Stability of Nonlinear Functional Differential Equations by the Contraction Mapping Principle

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Applied Mathematics

Waterloo, Ontario, Canada, 2016

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any final revisions, as accepted by my examiners.

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Abstract

Fixed point theory has a long history of being used in nonlinear differential equations, in order to prove existence, uniqueness, or other qualitative properties of solutions. However, using the contraction mapping principle for stability and asymptotic stability of solutions is of more recent appearance. Lyapunov functional methods have dominated the determination of stability for general nonlinear systems without solving the systems themselves. In particular, as functional differential equations (FDEs) are more complicated than ODEs, obtaining methods to determine stability of equations that are difficult to handle takes precedence over analytical formulas. Applying Lyapunov techniques can be challenging, and the Banach fixed point method has been shown to yield less restrictive criteria for stability of delayed FDEs. We will study how to apply the contraction mapping principle to stability under different conditions to the ones considered by previous authors. We will first extend a contraction mapping stability result that gives asymptotic stability of a nonlinear time-delayed scalar FDE which is linearly dominated by the last state of the system, in order to obtain uniform stability plus asymptotic stability. We will also generalize to the vector case. Afterwards we do further extension by considering an impulsively perturbed version of the previous result, and subsequently we shall use impulses to stabilize an unstable system, under a contraction method paradigm. At the end we also extend the method to a time dependent switched system, where difficulties that do not arise in non-switched systems show up, namely a dwell-time condition, which has already been studied by previous authors using Lyapunov methods. In this study, we will also deepen understanding of this method, as well as point out some other difficulties about using this technique, even for non-switched systems. The purpose is to prompt further investigations into this method, since sometimes one must consider more than one aspect other than stability, and having more than one stability criterion might yield benefits to the modeler.

Acknowledgments

I would like to thank Dr. Xinzhi Liu for his guidance, advice and suggestions during my stay. I am especially thankful for his patience and calm manner of approaching everything, I greatly appreciate his experience and great character. I thank my committee members, Dr. David Siegel and Dr. Edward Vrscay for their valuable comments and corrections.

I would like to thank the following people who made my graduate studies more fun and enlightening in their own ways, even if they did not notice it, essentially through interchange of opinions, mathematical/peer colloquy, funny remarks, socials, helped my code run well, and attended my Masters' defence: Hanzhe (Tom), Humeyra, Jason, Joanna, Jonathan, Kevin, Lorena, Marius, Mike, Mikhail, Minxin, Yangang and Yinan.

I am very grateful to Martha, for her ability of being omnipresent, through any communication device she is able to be my greatest company. I could not have done this so confidently, happily, cheerfully without knowing that she was always there, somewhere, always ready to hear me and to provide infinite love and support.

I would also like to thank, of course, my parents, for always supporting me, listening to me, their efforts invested in me, and all of the opportunities I have had, which in the end hinge on their achievements.

I would like to thank the Applied Mathematics Department of the University of Waterloo for support during my stay, especially financial support through teaching assistantships, and a highly efficient administration, especially I would like to thank Laura Frazee for her constant friendly attention and great efficiency.

Finally, I thank CONACyT Mexico for the financial support given to me for the entire duration of my Master degree studies.

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Chapter 1

Motivation

Introduction

One of the most important qualitative aspects of differential equations is determining the stability of a given model. The Lyapunov method for stability using a function

$$V : J \times \mathcal{C} \longrightarrow \mathbb{R}$$

where $J \subset \mathbb{R}$ is some interval, $\mathcal{C}$ is some subset of a metric space, is the most commonly used method to determine stability for nonlinear systems in ordinary differential equations (ODEs), where no simple criterion such as in the time invariant linear case exists. Nonetheless, when using systems that induce infinite-dimensional systems, such as in functional differential equations (FDEs), even linear systems represent a challenge, and of course nonlinear systems become even more complicated in FDEs. When using FDEs, common Lyapunov techniques divide into two main streams: one where $\mathcal{C}$ is some subset of $\mathbb{R}^n$, commonly known as Razumikhin techniques, or the other where $\mathcal{C}$ is some subset of an infinite dimensional function space.

Stability is an important concept originating from scientific studies such as the stability of our Solar System. In industrial applications, control design methods generally seek to operate around some equilibrium ideal solution. These design paradigms can be based on the Lyapunov method. The Lyapunov function, or functional in general, is a generalization of the concept of total energy from physical systems. It typically requires evaluating some functional that acts as a derivative type operator, such as a Dini-type derivative, and checking whether trajectories somehow do not not increase in energy (stability), or also strictly diminish in their energy (are asymptotically stable). The conditions required on the derivative-type operator in order to guarantee stability are generally *pointwise* conditions, and these can sometimes be restrictive.

Relatively recent studies have achieved stability results using the Banach fixed point theorem. To the best of the author's knowledge, these methods *for stability* of differential equations[1], began in papers [13, 14], in the year 2001. Further developments in [11, 12], [57, 58]

[1] We emphasize that this method has only recently been used for stability, in contrast to merely for proving existence of solutions, which has a longer history.

have achieved asymptotic stability results using the Banach fixed point theorem. It was shown that for delayed *scalar* FDEs, said fixed point method can be effective in the relaxation of some requirements of Lyapunov methods. In the aforementioned works, it was shown that for delayed functional differential equations, said fixed point method can be effective in the relaxation of some pointwise stability requirements that Lyapunov methods yield. To give an example, let us have the scalar delayed differential equation

$$x'(t) = -a(t)x(t) + b(t)x(t - r(t)) \tag{1.1}$$

where $b, r : [0, \infty) \longrightarrow \mathbb{R}$ are continuous functions such that $t - r(t) \longrightarrow \infty$ as $t \to \infty$. In a classical delayed FDE result in [20], sufficient conditions for stability of the previous were to have $a(t) \geq c$, where $c > 0$ is a constant, and $J|b(t)| \leq a(t)$ for all $t \geq 0$, for some $J > 1$ constant. In paper [57], using the Banach contraction principle, it was possible to show that a sufficient condition for stability of the same delayed DE is to have

$$\int_0^t e^{-\int_s^t a(u)du} |b(s)| ds \leq \alpha < 1, \tag{1.2}$$

as well as some Lipschitz type requirements that are easily fulfilled by this particular FDE, and $\int_0^t a(s)ds \longrightarrow \infty$ as $t \to \infty$. All of these conditions are satisfied under the sufficient requirements that were obtained in [20], through Lyapunov methods. Inequality (1.2) is an averaging condition that allows relaxation of the pointwise conditions $a(t) \geq c > 0$ and $J|b(t)| \leq a(t)$ for all $t \geq 0$. Thus conditions for stability have been improved for this particular delayed equation. Similar successes were obtained in other results in [57, 58, 11]. In [57], it is also shown how conditions for stability of a Volterra integro-differential equation can be improved with respect to a Lyapunov analysis done before in [25].

The contraction mapping principle has also been applied in more recent times to neutral functional differential equations in [17], to stochastic delayed impulsive differential equations in [18], to cite a few examples that are not necessarily deterministic or just delayed FDES.

The previous successes prompted in this thesis theoretical investigations into cases that, to the best of the author's knowledge, have not been considered before. The author's original contributions belong primarily to Chapters 6 to 8 of this work. Nonetheless, in Chapter 5, after an observation about a technical detail regarding the usage of fixed point methods for stability, most of the second half of the aforementioned chapter includes an original study of different cases that further dwell into the difficulties of the contraction mapping method. Chapter 5 also contains results proved by the author that improve the previous result of [57], by obtaining uniform stability. At the end of Chapter 5, we also extend the aforesaid result, which is for one-dimensional FDEs, to the vector case. With respect to achieving uniform stability in chapters 5 to 8, the majority, if not all previous results using the Banach fixed point method for stability achieve only stability plus convergence to zero. We slightly improve by including uniform stability and convergence to zero of solution curves. Afterwards, considering a perturbed version of the vector version by impulses in Chapter 6, we explore sufficient conditions for asmptotic stability of the impulsive version of the purely delayed system. In many stability results obtained by the Banach contraction principle, it is assumed that the linear portion of the differential equation is well behaved in the sense that it is sufficiently dominant in order to induce a contraction and also asymptotic convergence. In Chapter 7, we assumed that the linear portion was not well behaved, but by adding impulses, we can still obtain a sufficiently well behaved system and characterize stability conditions. In this last mentioned scenario, we will obviously see some inconveniences and considerations that were not necessary to dwell on in well behaved systems, though we will characterize analytically what said considerations entail. These studies further exhibit the difficulties of applying the contraction mapping principle to asymptotic stability, by pushing the Banach fixed point theorem to its limit by adding more complicated terms to nonlinear systems. In Chapter 8, we will also begin the study of applying the contraction principle to time dependent switching of FDEs. Here we consider that the linear portion is well behaved again, and purely continuous. This is perhaps the easiest case, but it is done in order to begin, from similar conditions to the non-switched continuous case of Chapter 5, the analysis of the difference between using the contraction principle for a single system as in previous results, and using it on switched system. We will be able to point out some of the main and fundamental differences and difficulties that occur in the transition to switched systems. In particular we will obtain a dwell time as a necessary condition for a certain class of subsystems, although this is not a new thing to consider in hybrid systems theory, as this has been studied, for example, in [33] for ODES, and in [35, 50] for switched FDEs, and references therein. Nonetheless, to the extent of our knowledge, dwell time conditions have not been studied using the contraction mapping principle before. By beginning the application of fixed point methods for stability in switched systems, we seek to motivate further investigations into the topic.

In the first chapters, namely Chapters 2 to 4, before arriving to the results mentioned in the previous paragraph, we develop the minimal necessary theory of differential equations, especially theory beyond the scope of the more basic ordinary differential equations, although in the following Chapter 2 we give a quick overview of theory from ODEs, so that we can subsequently study the corresponding versions of this theory for more general systems, such as delayed FDEs, switched FDEs, etc. The important ideas from ODEs are fundamental to the more general frameworks that we will need, and a sufficient understanding of the ODE principles is necessary for the further generalization into FDEs. We will go through the fundamental theory of general nonlinear FDEs, though this topic is vast, and due to the infinite dimensional systems induced, the topic cannot be covered as concisely as perhaps ODEs can. A working knowledge of impulsive FDE systems is given, along with the minimal necessary elements of switched functional differential equations that we will require in this thesis. We will not go deeply into stability results of hybrid systems, in other words, those of impulsive and switched systems, since we will not study a direct comparison between using the contraction mapping theorem and using Lyapunov theory in hybrid systems. We still study general stability results, especially in continuous delayed functional differential equations, because the comparison between the two aforesaid methods has already been more clearly covered for this particular case.

Chapter 2

Preliminaries

"Could one not ask whether one of the bodies will always remain in a certain region of the heavens, or if it could just as well travel further and further away forever; whether the distance between two bodies will grow or diminish in the infinite future, or if instead it remains bracketed between certain limits forever? Could one not ask a thousand questions of this kind which would all be solved once one understood how to construct qualitatively the trajectories of the three bodies?"

Henri Poincaré

2.1 Introduction

We begin with an overview of basic theory from ordinary differential equations. Ordinary differential equations have of course played a very important role in the development of science and mathematics. They are among the most basic ways of modeling dynamics, in other words, the evolution of systems under the relationship between the derivative of the state of the system, and a mapping which defines a vector field for this evolution. In the following chapters we will examine more general types of vector fields, such as functional differential equations, to model derivatives depending on function behavior in the past, impulsive differential equations and switched systems. Nonetheless, ordinary differential equations remain the basic building blocks for these more general ways of mathematical modeling. Also, because of the relative simplicity of ordinary differential equations, they are sometimes the preferred manner of representing processes, instead of using maybe more accurate models involving delays, for example. Nonetheless, ordinary differential equations are an idealization of a situation because we implicitly assume that the future of the system starting from a given initial condition at an initial instant t_0, depends solely on the present state measured, in other words, to determine the future state of the system starting from t_0, you just need to know the present state at t_0, the past states before t_0 will not be necessary to determine the future state.

2.2 Ordinary Differential Equations

Most of the definitions and results from this section are based on the books [19] and [30].

Remark 2.1. *In this thesis, for a function of t, $x(t)$, we denote by $\dot{x}$ or x' the derivative in the independent variable t, evaluated at t namely*

$$\dot{x} = x' = x'(t) = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}.$$

Thus, whenever we have a differential equation, we will use the informal notation $\dot{x}$ or x' to specify a derivative evaluated at t, where it will be understood from the given context that the derivative is evaluated at this time instant.

Let $\mathbb{R}^n$ denote n-dimensional Euclidean space, and let us denote for now the Euclidean norm of an element $y \in \mathbb{R}^n$ as $\|y\|$. Let $J \subset \mathbb{R}$ be an interval, and $D \subset \mathbb{R}^n$ an open set, $f : J \times D \longrightarrow \mathbb{R}^n$ be a sufficiently smooth mapping, where we come back to what we mean by "sufficiently smooth" in the existence-uniqueness results. By an *ordinary differential equation* (ODE) , we mean an equation of the form

$$\dot{x} = f(t, x(t)), \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the dependent variable of t (usually identified as time). $x(t)$ is usually called the state. Often we will denote the ordinary differential equation as $\dot{x} = f(t, x)$, where the dependence of the state variable (or vector of states, whichever convention the reader prefers) $x(t)$ on the variable t is tacitly assumed. Sometimes we will say that equation (2.1) is a *system of differential equations*, since the components of the state vector

$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

are what interests the mathematical modeler.

When a differential equation is used to model the evolution of a state variable in an applied problem such as in a physical process or an economic phenomenon, the fundamental problem is to determine the future values of the state variable from its initial value, in other words, from the first value measured at a given instant, say, at $t = t_0 \in J$. The mathematical model is thus given by a pair of equations

$$\begin{aligned} \dot{x} &= f(t, x) \\ x(t_0) &= x_0, \end{aligned} \tag{2.2}$$

and the determination of a solution $x(t)$ to this problem is called an *initial value problem*. By a *solution* to an initial value problem, we mean a function $x : J_0 \longrightarrow D$ given by $t \mapsto x(t)$, where $J_0 \subset J$ is an interval, such that

$$x'(t) = f(t, x(t)) \quad \text{for all } t \in J_0,$$

and at the initial time $t = t_0$, $x(t)$ satisfies $x(t_0) = x_0$. Sometimes the dependence of the solution on the initial value is denoted by $x(t; t_0, x_0) = x(t)$, when explicitness is required.

Basic Theory

Of course, one of the first and most important issues to be dealt with when dealing with a differential equation, be it an ordinary differential equation, functional differential equation, hybrid system, impulsive system or other forms of differential equations, is to determine existence and uniqueness of solutions to an initial value problem. By uniqueness we mean that if $x(t; t_0, x_0) = x(t)$ and $y(t; t_0, x_0) = y(t)$ are both solutions of the initial value problem (2.2), then necessarily $x(t) = y(t)$.

Regarding ordinary differential equations, there is a general existence uniqueness theory, which we present here, but before doing so, let us take a moment to reflect on the importance of existence and uniqueness. We take for example, the following words from p. 4 in Carmen Chicone's book [19]:

The existence and uniqueness theorem is so fundamental in science that it is sometimes called "the principle of determinism. The idea is that if we know the initial conditions, then we can predict the future states of the system."

Intuitively, uniqueness stresses on the empirical security that we gain in a sufficiently useful model through the fact that the repetition at other times of the same conditions, for example in a well set experiment, should give us the same results always. This is among the least we should expect from a good theory expressed mathematically. Chicone further adds:

Although the principle of determinism is validated by the proof of the existence and uniqueness theorem, the interpretation of this principle for physical systems is not as clear as it might seem. The problem is that solutions of differential equations can get very complicated. For example, the future state of the system might depend sensitively on the initial state of the system. Thus, if we do not know the initial condition exactly, the final state may be very difficult (if not impossible) to predict.

Of course, a famous example of the previous statement lies in chaos theory. Sensitivity with respect to initial conditions is of fundamental importance in applications, such as in climate prediction, engineering applications, finance, and as a mathematical curiosity in itself in general theory of differential equations. One way in which the sensitivity conditions are addressed is through theorems about continuity with respect to initial conditions, which is why these results are of theoretical interest. However, these kind of results state that two or more solutions of an initial value problem remain close together only on compact, in particular, *bounded time intervals*, and often we are interested in the *long term behavior* of solutions, supposing they exist on unbounded intervals of time. For long term behavior we will later on introduce the important notion of stability.

Perhaps the reader has already identified how the notions of existence, uniqueness and continuity with respect to initial conditions correspond to the notion of *well-posedness* of a mathematical model, a notion commonly attributed to the mathematician Jacques Hadamard which states that [31] :

1. A solution exists, and given some class of initial data, a unique solution exists to the given problem.
2. The solution varies continuously with respect to the initial data.

We now state the basic theory of ordinary differential equations. In the following, $J \subset \mathbb{R}$ and $D \subset \mathbb{R}^n$ are as above.

Definition 2.1. *We say that a mapping $f : J \times D \longrightarrow \mathbb{R}^n$ satisfies a local Lipschitz condition in the variable x if for each $(t_0, x_0) \in J \times D$ and for each $t_1 > t_0$ such that $[t_0, t_1] \subset J$ there is an $r > 0$ and an $L > 0$ constant such that*

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\|$$

for all (t, x) and (t, y) such that $x, y \in \{z \in \mathbb{R}^n : \|z - x_0\| \leq r\} \subset D$ and $t \in [t_0, t_1]$. The $L > 0$ constant is called a Lipschitz constant, or a local Lipschitz constant[1]*. We say that f satisfies a Lipschitz condition at a particular point $(t_0, x_0) \in J \times D$ if the previous holds for all x, y in some ball around x_0.*

Due to the slightly different notions of Lipschitz conditions that one encounters in the literature, caused by the the topologically distinct regions where the definition implies the aforementioned condition holds, we must exert some caution when dealing with Lipschitz conditions, or more general Lipschitz-type conditions that we will encounter later on. The important thing to notice is that a certain Lipschitz constant L_1 that works in a certain neighborhood of $J \times D$ might be different from an L_2 Lipschitz constant that works in a different region of $J \times D$. If $D = \mathbb{R}^n$ and *the same* Lipschitz constant $L > 0$ works for $x, y \in \mathbb{R}^n$ in the definition of Lipschitz function, then f is said to be *globally Lipschitz*. See [30] p. 89 for more details on these distinctions.

Theorem 2.1. (Local Existence and Uniqueness) *Let $f(t, x)$ be continuous in both variables (t, x) and satisfy a local Lipschitz condition at $(t_0, x_0) \in J \times D$, in other words there exists for each $[t_0, t_1]$ an $r > 0$ such that*

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\|$$

for all (t, x) and (t, y) such that $x, y \in \{z \in \mathbb{R}^n : \|z - x_0\| \leq r\} \subset D$ and $t \in [t_0, t_1]$. Then there exists a $\delta > 0$ such that the initial value problem (2.2) *has a unique solution over $[t_0, t_0 + \delta]$.*

[1] Local Lipschitz constant because the same constant L might not work at a different point $(\hat{t}_0, \hat{x}_0) \in J \times D$.

The local Lipschitz condition is sufficient for uniqueness, but not necessary, see the book by R.P. Agarwal and V. Lakshmikantham [1] for other conditions that can also be sufficient to guarantee uniqueness. It is well known that continuity of $f(t,x)$ on $J \times D$ is enough to ensure existence, however, for the purposes of this thesis we shall generally assume uniqueness of solutions.

In an application, asserting existence and uniqueness is an important question about a mathematical model being considered. If a computer is carelessly used to obtain the solution of an initial value problem, then if the solution is not unique, one must determine what this means for the application of interest, otherwise there could be great vagueness in the information trying to be derived from the given mathematical model, since non-uniqueness might render the model useless. Nonetheless, non-uniqueness might still be tolerable for particular types of problems, since the nonuniqueness of solutions can still be of physical significance in certain applications [56] p. 5. There are plenty of known examples of ordinary differential equations whose initial value problems can induce infinitely many solutions. One can imagine the limitations of using a computer, when it only plots one of these infinitely many solutions, without us realizing that the model is not well posed the moment the model gives non-unique solutions. A typical example is

$$f(t,x) = x^{2/3}, \qquad x(0) = 0.$$

f is continuous, but not Lipschitz continuous. This initial value problem has infinite solutions $x \equiv 0$ and

$$x(t) = \begin{cases} 0 & \text{if } 0 \le t \le c \\ \dfrac{(t-c)^3}{27} & \text{if } t > c. \end{cases} \tag{2.3}$$

There are far worse examples of how bad nonuniqueness can be: [1] gives an example with a continuous function $f(t,x)$ where in a given open rectangle $\mathcal{R}$ in the Euclidean plane, for any (t_0, x_0) in the interior of $\mathcal{R}$, there exist an infinite number of solutions going through (t_0, x_0) in any interval of the form $[t_0 - \epsilon, t_0]$ or $[t_0, t_0 + \epsilon]$.

The existence-uniqueness result given above provides an interval of existence of the solution over $[t_0, t_0 + \delta]$, and $\delta > 0$ might be very small. The theorem doe not say anything about how long the interval of existence of the solution may be. However, under the hypothesis of Theorem 2.1 (or even weaker hypothesis such that uniqueness is not guaranteed), it is proved in differential equations courses that there is a *maximal interval of existence.* If a solution can no longer be continued beyond an interval (α, β), then we say that (α, β) is a maximal interval of existence, where $\alpha \ge -\infty$, $\beta \le \infty$. The following result characterizes the behavior of solutions on maximal intervals of existence in the case when $\beta < \infty$. It also holds supposing f satisfies sufficient conditions for existence of a solution.

Theorem 2.2. (Extended Existence) *Let $f(t,x)$ satisfy the same hypotheses of Theorem 2.1. Let $(t_0, x_0) \in J \times D$ induce an initial value problem* (2.2)*, and suppose that the maximal interval of existence of the solution $t \mapsto x(t)$ is given as $\alpha < t < \beta$ with $\beta < \infty$. Then, for each compact subset $K \subset D$ there is some $t \in (\alpha, \beta)$ such that $x(t) \notin K$. In particular, either $\|x(t)\|$ becomes unbounded or $x(t)$ approaches the boundary of D as $t \to \beta^-$.*

If $\|x(t)\|$ becomes unbounded as $t \to \beta^- < \infty$, we say that the solution *blows up in a finite time*.

In general, there is no guarantee that a solution is defined for all $t \geq t_0$. This is important for analysis involving the long term behavior of solutions of differential equations. Conditions to guarantee the existence of solutions for all $t \geq t_0$ are an important topic of study in theory of differential equations. In fact, one of the central aspects of the nonlinear systems that we will study in this thesis shall revolve under conditions to guarantee that the solutions of our differential equations exist for an indefinite amount of time in the future. One simple criterion for the latter, but which can nonetheless be a formidable task to know if it is satisfied, is to prove that the solutions remain bounded for all future time. There are diverse methods to try to show this for a particular system, and these type of questions will be very important in our future study of stability of more general systems.

One simple way to obtain the existence of solutions on an interval $[t_0, t_1]$ for t_1 arbitrarily large, is to ask for a vector field f to satisfy a *global* Lipschitz condition, as we state in the following result [30].

Theorem 2.3. (Global Existence and Uniqueness) *Let $f(t, x)$ be piecewise continuous in t and suppose that there exists an $L > 0$ such that*

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\|$$

for all $x, y \in \mathbb{R}^n$ and $t \in [t_0, t_1]$. Then the initial value problem (2.2) *has a unique solution over $[t_0, t_1]$.*

Of course, the global Lipschitz condition is only sufficient, and not necessary for global existence of solutions. By a global solution, we mean a solution defined on the time interval J of definition of the vector field.

2.2.1 Linear Systems

A useful example of a particular type of ordinary differential equation which we shall use, is the following *linear homogeneous time-varying* IVP defined for $y \in \mathbb{R}^n$. In the following, $A(t)$ is an $n \times n$ continuous matrix-valued function defined on some interval $J \subset \mathbb{R}$.

$$\begin{aligned} y'(t) &= A(t)y \\ y(t_0) &= y_0. \end{aligned} \tag{2.4}$$

It can be proved, for example, using a Gronwall inequality (see [19] Section 2.1.2) that (2.4) has a unique solution defined wherever $A(t)$ is defined, even if it is an infinite-length interval. Associated to the time-varying system (2.4) is the IVP

$$\begin{aligned} y'(t) &= A(t)y \\ y(t_0) &= e_k \end{aligned} \tag{2.5}$$

where e_k is the k-th standard basis vector of $\mathbb{R}^n$ for $1 \leq k \leq n$. For each k, let $\varphi_k(t)$ denote the unique solution of (2.5). With these solutions φ_k, we define the *fundamental matrix* solution, or also known as the *state transition matrix* $\Phi(t, t_0)$ of the linear ordinary differential equation $y'(t) = A(t)y$ as the matrix whose k-th column is $\varphi_k(t)$. The solution of the particular initial value problem (2.4) can be expressed as

$$y(t) = \Phi(t, t_0)y_0. \tag{2.6}$$

The state transition matrix is useful for expressing solutions of initial value problems indexed under different initial conditions $y_0 \in \mathbb{R}^n$. The state transition matrix has the properties stated below [19].

Properties of the State Transition Matrix $\Phi(t, t_0)$:

(i) $\Phi(t, t_0)$ solves the matrix ODE IVP $\Phi'(t, t_0) = A(t)\Phi(t, t_0)$ with $\Phi(t_0, t_0) = I_d$, where I_d denotes the $n \times n$ identity matrix.

(ii) $\Phi(t, t_2)\Phi(t_2, t_1) = \Phi(t, t_1)$ for all $t_1, t_2, t \in \mathbb{R}$.

(iii) $[\Phi(t, t_0)]^{-1}$ exists and $[\Phi(t, t_0)]^{-1} = \Phi(t_0, t)$.

Remark. If the matrix $A(t) = A$ is constant, then the state transition matrix is known to be $\Phi(t, t_0) = e^{(t-t_0)A}$, the matrix exponential defined by

$$e^{(t-t_0)A} = \sum_{j=0}^{\infty} \frac{(t-t_0)^j A^j}{j!},$$

where diverse numerical linear algebra methods for obtaining the matrix exponential are known. Thus the solution to the initial value problem (2.4) is represented as $y(t) = e^{tA}y_0$.

For the special case when $n = 1$ we have the scalar time linear varying system, more commonly known in basic differential equations courses as the *scalar first order linear* ODE. Thus $A(t)$ reduces to a scalar function, and the fundamental matrix is well known to be

$$\Phi(t, t_0) = \exp\left(\int_{t_0}^{t} A(s)ds\right), \tag{2.7}$$

which we will commonly denote as

$$\Phi(t, t_0) = e^{\int_{t_0}^{t} A(s)ds}.$$

Variation of Parameters

A fundamental tool that is used in stability analysis of nonautonomous nonlinear systems is the variation of parameters method. There are many variation of parameters formulas, depending on what system is being studied, but the formulas that we will use involve the state transition matrix of a generally time-varying system. Although the state transition matrix is in general complicated to determine (except in some cases), the strength of the variation of parameters formulas lies in a theoretical and symbolic representation of solutions in order to obtain applicable results through the fact that it is very often possible to establish bounds on the state transition matrix using some operator norm. In this manner, since we cannot solve many differential equations in the first place anyways, we can obtain results that are applicable if we use norms to state our conclusions within a region necessary for the bounds to work. Applications do not always require analytically exact representations of solutions and measurement of variables, since error is always involved. It is often sufficient to know what happens within a bounded region, with a good margin of error.

The following formula will be indispensable for the work done here. We will base a lot of our analysis on this particular case, even when working with more complicated systems. The result is standard in ODE textbooks, and can be found for example in [19].

Proposition 2.1. (Variation of Parameters) *Consider the initial value problem*

$$\dot{x} = A(t)x + g(t,x), \qquad x(t_0) = x_0,$$

where we just assume sufficient hypothesis on $g : J \times D \longrightarrow \mathbb{R}^n$ *for a solution to exist and for* $s \mapsto g(s, x(s))$ *to be continuous. Let* $\Phi(t, t_0)$ *be the state transition matrix of the related homogeneous system* $\dot{y} = A(t)y$ *that is defined on some interval* J *containing* t_0*. If* $t \mapsto \varphi(t)$ *is the solution of the initial value problem defined on some subinterval of* J*, then* φ *has the representation formula*

$$\begin{aligned}\varphi(t) &= \Phi(t,t_0)x_0 + \Phi(t,t_0)\int_{t_0}^{t} \Phi^{-1}(s,t_0)g(s,x(s)) \\ &= \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,s)g(s,x(s)).\end{aligned} \tag{2.8}$$

2.3 Stability of Differential Equations

2.3.1 Motivation

A fundamental problem in the theory of differential equations is to study the motion of the system using the vector field that induces the differential equations. Qualitative analysis involves questions of the type: Do the solutions go to infinity, or do they remain bounded within a certain region? What conditions must a vector field satisfy in order for the solutions to remain within a given region? Do nearby solutions act similarly to a particular solution of interest? These are questions of qualitative type, in contrast with analytic methods which tend to search for a formula to express each solution of a differential equation.

As our vector fields get more complicated, such as when one goes from ordinary differential equations to functional differential equations, analytic methods go out the window, since solving the equations becomes even more impossible. Thus qualitative methods take the leading role.

It is widely regarded that Henri Poincaré was an important pioneer in the methods of qualitative analysis of differential equations. For a long period in mathematics after the invention of calculus, most of research and applications of differential equations was centered around the analytic expression of solutions of differential equations. By this we mean the expression of solutions in terms of formulas involving algebraic operations with known functions. To get a quick idea of the situation that permeated this area during the era between Newton and that of Poincaré, let us borrow the following quote, which the editor of the introduction to Poincaré's famous work, *New Methods of Celestial Mechanics* [45], attributes to Lagrange in the preface to his *Mécanique Analytique*:

"I have set myself the problem of reducing this science [mechanics], and the art of solving the problems appertaining to it, to general formulas whose simple development gives all the equations necessary for the solutions of each problem... No diagrams will be found in this work. The methods which I expound in it demand neither constructions nor geometrical or mechanical reasonings, but solely algebraic [analytic] operations subjected to a uniform and regular procedure. Those who like analysis will be pleased to see mechanics become a new branch of it, and will be obliged to me for having extended its domain."

Perhaps if Lagrange had known what was to come, for example, nonlinear functional differential equations, he might have changed his mind. Even when it was known that the great majority of differential equations could not be integrated in terms of known functions or expressed in terms of power series, the study of the properties of solutions of differential equations presented a heavy tendency towards local analysis.

The influence of Poincaré shifted the study of differential equations in terms of formulas, to the global properties of the solutions without solving the differential equations themselves. The important contributions of Poincaré came at around the same time as those of the Russian mathematician Aleksandr Lyapunov, who developed what are now among the most widely used methods for determining stability of nonlinear differential equations, of course, also without solving the challenging nonlinear equations. These methods use Lyapunov functions, which we will define below. Thanks to the contributions of these two mathematicians, important advances in the study of nonlinear differential equations were achieved. There are of course more simple criteria than Lyapunov methods for stability of relatively simple systems, such as linear time invariant systems. However, very often in applications a linear model is not good enough, since nonlinear systems possess qualitative features that a linear system will never capture, and these important features are of interest to the modeler. An example of this is in the human heart, which operates on the basis of a stable limit cycle, which is a dynamical behavior that a linear system cannot achieve. See [56] for more details, and [15] for more examples where nonlinear systems are preferable. Lyapunov methods are the preferred method when analyzing nonlinear systems.

The main global property that we shall address in this thesis is the property of stability of a differential equation, which we shall define below. In broad terms, stability is the property of being able to guarantee that solutions of differential equations with sufficiently close initial values remain close to each other over indefinite amounts of time in the future. It can also be viewed as a result about the long term behavior of solutions to initial value problems under perturbations of the initial condition. This is greatly important for applications because nothing is really exact, so we would like a model that will make long term predictions, to possess this characteristic in order for it to be of practical significance.

The development of stability theory begun at the end of the 19th century, has been influenced by the problem of determining the stability of our solar system. Nevertheless, stability applications have seeped into industry such as in engineering applications, where the common practice is to run a process in "steady state"[2]. For the system that the engineer is interested in, it is frequently of much greater importance to know that the system is approaching a stable equilibrium and will remain there for long time periods (mathematically, this means indefinitely in time), than to have an exact computation of short term transient behavior. Furthermore, the control engineer must take care that the parameters of the model do not fall into dangerous instability regions. Addressing the sensitivity issue, as [19] puts it, if the process does not stay near the steady state after a small disturbance, the engineer faces a problem. If under a small perturbation, we do not return to a stable state, then the model is useless, since applications are never exact, there is always an error when making a measurement. Computers make it possible to find approximately the solutions of differential equations on a *finite* interval of time, but they do not answer the qualitative aspects of the global behavior of phase curves.

We will now give some commonly known stability definitions, methods and criteria. Around these methods, control paradigms are designed, although we will not particularly address these control methods and theory. The importance of stability in control theory illustrates one of the central applications and one of the roles that stability has in numerous applications.

2.3.2 Stability Results

The following definitions are taken from [30], and are for nonautonomous systems, which occur when the vector field in (2.1) depends on the t-variable explicitly, and not implicitly through $x(t)$. We shall explain the necessity of nonautonomous systems below.

Let us define the nonautonomous system

$$\dot{x} = f(t, x) \tag{2.9}$$

where $f : J \times D \longrightarrow \mathbb{R}^n$ satisfies the existence-uniqueness hypotheses from the theorems above, and $D \subset \mathbb{R}^n$ is a domain (open connected set) containing the origin $x = 0$.[3]

Definition 2.2. *We say that the origin $x = 0$ is an equilibrium point or rest point for* (2.9) *at $t = 0$ if $f(t, 0) = 0$ for every $t \geq 0$. More generally, the solution $t \mapsto \varphi(t)$ is an equilibrium solution at $t = a$ if $f(t, \varphi(t)) = 0$ for every $t \geq a$.*

Intuitively, the equilibrium point corresponds to a state not moving away from the prescribed point, since the zero vector is attached to the equilibrium, causing the path to remain there indefinitely as time moves forward.

An equilibrium point at the origin could be viewed as the translation of a nonzero equilibrium

[2]This steady state notion corresponds to what is also known as an equilibrium point.

[3]It is not necessary for the vector field to satisfy uniqueness to define stability.

point, or even more generally, as a translation of a nonzero solution of the system. We can understand this better as follows. Suppose that the curve given by $s \mapsto \varphi(s)$ is a solution of the system

$$\frac{dy}{ds} = g(s, y)$$

defined for all $s \geq a$. By introducing the change of variables

$$x = y - \varphi(s); \quad t = s - a,$$

the previous system is transformed into the form

$$\begin{aligned} \dot{x} &= \dot{y}(t+a) - \dot{\varphi}(t+a) \\ &= g\big(t+a, y(t+a)\big) - \dot{\varphi}(t+a) \\ &= g\big(t+a, x+\varphi(t+a)\big) - \dot{\varphi}(t+a) \\ &= g\big(t+a, x+\varphi(t+a)\big) - g\big(t+a, \varphi(t+a)\big) \\ &=: f(t,x), \end{aligned}$$

so that we can view this as a way of defining the vector field $f(t,x)$ through a translation of a whole solution φ of the system induced by $g(s,y)$. Supposing φ is an equilibrium solution at $s = a$, then since we obtained

$$f(t,x) = g\big(t+a, x+\varphi(t+a)\big) - g\big(t+a, \varphi(t+a)\big),$$

the origin $x = 0$ becomes an equilibrium of the transformed system at $t = 0$. Notice that if the solution $\varphi(s)$ is not constant, then the transformed system will be nonautonomous even when the original system is autonomous, that is, even if $g(s, y(s)) = g(y(s))$. That is why we must study nonautonomous systems.

Notice that we provide the definitions of stability for the point $x = 0$ only. Because of the previous translation of a solution $\varphi(s)$ argument, by determining the stability behavior of the origin $x = 0$ as an equilibrium point of the transformed system, we are determining the stability behavior of the solution $\varphi(s)$ of the original system. Thus defining stability for the origin is sufficient.

Definition 2.3. (Stability for ODEs) *The equilibrium point $x = 0$ of (2.9) is said to be*

- *Stable if for each $\epsilon > 0$, there is a $\delta = \delta(\epsilon, t_0) > 0$ such that*

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \qquad \forall t \geq t_0 \geq 0. \tag{2.10}$$

- *Uniformly stable if, for each $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ independent of t_0, such that (2.10) is satisfied, for all $t_0 \geq 0$.*
- *Unstable if it is not stable.*
- *Asymptotically stable if it is stable and for each t_0 there is a constant $c = c(t_0) > 0$ such that $x(t) \to 0$ as $t \to \infty$, for all $\|x(t_0)\| < c$.*

- *Uniformly asymptotically stable if it is uniformly stable and there is a constant $c > 0$ independent of t_0, such that or all $\|x(t_0)\| < c$, $x(t) \to 0$ as $t \to \infty$, uniformly in t_0; that is, for each $\eta > 0$, there is $T = T(\eta) > 0$ such that*

$$\|x(t)\| < \eta, \quad \forall t \geq t_0 + T(\eta),\ \forall \|x(t_0)\| < c.$$

- *Globally uniformly asymptotically stable if it is uniformly stable, $\delta(\epsilon)$ can be chosen to satisfy $\lim_{\epsilon\to\infty} \delta(\epsilon) = \infty$, and for each pair of positive numbers η, c, there is $T = T(\eta, c) > 0$ such that*

$$\|x(t)\| < \eta, \quad \forall t \geq t_0 + T(\eta, c),\ \forall \|x(t_0)\| < c.$$

For autonomous systems, in other words, the case when $f(t, x) = f(x)$ we have that the δ of the definition of stability is independent of t_0, and there the notion of being uniformly stable or uniformly asymptotically stable is unnecessary. For the case of autonomous systems:

$$\dot{x} = f(x), \tag{2.11}$$

we have the following famous result by Lyapunov.

Theorem 2.4. (Lyapunov's Stability Theorem) *Let $x = 0$ be an equilibrium point for (2.11), and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : D \longrightarrow \mathbb{R}$ be a continuously differentiable function such that*

$$V(0) = 0 \qquad \text{and} \qquad V(x) > 0 \ \text{ in } \ D\backslash\{0\}$$

$$\frac{d}{dt}[V(x(t))] = \frac{\partial V}{\partial x}\big(x(t)\big) \cdot x'(t) = \frac{\partial V}{\partial x}\big(x(t)\big) \cdot f(x(t)) \leq 0 \ \text{ in } \ D.$$

Then, $x = 0$ is stable. Moreover, if

$$\frac{d}{dt}[V(x(t))] < 0 \ \text{ in } \ D\backslash\{0\},$$

then $x = 0$ is asymptotically stable.

The proof of the previous theorem and examples of its application can be seen in [19], [27] or [30]. A function $V(x)$ such that $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$ is said to be *positive definite*. If it satisfies the weaker condition $V(x) \geq 0$ for $x \neq 0$, it is called *positive semidefinite*. Negative definite and negative semidefinite are defined similarly with the inequalities reversed. A function satisfying the hypotheses of Lyapunov's stability theorem is called a *Lyapunov function*.

Lyapunov was interested in stability of mechanical systems, and Lyapunov functions generalize the total energy function in mechanical or electrical systems. That is why it is very often that in applications in these areas, a good candidate for a Lyapunov function tends to be the sum of the kinetic plus potential energy, or the Hamiltonian. When these energy functions fail to act as Lyapunov functions (or are not convenient enough, they might give us stability, but not asymptotic stability, for instance), a certain amount of ingenuity and experience is required to find a suitable function.

The hypersurfaces that we obtain when we define sets of the form

$$\{x : V(x) = c\}$$

for each $c \geq 0$ constants are called *Lyapunov surfaces*, and these generalize the notion of energy surfaces in classical physics. For sufficiently small positive values of c, namely $0 < c < \sup_{x \in D} V(x)$, we have that the level surfaces are closed, in fact, in [52] it is proved that when the derivative $\dot{V} < 0$ is negative definite, in other words, when asymptotic stability holds, the Lyapunov surfaces are homotopically equivalent to the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$. One can see intuitively how this makes sense, since Lyapunov surfaces are the boundaries of the sets $\Omega_c = \{x : V(x) \leq c\}$ and if D is an unbounded set and $c > \sup_{x \in D} V(x)$, then $\Omega_c = D$ becomes unbounded. Thus, when the derivative of V is negative definite, using the fact that the Lyapunov surfaces are topologically equivalent to spheres, it then makes sense to talk about closed manifolds similar to energy level sets surrounding the origin, so that the negative definiteness of the derivative captures the notion of a path losing "energy" and directing itself always moving inwards relative to closed surfaces with less energy, arriving arbitrarily near the origin as the solution punctures through smaller closed surfaces.

For the weaker case where the derivative $\dot{V}$ is merely negative semidefinite, a similar intuitive analysis holds, except that we cannot guarantee that the solutions move inside of the Lyapunov surfaces, but we can at least know that the solutions, for sufficiently small initial conditions, remain on suitably small Lyapunov surfaces. In this sense, at least the solutions remain bounded and within a certain suitably small region, which guarantees that the solution will not run off or blow up.

In general, Lyapunov type results for more complicated differential equations, be it for ordinary differential equations, functional differential equations, impulsive systems, etc., tend to require a notion of the Lyapunov function (or *functional* as we will see ahead) decreasing along the solution trajectories. Sometimes the conditions are not smooth enough and we must weaken the notion of derivative by using suitable generalizations of derivatives, for example as in [37], where a Dini-type derivative is used, but in the end all methods capture the essence of some type of derivative being somehow negative (for example, being bounded by a negative definite function) to denote that the Lyapunov function/functional is decreasing along trajectories of solutions, in a sufficiently small neighborhood of the rest point.

In this thesis, we will eventually capture the notion of some type of characteristic of the differential equation that limits some type of energy of the solutions, and guarantees that the vector field has the necessary conditions for asymptotic stability. The measure of how limited the energy must be will be captured through a contraction requirement in a metric space setting, and we will study this in later chapters.

We now state an extension of the previous Lyapunov theorem for nonautonomous systems. Using Dini derivatives, we may weaken the smoothness hypothesis of the Lyapunov function, for example, but since our objective is to select some of the existing theory solely for the purpose of creating a narrative by giving an idea about the established literature, the following theorem, taken from [30] p. 151 is sufficient for our purposes.

Theorem 2.5. *Let* $x = 0$ *be an equilibrium point for* (2.9) *and* $D \subset \mathbb{R}^n$ *be a domain containing* $x = 0$*. Let* $V : [0, \infty) \times D \longrightarrow \mathbb{R}$ *be a continuously differentiable function, and* $W_1, W_2 : D \longrightarrow \mathbb{R}$ *continuous positive definite functions such that*

$$W_1(x) \leq V(t, x) \leq W_2(x),$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq 0 \tag{2.12}$$

$\forall t \geq 0$*, and* $\forall x \in D$*. Then* $x = 0$ *is uniformly stable. If* (2.13) *is strengthened to*

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x) \tag{2.13}$$

for all $x \in D$*, where* $W_3 : D \longrightarrow \mathbb{R}$ *is a continuous positive definite function, then* $x = 0$ *is uniformly asymptotically stable.*

The previous theorem can be strengthened to achieve even global uniform asymptotic stability, see [30].

Chapter 3

Delayed Functional Differential Equations

3.1 Motivation

By a *delayed differential equation* (DDE), in the broadest sense we mean differential equations which somehow include information from the past. Incorporating values of functions from the past to define a vector field is also known as a particular type of what is known as a *functional differential equation* (FDE).

Implicit in the utilization of ordinary differential equations, is the assumption that the future of the system being modeled is completely independent of the past and only depends on the present state. Delayed information in dynamical systems can play a crucial role. To give us a quick idea of the importance, let us think about economics. The difficulty with the theoretical assumption of the invisible hand, famously postulated by Adam Smith, is that the hand acts like a controller that tries to stabilize supply and demand. However, the hand is clumsy because it is trying to stabilize dynamics that are reacting to information from the past. For a particular example of the faulty invisible hand in the oil industry, see [44], where they explain the difficulties of using a feedback loop model when a corrective mechanism is trying to be implemented in such a way that it balances out the unstabilizing effect of inevitable time lags of commercial investment.

As T. A. Burton in [15] says, when man devises a machine to run a machine, there is always a delay in the response. This is because a system involving a feedback control will most certainly involve time delays, because a finite amount of time is required to sense and measure information in order to react to it. Sometimes delays can be treated as negligible and thus there is no harm in approximation by ODEs. Nonetheless, this is not always the case, a famous example in control which does not involve delays as large as the economic system example above is from Nicolas Minorsky in his study in [42] of the control of the motion of a battleship. Minorsky introduced a delay (representing the time for the readjustment of the ballast) and observed that the motion was oscillatory if the delay was too large. Similar situations occur when piloting an aircraft, and these types of investigations spurred interest in military applications, which some authors, such as in [24], [3] and [21], point out as a decisive factor that lead to the rapid development of the theory for functional differential equations in the United States[1] during the middle of the 20th century. Delays may introduce oscillations that destroy stability of a similar non-delayed ODE version of the system, but as we will see below, delays can also create stability.

There are numerous other examples of applications where delays can give more useful results, such as in biological sciences, for example in neural networks, where delays can represent the synaptic processing time, in ecological models, epidemiology, to name some examples. See [10] for a discussion of delay differential equations (DDEs) applications to biosciences.

If you have ever been hit in the face by a ball and saw it coming, you might have noticed how your reflexes allow you to somewhat process the image of a ball getting larger continuously in time, and the next second, neighboring laughter confirms you got hit. The failure in your reaction time between sensing the ball and moving your body is accounted for by delay in your dynamics. In ODE world, we would dodge the ball akin to a scene from The Matrix. In the real world, in delayed humans world, we feel pain. Something similar occurs when you balance yourself and react to the tilting of your body, your decisions at each movement represented in your arms seem quite random. Reacting to delays can break stability, such as in pilot induced oscillations (PIOs) that result from delays in response time of the pilot above a certain threshold, where bifurcation behavior of the dynamics breaks stability, see [22].

There are numerical methods to approximate particular kinds of delay differential equations by an ordinary differential equations on a finite interval. Nonetheless, this is not always possible, and even when so, the long term dynamics of delay differential equations can differ substantially from the dynamics of the approximating ordinary differential equation, see for example [10].

For more examples of applications, see Chapter 1 of [3], where Jack K. Hale gives a brief historical perspective of functional differential equations in applications, [21] for a wide range of applications, as well as [15] for further examples and sources.

In all of the previous discussions, one can see how ordinary differential equations can become insufficient. Many models are better represented by more general differential equations known commonly as functional differential equations. Functional differential equations involve what are known as retarded, or perhaps more politically correct, delay functional differential equa-

[1] In contrast to results by Russian scientists working in control, see [9] for a general overview of the Russian school of control led by A. A. Andronov during the 1930's and 1940's. The distinguishing feature of the Russian school was the emphasis on nonlinear systems in control, and this included delay functional differential equations. The results of Russian scientists in nonlinear oscillations research during WWII, primarily those of the group of Andronov, led the Princeton mathematician Solomon Lefschetz, who introduced translations of main Russian works to the English-speaking world, to raise concern in the United States for the need of developing nonlinear systems research in the Cold War context of the late 1950's and 1960's (see [5], [9]). Lefschetz received heavy funding from the military of the United States to start a "Project on Differential Equations", specifically devoted to nonlinear equations. The objective of the project was to form applied mathematicians for industry, or in case of an emergency, for defense purposes, see [5] p. 19, 20.

tions. Functional differential equations (FDEs) involve in the defining vector field a dependence on the values of a function in a specified time interval, which may be finite or infinite. Here we will concentrate on previous values of a function, so that this captures the essence of delayed information affecting the present direction of the state $x(t)$. This is why, to stress that we consider only values of a function before a particular time t_0, the word "delayed" is attached to specify what kind of functional differential equation we are dealing with. In this sense "delayed differential equation" and "delayed functional differential equation" are used interchangeably.

We will present now some general theory of delayed FDEs, starting with the essential definitions, discussions of some basic differences with respect to ODEs, as well as foundational theoretical results. In the last section we touch upon stability definitions and give some stability results, which will be sufficient for a working knowledge in the topics of this thesis. For this last reason, we do not intend to give a broad comprehensive examination of stability results for FDEs.

3.2 Basic Concepts of Functional Differential Equations

The following concepts are based primarily on [24] and [3].

Let $0 < r \leq \infty$. *We will denote the Euclidean norm of a vector $x \in \mathbb{R}^n$ as $|x|$ from now on in order to avoid confusion with another norm we shall use.* Let us define the delayed functional differential equation

$$x'(t) = f(t, x_t), \qquad t \geq \sigma \tag{3.1}$$

where we explain below what x_t means. Here, we have that $x(t) \in \mathbb{R}^n$, $f : J \times C([-r, 0], D) \longrightarrow \mathbb{R}^n$ with $J \subset \mathbb{R}$ an interval, $D \subset \mathbb{R}^n$ an open set. $C([-r, 0], D)$ denotes the space of continuous functions[2] mapping the interval $[-r, 0]$ into $D \subset \mathbb{R}^n$, where we use the uniform convergence topology induced by the norm

$$\|\psi\|_r := \sup_{s\in[-r,0]} |\psi(s)|, \tag{3.2}$$

where of course for $r = \infty$ this norm is $\|\psi\|_r = \sup_{s\in(-\infty,0]} |\psi(s)|$. Wherever the norm symbol $\|\cdot\|$ is used, *we refer to the norm on $C([-r, 0], D)$.*

Remark 3.1. *We will on occasions denote $C = C[-r, 0]$ when no confusion should arise. In the case when $r = \infty$, we will consider the space $BC\big((-\infty, 0], D\big)$, of* ***bounded*** *continuous functions on the infinite interval $(-\infty, 0]$, to obtain a complete metric space.*

If for some $\sigma \in \mathbb{R}$, $A > 0$ we have a continuous function $x : [\sigma - r, \sigma + A] \longrightarrow \mathbb{R}^n$, then for each $t \in [\sigma, A]$ we denote by x_t the function in $C[-r, 0]$ defined explicitly as

$$x_t(\theta) := x(t + \theta) \qquad \text{for } \theta \in [-r, 0] \tag{3.3}$$

[2]This is a Banach space when r, the delay is finite, when $r = \infty$, in order to have a complete metric space, we choose $BC(-\infty, 0]$, since the bounded functions induce a Banach space.

Note that if $x : [\sigma - r, \sigma + A] \longrightarrow \mathbb{R}^n$, then for each $t \in [\sigma, A]$, x_t simply denotes the restriction of $s \mapsto x(s)$ to the interval $s \in [t-r, t]$. In this manner we make sense of what the notation means in (3.1), where we have that the vector field at a particular time t thus exhibits dependence on the past behavior of $s \mapsto x(s)$ for $s \in [t-r, t]$.

To mention some quick examples of delayed FDEs in the form (3.1), we have equations with a fixed delay (the simplest possible case) such as

$$x'(t) = f(t, x(t), x(t-r))$$

or equations with multiple time varying delays on the same state x

$$x'(t) = f\big(t, x(t-\tau_1(t)), ..., x(t-\tau_p(t))\big)$$

with $0 \leq \tau_i(t) \leq r$ for all $i = 1, ..., p$. We also have integrodifferential equations

$$x'(t) = \int_{-r}^{0} g\big(t, x(t+\theta)\big)\, d\theta,$$

where we see how in the integration process we need to know the values of x in $[t-r, t]$ for each t where the vector field is defined. Vito Volterra considered the past states of a system when studying predator-prey dynamics, and he investigated the integro-differential system of equations

$$\begin{aligned} x'(t) &= \left[\epsilon_1 - \gamma_1\, y(t) - \int_{-r}^{0} F_1(\theta) y(t+\theta)\, d\theta\right] x(t) \\ y'(t) &= \left[-\epsilon_1 + \gamma_2\, x(t) + \int_{-r}^{0} F_2(\theta) x(t+\theta)\, d\theta\right] y(t) \end{aligned}$$

where x, y are the number of prey and predators, respectively, and all constants and functions are nonnegative. The previous equations relating derivatives and integrals over past time intervals are particular cases of more general forms of integro-differential equations such as systems of the form

$$x'(t) = f\left(t, x(t), x(\alpha(t, x(t))), \int_{-r}^{0} \mathcal{K}\big(t, \theta, x(t), x(t+\theta)\big)\, d\theta\right),$$

where $\alpha(t, x(t)) \leq t$ represents a state dependent delay. As can be seen, a rich variety of differential equations are contained in the functional differential equations category. We will see that since the initial conditions must be functions along a prescribed interval of time, functional differential equations generate infinite dimensional dynamical systems, and are somewhere between ODEs and PDEs in some classification sense.

3.2.1 Motivating the IVP

Supposing the vector field (3.1) is defined for all $t \geq \sigma$, then given an initial time $t_0 \geq \sigma$ of interest, we wish to formulate a notion of an IVP. The minimal necessary information to obtain a cogent theory of existence and uniqueness of solutions of (3.1) is to have as given data the behavior of a function on the entire interval $[t_0 - r, t_0]$. In other words, we need to know the history of the function in a time delay of size r before the present time t_0. We will see why this is the minimal information required with an example taken from [24]. The simplest example of a linear delay differential equation could be the scalar DDE with a constant delay

$$x'(t) = Ax(t) + Bx(t - r) + f(t) \tag{3.4}$$

where A, B and $0 < r < \infty$ are constants, f is a given continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$.

Suppose we want to calculate, using (3.4), the derivative at $t = 0$, so that we need to know, given the form of this DDE, the values $x(0), x(-r)$ and $f(0)$. Suppose we have the initial value $x(-r)$. Once we advance, say to $x(\epsilon)$, with $0 < \epsilon < r$ small, notice that to calculate the derivative at $t = \epsilon$ so that we can advance the next step, we need to know

$$x'(\epsilon) = Ax(\epsilon) + Bx(\epsilon - r) + f(\epsilon)$$

where $\epsilon - r \in (-r, 0)$. In this manner, we realize that we need to know the values of $x(\cdot)$ on the whole interval $[-r, 0]$. On the other hand, if we do not specify these values, we obtain an unsatisfactory notion of uniqueness, as the particular ODE attempt of an IVP example with

$$x'(t) = -\frac{\pi}{2}x(t - 1), \qquad x(0) = \frac{1}{\sqrt{2}}$$

illustrates. Here $\varphi_1(t) = \sin\left[\frac{\pi}{2}\left(t + \frac{1}{2}\right)\right]$ and $\varphi_2(t) = \cos\left[\frac{\pi}{2}\left(t + \frac{1}{2}\right)\right]$ are both solutions to the above. But if we specify the initial behavior on the interval $[-1, 0]$, we obtain that only one solution exists to each IVP, by the existence-uniqueness result in Theorem 3.1 that we give below.

