
$DemandAdjustmentPeriod$	20 $[years]$
$Population$	$100{,}000$ $[-]$

$^{\dagger}lpcd$ denotes $liter\ per\ capita\ per\ day$

Table 2.2: Relationship of the average condition of the network on operational expenditures.

$AverageCondition^{n+1}$	$ConditionFactor^{n+1}$
0.00	0.00
10.00	1.500
20.00	3.500
30.00	6.500
40.00	11.00
50.00	18.00
60.00	26.00
70.00	38.00
80.00	55.00
90.00	75.00
100.00	100.00

2.1.2 Finance Sector

Details regarding the formulation and parameterization of the finance sector are described in Section 4.3 and Figure 3 of Rehan et al. (Rehan, et al. 2011) and are not repeated here for brevity. Once again, this presentation involves casting the system dynamics problem into primary and secondary variables.

2.1.2.1 Primary Variables and Equations

The essence of Rehan et al. (Rehan, et al. 2011) is to explore the financial sustainability of water conveyance networks. They achieve this objective by adjusting the amount that the utility charges per cubic meter of water consumed and then discharged (i.e. $UserFee^{n+1}$ [$\$/m^3$]) so that revenues equal expenditures and the utility maintains a zero funds balance, denoted as $FundBalance^{n+1}$ [$]. These two primary variables lead to the following two differential equations:

$$\frac{FundBalance^{n+1} - FundBalance^{n}}{\Delta t} = \theta(Revenue^{n+1} - Expenditures^{n+1}) + (1-\theta)(Revenue^{n} - Expenditures^{n}) \quad \textbf{2-14}$$

and

$$\frac{UserFee^{n+1} - UserFee^{n}}{\Delta t} = \theta(UserFeeHike^{n+1} - UserFeeDecline^{n+1}) + (1-\theta)(UserFeeHike^{n} - UserFeeDecline^{n}) \tag{2-15}$$

2.1.2.2 Secondary Variables and Equations

$Revenue^{n+1}$ $[\$/yr]$ represents the utilities annual income and is generated via the product of the unit cost of water consumed and then discharged (i.e. $UserFee^{n+1}$) and the total water consumption by residents of the municipality (i.e. $TotalWaterConsumption^{n+1}$ $[m^3/yr]$). This is expressed as:

$$Revenue^{n+1} = UserFee^{n+1} \times TotalWaterConsumption^{n+1} \tag{2-16}$$

The infrastructure sector of Rehan et al. (Rehan, et al. 2011) is limited to watermain and sanitary sewer pipes, with all $Expenditures^{n+1}[\$/yr]$ equal to the sum of operational and capital costs:

$$Expenditures^{n+1} = OpEx^{n+1} + CapEx^{n+1} \tag{2-17}$$

Operational expenditures (i.e. $OpEx^{n+1}$ $[\$/yr]$) involve the general maintenance of the water conveyance network and increase proportionally with the average age of the network as indicated by $ConditionFactor^{n+1}$ on Table 2. Operational expenditures are calculated as:

$$OpEx^{n+1} = UnitPriceOpEx \times TotalLengthofPipes \times \kappa_1 \times \left(1 + \frac{ConditionFactor^{n+1}}{100}\right) \tag{2-18}$$

where $\kappa_1 = 1000\ m/km$. Capital expenditures involve replacing deteriorated pipe with new construction as given by:

$$CapEx^{n+1} = UnitPriceCapEx \times RehabLength^{n+1} \times \kappa_1 \tag{2-19}$$

Financial sustainability, as expressed by a zero funds balance, is the primary control on the transient evolution of the system of equations. Hence, a zero funds balance provides the physical context for the simulated results in terms of how the utility manages the network. The control is used to constrain $FundBalance^{n+1}$ should its value deviate beyond a bounded interval demarked by $MinFundBalance$ and $MaxFundBalance$ [\$]. Specifically, if $FundBalance^{n+1}$ decreases below $MinFundBalance$ resulting in debt, then $RequiredRevenue$ [$\$/yr$] is calculated to eliminate this debt as:

$$\begin{aligned}
&\text{if } (FundBalance^{n+1} < MinFundBalance)\\
&\quad RequiredRevenue^{n+1} = Expenditures^{n+1} - FundBalance^{n+1} / \mathbf{1}\\
&\text{else}\\
&\quad RequiredRevenue^{n+1} = Expenditures^{\,n+1}
\end{aligned} \qquad \textbf{2-20}$$

where "$\mathbf{1}$" denotes a one-year period over which the debt is to be eliminated. In the context of Scenario 3C of Rehan et al. (Rehan, et al. 2011), all revenue is generated by increasing the unit cost of water $UserFee^{n+1}$ billed to the consumers. The rate of increase $UserFeeHike^{n+1}$ [$\$/m^3/yr$] is calculated as:

$$\begin{aligned}
&\text{if } (FundBalance^{n+1} > MaxFundBalance)\\
&\quad UserFeeHike^{n+1} = 0\\
&\text{else}\\
&\quad UserFeeHike^{n+1}\\
&\qquad = \min\left\{MaxFeeHike \times UserFee^{n+1} / \mathbf{1},\ \left(\frac{RequiredRevenue^{n+1}}{TotalWaterConsumtion^{n+1}} - UserFee^{n+1}\right) / \mathbf{1}\right\}
\end{aligned} \qquad \textbf{2-21}$$

and remains in place until $FundBalance^{n+1}$ exceeds $MaxFundBalance$ in an attempt to generate a slight surplus. However, once a surplus in excess of $MaxFundBalance$ is generated, the control then decreases the unit cost of water with the rate of decrease $UserFeeDecline^{n+1}$ [$\$/m^3/yr$] calculated as:

$$\begin{aligned}
&\text{if } (FundBalance^{n+1} > MaxFundBalance)\\
&\quad UserFeeDecline^{n+1} = UserFee^{n+1} / \mathbf{1}\\
&\qquad - \left(\frac{Expenditures^{n+1} - FundBalance^{n+1} / \mathbf{1}}{TotalWaterConsumption^{n+1}}\right)\\
&\text{Else}\\
&\quad UserFeeDecline^{n+1} = 0
\end{aligned} \qquad \textbf{2-22}$$

This control on the primary variable $UserFee^{n+1}$ as expressed by Equations 2-22 is highly non-linear and is strongly dependent on both the infrastructure and consumer sectors.

2.1.3 Consumer Sector

Details regarding the formulation and parameterization of the consumer sector are described in Section 4.2 and Figure 3 of Rehan et al. (Rehan, et al. 2011) and are not repeated here for brevity. This section presents the consumer sector in the context of a numerical DAE approach using primary and secondary variables.

2.1.3.1 Primary Variables and Equations

A key observation by Rehan et al. (Rehan, et al. 2011) is that individual consumers begin to conserve water as its unit price ($UserFee^{n+1}$) increases in a manner dictated by the price elasticity of demand. These conservation measures are observed by temporal changes in an individual consumers' $WaterDemand^{n+1}$ $[lpcd]$:

$$\frac{WaterDemand^{n+1} - WaterDemand^{n}}{\Delta t} = \theta(-DemandChange^{n+1}) + (1-\theta)(-DemandChange^{n}) \qquad \textbf{2-23}$$

2.1.3.2 Secondary Variables and Equations

The rate at which water conservation is implemented is governed by the secondary variable $DemandChange^{n+1}$ $[lpcd/yr]$:

if $(time < 1\ year)$

$$DemandChange^{n+1} = \min\left\{ \frac{UserFee^{n+1} - UserFee^{0}}{UserFee^{0}} \times ElasticityofDemand \times WaterDemand^{n+1}\ ,\ WaterDemand^{n+1} - MinimumDemand \right\} / \mathbf{1} \qquad \textbf{2-24}$$

else

$$DemandChange^{n+1} = \min\left\{ \frac{UserFee^{n+1} - \text{DELAY}\,(\,UserFee^{n+1}, 1\,year\,)}{\text{DELAY}\,(\,UserFee^{n+1}, 1\,year\,)} \times ElasticityofDemand \times WaterDemand^{n+1}\,,\ WaterDemand^{n+1} - MinimumDemand \right\} / \mathbf{1}$$

where the function DELAY($UserFee^{n+1}$, $1\ year$) returns the value of $UserFee^{n+1}$ from the current time less 1 year. The constant "**1**" denotes a one-year period over which residents adjust their consumption in response to price changes via $ElasticityofDemand$. More significantly, residents will purchase water efficient dish washers, toilets and other durable goods over a period of time denoted as $DemandAdjustmentPeriod$ $[years]$ to reduce their $TotalWaterConsumption^{n+1}$ $[m^3/yr]$:

$$TotalWaterConsumption^{n+1} = \text{SMTH}\,(\,WaterDemand^{n+1}, DemandAdjustmentPeriod\,) \times Population \times \kappa_3 / \kappa_2 \qquad \textbf{2-25}$$

where $\kappa_2 = 1000\ l/m^3$ and $\kappa_3 = 365\ days/year$. Because $TotalWaterConsumption^{n+1}$ has a direct impact on $Revenue^{n+1}$ in Equation 2-16, any lost revenue due to conservation must be recovered by increasing $UserFee^{n+1}$ leading to even greater conservation. This feedback loop is the main source of non-linearity in the numerical model. The function SMTH ($WaterDemand^{n+1}$, $DemandAdjustmentPeriod$) applies exponential smoothing to the secondary variable $WaterDemand^{n+1}$ over the length of time prescribed by $DemandAdjustmentPeriod$. For the case SMTH (X^{n+1}, T) where X^{n+1} is the secondary variable $WaterDemand^{n+1}$ in the interval $n + 1 - T < t < n + 1$, exponential smoothing is denoted by $\bar{X}^{n+1}$ as:

$\bar{X}^0 = X^0$ as the initial condition;

$\bar{X}^{n+1} = \alpha\, X^{n+1} + (1-\alpha)\, \bar{X}^n$ for subsequent timesteps;

which recursively leads to the following series that only includes those terms in the time interval $n + 1 - T < t < n + 1$;

$$\bar{X}^{n+1} = \alpha\, \{\, X^{n+1} + (1-\alpha)\, X^n + (1-\alpha)^2\, X^{n-1} + \cdots + (1-\alpha)^n\, X^0 \} \qquad \textbf{2-26}$$

Trial and error indicate that STELLA 7.0.2 uses a value of $\alpha = 0.1$.

