Exact Results in Supersymmetric Gauge Theory

by

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This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Statement of Contributions

Chapter 2 of this thesis consists of material from the paper "Irregular Singularities in the H_3^+ WZW Model" [39], co-authored with Davide Gaiotto.

Abstract

Exact results are a key component for understanding any physical theory. Unfortunately in the context of quantum field theory (QFT) they are in general impossible to obtain, and we need some sort of approximations. However there exists certain non-realistic theories rich in exactly computable quantities, and from those exact quantities we can infer various theoretical implications for realistic quantum field theories. Supersymmetric gauge theories stand out among these non-realistic theories as the best compromise between the contradicting requirements of realism and exact computability.

This thesis consists of three projects, in which we explore some exact results in supersymmetric quantum field theory.

In the first project we define and describe irregular vertex operators in the H_3^+ Wess-Zumino-Witten model. Irregular vertex operators are a QFT-equivalent of irregular singular points in the theory of differential equations, and their study is motivated by a relation to the partition functions of some asymptotically free four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories. The definition is shown to be compatible with previously defined irregular vertex operators in Liouville theory through a known duality between the H_3^+ and Liouville theories.

In the second project we use supersymmetric localization to compute the partition function of $\mathcal{N} = 2$ supersymmetric gauge theories on a four-sphere in the presence of a surface defect on a two-sphere subspace, taking the form of a two-dimensional gauged linear sigma model. The result generalizes the known results for separate supersymmetric gauge theories on the separate spaces. We obtain a partition function in the form of a standard partition function on S^4, with a modified instanton partition function and an additional insertion corresponding to a shifted version of the S^2 partition function.

In the third project we develop a new method for finding the ground states of fermions in the presence of BPS monopoles. We use it to find the ground states in the case of Abelian BPS monopoles in $\mathbb{R}^3$, which were previously unknown.

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Chapter 1

Introduction and summary

The standard model of particle physics describes the microscopic world with remarkable precision. It consists of three generations of quarks and leptons, each containing several elementary fields (or particles), subject to three types of fundamental interactions (electroweak, strong and Higgs), each mediated by its own set of fields. However extracting any accurate prediction from the theory is a difficult task, and often impossible given currently available methods and computing power. This is due to the complexity of the standard model as a quantum field theory, but also the difficulty of quantum field theory itself. In short we are left with the following dilemma: we can either keep pushing our knowledge of the standard model, knowing that some questions may never be answered properly, or we can look at simpler models where those questions can be answered, knowing that the results will not apply directly to the real world. Of course both approach should be taken, as they serve different purposes, however in this work we will follow exclusively the latter one.

In this thesis, we describe three different projects, motivated by a common goal: to obtain exact results in quantum field theory. In the first [39], we consider a special set of operators (or states) in two-dimensional conformal field theory (CFT), which induce unusual singularities in the current algebra. We describe such operators, called irregular vectors, in a specific Wess-Zumino-Witten (WZW) model. In the second project [64], we consider gauge theories with extended supersymmetry interacting with fields living on a two-dimensional subspace, and compute the main components of their partition function using supersymmetric localization. In the last project [65], we explore a certain class of magnetic monopoles, and describe a method for finding fermion zero modes in the presence of a fixed background of such monopoles. In the following sections we summarize each

project, but first we elaborate on our motivation for finding exact results in non-realistic theories, and explore a few aspects of supersymmetry, a common theme for all the projects.

1.1 Exact results

1.1.1 The limits of perturbation theory

Almost every successful method so far in particle physics involves perturbation theory. In that context one approximates the evolution of a small set of elementary excitations (scattering amplitudes), or particles, by considering the most probable intermediate states, also made of elementary excitations. These intermediate states are expressed schematically as a set of Feynman diagrams [80], and their contributions to scattering amplitudes are sorted into a formal series expansion of the coupling constants. As long as those coupling constants are small and only a few particles are involved, a small number of Feynman suffice to obtain a reasonably accurate result. Perturbation theory works relatively well in particle accelerator experiments, in which a pair of particles collide to form a new small set of particles.

Even in particle accelerators the perturbative approach has its limits, since as the collision energy increases, so does the number of resulting particles and thus the computational burden of comparing theory and experiment. Another problem concerns quantum chromodynamics (QCD), which has a relatively high coupling constant (at experimental energy scales), and perturbative methods give inaccurate results. The accuracy can be improved slightly by considering higher order terms in the approximation, however since the number of Feynman diagrams one needs to compute grows factorially with the order, the computation quickly becomes intractable[1]. The computational complexity can be made polynomial by statistical approximations ("Monte Carlo" sampling), although these methods remain a challenge even for modern supercomputers.

In any case, perturbation theory can fail completely when we consider experimental setups other than the well-suited particle collision experiments. This can happen because some phenomena, dubbed "non-perturbative", leave no trace at all in any order of perturbation theory. Non-perturbative phenomena are in general due to a fundamental limitation of the perturbative approach, in which there is a distinguished set of "elementary particles". It is known from many examples that there is in general no unique fundamental set

[1] One remembers that for a computational problem to be possible for large value of a parameter n, here the order in perturbation theory, the amount of computation needed as a function of n must be bounded by a polynomial in n.

of particles, and there can be various such sets at different energy levels, and the basic excitation will appear completely different as we go through various phases of a theory. The main example of such phenomenon in the standard model concerns QCD: at sufficiently high energy we find quarks and gluons, while at lower energy the quarks are "confined" into hadrons (for example protons and neutrons), which then appear elementary. A similar phenomenon happens in the context of the Higgs mechanism, in which the Higgs boson disappears at low energy while giving mass to the other elementary particles.

As stated previously, perturbation theory requires the number of particles to be small. Therefore it will fail to describe most phenomena involving a large number of particles, for the same reason that classical dynamics fails in the analogous case and must be replaced by a different approach (thermodynamics). It is worth noting that for some large systems one can find a different perturbative approach, in which an effective field theory is built from some "collective excitations" of the fields (in a QFT equivalent of the classical thermodynamical approach), however this is not always the case. The worst case concerns critical phenomena, in which all the excitations contribute at every scale, which is the exact opposite of the assumptions of perturbation theory. These critical phenomena actually occur mainly at phase transitions, i.e. at the boundary between two different (perturbative) descriptions in terms of elementary particles.

Given the above limitations, it is essential to look for alternatives to perturbation theory, to be able to describe the physical phenomena for which perturbative methods fail. In the following we explore two such approach, Lattice QCD and exact computations in approximate theories.

1.1.2 The limits of non-perturbative numerical methods

The most basic numerical approach to understand a physical system is to simulate it on a computer. Quantum field theory is no exception to this; however as for any continuous system a perfect simulation is impossible, since by construction computers are discrete. The space and time dimensions must be approximated by discrete points separated by finite intervals, in such a way that the approximation approaches the true result as the size of the intervals is sent to zero (and the number of points is sent to infinity). The points form a lattice, and are given a set of local degrees of freedom. We designate the discretized theory by the term "lattice quantum field theory" (or "lattice QFT").

Simulations on a lattice QFT require very intensive computations. This is due in large part by the large number of degrees of freedom involved. Indeed, since the space is three-dimensional even a small number of points per dimension will result in a very large lattice.

This lattice size multiplies the degrees of freedom at each point, which may already be large, for example the standard model has over a hundred degrees of freedom per point. This means simulating the standard model on a lattice of a thousand points per dimension will already involve over 100 billion degrees of freedom. However the main issue comes from the fact that the theory is quantum, and the size of the Hilbert space grows exponentially with respect to the number of degrees of freedom, so even for tiny lattices it is already impossible to enumerate all the states, and a perfect simulation is hopeless[2]. For example we may be interested in computing a transition matrix relating the initial and final states. For a Hilbert space of size N, this is a $N \times N$ matrix, and it is obtained by a set of matrix multiplications, one for each time step. Computing even one such element shows the same problem, since we still need to consider all N intermediate states.

Despite these issues there has been some progress in lattice QFT computations. It is actually possible to partially overcome the difficulties associated with the exponentially large Hilbert spaces with further approximations. For example by estimating the matrix elements through a statistical sample of the intermediate states. It is also possible to obtain meaningful results from relatively small lattices (i.e. too small to give accurate results) by estimating the required quantity as a function of the lattice separation and extrapolating to zero-separation (infinite lattice). However even with these methods realistic lattice QFT computations remain a big computational challenge. There are many open questions, although most of them will require more computational power than is available today before we can hope for an answer.

1.1.3 The advantages of non-realistic theories

In the previous two sections we obtained the same conclusion concerning numerical results: we need better computers. However numerical results are also limited in their explanatory power: while numbers are well-suited for comparing to experiments, and it may be possible to "observe" or "describe" physical phenomena through a numerical simulation, it is in general not possible to give a concrete explanation for what is going on. To see this consider the example of confinement. As mentioned previously, confinement cannot be described using perturbation theory. It can be observed in lattice QFT, and we can observe hadron-like states on a lattice, and estimate their masses. It is however impossible to formally prove

[2]This comment applies to transistor-based "classical" computers. Quantum computers, involving the simulation of a quantum system, may eventually allow to partially overcome this difficulty by providing Hilbert spaces of exponentially growing size (with respect to the number of quantum bits). It is however unknown to this day whether it is feasible to build quantum computers powerful enough to be relevant for lattice QFT.

the existence of hadrons from this framework, and even finding a concrete explanation is tricky.