With the previous discussion as a guide, let us now define the delayed FDE IVP problem for a given fixed t_0 in the interval of definition

$$\begin{aligned} x'(t) &= f(t, x_t), \qquad t \geq t_0 \\ x_{t_0} &= \phi \end{aligned} \tag{3.5}$$

where remembering the notation, $x_{t_0} = \phi$ is an abbreviation for

$$x(t_0 + s) = \phi(s) \qquad \forall s \in [-r, 0],$$

so that $x_{t_0} = \phi$ in the sense of elements of $C = C([-r, 0], D)$. We have $f : J \times C([-r, 0], D) \longrightarrow \mathbb{R}^n$ with $J \subset \mathbb{R}$ an interval such that $[t_0 - r, t_0] \subset J$. t_0 is the particular initial moment of interest for the initial value problem, and by going backwards an interval of size r, we induce an initial interval. $D \subset \mathbb{R}^n$ is an open connected set. We let f define the vector field for times $t \geq t_0$, and before t_0, on $[t_0 - r, t_0]$, ϕ defines the solution, and not f. Note that the case $r = \infty$ means we have $x(t) = \phi(t) \ \ \forall t \in (-\infty, t_0]$.

Let us now clearly state what a solution of a delayed FDE IVP is.

Definition 3.1. (Right-hand Derivative) *We define the right hand derivative of a function* $\varphi(t)$ *at a value* $t = t_0$ *as*

$$\lim_{h\to 0^+} \frac{\varphi(t_0+h)-\varphi(t_0)}{h}.$$

Remark 3.2. *We will specify whenever the right-hand derivative is being used. We will still denote a right-hand derivative of a function* $\varphi(t)$ *at a value* t *as* $\varphi'(t)$*, in order to not complicate the notation and in this way we avoid making it cumbersome to switch notation for derivatives when it is right-hand, and change it again when it is the normal derivative.*

Definition 3.2. (Solution of a FDE IVP) *Given* (3.5)*, a continuous function* $x : [t_0-r, t_0+A] \longrightarrow \mathbb{R}^n$*, for some* $A > 0$ *is called a solution of* (3.5) *through* $(t_0, \phi) \in \mathbb{R}_+ \times C$ *if* $x_{t_0} = \phi$ *and* $t \mapsto x(t)$ *satisfies the differential condition* (3.5) *for* $t \in [t_0, A]$*. At* $t = t_0$*, the derivative in* (3.5) *refers to the* ***right-hand*** *derivative*[3]*. Sometimes the dependence of* x *on* (t_0, ϕ) *is written explicitly through notation such as* $x(t) = x(t_0, \phi)(t)$ *or* $x(t) = x(t; t_0, \phi)$*.*

Let us continue to work on our first example.

Remark 3.3. *We use the symbol* $\triangle$ *to denote the end of an example.*

Example 3.1. Going back to the particular equation (3.4), let us append an initial condition at $t_0 = 0$ to get an IVP:

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bx(t-r) + f(t), \qquad t \geq 0 \\ &x_0 = \phi \end{aligned} \tag{3.6}$$

Remember that $r < \infty$. Notice that we can solve this in a particularly simple and straightforward manner, by using a variation of parameters type approach, or multiplying by the integrating function $\mu(t) = e^{-At}$ to obtain in a manner similar to the basic first order linear ODE that

$$\begin{aligned} &x(t) = \phi(t) \qquad t \in [-r, 0], \\ &x(t) = e^{At}\phi(0) + \int_0^t e^{A(t-s)}[Bx(s-r) + f(s)]ds, \qquad t \geq 0. \end{aligned} \tag{3.7}$$

We can directly show that the solution is unique in this case by explicitly evaluating the formula, since we have a fixed delay and a particularly simple equation for this case. The right hand side of (3.7) involves information that we know if we backtrack in intervals of length r, this is essentially what the *method of steps* for fixed delays is all about. On the interval $t \in [0, r]$, we have that we need to plug into the equation (3.7) the values of $x(s-r)$ for $s \in [0, r]$, this means, in other words, that we need the values of $t \mapsto x(t)$ for $t \in [-r, 0]$, which is the information contained in the initial function ϕ.[4] Thus for $t \in [0, r]$ we have that

[3] Since the initial function ϕ need not be differentiable or have its derivative related to $f(t_0, x_{t_0})$, the right hand derivative $\dot{x}(t_0)$.

[4] Notice that we can only do this on a forward interval of length r, we cannot do this further than that into the future, say for $t \in [0, r_1]$ with $r_1 > r$, since we would have to backstep a larger interval, as far as $[-r_1, 0]$, and we do not have that much information in our initial condition. Thus we must be patient and do our steps "r-length at a time".

$$x(t) = e^{At}\phi(0) + \int_0^t e^{A(t-s)}[B\,\phi(s-r) + f(s)]ds, \qquad t \in [0,r] \tag{3.8}$$

Now $x(t)$ is known on $[-r,r]$, we can advance the solution to $[r,2r]$, by using the values of x on $[0,r]$ to get that for $r \le t \le 2r$

$$\begin{aligned}
x(t) &= e^{At}\phi(0) + \int_0^t e^{A(t-s)}[B\,x(s-r) + f(s)]ds \\
&= e^{At}\phi(0) + \int_0^r e^{A(t-s)}[Bx(s-r) + f(s)]ds + \int_r^t e^{A(t-s)}[Bx(s-r) + f(s)]ds \\
&= e^{At}\phi(0) + \int_0^r e^{A(t-s)}[B\,\phi(s-r) + f(s)]ds + \int_r^t e^{A(t-s)}[Bx(s-r) + f(s)]ds
\end{aligned}$$

and notice that the integral from $[r,t]$ has $s-r \in [0,r]$ so that we use plug in formula (3.8) with $s-r$ in place of t to get

$$\begin{aligned}
x(t) = e^{At}\phi(0) &+ \int_0^r e^{A(t-s)} B\,\phi(s-r) + f(s)]ds \\
&+ \int_r^t e^{A(t-s)} \left[B \left(e^{A(s-r)}\phi(0) + \int_0^{s-r} e^{A(s-r-\xi)}[B\,\phi(\xi - r) + f(\xi)]d\xi \right) + f(s) \right] ds.
\end{aligned}$$

Thus we have now extended the solution x to the interval $[r,2r]$, and we now have a formula for $x(t)$ when $t \in [-r,2r]$. We can continue this process indefinitely, showing that the uniquely defined $x(t)$ exists on $[-r,\infty)$.

Notice, as can be seen from the previous equation, that if f has derivatives of all orders, or if $f \equiv 0$, then the solution gets smoother and smoother as you make t larger. This is because every interval of length r that you go forward, more integrals pile up on the last time interval $[(k-1)r, kr]$, and the previous intervals give rise to constants, such as in the last equation where $\int_0^r e^{A(t-s)} B\,\phi(s-r) + f(s)]ds = e^{At}\int_0^r e^{-As} B\,\phi(s-r) + f(s)]ds$ is simply $t \mapsto e^{At}$ multiplied by a now constant term. △

Notice that in the previous example, the calculations for $t \in [r,2r]$ quickly became larger, and this is why the method of steps may not give us much qualitative information about the solution; it might give an explicit formula, but in general no essential properties of the solution are revealed.

3.2.2 No more going back

In ODEs, results for a backward extension in the variable t are easy to obtain with mild assumptions on the vector fields, time has a very symmetric role there. However this is no longer the case in delayed FDEs. Delay differential equations, by imposing past conditions along intervals as initial conditions, induce a type of "arrow of time".

Boundary conditions and differentiability requirement

The following example is based on [24]. Take again (3.4). We will extend the solution to the left of $-r$ using the vector field. Notice that x is differentiable at $t_0 = 0$ if and only if

$$\frac{d\phi}{dt}(0) = A\phi(0) + B\phi(-r) + f(0).$$

This gives a sort of boundary condition. Something similar will happen as we extend the solution further to the left. Suppose, for example that in (3.5), ϕ is differentiable in $[-\epsilon, 0]$ for $0 < \epsilon < r$. Let $B \neq 0$ so that the delay is present. Then notice that if we wish to extend the solution to the left of $-r$, using (3.5) we have necessarily

$$x(t-r) = \frac{1}{B}[\dot{x}(t) - Ax(t) - f(t)]. \tag{3.9}$$

This means that to extend to the left of $-r$, we use the previous formula with $t \in [-\epsilon, 0]$ to define $x(s)$ for $s \in [-r-\epsilon, -r]$. Notice that since the right hand side of (3.9) will use $t \in [-\epsilon, 0]$, we will thus need the derivative $\dot{\phi}(t)$ on $[-\epsilon, 0]$, which is why we asked for this differentiability condition on ϕ.

Suppose ϕ has a derivative on $[-r, 0]$, so that x is defined using (3.9) on $[-2r, \infty)$. To extend $x(t)$ to $[-2r-\epsilon, \infty)$ with $0 < \epsilon \leq r$, we would need $x(s)$ as defined by (3.9) to be differentiable for $s \in [-r-\epsilon, -r]$. Also, by the DDE we would need to satisfy

$$\dot{x}(-r) = Ax(-r) + Bx(-2r) + f(-r). \tag{3.10}$$

To be consistent with our previous extension, then both sides of (3.9) should be differentiable, with $t \in [-\epsilon, 0]$. This implies that ϕ must be two times differentiable, and f at least once, since (3.9) implies

$$\begin{aligned} \dot{x}(-r) &= \frac{1}{B}[\ddot{x}(0) - A\dot{x}(0) - \dot{f}(0)] \\ &= \frac{1}{B}[\ddot{\phi}(0) - A\dot{\phi}(0) - \dot{f}(0)]. \end{aligned} \tag{3.11}$$

Using $\dot{\phi}(0) = A\phi(0) + B\phi(-r) + f(-r)$ in the previous, $\dot{\phi}(-r) = \dot{x}(-r)$ in (3.11) forces on $\ddot{\phi}(0)$ a certain value relating values of $\dot{\phi}, \dot{f}, \phi$ and f at 0 and $-r$, giving even more boundary conditions on the derivatives of ϕ. Also note $\dot{\phi}(-r)$ must equal a value given by the DDE. As we continue to extend further, things get more complicated, and higher order derivatives are required of ϕ, as well as on f.

Loss of backward uniqueness

The following is an example presented in [53]. Let

$$\begin{aligned} \dot{x}(t) &= b(t)x(t-1), \qquad t \geq 0 \\ x(t) &= 0, \qquad\qquad t \in [1, 2], \end{aligned} \tag{3.12}$$

so that $t_0 = 2$. We have that

$$b(t) = \begin{cases} 0 & \text{if } t \le 0 \\ \cos(2\pi t) - 1 & \text{if } 0 \le t \le 1 \\ 0 & \text{if } t \ge 1. \end{cases}$$

Using the method of steps, it can easily be shown that the unique solution is $x(t) \equiv 0$ for $t \ge 1$. One backward extension is $x(t) \equiv 0$ for all $t \in \mathbb{R}$. Another one is

$$x(t) = \begin{cases} c & \text{if } t \le 0 \\ c + c\displaystyle\int_0^t [\cos(2\pi\xi) - 1]d\xi & \text{if } 0 \le t \le 1 \\ 0 & \text{if } t \ge 1. \end{cases}$$

and the previous satisfies the DDE IVP for any constant c.

3.2.3 Delays can sometimes be good for us

Not everything is worse-off with delays, sometimes delays in a differential equation can make some work easier. The following examples are taken from [15].

A DDE behaving better than its similar ODE counterpart

The scalar ODE IVP

$$\dot{x}(t) = x^2(t) \qquad t \ge 0, \qquad\qquad x(0) = 1 \tag{3.13}$$

has the unique solution $x(t) = \dfrac{1}{1-t}$, which blows up at $t = 1$. However, introducing even the smallest bounded away from zero delay avoids a blowup.

Proof. Suppose we introduce the variable but strictly positive $0 < \gamma \le r(t) \le \alpha \ \forall t \ge 0$, for some $\alpha > 0$, $\gamma > 0$ and modify the ODE dynamics of (3.13) to

$$\dot{x}(t) = x^2(t - r(t)) \qquad t \ge 0.$$

This DDE with a bounded delay has solutions existing on all of $[0, \infty)$, no matter the initial condition. To prove this, let an initial condition be given through some $\phi \in C([-\alpha, 0], \mathbb{R})$. Suppose $x(t) = x(t; t_0, \phi)$ solves the IVP induced by the DDE, with $t_0 = 0$. Suppose for contradiction that there exists a finite blowup time T, so that the solution is defined on $[0, T)$ and satisfies $\limsup_{t \to T^-} |x(t)| = \infty$. We have $t - r(t) \le T - \gamma$ for $t \in [0, T]$. Notice $T - \gamma > 0$. Since $s \mapsto x(s)$ is continuous for $s \in [0, T - \gamma]$, the solution is thus bounded on this interval by some positive constant M. Thus, since $t - r(t) \in [0, T - \gamma]$, then by definition of the vector field, we have that $|\dot{x}(t)| = |x^2(t - r(t))| \le M^2$, which can be written as $|\dot{x}(t)| \le M^2 \ \forall t \in [0, T]$. Thus an integration of $\dot{x}$ on $[0, T]$ yields $|x(t) - x(0)| \le M^2 T$, implying that the solutions are bounded on $[0, T]$, contradicting blowup.

□

Delays can give explicit analytical formulas

Sometimes the delayed values can send us back to intervals where we know how the function behaved.

Example 3.2. (Kaplansky, 1957) Let us have

$$\ddot{x}(t) + tx = 0 \qquad t \geq 0.$$

This scalar ODE cannot be integrated to obtained a closed form solution. One can prove using algebraic techniques from ideal theory that you cannot solve it exactly using known functions (see [29]). But again, if $\alpha \geq r(t) \geq \gamma > 0$ on $[0, \infty)$, we define the related DDE

$$\ddot{x}(t) + tx(t - r(t)) = 0 \qquad t \geq 0.$$

In this way, given an initial function $\phi \in C([-\alpha, 0], \mathbb{R})$ for suitable $\alpha \in (0, \infty]$, if ϕ can have its integral evaluated, then, notice for example that by using a step method, for $t \in [0, \alpha]$, for example, that $x(t - r(t)) = \phi(t - r(t))$, since $t - r(t) \in [-\alpha, 0]$. Substituting into the previous DDE:

$$\ddot{x}(t) = -t\phi(t - r(t)) \qquad 0 \leq t \leq \alpha,$$

and we can directly integrate the previous twice to obtain an explicit formula. Continuing in this step-wise manner, we can obtain the solution. △

Example 3.3. (Predator-Prey) The predator prey system

$$\begin{aligned}\dot{x} &= ax - bx^2 - cxy \\ \dot{y} &= -ky + dxy\end{aligned}$$

This nonlinear coupled ODE has never been integrated in closed form. All constants a, b, c, d, k are positive The term $+dxy$ represents the utilization by the predator y of the prey x it has consumed (see [4]). It means that the predator y grows in numbers proportional to the number of interactions of predator and prey. Of course, the predator in the present grows in numbers according to prey it eats, which does not immediately translate into growth. Assuming this utilization does not occur right away, suppose there is an average time $T > 0$ for this, then we can use the modified model

$$\begin{aligned}\dot{x} &= ax(t) - bx^2(t) - cx(t)y(t) \\ \dot{y} &= -ky(t) + d\,x(t - T)y(t - T)\end{aligned}$$

with $d\,x(t - T)y(t - T)$ representing the contribution that took T amount of time to process.[5] Thus given an initial two variable function $\phi(t) = (\phi_1(t), \phi_2(t))$ on an initial time interval $[-T, 0]$, we can substitute ϕ into the second equation for y to integrate for $0 \leq t \leq T$:

$$\dot{y} = -ky(t) + d\,\phi_1(t - T)\phi_2(t - T).$$

[5] One could of course, suppose that there is a random variable with expected value T and use as the gain in predator population an integration term $d \int_{t_0}^{t} x(t - s)y(t - s)g(s)ds$ with $g(s)$ the density function of the random processing time, but a similar analysis can be done. We thus assume this simplicity of constant delay T just to illustrate.

Notice that here we now have an uncoupled equation for y since it is just the scalar first order linear ODE in y. Using a variation of parameters approach we can now easily solve the IVP with $y(0) = \phi_2(0)$, so we obtain $y = \eta(t)$ and plug into the equation for $\dot{x}$ the value of $y(t)$. Now we have a Bernoulli equation for x on $0 \leq t \leq T$

$$\dot{x} = ax - bx^2 - c\eta(t)x, \qquad x(0) = \phi_1(0)$$

and this equation is studied in ODE courses. △

3.3 Fundamental Theoretical Results

As we have seen in the previous sections, it is possible to prove existence and uniqueness results directly for some FDEs. We will now review some of these fundamental results for a more comprehensive theory of FDEs. The following theoretical development is based on the paper [20] by Rodney D. Driver, since the existence and uniqueness results developed there are more adequate for infinite delays, with finite delays included as well. On the other hand, the concepts that Driver uses, run parallel to the subsequent development of the theory for impulsive FDE systems in the paper by G. Ballinger and X. Liu in [6], which in turn leads to the paper [39] by X. Liu and P. Stechlinski for switched FDE systems.

Infinite delays are of course, a theoretical condition that occurs when we look at equations such as $x' = x(t - t^2)$, where $t - t^2 \to -\infty$ as $t \to \infty$. We might want to study a system's stability, so we need large values of t, which might require going arbitrarily far into the past. No system runs forever of course, but stability is a mathematical idealization in this context, since it is a concept about running systems for arbitrarily large times $t \to \infty$, and this has been useful for applications.

Another good source for basic results is the book by J. Hale [24], however this is so for *finite* delays. This is because Hale develops a more complicated theory for infinite delays, due to the fact that there is a certain vagueness about what could the phase space for an FDE be (see the last chapter in [24] for more details). In some cases, treating the infinite dimensional $C([-r, 0], D)$ as the phase space is useful, and as Hale in [3] puts it, sometimes ODE results are better generalized to FDEs if this functional space is used as the phase space. Thus, for infinite delayed FDEs, Hale and J. Kato developed a theory where they impose certain restrictive conditions on Banach spaces that are candidates for being a phase space, see for example, [23]. However, sometimes other results are more easy to formulate and understand if $\mathbb{R}^n$ is taken as the space of interest. For example, for stability of infinitely delayed FDEs, since the initial condition ϕ is always part of the definition of the solution $x(t; t_0, \phi)$, then defining stability concepts for the path $t \mapsto x_t$ can be senseless when $r = \infty$, since the norm $\|x_t\|_r = \sup_{s \in (-\infty, t]} |x(s)|$ always includes the generally nonzero initial condition, and in applications, we are interested in the final values of $s \mapsto x(s)$ being sufficiently small, not all of the history of the solution, represented as $s \mapsto x_s$.

The approach that works best for us takes into account the fact that it makes sense to work with only bounded initial conditions for infinite delay, which means that

$$\{\phi \in C\big((-\infty,0],D\big) : \phi \text{ is bounded on } (-\infty,0]\}.$$

For an interval $[a,b]$, with $-\infty \leq a < b \leq \infty$, if $a = -\infty$ we denote $[a,b]$ as $(-\infty,b]$, and similarly if $b = \infty$. For any type of region $R \subset \mathbb{R}^n$, let us denote by

$$C\big([a,b],R\big) = \{\phi : [a,b] \longrightarrow R : \phi \text{ is continuous on } [a,b]\}. \tag{3.14}$$

taking special care of the target set. If $a = -\infty$, and $b < \infty$ we will use the space of *bounded* continuous functions

$$BC\big((-\infty,b],R\big) = \{\phi \in C\big((-\infty,b],D\big) : \phi \text{ is bounded on } (-\infty,b]\}. \tag{3.15}$$

Remark 3.4. *Notice that in [20], the author implicitly assumes the same $BC\big((-\infty,b],R\big)$ space when the infinite delay is used, since he states on p. 402 that $\phi \in C\big((-\infty,t],D\big)$ means that there exists a compact set $F_\phi \subset D$ such that $\phi \in C\big((-\infty,t],F_\phi\big)$, which implies the boundedness of ϕ on $(-\infty,t]$.*

We now introduce a notion which Driver uses and differs from what Hale in [24] uses. This is the definition that allows the existence result of Driver to work for infinite delay as well as finite delay. It turns out that with a few modifications, this notion will also work when we move on to more general FDEs such as impulsive.

In the following, we remind the reader that $J \subset \mathbb{R}$ is an open interval (a,b) with $-\infty \leq a < b \leq \infty$ and $D \subset \mathbb{R}^n$ is an open connected set. $0 < r \leq \infty$ represents the fixed bound on the delay.

Definition 3.3. *We say that $f : J \times C([-r,0],D) \longrightarrow \mathbb{R}^n$ **is continuous**[6] **in t**, or **composite continuous** if for each $t_0 \in J$, and $\alpha > 0$ such that $[t_0, t_0+\alpha] \subset J$, if $\psi \in C([t_0-r, t_0+\alpha], D)$, then the composite mapping $t \mapsto f(t,\psi_t)$ is a continuous function from J to $\mathbb{R}^n$. In other words, $t \mapsto f(t,\psi_t)$ belongs to $C([t_0,t_0+\alpha],\mathbb{R}^n)$.*

Definition 3.4. *We say that F is **locally Lipschitz with respect to the second variable** $\phi \in C = C([-r,0],D)$ if for every if for each $t_0 \in J$, and $\alpha > 0$ such that $[t_0,t_0+\alpha] \subset J$ and each compact set $F \subset D$, there exists a constant $L = L_{t_0,\alpha,F}$ such that whenever $t \in [t_0,t_0+\alpha]$ and $\phi,\psi \in C([-r,0],F)$ then*

$$|F(t,\phi) - F(t,\psi)| \leq L\|\phi - \psi\|_r.$$

Remark 3.5. *Notice the images of the elements ϕ,ψ are contained in the particular fixed compact set F. Do not forget the dependence of the Lipschitz constant on t_0,α,F. Notice that given the set D, we should be able to find a Lipschitz constant that works for any given subinterval of J of the given form, along with a particular compact set. The same Lipschitz constant need not work elsewhere.*

[6] We are using the same terminology as Driver uses in his [20] paper. However, we can also say that f is *composite continuous* or *composite C* to suggest the parallelism with the terminology introduced in the paper by Ballinger and Liu [6]. Sometimes we might use these interchangeably.

The following existence-uniqueness result is taken from [20] p. 406.

Theorem 3.1. (Existence-Uniqueness for FDE) *Let the functional $f(t, \psi)$ be continuous in t and locally Lipschitz in $\phi \in C([-r, 0], D)$, with $r = \infty$ possible. Then for any initial condition $(t_0, \phi) \in J \times C([-r, 0], D)$, we have that there exists an $h > 0$ such that a unique solution to (3.5) through $(t_0, \phi) \in J \times C([-r, 0], D)$ exists on $[t_0, t_0 + h]$.*

There are as mentioned, other existence and uniqueness theorems. One can prove existence of solutions assuming only continuity in t and continuity in the second variable $\phi \in C([-r, 0], D)$, as is done in [39] following the proof verbatim. However the previous result is sufficient for our purposes. Lemma 1.1 in [24] follows from supposing the composite continuity of $t \mapsto f(t, \psi_t)$, instead of following from Lemma 2.1 in the same reference, which does not work for infinite delays, due to loss of uniform continuity on a noncompact set.

As in ODEs, we have the following forward extension result for FDEs, though it is slightly different from the ODE one. It is also from [20].

Theorem 3.2. (Extended Existence-Uniqueness for FDE) *Let the functional $f(t, \psi)$ be continuous in t and locally Lipschitz in $\phi \in C([-r, 0], D)$, with $r = \infty$ possible. Let some initial condition $(t_0, \phi) \in J \times C([-r, 0], D)$. Then there exists a $\beta > 0$ such that there is a unique solution $x(t) = x(t; t_0, \phi)$ defined on $[-r, \beta)$, with $b \geq \beta > t_0$. If $\beta < b$, and β can no longer be increased, then, for any compact set $F \subset D$, there is a sequence of numbers $t_0 < t_1 < t_2 < \cdots$ such that $t_k < \beta$ for every k, $\lim_{k \to \infty} t_k \uparrow \beta$ and*

$$x(t_k) \in D \setminus F \qquad \text{for } k \geq 1.$$

Remark 3.6. *Notice that it is only at a sequence of times such that the solution $x(t)$ leaves the compact set F. This is weaker than what we can assert for ODEs, where we can even affirm that $x(t)$ approaches the boundary of D as $t \uparrow \beta$.*

As is the case for ODEs, theorems to guarantee boundedness of solutions and indefinite forward existence are not that easy to obtain, we will return to some of these questions later on. For this thesis, we will not require results on continuity with respect to initial conditions, since we will study stability. For smoothness with respect to initial conditions, as well as continuity results, we refer the reader to [15], [24].

3.4 Stability of Delayed FDEs

We will state the main definitions for stability of delayed functional differential equations. We will state sufficient stability results for our purposes, though there are many small variations of similar results for each different type of stability behavior we define, such as for stability, asymptotic stability, etc. We will concentrate on asymptotic stability results, since that is the kind of stability that the results developed ahead will treat. For a greater amount of results, we refer to [15], [20], [24] as main sources.

Let each $t_0 \in J, \phi \in C([-r,0],D)$ induce an initial value problem

$$\begin{aligned} x'(t) &= f(t,x_t), \qquad t \geq t_0 \\ x_{t_0} &= \phi \end{aligned} \tag{3.16}$$

We have $f : J \times C([-r,0],D) \longrightarrow \mathbb{R}^n$ with $J \subset \mathbb{R}$ an infinite interval of the form $[a,\infty)$ $a \geq -\infty$, we can assume $J = \mathbb{R}_+ = [0,\infty)$ for simplicity.

For stability analysis, we assume that $0 \in D$, which implies that $0 \in C([-r,0],D)$ and that $f(t,0) \equiv 0$ for all $t \in J$. Thus 0 is an equilibrium solution.

Remark 3.7. *Of course, just as we did for ODEs, we can study the translation of a nonzero equilibrium solution $t \mapsto \varphi(t)$ of an FDE $y' = g(t,y_t)$ by defining the change of variable $x(t) = y(t) - \varphi(t)$ and obtaining a new vector field $f(t,x_t)$ with a zero equilibrium. Thus, studying the stability of the trivial solution $\varphi(t) \equiv 0$ is sufficient.*

Remember that the Euclidean norm is denoted $|\cdot|$.

Definition 3.5. (Stability Definitions for FDEs) *The zero solution of* (3.16) *is said to be*

- ***Stable*** *if for each $\epsilon > 0$ and $t_0 \in J$, there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that if $\phi \in C([-r,0],D)$ with $\|\phi\|_r < \delta$, and $x(t) = x(t;t_0,\phi)$ is any solution of the induced IVP* (3.16)*, then $x(t;t_0,\phi)$ satisfies*

$$|x(t;t_0,\phi)| < \epsilon, \qquad \forall t \geq t_0. \tag{3.17}$$

- ***Uniformly stable*** *if, for each $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ independent of t_0, such that* (3.17) *is satisfied when $x(t) = x(t;t_0,\phi)$ is any solution of the induced IVP* (3.16).

- ***Unstable*** *if it is not stable.*

- ***Asymptotically stable*** *if it is stable and for every $t_0 \in J$ there is a constant $c = c(t_0) > 0$ such that if $\phi \in C([-r,0],D)$ with $\|\phi\|_r < c$, then $x(t;t_0,\phi) \to 0$ as $t \to \infty$.*

- ***Uniformly asymptotically stable*** *if it is uniformly stable and there is a constant $c > 0$ independent of t_0, such that or all $\phi \in C([-r,0],D)$ with $\|\phi\|_r < c$, $x(t) \to 0$ as $t \to \infty$, uniformly in t_0; that is, for each $\eta > 0$, there is $T = T(\eta) > 0$ such that*

$$|x(t)| < \eta, \qquad \forall t \geq t_0 + T(\eta),\ \forall \|\phi\|_r < c.$$

3.4.1 Some Stability Results

In general, Lyapunov type methods are applied for stability of functional differential equations, even if the system is linear. This is because in general, even constant linear FDEs can have infinitely many roots. The location of the roots in the complex plane determines the long term dynamics of solutions, and obtaining these roots is not an easy task. See for example, [24, 3].

The two main streams in Lyapunov type stability results consist in either using Lyapunov *functionals* $V : J \times C([-r,0],D) \longrightarrow \mathbb{R}$, since the derivative depends on $\psi \in C([-r,0],D)$; the other methods, commonly known as *Razumikhin techniques*, involve the use of Lyapunov *functions* $V : J \times \mathbb{R}^n \longrightarrow \mathbb{R}$.

A fading memory condition for infinite delays

Driver in [20] p. 422 adds the following notational remark, which we explain. If the functional $f(t,\psi)$ depends only on the value of $\psi(s)$ for $s \in [g(t), t]$, where $-r \leq g(t) \leq t$ for every $t \geq t_0$, then the notation

$$f(t,\psi,g(t)) \tag{3.18}$$

will be used to indicate this.

Driver does this because the results he proves hold for infinite delays ($r = \infty$) as well. Throughout the paper, Driver points out many times in asymptotic stability results, that some type of "fading memory" condition is needed to obtain asymptotic stability for infinite delays. Driver adds right after his remark that we shall be interested in the case when $\lim_{t\to\infty} g(t) = \infty$. Thus, we need not worry for finite delays. In many cases, for example

$$x' = x(t - r(t)),$$

$g(t) = t - r(t)$ with $0 \leq r(t) \leq r$, so that for finite delays, for example, we easily see that $g(t) \to \infty$ as $t \to \infty$, since we can always find a $g(t) \geq t - r \to \infty$ as $t \to \infty$.

The previous memoryless type condition expresses our intuitive understanding that a system can eventually stabilize or reach equilibrium as long as initial disturbances eventually die out.

These type of conditions are labeled *fading memory conditions*, and they appear in different guises in many results about asymptotic stability for delayed FDEs. These considerations will come back to haunt us later on.

Lyapunov Functional for Stability

Given that the vector field depends on a function space, in the case of delays on $C([-r,0],D)$, it might seem more natural to generalize Lyapunov's method for stability for ODEs using a *functional* $V : J \times C([-r,0],D) \longrightarrow \mathbb{R}$. An example of a Lyapunov functional could be something like

$$V(t,\psi) = \psi^2(0) + c\int_{-r}^{0} \psi^2(s)\, ds,$$

so that the information of $\psi \in C([-r,0],D)$ on the whole interval $[-r,0]$ is always used to define the value of V. Of course along a solution path $x(t)$ this is reduces to

$$V(t,x_t) = x^2(t) + c\int_{-r}^{0} x^2(t+s)\, ds.$$

We will again use the analogy with a Lyapunov function in ODEs representing some sort of energy of a system, so we will need to somehow encapsulate the notion of increasing or decreasing energy in order to generalize Lyapunov theory from ODEs. The most commonly used notion is the Dini type derivative along the solutions of (3.16) for *functionals* $V : J \times C([-r,0],D) \longrightarrow \mathbb{R}$ defined as

$$D^+V_{(3.16)}(t,\psi) := \limsup_{h\to 0^+} \frac{1}{h}\big[V(t+h,\,\psi^*) - V(t,\psi)\big], \tag{3.19}$$

where

$$\psi^*(s) = \begin{cases} \psi(s) & \text{if } s \in [-r, 0] \\ \psi(0) + hf(t, \psi) & \text{if } s \in [0, h]. \end{cases}$$

Of course, the reason we use this Dini-type derivative is because we will want Lyapunov functions that are not in general differentiable. Others use different definitions for the derivative. Actually, one can prove the following, see [15, 20].

Lemma 3.1. *Let $f : J \times C([-r, 0], D) \longrightarrow \mathbb{R}$ and $V : J \times C([-r, 0], D) \longrightarrow \mathbb{R}$ be locally Lipschitz in $\psi \in C([-r, 0], D)$. Then for every $t \geq t_0$, and every $\psi \in C([-r, 0], D)$, if $x(s; t, \psi) = x(s)$ is the unique solution of* (3.16) *through $(t, \psi) \in J \times C([-r, 0], D)$, then*

$$D^+V_{(3.16)}(t, x_t) = \limsup_{h \to 0^+} \frac{1}{h}\big[V(t + h, x_{t+h}) - V(t, x_t)\big].$$

Because of the previous lemma, many authors define the derivative differently. When working with functionals, perhaps the definition implied through the previous lemma is more commonly used.

We have the following result for asymptotic stability using a Lyapunov functional. The result is taken from Driver [20]. $B_H \subset \mathbb{R}^n$ denotes the open Euclidean ball of radius H.

Theorem 3.3. (Asymptotic Stability using a Lyapunov Functional) *Suppose in* (3.16) *that $|f(t, \psi)| \leq M$ for all $t \geq t_0$ and $\|\psi\|_r \leq H_1$ for some $H_1 > 0$ constant. If there exists a functional $V(t, \psi)$, $V : J \times C([-r, 0], B_H) \longrightarrow \mathbb{R}$ defined whenever $t \geq t_0$ and for $\|\psi\|_r < H$, H constant, $0 < H_1 < H$ such that*

(a) $V(t, 0) \equiv 0$, $V(t, \psi)$ is continuous in t and locally Lipschitz with respect to ψ.

(b) $V(t, \psi) \geq w(\psi(0))$ where $w(x)$ is a positive definite continuous function on B_H

(c)

$$D^+V_{(3.16)}(t, \psi) < -w_1(\psi(0)) \tag{3.20}$$

where $w_1(x)$ is another positive definite continuous function on B_H.

Then the zero solution of (3.16)*is asymptotically stable.*

Remark 3.8. *Driver mentions that in the proof of the previous result, it is strongly hinted that a fading memory type condition, similar to the one explained above, must necessarily hold.*

Using a Lyapunov Function for Stability (Razumikhin Technique)

Razumikhin techniques come from the observation that if a solution of a delayed FDE (3.16) were to be unstable, then suppose that it starts off in a small ball around the origin and is about to leave the ball at some time $t^* > t_0$. Then, since this is the *first time* it leaves the ball, one makes the observation that

$$\|x_t^*\| = |x(t^*)| = |x_{t^*}(0)|,$$

since $|x(t^*+s)| \leq |x(t^*)|$ for all $s \in [-r, 0]$. Since we assumed that the solution $x(t)$ is about to leave the ball at time t^*, then at that moment

$$\left.\frac{d(|x(t)|)}{dt}\right|_{t=t^*} \geq 0.$$

In the end, some sort of energy must be increasing when a solution is leaving a ball, which can be reflected in the norm, or more generally, in a properly chosen V, remembering the analogy of Lyapunov functions in ODEs with the total energy of a physical system. Therefore we must consider initial data satisfying the previous conditions. The previous analysis motivates us to concentrate, given $\psi \in C([-r, 0], D)$, on the final value $\psi(0)$. This motivates the definition of the derivative along the solutions of (3.16) for Lyapunov functions $V : J \times \mathbb{R}^n \longrightarrow \mathbb{R}$ as

$$D^+V_{(3.16)}(t, \psi(0)) := \limsup_{h \to 0^+} \frac{1}{h}\big[V(t+h,\, \psi(0) + hf(t, \psi)) - V(t, \psi(0)\,)\big]. \tag{3.21}$$

Notice that the previous is a functional even though V is a function, since it is taking $\psi \in C([-r, 0], D)$ but evaluating it at the final point $s = 0$. Also we will be interested in using the previous derivative when $\psi = x_t$ so that the derivative is

$$\begin{aligned} D^+V_{(3.16)}(t, x_t(0)) &= \limsup_{h \to 0^+} \frac{1}{h}\big[V(t+h,\, x_t(0) + hf(t, x_t)) - V(t, x_t(0)\,)\big] \\ &= \limsup_{h \to 0^+} \frac{1}{h}\big[V(t+h,\, x(t) + hf(t, x_t)) - V(t, x(t)\,)\big]. \end{aligned}$$

Remark 3.9. *The previous definition* (3.21) *of the derivative for a Lyapunov function is actually the same thing as the first Dini derivative* (3.19) *that we defined in the previous study for Lyapunov functionals, except that we are using the particular Lyapunov functional* $V_1(t, \psi) \equiv V(t, \psi(0))$*, which reduces to a function when we just say that* $\phi(0) = x \in \mathbb{R}^n$*, and make this function explicitly depend on* $x \in \mathbb{R}^n$*.*

It can be proved that if $V : J \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is C^1 in both variables of $J \times \mathbb{R}^n$, then

$$D^+V_{(3.16)}(t, \psi(0)) = \frac{\partial V}{\partial t}(t, \psi(0)) + \frac{\partial V}{\partial x}(t, \psi(0)) \cdot f(t, \psi) \tag{3.22}$$

Where $\cdot$ denotes the dot product with the gradient $\frac{\partial V}{\partial x}(t, \psi(0))$.

We have the following result from Driver in [20]. It could be considered a Razumikhin type theorem. In the following result $B_H \subset \mathbb{R}^n$ denotes the open Euclidean ball of radius H.

Theorem 3.4. (Asymptotic Stability using a Lyapunov Function) *Let $f(t,\psi) = f(t,\psi,g(t))$ in (3.16), where $\lim_{t\to\infty} g(t) = \infty$. If there exists a function $V(t,x)$, $V : J \times B_H \longrightarrow \mathbb{R}$ defined whenever $t \geq -r$ and for $|x| < H$ with $H > 0$ constant such that*

(a) $W_1(x) \leq V(t,x) \leq W_2(x)$, with W_1, W_2 positive definite[7] continuous functions on $|x| < H$.

(b) $V(t,x)$ is continuous in t and locally Lipschitz[8] with respect to x.

(c) There exists a continuous nondecreasing function $h(d) > d$ for all $d > 0$ and a continuous function $w_1(x) > 0$ for all $0 < |x| < H$ such that

$$D^+V_{(3.16)}(t,\psi(0)) < -w_1(\psi(0)) \tag{3.23}$$

whenever $t \geq t_0$, $\|\psi\|_r < H$, and

$$V(s,\psi(s)) < h(V(t,\psi(t))) \qquad \text{for all} \qquad s \in [g(t),t]. \tag{3.24}$$

Then the zero solution of (3.16) is uniformly stable and asymptotically stable. If $g(t) \geq t - p$ for $t \geq t_0$ and some constant $p \geq 0$, then the asymptotic stability is uniform, in other words, we have uniform asymptotic stability.

Notice that the previous theorem gives uniform asymptotic stability when there are finite delays. More importantly, notice the last conditions given in (3.23),(3.24). These are the conditions that capture the "Razumikhin spirit" of approach. The two aforementioned conditions combined are a way of saying that a certain rate of change of the Lyapunov function (this change captured in the Dini type derivative (3.21)) is decreasing *whenever* (3.24) *holds.* (3.24) uses a nondecreasing function h to capture a certain measure of the behavior of the history of the of the trajectory on an interval $s \in [g(t),t]$ being dominated by what is happening at the immediate final time t. In the intuitive example of x leaving a ball for the first time, the norm $|x(t)|$ acts as V if $V(t,x(t)) = |x(t)|$. Thus $h(\eta) = \eta$, the identity, plays the role of $h(V(t,x(t)))$ in the aforesaid intuitive explanation. The fading memory condition makes the disturbances from initial times die out, so that we just focus on "the lastest" behavior. This will perhaps be better understood in the example below.

Remark 3.10. *Notice that uniform stability plus asymptotic stability is not the same as uniform asymptotic stability.*

The following example uses the previous result. It is taken from Driver [20].

Example 3.4. The trivial solution of

$$x' = -a(t)x(t) + b(t)x(t-r(t)) \tag{3.25}$$

is asymptotically stable provided that the functions involved are continuous, $a(t) \geq c > 0$ with c a constant, $J|b(t)| \leq a(t)$ with $J > 1$ constant and $t - r(t) \to \infty$ as $t \to \infty$.

[7] We remind the reader that positive definite means $W(0) = 0$ and $W > 0$ for $x \neq 0$ in the region of interest, in this case the open ball of radius H.

[8] Notice locally Lipschtiz with respect to $x \in \mathbb{R}^n$, not with respect to a variable in $C([-r,0],D)$, so use the appropriate Lipschitz notion.

Proof. We will use $h(d) = Jd$, let

$$w_1(x) = 2c\left(1 - \frac{1}{\sqrt{J}}\right)x^2.$$

Take $V(t,x) = x^2$. Verifying the first two conditions (a) and (b) of the theorem is immediate. Let us make sense of the last condition (c). We have that the derivative $D^+V_{(3.16)}(t, x_t(0))$ must satisfy a certain condition whenever $t \geq t_0$ and $V(s, x(s)) < h(V(t, x(t)))$ for all $s \in [t - r(t), t]$. This means that this derivative should be bounded by $-w_1(x(t))$ whenever

$$V(s, x(s)) = x^2(s) \leq J\,x^2(t) = h(V(t, x(t))) \qquad \forall s \in [t - r(t), t],$$

or equivalently, whenever

$$|x(s)| \leq \sqrt{J}\,|x(t)| \qquad \forall s \in [t - r(t), t]. \tag{3.26}$$

So now that we have clarified what condition (3.24) means for this particular case, now let us calculate the derivative of the Lyapunov function we use. Assuming V is C^1, so as mentioned before the derivative along the trajectories of the FDE is

$$\begin{aligned} D^+V_{(3.16)}(t, x_t(0)) &= 2x(t)x'(t) \\ &= 2x(t)\big[-a(t)x(t) + b(t)x(t - r(t))\big] \\ &\leq -2a(t)x^2(t) + 2|b(t)|\sqrt{J}x^2(t) \\ &\leq -2a(t)\left(1 - \frac{1}{\sqrt{J}}\right)x^2(t) \leq -2c\left(1 - \frac{1}{\sqrt{J}}\right)x^2(t) = -w_1(x(t)), \end{aligned}$$

where we have used (3.26) in the first inequality of the previous, and $|b(t)| \leq \frac{1}{J}a(t)$ for the second inequality. Thus, since $t - r(t) \to \infty$ as $t \to \infty$, we have all of the sufficient conditions to conclude asymptotic stability of the zero solution. □

Remark 3.11. *Notice how the Razumikhin type condition* (3.24) *was translated into the last state $|x(t)|$ dominating all of the previous* latest *states $|x(s)|$ for all $s \in [t - r(t), t]$. This means that if the last state is the largest, then the derivative must be in some sense decreasing so as to pull the solution back in, whereas if the last state $|x(t)|$ does not dominate the previous, then the solution is already somehow decreasing anyways, since the last state does not dominate in norm.*

Chapter 4

Impulsive and Switched FDEs

4.1 Introduction

In this chapter we will give an introduction to hybrid systems. As can be seen in [51], the term "hybrid system" has different meanings according to the specialist who needs them. In the broadest sense, which is possibly not mathematically describable or expressible in a single statement such as an ODE is by $\dot{x} = f(t, x)$, a hybrid system is a dynamical system that exhibits a coupling between variables that take values in a continuum, such as metric space or $\mathbb{R}^n$, and variables that take on discrete values. There are many technological examples of hybrid systems around us. When a computer or some other digital device which takes on Boolean values or switches, interacts with some process which is modeled in terms of constituent elements exhibiting continuous dynamics, the resulting "closed" system, product of the combination in some mathematical model that reflects the evolution of all of the states of this system (digital device + process modeled) as they each affect each other, is a hybrid system. Examples of the previous are: cars or flying machines modeled by Newton's Laws, but with discrete-valued mechanisms such as the gear transmission or computer flight controllers.

The previous are examples from engineering applications, but there are also examples from natural sciences. This is because humans have invented in their language discrete variables, such as wherever they introduce a dichotomy to conceptualize what they experience in the world. In a sense, through our language, we act somewhat like computers interacting in a world modeled by continuous dynamical elements, which is itself already an idealization. Like when they[1] say "on" or "off" or something similar, when modeling impacts, or when a surface reflects light, conceptually there is a "before reflection" and "after reflection". For this last example of light, see for instance [34], where for light reflection off a surface, the classical Snell's Law of reflection can be derived from modifying the classical ODE version of the Pontryagin maximum principle, whose hypotheses break down for this simple problem, by extending the aforesaid principle to a hybrid system version. So even classical physics can have hybrid system approaches, as this hybrid optimization approach to a variational problem illustrates.

[1] The humans.

The student from, say physics, will notice that modeling impacts, for example, such as a ball bouncing off the floor, is not a big issue for classical ODE methods, since, as in other areas like engineering, the approach to these problems was to simply work around these troublesome small time instants, and in between impacts simply use ODEs. The engineering solution to similar problems was either to adopt a purely discrete system model, or using a purely continuous model. All of these workarounds have been used before. Nonetheless, applications have caught up to us, in the sense that we must consider not avoiding these issues and integrating all aspects of the dynamics into a single coherent system. Thus, hybrid systems have become compelling for applications, such as in fuzzy logic control, where a single controller can be difficult or impossible to implement in a single closed loop system.

Before focusing on the particular types of hybrid systems that we shall adopt, let us quickly give some examples to motivate and illustrate what hybrid systems are about.

Example 4.1. (Hybrid Controller, [26]) In this problem, a controller is designed to deal with a complex system for which traditional approaches using a single continuous controller do not provide satisfactory performance, due to the continuous system presenting different possible classified modes of evolving. A hybrid controller, by encoding different modes of proceeding into discrete logical states, can provide a possible solution to this problem. The basic elements are a set of alternate candidate controllers and switches to adjust to possible scenarios. The adjustment is done by what is commonly called a “supervisor”. This discrete logical decision unit works by a specifically designed logic that uses measurements collected online to determine which controller is best suited for the given situation, and by picking the corresponding control strategy, it closes the feedback loop. The following Figure 4.1 illustrates the elements of this hybrid control architecture. Here u represents the control input, w is an external disturbance or measurement noise caused by the environment, and y is the measured output.

The supervisor chooses from the given subset of controllers labeled from 1 to M, by sending a given signal $\sigma(t)$ at time t. Or if the signal also depends on the state y, the signal is $\sigma(t, y(t))$. This example is taken from [26], where it is explained that this is a simplified diagram, since switching controllers in practice are applied differently. $\triangle$

Example 4.2. (Transmission in a Car, [51]) Consider a model of manual transmission in a car

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{-ax_2 + u}{1 + v} \end{aligned}$$

where x_1 is the position, x_2 the velocity, $v \in \{1, 2, 3, 4\}$ is the gear shift position, u is the acceleration input and a is some system parameter. This is a hybrid system having 4 different operational modes, where shifting gears to different positions $v \in \{1, 2, 3, 4\}$ represents a switch to a different mode, or vector field. The continuous state involved is 2-dimensional. Notice that

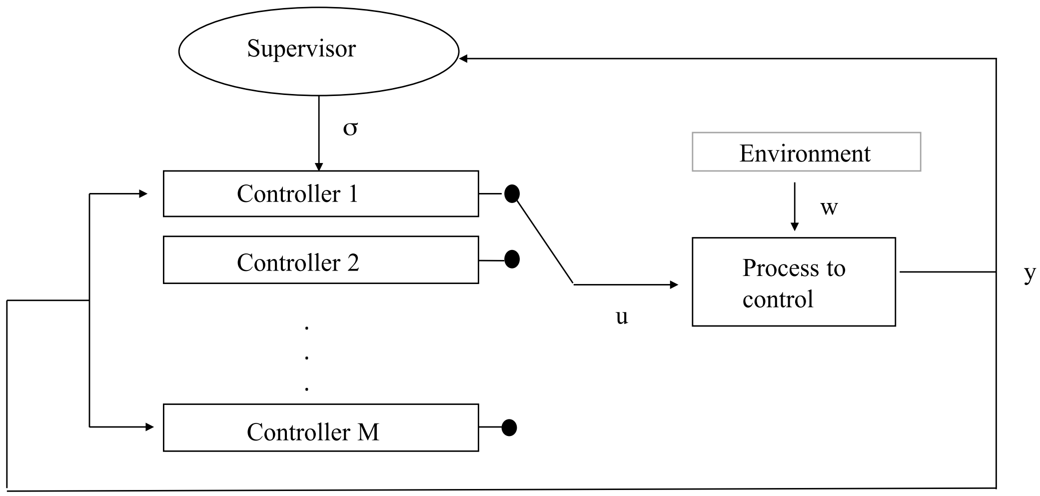

Figure 4.1: Hybrid Control

in this case the supervisor is the driver. △

The next example is quite different from the previous ones, in that the discrete transition occurs in one of the state variables to be controlled, leading to discontinuities in the trajectories of solutions, whereas the previous do not necessarily entail discontinuities in the state variables of the respective systems.

Example 4.3. (R. Bellman, [7]) Let us consider a control process in pharmacokinetics ruled by the linear differential equation

$$u' = -au + v \qquad u(0) = c_0 \tag{4.1}$$

a "one-compartment model". Here u represents the concentration of a drug in an organ at time $t > 0$, c_0 represents the initial concentration, and $v(t)$ the rate of injection of the drug. Suppose we modify the previous in the following manner. Let us at some moment $\tau_1 > 0$ add an additional dosis c_1 to the initial c_0 concentration. At τ_j we add another dosis c_j. Thus we have the initial condition $u(0) = c_0$, but the differential equation is modified to

$$\begin{aligned} u' &= -au + v \qquad && t \neq \tau_j \\ \Delta u &= c_j && t = \tau_j. \end{aligned} \tag{4.2}$$

The previous is an *impulsive differential equation*, representing impulsive control with control input u. We model the sudden injection of the drug as a discontinuity. This, of course, is an idealization, though notice that relative to time scales of interest, an injection takes place in a relatively negligible amount of time. The previous implies that in larger time scales, the rate of increase of the vector field attains very high norm values precisely at injection moments, caused by this relatively very quick rate of change. This of course might remind the reader of the Dirac delta function representing a pulse at an instant, since as the time of application of the pulse gets shorter, the slope at these moments seems to go to infinity as the slope gets larger by dividing by smaller time lengths. Thus simplification can be achieved by assuming impulsive behavior of trajectories, at the cost of breaking continuity of solutions. This is not such a big deal, considerable simplification might yield benefits even if we must lose continuity. The reason to consider impulsive, in contrast to continuous control, is that the latter can be more complicated or less cost efficient than impulsive control. Short time instant intervention of a controller can yield more cost efficient solutions. See the paper by Richard Bellman [7] for a dynamic programming approach in an optimal control framework of the previous, since we would of course like to achieve higher efficiency, so cost functionals are introduced. [2] $\triangle$

Impulsive systems have found uses in applications, ranging from cost optimal methods to control rockets [36], impulsive control of interest rates in stochastic processes [43], to management inspection in operations research and quality control [8].

As mentioned earlier, hybrid systems encompass a broad array of different problems, unified by the underlying philosophy of discrete dynamics interacting with continuous ones. In the results of this thesis however, we will require only two main types of hybrid systems, or more precisely, hybrid delayed systems. These will be impulsive delayed FDEs and time dependent continuous switched FDEs with delays.

4.2 Impulsive FDEs

Here we will give an introductory overview of discontinuous, or impulsive systems. In general terms, these involve an immediate reinitialization of the state, in other words, instead of the vector field f being changed or switched, the initial condition is the one that is immediately changed, causing a break in continuity. Impulsive systems are used to model systems where rapid changes in the state occur, and it can be preferable to model these state differences as occurring instantly if the time scale of the change is small enough. This can occur, for example, when modeling vaccination schemes that are done at particular moments of time that are very small compared to the time scale. Another example occurs in impulsive control, where it may be cheaper to very briefly intervene in a process to be controlled, rather than apply a continuous control. In delayed neural networks, in [28], since there are unavoidable delays when a neuron processes information, this leads to instability. With an impulsive control method, the driven network receives signals from the driving system only during short negligible time dura-

[2] Supposing, say, that u_D is a fixed constant representing the desired drug level we would ideally desire over the time interval $[0, T]$. Thus we can set up an optimization problem of adequate cost functionals, with target set for the state u_D in a finite time horizon framework $[0, T]$.

tion, and in this manner the amount of conveyed information is decreased, thus reducing delays.

Of course, in the previous models, one can argue that discontinuities are theoretical idealizations, but one may very well argue that so is continuity.

4.2.1 Fundamental Theory

We will start off directly with impulsive delayed FDEs, in contrast to starting with the ODE version of these, or rather impulsive ODEs, as is done in [32]. For our purposes, this is no big difference if one takes into account that initial conditions in ODEs are vectors in $\mathbb{R}^n$, whereas initial conditions in FDEs are functions on prescribed intervals. The state vectors $x(t)$ of the underlying differential equations are the ones that are reset by a jump operator, or difference functional $I(t, x_t)$, where we remember the notation introduced in Chapter 3 for a function $x : [\sigma - r, \sigma + A] \longrightarrow \mathbb{R}^n$, such that for each $t \in [\sigma, A]$ we denote by x_t the function defined explicitly as

$$x_t(\theta) := x(t + \theta) \qquad \text{for } \theta \in [-r, 0]. \tag{4.3}$$

But first of all, we cannot continue to use the space $C[-r, 0]$ or $BC[-r, 0]$ for our functions x_t, since we now have discontinuities. The preferred function spaces will be given first.

