The Model Theory of Algebraically Closed Fields

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Model theory can express properties of algebraic subsets of complex n-space. The constructible subsets are precisely the first order definable subsets, and varieties correspond to maximal consistent collections of formulas, called types. Moreover, the topological dimension of a constructible set is equal to the Morley rank of the formula which defines it.

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Contents

Chapter 1

Introduction

Algebraic geometry is the mathematics developed to use algebraic results in the study of geometric objects. In particular, it establishes a one-to-one correspondence between radical ideals in a polynomial ring and the closed sets of the Zariski topology on complex n-space. The irreducible closed sets in this topology, which correspond to the prime ideals of the ring, will be of particular interest, as they form a set of building blocks for all the constructible subsets of n-space.

The mathematics of model theory also has a place in this relationship. If we create a mathematical setting in which we can model algebraically closed fields, many of the objects and notions from algebraic geometry develop model theoretic analogues. First, a one-to-one correspondence between the first order definable subsets and the constructible sets can be established. Indeed, the subsets of complex n-space that are first order definable are precisely the constructible sets! Perhaps even more striking is the bijection between the irreducible closed sets and maximal consistent collections of formulas, which we call types. Also, the topological dimension of a constructible set is preserved in a model theoretic notion called Morley rank.

Apart from chapter 2, this thesis is model theoretic mathematics. However, its focus is not on results in model theory, but rather on algebraically closed fields, and in particular on some of the things that can be said about algebraically closed fields with model theory.

This thesis arose out of reading an article by David Marker entitled 'Introduction to the Model Theory of Fields'. This article forms the first chapter of the book *Model theory of Fields* [9], and is thus referenced. From his paper several questions arose. The primary question was why his definition of Morley rank, given in the context of algebraically closed fields, was equivalent to the standard definition? My contribution in this thesis is to give a more thorough and correct exposition of these known results, motivated by the results of Marker's paper.

Chapter 2 introduces the necessary fundamentals of algebraic geometry. In particular, the varieties, algebraic sets and constructible sets in F^n are defined. Furthermore, several different notions of the dimension of such sets are introduced and equated.

In Chapter 3 the model theoretic side of this thesis begins. The theory ACF of algebraically closed fields is defined, and many of the important model theoretic properties of ACF are proven. The most important of these properties is quantifier elimination, which states that every formula is equivalent to a formula with no quantifiers. Two important implications of the quantifier elimination of ACF are discussed. The first is the first main goal of this paper, that the subsets of F^n that are first order definable are precisely the constructible sets of algebraic geometry. Secondly, it allows a model theoretic proof of Hilbert's Nullstellensatz.

Chapter 4 is about Morley rank. For the results in chapter 5, an alternative and more workable definition of Morley rank in the setting of algebraically closed fields is required. This characterization of Morley rank is the purpose of this chapter, and several technical results about algebraically closed fields that are needed for this characterization are proven.

Chapter 5 is devoted to proving the second main goal of this paper, to show that the Morley rank of a formula and the dimension of the constructible set it defines are equal. In proving this result, the model theoretic notion of types is introduced, and the bijection between varieties and types is established.

I do not assume any familiarity on the reader's part with the subject of algebraic geometry. However, I will assume some acquaintance with the fundamental definitions and motivation behind model theory. The reader should be familiar with what is a first order language $\mathcal{L}$. Terms, atomic formulas, quantifier free formulas, sentences and the general process of constructing formulas in a language $\mathcal{L}$ should be understood. I will freely use the words structure, substructure and extension of the language $\mathcal{L}$. The reader should know what is meant by a theory in the language $\mathcal{L}$, and when a structure $\mathfrak{A}$ is a model of a theory. Lastly, consistency, the Compactness Theorem of first order logic, and notions from logic about truth and consequence, such as when a formula ϕ is a consequence of a theory T, written $T \vDash \phi$, are requisite.

Chapter 2

Fundamentals of Algebraic Geometry

This chapter gives a detailed introduction to the field of algebraic geometry, providing the geometric grounding necessary for the rest of the thesis. The results in this chapter can be found, in somewhat less detail, in Hartshorne [5, section 1.1].

2.1 Varieties, Algebraic Sets, and Constructible Sets

Let F be an algebraically closed field. We define $n-space$ over F, denoted F^n, to be the set of n-tuples of elements of F. Let $F[X_1, ..., X_n]$ be the ring of polynomials in n variables over F. We will establish some useful connections between subsets of F^n and subsets of $F[X_1, ..., X_n]$. This gives us a powerful way to translate between geometric figures in F^n and algebraic objects in $F[X_1, ..., X_n]$.

Let f be a polynomial from the ring $F[X_1, ..., X_n]$. We can interpret f as a function from F^n to F, where $(a_1, ..., a_n) \mapsto f(a_1, ..., a_n)$. This function generates a well defined subset of F^n, namely $\{(a_1, ..., a_n) \in F^n : f(a_1, ..., a_n) = 0\}$.

We can extend this idea to any subset of $F[X_1, ..., X_n]$ in a natural way.

Definition 2.1.1 *Let Γ be a subset of $F[X_1, ..., X_n]$. The zero set of Γ, denoted $Z(\Gamma)$, is the set of points in F^n where all the polynomials in Γ vanish. Symbolically $Z(\Gamma) = \{(a_1, ..., a_n) \in F^n : g(a_1, ..., a_n) = 0$ for every $g \in \Gamma\}$.*

This gives us a well defined map $\Gamma \mapsto Z(\Gamma)$ which takes subsets of $F[X_1, ..., X_n]$ to certain subsets of F^n. The images in F^n under this map motivate the next definition.

Definition 2.1.2 *We say a subset $X \subseteq F^n$ is an algebraic set in F^n if $X = Z(\Gamma)$ for some $\Gamma \subseteq F[X_1, ..., X_n]$.*

Therefore the map $\Gamma \mapsto Z(\Gamma)$ takes subsets of $F[X_1, ..., X_n]$ to the algebraic sets in F^n. Next, we show that every algebraic set $Y = Z(\Gamma)$ is in fact the zero set of some finite subset $\{f_1, ..., f_r\} \subseteq F[X_1, ..., X_n]$.

Claim 2.1.3 *If J is the ideal of $F[X_1, ..., X_n]$ generated by $\Gamma \subseteq F[X_1, ..., X_n]$, then $Z(J) = Z(\Gamma)$.*

Proof. If $\overline{a} \in Z(J)$ then $g(\overline{a}) = 0$ for every $g \in J$. Since $\Gamma \subseteq J$, $g(\overline{a})$ for every $g \in \Gamma$, thus $\overline{a} \in Z(\Gamma)$. So $Z(J) \subseteq Z(\Gamma)$.

If $\overline{a} \in Z(\Gamma)$ then $g(\overline{a}) = 0$ for every $g \in \Gamma$. Take $h_1, h_2, k \in F[X_1, ..., X_n]$ such that both of h_1 and h_2 vanish at $\overline{a}$. Then $(h_1 + h_2)(\overline{a}) = h_1(\overline{a}) + h_2(\overline{a}) = 0$ and $(h_1 \cdot g)(\overline{a}) = (g \cdot h_1)(\overline{a}) = g(\overline{a}) \cdot h_1(\overline{a}) = 0$. Since J is the closure of Γ under polynomial addition and under multiplication by polynomials from $F[X_1, ..., X_n]$, we see that if $g \in J$, then $g(\overline{a}) = 0$. Thus $Z(\Gamma) \subseteq Z(J)$, and we have the desired equality $Z(J) = Z(\Gamma)$. ■

Thus any generating set of the ideal J in $F[X_1, ..., X_n]$ will have the same zero set in F^n. It remains to show that every ideal in $F[X_1, ..., X_n]$ is finitely generated.

Definition 2.1.4 *A ring R is <u>noetherian</u> if it satisfies the ascending chain condition for ideals of R: for any sequence $J_1 \subseteq J_2 \subseteq ...$ of ideals in R, there is an integer r such that $J_r = J_{r+1} = ...$*

Claim 2.1.5 *If R is a noetherian ring, then every ideal J in R is finitely generated.*

Proof. We prove this claim using the contrapositive method. Suppose the ring R contains the ideal J which is not finitely generated. Let $\{f_1, f_2, ...\}$ be a minimal generating set for J. Then $\langle f_1 \rangle \subsetneqq \langle f_1, f_2 \rangle \subsetneqq ...$ is a sequence of ideals in R. So R does not satisfy the ascending chain condition for ideals, and thus R is not noetherian. ■

Theorem 2.1.6 *Hilbert Basis Theorem: If a ring R is noetherian, then so is the polynomial ring $R[X]$.*

Proof. For a proof of the Hilbert Basis Theorem, see Artin [1, page 469]. ■

As F is a field, it has no non-trivial ideals, and is therefore a noetherian ring. Inductively, the Hilbert Basis Theorem shows that $F[X_1, ..., X_n]$ is also a noetherian ring, and therefore every ideal in $F[X_1, ..., X_n]$ is indeed finitely generated. So for every algebraic set $Y \subseteq F^n$, we can find a finite set of polynomials $\{f_1, ..., f_r\} \in F[X_1, ..., X_n]$ such that $Y = Z(\{f_1, ..., f_r\})$.

In fact, the algebraic sets in F^n induce a topology on the space F^n.

Definition 2.1.7 *The Zariski topology on F^n is defined by taking the closed subsets to be the algebraic sets in F^n.*

Claim 2.1.8 *The Zariski topology is indeed a topology on F^n.*

Proof. If Y_1 and Y_2 are algebraic sets, then $Y_1 = Z(\Gamma_1)$ and $Y_2 = Z(\Gamma_2)$, where $\Gamma_1, \Gamma_2 \subseteq F[X_1, ..., X_n]$. Let $\Gamma_1 \cdot \Gamma_2 = \{g \in F[X_1, ..., X_n] : g = f_1 \cdot f_2$, where $f_1 \in \Gamma_1$ and $f_2 \in \Gamma_2\}$, the set of all products of an element of Γ_1 by an element of Γ_2. If the point $\overline{a}$ is a root of the polynomial $f \in \Gamma_1$, then it is a root of any polynomial $f \cdot g$, where $g \in F[X_1, ..., X_n]$, since $(f \cdot g)(\overline{a}) = f(\overline{a}) \cdot g(\overline{a}) = 0$. Thus $Y_1 = Z(\Gamma_1) = \{\overline{a} \in F^n : f(\overline{a}) = 0$ for every $f \in \Gamma_1\} \subseteq \{\overline{a} \in F^n : (f \cdot g)(\overline{a}) = 0$ for every $f \cdot g \in \Gamma_1 \cdot \Gamma_2\} = Z(\Gamma_1 \cdot \Gamma_2)$. Likewise $Y_2 \subseteq Z(\Gamma_1 \cdot \Gamma_2)$, therefore $Y_1 \cup Y_2 \subseteq Z(\Gamma_1 \cdot \Gamma_2)$. Conversely, if $\overline{a} \in Z(\Gamma_1 \cdot \Gamma_2)$, and if $\overline{a} \notin Y_1$, then there is a polynomial f in Γ_1 such that $f(\overline{a}) \neq 0$. But since $\overline{a} \in Z(\Gamma_1 \cdot \Gamma_2)$, we know that $(f \cdot g)(\overline{a}) = 0$ for every $g \in \Gamma_2$. Now since $f(\overline{a}) \neq 0$, and since $f(\overline{a}) \cdot g(\overline{a}) = (f \cdot g)(\overline{a}) = 0$, it must be that $g(\overline{a}) = 0$ for every $g \in \Gamma_2$. So $\overline{a} \in Z(\Gamma_2) = Y_2$, and therefore $Z(\Gamma_1 \cdot \Gamma_2) \subseteq Y_1 \cup Y_2$. So in fact $Y_1 \cup Y_2 = Z(\Gamma_1 \cdot \Gamma_2)$, which tells us that the union of a pair of algebraic sets is itself an algebraic set.

If Y_i $(i \in I)$ is a family of algebraic sets, then $Y_i = Z(\Gamma_i)$, where $\Gamma_i \subseteq F[X_1, ..., X_n]$, for each i in the index set I. If $\overline{a} \in \bigcap_{i \in I} Y_i = \bigcap_{i \in I} \{\overline{b} \in F^n : f(\overline{b}) = 0$

for every $f \in \Gamma_i\}$, then $\overline{a} \in \{\overline{b} \in F^n : f(\overline{b}) = 0$ for every $f \in \Gamma_i\}$, for every $i \in I$. So $f(\overline{a}) = 0$ for every $f \in \bigcup_{i\in I} \Gamma_i$, hence $\overline{a} \in Z(\bigcup_{i\in I} \Gamma_i)$. Therefore $\bigcap_{i\in I} Y_i \subseteq Z(\bigcup_{i\in I} \Gamma_i)$. Conversely, if $\overline{a} \in Z(\bigcup_{i\in I} \Gamma_i)$, then for every $i \in I$, $f(\overline{a}) = 0$ for every $f \in \Gamma_i$, so $\overline{a} \in Z(\Gamma_i) = Y_i$. So $\overline{a} \in \bigcap_{i\in I} Y_i$, and hence $Z(\bigcup_{i\in I} \Gamma_i) \subseteq \bigcap_{i\in I} Y_i$. Therefore $\bigcap_{i\in I} Y_i = Z(\bigcup_{i\in I} \Gamma_i)$, and thus any intersection of algebraic sets is itself an algebraic set.

Furthermore, observe that the empty set $\emptyset = Z(\{1\})$, where $\{1\} \subseteq F[X_1, ..., X_n]$, and that the whole space $F^n = Z(\{0\})$, where $\{0\} \subseteq F[X_1, ..., X_n]$. So the empty set and the whole space are indeed algebraic sets. So the Zariski topology is indeed a topology on the space F^n. ■

For the rest of the paper we will use the terms algebraic set and Zariski closed set interchangably.

The map $\Gamma \mapsto Z(\Gamma)$ takes subsets of $F[X_1, ..., X_n]$ to Zariski closed sets in F^n. We have seen that any two generating sets of the same ideal in $F[X_1, ..., X_n]$ will map to the same Zariski closed set in F^n. Thus $\Gamma \mapsto Z(\Gamma)$ does not have a well defined inverse. However, we can establish a Galois connection between the subsets of F^n and the subsets of $F[X_1, ..., X_n]$, given by the map $\Gamma \mapsto Z(\Gamma)$ and the map $X \mapsto I(X)$ defined below.

Definition 2.1.9 *Let X be a subset of F^n. The ideal of X in $F[X_1, ..., X_n]$, denoted $I(X)$, is the set of polynomials in $F[X_1, ..., X_n]$ that vanish at every point in X. Symbolically $I(X) = \{f \in F[X_1, ..., X_n] : f(a_1, ..., a_n) = 0$ for all $(a_1, ..., a_n) \in X\}$.*

Claim 2.1.10 *If X is a subset of F^n, then $I(X)$ is indeed an ideal of $F[X_1, ..., X_n]$.*

Proof. Take $h_1, h_2 \in I(X)$, $g \in F[X_1, ..., X_n]$, and an arbitrary point $\overline{a} = (a_1, ..., a_n) \in X$. Then $(h_1 + h_2)(\overline{a}) = h_1(\overline{a}) + h_2(\overline{a}) = 0$, and $(h_1 \cdot g)(\overline{a}) = (g \cdot h_1)(\overline{a}) = g(\overline{a}) \cdot h_1(\overline{a}) = 0$. So both $h_1 + h_2$ and $h_1 \cdot g$ are in $I(X)$, showing that $I(X)$ is an ideal of $F[X_1, ..., X_n]$. ∎

We will look at some of the properties relating the two maps $\Gamma \mapsto Z(\Gamma)$ and $X \mapsto I(X)$. In particular we will look at when these two maps are inverses of one another. First, we need to introduce a few more things.

Definition 2.1.11 *If J is an ideal in a commutative ring R, the radical of J, denoted $\sqrt{J}$, is the set of all elements of R that, raised to some positive power, are in J. Symbolically, $\sqrt{J} = \{s \in R : s^r \in J \text{ for some } r > 0\}$. We say the ideal J is a radical ideal if $J = \sqrt{J}$.*

Claim 2.1.12 *If J is an ideal in a commutative ring R, then $\sqrt{J}$ is equal to the ideal $\bigcap P_i$, where the intersection is taken over all the prime ideals P which contain J.*

Proof. See Hungerford [8, page 379]. ∎

Theorem 2.1.13 *(Hilbert's Nullstellensatz). Let F be an algebraically closed field, J an ideal in $F[X_1, ..., X_n]$, and $f \in F[X_1, ..., X_n]$ a polynomial which vanishes at all points of $Z(J)$. Then $f \in \sqrt{J}$.*

An algebraic proof of Hilbert's Nullstellensatz is available in several sources: Atiyah-Macdonald [2, page 85] or Artin [1, page 375]. Instead, we will later in this paper develop the model theory required to present a model theoretic proof of the theorem.

We are now ready to summarize some of the properties of the maps $\Gamma \mapsto Z(\Gamma)$ and $X \mapsto I(X)$.

Theorem 2.1.14 *Let F be an algebraically closed field.*

(a) If $\Gamma_1 \subseteq \Gamma_2$ are subsets of $F[X_1, ..., X_n]$, then $Z(\Gamma_1) \supseteq Z(\Gamma_2)$.

(b) If $Y_1 \subseteq Y_2$ are subsets of F^n, then $I(Y_1) \supseteq I(Y_2)$.

(c) If Y_1, Y_2 are subsets of F^n, then $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

(d) For any ideal $J \subseteq F[X_1, ..., X_n]$, $I(Z(J)) = \sqrt{J}$.

