Unavoidable Minors of Large $5-$Connected Graphs

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

This thesis shows that, for every positive integer $n \geq 5$, there exists a positive integer N such that every $5-$connected graph with at least N vertices has a minor isomorphic to one of thirty explicitly defined $5-$connected graphs $H_1(n), ..., H_{30}(n)$, each with at least n vertices.

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I would like thank my supervisor Professor Jim Geelen without whose guidance, support and trust, this project would not have been possible. I am lucky to have had him to teach me the ropes during these initial years, and to have been exposed to his perspective on and philosophy of research, in part, in the process. I hope I am able to appreciate both to the fullest in time to come.

I would also like to thank Professor Joseph Cheriyan for his advice and help at C&O, and my friends Nishad, Anand and Akash for their indulgence.

Finally, I would like to thank my family - Ma, Papa, Priyam, for their constant love and support, and Ambica, for being my first bona fide collaborator.

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Chapter 1

Introduction

For every positive integer n, there exists, in each of the following cases, a positive integer N such that

(i) every connected graph with at least N vertices contains either a vertex of degree n or a path of length n;

(ii) every 2−connected graph with at least N vertices either has a $K_{2,n}$−minor or contains a cycle of length n;

(iii) every 3−connected graph with at least N vertices has either a $K_{3,n}$−minor or a minor isomorphic to a wheel of length n (proved by Oporowski, Oxley and Thomas ([OOT93]));

(iv) every 4−connected graph with at least N vertices has a minor isomorphic to one of $K_{4,n}$ and three other 4−connected graphs, each with at least n vertices (Figure 1.2) (proved indirectly by Geelen and Joeris ([GJ16]), and Oporowski, Oxley and Thomas ([OOT93])).

This thesis establishes a similar extremal result giving thirty explicitly defined 5−connected graphs, each with at least n vertices, as unavoidable minors of every 5−connected graph with at least N vertices.

For every positive integer θ, a graph G is said to be $\theta-$*connected* if at least θ vertices must be deleted from G in order to disconnect it, i.e., for each subset $Y \subseteq V(G)$ with $|Y| \leq \theta - 1, G \backslash Y$ is connected.

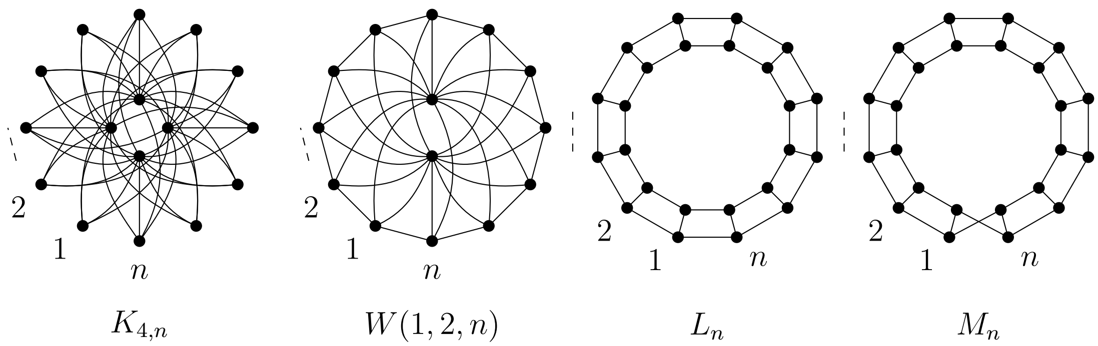

Figure 1.1: Unavoidable minors of large internally 4−connected graphs and graphs with large 4−connected sets.

A graph H is said to be a *minor* of another graph G if H can be obtained from a subgraph G' of G by contracting connected subgraphs in G'−a graph obtained by contracting a connected subgraph of G' is one that is obtained by identifying all of the former's vertices into a single vertex and deleting all its edges. We say that G has an H−*minor* if there is a minor of G that is isomorphic to H.

1.1 Two Proofs for Large 4−Connected Graphs

Unavoidable minors of "sufficiently large" 4−connected graphs are easily derived from the results obtained in each of [GJ16] and [OOT93], both of which find unavoidable minors of large graphs with slightly weaker but distinct connectivity properties. Incidentally, both of them list graphs from the same four infinite families as unavoidable minors. These are $K_{4,n}$, the $2n$−spoke double wheel $W(1,2,n)$, the n−wrung circular ladder L_n and the n−rung Möbius ladder M_n (see Figure 1.1). We state the corollary describing avoidable minors of sufficiently large 4−connected graphs after discussing these two results.

1.1.1 Large Internally 4−Connected Graphs

A *separation* in a graph G is a pair (G_1, G_2) of subgraphs of G such that $G_1 \cup G_2 = G$ and $E(G_1 \cap G_2) = \emptyset$, the *order* of the separation being $|V(G_1 \cap G_2)|$. Note that, for every positive integer θ, a graph G is θ−connected if there does not exist in G a separation of order $\theta - 1$ or less.

A graph is G said to be *internally* $4-$*connected* if it is $3-$connected and, for every separation (G_1, G_2) in G of order 3, one of $V(G_1) - V(G_2)$ and $V(G_2) - V(G_1)$ contains at most one vertex. Every $4-$connected graph is internally $4-$connected.

Oporowski, Oxley and Thomas show in ([OOT93]) that every sufficiently large internally $4-$connected graph has a minor isomorphic to one of $K_{4,n}, W(1,2,n), L_n$ and M_n:

Theorem 1.1.1. *For every integer* $n \geq 4$*, there is an integer* $N_1(n)$ *such that every internally* $4-$*connected graph* G *with at least* $N_1(n)$ *vertices has a minor isomorphic to one of* $K_{4,n}, W(1,2,n), L_n$ *and* M_n*.*

In particular, they prove, as an alternative statement of the above theorem, that every graph with no minor isomorphic to any of $K_{4,n}, W(1,2,n), L_n$ and M_n admits a tree-decomposition of width at most $N_1(n)$ and edge-width at most 3. A similar duality observed by Geelen and Joeris ([GJ16]) in the context of a graph containing a large highly connected set is what forms the basis of our proof. We discuss tree-decompositions and the results obtained by Geelen and Joeris in greater detail in Chapter 2.

Remark: The result describing unavoidable minors of sufficiently large $3-$*connected graphs is observed as a corollary of the above theorem in [OOT93].*

1.1.2 Graphs with Large $4-$Connected Sets

A $\theta-$*connected set* in a graph G is a subset X of vertices such that, for all subsets $Y, Z \subseteq X$ with $|Y| = |Z| \leq \theta$, there exist θ vertex-disjoint $(Y, Z)-$paths in G. If G is $\theta-$connected, then $V(G)$ forms a $\theta-$connected set in G.

Unavoidable minors of graphs with sufficiently large $\theta-$connected sets is one aspect of the duality that Geelen and Joeris observe in [GJ16]. We give a general graph construction for these minors in the next chapter; for graphs with sufficiently large $4-$connected sets, these are $K_{4,n}, W(1,2,n), L_n$ and M_n, as the following corollary of the theorem by Geelen and Joeris enumerates.

Corollary 1.1.2. *For every integer* $n \geq 4$*, there is an integer* $N_2(n)$ *such that every graph* G *with a* $4-$*connected set of size at least* $N_2(n)$ *has a minor isomorphic to one of* $K_{4,n}, W(1,2,n), L_n$ *and* M_n*.*

Observe that each of $K_{4,n}, W(1,2,n), L_n$ and M_n is internally $4-$connected and has a $4-$connected set of at least n vertices. We denote by $W(2,0,n)$ and $TW(2,0,n)$ the

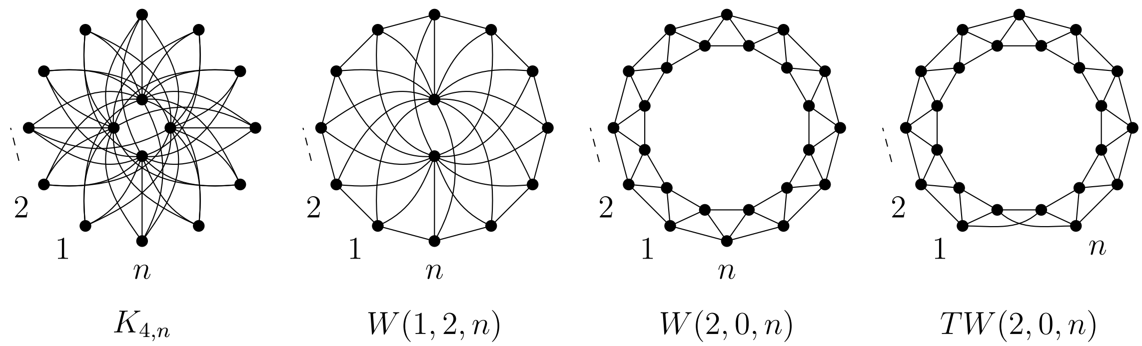

Figure 1.2: Unavoidable minors of large 4−connected graphs.

minors of L_{2n} and M_{2n+2}, respectively, shown in Figure 1.2. The following corollary then follows directly from Thorem 1.1.1 and Corollary 1.1.2.

Corollary 1.1.3. *For every integer* $n \geq 4$*, there is an integer* $N(n)$ *such that every* $4-$*connected graph* G *with at least* $N(n)$ *vertices has a minor isomorphic to one of* $K_{4,n}$, $W(1,2,n), W(2,0,n)$ *and* $TW(2,0,n)$.

Proof. $N(n) = \min\{\, N_1(2n+2), N_2(2n+2)\,\}$ suffices. □

1.2 Large 5−Connected Graphs: The Two Cases

We find, in this thesis, a set $\{\, H_i(n) : i \in \{\, 1, ..., 30\,\}\,\}$ of unavoidable minors of sufficiently large graphs that are 5−connected. Other than the complete bipartite graph $K_{5,n}$, where $n \geq 5$ is a postive integer, the said set includes the graphs depicted in the Figures 1.3, 1.4 and 1.5. That each of these graphs is 5−connected is something that can be easily checked. We give explicit constructions for these graphs in the appendix. Our main result is the following.

Theorem 1.2.1. *For each* $n \in \mathbb{N}$ *with* $n \geq 5$*, there exists* $N \in \mathbb{N}$ *such that, if* G *is a* $5-$*connected graph with at least* N *vertices, then* G *has a minor isomorphic to* $K_{5,n}$, $W(1,3,n)$, $W_j(2,1,n)$, $TW_j(2,1,n)$, $CW_{k(a)}(2,1,n)$, $CW_{k(b)}(2,1,n)$, $W(2,2,n)$, $TW(2,2,n)$, $W_1^-(3,0,n)$, $TW_1^-(3,0,n)$, $W_{2(a)}^-(3,0,n)$, $W_{2(b)}^-(3,0,n)$, $TW_{2(a)}^-(3,0,n)$, $TW_{2(b)}^-(3,0,n)$, $W(3,0,n)$ *or* $TW_i(3,0,n)$*, where* $i \in \{\, 1,2,3\,\}, j \in \{\, 1,2\,\}, k \in \{\, 1,...,6\,\}$.

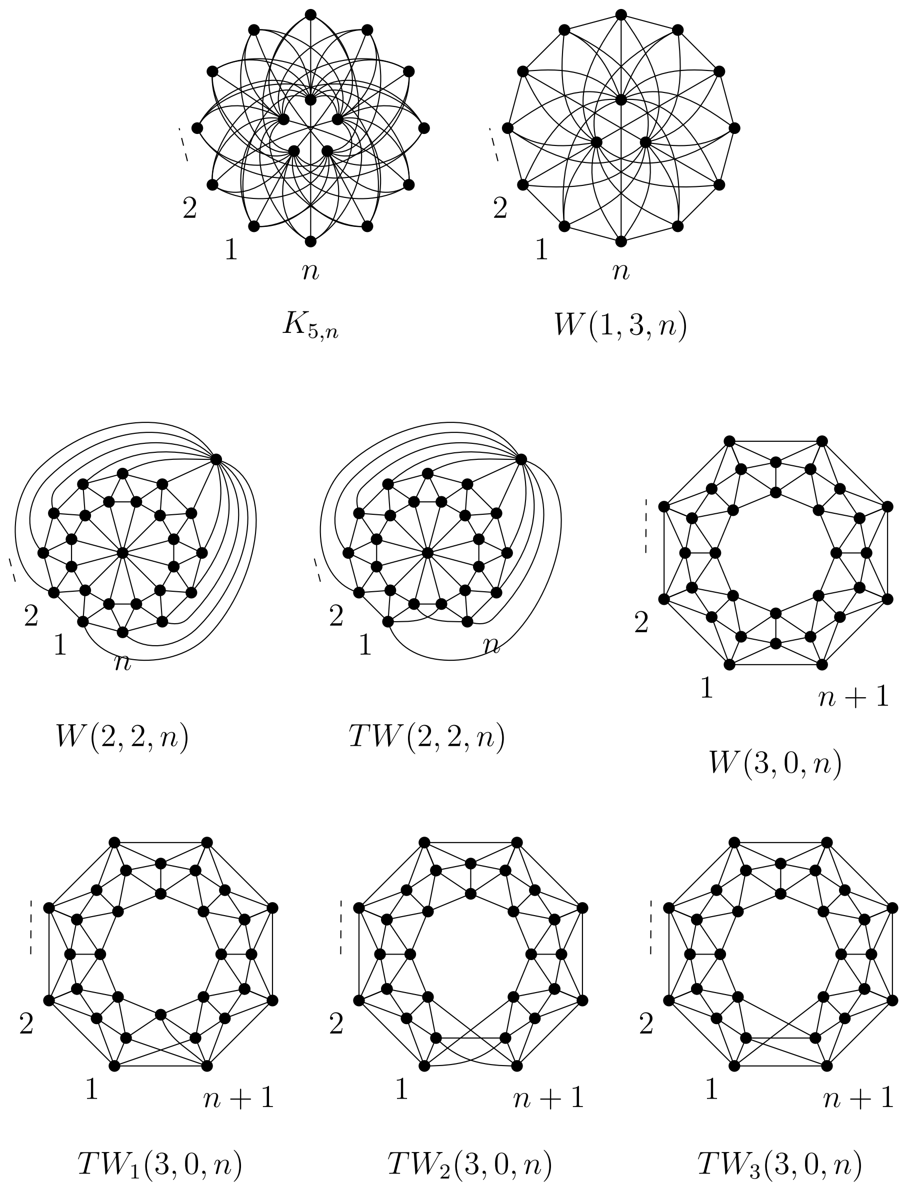

Figure 1.3: Unavoidable minors of large 5−connected graphs.

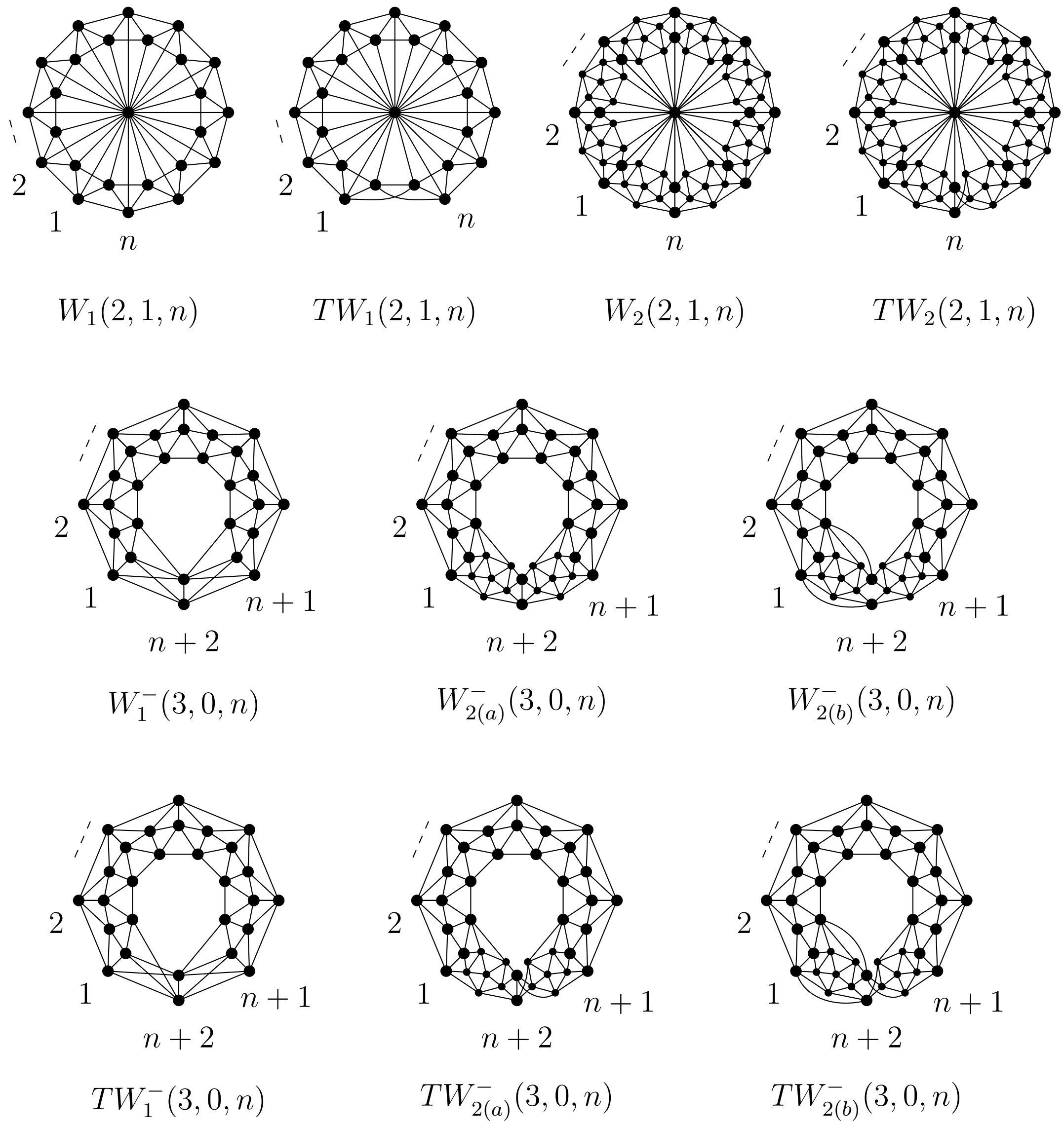

Figure 1.4: Unavoidable minors of large 5−connected graphs (contd.).

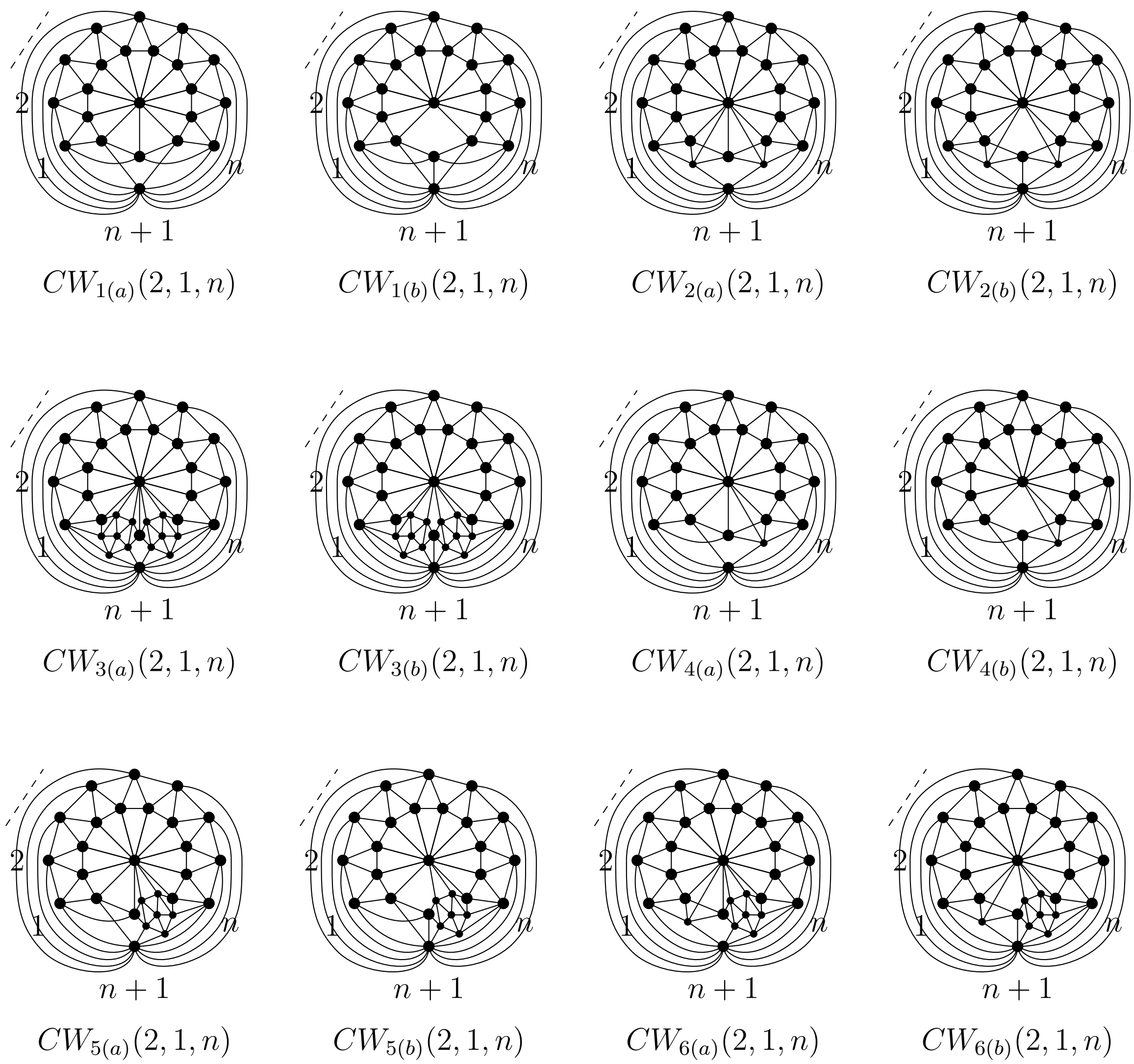

Figure 1.5: Unavoidable minors of large 5−connected graphs (contd.).

A similar result was claimed by Kawarabayashi in 2006 (see [KM07]).

Geelen and Joeris prove in [GJ16] (see Chapter 2 of this thesis for more details) that a graph either contains a sufficiently large $\theta-$connected set (and, hence, one of a set of unavoidable minors) or admits a tree decomposition of bounded width that has edge-width at most $\theta - 1$. Our proof uses these results to find a complete set of unavoidable minors of sufficiently large $5-$connected graphs by combining two cases: when the said graph has a large $6-$connected set and when it does not.

The first case is straightforward as all of the unaovidable minors found in [GJ16] for $\theta = 6$ have, in turn, minors that are $5-$connected. We find these minors in Chapter 2. For the latter case, we use the dual result in [GJ16] which guarantees a tree-decomposition of bounded width and edge-width at most 5 for graphs that do not contain a large $6-$connected set:

(1) we find a set of possible rooted minors of the "smaller" side of a $5-$separation in the graph (Chapter 3) as well as

(2) one of the intersection of the "larger" sides of two non-crossing separations in the graph (such an intersection is, in turn, separated by each one of a large family of nested separations in the graph) (Chapter 4), and then

(3) patch members of the two sets together (Chapter 4).

Enumerating all possible triples of rooted minors, each containing one rooted minor of the intersection of the larger sides and two rooted minors for the disjoint smaller sides of the of the two non-crossing separations considered, then gives us the remainder of the set of unavoidable minors mentioned in Theorem 1.2.1.

We conclude with a short proof of Theorem 1.2.1 in Chapter 5 that puts the two cases together.

Chapter 2

Large $\theta-$Connected Sets

In this chapter, we review the results concerning large $\theta-$connected sets obtained by Geelen and Joeris in [GJ16].

2.1 Graphs with Large $\theta-$Connected Sets

The main result obtained in [GJ16] is an unavoidable-minor characterization of graphs with sufficiently large $\theta-$connected sets, for each positive integer $\theta \geq 2$.

Let $r, \ell, n \in \mathbb{N}$ with $r \geq 1$ and $n \geq 3$. Now, let T be a tree with r vertices, let Z be an $\ell-$element set, let $\pi : V(T) \to V(T)$ be a permutation, and let $\psi : Z \to V(T)$ be a function. Then the $(r, \ell, n)-$*wheel defined by* (T, Z, π, ψ) is the graph G constructed as follows:

(1) Let G' be the disjoint union of n copies of T, named $T_1, ..., T_n$, where, for each $v \in V(T)$ and $i \in \{ 1, ..., n \}$, the copy of v in T_i is labelled v_i.

(2) Let G'' be the graph obtained from G' by adding an edge between v_i and v_{i+1} for each $v \in V(T)$ and each $i \in \{ 1, ..., n-1 \}$.

(3) Let G''' be the graph obtained from G'' by adding an edge between v_n and $\pi(v)_1$ for each $v \in V(T)$.

(4) Then G is obtained from G''' by adding Z as a set of isolated vertices (or *hubs* of the wheel) and then, for each $z \in Z$ and each $i \in \{ 1, ..., n \}$, adding an edge between z and $\psi(z)_i$.

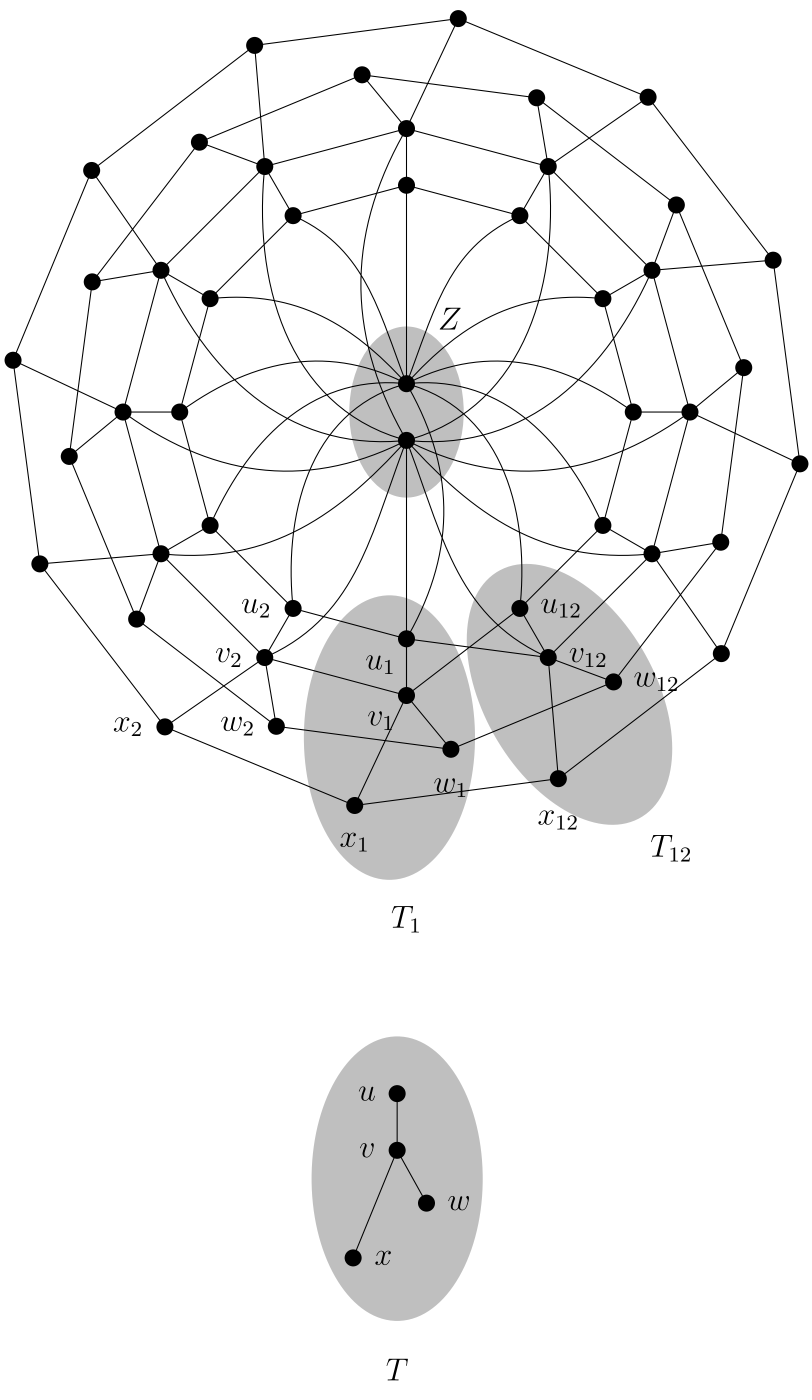

Figure 2.1: A $(4, 2, 12)-$wheel together with its tree T.

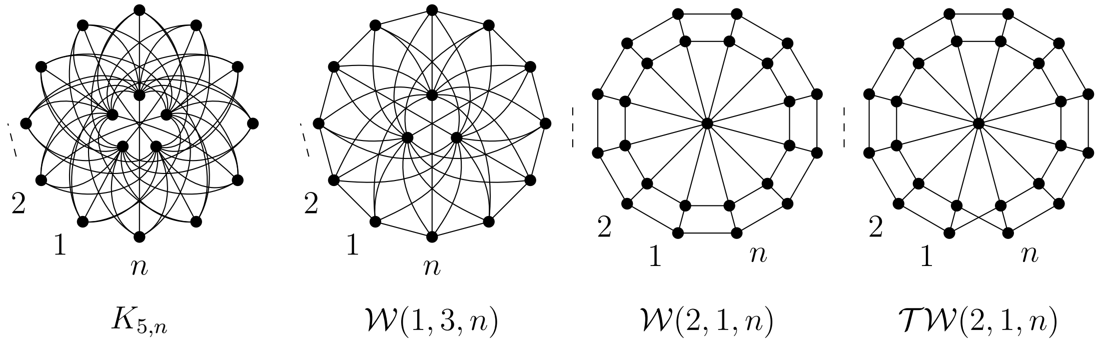

Figure 2.2: Unavoidable minors of graphs with large 5−connected sets.