An obvious choice of function space is the space of piecewise continuous functions. In the following $a < b$ with $a, b \in \mathbb{R}$ and $D \subset \mathbb{R}^n$. For finite delays the following PC-spaces are the most used, since for infinite delays we need boundedness assumptions on functions. Let us have $x(t^+) = \lim_{s \to t^+} x(s)$, and $x(t^-) = \lim_{s \to t^-} x(s)$.

$$\begin{aligned} PC([a,b], D) = \{x : [a,b] \longrightarrow D \mid x(t) = x(t^+)\ \forall t \in [a,b);\ x(t^-) \text{ exists } \forall t \in (a,b]; \\ x(t^-) = x(t) \text{ for all but at most a \textit{finite} number of points } t \in (a,b]\} \end{aligned}$$

$$\begin{aligned} PC([a,b), D) = \{x : [a,b) \longrightarrow D \mid x(t) = x(t^+)\ \forall t \in [a,b);\ x(t^-) \text{ exists } \forall t \in (a,b); \\ x(t^-) = x(t) \text{ for all but at most a \textit{finite} number of points } t \in (a,b)\} \end{aligned}$$

These classes describe spaces that are right-continuous with left limits everywhere, and they are left continuous except possibly on a finite number of points where they are defined. Notice the previous intervals of definition are finite in length. For infinite intervals we have

$$PC([a,\infty), D) = \{x : [a,\infty) \longrightarrow D \mid \forall c > a, x|_{[a,c]} \in PC([a,c], D)\}$$

$$\begin{aligned} PC\big((-\infty,b], D\big) = \{x : (-\infty,b] \longrightarrow D \mid x(t) = x(t^+)\ \forall t \in (-\infty,b); \\ x(t^+) \text{ exists in } D\ \forall t \in (-\infty,b];\ x(t^-) = x(t) \\ \text{for all but a \textit{countable} number of points } t \in (-\infty,b], \\ \text{and discontinuities do not have finite accumulation points.}\} \end{aligned}$$

$$PC(\mathbb{R}, D) = \{x : \mathbb{R} \longrightarrow D \mid \forall b \in \mathbb{R}, x|_{(-\infty,b]} \in PC\big((-\infty, b], D\big)\}$$

Thus, whatever the case of PC-space defined by the respective domain subinterval of $\mathbb{R}$, we ask for at most finite discontinuities on compact subsets of $\mathbb{R}$. To complete the characterization of a function space for our initial conditions, if $r < \infty$ is the delay, we will equip the space $PC([-r, 0], D)$ with the supremum norm

$$\|\psi\|_r := \sup_{s\in[-r,0]} |\psi(s)|, \tag{4.4}$$

where $|\cdot|$ denotes the Euclidean norm. In the case of infinite delays $r = \infty$, which occur as theoretical convenience, say in Volterra integro-differential equations, we would like to consider the norm

$$\|\psi\|_r = \sup_{s\in(-\infty,0]} |\psi(s)|. \tag{4.5}$$

But of course, we need boundedness requirements in order for such a theory of differential equations to be of practical significance, so similar to the delayed case, we consider bounded functions on infinite intervals. This is what motivates the use of PCB-spaces, or piecewise continuous bounded function spaces. In the following, $a < b$ are finite real numbers:

$$\begin{aligned}
&PCB([a, b], D) = PC([a, b], D)\\
&PCB([a, b), D) = \{x \in PC([a, b), D) \mid x \text{ is bounded on } [a, b)\}\\
&PCB([a, \infty), D) = \{x : [a, \infty) \longrightarrow D \mid \forall c > a, x|_{[a,c]} \in PC([a, c], D),\ x \text{ is bounded on } [a, \infty)\}\\
&PCB\big((-\infty, b], D\big) = \{x \in PC\big((-\infty, b], D\big) \mid x \text{ is bounded on } (-\infty, b]\}\\
&PCB(\mathbb{R}, D) = \{x \in PC(\mathbb{R}, D) \mid\ x \text{ is bounded on } \mathbb{R}\}.
\end{aligned}$$

Remark 4.1. *Since we will be interested in both cases, finite delays and infinite delays, we will frequently just use the space $PCB[-r, 0]$ whether $r < \infty$ or $r = \infty$, for notational convenience. This is becauses if r is finite then $PCB[-r, 0] = PC[-r, 0]$. The norm will be the one defined in* (4.4), (4.5)*, where $PCB[-r, 0]$ with $r = \infty$ is of course $PCB(-\infty, 0]$, with norm* (4.5)*, and $[-r, 0]$ denotes $(-\infty, 0]$ for this case.*

As in the case of delayed FDEs, if for some $t_0 \in \mathbb{R}$, $A > 0$ we have $x \in PCB[t_0 - r, t_0 + A]$, then for each $t \in [t_0, A]$ we denote by x_t the function in $PCB[-r, 0]$ defined as $x_t(\theta) := x(t + \theta)$ for $\theta \in [-r, 0]$. Thus x_t simply denotes the restriction of $s \mapsto x(s)$ to the interval $s \in [t-r, t]$. But now we have a second convention to take into account for these possible left-hand discontinuities. By x_{t^-} we refer to the function defined by a given $x \in PCB([t_0-r, b], D)$ through the assignment

$$\begin{aligned}
x_{t^-}(s) &= x_t(s) \quad \text{ for } s \in [-r, 0)\\
x_{t^-}(0) &= \lim_{u\to t^-} x(u) = x(t^-).
\end{aligned} \tag{4.6}$$

This is a way of getting a well defined function in $PCB[-r, 0]$, that takes into account only the information available right until before the jump occurs right at an impulse moment $t = \tau_k$. In this way, we will be able to define a difference operator $I(t, x_{t^-})$ that reflects that an impulse from $x(t^-)$ to a value $x(t)$, depends only on the information available until just *before* the impulse occurs at time t.

With all of the previous in mind, and since we will be interested about future values of a system, let $J \subset \mathbb{R}^+ = [0, \infty)$ be an interval of the form $[a, b)$ with $0 \leq a < b \leq \infty$. The general form of a *time-dependent* impulsive delayed nonautonomous system, or IFDE for short, will be given, for some initial time $t_0 \in J$ of interest, as

$$x'(t) = f(t, x_t), \qquad t \neq \tau_k, t \geq t_0 \tag{4.7}$$

$$\Delta x(t) = I(t, x_{t^-}), \qquad t = \tau_k, t > t_0. \tag{4.8}$$

Here, we have that $x(t) \in \mathbb{R}^n$, $f, I : J \times PCB([-r, 0], D) \longrightarrow \mathbb{R}^n$ with $J \subset \mathbb{R}^+$ an interval, $D \subset \mathbb{R}^n$ an open set and $\Delta x(t) = x(t) - x(t^-)$. The impulse times τ_k are assumed fixed constants that satisfy $0 = \tau_0 < \tau_1 < \cdots$ and $\lim_{k\to\infty} \tau_k = \infty$.

Remark 4.2. *Take note that the difference operator I is also delayed, not just the vector field functional f. This is reflected by the dependence of $I(t, x_{t^-})$ on the functional x_{t^-} as defined in* (4.6) *above. This captures the fact that an impulse should depend on the immediate values previous to it. That is why we defined x_{t^-} in* (4.6)*.*

Remark 4.3. *Here, due to the discontinuous nature of the system, which we will further elucidate below with an example, $x'(t)$ denotes the **right-hand** derivative of $x(t)$ with respect to t.*

Remark 4.4. *Notice how this exhibits a hybrid behavior of a discrete evolution system interacting with a continuous one in the following sense: Equation* (4.7) *represents a continuous transition scheme, which is essentially a delayed differential equation, while Equation* (4.8) *represents abrupt discrete changes in the dynamics at impulse moments τ_k, or a differential difference equation.*

Remark 4.5. *One can of course, define state-dependent impulsive FDE systems, but in this thesis we will restrict ourselves to the class of fixed-time dependent delays.*

We further assume, for the sake of formality, that $\psi(0) + I(\tau_k, \psi) \in D$ for all $(\tau_k, \psi) \in J \times PCB([-r, 0], D)$ for which $\psi(0^-) = \psi(0)$. This assumption is so that the solutions of (4.7) may be continued after an impulse moment τ_k, otherwise you leave the region where the vector field is defined. For the purposes of existence and uniqueness of solutions to (4.7), no further assumptions need to be imposed on the impulsive functional I, just that it does not jump out of the region where the vector field is mathematically defined.

We will impose, in the spirit of delay differential equations of Chapter 3, that the initial condition for equation (4.7) will be given for $t_0 \geq 0$ as

$$x_{t_0} = \phi \tag{4.9}$$

for $t_0 \in J$, and $\phi \in PCB([-r, 0], D)$.

Definition 4.1. *A function $x \in PCB([t_0 - r, t_0 + \gamma], D)$, where $\gamma > 0$ and $[t_0, t_0 + \gamma] \subset J$ is said to be a solution of* (4.7)-(4.8) *with initial condition* (4.9) *if*

(i) *x is continuous at each $t \neq \tau_k$ in $(t_0, t_0 + \gamma]$;*

(ii) *the derivative of x exists and is continuous at all but at most a finite number of points t in $(t_0, t_0 + \gamma)$;*

(iii) *the right-hand derivative of x exists and satisfies the delay differential equation* (4.7) *for all $t \in [t_0, t_0 + \gamma)$;*

(iv) *x exists satisfies the delay difference equation* (4.8) *at each $\tau_k \in (t_0, t_0 + \gamma]$; and*

(v) *x satisfies the initial condition* (4.9).

The previous definition is given because local existence uniqueness results posit the existence of solutions defined on a compact time interval, such as $[t_0, t_0 + \beta]$, while we will be interested in extensions to maximal *open* intervals. For this reason, a second definition of solution for open intervals is given now.

Definition 4.2. *A function $x \in PCB([t_0 - r, t_0 + \beta), D)$ where $0 < \beta \leq \infty$ and $[t_0, t_0 + \beta) \subset J$ is said to be a solution of* (4.7)-(4.8) *with initial condition* (4.9) *if for each $0 < \gamma < \beta$, the restriction of x to $[t_0 - r, t_0 + \gamma]$ is a solution of* (4.7)-(4.9).

We will explain the right hand derivative conventions of our definitions. There are quite a few technical details that one must consider when moving to discontinuous systems. In the classical theory of delay differential equations, as seen previously, solutions are continuously differentiable for $t > t_0$ (for delayed FDEs, the derivative is a right hand derivative at $t = t_0$) and satisfy (4.7) for all $t \geq t_0$. At t_0 they are continuous though. We must allow our definition to accommodate for discontinuities at impulse times $\tau_k > t_0$. Nonetheless, we will not force an impulse condition, or for the delay difference equation (4.8) to be satisfied in case an initial time $t_0 = \tau_k$ for some k. This is because this would impose an unnecessary restriction on the initial conditions ϕ since they would have values dependent on the functional I.

Let us see an example to understand why the derivative can be discontinuous at non-impulsive moments.

Example 4.4. (G. Ballinger, X. Liu [6]) Suppose that we have a scalar differential equation without impulses, but with a discontinuous initial function. Let

$$x'(t) = x(t-1) \tag{4.10}$$

with $r = 1$ representing an upper bound on the delay, $t_0 = 0$ and a piecewise continuous initial function $s \mapsto \phi(s)$ with a single discontinuity at $t^* \in (-1, 0]$. This equation can be solved by the method of steps[3], but notice that the delayed differential equation itself lets us know that for $t \in [0, 1]$:

$$x'(t) = \phi(t-1) \qquad t \in [0, 1].$$

[3] As shown in Chapter 3, for instance.

But notice that then the derivative is defined in terms of a discontinuous function, so for instance, at $t = t^* + 1 \in (0, 1]$, which is not even an impulsive moment, the discontinuity of ϕ forces one on the derivative of x at $t^* + 1 \in (0, 1]$. Nonetheless, the right hand derivative exists, since ϕ is right-continuous. $x(t)$ is still continuous at $t = t^* + 1$ though. △

Restricting the class of of initial functions to $C[-r, 0]$ or $BC[-r, 0]$ will not help us either, because in general if there is a difference operator I causing impulses, then this will cause a discontinuity in the solution, which in case of a differential delay equation like (4.10), would now cause a discontinuity of the derivative of $x(t)$ in the next interval in the method of steps. Once a solution undergoes an impulse, its history reflected through the delay would now be discontinuous. Notice that in case of infinitely delayed impulsive FDEs, the history is never erased, you always go back to the discontinuity in order to define the vector field forward in time. These deliberations motivate the definition of solution of impulsive FDEs given above.

We now point out another important difference between delayed FDEs and impulsive FDEs, which prohibits the application of the approach to the fundamental theory of existence of solutions as given in [24], and makes us prefer the approach constructed in [20], which is essentially the approach to FDEs that we gave in the previous Chapter 3.

The following lemma is essential for [24] to develop the theory of FDEs. In the following, $r < \infty$ is also crucial.

Lemma 4.1. (J.K Hale, [24]) *Let $x \in C([t_0 - r, t_0 + \gamma], \mathbb{R}^n)$ where $\gamma > 0$, $r < \infty$. Then the mapping $t \mapsto x_t \in C[-r, 0]$ is a continuous mapping of $t \in [t_0, t_0 + \gamma]$ to $C([-r, 0], \mathbb{R}^n)$, in other words, continuous with respect to the uniform norm $\|\cdot\|_r$ of $C[-r, 0]$*

If $x \in PC([t_0 - r, t_0 + \gamma], D)$ with $r < \infty$, the previous lemma does not apply. In fact, $t \mapsto x_t \in PCB([-r, 0], D)$ may be discontinuous on a whole continuum (contrasting with the requirement that discontinuities in PCB-spaces must be discrete), for example discontinuous at all $t \in [t_0, t_0 + \gamma]$. This can be seen by the following counterexample.

Example 4.5. (G. Ballinger, X. Liu, [6]) Suppose we have

$$x(t) = \begin{cases} 0 & \text{if } t \in [-1, 0) \\ 1 & \text{if } t \in [0, 1], \end{cases}$$

where $t_0 = 0$, $r = 1$ and $\gamma = 1$. Suppose that $t_1, t_2 \in [0, 1]$, and let $\delta > 0$ such that $0 < t_1 - t_2 < \delta$. Then for $s = -t_1 \in [-r, 0]$ we have that

$$|x(t_1 + s) - x(t_2 + s)| = |x(0) - x(t_2 - t_1)| = 1$$

because $0 < t_1 - t_2$ implies that $t_2 - t_1 < 0$. This last equality implies that $\|x_{t_1} - x_{t_2}\|_r = 1$, no matter how close t_1 and t_2 are, and at each point $t_1 \in [0, 1]$ we can use $s = -t_1 \in [-r, 0]$ to make the previous functional mapping discontinuous. Therefore, $t \mapsto x_t$ is discontinuous at each $t \in [0, 1]$. △

In order to obtain satisfactory results for delayed FDEs that generalize those of classical ODE theory, some authors, such as in [24], assume that the FDE evolution takes place as trajectories $t \mapsto (t, x_t)$ in the infinite dimensional space $(t, x_t) \in \mathbb{R} \times C([-r, 0], D)$ or a suitable subset thereof. In ODEs the space used corresponds to the extended state space or finite dimensional $(t, x(t)) \in \mathbb{R} \times \mathbb{R}^n$. J. K. Hale and V. Lunel in [24] discuss how for FDES we may use either the infinite-dimensional $\mathbb{R}_+ \times C([-r, 0], D)$ or the finite dimensional $\mathbb{R}_+ \times \mathbb{R}^n$ whenever convenient, though they dwell on the advantages and disadvantages of each approach. Nonetheless, the previous Example 4.5 illustrates how $t \mapsto x_t$ for $x \in PCB[-r, 0]$ is not in general even a well defined mapping. We would like to have at least piecewise continuity of said mapping, but this is not possible. Thus imagine trying to model trajectories in $\mathbb{R} \times PCB([-r, 0], D)$, where having $|t_1 - t_2| < \delta$ does not imply trajectories x_{t_1} and x_{t_2} are near to each other. Thus this infinite dimensional approach of Hale and Lunel has a serious difficulty, and given that many important ODE results are generalized in this way to FDEs, this implies that much of the theory for continuous delayed systems cannot be applied indiscriminately to impulsive delayed FDEs.

Also, in approaches such as in [24], in order to prove existence of solutions, the functional f in $x' = f(t, x_t)$ is assumed to be continuous in both variables $(t, \psi) \in J \times C([-r, 0], D)$. Nonetheless, for impulsive FDEs, this may be a bad idea. Even simple continuous functionals on $\mathbb{R}_+ \times C([-r, 0], \mathbb{R}^n)$ may not be extended continuously to $\mathbb{R}_+ \times PCB([-r, 0], \mathbb{R}^n)$, simply because the composite mapping $t \mapsto x_t$ may be highly discontinuous. Thus, we will need a suitable weakening of continuity conditions on the vector field functionals $f(t, x_t)$. An example taken from [6] to illustrate siutable weakening continuity requirements is given by

$$f(t, \psi) = \psi(-1 - e^{-t}) \tag{4.11}$$

with corresponding FDE

$$x'(t) = x_t(-1 - e^{-t}) = x(t - 1 - e^{-t}) \tag{4.12}$$

with $r = 2$. f here is discontinuous on $\mathbb{R}_+ \times PCB([-r, 0], \mathbb{R}^n)$, although continuous on $\mathbb{R}_+ \times C([-r, 0], \mathbb{R}^n)$. Nonetheless, given an initial condition $\phi \in PCB[-r, 0]$, and a delay *difference equation* satisfied at impulse times, this system can be solved by the method of steps, and satisfies an existence of solutions result proved in [6], for an appropriately constructed theory of impulsive FDEs, under the definition of solution given in Definitions 4.1-4.2. We will give said existence result below.

The appropriate first step to developing a fundamental theory of impulsive FDEs is to take the approach of D. Driver in [20], which we used in Chapter 3, which holds for infinite delays inclusive. Let us remember that the approach of Hale and Lunel in [24] does not work for infinitely delayed continuous FDEs, which is why we gave the adequate theoretical perspective when we studied FDEs in this thesis. Namely, let us assume something similar to *composite continuity* of f, which means that the mapping $t \mapsto f(t, x_t)$ is continuous when x is continuous. A suitable modification for discontinuous x is given below. For continuous FDE systems, $f : \mathbb{R}_+ \times C([-r, 0], \mathbb{R}^n) \longrightarrow \mathbb{R}^n$ is continuous in its two variables, so if $r < \infty$, since $t \mapsto x_t$ is continuous if x is (thanks to Lemma 4.1), then $f : \mathbb{R}_+ \times C([-r, 0], \mathbb{R}^n) \longrightarrow \mathbb{R}^n$ is also composite continuous. So no harm is done in that case.[4]

As mentioned in [6], the functional f in (4.11) is composite piecewise continuous in the sense that if x is piecewise continuous with respect to t, then $t \mapsto f(t, x_t)$ is a piecewise continuous mapping. This is what we will formally define as being *composite-PC* or *composite-PCB*, in the case of infinite delays. We define the following using PCB-function spaces, as in [50], instead of PC-spaces like in [6], due to the former being adequate to include infinitely delayed impulsive FDEs.

Definition 4.3. (Composite-PCB) *A functional $f : J \times PCB([-r, 0], D) \longrightarrow \mathbb{R}^n$ is called composite-PCB on J if for each $t_0 \in J$, $\beta > 0$ such that $[t_0, t_0 + \beta] \subset J$, if whenever $x \in PCB([t_0 - r, t_0 + \beta], D)$ and x is continuous at each $t \neq \tau_k$ in $(t_0, t_0 + \beta]$, then $t \mapsto f(t, x_t)$ is an element of the function class $PCB([t_0, t_0 + \beta], \mathbb{R}^n)$.*

The previous definition is important so that we can begin to prove an existence result for solutions of impulsive FDEs, since the first step requires to integrate the vector field. Thanks to the previous definition, we immediately have the following lemma. As usual, when $r = \infty$, then $[-r, c] = (-\infty, c]$ for $c \in \mathbb{R} \cup \{\infty\}$.

Lemma 4.2. *Suppose f is composite-PCB. Then a function $x \in PCB([t_0 - r, t_0 + \beta])$, where $\beta > 0$ such that $[t_0, t_0 + \beta] \subset J$, is a solution of (4.7)-(4.9) if and only if x satisfies*

$$x(t) = \begin{cases} \phi(t - t_0) & \text{if } t \in [t_0 - r, t_0] \\ \phi(0) + \displaystyle\int_{t_0}^{t} f(s, x_s)ds + \sum_{k:\tau_k \in (t_0, t]} I\big(\tau_k, x_{\tau_k^-}\big) & \text{if } t \in (t_0, t_0 + \beta]. \end{cases}$$

Of course one could just start to define the integral equation of Lemma 4.2, and weaken conditions on f so that the integral merely need exist, have definition of solutions weakened so that they just be piecewise *absolutely* continuous and satisfy the previous integral equation, or the FDE a.e. with respect to Lebesgue measure, to get Carathéodory type solutions. Nonetheless, this is unnecessary for our purposes.

It stands out to mention that even if $f : J \times PCB([-r, 0], \mathbb{R}^n) \longrightarrow \mathbb{R}^n$ is continuous on its domain of definition, we cannot confirm piecewise continuity of the composite map $t \mapsto f(t, x_t)$, in other words, we cannot confirm that is is composite- PCB as defined above. An example from [6] is for the functional on $\mathbb{R}_+ \times PCB([-1, 0], \mathbb{R}^n)$ $(r = 1)$

$$f(t, \psi) = \sum_{n=1}^{\infty} \frac{\psi(q_n)}{2^n},$$

where $\{q_n\}$ denotes some enumeration of the rational numbers in $[-1, 0)$. See [6] for further details.

[4]Driver in [20] notes that composite continuity plus a suitable local Lipschitz condition gives continuity of $f : \mathbb{R}_+ \times C([-r, 0], \mathbb{R}^n) \longrightarrow \mathbb{R}^n$ as well.

We now introduce further useful notions for the fundamental theory of impulsively delayed FDEs.

Definition 4.4. (Quasibounded) *A functional $f : J \times PCB([-r, 0], D) \longrightarrow \mathbb{R}^n$ is said to be quasibounded if for each $t_0 \in J$ and $\beta > 0$ such that $[t_0, t_0 + \beta] \subset J$, and for each compact set $F \subset D$ there exists some constant $M = M(t_0, \beta, F) > 0$ such that $|f(t, \psi)| \leq M$ for all $(t, \psi) \in [t_0, t_0 + \beta] \times PCB([-r, 0], F)$.*

Remark 4.6. *Notice that $\psi \in PCB([-r, 0], F)$, so that the image of ψ is bounded within the compact set $F \subset D$. Thus this boundedness condition holds in a local sense, on any forward compact time interval contained in J with functionals with bounded images in any given compact set $F \subset D \subset \mathbb{R}^n$. It is akin to saying that "f maps compacts to compacts".*

Definition 4.5. (Continuity in 2nd Variable) *A functional $f : J \times PCB([-r, 0], D) \longrightarrow \mathbb{R}^n$ is said to be continuous in its second variable if for each fixed $t \in J$, $\psi \mapsto f(t, \psi)$ is a continuous function of ψ on $PCB([-r, 0], D)$.*

Definition 4.6. (Locally Lipschitz) *A functional $f : J \times PCB([-r, 0], D) \longrightarrow \mathbb{R}^n$ is said to be locally Lipschitz in its second variable if for each $t_0 \in J$ and $\beta > 0$ such that $[t_0, t_0 + \beta] \subset J$, and for each compact set $F \subset D$ there exists some constant $L = L(t_0, \beta, F) > 0$ such that $|f(t, \psi_1)| - f(t, \psi_2)| \leq L\|\psi_1 - \psi_2\|_r$ for all $t \in [t_0, t_0 + \beta]$ and $\psi_1, \psi_2 \in PCB([-r, 0], F)$.*

Remark 4.7. *Notice that $\psi_1, \psi_2 \in PCB([-r, 0], F)$, so that the images of ψ_1, ψ_2 are bounded within the compact set $F \subset D$. Also notice that the Lipschitz condition holds in a local sense, on any forward compact time interval contained in J with functionals with bounded images in any given compact set $F \subset D \subset \mathbb{R}^n$.*

If f is locally Lipschitz in its second variable, then automatically it is continuous in its second variable. In addition, if f is composite-PCB, then it is also quasibounded, since $|f(t, \psi)| \leq L\|\psi\|_r + |f(t, 0)|$ where $t \in [t_0, t_0 + \beta]$, $\text{im}(\psi) \in F$ and the mapping in the variable t defined by $t \mapsto f(t, 0)$, is bounded by a constant on the compact set $t \in [t_0, t_0 + \beta]$ due to f being composite-PCB.

The following existence result was proved by G. Ballinger and X. Liu in [6] for finite delay, and subsequently generalized to infinite delayed and switched systems by X. Liu and P. Stechlinski in [39]. The version we state below is for infinite delay, and it is a somewhat intermediate result between that of [6] and that of [39] which follows from the latter since this is a particular case when no switching is included.

Theorem 4.1. (Local Existence) *Assume f is composite-PCB, quasibounded and continuous in its second variable. Then for each $(t_0, \phi) \in J \times PCB([-r, 0], D)$, there exists a solution $x(t) = x(t; t_0, \phi)$ of (4.7)-(4.9) on $[t_0 - r, t_0 + \beta]$ for some $\beta > 0$.*

Typical examples that conform to the previous existence result are:

(i) $f(t, x_t) = g\big(t, x(t-h_1(t)), ..., x(t-h_m(t))\big)$ where $g \in C(\mathbb{R}^+ \times \mathbb{R}^{n\times(m+1)}, \mathbb{R}^n)$, the functions h_k are continuous and satisfy $0 \leq h_k(t) \leq r$, $\forall t$ for fixed $r < \infty$, and the functions $t - h_k(t)$ are strictly increasing on $\mathbb{R}^+$;

(ii) $f(t, x_t) = g\left(t, x(t), \int_{t-r}^{t} G(t, s, x(s))ds\right)$ where $g \in C(\mathbb{R}^+ \times \mathbb{R}^{2n}, \mathbb{R}^n)$ and $G \in C(\mathbb{R}^+ \times [-r, \infty) \times \mathbb{R}^n, \mathbb{R}^n)$,

to name a few.

Let us discuss forward extension of solutions.

Definition 4.7. (Forward Continuation) *If x and y are solutions on the intervals J_1 and J_2 respectively, where J_2 properly contains J_1 and both intervals have the same closed left endpoint, and if $x(t) = y(t)$ $\forall t \in J_1$, then y is said to be a **proper continuation** of x to the right, or simply a **continuation** of x, and x is said to be continuable to J_2.*

As in delayed FDEs, we mentioned previously that backward continuation is not necessarily unique, and complications arise for continuous delayed systems, as mentioned in Chapter 3. The same carries on to impulsive systems of course. Thus our focus on forward continuation. The following result is proved in the finite delay case in [6], and a version that includes infinitely delayed switched in [50, 39].

Theorem 4.2. *Suppose f is composite-PCB, quasibounded and continuous in its second variable. Let $(t_0, \phi) \in J \times PCB([-r, 0], D)$, with corresponding solution $x(t) = x(t; t_0, \phi)$ of (4.7)-(4.9) on $[t_0 - r, t_0 + \beta] \subset J$ for some $\beta > 0$. Then x is continuable. If x is defined on an interval of the form $[t_0 - r, t_0 + \beta)$, where $0 < \beta < \infty$, and $[t_0, t_0 + \beta] \subset J$, and if x is non-continuable, then for every compact set $G \subset D$, there exists a sequence of numbers $\{t_k\}$ with $t_0 < t_k < t_{k+1} < t_0 + \beta$ for $k \geq 1$ such that $\lim_{k\to\infty} t_k = t_0 + \beta$ and $x(t_k) \notin G$.*

Definition 4.8. *A solution x of (4.7)-(4.9) is said to be unique if given any other solution y of (4.7)-(4.9), then $x(t) = y(t)$ on their commmon interval of existence.*

With the Lipschitz condition, as expected, we obtain uniqueness of solutions. One must be careful with the notion of uniqueness, nonetheless. This is because solutions with distinct initial conditions may merge, say, if the impulsive functional $I(t, x_{t^-})$ is not injective. For example if $I(t, x_{t^-}) = I(t, x(t^-))$, and even though $x(t^-) \neq y(t^-)$, $I(t, x(t^-)) = I(t, y(t^-))$ may happen and afterwards solutions merge. The following uniqueness result is a version of the result in [50].

Theorem 4.3. (Uniqueness of Solutions) *Assume that f is composite-PCB and locally Lipschitz in its second variable. Then there exists at most one solution of (4.20) on $[t_0 - r, t_0 + \beta)$ where $0 < \beta \leq \infty$ and $[t_0 - r, t_0 + \beta) \subset J$.*

This will be sufficient fundamental theory of impulsive FDEs. We have omitted continuous dependence on initial values, for instance, because we will be interested in stability, which is essentially a stronger form of being continuous with respect to initial values, when there is a particular equilibrium solution of interest. This is because in stability, we state that for initial conditions near the equilibrium solution, the future values of the solution remain near the equilibirium solution indefinitely in future time. For further details concerning continuous dependence on initial conditions, as well as remarks about particularities of this quality in impulsive FDEs, see [38], for instance.

4.2.2 Global Existence

Notice that global existence in time, supposing that $J \subset \mathbb{R}^+$ is an infinite forward interval, is quite different from the corresponding theory of ODEs. Forward global existence is important of course for stability. As mentioned in [47], earlier impulsive FDE results assumed global existence of solutions of the continuous portion (4.7) of the system as part of a sufficient condition for global existence when the impulses were added through the difference operator I in (4.8). Nonetheless, with impulsive systems it is possible for solutions to exist for all future times, whereas the purely continuous portion blows up in finite time. An example is given by

$$x'(t) = 1 + x^2, \qquad t \geq 0,\ t \neq \frac{k\pi}{4}, \tag{4.13}$$

$$\Delta x(t) = -1, \qquad t = \frac{k\pi}{4},\ k = 1, 2, ... \tag{4.14}$$

Notice that in this particular case we have an impulsed ODE. Suppose we have the initial condition $x(0) = 0$. Notice that the purely continuous portion (4.13) blows up, whereas incorporating the discrete impulsive moments gives us global existence of solution. This is because if we just consider $y'(t) = 1+y^2$ for $t \geq 0$, $y(0) = 0$, then $y(t) = \tan(t)$ exists on the maximal time interval $[0, \pi/2)$, and blows up at $t = \pi/2$. Nonetheless, the solution of (4.13)-(4.14) with $x(0) = 0$ is

$$x(t) = \tan\left(t - \frac{n\pi}{4}\right), \qquad t \in \left(\frac{n\pi}{4}, \frac{(n+1)\pi}{4}\right],\ \ n = 0, 1, 2, ... \tag{4.15}$$

and the solution exists for all future times. Therefore global existence of solutions of the continuous portion is a poor choice as part of sufficient hypotheses to obtain global existence of impulsive FDEs.

In papers [47, 41], global existence criteria are obtained that are independent of the global existence of solutions of the continuous portion. Nevertheless, fixed point methods are also invoked in the aforementioned papers, so thus it is perhaps no surprise that the particular fixed point theorem of Banach will provide us, in future results in this thesis, with a global existence result.

4.2.3 Stability of Impulsive FDEs

We now give the definitions of stability for impulsive FDEs. We will use the terminology and approach of X. Liu and G. Ballinger in [37]. Let us have

$$\begin{aligned} x'(t) &= f(t, x_t), \qquad t \neq \tau_k, t \geq t_0 \\ \Delta x(t) &= I(t, x_{t^-}), \qquad t = \tau_k, t > t_0. \end{aligned} \tag{4.16}$$

We have $f : J \times PCB([-r, 0], D) \longrightarrow \mathbb{R}^n$ with $J \subset \mathbb{R}$ an infinite interval of the form $[a, \infty)$ $a \geq -\infty$, we can assume $J = \mathbb{R}_+ = [0, \infty)$ for simplicity. Let each $t_0 \in J, \phi \in PCB([-r, 0], D)$ induce an initial value problem by appending to (4.16) the initial condition

$$x_{t_0} = \phi. \tag{4.17}$$

Remark 4.8. *For stability analysis, we assume that* $0 \in D$*, which implies that* $0 \in PCB([-r,0],D)$ *and that* $f(t,0) \equiv 0$ *for all* $t \in J$*, and* $I(\tau_k,0) \equiv 0$ *for all* k*. Thus* 0 *is an equilibrium solution.*

Remark 4.9. *As we did for ODEs and FDEs, we can study the translation of a nonzero equilibrium solution* $t \mapsto \varphi(t)$ *of an impulsive FDE by defining a change of variable* $x(t) = y(t) - \varphi(t)$ *and obtaining a new vector field and impulsive difference operator. Thus, studying the stability of the trivial solution* $\varphi(t) \equiv 0$ *is sufficient.*

Remember that the Euclidean norm is denoted $|\cdot|$.

Definition 4.9. (Stability Definitions for IFDEs) *The zero solution of* (4.16) *is said to be*

- ***Stable*** *if for each* $\epsilon > 0$ *and* $t_0 \in J$*, there exists a* $\delta = \delta(\epsilon, t_0) > 0$ *such that if* $\phi \in PCB([-r,0],D)$ *with* $\|\phi\|_r \leq \delta$*, and* $x(t) = x(t;t_0,\phi)$ *is any solution of the induced IVP* (4.16)-(4.17)*, then* $x(t;t_0,\phi)$ *is defined and satisfies*

$$|x(t;t_0,\phi)| \leq \epsilon, \quad \forall t \geq t_0. \tag{4.18}$$

- ***Uniformly stable*** *if, for each* $\epsilon > 0$ *there is a* $\delta = \delta(\epsilon) > 0$ *independent of* t_0*, such that* (4.18) *is satisfied when* $x(t) = x(t;t_0,\phi)$ *is any solution of the induced IVP* (4.16)-(4.17) *if* $\|\phi\|_r \leq \delta$*.*

- ***Unstable*** *if it is not stable.*

- ***Asymptotically stable*** *if it is stable and for every* $t_0 \in J$ *there is a constant* $c = c(t_0) > 0$ *such that if* $\phi \in PCB([-r,0],D)$ *with* $\|\phi\|_r \leq c$*, then* $x(t;t_0,\phi) \to 0$ *as* $t \to \infty$*.*

- ***Uniformly asymptotically stable*** *if it is uniformly stable and there is a constant* $c > 0$ *independent of* t_0*, such that or all* $\phi \in PCB([-r,0],D)$ *with* $\|\phi\|_r \leq c$*,* $x(t) \to 0$ *as* $t \to \infty$*, uniformly in* t_0*; that is, for each* $\eta > 0$*, there is* $T = T(\eta) > 0$ *such that*

$$|x(t)| \leq \eta, \quad \forall t \geq t_0 + T(\eta), \ \forall \|\phi\|_r \leq c.$$

Notice that we use the "$\leq$" inequality symbol instead of a strict inequality in our definitions of stability for impulsive FDEs above. Thus we ask, in IFDEs, that $\|\phi\|_r \leq \delta$ implies $|x(t;t_0,\phi)| \leq \epsilon$ be satisfied, instead of the usual $\|\phi\|_r < \delta$ implies $|x(t;t_0,\phi)| < \epsilon$ for continuous systems. Of course, these two ways of defining stability are equivalent. The advantage of the non-strict inequality when dealing with piecewise continuous systems is that if $\phi \in PCB([-r,0],D)$ and $|\phi(s)| < \delta$ for all $s \in [-r,0]$, then although it is also true that $\|\phi\|_r \leq \delta$, we cannot conclude that $\|\phi\|_r < \delta$, since it is possibly discontinuous from the left. For example if $\lim_{s\to 0^-} |\phi(s)| = \delta$, and before $s = 0$, $|\phi(s)| < \delta$ on $[-r,0)$ and is increasing in norm to size δ, but by a discontinuity $|\phi(0^+)| = |\phi(0)| < \delta$. Thus in this case it is true that $|\phi(s)| < \delta$ for all $s \in [-r,0]$, but $\lim_{s\to 0^-} |\phi(s)| = \delta$ implies that in this example $\|\phi\|_r = \delta$, even though all of the values of ϕ on $[-r,0]$ had norm strictly less than δ. Notice that if ϕ were continuous, then we would be able to conclude that $\|\phi\|_r < \delta$. Thus we take the convention of X. Liu and G. Ballinger in [37] of relaxing strict inequality requirements.

In stability results for IFDEs, there are, as in delayed continuous FDEs, two main types of Lyapunov results. One is stability with Lyapunov functions (Razumikhin technique), where one considers Dini derivatives of the type

$$D^+V_{(4.16)}(t,\psi(0)) := \limsup_{h\to 0^+} \frac{1}{h}\big[V(t+h,\ \psi(0)+hf(t,\psi)) - V(t,\psi(0)\,)\big]. \qquad (4.19)$$

for $V : J \times \mathbb{R}^n \longrightarrow \mathbb{R}$ a function. Stability results for IFDEs using Lyapunov functions can be seen, for example, in [55], [40], where the former is for finite delays and the latter for infinite delays, though notice none of those results include delays in the impulse difference operator. For a uniform asymptotic stability result with delayed impulse functions I, see the paper [37], which uses a Razumikhin type technique. The other main stability results use Lyapunov *functionals*, with derivatives for functionals $V : J \times PCB([-r,0],D) \longrightarrow \mathbb{R}$ such as

$$D^+V_{(4.16)}(t,x_t) = \limsup_{h\to 0^+} \frac{1}{h}\big[V(t+h,x_{t+h}) - V(t,x_t)\big].$$

Results using Lyapunov functionals can be seen in [48], [16], for instance. Again the last results do not include delays in the impulsive operator I.

We will not go into particular stability results for IFDEs, since there are numerous variations of results, due to different formulations of IFDEs according to distinct authors, and we will not require them. The papers cited above follow a similar theoretical convention as we do in this thesis. For more theory of impulsive differential equations, including stability, the monograph [32] is a good source, particularly for non-delayed impulsive differential equations.

4.3 Switched FDE Systems with Delays

We will now give an introduction to switched systems of FDEs. As in the multiple hybrid controller architecture given in the examples at the beginning of this chapter, switched systems occur when the vector field is changed according to some logical (deterministic or nondeterministic) rule. Switched systems can have state-dependent switching or switching at prespecified time instants. Also, they may include impulses in their various forms. For the purposes of the results obtained in this thesis though, we will only require the theory for continuous (non-impulsive) switched systems, though they will include delays. Also we will assume that switches are only time-dependent. In this section, we will only fulfill the necessary theoretical requirements for the results of this thesis. For further background and information, starting, for example, from non-delayed (switched ODEs essentially, but with control), we recommend the monograph [33]. For hybrid FDEs, switched systems including delays and/or impulsive functionals see [35], [2], [50], and references therein, where the former two include detailed analysis of stochastic versions of switched systems, and [2] includes not only stochastic state vectors, but also Markovian switching rules.

4.3.1 Fundamental Theory

Suppose we have a finite family of vector fields $\{f_i\}_{i\in\mathcal{P}}$, where $f_i : J \times BC([-r, 0], D) \longrightarrow \mathbb{R}^n$ with $J \subset \mathbb{R}$ an interval and $D \subset \mathbb{R}^n$ is an open set. We will allow for the case $r = \infty$ as well as finite r. The indexing set for the vector fields $f_i(t, x_t)$ is $\mathcal{P} = \{1, ..., M\}$ for some *finite* positive integer M. Each vector field induces a delayed FDE, and sometimes each f_i is called a *subsystem*. The general type of non-autonomous switched time-dependent FDE IVP that we will consider in this work will be in the form of

$$x'(t) = f_{\sigma(t)}(t, x_t) \qquad t \geq t_0 \tag{4.20}$$

$$x_{t_0} = \phi \tag{4.21}$$

where $t_0 \in J$. Essentially this is a delayed FDE as seen in Chapter 3. The main new element here is the switching rule $\sigma : J \longrightarrow \mathcal{P}$, which takes on values in the indexing set $\mathcal{P} = \{1, ..., M\}$ for the vector fields, so that when $\sigma(t) = i \in \mathcal{P}$ we have $f_{\sigma(t)}(t, x_t) := f_i(t, x_t)$. We will use only deterministic time dependent *admissible switching rules*, so that σ is a piecewise constant right continuous mapping, with a *dwell time*, in other words, there is a positive time of at least $\eta > 0$ between switching occurrences. More precisely, a switching signal σ comes with a sequence of switching times $\{t_k\}_{k=1}^N$ with $1 \leq N \leq \infty$. Thus we ask for $t_k - t_{k-1} \geq \eta$ for all k.

Definition 4.10. *A deterministic time dependent switching rule $\sigma : J \longrightarrow \mathcal{P}$ with associated switching moments $\{t_k\}_{k=1}^N$ will be called an* ***admissible switching rule****, if σ is a piecewise constant right continuous mapping, and there exists a* ***dwell time*** *$\eta > 0$ such that*

$$\inf_k \{t_k - t_{k-1}\} \geq \eta.$$

Given a family of vector fields $\{f_i\}_{i\in\mathcal{P}}$, denote the set of admissible switching rules by $\mathcal{S}$.[5]

The role of $\sigma \in \mathcal{S}$ is to select vector fields, and is also called a *switching signal*. In applications, one constructs the switching signal in order to achieve a desired objective.

Remark 4.10. *Supposing that there is a dwell time is done because we want to avoid* ***Zeno behavior*** *considerations, which occur when an infinite number of switching instants accumulate at a finite point. This happens, for instance, in the mathematical model of a bouncing ball that switches vector field each time it bounces. The ball satisfies Newton's laws, and clearly each bounce gets closer and closer in time, leading to a finite accumulation point of infinitely smaller bounces, according to the model, which we remember is an idealization in the first place. See [51, 33] for further details on this example.*

Suppose we have explicit switching times $\{t_k\}_{k=1}^N$ with $1 \leq N \leq \infty$, and $t_1 < t_2 < ...$ such that $\min\{t_k - t_{k-1}\} \geq \eta, \quad \forall k$. The switching index varies according to the switching rule σ. Suppose $t_0 \in J$ is the initial instant such that no switching occurs at t_0. Then, if

[5] We will work on stability of a switched FDE, where we will determine a dwell time $\eta > 0$ that guarantees stability of all admissible switching rules.

$\sigma\big|_{[t_{k-1},t_k)} = i_k \in \mathcal{P}$, we activate system i_k during this interval. Thus for $t \in [t_{k-1}, t_k)$, the dynamics are orchestrated by the selected f_{i_k} and

$$x'(t) = f_{i_k}(t, x_t) \qquad t \in [t_{k-1}, t_k).$$

At time t_k, we disengage system f_{i_k} and activate system $f_{i_{k+1}}$ for $t \in [t_k, t_{k+1})$. For the purposes of stability, we focus of course, on the more interesting case of an infinite number of switches, in other words $N = \infty$. In an application, this corresponds to indefinite switching, or the controller continuously being able to switch operational mode.

Remark 4.11. *We define a solution as in the non-switched case for continuous delayed FDEs in Chapter 3, except that we must satisfy the switched differential equation* (4.20) *at all times, with initial condition* (4.21)*. Notice that once a switching rule* $\sigma : J \longrightarrow \mathcal{P}$ *is made explicit along with switching times* $\{t_k\}_{k=1}^N$*, we have defined the vector field by* (4.20) *as* $f_{\sigma(t)}$*, and solutions are parametrized not solely by initial condition, but also by switching law* σ*. Thus* $x(t) = x(t; t_0, \phi, \sigma)$ *explicitly denotes the dependence on the switching rule. Nonetheless, we frequently drop the* σ *variable in* $x(t; t_0, \phi, \sigma)$ *when it is clear that a corresponding switching rule defined the vector field for the solution.*

The fundamental theory of existence and uniqueness of solutions is essentially derived from that of non-switched systems, because we have dwell times, we can integrate the vector fields in between the switches. In between switches, in a local time sense that is, all is just as before essentially. Long term dynamics such as stability are what may change, as we will see below.

Thus, we can obtain existence by assuming that each vector field of $\{f_i\}_{i\in\mathcal{P}}$ is *composite-continuous*, *quasibounded* and *continuous in the second variable*, using the same definitions as above for IFDEs in Section 4.2, except that instead of PCB-spaces we substitute bounded continuous BC-spaces, since we will settle for continuous switched FDEs in this thesis. The composite continuity hypothesis guarantees the following lemma immediately.

Lemma 4.3. *Suppose* f *is composite-BC,* $\sigma \in \mathcal{S}$ *a switching rule. Then a function* $x \in BC([t_0 - r, t_0 + \beta])$*, where* $\beta > 0$ *such that* $[t_0, t_0 + \beta] \subset J$*, is a solution of* (4.20) *with initial condition* (4.21) *if and only if* x *satisfies*

$$x(t) = \begin{cases} \phi(t - t_0) & \text{if } t \in [t_0 - r, t_0] \\ \phi(0) + \displaystyle\int_{t_0}^{t} f_{\sigma(s)}(s, x_s)ds & \text{if } t \in (t_0, t_0 + \beta]. \end{cases}$$

To obtain uniqueness, we simply add the Lipschitz continuity hypothesis, as defined above, with suitable replacement of PCB by BC. We can also obtain existence-uniqueness of solutions from theorems 3.4.3-3.5.1 in [50], which also hold if we add impulses. We enunciate an existence-uniqueness result for switched continuous FDEs, as it is the only one we will need in this work, and can be taken as a corollary of Theorem 3.5.1 in [50].

Theorem 4.4. (Uniqueness of Solutions (Switched)) *Assume that each* $f_j \in \{f_i\}_{i\in\mathcal{P}}$ *is composite-BC and locally Lipschitz in its second variable. Then there exists at most one solution of* (4.20)-(4.21) *on* $[t_0 - r, t_0 + \beta)$ *where* $0 < \beta \leq \infty$ *and* $[t_0 - r, t_0 + \beta) \subset J$.

To define forward continuation of solutions of a switched FDE (4.20), we define the following additional concept, product of considerations on the switching rule. This is because remember that we have the vector field defined for $\phi \in BC([-r,0],D)$ with $D \subset \mathbb{R}^n$ an open set.[6]

Definition 4.11. *A switching time $t_k \in J$ is called a* ***terminating switching time*** *or* ***terminating switch*** *if $x(t_k) \notin D$.*

The following result is a modified version of Theorem 3.6.1 in [50], considered as a corollary of it. Similar notions of continuable solutions and maximal interval of existence hold for switched systems, as defined above in Section 4.2 for IFDEs.

Theorem 4.5. *Suppose each f_i in (4.20) is composite-PCB, quasibounded and continuous in its second variable. Let us have an admissible switching rule $\sigma \in \mathcal{S}$, with corresponding switching instants $\{t_k\}$. Then, for each $(t_0, \phi) \in J \times PCB([-r,0],D)$, there exists a $\beta > 0$ such that $[t_0 - r, t_0 + \beta) \subset J$, and that the induced switched FDE IVP (4.20)-(4.21) has a corresponding non-continuable solution $x(t) = x(t; t_0, \phi, \sigma)$ on $[t_0 - r, t_0 + \beta)$. If $t_0 + \beta \in \operatorname{int}(J)$ is a finite time, then at least one of the following statements is true:*

(i) $t_0 + \beta$ is a terminating switch time;

(ii) For every compact set $G \subset D$, there exists a time $t \in (t_0, t_0 + \beta)$ such that $x(t) \notin G$.

There are distinct criteria to show global existence of solutions, but we will not study these results, since in our future results we will obtain global existence of solutions using the Banach fixed point theorem.

4.3.2 Stability of Switched FDEs

Suppose in (4.20) that $J = \mathbb{R}_+$ for simplicity, and that all subsystems have a zero equilibrium point, $f_i(t, 0) \equiv 0$ $\forall t \geq 0$, $\forall i \in \mathcal{P}$. Let σ be the switching rule with corresponding switching instants $t_1 < t_2 < ... < t_k < ...$ with $t_k \to \infty$ as $k \to \infty$.

Stability theory of switched FDEs is more complicated than for non-switched systems, mainly because of the different notions of stability that are possible when you introduce a switching signal. Stability notions can now be dependent on the switching signal, not just on the initial time t_0. Some stability definitions ask for uniformity with respect to the switching signal, be it in the set of all switching rules (stability under arbitrary switching), or in some subset $\mathcal{S}$ of these (constrained switching). For the latter, there are two subtypes of stability depending on whether the subsystems involved are all stable or not. The previous considerations can be better understood if one realizes that:

[6]In [50], it is assumed that each $f_i : J \times PCB([-r,0],D_i) \longrightarrow \mathbb{R}^n$, where each D_i of $PCB([-r,0],D_i)$ is an open set in $\mathbb{R}^n$ for each $i \in \mathcal{P}$, and solutions are defined such that $x(t) \in \bigcup_{i \in \mathcal{P}} D_i$ for all t where the solution is defined. Nonetheless, we can assume $D = \bigcup_{i \in \mathcal{P}} D_i$ for instance, or simply that the vector fields f_i have common set D, and thus common $PCB([-r,0],D)$ space of definition in the second variable.

(a) It is possible to have two subsystems defining a switched system of differential equations, such that both systems individually possess stability of the trivial solution. Yet it is possible to construct a switching rule such that alternating between these two systems causes instability. See Example 2.2.1 in [35] or Example 2.3.5 in [50]. Also [51, 33] Thus, the lesson here is that unconstrained switching can destabilize a switched system even if all the involved subsystems are stable.

(b) In somewhat of the opposite direction to the previous, it is possible to have all of the involved subsystems unstable, yet one can design a switching rule such that the resulting switched system presents stability of the trivial solution. See Example 2.3.3 of [49] to illustrate this. The lesson here is that it is possible to construct a switching signal σ such that stability is achieved even if all subsystems involved are unstable.

In applications there may be many limitations as to how to choose a switching rule. If one subsystem is stable, then by letting the switching rule decide to stay in this subsystem forever, stability is achieved. Nonetheless, this is not always possible.

Given a specific switching rule σ, stability definitions for the continuous switched FDE that this fixed switching law induces are the same as in Definition 3.5 in Section 3.4 of Chapter 3. The definition given below is uniform with respect to a set of switching rules $\sigma \in \mathcal{S}$, which will be our set of admissible switching signals that are characterized by a corresponding dwell time.

Definition 4.12. (Stability Definitions for Switched FDEs) *The zero solution of* (4.20) *is said to be*

- ***Stable over*** $\mathcal{S}$ *if for each* $\epsilon > 0$ *and* $t_0 \in J$*, there exists a* $\delta = \delta(\epsilon, t_0) > 0$ *independent of* $\sigma \in \mathcal{S}$ *such that if* $\phi \in BC([-r, 0], D)$ *with* $\|\phi\|_r < \delta$, $\sigma \in \mathcal{S}$ *and* $x(t) = x(t; t_0, \phi, \sigma)$ *is any solution of the switched IVP* (4.20)-(4.21) *induced by* σ*, then* $x(t)$ *is defined for all* $t \geq t_0$ *and satisfies*
$$|x(t; t_0, \phi)| < \epsilon, \quad \forall t \geq t_0. \tag{4.22}$$

- ***Uniformly over*** $\mathcal{S}$ *if, for each* $\epsilon > 0$ *there is a* $\delta = \delta(\epsilon) > 0$ *independent of* $t_0 \in J, \sigma \in \mathcal{S}$, *such that* (4.22) *is satisfied if* $\|\phi\|_r < \delta$ *when* $x(t) = x(t; t_0, \phi, \sigma)$ *is any solution of the IVP* (4.20)-(4.21) *induced by* σ.

- ***Unstable*** *if it is not stable.*

- ***Asymptotically stable over*** $\mathcal{S}$ *if it is stable and for every* $t_0 \in J$ *there is a constant* $c = c(t_0) > 0$ *independent of* $\sigma \in \mathcal{S}$ *such that if* $\phi \in BC([-r, 0], D)$ *with* $\|\phi\|_r < c$*, then* $x(t; t_0, \phi, \sigma) \to 0$ *as* $t \to \infty$.

- ***Uniformly asymptotically stable over*** $\mathcal{S}$ *if it is uniformly stable and there is a constant* $c > 0$ *independent of* $t_0 \in J, \sigma \in \mathcal{S}$ *such that or all* $\phi \in BC([-r, 0], D)$ *with* $\|\phi\|_r < c$, $x(t; t_0, \phi, \sigma) \to 0$ *as* $t \to \infty$*, uniformly in* t_0*; that is, for each* $\eta > 0$*, there is* $T = T(\eta) > 0$ *such that*
$$|x(t; t_0, \phi, \sigma)| < \eta, \quad \forall t \geq t_0 + T(\eta),\ \forall \|\phi\|_r < c.$$

We now give a brief overview of the main stability problems that switched systems can induce, according to the broad classification in [33].

Stability by Arbitrary Switching

In this case, we wish to find conditions that guarantee stability under any switching rule (with or without dwell time). This necessarily entails that all subsystems involved possess good stability properties, otherwise, if f_i is an unstable system, choose the switching rule $\sigma \equiv i$. For this method, in general one finds a Lyapunov functional that works for all of the subsystems, and the functional is decreasing always in every subsystem. Actually, the systems should be very well behaved, it is not too practical to look for this type of stability. Besides, supposing you can find a Lyapunov functional that works for every subsystem, the corresponding stability theory is almost the same as for switched systems, and thus we will not consider this.

Stability by Constrained Switching

We will consider constrained switching in this thesis. Here the topic is divided into two subtopics, where we will only pursue the first one, namely:

- If a switched system is not stable for arbitrary switching, identify those switching signals for which it is stable (asymptotically stable). In this context, there can be unstable subsystems along with stable ones. Lyapunov function or functional methods dominate in this area. Common criteria involve identifying dwell time conditions, or in more relaxed occasions, average dwell time conditions. See [33, 35, 50] for more information. Of course, if, say, there are asymptotically stable subsystems, it is easy to conclude that many switching rules identified in this problem will depend on remaining more time within the "good" subsystems than within the unstable ones, allowing for the asymptotically stable ones to alleviate any transient non-stable behavior. These methods generally involve "slow switching", because of the prolongation of the action of stable systems by dwell times, or average dwell times.

- If all individual subsystems are unstable, or it is not possible to remain enough time withing well-behaved subsystems, construct a switching signal that obtains the desired stability properties. When all subsystems are unstable, this problem is perhaps of the hardest type, compared to the previously mentioned. These methods generally involve "fast switching", because we try to avoid the destabilizing influences of the bad systems. See [35, 50] and references therein for more details.

We will not go too deeply into studying stability of switched systems, because we will only achieve one type of stability result for constrained switching. In particular, we will develop a dwell time type criterion for stability. Besides, we will not use Lyapunov theory, since we will achieve stability by a contraction method, and in this work we will not compare these two methods *for switched systems*, or any others.

Chapter 5

Contraction Mapping Principle in Stability of a Delayed FDE

5.1 Introduction

In brief, achieving asymptotic stability results by using the Banach fixed point theorem can sometimes provide better conditions for convergence to zero of solutions, than Lyapunov methods. The advantages of this particular fixed point method have been achieved thanks to contraction mapping methods requiring averaging conditions of the vector field, by using appropriately chosen variation of parameters type formulas to invert the differential equation into an integral form. As is known in differential equations theory, a common method for proving existence of solutions is through fixed point methods. However, in fairly recently times, the contraction mapping principle has been used to obtain further properties of the solution, namely attractivity of solutions to an equilibrium, and not merely the existence of these solution curves, as is normally done in classical differential equations theory. The aforementioned method for stability of differential equations has been applied successfully in [12, 57, 58] for delayed differential equations, [17] for neutral delay differential equations, in neutral stochastic differential equations [54], and in delayed stochastic FDEs with impulses [18].

We will illustrate how to use the Banach fixed point theorem in the asymptotic stability of nonlinear delay differential equations (DDEs), based primarily on the paper [57]. Nonetheless, we will obtain suitable generalizations, and stronger forms of some of the results in [57]. Namely, in the aforementioned paper, asymptotic stability is achieved, while we will discuss how to obtain uniform stability plus stability by making a simple observation. We will also generalize the previous asymptotic stability result to systems of FDEs, not just to scalar FDEs as is done in the aforesaid paper. This raises the question as to how far this particular fixed point method can carry us, and what are the limitations of this technique. We will point out the important limitation that the Banach fixed point theorem gives uniqueness of solutions *only* within the complete metric space where it is defined. If the metric space onto which we apply the contraction mapping principle is too small, then we are not obtaining a satisfactory uniqueness result. We will discuss this in detail below.

We repeat that, only in relatively recent times, in [13, 14] (both in the year 2001), this method has begun to receive attention, and has had recent successful applications, which to us justifies further study into this method.

5.2 The Basic Idea

We will make use of the following fixed point theorem.

Theorem 5.1. (Banach Contraction Principle) *Let* (X, d) *be a nonempty complete metric space and let* $T : X \longrightarrow X$ *be a continuous mapping such that there is real number* $0 < \alpha < 1$ *satisfying*

$$d(T(x), T(y)) < \alpha d(x, y) \quad \text{for } x, y \in X.$$

Then there is a unique point $x_0 \in X$ *such that* $T(x_0) = x_0$.