Chapter 3: Results of the Numerical DAE Approach and Discussion

In the context of this work, I restrict our attention to discussing the significance of any potential numerical errors in the solution of the Rehan et al. (Rehan, et al. 2011) system dynamics problem. Note that interpretation of the physical relevance of the simulation results to the financially sustainable management of a water distribution network was the subject of Rehan et al. (Rehan, et al. 2011) and is not repeated here for brevity. Earlier, I hypothesised that the Rehan et al. (Rehan, et al. 2011) simulation results for Scenario 3C include local minima or maxima and hence may not be monotone. This issue is examined by re-simulating Scenario 3C with STELLA 7.0.2 using their fourth-order Runge-Kutta algorithm to maximize numerical accuracy. In addition, Scenario 3C is also simulated using the numerical DAE approach outlined in Section 2. Initially, I would like to ensure that the two algorithms are point-wise stable in that they converge to a single solution as the timestep is progressively reduced, along with the numerical error in approximating the time derivative. To illustrate this discussion, Figure 3.2 and Figure 3.3 present all primary and some secondary variables over the 100-year simulation period with a logarithmic time scale to focus on the key dynamics. The largest timestep is $\Delta t = 2^{-2}\ year$ to conform to the Rehan et al (Rehan, et al. 2011) simulation results. The smallest timestep STELLA 7.0.2 would accept was $\Delta t = 2^{-8}\ year$. For the numerical DAE approach, the timestep was progressively halved to $\Delta t = 2^{-9}\ year$. Later in this work I will re-examine the sensitivity of the two primary variables $UserFee^{n+1}$ and $FundBalance^{n+1}$ to the STELLA 7.0.2 and VENSIM 6.0b fourth-order Runge-Kutta and first-order Euler integration algortihms.

3.1 Results of the Numerical Modeling

Figure 3.2(a) shows the condition of the pipes as defined by Equations 2-3 to 2-7. Clearly, both Runge-Kutta and the numerical DAE approach produced visually-identical answers over the entire simulation period. This result appears to be largely a consequence of the fact that the infrastructure sector is independent of the finance and consumer sectors, with pipe condition behaving as a series of DAEs. $RehabFraction$ is the single control isolated to this sector, and is manually adjusted independent of the remainder of the system to ensure that no more than 5% of the network has pipes in $PipeCondition^{n}_{80-100}$ for the 100-year simulation period. I conclude that the fourth-order

Runge-Kutta approach employed by STELLA 7.0.2 is virtually identical to a numerical DAE approach for linear systems.

Figure 3.2 (b) presents simulation results for the primary variable $FundBalance^{n+1}$. These results indicate that the funds balance does deviate from "zero" despite the intent of Scenario 3C to signal financial sustainability though a zero funds balance. This deviation occurs because the two controls $UserFeeHike^{n+1}$ (see Equation 2-21 and Figure 3.3 (a)) and $UserFeeDecline^{n+1}$ (see Equation 2-22 and Figure 3.3 (b)) impose their effect over a 1-year implementation period, as signaled by $RequiredRevenue^{n+1}$ in Equation 2-20. As such, both Runge-Kutta and the numerical DAE approach show the same qualitative trends: a peak deficit of approximately \$4-to-\$5 million around year 2, and; a peak surplus of \$0.5-to-\$1 million around year 6. As the timestep is refined, Runge-Kutta shows virtually no change in its solution. However, the numerical DAE approach shows a progressive convergence to a single solution. The converged numerical DAE solution is absent the spurious oscillations that occur in both the Runge-Kutta solution as well as the numerical DAE solution with the coarsest timestep, as the peak surplus is spend down to a late-time zero funds balance. Examination of the two controls show that they are directly related to the two primary variables $UserFee^{n+1}$ and $FundBalance^{n+1}$ and indirectly to many others through the secondary variables. Therefore, the accuracy of the control variables is also $\mathcal{O}(\Delta t^k)$. Note that that the solutions proceed from the end of the first time step which varies due to refinement from $\Delta t = 2^{-2}\ year$ to $\Delta t = 2^{-9}\ year$. The initial condition is not shown on the logarithmic time scale because it occurs at time $t = 0\ year$. This explains the discontinuity that is seen at the beginning f the graph for coarse time steps.

Figure 3.2 (c) and (d) present simulation results for the primary variables $UserFee^{n+1}$ (see Equation 2-15) and $WaterDemand^{n+1}$ (see Equation 2-23), respectively. These two variables have a strong non-linear dependence upon one another through the secondary variable $DemandChange^{n+1}$ (see Figure 3.3 (c) and Equation 2-24). As the $UserFee^{n+1}$ increases rapidly from year 0.1 to approximately year 12, there is a corresponding decrease in $WaterDemand^{n+1}$ and $TotalWaterConsumption^{n+1}$ (see Figure 3.3 (d) and Equation 2-25). The Runge-Kutta and numerical DAE solutions deviate markedly beyond year 1. As the timestep is refined, neither Runge-Kutta nor the numerical DAE solutions appear to converge to the same solution. However, the numerical DAE solution does exhibit the desired point-wise stability behavior of converging to a single solution over the entire simulation period as the timestep is refined.

Figure 3.3 (e) depicts $Revenue^{n+1}$ which is the product of $UserFee^{n+1}$ and $TotalWaterConsumption^{n+1}$ and is quantified by Equation 2-16. Despite the divergence between the Runge-Kutta and the numerical DAE solutions for $UserFee^{n+1}$ and $TotalWaterConsumption^{n+1}$, the solution for $Revenue^{n+1}$ remains similar in shape over the entire simulation period, with late-time convergence to nearly the same identical solution. This similarity is not fortuitous, and is a consequence of two aspects of the Rehan et al. (Rehan, et al. 2011) system dynamics problem. First, a fundamental attribute of the infrastructure sector is the conservation of pipe lengths in that $TotalLengthofPipes$ remains constant over the simulation period. Second, the financial sustainability constraint of a zero funds balance ensures that in the long term revenues equal expenses. Total expenses involve the sum of operational and capital expenditures on the pipe network (see Figure 3.3 (f), (g) and (h)) and are largely a function of the average condition of the network. Deterioration of the pipes is effectively a linear problem that can be solved by either the Runge-Kutta or the numerical DAE approach. Therefore, $Revenue^{n+1}$ ultimately becomes constrained by the physical deterioration of the pipes and the need for the utility to maintain the system independent of numerical details associated with the Runge-Kutta and numerical DAE algorithms. As a result, at the end of the 100-year simulation period, Runge-Kutta over-estimates $UserFee^{n+1}$ and proportionally underestimates $TotalWaterConsumption^{n+1}$ in order to maintain the same revenue stream relative to the numerical DAE solution. However, only one solution can be physically correct.

To demonstrate that the numerical DAE solution is point-wise stable and converging to the correct solution, I analyze the order of the timestep error $\mathcal{O}(\Delta t^k)$ for the $UserFee^{n+1}$ and $FundBalance^{n+1}$ primary variables at year 10 within the 100-year simulation period. This point in time was chosen because the Runge-Kutta and the numerical DAE solutions show a clear divergence in their values. Table 3.1 and Table 3.2 provide values of these primary variables for the numerical DAE solution as the timestep was sequentially halved from $\Delta t = 2^{-2}\ years$ down to $\Delta t = 2^{-9}\ years$. For all timestep sizes, the fixed-point iteration continued for a given timestep until a convergence tolerance of $ctol = 10^{-5}$on the primary variable $UserFee^{n+1}$ was achieved, where:

$$ctol_{UserFee^{n+1}} = \frac{|\, UserFee^{n+1,k+1} - UserFee^{n+1,k}|}{|\, UserFee^{n+1,k}|} \quad \textbf{3-1}$$

and k denotes the fixed-point iteration level. Figure 3.4 (a) and (b) show the convergence of the primary variables $UserFee^{n+1}$ and $FundBalance^{n+1}$, respectively, at year 10 for the numerical DAE and Runge-Kutta algorithms. Visually it appears that they are converging to entirely different values of $UserFee^{n+1}$. However, they seem to be converging to the same value of $FundBalance^{n+1}$. This latter observation is entirely fortuitous. On Figure 3.4 (b) I see that at year 10, the numerical DAE $FundBalance^{n+1}$ solution using a timestep of $\Delta t = 2^{-9}\ years$ exhibits a downward fluctuation as it recovers from the peak surplus at 7 years, while the Runge-Kutta solution exhibits an upward fluctuation as it recovers from the peak surplus at 5 years. The recovery pattern of both solutions is entirely different towards a zero funds balance at the end of the 100-year simulation period. On Tables 3 and 4, the relative error δ_X is calculated as:

$$\delta_{X_{\Delta t=2^{-m}}} = \frac{|X_{\Delta t=2^{-m}} - X_{\Delta t=2^{-m+1}}|}{|X_{\Delta t=2^{-m}}|} \quad m \in \{3 \ldots 9\} \quad \textbf{3-2}$$

while the ratio of changes Λ_X is calculated as:

$$\Lambda_{X_{\Delta t=2^{-m}}} = \frac{|X_{\Delta t=2^{-m}} - X_{\Delta t=2^{-m+1}}|}{|X_{\Delta t=2^{-m+1}} - X_{\Delta t=2^{-m+2}}|} \quad m \in \{4 \ldots 9\} \quad \textbf{3-3}$$

The ratio of changes divided by the sequential reduction in timestep size (which is “2” for this analysis) effectively yields the order of the algorithm k as specified in Equation 2-2. Examination of the relative error for both $UserFee^{n+1}$ and $FundBalance^{n+1}$ indicate that an even finer timestep than $\Delta t = 2^{-9}\ years$ would benefit the solution. The ratio of changes indicates that fully-explicit, fully-implicit and even Crank-Nicolson temporal weighting are all first-order accurate solutions. In the case of Crank-Nicolson, I surmise that the non-smooth behavior of the controls $UserFeeHike^{n+1}$ and $UserFeeDecline^{n+1}$ is impeding the ability of Crank-Nicolson to achieve second-order accuracy. This perturbation occurs when the $\theta(\boldsymbol{Y}^{n+1})$ and $(1-\theta)(\boldsymbol{Y}^{n})$ solutions straddle a discontinuity in the controls.