To describe confinement, we may wish to turn to non-realistic theories, and indeed it can be described by an exact computation in some supersymmetric theories [86, 87]. The result itself may be not entirely relevant for the standard model, but we can use it to make the following conclusion: it is indeed possible to observe confinement in a quantum field theory. We also gain some insight into a possible explanation for the phenomenon (a certain form of the Higgs mechanism, in which confinement is a consequence of a certain set of quasiparticles acquiring a mass), and prove that it is a theoretically possible explanation. This goes further than any known result in realistic theories.

Confinement is one of many examples in which simpler, non-realistic theories help understanding the standard model (and other realistic quantum field theories, such as those describing condensed matter systems and hypothetical theories for cosmology and physics beyond the standard model) by providing exact results. Therefore we may want to look for more such results in theories simpler than the standard model. Some comments are in order here concerning the term "non-realistic". Here we are interested in theories showing some similarities to the real world in their phenomenological behavior. We are not necessarily interested actual numerical accuracy, which would be the domain of "realistic" approximate theories (such as some forms of lattice QFT and some effective field theories). We now turn to some of the most important theories in which we can hope for exact results.

To reduce the complexity of a quantum field theory, one may want to reduce the number of elementary fields and simplify their interactions. However in the context of non-perturbative physics, this method does not provide enough simplification, and it may actually make the non-perturbative phenomena trivial. An alternative, more suitable approach is to consider theories with more structure, or symmetry. In four spacetime dimensions this kind of simplification almost always involve supersymmetry in some form. As described in the next section, the reason for this is as follows: it allows for using powerful mathematical tools from geometry and topology, and as a consequence it is in some cases possible to obtain non-trivial exact results, which is generally impossible in non-supersymmetric interacting theories. We may also look for conformally invariant theories, although most known four-dimensional conformal theories are also supersymmetric. Theories with both symmetries, or superconformal theories, are a particularly good source of exact results. A different approach is to vary the dimensionality, as theories in fewer dimensions are much easier to handle.

1.2 Why supersymmetry?

We now turn to supersymmetry, with the goal of justifying its importance for the computation of exact results. We begin with a brief review of supersymmetry.

In particle physics, supersymmetry is described as a symmetry of a quantum field theory relating bosonic (integer spin) and fermionic (half-integer spin) particles. However in this work a more formal approach will be needed, and instead of describing supersymmetry by some symmetry transformation we will describe it through a symmetry algebra and its representations.

The main building block of a supersymmetry algebra consists of a pair of anticommuting (Grassmann odd) symmetry charges Q and $Q^\dagger$, called supercharges, and a commuting (Grassmann even) charge H[3]. They satisfy the algebra

$$\{Q, Q^\dagger\} = 2H, \qquad [Q, H] = [Q^\dagger, H] = 0, \qquad Q^2 = (Q^\dagger)^2 = 0 \tag{1.1}$$

where H is a commuting conserved charge. (In supersymmetric quantum mechanics, H is the Hamiltonian.) Even in the simplest case, supersymmetry has some interesting consequences: H must be positive semidefinite, $H \geq 0$, and any eigenstate of H with nonzero eigenvalue E must be part of a multiplet containing a boson $|\phi\rangle$ and a fermion $|\psi\rangle$, satisfying either

$$\begin{aligned} Q|\phi\rangle &= \sqrt{E}|\psi\rangle, \qquad & Q^\dagger|\phi\rangle &= 0, \\ Q^\dagger|\psi\rangle &= \sqrt{E}|\phi\rangle, \qquad & Q|\psi\rangle &= 0, \end{aligned} \tag{1.2}$$

or the equivalent with $Q \leftrightarrow Q^\dagger$. The states annihilated by H play a special role in supersymmetry, as they are the ground states with respect to H, and they are not required to appear in multiplets. One of the most important results in supersymmetric quantum mechanics is based on these facts [97]: suppose we want to know if supersymmetry is spontaneously broken. (The question arises in any realistic application of supersymmetry, as if supersymmetry is present in nature, it must be broken.) This can only happen if there is no supersymmetric ground state, and in particular every state is part of a multiplet. From this fact we obtain a simple constraint on supersymmetry breaking: if the number of fermionic and bosonic states is different, then there exists at least one ground state and supersymmetry is not spontaneously broken. To formalize this result and to make sense of it for infinite-dimensional Hilbert spaces, one introduces the quantity $\mathrm{Tr}[(-1)^F e^{-\beta H}]$, called Witten index. Here F is a fermion number operator, and $(-1)^F$ acts as $+1$ on

[3]Most of the material in this section follows [55]

bosonic states and -1 on fermionic states, and β is a positive constant. The Witten index can be interpreted as a partition function for a supersymmetric theory compactified on a cylinder of circumference β with antiperiodic boundary conditions for the fermions. By supersymmetry, the index reduces to the trace of $(-1)^F$ over the ground states (an integer), and if it is nonzero then there must be at least one supersymmetric ground state. The above result is not isolated, and is in fact the prototype for a wide variety of exact results in supersymmetric quantum field theory: if an operator $\mathcal{O}$ is invariant under the action of a supercharge Q (and by extension $Q^\dagger$), then its correlator $\langle\mathcal{O}\rangle$ depends only on Q-invariant states, allowing for drastic simplifications. In the Hilbert space formalism the statement follows from a simple fact: consider the projection of an operator to the subspace spanned by a multiplet of the form (1.1), taking the explicit form

$$\begin{pmatrix} |\phi\rangle & |\psi\rangle \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \langle\phi| \\ \langle\psi| \end{pmatrix}. \tag{1.3}$$

Invariance under Q forces $a = b = 0$, while invariance under $Q^\dagger$ imposes $c = d = 0$, i.e. the operator vanishes in that subspace. Before exploring the consequences of this result, we proceed with a review of supersymmetric quantum field theory.

In supersymmetric quantum field theory, supersymmetry follows the same principles as outlined above, but we now require compatibility with the spacetime symmetries. In four dimensions, Poincaré invariance forces the supercharges to appear in sets of four. The basic supersymmetry algebra consists of a pair of (left-handed and right-handed Weyl) spinor supercharges Q, $\bar{Q}$, satisfying $\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma_{\alpha\dot{\alpha}} \cdot P$, where P is the momentum four-vector and σ is the set of Pauli matrices [96]. The supersymmetry algebra generalizes to sets of $4\mathcal{N}$ supercharges Q^I, $\bar{Q}^I$, $I = 1, \cdots, \mathcal{N}$ (usually with $\mathcal{N} \in \{1, 2, 4, 8\}$), taking the form

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = 2\delta^{IJ}\sigma_{\alpha\dot{\alpha}} \cdot P, \qquad \{Q^I_\alpha, Q^J_\beta\} = 2\epsilon_{\alpha\beta}Z^{IJ}, \qquad \{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\} = 2\bar{\epsilon}_{\dot{\alpha}\dot{\beta}}\bar{Z}^{IJ}. \tag{1.4}$$

Here ϵ, $\bar{\epsilon}$ are two-dimensional antisymmetric tensors, and Z^{IJ}, $\bar{Z}^{IJ}$ are a set of central charges (i.e. they commute with all symmetry generators). There is a $SU(\mathcal{N}) \times U(1)$ symmetry acting on the supercharge indices (I, J, $\cdots$), called R-symmetry, although it is often broken. A generic supermultiplet consists of $2^{2\mathcal{N}}$ states, since all states must be part of a basic multiplet under each pair of supercharges (although CPT invariance may further double the state count). However, there can be smaller multiplets, as described below. In the present work we are mainly interested in the case $\mathcal{N} = 2$, where we can write $Z^{IJ} = \epsilon^{IJ}Z$, where Z is simply $\mathbb{C}$-valued.

The central charge is the main new feature of extended supersymmetry, and gives rise to a variety of new phenomena. In particular, it separates the Hilbert space into a variety

of disconnected subspaces with different values for Z (since no local operator can change Z). For each sector, we can obtain a lower bound on the energy: consider a particle of mass M in its rest frame, or equivalently a zero-momentum state $|\psi\rangle$, with $P^0|\psi\rangle = M|\psi\rangle$ and $\mathbf{P}|\psi\rangle = 0$). Define $a^{\pm}_{\alpha} = \frac{1}{2}(Q^1_{\alpha} \pm \sigma^0_{\alpha\dot{\alpha}}(\bar{Q}^2)^{\dot{\alpha}})$, satisfying $\{a^{\pm}_{\alpha}, (a^{\pm})^{\dagger}_{\dot{\alpha}}\}|\psi\rangle = \sigma^0_{\alpha\dot{\alpha}}(M \pm Z)|\psi\rangle$. As a consequence the states $(a^{\pm})^{\dagger}_{\dot{\alpha}}|\psi\rangle$ have squared norm $(M \pm Z)\langle\psi|\psi\rangle$. Since all physical states must have positive norm, we find the constraint $M \geq |Z|$, or BPS bound[4]. For $M = |Z|$, the state is annihilated by either $(a^{+})^{\dagger}$ or $(a^{-})^{\dagger}$ implying that it can be part of a "short" multiplet, containing a quarter of the states states. The analysis generalizes to nonzero momentum, and a short multiplet with $M = |Z|$ is called "BPS state". Massless states correspond to a special kind of short multiplet, corresponding to the $Z = 0$ sector. Supersymmetric ground states, annihilated by all the supercharges, further specialize the $Z = 0$ sector. Related to BPS stated are BPS operators, which are invariant under half the supercharges, and generate BPS states when acting on supersymmetric ground states.