In general, the basic idea is that given a delay differential equation of the form

$$\begin{aligned} x' &= f(t, x_t) \qquad t \geq t_0 \\ x_{t_0} &= \phi \end{aligned} \tag{5.1}$$

we try to build a mapping that inverts (5.1), in other words, we perform some operation of integration such that

$$x(t) = a(t) + \int_{t_0}^{t} G\big(t, \phi, s, x(\cdot)\big)\, ds. \tag{5.2}$$

The right hand side of the previous equation defines a mapping P on a function space. We then proceed to decide which complete metric space $\mathcal{M}$ could be a good candidate for our purposes. Then we restrict P to the space $\mathcal{M}$ and then we try to make P map $\mathcal{M}$ to itself $P : \mathcal{M} \longrightarrow \mathcal{M}$. The mapping in general must not be something as obvious as for example, integrating the vector field directly and using $(P^*x)(t) = \phi(0) + \int_{t_0}^{t} f(s, x_s)\, ds$. The mapping will be built to exploit properties useful for us. The solution to the functional differential equation will be given by the fixed point of the mapping

$$\begin{aligned} (Py)(t) &= \phi(t - t_0) \quad t \leq t_0 \\ (Py)(t) &= a(t) + \int_{t_0}^{t} G\big(t, \phi, s, y(\cdot)\big)\, ds. \end{aligned}$$

In this way we determine that the mapping P is actually a viable representation of the solution by proving existence of the solution of (5.1). Finally, the integral form of the solution will provide us with an aid to determining stability of (5.1). Moreover, the complete metric space $\mathcal{M}$ will provide us with properties that will aid with the asymptotic stability. The $\mathcal{M}$ plays an important role for this, as we shall see ahead.

5.3 A General Result for a Nonlinear FDE

The differential model studied here is given by the scalar delayed differential equation

$$x'(t) = -a(t)x(t) + g(t, x_t), \tag{5.3}$$

where $a : \mathbb{R}_+ \longrightarrow \mathbb{R}$ and $g : \mathbb{R}_+ \times BC \longrightarrow \mathbb{R}$ are continuous, where here we denote

$$BC = \{\phi \in C[\mathbb{R}_-; \mathbb{R}] : \phi \text{ bounded}\}.$$

By $\mathbb{R}_+$ and $\mathbb{R}_-$ we mean $[0, \infty)$ and $(-\infty, 0]$ respectively. We endow the normed space BC with the uniform norm $\|\cdot\|_r$ defined on $\mathbb{R}_-$, which we will simply denote by $\|\cdot\|$.

If we have a continuous function $x : \mathbb{R} \longrightarrow \mathbb{R}$, we denote by x_t the function in BC defined explicitly as

$$x_t(\theta) := x(t + \theta) \qquad \text{for } \theta \in (-\infty, 0] = \mathbb{R}_-$$

As mentioned in the theoretical background on FDEs that we gave in earlier chapters, if $x : \mathbb{R} \longrightarrow \mathbb{R}$, then x_t simply denotes the restriction of x to the interval $(-\infty, t]$.

With the aid of the previous fixed point theorem, we have the following result from [57] concerning the stability of (5.3).

For each $\gamma > 0$ define $C(\gamma) := \{\phi \in BC : \|\phi\|_r \leq \gamma\}$. For a function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$, we define

$$\|\psi\|^{[s,t]} := \sup_{u \in [s,t]} |\psi(u)|.$$

Theorem 5.2. (B. Zhang, [57]) *Suppose that there exist positive constants α, L and a continuous function $b : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that the following conditions hold:*

(i) $\liminf_{t\to\infty} \int_0^t a(s)ds > -\infty$.

(ii) $\int_0^t e^{-\int_s^t a(u)du} b(s)ds \leq \alpha < 1$ *for all* $t \geq 0$.

(iii) $|g(t, \phi) - g(t, \psi)| \leq b(t)\|\phi - \psi\|$ *for all* $\phi, \psi \in C(L)$, *and* $g(t, 0) = 0$.

(iv) $\forall \epsilon > 0$ *and* $t_1 \geq 0$ *given, there exists a* $t_2 > t_1$ *such that* $t \geq t_2$ *and* $x_t \in C(L)$ *imply* [1]

$$|g(t, x_t)| \leq b(t)\Big(\epsilon + \|x\|^{[t_1,t]}\Big). \tag{5.4}$$

Then the zero solution of (5.3) *is asymptotically stable if*

(v) $\int_0^t a(s)ds \longrightarrow \infty$ *as* $t \to \infty$.

[1] This is a *fading memory condition*, we saw this type of condition in stability theory of delayed FDEs in Chapter 3, which Driver in [20] states that this aforesaid characteristic is necessary for asymptotic stability in *infinitely* delayed FDEs. It is automatically satisfied for finite delays, see Lemma 5.2 below. We further characterize this property in examples below.

Remark 5.1. *Notice that the author B. Zhang in [57] Theorem 2.1 has a necessary and sufficient condition for asymptotic stability. We will only focus on the sufficient conditions for stability in this work. In the aforementioned result, it says that the zero is asymptotically stable if and only if* $\int_0^t a(s)ds \longrightarrow \infty$ *as* $t \to \infty$.

Proof. Suppose that condition (v) holds. Let $t_0 \geq 0$ and since $\alpha < 1$, find $0 < \delta_0 \leq L$ such that $\delta_0 K + \alpha L \leq L$, where

$$K := \sup_{t \geq t_0} e^{-\int_{t_0}^t a(u)du}. \tag{5.5}$$

This means that

$$\delta_0 \leq \min\left\{L, \frac{L}{K}(1-\alpha)\right\}. \tag{5.6}$$

Thanks to (i), K is well defined. We will be particularly interested in small values of δ_0, so let us choose $\delta_0 < L$. Let $\phi \in C(\delta_0)$ fixed, so that we have an initial value problem for (5.3) through $(t_0, \phi) \in \mathbb{R}_+ \times C$. With this ϕ, set

$$S := \big\{x : \mathbb{R} \longrightarrow \mathbb{R} \mid x_{t_0} = \phi, x_t \in C(L) \text{ for } t \geq t_0 \text{ and } x(t) \to 0 \text{ as } t \to \infty\big\}. \tag{5.7}$$

S is a complete metric space under the metric[2]

$$\rho(x, y) := \sup_{t \geq t_0} |x(t) - y(t)|.$$

Note that in using this metric on S, we are not interested in what happens before t_0, since $x_{t_0} = \phi = y_{t_0}$ by definition of S. It is easy to see that S is nonempty by simply defining a function $z(t)$ which equals ϕ for $t \leq t_0$, letting $\delta_0 < L$ and pasting together with a function that decays to zero as $t \to \infty$ (such as an exponential function), or even one that becomes zero at a finite time and remains at the constant value of zero.

By analyzing the DDE (5.3) $x'(t) = -a(t)x(t) + g(t, x_t)$, we realize that we can give an equivalent integral formulation of this problem by doing something similar to what we do when we solve a linear first order ODE (ordinary differential equation): A solution to (5.3), if it exists, would have to satisfy the following after we multiply by the integrating factor

$$\mu(t) := e^{\int_{t_0}^t a(s)ds}$$

to obtain

$$\frac{d}{dt}\left(x(t)\mu(t)\right) = \mu(t)g(t, x_t), \quad \text{so that}$$

$$x(t)\mu(t) - x(t_0) = \int_{t_0}^t e^{\int_{t_0}^s a(u)du} g(s, x_s)ds,$$

[2]The boundedness assumption on $BC[\mathbb{R}_-, \mathbb{R}]$ gives completeness. This subset S is a closed subset of $BC[\mathbb{R}_-, \mathbb{R}]$, because if the limit does not converge to zero, then the sequence in S would not even be able to converge to this limit function. Something similar occurs for boundedness by the uniform constant L.

and the previous implies that along with the initial condition for the DDE:

$$x(t) = \phi(0)e^{-\int_{t_0}^{t} a(s)ds} + \int_{t_0}^{t} e^{-\int_s^t a(u)du} g(s, x_s)ds. \tag{5.8}$$

We thus have an equivalent integral expression for the solution $x(t)$. This suggests that we define the following mapping P defined on S:

$$(Px)(t) := \begin{cases} \phi(t - t_0) & \text{if } t \le t_0 \\ \phi(0)e^{-\int_{t_0}^{t} a(s)ds} + \int_{t_0}^{t} e^{-\int_s^t a(u)du} g(s, x_s)ds & \text{if } t \ge t_0 \end{cases} \tag{5.9}$$

In order to apply the Banach fixed point theorem, we need to prove that P maps S to itself. Clearly $Px : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, and by definition $(Px)_{t_0} = \phi$. Let us prove that $(Px)_t \in C(L)$ for $t \ge t_0$: Using the fact that $\phi \in C(\delta_0)$ along with conditions (ii), (iii) and that $\|x_s\| \le L$, $\forall s$, we have that

$$\begin{aligned} |(Px)(t)| &\le |\phi(0)|e^{-\int_{t_0}^{t} a(s)ds} + \int_{t_0}^{t} e^{-\int_s^t a(u)du} |g(s, x_s)|ds \\ &\le \delta_0 e^{-\int_{t_0}^{t} a(s)ds} + \int_{t_0}^{t} e^{-\int_s^t a(u)du} b(s)\|x_s\|ds \\ &\le \delta_0 K + L\int_{t_0}^{t} e^{-\int_s^t a(u)du} b(s)ds \\ &\le \delta_0 K + L\alpha \le L \end{aligned}$$

by the choice of δ_0. This shows that $(Px)_t \in C(L)$ for $t \ge t_0$.

Now we show that $(Px)(t) \to 0$ as $t \to \infty$. Since $x(t) \to 0$ as $t \to \infty$, given $\epsilon > 0$ there exists a $t_1 > t_0$ such that $|x(t)| < \epsilon$ for all $t \ge t_1$. Since $|x(t)| \le L$ for all $t \in \mathbb{R}$, by condition (iv) there exists $t_2 > t_1$ such that $t \ge t_2$ implies

$$|g(t, x_t)| \le b(t)\Big(\epsilon + \|x\|^{[t_1,t]}\Big).$$

For $t \ge t_2$ we have:

$$\begin{aligned} &\left|\int_{t_0}^{t} e^{-\int_s^t a(u)du} g(s, x_s)ds\right| \\ &\le \int_{t_0}^{t_2} e^{-\int_s^t a(u)du} |g(s, x_s)|ds + \int_{t_2}^{t} e^{-\int_s^t a(u)du} |g(s, x_s)|ds \\ &\le \int_{t_0}^{t_2} e^{-\int_s^t a(u)du} b(s)\|x_s\|ds + \int_{t_2}^{t} e^{-\int_s^t a(u)du} b(s)\Big(\epsilon + \underbrace{\|x\|^{[t_1,t]}}_{\le\epsilon \text{ since } t\ge t_2}\Big)ds \\ &\le \|x_t\| \int_{t_0}^{t_2} e^{-\int_s^t a(u)du} b(s)ds + \int_{t_2}^{t} e^{-\int_s^t a(u)du} b(s)(2\epsilon)ds \\ &\le L\int_{t_0}^{t_2} e^{-\int_s^{t_2} a(u)du - \int_{t_2}^{t} a(u)du} b(s)ds + 2\epsilon\alpha \\ &\le \alpha L e^{-\int_{t_2}^{t} a(u)du} + 2\alpha\epsilon \end{aligned}$$

By (v) there exists $t_3 > t_2$ such that

$$\delta_0 e^{-\int_{t_0}^{t} a(u)du} + Le^{-\int_{t_2}^{t} a(u)du} < \epsilon$$

The previous two estimates yield that for $t \geq t_3$:

$$\begin{aligned}|(Px)(t)| &= \left|\phi(0)e^{-\int_{t_0}^{t} a(s)ds} + \int_{t_0}^{t} e^{-\int_s^t a(u)du} g(s, x_s)ds\right| \\ &\leq \delta_0 e^{-\int_{t_0}^{t} a(u)du} + \alpha Le^{-\int_{t_2}^{t} a(u)du} + 2\alpha\epsilon < 3\epsilon\end{aligned}$$

This proves that $(Px)(t) \xrightarrow[\infty]{t} 0$. This proves that $Px \in S$ for every $x \in S$. This implies that $P : S \longrightarrow S$ is well defined. To prove that P is a contraction on S is straightforward, since for $x, y \in S$:

$$\begin{aligned}|(Px)(t) - (Py)(t)| &\leq \int_{t_0}^{t} e^{-\int_s^t a(u)du} |g(s, x_s) - g(s, y_s)|ds \\ &\leq \int_{t_0}^{t} e^{-\int_s^t a(u)du} b(s)\|x_s - y_s\|ds \\ &\leq \sup_{s \geq t_0} |x(s) - y(s)| \int_{t_0}^{t} e^{-\int_s^t a(u)du} b(s)ds \\ &\leq \alpha\rho(x, y)\end{aligned}$$

where the last inequality follows from the fact that $\rho(x, y)$ takes into account the difference $|x(u) - y(u)|$ with $u \in [t_0, \infty)$, but by definition of S we have $x(u) = \phi(u - t_0) = y(u)$ for $u \leq t_0$, so that we can disregard any contribution to the difference before t_0.

By the contraction mapping theorem[3] there exists a unique fixed point $x \in S$, which solves (5.3), for each $\phi \in C(\delta_0)$, and by definition of S we have that

$$x(t) = x(t, t_0, \phi) \xrightarrow[\infty]{t} 0.$$

In order to prove asymptotic stability, since we already proved that the solution $x(t, t_0, \phi)$ converges to zero for $\|\phi\| \leq \delta$, what is left to prove in order to conclude asymptotic stability is that the solution is stable. Let $\epsilon > 0$, $\epsilon < L$ be given. We will find a $\delta < \epsilon$ such that $\delta K + \alpha\epsilon < \epsilon$ (since we are interested in small values of δ). If $x(t, t_0, \phi)$ is a solution with $\|\phi\| < \delta$, then using the representation

$$x(t) = x(t, t_0, \phi) = \phi(0)e^{-\int_{t_0}^{t} a(s)ds} + \int_{t_0}^{t} e^{-\int_s^t a(u)du} g(s, x_s)ds,$$

[3] See the comments section after this proof, for a comment on uniqueness.

we prove that $|x(t)| < \epsilon$ for all $t \geq t_0$. Notice that $|x(t_0)| < \delta < \epsilon$. Suppose for the sake of contradiction that there exists $t^* > t_0$ such that $|x(s)| < \epsilon$ for $t_0 \leq s < t^*$ but $|x(t^*)| = \epsilon$. Then

$$\begin{aligned} |x(t^*)| &\leq \delta e^{-\int_{t_0}^{t^*} a(s)ds} + \int_{t_0}^{t*} e^{-\int_s^{t^*} a(u)du} b(s)\|x_s\| ds \\ &\leq \delta K + \alpha\epsilon < \epsilon, \end{aligned}$$

which contradicts the definition of t^*. Thus no such t^* exists and $|x(t)| < \epsilon$ for all $t \geq t_0$. Thus, the zero solution of (5.3) is asymptotically stable.

□

Remark 5.2. *See Section 5.4, where we discuss an important detail about uniqueness of solutions.*

5.4 Comment on Uniqueness

Notice that for each $\phi \in C(\delta_0)$, using a fixed point theorem we obtained existence and uniqueness of a solution of (5.3) *in* S, where

$$S := \big\{x : \mathbb{R} \longrightarrow \mathbb{R} \mid x_{t_0} = \phi, x_t \in C(L) \text{ for } t \geq t_0 \text{ and } x(t) \to 0 \text{ as } t \to \infty\big\}.$$

The Banach fixed point theorem works as long as $P|_S$, but P doesn't have to be restricted to S, and we might as well ask if there might be another fixed point of P outside of S. This is because the Banach contraction principle gives a unique solution *within the complete metric space S where the mapping is restricted to.* The space S used in the proof of Theorem 5.2 is such that $S \subset BC\big([t_0 - r, \infty), D\big)$ (*strict containment*), so one might argue that there might be a solution $x_2(t; t_0, \phi) \in BC\big([t_0 - r, \infty), D\big) \setminus S$, say, that does not converge to zero. Now, by definition, when speaking of "uniqueness", one must take note of *where is this uniqueness statement being held.* For delayed FDEs, by the general convention that uses BC-spaces, which is the one we gave in the theoretical background in Section 3.3, solutions must be unique within the respective BC-space where the solution is defined. We do not ask for uniqueness in an L^p-space, for instance, as in Carátheodory solutions, since this space is too big. And uniqueness within $S \subset BC\big([t_0 - r, \infty), D\big)$ (*strict containment*) is obviously not satisfactory, because this space is too small to be useful. Thus we see here a caveat about what uniqueness by this particular fixed point theorem really means. One must be careful in this sense.
One possible remedy for this would be to prove that any solution of the delay differential equation (5.3) through (t_0, ϕ) has to lie in S. This is something which some authors overlook when using fixed point theory, but other authors do not, such as Hassan Khalil in [30] p. 659, when proving existence-uniqueness for ODEs, where he points out that the fixed point theorem alone is not enough to conclude the local uniqueness of the solution to an ordinary differential equation, so he proves that any other solution satisfies the properties of the defining complete metric space, thus any other solution would necessarily be within the given complete metric space. In the aforesaid ODE case, it is easy to prove that any other solution is in the constructed complete metric space, by a simple continuity argument, which hinges on local uniqueness (so

for a *small enough* interval). Nonetheless, this is not so trivial to do for this FDE, because we have to guarantee an infinite time interval of existence, and we would have to prove that any other solution converges asymptotically to zero. But in our proof above, that the solution converges to zero depends on the fact that we are restricting our operator on functions which *a priori* converge to zero. Thus to do an argument similar to Khalil's in [30], we would have to do our argument without contingency on the convergence to zero of general elements $\tilde{x} \in BC[\mathbb{R};\mathbb{R}]$.

On the other hand, for this delay differential equation, by using fixed point theory, we are only proving existence of solutions for *small initial functions* $\phi \in C(\delta_0)$, since at the beginning of the proof we had to find a δ_0 so small such that $\delta_0 K + \alpha L \leq L$, this is seen in (5.6). Thus we are determining a region of attraction around the zero equilibrium solution, which under the conditions stated in the theorem, is an attractor. The bound on δ_0 acts as a type of upper limit on how large initial conditions may be, or how large can a perturbation be from the zero equilibrium. Notice in (5.6) that the larger K is, the smaller the initial condition may be, we shall come back to this later on, and in more general results.

Thus, we are not proving existence and uniqueness in general, or for larger initial functions, since we do not have any Lipschitz hypotheses guaranteed elsewhere. That is why the author of [57] uses a sufficiently small initial condition: so that we can guarantee, given the hypotheses on the vector field, that the solutions, which necessarily satisfy the variation of parameters type formula given in the proof, will never leave the ball centered at the origin with radius L, where the Lipschitz condition holds.

We can prove existence and uniqueness of solutions in a local time sense, without using fixed point theory. Nonetheless, before applying alternate theory of existence and uniqueness, we first arrive at a small technical issue of whether the vector field is well defined in the following sense. The open set $D \subset \mathbb{R}^n$ need not be bounded by L, and one can argue that the vector field defined by (5.3) might eventually evolve the state to norm sizes greater than L, where we do not have the Lipschitz type condition guaranteed. However, we will now show that given the differential equation (5.3), the solution $x(t)$ cannot leave a ball of radius L centered at 0, which in Euclidean space we denote $B_L(0)$, so that the function space $BC\big([-r,0], B_L(0)\big)$ is enough, which is equivalent to *the function space* ball centered at the zero function, denoted $B(L)$ as defined above. Thus the vector field would be well defined and remains in a ball of norm L, so that we can always guarantee the Lipschitz condition. We do so below.

Notice that the fact that the solutions of the impulsive FDE remain bounded by L, is independent of the contraction mapping being restricted to S. It is a property that depends solely on the variation or parameters formula, which necessarily any solution satisfies.

Lemma 5.1. *Under the hypotheses stated in Theorem 5.2, we have that if* $\sup_{s_2 \geq s_1 \geq 0} \left(e^{-\int_{s_1}^{s_2} a(s)ds}\right) \leq K < \infty$ *then the solutions of* (5.3) *with initial condition* $\|\phi\| < \delta_0 := \dfrac{(1-\alpha)}{K} L$ *remain bounded by L, i.e.,* $|x(t)| \leq L$ *for every* $t \geq t_0$.

Proof. The proof is completely similar to the way in which we prove stability of the solution in

Theorem 5.2, with the role of ϵ played by L this time.

For $\|\phi\| < \delta_0$, we claim that the solution $x(t)$ satisfies $|x(t)| \leq L$ for all $t \geq t_0$. Note that if x solves the FDE corresponding to the initial condition ϕ, then $|x(t_0)| = |\phi(0)| < L$. For the sake of contradiction suppose that there exists a $\hat{t} > t_0$ such that $|x(\hat{t})| > L$. Let

$$t^* = \inf\{\hat{t} : |x(\hat{t})| > L\}.$$

Now, by continuity, and by definition of t^*: $|x(t^*)| = L$. We thus have $|x(s)| \leq L$ for $s \in [t_0 - r, t^*]$. By the integral representation of $x(t)$, which all solutions to (5.3) satisfy with initial condition ϕ, we have that, since before t^* the paths are bounded by L, we can apply the Lipschitz condition (iii), so that

$$\begin{aligned} |x(t^*)| &\leq e^{-\int_{t_0}^{t^*} a(s)ds}|\phi(0)| + \int_{t_0}^{t^*} e^{-\int_s^{t^*} a(s)ds}|g(s, x_s)|ds \\ &< \delta_0 e^{-\int_{t_0}^{t^*} a(s)ds} + \int_{t_0}^{t^*} b(s)e^{-\int_s^{t^*} a(s)ds}\|x_s\|ds \\ &\leq \delta_0 K + \sup_{\theta \in [t_0 - r, t^*]} |x(\theta)| \left(\int_{t_0}^{t^*} b(s)e^{-\int_s^{t^*} a(s)ds} ds \right) \\ &\leq \delta_0 K + \alpha L = L. \end{aligned}$$

Thus we have that $|x(t^*)| < L$, contradicting the definition of t^*. ☐

We now prove easily that solutions to FDE (5.21) are unique, if $g(s, x_s)$ is *composite continuous*, as defined in Definition 3.3. Now, in [57], it is assumed that $g : J \times BC([-r, 0], D) \longrightarrow \mathbb{R}^n$ is continuous, but let us note that in the paper by D. Driver [20], it is noted on p. 403 that we can weaken this continuity condition by using the concept of composite continuity of g, which we gave in Chapter 3, and which we repeat below in Definition 5.1.

Proposition 5.1. *Supposing* $g : J \times BC([-r, 0], D) \longrightarrow \mathbb{R}^n$ *is composite continuous, and satisfies* $|g(t, \phi) - g(t, \psi)| \leq b(t)\|\phi - \psi\|$ *for all* $\phi, \psi \in C(L)$, *then solution to the IVP induced by* (5.3) *with initial condition* ϕ *is unique, if* $\|\phi\| < \delta_0 := \dfrac{(1-\alpha)}{K}L$.

Proof. By Lemma 5.1, we can guarantee now that we have a well defined delayed FDE (in the sense that the solutions to FDEs induced by the vector field (5.3) remain in a ball of radius L at all times (which is where the given Lipschitz-type condition holds), as long as the initial condition ϕ satisfies

$$\|\phi\| < \delta_0,$$

where δ_0 clearly gives an upper threshold on the initial conditions for an initial value problem. By the previous reasons, we have a local Lipschitz condition, in the sense of Definition 3.4 in Chapter 3, since if t is in a compact set, then $b(t)$ is bounded and gives us necessary Lipschitz constants, since any closed subset of the closed ball $B_L(0)$ would give us a compact subset. We also satisfy that the vector field is composite-continuous. Thus we are satisfying the hypotheses required in the local existence-uniqueness result of Theorem 3.1, that guarantees uniqueness in $BC\big([-r, 0], B_L(0)\big)$, even for infinite delay. ☐

Therefore, the additional information that we are obtaining from using the contraction mapping is the asymptotic stability of the unique solutions to each initial value problem.

5.5 Examples

Example 5.1. ([57]) We consider the delay differential equation

$$x'(t) = -a(t)x(t) + b(t)q(x(t - r(t))) \tag{5.10}$$

where $b, r : \mathbb{R}_+ \longrightarrow \mathbb{R}$ and $q : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous with

(i*) $\liminf_{t\to\infty} \int_0^t a(s)ds > -\infty$.

(ii*) $\sup_{t\geq 0} \int_0^t e^{-\int_s^t a(u)du}|b(s)|ds < 1$,

(iii*) $q(0) = 0$ and there exists an $L > 0$ so that $|x|, |y| \leq L$ implies

$$|q(x) - q(y)| \leq |x - y|.$$

(iv*) $r(t) \geq 0$, $t - r(t) \longrightarrow \infty$ as $t \to \infty$.

Then the zero solution of (5.10) is asymptotically stable if

(v*)

$$\int_0^t a(s)ds \longrightarrow \infty \quad \text{as} \quad t \to \infty$$

Proof. We simply check that the hypothesis for applying Theorem 5.2 are satisfied. Note that substituting $|b(t)|$ for $b(t)$, by (ii*) we get condition (ii). Here we have that

$$g(t, x_t) := b(t)q(x(t - r(t))) = b(t)q(x_t(-r(t))),$$

or $g(t, \phi) = b(t)q(\phi(-r(t)))$ for $\phi \in C(L)$. Thus by condition (iii*) we have that $g(t, 0) = 0$ and for $\phi, \psi \in C(L)$:

$$\begin{aligned}|g(t, \phi) - g(t, \psi)| &= |b(t)||q(\phi(-r(t))) - q(\psi(-r(t)))| \\ &\leq |b(t)||\phi(-r(t)) - \psi(-r(t))| \quad \text{(by condition (iii*))} \\ &\leq |b(t)|\|\phi - \psi\|,\end{aligned}$$

so that condition (iii) is satisfied. Finally, condition (iv*) of this example implies (iv) of Theorem 5.2 in the following way: Let $\epsilon > 0$ and $t_1 \geq 0$ be given. By hypothesis (iv*) we have that $t - r(t) \longrightarrow \infty$ as $t \to \infty$ implies that there exists $t_2 > t_1$ such that $t - r(t) \geq t_1$ for all $t \geq t_2$. Given that $r(t) \geq 0$, this implies that for t_2 as defined, it is true that $t - r(t) \in [t_1, t]$ for every $t \geq t_2$. Putting together the information we have so far, we have that given $\epsilon > 0$ and $t_1 \geq 0$, it is true that there exists a $t_2 > t_1$ such that using $\|x\|^{[t_1,t]} = \sup_{\theta\in[t_1,t]} |x(\theta)|$:

$$|x_t(-r(t))| \leq \|x\|^{[t_1,t]} \leq \epsilon + \|x\|^{[t_1,t]} \qquad \text{for } t \geq t_2,$$

which implies (using (iii*)) that

$$|g(t,x_t)| = |b(t)||q\big(x_t(-r(t))\big)| \le |b(t)||(x_t(-r(t))| \\ \le |b(t)|\Big(\epsilon + \|x\|^{[t_1,t]}\Big).$$

By Theorem 5.2 the stability of (5.10) follows given condition $\int_0^t a(s)ds \longrightarrow \infty$ as $t \to \infty$. □

Example 5.2. ([20, 57]) Now, if $q(x) := x$ in Example 5.1, then equation (5.10) reduces to

$$x'(t) = -a(t)x(t) + b(t)x(t - r(t)), \tag{5.11}$$

which is the same delay differential equation that we previously studied for stability using a Lyapunov function Razumikhin type technique in Example 3.4. In order to be able to apply this method, the Lyapunov function $V(x) = x^2$ was used under the following restrictions on $a(t), b(t)$: There exist constants $c > 0$ and $J > 1$ such that

$$a(t) \ge c \quad \text{and} \quad J|b(t)| \le a(t) \tag{5.12}$$

Conditions (5.12) imply (i*)-(ii*) of Example 5.1 along with (v*), and (iii*) (iv*) follow easily. Condition (ii*) follows from (5.12) because

$$\int_0^t e^{-\int_s^t a(u)du}|b(s)|ds \le \frac{1}{J}\int_0^t e^{-\int_s^t a(u)du}a(s)ds \\ = \frac{1}{J}e^{-\int_s^t a(u)du}\Big|_{s=0}^{s=t} = \frac{1}{J}\left(1 - e^{-\int_0^t a(u)du}\right).$$

Thus $\sup_{t\ge0}\int_0^t e^{-\int_s^t a(u)du}|b(s)|ds \le \frac{1}{J} < 1$.

Thus, the conditions (i*)-(v*) are less restrictive than (5.12), which are pointwise conditions on a and b, whereas conditions (i*)-(ii*) and (v*) are averaged conditions. $a(t)$ can be negative some of the time under these improved conditions, and a and b are related on average. △

Remark 5.3. *Remember in the proof of Theorem 5.2, that δ_0 in (5.6) depends also on L proportionally. The role of L is to guarantee a neighborhood of zero where the local Lipschitz condition holds, so that L can be arbitrarily large for linear systems, like in this example 5.2. Thus the result of Theorem 5.2 holds for arbitrarily large initial conditions in these cases.*

Example 5.3. ([57]) Now we have the Volterra equation

$$x'(t) = -a(t)x(t) + \int_{-\infty}^t E(t,s,x(s))ds \tag{5.13}$$

where $a : \mathbb{R}_+ \longrightarrow \mathbb{R}$ and $E : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$, where $\Omega = \{(t,s) \in \mathbb{R}^2 : t \ge s\}$ are continuous. Suppose there exist a constant $L > 0$ and a continuous function $q : \Omega \longrightarrow \mathbb{R}_+$ such that (i*) $\liminf_{t\to\infty} \int_0^t a(s)ds > -\infty$.

(ii*)
$$\sup_{t\geq 0} \int_0^t e^{-\int_s^t a(u)du} \left(\int_{-\infty}^s q(s,\tau)d\tau \right) ds < 1$$

(iii*) $E(t,s,0) = 0$ for all $(t,s) \in \Omega$ and there exists an $L > 0$ so that $|x|, |y| \leq L$ implies
$$|E(t,s,x) - E(t,s,y)| \leq q(t,s)|x-y|.$$

(iv*) Given $\epsilon > 0$ and $t_1 \geq 0$, there exists a $t_2 > t_1$ such that $t \geq t_2$ implies
$$\int_{-\infty}^{t_1} q(t,s)ds \leq \epsilon \int_{-\infty}^{t} q(t,s)ds.$$

Then the zero solution of (5.13) is asymptotically stable if

(v*) $\int_0^t a(s)ds \longrightarrow \infty$ as $t \to \infty$.

Proof. Here we have $g(t,\phi) = \int_{-\infty}^0 E(t, t+s, \phi(s))ds$, where we have merely used a change of variable to translate the interval of integration and show how functions $\phi \in \mathcal{C}$ are used in the definition of g. Also, let $b(t) = \int_{-\infty}^t q(t,s)ds$. We have for all $\phi, \psi \in C(L)$:
$$\begin{aligned}|g(t,\phi) - g(t,\psi)| &= \left| \int_{-\infty}^0 E(t,t+s,\phi(s))ds - \int_{-\infty}^0 E(t,t+s,\psi(s))ds \right| \\ &\leq \int_{-\infty}^0 q(t,t+s)\|\phi - \psi\| ds = b(t)\|\phi-\psi\|.\end{aligned}$$
Thus condition (iii) of Theorem 5.2 holds. Now let $\epsilon > 0$ and $t_1 \geq 0$ be given. By (iv*) there exists a $t_2 > t_1$ such that
$$L \int_{-\infty}^{t_1} q(t,s)ds < \epsilon \int_{-\infty}^{t} q(t,s)ds$$
for all $t \geq t_2$. Let $x : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous with $x_t \in C(L)$. If $t \geq t_2$, then
$$\begin{aligned}|g(t,x_t)| &\leq \int_{-\infty}^{t_1} |E(t,s,x(s))|ds + \int_{t_1}^{t} |E(t,s,x(s))|ds \\ &\leq \int_{-\infty}^{t_1} L\, q(t,s)ds + \int_{t_1}^{t} q(t,s)|x(s)|ds \\ &\leq \epsilon \int_{-\infty}^{t} q(t,s)ds + \|x\|^{[t_1,t]} \int_{t_1}^{t} q(t,s)ds \\ &\leq b(t)\Big(\epsilon + \|x\|^{[t_1,t]}\Big),\end{aligned}$$
by definition of $b(t)$. This implies that condition (iv) of Theorem 5.2 is satisfied. Thus the solution of (5.13) is asymptotically stable if (v*) holds. □

Example 5.4. ([25, 57]) When the Volterra integrodifferential equation from the previous example is linear, the authors of [25] give conditions for stability based on Lyapunov considerations for

$$x'(t) = -a(t)x(t) + \int_{-\infty}^{t} C(t,s)x(s)ds. \tag{5.14}$$

The required conditions for stability are $\int_0^\infty a(s)ds = \infty$,

$$\sup_{t\geq 0}\left\{\frac{1}{a(t)}\int_{-\infty}^{t}|C(t,s)|ds\right\} \leq \frac{1}{J}, \tag{5.15}$$

for some constant $J > 1$, and

$$\lim_{t\to\infty}\frac{1}{a(t)}\int_{-\infty}^{t_1}|C(t,s)|ds = 0, \tag{5.16}$$

for each $t_1 \geq 0$. Supposing the condition (iv*) of the previous example 5.3 holds, then conditions $\int_0^\infty a(s)ds = \infty$ and (5.15) imply the conditions of example 5.3. This is seen by letting $q(t,s) = |C(t,s)|$, and (5.15) implies

$$\int_0^t e^{-\int_s^t a(u)du}\left(\int_{-\infty}^{s}|C(s,\tau)|d\tau\right)ds \leq \int_0^t e^{-\int_s^t a(u)du}\frac{1}{J}a(s)ds \leq \frac{1}{J} < 1$$

by a similar calculation to example 2. Therefore the pointwise condition (5.15) from [25] can be relaxed and we can ask for an averaging condition of (5.15) in condition (ii*). △

5.6 Contraction Method for Uniform Stability

Notice in Example 5.2 that something important is elucidated. Remember the expression (5.5) we used in the proof of the theorem, namely

$$K := \sup_{t\geq t_0}\left\{e^{-\int_{t_0}^t a(u)du}\right\}\ ?$$

As we can see, in reality, the K constant given here *depends on* t_0. So the previous result holds by fixing a t_0. This makes us unable to conclude uniform stability in the previous result. Notice that $\int_0^t a(s)ds \longrightarrow \infty$ as $t \to \infty$, which is what ultimately pulls the term $e^{-\int_{t_0}^t a(u)du} \to 0$ as $t \to \infty$, which was a very important part of the proof in order to achieve asymptotic stability. Thus if $\lim_{t\to\infty} e^{-\int_{t_0}^t a(u)du} = M \neq 0$, then

$$\limsup_{t\to\infty}\int_0^t a(s)ds < \infty,$$

which can happen, for example if the integral diverged to $-\infty$, since $a(s)$ was negative "too often", which would cause the magnitude of K as defined above to be $+\infty$; or maybe K remains finite, but $a(s)$ never manages to obtain an integral that diverges to $+\infty$, such as $a(t) = \frac{1}{1+t^2}$, which gives a convergent integral to a finite number, instead of diverging to $+\infty$ as required in Theorem 5.2. This would mean that we cannot achieve asymptotic stability. One realizes that if in a certain sense, $a(t)$ is “too negative”, then one gets a bigger K. But since the integral, or rather the term

$$\inf_{t\in[0,\infty)} \int_0^t a(s)ds \tag{5.17}$$

is what determines the magnitude of K, one can have an idea of many scenarios that cause K to be bigger, which is what may limit in an application how negative $a(t)$ may be, since as seen in (5.6), we can only allow for smaller initial perturbations from the equilibrium when K is large. For example, if initially $a(t)$ starts off negative, we see that this is a scenario that causes the term (5.17) to quickly get large, making δ_0 in (5.6) smaller. If $a(t)$ is negative from the very beginning, or maybe $a(t)$ is negative for long periods of time sufficient to offset previous initial positive contributions to the integral in (5.17), then again $K = K(t_0)$ quickly grows in magnitude as you vary t_0 in regions where $a(t)$ is negative.

The averaging condition $\int_0^t e^{-\int_s^t a(u)du} b(s)ds \leq \alpha < 1$ for all $t \geq 0$ also means that somehow $b(t)$ must make up for this misbehavior in $a(t)$, by reducing the value of the integral in this interval.

Let us see some examples to understand the previous.

Example 5.5. In the Example (5.2) from [20], which we covered when analyzing Example 3.25 in Chapter 3, we obtained that $a(t) \geq c$ and $J|b(t)| \leq a(t)$ for some constants $c > 0$, $J > 1$ achieve asymptotic stability. With the fixed point theorem we can violate these conditions, but of course under limitations. Suppose for simplicity that

$$a(t) = \begin{cases} -1 & \textit{if } s \in [0,1] \\ -1 + 2(t-1) & \textit{if } t \geq 1. \end{cases}$$

and that $b(t) = \epsilon_0 > 0$ is a constant. From the very beginning, $a(t) < b(t)$, and until $a(t)$ surpasses $J|b(t)|$, $b(t)$ is what must keep the value of the given averaging integral less than some $\alpha < 1$. This can be seen through the following. First of all, since $-1 + 2(t-1) = J\epsilon_0$ at $t_1 = \frac{1}{2}(J\epsilon_0 + 3)$, then since $-a(u) \leq 1$ for all $u \geq 0$, in particular for $u \in [0, t_1]$, then for $t \in [0, t_1]$, using the bounds on $a(t), b(t)$:

$$\int_0^t e^{-\int_s^t a(u)du} b(s)ds \leq \int_0^t e^{\int_s^t du} b(s)ds \leq \epsilon_0(e^{t_1} - 1) \qquad t \in [0, t_1].$$

After t_1, $a(t) > J|b(t)|$. We can even allow $|b(t)|$ to become unbounded on $[t_1, \infty)$, as long as $a(t) > J|b(t)|$ holds. Thus the following analysis also allows for $b(t)$ to be unbounded on $[t_1, \infty)$

as long as $a(t) > J|b(t)|$. For $t \geq t_1$ we have that

$$\begin{aligned}
\int_0^t e^{-\int_s^t a(u)du} b(s)ds &\leq \epsilon_0(e^{t_1} - 1) + \int_{t_1}^t e^{-\int_s^t a(u)du} b(s)ds \\
&\leq \epsilon_0(e^{t_1} - 1) + \frac{1}{J}\int_{t_1}^t e^{-\int_s^t a(u)du} a(s)ds \\
&= \epsilon_0(e^{t_1} - 1) + \frac{1}{J}\left(1 - e^{-\int_{t_1}^t a(u)du}\right) \qquad (t \geq t_1) \\
&\leq \epsilon_0(e^{t_1} - 1) + \frac{1}{J}
\end{aligned}$$

Therefore, since

$$\epsilon_0(e^{t_1} - 1) + \frac{1}{J} < 1$$

if

$$\epsilon_0 < \frac{J-1}{J(e^{t_1} - 1)},$$

we can see how we have a requirement on how small $b(t)$ must be whenever $a(t)$ is the one that is violating the condition $a(t) > J|b(t)|$, even if $b(t)$ is allowed to be unbounded after t_1, where it must only satisfy $a(t) > J|b(t)|$. t_1 is related to how long was $a(t)$ negative and below $|b(t)|$, so the longer $a(t)$ is misbehaved, the larger penalty $b(t)$ must pay on this interval, since e^{t_1} gets larger. This gives us an idea as to how well behaved $b(t)$ must be whenever $a(t)$ violates the conditions given by Lyapunov stability. $\triangle$

The previous example will be crucial to our understanding of what happens in more general differential systems.

Example 5.6. Notice that in the proof of Theorem 5.2, the K in (5.5) depends on t_0. But notice that we can make it independent of t_0 as follows. Let us take another example. Suppose

$$a(t) = \begin{cases} 5\sin(t) & \text{if } t \in [0, \pi] \\ \sin(2t - \pi) & \text{if } t \in [\pi, 2\pi] \\ t - 2\pi & \text{if } t \geq 2\pi. \end{cases}$$

We have that $a(t) < 0$ if $t \in (\pi, \frac{3}{2}\pi)$. Nonetheless, the most negative contribution of $\int_\pi^t a(s)ds$ for $t \in (\pi, \frac{3}{2}\pi)$, does not affect if $t_0 = 0$, since

$$\int_\pi^{\frac{3\pi}{2}} a(t)dt = \int_\pi^{\frac{3\pi}{2}} \sin(2t - \pi)dt = -\frac{1}{2}\cos(2t - \pi)\Big|_{t=\pi}^{t=\frac{3\pi}{2}} = -1 \tag{5.18}$$

is canceled out by the positive contribution from the interval $[0, \pi]$ of $\int_0^\pi a(s)ds$

$$\int_0^\pi a(t)dt = 5\int_0^\pi \sin(t)dt = 10. \tag{5.19}$$

This makes, if $t_0 = 0$, $K := K(0) = \sup_{t\geq 0}\left(e^{-\int_0^t a(u)du}\right) = 1$, since afterwards, on the interval $[\frac{3}{2}\pi, \infty)$, we only have positive contributions to the integral.

However, the case is different if we now take $t_0 = \pi$. This is because of (5.18), so that we have

$$K = K(\pi) = \sup_{t\geq\pi}\left(e^{-\int_\pi^t a(u)du}\right) = e > 1$$

with the maximum value achieved at $t = \frac{3\pi}{2}$, since the integral $\int_\pi^t a(u)du$ is decreasing on $(\pi, \frac{3\pi}{2})$, and afterwards, positive contributions come to the integral after this time, making $t \mapsto \int_\pi^t a(s)ds$ increasing on $(\frac{3\pi}{2}, \infty)$. On $(2\pi, \infty)$ it is positive and increasing such that the overall dominant behavior of the positiveness causes $\int_\pi^t a(s)ds \longrightarrow \infty$ as $t \to \infty$ so that $e^{-\int_\pi^t a(u)du} \longrightarrow 0$. Thus, this is how K depends on the initial time t_0 taken into account. $\triangle$

Remark 5.4. *The previous example gives insight into how to calculate a K that is independent of the initial time t_0, by focusing on the longest interval where $a(t)$ in* (5.3) *is negative. Of course, the condition $\int_0^t a(s)ds \longrightarrow \infty$ as $t \to \infty$ makes it clear that overall $a(t)$ is positive and in the long run overcomes any unstable behavior, the constant K being a measure of how bad things get before the goodness of $a(t)$ overtakes. We repeat that the K is important because it can determine how large the initial condition δ_0 as defined in* (5.6) *can be, where the role of δ_0 can be seen in the proof above. Nonetheless, the role of K is more important in other cases we will study in following chapters.*

Perhaps condition (5.4) in Theorem 5.2 may seem somewhat contrived and difficult to identify in a system. Nonetheless, this comes from a familiar previous concept. This aforementioned condition is what we saw in Chapter 3 about infinitely delayed FDEs requiring *fading memory conditions* in order to achieve asymptotic stability. Also, B. Zhang [57] on p. 5 denotes this type of requirement as a "fading memory" condition. In an earlier work by Seifert [46] it is pointed out that some sort of decaying condition is required for the asymptotic stability of a general delay equation. For a physical system this can be interpreted as a system remembering its past (through the delay), but the influence of the past as time increases should diminish, which can be interpreted as "the memory fades with time". Intuitively, for finite delay dynamics, a fading memory condition such as (5.4) should be satisfied, since after a finite time length, in this case, the maximum bound on the delay, the information from the past is left out. We quickly prove this in the following lemma.

Lemma 5.2. *Under the conditions of Theorem* (5.2)*, if the delay $r < \infty$, then condition* (*iii*) *implies condition* (*iv*).

Proof. Let $\epsilon > 0$ and t_1 be given. Then if $t_2 = t_1 + r$ (which is finite, so well defined) then for any $t \geq t_2$, condition (i) along with $g(t,0) = 0$ implies

$$\begin{aligned} |g(t,x_t)| \leq b(t)\|x_t\| &= b(t)\Big(\sup_{s\in[-r,0]} |x_t(s)|\Big) \\ &= b(t)\Big(\sup_{s\in[t-r,t]} |x(s)|\Big) \\ &\leq b(t)\Big(\sup_{s\in[t_2-r,t]} |x(s)|\Big) = b(t)\|x\|^{[t_1,t]} \end{aligned}$$

□

Thus Theorem 5.2 can include finite delays.

Now, in [57], it is assumed that $g : J \times BC([-r,0],D) \longrightarrow \mathbb{R}^n$ is continuous, but let us note that in the paper by D. Driver [20], it is noted on p. 403 that we can weaken this continuity condition with the following definition below of composite continuity of g, which we gave in Chapter 3. In this sense, we can agree at least in a local existence sense with the material in Chapter 3, which was itself based on the paper by D. Driver. The relevance of this lies in the fact that posing the FDEs in the terminology of [20] allows us to also apply the result theirin, which also applies for infinitely delayed FDEs, contrary to the existence result of J. K. Hale in [24], which only works with finite delays. This also makes the theory run parallel in definitions and requirements on the vector field, with respect to the generalization of the existence result to impulsive FDEs (which is for finite delays) in [6] of G. Ballinger and X. Liu, and with the subsequent result by X. Liu and P. Stechlinksi in [39], which includes infinitely delayed switched FDEs.

Definition 5.1. *We say that $g : J \times BC([-r,0],D) \longrightarrow \mathbb{R}^n$ is* ***composite continuous*** *if for each $t_0 \in J$, and $\gamma > 0$ such that $[t_0, t_0+\gamma] \subset J$, if $\psi \in BC([t_0-r, t_0+\gamma], D)$, then the composite mapping $t \mapsto g(t,\psi_t)$ is a continuous function from J to $\mathbb{R}^n$. In other words, $t \mapsto g(t,\psi_t)$ belongs to $BC([t_0, t_0+\gamma], \mathbb{R}^n)$.*

Remark 5.5. *Of course, when $r < \infty$, $BC[-r,0] = C[-r,0]$.*

No harm is done in weakening the continuity requirement on g to composite continuity, since continuity on $J \times BC$ is implied by the composite continuity condition plus the local Lipschitz condition implied by our weighted Lipschitz condition.
The previous considerations motivate the following version of Theorem 5.2, which includes finite delays.

Theorem 5.3. *Suppose the conditions of Theorem 5.2 hold except that now assume $g : J \times BC([-r,0],D) \longrightarrow \mathbb{R}^n$ is composite continuous, and instead of (i) suppose that*

(i) For every $s_1 \leq s_2 \in [0,\infty)$ we have that $e^{-\int_{s_1}^{s_2} a(u)du} \leq K < \infty$ for some constant[4] $K > 0$, in other words,

$$\sup_{0\leq s_1\leq s_2} \left(e^{-\int_{s_1}^{s_2} a(u)du}\right) \leq K < \infty. \tag{5.20}$$

[4] It is clear that $K \geq 1$.

Then the zero solution of (5.3) *is uniformly stable and asymptotically stable.*

Proof. The proof is the same as for Theorem 5.2, except that we now have that for stability, the δ used is now independent of t_0, where δ depends on t_0 implicitly through K in (5.5). □

As we saw in Example 5.6, and in the Remark 5.4 above, if we focus on a particular t_0 of interest and are not interested in other possible t_0's as initial times, then we can possibly make $K = K(t_0)$ smaller, with the price of not concluding uniform stability, but perhaps a smaller K could be more useful for the particular problem of interest.

5.7 Generalization to System of Delayed Differential Equations

As the reader may have noticed, the previous results can be generalized to the following system of delayed functional differential equations

$$\begin{aligned} x'(t) &= A(t)x(t) + g(t, x_t), \quad t \geq 0. \\ x_{t_0} &= \phi \end{aligned} \tag{5.21}$$

Here, we have that $x(t) \in \mathbb{R}^n$, $g : J \times BC([-r, 0], D) \longrightarrow \mathbb{R}^n$ with $J \subset \mathbb{R}^+$ an interval, $D \subset \mathbb{R}^n$ an open set. $A(t)$ is an $n \times n$ continuous matrix function, in the sense that each entry of A is a continuous function in the interval of definition of the functional differential equation (5.21).

Remark 5.6. *In the case when $r = \infty$, we still denote the space $BC(-\infty, 0]$ by the notation $BC[-r, 0]$, by considering for this special case $[-r, 0]$ to mean the* infinite *interval $(-\infty, 0]$. Of course, when $r < \infty$, $BC[-r, 0] = C[-r, 0]$.*

For stability analysis, it is assumed that $0 \in D$, $J = \mathbb{R}^+, g(t, 0) = 0$ for all $t \in \mathbb{R}^+$. This guarantees that system (5.21) has a trivial solution $x(t) = 0$.

We will use the fundamental solution $\Phi(t, t_0)$ of the associated linear ordinary differential equation

$$\begin{aligned} y'(t) &= A(t)y(t) \\ y(t_0) &= y_0 \end{aligned} \tag{5.22}$$

such that the solution of IVP (5.22) is

$$y(t) = \Phi(t, t_0)y_0.$$

For a matrix M we use the standard linear operator norm induced by the Euclidean norm $|\cdot|$ on $\mathbb{R}^n$:

$$\|M\| := \|M\|_{\mathcal{L}(\mathbb{R}^n)} = \sup_{|y|=1} |My|.$$

We will use the inequality $|My| \leq \|M\| |y|$ for $y \in \mathbb{R}^n$.

The generalization can be given because the fundamental matrix of a linear system in the scalar time varying case when $A(t) = -a(t)$ is a scalar valued function is

$$\Phi(t_2, t_1) = e^{-\int_{t_1}^{t_2} a(u)du}.$$

Notice that the condition $\int_0^t a(s)ds \longrightarrow \infty$ as $t \to \infty$ implies that

$$\|\Phi(t,0)\| = \sup_{|x|=1} \left| e^{-\int_0^t a(u)du} x \right| = e^{-\int_0^t a(u)du} \longrightarrow 0$$

as $t \to \infty$. Fundamental matrices of higher dimensional systems are of course much more difficult to characterize, so we will essentially ask for the norm of the matrix to converge to zero as $t \to \infty$.

Theorem 5.4. *Suppose that $g : J \times BC([-r,0], D) \longrightarrow \mathbb{R}^n$ in (5.21) is composite continuous, and that there exist positive constants α, L and a continuous function $b : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that the following conditions hold:*

(i) *Let* $\sup_{s_2 \geq s_1 \geq 0} \left(\|\Phi(s_2, s_1)\| \right) \leq K < \infty$.[5]

(ii) $\int_0^t \|\Phi(t,s)\| b(s) ds \leq \alpha < 1$ *for all* $t \geq 0$.

(iii) $|g(t,\phi) - g(t,\psi)| \leq b(t)\|\phi - \psi\|$ *for all* $\phi, \psi \in C(L)$*, and* $g(t,0) = 0$.

(iv) $\forall \epsilon > 0$ *and* $t_1 \geq 0$ *given, there exists a* $t_2 > t_1$ *such that* $t \geq t_2$ *and* $x_t \in C(L)$ *imply*

$$|g(t, x_t)| \leq b(t) \Big(\epsilon + \|x\|^{[t_1, t]} \Big). \tag{5.23}$$

(v) $\|\Phi(t,0)\| \longrightarrow 0$ *as* $t \to \infty$.

Then zero solution of (5.21) *is uniformly stable and asymptotically stable.*

Proof. We show that if

$$\delta_0 < \frac{(1-\alpha)}{K} L, \tag{5.24}$$

then for an initial condition[6] $\|\phi\| \leq \delta_0$, the zero solution of (5.21) is uniformly stable and asymptotically stable.

Let $\phi \in C(\delta_0)$ fixed, so that we have an initial value problem for (5.21) through $(t_0, \phi) \in \mathbb{R}_+ \times BC[-r, 0]$. With this ϕ, set

$$S := \big\{ x : \mathbb{R} \longrightarrow \mathbb{R}^n \mid x_{t_0} = \phi, x_t \in C(L) \text{ for } t \geq t_0 \text{ and } x(t) \to 0 \text{ as } t \to \infty \big\}. \tag{5.25}$$

[5] Notice that $\Phi(t,t) = I_d$ for every t implies $K \geq 1$.

[6] Notice that $\delta_0 < L$ since $K \geq 1$ and $1 - \alpha < 1$.

S is a complete metric space under the metric

$$\rho(x, y) := \sup_{t \geq t_0} |x(t) - y(t)|.$$

Note that in using this metric on S, we are not interested in what happens before t_0, since $x_{t_0} = \phi = y_{t_0}$ by definition of S. It is easy to see that S is nonempty, and that the variation of parameters formula in this case is

$$x(t) = \Phi(t, t_0)\phi(0) + \int_{t_0}^{t} \Phi(t, s) g(s, x_s) ds. \tag{5.26}$$

We thus have an equivalent integral expression for the solution $x(t)$. This suggests that we define the following mapping P defined on S:

$$(Px)(t) := \begin{cases} \phi(t - t_0) & \text{if } t \leq t_0 \\ \Phi(t, t_0)\phi(0) + \int_{t_0}^{t} \Phi(t, s) g(s, x_s) ds & \text{if } t \geq t_0 \end{cases} \tag{5.27}$$

In order to apply the Banach fixed point theorem, we need to prove that P maps S to itself. Clearly $Px : \mathbb{R} \longrightarrow \mathbb{R}^n$ is continuous, and by definition $(Px)_{t_0} = \phi$. Let us prove that $(Px)_t \in C(L)$ for $t \geq t_0$: Using the fact that $\phi \in C(\delta_0)$ along with conditions (ii), (iii) and that $\|x_s\| \leq L \; \forall s$, we have that

$$\begin{aligned} |(Px)(t)| &\leq \|\Phi(t, t_0)\| |\phi(0)| + \int_{t_0}^{t} \|\Phi(t, s)\| |g(s, x_s)| ds \\ &\leq \delta_0 K + \int_{t_0}^{t} \|\Phi(t, s)\| b(s) \|x_s\| ds \\ &\leq \delta_0 K + L \int_{t_0}^{t} \|\Phi(t, s)\| b(s) ds \\ &\leq \delta_0 K + L\alpha \leq L, \end{aligned}$$

by the choice of δ_0. This shows that $(Px)_t \in C(L)$ for $t \geq t_0$, in other words, the solution is bounded by L.