Figure 2.1 depicts a possible $(4, 2, 12)-$wheel. A $(\theta; n)-$*wheel* is an $(r, \ell, n)-$wheel where $2r + \ell = \theta$. In any $(\theta; n)-$wheel W, every set of n vertices which contains exactly one vertex from each of the trees $T_1, ..., T_n$ forms a $\theta-$connected set in W. Note that, in the complete bipartite graph $K_{\theta,n}$, the set of vertices in the θ-partition forms a $\theta-$connected set whenever $n \geq \theta$.

Geelen and Joeris showed in [1] that $(\theta; n)-$wheels together with complete bipartite graphs constitute a set of unavoidable minors of graphs with large $\theta-$connected sets. In particular, they proved the following theorem.

Theorem 2.1.1. *There exists a function $f_{2.2.2} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that, for all $\theta, n \in \mathbb{N}$ with $\theta \geq 2, n \geq 3$, if G is a graph containing a $\theta-$connected set of size at least $f_{2.2.2}(\theta, n)$, then G has a $K_{\theta,n}-$minor or a $(\theta; n)-$wheel-minor.*

For all $\ell, n \in \mathbb{N}$ with $n \geq 3$, we denote the unique $(1, \ell, n)-$wheel by $\mathcal{W}(1, \ell, n)$. For $\theta = 5$, we denote the two distinct $(2, 1, n)-$wheels by $\mathcal{W}(2, 1, n)$ and $\mathcal{TW}(2, 1, n)$, as depicted in Figure 2.2. For $\theta = 6$, we denote the four distinct $(2, 2, n)-$wheels by $\mathcal{W}_1(2, 2, n)$, $\mathcal{TW}_1(2, 2, n)$, $\mathcal{W}_2(2, 2, n)$ and $\mathcal{TW}_2(2, 2, n)$, and the four distinct $(3, 0, n)-$wheels by $\mathcal{W}(3, 0, n)$, $\mathcal{TW}_1(3, 0, n)$, $\mathcal{TW}_2(3, 0, n)$ and $\mathcal{TW}_3(3, 0, n)$, as depicted in Figure 2.3. Then, for $\theta = 5, 6$, Theorem 2.2.2 can be restated individually as follows.

Corollary 2.1.2. *There exists a function $f_{2.1.2} : \mathbb{N} \to \mathbb{N}$ such that, for all $n \in \mathbb{N}$ with $n \geq 3$, if G is a graph containing a $5-$connected set of size at least $f_{2.1.2}(n)$, then G has a minor isomorphic to $K_{5,n}$, $\mathcal{W}(1, 3, n)$, $\mathcal{W}(2, 1, n)$ or $\mathcal{TW}(2, 1, n)$.*

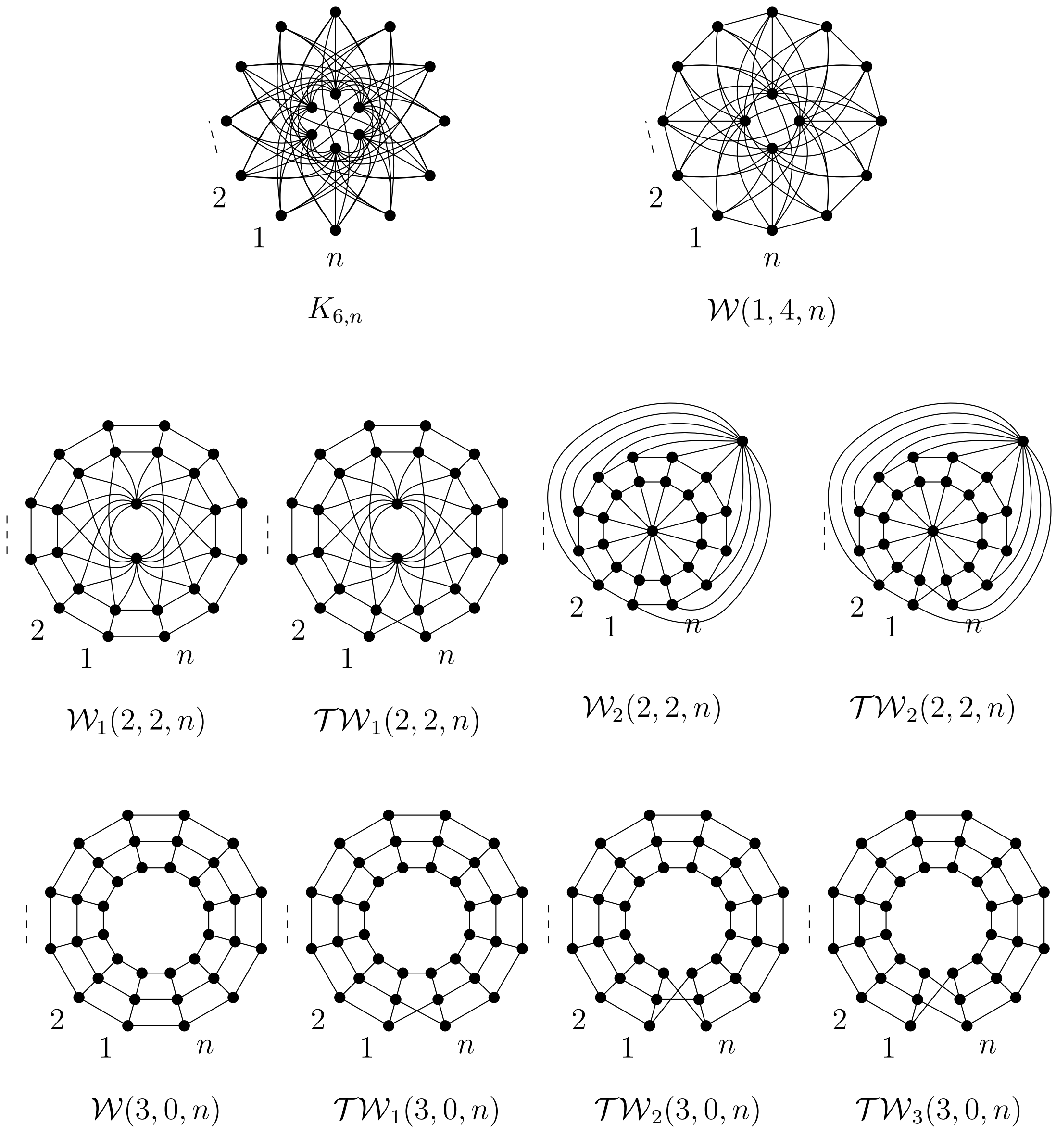

Figure 2.3: Unavoidable minors of graphs with large 6−connected sets.

Corollary 2.1.3. *There exists a function $f_{2.1.3} : \mathbb{N} \to \mathbb{N}$ such that, for all $n \in \mathbb{N}$ with $n \geq 3$, if G is a graph containing a $6-$connected set of size at least $f_{2.1.3}(n)$, then G has a minor isomorphic to $K_{6,n}$, $\mathcal{W}(1,4,n)$, $\mathcal{W}_1(2,2,n)$, $\mathcal{TW}_1(2,2,n)$, $\mathcal{W}_2(2,2,n)$, $\mathcal{TW}_2(2,2,n)$, $\mathcal{W}(3,0,n)$, $\mathcal{TW}_1(3,0,n)$, $\mathcal{TW}_2(3,0,n)$ or $\mathcal{TW}_3(3,0,n)$.*

The latter directly accounts for the case when the said sufficiently large $5-$connected graph has a large $6-$connected set and, as the following corollary shows, gives us our first batch of the unavoidable minors of sufficiently large $5-$connected graphs listed in Theorem 1.2.1 : $\{ K_{5,n}, W(1,3,n), W(2,2,n), TW(2,2,n), W(3,0,n), TW_1(3,0,n), TW_2(3,0,n), TW_3(3,0,n) \}$ (see Figure 1.3). Explicit graph constructions for $W(2,2,n)$, $TW(2,2,n)$, $W(3,0,n)$ and $TW_i(3,0,n)$, for each $i \in \{1,2,3\}$, are given in the appendix (see A.1).

Corollary 2.1.4. *For all $n \in \mathbb{N}$ with $n \geq 5$, if G is a $5-$connected graph that has a $6-$connected set of size at least $f_{2.1.3}(4n+4)$, then G has a minor isomorphic to $K_{5,n}$, $W(1,3,n)$, $W(2,2,n)$, $TW(2,2,n)$, $W(3,0,n)$ or $TW_i(3,0,n)$, where $i \in \{1,2,3\}$.*

Proof. Let G be a $5-$ connected graph that has a $6-$connected set of size at least $f_{2.1.3}(4n+4)$. The proof then follows from Corollary 2.1.3 and the observations that $W(1,3,n)$ is a $5-$connected minor of each of $\mathcal{W}(1,3,n)$, $\mathcal{W}(1,4,n)$, $\mathcal{W}_1(2,2,n)$ and $\mathcal{TW}_1(2,2,n)$, and $K_{5,n}$, $W(2,2,n)$, $TW(2,2,n)$, $W(3,0,n)$, $TW_1(3,0,n)$, $TW_2(3,0,n)$ and $TW_3(3,0,n)$ are, respectively, $5-$connected minors of $K_{6,n}$, $\mathcal{W}_2(2,2,2n+2)$, $\mathcal{TW}_2(2,2,2n+2)$, $\mathcal{W}(3,0,4n+4)$, $\mathcal{TW}_1(3,0,4n+3)$, $\mathcal{TW}_2(3,0,4n+2)$ and $\mathcal{TW}_3(3,0,4n+2)$. □

2.2 Large $\theta-$Connected Sets as Obstructions to Tree-decomposition

In this section, we review another general result obtained by Geelen and Joeris in [GJ16], which establishes an important property about the structure of graphs without a large $\theta-$connected set. For the sake of completeness, we will revisit a few definitions in the present context before we discuss the relevant result.

Earlier, we defined a *separation* in a graph G as a pair (G_1, G_2) of subgraphs of G such that $G_1 \cup G_2 = G$ and $E(G_1 \cap G_2) = \emptyset$. Observe that if G does not contain any isolated vertices, a separation in G can be also defined as a bipartition (A, B) of $E(G)$. The *order* of such a separation (denoted $\lambda(A)$ or $\lambda(B)$) is defined as the number of vertices v in G that are incident with both an edge in A and an edge in B (the set U of such vertices v

is called the *separating set* of (A, B); we also say that U $\lambda(A)-$*separates* $V(A) - U$ from $V(B) - U$ and vice-versa). For any subset F of $E(G)$, let $V(F)$ denote the set of all vertices v in G such that v is an end of an edge in F. Then $\lambda(A) = \lambda(B) = |V(A) \cap V(B)|$.

For every positive integer θ, a separation of order at most θ is called a $\theta-$*separation*, and a graph G is $\theta-$*connected* if, for every $(\theta-1)-$separation (A, B) in G, either $V(A) = V(G)$ or $V(B) = V(G)$.

A *tree-decomposition* of a graph G is a tree T such that the set of edges of G forms a subset of the set of leaves of T. For each vertex $v \in V(G)$, we define a subtree T_v of T as the minimum subtree containing the set of leaves in T that correspond to the edges in G incident with v. Each node $t \in V(T)$, then, corresponds naturally to a set of vertices in G: the vertices $v \in V(G)$ for which $t \in V(T_v)$. We call this set of vertices the *node-bag* of t and denote it by $V_G(T, t)$. Similarly, each edge f in T corresponds to the set of vertices $v \in V(G)$ for which T_v contains f. We call this set of vertices the *edge-bag* of f. Note that f also corresponds to the separation in G given by $(A_f^{(1)},\ A_f^{(2)})$, the bipartition of $E(G)$ induced by the leaves of the components $T^{(1)}$ and $T^{(2)}$ of $T\backslash f$. The order of this separation equals the size of the edge-bag of f.

The *node-width* of a tree-decomposition T is the size of the largest node-bag of a node in T. The *tree-width* of a graph G, denoted $tw(G)$, is the minimum node-width of a tree-decomposition of G *minus* 1. The *edge-width* of a tree-decomposition T is the size of the largest edge-bag of an edge in T. The *degree* of a tree-decomposition T is the largest degree of a node in T.

Robertson, Seymour and Thomas first observed in [RST94] that the existence of a large highly-connected set of vertices in a graph forces a large tree-width. This connection was later refined by Diestel, Jensen, Gorbonov and Thomassen ([DJGT99]) who proved, for each graph G and each $\theta \in \mathbb{N}$, that

(i) if G contains a $(\theta+1)-$connected set of size at least 3θ, then G has tree-width at least θ, and

(ii) conversely, if G has no $(\theta+1)-$connected set of size at least 3θ, then G has tree-width less than 4θ.

In [GJ16], Geelen and Joeris define a refinement of tree-width and relate it similarly to the existence of a large highly-connected set in the graph.

For each $\theta \in \mathbb{N}$, a $\theta-$*tree-decomposition* of a graph G is a tree-decomposition of G that has edge-width at most θ; the $\theta-$*tree-width* of G, denoted $tw_\theta(G)$, is the minimum node-width of a $\theta-$tree-decomposition of G minus 1. Geelen and Joeris prove the following theorem in [GJ16].

Theorem 2.2.1. *For each integer $\theta \geq 3$, if U is a maximum cardinality $(\theta+1)-$connected set in a graph G, then*

$$tw_\theta(G) < |U| \leq \binom{tw_\theta(G)}{\theta}\theta.$$

We combine Theorem 2.2.1 with the following theorem by Joeris ([Joe15]) to bound the minimum degree of a $\theta-$tree-decomposition of a graph G (also called the $\theta-$*branch-degree* of G, denoted $bd_\theta(G)$) which does not contain a large $(\theta+1)-$connected set. Corollary 2.2.3 states this bound explicitly.

Theorem 2.2.2. *For each $\theta \in \mathbb{N}$, if G is a graph with $bd_\theta(G) \geq 3$, then $tw_\theta(G) \leq bd_\theta(G)\theta$ and $bd_\theta(G) \leq \binom{tw_\theta(G)}{\theta}$.*

Corollary 2.2.3. *There exists a function $f_{2.2.3} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that, if G is a graph that has no $(\theta+1)-$connected set of size n, then $bd_\theta(G) < f_{2.2.3}(\theta, n)$.*

Proof. By Theorems 2.2.1 and 2.2.2, $f_{2.2.3}(\theta, n) = \binom{n}{\theta}$ suffices. □

Chapter 3

Smaller Sides of Separations

In this chapter, we find different sets of unavoidable rooted minors of the "smaller" side of a separation (A, B) in a sufficiently large $5-$connected graph, each with each minor rooted in the separating set of (A, B), that satisfy different prevailing conditions. As explained earlier, pairs of such minors can be patched together with the unavoidable rooted minors of the intersection of the "larger" sides of two non-crossing separations in the graph to get a set of unavoidable minors for the complete graph.

3.1 $2-$Linkages in Graphs

Let (u_1, u_2, v_1, v_2) be an ordered quadruple of distinct vertices in a graph G. A $2-$*linkage defined by* (u_1, u_2, v_1, v_2) (also called a $(u_1, u_2, v_1, v_2)-$*linkage*) in G is a pair of disjoint paths P_1 and P_2 such that P_i connects u_i with v_i for $i = 1, 2$.

Seymour ([Sey80]) and Thomassen ([Tho80]) independently gave complete characterization of a graph G that does not contain a $(u_1, u_2, v_1, v_2)-$linkage. Thomassen, in particular, gave an exact structural description of a graph G which contains no $(u_1, u_2, y_1, v_2)-$linkage and is edge-maximal under this restriction. Under another added assumption about G, his result can be stated as the following theorem which he observed as a corollary. The theorem directly follows from a similar result obtained by Jung ([Jun70]).

Theorem 3.1.1. *Let u_1, u_2, v_1 and v_2 be distinct vertices of a graph G. If G has no $(u_1, u_2, v_1, v_2)-$linkage and there does not exist in G a $3-$separation (A, B) with $\{ u_1, u_2, y_1, v_2 \} \subseteq V(A)$ and $|V(B) - V(A)| \geq 2$, then G has a planar embedding with u_1, u_2, v_1 and v_2, in cyclic order, on the boundary of the infinite face.*

An immediate corollary of this theorem that will be useful in our proof is as follows.

Corollary 3.1.2. *Let x_1, x_2, x_3, x_4 and x_5 be distinct vertices of a graph G with $|V(G)| \geq 6$. If G has no $(x_1, x_2, x_4, x_5)-$linkage and there does not exist in G a $4-$separation (A, B) with $|\{ x_1, x_2, x_3, x_4, x_5 \} - V(A)| + \lambda(A) \leq 4$, then G has a planar embedding with x_1, x_2, x_4 and x_5, in cyclic order, on the boundary of the infinite face.*

Proof. It suffices to observe that there does not exist a $3-$separation (A', B') with $\{ x_1, x_2, x_4, x_5 \} \subseteq V(A')$ and $|V(B') - V(A')| \geq 2$. □

3.2 Non-crossing Separations

Two separations (A, B) and (C, D) in a graph G are *distinct* if $A \neq C$ and $A \neq D$. They *cross* if $A \cap C \neq \emptyset$, $A \cap D \neq \emptyset$, $B \cap C \neq \emptyset$ and $B \cap D \neq \emptyset$. The separations are *non-crossing* if either $A \subseteq C$ and $D \subseteq B$, or $C \subseteq A$ and $B \subseteq D$.

Given a pair $((A, B), (C, D))$ of distinct non-crossing separations in G such that $A \subseteq C$ and $D \subseteq B$, and $\lambda(A) \geq \lambda(C)$, it is possible to *slide* from (A, B) to (C, D) if

(a) there exists an edge $e = uv$ in $C - A$, where $V(C) = V(A) \cup \{ v \}$ and $V(B) = V(D) \cup \{ u \}$, such that $C = A \cup \{ e \}$ (*single-step slide*), or

(b) for some $r \in \mathbb{N}$, where $r \geq 2$, there exists a sequence $(X_0, Y_0), ..., (X_r, Y_r)$ of distinct non-crossing separations in G, where $(X_0, Y_0) = (A, B)$ and $(X_r, Y_r) = (C, D)$, such that, for each $i \in \{ 1, ..., r \}$, $X_{i-1} \subseteq X_i$ and $Y_i \subseteq Y_{i-1}$, $\lambda(X_{i-1}) \geq \lambda(X_i)$, and it is possible to single-step-slide from (X_{i-1}, Y_{i-1}) to (X_i, Y_i) (*multi-step slide*).

When that is true, it is easy to show (by induction, if needed) that there exist $\lambda(C)$ pairwise disjoint paths $P_1, ..., P_{\lambda(C)}$ in $G(V(B), B - D)$ such that each path meets $V(A) \cap V(B)$ in one end and $V(C) \cap V(D)$ in the other. If, additionally, $\lambda(C) = \lambda(A)$, then $V(B) = (\bigcup_{i=1}^{5} V(P_i)) \cup (V(D) - V(C))$.

Proposition 3.2.1. *If (A, B) is a separation in a graph G with $|V(B) - V(A)| \geq 2$, then there exists a separation (C, D) in G, non-crossing with (A, B) and of order at most $\lambda(A)$, such that $A \subseteq C$ and $D \subseteq B$, it is possible to slide from (A, B) to (C, D), and either $V(D) \subseteq V(C)$ or, for each $v \in V(C) \cap V(D)$, $|N_{G(V(D),D)}(v) - V(C)| \geq 2$.*

Proof. Let (A, B) be a separation in a graph G such that $|V(B) - V(A)| \geq 2$. If, for each $u \in V(A) \cap V(B)$, $|N_{G(V(B),B)}(u) - V(A)| \geq 2$, then $C = A$ and $D = B$, and we are done. So we may assume that there exist $u \in V(A) \cap V(B)$ and $v \in V(B) - V(A)$ such that $N_{G(V(B),B)}(u) - V(A) = \{ v \}$. It is possible, then, to slide from (A, B) to $(A \cup \{ uv \}, B - \{ uv \})$ which has order at most $\lambda(A)$. Let (A', B') be a separation in G, non-crossing with (A, B), for which $A \subseteq A'$, $B' \subseteq B$ and $\lambda(A) \geq \lambda(A')$, such that it is possible to slide from (A, B) to (A', B'), and such that $|V(B') - V(A')|$ is minimal and, subject to that, $|B - B'|$ is minimal. Then, either $|V(B') - V(A')| = 0$ and $V(B') \subseteq V(A')$, or $|V(B') - V(A')| \geq 2$ and, for each $u' \in V(A') \cap V(B')$, $|N_{G(V(B'),B')}(u') - V(A')| \geq 2$. Thus, $C = A'$ and $D = B'$. □

3.3 Rooted Minors of Small Sides of Separations

The goal of this section is to find different sets of unavoidable rooted minors of one side of a 5−separation (A, B) in a 5−connected graph G. Each minor thus found is rooted in the separating set $\{ x_1, x_2, x_3, x_4, x_5 \}$ of (A, B), and different sets of unavoidable minors satisfy different additional requirements. We give labeled graph descriptions of these minors below in order to be able to match the roots in a minor to the roots in the graph directly.

Recall that $\mathcal{W}(1,1,5)$ (see A.1) is the unique $(1,1,5)$−wheel with $V(\mathcal{W}(1,1,5)) = \{ v_1, v_2, v_3, v_4, v_5, v_h \}, v_h$ being the lone hub-vertex and $v_1, ..., v_5$ being the vertices of the 5−cycle $\mathcal{W}(1,1,5) \backslash \{ v_h \}$ in that order; let $\mathcal{WW}(1,1,5)$ be the graph obtained from $\mathcal{W}(1,1,5)$ by subdividing the edge v_3v_4 with an additional vertex v_6, adding the edge v_hv_6, and splitting the vertex v_h into adjacent vertices v_{h_1}, v_{h_2} such that $N_{\mathcal{WW}(1,1,5)}(v_{h_2}) = \{ v_1, v_2, v_3, v_6, v_{h_1} \}$ and $N_{\mathcal{WW}(1,1,5)}(v_{h_1}) = \{ v_1, v_4, v_5, v_6, v_{h_2} \}$. For $i \in \{ 1, ..., 10 \}$, we define the graph G_i as follows (see Figure 3.1):

(a) $G_1 := (X, \{ x_1x_4, x_2x_4, x_2x_5 \})$;

(b) $G_2 := G_1 \cup \{ x_3x_4 \}$;

(c) $G_3 := G_1 \cup \{ x_4x_5 \}$;

(d) $G_4 := (X \cup \{ v \}, \{ vx_1, vx_2, vx_3, vx_4, vx_5, x_2x_4 \})$;

(e) $G_5 := G_4 \cup \{ x_3x_4 \}$;

(f) $G_6 := G_4 \cup \{ x_4x_5 \}$;

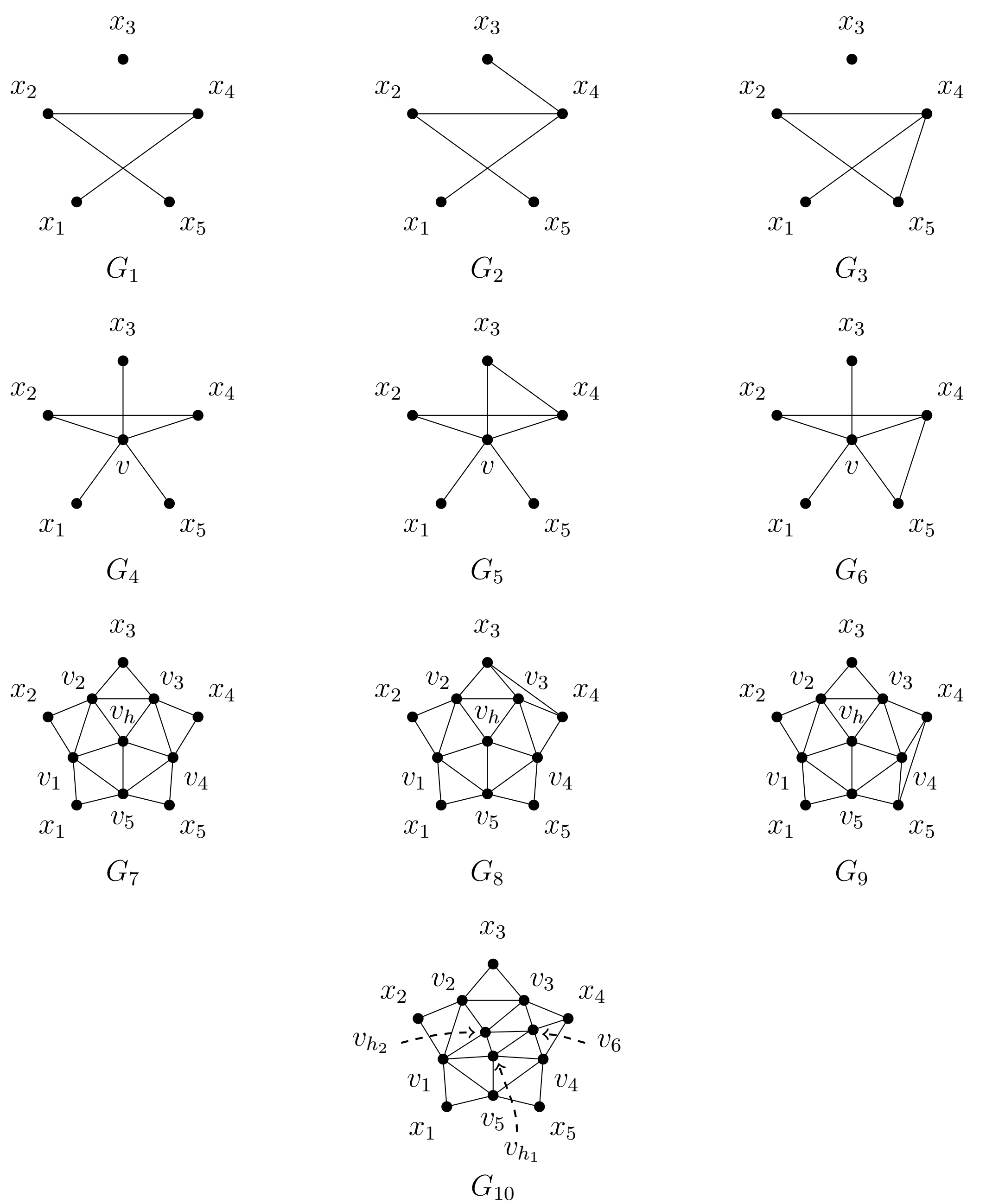

Figure 3.1: Rooted minors of one side of a $5-$separation in a $5-$connected graph.

(g) $G_7 := (V(\mathcal{W}(1,1,5)) \cup X,\ E(\mathcal{W}(1,1,5)) \cup \{\, v_{p+1}x_{q+1} : q = p; p, q \in \mathbb{Z}_5 \,\} \cup \{\, v_{p+1}x_{q+1} : q = p+1; p, q \in \mathbb{Z}_5 \,\})$;

(h) $G_8 := G_7 \cup \{\, x_3x_4 \,\}$;

(i) $G_9 := G_7 \cup \{\, x_4x_5 \,\}$;

(j) $G_{10} := (V(\mathcal{WW}(1,1,5)) \cup X,\ E(\mathcal{WW}(1,1,5)) \cup \{\, v_{p+1}x_{q+1} : q = p; p, q \in \mathbb{Z}_5 \,\} \cup \{\, v_{p+1}x_{q+1} : q = p+1; p, q \in \mathbb{Z}_5 \,\} \cup \{\, v_6x_4 \,\})$;

Additionally, $G_{8+(12)} := G_8 \cup \{\, x_1x_2 \,\}$; $G_{8+(15)} := G_8 \cup \{\, x_1x_5 \,\}$; $G_{9+(12)} := G_9 \cup \{\, x_1x_2 \,\}$; $G_{9+(15)} := G_9 \cup \{\, x_1x_5 \,\}$ (see Figure 3.2). For each $G \in \{\, G_1, ..., G_{10}, G_{8+(12)}, G_{8+(15)}, G_{9+(12)}, G_{9+(15)} \,\}$, we denote by $G^{(j_1k_1)...(j_sk_s)}$, where $j_{s'}, k_{s'} \in \{\, 1, ..., 5 \,\}$ and $j_{s'} \neq k_{s'}$ for each $s' \in \{\, 1, ..., s \,\}$ $(s \in \mathbb{N})$, the graph obtained from G by switching, in order, the vertex labels given by the pairs $(x_{j_1}, x_{k_1}), ..., (x_{j_s}, x_{k_s})$. Using this notation, we define the graphs $G_7^{(24)}$, $G_8^{(13)(24)(25)}$, $G_7^{(24)(25)}$, $G_8^{(13)(24)}$, $G_8^{(24)}$, $G_{8+(15)}^{(24)}$, $G_8^{(24)(25)}$, $G_{8+(15)}^{(24)(25)}$, $G_9^{(24)}$ and $G_9^{(24)(25)}$ as shown in Figure 3.2. Finally, for each graph G described above, we denote by $G(z)$ the graph G with the vertex label x_i replaced by z_i for each $i \in \{\, 1, ..., 5 \,\}$; thus, $G(x) = G$.

In the propositions and lemmas that follow we identify different subsets of the graphs described above as sets of unavoidable rooted minors of a side of (A, B) under different prevailing assumptions. One of these assumptions is a choice between the two possible separations (C, D) identified in Proposition 3.2.1 that one must be able to slide to from (A, B), while another considers the possibility of the side of (A, B) we're looking at being planar. Propositions 3.3.1 and 3.3.2 identify one such set each when it is possible to "slide off" the graph starting from (A, B) (the case when $V(D) \subseteq V(C)$); Corollary 3.3.3 explains why the planarity condition does not play any role in this case. Lemmas 3.3.4, 3.3.5 and 3.3.6 then deal with the specific case when one cannot slide off the graph and the side of (A, B) we're looking at is planar, coupled with different degree requirements at one or more of the root vertices. Finally, Lemmas 3.3.7 and 3.3.8 treat the case when it is not possible to slide off the graph without the planarity assumption and put the no-slide-off case together in slightly differing details.