The presence of BPS states and operators is what makes supersymmetric theories interesting toy models for non-perturbative computations and phenomena. A BPS operator is invariant under some supercharges, so as mentioned before its correlator depends only on field configurations invariant under those supercharges. In many cases the resulting simplifications allow for an exact computation, and thus we can find a wide variety of non-trivial exact results. A canonical example concerns the computation of instanton corrections to the metric of the moduli space of vacua in $\mathcal{N} = 2$ supersymmetric gauge theories [86, 87]. The metric determines the low-energy effective action for the theory, and thus encodes its low-energy behavior. By a non-renormalization theorem, the metric cannot receives perturbative corrections (except for a simple one at first order), but receives non-perturbative corrections. Such corrections are caused by instantons, i.e. critical points of the Lagrangian with nonzero finite action. By supersymmetry only BPS instantons (being the BPS states) contribute to the metric , and their contribution can be computed exactly. The result has a direct application: it provides an explicit example of confinement, which is extremely difficult to study in realistic theories.

With these remarks, we can now provide our main justification for supersymmetry: it makes exact computations possible. It is however worth noting that there are other motivations for supersymmetry, although they are less relevant to the present work. Supersymmetry is an essential component of realistic string theories, as a requirement for the presence of fermions [82]. Approximate supersymmetry is also present in many hypothetical extensions of the standard model, although only $\mathcal{N} = 1$ is possible since extended supersymmetry prevents chiral symmetry breaking, a key feature of the standard model.

[4]The term "BPS", named after the works of Bogomoln'yi [14], Prasad and Sommerfeld [84], is used in various contexts, all related to some extent to the bound described here.

1.3 Four dimensional $\mathcal{N} = 2$ supersymmetric gauge theories

We now briefly review some aspects of four dimensional $\mathcal{N} = 2$ supersymmetric gauge theories [8, 90]. There are two main types of $\mathcal{N} = 2$ supersymmetric multiplets: vector multiplets and hypermultiplets. Note that unlike in the previous section, here the multiplets are formed at the level of fields rather than states, but the principle is identical. In Minkowski signature, a $\mathcal{N} = 2$ vector multiplet consists of a vector field, a R-symmetry pair of fermions, and a complex scalar, all in the adjoint representation of a gauge group G. A hypermultiplet consists of a pair of fermion and a R-symmetry doublet of complex scalars. In some (real) representations hypermultiplets are not fundamental, and are in fact composed of two fundamental multiplets with half the matter content, or half-hypermultiplets. In a $\mathcal{N} = 2$ theory, vector multiplets plays the role of gauge fields, and hypermultiplets that of matter fields. In particular, hypermultiplets can have masses, and couple to vector multiplets through their representation with respect to the gauge groups. These are in fact the only couplings allowed by supersymmetry.

There is a special class of $\mathcal{N} = 2$ supersymmetric gauge theories, called class S theories, which are also superconformal and benefit from additional integrability properties [36, 87]. These are obtained by compactification of six-dimensional $\mathcal{N} = (2, 0)$ superconformal gauge theories on punctured Riemann surfaces [99]. The resulting theory depends only on the (6d) gauge group, the punctured Riemann surface $\mathcal{T}$ and its complex structure (but not its metric), and the boundary condition for each puncture. Given such data, and restricting to the gauge (and flavour) group $G = SU(2)$, a class S theory is constructed as follow.

The basic ingredient for the construction is a three-punctured sphere. In isolation, it corresponds to a set of eight half-hypermultiplets, forming a trifundamental representation of a $SU(2) \times SU(2) \times SU(2)$ flavor symmetry. Each $SU(2)$ group corresponds to a puncture, with boundary condition corresponding to a (Cartan-valued) mass parameter m for the group. To construct more complicated theories, we consider multiple three-punctured spheres and proceed in the following way. Taking two distinct punctures on separate spheres, we "gauge" their flavor symmetries by replacing them with a $SU(2)$ gauge group, for which both hypermultiplets are in the fundamental representation. Note that the "reverse" process amounts to freezing the gauge group to a vacuum expectation value m by removing its kinetic term. In terms of Riemann surfaces, the process amounts to "sewing" the spheres together by replacing the punctures with a thin tube connecting the sphere. One can repeat the process to generate a network of spheres, and by conformal invariance we can fatten the tubes and reshape the result to obtain a complicated punctured

Riemann surface. If fact every punctured Riemann surface can be obtained in this way, as given such surface one can easily reverse the process to obtain a set of spheres with three punctures. We thus have a description of the four-dimensional theory, depending on the punctured Riemann surface and a decomposition into three-punctured spheres. Note that the decomposition is not unique, and in fact a single Riemann surface can give rise to various four-dimensional theories, and as a consequence we obtain dualities between such theories.

The above construction has an interesting consequence: many BPS quantities in class S theories correspond to quantities on the corresponding Riemann surfaces. The Alday-Gaiotto-Tachikawa (AGT) duality provides the relation for several cases [3, 101]. For example, the partition function of a class S theory on a four-sphere corresponds to a correlation function for Toda conformal field theory on the corresponding Riemann surface, with the insertions in the correlation function corresponding to the punctures. The quantities on each side are described in the following sections.

In this work we are particularly interested in a specific class of non-local operators, living on a two dimensional subspace of the four-dimensional theory [37, 46, 47]. These are called surface operators, or surface defects. Such operator breaks some translation and rotation symmetry, and as a consequence at least half the supercharges are broken by the operator. In the most supersymmetric case, and the only case considered in this work, the remaining four supercharges are preserved, and their algebra corresponds to the two-dimensional $\mathcal{N} = (2, 2)$ supersymmetry algebra.

In a class S theory, the behavior of a surface operator depends on its six-dimensional interpretation, and in particular how it appears in the compactified Riemann surface $\mathcal{T}$ [32, 34, 43]. A "codimension four" defect correspond to a point in $\mathcal{T}$. On the four-dimensional theory it corresponds to the insertion of a non-local operator on a surface, and on the CFT side it results in the insertion of an additional vertex operator of a special kind. In contrast, a "codimension two" defect fills $\mathcal{T}$, and couples the four-dimensional theory to extra degrees of freedom on the surface. On the CFT side it generates a space-filling insertion, i.e. it modifies the theory. It is widely believed that the modified theory is a WZW model, as supported by several computations [7, 16, 62]. Although there is no formal proof of the result, in this work we will assume it to be true. In this work we are mainly interested in codimension two defects.

1.4 Two dimensional $\mathcal{N} = (2, 2)$ supersymmetric gauge theories

As mentioned in the previous section, surface defects in four dimensional $\mathcal{N} = 2$ theories behave as two dimensional $\mathcal{N} = (2, 2)$ theories. Here we review some basic facts about such theories [55]. While the $\mathcal{N} = (2, 2)$ supersymmetry can arise as a subalgebra of four-dimensional extended supersymmetry, it is best seen as the dimensional reduction of the four-dimensional $\mathcal{N} = 1$ supersymmetry algebra. The four supercharges satisfy the same algebra $\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma_{\alpha\dot{\alpha}} \cdot P$, where the momenta in the reduced dimensions play the role of central charges. In this work we consider two types of multiplets, vector and chiral multiplets, respectively playing the role of gauge and matter fields.

1.5 AGT duality and two-dimensional conformal field theory

In this section we review the two dimensional side of AGT duality, and complete the presentation of the first project of this thesis. We restrict our focus to the dual of $SU(2)$ gauge theories, which in the absence of codimension two defects correspond to Liouville theory.

In the Lagrangian formulation, Liouville theory is a nonlinear modification of the theory of a free boson ϕ, with action $S = \frac{1}{4\pi} \int d^2x (\partial\phi \cdot \partial\phi + 4\pi\mu e^{2b\phi})$, where μ and b are two parameters of the theory (see for example [50, 93]). However it is best described as a conformal field theory, i.e. in terms of its primary operators (punctures) and their correlation functions. The primaries V_α are parametrized by a "Liouville momentum" α, and have conformal weight $\Delta_\alpha = \alpha(Q - \alpha)$, $Q = b + b^{-1}$. Classically they are given by $\tilde{V}_\alpha = e^{2\alpha\phi}$. The Liouville momentum specifies a boundary condition for the field $\partial\phi$ near the puncture:

$$\partial\phi(w)V_\alpha(z) \sim \frac{\alpha}{w - z} V_\alpha(z), \tag{1.5}$$

up to terms regular near $w = z$.