Table 3.1: **Point-wise stability analysis of $X = UserFee^{n+1}$ evaluated using the numerical DAE method at simulation time $t = 10\ years$, as the timestep size is continuously refined.**

Δt [$years$]	Crank Nicolson			Fully-implicit			Fully-explicit		
	X [$\$/m^3$]	δ_X	Λ_X	X [$\$/m^3$]	δ_X	Λ_X	X [$\$/m^3$]	δ_X	Λ_X
2^{-2}	4.79695			4.77238			4.76565		
2^{-3}	4.95028	0.03097		4.93508	0.03297		4.93378	0.03408	
2^{-4}	5.01210	0.01233	2.48	5.01000	0.01500	2.16	5.01043	0.01530	2.19
2^{-5}	5.04424	0.00637	1.92	5.04253	0.00640	2.33	5.04143	0.00615	2.47
2^{-6}	5.08784	0.00860	0.74	5.08604	0.00856	0.74	5.08488	0.00854	0.71
2^{-7}	5.11760	0.00581	1.46	5.11634	0.0059	1.44	5.11556	0.00510	1.41
2^{-8}	5.13471	0.00330	1.74	5.13397	0.00343	1.72	5.13353	0.00350	1.71
2^{-9}	5.14383	0.00177	1.88	5.14343	0.00184	1.86	5.14320	0.00188	1.86

Table 3.2: **Point-wise stability analysis of $X = FundBalance^{n+1}$ evaluated using the numerical DAE method at simulation time $t = 10\ years$, as the timestep size is continuously refined.**

Δt [$years$]	Crank Nicolson			Fully-implicit			Fully-explicit		
	X [$]	δ_X	Λ_X	X [$]	δ_X	Λ_X	X [$]	δ_X	Λ_X
2^{-2}	-523,871.90			-441,529.01			-451,688.84		
2^{-3}	-280,303.49	0.87		-269,406.10	0.64		-278,679.60	0.62	
2^{-4}	-62,616.74	3.48	1.12	-61,945.18	3.35	0.83	-69,282.36	3.02	0.83
2^{-5}	-13,972.03	3.48	4.47	-8,908.15	5.95	3.91	-58,078.02	10.93	3.30
2^{-6}	-71,776.25	0.81	0.84	-66,490.95	0.87	0.92	-64,019.72	0.91	1.09
2^{-7}	-111,429.25	0.36	1.46	-107,901.70	0.38	1.39	-106,433.71	0.40	1.37
2^{-8}	-133,329.47	0.16	1.81	-131,328.84	0.18	1.76	-130,550.25	0.18	1.76
2^{-9}	-144,593.03	0.08	1.94	-143,533.84	0.08	1.92	-143,137.52	0.09	1.92

In summary, I conclude that the numerical DAE solution is behaving as expected. Therefore, the numerical DAE solution is physically correct. On a qualitative note, the numerous oscillations in the Runge-Kutta solution for $FundBalance^{n+1}$ (see Figure 3.2 (b)) during the recovery from the peak surplus at 5 years are disconcerting, especially because the converged numerical DAE solution is much smoother during the same period. Furthermore, the spurious spike in $CapEx^{n+1}$ at 11 years (see Figure 3.3 (h)) occurs only with Runge-Kutta and cannot be rationally explained based on the expected system behavior.

Figure 3.5 exhibits the sensitivity of the two primary variables $UserFee^{n+1}$ and $FundBalance^{n+1}$ to the STELLA 7.0.2 and VENSIM 6.0b fourth-order Runge-Kutta and first-order Euler integration algorithms, as well as our numerical DAE approach. In all cases, solutions are presented using the smallest timestep available ($\Delta t = 2^{-9}\, years$ for DAE and VENSIM, $\Delta t = 2^{-8}\, years$ for STELLA) to minimize numerical errors. Results indicate that the Runge-Kutta and Euler integration algorithms within each of STELLA and VENSIM yield identical answers for their respective refined timesteps. However, while STELLA and VENSIM show consistency at early time, their answers diverge during periods of non-linearity at around 10 years when the control is required to enforce a zero funds balance. Neither STELLA nor VENSIM is an accurate representation of the numerical DAE solution. Without having direct access to the source code for STELLA and VENSIM it is impossible to definitively state the source of this discrepancy. Based on our attempts to track the transient values of primary and secondary variables during a STELLA simulation, I speculate that STELLA does not invoke a fixed-point iteration to resolve the nonlinearity. Apart from differences in how the equations are formulated and discretized between the numerical DAE and system dynamics approaches, which can be found by comparing this paper against Appendix A of Rehan et al. (Rehan, et al. 2011), I speculate that the lack of a fixed-point iteration is a significant contribution to the observed discrepancy. This speculation is based on our earlier discussion surrounding Figure 3.2 (a) in which I observed the numerical DAE approach and STELLA yielded virtually identical answers for linear systems. Linear systems do not require a fixed point iteration.

Having established that for the numerical DAE approach, the fully-explicit, fully-implicit and even Crank-Nicolson temporal weighting all converge to the same identical solution with Δt in the range of 2^{-2} to $2^{-9}\, years$, the remaining question then becomes which temporal weighting strategy provides the least simulation time for a given Δt. This question is evaluated on Figure 3.6 which provides the normalized runtime per timestep and per fixed-point iteration using the numerical DAE algorithm. With $\Delta t = 2^{-2}\ years$, all temporal weighting schemes required approximately 8 fixed-point iterations per timestep for approximately the first twenty years of the simulation. This is the time interval during which the non-linearity is greatest. After this time, the number of fixed-point iterations diminishes to 2 per timestep as the problem reaches a steady-state. With $\Delta t = 2^{-9}\ years$, approximately 2 fixed-point iterations are required per timestep for all temporal weighting schemes for the entire simulation time. I conclude that fully-explicit is the optimal temporal weighting scheme for all timestep sizes for the Rehan et al. (Rehan, et al. 2011) system dynamics problem. The optimal timestep size depends on the desired level of accuracy and is the subjective choice of the professional engineer.

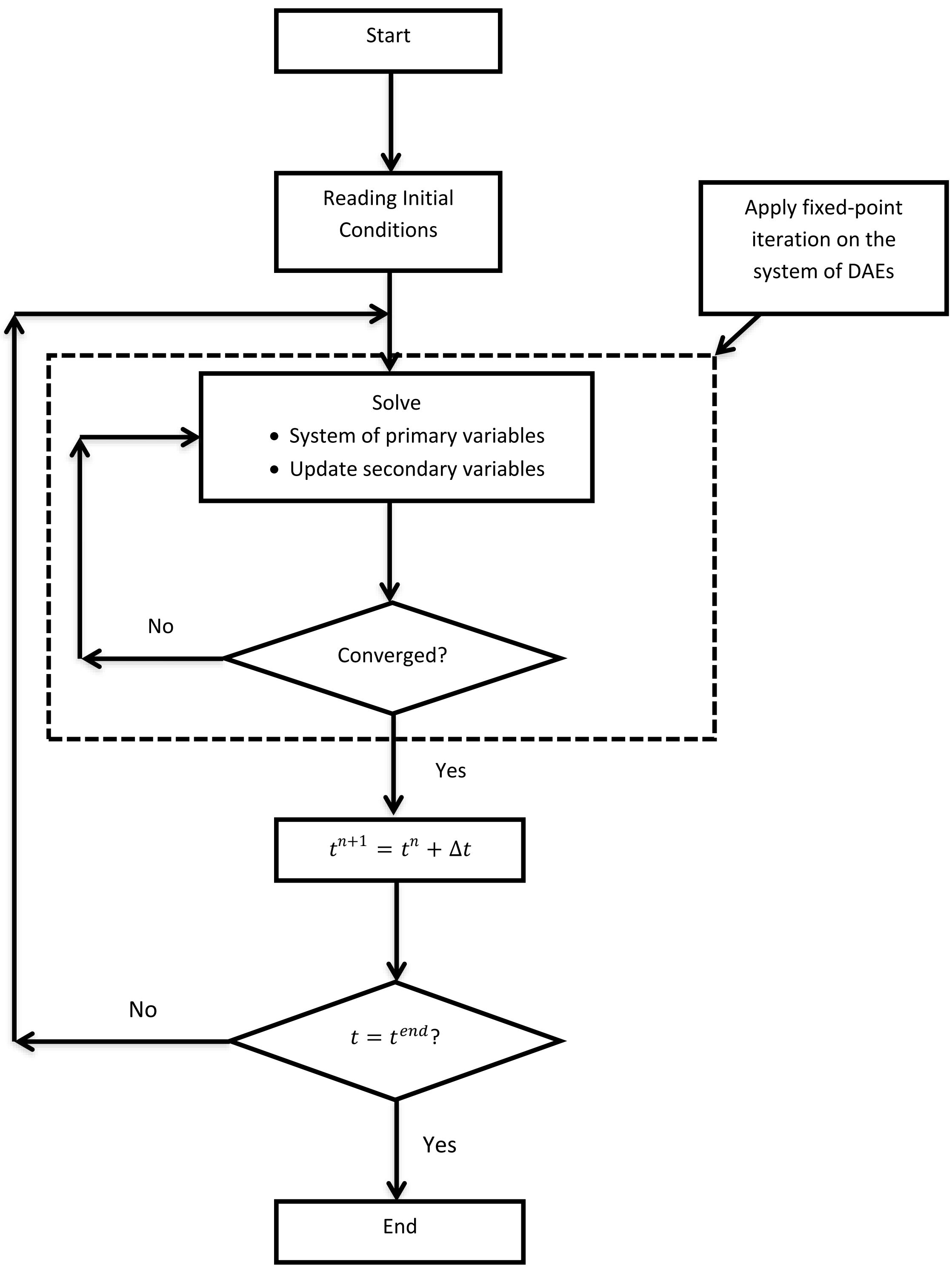

Figure 3.1: Flow-chart for implementing a fixed-point iteration.

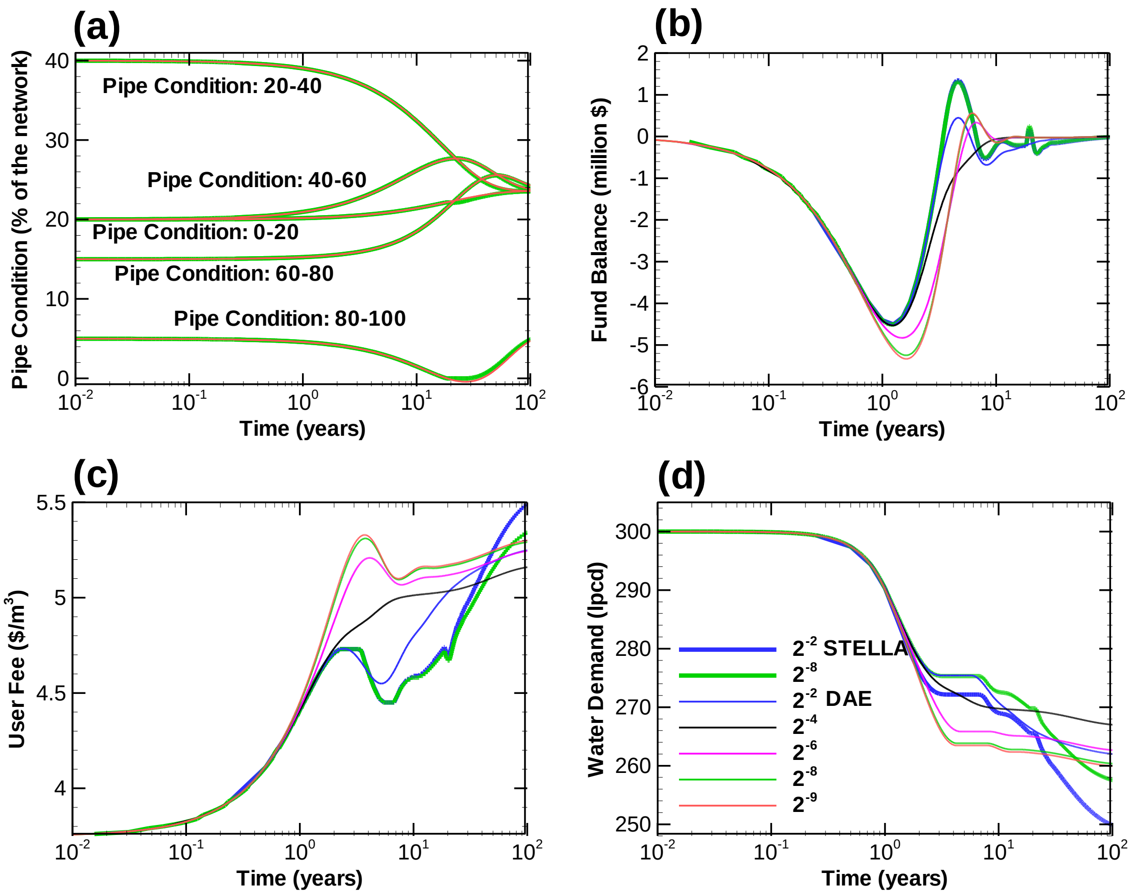

Figure 3.2: Primary variables over the 100-year simulation period. All numerical DAE results obtained using Crank-Nicolson weighting. The timestep size Δ*t* for each line is enumerated in the legend, with STELLA simulation results provided by thick lines and numerical DAE results by thin lines.