Proposition 3.3.1. *If (A, B) is a $5-$separation in a $5-$connected graph G with $V(A) \cap V(B) = \{\, x_1, x_2, x_3, x_4, x_5 \,\}$, $|V(A) - V(B)| \geq 1$ and $|V(B) - V(A)| \geq 4$, and there does not exist a separation (C, D) in G, non-crossing with (A, B) and of order at most $\lambda(A)$, such that*

(i) $A \subseteq C$ and $D \subseteq B$,

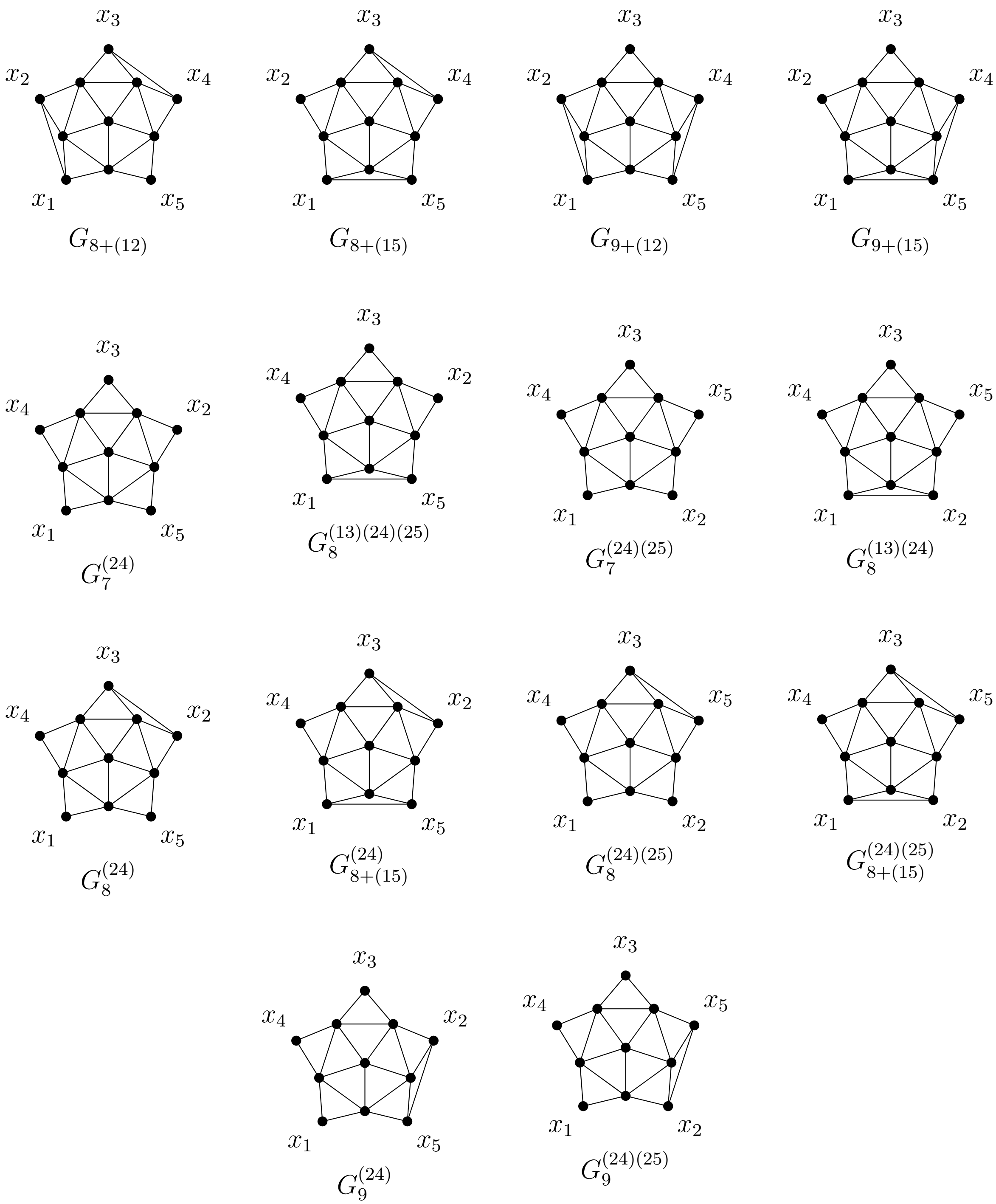

Figure 3.2: Rooted minors of one side of a 5−separation in a 5−connected graph (contd.).

(ii) it is possible to slide from (A, B) to (C, D), and

(iii) for each $u \in V(C) \cap V(D)$, $|N_{G(V(D),D)}(u) - V(C)| \geq 2$,

then $G(V(B), B)$ contains two disjoint connected subgraphs H_1 and H_2 that span, respectively, either $\{ x_2, x_4 \}$ and $\{ x_1, x_3, x_5 \}$, or $\{ x_4, x_5 \}$ and $\{ x_1, x_2, x_3 \}$.

Proof. Let (A, B) be a $5-$separation in a $5-$connected graph G with $V(A) \cap V(B) = \{ x_1, x_2, x_3, x_4, x_5 \}$, $|V(A) - V(B)| \geq 1$ and $|V(B) - V(A)| \geq 2$, and let it be the case that there does not exist a separation (C, D) in G, non-crossing with (A, B) and of order at most $\lambda(A)$, that has properties $(i) - (iii)$ described above. Then, by Proposition 3.2.1, there exist separations $(C_1, D_1), (C_2, D_2), (C_3, D_3)$ and (C_4, D_4) in G, each non-crossing with the other three and (A, B) and of order at most $\lambda(A)$, such that $A \subseteq C_4 \subseteq C_3 \subseteq C_2 \subseteq C_1$ and $D_1 \subseteq D_2 \subseteq D_3 \subseteq D_4 \subseteq B$, it is possible to slide from (A, B) to (C_4, D_4) and from (C_i, D_i) to (C_{i-1}, D_{i-1}) for each $i \in \{ 2, 3, 4 \}$, and $|V(D_i) - V(C_i)| = i$ for each $i \in \{ 1, 2, 3, 4 \}$. For each $i \in \{ 1, 2, 3, 4 \}$, since $V(D_i) - V(C_i) \neq \emptyset$, we have that $\lambda(C_i) = 5$. Further, since it is possible to slide from (A, B) to (C_4, D_4) and from (C_i, D_i) to (C_{i-1}, D_{i-1}) for each $i \in \{ 2, 3, 4 \}$, there exist five pairwise disjoint paths P_1, P_2, P_3, P_4, P_5 in $G(V(B), B - D_1)$ connecting the vertex-sets $V(A) \cap V(B), V(C_4) \cap V(D_4), V(C_3) \cap V(D_3), V(C_2) \cap V(D_2)$ and $V(C_1) \cap V(D_1)$.

Let $V(D_1) - V(C_1) = \{ x \}$ and $V(D_2) - V(C_2) = \{ x, y \}$ so that $y \in V(C_1) \cap V(D_1)$. Then x is adjacent to every vertex in $V(C_1) \cap V(D_1)$, and y is adjacent to at least four of the vertices in $V(C_2) \cap V(D_2)$. Let $V(C_2) \cap V(D_2) = \{ y_1, y_2, y_3, y_4, y_5 \}$, and let $Y_1 := \{ y_1, y_2 \}, Y_2 := \{ y_3, y_4, y_5 \}$, so that $\{ y_1x, y_2y \} \subseteq D_2$, and, for each $u \in Y_2, \{ ux, uy \} \subseteq D_2$. Further, let $X := \{ x_1x_2, x_3, x_4, x_5 \}, Y' := \{ y'_1, y'_2, y'_3, y'_4, y'_5 \}$, where $Y' = Y$, so that, for each $i \in \{ 1, ..., 5 \}, P_i$ connects x_i with y'_i. Then, unless $\{ y'_2, y'_4, y'_5 \} = Y_2, G(V(D_2), D_2)$ contains two disjoint connected subgraphs that span, respectively, either $\{ y'_2, y'_4 \}$ and $\{ y'_1, y'_3, y'_5 \}$, or $\{ y'_4, y'_5 \}$ and $\{ y'_1, y'_2, y'_3 \}$, and we are done. So we may assume that $\{ y'_2, y'_4, y'_5 \} = Y_2$. Now, if $D_3 \cap \{ y_1y_2, y_1y'_2, y_1y'_5, y_2y'_2, y_2y'_5, y'_2y'_4, y'_4y'_5 \} \neq \emptyset$, then $G(V(D_3), D_3)$ contains two disjoint connected subgraphs that span, respectively, either $\{ y'_2, y'_4 \}$ and $\{ y'_1, y'_3, y'_5 \}$, or $\{ y'_4, y'_5 \}$ and $\{ y'_1, y'_2, y'_3 \}$, and we are done. So we may assume that $V(C_3) \cap V(D_3) = Y \cup \{ y''_4 \} - \{ y'_4 \}$, where $y''_4y'_4 \in D_3 - D_2 \cap P_4$, and that $\{ y'_4y_1, y'_4y_2 \} \subseteq D_3$. Then $D_4 \cap \{ y_1y_2, y_1y'_2, y_1y'_5, y_2y'_2, y_2y'_5, y'_2y'_4, y'_4y'_5 \} \neq \emptyset$ and $G(V(D_4), D_4)$ contains two disjoint connected subgraphs that span, respectively, either $\{ y'_2, y''_4 \}$ and $\{ y'_1, y'_3, y'_5 \}$, or $\{ y''_4, y'_5 \}$ and $\{ y'_1, y'_2, y'_3 \}$. Consequently, $G(V(B), B)$ contains two disjoint connected subgraphs H_1 and H_2 that span, respectively, either $\{ x_2, x_4 \}$ and $\{ x_1, x_3, x_5 \}$, or $\{ x_4, x_5 \}$ and $\{ x_1, x_2, x_3 \}$. □

Proposition 3.3.2. *If (A, B) is a $5-$separation in a $5-$connected graph G with $V(A) \cap V(B) = \{ x_1, x_2, x_3, x_4, x_5 \}, |V(A) - V(B)| \geq 1$ and $|V(B) - V(A)| \geq 2$, and there does not exist a separation (C, D) in G, non-crossing with (A, B) and of order at most $\lambda(A)$, such that*

(i) $A \subseteq C$ *and* $D \subseteq B$,

(ii) it is possible to slide from (A, B) *to* (C, D), *and*

(iii) for each $u \in V(C) \cap V(D)$, $|N_{G(V(D),D)}(u) - V(C)| \geq 2$,

then $G(V(B), B)$ has a rooted G_1- or G_4-minor. If, additionally, x_4 has degree at least 3 in $G(V(B), B)$, then $G(V(B), B)$ has a rooted G_3-, G_5- or G_6-minor.

Proof. Let (A, B) be a $5-$separation in a $5-$connected graph G with $V(A) \cap V(B) = \{ x_1, x_2, x_3, x_4, x_5 \}, |V(A) - V(B)| \geq 1$ and $|V(B) - V(A)| \geq 2$, and let it be the case that there does not exist a separation (C, D) in G, non-crossing with (A, B) and of order at most $\lambda(A)$, that has properties $(i) - (iii)$ described above. Then, by Proposition 3.2.1, there exist separations (C_1, D_1) and (C_2, D_2) in G, each non-crossing with the other and (A, B) and of order at most $\lambda(A)$, such that $A \subseteq C_2 \subseteq C_1$ and $D_1 \subseteq D_2 \subseteq B$, it is possible to slide from (A, B) to (C_2, D_2) and from (C_2, D_2) to (C_1, D_1), $|V(D_2) - V(C_2)| = 2$ and $|V(D_1) - V(C_1)| = 1$, and, subject to that, $|C_2 - A|$ and $|C_1 - C_2|$ are both minimal. Clearly, it is possible to slide from (C_2, D_2) to (C_1, D_1) in a single step. Since, for each $i \in \{ 1, 2 \}$, $V(D_i) - V(C_i) \neq \emptyset$, we have that $\lambda(C_2) = \lambda(C_1) = 5$, and, since it is possible to slide from (A, B) to (C_2, D_2), that there exist five pairwise disjoint paths P_1, P_2, P_3, P_4, P_5 in $G(V(B), B - D_2)$, each meeting $V(A) \cap V(B)$ in one end and $V(C_2) \cap V(D_2)$ in the other, such that $V(B) = (\bigcup_{i=1}^{5} V(P_i)) \cup (V(D_2) - V(C_2))$. Let $V(D_1) - V(C_1) = \{ x \}$ and $V(D_2) - V(C_2) = \{ x, y \}$. Then, $y \in V(C_1) \cap V(D_1)$, x is adjacent to every vertex in $V(C_1) \cap V(D_1)$, and y is adjacent to at least four of the vertices in $V(C_2) \cap V(D_2)$.

Let $V(C_2) \cap V(D_2) = \{ y_1, y_2, y_3, y_4, y_5 \}$, and let $Y_1 := \{ y_1, y_2 \}, Y_2 := \{ y_3, y_4, y_5 \}$, so that $\{ y_1 x, y_2 y \} \subseteq D_2$, and, for each $u \in Y_2, \{ ux, uy \} \subseteq D_2$. Also, let $X := \{ x_1 x_2, x_3, x_4, x_5 \}, Z := \{ z_1, z_2, z_3, z_4, z_5 \}$, where $Z = Y$, so that, for each $i \in \{ 1, ..., 5 \}, P_i$ connects x_i with z_i. Contract all edges in $\bigcup_{i=1}^{5} E(P_i)$ to identify x_i with z_i, for each $i \in \{ 1, ..., 5 \}$, and, thus, reduce $G(V(B), B)$ to $G(V(D_2), D_2 \cup F)$, where $V(F) \subseteq Y$. In the case when x_4 has degree at least 3 in $G(V(B), B)$, we have that $|F| \geq 1$ with $z_4 \in V(F)$.

Case 1: $z_4 \in Y_1$. Then, if $z_2 \in Y_1$, contract the edges y_1x, y_2y to reduce $G(V(D_2), D_2)$ to the graph $G_1 \cup \{ x_1x_2, x_2x_3, x_3x_4, x_4x_5 \}$, a supergraph of both G_2, G_3; otherwise if $z_2 \in Y_2$, contract the only edge $e \in D_2$ incident with z_4 to reduce $G(V(D_2), D_2)$ to a supergraph of one of G_5, G_6 with $v = \{ x, y \} - V(\{ e \})$.

Case 2: $z_4 \in Y_2$ and $z_2 \in Y_1$. Then, if $z_5 \in Y_1$ and $F \neq \{ z_2z_4 \}$, contract the only edge $e \in D_2$ incident with z_2 to reduce $G(V(D_2), D_2 \cup F)$ to a supergraph G' of G_4 with $v = \{ x, y \} - V(\{ e \})$. In the case when x_4 has degree at least 3 in $G(V(B), B)$, either $\{ z_3z_4, z_4z_5 \} \cap F \neq \emptyset$ so that G' is also a supergraph of one of G_5, G_6, or $F = \{ z_1z_4 \}$ so that G' can be made a supergraph of G_3 by contracting vz_5. If $z_5 \in Y_1$ and $F = \{ z_2z_4 \}$, contract the only edge $f \in D_2$ incident with z_5 to reduce $G(V(D_2), D_2 \cup F)$ to a supergraph of G_6. If, on the other hand, $z_5 \in Y_2$, contract the only edge $e \in D_2$ incident with z_2 and the edge $uz_4 \in D_2$, where $u = \{ x, y \} - V(\{ e \})$, to reduce $G(V(D_2), D_2)$ to a supergraph of G_3 (contracting e alone reduces $G(V(D_2), D_2)$ to $G_4 \cup \{ x_2x_5, e' \}$, where $e' \in \{ x_2x_1, x_2x_3 \}$).

Case 3: $z_4 \in Y_2$ and $z_2 \in Y_2$. Then, if $z_5 \in Y_1$, contract the only edge $f \in D_2$ incident with z_5 (and the edge $uz_4 \in D_2$, where $u = \{ x, y \} - V(\{ f \})$, to reduce $G(V(D_2), D_2)$ to a supergraph of G_3, otherwise if $z_5 \in Y_2$, contract the edges $uz_4, u'z_2 \in D_2$, such that $uz_1 \in D_2$ is the only edge incident with z_1 and $u' = \{ x, y \} - \{ u \}$, to reduce $G(V(D_2), D_2)$ to a supergraph of G_3.

□

Corollary 3.3.3. *If (A, B) is a $5-$separation in a $5-$connected graph G with $V(A) \cap V(B) = \{ x_1, x_2, x_3, x_4, x_5 \}, |V(A) - V(B)| \geq 1$ and $|V(B) - V(A)| \geq 2$, such that $G(V(B), B)$ has a planar embedding with x_1, x_2, x_3, x_4, x_5, in cyclic order, on the boundary of the infinite face, then there does not exists a separation (C, D) in G, non-crossing with (A, B) and of order at most $\lambda(A)$, such that $A \subseteq C, D \subseteq B$, it is possible to slide from (A, B) to (C, D), and such that $V(D) \subseteq V(C)$.*

Proof. Suppose there does exist a separation (C, D) in addition to the separation (A, B) in a $5-$ connected graph G as described above. Then, by Proposition 3.3.2, $G(V(B), B)$ has a rooted G_1- or G_4-minor, a contradiction to the hypothesis that $G(V(B), B)$ has a planar embedding with x_1, x_2, x_3, x_4, x_5, in cyclic order, on the boundary of the infinite face. □

Remark: In subsequent proofs, for any subset U of vertices of a graph G, we denote by $E(U)$ the set of all edges f in G such that f has both ends in U.

Lemma 3.3.4. *If (A,B) is a $5-$separation in a $5-$connected graph G with $V(A)\cap V(B) = \{ x_1, x_2, x_3, x_4, x_5 \}, |V(A)-V(B)| \geq 1$ and $|V(B)-V(A)| \geq 2$, such that $G(V(B),B)$ has a planar embedding with x_1, x_2, x_3, x_4, x_5, in cyclic order, on the boundary of the infinite face, then $G(V(B),B)$ has a rooted G_7-minor.*

Proof. Let (A,B) be a $5-$separation in a $5-$connected graph G with $V(A) \cap V(B) = \{ x_1, x_2, x_3, x_4, x_5 \}, |V(A)-V(B)| \geq 1$ and $|V(B)-V(A)| \geq 2$, such that $G(V(B),B)$ has a planar embedding with x_1, x_2, x_3, x_4, x_5, in cyclic order, on the boundary of the infinite face. By Corollary 3.3.3 and Proposition 3.2.1, there exists a $5-$separation (C,D) in G, non-crossing with (A,B) such that $A \subseteq C, D \subseteq B$, it is possible to slide from (A,B) to (C,D), and such that, for each $u' \in V(C) \cap V(D), |N_{G(V(D),D)}(u') - V(C)| \geq 2$. Let $Y := \{ y_1, y_2, y_3, y_4, y_5 \} = V(C) \cap V(D)$; notice that $G(V(D),D) \backslash Y'$ is connected for every $Y' \subsetneq Y$. There also exist five pairwise disjoint paths P_1, P_2, P_3, P_4, P_5 in $G(V(B), B-D)$ such that, for each $i \in \{ 1, ..., 5 \}, P_i$ connects y_i with x_i and meets $G(V(D),D)$ only in y_i, and such that $V(B) = (\bigcup_{i=1}^{5} V(P_i)) \cup (V(D)-V(C))$. Then, since $G(V(B),B)$ has a planar embedding with x_1, x_2, x_3, x_4, x_5, in cyclic order, on the boundary of the infinite face, we have that

3.3.4.1. *$G(V(D),D)$ has a planar embedding with y_1, y_2, y_3, y_4, y_5, in cyclic order, on the boundary of the infinite face.*

Moreover, $|V(D)-V(C)| > 2$, for otherwise $G(V(D),D)$ contains a copy of $K_{2,3}$ as a subgraph with the larger partition contained in Y and the smaller partition formed by $V(D)-V(C)$, a contradiction to the fact that $K_{2,3}$ does not have a planar embedding with all the vertices in the larger partition on the boundary of the infinite face.

3.3.4.2. *$G(V(D),D) \backslash Y$ is $2-$connected.*

Proof of claim. Suppose that the planar graph $G(V(D),D) \backslash Y$ contains a $1-$separation (C',D') such that $V(C')-V(D') \neq \emptyset, V(D')-V(C') \neq \emptyset$. Then, since G is $5-$connected, $|N_{G(V(D),D)}(V(C')-V(D')) \cap Y| \geq 4$ and $|N_{G(V(D),D)}(V(D')-V(C')) \cap Y| \geq 4$. Let $V_1 := N_{G(V(D),D)}(V(C')-V(D')) \cap Y \cap N_{G(V(D),D)}(V(D')-V(C')), V_2 := \{ c, d \}$, where $c \in V(G(V(C')-V(D'), E(V(C')-V(D')))/E(V(C')-V(D')), d \in V(G(V(D')-V(C'), E(V(D')-V(C')))/E(V(D')-V(C'))$; $|V_1| \geq 3$. Then $G(V(D),D)/(E(V(C')-V(D')) \cup E(V(D')-V(C')))$ contains a copy of $K_{2,3}$ as a subgraph with the larger partition contained in V_1 and the smaller partition formed by V_2, and it has a planar embedding with every vertex in V_1 on the boundary of the infinite face, a contradiction. □

Thus, $G(V(D), D)\backslash Y$ is a planar graph with the infinite face bounded by a cycle S.

3.3.4.3. $|V(S)| \geq 5$.

Proof of claim. Suppose, for some $u' \in V(S)$, $|N_{G(V(D),D)}(u') \cap Y| \geq 3$. Then $G' := G(V(D), D)/(E(V(D) - Y - \{ u' \}))$ contains a copy of $K_{2,3}$ as a subgraph with the larger partition contained in Y and the smaller partition formed by $V(G') - Y$, and it has a planar embedding with y_1, y_2, y_3, y_4, y_5 on the boundary of the infinite face, a contradiction. Thus, for each $u' \in V(S), |N_{G(V(D),D)}(u') \cap Y| \leq 2$. The claim then follows from the fact that, for each $i \in \{ 1, ..., 5 \}, |N_{G(V(D),D)}(y_i) - V(C)| \geq 2$ established above. □

Since G is 5−connected, we also have from the proof of 3.3.4.3 that $N_{G(V(D),D)}(u') - Y - V(S) \neq \emptyset$, for each $u' \in V(S)$. If $G(V(D), D)\backslash(Y \cup V(S))$ is connected, then we are done since we can reduce $(G(V(B), B)$ to a supergraph of G_7 by contracting $G(V(D), D)\backslash(Y \cup V(S))$ to a single vertex v_h, S to a 5−cycle with vertices v_1, v_2, v_3, v_4, v_5 such that, for each $p \in \{ 1, .., 5 \}, q \in \{p-1 \pmod 5, p \pmod 5\}, v_p$ is adjacent to y_{q+1}, and each of the paths P_1, P_2, P_3, P_4, P_5 to a single vertex. So we may assume that $G(V(D), D)\backslash(Y \cup V(S))$ is not connected and, hence, that there exists a 2−separation (S_1, S_2) in $G(V(D), D)\backslash Y$ such that $V(S_1) - V(S_2) \neq \emptyset, V(S_2) - V(S_1) \neq \emptyset$, and such that $V(S_1) \cap V(S_2) = \{ s_1, s_2 \}$, where $s_1, s_2 \in V(S)$. Again, since G is 5−connected, we have that $|N_{G(V(D),D)}(V(S_1) - V(S_2)) \cap Y| \geq 3$ and $|N_{G(V(D),D)}(V(S_2) - V(S_1)) \cap Y| \geq 3$, and that at least one of these holds with equality, for otherwise $G' := G(V(D), D)/(E(V(S_1) - V(S_2)) \cup E(V(S_2) - V(S_1)))$ contains a copy of $K_{2,3}$ as a subgraph with the larger partition contained in $V_1 := N_{G(V(D),D)}(V(S_1) - V(S_2)) \cap Y \cap N_{G(V(D),D)}(V(S_2) - V(S_1))$ and the smaller partition contained in $V(G') - Y - \{ s_1, s_2 \}$, and it also has a planar embedding with every vertex in V_1 on the boundary of the infinite face, a contradiction.

Without loss of generality, let $|N_{G(V(D),D)}(V(S_1) - V(S_2)) \cap Y| = 3$ so that $(C', D') := (E(G) - F, F)$ is a 5−separation in G, where $F := \bigcup_{u' \in V(S_1) - V(S_2)} \delta_{G(V(D),D)}(u')$. Let $Y' := \{ y_1', y_2', y_3', y_4', y_5' \} = V(C') \cap V(D')$ so that $|Y' \cap Y| = 3$ and $Y' - Y = \{ s_1, s_2 \}$; notice that $G(V(D'), D')\backslash Y''$ is connected for every $Y'' \subsetneq Y'$. Further notice that there exist two disjoint paths between $Y' - Y$ and $Y - Y'$ in $G(V(D) - (V(S_1) - V(S_2)), D - D')$ which meet $G(V(D'), D')$ only in $Y' - Y$. That, together with 3.3.4.1, gives us that

3.3.4.4. *$G(V(D'), D')$ has a planar embedding with $y_1', y_2', y_3', y_4', y_5'$, in cyclic order, on the boundary of the infinite face, where, for each $i \in \{ 1, ...5 \}$, there exists a path Q_i in $G(V(D), D)$ (possibly of zero length) connecting y_i' with y_i which meets $G(V(D'), D')$ only in y_i' and is disjoint with the path Q_j, for each $j \in \{ 1, ..., 5 \}, j \neq i$.*

Also, since $|N_{G(V(D),D)}(V(S_1) - V(S_2)) \cap Y| = 3$ and, for each $u' \in V(S), |N_{G(V(D),D)}(u') \cap Y| \leq 2$, we have that $|V(D') - V(C')| \geq 2$. Then, by Corollary 3.3.3 and Proposition 3.2.1, there exists a 5−separation (C'', D'') in G, non-crossing with $(C'D')$, such that $C' \subseteq C'', D'' \subseteq D'$ it is possible to slide from (C', D') to (C'', D''), and such that, for each $u' \in V(C'') \cap V(D''), |N_{G(V(D''),D'')}(u') - V(C'')| \geq 2$. Let $Y'' := \{ y''_1, y''_2, y''_3, y''_4, y''_5 \} = V(C'') \cap V(D'')$; notice that $G(V(D''), D'') \backslash Y'''$ is connected for every $Y''' \subsetneq Y''$. There also exist five pairwise disjoint paths $Q'_1, Q'_2, Q'_3, Q'_4, Q'_5$ in $G(V(D'), D' - D'')$ such that, for each $i \in \{ 1, ..., 5 \}, Q'_i$ connects y''_i with y'_i and meets $G(V(D''), D'')$ only in y''_i. Notice that the path $Q''_i := Q'_i \cup Q_i$ connects y''_i with y_i, for each $i \in \{ 1, ..., 5 \}$, and is disjoint with the path Q''_j, for each $j \in \{ 1, ..., 5 \}, j \neq i$. As before, that, together with 3.3.4.4, gives us that

3.3.4.5. *$G(V(D''), D'')$ has a planar embedding with $y''_1, y''_2, y''_3, y''_4, y''_5$ on the boundary of the infinite face.*

Consider such a separation (C'', D'') in G with $|D''|$ minimal. Then we have, as we did with (C, D), that $|V(D'') - V(C'')| > 2$, that $G(V(D''), D'') \backslash Y''$ is 2−connected, that $G(V(D''), D'') \backslash Y''$ is a planar graph with the infinite face bounded by a cycle S'' such that $|V(S'')| \geq 5$, and that, for each $u' \in V(S''), N_{G(V(D''),D'')}(u') - Y'' - V(S'') \neq \emptyset$. Suppose, now, that $G(V(D''), D'') \backslash (Y'' \cup V(S''))$ is not connected so that there exists a 2−separation (S''_1, S''_2) in $G(V(D''), D'') \backslash Y''$ such that $V(S''_1) - V(S''_2) \neq \emptyset, V(S''_2) - V(S''_1) \neq \emptyset$, and such that $V(S''_1) \cap V(S''_2) \subseteq V(S'')$; let $X'_1 := \{ s''_1, s''_2 \} = V(S''_1) \cap V(S''_2), X_2 := (N_{G(V(D''),D'')}(V(S''_1) - V(S''_2)) \cap Y'')$. Then, (assuming, without loss of generality, that $|X'_2| = 3$,) $(A', B') := (E(G) - H, H)$, where $H := \bigcup_{u' \in V(S''_1) - V(S''_2)} \delta_{G(V(D''),D'')}(u')$, is a 5−separation in G, with $V(A') \cap V(B') = X'_1 \cup X'_2$, such that $|V(B') - V(A')| \geq 2$ and $G(V(B'), B')$ has a planar embedding with $x'_1, x'_2, x'_3, x'_4, x'_5$, in cyclic order, on the boundary of the infinite face, where $\{ x'_1, x'_2, x'_3, x'_4, x'_5 \} = X'_1 \cup X'_2$; the last property is due to 3.3.4.5, the two disjoint paths between X'_1 and $Y'' - X'_2$ in $G(V(D'') - (V(S''_1) - V(S''_2)), D'' - B')$ which meet $G(V(B'), B')$ only in X'_1, and $G(V(B'), B') \backslash X''$ being connected for every $X'' \subsetneq X'_1 \cup X'_2$. By Corollary 3.3.3 and Proposition 3.2.1, there exists another 5−separation (A'', B'') in G, non-crossing with (A', B'), such that $A' \subseteq A'', B'' \subseteq B'$, it is possible to slide from (A', B') to (A'', B''), and, for each $u' \in V(A'') \cap V(B''), |N_{G(V(B'',B'')}(u') - V(A'')| \geq 2$; let $X'' := \{ x''_1, x''_2, x''_3, x''_4, x''_5 \} = V(A'') \cap V(B'')$; notice that $G(V(B''), B'') \backslash X'''$ is connected for every $X''' \subsetneq X''$. There also exist five pairwise disjoint paths between X'' and X' in $G(V(B'), B' - B'')$ connecting x''_i with x'_i and, hence, with y''_i (and, ultimately, with y_i), for each $i \in \{ 1, ..., 5 \}$. That, together with 3.3.4.5, gives us that $G(V(B''), B'')$ has a planar embedding with $x''_1, x''_2, x''_3, x''_4, x''_5$, in cyclic order, on the boundary of the infinite face. But $|B''| < |D''|$, a contradiction to the minimality of $|D''|$. Thus,

$G(V(D''), D'')\backslash(Y'' \cup V(S''))$ is connected and, for some $v_h \in V(D'') - Y'' - V(S''), \{ v_1, v_2, v_3, v_4, v_5 \} \subseteq V(S''), G(V(D''), D'')$ has a rooted $G_7(y'')-$minor; let $U'' \subseteq V(D''), F'' \subseteq D''$ be such that $G(V(D''), D'')\backslash U''/F'' \supseteq G'_7$. Then $G(V(B), B)\backslash(U' \cup U'')/(F' \cup F'') \supseteq G_7$, where $U' := V(D) - V(D'') - \bigcup_{i \in \{ 1,...,5 \}} V(Q''_i), F' := \bigcup_{i \in \{ 1,...,5 \}} E(Q''_i \cup P_i)$. □

Lemma 3.3.5. *If (A, B) is a 5−separation in a 5−connected graph G with $V(A) \cap V(B) = \{ x_1, x_2, x_3, x_4, x_5 \}, |V(A) - V(B)| \geq 1, |V(B) - V(A)| \geq 2$ and $|N_{G(V(B),B)}(x_4)| \geq 3$, such that $G(V(B), B)$ has a planar embedding with x_1, x_2, x_3, x_4, x_5, in cyclic order, on the boundary of the infinite face, then $G(V(B), B)$ has a rooted G_8- or G_9-minor.*

Proof. Let (A, B) be a 5−separation in a 5−connected graph G with $V(A) \cap V(B) = \{ x_1, x_2, x_3, x_4, x_5 \}, |V(A) - V(B)| \geq 1, |V(B) - V(A)| \geq 2$ and $|N_{G(V(B),B)}(x_4)| \geq 3$, such that $G(V(B), B)$ has a planar embedding with x_1, x_2, x_3, x_4, x_5, in cyclic order, on the boundary of the infinite face. By Corollary 3.3.3 and Proposition 3.2.1, there exists a 5−separation (C, D) in G, non-crossing with (A, B) such that $A \subseteq C, D \subseteq B$, it is possible to slide from (A, B) to (C, D), and such that, for each $u' \in V(C) \cap V(D), |N_{G(V(D),D)}(u') - V(C)| \geq 2$. Let $Y := \{ y_1, y_2, y_3, y_4, y_5 \} = V(C) \cap V(D)$; notice that $G(V(D), D)\backslash Y'$ is connected for every $Y' \subsetneq Y$. There also exist five pairwise disjoint paths P_1, P_2, P_3, P_4, P_5 in $G(V(B), B - D)$ such that, for each $i \in \{ 1, ..., 5 \}, P_i$ connects y_i with x_i and meets $G(V(D), D)$ only in y_i, and such that $V(B) = (\bigcup_{i=1}^{5} V(P_i)) \cup (V(D) - V(C))$. Then, since $G(V(B), B)$ has a planar embedding with x_1, x_2, x_3, x_4, x_5, in cyclic order, on the boundary of the infinite face, we have that

3.3.5.1. *$G(V(D), D)$ has a planar embedding with y_1, y_2, y_3, y_4, y_5, in cyclic order, on the boundary of the infinite face.*

Contract all edges in $\bigcup_{i=1}^{5} E(P_i)$ to identify x_i with y_i, for each $i \in \{ 1, ..., 5 \}$, and, thus, reduce $G(V(B), B)$ to $G(V(D), D \cup F)$, where $V(F) \subseteq Y$. Notice that, by Lemma 3.3.4, $G(V(D), D)$ already has a rooted $G_7(y)-$minor. Further, since $|N_{G(V(B),B)}(x_4)| \geq 3$, we may assume that $|N_{G(V(D),D)}(y_4) - V(C)| \geq 3$ for otherwise $|N_{G(V(D),D)}(y_4) - V(C)| = 2$ and, hence, $|(D \cup F) \cap \{ y_3y_4, y_4y_5 \}| \geq 1$, and we are done. By the proof of Lemma 3.3.4, we have that $G(V(D), D)\backslash Y$ is a 2−connected planar graph with the infinite face bounded by a cycle S such that $|N_{G(V(D),D)}(u) \cap Y| \leq 2$, for each $u \in V(S)$; since, for each $u \in V(C) \cap V(D), |N_{G(V(D),D)}(u) - V(C)| \geq 2$, and, additionally, $|N_{G(V(D),D)}(y_4) - V(C)| \geq 3$, we have that $|V(S)| \geq 6$. Consider such a separation (C, D) with $|D|$ minimal.