The AGT duality for $SU(2)$ symmetry groups states the following [3]. Consider a theory of class S obtained from the compactification of the $SU(2)$ six-dimensional $\mathcal{N} = (2, 0)$ superconformal theory on a n-punctured Riemann surface $\mathcal{T}$, with boundary conditions m_i at each puncture, $i = 1 \cdots n$. Then the four-sphere partition function of the theory is

equal to the Liouville correlator

$$Z_{S^4} = \langle V_{m_1} \cdots V_{m_n} \rangle_{\mathcal{T}} \tag{1.6}$$

We will give a precise definition of Z_{S^4} in section 1.6.

1.5.1 Asymptotically free theories and irregular vertex operators

The theories of class S obtained by compactification of the six-dimensional superconformal field theories are all conformal; however one can also recover theories with a negative β-function, or asymptotically free theories, by a slight generalization of the formalism. Specifically one allows for more singular boundary conditions near the punctures. On the CFT side we let [41]

$$\partial\phi(w) V_{\mathbf{c}}(z) \sim \sum_{k=0}^{r} \frac{c_k}{(w-z)^{k+1}} V_\alpha(z). \tag{1.7}$$

Here r is called the "rank" of the puncture, and $\mathbf{c}$ is the vector $(c_0 = \alpha, c_1, \cdots, c_r)$. A puncture with rank ≥ 1 is called irregular, and is in fact a "quantum equivalent" for an irregular singular point in the theory of ordinary differential equations.

It is not always possible to have irregular vertex operators in a CFT, however in many non-compact CFTs (characterized by a continuous spectrum) one can derive their existence from a "collision limit" of regular vertex operators (or primary fields). The method consists of taking the parameters of a set of punctures to infinity while simultaneously taking their separation to zero, in such a way that the limit is finite but nontrivial. In Liouville theory this corresponds to taking the Liouville momenta to infinity while constraining their sum, and in the simplest example we find

$$\lim_{z\to 0} V_{-cz^{-1}}(z) V_{cz^{-1}+\alpha}(0) = V_{(\alpha,c)}(0). \tag{1.8}$$

To prove the above result we need to show that the correlation functions have a finite limit, however as a simple check we find the correct boundary conditions:

$$\begin{aligned} \partial\phi(w) \left(\lim_{z\to 0} V_{-cz^{-1}}(z) V_{cz^{-1}+\alpha}(0) \right) &\sim \lim_{z\to 0} \left(-\frac{cz^{-1}}{w-z} + \frac{cz^{-1}+\alpha}{w} \right) V_{-cz^{-1}}(z) V_{cz^{-1}+\alpha}(0) \\ &\sim \left(\frac{c}{w^2} + \frac{\alpha}{w} \right) \lim_{z\to 0} V_{-cz^{-1}}(z) V_{cz^{-1}+\alpha}(0), \end{aligned} \tag{1.9}$$

assuming the above limits exist.

On the four-dimensional side, the collision limit has the following interpretation: one reduces a set of flavor symmetry groups to a single one by taking their mass parameters to infinity while keeping their sum finite. In the generic case this removes some hypermultiplets from the theory while keeping the vector multiplets unchanged, thus breaking conformal invariance (since a hypermultiplet has a positive contribution to the β-function). By construction AGT duality can be extended to the resulting theory, provided we use the appropriate irregular vertex operators in the CFT side [35].

1.5.2 Irregular vertex operators in WZW models

As mentioned previously in the presence of a codimension two surface defect the CFT is replaced by a WZW model. In this work we are interested in the simplest case, involving a $\widehat{sl}(2)$ WZW model, which we treat as a toy model for more complicated theories. Specifically we aim to find and describe irregular vertex operators in that theory, and chapter 2 is dedicated to this goal. In that project we find and describe such irregular punctures by considering a free-field realization of the theory, and confirm the result by comparing to the collision limit of primary fields. The free-field approach is a simpler alternative to the collision limit, and easily generalizes to many CFTs admitting a free-field realization. We also explore a duality [85] between the model and Liouville theory in the context of irregular punctures, related to an infrared duality [34] between codimension two and codimension four defects. In particular we find that our description is compatible with the duality and the description of irregular vectors in Liouville theory [41]. Finally we explore the semiclassical limit of correlation functions and observe the Stokes phenomenon, a key property related to irregular singular points in differential equations.

1.6 Supersymmetric localization and exact computations of partition functions

In this section we describe supersymmetric localization, and complete the presentation of the second project of this thesis.

As described in section 1.2, the correlator $\langle \mathcal{O} \rangle$ for an operator $\mathcal{O}$ invariant under a supercharge $\mathcal{Q}$ depends only on $\mathcal{Q}$-invariant field configurations. In the path integral formalism,

this property is realized as follow [55]. Consider the integral[5] over a bosonic variable X and a fermionic variable ψ, of a function $F(X,\psi)$ (playing the role of $\exp(-iS(X,\psi))\mathcal{O}(X,\psi)$

$$\int dXd\psi F(X,\psi) = \int dXd\psi(F_1(X)+\psi F_2(X)) = \int dXF_2(X). \tag{1.10}$$

In this system the basic supersymmetry algebra (1.1) is realized on the variables as

$$\begin{aligned} \delta_{\mathcal{Q}}X = \sqrt{h(X)}\psi, \qquad & \delta_{\mathcal{Q}^\dagger}X = 0, \\ \delta_{\mathcal{Q}}\psi = 0, \qquad & \delta_{\mathcal{Q}^\dagger}\psi = \sqrt{h(X)}X. \end{aligned} \tag{1.11}$$

If $F(X,\psi)$ is $\mathcal{Q}$-invariant, we find

$$\begin{aligned} 0 = \delta_{\mathcal{Q}}F(X,\psi) = \sqrt{h(X)}\psi F_1'(X), \\ 0 = \delta_{\mathcal{Q}^\dagger}F(X,\psi) = \sqrt{h(X)}F_2(X), \end{aligned} \tag{1.12}$$

so as long as $h(X)$ is nonzero $F(X,\psi)$ must be a constant, and in particular the fermionic integral vanishes. However at a point where $h(X)=0$ there is no such constraint. Summing up, we find

$$F(X,\psi) = F_0 + \delta(h(X))\left(\tilde{F}_1(X)+\psi\tilde{F}_2(X)\right), \tag{1.13}$$

and the path integral becomes

$$\int dXd\psi F(X,\psi) = \sum_{n,\ h(X_n)=0} \frac{F_2(X_n)}{|h'(X_n)|}, \tag{1.14}$$

i.e. it only receives contributions from the zeros of $h(X)$, i.e. it localizes to $\mathcal{Q}$-invariant points (configurations). This principle is called supersymmetric localization.

Under certain regularity assumptions the above analysis can be applied directly to a $\mathcal{Q}$-invariant QFT correlator $\langle\mathcal{O}\rangle$, written in the (Euclidian) path integral formalism as

$$\langle\mathcal{O}\rangle = \int \mathcal{D}\Phi e^{-S[\Phi]}\mathcal{O}[\Phi], \tag{1.15}$$

[5] Here we remind the rules for Grassmann integration over a fermionic variable ψ,

$$\int d\psi = 0, \quad \int d\psi\psi = 0$$

.

where Φ is a collective coordinate for the fields. As before the integral depends only on $\mathcal{Q}$-invariant field configurations, since it is essentially a (infinite) product of integrals of the type considered above. However the invariant configurations may be difficult to find, and we can deform the theory to simplify the computation. We deform the action with $S \to S + t\delta_{\mathcal{Q}} \cdot V$, and as long as $\delta_{\mathcal{Q}}^2 \cdot V = 0$ the path integral is modified by a $\mathcal{Q}$-exact term and is thus unchanged. If V is non-negative we can take $t \to \infty$, and in that limit the saddle-point approximation becomes exact, and from it we recover the original path integral.

The saddle-point approximation is used as follow. We aim to compute

$$\langle \mathcal{O} \rangle = \lim_{t\to\infty} \int \mathcal{D}\Phi e^{-S[\Phi]-t\delta_{\mathcal{Q}}\cdot V[\Phi]} \mathcal{O}[\Phi]. \tag{1.16}$$

By construction the integrand contributes only near the zeros of $\delta_{\mathcal{Q}} \cdot V$, or "classical configurations". The contribution from a zero at Φ_0 is

$$e^{-S[\Phi_0]} \lim_{t\to\infty} \int \mathcal{D}\Phi e^{-t(\mathcal{Q}\cdot V)[\Phi+\Phi_0]}. \tag{1.17}$$

After rescaling the fields and taking the limit we find that only the quadratic terms in the fields Φ contribute, and we obtain

$$e^{-S[\Phi_0]} \int \mathcal{D}\Phi e^{-(\mathcal{Q}\cdot V)[\Phi+\Phi_0]_{\text{Quad}}} = e^{-S[\Phi_0]} Z_{1-\text{loop}}[\Phi_0]. \tag{1.18}$$

The quantity $Z_{1-\text{loop}}[\Phi_0]$ is a one-loop determinant, and in many cases can be computed exactly, either by direct computation or using the Atiyah-Singer index theorem [10]. The full path integral is obtained by summing or integrating over the zeros of $\delta_Q \cdot V$.