(e) For any subset $Y \subseteq F^n$, $Z(I(Y)) = \overline{Y}$, the Zariski closure of Y.

Proof. (a). Let $\Gamma_1 \subseteq \Gamma_2$ be subsets of $F[X_1, ..., X_n]$. Take $\overline{a} \in Z(\Gamma_2)$, implying $g(\overline{a}) = 0$ for every polynomial $g \in \Gamma_2$. As $\Gamma_1 \subseteq \Gamma_2$, $g(\overline{a}) = 0$ for every polynomial $g \in \Gamma_1$, thus $\overline{a} \in Z(\Gamma_1)$. Therefore $Z(\Gamma_1) \supseteq Z(\Gamma_2)$.

(b). Let $Y_1 \subseteq Y_2$ be subsets of F^n. Take a polynomial $f \in I(Y_2)$, implying f vanishes at every point in Y_2. As $Y_1 \subseteq Y_2$, f vanishes at every point in Y_1, and thus $f \in I(Y_1)$. Therefore $I(Y_1) \supseteq I(Y_2)$.

(c). Let Y_1, Y_2 be two subsets of F^n. The polynomial $f \in I(Y_1 \cup Y_2) \Leftrightarrow f$ vanishes at every point of Y_1 and at every point of $Y_2 \Leftrightarrow f \in I(Y_1) \cap I(Y_2)$. Therefore $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

(d). Let J be an ideal in $F[X_1, ..., X_n]$. Hilbert's Nullstellensatz states that if $f \in I(Z(J))$, then $f \in \sqrt{J}$. So immediately we get that $I(Z(J)) \subseteq \sqrt{J}$. If $f \in \sqrt{J}$, then for some $r > 0$, $f^r \in J$. Take $\overline{a} \in Z(J)$, meaning that $g(\overline{a}) = 0$ for every $g \in J$, including f^r. Since f and f^r share the same roots, $f(\overline{a}) = 0$ as well. So f vanishes at every point in $Z(J)$, implying $f \in I(Z(J))$. Therefore

$\sqrt{J} \subseteq I(Z(J))$, and we get the desired equality $I(Z(J)) = \sqrt{J}$.

(e). Let Y be a subset of F^n. First we want to show that $Y \subseteq Z(I(Y))$. If we take $\overline{a} \in Y$, and $f \in I(Y)$, then f is a polyniomial that vanishes at every point in Y, in particular $f(\overline{a}) = 0$. Since f was chosen arbitrarily from $I(Y)$, we see that the point $\overline{a} \in Y$ is a root of every polynomial in $I(Y)$, thus $\overline{a} \in Z(I(Y))$. So indeed $Y \subseteq Z(I(Y))$. By definition, $Z(I(Y))$ is a closed set in the Zariski topology on F^n. Since the closure $\overline{Y}$ of Y is the smallest closed set containing Y, we get that $\overline{Y} \subseteq Z(I(Y))$.

On the other hand, let W be any closed set containing Y. Then by definition $W = Z(\Gamma)$, where Γ is some subset of $F[X_1, ..., X_n]$. So $Y \subseteq Z(\Gamma)$, and thus by (b) we get that $I(Y) \supseteq I(Z(\Gamma))$. Now we want to show that $\Gamma \subseteq I(Z(\Gamma))$. If we take $f \in \Gamma$, and $\overline{a} \in Z(\Gamma)$, then $\overline{a}$ is a root of every polynomial if Γ, in particular $f(\overline{a}) = 0$. Since $\overline{a}$ was chosen arbitrarily from $Z(\Gamma)$, we see that the polynomial f vanishes at every point in $Z(\Gamma)$, thus $f \in I(Z(\Gamma))$. So indeed $\Gamma \subseteq I(Z(\Gamma))$. So we now have $I(Y) \supseteq I(Z(\Gamma)) \supseteq \Gamma$, and by (a) we get that $Z(I(Y)) \subseteq Z(\Gamma) = W$. So $Z(I(Y))$ is contained in every closed set containing Y, in particular $\overline{Y}$. Thus $Z(I(Y)) = \overline{Y}$. ∎

This establishes the promised Galois connection between the subsets of F^n and the subsets of $F[X_1, ..., X_n]$, given by the maps $\Gamma \mapsto Z(\Gamma)$ and $X \mapsto I(X)$. These maps form a one-to-one inclusion reversing correspondence between the algebraic sets in F^n and the radical ideals in $F[X_1, ..., X_n]$.

For our purposes, the set of algebraic sets in F^n is not yet simple enough. Since the algebraic sets can be interpreted as the closed sets of a topology on F^n, we can always form algebraic sets out of finite unions of perhaps smaller algebraic sets. This motivates us to find a set of simpler algebraic sets in F^n which can be used as building blocks for all the algebraic sets. To do this, we need to introduce the notion of topological irreducibility, and the notion of a noetherian topological space.

Definition 2.1.15 *A non-empty closed subset Y of a topological space X is called irreducible if it cannot be expressed as the union of two proper subsets, each of which is closed in Y. By convention, the empty set is not considered to be irreducible.*

Definition 2.1.16 *A variety is an algebraic set in F^n which is irreducible in the Zariski topology.*

So a variety Y is an algebraic set which, in particular, cannot be built out of smaller algebraic sets. We will use the terms variety and irreducible Zariski closed set interchangably. It remains to show that every algebraic set in F^n can be formed out of a finite union of varieties.

Definition 2.1.17 *A topological space X is called noetherian if it satisfies the descending chain condition for closed subsets: for any sequence $Y_1 \supseteq Y_2 \supseteq \ldots$ of closed subsets, there is an integer r such that $Y_r = Y_{r+1} = \ldots$.*

Theorem 2.1.18 *In a noetherian topological space X, every nonempty closed subset Y can be expressed as a finite union $Y = Y_1 \cup \ldots \cup Y_r$ of irreducible closed subsets Y_i. If we require that $Y_i \subsetneqq Y_j$ for $i \neq j$, then the Y_i are uniquely determined. They are called the irreducible components of Y.*

Proof. Let Θ be the set of nonempty closed subsets of X which cannot be written as a finite union of irreducible closed subsets. Thus we want to show that Θ is the empty set. So suppose Θ is nonempty. If Θ contained no minimal element, then for every $Y \in \Theta$ we could find a $Y' \in \Theta$ such that $Y \underset{\neq}{\supseteq} Y'$, contradicting the descending chain condition of X. So Θ contains some minimal element Y. By the construction of Θ, Y is not irreducible, so we can write $Y = Y' \cup Y''$, where Y' and Y'' are proper closed subsets of Y. But since Y is a minimal element of Θ, both of Y' and Y'' are not in Θ, implying they can both be written as a finite union of irreducible closed subsets, hence $Y = Y' \cup Y''$ can also be written as a union of irreducible closed subsets. This contradicts Θ being nonempty.

So we can write any closed subset Y as a union $Y = Y_1 \cup ... \cup Y_r$ of irreducible sets. If $Y_i \subseteq Y_j$ for any $i \neq j$, then we can throw away the set Y_i, and write $Y = Y_1 \cup ... \cup Y_{i-1} \cup Y_{i+1} \cup ... \cup Y_r$. Thus it is safe to assume that $Y_i \nsubseteq Y_j$ for $i \neq j$. It remains to show that the Y_i are uniquely determined.

Suppose $Y = Y_1 \cup ... \cup Y_r$ and $Y = Y_1' \cup ... \cup Y_s'$ are two representations of Y. Then $Y_1' \subseteq Y = Y_1 \cup ... \cup Y_r$, so

$$Y_1' = Y_1' \cap Y = Y_1' \cap (Y_1 \cup ... \cup Y_r) = \bigcup_{i=1}^{r} (Y_1' \cap Y_i).$$

But since Y_1' is irreducible, it cannot be written as the finite union of closed proper subsets. As each $Y_1' \cap Y_i$ is indeed a closed subset of Y_1', we can conclude that $Y_1' = Y_1' \cap Y_i$ for some $1 \leq i \leq r$, and thus $Y_1' \subseteq Y_i$ for some i. Without loss of generality we can take $i = 1$, and we have $Y_1' \subseteq Y_1$. Similarly, we can show that $Y_1 \subseteq Y_j'$ for some $1 \leq j \leq s$. So $Y_1' \subseteq Y_1 \subseteq Y_j'$, and from our assumption that

$Y_i \nsubseteq Y_j$ for $i \neq j$, we get that $Y_1 = Y_1'$.

Let $Z = \overline{Y \setminus Y_1}$, the topological closure of the set $Y \setminus Y_1$. Observe that for $i \in \{1, ..., r\}$, $Y_i = \overline{Y_i} = \overline{(Y_i \setminus Y_1) \cup (Y_i \cap Y_1)} = \overline{Y_i \setminus Y_1} \cup \overline{Y_i \cap Y_1} = \overline{Y_i \setminus Y_1} \cup (Y_i \cap Y_1)$, a union of two closed subsets. Because Y_i is irreducible, either $Y_i = \overline{Y_i \setminus Y_1}$ or $Y_i = Y_i \cap Y_1$. However, if $Y_i = Y_i \cap Y_1$, then $Y_i \subseteq Y_1$, a contradiction. Thus $Y_i = \overline{Y_i \setminus Y_1}$. Therefore

$$\begin{aligned} Z = \overline{Y \setminus Y_1} &= \overline{(Y_1 \cup ... \cup Y_r) \setminus Y_1} \\ &= \overline{(Y_1 \setminus Y_1) \cup ... \cup (Y_r \setminus Y_1)} \\ &= \overline{(Y_2 \setminus Y_1) \cup ... \cup (Y_r \setminus Y_1)} \\ &= \overline{(Y_2 \setminus Y_1)} \cup ... \cup \overline{(Y_r \setminus Y_1)} \\ &= Y_2 \cup ... \cup Y_r. \end{aligned}$$

So $Z = Y_2 \cup ... \cup Y_r$ and also $Z = Y_2' \cup ... \cup Y_s'$. Proceeding by induction on r, we obtain the uniqueness of the irreducible components Y_i of Y. ■

Theorem 2.1.19 *F^n with the Zariski topology is a noetherian topological space.*

Proof. Let $Y_1 \supseteq Y_2 \supseteq ...$ be a descending chain of closed subsets in F^n. Then, from the properties of the maps $Y \mapsto I(Y)$ and $\Gamma \mapsto Z(\Gamma)$, $I(Y_1) \subseteq I(Y_2) \subseteq ...$ is an ascending chain of ideals in the ring $F[X_1, ..., X_n]$. Since the ring $F[X_1, ..., X_n]$ is a noetherian ring, this chain of ideals is eventually stationary. But for each i, $Y_i = Z(I(Y_i))$, so the descending chain $Y_1 \supseteq Y_2 \supseteq ...$ is also eventually stationary. Thus F^n satisfies the descending chain condition, and is a noetherian topological space. ■

Corollary 2.1.20 *Every algebraic set in F^n can be uniquely written as a finite union of varieties in F^n, no one contained in another.*

So the varieties are a good choice of building blocks for the algebraic sets. Since algebraic sets in F^n correspond to radical ideals in $F[X_1, ..., X_n]$, we can ask what kind of ideals do varieties correspond to?

Claim 2.1.21 *Y is a variety in F^n if and only if $I(Y)$ is a prime ideal of the ring $F[X_1, ..., X_n]$. Equivalently, J is a prime ideal if and only if $Z(J)$ is a variety.*

Proof. Suppose Y is a variety in F^n, hence it is an irreducible closed subset in the Zariski topology on F^n. We want to show that $I(Y)$, the set of all polynomials in $F[X_1, ..., X_n]$ which vanish at every point in Y, is a prime ideal. Consider the polynomial $fg \in I(Y)$. Since $\{fg\} \subseteq I(Y)$, we get $Z(\{fg\}) \supseteq Z(I(Y)) = Y$. Since the roots of the polynomial fg are the roots of f together with the roots of g, we have that $Y \subseteq Z(\{fg\}) = Z(\{f\}) \cup Z(\{g\})$. Thus $Y = Y \cap (Z(\{f\}) \cup Z(\{g\})) = (Y \cap Z(\{f\})) \cup (Y \cap Z(\{g\}))$. Since Y is irreducible, and since both of $Y \cap Z(\{f\})$, and $Y \cap Z(\{g\})$ are closed subsets of Y, it must be that $Y = Y \cap Z(\{f\})$ or $Y = Y \cap Z(\{g\})$. If $Y = Y \cap Z(\{f\})$, then $Y \subseteq Z(\{f\})$, implying $I(Y) \supseteq I(Z(\{f\})) \ni f$. Otherwise $Y = Y \cap Z(\{g\})$, and thus $g \in I(Y)$. So either $f \in I(Y)$ or $g \in I(Y)$, proving $I(Y)$ is a prime ideal.

Conversely, let J be a prime ideal. Now suppose $Z(J) = Y_1 \cup Y_2$, where Y_1 and Y_2 are closed subsets in F^n. Then $J = I(J) = I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$. Since J is prime, either $J = I(Y_1)$ or $J = I(Y_2)$. So either $Z(J) = Z(I(Y_1)) = Y_1$ or $Z(J) = Z(I(Y_2)) = Y_2$, hence $Z(J)$ is irreducible. ∎

So far we have defined the algebraic sets in F^n, and a nice subset of them, the varieties in F^n. We can also extend the set of algebraic sets to the set of constructible sets in F^n, defined below.

Definition 2.1.22 *Let Σ be a collection of subsets from a topological space X. We say the subset $S \subseteq X$ is a <u>boolean combination of Σ</u> if S is in the smallest set Φ such that (1) Φ contains Σ, and (2) Φ is closed under taking complements, unions and intersections. S is a <u>finite boolean combination</u> of Σ if we restrict ourselves to finite unions, finite intersections and complements.*

Indeed, we can define the set of boolean combinations of a subset of any boolean algebra.

Definition 2.1.23 *A subset of F^n is called <u>constructible</u> if it is a finite boolean combination of closed sets in the Zariski topology on F^n.*

Since any Zariski closed set can be written uniquely as a finite union of varieties, the constructible sets in F^n are equivalently the finite boolean combinations of varieties.

2.2 Dimension of Constructible sets

We now introduce the dimension of a set in F^n. We start by defining the dimension of algebraic sets, and then extend this to the constructible sets.

Definition 2.2.1 *If Y is a closed subset of a topological space X, then the dimension of Y, denoted $\dim(Y)$, is the supremum of all integers n such that there is a chain $Y_0 \subset Y_1 \subset ... \subset Y_n \subseteq Y$ of distinct irreducible closed subsets of X.*

This definition extends naturally to an algebraic set Y in F^n, where $\dim(Y)$ is the supremum of all integers n such that there is a chain $Y_0 \subset Y_1 \subset ... \subset Y_n \subseteq Y$ of distinct varieties of F^n. We can also define the dimension of an algebraic set Y in completely algebraic terms, by defining the dimension of Y to be the Krull dimension of the ring F[Y].

Definition 2.2.2 *If Y is an algebraic set in F^n, then the coordinate ring of Y, denoted $F[Y]$, is the quotient ring $F[X_1, ..., X_n]/I(Y)$.*

Intuitively we think of the ring $F[Y]$ as the ring of polynomial functions with domain Y, since two polynomial functions agree everywhere on Y if and only if they are equivalent in the ring $F[Y]$.

Definition 2.2.3 *The Krull dimension of a commutative ring R, denoted* $\dim(R)$, *is the supremum of all integers n such that there is a chain $J_0 \subset J_1 \subset ... \subset J_n \subseteq R$ of distinct prime ideals of R.*

Claim 2.2.4 *If Y is an algebraic set in F^n, then* $\dim(Y) = \dim(F[Y])$.

Proof. Let Y be an algebraic set in F^n, and suppose $\dim(Y) = d$. Then there is a chain of $Y_0 \subset Y_1 \subset ... \subset Y_d \subseteq Y$ of distinct varieties of F^n. Thus we have a chain $I(Y_0) \supset I(Y_1) \supset ... \supset I(Y_d) \supseteq I(Y)$ of prime ideals in $F[X_1, ..., X_n]$. Also since the maps $Y \mapsto I(Y)$ and $\Gamma \mapsto Z(\Gamma)$ are one-to-one for algebraic sets and radical ideals, the inclusions $I(Y_0) \supset I(Y_1) \supset ... \supset I(Y_d)$ are indeed proper inclusions. Moreover, $I(Y_0)/I(Y) \supset I(Y_1)/I(Y) \supset ... \supset I(Y_d)/I(Y)$ is proper chain of ideals in the ring $F[Y] = F[X_1, ..., X_n]/I(Y)$. So $\dim(F[Y]) \geq d$.

Suppose $\dim(F[Y]) = d$. Then there is a chain of distinct prime ideals of $F[Y]$ of length d. So we get a chain $I(Y) \subseteq J_0 \subset J_1 \subset ... \subset J_d$ of distinct prime ideals of $F[X_1, ..., X_n]$. Taking the zero sets of these ideals we have a chain $Y = Z(I(Y)) \supseteq Z(J_0) \supset Z(J_1) \supset ... \supset Z(J_d)$ of varieties on F^n. Again, these inclusions are proper inclusions. Thus $\dim(Y) \geq d$. So we get the desired equality $\dim(Y) = \dim(F[Y])$. ■

If the algebraic set Y is a variety, then the ideal $I(Y)$ is a prime ideal, and so the ring $F[Y] = F[X_1, ..., X_n]/I(Y)$ is an integral domain. Thus we can extend $F[Y]$ to its quotient field.