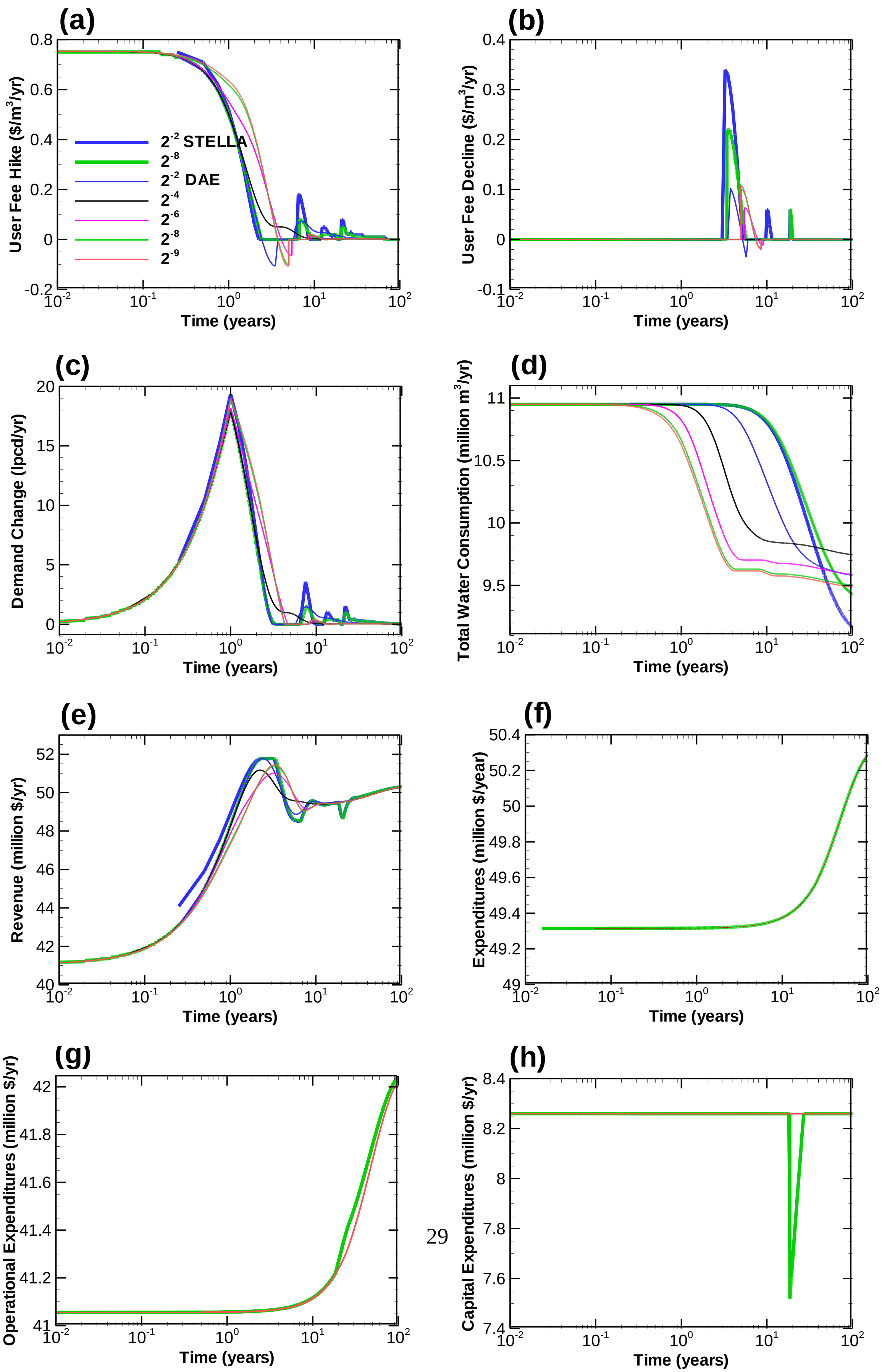
(a)
User Fee Hike ($/m³/yr)
Time (years)
2-2 STELLA
2-8
2-2 DAE
2-4
2-6
2-8
2-9
(b)
User Fee Decline ($/m³/yr)
Time (years)
(c)
Demand Change (lpcd/yr)
Time (years)
(d)
Total Water Consumption (million m³/yr)
Time (years)
(e)
Revenue (million $/yr)
Time (years)
(f)
Expenditures (million $/year)
Time (years)
(g)
Operational Expenditures (million $/yr)
Time (years)
(h)
Capital Expenditures (million $/yr)
Time (years)

Figure 3.3: Secondary variables over the 100-year simulation period. All numerical DAE results obtained using Crank-Nicolson weighting. The timestep size Δt for each line is enumerated in the legend, with STELLA simulation results provided by thick lines and numerical DAE results by thin lines.

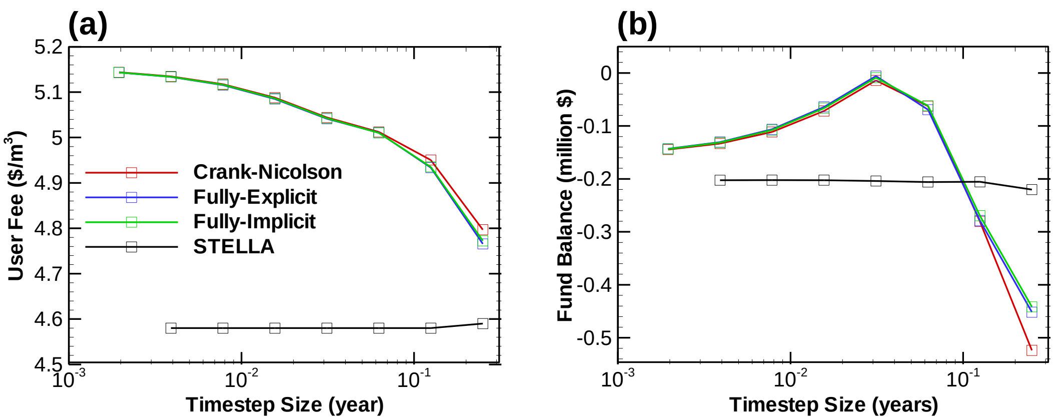

Figure 3.4: Convergence of the numerical DAE and STELLA 7.0.2 algorithms as the timestep is sequentially halved, for the primary variables $UserFee^{n+1}$ and $FundBalance^{n+1}$ at year 10. The numerical DAE solution was obtained using Crank-Nicolson, fully-implicit and fully-explicit temporal weighting.

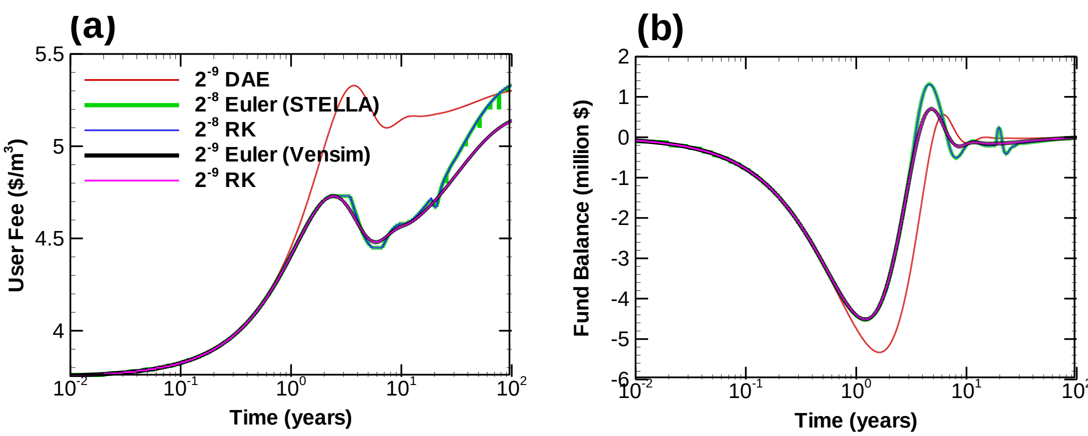

Figure 3.5: Sensitivity of the two primary variables (a) $UserFee^{n+1}$ and (b) $FundBalance^{n+1}$ to the STELLA 7.0.2 and VENSIM 6.0b fourth-order Runge-Kutta and first-order Euler integration algorithms, and the numerical DAE approach.

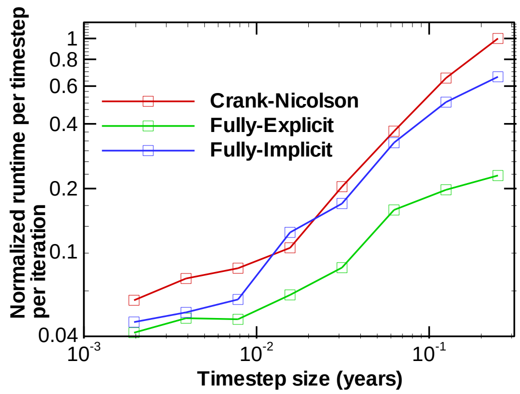

Figure 3.6: Normalized runtime per timestep and per fixed-point iteration using the numerical DAE algorithm with Crank-Nicolson, fully-explicit and fully-implicit temporal weighting.