Now we show that $(Px)(t) \to 0$ as $t \to \infty$. Since $x(t) \to 0$ as $t \to \infty$, given $\epsilon > 0$ there exists a $t_1 > t_0$ such that $|x(t)| < \epsilon$ for all $t \geq t_1$. Since $|x(t)| \leq L$ for all $t \in \mathbb{R}$, by condition (iv) there exists $t_2 > t_1$ such that $t \geq t_2$ implies

$$|g(t, x_t)| \leq b(t)\Big(\epsilon + \|x\|^{[t_1, t]}\Big).$$

For $t \geq t_2$ we have:

$$
\begin{aligned}
&\left|\int_{t_0}^{t} \Phi(t,s)g(s,x_s)ds\right| \\
&\leq \int_{t_0}^{t_2} \|\Phi(t,s)\||g(s,x_s)|ds + \int_{t_2}^{t} \|\Phi(t,s)\||g(s,x_s)|ds \\
&\leq \int_{t_0}^{t_2} \|\Phi(t,s)\|b(s)\|x_s\|ds + \int_{t_2}^{t} \|\Phi(t,s)\|b(s)\Big(\epsilon + \underbrace{\|x\|^{[t_1,t]}}_{\leq \epsilon \text{ since } t\geq t_2}\Big)ds \\
&\leq \|x_t\| \int_{t_0}^{t_2} \|\Phi(t,s)\|b(s)ds + \int_{t_2}^{t} \|\Phi(t,s)\|b(s)(2\epsilon)ds \\
&\leq L \int_{t_0}^{t_2} \|\Phi(t,t_2)\Phi(t_2,s)\|b(s)ds + 2\epsilon\alpha \\
&\leq L\|\Phi(t,t_2)\| \int_{t_0}^{t_2} \|\Phi(t_2,s)\|b(s)ds + 2\epsilon\alpha \\
&\leq \alpha L\|\Phi(t,t_2)\| + 2\alpha\epsilon
\end{aligned}
$$

By (v) there exists $t_3 > t_2$ such that[7] if $t \geq t_3$

$$\|\Phi(t,t_0)\|\delta_0 + \alpha L\|\Phi(t,t_2)\| < \epsilon$$

The previous two estimates yield that for $t \geq t_3$:

$$
\begin{aligned}
|(Px)(t)| &= \left|\Phi(t,t_0)\phi(0) + \int_{t_0}^{t} \Phi(t,s)g(s,x_s)ds\right| \\
&\leq \|\Phi(t,t_0)\|\delta_0 + \alpha L\|\Phi(t,t_2)\| + 2\alpha\epsilon < 3\epsilon.
\end{aligned}
$$

This proves that $(Px)(t) \xrightarrow[\infty]{t} 0$. This proves that $Px \in S$ for every $x \in S$. This implies that $P : S \longrightarrow S$ is well defined. To prove that P is a contraction on S is straightforward, since for $x, y \in S$:

$$
\begin{aligned}
|(Px)(t) - (Py)(t)| &\leq \int_{t_0}^{t} \|\Phi(t,s)\||g(s,x_s) - g(s,y_s)|ds \\
&\leq \int_{t_0}^{t} \|\Phi(t,s)\|b(s)\|x_s - y_s\|ds \\
&\leq \sup_{s\geq t_0} |x(s) - y(s)| \int_{t_0}^{t} \|\Phi(t,s)\|b(s)ds \\
&\leq \alpha\rho(x,y).
\end{aligned}
$$

[7] $\|\Phi(t,t_2)\| \longrightarrow 0$ because otherwise, suppose that the operator $\Phi(t,t_2)$ does not decay to zero. Then $\|\Phi(t,0)\| = \|\Phi(t,t_2)\Phi(t_2,0)\|$ cannot decay to the zero operator, contrary to the supposition.

where the last inequality follows from the definition of S and the metric that we defined there.

By the contraction mapping theorem there exists a unique fixed point $x \in S$, which solves (5.21), for each $\phi \in C(\delta_0)$, and by definition of S we have that

$$x(t) = x(t, t_0, \phi) \xrightarrow[\infty]{t} 0.$$

In order to prove asymptotic stability, since we already proved that the solution $x(t, t_0, \phi)$ converges to zero for $\|\phi\| \leq \delta_0$, what is left to prove in order to conclude asymptotic stability is that the solution is stable. Let $\epsilon > 0$, $\epsilon < L$ be given. We will find a $\delta < \epsilon$ such that $\delta K + \alpha\epsilon < \epsilon$, so that $\delta < \min\{\epsilon, \frac{\epsilon}{K}(1-\alpha)\}$. If $x(t, t_0, \phi)$ is a solution with $\|\phi\| < \delta$, then we prove that $|x(t)| < \epsilon$ for all $t \geq t_0$. Notice that $|x(t_0)| < \delta < \epsilon$. Suppose for the sake of contradiction that there exists $t^* > t_0$ such that $|x(s)| < \epsilon$ for $t_0 \leq s < t^*$ but $|x(t^*)| = \epsilon$. Notice $\epsilon < L$ allows application of the Lipschitz-type bounds. Then

$$\begin{aligned}|x(t^*)| &< \delta\|\Phi(t^*, t_0)\| + \int_{t_0}^{t^*} \|\Phi(t^*, s)\| b(s) \|x_s\| ds \\ &\leq \delta K + \alpha\epsilon < \epsilon,\end{aligned}$$

which contradicts the definition of t^*. Thus no such t^* exists, and so $|x(t)| < \epsilon$ for all $t \geq t_0$. Thus, the zero solution of (5.21) is asymptotically stable.

□

Remark 5.7. *We have the same uniqueness comments as in Section 5.4, and we can obtain a result similar to Lemma 5.1, and corresponding uniqueness result as in Proposition 5.1. We only obtain Lemma 5.3, to illustrate the same principle holds.*

Lemma 5.3. *Under the hypotheses stated in Theorem 5.4, we have that if* $\sup_{s_2 \geq s_1 \geq 0} \left(\|\Phi(s_2, s_1)\|\right) \leq K < \infty$ *then the solutions of* (5.21) *with initial condition* $\|\phi\| < \delta_0 := \dfrac{(1-\alpha)}{K} L$ *remain bounded*[8] *by* L*, i.e.,* $|x(t)| \leq L$ *for every* $t \geq t_0$*.*

Proof. The proof is completely similar to the way in which we prove stability of the solution in Theorem 5.4, with the role of ϵ played by L this time.

For $\|\phi\| < \delta_0$, we claim that the solution $x(t)$ satisfies $|x(t)| \leq L$ for all $t \geq t_0$. Note that if x solves the FDE corresponding to the initial condition ϕ, then $|x(t_0)| = |\phi(0)| < L$. For the sake of contradiction suppose that there exists a $\hat{t} > t_0$ such that $|x(\hat{t})| > L$. Let

$$t^* = \inf\{\hat{t} : |x(\hat{t})| > L\}.$$

Now, by continuity, and by definition of t^*: $|x(t^*)| = L$. We thus have $|x(s)| \leq L$ for $s \in [t_0 - r, t^*]$. By the integral representation of $x(t)$, which all solutions to (5.3) satisfy with initial

[8] Note $\delta_0 < L$.

condition ϕ, we have that, since before t^* the paths are bounded by L, we can apply the Lipschitz condition (iii), so that

$$\begin{aligned} |x(t^*)| &\leq \|\Phi(t^*,t_0)\||\phi(0)| + \int_{t_0}^{t^*} \|\Phi(t^*,s)\||g(s,x_s)|ds \\ &< \delta_0 K + \int_{t_0}^{t^*} \|\Phi(t^*,s)\|b(s)\|x_s\|ds \\ &\leq \delta_0 K + \sup_{\theta\in[t_0-r,t^*]} |x(\theta)| \left(\int_{t_0}^{t^*} \|\Phi(t^*,s)\|b(s)ds\right) \\ &\leq \delta_0 K + \alpha L = L. \end{aligned}$$

Thus we have that $|x(t^*)| < L$, contradicting the definition of t^*. □

We can now prove easily that solutions to FDE (5.21) are unique, following verbatim the argument in Proposition 5.1.

In the following chapters, we will obtain asymptotic stability results using the Banach contraction principle. We will consider different cases, such as discontinuous systems, also systems that are not well behaved as the ones here, and finally switched systems. Nonetheless, many of the considerations from this chapter will carry on to the different cases that we study.

Chapter 6

Stability by Banach Contraction in System of Impulsive FDEs

6.1 Introduction

We will now consider the perturbed version of the main results of Chapter 5, namely, we will perturb with jumps that give rise to discontinuous FDEs. We thus generalize the previous result to a theoretical framework of impulsive FDEs, and in doing so we will obtain some insight into how difficult it may be to fit in these "harmless" perturbations of the previous result. Harmless in the sense that they do not break the contraction requirement of the previous result by [57]. We will notice that we also obtain global existence of solutions as a by-product, just like Lyapunov stability methods can do. The fixed point method here gives a global existence and uniqueness result, whereas existence results such as those of [6, 39] give local existence and uniqueness. Determining global existence is not a trivial matter for impulsive DEs, as is illustrated in [47, 41]. Although perhaps, it is not a surprise that in both of these aforementioned results, fixed point methods are used in order to prove the existence of global solutions. However, instead of the Banach contraction principle, which we shall use, the aforesaid papers use the fixed point theorem by Schaefer, which does not necessarily conclude unique solutions.

6.2 Preliminaries

Using fixed point theory, conditions for stability of the impulsive delayed differential equation

$$\begin{aligned} x'(t) &= A(t)x(t) + g(t, x_t), \qquad && t \neq \tau_k, t \geq 0 \\ \Delta x(t) &= I(t, x_{t^-}), && t = \tau_k, t \geq 0 \end{aligned} \tag{6.1}$$

are given. Here, we have that $x(t) \in \mathbb{R}^n$, $g, I : J \times PCB([-r, 0], D) \longrightarrow \mathbb{R}^n$ with $J \subset \mathbb{R}_+$ an interval, $D \subset \mathbb{R}^n$ an open set and $\Delta x(t) = x(t) - x(t^-)$. The impulse times τ_k satisfy $0 = \tau_0 < \tau_1 < \cdots$ and $\lim_{k\to\infty} \tau_k = \infty$. $A(t)$ is an $n \times n$ continuous matrix function, in the sense that each entry of A is a continuous function in the interval of definition of the functional differential equation (6.1). We state and explain the conditions on system (6.1) in the paragraphs below.

As in the convention used in Ballinger & Liu [6], we do not ask for the jump condition in (6.1) to be satisfied at t_0, since this imposes an unnecessary restriction on the initial condition.

Remark 6.1. *In the case when $r = \infty$, we still denote the space $PCB(-\infty, 0]$ by the notation $PCB[-r, 0]$, by considering for this special case $[-r, 0]$ to mean the* infinite *interval $(-\infty, 0]$, and using the piecewise continuous bounded functions on $(-\infty, 0]$. Of course, $PCB[-r, 0] = PC[-r, 0]$ when $r < \infty$.*

By x_{t^-} in (6.1) we refer to the function defined by a given $x \in PCB([t_0 - r, b], D)$ through the assignment

$$x_{t^-}(s) = x_t(s) \quad \text{for } s \in [-r, 0)$$
$$x_{t^-}(0) = \lim_{u \to t^-} x(u) = x(t^-).$$

This is a way of getting a well defined function in $PCB[-r, 0]$, that takes into account only the information available right until before the jump occurs right at $t = \tau_k$. In this way, the mapping I induces a jump from $x(t^-)$ to a value $x(t)$, using the information available until just before the impulse occurs at time t.

The norm that we use on $PCB([-r, 0], D)$ will be

$$\|\psi\|_r := \sup_{s \in [-r, 0]} |\psi(s)|,$$

where of course for $r = \infty$ this norm is $\|\psi\|_r = \sup_{s \in (-\infty, 0]} |\psi(s)|$. Wherever the norm symbol $\|\cdot\|$ is used, **we refer to the norm on** $PCB([-r, 0], D)$. We will denote the Euclidean norm by $|x|$ whenever no confusion should arise.

The initial condition for equation (6.1) will be given for $t_0 \geq 0$ as

$$x_{t_0} = \phi \tag{6.2}$$

for $t_0 \in J$, and $\phi \in PCB([-r, 0], D)$. For stability analysis, it is assumed that $0 \in D$, $J = \mathbb{R}^+, g(t, 0) = I(t, 0) = 0$ for all $t \in \mathbb{R}^+$. This guarantees that system (6.1) has a trivial solution $x(t) = 0$.

6.3 Main Results

In order for the necessary integrals to exist (namely those of nonlinear part g), we will assume that g is composite-PC. The precise definition is given as

Definition 6.1. A mapping $g : J \times PCB([-r, 0], D) \longrightarrow \mathbb{R}^n$, where $0 \leq r \leq \infty$, is said to be *composite-PCB* if for each $t_0 \in J$ and $\beta > 0$ where $[t_0, t_0+\beta] \subset J$, if $x \in PCB([t_0-r, t_0+\beta], D)$, and x is continuous at each $t \neq \tau_k$ in $(t_0, t_0 + \beta]$ then the composite function $t \mapsto g(t, x_t)$ is an element of the function class $PCB([t_0, t_0 + \beta], \mathbb{R}^n)$.

Let us define

$$\|\psi\|^{[s,t]} := \sup_{u\in[s,t]} |\psi(u)|.$$

Remark 6.2. *We denote by $B(L) \subset PCB[-r,0]$ the closed ball of radius L in $PCB[-r,0]$:*

$$B(L) = \{\psi \in PCB[-r,0] : \|\psi\|_r \leq L\}.$$

6.3.1 One-dimensional case.

We will first focus on the scalar version of (6.1), because this gives us insight into what we need for the vector version.

For the special case of one dimension, in order to agree with common notational convention for the scalar case[1], we let $A(t) = -a(t)$ be a continuous function, so that the equation is of the form

$$\begin{aligned} x'(t) &= -a(t)x(t) + g(t,x_t), \qquad && t \neq \tau_k, t \geq 0 \\ \Delta x(t) &= I(t, x_{t^-}), && t = \tau_k, t \geq 0 \end{aligned} \tag{6.3}$$

The main result in scalar form is the following. We remind the reader that $g(t,0) = I(t,0) = 0$, and that the fading type memory condition for infinitely delayed impulsive FDEs is expressed in condition (6.14), which holds automatically for finite delays $r < \infty$, as mentioned in the previous chapter, in Lemma 5.2. Notice we have the decaying memory condition on the jump functional as well.

Theorem 6.1. *Suppose that there exists positive constants α, L and continuous functions $b, c : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that the following conditions hold:*

(i) For all $s_2 \geq s_1 \in [0,\infty)$, let us have the uniform bound $e^{-\int_{s_1}^{s_2} a(s)ds} \leq K < \infty$, in other words let $\sup_{s_2\geq s_1\geq 0}\left(e^{-\int_{s_1}^{s_2} a(s)ds}\right) \leq K < \infty$.[2]

(ii) $|g(t,\phi) - g(t,\psi)| \leq b(t)\|\phi - \psi\|$ for all $\phi,\psi \in B(L)$, and $g(t,0) = 0$.

(iii) $|I(t,\phi) - I(t,\psi)| \leq c(t)\|\phi - \psi\|$ for all $\phi,\psi \in B(L)$, and $I(t,0) = 0$.

(iv) For all $t \geq 0$

$$\int_0^t e^{-\int_s^t a(u)du} b(s)ds + \sum_{0<\tau_k\leq t} c(\tau_k)e^{-\int_{\tau_k}^t a(u)du} \leq \alpha < 1.$$

(v) For every $\epsilon > 0$ and $t_1 \geq 0$, there exists a $t_2 > t_1$ such that $t \geq t_2$ and $x_t \in B(L)$ implies

$$\begin{aligned} |g(t,x_t)| &\leq b(t)\left(\epsilon + \|x\|^{[t_1,t]}\right) \\ |I(t,x_{t^-})| &\leq c(t)\left(\epsilon + \|x\|^{[t_1,t]}\right). \end{aligned} \tag{6.4}$$

[1] Such as in [20, 24, 57].

[2] Notice that $e^{-\int_t^t a(s)ds} = 1$ for every t implies $K \geq 1$. From the previous chapter, we already have a way to determine a candidate for a K independent of t_0, see Remark 5.4.

(vi) $\int_0^t a(s)ds \longrightarrow \infty$ *as* $t \to \infty$.

Then zero solution of (6.3) *is uniformly stable and asymptotically stable.*

Proof. We show that if

$$\delta_0 < \frac{(1-\alpha)}{K} L, \tag{6.5}$$

then for an initial condition[3] $\|\phi\| \leq \delta_0$, the zero solution of (6.3) is uniformly stable and asymptotically stable. For any initial condition ϕ let us define, using the *fixed* impulse moments $\{\tau_k\}$ that define the impulsive operator, the space

$$\begin{aligned}\mathcal{S} = \{x \in PCB([t_0 - r, \infty), D) &: x_{t_0} = \phi,\ x_t \in B(L) \text{ for } t \geq t_0,\\ &x \text{ is discontinuous only at impulsive moments } t = \tau_k, \text{ and } x(t) \to 0 \text{ as } t \to \infty\}.\end{aligned}$$

$\mathcal{S}$ is a nonempty complete metric space under the metric[4].

To obtain a mapping suitable for the Banach fixed point method, we make the following observation. For $s \in [\tau_{k-1}, \tau_k)$, we have that, using the differential equation (6.3):

$$x'(s)e^{\int_{\tau_{k-1}}^s a(u)du} + a(s)x(s)e^{\int_{\tau_{k-1}}^s a(u)du} = \frac{d}{ds}\left(x(s)e^{\int_{\tau_{k-1}}^s a(u)du}\right) = g(s, x_s)e^{\int_{\tau_{k-1}}^s a(u)du}.$$

This implies that

$$\begin{aligned}x(t) &= e^{-\int_{\tau_{k-1}}^t a(u)du} x(\tau_{k-1}) + e^{-\int_{\tau_{k-1}}^t a(u)du} \int_{\tau_{k-1}}^t g(s, x_s) e^{\int_{\tau_{k-1}}^s a(u)du} ds\\ &= \left[x(\tau_{k-1}^-) + I\left(\tau_{k-1}, x_{\tau_{k-1}^-}\right)\right] e^{-\int_{\tau_{k-1}}^t a(u)du} + \int_{\tau_{k-1}}^t g(s, x_s) e^{-\int_s^t a(u)du} ds.\end{aligned}$$

Thus, for $t \in [\tau_{k-1}, \tau_k)$

$$\begin{aligned}x(t) = x(\tau_{k-1}^-)e^{-\int_{\tau_{k-1}}^t a(u)du} &+ \int_{\tau_{k-1}}^t e^{-\int_s^t a(u)du} g(s, x_s) ds\\ &+ I\left(\tau_{k-1}, x_{\tau_{k-1}^-}\right) e^{-\int_{\tau_{k-1}}^t a(u)du}.\end{aligned} \tag{6.6}$$

[3]Notice that $\delta_0 < L$ since $K \geq 1$ and $1 - \alpha < 1$.

[4]The space is complete because we have *fixed* discontinuity moments τ_k, and the functions are bounded (uniformly bounded by L), and convergent to zero. To prove completeness is similar to the way we prove completeness of the bounded functions defined on a metric space: We can define a Cauchy sequence in $\mathbb{R}$, and use this sequence to define a pointwise limit function, which we prove is the uniform limit. The key point is that the discontinuities are fixed, so by focusing on the fixed impulsive moments that the impulsive FDE defines, we can build appropriate Cauchy sequences at these points, since the corresponding limits at discontinuities are well defined por PCB-spaces. Boundedness by L and and convergence to zero are also immediate, since if the limit function does not satisfy these properties, there cannot be uniform convergence of the sequence of functions.

We stress that this formula holds for $t \in [\tau_{k-1}, \tau_k)$ only, but by backstepping we can express $x(t_{k-1}^-)$ using the analogous formula to (6.16) but for $t \in [\tau_{k-2}, \tau_{k-1})$, since $x(\tau_{k-1}^-)$ uses the expression for $x(t)$ for $t \in [\tau_{k-2}, \tau_{k-1})$, as $t \to \tau_{k-1}^-$. Backstepping in this way we get:

$$\begin{aligned}
x(\tau_{k-1}^-) &= x(\tau_{k-2}^-)e^{-\int_{\tau_{k-2}}^{\tau_{k-1}} a(u)du} + \int_{\tau_{k-2}}^{\tau_{k-1}} e^{-\int_s^{\tau_{k-1}} a(u)du} g(s, x_s)ds + I\left(\tau_{k-2}, x_{\tau_{k-2}^-}\right) e^{-\int_{\tau_{k-2}}^{\tau_{k-1}} a(u)du} \\
&\vdots \\
x(\tau_2^-) &= x(\tau_1^-)e^{-\int_{\tau_1}^{\tau_2} a(u)du} + \int_{\tau_1}^{\tau_2} e^{-\int_s^{\tau_2} a(u)du} g(s, x_s)ds + I\left(\tau_1, x_{\tau_1^-}\right) e^{-\int_{\tau_1}^{\tau_2} a(u)du} \\
x(\tau_1^-) &= \phi(0)e^{-\int_{t_0}^{\tau_1} a(u)du} + \int_{t_0}^{\tau_1} e^{-\int_s^{\tau_1} a(u)du} g(s, x_s)ds
\end{aligned}$$

By recursive substitution into (6.16) we get that in general, the solution $x(t)$ must satisfy:

$$x(t) = \phi(0)e^{-\int_{t_0}^{t} a(u)du} + \int_{t_0}^{t} e^{-\int_s^{t} a(u)du} g(s, x_s)ds + \sum_{t_0 < \tau_k \le t} I\left(\tau_k, x_{\tau_k^-}\right) e^{-\int_{\tau_k}^{t} a(u)du}.$$

This makes us define the mapping P by

$$(Px)(t) = \phi(0)e^{-\int_{t_0}^{t} a(u)du} + \int_{t_0}^{t} e^{-\int_s^{t} a(u)du} g(s, x_s)ds + \sum_{t_0 < \tau_k \le t} I\left(\tau_k, x_{\tau_k^-}\right) e^{-\int_{\tau_k}^{t} a(u)du}. \tag{6.7}$$

To prove that P defines a contraction mapping on $\mathcal{S}$, we must prove first that P maps $\mathcal{S}$ to itself.

Clearly, Px has left limits well defined, since $\sum_{t_0<\tau_k\le t} I\left(\tau_k, x_{\tau_k^-}\right) e^{-\int_{\tau_k}^{t} a(u)du}$ has limit from the left, since $e^{-\int_{\tau_k}^{t} a(u)du}$ is continuous and each $I\left(\tau_k, x_{\tau_k^-}\right)$ is well defined thanks to x having limit from the left at each τ_k. Clearly the

$$\phi(0)e^{-\int_{t_0}^{t} a(u)du} + \int_{t_0}^{t} e^{-\int_s^{t} a(u)du} g(s, x_s)ds$$

part has well defined limits, since this part is even continuous at τ_l, by continuity of the Riemann integral. Right continuity at each impulse moment τ_l is reduced to verifying right continuity of

$$Q(t) := \sum_{t_0 < \tau_k \le t} I\left(\tau_k, x_{\tau_k^-}\right) e^{-\int_{\tau_k}^{t} a(u)du}$$

at τ_l. Choose $\eta > 0$ small enough such that $\tau_l + \eta < \tau_m$ for any $m > l$. Then

$$Q(\tau_l + \eta) - Q(\tau_l) =$$

$$\sum_{t_0<\tau_k\le\tau_l+\eta} I\left(\tau_k, x_{\tau_k^-}\right) e^{-\int_{\tau_k}^{\tau_l+\eta} a(u)du} - \sum_{t_0<\tau_k\le\tau_l} I\left(\tau_k, x_{\tau_k^-}\right) e^{-\int_{\tau_k}^{\tau_l} a(u)du}$$
$$= \sum_{t_0<\tau_k\le\tau_l} I\left(\tau_k, x_{\tau_k^-}\right) \left[e^{-\int_{\tau_k}^{\tau_l+\eta} a(u)du} - e^{-\int_{\tau_k}^{\tau_l} a(u)du}\right] \xrightarrow{\eta\to 0} 0$$

where we note that both sums have the same number of elements, due to $\tau_l + \eta < \tau_m$ for any $m > l$. Therefore for each $x \in \mathcal{S}$, we have that Px is right continuous and has left limits at impulse times, clearly it is continuous at nonimpulsive moments.

By definition of $\mathcal{S}$, we have that $(Px)_{t_0} = \phi$. We must show that $|(Px)(t)| \le L$ for every $t \ge 0$. We remind the reader that $\|\phi\| \le \delta_0$, with δ_0 as defined in (6.5). We claim that $|(Px)(t)| \le L$ for all $t \ge t_0$. We have that, since $|x(s)| \le L$ by definition of $\mathcal{S}$, so that the Lipschitz properties $(ii), (iii)$ hold, then

$$\begin{aligned}|(Px)(t)| &\le |\phi(0)|e^{-\int_{t_0}^t a(u)du} + \int_{t_0}^t e^{-\int_s^t a(u)du}|g(s,x_s)|ds + \sum_{t_0<\tau_k\le t} |I\left(\tau_k, x_{\tau_k^-}\right)|e^{-\int_{\tau_k}^t a(u)du} \\ &\le \delta_0 e^{-\int_{t_0}^t a(u)du} + \int_{t_0}^t e^{-\int_s^t a(u)du} b(s)|\|x_s\|ds + \sum_{t_0<\tau_k\le t} c(\tau_k)e^{-\int_{\tau_k}^t a(u)du}\|x_{\tau_k^-}\| \\ &\le \delta_0 K + \sup_{\theta\in[t_0-r,t]} |x(\theta)| \left(\int_{t_0}^t e^{-\int_s^t a(u)du} b(s)ds + \sum_{t_0<\tau_k\le t} c(\tau_k)e^{-\int_{\tau_k}^t a(u)du}|\right) \\ &\le \delta_0 K + \alpha L < L.\end{aligned}$$

Thus $|(Px)(t)| \le L$ for every $t \ge 0$.

Now we show what $(Px)(t) \longrightarrow 0$ as $t \to \infty$.

For this, note that we can divide Px into

$$(Px)(t) = (P_1x)(t) + (P_2x)(t)$$

with $(P_1x)(t) = \phi(0)e^{-\int_{t_0}^t a(u)du} + \sum_{t_0<\tau_k\le t} I\left(\tau_k, x_{\tau_k^-}\right) e^{-\int_{\tau_k}^t a(u)du}$ and
$(P_2x)(t) = \int_{t_0}^t e^{-\int_s^t a(u)du} g(s, x_s)ds$.

By definition of $\mathcal{S}$, $x(t) \longrightarrow 0$ as $t \to \infty$. Thus we have that for any $\epsilon > 0$ there exists $T_1 > t_0$ such that

$$|x(t)| < \epsilon \qquad \text{for all } \; t \ge T_1. \tag{6.8}$$

By hypothesis (v), given this ϵ and T_1, there exists $T_2 > T_1$ such that $t \ge T_2$ implies

$$\begin{aligned}|g(t, x_t)| &\le b(t)\left(\epsilon + \|x\|^{[T_1,t]}\right) \\ |I(t, x_{t^-})| &\le c(t)\left(\epsilon + \|x\|^{[T_1,t]}\right)\end{aligned} \tag{6.9}$$

Let us first analyze the term $(P_2x)(t)$. If $s \geq T_2 > T_1$, by (6.8) we get

$$\|x\|^{[T_1,s]} < \epsilon. \tag{6.10}$$

By definition of $\mathcal{S}$, $\|x_t\| \leq L$ for all $t \geq t_0$, $x \in \mathcal{S}$, and using the first inequality in (6.9) and inequality (6.10), we obtain that for $t > T_2$:

$$\begin{aligned}
|(P_2x)(t)| &= \left| \int_{t_0}^{t} e^{-\int_s^t a(u)du} g(s, x_s) ds \right| \\
&\leq \int_{t_0}^{T_2} e^{-\int_s^t a(u)du} |g(s, x_s)| ds + \int_{T_2}^{t} e^{-\int_s^t a(u)du} |g(s, x_s)| ds \\
&\leq \int_{t_0}^{T_2} e^{-\int_s^t a(u)du} |g(s, x_s)| ds + \int_{T_2}^{t} e^{-\int_s^t a(u)du} |g(s, x_s)| ds \\
&\leq \int_{t_0}^{T_2} e^{-\int_s^t a(u)du} b(s) \|x_s\| ds + \int_{T_2}^{t} e^{-\int_s^t a(u)du} b(s) \left(\epsilon + \|x\|^{[T_1,t]}\right) ds \\
&\leq L \int_{t_0}^{T_2} e^{-\int_s^t a(u)du} b(s) ds + \int_{T_2}^{t} e^{-\int_s^t a(u)du} b(s)(2\epsilon) ds \\
&= L e^{-\int_{T_2}^t a(u)du} \int_{t_0}^{T_2} e^{-\int_s^{T_2} a(u)du} b(s) ds + 2\epsilon \int_{T_2}^{t} e^{-\int_s^t a(u)du} b(s) ds \\
&\leq \alpha L e^{-\int_{T_2}^t a(u)du} + 2\alpha\epsilon
\end{aligned} \tag{6.11}$$

Since we have assumed that $e^{-\int_0^t a(u)du} \longrightarrow \infty$ as $t \to \infty$, we see that given ϵ we can find $T > T_2$ such that

$$\alpha L e^{-\int_{T_2}^t a(u)du} < \epsilon \quad \text{for} \;\; t \geq T.$$

Substituting this last inequality into (6.11), we get that for $t > T$

$$|(P_2x)(t)| \leq \epsilon + 2\alpha\epsilon = \epsilon(1 + 2\alpha)$$

This proves that $(P_2x)(t) \longrightarrow 0$ as $t \to \infty$.

We now prove $(P_1x)(t) \longrightarrow 0$ as $t \to \infty$. It is similar to the way we proved this for P_2. Notice that using (6.9), (6.10) and (v) we have that for $t > T_2$:

$$\left| \sum_{t_0 < \tau_k \leq t} e^{-\int_{\tau_k}^t a(u)du} I\left(\tau_k, x_{\tau_k^-}\right) \right|$$

$$\begin{aligned}
&\le \sum_{t_0<\tau_k\le T_2} e^{-\int_{\tau_k}^t a(u)du} \left| I\left(\tau_k, x_{\tau_k^-}\right)\right| + \sum_{T_2<\tau_k\le t} e^{-\int_{\tau_k}^t a(u)du} \left| I\left(\tau_k, x_{\tau_k^-}\right)\right| \\
&\le \sum_{t_0<\tau_k\le T_2} c(\tau_k) e^{-\int_{\tau_k}^t a(u)du} \left\| x_{\tau_k^-}\right\| + \sum_{T_2<\tau_k\le t} c(\tau_k) e^{-\int_{\tau_k}^t a(u)du} \left(\epsilon + \|x\|^{[T_1,\tau_k]}\right) \\
&= e^{-\int_{T_2}^t a(u)du} \sum_{t_0<\tau_k\le T_2} c(\tau_k) e^{-\int_{\tau_k}^{T_2} a(u)du} \left\| x_{\tau_k^-}\right\| + \sum_{T_2<\tau_k\le t} c(\tau_k) e^{-\int_{\tau_k}^t a(u)du} \left(\epsilon + \|x\|^{[T_1,\tau_k]}\right) \\
&\le L e^{-\int_{T_2}^t a(u)du} \sum_{t_0<\tau_k\le T_2} c(\tau_k) e^{-\int_{\tau_k}^{T_2} a(u)du} + 2\epsilon \sum_{T_2<\tau_k\le t} c(\tau_k) e^{-\int_{\tau_k}^t a(u)du} \\
&\le \alpha L e^{-\int_{T_2}^t a(u)du} + 2\alpha\epsilon
\end{aligned}$$

In a similar way as we did for (P_2x), we can find some $T^* > T_2$, such that $t > T^*$ implies, adding the $e^{-\int_{t_0}^t a(u)du}\phi(0)$ term:

$$e^{-\int_{t_0}^t a(u)du}|\phi(0)| + \alpha L e^{-\int_{T_2}^t a(u)du} < \epsilon.$$

This proves $(P_1x)(t) \longrightarrow 0$ as $t \to \infty$. Therefore choosing $\max\{T, T^*\}$ we have $(Px)(t) \longrightarrow 0$ as $t \to \infty$.

Thus $P : \mathcal{S} \longrightarrow \mathcal{S}$. What is left to prove is that P is a contraction. For this, let $x, y \in \mathcal{S}$. By definition of $\mathcal{S}$ we have that $(Px)(t) - (Py)(t) = 0$ for $t \in [t_0 - r, t_0]$. For $t \ge t_0$ we get:

$$\begin{aligned}
&|(Px)(t) - (Py)(t)| = \\
&\left| \int_{t_0}^t e^{-\int_s^t a(u)du}[g(s,x_s) - g(s,y_s)]ds + \sum_{t_0<\tau_k\le t}\left[I\left(\tau_k, x_{\tau_k^-}\right) - I\left(\tau_k, y_{\tau_k^-}\right)\right] e^{-\int_{\tau_k}^t a(u)du}\right| \\
&\qquad \le \int_{t_0}^t e^{-\int_s^t a(u)du} b(s)\|x_s - y_s\| ds + \sum_{t_0<\tau_k\le t} c(\tau_k)\left\| x_{\tau_k^-} - y_{\tau_k^-}\right\| e^{-\int_{\tau_k}^t a(u)du} \\
&\qquad \le d_{\mathcal{S}}(x,y)\left(\int_{t_0}^t e^{-\int_s^t a(u)du} b(s)ds + \sum_{t_0<\tau_k\le t} c(\tau_k) e^{-\int_{\tau_k}^t a(u)du}\right) \le \alpha d_{\mathcal{S}}(x,y)
\end{aligned}$$

where recall that the norm $\|\cdot\|$ denotes the norm on $PCB([-r,0],D)$, and $d_{\mathcal{S}}(x,y) = \sup_{s\in[t_0,\infty)} |x(s) - y(s)|$.

Thus P is a contraction on $\mathcal{S}$. This implies that there is a unique solution to (6.3) with initial condition ϕ.[5]

To prove uniform stability, assume that we are given an $\epsilon > 0$. Choose $\delta < \epsilon$ such that $\delta K + \alpha\epsilon < \epsilon$, in other words, $\delta < \min\{\epsilon, (1-\alpha)\epsilon/K\}$. Notice that K is independent of t_0, thus so is δ. This will give us uniform stability.

For $\|\phi\| \le \delta$, we claim that $|x(t)| \le \epsilon$ for all $t \ge t_0$. Note that if x is the unique solution

[5] See Section 6.4 below for a clarification about uniqueness.

corresponding to the initial condition ϕ, then $|x(t_0)| = |\phi(0)| < \epsilon$. For the sake of contradiction suppose that there exists a $\hat{t} > t_0$ such that $|x(\hat{t})| > \epsilon$. Let

$$t^* = \inf\{\hat{t} : |x(\hat{t})| > \epsilon\}.$$

By right continuity, either $|x(t^*)| = \epsilon$ if there is no impulsive moment at t^*, or $|x(t^*)| \geq \epsilon$ as a consequence of a jump at t^*. Whatever the case, we have $|x(s)| \leq \epsilon$ for $s \in [t_0 - r, t^*)$, where $|x(t^*)| = \epsilon$ if this occurs at a non-impulsive moment. By the integral representation of $x(t)$, we have that

$$\begin{aligned}
|x(t^*)| &= \left| \phi(0)e^{-\int_{t_0}^{t^*} a(u)du} + \int_{t_0}^{t^*} e^{-\int_s^{t^*} a(u)du} g(s, x_s)ds + \sum_{t_0<\tau_k\leq t^*} I\left(\tau_k, x_{\tau_k^-}\right) e^{-\int_{\tau_k}^{t^*} a(u)du} \right| \\
&\leq \delta e^{-\int_{t_0}^{t^*} a(u)du} + \int_{t_0}^{t^*} e^{-\int_s^{t^*} a(u)du} b(s)\|x_s\|ds + \sum_{t_0<\tau_k\leq t^*} c(\tau_k)e^{-\int_{\tau_k}^{t} a(u)du}\|x_{\tau_k^-}\| \\
&\leq \delta K + \sup_{\theta\in[t_0-r,t^*)} |x(\theta)| \left(\int_{t_0}^{t^*} e^{-\int_s^{t^*} a(u)du} b(s)ds + \sum_{t_0<\tau_k\leq t^*} c(\tau_k)e^{-\int_{\tau_k}^{t^*} a(u)du} \right) \\
&\leq \delta K + \alpha\epsilon < \epsilon
\end{aligned}$$

and this gives us the desired contradiction, by the definition of t^*. Therefore the solution is uniformly stable, and since $x(t)$ converges to zero as $t \to \infty$, we get uniform stability and asymptotic stability of trajectories. □

See Lemma 6.1 below, which we prove for a more general version of the previous result, and further comments in Section 6.4.

6.3.2 Vector Version

We will use the fundamental solution $\Phi(t, t_0)$ of the associated linear ordinary differential equation

$$\begin{aligned} y'(t) &= A(t)y(t) \\ y(t_0) &= y_0 \end{aligned} \tag{6.12}$$

such that the solution of IVP (6.12) is

$$y(t) = \Phi(t, t_0)y_0.$$

For a matrix M we use the standard linear operator norm induced by the Euclidean norm $|\cdot|$ on $\mathbb{R}^n$:

$$\|M\| := \|M\|_{\mathcal{L}(\mathbb{R}^n)} = \sup_{|y|=1} |My|.$$

We will use the inequality $|My| \leq \|M\||y|$ for $y \in \mathbb{R}^n$.
Notice that the previous result can be generalized to n-dimensional case by noticing that we have that for t_1, t_2 in the scalar case:

$$\Phi(t_2, t_1) = e^{-\int_{t_1}^{t_2} a(u)du}.$$

Therefore it follows that a way of determining that $\|\Phi(t,0)\| \longrightarrow 0$ as $t \to \infty$ is by observing that $\int_0^t a(s)ds \longrightarrow \infty$ as $t \to \infty$ is a sufficient condition.

Theorem 6.2. *Suppose that there exists positive constants α, L and continuous functions $b, c : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that the following conditions hold:*

(i) *For all $s_2 \geq s_1 \in [0,\infty)$, let us have the uniform bound $\|\Phi(s_2,s_1)\| \leq K < \infty$, in other words let $\sup_{s_2 \geq s_1 \geq 0} (\|\Phi(s_2,s_1)\|) \leq K < \infty$.*[6]

(ii) $|g(t,\phi) - g(t,\psi)| \leq b(t)\|\phi - \psi\|$ *for all $\phi, \psi \in B(L)$, and $g(t,0) = 0$.*

(iii) $|I(t,\phi) - I(t,\psi)| \leq c(t)\|\phi - \psi\|$ *for all $\phi, \psi \in B(L)$ and $I(t,0) = 0$.*

(iv) *For all $t \geq 0$*

$$\int_0^t b(s)\|\Phi(t,s)\|ds + \sum_{0<\tau_k \leq t} c(\tau_k)\|\Phi(t,\tau_k)\| \leq \alpha < 1. \tag{6.13}$$

(v) *For every $\epsilon > 0$ and $t_1 \geq 0$, there exists a $t_2 > t_1$ such that $t \geq t_2$ and $x_t \in B(L)$ implies*

$$\begin{aligned} |g(t,x_t)| &\leq b(t)\left(\epsilon + \|x\|^{[t_1,t]}\right) \\ |I(t,x_{t^-})| &\leq c(t)\left(\epsilon + \|x\|^{[t_1,t]}\right). \end{aligned} \tag{6.14}$$

(vi) $\|\Phi(t,0)\| \longrightarrow 0$ *as $t \to \infty$.*

Then zero solution of (6.1) *is uniformly stable and asymptotically stable.*

Proof. We show that if

$$\delta_0 < \frac{(1-\alpha)}{K}L, \tag{6.15}$$

then for an initial condition[7] $\|\phi\| \leq \delta_0$, the zero solution of (6.1) is uniformly stable and asymptotically stable.

For an initial condition $\|\phi\| \leq \delta_0$, let us define, using the *fixed* impulse moments $\{\tau_k\}$ that define the impulsive operator, the space

$$\begin{aligned} \mathcal{S} = \{x \in PCB([t_0 - r,\infty), D) : x_{t_0} = \phi,\ x_t \in B(L) \text{ for } t \geq t_0, \\ x \text{ is discontinuous only at impulsive moments } t = \tau_k, \text{ and } x(t) \to 0 \text{ as } t \to \infty\}. \end{aligned}$$

$\mathcal{S}$ is a nonempty complete metric space under the metric

$$d_{\mathcal{S}}(x,y) = \sup_{s \in [t_0 - r,\infty)} |x(s) - y(s)| = \sup_{s \in [t_0,\infty)} |x(s) - y(s)| \text{ for } x, y \in \mathcal{S},$$

[6]Notice that $\Phi(t,t) = I_d$ for every t implies $K \geq 1$. From the previous chapter, we already have a way to determine a candidate for a K independent of t_0, see Remark 5.4.

[7]Notice that $\delta_0 < L$ since $K \geq 1$ and $1 - \alpha < 1$.

where we note that we can disregard the contribution on the subinterval $[t_0 - r, t_0]$ because of the definition of $\mathcal{S}$, and we remind the reader that $[t_0 - r, t_0] = (-\infty, t_0]$ when $r = \infty$.

To obtain a mapping suitable for the Banach fixed point method, we make the following observation. For $s \in [\tau_{k-1}, \tau_k)$, we have that, using the fundamental matrix and the functional differential equation (6.1):

$$
\begin{aligned}
x(t) &= \Phi(t, \tau_{k-1})x(\tau_{k-1}) + \int_{\tau_{k-1}}^{t} \Phi(t,s)g(s,x_s)ds \\
&= \Phi(t, \tau_{k-1}) \left[x(\tau_{k-1}^-) + I\left(\tau_{k-1}, x_{\tau_{k-1}^-}\right)\right] + \int_{\tau_{k-1}}^{t} \Phi(t,s)g(s,x_s)ds
\end{aligned}
$$

Note that the necessary integrals will exist because $g(t, x_t)$ is composite-PCB as defined above.

The first line follows from variation of parameters for ordinary differential equations, as follows. Assume that a solution in the interval $s \in [\tau_{k-1}, \tau_k)$ is given by $x(t) = \Phi(t, \tau_k)m(t)$, where $m(t)$ is a differentiable vector valued function to be determined in the following fashion. By the product rule for differentiation we have that

$$
\begin{aligned}
x'(t) =& \Phi'(t, \tau_k)m(t) + \Phi(t, \tau_{k-1})m'(t) \\
=& A(t)\Phi(t, \tau_k)m(t) + \Phi(t, \tau_{k-1})m'(t)
\end{aligned}
$$

By the differential equation that $x(t)$ satisfies on $[\tau_{k-1}, \tau_k)$, this implies

$$A(t)\Phi(t, \tau_{k-1})m(t) + \Phi(t, \tau_{k-1})m'(t) = A(t)\Phi(t, \tau_{k-1})m(t) + g(t, x_t).$$

Thus

$$m'(t) = [\Phi(t, \tau_{k-1})]^{-1}g(t, x_t) = \Phi(\tau_{k-1}, t)g(t, x_t)$$

The previous expression implies, after integrating from τ_{k-1} to t and using $m(\tau_{k-1}) = x(\tau_{k-1})$ that

$$m(t) = x(\tau_{k-1}) + \int_{\tau_{k-1}}^{t} \Phi(\tau_{k-1}, s)g(s, x_t)$$

so that

$$x(t) = \Phi(t, \tau_{k-1})x(\tau_{k-1}) + \int_{\tau_{k-1}}^{t} \Phi(t,s)g(s,x_s)ds.$$

Thus, for $t \in [\tau_{k-1}, \tau_k)$, we obtain the formula

$$x(t) = \Phi(t, \tau_{k-1})x(\tau_{k-1}^-) + \int_{\tau_{k-1}}^{t} \Phi(t,s)g(s,x_s)ds + \Phi(t, \tau_{k-1})I\left(\tau_{k-1}, x_{\tau_{k-1}^-}\right). \tag{6.16}$$

We stress that this formula holds for $t \in [\tau_{k-1}, \tau_k)$ only, but by backstepping we can express $x(t_{k-1}^-)$ using the analogous formula to (6.16) but for $t \in [\tau_{k-2}, \tau_{k-1})$, since $x(\tau_{k-1}^-)$ uses the expression for $x(t)$ for $t \in [\tau_{k-2}, \tau_{k-1})$, as $t \to \tau_{k-1}^-$. Backstepping in this way we get:

$$\begin{aligned}
x(\tau_{k-1}^-) &= \Phi(\tau_{k-1}, \tau_{k-2})x(\tau_{k-2}^-) + \int_{\tau_{k-2}}^{\tau_{k-1}} \Phi(\tau_{k-1}, s)g(s, x_s)ds + \Phi(\tau_{k-1}, \tau_{k-2})I\left(\tau_{k-2}, x_{\tau_{k-2}^-}\right) \\
&\vdots \\
x(\tau_2^-) &= \Phi(\tau_2, \tau_1)x(\tau_1^-) + \int_{\tau_2}^{\tau_1} \Phi(\tau_2, s)g(s, x_s)ds + \Phi(\tau_2, \tau_1)I\left(\tau_1, x_{\tau_1^-}\right) \\
x(\tau_1^-) &= \Phi(\tau_1, t_0)\phi(0) + \int_{t_0}^{\tau_1} \Phi(\tau_1, s)g(s, x_s)ds,
\end{aligned}$$

where we remind ourselves that $x(t_0) = \phi(0)$ and $t_0 > 0 = \tau_0$. By recursive substitution into (6.16) we get that in general, the solution $x(t)$ must satisfy:

$$x(t) = \Phi(t, t_0)\phi(0) + \int_{t_0}^{t} \Phi(t, s)g(s, x_s)ds + \sum_{t_0 < \tau_k \leq t} \Phi(t, \tau_k)I\left(\tau_k, x_{\tau_k^-}\right)$$

This makes us define the mapping P by

$$(Px)_{t_0} = \phi,$$

and for $t \geq t_0$:

$$(Px)(t) = \Phi(t, t_0)\phi(0) + \int_{t_0}^{t} \Phi(t, s)g(s, x_s)ds + \sum_{t_0 < \tau_k \leq t} \Phi(t, \tau_k)I\left(\tau_k, x_{\tau_k^-}\right). \tag{6.17}$$

To prove that P defines a contraction mapping on $\mathcal{S}$, we must prove first that P maps $\mathcal{S}$ to itself.

Clearly, Px has left limits well defined, since $\sum_{t_0 < \tau_k \leq t} \Phi(t, \tau_k)I\left(\tau_k, x_{\tau_k^-}\right)$ has limit from the left, since $\Phi(t, \tau_k)$ is continuous and each $I\left(\tau_k, x_{\tau_k^-}\right)$ is well defined thanks to x having limit from the left at each τ_k. Clearly the term

$$\Phi(t, t_0)\phi(0) + \int_{t_0}^{t} \Phi(t, s)g(s, x_s)ds$$

has well defined limits at impulse times, since this part is even continuous at impulse moment τ_l, by continuity of the Riemann integral. Right continuity at each impulse time τ_l is reduced to verifying right continuity of

$$Q(t) := \sum_{t_0 < \tau_k \leq t} \Phi(t, \tau_k)I\left(\tau_k, x_{\tau_k^-}\right)$$

at τ_l. Choose $\eta > 0$ small enough such that $\tau_l + \eta < \tau_{l+1}$. Then

$$Q(\tau_l + \eta) - Q(\tau_l) =$$

$$\sum_{t_0<\tau_k\le\tau_l+\eta}\Phi(\tau_l+\eta,\tau_k)I\left(\tau_k,x_{\tau_k^-}\right)-\sum_{t_0<\tau_k\le\tau_l}\Phi(\tau_l,\tau_k)I\left(\tau_k,x_{\tau_k^-}\right)$$
$$=\sum_{t_0<\tau_k\le\tau_l}\left[\Phi(\tau_l+\eta,\tau_k)-\Phi(\tau_l,\tau_k)\right]I\left(\tau_k,x_{\tau_k^-}\right)\xrightarrow{\eta\to 0}0$$

where we note that both sums have the same number of elements, due to $\tau_l+\eta<\tau_{l+1}$. Therefore for each $x\in\mathcal{S}$, we have that Px is right continuous and has left limits at impulse times, clearly it is continuous at nonimpulsive moments.

By definition of $\mathcal{S}$, we must show that $|(Px)(t)|\le L$ for every $t\ge 0$.
We remind the reader that $\|\phi\|\le\delta_0$, with δ_0 as defined in (6.15). We claim that $|(Px)(t)|\le L$ for all $t\ge t_0$. We have that, noticing that $|x(s)|\le L$ by definition of $\mathcal{S}$, so that the Lipschitz properties $(ii),(iii)$ hold, so that

$$\begin{aligned}|(Px)(t)|&\le\|\Phi(t*,t_0)\||\phi(0)|+\int_{t_0}^t\|\Phi(t,s)\||g(s,x_s)|ds+\sum_{t_0<\tau_k\le t}\|\Phi(t,\tau_k)\||I\left(\tau_k,x_{\tau_k^-}\right)|\\&\le\delta_0\|\Phi(t,t_0)\|+\int_{t_0}^t b(s)\|\Phi(t,s)\|\|x_s\|ds+\sum_{t_0<\tau_k\le t}c(\tau_k)\|\Phi(t,\tau_k)\|\|x_{\tau_k^-}\|\\&\le\delta_0K+\sup_{\theta\in[t_0-r,t]}|x(\theta)|\left(\int_{t_0}^t b(s)\|\Phi(t,s)\|ds+\sum_{t_0<\tau_k\le t}c(\tau_k)\|\Phi(t,\tau_k)\|\right)\\&\le\delta_0K+\alpha L<L.\end{aligned}$$

Thus $|(Px)(t)|\le L$ for every $t\ge 0$.

By definition of $\mathcal{S}$, we have that $(Px)_{t_0}=\phi$. Now we show what $(Px)(t)\longrightarrow 0$ as $t\to\infty$.

For this, note that we can divide Px into

$$(Px)(t)=(P_1x)(t)+(P_2x)(t)$$

with

$$(P_1x)(t)=\Phi(t,t_0)\phi(0)+\sum_{t_0<\tau_k\le t}\Phi(t,\tau_k)I\left(\tau_k,x_{\tau_k^-}\right)$$

and

$$(P_2x)(t)=\int_{t_0}^t\Phi(t,s)g(s,x_s)ds.$$

By definition of $\mathcal{S}$, $x(t)\longrightarrow 0$ as $t\to\infty$. Thus we have that for any $\epsilon>0$ there exists $T_1>t_0$ such that

$$|x(t)|<\epsilon\qquad\text{for all }\ t\ge T_1.\tag{6.18}$$

By hypothesis (iv), given this ϵ and T_1, there exists $T_2 > T_1$ such that $t \geq T_2$ implies

$$\begin{aligned} |g(t, x_t)| &\leq b(t)\left(\epsilon + \|x\|^{[T_1,t]}\right) \\ |I(t, x_{t-})| &\leq c(t)\left(\epsilon + \|x\|^{[T_1,t]}\right) \end{aligned} \tag{6.19}$$

Let us first analyze the term $(P_2x)(t)$. If $s \geq T_2 > T_1$, by (6.18) we get

$$\|x\|^{[T_1,s]} < \epsilon. \tag{6.20}$$

By definition of $\mathcal{S}$, $\|x_t\| \leq L$ for all $t \geq t_0$, $x \in \mathcal{S}$, and using the first inequality in (6.19) and inequality (6.20), we obtain that for $t > T_2$:

$$\begin{aligned} |(P_2x)(t)| &= \left|\int_{t_0}^{t} \Phi(t,s)g(s,x_s)ds\right| \\ &\leq \int_{t_0}^{T_2} |\Phi(t,s)g(s,x_s)|ds + \int_{T_2}^{t} |\Phi(t,s)g(s,x_s)|ds \\ &\leq \int_{t_0}^{T_2} \|\Phi(t,s)\||g(s,x_s)|ds + \int_{T_2}^{t} \|\Phi(t,s)\||g(s,x_s)|ds \\ &\leq \int_{t_0}^{T_2} b(s)\|\Phi(t,s)\|\|x_s\|ds + \int_{T_2}^{t} b(s)\|\Phi(t,s)\|\left(\epsilon + \|x\|^{[T_1,t]}\right)ds \\ &\leq L\int_{t_0}^{T_2} b(s)\|\Phi(t,s)\|ds + \int_{T_2}^{t} b(s)\|\Phi(t,s)\|(2\epsilon)ds \\ &= L\|\Phi(t,T_2)\|\int_{t_0}^{T_2} b(s)\|\Phi(T_2,s)\|ds + 2\epsilon\int_{T_2}^{t} b(s)\|\Phi(t,s)\|ds \\ &\leq \alpha L\|\Phi(t,T_2)\| + 2\alpha\epsilon \end{aligned} \tag{6.21}$$

Since we have assumed that $\|\Phi(t,0)\| \longrightarrow \infty$ as $t \to \infty$, we see that given ϵ we can find $T > T_2$ such that

$$\alpha L\|\Phi(t,T_2)\| < \epsilon \quad \text{for} \quad t \geq T.$$

Substituting this last inequality into (6.21), we get that for $t > T$

$$|(P_2x)(t)| \leq \epsilon + 2\alpha\epsilon = \epsilon(1 + 2\alpha)$$

This proves that $(P_2x)(t) \longrightarrow 0$ as $t \to \infty$. We now prove $(P_1x)(t) \longrightarrow 0$ as $t \to \infty$. It is similar to the way we proved this for P_2. Notice that using (6.19), (6.20) and (iv) we have that for $t > T_2$:

$$\left|\sum_{t_0<\tau_k\leq t} \Phi(t,\tau_k)I\left(\tau_k, x_{\tau_k^-}\right)\right|$$

$$
\begin{aligned}
&\leq \sum_{t_0<\tau_k\leq T_2} \|\Phi(t,\tau_k)\| |I\left(\tau_k, x_{\tau_k^-}\right)| + \sum_{T_2<\tau_k\leq t} \|\Phi(t,\tau_k)\| |I\left(\tau_k, x_{\tau_k^-}\right)| \\
&\leq \sum_{t_0<\tau_k\leq T_2} c(\tau_k)\|\Phi(t,\tau_k)\| \left\|x_{\tau_k^-}\right\| + \sum_{T_2<\tau_k\leq t} c(\tau_k)\|\Phi(t,\tau_k)\| \left(\epsilon + \|x\|^{[T_1,\tau_k]}\right) \\
&= \|\Phi(t,T_2)\| \sum_{t_0<\tau_k\leq T_2} c(\tau_k)\|\Phi(T_2,\tau_k)\| \left\|x_{\tau_k^-}\right\| + \sum_{T_2<\tau_k\leq t} c(\tau_k)\|\Phi(t,\tau_k)\| \left(\epsilon + \|x\|^{[T_1,\tau_k]}\right) \\
&\leq L\|\Phi(t,T_2)\| \sum_{t_0<\tau_k\leq T_2} c(\tau_k)\|\Phi(T_2,\tau_k)\| + 2\epsilon \sum_{T_2<\tau_k\leq t} c(\tau_k)\|\Phi(t,\tau_k)\| \\
&\leq \alpha L\|\Phi(t,T_2)\| + 2\alpha\epsilon
\end{aligned}
$$

In a similar way as we did for (P_2x), we can find some $T^* > T_2$, such that $t > T^*$ implies, adding the $\Phi(t,t_0)\phi(0)$ term, that

$$\|\Phi(t,t_0)\| |\phi(0)| + \alpha L \|\Phi(t,T_2)\| < \epsilon.$$

This proves $(P_1x)(t) \longrightarrow 0$ as $t \to \infty$. Therefore choosing $\max\{T, T^*\}$ we have $(Px)(t) \longrightarrow 0$ as $t \to \infty$.

Thus $P : \mathcal{S} \longrightarrow \mathcal{S}$. We now prove that P is a contraction. For this, let $x, y \in \mathcal{S}$. By definition of $\mathcal{S}$ we have that $(Px)(t) - (Py)(t) = 0$ for $t \in [t_0 - r, t_0]$. For $t \geq t_0$ we get:

$$
\begin{aligned}
&|(Px)(t) - (Py)(t)| = \\
&\left| \int_{t_0}^{t} \Phi(t,s)[g(s,x_s) - g(s,y_s)]ds + \sum_{t_0<\tau_k\leq t} \Phi(t,\tau_k) \left[I\left(\tau_k, x_{\tau_k^-}\right) - I\left(\tau_k, y_{\tau_k^-}\right) \right] \right| \\
&\qquad \leq \int_{t_0}^{t} b(s)\|\Phi(t,s)\| \|x_s - y_s\| ds + \sum_{t_0<\tau_k\leq t} c(\tau_k)\|\Phi(t,\tau_k)\| \left\|x_{\tau_k^-} - y_{\tau_k^-}\right\| \\
&\qquad \leq d_{\mathcal{S}}(x,y) \left(\int_{t_0}^{t} b(s)\|\Phi(t,s)\| ds + \sum_{t_0<\tau_k\leq t} c(\tau_k)\|\Phi(t,\tau_k)\| \right) \leq \alpha d_{\mathcal{S}}(x,y)
\end{aligned}
$$

where recall that $d_{\mathcal{S}}(x,y) = \sup_{s\in[t_0,\infty)} |x(s) - y(s)|$.