3.3.5.2. *If (A', B') is a $5-$separation in G, non-crossing with (C, D) and with $V(A') \cap V(B') = \{ x'_1, x'_2, x'_3, x'_4, x'_5 \}$, such that*

(i) $C \subseteq A', B' \subseteq D, (A', B') \neq (C, D)$,

(ii) $|V(B') - V(A')| \geq 2$,

(iii) there exist five pairwise disjoint paths $P'_1,\ P'_2,\ P'_3,\ P'_4,\ P'_5$ *in* $G(V(D), D - B')$ *such that, for each* $i \in \{ 1, ..., 5 \}$, P'_i *connects* x'_i *with* y_i *and meets* $G(V(B'), B')$ *only in* x'_i, *and*

(iv) $|N_{G(V(B'),B')}(x'_4)| \geq 3$,

then $G(V(B), B)$ *has a rooted* G_8- *or* G_9- *minor.*

Proof of claim. Let (A', B') be a $5-$separation in G, non-crossing with (C, D) and with $V(A') \cap V(B') = \{ x'_1, x'_2, x'_3, x'_4, x'_5 \}$, such that it has the properties $(i) - (iv)$ described above; let $X' := \{ x'_1, x'_2, x'_3, x'_4, x'_5 \}$. Since $G(V(B'), B') \backslash X''$ is connected for every $X'' \subsetneq X'$, we have, by (iii) and 3.3.5.1, that $G(V(B'), B')$ has a planar embedding with $x'_1, x'_2, x'_3, x'_4, x'_5$ on the boundary of the infinite face. By Corollary 3.3.3 and Proposition 3.2.1, there exists another $5-$separation (A'', B'') in G, non-crossing with (A', B') and with $V(A'') \cap V(B'') = \{ x''_1, x''_2, x''_3, x''_4, x''_5 \}$, such that $A' \subseteq A'', B'' \subseteq B'$, it is possible to slide from (A', B') to (A'', B''), and, for each $u \in V(A'') \cap V(B'')$, $|N_{G(V(B''),B'')}(u) - V(A'')| \geq 2$; let $X'' := \{ x''_1, x''_2, x''_3, x''_4, x''_5 \}$; notice that $G(V(B''), B'') \backslash X'''$ is connected for every $X''' \subsetneq X''$. There also exist five pairwise disjoint paths $P''_1,\ P''_2,\ P''_3,\ P''_4,\ P''_5$ in $G(V(B'), B' - B'')$ such that, for each $i \in \{ 1, ..., 5 \}$, P''_i connects x''_i with x'_i and meets $G(V(B''), B'')$ only in x''_i; notice that the path $P'_i \cup P''_i$ connects x''_i with y_i, for each $i \in \{ 1, ..., 5 \}$. As before, that, together with 3.3.5.1, gives us that $G(V(B''), B'')$ has a planar embedding with $x''_1, x''_2, x''_3, x''_4, x''_5$ on the boundary of the infinite face. Clearly, $|B''| < |D|$ and, hence, $|N_{G(V(B''),B'')}(x''_4) - V(A'')| = 2$, for otherwise (A'', B'') contradicts the minimality of $|D|$. Delete all vertices in $V(D) - \bigcup_{i=1}^{5} V(P'_i \cup P''_i)$ and contract all edges in $\bigcup_{i=1}^{5} E(P'_i \cup P''_i)$ to identify x_i with x''_i, for each $i \in \{ 1, ..., 5 \}$, and, thus, reduce $G(V(D), D \cup F)$ to $G(V(B''), B'' \cup F' \cup F)$, where $V(F) \cup V(F') \subseteq X''$. Then we are done, since, by Lemma 3.3.4, $G(V(B''), B'' \cup F' \cup F)$ already has a rooted $G_7(x'')-$minor, and $|(B'' \cup F' \cup F) \cap \{ x''_3 x''_4, x''_4 x''_5 \}| \geq 1$. □

Without loss of generality, let G_D be a plane graph embedding $G(V(D), D)$ in the plane with y_1, y_2, y_3, y_4, y_5, in clockwise order, on the boundary of the infinite face. For any two vertices $u, u' \in V(S)$, we denote by $S[u, u']$ the set of all vertices $v \in V(S)$ such that v is seen while traversing S from u to u' (both inclusive) in clockwise direction in G_D without repeating any vertices; correspondingly, $S[u, u') := S[u, u'] - \{ u' \}$, $S(u, u'] := S[u, u'] - \{ u \}$, $S(u, u') := S[u, u'] - \{ u, u' \}$. We use identical notation for analogous sets of vertices in a path P in G_D, only, unlike a cycle, for any two vertices $u, u' \in V(P)$, $P(u, u') = P(u', u)$. For each $i \in \{ 1, ..., 5 \}$, let y_i^-, y_i^+ denote the vertices in $V(S) \cap N_{G_D}(y_i)$ such that no vertex in $S(y_i^-, y_i^+)$ has a neighbor $y \in Y, y \neq y_i$. Since $|N_{G(V(D),D)}(y_4) - V(C)| \geq 3$, we have that $S(y_4^-, y_4^+) \neq \emptyset$.

Suppose there does not exist a path between $S(y_4^-, y_4^+)$ and $S(y_1^-, y_2^+)$ in $G_D \backslash Y$ that is internally disjoint with S. Let w be a vertex in $S(y_4^-, y_4^+)$. There exists a vertex $a \in S[y_2^+, y_3^+)$ which is connected to w by a path in $G_D \backslash Y$ that is internally disjoint with S. Consider such a vertex a for which $|S[y_2^+, a)|$ is minimal. Similarly, consider a vertex $b \in S(y_5^-, y_1^-]$ which is connected to w by a path in $G_D \backslash Y$ that is internally disjoint with S, and for which $|S(b, y_1^-]|$ is minimal. Notice that $\{ a, b \}$ 2−separates $G_D \backslash Y$ as (S_1, S_2) with $V(S_1) - V(S_2) \neq \emptyset, V(S_2) - V(S_1) \neq \emptyset$ and $V(S_1) \cap V(S_2) = \{ a, b \}$; let $w \in V(S_1)$. In turn, $(C', D') := (E(G) - H, H)$, where $H := \bigcup_{u \in V(S_1) - V(S_2)} \delta_{G(V(D),D)}(u)$, is a 5−separation in G, non-crossing with (C, D) and with $V(C') \cap V(D') = \{a, y_3, y_4, y_5, b\}$, such that $C \subsetneq C', D' \subsetneq D, |N_{G(V(D'),D')}(y_4) - V(C')| \geq 3$ (and, hence, $|V(D') - V(C')| \geq 3$), and such that there exist two disjoint paths in $G(V(D), D - D')$ connecting a with y_2 and b with y_1 that meet $G(V(D'), D')$ only in a and b, respectively, and we are done by 3.3.5.2. So we may assume that there exists at least one such path Q with ends $w \in S(y_4^-, y_4^+), q \in S(y_1^-, y_2^+)$, and one for which both $|S(y_4^-, w]|$ and $|S[q, y_2^+)|$ are minimum. Moreover, we may assume for any such path Q that $\{ w, q \}$ does not 2−separate $G_D \backslash Y$ (proof follows) and, hence, that $|V(Q)| \geq 3$.

3.3.5.3. $\{ w, q \}$ *does not* 2−*separate* $G_D \backslash Y$*.*

Proof of claim. Suppose that $\{ w, q \}$ 2−separates $G_D \backslash Y$ as (S_1, S_2) so that $V(S_1) \cap V(S_2) = \{ w, q \}$. Without loss of generality, let $y_4^- \in V(S_1), y_4^+ \in V(S_2)$. Then, assuming that $q \in S[y_1^+, y_2^+)$ (the case when $q \in S(y_1^-, y_1^+)$ is symmetrically analogous), $(C', D') := (E(G) - H, H)$, where $H := \bigcup_{u \in V(S_1) - V(S_2)} \delta_{G(V(D),D)}(u)$, is a 5−separation in G, non-crossing with (C, D) and with $V(C') \cap V(D') = \{ q, y_2, y_3, y_4, w \}$, such that $C \subseteq C', D' \subseteq D, |V(D') - V(C')| \geq 2$, and such that $G(V(D'), D')$ has a planar embedding with q, y_2, y_3, y_4, w, in cyclic order, on the boundary of the infinite face; the last property is due to 3.3.5.1 combined with the facts that $G(V(D'), D') \backslash Y'$ is connected for every $Y' \subsetneq \{ q, y_2, y_3, y_4, w \}$, and there exist disjoint paths P_w and P_q in $G(V(D), D - D')$ connecting w with y_5 and q with y_1, and meeting $G(V(D'), D')$ only in w and q, respectively, such that $v_4^+ \in V(P_w)$.

By Lemma 3.3.4, $G(V(D), D \cup F)\backslash(V(D) - V(D') - V(P_w) - V(P_q))/(E(P_w) \cup E(P_q))$ has a rooted $G_7(y)-$minor which, together with the edge y_4y_5, gives us a rooted G_9-minor in $G(V(B), B)$. ☐

Now suppose that there exists another path Q' between $S(y_4^-, y_4^+)$ and $S(y_1^-, y_2^+)$ in $G_D\backslash Y$, with ends $w' \in S(y_4^-, y_4^+), q' \in S(y_1^-, y_2^+)$, which is internally disjoint with S, Q, and which lies in the face of G_D bounded by the cycle formed by the vertices $V(Q) \cup S(w, q)$ (the case when it lies in the face of G_D bounded by the cycle formed by the vertices $V(Q) \cup S(q, w)$ is symmetrically analogous). Consider such a path Q' for which both $|S[w', y_4^+)|$ and $|S(y_1^-, q']|$ are minimum. By 3.3.5.3, there exists a path Q_1 between $Q'(w', q')$ and $S[y_4^+, y_1^-]$, with ends $q_1' \in Q'(w', q')$ and $q_1 \in S[y_4^+, y_1^-]$, that is internally disjoint with S, Q'. We may assume that $q_1 \in S(y_5^-, y_1^-]$, for otherwise $q_1 \in S[y_4^+, y_5^-]$ and, unless $S(y_5^-, y_1^-]$ is 4$-$separated from the rest of the graph by $\{ y_5^-, y_5, y_1, q' \}$ $S(y_5^-, y_1^-]$, $q' \in S(y_2^-, y_2^+)$; in this case, $\{ y_1, y_2, q', y_5^-, y_5 \}$ forms the separating set of a 5$-$separation (A', B') in G with $V(A') \cap V(B') = \{ x_1', ..., x_5' \}$ (where $x_1' = y_1, x_2' = y_2, x_3' = q', x_4' = y_5^-, x_5' = y_5$) and $S[y_5^+, y_2^-] \subseteq V(B') - V(A')$, such that $G(V(B'), B')$ has a planar embedding with $y_1, y_2, q', y_5^-, y_5'$, in cyclic order, on the boundary of the infinite face, so that, by Lemma 3.3.4, $G(V(B'), B')$ has a rooted $G_7(x')-$minor which, together with the path between q' and y_3 along S using the edge $y_3^-y_3$, the path between y_5^- and y_4 along S using the edge $y_4^+y_4$ and the path between y_3 and y_4 along S using the edges $y_3y_3^+$ and $y_4^-y_4$, ensures that $G(V(D), D)$ has a rooted $G_8(y)-$minor and we are done. In turn, there exists another path Q_5 between $S[y_4^+, q_1)$ and $Q'(w', q_1'] \cup Q_1[q_1', q_1)$, with ends $q_5' \in Q'(w', q_1'] \cup Q_1[q_1', q_1)$ and $q_5 \in S[y_4^+, q_1)$, that is internally disjoint with S, Q' and Q_1, for otherwise $\{ w', y_4, y_5, q_1 \}$ 4$-$separates $S(w', q_1)$ from the rest of the graph G. Similarly, there exists a path Q_2 between $Q(w, q)$ and $S[y_2^+, y_3^+)$, with ends $q_2' \in Q(w, q)$ and $q_2 \in S[y_2^+, y_3^+)$, that is internally disjoint with S, Q, and a path Q_3 between $S(q_2, y_4^-]$ and $Q(w, q_2'] \cup Q_2[q_2', q_2)$, with ends $q_3' \in Q(w, q_2'] \cup Q_2[q_2', q_2)$ and $q_3 \in S(q_2, y_4^-]$, that is internally disjoint with S, Q and Q_2. By 3.3.5.3, there also exists a path Q'' between $Q(w, q)$ and $Q'(w', q')$ that is internally disjoint with S, Q, Q'. Contract the edges in S to identify $S[q_1, y_1^-]$ into v_5, $S(y_1^-, y_2^+)$ into v_1, $S[y_2^+, q_2]$ into v_2, $S(q_2, y_4^-]$ into v_3, $S(y_4^-, y_4^+)$ into v_6, $S[y_4^+, q_1)$ into v_4, the edges in Q, Q_2, Q_3 to identify $Q(w, q) \cup Q_2(q_2, q_2'] \cup Q_3(q_3, q_3']$ into v_{h_2}, and the edges in Q', Q_1, Q_5 to identify $Q'(w', q') \cup Q_1(q_1, q_1'] \cup Q_5(q_5, q_5']$ into v_{h_1}. Finally, contract all but one edges in Q'' to get a graph that contains G_{10} as a subgraph, and we are done because the latter has a rooted G_8- as well as a rooted G_9-minor (e.g. contract v_4x5 and v_5x_1 to identify the respective vertex-pairs and relabel v_{h_2}, v_6, v_{h_1} as v_h, v_4, v_5, respectively, to get G_9 from G_{10}). So we may assume that there do not exist two internally disjoint paths between $S(y_4^-, y_4^+)$ and $S(y_1^-, y_2^+)$ in $G_D\backslash Y$ that are both internally disjoint with S.

Since S bounds the infinite face of $G_D \backslash Y$, there do not exist four pairwise internally disjoint paths between $S(y_4^-, y_4^+)$ and $S(y_1^-, y_2^+)$ in $G_D \backslash Y$, and, hence, there exists a 3−separation (S_1, S_2) in $G_D \backslash Y$ such that $V(S_1) \cap V(S_2) = \{ z, a, b \}$, where $z \in Q(w, q), a \in S[y_2^+, y_4^-], b \in S[y_4^+, y_1^-], S(y_4^-, y_4^+) \subseteq V(S_1)$, and $S(y_1^-, y_2^+) \subseteq V(S_2)$. It cannot be that $a \in S[y_3^+, y_4^-]$ and $b \in S[y_4^+, y_5^-]$, for otherwise $\{ a, y_4, b, z \}$ 4−separates $S(y_4^-, y_4^+)$ from the rest of the graph G. Suppose $a \in S[y_3^+, y_4^-]$ and $b \in S(y_5^-, y_1^-]$. Then $(C', D') := (E(G) - H, H)$, where $H := \bigcup_{u \in V(S_1) - V(S_2)} \delta_{G(V(D),D)}(u)$, is a 5−separation in G, non-crossing with (C, D) and with $V(C') \cap V(D') = \{a, y_4, y_5, b, z\}$, such that $C \subsetneq C', D' \subsetneq D, |V(D') - V(C')| \geq 2$, and such that $G(V(D'), D')$ has a planar embedding with a, y_4, y_5, b, z, in cyclic order, on the boundary of the infinite face; the last property is due to 3.3.5.1 combined with the facts that $G(V(D'), D') \backslash Y'$ is connected for every $Y' \subsetneq \{ q, y_2, y_3, y_4, w \}$, and that there exist disjoint paths P_a, P_b, P_z in $G(V(D), D - D')$, connecting a with y_3, b with y_1, z with y_2, that meet $G(V(D'), D')$ only in a, b, z, respectively. If $a = y_4^-$, then, by Lemma 3.3.4, $G(V(D), D \cup F) \backslash (V(D) - V(D') - V(P_a) - V(P_b) - V(P_z))/(E(P_A) \cup E(P_b) \cup E(P_z))$ has a rooted $G_7(y)$−minor which, together with the edge y_3y_4, gives us a rooted G_8−minor in $G(V(B), B)$. If, on the other hand, $a \in S[y_3^+, y_4^-)$, then $|N_{G(V(D'),D')}(y_4) - V(C')| \geq 3$ and we are done by 3.3.5.2. So we may assume that $a \in S[y_2^+, y_3^+)$. Similarly, we may assume that $b \in S(y_5^-, y_1^-]$. Then, (C', D') is a 6−separation in G, non-crossing with (C, D) and with $V(C') \cap V(D') = \{ a, y_3, y_4, y_5, b, z \}$, such that $C \subseteq C', D' \subseteq D, |N_{G(V(D'),D')}(y_4) - V(C')| \geq 3, G(V(D'), D')$ has a planar embedding with a, y_3, y_4, y_5, b, z, in cyclic order, on the boundary of the infinite face, and such that there exist disjoint paths P_a, P_b, P_z in $G(V(D), D - D')$ connecting a with y_2, b with y_1, z with $S(y_1^-, y_2^+)$ that meet $G(V(D'), D')$ only in a, b, z, respectively.

Consider such a separation (C', D') with $|D'|$ minimal. Notice that $|N_{G_D}(w) - V(S) - \{ y_4 \}| \geq 2$ so that there exists a vertex $z'' \in N_{G_D}(w) - V(S) - \{ y_4, z \}$. Analogous to 3.3.5.3, we may assume that

3.3.5.4. $\{ w, z \}$ *does not* 2−*separate* $G_D(V(S_1), S_1)$, *where* $w \in S(y_4^-, y_4^+)$ *such that there exists a path between* w *and* z *in* $G(V(D'), D')$ *disjoint with* S.

For otherwise, if (S_1', S_2') is a 2−separation in $G_D(V(S_1), S_1)$, with $V(S_1') \cap V(S_2') = \{ w, z \}$, such that $y_4^- \in V(S_2')$ and $\{ y_4^+, z'' \} \subseteq V(S_1')$, then $(C'', D'') := (E(G) - H', H')$, where $H' := \bigcup_{u \in V(S_1') - V(S_2') - \{ b \}} \delta_{G(V(D'),D')}(u)$, is a 5−separation in G that is similar to the 5−separation (C', D') observed when $a = y_4^-$ and $b \in S(y_5^-, y_1^-]$ and, thus, yields a rooted G_8−minor in $G(V(B), B)$, and we are done (likewise, when $\{ y_4^-, z'' \} \subseteq V(S_1')$ and $y_4^+ \in V(S_2')$, we get a 5−separation (C'', D'') that is similar to the 5−separation (C', D') observed when $b = y_4^+$ and $a \in S[y_2^+, y_3^+)$ and, thus, yields a rooted G_9−minor in $G(V(B), B)$, and we are done again). Continuing with the analogy, suppose, now, that there exists another path Q' between $S(y_4^-, y_4^+)$ and z in $G_D(V(S_1), S_1)$, with $w' \in S(y_4^-, y_4^+)$ as its other end, which is internally disjoint with S, Q (the subpath of Q between $S(y_4^-, y_4^+)$ and z, including w, may be chosen differently for this purpose, if required), and which lies in the face of G_D bounded by the cycle formed by the vertices $V(Q) \cup S(w, q)$ (the case when it lies in the face of G_D bounded by the cycle formed by the vertices $V(Q) \cup S(q, w)$ is symmetrically analogous). Consider such a path Q' for which $|S[w', y_4^+)|$ is minimum. Then, by 3.3.5.4, there exists a path Q_1 between $Q'(w', z)$ and $S[y_4^+, b]$ with ends $q_1' \in Q'(w', z)$ and $q_1 \in S[y_4^+, b]$. We may assume that $q_1 \in S(y_5^-, b]$, for otherwise $\{ z, b, y_3, y_4, y_5^- \}$ forms the separating set of a $5-$separation (A', B') in G with $\{ w \} \cup S[y_3^+, y_4^-] \subseteq V(B') - V(A')$ which satisfies the hypotheses of 3.3.5.2 and we are done. In turn, there exists another path Q_5 between $S[y_4^+, q_1)$ and $Q'(w', q_1'] \cup Q_1[q_1', q_1)$ with ends $q_5' \in Q'(w', q_1'] \cup Q_1[q_1', q_1)$ and $q_5 \in S[y_4^+, q_1)$, that is internally disjoint with S, Q' and Q_1, for otherwise $\{ w', y_4, y_5, q_1 \}$ $4-$separates $S(w', q_1)$ from the rest of the graph G. Similarly, there exists a path Q_2 between $Q(w, z)$ and $S[a, y_3^+)$ with ends $q_2' \in Q(w, z)$ and $q_2 \in S[a, y_3^+)$, and a path Q_3 between $S(q_2, y_4^-]$ and $Q(w, q_2'] \cup Q_2[q_2', q_2)$ with ends $q_3' \in Q(w, q_2'] \cup Q_2[q_2', q_2)$ and $q_3 \in S(q_2, y_4^-]$. Finally, by 3.3.5.4, there also exists a path Q'' between $Q(w, z)$ and $Q'(w', z)$ which, together with the paths Q_1, Q_5, Q_2, Q_3 and the paths P_a, P_b, P_z, yields a rooted $G_{10}-$minor in $G(V(B), B)$ and we are done again. So we may assume that there does not exist such a path Q' and, hence, that, for some $z' \in Q(w, z), a' \in S[a, y_3^+), b' \in S(y_5^-, b], \{ z', a', b' \}$ $3-$separates $G_D(V(S_1), S_1)$ as (S_1', S_2') such that $V(S_1') \cap V(S_2') = \{ z', a', b' \}, S(y_4^-, y_4^+) \subseteq V(S_1'), z \in V(S_2')$. Then, $(C'', D'') := (E(G) - H', H')$, where $H' := \bigcup_{u \in V(S_1'') - V(S_2'')} \delta_{G(V(D), D)}(u)$, is a $6-$ separation in G, non-crossing with (C, D) and with $V(C'') \cap V(D'') = \{ a', y_3, y_4, y_5, b', z' \}$, such that $C \subseteq C''$, $D'' \subseteq D$, $|N_{G(V(D''),\ D'')}(y_4) - V(C'')| \geq 3, G(V(D''), D')$ has a planar embedding with $a', y_3, y_4, y_5, b', z'$, in cyclic order, on the boundary of the infinite face, and such that there exist disjoint paths $P_{a'}, P_{b'}, P_{z'}$ in $G(V(D), D - D'')$ connecting a' with y_2, b' with y_1, z' with $S(y_1^-, y_2^+)$ (using paths P_a, P_b, P_z as subpaths, respectively) that meet $G(V(D''), D'')$ only in a', b', z', respectively. But $|D''| < |D'|$, a contradiction. □

Corollary 3.3.6. *Let (A, B) be a $5-$separation in a $5-$connected graph G, with $V(A) \cap V(B) = \{ x_1, x_2, x_3, x_4, x_5 \}, |V(A) - V(B)| \geq 1, |V(B) - V(A)| \geq 2, |N_{G(V(B), B)}(x_4)| \geq 3$, such that $G(V(B), B)$ has a planar embedding with x_1, x_2, x_3, x_4, x_5, in cyclic order, on the boundary of the infinite face. If, additionally, $|N_{G(V(B), B)}(x')| \geq 3$, for some $x' \in \{ x_1, x_5 \}$, then $G(V(B), B)$ has a rooted $G_{8+(12)}-, G_{8+(15)}-, G_{9+(12)}-$ or $G_{9+(15)}-$minor when $x' = x_1$, and a rooted $G_{8+(15)}-$ or G_9-minor when $x' = x_5$.*

Proof. Let (A, B) be a $5-$separation in a $5-$connected graph G, with $V(A) \cap V(B) = \{ x_1, x_2, x_3, x_4, x_5 \}$, such that $|V(A) - V(B)| \geq 1, |V(B) - V(A)| \geq 2, |N_{G(V(B),B)}(x_4)| \geq 3$, and such that $G(V(B), B)$ has a planar embedding with x_1, x_2, x_3, x_4, x_5, in cyclic order, on the boundary of the infinite face. By Corollary 3.3.3 and Proposition 3.2.1, there exists a $5-$separation (C, D) in G, non-crossing with (A, B) such that $A \subseteq C, D \subseteq B$, it is possible to slide from (A, B) to (C, D), and such that, for each $u' \in V(C) \cap V(D), |N_{G(V(D),D)}(u') - V(C)| \geq 2$. Let $Y := \{ y_1, y_2, y_3, y_4, y_5 \} = V(C) \cap V(D)$; notice that $G(V(D), D) \backslash Y'$ is connected for every $Y' \subsetneq Y$. There also exist five pairwise disjoint paths P_1, P_2, P_3, P_4, P_5 in $G(V(B), B - D)$ such that, for each $i \in \{ 1, ..., 5 \}, P_i$ connects y_i with x_i and meets $G(V(D), D)$ only in y_i, and such that $V(B) = (\bigcup_{i=1}^{5} V(P_i)) \cup (V(D) - V(C))$; let $y' \in \{ y_1, y_5 \}$ be the other end of the path $P' \in \{ P_1, ..., P_5 \}$ that has x' as one of its ends. Then, since $G(V(B), B)$ has a planar embedding with x_1, x_2, x_3, x_4, x_5, in cyclic order, on the boundary of the infinite face, we have that

3.3.6.1. *$G(V(D), D)$ has a planar embedding with y_1, y_2, y_3, y_4, y_5, in cyclic order, on the boundary of the infinite face.*

Contract all edges in $\bigcup_{i=1}^{5} E(P_i)$ to identify x_i with y_i, for each $i \in \{ 1, ..., 5 \}$, and, thus, reduce $G(V(B), B)$ to $G(V(D), D \cup F)$, where $V(F) \subseteq Y$. Notice that by Lemma 3.3.4, $G(V(D), D)$ already has a rooted $G_7(y)-$minor. Further, since $|N_{G(V(B),B)}(x')| \geq 3$, we may assume that $|N_{G(V(D),D)}(y') - V(C)| \geq 3$: if $|N_{G(V(D),D)}(y') - V(C)| = 2$, then either $|(D \cup F) \cap \{ y_1y_2, y_1y_5 \}| \geq 1 (y' = y_1)$ or $|(D \cup F) \cap \{ y_1y_5, y_4y_5 \}| \geq 1 (y' = y_5)$; if, additionally, $|N_{G(V(D),D)}(y_4) - V(C)| = 2$, then, since $|N_{G(V(B),B)}(x_4)| \geq 3, |(D \cup F) \cap \{ y_3y_4, y_4y_5 \}| \geq 1$ and we are done; on the other hand, if $|N_{G(V(D),D)}(y_4) - V(C)| \geq 3$, then, by Lemma 3.3.5, $G(V(D), D)$ has a rooted $G_8(y)-$ or $G_9(y)-$minor, and we are done again. Similarly, we may assume that $|N_{G(V(D),D)}(y_4) - V(C)| \geq 3$. By the proof of Lemma 3.3.4, we have that $G(V(D), D) \backslash Y$ is a $2-$connected planar graph with the infinite face bounded by a cycle S such that, for each $u \in V(S), |N_{G(V(D),D)}(u) \cap Y| \leq 2$; since $|N_{G(V(D),D)}(y') - V(C)| \geq 3$ and $|N_{G(V(D),D)}(y_4) - V(C)| \geq 3$, and, additionally, $|N_{G(V(D),D)}(u) - V(C)| \geq 2$ for each $u \in V(C) \cap V(D)$, we have that $|V(S)| \geq 6$. Consider such a separation (C, D) with $|D|$ minimal.