Definition 2.2.5 *If Y is a variety in F^n, the function field of Y, denoted $F(Y)$, is the quotient field of the integral domain $F[Y]$. Symbolically $F(Y) = \{f/g : f, g \in F[Y] = F[X_1, ..., X_n]/I(Y)$, and $g \neq 0\}$.*

We can now state an important classical result about the dimension of a variety.

Theorem 2.2.6 *If Y is a variety of F^n, then the dimension of Y is equal to the transcendence degree of $F(Y)$ over F.*

Proof. See Atiyah-MacDonald [2, Chapter 11]. ■

Next we extend the definition of dimension from varieties to constructible sets. Here we use the fact that a constructible set is a finite boolean combination of varieties.

Definition 2.2.7 *If Y is a constructible set in F^n, the dimension of Y is the maximal dimension of an irreducible component of $\overline{Y}$, the Zariski closure of Y.*

Recall that the irreducible components of $\overline{Y}$ are the varieties $Y_1, ..., Y_r$, where $\overline{Y}$ is written uniquely as the union $\overline{Y} = Y_1 \cup ... \cup Y_r$, with no Y_i contained in another. We need to check that this definition of dimension restricted to algebraic sets is equivalent to the original topological definition of dimension of algebraic sets given above.

Claim 2.2.8 *If Y is an algebraic set in F^n, then the dimension of Y is equal to the maximal dimension of an irreducible component of $\overline{Y}$, the Zariski closure of Y.*

Proof. Let Y be an algebraic set, and write $Y = Y_1 \cup ... \cup Y_r$, where $Y_1, ..., Y_r$ are the irreducible components of Y, no one contained in another. Since Y is a Zariski closed set, $Y = \overline{Y}$. So we need to show that

$$\dim(Y) = \max\{\dim(Y_1), ..., \dim(Y_r)\}.$$

Without loss of generality we can assume that

$$\dim(Y_1) = \max\{\dim(Y_1), ..., \dim(Y_r)\} = d.$$

This means we can find a proper chain of irreducible closed subsets of length d in Y_1. But because $Y \supseteq Y_1$, this chain is also contained in Y, and therefore

$$\dim(Y) \geq \dim(Y_1) = \max\{\dim(Y_1), ..., \dim(Y_r)\}.$$

For the other dirrection, suppose $\dim(Y) = d$. Then there exists a chain $W_0 \subset W_1 \subset ... \subset W_d \subseteq Y$ of irreducible closed subsets. Let $Z = \overline{Y \setminus W_d}$, the Zariski closure of $Y \setminus W_d$. Since Z is a closed set, we can write Z uniquely as the union of its irreducible components, $Z = Z_1 \cup ... \cup Z_s$. Observe that $Y = W_d \cup Z_1 \cup ... \cup Z_s$. We want to show that none of these irreducibles is contained in another.

- Clearly $Z_i \nsubseteq Z_j$ for $i \neq j$.
- If $W_d \subset Z_j$ for some j, then $W_0 \subset ... \subset W_d \subset Z_j \subseteq Z \subseteq Y$, contradicting $\dim(Y) = d$. Thus W_d is not properly contained in any Z_j.
- If $W_d \supseteq Z_j$ for some j, then

$$\begin{aligned}
Z_j &\subseteq Z = \overline{Y \setminus W_d} && \text{(definition)} \\
&= \overline{(\overline{Y \setminus W_d}) \setminus W_d} && \text{(porperties of sets)} \\
&\subseteq \overline{(\overline{Y \setminus W_d}) \setminus Z_j} && \text{(since } W_d \supseteq Z_j\text{)} \\
&= \overline{Z \setminus Z_j} && \text{(definition)} \\
&= Z_1 \cup ... \cup Z_{j-1} \cup Z_{j+1} \cup ... \cup Z_s. && \text{(from proof of theorem 2.1.17)}
\end{aligned}$$

  So Z_j is redundant in the representation $Z = Z_1 \cup ... \cup Z_s$, contradicting the uniqueness of this representation.

Therefore $Y = W_d \cup Z_1 \cup ... \cup Z_s$ and $Y = Y_1 \cup ... \cup Y_r$ two representations of Y as a union of irreducible closed sets, no one contained in another. As this representation is unique, we have $W_d = Y_i$, for some i. So the chain $W_0 \subset W_1 \subset ... \subset W_d = Y_i$ is contained in Y_i and thus $\dim(Y_i) \geq \dot{d}$. Thus

$$\max\{\dim(Y_1), ..., \dim(Y_r)\} \geq \dim(Y_i) \geq d = \dim(Y\dot{)}.$$

This proves the desired equality $\dim(Y) = \max\{\dim(Y_1), ..., \dim(Y_r)\}$. ■

Chapter 3

The Theory of Algebraically Closed Fields

3.1 The Theory ACF

In this section, we introduce the model theoretic setting of algebraically closed fields. Our language is the language of rings, denoted $\mathcal{L}_r$, consisting of the two binary function symbols $+$ and $\cdot$, the unary function symbol $-$, and the two nullary (or constant) symbols 0 and 1. In short we write $\mathcal{L}_r = \{+, \cdot, -, 0, 1\}$, and we call this the *signature* of the language $\mathcal{L}_r$.

The theory of integral domains consists of the sentences

$$\forall x \forall y \forall z (x + (y + z) = (x + y) + z),$$
$$\forall x (x + 0 = x),$$
$$\forall x (x + (-x) = 0),$$
$$\forall x \forall y (x + y = y + x),$$
$$\forall x \forall y \forall z (x \cdot (y \cdot z) = (x \cdot y) \cdot z),$$
$$\forall x \forall y \forall z (x \cdot (y + z) = (x \cdot y) + (x \cdot z)),$$
$$\forall x (x \cdot 1 = x),$$
$$\forall x \forall y (x \cdot y = y \cdot x),$$
$$\forall x \forall y (x \cdot y = 0 \rightarrow (x = 0 \vee y = 0)).$$

The theory of fields consists of the theory of integral domains together with the sentence

$$\forall x \exists y (x = 0 \vee xy = 1).$$

The theory algebraically closed fields, denoted ACF, consists of the theory of fields together with, for each $1 \leq n < \omega$, the sentence

$$\forall a_0 \forall a_1 ... \forall a_{n-1} \exists x \left(x^n + \sum_{i=0}^{n-1} a_i x^i = 0 \right).$$

Remark 3.1.1 *The models of ACF are precisely the algebraically closed fields. We can also characterize the substructures of a model of ACF. Since every integral* *domain D can be extended to its quotient field $F_D = \{a/b : a, b \in D, b \neq 0\}$, every integral domain is a subset of a field. Moreover, the axioms of integral domains are all $\forall_1$ formulas, and are thus preserved in substructures. Hence the integal domains are precisely the substructures of fields.*

The theory ACF says nothing about the characteristic of its models. For each $1 \leq n < \omega$, let ϕ_n be the sentence

$$\forall x(\underbrace{x + ... + x}_{\text{n times}} = 0).$$

Now for every prime p, let $ACF_p = ACF \cup \{\phi_p\}$, and let $ACF_0 = ACF \cup \{\neg\phi_n : 1 \leq n < \omega\}$. Thus when p is prime or zero, ACF_p denotes the theory of algebraically closed fields of characteristic p.

We will look at several properties of ACF_p. First we need to recall a classical theorem about algebraically closed fields.

Theorem 3.1.2 *Algebraically closed fields are descibed up to isomorphism by their characteristic and their transcendence degree.*

Proof. See Hungerford [8, page 317]. ■

Definition 3.1.3 *For a cardinal κ, a theory T is said to be κ-categorical if there is, up to isomorphism, a unique model of T of cardinality κ.*

For the remainder of the paper, the symbols $\mathfrak{A}$ and $\mathfrak{B}$ will be reserved to represent models, and the symbols A and B will be reserved to represent their respective domains. A reference of the following proposition can be found in Hodges [6, page 612].

Proposition 3.1.4 *Let p be a prime or zero and let κ be an uncountable cardinal. Then the theory ACF_p is κ-categorical.*

Proof. Let p be a prime or zero, let κ be an uncountable cardinal, and let $\mathfrak{A}$ and $\mathfrak{B}$ to be two models of ACF_p of cardinality κ. Thus $\mathfrak{A}$ and $\mathfrak{B}$ are algebraically closed fields of characteristic p and cardinality κ. The cardinality of an algebraically closed field of transcendence degree λ is equal to $\aleph_0 + \lambda$. Thus in an uncountable algebraically closed field, transcendence degree equals cardinality. So $\mathfrak{A}$ and $\mathfrak{B}$ both have transcendence degree κ. Thus from theorem 3.1.2, $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$. Hence there is, up to isomorphism, a unique model of ACF_p of cardinality κ, and therefore ACF_p is κ-categorical. ∎

Definition 3.1.5 *Two $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent, denoted $\mathfrak{A} \equiv \mathfrak{B}$, if for every sentence ϕ of $\mathcal{L}$, $\mathfrak{A} \models \phi$ if and only if $\mathfrak{B} \models \phi$.*

Definition 3.1.6 *We say a theory T is complete if any two models of T are elementarily equivalent. Equivalently, a theory T is complete if for every sentence ϕ, either $T \models \phi$ or $T \models \neg\phi$.*

Theorem 3.1.7 *(Upward Lowenheim-Skolem-Tarski Theorem) If an $\mathcal{L}$-theory T has infinite models, then it has infinite models of any given cardinality $\alpha \geq \|\mathcal{L}\|$, where $\|\mathcal{L}\|$ denotes the least infinite cardinal greater than or equal to the number of symbols in $\mathcal{L}$.*

Proof. See Hodges [6, page 267]. ■

Corollary 3.1.8 *(Łoś-Vaught Test) Suppose that an $\mathcal{L}$-theory T has only infinite models and T is α-categorical for some infinite cardinal $\alpha \geq \|\mathcal{L}\|$. Then T is complete.*

Proof. Let $\mathfrak{A}$ and $\mathfrak{B}$ be models of T. Since T has only infinite models, both are infinite. Define the $\mathcal{L}$-theory of $\mathfrak{A}$, denoted $\mathrm{Th}(\mathfrak{A})$, to be the set of all $\mathcal{L}$-sentences that are true in the model $\mathfrak{A}$. Thus by an elementary argument it can be shown that any model of $\mathrm{Th}(\mathfrak{A})$ is elementarily equivalent to $\mathfrak{A}$, showing that $\mathrm{Th}(\mathfrak{A})$ is a complete theory having $\mathfrak{A}$ as a model. By the Upward Lowenheim-Skolem-Tarski Theorem, there exists a model $\mathfrak{A}'$ of $\mathrm{Th}(\mathfrak{A})$ of cardinality α. Since $\mathrm{Th}(\mathfrak{A})$ is complete, $\mathfrak{A}' \equiv \mathfrak{A}$, and since $T \subseteq \mathrm{Th}(\mathfrak{A})$, $\mathfrak{A}'$ is a model of T. Likewise, there exists a model $\mathfrak{B}'$ of T of cardinality α such that $\mathfrak{B}' \equiv \mathfrak{B}$. As T is α-catagorical, $\mathfrak{A}' \cong \mathfrak{B}'$, and in particular $\mathfrak{A}' \equiv \mathfrak{B}'$. Therefore $\mathfrak{A} \equiv \mathfrak{B}$, implying T is a complete theory. ■

Proposition 3.1.9 *Let p be prime or zero. The theory ACF_p is complete.*

Proof. This is now a simple application of the Łoś-Vaught Test. Since the theory ACF_p is κ-categorical for any given uncountable cardinal κ, and since there are no finite algebraically closed fields, ACF_p is complete. ■

3.2 Definable Sets and Quantifier Elimination

In this section we introduce definable sets and prove the main goal of this chapter: that the definable subsets of F^n are precisely the constructible sets of algebraic geometry. We begin with a characterization of the sets defined by quantifier free formulas in $\mathcal{L}_r$ with parameters from an arbitrary integral domain D.

Definition 3.2.1 *Let $\mathfrak{A}$ be a model of the $\mathcal{L}$-theory T, and let X be a subset of A, the domain of $\mathfrak{A}$. We say the subset $Y \subseteq A^n$ is* <u>*first order definable with parameters from X*</u> *if and only if there is an $\mathcal{L}$-formula $\phi(x_1, ..., x_n, y_1, ..., y_m)$ and $a_1, ..., a_m \in X$ such that $Y = \{(b_1, ..., b_n) \in A^n : \mathfrak{A} \vDash \phi(b_1, ..., b_n, a_1, ..., a_m)\}$. We may refer to the subset of A^n defined by the formula ϕ as $\phi(A^n)$.*

Example 3.2.2 *Consider the locus of $x^2y + x\sqrt{2}i = \pi$ in $\mathbb{C}^2$. This set is defined by the formula $\phi(x_1, x_2, y_1, y_2) \equiv (x_1 \cdot x_1 \cdot x_2 + x_1 \cdot y_1 = y_2)$ with parameters $\sqrt{2}i, \pi \in \mathbb{C}$.*

Proposition 3.2.3 *Let $\mathfrak{A}$ be a model of ACF containing the integral domain D. If $\phi(x_1, ..., x_n)$ is a quantifier free formula in $\mathcal{L}_r$ with parameters from D, then there are polynomials $f_{i,j}, g_{i,j} \in D[x_1, ..., x_n]$ such that $\phi(x_1, ..., x_n)$ defines the same set as $\bigvee_{i=1}^{l} (\bigwedge_{j=1}^{m_i} f_{i,j}(x_1, ..., x_n) = 0 \wedge \bigwedge_{j=1}^{p_i} g_{i,j}(x_1, ..., x_n) \neq 0)$.*

Proof. The terms of the language $\mathcal{L}_r$ are built (finitely) from the variables $\{x_1, x_2, ...\}$, the constant symbols $\{0, 1\}$, and the function symbols $\{\cdot, +, -\}$. In any model of ACF, the symbols $\{\cdot, +, -, 0, 1\}$ satisfy the field axioms in ACF, and thus each term in $\mathcal{L}_r$ defines a polynomial in the variables $\{x_1, x_2, ...\}$ and with coefficients from $\mathbb{Z}$.

An atomic formula in $\mathcal{L}_r$ is of the form $t_1 = t_2$, where t_1, t_2 are terms from $\mathcal{L}_r$ with variables from say $\{x_1, ..., x_n\}$. Thus in any model $\mathfrak{A}$ of ACF, the atomic formulas $t_1 = t_2$ and $t_1 - t_2 = 0$ are logically equivalent, and thus will define the same set of points in A^n. So the set defined in a model $\mathfrak{A}$ of ACF by an atomic formula $\phi(x_1, ..., x_n)$ in $\mathcal{L}_r$ is precisely the set of roots in A^n of some polynomial in $\mathbb{Z}[X_1, ..., X_n]$.

If we want to talk about the set defined by an atomic formula in $\mathcal{L}_r$ with parameters from the integral domain D, we simply extend the set of constants to include $\{d_1, d_2, ...\}$ representing the elements of D. So if $\mathfrak{A}$ is a model of ACF containing D, an atomic formula in $\mathcal{L}_r$ with parameters from D and variables among $\{x_1, ..., x_n\}$ defines exactly the set of roots in A^n of some polynomial in $D[x_1, ..., x_n]$.

The quantifier free formulas in $\mathcal{L}_r$ are built (finitely) from the atomic formulas and the logical connectives $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$. Every such formula is logically equivalent to a formula in disjunctive normal form, hence the quantifier free formula $\phi(x_1, ..., x_n)$ with parameters from D is logically equivalent to $\bigvee_{i=1}^{l}(\bigwedge_{j=1}^{m_i} \psi_{i,j} \wedge \bigwedge_{j=1}^{p_i} \neg\gamma_{i,j})$, where $\psi_{i,j}, \gamma_{i,j}$ are atomic formulas with free variables among $\{x_1, ..., x_n\}$, and parameters from D. Thus there are polynomials $f_{i,j}, g_{i,j} \in D[x_1, ..., x_n]$ such that $\phi(x_1, ..., x_n)$ defines the same set as $\bigvee_{i=1}^{l}(\bigwedge_{j=1}^{m_i} f_{i,j}(x_1, ..., x_n) = 0 \wedge \bigwedge_{j=1}^{p_i} g_{i,j}(x_1, ..., x_n) \neq 0)$. ■

To characterize the sets defined by arbitrary formulas in $\mathcal{L}_r$, we must introduce the notion of quantifier elimination.

Definition 3.2.4 *We say that an $\mathcal{L}$-theory T has quantifier elimination if and only if for every $\mathcal{L}$-formula $\phi(\overline{x})$, there is a quantifier free $\mathcal{L}$-formula $\psi(\overline{x})$ such that $T \vDash \forall\overline{x}(\phi(\overline{x}) \leftrightarrow \psi(\overline{v}))$.*

In words, the theory T has quantifier elimination if and only if every formula is provably equivalent to a quantifier free formula. By provably equivalent here we mean equivalent in any model of T, and we say the two formulas are equivalent modulo the theory T.

We will prove that the theory ACF has quantifier elimination. This result was first proved by Tarski, although by different means. The proof presented below is my interpretation of the proof given by Marker in his paper [9, chapter 1], and it comes in three parts. The first is the following theorem [9, theorem 1.4] that states that a formula is equivalent to a quantifier free formula modulo a theory T if and only if the subset defined by the formula is invariant over all models of T.