1.6.1 Supersymmetric localization on S^4 and S^2

The method described above was successfully used for several exact computations. In this work we are interested in two such results, the partition functions of $\mathcal{N} = 2$ supersymmetric gauge theories on S^4 [81] and $\mathcal{N} = (2, 2)$ supersymmetric gauge theories on S^2 [12, 31]. We restrict to localization on the Coulomb branch, in which the vector fields acquire a vacuum expectation value and gauge invariance is broken to an Abelian subgroup[6].

[6]One can also localize to the Higgs branch, where the hypermultiplet scalars acquire a vacuum expectation value and gauge invariance is completely broken. This was done in the two dimensional case in [12, 31]. The branch for localization is determined through the choice of the supercharge Q and the deformation term V.

In both cases, the supercharge Q is chosen so that it generates an algebra leaving the poles of the sphere invariant. Given this property, the Atiyah-Singer index theorem states that the one-loop determinant receives only contributions from the poles (the fixed points of the algebra), and provides these contributions. The rest of the computation follows as outlined above. In two dimensions the set of classical configurations is parametrized by a Cartan-valued real parameter a and a Cartan-valued integer parameter B (magnetic flux), and the partition function takes the form[7]

$$Z_{2d} = \sum_B \int_{2d \text{ Coulomb}} da e^{-S_{cl}^{2d}(a,B)} Z_{1-\text{loop}}^{2d}(a, B). \tag{1.19}$$

The four-dimensional case is more complex, as the presence of instantons (and anti-instantons) makes the set of classical configurations much larger. The instantons form a full quantum mechanical theory by themselves, and localization as described above leaves a part of the path integral unevaluated. However instantons do not affect the one-loop computation, so the two computations can be done independently. Schematically we find (again, a is a Cartan-valued real parameter for the Coulomb branch)

$$Z_{4d} = \int_{4d \text{ Coulomb}} da e^{-S_{cl}^{4d}(a)} Z_{1-\text{loop}}^{4d}(a) \int_{\text{inst, anti-inst}} \mathcal{D}\Phi e^{S[a;\Phi]}. \tag{1.20}$$

In the localization framework instantons localize to the North pole, while anti-instantons localize to the South pole. The two contributions are related by complex conjugation, and we write their combined contribution as

$$\int_{\text{inst, anti-inst}} \mathcal{D}\Phi e^{S[a;\Phi]} = |Z_{\text{inst}}^{4d}(a)|^2. \tag{1.21}$$

The quantity $Z_{\text{inst}}^{4d}(a)$ is an instanton partition function [76, 77, 81]. We will not consider the details of instanton partition functions in this work.

1.6.2 Supersymmetric localization involving a surface defect

Given the known results on S^2 and S^4, we can now consider localization of the four-dimensional theory in the presence of a codimension two surface defect. The defect is located on a great sphere of S^4 (i.e. it has maximal radius). In the first time we consider the case where the defect does not interact with the bulk theory. If we define the poles

[7] In this section we omit the dependence on the parameters of the theory for simplicity.

of S^4 in such a way that they are located on the defect, then the (Coulomb branch) localization computations on S^2 and S^4 are actually compatible. By compatible we mean that we can embed the $\mathcal{N}=(2,2)$ supersymmetry algebra on S^2 inside the $\mathcal{N}=2$ supersymmetry algebra on S^4, and that we can pick the same supercharge for both localization computations. The result still holds when the defect interacts (in a supersymmetric way) with the bulk theory, and in chapter 3 we show it explicitly. We describe the embedding explicitly by writing the four-dimensional multiplets as sets of $\mathcal{N}=(2,2)$ multiplets, and use it to find the allowed interactions. For localization only one coupling is relevant, in which a chiral multiplet on S^2 interacts with a vector multiplet on S^4 (or more precisely, a subset of its fields forming a $\mathcal{N}=(2,2)$ vector multiplet on S^2) through its representation of the gauge group. We generalize the localization formula to include the interaction, and find a partition function of the form

$$Z_{2d-4d} = \int_{4d\ \text{Coulomb}} da\ e^{-S_{cl}^{4d}(a)} Z_{1-\text{loop}}^{4d}(a) \\ \times \sum_B \int_{2d\ \text{Coulomb}} d\tilde{a}\ e^{-S_{cl}^{2d}(\tilde{a},B)} |Z_{\text{inst}}^{2d-4d}(\tilde{a},B;a)|^2 Z_{1-\text{loop}}^{2d}(\tilde{a},B;a). \tag{1.22}$$

We find the Coulomb branch integral and the one-loop determinants to be similar to those of the separate computations, in which the 2d fields see the 4d Coulomb branch in a way identical to a 2d branch with zero magnetic flux (as a "twisted mass"). The instanton partition function is modified by the presence of the defect, however its computation is left for future work.

1.7 BPS states, magnetic monopoles and zero modes

In this section we introduce magnetic monopoles, and complete the presentation of the third project of this thesis.

1.7.1 Monopoles in four dimensions

In four-dimensional gauge theory, a magnetic monopole is a time-independent solution of the equations of motion with finite energy. As a simple example [54, 83] (see also [94]), consider the theory of a gauge field A_μ coupled with an adjoint scalar field ϕ. (This is a prototype for a $\mathcal{N}=2$ supersymmetric gauge theory.) To make time invariance manifest we use a three dimensional language, in which[8] $A_\mu = (A_0, \mathbf{A})$, and use gauge invariance

[8] Here and in the following we use the three dimensional vector notation $\mathbf{A} = (A_i) = (A_1, A_2, A_3)$.

to set $A_0 = 0$. The resulting field strength can be written in terms of a magnetic field $\mathbf{B}$, with $B_i = \epsilon_{ijk} F_{jk}$. We can find a bound on the energy (mass) of a monopole by a simple computation [14]:

$$\begin{aligned} E &= \frac{1}{e^2} \int d^3x \mathrm{Tr} \left(\mathbf{B}^2 + (\mathbf{D}\phi)^2 \right) = \frac{1}{e^2} \int d^3x \mathrm{Tr} \left((\mathbf{B} \mp \mathbf{D}\phi)^2 \pm 2\mathbf{B} \cdot \mathbf{D}\phi \right) \\ &\geq \pm \frac{2}{e^2} \int d^3x \mathrm{Tr} \left(\mathbf{B} \cdot \mathbf{D}\phi \right), \end{aligned} \tag{1.23}$$

where the last quantity is a topological quantity (magnetic charge). The bound is attained if and only if the Bogomoln'yi equations

$$\mathbf{B} = \pm \mathbf{D}\phi \tag{1.24}$$

are satisfied. The Bogomoln'yi equations are particularly important to $\mathcal{N} = 2$ supersymmetric gauge theories [86], as they correspond to a condition for invariance under some supersymmetry. Its solutions thus correspond to BPS states, and they are called BPS monopoles. In this work we are exclusively concerned with BPS monopoles.

1.7.2 Monopoles from supersymmetric quantum mechanics

BPS monopoles also arise naturally in supersymmetric quantum mechanics with four supercharges. Consider the reduction of a four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theory to one dimension (or equivalently the reduction of a two-dimensional $\mathcal{N} = (2, 2)$ gauge theory). The field content consists of vector multiplets and chiral multiplets. In the one-dimensional theory, a vector multiplet consists of a vector $\mathbf{x} = \mathbf{x}(t)$ (the reduction of the gauge field, $A_\mu = (0, \mathbf{x})$), fermions λ, $\bar{\lambda}$ and an auxiliary field D. We are interested in a low-energy effective theory, obtained by taking the gauge coupling e to zero. In that limit we can integrate away the chiral multiplets [79, 88, 89], resulting in an effective interaction term

$$\mathcal{L}_{\text{eff}} = \mathbf{A}(\mathbf{x}) \cdot \dot{\mathbf{x}} - \Phi(\mathbf{x}) D + \mathbf{C}(\mathbf{x}) \cdot \bar{\lambda} \boldsymbol{\sigma} \lambda \tag{1.25}$$

for the vector multiplet. The potentials Φ and $\mathbf{C}$ correspond to one-loop corrections to the theory, while $\mathbf{A}$ is a Berry connection. In quantum mechanics, a Berry connection describes how the ground states vary as the parameters of the theory (here the potentials in $\mathcal{L}_{\text{eff}}$) are varied adiabatically. Supersymmetry constrains the potentials with [25]

$$\mathbf{C} = \mathbf{D}\Phi = *\mathbf{F}, \tag{1.26}$$

where $\mathbf{F}$ is the field strength. The second equality in (1.26) is precisely the Bogomoln'yi equation, so Φ and $\mathbf{A}$ describe a monopole configuration. For example, in a $U(1)$ theory, a chiral multiplet of charge q results in the configuration ($x = ||\mathbf{x}||$)

$$\Phi = \frac{q}{2x}, \qquad \mathbf{A} = q\frac{x^2\mathbf{d}x^1 - x^1\mathbf{d}x^2}{2x(x+x^3)}, \tag{1.27}$$

corresponding to a monopole of charge q centered at the origin. The method described above is actually a form of supersymmetric localization (since varying e modifies the action by a Q-exact term), and many BPS quantities computed in the effective theory are exact (in the original theory).