3.3.6.2. *If (A', B') is a $5-$separation in G, non-crossing with (C, D) and with $V(A') \cap V(B') = \{ x'_1, x'_2, x'_3, x'_4, x'_5 \}$, such that*

(i) $C \subseteq A', B' \subseteq D, (A', B') \neq (C, D)$,

(ii) $|V(B') - V(A')| \geq 2$,

(iii) *there exist five pairwise disjoint paths* $P'_1,\ P'_2,\ P'_3,\ P'_4,\ P'_5$ *in* $G(V(D), D - B')$ *such that, for each* $i \in \{ 1, ..., 5 \}, P'_i$ *connects* x'_i *with* y_i *and meets* $G(V(B'), B')$ *only in* x'_i,

(iv) $|N_{G(V(B'),B')}(x'_4)| \geq 3$, *and,*

(v) $|N_{G(V(B),B)}(x'')| \geq 3$, *where* $x'' \in \{ x'_1, x'_5 \}$ *is the other end of the path* $P'' \in \{ P'_1, ..., P'_5 \}$ *that has* y' *as one of its ends,*

then $G(V(B), B)$ *has a rooted* $G_{8+(12)}-, G_{8+(15)}-, G_{9+(12)}-$ *or* $G_{9+(15)}-$*minor when* $x' = x_1$, *and a rooted* $G_{8+(15)}-$ *or* G_9-*minor when* $x' = x_5$.

Proof of claim. Let (A', B') be a 5−separation in G, non-crossing with (C, D) and with $V(A') \cap V(B') = \{ x'_1, x'_2, x'_3, x'_4, x'_5 \}$, such that it has the properties $(i) - (v)$ described above; let $X' := \{ x'_1, x'_2, x'_3, x'_4, x'_5 \}$. Since $G(V(B'), B')\backslash X''$ is connected for every $X'' \subsetneq X'$, we have, by (iii) and 3.3.6.1, that $G(V(B'), B')$ has a planar embedding with $x'_1, x'_2, x'_3, x'_4, x'_5$ on the boundary of the infinite face. By Corollary 3.3.3 and Proposition 3.2.1, there exists another 5−separation (A'', B'') in G, non-crossing with (A', B') and with $V(A'') \cap V(B'') = \{ x''_1, x''_2, x''_3, x''_4, x''_5 \}$, such that $A' \subseteq A'', B'' \subseteq B'$, it is possible to slide from (A', B') to (A'', B''), and, for each $u \in V(A'') \cap V(B'')$, $|N_{G(V(B''),B'')}(u) - V(A'')| \geq 2$; let $X'' := \{ x''_1, x''_2, x''_3, x''_4, x''_5 \}$; notice that $G(V(B''), B'')\backslash X'''$ is connected for every $X''' \subsetneq X''$. There also exist five pairwise disjoint paths $P''_1,\ P''_2,\ P''_3,\ P''_4,\ P''_5$ in $G(V(B'), B' - B'')$ such that, for each $i \in \{ 1, ..., 5 \}, P''_i$ connects x''_i with x'_i and meets $G(V(B''), B'')$ only in x''_i; let x''' be the other end of the path $P''' \in \{ P''_1, ..., P''_5 \}$ that has x'' as one of its ends. Notice that the path $P'_i \cup P''_i$ connects x''_i with y_i, for each $i \in \{ 1, ..., 5 \}$. That, together with 3.3.6.1, gives us that $G(V(B''), B'')$ has a planar embedding with $x''_1, x''_2, x''_3, x''_4, x''_5$ on the boundary of the infinite face. As before, we may assume that $|N_{G(V(B''),B'')}(x''') - V(A'')| \geq 3$ and $|N_{G(V(B''),B'')}(x''_4) - V(A'')| \geq 3$, for otherwise we are done. But now $|B''| < |D|$, a contradiction to the minimality of $|D|$. □

Without loss of generality, let G_D be a plane graph embedding $G(V(D), D)$ in the plane with y_1, y_2, y_3, y_4, y_5, in clockwise order, on the boundary of the infinite face. For each $i \in \{ 1, ..., 5 \}$, let y_i^-, y_i^+ denote the vertices in $V(S) \cap N_{G_D}(y_i)$ such that no vertex in $S(y_i^-, y_i^+)$ has a neighbor $y \in Y, y \neq y_i$, where $S(y_i^-, y_i^+)$ is defined as before. Since $|N_{G(V(D),D)}(y_4) - V(C)| \geq 3$, we have that $S(y_4^-, y_4^+) \neq \emptyset$. Similarly, either $S(y_1^-, y_1^+) \neq \emptyset(x' = x_1)$ or $S(y_5^-, y_5^+) \neq \emptyset(x' = x_5)$.

Suppose there does not exist a path between $S(y_4^-, y_4^+)$ and $S(y_1^-, y_2^+)$ in $G_D \backslash Y$ that is internally disjoint with S. Let w be a vertex in $S(y_4^-, y_4^+)$. There exists a vertex $a \in S[y_2^+, y_3^+)$ which is connected to w by a path in $G_D \backslash Y$ that is internally disjoint with S. Consider such a vertex a for which $|S[y_2^+, a)|$ is minimal. Similarly, consider a vertex $b \in S(y_5^-, y_1^-]$ which is connected to w by a path in $G_D \backslash Y$ that is internally disjoint with S, and for which $|S(b, y_1^-]|$ is minimal. Notice that $\{ a, b \}$ 2−separates $G_D \backslash Y$ as (S_1, S_2) with $V(S_1) - V(S_2) \neq \emptyset, V(S_2) - V(S_1) \neq \emptyset$ and $V(S_1) \cap V(S_2) = \{ a, b \}$; let $w \in V(S_1)$. In turn, $(C', D') := (E(G) - H, H)$, where $H := \bigcup_{u \in V(S_1) - V(S_2)} \delta_{G(V(D), D)}(u)$, is a 5−separation in G, non-crossing with (C, D) and with $V(C') \cap V(D') = \{ b, a, y_3, y_4, y_5 \}$, such that $C \subsetneq C', D' \subsetneq D, |N_{G(V(D'), D')}(y_4) - V(C')| \geq 3$ (and, hence, $|V(D') - V(C')| \geq 3$), and such that there exist two disjoint paths in $G(V(D), D - D')$ connecting a with y_2 (using the edge $y_2 y_2^+$) and b with y_1 (using the edge $y_1 y_1^-$) that meet $G(V(D'), D')$ only in a and b, respectively; let $y'_1 = b, y'_2 = a, y'_3 = y_3, y'_4 = y_4$ and $y'_5 = y_5$. Further, there exists a path connecting y_1 with y_2 (using the edges $y_1 y_1^+$ and $y_2 y_2^-$) in $G(V(D), D - D')$ that meets the first two paths only in y_2 and y_1, respectively. Now, if $b \notin S(y_5^-, y_5^+]$ and $x' = x_5$, then (C', D') satisfies the hypotheses of 3.3.6.2 and we are done; if, on the other hand, $b \in S(y_5^-, y_5^+]$, then, by Lemma 3.3.5, $G(V(D'), D')$ has a rooted $G_8(y')-$ or $G_9(y')-$minor which, together with the three paths and the edge $y_5 y_5^+$ yields in $G(V(B), B)$ a rooted $G_{8+(12)}-$ or $G_{9+(12)}-$minor when $x' = x_1$, and a rooted $G_{8+(15)}-$ or G_9-minor when $x' = x_5$. So we may assume that there exists at least one such path Q with ends $w \in S(y_4^-, y_4^+)$ and $q \in S(y_1^-, y_2^+)$, and one for which both $|S(y_4^-, w]|$ and $|S[q, y_2^+)|$ are minimum. Moreover, we may assume for any such path Q that $\{ w, q \}$ does not 2−separate $G_D \backslash Y$ (proof follows) and, hence, that $|V(Q)| \geq 3$.

3.3.6.3. $\{ w, q \}$ *does not* 2−*separate* $G_D \backslash Y$.

Proof of claim. Suppose that $\{ w, q \}$ 2−separates $G_D \backslash Y$ as (S_1, S_2) so that $V(S_1) \cap V(S_2) = \{ w, q \}$. Without loss of generality, let $y_4^- \in V(S_1), y_4^+ \in V(S_2)$. If $q \in S[y_1^+, y_2^+)$, then $(C', D') := (E(G) - H, H)$, where $H := \bigcup_{u \in V(S_1) - V(S_2)} \delta_{G(V(D), D)}(u)$, is a 5−separation in G, non-crossing with (C, D) and with $V(C') \cap V(D') = \{ q, y_2, y_3, y_4, w \}$, such that $C \subseteq C', D' \subseteq D, |V(D') - V(C')| \geq 2$, and such that $G(V(D'), D')$ has a planar embedding with q, y_2, y_3, y_4, w, in cyclic order, on the boundary of the infinite face; the last property is due to 3.3.6.1 combined with the facts that $G(V(D'), D') \backslash Y'$ is connected for every $Y' \subsetneq \{ q, y_2, y_3, y_4, w \}$, and there exist two disjoint paths in $G(V(D), D - D')$ connecting w with y_5 (using the edge $y_5 y_5^-$ and the vertex y_4^+) and q with y_1 (using the edge $y_1 y_1^+$), and meeting $G(V(D'), D')$ only in w and q, respectively; let $y'_1 = q, y'_2 = y_2, y'_3 = y_3, y'_4 = y_4$ and $y'_5 = w$. Further, there exists a path connecting y_1 with y_5 (using the edges $y_1 y_1^-$ and

$y_5y_5^+$) in $G(V(D), D - D')$ that meets the first two paths only in y_5 and y_1, respectively. By Lemma 3.3.4, $G(V(D'), D')$ has a rooted $G_7(y')-$minor which, together with the three paths and the edge $y_5y_5^+$ yields a rooted $G_{9+(15)}-$minor in $G(V(B), B)$.

Similarly, if $q \notin S[y_1^+, y_2^+)$ then, with $H := \bigcup_{u \in V(S_2)-V(S_1)} \delta_{G(V(D),D)}(u)$, $(C', D') := (E(G) - H, H)$ is a $5-$separation in G, non-crossing with (C, D) and with $V(C') \cap V(D') = \{ y_1, q, w, y_4, y_5 \}$, such that $C \subseteq C', D' \subseteq D, |V(D') - V(C')| \geq 2$, and such that $G(V(D'), D')$ has a planar embedding with q, w, y_4, y_5, y_1, in cyclic order, on the boundary of the infinite face; the last property is due to 3.3.6.1 combined with the facts that $G(V(D'), D') \backslash Y'$ is connected for every $Y' \subsetneq \{ y_1, q, w, y_4, y_5 \}$, and there exist two disjoint paths in $G(V(D), D - D')$ connecting w with y_3 (using the edge $y_3y_3^+$ and the vertex y_4^-) and q with y_2 (using the edge $y_2y_2^-$), and meeting $G(V(D'), D')$ only in w and q, respectively; let $y_1' = y_1, y_2' = q, y_3' = w, y_4' = y_4$ and $y_5' = y_5$. Further, there exists a path connecting y_2 with y_3 (using the edges $y_2y_2^+$ and $y_3y_3^-$) in $G(V(D), D - D')$ that meets the first two paths only in y_3 and y_2, respectively. By Lemma 3.3.4, $G(V(D'), D')$ has a rooted $G_7(y')-$minor (or, by Lemma 3.3.5, a rooted $(G_7(y') \cup \{ y_1'y_5' \})-$ or $G_9(y')-$minor, when $x' = x_5$) which, together with the three paths and the edges $y_1y_1^+$ and $y_4y_4^-$, yields in $G(V(B), B)$ a rooted $(G_8 \cup \{ x_1x_5, x_1x_2 \})-$minor when $x' = x_1$, and a rooted $(G_9 \cup \{ x_1x_2, x_2x_3, x_3x_4 \})-$ or $(G_8 \cup \{ x_1x_5, x_1x_2, x_2x_3 \})-$minor when $x' = x_5$. □

Now suppose that there exists another path Q' between $S(y_4^-, y_4^+)$ and $S(y_1^-, y_2^+)$ in $G_D \backslash Y$, with ends $w' \in S(y_4^-, y_4^+), q' \in S(y_1^-, y_2^+)$, which is internally disjoint with S and Q, which lies in the face of G_D bounded by the cycle formed by the vertices $V(Q) \cup S(w, q)$ (the case when it lies in the face of G_D bounded by the cycle formed by the vertices $V(Q) \cup S(q, w)$ is symmetrically analogous), and for which both $|S[w', y_4^+)|$ and $|S(y_1^-, q']|$ are minimum. Then, by the proof of Lemma 3.3.5, $G(V(B), B)$ has a rooted $G_{10}-$minor which, in turn, has a rooted $(G_8 \cup \{ x_2x_3 \})-$minor as well as a rooted $G_{9+(15)}-$minor, and we are done. So we may assume that there do not exist two internally disjoint paths between $S(y_4^-, y_4^+)$ and $S(y_1^-, y_2^+)$ in $G_D \backslash Y$ that are both internally disjoint with S.

Since S bounds the infinite face of $G_D \backslash Y$, there do not exist four pairwise internally disjoint paths between $S(y_4^-, y_4^+)$ and $S(y_1^-, y_2^+)$ in $G_D \backslash Y$, and, hence, there exists a $3-$separation (S_1, S_2) in $G_D \backslash Y$ such that $V(S_1) \cap V(S_2) = \{ z, a, b \}$, where $z \in Q(w, q), a \in S[y_2^+, y_4^-], b \in S[y_4^+, y_1^-], S(y_4^-, y_4^+) \subseteq V(S_1)$, and $S(y_1^-, y_2^+) \subseteq V(S_2)$. It cannot be that $a \in S[y_3^+, y_4^-]$ and $b \in S[y_4^+, y_5^-]$, for otherwise $\{ a, y_4, b, z \}$ $4-$separates $S(y_4^-, y_4^+)$ from the rest of the graph G. Suppose $a \in S[y_3^+, y_4^-], b \in S(y_5^-, y_1^-]$. Then $(C', D') := (E(G) - H, H)$, where $H := \bigcup_{u \in V(S_1)-V(S_2)} \delta_{G(V(D),D)}(u)$, is a $5-$separation in G, non-crossing with (C, D) and with $V(C') \cap V(D') = \{a, y_4, y_5, b, z\}$, such that $C \subsetneq C', D' \subsetneq D, |V(D') - V(C')| \geq 2$, and such that $G(V(D'), D')$ has a planar embedding with a, y_4, y_5, b, z, in cyclic order, on the boundary of the infinite face; the last property is due to 3.3.6.1 combined with the facts that $G(V(D'), D')\backslash Y'$ is connected for every $Y' \subsetneq \{ b, z, a, y_4, y_5 \}$, and there exist pairwise disjoint paths P_a, P_b and P_z in $G(V(D), D - D')$ connecting a with y_3 (using the edge $y_3 y_3^+$), b with y_1 (using the edge $y_1 y_1^-$) and z with y_2 (using the edge $y_2 y_2^-$), that meet $G(V(D'), D')$ only in a, b and z, respectively; let $y_1' = b, y_2' = z, y_3' = a, y_4' = y_4$ and $y_5' = y_5$. Further, there exists a path connecting y_2 with y_3 (using the edges $y_2 y_2^+$ and $y_3 y_3^-$) in $G(V(D), D - D')$ that meets the paths P_a and P_z only in y_3 and y_2, respectively. If $a = y_4^-$, then we are done since, by Lemma 3.3.4, $G(V(D'), D')$ has a rooted $G_7(y')-$minor (or, by Lemma 3.3.5, a rooted $(G_7(y') \cup \{ y_1' y_5' \})-$ or $G_9(y')-$minor, when $x' = x_5$ and $b \notin S(y_5^-, y_5^+])$ which, together with the four paths and the edges $y_4 y_4^-$ and $y_1 y_1^+$ (and $y_5 y_5^+$ when $x' = x_5$ and $b \in S(y_5^-, y_5^+])$, yields in $G(V(B), B)$ a rooted $(G_8 \cup \{ x_1x_2, x_2x_3 \})-$minor when $x' = x_1$, and a rooted $(G_9 \cup \{ x_1x_2, x_2x_3, x_3x_4 \})-$ or $(G_8 \cup \{ x_1x_5, x_1x_2, x_2x_3 \})-$minor when $x' = x_5$. If, on the other hand, $a \in S[y_3^+, y_4^-)$, then $|N_{G(V(D'),D')}(y_4) - V(C')| \geq 3$; if, now, $x' = x_5$ and $b \notin S(y_5^-, y_5^+]$, then we are done by 3.3.6.2; otherwise, by Lemma 3.3.5, $G(V(D'), D')$ has a rooted $G_8(y')-$ or $G_9(y')-$minor which, together with the four paths and the edge(s) $y_1 y_1^+$ (and $y_5 y_5^+$ when $x' = x_5$ and $b \in S(y_5^-, y_5^+])$, yields in $G(V(B), B)$ a rooted $(G_8 \cup \{ x_1x_2, x_2x_3 \})-$ or $(G_9 \cup \{ x_1x_2, x_2x_3 \})-$minor when $x' = x_1$, and a rooted $(G_8 \cup \{ x_1x_2, x_2x_3, x_1x_5 \})-$ or $(G_9 \cup \{ x_1x_2, x_2x_3, x_1x_5 \})-$minor when $x' = x_5$ and $b \in S(y_5^-, y_5^+]$, and we are done again. So we may assume that $a \in S[y_2^+, y_3^+)$. Similarly, we may assume that $b \in S(y_5^-, y_1^-]$, for otherwise $G(V(B), B)$ has a rooted $(G_8 \cup \{ x_1x_2, x_1x_5 \})-$ or $(G_9 \cup \{ x_1x_2, x_1x_5 \})-$minor and we are done again. Then, (C', D') is a $6-$separation in G, non-crossing with (C, D) and with $V(C') \cap V(D') = \{ a, y_3, y_4, y_5, b, z \}$, such that $C \subseteq C', D' \subseteq D, |N_{G(V(D'),D')}(y_4) - V(C')| \geq 3, G(V(D'), D')$ has a planar embedding with a, y_3, y_4, y_5, b, z, in cyclic order, on the boundary of the infinite face, and such that there exist pairwise disjoint paths P_a, P_b and P_z in $G(V(D), D - D')$ connecting a with y_2 (using the edge $y_2 y_2^+$), b with y_1 (using the edge $y_1 y_1^-$) and z with $S(y_1^-, y_2^+)$ that meet $G(V(D'), D')$ only in a, b and z, respectively.

Consider such a separation (C', D') with $|D'|$ minimal. Notice that $|N_{G_D}(w) - V(S) - \{ y_4 \}| \geq 2$ so that there exists a vertex $z'' \in N_{G_D}(w) - V(S) - \{ y_4, z \}$. Analogous to 3.3.6.3, we may assume that

3.3.6.4. $\{ w, z \}$ *does not* $2-$*separate* $G_D(V(S_1), S_1)$, *where* $w \in S(y_4^-, y_4^+)$ *such that there exists a path between* w *and* z *in* $G(V(D'), D')$ *disjoint with* S.

For otherwise, if (S_1', S_2') is a $2-$separation in $G_D(V(S_1), S_1)$, with $V(S_1') \cap V(S_2') = \{ w, z \}$, such that $y_4^- \in V(S_2')$ and $\{ y_4^+, z'' \} \subseteq V(S_1')$, then $(C'', D'') := (E(G) - H', H')$, where $H' := \bigcup_{u \in V(S'_1) - V(S'_2) - \{ b \}} \delta_{G(V(D'),D')}(u)$, is a 5−separation in G that is similar to the 5−separation (C', D') observed when $a = y_4^-$ and $b \in S(y_5^-, y_1^-]$ and, thus, yields in $G(V(B), B)$ a rooted $(G_8 \cup \{ x_1x_2, x_2x_3 \})$−minor when $x' = x_1$, and a rooted $(G_9 \cup \{ x_1x_2, x_2x_3, x_3x_4 \})$− or $(G_8 \cup \{ x_1x_5, x_1x_2, x_2x_3 \})$−minor when $x' = x_5$, and we are done (likewise, when $\{ y_4^-, z'' \} \subseteq V(S'_1)$ and $y_4^+ \in V(S'_2)$, we get a 5−separation (C'', D'') that is similar to the 5−separation (C', D') observed when $b = y_4^+$ and $a \in S[y_2^+, y_3^+)$ and, thus, yields a rooted $(G_8 \cup \{ x_1x_2, x_1x_5 \})$− or $(G_9 \cup \{ x_1x_2, x_1x_5 \})$−minor in $G(V(B), B)$, and we are done again). Continuing with the analogy, suppose, now, that there exists another path Q' between $S(y_4^-, y_4^+)$ and z in $G_D(V(S_1), S_1)$, with $w' \in S(y_4^-, y_4^+)$ as its other end, which is internally disjoint with S, Q (the subpath of Q between $S(y_4^-, y_4^+)$ and z, including w, may be chosen differently for this purpose, if required), and which lies in the face of G_D bounded by the cycle formed by the vertices $V(Q) \cup S(w, q)$ (the case when it lies in the face of G_D bounded by the cycle formed by the vertices $V(Q) \cup S(q, w)$ is symmetrically analogous). Consider such a path Q' for which $|S[w', y_4^+)|$ is minimum. Then, as in the proof of Lemma 3.3.5, $G(V(B), B)$ has a rooted G_{10}−minor which, in turn, has a rooted $(G_8 \cup \{ x_2x_3 \})$−minor as well as a rooted $G_{9+(15)}$−minor, and we are done. So we may assume that there does not exist such a path Q' and, hence, that, for some $z' \in Q(w, z), a' \in S[a, y_3^+), b' \in S(y_5^-, b], \{ z', a', b' \}$ 3−separates $G_D(V(S_1), S_1)$ as (S'_1, S'_2) such that $V(S'_1) \cap V(S'_2) = \{ z', a', b' \}, S(y_4^-, y_4^+) \subseteq V(S'_1)$ and $z \in V(S'_2)$. Then, $(C''', D'') := (E(G) - H', H')$, where $H' := \bigcup_{u \in V(S''_1) - V(S''_2)} \delta_{G(V(D),D)}(u)$, is a 6−separation in G, non-crossing with (C, D) and with $V(C'') \cap V(D'') = \{ a', y_3, y_4, y_5, b', z' \}$, such that $C \subseteq C''$, $D'' \subseteq D$, $|N_{G(V(D''),\ D'')}(y_4) - V(C'')| \geq 3, G(V(D''), D')$ has a planar embedding with $a', y_3, y_4, y_5, b', z'$, in cyclic order, on the boundary of the infinite face, and such that there exist disjoint paths $P_{a'}, P_{b'}, P_{z'}$ in $G(V(D), D - D'')$ connecting a' with y_2, b' with y_1, z' with $S(y_1^-, y_2^+)$ (using paths P_a, P_b, P_z as subpaths, respectively) that meet $G(V(D''), D'')$ only in a', b', z', respectively. But $|D''| < |D'|$, a contradiction. □

Lemma 3.3.7. *If (A, B) is a 5−separation in a 5−connected graph G, with $V(A) \cap V(B) = \{ x_1, x_2, x_3, x_4, x_5 \}$, such that $|V(A) - V(B)| \geq 1$ and, for each $u \in V(A) \cap V(B), |N_{G(V(B),B)}(u) - V(A)| \geq 2$, then $G(V(B), B)$ has a rooted G_1−, G_4− or G_7−minor. If, additionally, x_4 has degree at least 3 in $G(V(B), B)$, then $G(V(B), B)$ has a rooted G_2−, G_3−, G_5−, G_6−, G_8− or G_9−minor.*

Proof. Let (A, B) be a 5−separation in a 5−connected graph G, with $V(A) \cap V(B) = \{ x_1, x_2, x_3, x_4, x_5 \}$, such that $|V(A) - V(B)| \geq 1$ and $|N_{G(V(B),B)}(u) - V(A)| \geq 2$ for each $u \in V(A) \cap V(B)$. Since G is 5−connected, there does not exist a separation (C, D) in

$G(V(B),B)$ such that $|\{ x_1,x_2,x_3,x_4,x_5 \} - V(C)| + \lambda(C) \leq 4$, and, hence, $G(V(B),B)\backslash x_3$ is $2-$connected.

If $G(V(B),B)\backslash x_3$ does not have an $(x_1,x_2,x_4,x_5)-$linkage, then, by Corollary 3.1.2, $G(V(B),B)\backslash x_3$ has a planar embedding with x_1,x_2,x_4,x_5, in cyclic order, on the boundary of the infinite face. Without loss of generality, let $G_B^{-x_3}$ be a plane graph embedding $G(V(B),B)\backslash x_3$ in the plane with x_1,x_2,x_4,x_5, in clockwise order, on the boundary of the infinite face. Since $G(V(B),B)\backslash x_3$ is $2-$connected, the infinite face in $G_B^{-x_3}$ is bounded by a cycle S^{-x_3}. Suppose, now, that $N_{G(V(B),B)}(x_3) \subseteq V(S^{-x_3})$.

If there exists a vertex $b \in N_{G(V(B),B)}(x_3) \cap S^{-x_3}(x_5,x_1)$, then there exist in $G_B^{-x_3}$ two internally disjoint paths between $\{ b \}$ and $\{ x_2,x_4 \}$ such that the path between b and x_2 is disjoint with $S^{-x_3}[x_4,b) \cup \{ x_3 \} \cup S^{-x_3}(b,x_1]$ and the path between b and x_4 is disjoint with $S^{-x_3}(b,x_2] \cup \{ x_3 \} \cup S^{-x_3}[x_5,b)$; as a result, $G(V(B),B)$ has a rooted G_6-minor and we are done. Similarly, $G(V(B),B)$ has a rooted G_6-minor if there exists a vertex $b \in N_{G(V(B),B)}(x_3) \cap S^{-x_3}(x_1,x_2)$. If $N_{G(V(B),B)}(x_3) - V(A) \subseteq S^{-x_3}(x_2,x_4)$, then $G(V(B),B)\backslash \{ x_1x_3, x_3x_5 \}$ has a planar embedding with x_1,x_2,x_3,x_4,x_5, in clockwise order, on the boundary of the infinite face and, by Lemma 3.3.4, has a rooted G_7-minor; if, additionally, x_4 has degree at least 3 in $G(V(B),B)$, then, by Lemma 3.3.5, $G(V(B),B)$ has a rooted G_8- or G_9-minor and we are done again. Similarly, if $N_{G(V(B),B)}(x_3) - V(A) \subseteq S^{-x_3}(x_4,x_5)$, then $G(V(B),B)\backslash \{ x_1x_3, x_2x_3 \}$ has a planar embedding with x_1,x_2,x_4,x_3,x_5, in clockwise order, on the boundary of the infinite face and, by Lemma 3.3.4, has a rooted $G_7^{(34)}-$minor which, in turn, has a rooted G_5-minor.

So we may assume that there exist $a,b \in N_{G(V(B),B)}(x_3)$ such that $a \in S^{-x_3}(x_2,x_4)$ and $b \in S^{-x_3}(x_4,x_5)$. Then there exist in $G_B^{-x_3}$ two internally disjoint paths between $\{ b \}$ and $\{ x_1,x_2 \}$ such that the path between b and x_1 (say $P_{bx_1}^{-x_3}$) is disjoint with $S^{-x_3}[x_2,b) \cup \{ x_3 \} \cup S^{-x_3}(b,x_5]$ and the path between b and x_2 is disjoint with $S^{-x_3}[x_4,b) \cup \{ x_3 \} \cup S^{-x_3}(b,x_1]$. We may assume that $P_{bx_1}^{-x_3}$ is disjoint with $S(x_1,x_2)$, for otherwise $G(V(B),B)$ contains a rooted G_5-minor and we are done. There also exists a path $P_{bx_2}^{-x_3}$ in $G_B^{-x_3}$ between b and x_2 disjoint with $\{ a,x_1 \}$, for otherwise $\{ a,x_1 \}$ separates x_2 from b in $G_B^{-x_3}$ and $N_{G(V(B),B)}(x_2) \subseteq \{ a,x_1,x_3 \}$, a contradiction. If there exists such a path $P_{bx_2}^{-x_3}$ that is also disjoint with $S^{-x_3}(a,x_4]$, then $G(V(B),B)$ contains a rooted G_5-minor and we are done; so we may assume that that is not the case. Let c be the first vertex in $S^{-x_3}(a,x_4]$ that $P_{bx_2}^{-x_3}$ meets going from x_2 to b. Then $P_{bx_2}^{-x_3}[x_2,c]$ is disjoint with $S^{-x_3}[x_4,x_1] \cup P_{bx_1}^{-x_3}[b,x_1]$. Now, there exist in $G_B^{-x_3}$ two internally disjoint paths between $\{ a \}$ and $\{ x_1,x_5 \}$ such that the path between a and x_1 is disjoint with $S^{-x_3}[x_2,a) \cup \{ x_3 \} \cup S^{-x_3}(a,x_5]$ and the path between a and x_5 (say $P_{ax_5}^{-x_3}$) is disjoint with $S^{-x_3}[x_1,a) \cup \{ x_3 \} \cup S^{-x_3}(a,x_4]$. Since $V(P_{ax_5}^{-x_3}) \cap (P_{bx_2}^{-x_3}[x_2,c) - S^{-x_3}[x_1,a)) \neq \emptyset$, there exists a path between b and x_2 contained in $P^{-x_3}_{bx_2} \cup P^{-x_3}_{ax_5} \cup P^{-x_3}_{bx_1}$ that is disjoint with $\{ a, x_1 \} \cup S^{-x_3}(a, x_4]$, a contradiction.