Thus P is a contraction on $\mathcal{S}$. This implies that there is a unique solution to (6.1) with initial condition (6.2).[8]

To prove uniform stability, assume that we are given an $\epsilon > 0$. Choose $\delta < \epsilon$ such that $\delta K + \alpha\epsilon < \epsilon$, in other words, $\delta < \min\{\epsilon, (1-\alpha)\epsilon/K\}$. Notice that K is independent of t_0, thus so is δ. This will give us uniform stability.

For $\|\phi\| \leq \delta$, we claim that $|x(t)| \leq \epsilon$ for all $t \geq t_0$. Note that if x is the unique solution corresponding to the initial condition ϕ, then $|x(t_0)| = |\phi(0)| < \epsilon$. For the sake of contradiction suppose that there exists a $\hat{t} > t_0$ such that $|x(\hat{t})| > \epsilon$. Let

$$t^* = \inf\{\hat{t} : |x(\hat{t})| > \epsilon\}.$$

[8]More on what we mean by uniqueness below in Section 6.4.

By right continuity, either $|x(t^*)| = \epsilon$ if there is no impulsive moment at t^*, or $|x(t^*)| \geq \epsilon$ as a consequence of a jump at t^*. Whatever the case, we have $|x(s)| \leq \epsilon$ for $s \in [t_0 - r, t^*)$, where $|x(t^*)| = \epsilon$ if this occurs at a non-impulsive moment. Notice $\epsilon < L$ allows application of the Lipschitz-type bounds. By the integral representation of $x(t)$, we have that

$$\begin{aligned}
|x(t^*)| &\leq \|\Phi(t^*, t_0)\||\phi(0)| + \int_{t_0}^{t^*} \|\Phi(t^*, s)\||g(s, x_s)|ds + \sum_{t_0 < \tau_k \leq t^*} \|\Phi(t^*, \tau_k)\||I\left(\tau_k, x_{\tau_k^-}\right)| \\
&\leq \delta\|\Phi(t^*, t_0)\| + \int_{t_0}^{t^*} b(s)\|\Phi(t^*, s)\|\|x_s\|ds + \sum_{t_0 < \tau_k \leq t^*} c(\tau_k)\|\Phi(t^*, \tau_k)\|\|x_{\tau_k^-}\| \\
&\leq \delta K + \sup_{\theta \in [t_0 - r, t^*)} |x(\theta)| \left(\int_{t_0}^{t^*} b(s)\|\Phi(t^*, s)\|ds + \sum_{t_0 < \tau_k \leq t^*} c(\tau_k)\|\Phi(t^*, \tau_k)\| \right) \\
&\leq \delta K + \alpha\epsilon < \epsilon
\end{aligned}$$

and this gives us the desired contradiction, by the definition of t^*. Therefore the solution is uniformly stable, and since $x(t)$ converges to zero as $t \to \infty$, we get uniform stability and asymptotic stability of trajectories. □

Remark 6.3. *Notice that the fact that the solutions of the impulsive FDE remain bounded by L, is independent of the contraction mapping being restricted to $\mathcal{S}$. It is a property that depends solely on the variation or parameters formula, which necessarily any solution satisfies. This can be seen similar to the way we proved stability. When proving that $|(Px)(t)| \leq L$ above, we did assume that $|x(t)| \leq L$ for all t and $x \in \mathcal{S}$ so that we could apply the Lipschitz conditions $(ii), (iii)$, but we can still modify this.*

Lemma 6.1. *Under the hypotheses stated in Theorem 6.2, we have that if $\sup_{s_2 \geq s_1} (\|\Phi(s_2, s_1)\|) \leq K < \infty$ then the solutions of (6.1) with initial condition $\|\phi\| \leq \delta_0 < \frac{(1-\alpha)}{K} L$ remain bounded[9] by L, i.e., $|x(t)| \leq L$ for every t where x is defined .*

Proof. The proof is completely similar to the way in which we prove stability of the solution in Theorem 6.2, with the role of ϵ played by L this time.

For $\|\phi\| \leq \delta_0$, we claim that the solution $x(t)$ satisfies $|x(t)| \leq L$ for all $t \geq t_0$. Note that if x solves the impulsive FDE corresponding to the initial condition ϕ, then $|x(t_0)| = |\phi(0)| < L$. For the sake of contradiction suppose that there exists a $\hat{t} > t_0$ such that $|x(\hat{t})| > L$. Let

$$t^* = \inf\{\hat{t} : |x(\hat{t})| > L\}.$$

By right continuity, either $|x(t^*)| = L$ if there is no impulsive moment at t^*, or $|x(t^*)| \geq L$ as a consequence of a jump at t^*. Whatever the case, using right continuity, we have $|x(s)| \leq L$ for $s \in [t_0 - r, t^*)$, where $|x(t^*)| = L$ if this occurs at a non-impulsive moment. By the integral

[9] Note that $\frac{(1-\alpha)}{K} L < L$, so that $\delta_0 < L$.

representation of $x(t)$, which all solutions to (6.1) satisfy with initial condition ϕ, we have that, since before t^* the paths are bounded by L, we can apply the Lipschitz conditions $(ii), (iii)$, so that

$$\begin{aligned}|x(t^*)| &\leq \|\Phi(t^*,t_0)\||\phi(0)| + \int_{t_0}^{t^*} \|\Phi(t^*,s)\||g(s,x_s)|ds + \sum_{t_0<\tau_k\leq t^*} \|\Phi(t^*,\tau_k)\||I\left(\tau_k, x_{\tau_k^-}\right)| \\ &\leq \delta_0\|\Phi(t^*,t_0)\| + \int_{t_0}^{t^*} b(s)\|\Phi(t^*,s)\|\|x_s\|ds + \sum_{t_0<\tau_k\leq t^*} c(\tau_k)\|\Phi(t^*,\tau_k)\|\|x_{\tau_k^-}\| \\ &\leq \delta_0 K + \sup_{\theta\in[t_0-r,t^*)} |x(\theta)| \left(\int_{t_0}^{t^*} b(s)\|\Phi(t^*,s)\|ds + \sum_{t_0<\tau_k\leq t^*} c(\tau_k)\|\Phi(t^*,\tau_k)\|\right) \\ &\leq \delta_0 K + \alpha L < L\end{aligned}$$

and this gives us the desired contradiction, since we proved $|x(t^*)| < L$, and we assumed $|x(t^*)| = L$ if t^* is a continuity point, or $|x(t^*)| \geq L$ if t^* is a discontinuity point. □

6.4 An Observation on Uniqueness

The importance of the previous Lemma 6.1 lies in the fact that the Lipschitz type conditions (i) and (ii) that we use in Theorem 6.2 are guaranteed only for ϕ, ψ contained in a ball of radius L centred at the zero function in the function space $PCB\big([-r,0],D\big)$.

Now, the Banach Contraction Principle gives a unique solution *within the complete metric space $\mathcal{S}$ where the mapping is restricted to.* The space $\mathcal{S}$ used in the proof of Theorem 6.2 is such that $\mathcal{S} \subset PCB\big([t_0-r,\infty),D\big)$ (*strict containment*), so one might argue that there might be a solution $x_2(t;t_0,\phi) \in PCB\big([t_0-r,\infty),D\big) \setminus \mathcal{S}$, say, that does not converge to zero. Now, by definition, when speaking of "uniqueness", one must take note of *where is this uniqueness statement being held.* For impulsive FDEs, by the general convention that uses PCB-spaces, which is the one we gave in the theoretical background in Section 4.2, solutions must be unique within the respective PCB-space where the solution is defined. We do not ask for uniqueness in an L^p-space, for instance, as in Carátheodory solutions, since this space is too big. And uniqueness within $\mathcal{S} \subset PCB\big([t_0-r,\infty),D\big)$ (*strict containment*) is obviously not satisfactory, because this space is too small to be useful. Thus we see here a caveat about what uniqueness by this particular fixed point theorem really means. One must be careful in this sense.

To remedy this, we proved that independently of any contraction mapping argument, the solutions are all bounded by L in Lemma 6.1. We can argue that the hypotheses supposed on the vector field are sufficient to establish uniqueness by other uniqueness results, such as that in a previous result of X. Liu and P. Stechlinski [39]. But first we arrive at an issue of whether the vector field is well defined in the following sense. The open set $D \subset \mathbb{R}^n$ need not be bounded by L, and one can argue that the vector field defined by (6.1) might eventually evolve the state to norm sizes greater than L, where we do not have the Lipschitz type condition guaranteed. However, the result in Lemma 6.1 proved just now shows us that, given the differential equation

(6.1), the solution $x(t)$ with initial condition ϕ satisfying

$$\|\phi\| < \delta_0 < \frac{(1-\alpha)}{K} L =: \delta_{L,K},$$

the solution will remain in a ball of size L.

Thus $\delta_{L,K}$ clearly gives an upper threshold on the initial conditions for an initial value problem, because solutions with these types of initial conditions cannot leave the closed ball of radius L centered at 0, which[10] in Euclidean space we denote $B_L(0)$, so that the function space $PCB\big([-r,0], B_L(0)\big)$ is enough, which is equivalent to *the function space* ball centered at the zero function, denoted $B(L)$ as defined above. Thus the vector field is well defined and remains in a ball of norm L.

Now we can apply standard existence-uniqueness theory, as follows:

Proposition 6.1. *Supposing $g : J \times PCB([-r,0], D) \longrightarrow \mathbb{R}^n$ is composite continuous, and satisfies $|g(t,\phi) - g(t,\psi)| \leq b(t)\|\phi - \psi\|$ for all $\phi, \psi \in C(L)$, then solution to the IVP induced by* (6.1) *with initial condition ϕ is unique, if $\|\phi\| \leq \delta_0 < \dfrac{(1-\alpha)}{K} L =: \delta_{L,K}$.*

Proof. We have a local Lipschitz condition in a ball of size L, as defined in Definition 4.6 in Section 4.2, since if t is in a compact set, then $b(t)$ is bounded and gives us necessary Lipschitz constants, since any closed subset of the closed ball $B_L(0)$ would give us a compact subset. Since we assumed $g(s, x_s)$ is composite-PCB, we are actually satisfying the hypotheses required in the uniqueness result of Theorem 4.3. This guarantees uniqueness in $PCB\big([-r,0], B_L(0)\big)$, even for infinite delay. □

Thus the solution found by the contraction mapping principle is unique in a satisfactory way, and whatever we achieve through the contraction method, must hold for each unique solution. Notice that the proof of uniform stability of the zero solution of (6.1) also depended only on the variation of parameters formula. Therefore, the additional information that we are obtaining from using the contraction mapping is the asymptotic stability of the unique solutions to each initial value problem.

6.5 An Example

Notice that the condition

$$\int_0^t b(s)\|\Phi(t,s)\| ds + \sum_{0<\tau_k \leq t} c(\tau_k)\|\Phi(t,\tau_k)\| \leq \alpha < 1$$

is not easy to evaluate, unless we know some bounds. For the scalar case, let us concentrate on guaranteeing

[10] We can make it be an open ball by suitably using a strict inequality when defining the δ_0.

$$\sum_{0<\tau_k\leq t} c(\tau_k)e^{-\int_{\tau_k}^t a(u)du} \leq \frac{\alpha}{2} \tag{6.22}$$

for a given $\alpha < 1$. We already know, from Example 5.2 of the previous chapter, how to make the first contribution from the integral less than $\alpha/2$, by a simple rescaling by the 1/2 factor. Notice that if $t \in [\tau_{n-1}, \tau_n)$, for $n \geq 2$ (since for $n = 1$, $t \in [0, \tau_1)$, so no jumps have occurred, we do not even need to worry about this contribution at $n = 1$) we have that[11]

$$\begin{aligned}
&\sum_{0<\tau_k\leq t} c(\tau_k)e^{-\int_{\tau_k}^t a(u)du} \\
&\quad = c(\tau_1)e^{-\int_{\tau_1}^t a(u)du} + c(\tau_2)e^{-\int_{\tau_2}^t a(u)du} + \cdots + c(\tau_{n-2})e^{-\int_{\tau_{n-2}}^t a(u)du} + c(\tau_{n-1})e^{-\int_{\tau_{n-1}}^t a(u)du} \\
&\quad = e^{-\int_{\tau_{n-1}}^t a(u)du} \sum_{m=1}^{n-2} c(\tau_{n-1-m}) \left(\prod_{j=1}^{m} e^{-\int_{\tau_{n-1-j}}^{\tau_{n-j}} a(u)du} \right) + c(\tau_{n-1})e^{-\int_{\tau_{n-1}}^t a(u)du},
\end{aligned} \tag{6.23}$$

where we have used that for each m:

$$e^{-\int_{\tau_{n-1}}^t a(u)du} \left(\prod_{j=1}^{m} e^{-\int_{\tau_{n-1-j}}^{\tau_{n-j}} a(u)du} \right) = e^{-\int_{\tau_{n-1-m}}^t a(u)du}.$$

Suppose that we allow sufficient time between jumps so that the "good" behavior of $a(t)$ dominates on each continuous subinterval so that $e^{-\int_{\tau_j}^{\tau_{j+1}} a(u)du} \leq \beta < \frac{1}{2}$. Now, notice how we always obtain a left over term

$$+\, c(\tau_{n-1})e^{-\int_{\tau_{n-1}}^t a(u)du}, \tag{6.24}$$

and that $e^{-\int_{\tau_{n-1}}^t a(u)du}$ might be relatively large, at least not smaller than β, for example, if $a(u)$ is negative at the beginning of the impulse at τ_{n-1}. Maybe there still has not been enough time for the good behavior of $a(u)$ to have the good effects that allow for asymptotic stability. Suppose that the worst that can happen is captured as $e^{-\int_{s_1}^{s_2} a(u)du} \leq K$ for every $s_1 \leq s_2 \in [0, \infty)$.

Remark 6.4. *Notice that* $K \geq 1$, *since* $e^{-\int_{s_2}^{s_2} a(u)du} = 1$. *In case that* $a(u) \geq 0$ *always, then* $K = 1$ *automatically.*

Thus we have that if, say, $c(\tau_m) \leq \frac{\alpha}{4K}$, and $\beta < \frac{1}{2}$, then

[11] For $n = 2$, we use as notational convention $\sum_{m=1}^{0} (\cdot) = 0$, so that only the term $c(\tau_1)e^{-\int_{\tau_1}^t a(u)du}$ is left for this special case.

$$
\begin{aligned}
e^{-\int_{\tau_{n-1}}^{t} a(u)du} \sum_{m=1}^{n-2} c(\tau_{n-1-m}) & \left(\prod_{j=1}^{m} e^{-\int_{\tau_{n-1-j}}^{\tau_{n-j}} a(u)du} \right) + c(\tau_{n-1}) e^{-\int_{\tau_{n-1}}^{t} a(u)du} \\
& \leq K \sum_{m=1}^{n-2} c(\tau_{n-1-m}) \beta^m + K c(\tau_{n-1}) \\
& \leq \frac{\alpha}{4} \sum_{m=1}^{\infty} \beta^m + \frac{\alpha}{4} \\
& \leq \frac{\alpha}{4} \frac{\beta}{1-\beta} + \frac{\alpha}{4} < \frac{\alpha}{2}
\end{aligned} \tag{6.25}
$$

where we have used that $\frac{\beta}{1-\beta} < 1$ because $\beta < \frac{1}{2}$. So we have shown that

$$
\sum_{0<\tau_k \leq t} c(\tau_k) e^{-\int_{\tau_k}^{t} a(u)du} \leq \frac{\alpha}{2},
$$

as long as the intervals $[\tau_j, \tau_{j+1})$ between jumps allow sufficient time for $e^{-\int_{\tau_j}^{\tau_{j+1}} a(u)du} \leq \beta < \frac{1}{2}$, the condition $e^{-\int_{s_1}^{s_2} a(u)du} \leq K$ holds for every $s_1 \leq s_2 \in [0, \infty)$ for and the Lipschitz weighting function of the jumps satisfies $c(\tau_m) \leq \frac{\alpha}{4K}$ for all $m \geq 1$. △

Remark 6.5. *Through a similar analysis to the one we did in the previous chapter for continuous delayed functions, we can have an idea of how to calculate the maximum bound K. As can be remembered from the previous chapter, such as in Example 5.6 and ensuing remarks there, a good candidate to finding K to obtain a uniform bound in t_0 is to look for the longest interval where $a(t)$ is negative.*

The previous example motivates the following corollary, which could serve as a criterion to determine if the hypotheses of Theorem 6.2 hold. Of course, different criteria can be obtained, this is just one of many possible that give sufficient conditions for the application of Theorem 6.2.

Corollary 6.1. *Suppose that the conditions of Theorem 6.2 hold, except that instead of condition* (6.13), *we have that there exists an $\alpha \in (0, 1)$ such that*

$$
\sup_{t \geq 0} \left(\int_0^t b(s) \|\Phi(t, s)\| ds \right) \leq \frac{\alpha}{2} \tag{6.26}
$$

and the following conditions hold. The intervals $[\tau_j, \tau_{j+1})$ between impulses satisfy that for every $j \geq 1$

$$
\|\Phi(\tau_{j+1}, \tau_j))\| \leq \beta < \frac{1}{2}, \tag{6.27}
$$

$\|\Phi(s_2, s_1)\| \leq K$ holds for every $s_1 \leq s_2 \in [0, \infty)$, and the Lipschitz weighting function of the impulses satisfies $c(\tau_m) \leq \frac{\alpha}{4K}$ for all $m \geq 1$. Then the trivial solution of (6.1) *is uniformly stable and asymptotically stable.*

Proof. We just need to prove that the hypotheses of this proposition imply that

$$\sum_{0<\tau_k\leq t} c(\tau_k)\|\Phi(t,\tau_k)\| \leq \frac{\alpha}{2},$$

so that, along with (6.26) we have that condition (6.13) of Theorem 6.2 holds. If $t \in [\tau_{n-1}, \tau_n)$, then

$$\begin{aligned}
&\sum_{0<\tau_k\leq t} c(\tau_k)\|\Phi(t,\tau_k)\| \\
&\qquad = \sum_{m=1}^{n-2} c(\tau_{n-1-m}) \left\| \Phi(t,\tau_{n-1}) \left(\prod_{j=1}^{m} \Phi(\tau_{n-j},\tau_{n-1-j}) \right) \right\| + c(\tau_{n-1})\|\Phi(t,\tau_{n-1})\| \\
&\qquad \leq \|\Phi(t,\tau_{n-1}\| \sum_{m=1}^{n-2} c(\tau_{n-1-m}) \left(\prod_{j=1}^{m} \|\Phi(\tau_{n-j},\tau_{n-1-j})\| \right) + c(\tau_{n-1})\|\Phi(t,\tau_{n-1})\| \\
&\qquad \leq K \sum_{m=1}^{n-2} c(\tau_{n-1-m})\beta^m + Kc(\tau_{n-1}) \\
&\qquad \leq \frac{\alpha}{4} \sum_{m=1}^{\infty} \beta^m + \frac{\alpha}{4} \leq \frac{\alpha}{4}\frac{\beta}{1-\beta} + \frac{\alpha}{4} < \frac{\alpha}{2}.
\end{aligned}$$

The rest follows from the main result, Theorem 6.2.

□

Chapter 7

Impulsive Stabilization of an FDE by Contraction Principle

7.1 Introduction

In the previous chapters, we have assumed that the functional differential equations considered were well behaved, in the sense that the linear portion of the system, which depended on the last state $x(t)$ as $A(t)x(t)$, was sufficiently well behaved to dominate the whole behavior of the system somehow, including the nonlinear portion $g(t, x_t)$, in order to achieve asymptotic stability. Here we shall assume that the system is not well behaved, and rather, this time impulses will have the stabilizing role. There are different conditions to consider, which we state in the main result of this chapter. In examples below we shall be able to understand the role of the conditions from the main theorem, depending on how badly behaved the linear part of the system is. Time spacing between impulses will play a crucial role. If the fundamental matrix has operator norm converging in time to infinity, contrary to the previous chapters where the norm converges to zero, then we will need to use impulses to break up the acting of the operator before it gets "too big" again.

7.2 Preliminaries

Using the Banach contraction principle, conditions for stability of the impulsive delayed differential equation

$$\begin{aligned} x'(t) &= A(t)x(t) + g(t, x_t), \qquad t \neq t_k, t \geq 0 \\ \Delta x(t) &= I(t, x_{t^-}) = [B(t) - I_d]x(t^-) + w(t, x_{t^-}), \qquad t = t_k, t \geq 0 \end{aligned} \tag{7.1}$$

are given. Here, we have that $x(t) \in \mathbb{R}^n$, $g, I, w : J \times PCB([-r, 0], D) \longrightarrow \mathbb{R}^n$ with $J \subset \mathbb{R}_+$ an interval, $D \subset \mathbb{R}^n$ an open set and $\Delta x(t) = x(t) - x(t^-)$. The impulse times t_k satisfy $t_1 < \cdots$ and $\lim_{k\to\infty} t_k = \infty$. $A(t), B(t)$ are $n \times n$ continuous matrix functions, in the sense that all entries of these matrices are continuous functions in the interval of definition of the functional differential equation (7.1). We state and explain the conventions and conditions assumed on system (7.1) in the paragraphs below.

As in the convention used in Ballinger & Liu [6], we do not ask for the jump condition in (7.1) to be satisfied at t_0, the initial time, where we remind the reader that the first impulse moment is t_1, since this imposes an unnecessary restriction on the initial condition.

Remark 7.1. *In the case when $r = \infty$, we still denote the space $PCB(-\infty, 0]$ by the notation $PCB[-r, 0]$, by considering for this special case $[-r, 0]$ to mean the* infinite *interval $(-\infty, 0]$, and using the piecewise continuous bounded functions on $(-\infty, 0]$. Of course, $PCB[-r, 0] = PC[-r, 0]$ when $r < \infty$.*

By x_{t^-} in (7.1) we refer to the function defined by a given $x \in PCB([t_0 - r, b], D)$ through the assignment

$$x_{t^-}(s) = x_t(s) \quad \text{for } s \in [-r, 0)$$
$$x_{t^-}(0) = \lim_{u \to t^-} x(u) = x(t^-).$$

This is a way of getting a well defined function in $PCB[-r, 0]$, that takes into account only the information available right until before the jump occurs right at $t = t_k$. In this way, the mapping I induces a jump from $x(t^-)$ to a value $x(t)$, using the information available until just before the impulse occurs at time t.

The norm that we use on $PCB([-r, 0], D)$ will be

$$\|\psi\|_r := \sup_{s \in [-r, 0]} |\psi(s)|,$$

where of course for $r = \infty$ this norm is $\|\psi\|_r = \sup_{s \in (-\infty, 0]} \|\psi(s)\|$. Wherever the norm symbol $\|\cdot\|$ is used, **we refer to the norm on** $PCB([-r, 0], D)$. We will denote the Euclidean norm by $|x|$ whenever no confusion should arise.

The initial condition for equation (7.1) will be given for $t_0 \geq 0$ as

$$x_{t_0} = \phi \tag{7.2}$$

for $t_0 \in J$, and $\phi \in PCB([-r, 0], D)$. For stability analysis, it is assumed that $0 \in D$, $J = \mathbb{R}_+$, $g(t, 0) = w(t, 0) = 0$ for all $t \in \mathbb{R}_+$. This guarantees that system (7.1) has a trivial solution $x(t) = 0$.

In other papers, for example [57], it is assumed that the continuous matrix $A(t)$ is well behaved in the sense that its induced linear system (7.3) has a fundamental matrix $\Phi(t, t_0)$ that converges to zero in operator norm as $t \to \infty$.[1] However, here we will assume that $A(t)$ is not as well behaved, but still remains bounded by a small enough constant.

[1] In [57], the one-dimensional case is treated only, where the matrix $A(t)$ is reduced to a scalar function.

7.3 Main Results

In order for the necessary integrals to exist (namely those of nonlinear part g), we will assume that g is composite-PCB. The precise definition is given below.

Definition 7.1. A mapping $g : J \times PCB([-r, 0], D) \longrightarrow \mathbb{R}^n$, where $0 \leq r \leq \infty$, is said to be *composite-PCB* if for each $t_0 \in J$ and $\beta > 0$ where $[t_0, t_0+\beta] \subset J$, if $x \in PCB([t_0-r, t_0+\beta], D)$, and x is continuous at each $t \neq t_k$ in $(t_0, t_0 + \beta]$ then the composite function $t \mapsto g(t, x_t)$ is an element of the function class $PCB([t_0, t_0 + \beta], \mathbb{R}^n)$.

We will use the fundamental solution $\Phi(t, t_0)$ of a linear ordinary differential equation

$$\begin{aligned} y'(t) &= A(t)y(t) \\ y(t_0) &= y_0 \end{aligned} \tag{7.3}$$

such that the solution of IVP (7.3) is

$$y(t) = \Phi(t, t_0)y_0.$$

For a matrix M we use the standard linear operator norm induced by the Euclidean norm $|\cdot|$ on $\mathbb{R}^n$:

$$\|M\| := \|M\|_{\mathcal{L}(\mathbb{R}^n)} = \sup_{|y|=1} |My|.$$

We will use the inequality $|My| \leq \|M\||y|$ for $y \in \mathbb{R}^n$.

From the IFDE (impulsive functional differential equation) (7.1), we have that we can also represent the evolution of the system as

$$\begin{aligned} x'(t) &= A(t)x(t) + g(t, x_t), \qquad t \neq t_k, t \geq 0 \\ x(t_k^+) &= B(t_k)x(t_k^-) + w(t_k, x_{t_k^-}) \end{aligned} \tag{7.4}$$

First off, we begin characterizing what the solution looks like, using a variation of parameters type formula.

Lemma 7.1. *The solution to the IVP IFDE* (7.4) *with initial condition* (7.2) *satisfies, for* $t \in [t_{n-1}, t_n)$ *with* $n \geq 1$*:*

$$\begin{aligned} x(t) = {} & \Phi(t, t_{n-1}) \left(\prod_{k=1}^{n-1} B(t_{n-k})\Phi(t_{n-k}, t_{n-k-1}) \right) \phi(0) + \int_{t_{n-1}}^{t} \Phi(t, s)g(s, x_s)ds \\ & + \Phi(t, t_{n-1}) \sum_{m=0}^{n-2} \int_{t_{n-m-2}}^{t_{n-m-1}} \left(\prod_{k=1}^{m} B(t_{n-k})\Phi(t_{n-k}, t_{n-k-1}) \right) B(t_{n-m-1})\Phi(t_{n-m-1}, s)g(s, x_s)ds \\ & + \Phi(t, t_{n-1}) \sum_{m=0}^{n-2} \left(\prod_{k=1}^{m} B(t_{n-k})\Phi(t_{n-k}, t_{n-k-1}) \right) w(t_{n-m-1}, x_{t_{n-m-1}^-}) \end{aligned} \tag{7.5}$$

where for $m = 0$, we define $\prod_{k=1}^{0} B(t_{n-k})\Phi(t_{n-k}, t_{n-k-1}) \equiv I_d$, the identity operator on $\mathbb{R}^n$, and for $n = 1$ we define $\sum_{m=0}^{-1} \equiv 0$.[2]

Proof. We have that if $t \in [t_{n-1}, t_n)$ with $n \geq 1$, then the dynamical system (7.4) evolves continuously in this time interval, so we evolve, assuming right continuity, the state $x(t_{n-1}^+) = x(t_{n-1})$, where sometimes we will write $x(t_{k-1}^+)$ to emphasize that the system evolved according to the vector field that acted on the previous interval $[t_{k-2}, t_{k-1})$. We have

$$x(t) = \Phi(t, t_{n-1})x(t_{n-1}^+) + \int_{t_{n-1}}^{t} \Phi(t, s)g(s, x_s)ds. \tag{7.6}$$

We substitute into equation (7.6) the expression $x(t_{n-1}^+) = B(t_{n-1})x(t_{n-1}^-) + w(t_{n-1}, x_{t_{n-1}^-})$ from (7.1). Thus

$$\begin{aligned} x(t) =&\Phi(t, t_{n-1}) \left[B(t_{n-1})x(t_{n-1}^-) + w(t_{n-1}, x_{t_{n-1}^-})\right] \\ &+ \int_{t_{n-1}}^{t} \Phi(t, s)g(s, x_s)ds \\ =&\Phi(t, t_{n-1})B(t_{n-1})x(t_{n-1}^-) + \Phi(t, t_{n-1})w(t_{n-1}, x_{t_{n-1}^-}) \\ &+ \int_{t_{n-1}}^{t} \Phi(t, s)g(s, x_s)ds. \end{aligned} \tag{7.7}$$

Now we expand the previous line by substituting the value

$$x(t_{n-1}^-) = \Phi(t_{n-1}, t_{n-2})x(t_{n-2}^+) + \int_{t_{n-2}}^{t_{n-1}} \Phi(t_{n-1}, s)g(s, x_s)ds. \tag{7.8}$$

These steps will be done successively, where we will evolve the continuous part of the dynamics using the fundamental matrix in a variation of parameters formula in the respective time interval where only the continuous dynamics play a role, say, during time interval $[t_k, t_{k+1})$ starting from $x(t_k^+)$. Then we substitute the value of the state $x(t_k^+)$ itself:

$$x(t_k^+) = B(t_k)x(t_k^-) + w(t_k, x_{t_k^-}) \qquad k = 1, \cdots, n-1. \tag{7.9}$$

Afterwards we substitute into (7.9) the previous continuous dynamics from the interval $[t_{k-1}, t_k)$. This will lead us to require to plug in the value of $x(t_k^-)$, in other words, we obtain the contribution from the previous interval $[t_{k-1}, t_k)$ using

$$x(t_k^-) = \Phi(t_k, t_{k-1})x(t_{k-1}^+) + \int_{t_{k-1}}^{t_k} \Phi(t_k, s)g(s, x_s)ds \qquad k = 1, \cdots, n-1. \tag{7.10}$$

We repeat this process successively. We will do this a couple of times to get an idea of a general formula by observing what type of terms we can group together as we continue this process.

[2]Notice that when $n = 2$, $\sum_{m=0}^{0}$ indicates the sum when m only takes on the value zero, and similarly for the product $\prod_{k=1}^{n-1} = \prod_{k=1}^{1}$, when k only takes on the unique value $k = 1$.

All of this is done in order to obtain a variation of parameters formula that puts together all of the elements involved in the definition of IFDE (7.1), in other words, the continuous portions of the system plus the discrete contributions from (7.4).

Substitution of (7.8) into (7.7) gives us

$$\begin{aligned} x(t) =& \Phi(t,t_{n-1})B(t_{n-1})\left[\Phi(t_{n-1},t_{n-2})x(t^+_{n-2}) + \int_{t_{n-2}}^{t_{n-1}} \Phi(t_{n-1},s)g(s,x_s)ds\right] \\ &+ \Phi(t,t_{n-1})w(t_{n-1},x_{t^-_{n-1}}) + \int_{t_{n-1}}^{t} \Phi(t,s)g(s,x_s)ds, \end{aligned}$$

thus

$$\begin{aligned} x(t) =& \Phi(t,t_{n-1})B(t_{n-1})\Phi(t_{n-1},t_{n-2})x(t^+_{n-2}) \\ &+ \Phi(t,t_{n-1})B(t_{n-1})\int_{t_{n-2}}^{t_{n-1}} \Phi(t_{n-1},s)g(s,x_s)ds \\ &+ \Phi(t,t_{n-1})w(t_{n-1},x_{t^-_{n-1}}) + \int_{t_{n-1}}^{t} \Phi(t,s)g(s,x_s)ds. \end{aligned} \tag{7.11}$$

Into the previous equation (7.11), we plug in the value

$$x(t^+_{n-2}) = B(t_{n-2})x(t^-_{n-2}) + w(t_{n-2},x_{t^-_{n-2}})$$

to get

$$\begin{aligned} x(t) =& \Phi(t,t_{n-1})B(t_{n-1})\Phi(t_{n-1},t_{n-2})B(t_{n-2})x(t^-_{n-2}) \\ &+ \Phi(t,t_{n-1})B(t_{n-1})\Phi(t_{n-1},t_{n-2})w(t_{n-2},x_{t^-_{n-2}}) \\ &+ \Phi(t,t_{n-1})B(t_{n-1})\int_{t_{n-2}}^{t_{n-1}} \Phi(t_{n-1},s)g(s,x_s)ds \\ &+ \Phi(t,t_{n-1})w(t_{n-1},x_{t^-_{n-1}}) + \int_{t_{n-1}}^{t} \Phi(t,s)g(s,x_s)ds. \end{aligned} \tag{7.12}$$

Using

$$x(t^-_{n-2}) = \Phi(t_{n-2},t_{n-3})x(t^+_{n-3}) + \int_{t_{n-3}}^{t_{n-2}} \Phi(t_{n-2},s)g(s,x_s)ds$$

in (7.12) we have that

$$\begin{aligned} x(t) =& \Phi(t,t_{n-1})B(t_{n-1})\Phi(t_{n-1},t_{n-2})B(t_{n-2})\left[\Phi(t_{n-2},t_{n-3})x(t^+_{n-3}) + \int_{t_{n-3}}^{t_{n-2}} \Phi(t_{n-2},s)g(s,x_s)ds\right] \\ &+ \Phi(t,t_{n-1})B(t_{n-1})\Phi(t_{n-1},t_{n-2})w(t_{n-2},x_{t^-_{n-2}}) \\ &+ \Phi(t,t_{n-1})B(t_{n-1})\int_{t_{n-2}}^{t_{n-1}} \Phi(t_{n-1},s)g(s,x_s)ds \\ &+ \Phi(t,t_{n-1})w(t_{n-1},x_{t^-_{n-1}}) + \int_{t_{n-1}}^{t} \Phi(t,s)g(s,x_s)ds, \end{aligned}$$

which rearranging, gives us that so far

$$\begin{aligned} x(t) =&\Phi(t,t_{n-1})B(t_{n-1})\Phi(t_{n-1},t_{n-2})B(t_{n-2})\Phi(t_{n-2},t_{n-3})x(t_{n-3}^{+}) \\ &+\Phi(t,t_{n-1})B(t_{n-1})\Phi(t_{n-1},t_{n-2})B(t_{n-2})\int_{t_{n-3}}^{t_{n-2}}\Phi(t_{n-2},s)g(s,x_s)ds \\ &+\Phi(t,t_{n-1})B(t_{n-1})\Phi(t_{n-1},t_{n-2})w(t_{n-2},x_{t_{n-2}^{-}}) \\ &+\Phi(t,t_{n-1})B(t_{n-1})\int_{t_{n-2}}^{t_{n-1}}\Phi(t_{n-1},s)g(s,x_s)ds \\ &+\Phi(t,t_{n-1})w(t_{n-1},x_{t_{n-1}^{-}})+\int_{t_{n-1}}^{t}\Phi(t,s)g(s,x_s)ds. \end{aligned} \tag{7.13}$$

After inserting the value of $x(t_{n-3}^{+})$ into the previous equation, we obtain

$$\begin{aligned} x(t) =&\Phi(t,t_{n-1})B(t_{n-1})\Phi(t_{n-1},t_{n-2})B(t_{n-2})\Phi(t_{n-2},t_{n-3})B(t_{n-3})x(t_{n-3}^{-}) \\ &+\Phi(t,t_{n-1})B(t_{n-1})\Phi(t_{n-1},t_{n-2})B(t_{n-2})\Phi(t_{n-2},t_{n-3})w(t_{n-3},x_{t_{n-3}^{-}}) \\ &+\Phi(t,t_{n-1})B(t_{n-1})\Phi(t_{n-1},t_{n-2})B(t_{n-2})\int_{t_{n-3}}^{t_{n-2}}\Phi(t_{n-2},s)g(s,x_s)ds \\ &+\Phi(t,t_{n-1})B(t_{n-1})\Phi(t_{n-1},t_{n-2})w(t_{n-2},x_{t_{n-2}^{-}}) \\ &+\Phi(t,t_{n-1})B(t_{n-1})\int_{t_{n-2}}^{t_{n-1}}\Phi(t_{n-1},s)g(s,x_s)ds \\ &+\Phi(t,t_{n-1})w(t_{n-1},x_{t_{n-1}^{-}})+\int_{t_{n-1}}^{t}\Phi(t,s)g(s,x_s)ds, \end{aligned}$$

which after evaluating $x(t_{n-3}^{-})$ becomes

$$\begin{aligned} x(t) =&\Phi(t,t_{n-1})B(t_{n-1})\Phi(t_{n-1},t_{n-2})B(t_{n-2})\Phi(t_{n-2},t_{n-3})B(t_{n-3})\Phi(t_{n-3},t_{n-4})x(t_{n-4}^{+}) \\ &+\Phi(t,t_{n-1})B(t_{n-1})\Phi(t_{n-1},t_{n-2})B(t_{n-2})\Phi(t_{n-2},t_{n-3})B(t_{n-3})\int_{t_{n-4}}^{t_{n-3}}\Phi(t_{n-3},s)g(s,x_s)ds \\ &+\Phi(t,t_{n-1})B(t_{n-1})\Phi(t_{n-1},t_{n-2})B(t_{n-2})\Phi(t_{n-2},t_{n-3})w(t_{n-3},x_{t_{n-3}^{-}}) \\ &+\Phi(t,t_{n-1})B(t_{n-1})\Phi(t_{n-1},t_{n-2})B(t_{n-2})\int_{t_{n-3}}^{t_{n-2}}\Phi(t_{n-2},s)g(s,x_s)ds \\ &+\Phi(t,t_{n-1})B(t_{n-1})\Phi(t_{n-1},t_{n-2})w(t_{n-2},x_{t_{n-2}^{-}}) \\ &+\Phi(t,t_{n-1})B(t_{n-1})\int_{t_{n-2}}^{t_{n-1}}\Phi(t_{n-1},s)g(s,x_s)ds \\ &+\Phi(t,t_{n-1})w(t_{n-1},x_{t_{n-1}^{-}})+\int_{t_{n-1}}^{t}\Phi(t,s)g(s,x_s)ds. \end{aligned} \tag{7.14}$$

At this point we are able to notice how a general formula could be defined. The first term on the right hand side of (7.14), namely

$$\Phi(t,t_{n-1})B(t_{n-1})\Phi(t_{n-1},t_{n-2})B(t_{n-2})\Phi(t_{n-2},t_{n-3})B(t_{n-3})\Phi(t_{n-3},t_{n-4})x(t_{n-4}^+)$$

will continue to be expanded successively until we reach the term

$$\begin{aligned}&\Phi(t,t_{n-1})B(t_{n-1})\cdots B(t_2)\Phi(t_2,t_1)x(t_1^+)\\&\qquad=\Phi(t,t_{n-1})B(t_{n-1})\cdots B(t_2)\Phi(t_2,t_1)\left[B(t_1)x(t_1^-)+w(t_1,x_{t_1^-})\right]\end{aligned}$$

$$\begin{aligned}=&\Phi(t,t_{n-1})B(t_{n-1})\cdots B(t_2)\Phi(t_2,t_1)B(t_1)\left[\Phi(t_1,t_0)x(t_0)+\int_{t_0}^{t_1}\Phi(t_1,s)g(s,x_s)ds\right]\\&+\Phi(t,t_{n-1})B(t_{n-1})\cdots B(t_2)\Phi(t_2,t_1)w(t_1,x_{t_1^-}).\end{aligned}$$

Thus, the final term to add to the equation for $x(t)$ as we continue to expand equation (7.14) as we backstep in time all the way back to the contribution in the interval $[t_0,t_1)$ is:

$$\begin{aligned}&\Phi(t,t_{n-1})B(t_{n-1})\cdots B(t_2)\Phi(t_2,t_1)B(t_1)\Phi(t_1,t_0)x(t_0)\\&+\Phi(t,t_{n-1})B(t_{n-1})\cdots B(t_2)\Phi(t_2,t_1)w(t_1,x_{t_1^-})\\&+\Phi(t,t_{n-1})B(t_{n-1})\cdots B(t_2)\Phi(t_2,t_1)B(t_1)\int_{t_0}^{t_1}\Phi(t_1,s)g(s,x_s)ds.\end{aligned}\tag{7.15}$$

In this manner, we have that

$$\begin{aligned}x(t)=&\Phi(t,t_{n-1})B(t_{n-1})\cdots B(t_2)\Phi(t_2,t_1)B(t_1)\Phi(t_1,t_0)x(t_0)\\&+\Phi(t,t_{n-1})B(t_{n-1})\cdots B(t_2)\Phi(t_2,t_1)w(t_1,x_{t_1^-})\\&+\Phi(t,t_{n-1})B(t_{n-1})\cdots B(t_2)\Phi(t_2,t_1)B(t_1)\int_{t_0}^{t_1}\Phi(t_1,s)g(s,x_s)ds\\&+\cdots\\&+\cdots\\&+\Phi(t,t_{n-1})B(t_{n-1})\Phi(t_{n-1},t_{n-2})B(t_{n-2})\Phi(t_{n-2},t_{n-3})B(t_{n-3})\int_{t_{n-4}}^{t_{n-3}}\Phi(t_{n-3},s)g(s,x_s)ds\\&+\Phi(t,t_{n-1})B(t_{n-1})\Phi(t_{n-1},t_{n-2})B(t_{n-2})\Phi(t_{n-2},t_{n-3})w(t_{n-3},x_{t_{n-3}^-})\\&+\Phi(t,t_{n-1})B(t_{n-1})\Phi(t_{n-1},t_{n-2})B(t_{n-2})\int_{t_{n-3}}^{t_{n-2}}\Phi(t_{n-2},s)g(s,x_s)ds\\&+\Phi(t,t_{n-1})B(t_{n-1})\Phi(t_{n-1},t_{n-2})w(t_{n-2},x_{t_{n-2}^-})\\&+\Phi(t,t_{n-1})B(t_{n-1})\int_{t_{n-2}}^{t_{n-1}}\Phi(t_{n-1},s)g(s,x_s)ds\\&+\Phi(t,t_{n-1})w(t_{n-1},x_{t_{n-1}^-})+\int_{t_{n-1}}^{t}\Phi(t,s)g(s,x_s)ds.\end{aligned}\tag{7.16}$$

This formula can be simplified by grouping similar terms.

-TERMS OF TYPE I

Notice the following type of term in expression (7.16): The terms containing as a "factor" an integral of the form $\int_{t_{k-1}}^{t_k} \Phi(t_k, s)g(s, x_s)ds$, for $k = 1, .., n-1$.[3] The terms are enlisted below for clarity, notice how we have grouped some of the factors to the inside of the integral sign for a convenient form of identification that we will adapt, in order to synthesize the expression for $x(t)$ in a formula with product and summation notation. This is possible because the matrices are independent of the variable of integration. For example, the term

$$\Phi(t, t_{n-1})B(t_{n-1}) \int_{t_{n-2}}^{t_{n-1}} \Phi(t_{n-1}, s)g(s, x_s)ds$$

will be rewritten as

$$\Phi(t, t_{n-1}) \int_{t_{n-2}}^{t_{n-1}} \underbrace{B(t_{n-1})\Phi(t_{n-1}, s)}_{\text{1 couple}} g(s, x_s)ds,$$

and the next term of this type, with 2 couples will be written

$$\Phi(t, t_{n-1}) \int_{t_{n-3}}^{t_{n-2}} \underbrace{B(t_{n-1})\Phi(t_{n-1}, t_{n-2})} \underbrace{B(t_{n-2})\Phi(t_{n-2}, s)} g(s, x_s)ds$$

We continue in this way to identify up to the longest term with $n-1$ couples of this type (notice how the following term comes from the last expression (7.15) that we added):

$$\Phi(t, t_{n-1}) \int_{t_0}^{t_1} \underbrace{B(t_{n-1})\Phi(t_{n-1}, t_{n-2})} \underbrace{B(t_{n-2})\Phi(t_{n-2}, t_{n-3})} \cdots \underbrace{B(t_1)\Phi(t_1, s)} g(s, x_s)ds.$$

Notice, for example for the term with $n-1$ couples, that we can write this term in product notation (product of linear operators) as

$$\Phi(t, t_{n-1}) \int_{t_0}^{t_1} \left(\prod_{k=1}^{n-2} B(t_{n-k})\Phi(t_{n-k}, t_{n-k-1}) \right) B(t_1)\Phi(t_1, s)g(s, x_s)ds$$

where we leave the last couple out of the product symbol because of the s-variable in the $B(t_1)\Phi(t_1, s)$ factor or rightmost pair.[4] We will do something analogous for the rest of the terms.

Special care must be taken for the case of the shortest length term of this type (with one "couple")

$$\Phi(t, t_{n-1}) \int_{t_{n-2}}^{t_{n-1}} \underbrace{B(t_{n-1})\Phi(t_{n-1}, s)}_{\text{1 couple}} g(s, x_s)ds,$$

[3] Notice that this excludes the last term $\int_{t_{n-1}}^{t} \Phi(t, s)g(s, x_s)ds$ on the right hand side of (7.16), since it is not in general evaluated up to an impulse moment t_k.

[4] We use the product symbol as
$\prod_{k=1}^{n-2} B(t_{n-k})\Phi(t_{n-k}, t_{n-k-1}) = B(t_{n-1})\Phi(t_{n-1}, t_{n-2}) \cdots B(t_2)\Phi(t_2, t_1)$.

as we explain right now. For this purpose, notice that for 3 couples we have

$$\Phi(t,t_{n-1})\int_{t_{n-4}}^{t_{n-3}}\left(\prod_{k=1}^{2}B(t_{n-k})\Phi(t_{n-k},t_{n-k-1})\right)B(t_{n-3})\Phi(t_{n-3},s)g(s,x_s)ds,$$

for 2 couples we have

$$\Phi(t,t_{n-1})\int_{t_{n-3}}^{t_{n-2}}\left(\prod_{k=1}^{1}B(t_{n-k})\Phi(t_{n-k},t_{n-k-1})\right)B(t_{n-2})\Phi(t_{n-2},s)g(s,x_s)ds.$$

The product notation has already exhausted the possibility of writing the term for one couple, unless we simply define the notation $\prod_{k=1}^{0}$ to mean

$$\prod_{k=1}^{0}B(t_{n-k})\Phi(t_{n-k},t_{n-k-1})\equiv I_d,$$

the identity linear operator on $\mathbb{R}^n$.
In this notation we group together all of the terms containing couples plus a final integral factor $\int_{t_{k-1}}^{t_k}\Phi(t_k,s)g(s,x_s)ds$ under a single summation symbol

$$\Phi(t,t_{n-1})\sum_{m=0}^{n-2}\int_{t_{n-m-2}}^{t_{n-m-1}}\left(\prod_{k=1}^{m}B(t_{n-k})\Phi(t_{n-k},t_{n-k-1})\right)B(t_{n-m-1})\Phi(t_{n-m-1},s)g(s,x_s)ds \tag{7.17}$$

where we take note that for $n=1$ we define $\sum_{m=0}^{-1}\equiv 0$, so that this summation contribution will not be seen in the case for $n=1$, where $t\in[t_0,t_1)$, in other words, no impulse has ocurred yet.

-TERMS OF TYPE II

The second type of terms that we group together come from observing in expression (7.16) that we have the following list of terms containing a final vector "factor" of the form $w(t_k,x_{t_k^-})$, for $k=1,...,n-1$:

- $\Phi(t,t_{n-1})w(t_{n-1},x_{t_{n-1}^-})$
- $\Phi(t,t_{n-1})\underbrace{B(t_{n-1})\Phi(t_{n-1},t_{n-2})}_{1\text{ "middle" couple}}w(t_{n-2},x_{t_{n-2}^-})$
- $\Phi(t,t_{n-1})\underbrace{B(t_{n-1})\Phi(t_{n-1},t_{n-2})B(t_{n-2})\Phi(t_{n-2},t_{n-3})}_{2\text{ "middle" couples}}w(t_{n-3},x_{t_{n-3}^-})$

  ....
  ....
- $\Phi(t,t_{n-1})\underbrace{B(t_{n-1})\Phi(t_{n-1},t_{n-2})B(t_{n-2})\cdots B(t_2)\Phi(t_2,t_1)}_{n-2\text{ "middle" couples}}w(t_1,x_{t_1^-}).$

In order to group these terms together using the product notation, we define in a similar fashion as we did for the previous terms of type I the notation

$$\prod_{k=1}^{0} B(t_{n-k})\Phi(t_{n-k}, t_{n-k-1}) \equiv I_d.$$

Thus for zero "middle couples", the shortest term of type II we have

$$\Phi(t, t_{n-1}) \left(\prod_{k=1}^{0} B(t_{n-k})\Phi(t_{n-k}, t_{n-k-1}) \right) w(t_{n-1}, x_{t_{n-1}^-}).$$

for one "middle couple":

$$\Phi(t, t_{n-1}) \left(\prod_{k=1}^{1} B(t_{n-k})\Phi(t_{n-k}, t_{n-k-1}) \right) w(t_{n-2}, x_{t_{n-2}^-}).$$

$$\vdots$$

for $n-2$ "middle couples":

$$\Phi(t, t_{n-1}) \left(\prod_{k=1}^{n-2} B(t_{n-k})\Phi(t_{n-k}, t_{n-k-1}) \right) w(t_1, x_{t_1^-}).$$

Similar to the terms of type I, we group the previous terms together under a single summation symbol:

$$\Phi(t, t_{n-1}) \sum_{m=0}^{n-2} \left(\prod_{k=1}^{m} B(t_{n-k})\Phi(t_{n-k}, t_{n-k-1}) \right) w(t_{n-m-1}, x_{t_{n-m-1}^-}). \tag{7.18}$$

Finally notice that in expression (7.16), we can write the first term as

$$\Phi(t, t_{n-1})B(t_{n-1}) \cdots B(t_2)\Phi(t_2, t_1)B(t_1)\Phi(t_1, t_0)x(t_0) =$$

$$\Phi(t, t_{n-1}) \left(\prod_{k=1}^{n-1} B(t_{n-k})\Phi(t_{n-k}, t_{n-k-1}) \right) \phi(0).$$

By simply adding the left out term $\int_{t_{n-1}}^{t} \Phi(t, s)g(s, x_s)ds$, we obtain the lemma.

□

We now state and prove the main result of this section. Notice how we will focus on sufficiently small initial conditions, similar to the way B. Zhang in [57] bounds initial conditions. This is necessary to obtain the contraction mapping. This is because stability is similar to continuity, but in the sense of global behavior of a solution, and with respect to initial conditions that start sufficiently close to the stable zero solution.

Remark 7.2. *In the following Theorem 7.1, notice that the first condition implies a Lipschitz condition on the nonlinearity of the continuous portion, in the sense of Definition 4.6. We have a situation that is completely similar to the case discussed in Section 6.4, and we can similarly prove a result similar to Lemma 6.1 and Proposition 6.1. We can prove existence-uniqueness of solutions of system* (7.1)-(7.2)*, by general theory of Chapter* **??***, so that the following contraction mapping result is finding the unique solution in a satisfactory way. Remember in Section 6.4 that there can be a caveat about merely using this fixed point method to prove existence of solutions.*

Theorem 7.1. *Suppose that there exist positive constants α, L and continuous functions $b, c : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that the following conditions hold:*

(i) *Let $g(t, 0) \equiv 0$, $|g(t, \phi) - g(t, \psi)| \leq b(t)\|\phi - \psi\|$ for all $\|\phi\|, \|\psi\| \leq L$.*

(ii) *Let $w(t, 0) \equiv 0$, $|w(t, \phi) - w(t, \psi)| \leq c(t)\|\phi - \psi\|$ for all $\|\phi\|, \|\psi\| \leq L$.*

(iii)
$$\int_{t_{k-1}}^{t_k} b(s)ds \leq \alpha \quad \text{for all } k, \qquad \text{and} \qquad c(t) \leq \alpha \quad \text{for all } t \geq 0. \tag{7.19}$$

(iv) *The fundamental matrix of the induced linear system* (7.3) *is bounded, in the sense that for every $k \geq 1$:*
$$\|\Phi(s_2, s_1)\| \leq K, \qquad \text{for every } s_1, s_2 \in [t_{k-1}, t_k) \tag{7.20}$$
for some $K > 0$ constant.[5]

(v) *For all $s_1, s_2 \in [t_{k-1}, t_k)$, $s_1 \leq s_2$ for every $k \geq 1$, we have that the fundamental matrix of the induced linear system* (7.3) *together with the operator $B(\cdot)$ satisfies*
$$\|B(s_2)\Phi(s_2, s_1)\| \leq \alpha. \tag{7.21}$$

(vi) *For every $\epsilon > 0$ and $T_1 \geq 0$, there exists a $T_2 > T_1$ such that $t \geq T_2$ and $\|x_t\| \leq L$ implies*
$$\begin{aligned} |g(t, x_t)| &\leq b(t)\left(\epsilon + \|x\|^{[T_1,t]}\right) \\ |w(t, x_{t^-})| &\leq c(t)\left(\epsilon + \|x\|^{[T_1,t]}\right). \end{aligned} \tag{7.22}$$

(vii) $\alpha < \min\{\frac{1}{3}, \frac{1}{2K+1}\}$.

Then the zero solution of (7.1) *is uniformly stable and asymptotically stable.*

Remarks.

- Notice that condition *(iii)* in (7.19) places a type of bound on the nonlinearities of the operator that defines the impulses. Condition (7.21) forces the linear portion of the jump to bring the image of the fundamental matrix Φ down to operator norm less than α, with the latter suitably defined. This gives a "contractive" requirement between jumps.

[5] The same K for every $k \geq 1$.

- $K \geq 1$ in (7.20), since $\Phi(t,t) = I_d$.
- Notice that in the definition of δ_0 below in the proof at (7.23), necessarily

$$1 - \frac{2K\alpha}{1-\alpha} > 0,$$

which implies that

$$K < \frac{1-\alpha}{2\alpha}.$$

Now, if $\frac{1-\alpha}{2\alpha} < 1$, then $K < 1$, which as previously mentioned, is not possible. Thus, necessarily $\frac{1-\alpha}{2\alpha} > 1$ which happens if and only if $\alpha < 1/3$. Also, $K < \frac{1}{\alpha}$ follows from the definition of δ_0, but notice that $\frac{1-\alpha}{2\alpha} < \frac{1}{\alpha}$.

- We will see in an example below, that condition (7.20) can be determined by how often impulses occur, and considerations similar to comments in Example 5.6 and ensuing Remark 5.4.
- This result holds for infinite delays $r = \infty$, or for finite delays $r < \infty$. For the latter case, by Lemma 5.2, condition (7.22) holds automatically for finite delays, so we can leave it out in this case.