1.7.3 Periodic monopoles from two-dimensional gauge theory

BPS monopoles also occur in two dimensional $\mathcal{N} = (2,2)$ supersymmetric gauge theory on a cylinder. The cylindrical geometry encodes many non-perturbative phenomena, including the spectrum of BPS states. In this setup we can use localization to find a monopole description similar to the one-dimensional case, however the monopoles are now periodic. Indeed, in the $e \to 0$ (zero coupling) limit the theory becomes effectively one dimensional, and the result is a set of one-dimensional vector multiplets coupled to towers of chiral multiplets[9]. The chiral multiplets are related by a momentum translation, and once they are integrated out this leads to a translation in $\mathbf{x}$ in the potential, hence the periodicity.

By the above construction, the BPS spectrum of two dimensional theories can be obtained from a periodic monopole geometry. This requires finding the supersymmetric ground states, to which we now turn.

1.7.4 Supersymmetric ground states

One of the most important properties of a supersymmetric theory is its set of supersymmetric ground states. In the monopole quantum mechanics (periodic or not) the ground states are highly nontrivial configurations. To find them we first describe the Hilbert space of the theory. Once the auxiliary field D has been integrated out, the bosonic field content is the vector $\mathbf{x}(t)$, describing the motion of a particle in three-dimensional space. The bosonic part H_b of the Hilbert space is the space of square integrable complex functions on $\mathbb{R}^3$

[9]This is essentially a Kaluza-Klein reduction. Note that the reduction leads to single vector multiplets instead of towers of them, as gauge invariance can be used to remove the nonzero modes.

(defined outside the monopole singularities). The fermionic part H_f is four-dimensional, and we pick a basis $\{|-\rangle, |0\rangle^\alpha, |+\rangle\}$, on which the fermionic operators act as

$$\begin{aligned} \lambda_\alpha|-\rangle &= 0, & \bar\lambda^\alpha|-\rangle &= |0\rangle^\alpha, \\ \lambda_\alpha|0\rangle^\beta &= \delta_\alpha^\beta|-\rangle, & \bar\lambda^\alpha|0\rangle^\beta &= \epsilon^{\alpha\beta}|+\rangle, \\ \lambda_\alpha|+\rangle &= \epsilon_{\alpha\beta}|0\rangle^\beta = -|0\rangle_\alpha, & \bar\lambda^\alpha|+\rangle &= 0. \end{aligned} \tag{1.28}$$

The total Hilbert space is $H_b \otimes H_f$, and a generic state is described by the set of wave-functions $\langle \mathbf{x}|\psi\rangle = \psi_-(\mathbf{x})|-\rangle + \psi_\alpha(\mathbf{x})|0\rangle^\alpha + \psi_+(\mathbf{x})|+\rangle$. The supercharges act on $H_b \otimes H_f$ as the operators

$$Q^\alpha = (\boldsymbol{P} - \boldsymbol{A})(\bar\lambda\sigma)^\alpha - \Phi\bar\lambda^\alpha, \qquad \bar Q_\alpha = (\boldsymbol{P} - \boldsymbol{A})(\sigma\lambda)_\alpha - \Phi\lambda_\alpha, \tag{1.29}$$

where $\boldsymbol{P} = \dot{\boldsymbol{x}} + \boldsymbol{A}$ is the canonical momentum for $\mathbf{x}$. The ground states are described by the conditions $Q|\psi\rangle = \bar Q|\psi\rangle = 0$, or

$$\begin{aligned} ((\boldsymbol{\partial}_{\mathbf{x}} - i\mathbf{A})\cdot\sigma - \Phi)\,\psi &= 0, \\ (\boldsymbol{\partial}_{\mathbf{x}} - i\mathbf{A})\psi_+ = \Phi\psi_+ &= 0, \\ (\boldsymbol{\partial}_{\mathbf{x}} - i\mathbf{A})\psi_- = \Phi\psi_- &= 0, \end{aligned} \tag{1.30}$$

These imply $\psi_- = \psi_- = 0$, and we are left with the equation $((\boldsymbol{\partial}_{\mathbf{x}} - i\mathbf{A})\cdot\sigma - \Phi)\,\psi = 0$ for ψ. This equation describes the zero modes of a fermion in a monopole background.

1.7.5 Fermion zero modes

In the previous section we showed that finding the ground states of the monopole quantum mechanics amounts to obtaining the fermion zero modes for the appropriate monopole configuration. Such zero mode have been the subject of much research, in particular for their relation to the Nahm transform [57, 74]. The Nahm transform relates solutions to the Bogomoln'yi equations to solution of a set of topological-anti-topological (tt^*) equations [19, 20], describing the relation between two topological constructions for the supersymmetric ground states in some supersymmetric theories. In the $\mathbb{R}^3$ case the tt^* equations correspond to the Nahm equations [71–74], while in the periodic case they correspond to a Hitchin system.

Despite the relevance of fermion zero modes, no generic solution for multiple monopoles was known prior to this work. In chapter 4 we present a solution for the case of Abelian monopole configurations on $\mathbb{R}^3$. The solution relies on a new method, in which the zero modes are obtained in terms of a residue integral over a set of flat sections for an auxiliary pair of connections (a Lax pair).

Chapter 2

Irregular Singularities in the H_3^+ WZW Model

2.1 Introduction and Conclusions

As seen the introduction, there are several protected quantities (BPS operators) which are exactly computable in $\mathcal{N} = 2$ four-dimensional gauge theories, and have a rich physical and mathematical content. Many four-dimensional gauge theories can be engineered from the twisted compactification of a six-dimensional CFT on a Riemann surface [36, 40, 99]. Protected quantities associated to such "class S" gauge theories can be usually given an interpretation in terms of mathematical structures attached to the Riemann surface. In particular, the Ω-deformed partition function of the four-dimensional theories takes the form of two-dimensional conformal blocks for Virasoro or W-algebras [3, 101]. Other current algebras arise from modifications of the four-dimensional setup: super-Virasoro and para-Virasoro (or W-algebras) [15, 78] arise from geometric quotients of the Ω background, WZW current algebras and their Hamiltonian reductions arise in the presence of extra surface defects [7, 62]. Furthermore, the partition function on a four-dimensional ellipsoid [48, 81] coincides with the correlation functions of the Liouville-like conformal field theories directly associated to the appropriate current algebra: Liouville and Toda, super or para-Liouville or super or para-Toda, the H_3^+ WZW model and its higher rank $SL(N, \mathbb{C})/SU(N)$ generalizations.

There is an intricate dictionary which pairs up a specific class S theory with a specific choice of correlation function in the two-dimensional conformal field theories. Standard superconformal field theories in the class S can be matched to generic correlation functions

of standard vertex operators in the two-dimensional CFTs. Asymptotically free gauge theories and Argyres-Douglas-like theories are matched with conformal blocks and correlation functions involving more exotic vertex operators, dubbed irregular vertex operators [35, 40, 41]. Indeed, asymptotically free gauge theories and Argyres-Douglas-like theories can be derived by a careful degeneration limit of the standard superconformal field theories. Correspondingly, on the two-dimensional CFT side, irregular vertex operators arise from a collision of standard vertex operators, whose conformal dimension is sent to infinity in a specific way.

The definition of irregular vectors has been given systematically for the Virasoro current algebra and Liouville theory correlation functions in [41]. The main purpose of this chapter is to give a systematic definition of irregular conformal blocks for the $\hat{sl}(2)$ current algebra and H_3^+ WZW theory. Similar, but sometimes more restrictive definitions appeared before in the mathematical literature, see for example [58, 69, 70] or [33].

We have a good reason for picking this example, among all possible generalizations of the Virasoro problem: it is a toy model for a different, deep problem which arises in the calculation of scattering amplitudes at strong coupling in planar $\mathcal{N} = 4$ SYM theory. Scattering amplitudes in planar $\mathcal{N} = 4$ SYM theory can be related to correlation functions of polygonal Wilson loops with null edges [2, 4–6]. At strong coupling, the Wilson loop correlation function should be computable in terms of a string world-sheet in AdS_5, ending on the null polygon at the boundary. At leading order, the calculation is classical, and one has to compute the area of a minimal-area surface bounded by the null polygon. In order to go beyond the leading order, one has to do a full quantum computation on the world-sheet theory describing a superstring moving in $AdS_5 \times S^5$. In particular, one has to find out how to impose appropriate boundary conditions at infinity, encoding in the full quantum theory the shape of the null Wilson loop at the boundary.