3.2 Conclusions

The system dynamics model of Rehan et al. (Rehan, et al. 2011), which focuses on modeling of a financially sustainable water and wastewater distribution network using the software package STELLA 7.0.2, provides the prototype problem for this work. Specifically, I illustrate a strategy for converting system dynamics mathematical vernacular (i.e. stocks, flows, connectors and convertors) into a semi-explicit index-1 DAE system. The differential equations capture the transient evolution of the modeled system via a series of independent primary variables that all include a time derivative. In addition, a series of algebraic (secondary) equations and variables follow that are used to specify the non-linear behavior of the system as well as any controls on the physically relevant range (and hence temporal evolution) of the primary variables. In the case of the Rehan et al. (Rehan, et al. 2011) system dynamics problem, a secondary variable is identified that specifies the non-linear behavior by which consumers adjust their change in demand for water as its unit price increases, with this demand change involving the price elasticity of demand for water. Also, secondary variables are identified that specify the rate at which the utility should adjust the unit price of water in order to achieve financial sustainability, expressed by maintaining a zero funds balance with revenues equalling expenditures. These variables act as a control on the evolution of the system ensuring that the fee hike (and decline) rate on the unit cost of water never exceeds an upper bound presumably set by the municipal council, while maintaining the funds balance within specified upper (surplus) and lower (deficit) bounds. The set of equations is solved by discretizing the time derivatives using fully-implict, Crank-Nicolson and fully-explicit temporal weighting in a numerical Euler-based DAE scheme with a fixed-point iteration to resolve the non-linearity.

Values of primary and secondary variables are presented over the 100-year simulation period of the Rehan et al. (Rehan, et al. 2011) system dynamics problem using the numerical DAE approach as well as the fourth-order Runge-Kutta algorithm invoked by the STELLA 7.0.2 software package. Both algorithms deviate markedly once the non-linearity of the system becomes dominant and the control is required to enforce a zero funds balance. Analysis of the numerical DAE solution at this point of departure indicate that it is point-wise stable in the sense that the primary variables converge to a single solution as the timestep is progressively reduced. The rate of convergence to the single solution is proportional to the order of the temporal error used to discretize the time derivative. The converged numerical DAE solution obtained does not exhibit many of the spurious oscillations inherent in the Runge-Kutta solution. I therefore conclude that the numerical DAE solution is physically correct whereas the Runge-Kutta algorithm exhibits numerical errors that significantly compromise the relevance of the solution. Despite this, I believe system dynamics software packages are invaluable tools to help the civil engineer develop prototype numerical models of complex systems. However, it is the due diligence of the civil engineer to employ numerical DAE algorithms to ensure that solutions to these complex systems, when used in design or planning, conform to expected behaviour and are free from numerical aberrations.
The outcome of this research has been published in the ASCE Journal of Computing in Civil Engineering (Shadpour, et al. 2015).

References

APM. *APM Body of Knowledge. High Wycombe, Association for Project Management.* Association for Project Management, 2006.

Aven, T., and E. Zio. "Some considerations on the treatment of uncertainties in risk assessment for practical decision making." *Reliability Engineering & System Safety* 96, no. 1 (2011): 64-74.

Baccarini, D. *Estimating Project Cost Contingency – Beyond the 10%.* n.d.

Baird, G.M. "Baird, GM (2010) introduces a game plan." *American Water Works Association* 102, no. 4 (2010): 74.

Bakhshi, Payam, and Ali Touran. "Procedia Engineering 85." *An overview of budget contingency calculation methods in construction industry.* 2014. 52–60.

Bendtsen, Claus, and Per Grove Thomsen. *Numerical Solution of Differential Algebraic Equations.* IMM-REP-1999-8, 1999.

Bocchini, Paolo, Dan M. Frangopol, and George Deodatis. "A random field based technique for the efficiency enhancement of bridge network life-cycle analysis under uncertainty." *Engineering Structures* 33, no. 12 (2011): 3208-3217.

Brenan, K.E., S.L. Campbell, and L.R. Petzold. *Numerical Solution of Initial-Value Problems in Differential Algebraic Equations.* New York: North Holland, 1989.

Bryde, D.J., and J.M. Volm. "Perceptions of owners in German construction projects: congruence with project risk theory." *Construction Management and Economics* 27, no. 11 (2009): 1059-1071.

Choate, P., and S. Walter. *America in Ruins: Beyond the Public Works Pork Barrel.* Washington D.C.: Council of State Planning Agencies, 1981.

Clark, D.E. "Monte Carlo Analysis: ten years of experience." *Cost Engineering* 43, no. 6 (2001): 40-45.

Conha, M., and J. Sousa. "Robust design of water distribution networks for a proactive risk management." 136, no. 2 (2010): 227-236.

Cunha, M., and J. Sousa. "Water distribution network design optimization: simulated annealing approach." *Journal of Water Resources Planning and Management* 125, no. 4 (1999): 215-221.

Dandy, G.C., and M. Engelhardt. "Multi-objective trade-offs between cost and reliability in the replacement of water mains." *J. Water Resour. Plann. Manage. (ASCE)* 132, no. 2 (2006): 79-88.

Eldosouky, Ibrahim Adel, Ahmed Hussein Ibrahim, and Hossamn El-Deen Mohammed. "Management of construction cost contingency covering upside and downside risks." *Alexandria Engineering Journal* 53 (2014): 863-881.

Ellingwood, Bruce R., and Kursat Kinali. "Quantifying and communicating uncertainty in seismic risk assessment." *Structural Safety* 31, no. 2 (2009): 179–187.

EPA. Environmental Protection Agency, 2002.

Evengard, B., Berner, J., Brubaker, M. and Revich., B. "Climate change and water security with a focus on the Arctic." *Global Health Action* 4 (2011).

Fenner, R.A., and L. Sweeting. "A decision support model for the rehablitation of “non-critical” sewers." *Water Science and Technology* 39, no. 9 (1999): 193-200.

FHWA. Federal Highway Administration, 1999.

Forrester, Jay Wright. "Industrial Dynamics--A Major Breakthrough for Decision Makers." *Harvard Business Review* 36, no. 4 (1958): 37–66.

Franks, T. "Capacity building and institutional development: reflections on water." *Public Administration & Development* 19, no. 1 (1999): 51.

Giustolisi, O., Z. Kapelan, and D.A. Savic. "7th International Conference on Hydroinformatics (HIC)." *Multi-objective Evolutionary Polynomial Regression.* Nice, France: P. Gourbesville, J. Cunge, V. Guinot & S.Y. Liong (Eds), Research Publishing, 2006. 725-732.

GoldSim. n.d. http://www.goldsim.com/Home/ (accessed 7 18, 2016).

Grigg, Neil S. *Water, Wastewater, and Stormwater Infrastructure Management.* N.Y.: McGraw Hill, 2003.

Huang, C.L., J. Vause, H.W. Ma, and C.P. Yu. "Urban water metabolism efficiency assessment: Integrated analysis of available and virtual water." *Science of The Total Environment* 452-453 (2013): 19–27.

Huang, Xin, Hao Zhou, and Haibin Zhu. "A framework for assessing the systemic risk of major financial institutions." *Journal of Banking & Finance* 33, no. 11 (2009): 2036-2049.

isee systems. n.d. http://www.iseesystems.com/softwares/Stella-iThink.aspx (accessed 7 18, 2016).

Kim, G.H., S.N. An, and K.I. Kang. "Comparison of construction cost estimating models based on regression analysis, neutral networks, and case-based reasoning." *Building and Environment* 39 (2004): 1235-1242.

Knight, Lindsay, and World Health Organization. "The Right to Water, Health & human rights publication series." 2003.

Lalonde, E., and C. Bergeron. "A Decision Support Methodology for Asset Management." *Pipeline Engineering and Construction International Conference.* Baltimore, Maryland, 2003. 33-43.

Laryea, S., and W. Hughes. "22nd Annual ARCOM Conference." *The Price of Risk in Construction Projects.* D. Boyd. Birmingham, UK, Association of Researchers in Construction Management, 2006. 553-561.

Lipsey, R. and Chrystal, A. "Chapter 18." In *Principles of Economics,* edited by 9. Oxford University Press, 1999.

Liu, Chein-Shan. "Elastoplastic models and oscillators solved by a Lie-group differential algebraic equations method." *INTERNATIONAL JOURNAL OF NON-LINEAR MECHANICS* 69 (2014): 93-108.

Liu, Chein-Shan. "On-line detecting heat source of a nonlinear heat conduction equation by a differential algebraic equation method." *INTERNATIONAL JOURNAL OF HEAT AND MASS TRANSFER* 76 (2014): 153-161.

Mak, S., and D. Picken. "Using risk analysis to determine construction project contingencies." *Journal of Construction Engineering and Management* 126, no. 2 (2000): 130-136.

Mak, S., J. Wong, and D. Picken. "The effect on contingency allowances of using risk analysis in capital cost estimating: a Hong Kong case study." *Construction Management and Economics* 16 (1998): 615-619.

Male, J.W., T.M. Walski, and A.H. Slutsky. "Analyzing Water Main Replacement Policies." *J. Water Resour. Plann. Manage. ASCE* 116, no. 3 (1990): 362-274.

MathWorks. n.d. http://www.mathworks.com/products/matlab/?requestedDomain=www.mathworks.com (accessed 7 18, 2016).

McKay, N.K., and C.S.W. Vincent. "Unifying the theory and practice of production scheduling." *Journal of Manufacturing Systems* 18, no. 4 (1999): 241–255.

Mirza, M. S., and M. Haider. "The state of infrastructure in Canada: Implications for Infrastrcuture Planning and Policy." *Infrastructure Canada* 29, no. 1 (2003): 17-38.

Musa, M.M., and I.K. Zubairu. "Appraisal of the Performance of Contingency Cost Provision for Building Projects in Nigeria." *Journal of Environmental Technology* 4 (2011): 41-48.

Narni, Nageswara Rao. "2nd International Conference on Advances in Computing, Communications and Informatics (ICACCI) ." *Numerical solution of breakage population balance equations using differential algebraic equations form.* IEEE, 345 E 47TH ST, NEW YORK, NY 10017 USA , 2013. 1384-1388 .

Ng, M.W., Z. Zhang, and S.T. Waller. "The Price of Uncertainty in Pavement Infrastructure Management Planning: An Integer Programming Approach." *Transportation Research Part C* 19, no. 6 (2011): 1326-1338.

Oberlender, G.D., and S.M. Trost. "Predicting accuracy of early cost estimates based on estimate quality." *Journal of Construction Engineering and Management* 127, no. 3 (2001): 173-182.

Pellissetti, M.F., and G.I. Schueller. "Scalable uncertainty and reliability analysis by integration of advanced Monte Carlo simulation and generic finite element solvers." *COMPUTERS & STRUCTURES* 87, no. 13-14 (2009): 930-947.

Petzold, L.R., and U.M Ascher. *Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations.* 1998.

Pytlak, R., and T. Zawadzki. "On solving optimal control problems with higher index differential-algebraic equations." *OPTIMIZATION METHODS & SOFTWARE* 29, no. 5 (2014): 1139-1162.