Theorem 3.2.5 *Let $\mathcal{L}$ be a language containing at least one constant symbol, c. Let T be an $\mathcal{L}$-theory and let $\phi(\overline{x})$ be an $\mathcal{L}$-formula with free variables $x_1, ..., x_n$. The following are equivalent:*

i) *There is a quantifier free $\mathcal{L}$-formula $\psi(\overline{x})$ such that $T \vdash \forall\overline{x}(\phi(\overline{x}) \leftrightarrow \psi(\overline{x}))$.*

ii) *If $\mathfrak{A}$ and $\mathfrak{B}$ are models of T, and $\mathfrak{C}$ is a substructure of both $\mathfrak{A}$ and $\mathfrak{B}$, then $\mathfrak{A} \vDash \phi(\overline{a})$ if and only if $\mathfrak{B} \vDash \phi(\overline{a})$ for all $\overline{a} = (a_1, ..., a_n) \in C^n$.*

Proof. $(i \to ii)$ Suppose that $\psi(\overline{x})$ is a quantifier free $\mathcal{L}$-formula such that $T \vdash \forall\overline{x}(\phi(\overline{x}) \leftrightarrow \psi(\overline{x}))$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be models of T, and take $\mathfrak{C}$ to be a substructure of both $\mathfrak{A}$ and $\mathfrak{B}$. Consider $\overline{a} = (a_1, ..., a_n) \in C^n$. Since quantifier free formulas are preserved in substructures and extensions, and since ϕ and ψ are provably equivalent in any model of T, $\mathfrak{A} \vDash \phi(\overline{a}) \Leftrightarrow \mathfrak{A} \vDash \psi(\overline{a}) \Leftrightarrow \mathfrak{C} \vDash \psi(\overline{a}) \Leftrightarrow \mathfrak{B} \vDash \psi(\overline{a}) \Leftrightarrow \mathfrak{B} \vDash \phi(\overline{a})$. Thus $\mathfrak{A} \vDash \phi(\overline{a})$ if and only if $\mathfrak{B} \vDash \phi(\overline{a})$ for all $\overline{a} \in C$.

$(ii \to i)$ Let $\Gamma = \{\psi(\overline{x}) : \psi$ is quantifier free and $T \vdash \forall\overline{x}(\phi(\overline{x}) \to \psi(\overline{x}))\}$. Γ is not empty, as $T \vdash \forall\overline{x}(\phi(\overline{v}) \to (c = c))$. Let $d_1, ..., d_n$ be new constant symbols, and let us claim for now that $T \cup \{\psi(\overline{d}) : \psi(\overline{x}) \in \Gamma\} \vdash \phi(\overline{d})$. Thus, by the Compactness Theorem of first-order logic, there is a finite set of formulas $\psi_1, ..., \psi_m \in \Gamma$ such that $T \cup \{\psi_1(\overline{d}), ..., \psi_m(\overline{d})\} \vdash \phi(\overline{d})$. Therefore $T \vdash \bigwedge_{i=1}^{m} \psi_i(\overline{d}) \to \phi(\overline{d})$. Since $d_1, ..., d_n$ are new constant symbols, this is equivalent to $T \vdash \forall \overline{x}(\bigwedge_{i=1}^{m} \psi_i(\overline{x}) \to \phi(\overline{x}))$. Since the formulas ψ_i come from Γ, we know that for each $i = 1, ..., m$, ψ_i is quantifier free and $T \vdash \forall \overline{x}(\phi(\overline{x}) \to \psi_i(\overline{x}))$. So the formula $\bigwedge_{i=1}^{m} \psi_i(\overline{x})$ is also quantifier free, and $T \vdash \forall \overline{x}(\phi(\overline{x}) \to \bigwedge_{i=1}^{m} \psi_i(\overline{x}))$. Hence we get the desired result $T \vdash \forall \overline{x}(\bigwedge_{i=1}^{m} \psi_i(\overline{x}) \leftrightarrow \phi(\overline{x}))$.

It remains to prove the claim that $T \cup \{\psi(\overline{d}) : \psi(\overline{x}) \in \Gamma\} \vdash \phi(\overline{d})$. We proceed by contradiction. Suppose that $T \cup \{\psi(\overline{d}) : \psi(\overline{x}) \in \Gamma\} \nvdash \phi(\overline{d})$. Then we can find a model $\mathfrak{A}$ such that $\mathfrak{A} \models T \cup \{\psi(\overline{d}) : \psi(\overline{x}) \in \Gamma\} \cup \neg\phi(\overline{d})$. Let $\mathfrak{C}$ be the substructure of $\mathfrak{A}$ generated by the new constant symbols $\{d_1, ..., d_n\}$. This substructure is defined to be the intersection of all the substructures of $\mathfrak{A}$ whose domains contain $\{d_1, ..., d_n\}$. Let Diag($\mathfrak{C}$) be the Robinson diagram of $\mathfrak{C}$, the set of all atomic and negated atomic formulas with parameters from C that are true in $\mathfrak{C}$.

Let $\Sigma = T\cup$ Diag($\mathfrak{C}$) $\cup \{\phi(\overline{d})\}$. The formulas in Diag($\mathfrak{C}$) are quantifier free formulas that are true in $\mathfrak{C}$, and since quantifier free formulas are preserved in extensions, they are also true in $\mathfrak{A}$. As $\mathfrak{A}$ is a model of T, we see that $T\cup$ Diag($\mathfrak{C}$) is consistent. So if Σ is inconsistent, then $T\cup$ Diag($\mathfrak{C}$) $\models \neg\{\phi(\overline{d})\}$. Again by the Compactness Theorem, we would have a finite set of formulas $\psi_1(\overline{d}), ..., \psi_m(\overline{d}) \in$ Diag($\mathfrak{C}$) such that $T \cup \{\psi_1(\overline{d}), ..., \psi_m(\overline{d})\} \vdash \neg\phi(\overline{d})$. Hence $T \vdash \bigwedge_{i=1}^{m} \psi_i(\overline{d}) \to \neg\phi(\overline{d})$, and since $\{d_1, ..., d_n\}$ are new constant symbols, we get that $T \vdash \forall \overline{x}(\bigwedge_{i=1}^{m} \psi_i(\overline{x}) \to \neg\phi(\overline{x}))$. Applying the contrapositive to this formula gives us $T \vdash \forall \overline{x}(\phi(\overline{x}) \to$

$\bigvee_{i=1}^{m} \neg\psi_i(\overline{x}))$. So by definition, as the formulas ψ_i are quantifier free, $\bigvee_{i=1}^{m} \neg\psi_i(\overline{x}) \in \Gamma$. From our hypothesis, $A \vDash \{\psi(\overline{d}) : \psi(\overline{x}) \in \Gamma\}$. Moreover, as $\mathfrak{C}$ is a substructure of $\mathfrak{A}$ it preserves all quantifier free formulas, hence $\mathfrak{C} \vDash \{\psi(\overline{d}) : \psi(\overline{x}) \in \Gamma\}$. Thus $\mathfrak{C} \vDash \bigvee_{i=1}^{m} \psi_i(\overline{d})$. But as we have shown, $\bigvee_{i=1}^{m} \neg\psi_i(\overline{x}) \in \Gamma$, so $\mathfrak{C} \vDash \bigvee_{i=1}^{m} \neg\psi_i(\overline{d})$ also, which is a contradiction. So Σ must be consistent.

Let $\mathfrak{B} \vDash \Sigma$. Note that $\text{Diag}(\mathfrak{C}) \subseteq \Sigma$ so $\mathfrak{C} \subseteq \mathfrak{B}$. From the definition of Σ we know $\Sigma \vDash \phi(\overline{d})$, and hence $\mathfrak{B} \vDash \phi(\overline{d})$. However, from our hypothesis, since $\mathfrak{A}$ and $\mathfrak{B}$ are both models of T, and since $\mathfrak{C}$ is a substructure of both $\mathfrak{A}$ and $\mathfrak{B}$ containing $d_1, ..., d_n$, then $\mathfrak{A} \vDash \neg\phi(\overline{d})$ if and only if $\mathfrak{B} \vDash \neg\phi(\overline{d})$. This establishes the contradiction, and proves the claim that $T \cup \{\psi(\overline{d}) : \psi(\overline{x}) \in \Gamma\} \vdash \phi(\overline{d})$, finishing the proof of the theorem. ∎

The next lemma is part of a theorem from Hodges [6, Theorem 8.4.1], and it gives us a sufficient and more easily proven condition for a theory to have quantifier elimination.

Lemma 3.2.6 *Suppose that for every quantifier free $\mathcal{L}$-formula $\theta(\overline{x}, w)$, there is a quantifier free $\psi(\overline{x})$ such that $T \vdash \forall\overline{x}(\exists w\theta(\overline{x}, w) \leftrightarrow \psi(\overline{x}))$. Then in T every $\mathcal{L}$-formula is provably equivalent to a quantifier free $\mathcal{L}$-formula, and therefore T has quantifier elimination.*

Proof. We proceed by induction on the complexity of formulas, where complexity is defined as the number of logical connectives in a formula.

Suppose that for every quantifier free formula $\theta(\overline{x}, w)$, there is a quantifier free formula $\psi(\overline{x})$ such that $T \vdash \forall\overline{x}(\exists w\theta(\overline{x}, w) \leftrightarrow \psi(\overline{x}))$. Our base case is the atomic formulas of $\mathcal{L}$. If $\phi(\overline{x})$ is indeed an atomic formula, then it is quantifier free, and clearly equivalent to itself.

Now consider the formula $\phi(\overline{x})$, and suppose that any formula with complexity less than the complexity of $\phi(\overline{x})$ is provably equivalent to a quantifier free formula.

If $\phi(\overline{x})$ is of the form $\neg\theta_0(\overline{x})$, then by the induction hypothesis, $\theta_0(\overline{x})$ is provably equivalent to quantifier free formula $\psi_0(\overline{x})$. So $T \vdash \forall\overline{x}(\theta_0(\overline{x}) \leftrightarrow \psi_0(\overline{x}))$, implying $T \vdash \forall\overline{x}(\phi(\overline{x}) \leftrightarrow \neg\psi_0(\overline{x}))$, where $\neg\psi_0(\overline{x})$ is quantifier free. Therefore $\phi(\overline{x})$ is equivalent to a quantifier free formula.

If $\phi(\overline{x})$ is of the form $\theta_0(\overline{x}) \wedge \theta_1(\overline{x})$, then by the induction hypothesis, both of $\theta_0(\overline{x})$ and $\theta_1(\overline{x})$ are provably equivalent to quantifier free formulas $\psi_0(\overline{x})$ and $\psi_1(\overline{x})$ respectively. So $T \vdash \forall\overline{x}(\theta_0(\overline{x}) \leftrightarrow \psi_0(\overline{x}))$ and $T \vdash \forall\overline{x}(\theta_1(\overline{x}) \leftrightarrow \psi_1(\overline{x}))$, implying $T \vdash \forall\overline{x}(\phi(\overline{x}) \leftrightarrow \psi_0(\overline{x}) \wedge \psi_1(\overline{x}))$, where $\psi_0(\overline{x}) \wedge \psi_1(\overline{x})$ is quantifier free. Therefore $\phi(\overline{x})$ is equivalent to a quantifier free formula.

If $\phi(\overline{x})$ is of the form $\exists w\theta(\overline{x}, w)$, then by the induction hypothesis, $\theta(\overline{x}, w)$ is provably equivalent to a quantifier free formula $\psi_0(\overline{x}, w)$. So $T \vdash \forall\overline{x}(\theta(\overline{x}, w) \leftrightarrow \psi_0(\overline{x}, w))$, implying $T \vdash \forall\overline{x}(\phi(\overline{x}) \leftrightarrow \exists w\psi_0(\overline{x}, w))$. Since $\psi_0(\overline{x}, w)$ is quantifier free, we know from our hypothesis that there is a quantifier free formula $\psi(\overline{x})$ such that $T \vdash \forall\overline{x}(\exists w\psi_0(\overline{x}, w) \leftrightarrow \psi(\overline{x}))$. So we get $T \vdash \forall\overline{x}(\phi(\overline{x}) \leftrightarrow \psi(\overline{x}))$, and therefore $\phi(\overline{x})$ is equivalent to a quantifier free formula.

This shows that all formulas consisting only of logical symbols from $\{\neg, \wedge, \exists\}$ are equivalent to quantifier free formulas. However, since the logical symbols $\{\neg, \wedge, \exists\}$ form an adequate set of symbols for $\mathcal{L}$, every formula is logically equivalent to one consisting only of the symbols $\{\neg, \wedge, \exists\}$, and this proves the lemma. ∎

So to prove a theory has quantifier elimination, it suffices by theorem 3.2.5 and lemma 3.2.6 to show that if $\mathfrak{A}$ and $\mathfrak{B}$ are models of T, and $\mathfrak{C}$ is a substructure of both $\mathfrak{A}$ and $\mathfrak{B}$, then $\mathfrak{A} \models \phi(a_1, ..., a_n)$ if and only if $\mathfrak{B} \models \phi(a_1, ..., a_n)$ for all $a_1, ..., a_n \in C$, and for all formulas $\phi(\overline{x})$ of the form $\exists w \theta(\overline{x}, w)$, where $\theta(\overline{x}, w)$ is quantifier free.

Theorem 3.2.7 *The theory ACF has quantifier elimination.*

Proof. Let K and L be models of ACF and let D be a substructure of both K and L. Thus K and L are algebraically closed fields and from remark 3.1.1, D is an integral domain contained in both K and L. Let $\phi(\overline{x}, w)$ be a quantifier free formula, let $\overline{a} \in D$, and suppose that $K \models \exists w \phi(\overline{a}, w)$. This implies there is a $b \in K$ such that $K \models \phi(\overline{a}, b)$. From here, it suffices to show $L \models \exists w \phi(\overline{a}, w)$.

From our characterization of the sets defined by quantifier free formulas, we know that there are polynomials $f_{i,j}, g_{i,j} \in D[w]$ such that $\phi(\overline{a}, w)$ defines the same set as $\bigvee_{i=1}^{l} (\bigwedge_{j=1}^{m_i} f_{i,j}(w) = 0 \wedge \bigwedge_{j=1}^{p_i} g_{i,j}(w) \neq 0)$. Since $K \models \phi(\overline{a}, b)$, the element $b \in K$ must satisfy $\bigvee_{i=1}^{l} (\bigwedge_{j=1}^{m_i} f_{i,j}(w) = 0 \wedge \bigwedge_{j=1}^{p_i} g_{i,j}(w) \neq 0)$, implying that for some fixed i, $\bigwedge_{j=1}^{m_i} f_{i,j}(b) = 0 \wedge \bigwedge_{j=1}^{p_i} g_{i,j}(b) \neq 0$. Therefore, for this i, $f_{i,j}(b) = 0$ for every $1 \leq j \leq m_i$ and $g_{i,j}(b) \neq 0$ for every $1 \leq j \leq p_i$.

Let F be the quotient field of the integral domain D, and let $\overline{F}$ be the algebraic closure of F. As $\overline{F}$ is the smallest algebraically closed field containing D, there are subfields of both K and L isomorphic to $\overline{F}$. If any $f_{i,j}$ is not the zero polynomial, we see that b is a root of that particular $f_{i,j}$, thus $b \in \overline{F}$. So there is an isomorphic copy b^* of b in L, and therefore $L \models \phi(\overline{a}, b^*)$, hence we get $L \models \exists w \phi(\overline{a}, w)$.

So we can assume that for each $1 \leq j \leq m_i$, $f_{i,j}$ is simply the zero polynomial. Now, since for each $1 \leq j \leq p_i$, $g_{i,j}(b) \neq 0$ implies $g_{i,j}(w)$ is not the zero polynomial, and thus has finitely many roots. Let $\{c_1, ..., c_s\}$ be the collection of all the roots in L of the polynomials $g_{i,j}$ for $1 \leq j \leq p_i$. If we pick any $d \in L$, with $d \notin \{c_1, ..., c_s\}$, we get $L \vDash \phi(\overline{a}, d)$, and thus $L \vDash \exists w \phi(\overline{a}, w)$. This completes the proof that ACF has quantifier elimination. ∎

Corollary 3.2.8 *Let $\mathfrak{A}$ be a model of ACF containing the integral domain D. If $\phi(x_1, ..., x_n)$ is an arbitrary formula in $\mathcal{L}_r$ with parameters from D, then there are polynomials $f_{i,j}, g_{i,j} \in D[x_1, ..., x_n]$ such that $\phi(x_1, ..., x_n)$ defines the same set as*

$$\bigvee_{i=1}^{l}(\bigwedge_{j=1}^{m_i} f_{i,j}(x_1, ..., x_n) = 0 \wedge \bigwedge_{j=1}^{p_i} g_{i,j}(x_1, ..., x_n) \neq 0).$$

Moreover, if we take our $\mathcal{L}_r$-formulas to have parameters from the algebraically closed field F, then the definable subsets of F^n are precisely the constructible sets from algebraic geometry.

Proof. The first part of the corollary is immediate from the characterization of sets defined by quantifier free formulas in proposition 3.2.3, and the quantifier elimination of ACF.

For a fixed $i \in \{1, ..., l\}$ the formula $\bigwedge_{j=1}^{m_i} f_{i,j}(x_1, ..., x_n) = 0$ defines the zero set of the polynomials $f_{i,1}, ..., f_{i,m_i}$, which is an algebraic set in F^n. Thus

$$\bigvee_{i=1}^{l}(\bigwedge_{j=1}^{m_i} f_{i,j}(x_1, ..., x_n) = 0 \wedge \bigwedge_{j=1}^{p_i} g_{i,j}(x_1, ..., x_n) \neq 0)$$

expresses a finite boolean combination of algebraic sets in F^n, which is exactly a constructible set. ■

Remark 3.2.9 *The theory of fields does not have quantifier elimination. To see this, consider the two models* $\mathbb{R}$ *and* $\mathbb{Q}$ *of the theory of fields with the commmon substructure* $\mathbb{Q}$*, and consider the formula* $\phi(y) \equiv \exists x(x \cdot x = y)$*. Clearly* $\mathbb{R} \models \phi(2)$ *but* $\mathbb{Q} \not\models \phi(2)$*. Thus from theorem 3.2.5, the formula* ϕ *is not equivalent to any quantifier free formula modulo the theory of fields.*

3.3 A Model Theoretic Proof of the Nullstellensatz

Now that we have established the quantifier elimination of ACF, we can present a model theoretic proof of the weak form of Hilbert's Nullstellensatz.