So we may assume that $N_{G(V(B),B)}(x_3) \not\subseteq V(S^{-x_3})$. If there exists a vertex $v \in N_{G(V(B),B)}(x_3) - V(S^{-x_3})$, then there exist in $G^{-x_3}_B$ four pairwise internally disjoint paths between $\{ v \}$ and $\{ x_1, x_2, x_4, x_5 \}$ such that the paths between $\{ v \}$ and $\{ x_1, x_2 \}$ are both disjoint with $S^{-x_3}[x_4, x_5] \cup \{ x_3 \}$ and, hence, $G(V(B), B)$ has a rooted G_6-minor and we are done.

So we may assume that there exists an $(x_1, x_2, x_4, x_5)-$linkage in $G(V(B), B)\backslash x_3$. Repeating the argument with an $(x_1, x_2, x_5, x_4)-$ linkage, we may assume that $G(V(B), B)\backslash x_3$ has an $(x_1, x_2, x_5, x_4)-$ linkage as well, for otherwise $G(V(B), B)$ has a rooted $G^{(45)}_5-, G^{(45)}_6-$ or $G^{(45)}_7-$minor, each of which, in turn, has a rooted G_3-minor and we are done. Since the two linkages together ensure a rooted G_1-minor, we will also assume for the remainder of the proof that x_4 has degree at least 3 in $G(V(B), B)$.

Let P_{24}, P_{15} be the disjoint paths connecting x_2 with x_4 and x_1 with x_5, respectively, in a $(x_1, x_2, x_5, x_4)-$linkage in $G(V(B), B)\backslash x_3$. Then, for some $x'_2 \in P_{24}[x_2, x_4), x'_4 \in P_{24}(x'_2, x_4], x'_1 \in P_{15}[x_1, x_5), x'_5 \in P_{15}(x'_1, x_5]$, there exist disjoint paths P_{14}, P_{25} in $G(V(B), B)\backslash x_3$, connecting x'_1 with x'_4 and x'_2 with x'_5, respectively, each of which meets the paths P_{24}, P_{15} in exactly two of the four vertices x'_1, x'_2, x'_4, x'_5. The paths $P_{14}, P_{15}, P_{24}, P_{25}$ together ensure a rooted G_1-minor in $G(V(B), B)$. Now consider such a set of four paths for which $|P_{15}[x_1, x'_1)|$ is minimal. If there exists a path in $G(V(B), B)$ between $\{ x_3 \}$ and $P_{24}(x'_2, x_4] \cup P_{14}(x'_1, x'_4]$ that is disjoint with $P_{24}[x_2, x'_2] \cup P_{25}(x'_2, x'_5) \cup P_{15}[x_1, x_5]$, then $G(V(B), B)$ has a rooted G_2-minor and we are done. So we may assume that no such path exists. Similarly, we may assume that there does not exist a path in $G(V(B), B)$ between $P_{24}(x'_2, x_4] \cup P_{14}(x'_1, x'_4]$ and $P_{24}[x_2, x'_2) \cup P_{25}(x'_2, x'_5] \cup P_{15}(x'_1, x_5]$ that is disjoint with $P_{15}[x_1, x'_1] \cup \{ x'_2, x_3 \}$, for otherwise $G(V(B), B)$ has a rooted G_3-minor and we are done again. There cannot exist a path in $G(V(B), B)$ between $P_{24}(x'_2, x_4] \cup P_{14}(x'_1, x'_4]$ and $P_{15}[x_1, x'_1)$ (when $P_{15}[x_1, x'_1) \neq \emptyset$) that is disjoint with $P_{24}[x_2, x'_2] \cup P_{25}[x'_2, x'_5] \cup P_{15}[x'_1, x_5] \cup \{ x_3 \}$, for otherwise we can identify a path P'_{14} in $G(V(B), B)\backslash x_3$ connecting x''_4 with x''_1, where $x''_1 \in P_{15}[x_1, x'_1), x''_4 \in P_{24}(x'_2, x_4]$, disjoint with the path P_{25}, and meeting paths P_{15}, P_{24} in only x''_1, x''_4, respectively, such that $|P'_{15}[x_1, x''_1)| < |P_{15}[x_1, x'_1)|$, a contradiction to the minimality of $|P_{15}[x_1, x'_1)|$. So we may assume that $\{ x'_1, x'_2 \}$ $2-$separates $G(V(B), B)$ as (S_1, S_2), with $S_1 \cap S_2 = \{ x'_1, x'_2 \}$, such that $\{ x_1, x_2, x_3, x_5 \} \subseteq V(S_1)$ and $x_4 \in V(S_2)$; but then $\{ x'_1, x'_2 \} \cap \{ x_1, ..., x_5 \} = \emptyset, V(S_2) = \{ x'_1, x'_2, x_4 \}$ and $N_{G(V(B),B)}(x_4) = \{ x'_1, x'_2 \}$, a contradiction if x_4 has degree at least 3 in $G(V(B), B)$. □

Lemma 3.3.8. *If (A, B) is a $5-$separation in a $5-$connected graph G, with $V(A) \cap V(B) = \{ x_1, x_2, x_3, x_4, x_5 \}$, such that, for each $u \in V(A) \cap V(B), |N_{G(V(B),B)}(u) - V(A)| \geq 2$,* *then, either there exist two disjoint connected subgraphs $H_1, H_2 \subsetneq G(V(B), B)$ such that, for some $x \in \{ x_2, x_5 \}, \{ x_4, x \} \subseteq V(H_1)$ and $\{ x_1, x_2, x_3, x_5 \} - \{ x \} \subseteq V(H_2)$, or $G(V(B), B)$ has a rooted $G_7^{(24)}-$ or $G_7^{(24)(25)}-$minor. If there do not exist such subgraphs H_1 and H_2 and, for some $Y \subseteq \{ x_2, x_5 \}, Y \neq \emptyset, |N_{G(V(B),B)}(y)| \geq 3$ for each $y \in Y$, then $G(V(B), B)$ has a rooted $G_8^{(24)}-$, $G_9^{(24)}-$, $G_9^{(24)(25)}-$ or $G_8^{(13)(24)}-$minor when $Y = \{ x_2 \}$, a rooted $G_9^{(24)}-$, $G_8^{(13)(24)(25)}-$, $G_8^{(24)(25)}-$ or $G_9^{(24)(25)}-$minor when $Y = \{ x_5 \}$, and a rooted $G_{8+(15)}^{(24)}-$, $G_9^{(24)}-$, $G_{8+(15)}^{(24)(25)}-$ or $G_9^{(24)(25)}-$minor when $Y = \{ x_2, x_5 \}$.*

Proof. Let (A, B) be a $5-$separation in a $5-$connected graph G, with $V(A) \cap V(B) = \{ x_1, x_2, x_3, x_4, x_5 \}$, such that, for each $u \in V(A) \cap V(B), |N_{G(V(B),B)}(u) - V(A)| \geq 2$. It suffices to prove the theorem for such a separation (A, B) with $|B|$ minimal in the sense that there does not exist another $5-$separation (A', B') in G, non-crossing with (A, B) and with $V(A') \cap V(B') = \{ y_1, y_2, y_3, y_4, y_5 \}$, such that $A \subsetneq A', B' \subsetneq B$ and, for each $u \in V(A') \cap V(B'), |N_{G(V(B'),B')}(u) - V(A')| \geq 2$. If there exists a $5-$separation (A', B') in G, non-crossing with (A, B), as described, then there exist in $G(V(B), B - B')$ five pairwise disjoint paths connecting $V(A) \cap V(B)$ with $V(A') \cap V(B')$ (say one connecting x_i with y_i, for each $i \in \{ 1, ..., 5 \}$); in such a case, if there exist two disjoint connected subgraphs $H'_1, H'_2 \subsetneq G(V(B'), B')$ such that, for some $x' \in \{ y_2, y_5 \}, \{ y_4, x' \} \subseteq V(H'_1)$ and $\{ y_1, y_2, y_3, y_5 \} - \{ x' \} \subseteq V(H'_2)$, they can be extended to form the subgraphs H_1 and H_2, respectively; similarly, each of the planar graphs $G_7^{(24)}(y)$, $G_7^{(24)(25)}(y)$, $G_8^{(24)}(y)$, $G_9^{(24)}(y)$, $G_9^{(24)(25)}(y)$, $G_8^{(13)(24)}(y)$, $G_8^{(13)(24)(25)}(y)$, $G_8^{(24)(25)}(y)$, $G_{8+(15)}^{(24)}(y)$ and $G_{8+(15)}^{(24)(25)}(y)$, if found to be a rooted minor of $G(V(B'), B')$, ensures a corresponding rooted minor of $G(V(B), B)$. As before, since G is $5-$connected, there does not exist a separation (C, D) in $G(V(B), B)$ such that $|\{x_1, x_2, x_3, x_4, x_5\} - V(C)| + \lambda(C) \leq 4$, and, hence, $G(V(B), B) \backslash x_3$ is $2-$connected.

If $G(V(B), B) \backslash x_3$ does not have an $(x_1, x_2, x_5, x_4)-$linkage, then, by Corollary 3.1.2, $G(V(B), B) \backslash x_3$ has a planar embedding with x_1, x_2, x_5, x_4, in cyclic order, on the boundary of the infinite face. Without loss of generality, let $G_B^{-x_3}$ be a plane graph embedding $G(V(B), B) \backslash x_3$ in the plane with x_1, x_2, x_5, x_4, in clockwise order, on the boundary of the infinite face. Since $G(V(B), B) \backslash x_3$ is $2-$connected, the infinite face in $G_B^{-x_3}$ is bounded by a cycle S^{-x_3}. Suppose, now, that $N_{G(V(B),B)}(x_3) \subseteq V(S^{-x_3})$.

If there exists a vertex $b \in N_{G(V(B),B)}(x_3) \cap S^{-x_3}(x_4, x_5)$, then let $H_1 := G[S[x_5, x_4]]$ and $H_2 := G[V(B) - V(H_1)]$, and we are done. If $N_{G(V(B),B)}(x_3) - V(A) \subseteq S^{-x_3}(x_5, x_4)$, then $G(V(B), B) \backslash \{ x_1x_3, x_2x_3 \}$ has a planar embedding with x_1, x_2, x_5, x_3, x_4, in clockwise order, on the boundary of the infinite face and, by Lemma 3.3.4, has a rooted $G_7^{(24)(25)}-$minor; if, additionally, $|N_{G(V(B),B)}(x_2)| \geq 3$, then, by Lemma 3.3.5, $G(V(B), B)$

has a rooted $G_9^{(24)(25)}-$ or $G_8^{(13)(24)}-$minor. Similarly, if $|N_{G(V(B),B)}(x_5)| \geq 3$, then, by Lemma 3.3.5, $G(V(B), B)$ has a rooted $G_8^{(24)(25)}-$ or $G_9^{(24)(25)}-$minor; if $|N_{G(V(B),B)}(x_2)| \geq 3$ and $|N_{G(V(B),B)}(x_5)| \geq 3$, then, by Corollary 3.3.6, $G(V(B), B)$ has a rooted $G_{8+(15)}^{(24)(25)}-$ or $G_9^{(24)(25)}-$minor, and we are done again, in each case.

So we may assume that $N_{G(V(B),B)}(x_3) \not\subseteq V(S^{-x_3})$. If there exists a vertex $v \in N_{G(V(B),B)}(x_3) - V(S^{-x_3})$, then there exist in $G_B^{-x_3}$ four pairwise internally disjoint paths between $\{ v \}$ and $\{ x_1, x_2, x_4, x_5 \}$ such that the paths between $\{ v \}$ and $\{ x_1, x_2 \}$ are both disjoint with $S^{-x_3}[x_5, x_4] \cup \{ x_3 \}$; again, let $H_1 := G[S[x_5, x_4]]$ and $H_2 := G[V(B) - V(H_1)]$, and we are done.

So we may assume that there exists an $(x_1, x_2, x_5, x_4)-$linkage in $G(V(B), B)\backslash x_3$. Repeating the argument with an $(x_5, x_2, x_4, x_1)-$ linkage, we may assume that $G(V(B), B)\backslash x_3$ has an $(x_5, x_2, x_4, x_1)-$ linkage as well, for otherwise, either there exist two disjoint connected subgraphs $H_1,\ H_2 \subsetneq G(V(B), B)$ such that $\{ x_2, x_4 \} \subseteq V(H_1)$ and $\{ x_1, x_3, x_5 \} \subseteq V(H_2)$, or $G(V(B), B)$ has a rooted $G_7^{(24)}-$minor; in the case when there do not exist such subgraphs H_1 and H_2 and, for some $Y \subseteq \{ x_2, x_5 \}, Y \neq \emptyset, |N_{G(V(B),B)}(y)| \geq 3$ for each $y \in Y$, $G(V(B), B)$ has a rooted $G_8^{(24)}-$ or $G_9^{(24)}-$minor when $Y = \{ x_2 \}$, a rooted $G_9^{(24)}-$ or $G_8^{(13)(24)(25)}-$minor when $Y = \{ x_5 \}$, and a rooted $G_{8+(15)}^{(24)}-$ or $G_9^{(24)}-$minor when $Y = \{ x_2, x_5 \}$, and we are done.

Let P_{24}, P_{15} be the disjoint paths connecting x_2 with x_4 and x_1 with x_5, respectively, in a $(x_1, x_2, x_5, x_4)-$linkage in $G(V(B), B)\backslash x_3$. Then, for some $x_2' \in P_{24}[x_2, x_4), x_4' \in P_{24}(x_2', x_4], x_1' \in P_{15}[x_1, x_5), x_5' \in P_{15}(x_1', x_5]$, there exist disjoint paths P_{12}, P_{45} in $G(V(B), B)\backslash x_3$, connecting x_1' with x_2' and x_4' with x_5', respectively, each of which meets the paths P_{24}, P_{15} in exactly two of the four vertices x_1', x_2', x_4', x_5'. Consider such a set of four paths for which $|P_{24}(x_4', x_4]| + |P_{15}[x_1, x_1')|$ is minimal. If there exists a path in $G(V(B), B)$ between $\{ x_3 \}$ and $P_{12}[x_1', x_2') \cup P_{45}(x_4', x_5'] \cup P_{15}[x_1, x_5]$ that is disjoint with $P_{24}[x_2, x_4]$, then let $H_1 := P_{24}$ and $H_2 := G[V(B) - V(H_1)]$, and we are done. Similarly, if there exists a path in $G(V(B), B)$ between $\{ x_3 \}$ and $P_{12}[x_1', x_2'] \cup P_{24}[x_2, x_4') \cup P_{15}[x_1, x_5')$ that is disjoint with $P_{24}[x_4', x_4] \cup P_{45}[x_4', x_5'] \cup P_{15}[x_5', x_5]$, then let $H_1 := G[P_{24}[x_4', x_4] \cup P_{45}[x_4', x_5'] \cup P_{15}[x_5', x_5]]$ and $H_2 := G[V(B) - V(H_1)]$, and we are done again. So we may assume that there does not exist a path in $G(V(B), B)$ between $\{ x_3 \}$ and $P_{12}[x_1', x_2'] \cup P_{24}[x_2, x_4') \cup P_{45}(x_4', x_5'] \cup P_{15}[x_1, x_5]$ that is disjoint with $P_{24}[x_4', x_4]$. Since $|N_{G(V(B),B)}(x_3) - V(A)| \geq 2$, we may also assume that there exists a path in $G(V(B), B)$ between $\{ x_3 \}$ and $P_{24}(x_4', x_4)$ that is disjoint with $P_{12}[x_1', x_2'] \cup P_{24}[x_2, x_4'] \cup P_{45}[x_4', x_5'] \cup P_{15}[x_1, x_5] \cup \{ x_4 \}$ and, hence, that $P_{24}(x_4', x_4) \neq \emptyset$.

If there exists a path in $G(V(B), B)$ between $P_{24}(x_4', x_4]$ and $P_{15}(x_1', x_5] \cup P_{45}(x_4', x_5']$ that is disjoint with $\{ x_3 \} \cup P_{24}[x_2, x_4') \cup P_{12}(x_1', x_2'] \cup P_{15}[x_1, x_1')$, then we can choose a new path

P'_{45} connecting $x''_4 \in P_{24}(x'_4, x_4]$ and $x''_5 \in P_{15}(x'_1, x_5]$ such that $|P_{24}(x''_4, x_4]| + |P_{15}[x_1, x'_1)| < |P_{24}(x'_4, x_4]| + |P_{15}[x_1, x'_1)|$, a contradiction. So we may assume that such a path does not exist. Similarly, we may assume that there does not exist a path in $G(V(B), B)$ between $P_{24}(x'_4, x_4]$ and $P_{24}[x_2, x'_4) \cup P_{12}(x'_1, x'_2]$ that is disjoint with $\{ x_3 \} \cup P_{15}[x_1, x_5] \cup P_{45}(x'_4, x'_5]$. If $P_{15}[x_1, x'_1) \neq \emptyset$, we may assume by a similar logic that there does not exist a path in $G(V(B), B)$ between $P_{15}[x_1, x'_1)$ and $P_{24}[x_2, x'_4) \cup P_{12}(x'_1, x'_2] \cup P_{45}(x'_4, x'_5] \cup P_{15}(x'_1, x_5]$ that is disjoint with $\{ x_3 \} \cup P_{24}(x'_4, x_4]$. Thus, $\{ x'_1, x'_4 \}$ 2−separates $G(V(B), B)$ as $(B - D, D)$, with $V(B - D) \cap V(D) = \{ x'_1, x'_4 \}$, such that $\{ x_2, x_5 \} \subseteq V(B - D)$ and $\{ x_1, x_3, x_4 \} \subseteq V(D)$, and, as a result, $\{ x'_1, x'_4 \} \cap \{ x_1, ..., x_5 \} = \emptyset, V(B - D) = \{ x'_1, x_2, x'_4, x_5 \}$, $N_{G(V(B),B)}(x_2) \cup \{ x_5 \} = \{ x'_1, x'_4, x_5 \}$ and $N_{G(V(B),B)}(x_5) \cup \{ x_2 \} = \{ x'_1, x_2, x'_4 \}$.

Now, if $|V(B) - V(A)| \geq 6$, then (C, D) is a 5−separation in G, where $C := A \cup (B - D)$, and, by Proposition 3.2.1, either there exists a 5−separation (A', B') in G, non-crossing with (C, D) (and, hence, (A, B)), such that $C \subsetneq A', B' \subsetneq D$ and, for each $u \in V(A') \cap V(B'), |N_{G(V(B'),B')}(u) - V(A')| \geq 2$, contradicting the minimality of $|B|$, or we are done by Proposition 3.3.1. So we may assume that $|V(B) - V(A) - \{ x'_1, x'_4 \}| \leq 3$. Since, either there exists a vertex $v \in V(B) - V(A) - \{ x'_1, x'_4 \}$ which is connected to $\{ x_1, x'_1, x_3, x_4, x'_4 \}$ via five paths contained in $G(V(D), D)$ and pairwise sharing only the vertex v, or $\{ x'_1x_1, x'_1x_3, x'_4x_1, x'_4x_3 \} \subseteq D$, we may also assume that $N_{G(V(D),D)}(x_4) \cap \{ x'_1, x'_4 \} = \emptyset$ and, hence, that $|V(B) - V(A) - \{ x'_1, x'_4 \}| \geq 2$, for otherwise there exist two disjoint connected subgraphs $H'_1, H'_2 \subsetneq G(V(D), D)$ such that, for some $x' \in \{ x'_1, x'_4 \}, \{ x_4, x' \} \subseteq V(H'_1)$ and $\{ x_1, x'_1, x_3, x'_4 \} - \{ x' \} \subseteq V(H'_2)$, and they can be extended to form the subgraphs H_1 and H_2, respectively. Let $\{ a, b \} \subseteq N_{G(V(D),D)}(x_4) - \{ x_1, x_3 \}$. Then there exist in $G(V(D), D)$ two disjoint paths P_a and P_b disjoint with $\{ x_1, x_3, x_4 \}$ and connecting, respectively, a and b with $\{ x'_1, x'_4 \}$, for otherwise $\{ a, b \}$ is 4−separated from the rest of the graph G. If there exists a vertex $c \in V(B) - V(A) - \{ x'_1, x'_4, a, b \}$ such that $c \notin V(P_a) \cup V(P_b)$, we may assume that c is adjacent to at least one of a and b (say a, without loss of generality) for otherwise c is adjacent to every vertex in $\{ x_1, x'_1, x_3, x_4, x'_4 \}$ and we can find two disjoint connected subgraphs H'_1 and H'_2 in $G(V(D), D)$ that can be extended to form the subgraphs H_1 and H_2, respectively, as described earlier. Let $P'_a := P_a \cup \{ ac \}$ if there exists such a vertex c and $P'_a := P_a$ otherwise (assuming, without loss of generality, that if there exists a vertex $c \in V(B) - V(A) - \{ x'_1, x'_4, a, b \}$ such that $c \in V(P_a) \cup V(P_b)$, then $c \in V(P_a)$). If each of x_1 and x_3 has a neighbor in either P'_a or P_b then we are done since we can find two disjoint connected subgraphs H'_1 and H'_2 in $G(V(D), D)$ that can be extended to form the subgraphs H_1 and H_2, respectively, as described earlier. So we may assume, without loss of generality, that x_1 does not have a neighbor in P_b and x_3 does not have a neighbor in P'_a. Then each of x'_1 and x'_4 is adjacent to b and has a neighbor in $\{ a, c \}$, for otherwise either b or $\{ a, c \}$ is $4-$separated from the rest of the graph G. As a result, if $x_1' \in V(P_a')$ then there exist two disjoint connected subgraphs $H_1', H_2' \subsetneq G(V(D), D)$ such that $\{ x_1, a, c, x_3, x_4' \} \subseteq V(H_1')$ and $\{ x_4, b, x_1' \} \subseteq V(H_2')$, and if $x_1' \in V(P_b)$ then there exist two disjoint connected subgraphs $H_1', H_2' \subsetneq G(V(D), D)$ such that $\{ x_1, a, c, x_3, x_1' \} \subseteq V(H_1')$ and $\{ x_4, b, x_4' \} \subseteq V(H_2')$. In either case, H_1' and H_2' can be extended to form the subgraphs H_1 and H_2, respectively. $\square$

Chapter 4

Nested Separations in the Larger Sides of Separations

In this chapter, we find a set of unavoidable rooted minors of the intersection of the "larger" sides of two non-crossing separations in a $5-$connected graph which itself, in turn, is separated by each one of a large family of nested separations. As before, each minor that we find is rooted in the separating sets of the two non-crossing separations considered. These minors are then patched together with the unavoidable rooted minors of the smaller sides of the two separations to construct a set of unavoidable minors of the complete graph - the remaining unavoidable minors of large $5-$connected graphs mentioned in Theorem 1.2.1. We start by defining nested separations and proving an observation that relates a family of these to a bounded-degree tree-decomposition.

4.1 Nested Separations

Recall that two separations (A, B) and (C, D) in a graph G *cross* if $A \cap C \neq \emptyset$, $A \cap D \neq \emptyset$, $B \cap C \neq \emptyset$ and $B \cap D \neq \emptyset$. For each tree-decomposition T of G, the separations (A_f, B_f) corresponding to the edges f of T form a family of (pairwise) *non-crossing* separations: for every $e, f \in E(T)$, either $A_e \subseteq A_f$ and $B_f \subseteq B_e$, or $A_f \subseteq A_e$ and $B_e \subseteq B_f$. In particular, if $f_1, f_2, ..., f_\ell \in E(T)$ form a path P of length ℓ in that order in T, then either $A_{f_1} \subseteq A_{f_2} \subseteq ... \subseteq A_{f_\ell}$ and $B_{f_\ell} \subseteq ... \subseteq B_{f_2} \subseteq B_{f_\ell}$, or $B_{f_1} \subseteq B_{f_2} \subseteq ... \subseteq B_{f_\ell}$ and $A_{f_\ell} \subseteq ... \subseteq A_{f_2} \subseteq A_{f_\ell}$, and the family $\{ (A_f, B_f) : f \in E(P) \}$ is said to be one of *nested* separations. As before, the separations in a family are *distinct* if, for every pair of separations $((A_e, B_e), (A_f, B_f))$ in the family, $A_e \neq A_f$ and $A_e \neq B_f$.

Proposition 4.1.1. *There exists a function $f_{4.1.1} : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that, for all $\theta, \delta, n \in \mathbb{N}$ with $\theta > 0,\ \delta > 0,\ \delta \neq 2,\ n > 0$, if T is a $\theta-$tree-decomposition with degree δ of a graph G with $|V(G)| \geq f_{4.1.1}(\theta, \delta, n)$, then G contains a family of distinct nested $\theta-$separations of size at least n.*

Proof. Let $f_{4.1.1}(\theta, \delta, n) = 5\ \delta^{n+1}\ \theta$. Let T be a $\theta-$tree-decomposition with degree δ of a graph G with $|V(G)| \geq f_{4.1.1}(\theta, \delta, n)$. We may assume that $|V(T)|$ and $|E(T)|$ are both minimal so that, for each internal node $x \in V(T), T \backslash x$ contains at least three components each containing a distinct edge of G. By Theorem 2.2.2, $tw_\theta(G) \leq \theta\ \delta$. If T does not have a path of length at least n, then $|V(T)| \leq 2\ \delta^n$ and $|V(G)| \leq 2\ \delta^n\ (tw_\theta(G)+1)\ \leq 4\ \delta^{n+1}\ \theta$, a contradiction. The result then follows from the observation that each edge in any path of length at least n in T corresponds to a distinct separation in G. □

4.2 Unavoidable Minors in the Absence of Large 6−Connected Sets

The goal of this section is to find a set of unavoidable minors of every sufficiently large 5−connected graph that does not contain a large 6−connected set by finding a set of unavoidable rooted minors of the intersection of the "larger" sides of two non-crossing separations in the graph. Labelled graph descriptions and figures (Figures *A*.1 and *A*.2) of these rooted minors ($G^1, G^{1(a)}, G^{1(b)}, G^{1(e)}, G^2, G^{2(c)}$, etc.) and other intermediate structures can be found in the appendix (see A.2) where we give explicit graph constructions for the unavoidable minors we find in this section − the remaining unavoidable minors of large 5−connected graphs mentioned in Theorem 1.2.1 (which were not accounted for in Corollary 2.1.4).

Lemma 4.2.1. *There exists a function $f_{4.2.1} : \mathbb{N} \to \mathbb{N}$ such that, for all $n \in \mathbb{N}$ with $n \geq 5$, if G is a 5−connected graph that contains a family of distinct nested 5−separations of size at least $f_{4.2.1}(n)$ but does not have a minor isomorphic to the graph $W(1,3,n)$, then, G has a minor isomorphic to $W_j(2,1,n)$, $TW_j(2,1,n)$, $W_1^-(3,0,n)$, $TW_1^-(3,0,n)$, $W_{2(a)}^-(3,0,n)$, $W_{2(b)}^-(3,0,n)$, $TW_{2(a)}^-(3,0,n)$, $TW_{2(b)}^-(3,0,n)$, $CW_{k(a)}(2,1,n)$ or $CW_{k(b)}(2,1,n)$, where $j \in \{1,2\}$ and $k \in \{1,...,6\}$.*

Proof. Let $f_{4.2.1} = 11(8+5*48n*2(3n+10)*12*2(20n+14)(12n+13)(32n+71)) < 2^{34}n^5$. Let G be a 5−connected graph that contains a family $\mathcal{F}$ of distinct nested 5−separations of size at least $f_{4.2.1}(n)$ but does not have a minor isomorphic to the graph $W(1,3,n)$. In particular, let $\mathcal{F} := \{ (A_j, B_j) : j \in \{ 1, ..., N \} \}$, where $N \geq f_{4.2.1}(n)$, such that $A_{j_1} \subseteq A_{j_2}$ and $B_{j_2} \subseteq B_{j_1}$ whenever $j_1 < j_2$. Thus, we may also treat $\mathcal{F}$ as a sequence of separations ordered by the containment relation on the set of first partitions of the separations in $\mathcal{F}$. We will abuse the notation slightly in this way by treating a family of distinct nested $5-$separations in G both as a set and as a sequence.

Since any set of 12 distinct nested $5-$separations contains at least two $5-$separations such that the separating set of either is not contained in that of the other, there exists a subsequence $\mathcal{F}_1 \subseteq \mathcal{F}$ with $|\mathcal{F}_1| \geq \lfloor |\mathcal{F}|/11 \rfloor - 8$ such that the separating sets of any two $5-$separations in $\mathcal{F}_1$ differ in at least one vertex. Upto relabeling of separations, we may assume that $\mathcal{F}_1 = \{ (A_j, B_j) : j \in \{ 1, ..., N_1 \} \}$, where $N_1 \geq \lfloor N/11 \rfloor - 8$. Additionally, we may assume that $|V(A_1) - V(B_1)| \geq 4, |V(B_{N_1}) - V(A_{N_1})| \geq 4|$ and, since G is $5-$connected, that $|V(A_j) \cap V(B_j)| = 5$, for each $j \in \{ 1, ..., N_1 \}$. We may further assume that, for each $j \in \{ 1, ..., N_1 - 1 \}, V(A_j) \cap V(B_j)$ is connected to $V(A_{j+1}) \cap V(B_{j+1})$ by a set of 5 disjoint paths each of which is contained in $G(V(B_j \cap A_{j+1}), B_j \cap A_{j+1})$; the union of all such sets of paths gives us 5 disjoint paths $P_1, ..., P_5$ that connect $V(A_1) \cap V(B_1)$ with $V(A_{N_1}) \cap V(B_{N_1})$. Then there exists a subsequence $\mathcal{F}'_1 \subseteq \mathcal{F}_1$ with $|\mathcal{F}'_1| \geq \lfloor \mathcal{F}_1/5 \rfloor$ such that, for some $P' \in \{ P_1, ..., P_5 \}$, the separating sets of any two $5-$separations in $\mathcal{F}'_1$ meet $V(P')$ in distinct vertices. Without loss of generality, let $P' = P_1$. Again, upto relabeling of separations, we may assume that $\mathcal{F}'_1 = \{ (A_j, B_j) : j \in \{ 1, ..., N'_1 \} \}$, where $N'_1 \geq \lfloor N_1/5 \rfloor$. For each $j \in \{ 1, ..., N'_1 \}$, let $V(A_j) \cap V(B_j) \cap P_1 = \left\{ u_j^{(1)} \right\}$.