Proof. Let us apply the Banach contraction method for stability. For the purpose of this, first we define, given ϕ as an initial condition. Let $\delta_0 > 0$ such that

$$\delta_0 \leq \min\left\{L,\ \frac{L(1-K\alpha)}{K},\ \frac{L}{K\alpha}\left(1 - \frac{2K\alpha}{1-\alpha}\right)\right\}. \tag{7.23}$$

Let us now define a suitable complete metric space. Let us define, using the initial condition ϕ, and the *fixed* impulse moments $\{t_k\}$ that would define the impulsive operator of the FDE, the space

$$\begin{aligned}\mathcal{S} = \{&x \in PCB([t_0 - r, \infty), D) : x_{t_0} = \phi,\ x_t \in B(L) \text{ for } t \geq t_0,\\ &x \text{ is discontinuous only at impulsive moments } t = t_k, \text{ and } x(t) \to 0 \text{ as } t \to \infty\}.\end{aligned}$$

$\mathcal{S}$ is a nonempty complete metric space under the metric

$$d_{\mathcal{S}}(x,y) = \sup_{s \in [t_0 - r, \infty)} |x(s) - y(s)| = \sup_{s \in [t_0, \infty)} |x(s) - y(s)| \text{ for } x, y \in \mathcal{S},$$

where we note that when calculating the distance between two elements of $\mathcal{S}$, we can disregard the contribution on the subinterval $[t_0 - r, t_0]$ because of the definition of $\mathcal{S}$, and we remind the reader that $[t_0 - r, t_0] = (-\infty, t_0]$ when $r = \infty$.

To obtain a suitable mapping, based on the previous lemma proved, we define the mapping P on $\mathcal{S}$ as

$$(Px)_{t_0} = \phi,$$

and for $t \geq t_0$:

$$\begin{aligned}
(Px)(t) = \Phi(t, t_{n-1}) \left(\prod_{k=1}^{n-1} B(t_{n-k}) \Phi(t_{n-k}, t_{n-k-1}) \right) \phi(0) + \int_{t_{n-1}}^{t} \Phi(t, s) g(s, x_s) ds \\
+ \Phi(t, t_{n-1}) \sum_{m=0}^{n-2} \int_{t_{n-m-2}}^{t_{n-m-1}} \left\{ \left(\prod_{k=1}^{m} B(t_{n-k}) \Phi(t_{n-k}, t_{n-k-1}) \right) \right. \\
\left. \times B(t_{n-m-1}) \Phi(t_{n-m-1}, s) g(s, x_s) ds \right\} \\
+ \Phi(t, t_{n-1}) \sum_{m=0}^{n-2} \left(\prod_{k=1}^{m} B(t_{n-k}) \Phi(t_{n-k}, t_{n-k-1}) \right) w(t_{n-m-1}, x_{t^-_{n-m-1}}).
\end{aligned} \tag{7.24}$$

The operators involved in the definition of P are continuous, and the Riemann integral is a continuous function. We only have discontinuities from the left side of impulse moments, where discrete transitions occuring at these impulse times involve the application of the transition rule in (7.4) at the next impulse moment, which is what might generate a discontinuity, but retains right continuity. Thus the mapping P retains the piecewise continuity of $x \in \mathcal{S}$.

Let us now show that $|(Px)(t)| \leq L$ for all t, where we remind ourselves that in the notation used here, $|\cdot|$ denotes the Euclidean norm. Of course this is true for $t \leq t_0$. For $t \in [t_0, t_1)$, we have that no impulse has acted yet (thus Φ has not been yet controlled or pushed down in norm by an impulse moment, which is why this case is separate), so that the the mapping $(Px)(t)$ reduces in this interval to

$$(Px)(t) = \Phi(t, t_0)\phi(0) + \int_{t_0}^{t} \Phi(t, s) g(s, x_s) ds.$$

Therefore if $t \in [t_0, t_1)$, using $\|x_s\| \leq L$ for all s,

$$\begin{aligned}
|(Px)(t)| &\leq \|\Phi(t, t_0)\| \, |\phi(0)| + \int_{t_0}^{t} \|\Phi(t, s)\| \, |g(s, x_s)| ds \\
&\leq K\delta_0 + K \int_{t_0}^{t} b(s) \|x_s\| ds \\
&\leq K\delta_0 + KL \int_{t_0}^{t} b(s) ds \\
&\leq K\delta_0 + KL\alpha \leq K \frac{L(1 - K\alpha)}{K} + KL\alpha = L,
\end{aligned}$$

so that Px remains bounded by L for $t \in [t_0, t_1)$.

For $n \geq 2$, we have that

$$
\begin{aligned}
|(Px)(t)| \leq \|\Phi(t,t_{n-1})\| &\left(\prod_{k=1}^{n-1} \left\|B(t_{n-k})\Phi(t_{n-k},t_{n-k-1})\right\|\right)|\phi(0)| + \int_{t_{n-1}}^{t} \|\Phi(t,s)\|\,|g(s,x_s)|ds \\
+ \|\Phi(t,t_{n-1})\| \sum_{m=0}^{n-2} \int_{t_{n-m-2}}^{t_{n-m-1}} &\left\{ \left(\prod_{k=1}^{m} \left\|B(t_{n-k})\Phi(t_{n-k},t_{n-k-1})\right\|\right) \right. \\
&\left. \times \left\|B(t_{n-m-1})\Phi(t_{n-m-1},s)\right\|\,|g(s,x_s)|ds \right\} \\
+ \|\Phi(t,t_{n-1})\| \sum_{m=0}^{n-2} &\left(\prod_{k=1}^{m} \left\|B(t_{n-k})\Phi(t_{n-k},t_{n-k-1})\right\|\right) \left|w(t_{n-m-1},x_{t^-_{n-m-1}})\right|.
\end{aligned}
\tag{7.25}
$$

Now we use a combination of the hypotheses stated, so that from the previous inequality it follows that:

$$
\begin{aligned}
|(Px)(t)| &\leq K\alpha^{n-1}\delta_0 + K\int_{t_{n-1}}^{t} b(s)\|x_s\|ds \\
&+ K\sum_{m=0}^{n-2}\alpha^m \int_{t_{n-m-2}}^{t_{n-m-1}} \alpha\, b(s)\|x_s\|ds \\
&+ K\sum_{m=0}^{n-2}\alpha^m c(t_{n-m-1})\left\|x_{t^-_{n-m-1}}\right\| \\
&\leq K\alpha^{n-1}\delta_0 + KL\alpha + KL\sum_{m=0}^{n-2}\alpha^{m+2} + KL\sum_{m=0}^{n-2}\alpha^{m+1} \\
&\leq K\alpha^{n-1}\delta_0 + KL\alpha + KL\alpha^2\sum_{m=0}^{\infty}\alpha^{m} + KL\alpha\sum_{m=0}^{\infty}\alpha^{m} \\
&= K\alpha^{n-1}\delta_0 + KL\alpha + KL\frac{\alpha^2}{1-\alpha} + KL\frac{\alpha}{1-\alpha} \\
&= K\alpha^{n-1}\delta_0 + 2LK\frac{\alpha}{1-\alpha} \\
&\leq K\alpha\delta_0 + 2KL\frac{\alpha}{1-\alpha} \\
&\leq K\alpha\frac{L}{K\alpha}\left(1-\frac{2K\alpha}{1-\alpha}\right) + 2KL\frac{\alpha}{1-\alpha} = L.
\end{aligned}
$$

From this it follows that $|(Px)(t)| \leq L$ for every t.

Now we show that $(Px)(t) \to 0$ as $t \to \infty$. For this purpose, we must show that given any $\epsilon > 0$, there exists a t^* such that $t > t^*$ implies $|(Px)(t)| < \epsilon$. Here we will make use of condition

(7.22), the fading memory condition. Since the mapping P is defined for elements of $\mathcal{S}$, and so $|x(t)| \to 0$, then for any given $\epsilon^* > 0$ (This ϵ^* will be a scalar multiple of the ϵ mentioned before, but for the moment we will call it ϵ^* and then suitably rescale it[6], so that the inequality $|(Px)(t)| < \epsilon$ is nicely achieved), there exists a $T_1 \geq t_0$ such that

$$|x(t)| < \epsilon^* \qquad \text{for all } \; t \geq T_1. \tag{7.26}$$

For this given ϵ^* and corresponding T_1, by (vi) there exists a $T_2 > T_1$ such that $t \geq T_2$ and $\|x_t\| \leq L$ implies

$$\begin{aligned} |g(t, x_t)| &\leq b(t)\left(\epsilon^* + \|x\|^{[T_1,t]}\right) \\ |w(t, x_{t^-})| &\leq c(t)\left(\epsilon^* + \|x\|^{[T_1,t]}\right). \end{aligned} \tag{7.27}$$

Suppose that for a certain $k^* \geq 1$, $T_2 \in [t_{k^*}, t_{k^*+1})$. From inequality (7.25), we have that

$$\begin{aligned} |(Px)(t)| \leq & K\alpha^{n-1}\delta_0 + K\int_{t_{n-1}}^{t} |g(s, x_s)|ds \\ & + K\sum_{m=0}^{n-2} \alpha^{m+1}\int_{t_{n-m-2}}^{t_{n-m-1}} |g(s, x_s)|ds \\ & + K\sum_{m=0}^{n-2} \alpha^m \Big|w(t_{n-m-1}, x_{t^-_{n-m-1}})\Big|. \end{aligned} \tag{7.28}$$

Thus, supposing t sufficiently large so that $t_{n-1} \geq T_2$ (since $t \in [t_{n-1}, t_n)$, the larger t is, the larger n is). For now, let $1 \leq N < n-2$ be an integer, we will give further conditions on N so that this integer is convenient for us in order to break up some upcoming sums into two parts. Notice that there exists T_3 so that if $t > T_3$ then n is sufficiently large, so that

$$K\alpha^{n-1}\delta_0 < \epsilon^*, \tag{7.29}$$

since $\alpha < 1$. Using (7.26) and (7.27) in inequality (7.28), along with $t_{n-1} \geq T_2$ and the previous inequality (7.29), so that we will want $t \geq \max\{T_2, T_3\}$, we have

$$\begin{aligned} |(Px)(t)| \leq & K\alpha^{n-1}\delta_0 + K\int_{t_{n-1}}^{t} b(s)\left(\epsilon^* + \|x\|^{[T_1,s]}\right)ds \\ & + K\sum_{m=0}^{n-2} \alpha^{m+1}\int_{t_{n-m-2}}^{t_{n-m-1}} |g(s, x_s)|ds \\ & + K\sum_{m=0}^{n-2} \alpha^m \Big|w(t_{n-m-1}, x_{t^-_{n-m-1}})\Big|. \end{aligned}$$

[6]We will see that $\epsilon^* = \frac{1-\alpha}{2(1-\alpha)+4K\alpha}\epsilon < \epsilon$ is sufficient.

$$
\begin{aligned}
&\leq \quad \epsilon^* + 2K\alpha\epsilon^* \\
&+ K\sum_{m=0}^{N}\alpha^{m+1}\int_{t_{n-m-2}}^{t_{n-m-1}}|g(s,x_s)|ds + K\sum_{m=N+1}^{n-2}\alpha^{m+1}\int_{t_{n-m-2}}^{t_{n-m-1}}|g(s,x_s)|ds \\
&+ K\sum_{m=0}^{N}\alpha^{m}\Big|w(t_{n-m-1},x_{t^-_{n-m-1}})\Big| + K\sum_{m=N+1}^{n-2}\alpha^{m}\Big|w(t_{n-m-1},x_{t^-_{n-m-1}})\Big|
\end{aligned}
\tag{7.30}
$$

where of course, in the last inequality we just broke up the sums into two parts. Notice, for example in the term

$$\sum_{m=0}^{N}\alpha^{m+1}\int_{t_{n-m-2}}^{t_{n-m-1}}|g(s,x_s)|ds,$$

that the integrals involved in the sum are over the *last* intervals $[t_{n-N-2}, t_{n-N-1}), ..., [t_{n-2}, t_{n-1})$. Remembering we supposed that for a certain $k^* \geq 1$, $T_2 \in [t_{k^*}, t_{k^*+1})$, we will need to choose N not too large so that $t_{n-N-2} \geq t_{k^*+1} > T_2$ so that we can apply the fading memory condition (7.27). The fading memory condition along with (7.26), will essentially allow us to send this first part of the sum to zero. On the other hand, the second part of the sum, namely

$$\sum_{m=N+1}^{n-2}\alpha^{m+1}\int_{t_{n-m-2}}^{t_{n-m-1}}|g(s,x_s)|ds,$$

involves the *first* time intervals $[t_0, t_1), ..., [t_{n-N-3}, t_{n-N-2})$. These involve smaller times $t < t_{n-N-2}$, so this part does not allow the application of the fading memory condition. However, combining the Lipschitz type conditions along with the bounds (7.19), we realize that we end up with the tail of a convergent series. Thus for N sufficiently large, we can make this contribution sufficiently small. Something completely similar occurs for the terms

$$\sum_{m=0}^{N}\alpha^{m}\Big|w(t_{n-m-1},x_{t^-_{n-m-1}})\Big| \qquad \text{and} \qquad \sum_{m=N+1}^{n-2}\alpha^{m}\Big|w(t_{n-m-1},x_{t^-_{n-m-1}})\Big|,$$

respectively. Notice that we need a suitable N to fulfill both purposes. We need to satisfy N sufficiently large to make the tail of the series small, and also $n - N - 2 \geq k^* + 1$, to apply the fading conditions. $n - N - 2 \geq k^* + 1$ implies $N \leq n - k^* - 3$. Thus

$$N = n - k^* - 4$$

is enough. Notice that as t gets larger, n does, so N defined in this way also gets larger, guaranteeing that the tail of both of the convergent series involved can become arbitrarily small. With all of this in mind, we do as follows.

First off, with N as defined above, the fading memory condition together with (7.26) implies that for the sums involving the final time intervals $[t_{n-N-2}, t_{n-N-1}), ..., [t_{n-2}, t_{n-1})$, since $t_{n-N-2} >$

T_2:

$$\begin{aligned} K\sum_{m=0}^{N}\alpha^{m+1}\int_{t_{n-m-2}}^{t_{n-m-1}}|g(s,x_s)|ds &\le K\sum_{m=0}^{N}\alpha^{m+1}\int_{t_{n-m-2}}^{t_{n-m-1}}b(s)\left(\epsilon^*+\|x\|^{[T_1,s]}\right)ds \\ &\le 2K\epsilon^*\sum_{m=0}^{N}\alpha^{m+1}\int_{t_{n-m-2}}^{t_{n-m-1}}b(s)ds \\ &\le 2K\epsilon^*\sum_{m=0}^{N}\alpha^{m+2}\le 2K\epsilon^*\frac{\alpha^2}{1-\alpha}. \end{aligned} \tag{7.31}$$

In a similar way, we have that we can apply the memoryless condition so that

$$\begin{aligned} K\sum_{m=0}^{N}\alpha^{m}\left|w(t_{n-m-1},x_{t^-_{n-m-1}})\right| &\le K\sum_{m=0}^{N}\alpha^{m}c(t_{n-m-1})\left(\epsilon^*+\left\|x_{t^-_{n-m-1}}\right\|\right) \\ &\le 2K\epsilon^*\sum_{m=0}^{N}\alpha^{m+1}\le 2K\epsilon^*\frac{\alpha}{1-\alpha} \end{aligned} \tag{7.32}$$

For the other parts of the sums, we only use the Lipschitz type condition (without memoryless part), so that

$$\begin{aligned} K\sum_{m=N+1}^{n-2}\alpha^{m+1}\int_{t_{n-m-2}}^{t_{n-m-1}}|g(s,x_s)|ds &\le K\sum_{m=N+1}^{n-2}\alpha^{m+1}\int_{t_{n-m-2}}^{t_{n-m-1}}b(s)\|x_s\|ds \\ &\le KL\sum_{m=N+1}^{n-2}\alpha^{m+1}\int_{t_{n-m-2}}^{t_{n-m-1}}b(s)ds \\ &\le KL\sum_{m=N+1}^{n-2}\alpha^{m+2}\le KL\sum_{m=N+1}^{\infty}\alpha^{m+2}. \end{aligned} \tag{7.33}$$

In a similar fashion

$$\begin{aligned} K\sum_{m=N+1}^{n-2}\alpha^{m}\left|w(t_{n-m-1},x_{t^-_{n-m-1}})\right| &\le K\sum_{m=N+1}^{n-2}\alpha^{m}c(t_{n-m-1})\left\|x_{t^-_{n-m-1}}\right\| \\ &\le KL\sum_{m=N+1}^{n-2}\alpha^{m+1}\le KL\sum_{m=N+1}^{\infty}\alpha^{m+1}. \end{aligned} \tag{7.34}$$

Since $N = n - k^* - 4$, choosing t large enough makes n large enough, so that we can easily make contributions from (7.33) and (7.34) (which involve tails of convergent series) satisfy

$$KL\sum_{m=N+1}^{\infty}\alpha^{m+2}+KL\sum_{m=N+1}^{\infty}\alpha^{m+1}<\epsilon^* \tag{7.35}$$

Using inequalities (7.31),(7.32) and (7.35) in inequality (7.30), we conclude that for t large enough:

$$|(Px)(t)| < \epsilon^* + 2K\alpha\epsilon^* + 2K\epsilon^*\frac{\alpha^2}{1-\alpha} + 2K\epsilon^*\frac{\alpha}{1-\alpha} + \epsilon^* = 4K\epsilon^*\frac{\alpha}{1-\alpha} + 2\epsilon^*$$
$$= \epsilon^*\left(2 + \frac{4K\alpha}{1-\alpha}\right).$$

Thus, given $\epsilon > 0$, if $\epsilon^* = \frac{1-\alpha}{2(1-\alpha)+4K\alpha}\epsilon$ we can find $t^* = \max\{T_3, t_{k^*+1}\}$ sufficiently large so that $|(Px)(t)| < \epsilon$ for $t \geq t^*$.

This proves that P is a mapping from $\mathcal{S}$ to itself. We now prove that $P : \mathcal{S} \longrightarrow \mathcal{S}$ is a contraction. For this, let $x, y \in \mathcal{S}$. By definition of $\mathcal{S}$ we have that $(Px)(t) - (Py)(t) = 0$ for $t \in [t_0 - r, t_0]$. For $t \geq t_0$ we get:

$$|(Px)(t) - (Py)(t)| =$$

$$\Bigg| \int_{t_{n-1}}^{t} \Phi(t,s)\Big[g(s,x_s) - g(s,y_s)\Big]ds$$
$$+ \Phi(t,t_{n-1})\sum_{m=0}^{n-2}\int_{t_{n-m-2}}^{t_{n-m-1}}\Bigg\{\left(\prod_{k=1}^{m} B(t_{n-k})\Phi(t_{n-k},t_{n-k-1})\right)$$
$$\times B(t_{n-m-1})\Phi(t_{n-m-1},s)\Big[g(s,x_s) - g(s,y_s)\Big]ds\Bigg\}$$
$$+ \Phi(t,t_{n-1})\sum_{m=0}^{n-2}\left(\prod_{k=1}^{m} B(t_{n-k})\Phi(t_{n-k},t_{n-k-1})\right)\Big[w(t_{n-m-1},x_{t^-_{n-m-1}}) - w(t_{n-m-1},y_{t^-_{n-m-1}})\Big]\Bigg|$$

$$\leq K\int_{t_{n-1}}^{t}\Big|g(s,x_s) - g(s,y_s)\Big|ds$$
$$+ K\sum_{m=0}^{n-2}\alpha^{m+1}\int_{t_{n-m-2}}^{t_{n-m-1}}\Big|g(s,x_s) - g(s,y_s)\Big|ds$$
$$+ K\sum_{m=0}^{n-2}\alpha^{m}\Big[w(t_{n-m-1},x_{t^-_{n-m-1}}) - w(t_{n-m-1},y_{t^-_{n-m-1}})\Big]\Bigg|$$

$$\leq K\int_{t_{n-1}}^{t} b(s)\|x_s - y_s\|ds + K\sum_{m=0}^{n-2}\alpha^{m+1}\int_{t_{n-m-2}}^{t_{n-m-1}} b(s)\|x_s - y_s\|ds$$
$$+ K\sum_{m=0}^{n-2}\alpha^{m}c(t_{n-m-1})\left\|x_{t^-_{n-m-1}} - y_{t^-_{n-m-1}}\right\|$$

$$\leq \alpha K\, d_{\mathcal{S}}(x,y) + K\, d_{\mathcal{S}}(x,y) \sum_{m=0}^{\infty} \alpha^{m+2} + K\, d_{\mathcal{S}}(x,y) \sum_{m=0}^{\infty} \alpha^{m+1}$$
$$= \left(\frac{2K\alpha}{1-\alpha}\right) d_{\mathcal{S}}(x,y).$$

where recall that $d_{\mathcal{S}}(x,y) = \sup_{s\in[t_0,\infty)} |x(s) - y(s)|$. From the bound assumed on α, we have that $\overline{\alpha} := \frac{2K\alpha}{1-\alpha} < 1$ defines a contraction constant[7] for P on the complete metric space $\mathcal{S}$. Thus P is a contraction on $\mathcal{S}$. This implies that there is a unique solution to (7.1) with initial condition[8] (7.2).

By definition of $\mathcal{S}$, we already have that the solution to the initial value problem (7.1) converges to zero. We must prove that the solution is stable.

To prove uniform stability, assume that we are given an $0 < \epsilon < L$. Choose

$$\delta < \min\left\{\epsilon,\ \frac{\epsilon(1-K\alpha)}{K},\ \frac{\epsilon}{K\alpha}\left(1 - \frac{2K\alpha}{1-\alpha}\right)\right\}.$$

Notice that δ is independent of t_0. For $\|\phi\| \leq \delta < \epsilon$, we claim that $|x(t)| \leq \epsilon$ for all $t \geq t_0$. Note that if x is the unique solution corresponding to the initial condition ϕ, then $|x(t_0)| = |\phi(0)| < \epsilon$. For the sake of contradiction suppose that there exists a $\hat{t} > t_0$ such that $|x(\hat{t})| > \epsilon$. Let

$$t^* = \inf\{\hat{t} : |x(\hat{t})| > \epsilon\}.$$

By right continuity, $|x(t^*)| > \epsilon$ occurs in either a continuous way or as a consequence of a jump at t^*. Whatever the case, we have $|x(s)| \leq \epsilon$ for $s \in [t_0 - r, t^*)$. First suppose that $t_0 \leq t < t_1$, in other words, that no impulse has occurred yet. Notice $\epsilon < L$ allows application of the Lipschitz-type bounds. Then

$$\begin{aligned} |x(t^*)| &\leq \|\Phi(t^*,t_0)\||\phi(0)| + \int_{t_0}^{t^*} \|\Phi(t^*,s)\||g(s,x_s)|ds \\ &\leq K\delta + K\int_{t_0}^{t^*} b(s)\|x_s\|ds \\ &\leq K\delta + K \sup_{\theta\in[t_0-r,t^*)} |x(\theta)| \int_{t_0}^{t^*} b(s)ds \\ &\leq K\delta + K\alpha\epsilon < \epsilon \end{aligned}$$

and this gives us the desired contradiction, by the definition of t^*, since $|x(t^*)| = \epsilon$ at continuity points, and $|x(t^*)| \geq \epsilon$ otherwise. For $t \geq t_1$ we have, in a similar way that we obtained

[7]Because $\alpha < \frac{1}{2K+1}$ implies that with K and α given as above, $K < \frac{1-\alpha}{2\alpha}$.

[8]See Remark 7.2 above, for necessary clarity about "uniqueness". We have uniqueness in a satisfactory PCB-space, not just within $\mathcal{S}$, by general existence-uniqueness theory independent of the fixed point theorem used here, by considerations similar to Section 5.4 and Section 6.4 in previous chapters.

inequality (7.28) that

$$\begin{aligned}
|(x(t^*)| \leq & K\alpha^{n-1}\delta + K\int_{t_{n-1}}^{t^*} b(s)\|x_s\|ds \\
& + K\sum_{m=0}^{n-2}\alpha^{m+1}\int_{t_{n-m-2}}^{t_{n-m-1}} b(s)\|x_s\|ds \\
& + K\sum_{m=0}^{n-2}\alpha^{m}c(t_{n-m-1})\left\|x_{t^-_{n-m-1}}\right\| \\
& \leq K\alpha\delta + K\epsilon\alpha + K\epsilon\sum_{m=0}^{\infty}\alpha^{m+2} + K\epsilon\sum_{m=0}^{\infty}\alpha^{m+1} \\
& \quad = K\alpha\delta + \epsilon\frac{2K\alpha}{1-\alpha} < \epsilon,
\end{aligned}$$

by the choice of δ, and now this gives us the desired contradiction for $t \geq t_1$ (so that $n \geq 2$), by the definition of t^*. Therefore the solution is uniformly stable, and since $x(t)$ converges to zero as $t \to \infty$, we also get asymptotic stability of trajectories.

□

Remark 7.3. *We notice that, in order to obtain contraction conditions, we may suppose a host of different types of inequalities in order to obtain this. The problem is how to put all of the conditions together so that they do not contradict each other or make the proof too difficult due to requiring bounds that become very hard to calculate if you assume an inconvenient set of independent hypotheses.*

Remark 7.4. *Notice* $\|\Phi(s_2, s_1)\| \leq K$ *for every* $s_1, s_2 \in [t_{k-1}, t_k)$ *for all* $k \geq 1$ *gives a uniform bound. The requirement* $s_1, s_2 \in [t_{k-1}, t_k)$*, instead of saying that* $s_2 \geq s_1 \in [0, \infty)$ *as in Theorem 5.2 or Theorem 6.2 of previous chapters, is because in this situation our operator* Φ *is always interrupted by the impulse operator at impulsive moments* t_k*. The impulsive operator guarantees a contractive requirement before letting* Φ *continue again. This intervention by impulses was not something needed to consider in the aforementioned results of previous chapters. We will see in examples below how this interruption plays a role in determining* K *or making it smaller.*

7.4 A Particular Linear Case

Now, suppose we have the following simple version, where $g(t, x_t) = M(t)x(t - r(t))$, with $t - r(t) \longrightarrow \infty$ as $t \to \infty$, and $M(t)$ a continuous time-varying matrix of dimension $n \times n$. Also suppose that the impulsive operator has no nonlinearities.

$$\begin{aligned}
x'(t) &= A(t)x(t) + M(t)x(t - r(t)), && t \neq t_k, t \geq 0 \\
\Delta x(t) &= [B(t) - I_d]x(t^-), && t = t_k, t \geq 0 \\
x_{t_0} &= \phi,
\end{aligned} \tag{7.36}$$

or equivalently:

$$\begin{aligned} x'(t) &= A(t)x(t) + M(t)x(t-r(t)), && t \neq t_k, t \geq 0 \\ x(t_k^+) &= B(t_k)x(t_k^-), && t = t_k, t \geq 0 \\ x_{t_0} &= \phi. \end{aligned} \tag{7.37}$$

The next result is a linear version of Theorem 7.1.

Corollary 7.1. *Suppose that in the linear FDE* (7.36), $t - r(t) \longrightarrow \infty$ *as* $t \to \infty$, *that there exists a positive constant* α, *and a continuous function* $b : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *such that the following conditions hold:*

(i) $M(t)$ *has its operator norm bounded* $\|M(t)\| \leq b(t)$, *for all* $t \geq 0$, *and*[9]

$$\int_{t_{k-1}}^{t_k} b(s)ds \leq \alpha \quad \text{for all } k. \tag{7.38}$$

(ii) The fundamental matrix of the induced linear system (7.3) *is bounded, in the sense that for every* $k \geq 1$*:*

$$\|\Phi(s_2, s_1)\| \leq K, \qquad \text{for every } s_1, s_2 \in [t_{k-1}, t_k) \tag{7.39}$$

for some $K > 0$ *constant.*

(iii) For all $s_1, s_2 \in [t_{k-1}, t_k)$, $s_1 \leq s_2$ *for every* $k \geq 1$, *we have that the fundamental matrix of the induced linear system* (7.3) *together with the operator* $B(\cdot)$ *satisfies*

$$\|B(s_2)\Phi(s_2, s_1)\| \leq \alpha. \tag{7.40}$$

(iv) $\alpha < \min\{\frac{1}{3}, \frac{1}{2K+1}\}$.

Then the zero solution of (7.36) *is uniformly stable and asymptotically stable, for arbitrarily large initial conditions* ϕ. *Thus we have global asymptotic stability.*

Proof. Notice that the δ_0 in (7.23) depends on L proportionally, and L is where the Lipschitz condition (7.19) holds. But in this case, we do not have a nonlinearity that forces a local Lipschitz condition, so L can be arbitrarily large. Thus asymptotic convergence holds, no matter how large the initial condition is.

We now just need to prove that the fading memory condition holds in case of infinite delay. By Lemma 5.2, for finite delays this is automatically satisfied (and $t - r(t) \longrightarrow \infty$, if $0 \leq r(t) \leq r$). The proof that condition (7.22) holds is similar to how we did in Example 5.1, as we illustrate: By hypothesis, we have that $t - r(t) \longrightarrow \infty$ as $t \to \infty$. This divergence to infinity implies that given $\epsilon > 0$ and $T_1 \geq 0$, there exists $T_2 > T_1$ such that $t - r(t) \geq T_1$ for all $t \geq T_2$. Given that $r(t) \geq 0$, this implies that for T_2 as defined, it is true that $t - r(t) \in [T_1, t]$ for every

[9] Notice that $b(t) := \|M(t)\|$ also works, but perhaps knowing this exactly is too difficult, so using matrix bounds one can settle for an upper estimate.

$t \geq T_2$. Putting together the information we have so far, we have that given $\epsilon > 0$ and $T_1 \geq 0$, it is true that there exists a $T_2 > T_1$ such that using $\|x\|^{[T_1,t]} = \sup_{\theta \in [T_1,t]} |x(\theta)|$:

$$|x_t(-r(t))| = |x(t-r(t))| \leq \|x\|^{[T_1,t]} \leq \epsilon + \|x\|^{[T_1,t]} \quad \text{for } t \geq T_2,$$

because $t - r(t) \in [T_1, t]$ for every $t \geq T_2$. Thus

$$\begin{aligned} |g(t,x_t)| \leq \|M(t)\| |x_t(-r(t))| &\leq |b(t)||x_t(-r(t))| \\ &\leq |b(t)| \Big(\epsilon + \|x\|^{[T_1,t]} \Big). \end{aligned}$$

□

7.5 Scalar Cases

Suppose that we have the scalar version of the previous result:

$$\begin{aligned} x'(t) &= -a(t)x(t) + g(t,x_t), && t \neq t_k, t \geq 0 \\ \Delta x(t) &= [u(t)-1]x(t^-) + w(t, x_{t^-}), && t = t_k, t \geq 0 \\ x_{t_0} &= \phi, \end{aligned} \tag{7.41}$$

where all functions involved are scalar valued, or equivalently:

$$\begin{aligned} x'(t) &= -a(t)x(t) + g(t,x_t), && t \neq t_k, t \geq 0 \\ x(t_k^+) &= u(t_k)x(t_k^-) + w(t_k, x_{t_k^-}), && t = t_k, t \geq 0 \\ x_{t_0} &= \phi. \end{aligned} \tag{7.42}$$

The following is immediate, since $\Phi(s_2, s_1) = e^{-\int_{s_1}^{s_2} a(u)du}$ for this scalar case.

Corollary 7.2. *Suppose that there exist positive constants α, L and continuous functions $b, c : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that the following conditions hold:*

(i) Let $g(t,0) \equiv 0$, $|g(t,\phi) - g(t,\psi)| \leq b(t)\|\phi - \psi\|$ for all $\|\phi\|, \|\psi\| \leq L$.

(ii) Let $w(t,0) \equiv 0$, $|w(t,\phi) - w(t,\psi)| \leq c(t)\|\phi - \psi\|$ for all $\|\phi\|, \|\psi\| \leq L$.

(iii)

$$\int_{t_{k-1}}^{t_k} b(s)ds \leq \alpha \quad \text{for all } k, \qquad \text{and} \qquad c(t) \leq \alpha \quad \text{for all } t \geq 0. \tag{7.43}$$

(iv) For every $k \geq 1$, we have the bound:

$$e^{-\int_{s_1}^{s_2} a(u)du} \leq K, \qquad \text{for every } s_1, s_2 \in [t_{k-1}, t_k) \tag{7.44}$$

for some $K > 0$ constant.

(v) *For all* $s_1, s_2 \in [t_{k-1}, t_k)$*,* $s_1 \le s_2$ *for every* $k \ge 1$*, we have that the function* $a(t)$ *together with the function* $u(t)$ *satisfies*

$$|u(s_2)|e^{-\int_{s_1}^{s_2} a(u)du} \le \alpha. \tag{7.45}$$

(vi) *For every* $\epsilon > 0$ *and* $T_1 \ge 0$*, there exists a* $T_2 > T_1$ *such that* $t \ge T_2$ *and* $\|x_t\| \le L$ *implies*

$$\begin{aligned} |g(t, x_t)| &\le b(t)\left(\epsilon + \|x\|^{[T_1,t]}\right) \\ |w(t, x_{t^-})| &\le c(t)\left(\epsilon + \|x\|^{[T_1,t]}\right). \end{aligned} \tag{7.46}$$

(vii) $\alpha < \min\{\frac{1}{3}, \frac{1}{2K+1}\}$.

Then the zero solution of (7.41) *is uniformly stable and asymptotically stable.*

Remark 7.5. *Remember that the fading memory conditions in* (7.46) *are useful only for the infinite delay case. By Lemma 5.2, for finite delays we can throw out these hypotheses* (7.46).

Now, suppose we have the following simple scalar version, where $g(t, x_t) = b(t)x(t - r(t))$, with $t - r(t) \longrightarrow \infty$ as $t \to \infty$. Also suppose that there is no nonlinearity in the impulsive operator.

$$\begin{aligned} x'(t) &= -a(t)x(t) + b(t)x(t - r(t)), && t \ne t_k, t \ge 0 \\ \Delta x(t) &= [u(t) - 1]x(t^-), && t = t_k, t \ge 0 \\ x_{t_0} &= \phi, \end{aligned} \tag{7.47}$$

where all functions involved are scalar valued, or equivalently:

$$\begin{aligned} x'(t) &= -a(t)x(t) + b(t)x(t - r(t)), && t \ne t_k, t \ge 0 \\ x(t_k^+) &= u(t_k)x(t_k^-), && t = t_k, t \ge 0 \\ x_{t_0} &= \phi. \end{aligned} \tag{7.48}$$

We have the following result. Notice that the proof is completely similar to Corollary 7.1, and the stability properties will now hold *globally*, in other words, the initial condition can be arbitrarily large and we will still have asymptotic stability, because L in (7.19), (7.23) can be arbitrarily large.

Corollary 7.3. *Suppose that in the scalar FDE* (7.47)*,* $t - r(t) \longrightarrow \infty$ *as* $t \to \infty$*, and there exists a positive constant* α *such that the following conditions hold:*

(i)

$$\int_{t_{k-1}}^{t_k} |b(s)|ds \le \alpha \quad \text{for all } k. \tag{7.49}$$

(ii) *For every* $k \ge 1$*, we have the bound:*

$$e^{-\int_{s_1}^{s_2} a(u)du} \le K, \qquad \text{for every } s_1, s_2 \in [t_{k-1}, t_k) \tag{7.50}$$

for some $K > 0$ *constant.*

(iii) *For all* $s_1, s_2 \in [t_{k-1}, t_k)$, $s_1 \leq s_2$ *for every* $k \geq 1$, *we have that the function* $a(t)$ *together with the function* $u(t)$ *satisfies*

$$|u(s_2)|e^{-\int_{s_1}^{s_2} a(u)du} \leq \alpha. \tag{7.51}$$

(iv) $\alpha < \min\{\frac{1}{3}, \frac{1}{2K+1}\}$.

Then the zero solution of (7.47) *is uniformly stable and asymptotically stable, for arbitrarily large initial conditions* ϕ. *Thus we have global asymptotic stability.*

7.6 Examples

We will now give some examples for the scalar results. Notice that we use the notation of the most general result Theorem 7.1, because we want to emphasize how the vector versions of the results are inspired from scalar considerations.

Example 7.1. Let us have the delay differential equation similar to examples given previously, such as Example 5.2:

$$x' = -\frac{1}{1+t^2}x + \frac{3}{4+\sin t}x(t - r(t)), \qquad t \geq 0 \tag{7.52}$$

where $t - r(t) \longrightarrow \infty$ as $t \to \infty$. Here $a(t) = \frac{1}{1+t^2}$. First of all, it is not true that $a(t) > J|b(t)|$, with $J > 1$ some constant, where $\frac{3}{5} \leq b(t) = \frac{3}{4+\sin t} \leq 1$. It is also not true that $\int_0^t a(s)ds \longrightarrow \infty$ as $t \to \infty$, since

$$\lim_{t\to\infty} \int_0^t a(s)ds = \frac{\pi}{2}.$$

Therefore, conditions of previous results from earlier chapters do not apply here, as they did in similar examples, such as Example 5.1 or Example 3.4, with either Lyapunov techniques or with the contraction method for stability. Therefore we will apply impulses to correct this behavior.

Let us identify the elements of Theorem 7.1 or Corollary 7.3 in this particular example. For all $s_2 > s_1$, we have that the bound

$$\|\Phi(s_2, s_1)\| = e^{-\int_{s_1}^{s_2} a(s)ds} \leq 1,$$

because $a(s) > 0$, so letting $K = 1$ is sufficient. Let us take, for simplicity of illustration linear impulses dependent on the last state, modeled by

$$x(t_k^+) = B(t_k)x(t_k^-), \tag{7.53}$$

where we can add small nonlinear perturbations $w(t, x_t)$ that are Lipschitz continuous according to the conditions of Theorem 7.1 or Corollary 7.3. For (7.52) with impulses (7.53), the Lipschitz conditions are clearly satisfied, where L can be as large as we like. The fading memory type condition is implied by $t - r(t) \longrightarrow \infty$ as $t \to \infty$, as we proved in the scalar Corollary 7.3.

Now let us characterize $B(t)$. As long as for all $s_2 > s_1$

$$\|B(s_2)\Phi(s_2, s_1)\| = |B(s_2)|e^{-\int_{s_1}^{s_2} a(u)du} \leq \alpha < \min\left\{\frac{1}{3}, \frac{1}{2K+1}\right\} = \frac{1}{3},$$

we can conclude asymptotic stability for arbitrarily large initial conditions ϕ, since L can be arbitrarily large, by linearity of $g(t, x_t)$ in this case. If $B(t)x(t^-) = \frac{1}{4}x(t^-)$ is a constant impulse, for example, then since $K = 1$, then for all $s_2 > s_1$

$$\|B(s_2)\Phi(s_2, s_1)\| = |B(s_2)|e^{-\int_{s_1}^{s_2} a(u)du} \leq K\|B(s_2)\| \leq \frac{1}{4}.$$

So if $\alpha = \frac{1}{4}$, we satisfy the necessary condition for the combination of the impulse operator and the linear portion. Now, for the condition on $b(t)$ in (7.49), we have that $\frac{3}{5} \leq b(t) \leq 1$, so a sufficient condition is to have

$$\int_{t_{k-1}}^{t_k} b(s)ds \leq t_k - t_{k-1} \leq \alpha = \frac{1}{4},$$

this gives us an upper bound on how often the impulses must occur, in order to send the system to an asymptotically zero equilibrium. Therefore the conditions of the theorem hold. $\triangle$

Of course, in any application, maybe finding a suitable, for example, optimal $B(t)$ is the next step, and the condition $t_k - t_{k-1} \leq \alpha = \frac{1}{4}$ obtained above implicitly gives us an idea of a cost, since how often we apply impulses to stabilize the system induces a cost to us.

We give another simple example to illustrate elements of Theorem 7.1 and Corollary 7.3 for a different scenario which we considered as unmanageable by previous results from Chapter 5 in Example 5.5, with respect to how badly we can violate the conditions from examples 5.2 and 3.4 given before.

Example 7.2. Suppose now

$$x' = x(t) + (2 + \cos(t^2))x(t - r(t)), \qquad t \geq 0 \tag{7.54}$$

with $t - r(t) \longrightarrow \infty$ as $t \to \infty$. Here we have that $a(t) = -1$ for all $t \geq 0$, which is a bad case scenario that we studied in Example 5.5, since the fixed point method only allows for $a(t)$ to be negative on occasions, as long as on average it is positive, $a(t) > J|b(t)|$, with $J > 1$ some constant, and $\int_0^t a(s)ds \longrightarrow -\infty$ as $t \to \infty$. Here none of those conditions hold, and it is the worst case scenario because $a(t)$ is always negative, $\int_0^t a(s)ds \longrightarrow -\infty$, and so

$$\lim_{t\to\infty} \|\Phi(t, 0)\| = \lim_{t\to\infty} e^{-\int_0^t a(s)ds} = \infty.$$

Let us determine what impulses can correct this behavior, under the fixed point conditions we obtained. In this particular example, for all $s_2 > s_1$, we have that the operator norm for each $s_2 \geq s_1$

$$\|\Phi(s_2, s_1)\| = e^{s_2 - s_1},$$

which rapidly goes to large values of the operator norm if the difference $s_2 - s_1$ is too large. Notice in the bound on the operator norm given in Theorem 7.1, in (7.20), or for the scalar case, bound (7.50) in Corollary 7.3, the bound is for $s_1, s_2 \in [t_{k-1}, t_k)$, in other words, on the same interval between impulses. This consideration factors into how often the impulses must be applied, in order to make K finite, as a first step to achieve asymptotic stability. The other criterion for how often to apply impulses is

$$\int_{t_{k-1}}^{t_k} b(s)ds = \int_{t_{k-1}}^{t_k} (2 + \cos(s^2))ds \leq 3(t_k - t_{k-1}) \leq \alpha.$$

Then if, say $\alpha = 1/4$ again, we need $t_k - t_{k-1} \leq \frac{1}{12}$. Let us take, as in the previous example, the linear impulses defined by

$$x(t_k^+) = B(t_k)x(t_k^-).$$

Notice that now, with the impulses spaced as required above, $s_2 - s_1 \leq \frac{1}{12}$, so that we can take for each $s_2 \geq s_1$ such that $s_1, s_2 \in [t_{k-1}, t_k)$:

$$\|\Phi(s_2, s_1)\| = e^{-\int_{s_1}^{s_2} a(s)ds} = e^{s_2 - s_1} \leq e^{\frac{1}{12}} =: K. \tag{7.55}$$

We have that $\alpha < \min\left\{\frac{1}{3}, \frac{1}{2K+1}\right\} = \frac{1}{2K+1} \approx 0.315$. If $\|B(t)\| < \frac{1}{4}e^{-\frac{1}{12}}$, for example, $B(t)x(t^-) = \frac{1}{5}e^{-\frac{1}{12}}x(t^-)$ is a constant impulse, then for all $s_2 > s_1$

$$\|B(s_2)\Phi(s_2, s_1)\| \leq K\|B(s_2)\| \leq \frac{1}{5} < \alpha.$$

Thus we obtain uniform stability and asymptotic stability under these conditions. △

Remark 7.6. *Notice how we placed the impulses to intervene in the divergence of the operator norm* $\|\Phi(s_2, s_1)\|$ *and make it sufficiently small in* (7.55)*. We quickly see how applying impulses more frequently helps here, though of course in applications, this induces a greater cost.*

Chapter 8

Stability of a Switched FDE by Contraction Principle

8.1 Introduction

We now apply the contraction method for stability to a type of delayed nonlinear switched functional differential equation. We will see that a new difficulty arises as we try to apply the Banach contraction principle to obtain stability in the same spirit as the earlier chapters. We will extend the result of Theorem 5.2 to switched systems, so that contrary to the previous chapter, we will assume that all of the subsystems involved are well-behaved. The main new item that we will encounter is a dwell time condition, even for subsystems that are asymptotically stable under the result by B. Zhang in [57] that we studied in Theorem 5.2. We will see these difficulties in an example after the main result is proved.

8.2 Preliminaries

Using fixed point theory, conditions for stability of the switched delayed differential equation

$$\begin{aligned} x'(t) &= A_{\sigma(t)}(t)x(t) + g_{\sigma(t)}(t, x_t) \qquad t \geq t_0 \\ x_{t_0} &= \phi \end{aligned} \tag{8.1}$$

are given. Here, we have that $x(t) \in \mathbb{R}^n$, $g_{\sigma(t)} : J \times BC([-r, 0], D) \longrightarrow \mathbb{R}^n$ with $J \subset \mathbb{R}^+$ an interval $t_0 \geq 0$, and $D \subset \mathbb{R}^n$ is an open set. $A_{\sigma(t)}(t)$ are $n \times n$ continuous matrices, in the sense that all entries of these matrices are continuous functions in the interval of definition of the functional differential equation (8.1). The indexing set for the vector fields

$$f_i(t, x_t) := A_i(t)x(t) + g_i(t, x_t)$$

is $\mathcal{P} = \{1, ..., M\}$ for some *finite* positive integer M. The switching rule $\sigma : [t_{k-1}, t_k) \longrightarrow \mathcal{P}$ with $1 \leq k \leq N \leq \infty$ takes on values in the indexing set $\mathcal{P}$ for the vector fields, so that when $\sigma(t) = i \in \mathcal{P}$ we have $f_{\sigma(t)}(t, x_t) := f_i(t, x_t)$. We will use only deterministic time dependent admissible switching rules, so that σ is a piecewise constant right continuous mapping, with a dwell time, in other words, there is a positive time of at least $\eta > 0$ between switching occurrences. Thus we have switching times $t_1 < t_2 < ... < t_N$, such that $\min\{t_k - t_{k-1}\} \geq \eta$, and we focus of course, on an infinite number of switches, in other words $N = \infty$.

We state and explain the conventions and conditions assumed on system (8.1) in the paragraphs below.

Remark 8.1. *In the case when $r = \infty$, we still denote the space $BC(-\infty, 0]$ by the notation $BC[-r, 0]$, by considering for this special case $[-r, 0]$ to mean the* infinite *interval $(-\infty, 0]$, and we are only interested in bounded initial conditions. Of course, $BC[-r, 0] = C[-r, 0]$ when $r < \infty$.*

The norm that we use on $BC([-r, 0], D)$ will be the usual

$$\|\psi\|_r := \sup_{s \in [-r,0]} |\psi(s)|,$$

where of course for $r = \infty$ this norm is $\|\psi\|_r = \sup_{s\in(-\infty,0]} \|\psi(s)\|$. Wherever the norm symbol $\|\cdot\|$ is used, **we refer to the norm on** $BC([-r, 0], D)$. We will denote the Euclidean norm by $|x|$ whenever no confusion should arise.

8.3 Main Results

In order for the necessary integrals to exist, we will assume that each g_i is composite continuous, or composite-C. We defined this notion earlier in Chapter 3. Nonetheless we repeat the definition below.

Definition 8.1. *A mapping $g : J \times BC([-r, 0], D) \longrightarrow \mathbb{R}^n$, where $0 \leq r \leq \infty$, is said to be composite-C if for each $t_0 \in J$ and $\beta > 0$ where $[t_0, t_0 + \beta] \subset J$, if $x \in BC([t_0 - r, t_0 + \beta], D)$, then the composite function $t \mapsto g(t, x_t)$ is an element of the function class $BC([t_0, t_0 + \beta], \mathbb{R}^n)$.*

The initial condition for equation (8.1) will be given for $t_0 \geq 0$, and $\phi \in BC([-r, 0], D)$. For stability analysis, it is assumed that $0 \in D$, $J = \mathbb{R}^+, g_i(t, 0) = 0$ for all $t \in \mathbb{R}^+$, $i \in \mathcal{P}$. This guarantees that system (8.1) has a trivial solution $x(t) = 0$.

If for each k and $t \in [t_{k-1}, t_k)$, we have the value $\sigma(t) = i_k \in \mathcal{P}$, then (8.1) becomes

$$\begin{aligned} x'(t) &= A_{i_k}(t)x(t) + g_{i_k}(t, x_t), \\ x_{t_0} &= \phi \end{aligned} \tag{8.2}$$

We will use the fundamental solution $\Phi_{i_k}(t, t_0)$ of the k-th induced linear ordinary differential equation

$$\begin{aligned} y'(t) &= A_{i_k}(t)y(t) \\ y(t_{k-1}) &= y_{k-1} \end{aligned} \tag{8.3}$$

such that the solution of IVP (8.3) is

$$y(t) = \Phi_{i_k}(t, t_{k-1})y_{k-1}.$$

For a matrix M we use the standard linear operator norm induced by the Euclidean norm $|\cdot|$ on $\mathbb{R}^n$:

$$\|M\| := \sup_{|y|=1} |My|.$$

We will use the inequality $|My| \leq \|M\||y|$ for $y \in \mathbb{R}^n$.

First off, we begin characterizing what the solution looks like, using a variation of parameters type formula.

Lemma 8.1. *The solution to the IVP IFDE* (8.2) *satisfies, for $t \in [t_{n-1}, t_n)$ with $1 \leq n \leq N \leq \infty$:*

$$\begin{aligned} x(t) = {} & \Phi_{i_n}(t, t_{n-1}) \left(\prod_{k=0}^{n-2} \Phi_{i_{n-1-k}}(t_{n-1-k}, t_{n-2-k}) \right) \phi(0) \\ & + \Phi_{i_n}(t, t_{n-1}) \sum_{m=0}^{n-2} \int_{t_{n-m-2}}^{t_{n-m-1}} \Bigg\{ \left(\prod_{k=0}^{m-1} \Phi_{i_{n-1-k}}(t_{n-1-k}, t_{n-2-k}) \right) \\ & \qquad \times \Phi_{i_{n-1-m}}(t_{n-1-m}, s) g_{i_{n-1-m}}(s, x_s) \Bigg\} ds \\ & + \int_{t_{n-1}}^{t} \Phi_{i_n}(t, s) g_{i_n}(s, x_s) ds \end{aligned} \tag{8.4}$$

where we define for $n = 1$ the notation $\prod_{k=1}^{-1} \Phi_{i_{n-1-k}}(t_{n-1-k}, t_{n-2-k}) \equiv I_d$, the identity operator on $\mathbb{R}^n$, and for $n = 1$ we define $\sum_{m=0}^{-1} \equiv 0$.[1]

Proof. We have that if $t \in [t_{n-1}, t_n)$ with $n \geq 1$, then the dynamical system (8.1) evolves according to f_{i_n} in this time interval, so we evolve the state $x(t_{n-1}^+) = x(t_{n-1})$, where sometimes we will write $x(t_{k-1}^+)$ to emphasize that the system evolved according to the vector field that acted on the previous interval $[t_{k-2}, t_{k-1})$. We have

$$x(t) = \Phi_{i_n}(t, t_{n-1})x(t_{n-1}^+) + \int_{t_{n-1}}^{t} \Phi_{i_n}(t, s) g_{i_n}(s, x_s) ds. \tag{8.5}$$

Notice that we will need to use the previous contributions from earlier applied systems. We have that

$$x(t_1) = \Phi_{i_1}(t_1, t_0)\phi(0) + \int_{t_0}^{t_1} \Phi_{i_1}(t_1, s) g_{i_1}(s, x_s) ds,$$

[1] In the case of finite switches applied such that the final switching time is $t_N < \infty$, and $t \geq t_N$, this formula es still valid, of course, up to $t_{n-1} = t_N$, where afterwards, only the last system chosen continues to evolve.

and we plug this into

$$x(t_2) = \Phi_{i_2}(t_2, t_1)x(t_1) + \int_{t_1}^{t_2} \Phi_{i_2}(t_2, s)g_{i_2}(s, x_s)ds,$$

so that

$$\begin{aligned} x(t_2) &= \Phi_{i_2}(t_2, t_1)\left[\Phi_{i_1}(t_1, t_0)\phi(0) + \int_{t_0}^{t_1} \Phi_{i_1}(t_1, s)g_{i_1}(s, x_s)ds\right] + \int_{t_1}^{t_2} \Phi_{i_2}(t_2, s)g_{i_2}(s, x_s)ds \\ &= \Phi_{i_2}(t_2, t_1)\Phi_{i_1}(t_1, t_0)\phi(0) + \int_{t_0}^{t_1} \Phi_{i_2}(t_2, t_1)\Phi_{i_1}(t_1, s)g_{i_1}(s, x_s)ds + \int_{t_1}^{t_2} \Phi_{i_2}(t_2, s)g_{i_2}(s, x_s)ds. \end{aligned}$$

We substitute the value of $x(t_2)$ into

$$x(t_3) = \Phi_{i_3}(t_3, t_2)x(t_2) + \int_{t_2}^{t_3} \Phi_{i_3}(t_3, s)g_{i_3}(s, x_s)ds,$$

to get

$$\begin{aligned} x(t_3) = \Phi_{i_3}(t_3, t_2)\Phi_{i_2}(t_2, t_1)\Phi_{i_1}(t_1, t_0)\phi(0) + \int_{t_0}^{t_1} \Phi_{i_3}(t_3, t_2)\Phi_{i_2}(t_2, t_1)\Phi_{i_1}(t_1, s)g_{i_1}(s, x_s)ds \\ + \int_{t_1}^{t_2} \Phi_{i_3}(t_3, t_2)\Phi_{i_2}(t_2, s)g_{i_2}(s, x_s)ds + \int_{t_2}^{t_3} \Phi_{i_3}(t_3, s)g_{i_3}(s, x_s)ds. \end{aligned}$$

Continuing in this manner, we end up moving moving forward in time the initial condition $\phi(0)$ all the way to the term

$$\begin{aligned} \Phi_{i_n}(t, t_{n-1})\Phi_{i_{n-1}}(t_{n-1}, t_{n-2})\Phi_{i_{n-2}}(t_{n-2}, t_{n-3}) \cdots \Phi_{i_2}(t_2, t_1)\Phi_{i_1}(t_1, t_0)\phi(0) \\ = \Phi_{i_n}(t, t_{n-1})\left(\prod_{k=0}^{n-2} \Phi_{i_{n-1-k}}(t_{n-1-k}, t_{n-2-k})\right)\phi(0). \end{aligned} \tag{8.6}$$

We also have terms of the following form, from longest term to shortest (notice the indices of the farthest factor to the right in the following list). The longest one is

$$\Phi_{i_n}(t, t_{n-1}) \int_{t_0}^{t_1} \Phi_{i_{n-1}}(t_{n-1}, t_{n-2})\Phi_{i_{n-2}}(t_{n-2}, t_{n-3}) \cdots \Phi_{i_2}(t_2, t_1)\Phi_{i_1}(t_1, s)g_{i_1}(s, x_s)ds$$

The next longest is

$$\Phi_{i_n}(t, t_{n-1}) \int_{t_1}^{t_2} \Phi_{i_{n-1}}(t_{n-1}, t_{n-2})\Phi_{i_{n-2}}(t_{n-2}, t_{n-3}) \cdots \Phi_{i_3}(t_3, t_2)\Phi_{i_2}(t_2, s)g_{i_2}(s, x_s)ds$$

Until we reach the last two terms

$$\Phi_{i_n}(t, t_{n-1}) \int_{t_{n-3}}^{t_{n-2}} \Phi_{i_{n-1}}(t_{n-1}, t_{n-2})\Phi_{i_{n-2}}(t_{n-2}, s)g_{i_{n-2}}(s, x_s)ds$$

and the shortest term of this type being

$$\Phi_{i_n}(t,t_{n-1})\int_{t_{n-2}}^{t_{n-1}}\Phi_{i_{n-1}}(t_{n-1},s)g_{i_{n-1}}(s,x_s)ds.$$

We group all of these listed terms together under a single one as

$$\Phi_{i_n}(t,t_{n-1})\sum_{m=0}^{n-2}\int_{t_{n-m-2}}^{t_{n-m-1}}\Bigg\{\left(\prod_{k=0}^{m-1}\Phi_{i_{n-1-k}}(t_{n-1-k},t_{n-2-k})\right) \\ \times\Phi_{i_{n-1-m}}(t_{n-1-m},s)g_{i_{n-1-m}}(s,x_s)\Bigg\}ds.$$

Adding the left over term from expression (8.5), namely $\int_{t_{n-1}}^{t}\Phi_{i_n}(t,s)g_{i_n}(s,x_s)ds$, along with the evolved initial value $\phi(0)$ in (8.6), we obtain the lemma.