Theorem 3.3.1 *(Hilbert's Nullstellensatz - weak form). Let* F *be an algebraically closed field, and let* P *be a prime ideal contained in* $F[X_1, ..., X_n]$*. Then* $Z(P) \neq \emptyset$*.*

In our discussion on algebraic sets we used the strong form of Hilbert's Nullstellensatz, theorem 2.1.13. So before we present the model theoretic proof of 3.3.1, we will present the traditional proof that the strong Nullstellensatz and the weak Nullstellensatz are indeed equivalent.

Proof. First we prove that theorem 3.3.1 implies theorem 2.1.13. Let J be an ideal in $F[X_1, ..., X_n]$. From 2.1.12 we have that $\sqrt{J}$ is an intersection of prime ideals, $\sqrt{J} = \bigcap P_i$. If $\sqrt{J} \neq F[X_1, ..., X_n]$, then there is at least one prime ideal in the intersection, taken without loss of generality to be P_1. Then we can conclude that $Z(J) \neq \emptyset$, because

$$
\begin{aligned}
\emptyset &\neq Z(P_1) && \text{(From hypothesis, as } P_1 \text{ is prime)},\\
&\subseteq Z(\textstyle\bigcap P_i) && \text{(From 2.1.14(a), since } P_1 \supseteq \textstyle\bigcap P_i),\\
&= Z(\sqrt{J}) && \text{(since } \sqrt{J} = \textstyle\bigcap P_i),\\
&\subseteq Z(J) && \text{(From 2.1.14(a), since } J \subseteq \sqrt{J}).
\end{aligned}
$$

Let $f \in F[X_1, ..., X_n]$ be a polynomial that vanishes at all points in $Z(J)$, in other words $f \in I(Z(J))$. If $\sqrt{J} = F[X_1, ..., X_n]$, then clearly $f \in \sqrt{J}$, and we are done. So we can assume that $\sqrt{J} \neq F[X_1, ..., X_n]$.

Take $f_1, ..., f_m$ to be generating polynomials of the ideal J. Consider the $m+1$ new polynomials in the variables $X_1, ..., X_n, Y$ consisting of $f_1(\overline{X}), ..., f_m(\overline{X})$ and the polynomial $f(\overline{X}) \cdot Y - 1$. If the point $(a_1, ..., a_n, b) \in F^{n+1}$ is a root of each f_i, then $(a_1, ..., a_n) \in Z(f_1, ..., f_m) = Z(J)$, and thus $f(a_1, ..., a_n) = 0$ also. So the polynomial $f(\overline{X}) \cdot Y - 1$ evaluated at $(a_1, ..., a_n, b)$ is equal to -1. Thus the $m+1$ polynomials $f_1, ..., f_m, f \cdot Y - 1$ have no common roots in F^{n+1}. If we let K be the ideal generated by the polynomials $f_1, ..., f_m, f \cdot Y - 1$ in $K[X_1, ..., X_n, Y]$, we then get that $Z(K) = \emptyset$, implying $K = F[X_1, ..., X_n, Y]$, from the discussion in the first paragraph. Hence $1 \in K$, and since the polynomials $f_1, ..., f_m, f \cdot Y - 1$ generate K, we can find polynomials $p_1, ..., p_m, p \in K[X_1, ..., X_n, Y]$ such that

$$1 = \sum_{i=1}^{m} p_i(\overline{X}, Y) f_i(\overline{X}) + p(\overline{X}, Y)(f(\overline{X}) \cdot Y - 1).$$

If we substitute $Y = 1/f$ into this expression, evaluating the result in $F(X_1, \dots, X_n)$, we obtain

$$1 = \sum_{i=1}^{m} p_i(\overline{X}, 1/f) f_i(\overline{X}).$$

Set N to be equal to the maximum degree of the variable Y, ranging over all occurrences of Y in all the polynomials $p_i(\overline{X}, Y)$. Then by multiplying both sides of the equation by $f(\overline{X})^N$, we obtain

$$f(\overline{X})^N = \sum_{i=1}^{m} f(\overline{X})^N p_i(\overline{X}, 1/f) f_i(\overline{X}),$$

where each $f(\overline{X})^N p_i(\overline{X}, 1/f)$ is indeed a polynomial in $F[X_1, ..., X_n]$. Hence the polynomials $f_1, ..., f_m$ generate some power of f, and thus $f \in \sqrt{J}$.

Second, we prove that theorem 2.1.13 implies theorem 3.3.1. Let F be an algebraically closed field, and P a prime ideal properly contained in $F[X_1, ..., X_n]$. Suppose that $Z(P) = \emptyset$. Then every $f \in F[X_1, ..., X_n]$ vanishes at all points of $Z(P)$, and our hypothesis gives us $F[X_1, ..., X_n] = \sqrt{P}$. We note that prime ideals are radical ideals, since if $f^r = f \cdot f^{r-1} \in P$ for some $1 \leq r < \omega$, then either f or f^{r-1} is in P. Continuing inductively, we must get $f \in P$. Therefore $P = \sqrt{P} = F[X_1, ..., X_n]$, contradicting $P \subsetneq F[X_1, ..., X_n]$. Hence $Z(P) \neq \emptyset$. ■

To prove the weak form of the Nullstellensatz 3.3.1, we will use the model completeness of ACF, a direct consequence of the quantifier elimination of ACF.

Definition 3.3.2 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be models of the $\mathcal{L}$-theory T. $\mathfrak{B}$ is said to be an <u>elementary extension</u> of $\mathfrak{A}$, denoted $\mathfrak{A} \preccurlyeq \mathfrak{B}$, iff $\mathfrak{A} \subseteq \mathfrak{B}$ and for every $\mathcal{L}$-formula*

$\phi(x_1,...,x_n)$ and tuple $(a_1,...,a_n) \in A^n$, $\mathfrak{A} \models \phi(a_1,...,a_n) \Leftrightarrow \mathfrak{B} \models \phi(a_1,...,a_n)$. In this case we also say that $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$.

Definition 3.3.3 *A theory T is model complete if all substructures and extensions of models of T are elementary. Symbolically, if $\mathfrak{A} \models T$ and $\mathfrak{B} \models T$ such that $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \preccurlyeq \mathfrak{B}$.*

Claim 3.3.4 *If a theory T has quantifier elimination then T is also model complete.*

Proof. Let $\mathfrak{A} \subseteq \mathfrak{B}$ be models of a theory T which has quantifier elimination, let $\phi(x_1,...,x_n)$ be a formula, and let $\overline{a} = (a_1,...,a_n) \in A^n$. Since T has quantifier elimination, ϕ is provably equivalent to a quantifier free formula ψ, hence $T \models \forall\overline{x}(\phi(\overline{x}) \leftrightarrow \psi(\overline{x}))$. In particular, $T \models \phi(\overline{a}) \leftrightarrow \psi(\overline{a})$. Since quantifier free formulas are preserved in substructure and extension, we have that $\mathfrak{A} \models \psi(\overline{a})$ if and only if $\mathfrak{B} \models \psi(\overline{a})$. Therefore, $\mathfrak{A} \models \phi(\overline{a})$ if and only if $\mathfrak{B} \models \phi(\overline{a})$, implying $\mathfrak{A} \preccurlyeq \mathfrak{B}$. ■

Corollary 3.3.5 *The theory ACF is model complete.*

Note that $\mathfrak{A} \preccurlyeq \mathfrak{B}$ implies $\mathfrak{A} \equiv \mathfrak{B}$, but $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A} \equiv \mathfrak{B}$ is not enough to imply $\mathfrak{A} \preccurlyeq \mathfrak{B}$. Indeed, neither completeness nor model completeness implies the other. The theory ACF is an example of a model complete theory which is not complete. For examples of theories that are complete and yet not model complete, see Chang and Keisler [3, page 110].

Now, the proof of the weak Nullstellensatz 3.3.1.

Proof. Let P be a prime ideal properly contained in $F[X_1,...,X_n]$. Since P is prime, the ring $F[X_1,...,X_n]/P$ is an integral domain. Moreover, since $P \neq$

$F[X_1,...,X_n]$, the ring $F[X_1,...,X_n]/P$ is nontrivial. If any constant polynomial c was in P, then $c \cdot (1/c) = 1 \in P$, contradicting $P \neq F[X_1,...,X_n]$. Thus an isomorphic copy of F is contained in the ring $F[X_1,...,X_n]/P$.

Let K be the algebraic closure of the quotient field of $F[X_1,...,X_n]/P$. As $F \subseteq F[X_1,...,X_n]/P \subseteq K$, and since both F and K are algebraically closed fields, we have by the model completeness of ACF that F is an elementary substructure of K, $F \preccurlyeq K$.

Let $f_1,...,f_m$ be a generating set of polynomials of the ideal P. From our characterization of $\mathcal{L}_r$-formulas in Corollary 3.2.8, we can find a formula $\phi(y_1,...,y_k)$ and parameters $b_1,...,b_k \in F$ such that $\phi(b_1,...,b_k)$ asserts

$$\exists X_1...\exists X_n \bigwedge_{i=1}^{m} f_i(X_1,...,X_n) = 0.$$

If we evaluate the polynomial f_i at the point $(X_1/P,...,X_n/P) \in K^n$, we get $f_i(X_1/P,...,X_n/P) = f_i(X_1,...,X_n)/P$, since taking the quotient of a ring by an ideal is a ring homomorphism, preserving the ring operations $\cdot, +, -$. We also have that $f_i(X_1,...,X_n)/P = 0/P$, since the polynomial f_i is a generator of the ideal P. Hence, for each $1 \leq i \leq m$, the point $(X_1/P,...,X_n/P) \in K^n$ satisfies $f_i(\overline{X}) = 0$. So

$$K \vDash \exists X_1...\exists X_n \bigwedge_{i=1}^{m} f_i(X_1,...,X_n) = 0,$$

satisfied by the point $(X_1/P, ..., X_n/P)$, and thus $K \vDash \phi(b_1, ..., b_k)$. Since F is an elementary substructure of K, by definition we must also get that $F \vDash \phi(b_1, ..., b_k)$, and thus

$$F \vDash \exists X_1 ... \exists X_n \bigwedge_{i=1}^{m} f_i(X_1, ..., X_n) = 0.$$

Hence there is some point in F^n vanishing at each polynomial f_i, and therefore $Z(f_1, ..., f_m) = Z(P) \neq \emptyset$. ■

Chapter 4

Morley Rank

In his paper, Marker does not give a standard definition of Morley rank. Instead, he gives an alternative definition that holds only in the context of strongly minimal theories, which are defined at the end of this chapter. As we will see, the theory ACF is strongly minimal, and thus Marker's definition of Morley rank is enough for the purposes of his paper. However, it still needs to be established that his definition of Morley rank is indeed a characterization of Morley rank in the context of strongly minimal theories. It is the goal of this chapter to prove this characterization. However, in keeping with the focus of this thesis, it will be proved only for algebraically closed fields, and not for strongly minimal theories in general. The proofs presented in this chapter are based on work done by Ross Willard.

4.1 A Key Theorem on Dimension

This section is devoted to proving a theorem on dimension that is key to the characterization of Morley rank in algebraically closed fields. It begins with two technical lemmas necessary for the proof of this theorem.

Definition 4.1.1 *Let F be an algebraically closed field, and K an algebraically closed extension of F. If $(b_1, ..., b_n) \in K^n$, we define the* <u>*dimension of* $(b_1, ..., b_n)$</u> *over F, denoted $\dim_F(b_1, ..., b_n)$, to be the transcendence degree of $F(b_1, ..., b_n)$ over F.*

Definition 4.1.2 *Let F be an algebraically closed field, and K an algebraically closed extension of F. An* <u>$(l, m)_F$*-tuple*</u> *in K is a tuple $(b_1, ..., b_l, c_1, ..., c_m) \in K^{l+m}$ where $\dim_F(b_1, ..., b_l) = l$ and $c_1, ..., c_m$ are algebraic over the field $F(b_1, ..., b_l)$.*

Definition 4.1.3 *Let F be an algebraically closed field, and let $f_1, ..., f_m$ be a sequence of polynomials such that $f_i \in F[x_1, ..., x_l, y_1, ..., y_i]$. Let σ_0 denote the homomorphism which takes the ring $F[x_1, ..., x_l]$ canonically into the field $F_0 = F(x_1, ..., x_l)$. If $\sigma_0(f_1) = f_1^*$ is irreducible in $F_0[y_1]$, then let σ_1 be the composition of the induced homomorphism $\sigma_0 : F[\overline{x}, y_1] \to F_0[y_1]$ with the homomorphism which canonically takes $F_0[y_1]$ onto the field $F_1 = F_0[y_1]/\langle f_1^* \rangle$. Continuing inductively, assume we have a ring homomorphism $\sigma_i : F[\overline{x}, y_1, ..., y_i] \to F_i$. If $\sigma_i(f_{i+1}) = f_{i+1}^*$ is irreducible in the ring $F_i[y_{i+1}]$, let σ_{i+1} be the composition of the induced homomorphism $\sigma_i : F[\overline{x}, y_1, ..., y_{i+1}] \to F_i[y_{i+1}]$ with the homomorphism which canonically takes $F_i[y_{i+1}]$ onto the field $F_{i+1} = F_i[y_{i+1}]/\left\langle f_{i+1}^* \right\rangle$. If we can continue this process all the way to F_m, then we will call $f_1, ..., f_m$ an* <u>$(l, m)_F$*-polynomial sequence.*</u>

If $(b_1, ..., b_l, c_1, ..., c_m)$ is an $(l, m)_F$-tuple and $f_1, ..., f_m$ is an $(l, m)_F$-polynomial sequence, then we will say the $(l, m)_F$-tuple satisfies the $(l, m)_F$-polynomial sequence if $f_i(b_1, ..., b_l, c_1, ..., c_i) = 0$ for every $i = 1, ..., m$.

Lemma 4.1.4 *Suppose F, K are algebraically closed fields with $F \leq K$. Then every $(l, m)_F$-tuple satisfies some $(l, m)_F$-polynomial sequence. Moreover, if $(\overline{b}, \overline{c})$ and $(\overline{b}', \overline{c}')$ are $(l, m)_F$-tuples in K which satisfy a common $(l, m)_F$-polynomial sequence, then there exists an automorphism τ of K fixing F such that $\tau(b_i) = b'_i$ for $i = 1, ..., l$ and $\tau(c_j) = c'_j$ for $j = 1, ..., m$.*

Proof. Suppose F, K are algebraically closed fields with $F \leq K$, and let $(b_1, ..., b_l, c_1, ..., c_m)$ be an $(l, m)_F$-tuple in K. Since $b_1, ..., b_l$ are algebraically independent over F, there is an isomorphism $\mu_0 : F(x_1, ..., x_l) \cong F(b_1, ..., b_l)$, where $\mu_0(x_j) = b_j$ for all $j = 1, ..., l$, and $\mu_0|_F = id_F$.

For $i = 1, ..., m$ we can construct a $(l, m)_F$-polynomial sequence as follows.

Let $p_i(y_i) \in F(\overline{b}, c_1, ..., c_{i-1})[y_i]$ be the minimal polynomial of c_i in the ring $F(\overline{b}, c_1, ..., c_{i-1})[y_i]$. This polynomial exists, as c_i is algebraic over $F(\overline{b}, c_1, ..., c_{i-1})$. The coefficients $\alpha_0, ..., \alpha_t$ of p_i are elements of the form

$$\alpha_j = \frac{g_j(b_1, ..., b_l, c_1, ..., c_{i-1})}{h_j(b_1, ..., b_l, c_1, ..., c_{i-1})},$$

where $g_j, h_j \in F[\overline{x}, y_1, ..., y_{i-1}]$ and $h_j(\overline{b}, c_1, \ldots, c_{i-1}) \neq 0$ for each $j = 1, ..., t$. The polynomial $q_i = p_i h_1(\overline{b}, \overline{c}) \cdots h_t(\overline{b}, \overline{c})$ will still be irreducible in $F(\overline{b}, c_1, ..., c_{i-1})[y_i]$, will still have c_i as a root, and will have coefficients of the form $g'_j(\overline{b}, c_1, ..., c_{i-1})$, where $g'_j \in F[\overline{x}, y_1, ..., y_{i-1}]$.