Rausand, M., and A. Hoyland. *System Reliability Theory: Models, Statistical Methods, and Applications.* Wiley-Interscience, 2004.

Rehan, R., M.A. Knight, C.T. Haas, and A.J.A. Unger. "Application of System Dynamics for Developing Financially Self-Sustaining Management Policies for Water and Wastewater Systems." *Water Research* 45, no. 16 (2011): 4737.

Renzetti, Steven. "Municipal Water Supply and Sewage Treatment: Costs, Prices, and Distortions." *The Canadian Journal of Economics* 32, no. 3 (1999): 688-704.

Rizzo, D.M., P.J. Mouser, D.H. Whitney, C.D. Mark, R.D. Magarey, and A.A. Vinov. "The comparison of four dynamic systems-based software packages: Translation and sensitivity analysis." *Journal of Environmental Modelling & Software* 21 (2006): 1491-1502.

Robaka, A., P. Herabata, and A.D. Gupta. "Decision support system for water infrastructure planning under budget uncertainty: A case study from Thailand." *Water International* 32, no. 1 (2007): 822-843.

Rogers, J.W., and G.E. Louis. "Risk and opportunity in upgrading the US drinking water infrastructure system." *Journal of Environmental Management* 87, no. 1 (2008): 26-36.

Sadiq, R., B. Rajani, and Y. Kleiner. "Fuzzy-based method to evaluate soil corrosivity for prediction of water main deterioration." *Journal of Infrastructure Systems* 10, no. 4 (2004): 149-156.

Scott, C.A., C.J. Bailey, R.P. Marra, J.W. Woods, K.J. Ormerad, and K. Lansey . "Scenario Planning to Address Critical Uncertainties for Robust and Resilient Water–Wastewater Infrastructures under Conditions of Water Scarcity and Rapid Development." *Water* 4, no. 4 (2012): 848-868.

Sepplet, R., and O. Richter. "“It was an artefact not the result”: A note on system dynamics dynamic model development tools." *Journal of Environmental Modeling and Software* 20 (2005): 1543-1548.

Shadpour, Atefeh, Andre J.A. Unger, Mark A. Knight, and Carl T. Haas. "Numerical DAE Approach for Solving a System Dynamics Problem." *ASCE Journal of Computing in Civil Engineering* 29, no. 3 (2015).

Shamir, Uri, and Charles D.D. Howard. "An Analytic Approach to Scheduling Pipe Replacement." *Journal AWWA - Management and Operations*, 1979.

Silinis, P.G., and S.W. Franks. "Development of a Disaggregation Algorithm for Assessing the Spatial Variability of Water Supply Infrastructure Performance." *Water Challenge: Balancing the Risks: Hydrology and Water Resources Symposium.* 2002. 1048-1055.

Simulistics. n.d. http://www.simulistics.com/ (accessed 7 18, 2016).

Singh, Sukhvir, Manju Arora, and Ram Kishore. "Estimation of measurement uncertainty in characterization of materials using transmission electron microscope (TEM) at NPL, India." *MAPAN-JOURNAL OF METROLOGY SOCIETY OF INDIA* 22, no. 4 (2007): 263-268 .

Son, J., and E.M. Rojas. "Impact of optimism bias regarding organizational dynamics on project planning and control." *Journal of Construction Engineering and Management, ASCE* 137, no. 2 (2011): 147-157.

Stern, C.T., and D.R. Kendall. "Mobile solutions deliver improved utility infrastructure and operations management." *American Water Works Association* 93, no. 11 (2001): 69.

"Sustainable Water and Sewage Systems Act, Bill 175." 2002.

Thal Jr., Alfred E., Jason J. Cook, and Edward D. White. "Estimation of Cost Contingency for Air Force Cobstruction Projects." *ASCE JOURNAL OF CONSTRUCTION ENGINEERING AND MANAGEMENT* 136, no. 11 (2010): 1181-1188.

"the Canadian Political Science Association and the Canadian Historical Association." *The Municipal Corporations Act of 1849* . Halifax, 1949.

Tidwell, V., J. Cooper, and C. Silva. "Threat assessment of water supply systems using Markov latent effects modeling." *Water Resources Planning and Management* 131, no. 3 (May 2005): 218-227.

Ugarelli, R., G. Venkatesh, V. Di Federico, and S. Sægrov. "Historical analysis of blockages in wastewater pipelines in Oslo and diagnosis of causative pipeline characteristics." *Urban Water* 7, no. 6 (2010): 335–343.

"UN Documents - Gathering a body of global agreements." International Conference on Water and the Environment. 1 31, 1992. http://www.un-documents.net/h2o-dub.htm (accessed 7 12, 2016).

US Environemntal Protection Agency, Office of Water. "2001 Update of Ambient Water Quality Criteria for Cadmium." 2001.

"Water Opportunities and Water Conservation Act, 2010, S.O. 2010, c. 19 - Bill 72." *Ministry of Environment.* n.d. https://www.ontario.ca/laws/statute/s10019 (accessed 7 18, 2016).

Wirahadikusumah, R., D.M. Abraham, T. Iseley, and R.K. Prasanth. "Assessment technologies for sewer system rehabilitation." *Automation in Construction* 7, no. 4 (1998): 259-270.

WOLFRAM. n.d. https://www.wolfram.com/mathematica/ (accessed 7 18, 2016).

Younis, Rizwan. "Development of Wastewater Collection Network Asset Database, Deterioration Models and Management Framework." Thesis, Department of Civil and Environmental Engineering, University of Waterloo, Waterloo, 2010.

Yu, S., A.J. Unger, B. Parker, and T. Kim. "Allocating risk capital for a brownfields redevelopment project under hydrogeological and financial uncertainty." *Journal of Environmental Management* 100 (2012): 96-108.