Consider now, for some $j' \in \{ 1, ..., N'_1 - 12n + 1 \}$, a subsequence $\{ (A_{j'+j}, B_{j'+j}) : j \in \{ 0, 1, ..., 12n - 1 \} \} \subseteq \mathcal{F}'_1$ of $12n$ distinct nested $5-$separations in G. For each $j \in \{ 0, 1, ..., 12n - 1 \}$, suppose that the separating set of $(A_{j'+j}, B_{j'+j})$ meets $V(P_i)$ in the same vertex u_i, for each $i \in \{ 2, ..., 5 \}$. Then, for each $j \in \{ 0, 3, ..., 12n - 3 \}$, there exist 3 (internally) disjoint paths connecting $P_1(u^{(1)}_{j'+j}, u^{(1)}_{j'+j+2})$ with some $3-$subset $\left\{ a^{(j)}, b^{(j)}, c^{(j)} \right\}$ of $\{ u_2, u_3, u_4, u_5 \}$, each contained in $G(V(B_{j'+j} \cap A_{j'+j+2}), B_{j'+j} \cap A_{j'+j+2})$ and each disjoint with $\{ u_2, u_3, u_4, u_5 \} - \left\{ a^{(j)}, b^{(j)}, c^{(j)} \right\}$, for otherwise $P_1(u^{(1)}_{j'+j}, u^{(1)}_{j'+j+2})$ is $4-$separated in G. For some $\{ j_1, ..., j_n \} \subseteq \{ 0, 3, ..., 12n - 3 \}, a^{(j_1)} = ... = a^{(j_n)}, b^{(j_1)} = ... = b^{(j_n)}$ and $c^{(j_1)} = ... = c^{(j_n)}$; without loss of generality, let $\left\{ a^{(j_1)}, b^{(j_1)}, c^{(j_1)} \right\} = \{ u_2, u_3, u_4 \}$. Since u_5 is connected with $u^{(1)}_{j'+j_1}$ by a path (say Q') in $G(V(A_{j'+j_1}), A_{j'+j_1})$ and with $u^{(1)}_{j'+j_n+2}$ by a path (say Q'') contained in $G(V(B_{j'+j_n+2}), B_{j'+j_n+2})$, G can be reduced to a $W(1, 3, n)$-minor where u_2, u_3 and u_4 form the three hubs and $P_1 \cup Q' \cup Q''$ forms the rim, a contradiction. Thus, we may assume that in every subsequence of $12n$ distinct nested $5-$separations contained in $\mathcal{F}'_1$, there exist at least two whose separating sets meet at least one of the sets $V(P_2), V(P_3), V(P_4)$ and $V(P_5)$ in distinct vertices. Then, as before, there exists a subsequence $\mathcal{F}_2 \subseteq \mathcal{F}_1'$ with $|\mathcal{F}_2| \geq \lfloor \mathcal{F}_1'/(12n*4) \rfloor$ such that, for some $P' \in \{ P_2, ..., P_5 \}$, the separating sets of any two $5-$separations in $\mathcal{F}_2$ meet both $V(P_1)$ and $V(P')$ in distinct vertices. Without loss of generality, let $P' = P_2$. Upto relabeling of separations, we may assume that $\mathcal{F}_2 = \{ (A_j, B_j) : j \in \{ 1, ..., N_2 \} \}$, where $N_2 \geq \lfloor N_1'/(48n) \rfloor$. For each $i \in \{ 1, ..., 5 \}, j \in \{ 1, ..., N_2 \}$, let $V(A_j) \cap V(B_j) \cap P_i = \left\{ u_j^{(i)} \right\}$.

Suppose, for some $j' \in \{ 1, ..., N_2 - 3n + 1 \}$, there exists a subsequence $\{ (A_{j'+j}, B_{j'+j}) : j \in \{ 0, 1, ..., 3n-1 \} \} \subseteq \mathcal{F}_2$ of $3n$ distinct nested $5-$separations in G such that, for each $j \in \{ 0, 3, ..., 3n-3 \}$, there does not exist a path in $G(V(B_{j'+j} \cap A_{j'+j+2}), B_{j'+j} \cap A_{j'+j+2})$ between $P_a[u_{j'+j}^{(a)}, u_{j'+j+2}^{(a)}]$ and $P_b[u_{j'+j}^{(b)}, u_{j'+j+2}^{(b)}]$ that is disjoint with P_c whenever $u_{j'+j}^{(a)} \neq u_{j'+j+2}^{(a)}$ and $u_{j'+j}^{(b)} \neq u_{j'+j+2}^{(b)}$, for any $a, b \in \{ 1, ..., 5 \}, a \neq b, c \in \{ 1, ..., 5 \} - \{ a, b \}$. Then, for each $i \in \{ 3, 4, 5 \}, u_{j'}^{(i)} = ... = u_{j'+3n-1}^{(i)}$, and, for each $j \in \{ 0, 3, ..., 3n-3 \}$, there exist in $G(V(B_{j'+j} \cap A_{j'+j+2}), B_{j'+j} \cap A_{j'+j+2})$ two sets of three (internally) disjoint paths – one connecting $\left\{ u_{j'}^{(3)}, u_{j'}^{(4)}, u_{j'}^{(5)} \right\}$ with $P_1(u_{j'+j}^{(1)}, u_{j'+j+2}^{(1)})$ and the other connecting $\left\{ u_{j'}^{(3)}, u_{j'}^{(4)}, u_{j'}^{(5)} \right\}$ with $P_2(u_{j'+j}^{(2)}, u_{j'+j+2}^{(2)})$, for otherwise one of $P_1(u_{j'+j}^{(1)}, u_{j'+j+2}^{(1)})$ and $P_2(u_{j'+j}^{(2)}, u_{j'+j+2}^{(2)})$ is $4-$separated from the rest of the graph G, for some $j \in \{0, 3, ..., 3n-3\}$. Since two disjoint paths (say Q' and Q'', respectively) connect $u_{j'}^{(1)}$ with $u_{j'}^{(2)}$ in $G(V(A_{j'}), A_{j'})$ and $u_{j'+3n-1}^{(1)}$ with $u_{j'+3n-1}^{(2)}$ in $G(V(B_{j'+3n-1}), B_{j'+3n-1})$, G can be reduced, in this case, to a $W(1, 3, n)$-minor with $u_{j'}^{(3)}, u_{j'}^{(4)}$ and $u_{j'}^{(5)}$ forming the three hubs and $P_1[u_{j'}^{(1)}, u_{j'+3n-1}^{(1)}] \cup P_2[u_{j'}^{(2)}, u_{j'+3n-1}^{(2)}] \cup Q' \cup Q''$ forming the rim, a contradiction. So we may assume that, for some $j \in \{ 0, 3, ..., 3n-3 \}$, there exist $a_{j'}, b_{j'} \in \{ 1, ..., 5 \}, a_{j'} \neq b_{j'}$, such that $u_{j'+j}^{(a_{j'})} \neq u_{j'+j+2}^{(a_{j'})}, u_{j'+j}^{(b_{j'})} \neq u_{j'+j+2}^{(b_{j'})}$, and $P_{a_{j'}}[u_{j'+j}^{(a_{j'})}, u_{j'+j+2}^{(a_{j'})}]$ and $P_{b_{j'}}[u_{j'+j}^{(b_{j'})}, u_{j'+j+2}^{(b_{j'})}]$ are connected by a path that is contained in $G(V(B_{j'+j} \cap A_{j'+j+2}), B_{j'+j} \cap A_{j'+j+2})$ and is disjoint with P_c, for each $c \in \{ 1, ..., 5 \} - \{ a_{j'}, b_{j'} \}$. Then, as before, there exists (upto relabeling of separations) a subsequence $\mathcal{F}_3 := \{ (A_j, B_j) : j \in \{ 1, ..., 2N_3 \} \} \subseteq \mathcal{F}_2$ with $N_3 \geq \lfloor N_2/(3n*10) \rfloor$ such that, for some $a, b \in \{ 1, ..., 5 \}$, where $a \neq b$, and for each $j' \in \{ 1, 2, ..., N_3 \}$, we have that $|P_a[u_{2j'-1}^{(a)}, u_{2j'}^{(a)}]| \geq 3, |P_b[u_{2j'-1}^{(b)}, u_{2j'}^{(b)}]| \geq 3$, and $P_a[u_{2j'-1}^{(a)}, u_{2j'}^{(a)}]$ and $P_b[u_{2j'-1}^{(b)}, u_{2j'}^{(b)}]$ are connected by a path $Q_{j'}^{(ab)}$ that is contained in $G(V(B_{2j'-1} \cap A_{2j'}), B_{2j'-1} \cap A_{2j'})$ and is disjoint with P_c, for each $c \in \{ 1, ..., 5 \} - \{ a, b \}$. Without loss of generality, let $a = 1$ and $b = 2$.

For each $j' \in \{ 1, 2, ..., N_3 \}$, at least one of $P_1[u_{2j'-1}^{(1)}, u_{2j'}^{(1)}]$ and $P_2[u_{2j'-1}^{(2)}, u_{2j'}^{(2)}]$ is also connected to $P_c[u_{2j'-1}^{(c)}, u_{2j'}^{(c)}]$, for some $c \in \{ 3, 4, 5 \}$, via a path that is disjoint with the other and with $P_{c'}$, for each $c' \in \{ 3, 4, 5 \} - \{ c \}$, and is contained in $G(V(B_{2j'-1} \cap A_{2j'}), B_{2j'-1} \cap A_{2j'})$ for otherwise $P_1(u_{2j'-1}^{(1)}, u_{2j'}^{(1)}) \cup P_2(u_{2j'-1}^{(2)}, u_{2j'}^{(2)})$ is $4-$separated from the rest of the graph G. Note that such a path is also disjoint with $Q^{(12)}_{j''}$, for each $j'' \in \{ 1, 2, ..., N_3 \} - \{ j' \}$. Thus, there exists (upto relabeling of separations) a subsequence $\mathcal{F}_4 := \{ (A_j, B_j) : j \in \{ 1, ..., 2N_4 \} \} \subseteq \mathcal{F}_3$ with $N_4 \geq \lfloor N_3/(6 * 2) \rfloor$ such that, for some $b' \in \{ 1, 2 \}, c \in \{ 3, 4, 5 \}$ and for each $j' \in \{ 1, 2, ..., N_4 \}$, we have that $|P_1[u^{(1)}_{2j'-1}, u^{(1)}_{2j'}]| \geq 3$, $|P_2[u^{(2)}_{2j'-1}, u^{(2)}_{2j'}]| \geq 3$, and $P_{b'}[u^{(b')}_{2j'-1}, u^{(b')}_{2j'}]$ is connected with both $P_{a'}[u^{(a')}_{2j'-1}, u^{(a')}_{2j'}]$ and $P_c[u^{(c)}_{2j'-1}, u^{(c)}_{2j'}]$, where $a' \in \{ 1, 2 \} - \{ b' \}$, via (internally) disjoint paths $Q^{(a'b')}_{j'}$ and $Q^{(b'c)}_{j'}$, respectively, both contained in $G(V(B_{2j'-1} \cap A_{2j'}), B_{2j'-1} \cap A_{2j'})$, where $Q^{(a'b')}_{j'}$ is disjoint with P'_c for each $c' \in \{ 3, 4, 5 \}$ and $Q^{(b'c)}_{j'}$ is disjoint with P'_c for each $c' \in \{ 1, ..., 5 \} - \{ b, c \}$. Without loss of generality, let $a' = 1, b' = 2$ and $c = 3$.

Case 1: Suppose, for some $c' \in \{ 4, 5 \}$, $J' \subseteq \{ 1, 2, ..., N_4 \}$, where $J' := \{ j'_1, j'_2, ..., j'_{2n} \}$, $j'_1 < j'_2 < ... < j'_{2n}$, and for each $j' \in J'$, there exists a path $Q^{(2c')}_{j'}$ connecting $P_2[u^{(2)}_{2j'-1}, u^{(2)}_{2j'}]$ with $P_{c'}[u^{(c')}_{2j'-1}, u^{(c')}_{2j'}]$ that is disjoint with P_d, for each $d \in \{ 1, 3, 4, 5 \} - \{ c' \}$, and is contained in $G(V(B_{2j'-1} \cap A_{2j'}), B_{2j'-1} \cap A_{2j'})$. Then, as before, G can be reduced to a $W(1, 3, n)$-minor with the three hubs formed by contracting segments of the paths P_1, P_3 and $P_{c'}$ contained in $G[V(B_{2j'_1-1} \cap A_{2j'_{2n}}), B_{2j'_1-1} \cap A_{2j'_{2n}}]$, and the rim formed (in part) by the segment of the path P_2 contained in $G[V(B_{2j'_1-1} \cap A_{2j'_{2n}}), B_{2j'_1-1} \cap A_{2j'_{2n}}]$, a contradiction.

Case 2: Suppose, for some $c' \in \{ 4, 5 \}$, $J' \subseteq \{ 1, 2, ..., N_4 \}$, where $J' := \left\{ j'_1, j'_2, ..., j'_{8n+8} \right\}$, $j'_1 < j'_2 < ... < j'_{8n+8}$, and for each $j' \in J'$, there exists a path $Q^{(1c')}_{j'}$ connecting $P_1[u^{(1)}_{2j'-1}, u^{(1)}_{2j'}]$ with $P_{c'}[u^{(c')}_{2j'-1}, u^{(c')}_{2j'}]$ that is disjoint with P_d, for each $d \in \{ 2, 3, 4, 5 \} - \{ c' \}$, and is contained in $G(V(B_{2j'-1} \cap A_{2j'}), B_{2j'-1} \cap A_{2j'})$. Say $c' = 5$. Observe that if $|P_4[u^{(4)}_{2j'_1-1}, u^{(4)}_{2j'_{8n+8}}]| > 1$ then, either $u^{(4)}_r$ has degree at least 3 in $G[V(A_r), A_r]$, for some $r \leq 2j'_{4n+5} - 1$, or $u^{(4)}_r$ has degree at least 3 in $G[V(B_r), B_r]$, for some $r \geq 2j'_{4n+4}$. By Propositions 3.2.1 and 3.3.2 and Lemma 3.3.7, $G[V(A_r), A_r]$ (or $G[V(B_r), B_r]$, whichever $u^{(4)}_r$ has degree at least 3 in) can be reduced to one of the graphs G_2, G_3, G_5, G_6, G_8 and G_9; similarly, each of $G[V(A_{2j'_1-1}), A_{2j'_1-1}], G[V(B_{2j'_{4n+4}}), B_{2j'_{4n+4}}], G[V(A_{2j'_{4n+5}-1}), A_{2j'_{4n+5}-1}]$ and $G[V(B_{2j'_{8n+8}}), B_{2j'_{8n+8}}]$ can be reduced to one of the graphs G_1, G_4 and G_7. Since each of $G[V(B_{2j'_1-1} \cap A_{2j'_{4n+4}}), B_{2j'_1-1} \cap A_{2j'_{4n+4}}]$ and $G[V(B_{2j'_{4n+5}-1} \cap A_{2j'_{8n+8}}), B_{2j'_{4n+5}-1} \cap A_{2j'_{8n+8}}]$ can be reduced to the graph $G^{1(e)}$, G contains a minor isomorphic to either $CW_{k(a)}(2, 1, n)$ or $CW_{k(b)}(2, 1, n)$, for some $k \in \{ 1, ..., 6 \}$. The case when $c' = 4$ is identical upto relabeling the paths P_4 and P_5.

Case 3: Suppose, for some $j' \in \{ 1, ..., N_4 - 12n - 12 \}$, $P_3[u^{(3)}_{2j'-1}, u^{(3)}_{2(j'+12n+12)}] = \left\{ u^{(3)}_{j'} \right\}$. Additionally, suppose, for each $j \in \{ 0, 1, ..., 12n + 12 \}, \{ a'', b'' \} = \{ 1, 2 \}, \{ c'', d'' \} =$

$\{4,5\}$, there does not exist a path connecting $P_{a''}[u^{(a'')}_{2(j'+j)-1}, u^{(a'')}_{2(j'+j)}]$ with $P_{c''}[u^{(c'')}_{2(j'+j)-1}, u^{(c'')}_{2(j'+j)}]$ that is disjoint with $P_{b''}, P_3$ and $P_{d''}$ and is contained in $G[V(B_{2(j'+j)-1} \cap A_{2(j'+j)}), B_{2(j'+j)-1} \cap A_{2(j'+j)}]$, so that $\{ u^{(1)}_{2(j'+j)-1}, u^{(2)}_{2(j'+j)-1}, u^{(3)}_{j'}, u^{(2)}_{2(j'+j)}, u^{(1)}_{2(j'+j)} \}$ 5−separates G as (C_j, D_j) with $P_1[u^{(1)}_{2(j'+j)-1}, u^{(1)}_{2(j'+j)}] \cup P_2[u^{(2)}_{2(j'+j)-1}, u^{(2)}_{2(j'+j)}] \subseteq V(C_j)$. Then, by Propositions 3.2.1 and 3.3.2 and Lemma 3.3.7, for each $j \in \{1, 3, ..., 12n+11\}, G[V(C_j), C_j]$ can be reduced to one of G_1, G_4 and G_7, with $u^{(1)}_{2(j'+j)-1}, u^{(2)}_{2(j'+j)-1}, u^{(3)}_{j'}, u^{(2)}_{2(j'+j)}$ and $u^{(1)}_{2(j'+j)}$ forming the vertices x_1, x_2, x_3, x_4 and x_5, respectively, and for some $\{j_1, j_2, ..., j_{2n+2}\} \subseteq \{1, 3, ..., 12n+11\}$, $G[V(C_{j_r}), C_{j_r}]$ can be reduced to the same graph G_C for each $j_r \in \{j_1, j_2, ..., j_{2n+2}\}$, where $G_C \in \{G_1, G_4, G_7\}$. When $G_C = G_7, G[V(B_{2j'-1} \cap A_{2(j'+12n+12)}), B_{2j'-1} \cap A_{2(j'+12n+12)}]$ can be reduced to the graph $G^{1(b)}$; similarly, when $G_C \in \{G_4, G_7\}$, $G[V(B_{2j'-1} \cap A_{2(j'+12n+12)}), B_{2j'-1} \cap A_{2(j'+12n+12)}]$ can be reduced to the graph $G^{1(a)}$ (the case when $G_C = G_1$ requires swapping the labels $u^{(1)}_{2(j'+j_{2r-1})+i}$ and $u^{(2)}_{2(j'+j_{2r-1})+i}$ for each $r \in 1, 2, ..., n+1, i \in \{0, 1, ..., 2(j_{2r} - j_{2r-1})\}$). As a result, since, by Propositions 3.2.1 and 3.3.2 and Lemma 3.3.7, $\left\{ u^{(1)}_1, u^{(2)}_1 \right\}$ is connected with $\left\{ u^{(4)}_1, u^{(5)}_1 \right\}$ via two disjoint paths, each disjoint with P_3 and contained in $G(V(A_1), A_1)$, and $\left\{ u^{(1)}_{2N_4}, u^{(2)}_{2N_4} \right\}$ is connected with $\left\{ u^{(4)}_{2N_4}, u^{(5)}_{2N_4} \right\}$ via two disjoint paths, each disjoint with P_3 and contained in $G(V(B_{2N_4}), B_{2N_4})$, G contains a minor isomorphic to either $W_j(2,1,n)$ or $TW_j(2,1,n)$, for some $j \in \{1, 2\}$.

Case 4: Suppose, for some $j' \in \{1, ..., N_4 - 2(12n+13)(32n+71) + 1\}$ and for each $j \in \{0, 1, ..., 2(12n+13)(32n+71) - 1\}, \{a'', b''\} = \{1, 2\}, \{c'', d''\} = \{4, 5\}$, there does not exist a path connecting $P_{a''}[u^{(a'')}_{2(j'+j)-1}, u^{(a'')}_{2(j'+j)}]$ with $P_{c''}[u^{(c'')}_{2(j'+j)-1}, u^{(c'')}_{2(j'+j)}]$ that is disjoint with $P_{b''}, P_3$ and $P_{d''}$ and is contained in $G[V(B_{2(j'+j)-1} \cap A_{2(j'+j)}), B_{2(j'+j)-1} \cap A_{2(j'+j)}]$. Additionally, suppose, for each $j \in \{0, 1, ..., (12n+13)(2(32n+71)-1)\}, |P_3[u^{(3)}_{2(j'+j)-1}, u^{(3)}_{2(j'+j+12n+12)}]| > 1$, so that, for some $J' \subseteq \{j', j'+1, ..., j'+2(12n+13)(32n+71)-1\}$, where $J' := \{j'_1, j'_2, ..., j'_{32n+71}\}, j'_1 < j'_2 < ... < j'_{32n+71}$, $u^{(3)}_{2j'_r-1} \neq u^{(3)}_{2j'_r}$, for each $j'_r \in J'$, and $u^{(3)}_{2j'_r} \neq u^{(3)}_{2j'_{r+1}-1}$, for each $j'_r \in J' - \{j'_{32n+71}\}$. Then, either for some $J'' \subseteq J'$, where $|J''| \geq 16n+16$, and for each $j'_r \in J''$, $P_3[u^{(3)}_{2j'_r-1}, u^{(3)}_{2j'_r}]$ is connected with one of $P_4[u^{(4)}_{2j'_r-1}, u^{(4)}_{2j'_r}]$ and $P_5[u^{(5)}_{2j'_r-1}, u^{(5)}_{2j'_r}]$ via a path that is disjoint with the other and with P_1 and P_2, and that is contained in $G[V(B_{2j'_r-1} \cap A_{2j'_r}), B_{2j'_r-1} \cap A_{2j'_r}]$, or for some $J'' \subseteq J'$, where $|J''| \geq 16n+56$, and for each $j'_r \in J''$, there does not exist such a path between $P_3[u^{(3)}_{2j'_r-1}, u^{(3)}_{2j'_r}]$ and either of $P_4[u^{(4)}_{2j'_r-1}, u^{(4)}_{2j'_r}]$ and $P_5[u^{(5)}_{2j'_r-1}, u^{(5)}_{2j'_r}]$. In the former case, for some $J''' \subseteq J''$, where

$|J'''| \geq 8n+8$, $P_3[u^{(3)}_{2j'_r-1}, u^{(3)}_{2j'_r}]$ is connected with the same $P_{c'}[u^{(c')}_{2j'_r-1}, u^{(c')}_{2j'_r}]$, where $c' \in \{ 4, 5 \}$, for each $j'_r \in J'''$, and, hence, G contains a minor isomorphic to either $CW_{k(a)}(2,1,n)$ or $CW_{k(b)}(2,1,n)$, for some $k \in \{ 1, ..., 6 \}$ (similar to Case 2, upto relabeling the paths). In the latter case, for some $J''' \subseteq J''$, where $|J'''| \geq 4n+14$, not only does there not exist a path between $P_3[u^{(3)}_{2j'_r-1}, u^{(3)}_{2j'_r}]$ and either of $P_4[u^{(4)}_{2j'_r-1}, u^{(4)}_{2j'_r}]$ and $P_5[u^{(5)}_{2j'_r-1}, u^{(5)}_{2j'_r}]$, for each $j'_r \in J'''$, but there also exist $j'_s, j'_t \in J''$, with $j'_s \leq j'_r \leq j'_t$ for each $j'_r \in J'''$, such that, for some $j''_s \in \{ 2j'_s - 1, 2j'_s - 2 \}, j''_t \in \{ 2j'_t, 2j'_t + 1 \}$, either $u^{(4)}_{j''_s}$ has degree at least 3 in $G[V(A_{j''_s}), A_{j''_s}]$ or $u^{(4)}_{j''_t}$ has degree at least 3 in $G[V(B_{j''_t}), B_{j''_t}]$, and, either $u^{(5)}_{j''_s}$ has degree at least 3 in $G[V(A_{j''_s}), A_{j''_s}]$ or $u^{(5)}_{j''_t}$ has degree at least 3 in $G[V(B_{j''_t}), B_{j''_t}]$ (proof follows).

4.2.1.1. *There exists a triple (J''', j'_s, j'_t) as described above.*

Proof of claim. Let $J'' := \left\{ j''_1, ..., j''_{16n+56} \right\}$, where $j''_< ... < j''_{16n+56}$, and let $j''_s = j''_1, j''_t = j''_{16n+56}$. If each of $u^{(4)}_{2j''_s-1}$ and $u^{(5)}_{2j''_s-1}$ has degree at least 3 in $G[V(A_{2j''_s-1}), A_{2j''_s-1}]$, or each of $u^{(4)}_{2j''_t}$ and $u^{(5)}_{2j''_t}$ has degree at least 3 in $G[V(B_{2j''_t}), B_{2j''_t}]$, then we're done. Suppose each of $u^{(4)}_{2j''_s-1}$ and $u^{(5)}_{2j''_s-1}$ has degree at most 2 in $G[V(A_{2j''_s-1}), A_{2j''_s-1}]$, and each of $u^{(4)}_{2j''_t}$ and $u^{(5)}_{2j''_t}$ has degree at most 2 in $G[V(B_{2j''_t}), B_{2j''_t}]$ $(*)$. Then, if each of $u^{(4)}_{2j''_{s+4n+14}-1}$ and $u^{(5)}_{2j''_{s+4n+14}-1}$ has degree at most 2 in $G[V(A_{2j''_{s+4n+14}-1}), A_{2j''_{s+4n+14}-1}]$, we're done with $j'_s = j''_s$ and $j'_t = j''_{s+4n+13}$. So we may assume that for some $j''_s, j''_t \in J'', j''_t \geq j''_{s+12n+41}$, exactly one of $u^{(4)}_{2j''_s-1}$ and $u^{(5)}_{2j''_s-1}$ has degree at least 3 in $G[V(A_{2j''_s-1}), A_{2j''_s-1}]$ (a similar argument holds for the case when exactly one of $u^{(4)}_{2j''_t}$ and $u^{(5)}_{2j''_t}$ has degree at least 3 in $G[V(B_{2j''_t}), B_{2j''_t}]$). This is also true when $(*)$ does not hold. Without loss of generality, let $u^{(5)}_{2j''_s-1}$ have degree at least 3 in $G[V(A_{2j''_s-1}), A_{2j''_s-1}]$. Then, for each $j''_i \in J''$, where $i \geq s+4n+14$, $u^{(4)}_{2j''_i-2}$ has degree at most 2 in $G[V(B_{2j''_i-2}), B_{2j''_i-2}]$ and $u^{(4)}_{2j''_i-1}$ has degree at most 2 in $G[V(B_{2j''_i-1}), B_{2j''_i-1}]$ for otherwise we're done. In turn, for each $j''_k \in J''$, where $k \geq s+8n+28$, $u^{(5)}_{2j''_k-2}$ has degree at most 2 in $G[V(B_{2j''_k-2}), B_{2j''_k-2}]$ and $u^{(5)}_{2j''_k-1}$ has degree at most 2 in $G[V(B_{2j''_k-1}), B_{2j''_k-1}]$ for otherwise we're done again. But then each of $u^{(4)}_{2j''_r-1}$ and $u^{(5)}_{2j''_r-1}$ has degree at least 3 in $G[V(A_{2j''_r-1}), A_{2j''_r-1}]$, where $r = s+8n+28$, and we're done with $j'_s = j''_r, j'_t = j''_t$. $\square$

By Propositions 3.2.1 and 3.3.1 and Lemma 3.3.8, either $G[V(A_{2j'_s-1}), A_{2j'_s-1}]$ can be reduced to G', where $G' \in \left\{ G^{(24)}_7, G^{(24)(25)}_7 \right\}$ if each of $u^{(4)}_{2j'_s-1}$ and $u^{(5)}_{2j'_s-1}$ has degree at most 2 in $G[V(A_{2j'_s-1}), A_{2j'_s-1}]$ and $G' \in \{ G^{(13)(24)(25)}_8, G^{(13)(24)}_8, G^{(24)}_8, G^{(24)}_{8+(15)}, G^{(24)(25)}_8, G^{(24)(25)}_{8+(15)},$

$G_9^{(24)}, G_9^{(24)(25)}$ } otherwise, with the vertices $u(1)_{2j'_s-1}, u(2)_{2j'_s-1}, u(3)_{2j'_s-1}, u(4)_{2j'_s-1}$ and $u(5)_{2j'_s-1}$ identified with x_1, x_4, x_3, x_2 and x_5, respectively, or $G[V(A_{2j'_s-1}), A_{2j'_s-1}]$ contains two disjoint connected subgraphs H_1 and H_2 such that, for some $u \in \{ u(4)_{2j'_s-1}, u(5)_{2j'_s-1} \}$, $\{ u(1)_{2j'_s-1}, u(3)_{2j'_s-1}, u(4)_{2j'_s-1}, u(5)_{2j'_s-1} \} - \{ u \} \subseteq V(H_1)$ and $\{ u(2)_{2j'_s-1}, u \} \subseteq V(H_2)$; in the latter case, $G[V(A_{2(j'_s+2)}), A_{2(j'_s+2)}]$ can be reduced to one of the graphs $G_3 \cup \{ x_5x_1, x_1x_2, x_2x_3, x_3x_5 \}$ and $G_3^{(45)} \cup \{ x_4x_1, x_1x_2, x_2x_3, x_3x_4 \}$, with the vertices $u(1)_{2(j'_s+2)}$, $u(2)_{2(j'_s+2)}, u(3)_{2(j'_s+2)}, u(4)_{2(j'_s+2)}$ and $u(5)_{2(j'_s+2)}$ identified with x_1, x_2, x_3, x_4 and x_5, respectively. Likewise with the graphs $G[V(B_{2j'_t}), B_{2j'_t}]$ and $G[V(B_{2(j'_t-2)-1}), B_{2(j'_t-2)-1}]$. Then, since $G[V(B_{2(j'_s+3)-1} \cap A_{2(j'_t-3)}), B_{2(j'_s+3)-1} \cap A_{2(j'_t-3)}]$ can be reduced to the graph $G^{2(c)}, G$ contains a minor isomorphic to one of $W_1^-(3,0,n)$, $TW_1^-(3,0,n)$, $W_{2(a)}^-(3,0,n)$, $W_{2(b)}^-(3,0,n)$, $TW_{2(a)}^-(3,0,n)$ and $TW_{2(b)}^-(3,0,n)$.