In the classical problem, the boundary condition at infinity produces an "irregular singularity" in a certain auxiliary connection, which lead to Stokes phenomena. The Stokes data at the singularity encodes the shape of the null polygon. Thus one needs to define a vertex operator for the world-sheet theory, which inserted at infinity leads to similar singularities and Stokes phenomena. Irregular vectors in Liouville theory give rise to irregular singularities in the differential equations satisfied by degenerate fields, and to Stokes phenomena. It is our hope that irregular singularities in the H_3^+ WZW theory may provide a closer analogue to the irregular singularities which are needed in the world-sheet theory of a superstring moving in $AdS_5 \times S^5$.

In this paper we derive the Ward identities which define irregular vertex operators and conformal blocks for the $\widehat{sl}(2)$ current algebra. We define a collision limit for $r + 1$

standard highest weight vertex operators, which leads to a generic rank r irregular vertex operator. Our results generalize in a natural way the definitions given for the Virasoro algebra in [41], and allow us to formulate a simple conjecture on the structure of Ward identities for irregular vertex operators in a general current algebra which admits a free field realization. We also give a generalization of the known duality between H_3^+ WZW correlation functions and Liouville theory correlation functions, which includes irregular vertex operators on both sides of the relation. This duality can be used in conjunction with the results of [41] in order to understand the structure of WZW conformal blocks with irregular singularities. Finally, we derive the KZ equations satisfied by irregular conformal blocks and correlation functions, and we give a semiclassical description of irregular vertex operators.

2.2 Irregular vectors and current algebras

In this section we give the explicit current algebra Ward identities for irregular vectors. Our main result in this section is a parameterization of the singularity in the currents which is compatible with the current algebra commutation relations, and with collision limits. In section 2.2.1 we review the Virasoro algebra analysis from [41]. In section 2.2.2 we look at the collision limit for several colliding primary fields of a $\widehat{sl}(2)$ current algebra. We look for an appropriate set of parameters which can be kept independent of the collision parameters while giving rise to a finite limit. The final form of the irregular Ward identities parameterizes the singularity of the currents by a set of bosonic oscillators, with a structure which mimics the Wakimoto free-field representation. The simplicity of our result suggests a natural extension to generic current algebras which admit a free field realization, which we discuss in section 2.2.3.

2.2.1 The Virasoro case

The Virasoro Ward identity for irregular vertex operators has been studied in [41]. An irregular vertex operator of rank r is defined as an eigenvector of the Virasoro modes $\{L_{2r}, \cdots, L_r\}$, annihilated by the higher modes L_n, $n > 2r$:

$$L_m \Psi_{\mathbf{\Lambda}}^{(r)} = \begin{cases} \Lambda_m \Psi_{\mathbf{\Lambda}}^{(r)} & 2r \geq m \geq r \\ 0 & m > 2r \end{cases} , \tag{2.1}$$

where Λ is the set of eigenvalues $\mathbf{\Lambda} = (\Lambda_r, \cdots, \Lambda_{2r})$. The Virasoro algebra forbids the field to be an eigenvector of any other mode. The Virasoro Ward identity for a rank r field $\Psi^{(r)}_{\mathbf{\Lambda}}$ can thus be written in the form

$$T(w)\Psi^{(r)}_{\mathbf{\Lambda}}(z) \sim \left[\sum_{k=r}^{2r} \frac{\Lambda_k}{(w-z)^{k+2}} + \sum_{k=0}^{r-1} \frac{\mathcal{L}_k}{(w-z)^{k+2}} + \frac{\partial_z}{(w-z)}\right] \Psi^{(r)}_{\mathbf{\Lambda}}(z). \tag{2.2}$$

where the differential operators $\mathcal{L}_k$ need to be compatible with the Virasoro algebra. The general solution proposed in [41] requires one to express the eigenvalues Λ_k in terms of a new set of auxiliary parameters $\mathbf{c} = (c_0 = \alpha, c_1, \cdots, c_r)$

$$\Lambda_k = (k+1)Qc_k - \sum_{l=0}^{k} c_l c_{k-l}, \tag{2.3}$$

where $c_{n>r} \equiv 0$. This is a generalization of the parametrization $\Delta = \alpha(Q-\alpha)$ familiar from Liouville theory. In terms of these parameters, the differential operators read

$$\mathcal{L}_k = (k+1)Qc_k - \sum_{l=0}^{k} c_l c_{k-l} + \sum_{l=k+1}^{r} (l-k)c_l \partial_{c_{l-k}}. \tag{2.4}$$

(Note that the derivatives ∂_{c_k} are only defined for $1 \leq k \leq r$.) This allows to write the Ward identity in the form

$$T(w)\Psi^{(r)}_{\mathbf{\Lambda}}(z) \sim \left[\sum_{k=0}^{2r} \frac{(k+1)Qc_k - \sum_{l=0}^{k} c_l c_{k-l} + \sum_{l=k+1}^{r}(l-k)c_l\partial_{c_{l-k}}}{(w-z)^{k+2}} + \frac{\partial_z}{(w-z)}\right] \Psi^{(r)}_{\mathbf{\Lambda}}(z). \tag{2.5}$$

The parameterization of the Virasoro Ward identities is clearly inspired by the free-field realization of the Virasoro algebra, but it is conceptually distinct. The correct statement is that an irregular vertex operator for the Virasoro algebra is an object which satisfies the same Ward identities as a coherent state

$$\exp\left[\sum_k c_k a_{-k}\right] |0\rangle \tag{2.6}$$

in the free field theory which realizes the Virasoro algebra. The a_{-k} are the creation modes of the free scalar. In general, there is a linear space of solutions of the irregular Virasoro Ward identities, i.e. irregular conformal blocks. As for standard conformal blocks, the

naive free-field description only provides a special solution in that space, and intricate configurations of screening charges are required to give a free-field description of general solutions.

A more intrinsic way to understand the origin of the irregular Virasoro Ward identities and the meaning of irregular conformal blocks is to define a rank r irregular vertex operator as a collision limit of $r+1$ regular vertex operators. Starting from the Ward identity

$$T(w)\prod_j \Psi_{\alpha_j}(z_j) \sim \sum_i \left[\frac{\alpha_i(Q-\alpha_i)}{(w-z_i)^2} + \frac{1}{w-z_i}\partial_{z_i}\right]\Psi_{\alpha_j}(z_j) \tag{2.7}$$

and defining [1]

$$\Psi = \prod_{k,t}(z_k - z_t)^{2\alpha_k\alpha_t}\prod_j \Psi_{\alpha_j}(z_j) \tag{2.8}$$

we can bring the z_i to a common point z, while the α_i are sent to infinity in such a way that

$$\sum_i \frac{\alpha_i}{w-z_i} \to \sum_k \frac{c_k}{(w-z)^{k+1}} \tag{2.9}$$

This collision limit brings the Ward identities for Ψ to the ones for the irregular vector $\Psi^{(r)}_{\mathbf{\Lambda}}(z)$.

2.2.2 Irregular $\widehat{sl}(2)$ currents

The Ward identity for a standard spin j primary field and the $\widehat{sl}(2)$ valued current $J(w) = J^a(w)t_a$ reads

$$J^a(w)\Phi_j(\mu|z) \sim \frac{1}{w-z}\mathcal{D}^a\Phi_j(\mu|z), \tag{2.10}$$

for some $\mu \in \mathbb{C}$, where $\mathcal{D}^a$ are spin-j generators for $sl(2)$. We use the following realization of the generators: [2]

$$\mathcal{D}^- = \mu, \qquad \mathcal{D}^0 = j - \mu\partial_\mu, \qquad \mathcal{D}^+ = 2j\partial_\mu - \mu\partial_\mu^2. \tag{2.11}$$

[1] Note that this rescaling allows to obtain a finite limit for the Virasoro Ward identities. Virasoro conformal blocks will have a finite limit if properly normalized and Liouville theory correlation functions will also have a finite limit (see [41] for details). In order to extend the collision limits to correlation functions of other CFTs, one may need to add further prefactors which take into account the behavior of three-point functions and the normalization conventions for the vertex operators in that CFT.

[2] Here the fields are rescaled by a factor μ^j relative to the ones in [85], so the differential operators are slightly different.

Here the fields are related to a more common "$x-$basis" [7, 26, 91, 92] by a Fourier transform and a rescaling [85]

$$\Phi_j(\mu|z) = \frac{1}{\pi}|\mu|^{4j+2}\int_{\mathbb{C}} d^2x\; e^{\mu x-\bar{\mu}\bar{x}}\Phi_j(x|z) \tag{2.12}$$

In that basis the fields $\Phi_j(x|z)$ satisfy a Ward identity similar to (2.10), where the differential operators are replaced by $\tilde{\mathcal{D}}^a$:

$$\tilde{\mathcal{D}}^- = -\partial_x, \qquad \tilde{\mathcal{D}}^0 = x\partial_x - j, \qquad \tilde{\mathcal{D}}^+ = x^2\partial_x - 2jx. \tag{2.13}$$

The use of the μ basis is rather convenient both for the collision limit, and to make contact with the Liouville-H_3^+ duality.