If we replace all the occurrences of the values $b_1, ..., b_l, c_1, ..., c_{i-1}$ in the polynomial q_i with the variables $x_1, ..., x_l, y_1, ..., y_{i-1}$ — i.e., if we let $f_i = g'_t \cdot y_i^t + \cdots + g'_1 \cdot y_i + g'_0$ — we get a polynomial $f_i \in F[\overline{x}, y_1, ..., y_i]$ that vanishes at $(\overline{b}, c_1, ..., c_i)$. Moreover, since $c_1, ..., c_{i-1}$ are roots of $f_1^*, ..., f_{i-1}^*$, the homomorphism $\sigma_{i-1} : F[\overline{x}, y_1, ..., y_{i-1}] \to F_{i-1}$ takes f_i to $\sigma_{i-1}(f_i) = f_i^*$ which corresponds to q_i under the isomorphism $\mu_{i-1} : F_{i-1} \cong F(\overline{b}, c_1, ..., c_{i-1})$. Thus f_i^* is indeed irreducible in $F_{i-1}[y_i]$. Thus we can define the field F_i and the homomorphism σ_i as in the definition of a $(l, m)_F$-polynomial sequence, where $F_i = (F_{i-1}[y_i])/\langle f_i^* \rangle$ and the homomorphism $\sigma_i : F[\overline{x}, y_1, ..., y_i] \to F_i$ is the induced homomorphism $\sigma_{i-1} : F[\overline{x}, y_1, ..., y_i] \to F_{i-1}[y_i]$ composed with the homomorphism which canonically takes $F_{i-1}[y_i]$ onto the field $F_i = F_{i-1}[y_i]/ \langle f_i^* \rangle$.

If p is the minimal polynomial of c in the ring $F[y]$, then $F(c) \cong F[y]/\langle p \rangle$. Thus, since p_i and $q_i = \mu_{i-1}(f_{i-1}^*)$ generate the same ideal in $F(\overline{b}, c_1, ..., c_{i-1})[y_i]$, we get the isomorphism $\mu_i : F_i = (F_{i-1}[y_i])/\langle f_i^* \rangle \cong F(\overline{b}, c_1, ..., c_i)$ extending the isomorphism μ_{i-1}.

For the proof of the second part of the lemma, suppose $(\overline{b}, \overline{c})$ and $(\overline{b}', \overline{c}')$ are $(l, m)_F$-tuples which satisfy a common $(l, m)_F$-polynomial sequence. Since both $b_1, ..., b_l$ and $b'_1, ..., b'_l$ are algebraically independent over F, there are isomorphisms $\mu_0 : F_0 = F(x_1, ..., x_l) \cong F(b_1, ..., b_l)$ and $\mu'_0 : F_0 = F(x_1, ..., x_l) \cong F(b'_1, ..., b'_l)$, where $\mu_0(x_j) = b_j$ and $\mu'_0(x_j) = b'_j$ for each $j = 1, ..., l$. Thus we can construct the isomorphism $\tau_0 = \mu'_0 \circ \mu_0^{-1} : F(\overline{b}) \cong F(\overline{b}')$, where $\tau_0(b_j) = b'_j$ for each $j = 1, ..., l$.

Now consider the first polynomial $f_1 \in F[x_1, ..., x_l, y_1]$ in the $(l, m)_F$-polynomial sequence. Under the homomorphism $\sigma_0 : F[x_1, ..., x_l] \to F(x_1, ..., x_l) = F_0$, we get the polynomial $\sigma_0(f_1) = f_1^* \in F(x_1, ..., x_l)[y_1]$, which, since $f_1, ..., f_m$ is an $(l, m)_F$-polynomial sequence, is irreducible in $F(x_1, ..., x_l)[y_1]$. Thus $\mu_0(f_1^*)$ is an irreducible polynomial in $F(b_1, ..., b_l)[y_1]$, and since $f_1(b_1, ..., b_l, c_1) = 0$ it has c_1 as a root. Therefore, when we extend the domain of τ_0 to the field extension $F(\overline{b}, c_1) \geq F(\overline{b})$, τ_0 must map c_1 to a conjugate of c_1' in $F(\overline{b}', c_1')$, since $\tau_0(\mu_0(f_1^*)(c_1)) = \tau_0(\mu_0(f_1^*))(\tau_0(c_1)) = \mu_0'(f_1^*)(\tau_0(c_1)) = 0$, and since $\mu_0'(f_1^*)$ is just a constant multiple of the minimal polynomial of c_1' in $F(\overline{b})[y_1]$. So we can set $\tau_1 : F(\overline{b}, c_1) \cong F(\overline{b}', c_1')$, where $\tau_1|_{F(\overline{b})} = \tau_0$ and $\tau_1(c_1) = c_1'$.

Moreover, mimicking an argument above, $F(\overline{b}, c_1) \cong F_1 = (F(\overline{x})[y_1])/\langle f_1^* \rangle$. So if we denote this isomorphism by $\mu_1 : F_1 \cong F(\overline{b}, c_1)$, we can define the isomorphism $\mu_1' = \tau_1 \circ \mu_1 : F_1 \cong F(\overline{b}', c_1')$.

Continuing inductively, suppose $\tau_i = \mu_i' \circ \mu_i^{-1} : F(\overline{b}, c_1, ..., c_i) \cong F(\overline{b}', c_1', ..., c_i')$, where $\tau_i(b_j) = b_j'$ for each $j = 1, ..., l$, and $\tau_i(c_j) = c_j'$ for each $j = 1, ..., i$. Consider the $i + 1$ polynomial $f_{i+1} \in F[\overline{x}, y_1, ..., y_{i+1}]$ in the $(l, m)_F$-polynomial sequence. Under the homomorphism $\sigma_i : F[\overline{x}, y_1, ..., y_i] \to F_i$, we get the irreducible polynomial $\sigma_i(f_{i+1}^*) = f_{i+1}^* \in F_i[y_{i+1}]$, and thus $\mu_i(f_{i+1}^*)$ is irreducible in $F(\overline{b}, c_1, ..., c_i)[y_{i+1}]$. Again we find that if we extend the domain of τ_i to the field extension $F(\overline{b}, c_1, ..., c_{i+1})$, τ_i must map c_{i+1} to a conjagate of c_{i+1}' in $F(\overline{b}, c_1', ..., c_{i+1}')$. So we can pick $\tau_{i+1} : F(\overline{b}, c_1, ..., c_{i+1}) \cong F(\overline{b}, c_1', ..., c_{i+1}')$, where $\tau_{i+1}(\langle \overline{b}, c_1, ..., c_{i+1} \rangle) = \langle \overline{b}', c_1', ..., c_{i+1}' \rangle$.

Since $\mu_i : F_i \cong F(\overline{b}, c_1, ..., c_i)$, we have that $\mu_i(f_{i+1}^*)$ is irreducible in the ring $F(\overline{b}, c_1, ..., c_i)[y_{i+1}]$. Also, since c_{i+1} is a root of $\mu_i(f_{i+1}^*)$, we get the isomorphism $\mu_{i+1} : F_{i+1} = (F_i[y_{i+1}])/\langle f_{i+1}^* \rangle \cong F(b_1, ..., b_l, c_1, ..., c_{i+1})$.

Thus we can construct the desired isomorphism

$$\tau_m : F(\overline{b}, c_1, ..., c_m) \cong F(\overline{b}', c_1', ..., c_m'), \quad \text{s.t.}$$
$$\tau_m(\langle \overline{b}, c_1, ..., c_m \rangle) = \langle \overline{b}', c_1', ..., c_m' \rangle.$$

Furthermore, any isomorphism between subfields of an algebraically closed field K can be extended to an automorphism of K. This completes the proof. ■

Lemma 4.1.5 *Suppose F is an algebraically closed field, and let $\theta(\overline{x}, y)$ be an $\mathcal{L}_r$-formula in the free variables $x_1, ..., x_l, y$ and with parameters from F. Suppose also that there does not exist an algebraically closed extension $K \geq F$ and $(b_1, ..., b_l, c) \in K^{l+1}$ such that $K \vDash \theta(b_1, ..., b_l, c)$ and $\dim_F(b_1, ..., b_l, c) = l+1$. Then there exists a finite set S of nonzero polynomials in $F[x_1, ..., x_l, y]$ such that in any algebraically closed extension K, if $K \vDash \theta(b_1, ..., b_l, c)$ and $\dim_F(b_1, ..., b_l) = l$, then $g(b_1, ..., b_l, c) = 0$ for some $g \in S$.*

Example 4.1.6 *Consider the formula*

$$\phi(x, y, p) \equiv [(x \cdot y = 1) \wedge \neg(x = 0)] \vee [x \cdot x = y \cdot p],$$

with the parameter $p = i$ in our model $\mathbb{C}$. Clearly there is no algebraically closed extension K of $\mathbb{C}$ containing a tuple (b, c) such that $K \vDash \phi(b, c, i)$ and $\{b, c\}$ are algebraically independent over $\mathbb{C}$. Here, $S = \{xy - 1, x^2 - iy\}$ is a finite set of polynomials in $\mathbb{C}[x, y]$ satisfying the conditions in the lemma. In general, it is much more difficult to find the set S explicitly, as ϕ may be a finite disjunction of only negated atomic formulas with parameters from F.

Proof. First, let us extend the language $\mathcal{L}_r$ to include the constant symbols $\{c_a : a \in F\}$ representing the elements of F, and the constant symbols $b_1, ..., b_l, c$. We can denote this new language by $\mathcal{L}_F$. Let $\mathrm{Th}(F)$ be the collection of all $\mathcal{L}_F$-sentences φ such that $F \vDash \varphi$. Since $\mathcal{L}_F$ contains symbols for all the elements of F, any model of $ACF \cup \mathrm{Th}(F)$ is an algebraically closed extension of F.

For every nonzero polynomial $g \in F[x_1, ..., x_l, y]$, $\neg g(b_1, ..., b_l, c) = 0$ is a sentence in $\mathcal{L}_F$. Let

$$G = \{\varphi \in \mathcal{L}_F : \varphi = (\neg g(b_1, ..., b_l, c) = 0),\ \text{where } g \in F[x_1, ..., x_l, y]\}.$$

We want to consider the theory

$$T' = ACF \cup \mathrm{Th}(F) \cup \theta(b_1, ..., b_l, c) \cup G.$$

A model of this theory is an algebraically closed extension K of F, such that $K \vDash \theta(b_1, ..., b_l, c)$ and $\dim_F(b_1, ..., b_l, c) = l + 1$. We can see that G does indeed encode $\dim_F(b_1, ..., b_l, c) = l + 1$, since $\dim_F(b_1, ..., b_l, c) = l + 1$ is equivalent to the algebraic independence of the elements $b_1, ..., b_l, c$, which is also equivalent to the nonexistence of a nonzero polynomial $g \in F[x_1, ..., x_l, y]$ such that $g(b_1, ..., b_l, c) = 0$.

By our hypothesis, we can assume that the theory T' is unsatisfiable. By the Compactness Theorem of first order logic, there is a finite subset $S' \subseteq T'$ that is unsatisfiable. Let $S = S' \cap G$. So in particular, $ACF \cup \mathrm{Th}(F) \cup \theta(b_1, ..., b_l, c) \cup S$ is unsatisfiable. So if K is an algebraically closed extension of F, and if $(b_1, ..., b_l, c) \in K^{l+1}$ where $K \vDash \theta(b_1, ..., b_l, c)$, then $g(b_1, ..., b_l, c) = 0$ for some $g \in S$, which proves the lemma. ■

Now for the statement and proof of the key theorem.

Theorem 4.1.7 *Suppose F is an algebraically closed field, and let $\psi(\overline{x})$ be an $\mathcal{L}_r$-formula with free variables $\overline{x} = x_1, ..., x_n$ and with parameters from F. Suppose also that there does not exist an algebraically closed extension $K \geq F$ with $(b_1, ..., b_n) \in K^n$ such that $K \vDash \psi(b_1, ..., b_n)$ and $\dim_F(b_1, ..., b_n) \geq l + 1$, where $l \leq n$. Then in any algebraically closed extension $K \geq F$, if*

$$X = \{\overline{b} \in K^n : K \vDash \psi(\overline{b}) \text{ and } \dim_F(\overline{b}) = l\}$$

then the automorphisms of K fixing F partition X into finitely many orbits.

Proof. Set K to be an algebraically closed extension of F, and define X as in the statement of the theorem. For each $S \subseteq \{1, ..., n\}$ with $|S| = l$, let

$$X_S = \{\overline{b} \in X : \{b_i : i \in S\} \text{ is algebraically independent over } F\}.$$

Consider $(b_1, ..., b_n) \in X_S$, and suppose $\tau \in \mathrm{Aut}_F(K)$. We want to show that $\tau(b_1, ..., b_n) \in X_S$ also.

Let $\{s_1, ..., s_k\}$ be a k-element subset of $\{b_1, ..., b_n\}$, where $k \leq n$. If $\{s_1, ..., s_k\}$ is algebraically dependent over F, then there exists a nonzero polynomial $g \in F[x_1, ..., x_k]$ such that $g(s_1, ..., s_k) = 0$. Since τ fixes F and preserves the field operations, we have $g(\tau(b_1), ..., \tau(b_k)) = \tau(g(b_1, ..., b_k)) = \tau(0) = 0$. Thus the subset $(\tau(s_1), ..., \tau(s_k))$ of $(\tau(b_1), ..., \tau(b_n))$ is algebraically dependent over F as well. The same argument can be applied to τ^{-1}, showing that τ preserves the subsets of $(b_1, ..., b_n)$ which are algebraically independent when τ acts on this n-tuple coordinatewise.

Thus the set X_S is τ-invariant. This is true for each $\tau \in \mathrm{Aut}_F(K)$, so X_S is closed under the action of $\mathrm{Aut}_F(K)$. Since X is the union of the X_S sets, and since there are finitely many choices for S, it suffices to show that each X_S is partitioned into only finitely many orbits by the action of $\mathrm{Aut}_F(K)$. We shall show this for $S = \{1, ..., l\}$.

Observe that the elements of X_S are now precisely the $(l, m)_F$-tuples in K which satisfy ψ, where $m = n - l$. For ease of argument, rename the variables $x_{l+1}, ..., x_n$ as $y_1, ..., y_m$.

For each $i = 1, ..., m$, define the formula

$$\theta_i(x_1, .., x_l, y_i) = \exists y_1 ... \exists y_{i-1} \exists y_{i+1} ... \exists y_m \psi(x_1, ..., x_l, y_1, ..., y_m).$$

From lemma 4.1.5, we know that for each θ_i we get a finite set $S_i \subseteq F[\overline{x}, y_i] \setminus \{0\}$ such that if $(\overline{b}, \overline{c}) \in X_S$, then for each i we can find a $g_i \in S_i$ such that $g_i(\overline{b}, c_i) = 0$. We shall use these sets to choose, for each $(\overline{b}, \overline{c}) \in X_S$, an $(l, m)_F$-polynomial sequence $f_1, ..., f_m$ satisfied by $(\overline{b}, \overline{c})$.

Fix our $(l, m)_F$-tuple $(\overline{b}, \overline{c})$ from X_S. We have $g_1 \in S_1 \subseteq F[\overline{x}, y_1]$, where $g_1(\overline{b}, c_1) = 0$. We can choose $f_1 \in F[\overline{x}, y_1]$ such that $\sigma_0(f_1) = f_1^*$ is an irreducible factor of $\sigma_0(g_1) = g_1^*$ in $F_0[y_1]$ and $f(\overline{b}, c_1) = 0$. Next, we have $g_2(\overline{x}, y_2) \in S_2 \subseteq F[\overline{x}, y_2]$, where $g(\overline{b}, c_2) = 0$. Write

$$g_2(\overline{x}, y_2) = \frac{1}{q(\overline{x})} \left[h_1(\overline{x}, y_1, y_2) ... h_k(\overline{x}, y_1, y_2) + r(\overline{x}, y_1, y_2) \right],$$

where 1) $q(\overline{x}) \neq 0$, 2) $r \in \langle f_1 \rangle$, the ideal generated by f_1 in $F[\overline{x}, y_1, y_2]$, and 3) $h_1^*, ..., h_k^*$ are, up to constants, the irreducible factors of g_2^* in $F_1[y_2]$. Evaluating this equation at $(\overline{x}, y_1, y_2) = (\overline{b}, c_1, c_2)$ will give $h_i(\overline{b}, c_1, c_2) = 0$ for some $i \in \{1, ..., k\}$. Letting $f_2 = h_i$, we get the desired polynomial.

This process can be repeated inductively all the way through to g_m. Write

$$g_i(\overline{x}, y_i) = \frac{1}{q(\overline{x})} \left[h_1(\overline{x}, y_1, ..., y_i) ... h_k(\overline{x}, y_1, ..., y_i) + r(\overline{x}, y_1, ..., y_i) \right],$$

where 1) $q(\overline{x}) \neq 0$, 2) $r \in \langle f_1, ..., f_{i-1} \rangle$ in $F[\overline{x}, y_1, y_2]$, and 3) $h_1^*, ..., h_k^*$ are, up to constants, the irreducible factors of g_i^* in $F_{i-1}[y_i]$. We can continue in this way to construct a $(l, m)_F$-polynomial sequence from factors of polynomials in $S_1 \cup ... \cup S_m$ satisfied by $(\overline{b}, \overline{c})$.

Since each S_i is finite, this process yields a finite number of $(l, m)_F$-polynomial sequences. Every element in X_S, which is a $(l, m)_F$-tuple, satisfies at least one. From lemma 4.1.4, if two $(l, m)_F$-tuples satisfy the same $(l, m)_F$-polynomial sequence, then they are in the same orbit under the action of $\mathrm{Aut}_F(K)$. So X_S is partitioned into finitely many orbits under the action of $\mathrm{Aut}_F(K)$, and this completes the proof. ■

4.2 Characterization of Morley Rank in Algebraically Closed Fields

Recall that $\phi(A^n)$ denotes the subset of A^n defined by $\phi(x_1, ..., x_n)$.