□

We now state and prove the main result of this section.

Remark 8.2. *In the following Theorem 8.1, notice that the first condition implies a Lipschitz condition on the nonlinearity of the continuous portion of each subsystem involved, in the sense of Definition 4.6. We have a situation that is completely similar to the case discussed in Section 6.4, and we can similarly prove a result similar to Lemma 5.1 and Proposition 5.1, through suitable modifications, and using Theorem 4.4. We can prove existence-uniqueness of solutions of the switched IVP* (8.1) *by general theory of Chapter* **??**, *so that the following contraction mapping result is finding the unique solution in a satisfactory way. This is because boundedness properties depend only on the variation of parameters formula. Remember in Section 5.4 for the simplest case of a continuous FDE, that there can be a caveat about merely using this fixed point method to prove existence of solutions.*

Remark 8.3. *We included the case for finite switching $N<\infty$ just for completeness, although is is not too interesting because we know that the last system applied is asymptotically stable, though we still need some conditions to guarantee that we do not abandon the region bounded by a ball or radius L before we reach the last subsystem. We can only guarantee the Lipschitz type conditions on each g_i, if we do not abandon a region of size L. We do not know what behavior the nonlinear portions g_i can cause once we abandon this "safety" region.*

Remark 8.4. *The initial condition for equation* (8.1) *will be given for $t_0\geq 0$, and $\phi\in BC([-r,0],D)$. For stability analysis, it is assumed that $0\in D$, $J=\mathbb{R}^+, g_i(t,0)=0$ for all $t\in\mathbb{R}^+$, $i\in\mathcal{P}$. This guarantees that system* (8.1) *has a trivial solution $x(t)=0$.*

Theorem 8.1. *Suppose that there exist positive constants α, L and for each $i\in\mathcal{P}$, continuous functions $b_i:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ such that the following conditions hold:*

(i) $g_i(t,0)\equiv 0$ for all $i\in\mathcal{P}$ and $|g_i(t,\phi)-g_i(t,\psi)|\leq b_i(t)\|\phi-\psi\|$ for all $\phi,\psi\in BC([-r,0],D)$ such that $\|\phi\|,\|\psi\|\leq L$, and for every subsystem $i\in\mathcal{P}$.

(ii) The fundamental matrices of the induced linear systems (8.3) *indexed by* $i \in \mathcal{P}$ *are bounded:*

$$\|\Phi_i(s_2, s_1)\| \leq K, \qquad \text{for every } s_1 \leq s_2 \in \mathbb{R}, i \in \mathcal{P} \tag{8.7}$$

for some $K > 0$ *constant.*

(iii) For all switching moments, $t_{k-1} < t_k$ *for every* $2 \leq k \leq M$*,* $i \in \mathcal{P}$*, we have that the fundamental matrices of the induced linear systems* (8.3) *satisfy*

$$\|\Phi_i(t_k, t_{k-1})\| \leq \alpha < \beta_0, \tag{8.8}$$

where $\beta_0 = \frac{3-\sqrt{5}}{2}$*.*

(iv) The averaging condition holds: For every $i \in \mathcal{P}$*,* $t \geq 0$

$$\int_0^t \|\Phi_i(t,s)\| b_i(s)\, ds \leq \alpha. \tag{8.9}$$

(v) For every $\epsilon > 0$ *and* $T_1 \geq 0$*, there exists a* $T_2 > T_1$ *such that* $t \geq T_2$ *and* $\|x_t\| \leq L$ *implies*

$$|g_i(t, x_t)| \leq b_i(t) \left(\epsilon + \|x\|^{[T_1,t]}\right) \tag{8.10}$$

for every subsystem $i \in \mathcal{P}$*.*

(vi) $K < \frac{(1-\alpha)^2}{\alpha}$*.*

(vii) For every $i \in \mathcal{P}$*,* $\|\Phi_i(t,0)\| \longrightarrow 0$ *as* $t \to \infty$*.*

Then the zero solution of (8.1)*, is uniformly stable and asymptotically stable.*

Remarks

- Notice that

$$K < \frac{(1-\alpha)^2}{\alpha}$$

is necessary for $\left(1 - \alpha - \frac{K\alpha}{1-\alpha}\right) > 0$ in the upper bound on δ_0 in (8.11) below in the proof. This gives an upper bound on the how large the norm of the induced linear system of every subsystem that is involved in (8.1) can be, before eventually behaving well. This is a fundamental difference with respect to the result by B. Zhang in [57], where no bound is required when only one system is used. Also, notice that $K \geq 1$, because of the fact that $\Phi(0,0) = I_d$ implies $K \geq 1$. K is a measure of how bad the behavior can be, in other words, it depends on the maximum of the norms of the linear operators in the subsystems involved. Thus, necessarily

$$\frac{(1-\alpha)^2}{\alpha} > 1,$$

since otherwise $K < 1$. This implies that $\alpha \in (0, \beta_0)$, so that we can use a geometric series, with $\beta_0 = \frac{3-\sqrt{5}}{2}$. Also notice that $\lim_{\alpha \to 0^+} \frac{(1-\alpha)^2}{\alpha} = \infty$, so choosing smaller α, by allowing for longer dwell times, can allow to accommodate for a given K.

- Notice that every subsystem involved behaves well, in the sense that the linear portions of each dynamical system have a fundamental matrix converging to zero. However, being asymptotically stable does not immediately follow from concatenating individually asymptotically stable subsystems under the fixed point criterion for delayed functional differential equations by B. Zhang in [57], which we studied in Chapter 5, since in general only on average the systems behave well. There remains the possibility of switching throughout the entire process (infinite switching) and hitting the "bad" contributions of the subsystems, while not remaining enough time in the "good" part of each subsystem (switching is done too fast) in order for stability behavior to dominate the dynamics. Without sufficient dwell time conditions as the ones we specify in this theorem, it could also be possible that with finite switches, we leave the region bounded by L where the Lipschitz condition is guaranteed. By the conditions specified here, we do not know how badly the nonlinear portion behaves outside of the region of size L. Thus it is possible to destabilize the dynamics by switching frequently without guaranteeing enough contributions from the stable portions of the subsystems involved to attenuate for unstable behavior.

- Conditions (8.8) and (8.9) implicitly define a dwell time. When this theorem is used on a particular type of model, such as in an example we do afterwards, the dwell time condition can be explicitly known, thus characterizing an admissible set $\mathcal{S}$ of switching rules completely specified by a dwell time that guarantees (8.8) and (8.9) hold. Therefore we obtain stability that is uniform over the set $\mathcal{S}$, as we discussed in Definition 4.12 previously.

- Remember that the fading memory conditions in (8.10) are useful only for the infinite delay cases. By Lemma 5.2, for finite delays we can throw out these hypotheses (8.10).

Proof. **Step 1**

Let us apply the Banach contraction method for stability. For the purpose of this, first we define, given ϕ as an initial condition. Let $\delta_0 > 0$ such that

$$\delta_0 < \min\left\{L,\ \frac{L(1-\alpha)}{K},\ \frac{L}{K\alpha}\left(1-\alpha-\frac{K\alpha}{1-\alpha}\right)\right\}. \tag{8.11}$$

Let us now define a suitable complete metric space. Let

$$\mathcal{S} = \{x \in BC([t_0 - r, \infty), D) : x_{t_0} = \phi,\ \|x_t\| \leq L \text{ for } t \geq t_0, x(t) \to 0 \text{ as } t \to \infty\}.$$

$\mathcal{S}$ is clearly a nonempty complete metric space under the metric[2]

$$d_{\mathcal{S}}(x,y) = \sup_{s\in[t_0-r,\infty)} |x(s)-y(s)| = \sup_{s\in[t_0,\infty)} |x(s)-y(s)| \text{ for } x, y \in \mathcal{S},$$

[2]We discussed completeness of this space in Theorem 5.2.

where we note that when calculating the distance between two elements of $\mathcal{S}$, we can disregard the contribution on the subinterval $[t_0 - r, t_0]$ because of the definition of $\mathcal{S}$, and we remind the reader that $[t_0 - r, t_0] = (-\infty, t_0]$ when $r = \infty$.

Remember that there is no switch at initial instants t_0 in our convention, thus given t_0, by a suitable relabeling of the switching instants $\{t_k\}_{k=1}^N$, choose the time lag between the initial instant t_0 and t_1 such that $\|\Phi_i(t_1, t_0)\| \leq \alpha$ as well, so that (8.8) holds for $t_0 < t_1$, the initial instant inclusive.

To obtain a suitable mapping, based on the previous Lemma 8.1 proved, we define the mapping P on $\mathcal{S}$ as

$$(Px)_{t_0} = \phi,$$

and for $t \geq t_0$, supposing $t \in [t_{n-1}, t_n)$, with $n \geq 1$ [3]:

$$\begin{aligned}(Px)(t) = {} & \Phi_{i_n}(t, t_{n-1}) \left(\prod_{k=0}^{n-2} \Phi_{i_{n-1-k}}(t_{n-1-k}, t_{n-2-k}) \right) \phi(0) \\ & + \Phi_{i_n}(t, t_{n-1}) \sum_{m=0}^{n-2} \int_{t_{n-m-2}}^{t_{n-m-1}} \left\{ \left(\prod_{k=0}^{m-1} \Phi_{i_{n-1-k}}(t_{n-1-k}, t_{n-2-k}) \right) \right. \\ & \qquad \left. \times \Phi_{i_{n-1-m}}(t_{n-1-m}, s) g_{i_{n-1-m}}(s, x_s) \right\} ds \\ & + \int_{t_{n-1}}^{t} \Phi_{i_n}(t, s) g_{i_n}(s, x_s) ds.\end{aligned} \tag{8.12}$$

Notice that the mapping is well defined until a switching rule stating at what sequence of times $t_1 < t_2 < ... < t_N$ with $1 \leq N \leq \infty$ the switching occurs and what system $i \in \mathcal{P}$ is engaged during each interval between switches. Clearly the mapping defines Px as a continuous function of time.

Step 2

Let us now show that $|(Px)(t)| \leq L$ for all t, where we remind ourselves that in the notation used here, $|\cdot|$ denotes the Euclidean norm. Of course this is true for $t \leq t_0$. For $t \in [t_0, t_1)$, we have that no switch has occurred yet, so that the the mapping $(Px)(t)$ reduces in this interval to

$$(Px)(t) = \Phi_{i_1}(t, t_0)\phi(0) + \int_{t_0}^{t} \Phi_{i_1}(t, s) g_{i_1}(s, x_s) ds.$$

[3] In the case that the final switching time is $t_N < \infty$, and $t \geq t_N$, this formula es still used, as mentioned before the proof of the previous Lemma 8.1.

Therefore if $t \in [t_0, t_1)$, using the definition of $\mathcal{S}$, since $\|x_t\| \leq L$, we have that we can apply the Lipschitz condition (i):

$$\begin{aligned}|(Px)(t)| &\leq \|\Phi_{i_1}(t,t_0)\|\,|\phi(0)| + \int_{t_0}^{t} \|\Phi_{i_1}(t,s)\|\,|g_{i_1}(s,x_s)|ds \\ &\leq K\delta_0 + \int_{t_0}^{t} \|\Phi_{i_1}(t,s)\|\; b_{i_1}(s)\|x_s\|ds \\ &\leq K\delta_0 + L\alpha \\ &\leq K\delta_0 + L\alpha \leq K\frac{L(1-\alpha)}{K} + L\alpha = L,\end{aligned}$$

where the last inequality follows from the choice of δ_0 above, so that Px remains bounded by L for $t \in [t_0, t_1)$.

For $n \geq 2$, we have that

$$\begin{aligned}|(Px)(t)| \leq{}& \|\Phi_{i_n}(t,t_{n-1})\| \left(\prod_{k=0}^{n-2} \left\|\Phi_{i_{n-1-k}}(t_{n-1-k}, t_{n-2-k})\right\| \right) |\phi(0)| \\ &+ \|\Phi_{i_n}(t,t_{n-1})\| \sum_{m=0}^{n-2} \int_{t_{n-m-2}}^{t_{n-m-1}} \left\{ \left(\prod_{k=0}^{m-1} \left\|\Phi_{i_{n-1-k}}(t_{n-1-k}, t_{n-2-k})\right\| \right) \right. \\ &\qquad \left. \times \left\|\Phi_{i_{n-1-m}}(t_{n-1-m}, s)\right\| |g_{i_{n-1-m}}(s,x_s)|ds \right\} \\ &+ \int_{t_{n-1}}^{t} \|\Phi_{i_n}(t,s)\|\,|g_{i_n}(s,x_s)|ds,\end{aligned} \tag{8.13}$$

so that using the different hypotheses of this theorem, along with $\alpha^{n-1} \leq \alpha$ (since $n \geq 2$), we have that

$$\begin{aligned}|(Px)(t)| \leq{}& K\alpha^{n-1}\delta_0 \\ &+ K\sum_{m=0}^{n-2} \alpha^m \int_{t_{n-m-2}}^{t_{n-m-1}} \left\|\Phi_{i_{n-1-m}}(t_{n-1-m}, s)\right\| b_{i_{n-1-m}}(s)\|x_s\|ds \\ &+ \int_{t_{n-1}}^{t} \|\Phi_{i_n}(t,s)\|\, b_{i_n}(s)\,\|x_s\|\,ds \\ \leq{}& K\alpha^{n-1}\delta_0 + KL\sum_{m=0}^{n-2} \alpha^{m+1} + L\alpha \\ \leq{}& K\alpha\delta_0 + KL\frac{\alpha}{1-\alpha} + \alpha L \leq L,\end{aligned}$$

where the last inequality follows from the choice of δ_0 above. From this it follows that $|(Px)(t)| \leq L$ for every t.

Step 3

Now we show that $(Px)(t) \to 0$ as $t \to \infty$. For this purpose, we must show that given any $\epsilon > 0$, there exists a t^* such that $t > t^*$ implies $|(Px)(t)| < \epsilon$. Here we will make use of condition (8.10), the fading memory condition. Since the mapping P is defined for elements of $\mathcal{S}$, and so $|x(t)| \to 0$, then, given $\epsilon^* = \frac{\epsilon}{5}$, there exists a $T_1 \geq t_0$ such that

$$|x(t)| < \epsilon^* \qquad \text{for all } \ t \geq T_1. \tag{8.14}$$

For this given ϵ^* and corresponding T_1, by (v) there exists a $T_2 > T_1$ such that $t \geq T_2$ and $\|x_t\| \leq L$ implies that for every $i \in \mathcal{P}$

$$|g_i(t, x_t)| \leq b_i(t) \left(\epsilon^* + \|x\|^{[T_1,t]} \right). \tag{8.15}$$

From inequality (8.13), we have that

$$\begin{aligned} |(Px)(t)| \leq & \|\Phi_{i_n}(t, t_{n-1})\| \alpha^{n-1} \delta_0 \\ & + \|\Phi_{i_n}(t, t_{n-1})\| \sum_{m=0}^{n-2} \alpha^m \int_{t_{n-m-2}}^{t_{n-m-1}} \left\|\Phi_{i_{n-1-m}}(t_{n-1-m}, s)\right\| |g_{i_{n-1-m}}(s, x_s)| ds \\ & + \int_{t_{n-1}}^{t} \|\Phi_{i_n}(t, s)\| \, |g_{i_n}(s, x_s)| \, ds. \end{aligned} \tag{8.16}$$

We will have to divide into two cases: One where only a finite number of switches $N < \infty$ will be done and then after t_N we let the last system applied corresponding to $[t_N, \infty)$ take over for the rest of the dynamical process. The other case is when we will do switching behavior throughout the whole process, so that $t_k \to \infty$ as $k \to \infty$.

Finite Switching

For the case of finite switching applied, we will take $n = N + 1$ (since the N-th system is active during $[t_{N-1}, t_N)$) so that $t \in [t_N, \infty)$, in inequality (8.16). Since t_N is finite, using condition (vii), there exists a $T_3 > t_0$ so that[4]

$$\|\Phi_{i_{N+1}}(t, t_N)\| \frac{\alpha L}{1 - \alpha} < \epsilon^*$$

For a similar reason, we know there exists a T_4 so that if $t > T_4$ then

$$\|\Phi_{i_{N+1}}(t, t_N)\| \alpha^N \delta_0 < \epsilon^*.$$

[4]Since $t_N < \infty$, we can make the length of the interval between t_N and t large enough, so that as a consequence of $\|\Phi_{i_{N+1}}(t, 0)\| \to 0$ as $t \to \infty$, we also have $\|\Phi_{i_{N+1}}(t, t_N)\| \to 0$.

Thus let $t > \max\{T_4, T_3, T_2\} > T_1 > t_N$, so that applying the previous two inequalities to (8.16):

$$\begin{aligned}
|(Px)(t)| \leq & \|\Phi_{i_{N+1}}(t,t_N)\|\alpha^N \delta_0 \\
& + \|\Phi_{i_{N+1}}(t,t_N)\| \sum_{m=0}^{N-1} \alpha^m \int_{t_{N-m-1}}^{t_{N-m}} \left\|\Phi_{i_{N-m}}(t_{N-m},s)\right\| b_{i_{N-m}}(s)\|x_s\| ds \\
& + \int_{t_N}^{t} \|\Phi_{i_{N+1}}(t,s)\| \, |g_{i_{N+1}}(s,x_s)| \, ds \\
& \leq \epsilon^* + \|\Phi_{i_{N+1}}(t,t_N)\| \frac{\alpha L}{1-\alpha} + \int_{t_N}^{t} \|\Phi_{i_{N+1}}(t,s)\| \, |g_{i_{N+1}}(s,x_s)| ds \\
& \leq 2\epsilon^* + \int_{t_N}^{t} \|\Phi_{i_{N+1}}(t,s)\| \, |g_{i_{N+1}}(s,x_s)| \, ds.
\end{aligned} \tag{8.17}$$

For the last term in the final inequality, we will use the fading memory condition as follows:

$$\int_{t_N}^{t} \|\Phi_{i_{N+1}}(t,s)\| \, |g_{i_{N+1}}(s,x_s)| \, ds$$

$$\begin{aligned}
&= \int_{t_N}^{T_2} \|\Phi_{i_{N+1}}(t,s)\| \, |g_{i_{N+1}}(s,x_s)| \, ds + \int_{T_2}^{t} \|\Phi_{i_{N+1}}(t,s)\| \, |g_{i_{N+1}}(s,x_s)| \, ds \\
&\leq \|\Phi_{i_{N+1}}(t,T_2)\| \int_{t_N}^{T_2} \|\Phi_{i_{N+1}}(T_2,s)\| \, |g_{i_{N+1}}(s,x_s)| \, ds + \int_{T_2}^{t} \|\Phi_{i_{N+1}}(t,s)\| \, \underbrace{|g_{i_{N+1}}(s,x_s)|}_{\text{fading memory}} \, ds \\
&\leq \|\Phi_{i_{N+1}}(t,T_2)\| \int_{t_N}^{T_2} \|\Phi_{i_{N+1}}(T_2,s)\| \, |b_{i_{N+1}}(s)\|x_s\| \, ds + \int_{T_2}^{t} \|\Phi_{i_{N+1}}(t,s)\| \, b_{i_{N+1}}(s) \left(\epsilon^* + \|x\|^{[T_1,s]}\right) ds \\
&< \|\Phi_{i_{N+1}}(t,T_2)\|\alpha L + 2\epsilon^* \int_{T_2}^{t} \|\Phi_{i_{N+1}}(t,s)\| \, b_{i_{N+1}}(s) \, ds \\
&< \|\Phi_{i_{N+1}}(t,T_2)\|\alpha L + 2\alpha\epsilon^*.
\end{aligned}$$

Again, thanks to $\|\Phi_i(t,t_0)\| \longrightarrow 0$, we can make the first term in the last inequality satisfy $\|\Phi_{i_{N+1}}(t,T_2)\|\alpha L < \epsilon^*$ for $t \geq T_5$, for some $T_5 > t_0$. In this way

$$\int_{t_N}^{t} \|\Phi_{i_{N+1}}(t,s)\| \, |g_{i_{N+1}}(s,x_s)| \, ds < 3\epsilon^*$$

if $t \geq T_5$. Thus, if $t^* = \max_{2 \leq j \leq 5}\{T_j\}$, then by the previous inequality used in the last inequality of (8.17), we have that

$$|(Px)(t)| < 5\epsilon^* \qquad \text{for } t \geq t^*.$$

Therefore if $\epsilon^* = \frac{1}{5}\epsilon$, we have convergence to zero for t large enough.

Infinite Switching

We can now suppose that for a certain $k^* \geq 1$, we have $T_2 \in [t_{k^*}, t_{k^*+1})$, for T_2 as defined through (8.15). No matter how large T_2 is, we can always find a suitable k^*. For the first term we choose t large enough so that $t \in [t_{n-1}, t_n)$ for n large enough (since we have infinite switches). With this sufficiently large n, for the first term in inequality (8.16):

$$\|\Phi_{i_n}(t, t_{n-1})\| \alpha^{n-1} \delta_0 < \epsilon^*, \tag{8.18}$$

since $\alpha < 1$. We work essentially in the same manner to the case of finite switching for the last term of inequality (8.16), where we choose t large enough so that $t_{n-1} \geq T_2$ so that using the fading memory condition we can make it as small as necessary.

For the second term in (8.16) we can, using some N^* which we will suitably choose ahead, break up the sum as

$$\begin{aligned} &\|\Phi_{i_n}(t, t_{n-1})\| \sum_{m=0}^{n-2} \alpha^m \int_{t_{n-m-2}}^{t_{n-m-1}} \left\|\Phi_{i_{n-1-m}}(t_{n-1-m}, s)\right\| |g_{i_{n-1-m}}(s, x_s)| ds \\ &= \|\Phi_{i_n}(t, t_{n-1})\| \sum_{m=0}^{N^*} \alpha^m \int_{t_{n-m-2}}^{t_{n-m-1}} \left\|\Phi_{i_{n-1-m}}(t_{n-1-m}, s)\right\| |g_{i_{n-1-m}}(s, x_s)| ds \\ &\quad + \|\Phi_{i_n}(t, t_{n-1})\| \sum_{m=N^*+1}^{n-2} \alpha^m \int_{t_{n-m-2}}^{t_{n-m-1}} \left\|\Phi_{i_{n-1-m}}(t_{n-1-m}, s)\right\| |g_{i_{n-1-m}}(s, x_s)| ds. \end{aligned} \tag{8.19}$$

Notice that for the first term on the right-hand side of this last equality, the integrals involved in the sum are over the *last* intervals $[t_{n-N^*-2}, t_{n-N^*-1}), ..., [t_{n-2}, t_{n-1})$. Since we supposed that for a certain $k^* \geq 1$, $T_2 \in [t_{k^*}, t_{k^*+1})$, we will need to choose N^* not too large so that $t_{n-N^*-2} \geq t_{k^*+1} > T_2$ so that we can apply the fading memory condition (8.15). The fading memory condition along with (8.14), will essentially allow us to send this first part of the sum to zero. On the other hand, the second part of the sum, namely

$$\|\Phi_{i_n}(t, t_{n-1})\| \sum_{m=N^*+1}^{n-2} \alpha^m \int_{t_{n-m-2}}^{t_{n-m-1}} \left\|\Phi_{i_{n-1-m}}(t_{n-1-m}, s)\right\| |g_{i_{n-1-m}}(s, x_s)| ds,$$

involves the *first* time intervals $[t_0, t_1), ..., [t_{n-N^*-3}, t_{n-N^*-2})$. These involve smaller times $t < t_{n-N^*-2}$, so this part does not allow the application of the fading memory condition. However, using the Lipschitz type conditions, we realize that we end up with the tail of a convergent series. Thus for N^* sufficiently large, we can make this contribution sufficiently small.

Notice that we need a suitable N^* to fulfill both purposes. We need to satisfy N^* sufficiently large to make the tail of the series small, and also $n - N^* - 2 \geq k^* + 1$, to apply the fading conditions because this way $t_{n-N^*-2} \geq t_{k^*+1} > T_2$. Now, $n - N^* - 2 \geq k^* + 1$ implies $N^* \leq n - k^* - 3$. Thus

$$N^* = n - k^* - 4$$

is enough. Notice that as t gets larger, then n does, so N^* defined in this way also gets larger, guaranteeing that the tail of both of the convergent series involved can become arbitrarily small. With all of this in mind, we do as follows.

First off, with N^* as defined above, the fading memory condition together with (8.14) implies that for the sums involving the final time intervals $[t_{n-N^*-2}, t_{n-N^*-1}), ..., [t_{n-2}, t_{n-1})$, since $t_{n-N^*-2} > T_2$:

$$\begin{aligned}
\|\Phi_{i_n}(t, t_{n-1})\| \sum_{m=0}^{N^*} \alpha^m \int_{t_{n-m-2}}^{t_{n-m-1}} \left\|\Phi_{i_{n-1-m}}(t_{n-1-m}, s)\right\| |g_{i_{n-1-m}}(s, x_s)| ds \\
\leq K \sum_{m=0}^{N^*} \alpha^m \int_{t_{n-m-2}}^{t_{n-m-1}} \left\|\Phi_{i_{n-1-m}}(t_{n-1-m}, s)\right\| b_{i_{n-1-m}}(s) \left(\epsilon^* + \|x\|^{[T_1,s]}\right) ds \\
\leq 2K\epsilon^* \sum_{m=0}^{N^*} \alpha^{m+1} \leq 2K\epsilon^* \frac{\alpha}{1-\alpha}.
\end{aligned} \tag{8.20}$$

For the other part of the sum, we only use the Lipschitz type condition (without memoryless part), so that

$$\begin{aligned}
\|\Phi_{i_n}(t, t_{n-1})\| \sum_{m=N^*+1}^{n-2} \alpha^m \int_{t_{n-m-2}}^{t_{n-m-1}} \left\|\Phi_{i_{n-1-m}}(t_{n-1-m}, s)\right\| |g_{i_{n-1-m}}(s, x_s)| ds \\
\leq K \sum_{m=N^*+1}^{n-2} \alpha^m \int_{t_{n-m-2}}^{t_{n-m-1}} \left\|\Phi_{i_{n-1-m}}(t_{n-1-m}, s)\right\| b_{i_{n-1-m}}(s) \|x_s\| ds \qquad (8.21) \\
\leq KL \sum_{m=N^*+1}^{n-2} \alpha^{m+1} \leq KL \sum_{m=N^*+1}^{\infty} \alpha^{m+1}.
\end{aligned}$$

Since $N^* = n - k^* - 4$, choosing t large enough makes n large enough, so that we can easily make the contribution from (8.21), which involves the tail of a convergent series) satisfy

$$KL \sum_{m=N^*+1}^{\infty} \alpha^{m+1} < \epsilon^*. \tag{8.22}$$

From inequalities (8.20) , (8.21) and (8.22), we conclude that for t large enough we can make the second term in inequality (8.16) less than

$$2K\epsilon^* \frac{\alpha}{1-\alpha} + \epsilon^*$$

Adding the other terms of (8.16) to this last term and rescaling ϵ^* as in the finite switching case, we obtain that $|(Px)(t)| < \epsilon$ for $t \geq t^*$, for a suitable t^*, which we require to be large enough for (8.18) to happen and also large enough for inequalities (8.20) (8.21) and (8.22) to occur.

This proves that P is a mapping from $\mathcal{S}$ to itself.

Step 4

We now prove that $P : \mathcal{S} \longrightarrow \mathcal{S}$ is a contraction. For this, let $x, y \in \mathcal{S}$. By definition of $\mathcal{S}$ we have that $(Px)(t) - (Py)(t) = 0$ for $t \in [t_0 - r, t_0]$. For $t \geq t_0$ we get:

$$
\begin{aligned}
&|(Px)(t) - (Py)(t)| = \\
&\Bigg| \int_{t_{n-1}}^{t} \Phi_{i_n}(t,s)\Big[g_{i_n}(s,x_s) - g_{i_n}(s,y_s)\Big] ds \\
&\quad + \Phi_{i_n}(t,t_{n-1}) \sum_{m=0}^{n-2} \int_{t_{n-m-2}}^{t_{n-m-1}} \Bigg\{ \left(\prod_{k=0}^{m-1} \Phi_{i_{n-1-k}}(t_{n-1-k}, t_{n-2-k}) \right) \\
&\qquad \times \Phi_{i_{n-1-m}}(t_{n-1-m}, s)\Big[g_{i_{n-1-m}}(s,x_s) - g_{i_{n-1-m}}(s,y_s)\Big] \Bigg\} ds \Bigg| \\
&\leq \int_{t_{n-1}}^{t} \big\|\Phi_{i_n}(t,s)\big\|\, b_{i_n}(s) \|x_s - y_s\| ds \\
&\quad + K \sum_{m=0}^{n-2} \alpha^m \int_{t_{n-m-2}}^{t_{n-m-1}} \big\|\Phi_{i_{n-1-m}}(t_{n-1-m}, s)\big\|\, b_{i_{n-1-m}}(s) \|x_s - y_s\| ds \\
&\leq \alpha\, d_{\mathcal{S}}(x,y) + K\, d_{\mathcal{S}}(x,y) \sum_{m=0}^{\infty} \alpha^{m+1} \\
&= \left(\alpha + \frac{\alpha\, K}{1-\alpha} \right) d_{\mathcal{S}}(x,y),
\end{aligned}
$$

where recall that $d_{\mathcal{S}}(x,y) = \sup_{s\in[t_0,\infty)} |x(s) - y(s)|$. Since $\alpha + \frac{\alpha K}{1-\alpha} < 1$ is a consequence of $K < \frac{(1-\alpha)^2}{\alpha}$, we obtain that P is a contraction on $\mathcal{S}$. This implies that there is a unique solution to the initial value problem (8.1).

By definition of $\mathcal{S}$, we already have that the solution to the initial value problem (8.1) converges to zero. We must prove that the solution is stable.

Step 5

To prove stability, assume that we are given an $\epsilon > 0$. Choose

$$\delta < \min\left\{ \epsilon,\ \frac{\epsilon(1-\alpha)}{K},\ \frac{\epsilon}{K\alpha}\left(1 - \alpha - \frac{K\alpha}{1-\alpha}\right) \right\}.$$

For $\|\phi\| < \delta$, we claim that $|x(t)| < \epsilon$ for all $t \geq t_0$. Note that if x is the unique solution corresponding to the initial condition ϕ, then $|x(t_0)| = |\phi(0)| < \epsilon$. For the sake of contradiction suppose that there exists a $\hat{t} > t_0$ such that $|x(\hat{t})| \geq \epsilon$. Let

$$t^* = \inf\{\hat{t} : |x(\hat{t})| \geq \epsilon\}.$$

By continuity, we have that $|x(s)| < \epsilon$ for $s \in [t_0 - r, t^*)$ and $|x(t^*)| = \epsilon$. First suppose that $t_0 \le t < t_1$, in other words, that no switch has occurred yet. Then

$$\begin{aligned}
|x(t^*)| &\le \|\Phi_{i_1}(t^*, t_0)\||\phi(0)| + \int_{t_0}^{t^*} \|\Phi_{i_1}(t^*, s)\||g_{i_1}(s, x_s)|ds \\
&\le K\delta + \int_{t_0}^{t^*} \|\Phi_{i_1}(t^*, s)\|b_{i_1}(s)\|x_s\|ds \\
&\le K\delta + \left(\sup_{\theta\in[t_0-r,t^*]} |x(\theta)|\right) \int_{t_0}^{t^*} \|\Phi_{i_1}(t^*, s)\|b_{i_1}(s)ds \\
&\le K\delta + \alpha\epsilon < \epsilon
\end{aligned}$$

and this gives us the desired contradiction, by the definition of t^*.

For $t \ge t_1$ we have $n \ge 2$, and from an inequality similar to the one that we obtained in (8.16), we have that

$$\begin{aligned}
|x(t^*)| \le& K\alpha^{n-1}\delta + \left(\sup_{\theta\in[t_0-r,t^*]} |x(\theta)|\right) \int_{t_{n-1}}^{t^*} \|\Phi_{i_n}(t, s)\|b_{i_n}(s)ds \\
&+ K\left(\sup_{\theta\in[t_0-r,t^*]} |x(\theta)|\right) \sum_{m=0}^{n-2} \alpha^m \int_{t_{n-m-2}}^{t_{n-m-1}} \left\|\Phi_{i_{n-1-m}}(t_{n-1-m}, s)\right\| b_{i_{n-1-m}}(s)\, ds \\
&\le K\alpha\delta + \epsilon\alpha + K\epsilon \sum_{m=0}^{\infty} \alpha^{m+1} = K\alpha\delta + \alpha\epsilon + \frac{K\epsilon\alpha}{1-\alpha} < \epsilon,
\end{aligned}$$

by the choice of δ, and now this gives us the desired contradiction for $t \ge t_1$ (so that $n \ge 2$), by the definition of t^*. Therefore the solution is stable, and since $x(t)$ converges to zero as $t \to \infty$, we get asymptotic stability of trajectories.

□

Remark 8.5. *We notice that, in order to obtain contraction conditions, we may suppose a host of different types of inequalities in order to obtain this. The problem is how to put all of the conditions together so that they do not contradict each other or make the proof too difficult due to requiring bounds that become very hard to calculate if you assume an inconvenient set of independent hypotheses.*

Remark 8.6. *Similar to Theorem 7.1, though not the same, notice that in the interesting case when we have infinite switches, $\|\Phi_{i_k}(s_2, s_1)\| \le K$ for every $s_1, s_2 \in [t_{k-1}, t_k)$ for all $k \ge 1$ gives a uniform bound. The requirement $s_1, s_2 \in [t_{k-1}, t_k)$, instead of saying that $s_2 \ge s_1 \in [0, \infty)$ as in Theorem 5.2, Theorem 6.2 of previous chapters, or for finite switches, is because in this situation our operators Φ_{i_k} are always interrupted at the switching moments t_k, where the next subsystem is engaged. Still, we guarantee in condition (iii) of this theorem proved, a contractive requirement before letting the next linear portion $\Phi_{i_{k+1}}$ carry on. This interruption of each subsystem plays a role in determining K or making it smaller.*

8.4 A Particular Linear Case for the Delay

Now, suppose we have the following simple version, where for each $i \in \mathcal{P}$, $g_i(t, x_t) = M_i(t)x_i(t - r_i(t))$, with $t - r_i(t) \longrightarrow \infty$ as $t \to \infty$ for each $i \in \mathcal{P}$, and $M_i(t)$ are continuous time-varying matrices of dimension $n \times n$.

If for each k and $t \in [t_{k-1}, t_k)$, we have the value $\sigma(t) = i_k \in \mathcal{P}$, then let us have the switched system

$$x'(t) = A_{i_k}(t)x(t) + M_{i_k}(t)x_{i_k}(t - r_{i_k}(t)). \tag{8.23}$$

The next result is a linear version of Theorem 8.1.

Remark 8.7. *Notice that in the following result, the initial conditions do not have to be bounded by a δ_0 as in* (8.11)*, since in said inequality, we can make L arbitrarily large, since we have a global Lipschitz condition, thanks to the linearity of this system* (8.23)*. Thus the initial condition will be arbitrarily large, and we achieve a global convergence to zero result.*

Corollary 8.1. *Suppose that in* (8.23)*, $t - r_i(t) \longrightarrow \infty$ as $t \to \infty$ for each $i \in \mathcal{P}$, that there exists a positive constant α, and continuous functions $b_i : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ for each $i \in \mathcal{P}$ such that the following conditions hold*[5]

(i) For each $i \in \mathcal{P}$, $M_i(t)$ has its operator norm bounded $\|M_i(t)\| \leq b_i(t)$, for all $t \geq 0$.

$$\|\Phi_i(s_2, s_1)\| \leq K, \qquad \text{for every } s_1 \leq s_2 \in \mathbb{R}, i \in \mathcal{P} \tag{8.24}$$

for some $K > 0$ constant.

(ii) For all switching moments, $t_{k-1} < t_k$ for every $2 \leq k \leq M$, $i \in \mathcal{P}$, we have that the fundamental matrices of the induced linear systems (8.3) *satisfy*

$$\|\Phi_i(t_k, t_{k-1})\| \leq \alpha < \beta_0, \tag{8.25}$$

where $\beta_0 = \frac{3-\sqrt{5}}{2}$.

(iii) The averaging condition holds: For every $i \in \mathcal{P}$, $t \geq 0$

$$\int_0^t \|\Phi_i(t, s)\| b_i(s)\, ds \leq \alpha. \tag{8.26}$$

(iv) $K < \frac{(1-\alpha)^2}{\alpha}$.

(v) For every $i \in \mathcal{P}$, $\|\Phi_i(t, 0)\| \longrightarrow 0$ as $t \to \infty$.

Then the zero solution of (8.23) *is uniformly stable and asymptotically stable, for arbitrarily large initial conditions ϕ. Thus we have global asymptotic stability.*

[5] Notice that $b_i(t) := \|M_i(t)\|$ also works, but perhaps knowing this exactly is too difficult, so using matrix bounds one can settle for an upper estimate.

Proof. Notice that the δ_0 in (8.11) depends on L proportionally, and L is where the Lipschitz-type conditions of Theorem 8.1 hold. But in this case, we do not have a nonlinearity that forces a local Lipschitz condition, so L can be arbitrarily large. Thus asymptotic convergence holds, no matter how large the initial condition is.

We now just need to prove that the fading memory condition (8.10) holds in case of infinite delay. By Lemma 5.2, for finite delays this is automatically satisfied (and $t - r_i(t) \longrightarrow \infty$, if $0 \leq r_i(t) \leq r$ for all i). The proof that condition (8.10) holds is similar to how we did in Corollary 7.1, except that now let T_1 be the maximum of each of the subsystems[6] $T_1 = \max_{i\in\mathcal{P}}\{T_{1_i}\}$ of each subsystem and similarly for T_2. The rest follows from Theorem 8.1. □

8.5 One-dimensional Cases

Suppose we reduce to the one-dimensional case where $A(t) = -a(t)$ is a scalar valued function. Suppose for each (t_0, ϕ) we induce the switched FDE IVP

$$\begin{aligned} x'(t) &= -a_{\sigma(t)}(t)x(t) + g_{\sigma(t)}(t, x_t) \qquad t \geq t_0 \\ x_{t_0} &= \phi \end{aligned} \tag{8.27}$$

The following is immediate, since $\Phi_i(s_2, s_1) = e^{-\int_{s_1}^{s_2} a_i(u)du}$ for this scalar case, and $\int_0^t a_i(s)ds \longrightarrow +\infty$ as $t \to \infty$ implies convergence to zero of each fundamental matrix, as seen earlier in Chapter 5.

Corollary 8.2. *Suppose that there exist positive constants α, L and for each $i \in \mathcal{P}$, continuous functions $b_i : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that the following conditions hold:*

(i) $g_i(t, 0) \equiv 0$ for all $i \in \mathcal{P}$ and $|g_i(t,\phi) - g_i(t,\psi)| \leq b_i(t)\|\phi - \psi\|$ for all $\phi, \psi \in BC([-r, 0], D)$ such that $\|\phi\|, \|\psi\| \leq L$, and for every subsystem $i \in \mathcal{P}$.

(ii) We have the bound

$$e^{-\int_{s_1}^{s_2} a_i(u)du} \leq K, \qquad \text{for every } s_1 \leq s_2 \in \mathbb{R}, i \in \mathcal{P} \tag{8.28}$$

for some $K > 0$ constant.

(iii) For all switching moments, $t_{k-1} < t_k$ for every $2 \leq k \leq M$, $i \in \mathcal{P}$, we have that

$$e^{-\int_{t_{k-1}}^{t_k} a_i(u)du} \leq \alpha < \beta_0, \tag{8.29}$$

where $\beta_0 = \frac{3-\sqrt{5}}{2}$.

(iv) The averaging condition holds: For every $i \in \mathcal{P}$, $t \geq 0$

$$\int_0^t e^{-\int_s^t a_i(u)du} b_i(s)\, ds \leq \alpha. \tag{8.30}$$

[6] Remember the number of subsystems is finite.

(v) *For every* $\epsilon > 0$ *and* $T_1 \geq 0$*, there exists a* $T_2 > T_1$ *such that* $t \geq T_2$ *and* $\|x_t\| \leq L$ *imply*

$$|g_i(t, x_t)| \leq b_i(t) \left(\epsilon + \|x\|^{[T_1,t]}\right) \tag{8.31}$$

for every subsystem $i \in \mathcal{P}$*.*

(vi) $K < \frac{(1-\alpha)^2}{\alpha}$*.*

(vii) *For every* $i \in \mathcal{P}$*,* $\int_0^t a_i(s)ds \longrightarrow +\infty$ *as* $t \to \infty$*.*

Then the zero solution of (8.27)*, is uniformly stable and asymptotically stable.*

Now, suppose we have the following scalar version, where for each $i \in \mathcal{P}$, $g_i(t, x_t) = b_i(t)x_i(t - r_i(t))$, with $t - r_i(t) \longrightarrow \infty$ as $t \to \infty$ for each $i \in \mathcal{P}$, and $b_i(t)$ are continuous functions.

If for each k and $t \in [t_{k-1}, t_k)$, we have the value $\sigma(t) = i_k \in \mathcal{P}$, then let us have the switched system

$$x'(t) = -a_{i_k}(t)x(t) + b_{i_k}(t)x_{i_k}(t - r_{i_k}(t)). \tag{8.32}$$

The next result is a scalar version of Corollary 8.1, which follows from it.

Remark 8.8. *As mentioned before, in the following result, the initial conditions do not have to be bounded by a* δ_0 *as in* (8.11)*, since in said inequality, we can make* L *arbitrarily large, since we have a global Lipschitz condition. Thus the initial condition will be arbitrarily large, and we achieve a global convergence to zero result.*

Corollary 8.3. *Suppose that in* (8.23)*,* $t - r_i(t) \longrightarrow \infty$ *as* $t \to \infty$ *for each* $i \in \mathcal{P}$*,*

(i) *We have the bound*

$$e^{-\int_{s_1}^{s_2} a_i(u)du} \leq K, \qquad \text{for every } s_1 \leq s_2 \in \mathbb{R}, i \in \mathcal{P} \tag{8.33}$$

for some $K > 0$ *constant.*

(ii) *For all switching moments,* $t_{k-1} < t_k$ *for every* $2 \leq k \leq M$*,* $i \in \mathcal{P}$*, we have that*

$$e^{-\int_{t_{k-1}}^{t_k} a_i(u)du} \leq \alpha < \beta_0, \tag{8.34}$$

where $\beta_0 = \frac{3-\sqrt{5}}{2}$*.*

(iii) *The averaging condition holds: For every* $i \in \mathcal{P}$*,* $t \geq 0$

$$\int_0^t e^{-\int_s^t a_i(u)du} b_i(s)\, ds \leq \alpha. \tag{8.35}$$

(iv) $K < \frac{(1-\alpha)^2}{\alpha}$*.*

(v) *For every* $i \in \mathcal{P}$*,* $\int_0^t a_i(s)ds \longrightarrow +\infty$ *as* $t \to \infty$*.*

Then the zero solution of (8.32) *is uniformly stable and asymptotically stable, for arbitrarily large initial conditions* ϕ*. Thus we have global asymptotic stability.*

8.6 An Example

Let us use a simple example to illustrate the distinct features of Theorem 8.1, or rather, the particular case of it in Corollary 8.3. We will use some of the notation from the more general Theorem 8.1, to identify its elements.

Example 8.1. Suppose we have the following switched version of the Example 3.4, which is essentially the FDE (8.32).

$$x' = -a_{i_k}(t)x(t) + b_{i_k}(t)x(t - r_{i_k}(t)) \qquad t \geq 0 \tag{8.36}$$

under a given switching rule so that $\sigma(t) = i_k \in \mathcal{P}$ for $t \in [t_{k-1}, t_k)$, where suppose that the switching rule is indefinitely applied, in other words, $\lim_{k\to\infty} t_k = \infty$. Suppose that for every $i \in \mathcal{P}$, $t - r_i(t) \longrightarrow 0$ as $t \to \infty$. Let us suppose similar hypotheses to the mentioned single system example, which we analyzed how it agrees with the fixed point result of Chapter 5 in Example 5.2. In this manner we can compare and obtain fundamental differences. Let us begin with the observation that now we need $\alpha \in (0, \beta_0)$. Suppose that $\min_{i\in\mathcal{P}}\{a_i(t)\} \geq c > 0$ for some fixed positive constant c, and suppose that there exists a constant $J > 3$ such that $a_i(t) \geq J|b_i(t)|$ for all $t \geq 0$, for each $i \in \mathcal{P}$. Then

$$\begin{aligned}\int_0^t e^{-\int_s^t a_i(u)du}|b_i(s)|ds &\leq \frac{1}{J}\int_0^t e^{-\int_s^t a_i(u)du}a_i(s)ds \\ &= \frac{1}{J}e^{-\int_s^t a_i(u)du}\Big|_{s=0}^{s=t} = \frac{1}{J}\left(1 - e^{-\int_0^t a_i(u)du}\right).\end{aligned}$$

Thus $\sup_{t\geq 0}\int_0^t e^{-\int_s^t a_i(u)du}|b_i(s)|ds \leq \frac{1}{J} < \frac{1}{3} < \beta_0$ for every $i \in \mathcal{P}$. Let us take $\alpha = \frac{1}{3}$. In this manner, we have shown that each subsystem individually satisfies the hypotheses of Theorem 5.2, so that individually we have that they are uniformly stable and asymptotically stable. Nonetheless, we must satisfy a dwell time constraint. Now, we need condition

$$\|\Phi_i(t_k, t_{k-1})\| = e^{-\int_{t_{k-1}}^{t_k} a_i(s)ds} \leq \frac{1}{3}. \tag{8.37}$$

But notice that even though $a_i(t) \geq c > 0$ for all $i \in \mathcal{P}$, sufficient time must occur for $\int_{t_{k-1}}^{t_k} a_i(s)ds$ to be positive enough to guarantee that for every $i \in \mathcal{P}$, $e^{-\int_{t_{k-1}}^{t_k} a_i(s)ds} \leq e^{-c(t_k - t_{k-1})} \leq \alpha \leq \frac{1}{3}$, which implies that we need enough time for at least

$$t_k - t_{k-1} \geq \frac{\ln(3)}{c},$$

otherwise we cannot guarantee that condition (8.37) holds, which is necessary for the application of the previous theorem. Therefore, even for well behaved systems, it is necessary to concatenate them after a sufficiently long time has passed so that $\int_{t_{k-1}}^{t_k} a_i(s)ds$ is positive enough to guarantee (8.37). Notice that this is also a consequence of the fact that $0 < \alpha < \beta_0$, and not merely $\alpha \in (0, 1)$ as in the previous result proved in [57]. This was a consequence of necessary bounds on K such that

$$\|\Phi_i(s_2, s_1)\| \leq K, \qquad \text{for every } s_1 \leq s_2 \in \mathbb{R}, i \in \mathcal{P}.$$

Here $K = 1$ is enough, since every function $a_i(t) > 0$. We also note that $K < \frac{(1-\alpha)^2}{\alpha} = \frac{4}{3}$. $\min_{i\in\mathcal{P}}\{a_i(t)\} \geq c > 0$ implies divergence to infinity of the integrals in (v) of Corollary 8.3. Thus, sufficient conditions for stability of (8.36) are satisfied. △

Remark 8.9. *The previous example shows us how to obtain the set of admissible switching signals $\mathcal{S}$, so that we have the mentioned stability properties (uniform stability and asymptotic stability) with respect to $\mathcal{S}$, as we discussed in Definition 4.12. Namely, the switching signals, given the family of vector fields in Example 8.1 are characterized by having dwell time $\eta \geq \frac{\ln(3)}{c}$, for c as in the given example.*

Remark 8.10. *As obviously supposed, the longer you let the systems act, the smaller α gets, and the larger K can be.*

The previous example illustrates one difficulty that arises when obtaining stability criteria even for a well behaved system under the hypotheses of the results shown in Chapter 5. This is namely because for switched systems, we now require dwell time conditions, though these are difficulties that even Lyapunov theory faces, as is studied in [33] for ODEs, [35] and [50] for FDEs, or in general any other stability method must face. Here we have characterized the dwell time using the Banach contraction principle.

One can imagine that if in another FDE in the spirit of the example given here, but with some subsystems having $a_i(t) < 0$, then one must allow sufficient time for the positive contributions of $a_i(t)$ to dominate and make the integral $\int_{t_{k-1}}^{t_k} a_i(s)ds$ positive enough to guarantee (8.37) holds. Thus we have obtained a *slow-switching criterion*, under the theory studied in Subsection 4.3.2.

As mentioned before proving the more general Theorem 8.1, conditions (8.8) and (8.9) implicitly define a dwell time. As shown in Example 8.36, the dwell time condition can be explicitly known, thus characterizing an admissible set $\mathcal{S}$ of switching rules completely specified by a dwell time that guarantees (8.8) and (8.9) hold. Therefore we obtain stability that is uniform over the set $\mathcal{S}$, as we discussed in Definition 4.12 previously.

The examples given in Chapter 5 can be generalized to their switched version counterparts, with more stringent requirements, similar to how we obtained for Example 8.1 here.

Chapter 9

Conclusions and Future Research

9.1 Conclusions

We have studied a fixed point technique, particularly, using the Banach contraction principle for asymptotic stability of some general types of functional differential equations. In particular, we have studied in this work the discontinuous, or impulsive FDEs case, as well as the case where the system itself is not as well behaved as the cases considered previously, so we use impulses to stabilize it. Finally we considered a switched FDEs case, where all systems are well behaved. During these studies, we encountered some difficulties which the Banach fixed point method has. For all cases considered, obtaining a contraction mapping can be challenging, and the variation of parameters formulas may be complicated, and we must seek a suitable one, depending on what part of the system studied will do the stabilizing role. For the switched case, the result obtained prescribed a limitation which was reflected in a dwell time, which depends on how bad (how large the norm of the fundamental matrices involved) are.

Even for the case of simple delayed FDEs, we pointed out and deepened in some weak points of using the contraction principle for stability. These difficulties carried on to more complicated systems that we eventually considered. In particular, the Banach contraction principle, as can be seen through an analysis of the proofs done in this thesis, requires Lipschitz type conditions on nonlinearities. This is because in the end, a contraction must be able to do some type of metric comparison inequality whenever we require conditions such as

$$d(P\psi, P\varphi) \leq \beta d(\psi, \varphi)$$

for $\beta \in (0, 1)$. A Lipschitz type condition works almost perfectly with the previous requirement. However, this limits how untame the nonlinearities may be, and thus we require uniqueness hypotheses for these methods. Contrast this with a Lyapunov based method, for example, in [37], where uniqueness hypotheses are not necessary. Also, sometimes in order to obtain a contraction condition we must force strong conditions of a certain type onto the vector field. Perhaps this suggests trying different fixed point theorems, such as Schauder's fixed point theorem.
Another weak point is that one of the hypotheses we required for the application of the contraction mapping principle in systems of differential equations, was to be able to calculate bounds on the state transition matrices involved. Since there is no general method to characterize the

state transition matrix, this can be a highly nontrivial pursuit, especially for large time invariant systems, unless some analytical considerations are able to be applied to bound these, or some numerical technique can obtain useful bounds. For scalar equations of course, the previous does not apply.

In this thesis, we obtained results and were able to deepen the study of the contraction principle for asymptotic stability to cases not considered before. However, a comparison still remains to be done with Lyapunov techniques for the functional differential equations considered here which have not been studied earlier by authors who have used the Banach fixed point method for stability. Some comparisons have been done for delayed FDEs, where some advantages were shown in [57, 58, 11, 12]. We were able to compare how the contraction principle differs with respect to results from [57], for example, in its application to the different systems considered here, but a Lyapunov comparison through a revision of literature for impulsive FDEs, impulsive control based on Lyapunov paradigms, and Lyapunov methods for switched FDEs still remains to be done, in order to further appreciate this method for more general systems.

The advantages of the fixed point technique studied here over Lyapunov methods could clearly be seen for some particular examples in delayed FDEs, although we still lack a clear comparison with impulsive FDEs and switched FDEs. Nonetheless, the success of the contraction mapping method for delayed differential equations in [57, 58, 11, 12] was, in the author's opinion, sufficient justification to begin the study of this method in more complex systems.

9.2 Future Research

As mentioned previously, we have just begun the study of how to apply the contraction method for asymptotic stability of differential equations that have not been previously considered. We still need to do a more exhaustive revision of literature in impulsive FDEs, control methods to stabilize an unstable impulsive system, and Lyapunov stability results for switched systems, in order to compare with results obtainable from fixed point methods, as has been done for some delayed FDEs, such as in [57, 58, 11, 12].

The method of impulsive stabilization using a fixed point paradigm must surely be improvable when focused on a more particular model. The same can be said for the result for switched systems that we obtained. This is because of the general situation in mathematics where sometimes if a result is for a too general case, it can become easily possible to sharpen the result or get a better criterion if we concentrate on one single specific model. The purpose here was to initiate research in the direction of the fixed point method for asymptotic stability, since there is not as much research literature available as there is for Lyapunov methods. Thus, once these general results have been obtained, we can choose to move to more particular cases, for example, by focusing on more particular kinds of nonlinearities, or particular types of linear portions. We could also focus on particular impulse functions, or just reduce to linear impulses dependent on the last state. Afterwards perhaps we can accompany these results with a comparison with Lyapunov methods.

Within switched systems, there are still other cases to consider, in this thesis we did not mention the method for impulsive switched systems, and we focused on well behaved systems in the sense of the result obtained first for delayed FDEs. We could consider the switched case when only some of the subsystems are badly behaved, but we remain in these subsystems for not too long, and somehow consider remaining in stable subsystems longer so that these undesired subsystem behaviors are canceled out, and having all of this somehow reflected in conditions that still allow for a contraction to occur, in other words, obtain an average dwell condition. Something completely similar to this has already done, as is switched ODEs in [33], or for switched FDEs in [35], [50], and references therein. There remains to consider impulsive stabilization for switched systems with more complicated additional components, or a combination of the previous hypotheses in this paragraph, although we can already imagine the challenges of having even more conditions to consider.

We could also begin the study of using weaker fixed point methods that do not conclude uniqueness of solutions, in order to seek to possibly eliminate Lipschitz requirements on vector fields. Using further considerations, we might be able to have a useful result even if uniqueness of solutions is not guaranteed.

Of course we can also take a less theoretical approach and go on to analyze particular models that use delayed FDEs, impulsive delayed FDEs, switched FDEs, control of particular models under fixed point paradigms, to name other possible research directions. Nonetheless, for all that has been said, it is important to point out the importance of this subject greatly depends on a deeper understanding through direct comparison with Lyapunov methods which can show that fixed point theory can offer better stability conditions at least for some cases, or can be more convenient to apply, otherwise fixed point methods have little to offer in terms of stability of more general types of differential equations.

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