Appendix A: numerical modeling of the urban wastewater management system

Coding in MATLAB (Crank-Nicolson scheme):

```
%Linear System of Equations by Gaussian Elimination (LU Factorization)
%FULL ITERATION
clear all; close all; clc
tic
%INITIAL CONDITIONS*****************************************
Cond_20 = 140;
Cond_40 = 280;
Cond_60 = 140;
Cond_80 = 105;
Cond_100 = 35;
Cum_CapEx = 0;
Cum_OpEx = 0;
Funds_Balance = 0;
User_Fees = 3.75;
Water_Demand = 300;
TotalEx_ToDate = 0;
Demand_Change = 0;
%CONSTANTS**************************************************
alpha1 = 0.1;
alpha2 = 0.1;
alpha3 = 0.1;
deltaT = 1/64;
Population = 100000;
Elasticity_of_Demand = 0.35;
Maximum_Funds_Balance = 0;
Minimum_Funds_Balance = 0;
Minimum_Demand = 200;
```

```
Unit_Price_CapEx = 1000;
Unit_Price_OpEx = 50;
CapEx = 8260000;
OpEx = 41055000;
Rehab_Fraction = 1.18;
Max_Fee_Hike_Rate = 1000;
Total_Length_Pipes = Cond_20 + Cond_40 + Cond_60 + Cond_80 +
Cond_100;
Average_Cond = (Cond_20*20 + Cond_40*40 + Cond_60*60 + Cond_80*80
+ Cond_100*100)/Total_Length_Pipes;
Total_Water_Consumption = Water_Demand*Population*365/1000;
Revenue = User_Fees * Total_Water_Consumption;

    C20(1) = Cond_20/Total_Length_Pipes*100;
    C40(1) = Cond_40/Total_Length_Pipes*100;
    C60(1) = Cond_60/Total_Length_Pipes*100;
    C80(1) = Cond_80/Total_Length_Pipes*100;
    C100(1) = Cond_100/Total_Length_Pipes*100;
    F(1) = Funds_Balance;
    U(1) = User_Fees;
    O(1) = OpEx;
    W(1) = Water_Demand;
    R(1) = Revenue;
    C(1) = CapEx;
    E(1) = OpEx+CapEx;
    T(1) = Total_Water_Consumption;

time = deltaT;
c = 2;
while time <= 100
```

```
%Saving the variables of the previous time into new variable names;
%These variables are used in the "b" matrix to keep it unchanged during
%the iteration:
Co100 = Cond_100;
Co20 = Cond_20;
Co40 = Cond_40;
Co60 = Cond_60;
Co80 = Cond_80;
FB = Funds_Balance;
TD = TotalEx_ToDate;
UF = User_Fees;
WD = Water_Demand;

xx = [0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100];
y = [0, 1.5, 3.5, 6.5, 11, 18, 26, 38.5, 55, 75, 100];
Average_Cond = (Cond_20*20 + Cond_40*40 + Cond_60*60 + Cond_80*80 + Cond_100*100)/Total_Length_Pipes;
Condition_Factor = interp1(xx,y,Average_Cond);
AC(c) = Average_Cond;

Rehab_Length = Total_Length_Pipes * Rehab_Fraction/100;
Renewal = Rehab_Length;

Expenditures = CapEx + OpEx;

%Required_Revenue
if (Funds_Balance < Minimum_Funds_Balance)
    Required_Revenue = Expenditures-Funds_Balance;
else
    Required_Revenue = Expenditures;
```

```
    end

    %User_Fee_Hike
    if (Funds_Balance > Maximum_Funds_Balance)
        User_Fee_Hike = 0;
    else
        User_Fee_Hike    =    min(Max_Fee_Hike_Rate/100*User_Fees,
(Required_Revenue/Total_Water_Consumption-User_Fees));
    end

    %User_Fee_Decline
    if (Funds_Balance > Maximum_Funds_Balance)
        User_Fee_Decline    =    User_Fees    -    (Expenditures-
Funds_Balance)/Total_Water_Consumption;
    else
        User_Fee_Decline = 0;
    end

    AA = 1000*Unit_Price_OpEx*(1+Condition_Factor/100);
    BB = 1000*Unit_Price_CapEx*Rehab_Fraction/100;

    UFH = User_Fee_Hike;
    UFD = User_Fee_Decline;
    TWC = Total_Water_Consumption;
    DC = Demand_Change;


%*****************************************************************
******
    %Doolittle Algorithm with Data Structures
```

```
%***************************************************************
******
    %Dara structure of original matrix
    a = [1+deltaT*Rehab_Fraction/200, deltaT*Rehab_Fraction/200,
deltaT*Rehab_Fraction/200,      deltaT*Rehab_Fraction/200,      -
deltaT/40+deltaT*Rehab_Fraction/200,...
        -deltaT*Rehab_Fraction/200,                            1-
deltaT*Rehab_Fraction/200+deltaT/40, -deltaT*Rehab_Fraction/200, -
deltaT*Rehab_Fraction/200, -deltaT*Rehab_Fraction/200,...
        -deltaT/40,   1+deltaT/40,   -deltaT/40,   1+deltaT/40,   -
deltaT/40,   1+deltaT/40,   deltaT*(AA+BB)/2,   deltaT*(AA+BB)/2,
deltaT*(AA+BB)/2, deltaT*(AA+BB)/2, deltaT*(AA+BB)/2, 1,...
        -deltaT*Total_Water_Consumption/2,  -deltaT*(AA+BB)/2,  -
deltaT*(AA+BB)/2,    -deltaT*(AA+BB)/2,    -deltaT*(AA+BB)/2,    -
deltaT*(AA+BB)/2, 1, 1, 1];

    ia = [1 6 11 13 15 17 24 30 31 32];

    ja = [1 2 3 4 5 1 2 3 4 5 2 3 3 4 4 5 1  2 3 4 5 6 8 1 2 3 4 5
7 8 9];

    idiag = [1 7 12 14 16 22 29 30 31];

    %Data structure of factored matrix
    af = [1+deltaT*Rehab_Fraction/200, deltaT*Rehab_Fraction/200,
deltaT*Rehab_Fraction/200,      deltaT*Rehab_Fraction/200,      -
deltaT/40+deltaT*Rehab_Fraction/200, 0, 0, 0, 0,...
        -deltaT*Rehab_Fraction/200,                            1-
deltaT*Rehab_Fraction/200+deltaT/40, -deltaT*Rehab_Fraction/200, -
deltaT*Rehab_Fraction/200,  -deltaT*Rehab_Fraction/200,  0,  0,  0,
0,...
```

```
        0, -deltaT/40, 1+deltaT/40, 0, 0, 0, 0, 0, 0,...
        0, 0, -deltaT/40, 1+deltaT/40, 0, 0, 0, 0, 0,...
        0, 0, 0, -deltaT/40, 1+deltaT/40, 0, 0, 0, 0,...
        deltaT*(AA+BB)/2,   deltaT*(AA+BB)/2,   deltaT*(AA+BB)/2,
deltaT*(AA+BB)/2,      deltaT*(AA+BB)/2,      1,      0,      -
deltaT*Total_Water_Consumption/2, 0,...
        -deltaT*(AA+BB)/2, -deltaT*(AA+BB)/2, -deltaT*(AA+BB)/2, -
deltaT*(AA+BB)/2, -deltaT*(AA+BB)/2, 0, 1, 0, 0,...
        0, 0, 0 ,0, 0, 0, 0, 1, 0,...
        0, 0, 0, 0, 0, 0, 0, 0, 1];

    iaf = [1 10 19 28 37 46 55 64 73 82];

    jaf = [1 2 3 4 5 6 7 8 9 1 2 3 4 5 6 7 8 9 1 2 3 4 5 6 7 8 9 1
2 3 4 5 6 7 8 9 1 2 3 4 5 6 7 8 9 ...
        1 2 3 4 5 6 7 8 9 1 2 3 4 5 6 7 8 9 1 2 3 4 5 6 7 8 9 1 2
3 4 5 6 7 8 9];

    idiagf = [1 11 21 31 41 51 61 71 81];

    n = sqrt(size(af,2));
    work(1:n) = 0;
    marker(1:n) = 0;
    i = 2;
    while i <= n
        for k = ia(i):ia(i+1)-1
            work(ja(k)) = a(k);
        end
        for k = iaf(i):iaf(i+1)-1
            marker(jaf(k)) = 1;
        end
        %************************************************
```

```
        %Computation of L\U elements in order:work array*
        %***********************************************
        for k = iaf(i):idiagf(i)-1
            id = jaf(k);
            mult = work(id)/af(idiagf(id));
            work(id) = mult;
            for kk = idiagf(id)+1:iaf(id+1)-1
                idd = jaf(kk);
                if (marker(idd) ~= 0)
                    work(idd) = work(idd)-mult*af(kk);
                end
            end
        end
        for k = iaf(i):iaf(i+1)-1
            af(k) = work(jaf(k));
            work(jaf(k)) = 0;
            marker(jaf(k)) = 0;
        end
        i = i+1;
    end
    %******
    %Solve*
    %******
    b            =            [(1-deltaT*Rehab_Fraction/200)*Co100-
deltaT*Rehab_Fraction/200*Co20-deltaT*Rehab_Fraction/200*Co40-
deltaT*Rehab_Fraction/200*Co60+(deltaT/40-
deltaT*Rehab_Fraction/200)*Co80,...

deltaT*Rehab_Fraction/200*Co100+(1+deltaT*Rehab_Fraction/200-
deltaT/40)*Co20+deltaT*Rehab_Fraction/200*Co40+deltaT*Rehab_Fract
ion/200*Co60+deltaT*Rehab_Fraction/200*Co80,...
        (1-deltaT/40)*Co40+deltaT/40*Co20,...
```

```
        (1-deltaT/40)*Co60+deltaT/40*Co40,...
        (1-deltaT/40)*Co80+deltaT/40*Co60,...
        FB-deltaT*(AA+BB)/2*Co100-deltaT*(AA+BB)/2*Co20-
deltaT*(AA+BB)/2*Co40-deltaT*(AA+BB)/2*Co60-
deltaT*(AA+BB)/2*Co80+deltaT*TWC/2*UF,...

TD+deltaT*(AA+BB)/2*Co100+deltaT*(AA+BB)/2*Co20+deltaT*(AA+BB)/2*
Co40+deltaT*(AA+BB)/2*Co60+deltaT*(AA+BB)/2*Co80,...
        UF+deltaT*(1/2*(User_Fee_Hike-User_Fee_Decline)+1/2*(UFH-
UFD)),...
        WD+deltaT*(1/2*(-Demand_Change)+1/2*(-DC))];
    z = b;
    for i = 1:n
        for k = iaf(i):idiagf(i)-1
            z(i) = z(i)-z(jaf(k))*af(k);
        end
    end
    x = z;
    for i = n:-1:1
        for k = idiagf(i)+1:iaf(i+1)-1
            x(i) = x(i)-x(jaf(k))*af(k);
        end
        x(i) = x(i)/af(idiagf(i));
    end

%*****************************************************************
******
            Cond_100 = x(1);
            Cond_20 = x(2);
            Cond_40 = x(3);
            Cond_60 = x(4);
            Cond_80 = x(5);
```

```
            Funds_Balance = x(6);
            TotalEx_ToDate = x(7);
            User_Fees = x(8);
            Water_Demand = x(9);
    %**********
    %CALCULATE*
    %**********
    OpEx                                                              =
Unit_Price_OpEx*Total_Length_Pipes*1000*(1+Condition_Factor/100);
    CapEx = Renewal*1000*Unit_Price_CapEx;

        %Demand_Change
        U(c) = User_Fees;
        if time <= 1
            Demand_Change         =         min((User_Fees          -
U(1))/U(1)*Elasticity_of_Demand*Water_Demand,       (Water_Demand-
Minimum_Demand));
        else
            Demand_Change      =      min((User_Fees      -      U(c-
floor(1/deltaT)))/U(c-
floor(1/deltaT))*Elasticity_of_Demand*Water_Demand, (Water_Demand-
Minimum_Demand));
        end
        if Demand_Change < 0
            Demand_Change = 0;
        end

        %Total_Water_Consumption
        W(c) = Water_Demand;
        if time <= 20
            S(1) = W(1);
            for j = 2:c
```

```
        S(j) = alpha1 * W(j-1)+(1-alpha1) * S(j-1);
    end
    S1 = S;
    WW = S;
    for j = 2:c
        S(j) = alpha2 * WW(j-1)+(1-alpha2) * S(j-1);
    end
    S2  = S;
    WW = S;
    for j = 2:c
        S(j) = alpha3 * WW(j-1)+(1-alpha3) * S(j-1);
    end
    Total_Water_Consumption = S(c)*Population*365/1000;
else
    S(1) = W(c-floor(20/deltaT));
    for j = c+1-floor(20/deltaT):c
        S(j) = alpha1 * W(j-1)+(1-alpha1) * S(j-1);
    end
    S1 = S;
    WW = S;
    for j = c+1-floor(20/deltaT):c
        S(j) = alpha2 * WW(j-1)+(1-alpha2) * S(j-1);
    end
    S2  = S;
    WW = S;
    for j = c+1-floor(20/deltaT):c
        S(j) = alpha3 * WW(j-1)+(1-alpha3) * S(j-1);
    end
    Total_Water_Consumption = S(c)*Population*365/1000;
end

TWC = Total_Water_Consumption;
```

```
%         %Revenue
%         Revenue = User_Fees * Total_Water_Consumption;

        CapEx_Sum = CapEx;
        OpEx_Sum = OpEx;
        Cum_CapEx = Cum_CapEx + deltaT * CapEx_Sum;