Definition 4.2.1 *Let $\mathfrak{A}$ be an $\mathcal{L}$-structure, and let $\psi(\overline{x})$ be an $\mathcal{L}$-formula in the free variables $\overline{x} = x_1, ..., x_n$ and with parameters from A. The Cantor-Bendixon Rank of ψ (with respect to $\mathfrak{A}$), denoted $RCB_{\mathfrak{A}}(\psi)$, is either -1 or an ordinal or ∞, and is defined inductively as follows. First, define a relation on formulas ψ and ordinals α, denoted $RCB_{\mathfrak{A}}(\psi) \geq \alpha$, by*

$RCB_{\mathfrak{A}}(\psi) \geq 0$	*iff*	*$\psi(A^n)$ is not empty.*
$RCB_{\mathfrak{A}}(\psi) \geq \alpha + 1$	*iff*	*there are $\mathcal{L}$-formulas $\psi_i(\overline{x})$ $(i < \omega)$ with parameters from A, such that the sets $\psi_i(A^n)$ $(i < \omega)$ are pairwise disjoint and contained in $\psi(A^n)$, and $RCB_{\mathfrak{A}}(\psi_i) \geq \alpha$ for each $(i < \omega)$.*
$RCB_{\mathfrak{A}}(\psi) \geq \delta$ *(limit)*	*iff*	*for all $\alpha < \delta$, $RCB_{\mathfrak{A}}(\psi) \geq \alpha$.*

If $\psi(A^n)$ is empty, define $RCB_{\mathfrak{A}}(\psi) = -1$. If α is the greatest ordinal such that $RCB_{\mathfrak{A}} \geq \alpha$, then define $RCB_{\mathfrak{A}}(\psi) = \alpha$. If $RCB_{\mathfrak{A}}(\psi) \geq \alpha$ for every ordinal α, then set $RCB_{\mathfrak{A}}(\psi) = \infty$.

Definition 4.2.2 *Let $\mathfrak{A}$ be an $\mathcal{L}$-structure, and let $\psi(\overline{x})$ be an $\mathcal{L}$-formula in the free variables $\overline{x} = x_1, ..., x_n$ and with parameters from A. The Morley Rank of ψ (with respect to $\mathfrak{A}$), denoted $RM_{\mathfrak{A}}(\psi)$, is the supremum of all the values $RCB_{\mathfrak{B}}(\psi)$ as $\mathfrak{B}$ ranges over all elementary extensions of $\mathfrak{A}$.*

As an example of a formula with with Morley rank d, we will later see that if a variety V in $\mathbb{C}^n$ has topological dimension d, then $RM_{\mathbb{C}}(\phi) = d$, where ϕ defines the variety V.

Two considerably different definitions of Morley rank can be found in Hodges [6] and Chang and Keisler [4]. We will use the definition from Hodges presented above, because it is in my opinion more tractable and because Chang and Keisler only define Morley rank for formulas with one free varible.

Morley rank is a property of formulas, given with respect to some structure. However, the Morley rank of a formula ϕ is defined totally in terms of the subset $\phi(A^n)$ it defines, and hence if two different formulas ϕ and ψ define the same subset $\phi(A^n) = \psi(A^n)$, then $RM_{\mathfrak{A}}(\phi) = RM_{\mathfrak{A}}(\psi)$. Thus if X is a first order definable subset of A^n, we may refer to the Morley rank of the set X with respect to the structure $\mathfrak{A}$, denoted $RM_{\mathfrak{A}}(X)$. Here $RM_{\mathfrak{A}}(X) = RM_{\mathfrak{A}}(\psi)$, where ψ is any formula, with or without parameters from A, that defines X. Armed with the key theorem 4.1.7, we now have the tools necessary to prove the characterization of Morley rank in algebraically closed fields.

Theorem 4.2.3 *Let $\mathfrak{A}$ be a model of ACF, and let $\psi(\overline{x})$ be an $\mathcal{L}_r$-formula in the free variables $\overline{x} = x_1, ..., x_n$ and with parameters from A. Then* $RM_{\mathfrak{A}}(\psi) = \max\{\dim_{\mathfrak{A}}(b_1, ..., b_n)$: *for all* $(b_1, ..., b_n) \in B^n$ *such that* $\mathfrak{B} \models \psi(b_1, ..., b_n)$, *where* $\mathfrak{B} \succcurlyeq \mathfrak{A}\}$. *In particular,* $RM_{\mathfrak{A}}(\psi)$ *will be finite.*

For readability, the proof is presented in two lemmas.

Lemma 4.2.4 *Suppose $k \geq 0$, F is an algebraically closed field, K is an algebraically closed extension of F, and $(b_1, ..., b_n) \in K^n$. Also suppose that $\varphi(x_1, ..., x_n)$ is* *an $\mathcal{L}_r$-formula with parameters from F, $K \vDash \varphi(b_1, ..., b_n)$, and $\dim_F(b_1, ..., b_n) = k$. Then $RM_F(\varphi) \geq k$.*

Proof. The proof is by induction on k.

Suppose $k = 0$. Since $(b_1, ..., b_n) \in \varphi(K^n)$, it is clear that $RM_F(\varphi) \geq 0$, as K is an elementary extension of F and $RCB_K(\varphi) \geq 0$.

Assume $k > 0$, and the lemma is true for $k-1$. Without loss of generality, we can assume that $\{b_1, ..., b_k\}$ is a k-element subset of $\{b_1, ..., b_n\}$ which is algebraically independent over F. We can choose an algebraically closed extension $C \geq K$ containing a set of elements $X = \{c^{(i)}, i < \omega\} \subseteq C$ such that X is algebraically independent over K. For each $i < \omega$ let ψ_i be the formula $\varphi \wedge (x_1 = c^{(i)})$, which has parameters from C.

Thus $\psi_i(C^n) \subseteq \varphi(C^n)$ for each i and $\psi_i(C^n) \cap \psi_j(C^n) = \emptyset$ when $i \neq j$. To prove that $RM_F(\varphi) \geq k$, it will suffice to show that $RM_C(\psi_i) \geq k - 1$ for each i. This will suffice, for the following reason. Assume $RM_C(\psi_i) \geq k - 1$ for each i. There there exists a family $\{C_i : i < \omega\}$ of algebraically closed extensions of C with $RCB_{C_i}(\psi_i) \geq k - 1$ for each i. Since the parameters of ψ_i come from C, we can replace C_i with any field to which C_i is isomorphic over C. Thus with no loss of generality we can assume that we have an algebraically closed field C^* which contains every C_i as a subfield. As the Cantor-Bendixon rank of a formula can only increase when passing to an extension field, we have $RCB_{C^*}(\psi_i) \geq k - 1$ for all i. Furthermore, $\psi_i(C^n) \subseteq \varphi(C^n)$ implies $\psi_i((C^*)^n) \subseteq \varphi((C^*)^n)$ for all i, and $\psi_i(C^n) \cap \psi_j(C^n) = \emptyset$ implies $\psi_i((C^*)^n) \cap \psi_j((C^*)^n) = \emptyset$ for all $i \neq j$. So $RM_F(\varphi) \geq RCB_{C^*}(\varphi) \geq k$ as required.

We can choose an algebraically closed extension $D \geq C$ containing a set of elements $\{d_2, ..., d_k\}$ that are algebraically independent over C. Letting $d_1 = c^{(i)}$, we get that $\{d_1, ..., d_k\}$ are algebraically independent over F. Hence there exists an automorphism $\tau \in \mathrm{Aut}_F(D)$ such that $\tau(b_j) = d_j$ for each $j = 1, ..., k$. This is because both $(b_1, ..., b_k)$ and $(d_1, ..., d_k)$ are algebraically independent sets over F.

Let $d_j = \tau(b_j)$ for $j = k+1, ..., n$. Since $K \vDash \varphi(b_1, ..., b_n)$, then $D \vDash \varphi(b_1, ..., b_n)$, as $K \leq D$. Also, since τ fixes F, and since all the parameters in φ are from F, we have that $D \vDash \varphi(\tau(b_1), ..., \tau(b_n))$, i.e. $D \vDash \varphi(d_1, ..., d_n)$. As $d_1 = c^{(i)}$, we get that $D \vDash \psi_i(d_1, ..., d_n)$.

Since $d_2, ..., d_k$ are algebraically independent over C, the extension $C(d_1, ..., d_n)$ over C has transcendence degree at least $k-1$. So $\dim_C(d_1, ..., d_n) \geq k-1$, implying that $RM_C(\psi_i) \geq k-1$ from the induction hypothesis. ■

Lemma 4.2.5 *Suppose $k \geq 0$, F is an algebraically closed field, and $\varphi(x_1, ..., x_n)$ is an $\mathcal{L}_r$-formula with parameters from F. If $RM_F(\varphi) \geq k$ then there exists an algebraically closed extension $K \geq F$ and a tuple $(b_1, ..., b_n) \in K^n$ such that $K \vDash \varphi(b_1, ..., b_n)$ and $\dim_F(b_1, ..., b_n) \geq k$.*

In particular, this lemma proves that the Morley rank of any $\mathcal{L}_r$-formula with respect to an algebraically closed field is finite.

Proof. The proof is by induction on k.

Suppose $k = 0$. $RM_F(\varphi) \geq 0$ implies that we can find an algebraically closed extension $K \geq F$ and a tuple $(b_1, ..., b_n) \in K^n$ such that $K \vDash \varphi(b_1, ..., b_n)$. As dimension cannot be negative, $\dim_F(b_1, ..., b_n) \geq 0$.

Assume $k > 0$ and the claim is true for $k-1$. Assume also, for sake of contradiction, that there does not exist an algebraically closed extension $K \geq F$ and a tuple $(b_1, ..., b_n) \in K^n$ such that $K \vDash \varphi(b_1, ..., b_n)$ and $\dim_F(b_1, ..., b_n) \geq k$. Since $RM_F(\varphi) \geq k$, there exists an algebraically closed extension $K \geq F$ and formulas ψ_i, $i < \omega$, with parameters from K such that

- $\psi_i(K^n) \subseteq \varphi(K^n)$ for each i,
- $\psi_i(K^n) \cap \psi_j(K^n) = 0$ for all $i \neq j$,
- $RM_K(\psi_i) \geq k-1$ for each i.

By the induction hypothesis, there exists for each $i < \omega$ an algebraically closed extension $C_i \geq K$ and a tuple $(c_1^{(i)}, ..., c_n^{(i)}) \in C_i$ such that $C_i \vDash \psi_i(c_1^{(i)}, ..., c_n^{(i)})$ and $\dim_K(c_1^{(i)}, ..., c_n^{(i)}) \geq k-1$. For the purposes of this proof, the field C_i can be replaced with any extension of K which is isomorphic to C_i over K. Thus without loss of generality, we can choose an algebraically closed field $C \geq K$ with sufficiently large transcendence degree, such that $C_i \leq C$ for each $i < \omega$. Then $C \vDash \varphi(c_1^{(i)}, ..., c_n^{(i)})$ for each $i < \omega$, and $\dim_K(c_1^{(i)}, ..., c_n^{(i)}) = k-1$ from our assumption.

From Theorem 4.1.7 applied to K, C and φ, we get that the action of $\mathrm{Aut}_K(C)$ partitions $\{(c_1^{(i)}, ..., c_n^{(i)}) : i < \omega\}$ into finitely many orbits. Thus we have a $\tau \in \mathrm{Aut}_K(C)$ such that $\tau(c_1^{(i)}, ..., c_n^{(i)}) = (c_1^{(j)}, ..., c_n^{(j)})$, for some $i \neq j$. So $C \vDash \psi_i(c_1^{(j)}, ..., c_n^{(j)})$, contradicting $\psi_i(K^n) \cap \psi_j(K^n) = 0$ for all $i \neq j$. ■

It should be noted that this characterization of Morley rank holds in a more general setting beyond algebraically closed fields.

Definition 4.2.6 *A theory T is strongly minimal if for any model $\mathfrak{A}$ of T, every subset of A that is definable with parameters from A is either finite or cofinite.*

If $\mathfrak{A}$ is a model of a strongly minimal theory, and $a, b_1, ..., b_n \in A$ then we say a is algebraic over $b_1, ..., b_n$ if there is an $\mathcal{L}$-formula $\psi(x, y_1, ..., y_n)$ such that $\mathfrak{A} \models \psi(a, b_1, ..., b_n)$ and $\{x \in A : \mathfrak{A} \models \psi(x, b_1, ..., b_n)\}$ is finite. If $\mathfrak{A} \preccurlyeq \mathfrak{B}$ are models of the strongly minimal T, and if $b_1, ..., b_n \in B$, we would define the $\dim_{\mathfrak{A}}(b_1, ..., b_n)$ to be the cardinality of the largest subset $\{c_1, ..., c_m\} \subseteq \{b_1, ..., b_n\}$ so that no c_i is algebraic over any finite subset of $A \cup \{c_1, ..., c_m\} \setminus \{c_i\}$. In the case of algebraically closed fields, these are clearly equivalent to the definitions of algebraic and dimension given above.

With this definition of the $\dim_{\mathfrak{A}}(b_1, ..., b_n)$, our characterization of Morley rank in theorem 4.2.3 holds in any strongly minimal theory. However, for the purpose of this paper, the proof of this more general result is unnecessary.

Chapter 5

Equivalence of Morley Rank and Dimension

This chapter is devoted to proving the second main result of this paper: that the Morley rank of a definable subset of F^n is equal to its dimension. We begin with proving this result for varieties.

Theorem 5.0.7 *If V is a variety, then its Morley Rank is equal to its dimension.*

Proof. Fix F to be an algebraically closed field, and let V be a variety in F^n of dimension d. Recalling theorem 2.2.6 on the dimension of varieties, we have that $F(V)$ has transcendence degree d over F, where $F(V)$ denotes the quotient field of the ring $F[V] = F[X_1, ..., X_n]/I(V)$. In fact, we can characterize the ring $F[V] = F[X_1, ..., X_n]/I(V)$ equivalently as

$$F[V] = F[X_1/I(V), ..., X_n/I(V)].$$

In other words, $X_1/I(V), ..., X_n/I(V)$ generate the ring $F[V]$ over F, and thus it follows that elements $X_1/I(V), ..., X_n/I(V)$ generate the field $F(V)$ over F. Now, since $F(V)$ has transcendence degree d over F, there must exist a subset of the set $\{X_1/I(V), ..., X_n/I(V)\}$ with cardinality d which is algebraically independent over F, and thus

$$\dim_F(X_1/I(V), ..., X_n/I(V)) = d.$$

Set K to be the algebraic closure of the field $F(V)$. Let $f_1, ..., f_m$ be a generating set of polynomials of the prime ideal $I(V)$ in $F[X_1, ..., X_n]$. Thus $V = Z(f_1, ..., f_m)$, and equivalently V is defined by the $\mathcal{L}_r$-formula

$$\phi_V(\overline{x}) = \bigwedge_{i=1}^{m}(f_i(\overline{x}) = 0).$$

Now consider the point $(X_1/I(V), ..., X_n/I(V))$ in K^n and observe that

$$\begin{aligned} \phi_V(X_1/I(V), ..., X_n/I(V)) &= \textstyle\bigwedge_{i=1}^{m}(f_i(X_1/I(V), ..., X_n/I(V)) = 0) \\ &= \textstyle\bigwedge_{i=1}^{m}(f_i(\overline{X})/I(V) = 0). \end{aligned}$$

Since the polynomials $f_1, ..., f_m$ generate the ideal $I(V)$, $f_i(\overline{X})/I(V) \equiv 0/I(V)$ for each $1 \leq i \leq m$, and hence $f_i(\overline{X})/I(V) = 0$ is true in K. Therefore the point $(X_1/I(V), ..., X_n/I(V)) \in K^n$ satisfies $\phi_V(\overline{x})$, so $K \models \phi_V(X_1/I(V), ..., X_n/I(V))$. From the discussion above $\dim_F(X_1/I(V), ..., X_n/I(V)) \geq d$, and thus the Morley rank of V is greater than or equal to its dimension d.

Conversely, let L be a field extension of F, and let K denote the algebraic closure of L. Since K, L, and F are all fields of the same characteristic p, and since $F \subseteq L \subseteq K$, the model completeness of ACF_p gives us that K is an elementary extension of F, $K \succcurlyeq F$.

Suppose $\overline{a} = (a_1, ..., a_n)$ is an n-tuple in L^n such that $K \vDash \phi_V(\overline{a})$, where ϕ_V defines the variety V. If f is an element of the ring $F[V]$, then f is a well defined function with domain V. Since $\overline{a} \in \phi_V(K^n)$, $f(\overline{a})$ is a well defined point in the field $F(a_1, ..., a_n) \subseteq L \subseteq K$. Thus the map $f \mapsto f(\overline{a})$ defines an F-algebra homomorphism from the coordinate ring $F[V]$ to the field $F(a_1, ..., a_n)$. Let us call this function $\alpha_{\overline{a}} : F[V] \to K$, where $\alpha_{\overline{a}}(f) = f(\overline{a})$. This is indeed an F-algebra homomorphism, as $\alpha(f+g) = (f+g)(\overline{a}) = f(\overline{a}) + g(\overline{a})$ and $\alpha(f \cdot g) = (f \cdot g)(\overline{a}) = f(\overline{a}) \cdot g(\overline{a})$. Observe that the range of $\alpha_{\overline{a}}$ is precisely $F[a_1, \ldots, a_n]$.

So, if the Morley rank of V is d^*, then we can find an elementary extension K of F, an n-tuple $\overline{a} \in K^n$ with dimension d^* over F, and a surjective F-algebra homomorphism $\alpha : F[V] \to F[a_1, \ldots, a_n]$. Since the quotient field of the range $F[a_1, \ldots, a_n]$ has transcendence degree d^* over F, the quotient field $F(V)$ of the domain $F[V]$ must also have transcendence degree at least d^* over F, showing that the dimension of V is at least equal to its Morley rank. This establishes the desired equality. ■

The next lemma is a result about dimensions of sets in the Zariski topology.