Finally, since $N_4 \geq 2(20n+14)(12n+13)(32n+71)$, either, for some $j' \in \{ 1, ..., N_4 - 2(12n+13)(32n+71)+1 \}$ and for each $j \in \{ 0, 1, ..., 2(12n+13)(32n+71)-1 \}$, $\{ a'', b'' \} = \{ 1, 2 \}$, $\{ c'', d'' \} = \{ 4, 5 \}$, there does not exist a path connecting $P_{a''}[u^{(a'')}_{2(j'+j)-1}, u^{(a'')}_{2(j'+j)}]$ with $P_{c''}[u^{(c'')}_{2(j'+j)-1}, u^{(c'')}_{2(j'+j)}]$ that is disjoint with $P_{b''}, P_3$ and $P_{d''}$ and is contained in the graph $G[V(B_{2(j'+j)-1} \cap A_{2(j'+j)}), B_{2(j'+j)-1} \cap A_{2(j'+j)}]$, or, for some $J' \subseteq \{ 1, 2, ..., N_4 \}$, where $J' := \{ j'_1, j'_2, ..., j'_{20n+14} \}$, $j'_1 < j'_2 < ... < j'_{20n+14}$, and for each $j' \in J'$, there exists a path connecting $P_{a''_{j'}}[u^{(a''_{j'})}_{2j'-1}, u^{(a''_{j'})}_{2j'}]$ with $P_{c''_{j'}}[u^{(c''_{j'})}_{2j'-1}, u^{(c''_{j'})}_{2j'}]$ that is disjoint with $P_{b''_{j'}}, P_3$ and $P_{d''_{j'}}$, and is contained in $G(V(B_{2j'-1} \cap A_{2j'}), B_{2j'-1} \cap A_{2j'})$, where $\{ a''_{j'}, b''_{j'} \} = \{ 1, 2 \}$, $\{ c''_{j'}, d''_{j'} \} = \{ 4, 5 \}$. In the latter case, for some $J'' \subseteq J'$, where $|J''| \geq 16n+16$, and for each $j' \in J''$, $a''_{j'} = 1$ and we're done by Case 2, for otherwise we are in Case 1, a contradiction. In the former case, if, for each $j \in \{0, 1, ..., (12n+13)(2(32n+71)-1)\}$, $|P_3[u^{(3)}_{2(j'+j)-1}, u^{(3)}_{2(j'+j+12n+12)}]| > 1$ then we're done by Case 4, otherwise we're done by Case 3. $\square$

Chapter 5

Unavoidable Minors

We conclude by giving a short proof of Theorem 1.2.1 that puts the two cases together and mentioning a deterrent to this approach being extended to higher connectivities.

5.1 Proof of Theorem 1.2.1

Proof of Theorem 1.2.1. Let

$$N = \max\left\{ 25(f_{2.2.3}(6, f_{2.1.3}(4n+4)))^{f_{4.2.1}(n)+1}, f_{2.1.3}(4n+4), f_{2.1.2}(n) \right\}$$

and let G be a $5-$connected graph with at least N vertices. We may assume that G does not contain a $6-$connected set of size at least $f_{2.1.3}(4n+4)$, for otherwise we're done by Corollary 2.1.4. Similarly, by Corollary 2.1.2, we may assume that G does not have a minor isomorphic to $W(1,3,n)$. Then, by Corollary 2.2.3, $bd_5(G) \leq f_{2.2.3}(6, f_{2.1.3}(4n+4)) - 1$, and, since $|V(G)| \geq 25(f_{2.2.3}(6, f_{2.1.3}(4n+4))-1)^{f_{4.2.1}(n)+1}$, by Proposition 4.1.1, it contains a family of distinct nested $5-$separations of size at least $f_{4.2.1}(n)$. The rest of the proof follows from Lemma 4.2.1. □

5.2 A Deterrent

It is easy to see that the two cases underlying this approach do not both trivially extend to large $\theta-$connected graphs for $\theta \geq 6$.

In particular, the first case, when the large $\theta-$connected graph under consideration also has a large $(\theta+1)-$connected set, does not trivially produce a $\theta-$connected minor for each of the unavoidable minors of graphs with large $(\theta+1)-$connected sets proposed by Geelen and Joeris in [GJ16], as was true for $\theta = 5$. This is clear from the fact that the set of unavoidable minors proposed by Geelen and Joeris invariably contains a planar graph which cannot have a $\theta-$connected minor for any $\theta \geq 6$.

It is also understandable that the second case, which covers large $\theta-$connected graphs that do not contain a large $(\theta + 1)-$connected set, will possibly entail, for $\theta \geq 6$, the discovery of a larger number of unavoidable rooted minors both for the intersection of the larger sides of two non-crossing separations in the graph and the smaller sides of these separations, as well as the identification of the different conditions in which the two can be patched together. The sheer number and complexity of these unavoidable rooted minors could make finding an explicit set of unaoidable minors in this case a much taller order than it was for $\theta = 5$.

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Appendix A

Graph Constructions

A.1 Unavoidable Minors from the First Case

This is the case when the said sufficiently large 5−connected graph contains a large 6−connected set. We give here graph constructions for $W(1,3,n)$, $W(2,2,n)$, $TW(2,2,n)$, $W(3,0,n)$ and $TW_i(3,0,n)$, for each $i \in \{1,2,3\}$ (see Figure 1.3).

$W(1,\ell,n)$ and the $(1,\ell,n)-$ wheel $\mathcal{W}(1,\ell,n)$ are both one and the same graph. Each of the graphs $W(2,2,n)$ and $TW(2,2,n)$ can be constructed using $n-1$ disjoint copies of the homogenous $(1,1,5)-$wheel $\mathcal{W}(1,1,5)$ as follows.

(a) For each $j \in \{1,...,n-1\}$, let $v_1^{(j)}, v_2^{(j)}, v_3^{(j)}, v_4^{(j)}, v_5^{(j)}, v_h^{(j)}$ denote the vertices of the $j-$th copy $\mathcal{W}(1,1,5)^{(j)}$, with $v_h^{(j)}$ as the lone hub, such that $v_1^{(j)}, ..., v_5^{(j)}$ form the vertices of the 5−cycle $\mathcal{W}(1,1,5)^{(j)} \setminus \left\{ v_h^{(j)} \right\}$ in that order; let u be an additional disjoint vertex. Let G' be the graph obtained by identifying the vertex-pairs $(v_5^{(j)}, v_1^{(j+1)})$, $(v_4^{(j)}, v_h^{(j+1)})$, $(v_3^{(j)},\ v_3^{(j+1)})$ and $(v_h^{(j)}, v_2^{(j+1)})$, and adding the edges $(u, v_1^{(j)})$ and $(u, v_5^{(j+1)})$, for each $j \in \{1,...,n-2\}$.

(b) Then $W(2,2,n+1)$ is obtained from G' by adding another vertex v along with the edges (v,u), $(v, v_1^{(1)})$, $(v, v_2^{(1)})$, $(v, v_4^{(n-1)})$, $(v, v_5^{(n-1)})$ and $(v_2^{(1)}, v_4^{(n-1)})$, whereas $TW(2,2,n)$ is obtained from G' by adding only the edges $(v_1^{(1)}, v_4^{(n-1)})$, $(v_2^{(1)}, v_5^{(n-1)})$ and $(v_2^{(1)}, v_4^{(n-1)})$.

Each of the graphs $W(3,0,n)$ and $TW_i(3,0,n)$, where $i \in \{ 1,2,3 \}$, on the other hand, can be constructed using n disjoint copies of $\mathcal{W}(1,1,5)$ as follows.

(a) For each $j \in \{ 1,...,n \}$, let $v_1^{(j)}, v_2^{(j)}, v_3^{(j)}, v_4^{(j)}, v_5^{(j)}, v_h^{(j)}$ denote the vertices of the $j-$th copy $\mathcal{W}(1,1,5)^{(j)}$, with $v_h^{(j)}$ as the lone hub, such that $v_1^{(j)},...,v_5^{(j)}$ form the vertices of the $5-$cycle $\mathcal{W}(1,1,5)^{(j)} \backslash \left\{ v_h^{(j)} \right\}$ in that order. Let G' be the graph obtained by identifying the vertex-pairs $(v_4^{(j)}, v_2^{(j+1)})$ and $(v_5^{(j)}, v_1^{(j+1)})$, and adding the edge $(v_3^{(j)}, v_3^{(j+1)})$, for each $j \in \{ 1,...,n-1 \}$.

(b) Then $W(3,0,n)$ is obtained from G' by taking another copy $\mathcal{W}(1,1,5)^{(n+1)}$ (with vertices labeled $v_1^{(n+1)},...,v_5^{(n+1)}, v_h^{(n+1)}$, as described in (a)) of $\mathcal{W}(1,1,5)$, identifying the vertex-pairs $(v_1^{(1)}, v_5^{(n+1)})$, $(v_2^{(1)}, v_4^{(n+1)})$, $(v_4^{(n)}, v_2^{(n+1)})$ and $(v_5^{(n)}, v_1^{(n+1)})$, and adding the edges $(v_3^{(1)}, v_3^{(n+1)})$ and $(v_3^{(n)}, v_3^{(n+1)})$; $TW_1(3,0,n)$ is obtained from G' by adding another vertex v along with the edges $(v, v_2^{(1)}), (v, v_3^{(1)})$, $(v, v_3^{(n)})$, $(v, v_4^{(n)})$, $(v, v_5^{(n)})$, $(v_1^{(1)}, v_4^{(n)})$, $(v_1^{(1)}, v_5^{(n)})$ and $(v_2^{(1)}, v_5^{(n)})$; $TW_2(3,0,n)$ is obtained from G' by adding only the edges $(v_1^{(1)}, v_3^{(n)})$, $(v_1^{(1)}, v_4^{(n)})$, $(v_2^{(1)}, v_4^{(n)})$, $(v_2^{(1)}, v_5^{(n)})$ and $(v_3^{(1)}, v_5^{(n)})$; and $TW_3(3,0,n)$ is obtained from G' by adding only the edges $(v_1^{(1)}, v_3^{(n)})$, $(v_1^{(1)}, v_5^{(n)}), (v_2^{(1)}, v_4^{(n)})$, $(v_2^{(1)}, v_5^{(n)})$ and $(v_3^{(1)}, v_4^{(n)})$.

A.2 Unavoidable Minors from the Second Case

This is the case when the said sufficiently large $5-$connected graph does not contain a large $6-$connected set. We give here graph constructions for $W_j(2,1,n)$, $TW_j(2,1,n)$, $W_1^-(3,0,n), TW_1^-(3,0,n)$, $W_{2(a)}^-(3,0,n)$, $W_{2(b)}^-(3,0,n)$, $TW_{2(a)}^-(3,0,n)$ and $TW_{2(b)}^-(3,0,n)$, for each $j \in \{ 1,2 \}$ (see Figure 1.4), as well as for $CW_{k(a)}(2,1,n)$ and $CW_{k(b)}(2,1,n)$, for each $k \in \{ 1,...,6 \}$ (see Figure 1.5).

For each $i \in \{ 1,...5 \}$, let $P_i(n)$ be a path containing the vertices $v_1^{(i)},...,v_n^{(i)}$ in order; to the union $P_1(n) \cup ... \cup P_5(n)$ add the edges $v_j^{(1)} v_j^{(2)}$ and $v_j^{(2)} v_j^{(3)}$, for each $j \in \{ 1,...,n \}$, to form the graph $G(n)$. Let G^1 be the graph formed from $G(2n+2)$ by identifying the vertices $v_1^{(i)},...,v_{2n+2}^{(i)}$ into $v^{(i)}$, for each $i \in \{ 3,4,5 \}$, G^2 be the graph formed from $G(4n+8)$ by identifying the vertices $v_1^{(i)},...,v_{4n+8}^{(i)}$ into $v^{(i)}$, for each $i \in \{ 4,5 \}$.

(a) Let $G^{1(a)}$ be the graph formed from G^1 by identifying the vertex-pair $(v_{2j-1}^{(1)}, v_{2j}^{(1)})$ into $v_j'^{(1)}$ and adding the edge $v_j'^{(1)} v^{(3)}$, for each $j \in \{ 1,...,n+1 \}$, and identifying the

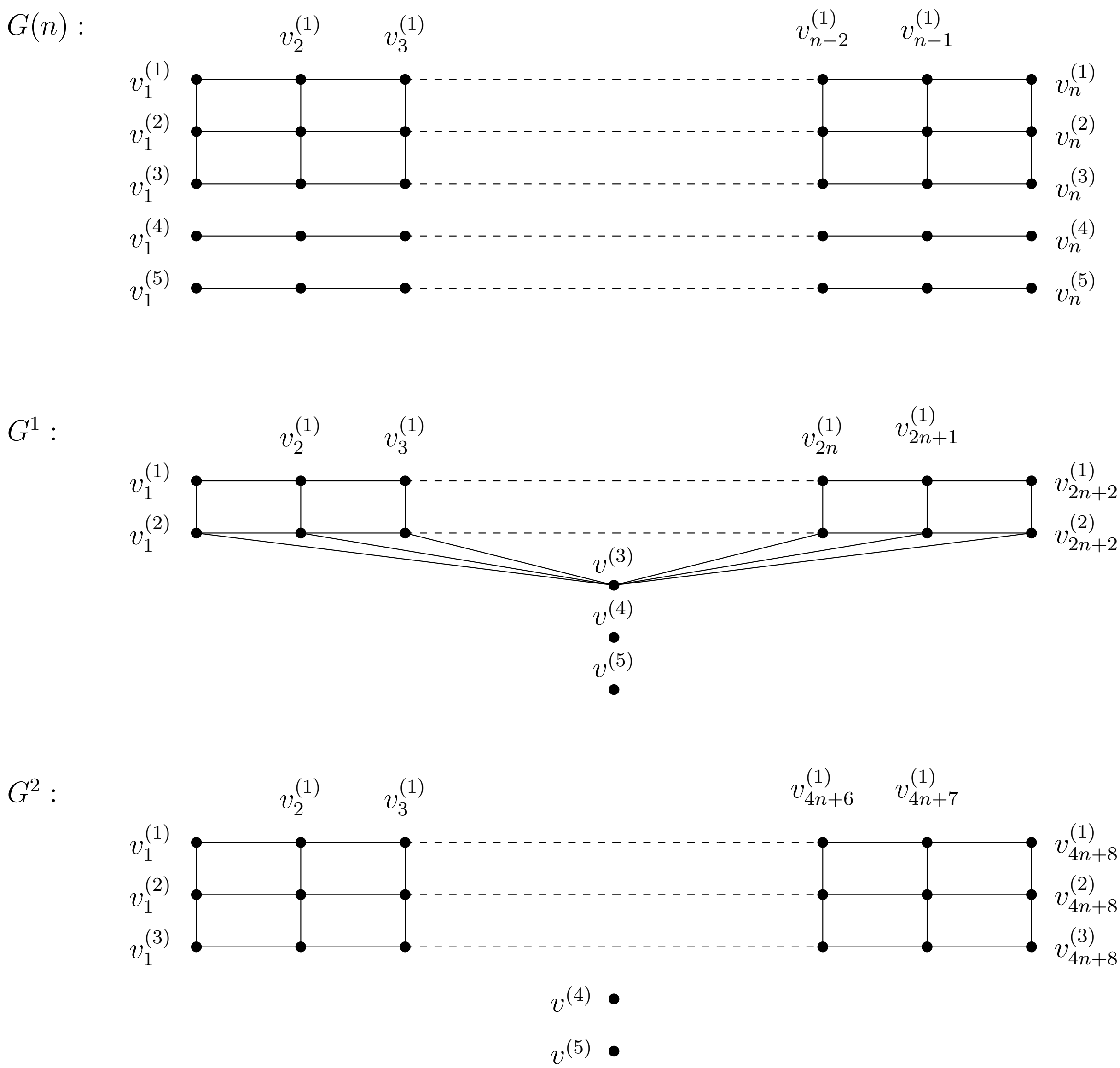

Figure A.1: $G(n), G^1$ and G^2.

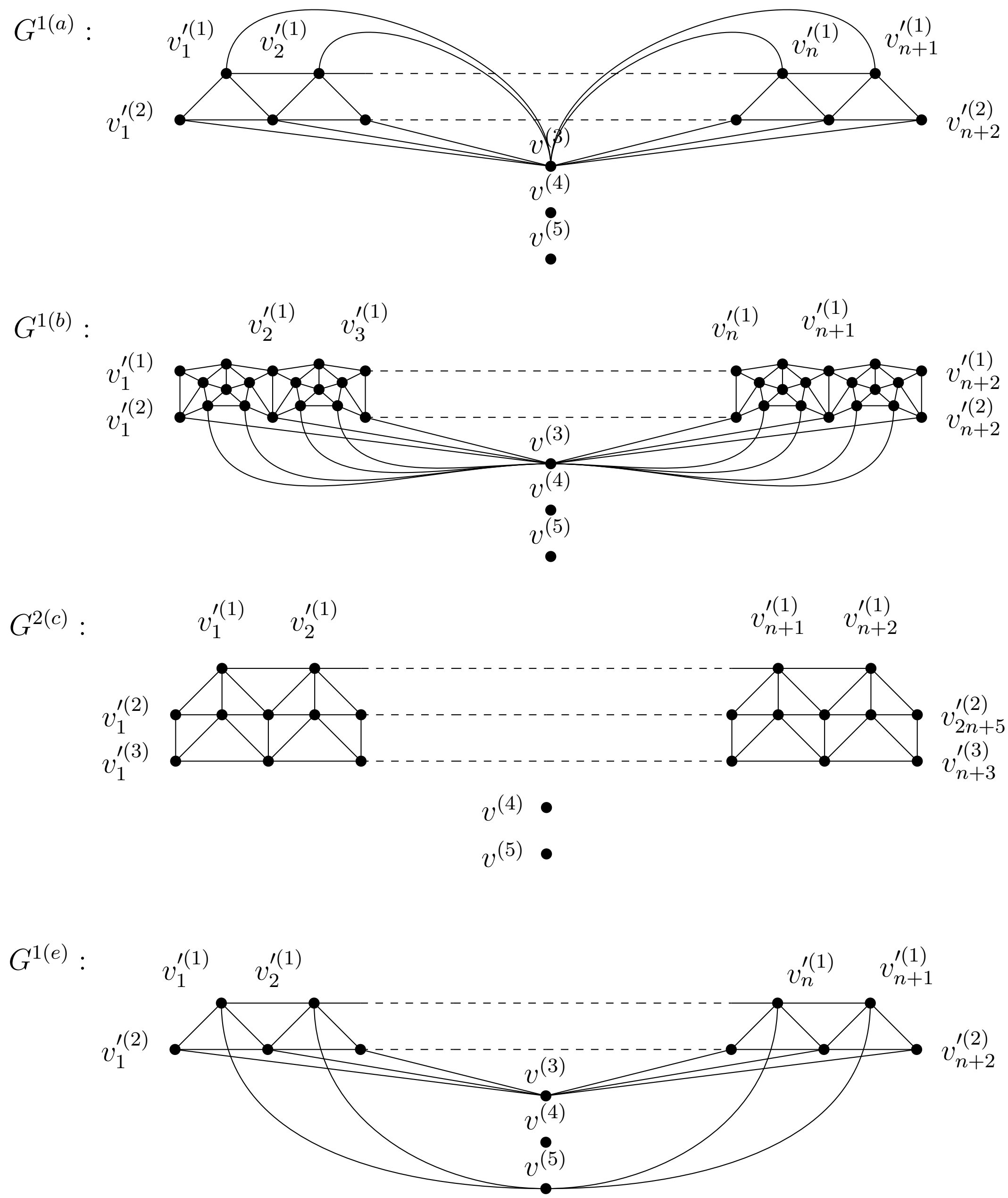

Figure A.2: Rooted minors of the intersection of the sides of two non-crossing 5−separations in a 5−connected graph.

vertex-pair $(v^{(2)}_{2j-2}, v^{(2)}_{2j-1})$ into $v'^{(2)}_j$, for each $j \in \{2, ..., n+1\}$; $v'^{(2)}_1 := v^{(2)}_1, v'^{(2)}_{n+2} := v^{(2)}_{2n+2}$. Then $W_1(2,1,n)$ is obtained from $G^{1(a)}$ by adding the edges $v'^{(1)}_1 v^{(5)}, v'^{(2)}_1 v^{(4)}, v'^{(1)}_{n+1} v^{(5)}, v'^{(2)}_{n+2} v^{(4)}$ and then contracting each of them except $v'^{(1)}_1 v^{(5)}$, while $TW_1(2,1,n)$ is obtained from $G^{1(a)}$ by adding the edges $v'^{(1)}_1 v^{(5)}, v'^{(2)}_1 v^{(4)}, v'^{(1)}_{n+1} v^{(4)}, v'^{(2)}_{n+2} v^{(5)}$ and then contracting each of them except $v'^{(1)}_{n+1} v^{(4)}$ along with $v'^{(2)}_1 v'^{(2)}_2$.

(b) Let $G_{7(j)}$ be the j−th of $n+1$ disjoint copies of G_7, having vertices $v_{1(j)}, ..., v_{5(j)}, v_{h(j)}, x_{1(j)}, ..., x_{5(j)}$. Let $G^{1(b)}$ be the graph formed from G^1 by deleting the edges $v^{(1)}_{2j-1} v^{(1)}_{2j}$, $v^{(2)}_{2j-1} v^{(2)}_{2j}$, and identifying the vertex-pairs $(x_{1(j)},\ v^{(1)}_{2j-1})$, $(x_{2(j)},\ v^{(2)}_{2j-1})$, $(x_{3(j)}, v^{(3)})$, $(x_{4(j)}, v^{(2)}_{2j}), (x_{5(j)}, v^{(1)}_{2j})$, for each $j \in \{1, ..., n+1\}$, and identifying the vertex-pairs $(v^{(1)}_{2j-2}, v^{(1)}_{2j-1})$ and $(v^{(2)}_{2j-2}, v^{(2)}_{2j-1})$ into $v'^{(1)}_j$ and $v'^{(2)}_j$, respectively, for each $j \in \{2, ..., n+1\}$; $v'^{(1)}_1 := v^{(1)}_1, v'^{(2)}_1 := v^{(2)}_1, v'^{(1)}_{n+2} := v^{(1)}_{2n+2}, v'^{(2)}_{n+2} := v^{(2)}_{2n+2}$. Then $W_2(2,1,n+1)$ is obtained from $G^{1(b)}$ by adding the edges $v'^{(1)}_1 v^{(5)}, v'^{(2)}_1 v^{(4)}, v'^{(1)}_{n+2} v^{(5)}$, $v'^{(2)}_{n+2} v^{(4)}$ and contracting each of them, while $TW_2(2,1,n+1)$ is obtained from $G^{1(b)}$ by adding the edges $v'^{(1)}_1 v^{(5)}, v'^{(2)}_1 v^{(4)}, v'^{(1)}_{n+2} v^{(4)}$, $v'^{(2)}_{n+2} v^{(5)}$ and contracting each of them.

(c) Let $G^{2(c)}$ be the graph formed from G^2 by identifying the vertices $v^{(1)}_{4j-3}, v^{(1)}_{4j-2}, v^{(1)}_{4j-1}$ and $v^{(1)}_{4j}$ to form the vertex $v'^{(1)}_j$, for each $j \in \{1, ..., n+2\}$, the vertex-pair $(v^{(2)}_{2j-2}, v^{(2)}_{2j-1})$ to form the vertex $v'^{(2)}_j$, for each $j \in \{2, ..., 2n+4\}$, the vertices $v^{(3)}_{4j-5}, v^{(3)}_{4j-4}, v^{(3)}_{4j-3}$ and $v^{(3)}_{4j-2}$ to form the vertex $v'^{(3)}_j$, for each $j \in \{2, ..., n+2\}$, and the vertex-pairs $(v^{(3)}_1, v^{(3)}_2)$ and $(v^{(3)}_{4n-7}, v^{(3)}_{4n-8})$ to form the vertices $v'^{(3)}_1$ and $v'^{(3)}_{n+3}$, respectively, and then contracting the edges $v'^{(2)}_1 v'^{(2)}_2, v'^{(3)}_1 v'^{(3)}_2, v'^{(2)}_{2n+4} v'^{(2)}_{2n+5}$ and $v'^{(3)}_{n+2} v'^{(3)}_{n+3}$ (note that $v'^{(2)}_1 := v^{(2)}_1, v'^{(2)}_{2n+5} := v^{(2)}_{4n+8}$). Let G' be a graph formed from $G^{2(c)}$ and two disjoint copies of K_4 by identifying the first copy with the vertices $v'^{(1)}_1, v'^{(2)}_1, v^{(4)}$ and $v^{(5)}$, and the second copy with the vertices $v'^{(1)}_1, v'^{(2)}_1, v^{(4)}$ and $v^{(5)}$. Then $W^-_1(3,0,n+1)$ is obtained from G' by adding the edges $v'^{(3)}_1 v^{(4)}$ and $v'^{(3)}_{n+3} v^{(4)}$, while $TW^-_1(3,0,n+1)$ is obtained from G' by adding the edges $v'^{(3)}_1 v^{(5)}$ and $v'^{(3)}_{n+3} v^{(4)}$.

(d) Let G'' be a graph formed from $G^{2(c)}$ and two disjoint copies $G^{(1)}_7$ and $G^{(2)}_7$ (containing vertices $x^{(1)}_1, ..., x^{(1)}_5$ and $x^{(2)}_1, ..., x^{(2)}_5$, respectively) of G_7 by identifying the vertex-pairs $(x^{(1)}_1, v'^{(1)}_1), (x^{(1)}_2, v'^{(2)}_1), (x^{(1)}_3, v'^{(3)}_1), (x^{(1)}_4, v^{(4)}), (x^{(1)}_5, v^{(5)}), (x^{(2)}_1,\ v'^{(1)}_{n+2}), (x^{(2)}_2,\ v'^{(2)}_{2n+5})$, $(x^{(2)}_3, v'^{(3)}_{n+3}), (x^{(2)}_4, v^{(4)})$ and $(x^{(2)}_5,\ v^{(5)})$. Then $W^-_{2(a)}(3,0,n+1)$ is obtained from G'' by adding the edge $v^{(4)} v^{(5)}$, while $W^-_{2(b)}(3,0,n+1)$ is obtained from G'' by adding the edges $v'^{(1)}_1 v^{(5)}$ and $v'^{(3)}_1 v^{(4)}$. Let G''' be a graph formed from $G^{2(c)}$ and two disjoint copies $G^{(1)}_7$

and $G_7^{(2)}$ (containing vertices $x_1^{(1)}, ..., x_5^{(1)}$ and $x_1^{(2)}, ..., x_5^{(2)}$, respectively) of G_7 by identifying the vertex-pairs $(x_1^{(1)},\ v_1'^{(1)}), (x_2^{(1)}, v_1'^{(2)}), (x_3^{(1)}, v_1'^{(3)}), (x_4^{(1)}, v^{(4)}), (x_5^{(1)}, v^{(5)}), (x_1^{(2)}, v_{n+2}'^{(1)}), (x_2^{(2)},\ v_{2n+5}'^{(2)}), (x_3^{(2)}, v_{n+3}'^{(3)}), (x_4^{(2)}, v^{(5)})$ and $(x_5^{(2)}, v^{(4)})$. Then $TW_{2(a)}^{-}(3, 0, n+1)$ is obtained from G''' by adding the edge $v^{(4)}v^{(5)}$, while $TW_{2(b)}^{-}(3, 0, n+1)$ is obtained from G'' by adding the edges $v_1'^{(1)}v^{(5)}$ and $v_1'^{(3)}v^{(4)}$.

(e) Let $G^{1(e)}$ be the graph formed from $G^{1(a)}$ by replacing the edge $v_j'^{(1)}v^{(3)}$ with the edge $v_j^{(1)}v^{(5)}$, for each $j \in \{ 1, ..., n+1 \}$. Let, for each $k \in \{ 1, ..., 6 \}$, with $H_1^k \in \{ G_1, G_4, G_7 \}$ containing vertices $x_1^{k(1)}, ..., x_5^{k(1)}$ and $H_2^k \in \{ G_1, G_4, G_7 \}$ containing vertices $x_1^{k(2)}, ..., x_5^{k(2})$, CW_k be the graph obtained from $G^{1(e)}, H_1^k$ and H_2^k by identifying the vertex-pairs $(x_1^{k(1)},\ v_1'^{(1)}), (x_2^{k(1)},\ v_1'^{(2)}), (x_3^{k(1)}, v^{(3)}), (x_4^{k(1)},\ v^{(4)}), (x_5^{k(1)},\ v^{(5)}), (x_1^{k(2)}, v_{n+1}'^{(1)}), (x_2^{k(2)},\ v_{n+2}'^{(2)}), (x_3^{k(2)}, v^{(3)}), (x_4^{k(2)},\ v^{(4)})$ and $(x_5^{k(2)},\ v^{(5)})$, where $(H_1^1,\ H_2^1) := (G_1, G_1), (H_1^2,\ H_2^2) := (G_4,\ G_4), (H_1^3,\ H_2^3) := (G_7,\ G_7), (H_1^4,\ H_2^4) := (G_1, G_4), (H_1^5, H_2^5) := (G_1, G_7)$ and $(H_1^6, H_2^6) := (G_4, G_7)$. Then, for each $k \in \{ 1, ..., 6 \}$, $CW_{k(a)}(2, 1, n+1)$ is obtained from CW_k by adding the edge $v^{(3)}v^{(4)}$ and $CW_{k(b)}(2, 1, n+1)$ is obtained from CW_k by adding the edge $v^{(4)}v^{(5)}$.