        Cum_OpEx = Cum_OpEx + deltaT * OpEx_Sum;

%*****************************************************************
******
    %Iteration

%*****************************************************************
******
    %error1:  The  absolute  difference  between  two  successive
iterations on
    %User_Fees
    %error1:  The  absolute  difference  between  two  successive
iterations on
    %Funds_Balance
    error1 = 1;
    error2 = 1;
    %error3 = 1;
    tol1 = 1e-5;
    tol2 = 1e-5;
    %tol3 = 1e-5;
    itt = 0;

    while (error1 > tol1 && error2 > tol2)

        X1 = User_Fees;
```

```
        X2 = Funds_Balance;
        %X3 = Water_Demand;

        %Required_Revenue
        if (Funds_Balance < Minimum_Funds_Balance)
            Required_Revenue = Expenditures-Funds_Balance;
        else
            Required_Revenue = Expenditures;
        end

        %User_Fee_Hike
        if (Funds_Balance > Maximum_Funds_Balance)
            User_Fee_Hike = 0;
        else
            User_Fee_Hike  =  min(Max_Fee_Hike_Rate/100*User_Fees,
(Required_Revenue/Total_Water_Consumption-User_Fees));
        end
        Y(c) = User_Fee_Hike;
        %User_Fee_Decline
        if (Funds_Balance > Maximum_Funds_Balance)
            User_Fee_Decline   =   User_Fees   -   (Expenditures-
Funds_Balance)/Total_Water_Consumption;
        else
            User_Fee_Decline = 0;
        end
        YY(c) = User_Fee_Decline;
        xx = [0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100];
        y = [0, 1.5, 3.5, 6.5, 11, 18, 26, 38.5, 55, 75, 100];
        Average_Cond  =  (Cond_20*20  +  Cond_40*40  +  Cond_60*60  +
Cond_80*80 + Cond_100*100)/Total_Length_Pipes;
        Condition_Factor = interp1(xx,y,Average_Cond);
```

```
        AA = 1000*Unit_Price_OpEx*(1+Condition_Factor/100);
        BB = 1000*Unit_Price_CapEx*Rehab_Fraction/100;


%*****************************************************************
**
        %Doolittle Algorithm with Data Structures

%*****************************************************************
**
        %Dara structure of original matrix
        a              =              [1+deltaT*Rehab_Fraction/200,
deltaT*Rehab_Fraction/200,              deltaT*Rehab_Fraction/200,
deltaT*Rehab_Fraction/200,                                        -
deltaT/40+deltaT*Rehab_Fraction/200,...
        -deltaT*Rehab_Fraction/200,                              1-
deltaT*Rehab_Fraction/200+deltaT/40, -deltaT*Rehab_Fraction/200, -
deltaT*Rehab_Fraction/200, -deltaT*Rehab_Fraction/200,...
        -deltaT/40,   1+deltaT/40,   -deltaT/40,   1+deltaT/40,   -
deltaT/40,   1+deltaT/40,   deltaT*(AA+BB)/2,   deltaT*(AA+BB)/2,
deltaT*(AA+BB)/2, deltaT*(AA+BB)/2, deltaT*(AA+BB)/2, 1,...
        -deltaT*Total_Water_Consumption/2,  -deltaT*(AA+BB)/2,   -
deltaT*(AA+BB)/2,    -deltaT*(AA+BB)/2,    -deltaT*(AA+BB)/2,    -
deltaT*(AA+BB)/2, 1, 1, 1];

    ia = [1 6 11 13 15 17 24 30 31 32];

    ja = [1 2 3 4 5 1 2 3 4 5 2 3 3 4 4 5 1  2 3 4 5 6 8 1 2 3 4 5
7 8 9];

    idiag = [1 7 12 14 16 22 29 30 31];
```

```
    %Data structure of factored matrix
    af = [1+deltaT*Rehab_Fraction/200, deltaT*Rehab_Fraction/200,
deltaT*Rehab_Fraction/200,      deltaT*Rehab_Fraction/200,      -
deltaT/40+deltaT*Rehab_Fraction/200, 0, 0, 0, 0,...
        -deltaT*Rehab_Fraction/200,                              1-
deltaT*Rehab_Fraction/200+deltaT/40, -deltaT*Rehab_Fraction/200, -
deltaT*Rehab_Fraction/200,  -deltaT*Rehab_Fraction/200,  0,  0,  0,
0,...
        0, -deltaT/40, 1+deltaT/40, 0, 0, 0, 0, 0, 0,...
        0, 0, -deltaT/40, 1+deltaT/40, 0, 0, 0, 0, 0,...
        0, 0, 0, -deltaT/40, 1+deltaT/40, 0, 0, 0, 0,...
        deltaT*(AA+BB)/2,   deltaT*(AA+BB)/2,   deltaT*(AA+BB)/2,
deltaT*(AA+BB)/2,      deltaT*(AA+BB)/2,      1,      0,      -
deltaT*Total_Water_Consumption/2, 0,...
        -deltaT*(AA+BB)/2, -deltaT*(AA+BB)/2, -deltaT*(AA+BB)/2, -
deltaT*(AA+BB)/2, -deltaT*(AA+BB)/2, 0, 1, 0, 0,...
        0, 0, 0 ,0, 0, 0, 0, 1, 0,...
        0, 0, 0, 0, 0, 0, 0, 0, 1];

    iaf = [1 10 19 28 37 46 55 64 73 82];

        jaf = [1 2 3 4 5 6 7 8 9 1 2 3 4 5 6 7 8 9 1 2 3 4 5 6 7 8
9 1 2 3 4 5 6 7 8 9 1 2 3 4 5 6 7 8 9 ...
            1 2 3 4 5 6 7 8 9 1 2 3 4 5 6 7 8 9 1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9];

        idiagf = [1 11 21 31 41 51 61 71 81];

        n = sqrt(size(af,2));
        work(1:n) = 0;
        marker(1:n) = 0;
        i = 2;
```

```
while i <= n
    for k = ia(i):ia(i+1)-1
        work(ja(k)) = a(k);
    end
    for k = iaf(i):iaf(i+1)-1
        marker(jaf(k)) = 1;
    end
    %************************************************
    %Computation of L\U elements in order:work array*
    %************************************************
    for k = iaf(i):idiagf(i)-1
        id = jaf(k);
        mult = work(id)/af(idiagf(id));
        work(id) = mult;
        for kk = idiagf(id)+1:iaf(id+1)-1
            idd = jaf(kk);
            if (marker(idd) ~= 0)
                work(idd) = work(idd)-mult*af(kk);
            end
        end
    end
    for k = iaf(i):iaf(i+1)-1
        af(k) = work(jaf(k));
        work(jaf(k)) = 0;
        marker(jaf(k)) = 0;
    end
    i = i+1;
end
%******
%Solve*
%******
```

```
        %b   =   [Co100,   Co20,   Co40,   Co60,   Co80,   FB,   TD,
UF+deltaT*(User_Fee_Hike-User_Fee_Decline),          WD+deltaT*(-
Demand_Change)];
        b         =         [(1-deltaT*Rehab_Fraction/200)*Co100-
deltaT*Rehab_Fraction/200*Co20-deltaT*Rehab_Fraction/200*Co40-
deltaT*Rehab_Fraction/200*Co60+(deltaT/40-
deltaT*Rehab_Fraction/200)*Co80,...

deltaT*Rehab_Fraction/200*Co100+(1+deltaT*Rehab_Fraction/200-
deltaT/40)*Co20+deltaT*Rehab_Fraction/200*Co40+deltaT*Rehab_Fract
ion/200*Co60+deltaT*Rehab_Fraction/200*Co80,...
        (1-deltaT/40)*Co40+deltaT/40*Co20,...
        (1-deltaT/40)*Co60+deltaT/40*Co40,...
        (1-deltaT/40)*Co80+deltaT/40*Co60,...
        FB-deltaT*(AA+BB)/2*Co100-deltaT*(AA+BB)/2*Co20-
deltaT*(AA+BB)/2*Co40-deltaT*(AA+BB)/2*Co60-
deltaT*(AA+BB)/2*Co80+deltaT*Total_Water_Consumption/2*UF,...

TD+deltaT*(AA+BB)/2*Co100+deltaT*(AA+BB)/2*Co20+deltaT*(AA+BB)/2*
Co40+deltaT*(AA+BB)/2*Co60+deltaT*(AA+BB)/2*Co80,...
        UF+deltaT*(1/2*(User_Fee_Hike-User_Fee_Decline)+1/2*(UFH-
UFD)),...
        WD+deltaT*(1/2*(-Demand_Change)+1/2*(-DC))];
        z = b;
        for i = 1:n
            for k = iaf(i):idiagf(i)-1
                z(i) = z(i)-z(jaf(k))*af(k);
            end
        end
        x = z;
        for i = n:-1:1
            for k = idiagf(i)+1:iaf(i+1)-1
```

```
            x(i) = x(i)-x(jaf(k))*af(k);
        end
        x(i) = x(i)/af(idiagf(i));
    end

%**************************************************************
    Cond_100 = x(1);
    Cond_20 = x(2);
    Cond_40 = x(3);
    Cond_60 = x(4);
    Cond_80 = x(5);
    Funds_Balance = x(6);
    TotalEx_ToDate = x(7);
    User_Fees = x(8);
    Water_Demand = x(9);

%**************************************************************
    %*******
    %UPDATE*
    %*******
    %Demand_Change
    U(c) = User_Fees;
    if time <= 1
        Demand_Change         =         min((User_Fees         -
U(1))/U(1)*Elasticity_of_Demand*Water_Demand,       (Water_Demand-
Minimum_Demand));
    else
        Demand_Change      =      min((User_Fees      -      U(c-
floor(1/deltaT)))/U(c-
floor(1/deltaT))*Elasticity_of_Demand*Water_Demand, (Water_Demand-
Minimum_Demand));
    end
```

```
if Demand_Change < 0
    Demand_Change = 0;
end
%Total_Water_Consumption
W(c) = Water_Demand;
if time <= 20
    S(1) = W(1);
    for j = 2:c
        S(j) = alpha1 * W(j-1)+(1-alpha1) * S(j-1);
    end
    S1 = S;
    WW = S;
    for j = 2:c
        S(j) = alpha2 * WW(j-1)+(1-alpha2) * S(j-1);
    end
    S2  = S;
    WW = S;
    for j = 2:c
        S(j) = alpha3 * WW(j-1)+(1-alpha3) * S(j-1);
    end
    Total_Water_Consumption = S(c)*Population*365/1000;
else
    S(1) = W(c-floor(20/deltaT));
    for j = c+1-floor(20/deltaT):c
        S(j) = alpha1 * W(j-1)+(1-alpha1) * S(j-1);
    end
    S1 = S;
    WW = S;
    for j = c+1-floor(20/deltaT):c
        S(j) = alpha2 * WW(j-1)+(1-alpha2) * S(j-1);
    end
    S2  = S;
```

```
        WW = S;
        for j = c+1-floor(20/deltaT):c
            S(j) = alpha3 * WW(j-1)+(1-alpha3) * S(j-1);
        end
        Total_Water_Consumption = S(c)*Population*365/1000;
    end

    %Revenue
    Revenue = User_Fees * Total_Water_Consumption;

    error1 = abs(User_Fees - X1);
    error2 = abs(Funds_Balance - X2);
    %error3 = abs(Water_Demand - X3);
    itt = itt + 1;

end
iteration(c) = itt;

%*****************************************************************
******
U(c) = User_Fees;
W(c) = Water_Demand;
F(c) = Funds_Balance;
O(c) = OpEx;
R(c) = Revenue;
C20(c) = Cond_20/Total_Length_Pipes*100;
C40(c) = Cond_40/Total_Length_Pipes*100;
C60(c) = Cond_60/Total_Length_Pipes*100;
C80(c) = Cond_80/Total_Length_Pipes*100;
C100(c) = Cond_100/Total_Length_Pipes*100;
C(c) = CapEx;
E(c) = Expenditures;
```

```
    T(c) = Total_Water_Consumption;

    c = c + 1;
    time = time + deltaT;
end

runtime = toc

RESULTS = [C20; C40; C60; C80; C100; F; U; W; R]';

X = linspace(0,100,c-1);
figure(1);      plot(X,C20,'r','LineWidth',2);      hold      on;
plot(X,C40,'g','LineWidth',2); hold on;
plot(X,C60,'b','LineWidth',2);              hold              on;
plot(X,C80,'y','LineWidth',2);              hold              on;
plot(X,C100,'k','LineWidth',2); grid on
legend('percent     20','percent     40','percent     60','percent
80','percent 100');
figure(2); plot(X,O,'g','LineWidth',2); grid on; title('OpEx');
figure(3);  plot(X,F,'g','LineWidth',2);  grid  on;  title('Funds
Balance');
figure(4);  plot(X,U,'g','LineWidth',2);  grid  on;  title('User
Fees','LineWidth',2);
figure(5); plot(X,R,'g','LineWidth',2); grid on; title('Revenue');
figure(6);  plot(X,W,'g','LineWidth',2);  grid  on;  title('Water
Demand');
%*****************************************************************
**********
```