Lemma 5.0.8 *If O is a nonempty open subset, V is a variety and $V \cap O \neq \emptyset$, then $V \setminus O$ has dimension strictly less than V.*

Proof. First, $V \setminus O = V \cap O^C$, an intersection of two closed sets. Thus $V \setminus O$ is an algebraic set. Moreover, $V \cap O$ is nonempty, and thus $V \setminus O \subsetneq V$. Now suppose that $\dim(V \setminus O) = d$. By one of the equivalent definitions of dimension for algebraic sets, there is a proper chain $V_0 \subset V_1 \subset ... \subset V_d \subseteq V \setminus O$ of varieties of length d contained in $V \setminus O$. Since $V \setminus O$ is properly contained in V, there is a proper chain $V_0 \subset ... \subset V_d \subseteq V \setminus O \subset V$ of varieties of length $d+1$ contained in V. Thus $\dim(V) > \dim(V \setminus O)$. ■

Before we get to the proof of our main result, we introduce the powerful model theoretic notion of types.

Definition 5.0.9 *Let $\mathfrak{A}$ be an $\mathcal{L}$-structure, and $X \subseteq A$. An n-type over X (with respect to $\mathfrak{A}$ and in the variables $x_1, ..., x_n$) is a maximal consistent set of $\mathcal{L}$-formulas with variables among $x_1, ..., x_n$ and with parameters from X which is consistent with $Th(\mathfrak{A})$.*

Definition 5.0.10 *Let $\mathfrak{A}$ be an $\mathcal{L}$-structure, let $X \subseteq A$, and let $\overline{b} = (b_1, ..., b_n) \in A$. The n-type of $\overline{b}$ over X (with respect to $\mathfrak{A}$ and in the variables $x_1, ..., x_n$), denoted $tp_{\mathfrak{A}}(\overline{b}/X)$, is the set of all $\mathcal{L}$-formulas $\phi(x_1, ..., x_n)$ with parameters from X such that $\mathfrak{A} \models \phi(\overline{b})$. In words, everything we can say about $\overline{b}$ in a first order $\mathcal{L}$-formula in terms of X.*

Definition 5.0.11 *Let $\mathfrak{A}$ be a model of a complete theory T. The Stone Space of $\mathfrak{A}$, denoted $S_n(\mathfrak{A})$, is the set of n-types over A with respect to $\mathfrak{A}$.*

To this point, we have established a correspondence between the $\mathcal{L}_r$-formulas and the constructible sets of algebraic geometry. We will now show a surprising bijection between the types in $S_n(F)$ and the varieties in F^n.

Claim 5.0.12 *Let V be a variety in F^n, where F is some algebraically closed field. The set Γ_V of $\mathcal{L}_r$-formulas with parameters from F which are first order consequences of $\{f(x_1, ..., x_n) = 0 : f \in I(V)\} \cup \{f(x_1, ..., x_n) \neq 0 : f \notin I(V)\}$ is a type in $S_n(F)$.*

Proof. Let ϕ be a $\mathcal{L}_r$-formula with free variables from $x_1, ..., x_n$ and parameters from F. Then, by quantifier elimination of ACF, ϕ is equivalent to some quantifier free formula in the variables $x_1, ..., x_n$, and thus expressable as a finite boolean combination of atomic formulas in the varibles $x_1, ..., x_n$ with parameters from F. The atomic formulas in those variables are all of the form $f(x_1, ..., x_n) = 0$, where $f \in F[X_1, ..., X_n]$. Taking the arbitrary atomic formula $\psi \equiv f(x_1, ..., x_n) = 0$, we get that either $f \in I(V)$ or $f \notin I(V)$. Thus exactly one of ψ or $\neg\psi$ is in Γ_V. Since ϕ is equivalent to a finite boolean combination of atomic formulas, all of which are either in Γ_V or inconsistent with Γ_V, either ϕ or $\neg\phi$ must be in Γ_V. So Γ_V is indeed maximal consistent.

It remains to show that Γ_V is consistent with Th(F). Let ϕ be a sentence with parameters from F such that $F \vDash \phi$. Mimicking the argument above, ϕ is equivalent to a finite boolean combination of variable free atomic formulas. Such formulas are of the form $t = 0$ or $t \neq 0$, where t is a term expressing an element of the field F. So if $F \vDash t = 0$, then t must express $0 \in F$, and $t = 0$ is in Γ_V as the polynomial 0 is in the ideal $I(V)$. If $F \vDash t \neq 0$, a similar argument shows that $t \neq 0$ is in Γ_V. Therefore $\phi \in \Gamma_V$, which completes the proof. ∎

Claim 5.0.13 *Let Γ be a type in $S_n(F)$, where F is an algebraically closed field. The zero set $Z(I_\Gamma)$, where I_Γ is the ideal $\{f \in F[X_1, ..., X_n]$: the atomic formula*

$f(x_1, ..., x_n) = 0$ is in $\Gamma\}$, is a variety in F^n.

Proof. It suffices to show that the ideal I_Γ is a prime ideal in $F[X_1, ..., X_n]$. Suppose $f \cdot g \in I_\Gamma$. Then the atomic formula $f(x_1, ..., x_n) \cdot g(x_1, ..., x_n) = 0$ is in the type Γ. If $f(x_1, ..., x_n) \neq 0$ and $g(x_1, ..., x_n) \neq 0$ were in Γ, then Γ would be inconsistent, a contradiction. Since Γ is consistent, one of $f(x_1, ..., x_n) = 0$ or $g(x_1, ..., x_n) = 0$ must be in Γ, implying one of f or g is in I_Γ. ∎

These maps between $S_n(F)$ and varieties in F^n establish our one-to-one correspondence. Indeed, these maps are inverses of one another, as the consequences of $\{f(x_1, ..., x_n) = 0 : f \in I_\Gamma\} \cup \{f(x_1, ..., x_n) \neq 0 : f \notin I_\Gamma\}$ are precisely the formulas in Γ.

Definition 5.0.14 *We define the <u>Morley rank of a type</u> $\Gamma(x_1, ..., x_n)$ over $\mathfrak{A}$ to be the least value of $RM_{\mathfrak{A}}(\phi)$ as ϕ ranges over all the formulas in Γ.*

Theorem 5.0.15 *The Morley rank of a variety V is equal to the Morley rank of its associated type, Γ_V.*

Proof. Recall that Γ_V is the set of consequences of $\{f(\overline{x}) = 0 : f \in I(V)\} \cup \{f(\overline{x}) \neq 0 : f \notin I(V)\}$. Let $f_1, ..., f_m$ be a generating set of polynomials of the ideal $I(V)$. Thus V is defined by the formula $\phi_V = \bigwedge_{i=1}^{m}(f_i(\overline{x}) = 0)$. The formula ϕ_V is clearly in Γ_V, as each $f_i \in I(V)$. Therefore, since the Morley rank of Γ_V is defined as the minimum Morley rank ranging over all formulas in Γ_V, we have $RM(V) = RM(\phi_V) \geq RM(\Gamma_V)$.

Conversely, suppose ψ is an arbitrary formula in Γ_V. From the quantifier elimination of ACF, we know that every formula is a finite boolean combination of atomic or negated atomic formulas, $f(\overline{x}) = 0$ or $f(\overline{x}) \neq 0$, where f is some polynomial in $F[X_1, ..., X_n]$. Thus ψ can be expressed as a finite boolean combination of formulas from $\{f(\overline{x}) = 0 : f \in I(V)\} \cup \{f(\overline{x}) \neq 0 : f \notin I(V)\}$. So we can find polynomials $f_{i,j} \in I(V)$, and $g_{i,j} \notin I(V)$ such that

$$\psi = \bigvee_{i=1}^{l} \left(\bigwedge_{j=1}^{r_i} f_{i,j}(\overline{x}) = 0 \wedge \bigwedge_{j=1}^{s_i} g_{i,j}(\overline{x}) \neq 0 \right).$$

We want to show that the Morley rank of ψ is at least as big as the Morley rank of V. Since $RM(\theta_1 \vee ... \vee \theta_t) = \max\{RM(\theta_1), ..., RM(\theta_t)\}$, it suffices to show that for some i, the formula $\bigwedge_{j=1}^{r_i} f_{i,j}(\overline{x}) = 0 \wedge \bigwedge_{j=1}^{s_i} g_{i,j}(\overline{x}) \neq 0$ has Morley rank at least as big as the Morley rank of V. It is therefore sufficient to consider an arbitrary formula of the form

$$\psi = \bigwedge_{i=1}^{r} f_i(\overline{x}) = 0 \wedge \bigwedge_{i=1}^{s} g_i(\overline{x}) \neq 0,$$

where $f_i \in I(V)$, and $g_i \notin I(V)$. Since the set defined by $\bigwedge_{i=1}^{r} f_i(\overline{x}) = 0$ contains V, it is safe to assume that $\bigwedge_{i=1}^{r} f_i(\overline{x}) = 0$ defines V precisely, as it can only decrease the Morley rank of the formula ψ. Let O denote the open set defined by $\bigwedge_{i=1}^{s} g_i(\overline{x}) \neq 0$. Thus we can assume ψ defines the set $V \cap O$.

If $V \cap O = \emptyset$, then $V \subseteq O^C$, the complement of O. Note that O^C is defined by $\bigvee_{i=1}^{s} g_i(\overline{x}) = 0$. Thus $O^C = C_1 \cup ... \cup C_s$, where C_i is the set defined by $g_i(\overline{x}) = 0$. As the irreducible set V is contained in the union of closed sets $C_1 \cup ... \cup C_s$, V must be fully contained in one of them, otherwise $V = (V \cap C_1) \cup ... \cup (V \cap C_s)$ would be a partition of V into a union of proper closed subsets of V, contradicting the irreducibility of V. So, without loss of generality, we have $V \subseteq C_1$. But then $g_1 \in I(V)$, contradicting out hypothesis. So $V \cap O \neq \emptyset$, and $V \setminus O$ is a proper subset of V.

Let us write $V = (V \cap O) \cup (V \setminus O)$. Written in terms of defining formulas, we have $\phi_V = \psi \vee \phi_{V \setminus O}$. Thus

$$\dim(V) = RM(V) = RM(\phi_V) = \max\{RM(\psi), RM(\phi_{V \setminus O})\}.$$

So to complete the proof, it suffices to show $RM(\phi_{V \setminus O}) < RM(V)$. Since $V \setminus O$ is a closed proper subset of V, it can be written as a union $V \setminus O = V_1 \cup ... \cup V_s$ of varieties all properly contained in V. Thus for each V_i, $\dim(V_i) < \dim(V)$. In terms of formulas, we can write $\phi_{V \setminus O} = \phi_{V_1} \vee ... \vee \phi_{V_s}$, where ϕ_{V_i} defines the variety V_i. Thus for each V_i, $\dim(V_i) < \dim(V)$. So

$$\begin{aligned}
RM(\phi_{V \setminus O}) &= \max\{RM(\phi_{V_1}), ..., RM(\phi_{V_s})\} \\
&= \max\{RM(V_1), ..., RM(V_s)\} \\
&= \max\{\dim(V_1), ..., \dim(V_s)\} && (\text{since } RM(V_i) = \dim(V_i)) \\
&< \dim(V) && (\text{since } \dim(V_i) < \dim(V)) \\
&= RM(V). && (\text{since } RM(V) = \dim(V))
\end{aligned}$$ ■

We can now prove the desired theorem.

Theorem 5.0.16 *If X is a constructible set, then its Morley rank is equal to its dimension.*

Proof. Let X be a nonempty constructible set. We can write the closure of X uniquely as a finite union $\overline{X} = V_1 \cup ... \cup V_m$ of varieties, no one contained in another. From the definition of dimension, $\dim(X) = \max\{\dim(V_1), ..., \dim(V_m)\}$. Moreover, since X and $\overline{X}$ have the same Zariski closures, $\dim(\overline{X}) = \dim(X)$. Let us assume without loss of generality that V_1 has maximum dimension among the V_i's, and therefore $\dim(X) = \dim(\overline{X}) = \dim(V_1)$.

Let us first consider the case when our nonempty constructible set X is of the form $X = V \cap O$, where V is a variety and O is an open set. We will show that $\overline{X} = V$.

Observe that $V \setminus (V \cap O) = V \setminus O$, since $V \setminus (V \cap O) = V \cap (V \cap O)^C = V \cap (V^C \cup O^C) = (V \cap V^C) \cup (V \cap O^C) = V \cap O^C = V \setminus O$. Since $V \cap O \subseteq V$, we can write $V = (V \setminus (V \cap O)) \cup (V \cap O)$, and now it immediately translates to $V = (V \setminus O) \cup X$. Taking the Zariski closures of both sides then gives $V = (V \setminus O) \cup \overline{X}$. Since V is irreducible, either $V = V \setminus O$ or $V = \overline{X}$. However, lemma 5.0.8 tells us that $\dim(V \setminus O) < \dim(V)$, and therefore $V = \overline{X}$. Equivalently, if V is a variety and O an open set, then $\overline{V \cap O} = V$, a fact we use later in the proof.

We have seen that a Zariski closed set C is the zero set of some finite collection of polynomials from $\{f_1, ..., f_s\} \subseteq F[X_1, ..., X_n]$. So the set C is defined by the $\mathcal{L}$-formula $\phi_C = (f_1(\overline{x}) = 0 \wedge ... \wedge f_s(\overline{x}) = 0)$. An open set is defined by $\neg\phi_{O^C} = (f_1(\overline{x}) \neq 0 \vee ... \vee f_s(\overline{x}) \neq 0)$. So our constructible set $X = V \cap O$ is defined by some formula $\phi_X = (g_1(\overline{x}) = 0 \wedge ... \wedge g_r(\overline{x}) = 0) \wedge (h_1(\overline{x}) \neq 0 \vee ... \vee h_t(\overline{x}) \neq 0)$, where V is defined by $\phi_V = (g_1(\overline{x}) = 0 \wedge ... \wedge g_r(\overline{x}) = 0)$ and O is defined by $\phi_O = (h_1(\overline{x}) \neq 0 \vee ... \vee h_t(\overline{x}) \neq 0)$.

Recall Γ_V is the type associated with the variety V. We know that the formula $\phi_V \in \Gamma_V$. We also want to show that the formula $\phi_O \in \Gamma_V$. Consider the formulas $h_i(\overline{x}) = 0$ for $1 \leq i \leq t$. If $(h_i(\overline{x}) = 0) \in \Gamma_V$ for each $1 \leq i \leq t$, then by definition each polynomial $h_i \in I(V)$, and thus $V \subseteq Z(h_i)$ for each $1 \leq i \leq t$. So $V \subseteq Z(h_1, ..., h_t) = O^C$, the complement of O. Thus $V \cap O = \emptyset$, a contradiction. So for some i, the formula $(h_i(\overline{x}) = 0) \notin \Gamma_V$. Since the type Γ_V is maximal consistent, it must be that the formula $\neg(h_i(\overline{x}) = 0) = (h_i(\overline{x}) \neq 0) \in \Gamma_V$, and hence the formula $\phi_O = (h_1(\overline{x}) \neq 0 \vee ... \vee h_t(\overline{x}) \neq 0) \in \Gamma_V$.

So $\phi_X = \phi_V \wedge \phi_O \in \Gamma_V$, and therefore $RM(\Gamma_V) \leq RM(\phi_X) = RM(X)$.

From theorem 5.0.15, we know that $RM(V) = RM(\Gamma_V)$, implying $RM(V) \leq RM(X)$. At the same time, since $X \subseteq V$, it must be that $RM(X) \leq RM(V)$. So $RM(X) = RM(V) = \dim(V) = \dim(X)$, and the proof is completed for nonempty constructible sets of the form $X = V \cap O$.

Now, let X be an arbitrary nonempty constructible set. As X is a finite boolean combination of Zariski closed sets, we write $X = \bigcup_{i=1}^{l} \left(\bigcap_{j=1}^{m_i} C_{i,j} \cap \bigcap_{j=1}^{p_i} O_{i,j} \right)$, where the $C_{i,j}$'s are closed sets and the $O_{i,j}$'s are open sets. Since finite intersections of closed sets are closed and finite intersections of open sets are open, we write $X = \bigcup_{i=1}^{l} (C_i \cap O_i)$. Now let us write the closed set C_i as a union $C_i = V_1^i \cup ... \cup V_{p_i}^i$ of varieties. Then we get $X = \bigcup_{i=1}^{l} \left((V_1^i \cap O_i) \cup ... \cup (V_{p_i}^i \cap O_i)\right)$. This can be simplified to $X = \bigcup_{i=1}^{l} (V_i \cap O_i)$, where the V_i's are varieties and the O_i's are open sets. Now we can write $\overline{X} = \bigcup_{i=1}^{l} \left(\overline{V_i \cap O_i}\right) = \bigcup_{i=1}^{l} (V_i)$, recalling from above that $\overline{V \cap O} = V$. So the V_i's are the irreducible components of $\overline{X}$.

By definition, $\dim(X) = \max\{\dim(V_1), ..., \dim(V_l)\}$. Let the set $V_i \cap O_i$ be defined by the formula ϕ_i. Then the set X is defined by the formula $\phi_X = \phi_1 \vee ... \vee \phi_l$. So we have $\dim(V_i) = \dim(\overline{V_i \cap O_i}) = \dim(V_i \cap O_i) = RM(V_i \cap O_i) = RM(\phi_i)$. Therefore $\dim(X) = \max\{RM(\phi_1), ..., RM(\phi_l)\} = RM(\phi_X) = RM(X)$. This completes the proof. ■

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