It will be useful to remember how such Ward identities arise in the context of a Wakimoto free-field realization of the current algebra [26]. In such realisation, we express the $\widehat{sl}(2)$ current in terms of a free boson ϕ and a bosonic ghost pair β, γ, in the form

$$\begin{aligned} J^-(w) =&\beta(w),\\ J^0(w) =&\partial\phi(w) + (\gamma\beta)(w),\\ J^+(w) =&-2(\partial\phi\gamma)(w) - k\partial\gamma - (\beta(\gamma\gamma))(w). \end{aligned} \tag{2.14}$$

The Ward identities (2.10) then follow by those of a a primary field $E_j(\mu|z)$ for the scalar ϕ and the $\beta\gamma$ system.

$$\begin{aligned} \beta(w)E_j(\mu|z) &\sim \frac{\mu}{w-z}E_j(\mu|z),\\ \partial\phi(w)E_j(\mu|z) &\sim \frac{j}{w-z}E_j(\mu|z),\\ \gamma(w)E_j(\mu|z) &\sim -\partial_\mu E_j(\mu|z). \end{aligned} \tag{2.15}$$

Our final answer for the irregular vector Ward identities will coincide with the Ward identities satisfied by an appropriate coherent state $E^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z)$ for ϕ and the $\beta\gamma$ system.

$$\begin{aligned} \beta(w)E^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z) &\sim \sum_{m=0}^{r}\frac{\mu_m}{(w-z)^{m+1}}E^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z),\\ \partial\phi(w)E^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z) &\sim \sum_{m=0}^{r}\frac{j_m}{(w-z)^{m+1}}E^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z),\\ \gamma(w)E^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z) &\sim -\sum_{m=0}^{r}(w-z)^m\partial_{\mu_m}E^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z), \end{aligned} \tag{2.16}$$

Combining these OPE with the Wakimoto realization we arrive to our proposal for the irregular Ward identities for an irregular vector of rank r $\Phi^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z)$, labeled by $\mathbf{j} = \{j_n\}$, $\boldsymbol{\mu} = \{\mu_n\}$, with $0 \leq n \leq r$.

$$
\begin{aligned}
J^-(w)\Phi^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z) &\sim \sum_{m=0}^{r} \frac{\mu_m}{(w-z)^{m+1}} \Phi^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z), \\
J^0(w)\Phi^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z) &\sim \sum_{m=0}^{r} \frac{j_m - \sum_n \mu_n \partial_{\mu_{n-m}}}{(w-z)^{m+1}} \Phi^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z), \\
J^+(w)\Phi^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z) &\sim \sum_{m=0}^{r} \frac{2\sum_n j_n \partial_{\mu_{n-m}} - \sum_{n,p} \mu_n \partial_{\mu_p} \partial_{\mu_{n-p-m+1}}}{(w-z)^{m}} \Phi^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z).
\end{aligned} \tag{2.17}
$$

As in the Virasoro case, the coherent state only provides a model to the Ward identities, but not the general solution of the Ward identities.

We obtain the same result if we define a rank r WZW irregular vertex operator in terms of the collision of $r+1$ regular vertex operators. For this purpose, consider $r+1$ primary fields $\Phi_{j^k}(\mu^k|z_k)$, using an upper index for j^k and μ^k to distinguish from the mode index. As each field approach the collision point z, we allow j^k and μ^k to diverge in the collision limit $q_k \to 0$, but we require the current $J^a(w)$ to remain finite. If we adjust the μ^k in such a way that

$$
\sum_i \frac{\mu^i}{w - z_i} \to \sum_{m=0}^{r} \frac{\mu_m}{(w-z)^{m+1}} \tag{2.18}
$$

then the Ward identity for $J^-(w)$

$$
J^-(w) \prod_k \Phi^{(r)}_{j^k}(\mu^k|z_k) \sim \sum_i \frac{\mu^i}{w - z_i} \prod_k \Phi^{(r)}_{j^k}(\mu^k|z_k) \tag{2.19}
$$

will obviously have the correct limit.

Note that the change of variables from μ^k to μ_m is generated by the Vandermonde matrix

$$
\mu_{m-1} = \sum_k M_{mk} \mu^k, \quad \text{where} \quad M_{mk} = (z_k - z)^{m-1} \tag{2.20}
$$

From the change of variables we read the relation between the derivatives, $\partial_{\mu^k} = \sum_{m=1}^{r+1} M_{mk} \partial_{\mu_{m-1}}$, and a straightforward calculation shows that the contribution from the $-\mu\partial_\mu$ terms in

$J^0(w)$ has a finite limit by itself. If we adjust the j^k in such a way that

$$\sum_i \frac{j^i}{w-z_i} \to \sum_{m=0}^{r} \frac{j_m}{(w-z)^{m+1}} \tag{2.21}$$

then we arrive to the desired $J^0(w)$ Ward identity. Finally, some tedious algebra shows that the $J^+(w)$ Ward identities have a finite limit, and we recover the Ward identities for a rank r WZW irregular vector. Notice that the relation between derivatives tells us how one would do the collision limit in the x basis: $x^k = \sum_{m=1}^{r+1} M_{mk} x_{m-1}$, i.e. the x^k parameters collide in a similar pattern as the z_k. The irregular vertex operators in the x_m and μ_m bases are again related by Fourier transform and a rescaling.

The WZW Ward identities are not sensitive to a rescaling by a function of the z_k in the collision limit. Such a factor can be fixed by requiring the stress tensor Ward identities to also have a specific finite limit. Choices which differ by a finite function of the j_m in the limit will give slightly different, albeit equivalent forms for the Virasoro Ward identities. A natural choice for the Ward identities is to mimic the form of the stress tensor in the Wakimoto free field realization of the theory.

$$T = -b^2(\partial\phi\partial\phi) + b^2\partial^2\phi - (\beta\partial\gamma) \tag{2.22}$$

This expression actually allows to write the whole Virasoro Ward identity, since the first r regular modes of $\partial\phi$ are fixed by its mode algebra. The result takes the form (2.5) with $c_m \to bj_m$, $Q \to -b$, plus $\mu\partial\mu$ terms which arise from the $\beta\partial\gamma$ term:

$$\begin{aligned} T(w)\Phi_{\mathbf{j}}^{(r)}(\boldsymbol{\mu}|z) \sim \Bigg(& -b^2 \sum_{m=0}^{2r} \frac{\sum_{n=0}^{m} j_n j_{m-n} + (m+1)j_m}{(w-z)^{m+2}} + \frac{\partial_z}{(w-z)} \\ & + \sum_{m=0}^{r-1} \frac{\sum_{n=1}^{r} n(\mu_{n+m}\partial_{\mu_n} + j_{n+m}\partial_{j_n})}{(w-z)^{m+2}} \Bigg) \Phi_{\mathbf{j}}^{(r)}(\boldsymbol{\mu}|z). \end{aligned} \tag{2.23}$$

In appendix 2.A.1, we derive the same result from the collision limit. It is an important building block for the generalization of the KZ equation, which is derived in appendix 2.B.

2.2.3 Generalization to other current algebras

It is pretty natural to take the Virasoro and $\widehat{sl}(2)$ irregular Ward identities as an example of a general proposal on how to describe irregular vectors for any current algebra which

admits a free-field realization: mimic the Ward identities satisfied by an appropriate coherent state for the free fields. For finite Lie algebras, the current algebra can be obtained from a Wakimoto realization, in a way similar to $\widehat{sl}(2)$. In this case we have one pair of bosonic ghosts β^a, γ^a per positive root, and one free boson $\partial\phi^i$ per element of the Cartan subalgebra. The irregular Ward identities for these fields work exactly in the same way as in the $\widehat{sl}(2)$ case, so the current Ward identities can be obtained by their Wakimoto representations analogously. For example, since the energy-momentum tensor in the Wakimoto representation is simply the sum of the one for the free fields, its irregular expression has to take the form

$$T(w)\Phi_{\mathbf{j}}^{(r)}(\boldsymbol{\mu}|z) \sim \left(-b^2 \sum_{m=0}^{2r} \frac{\sum_{n=0}^{m} \boldsymbol{j}_n \cdot \boldsymbol{j}_{m-n} + (m+1)\boldsymbol{\rho}\cdot\boldsymbol{j}_m}{(w-z)^{m+2}} + \frac{\partial_z}{(w-z)} \right.$$
$$\left. + \sum_{m=0}^{r-1} \frac{\sum_{n=1}^{r} n(\sum_a \mu^a_{n+m}\partial_{\mu^a_n} + \sum_i j^i_{n+m}\partial_{j^i_n})}{(w-z)^{m+2}} \right) \Phi_{\mathbf{j}}^{(r)}(\boldsymbol{\mu}|z), \qquad (2.24)$$

where $\boldsymbol{\rho}$ is the Weyl vector. (As in the $\widehat{sl}(2)$ case, this expression is not unique, as shifting the field by a function of the parameters j^i_m can modify the expression slightly.)

While the use of free-field realizations works well for affine Lie algebras, it can be used for other current algebras as well. We will give here a simple example: irregular vectors for the $\mathcal{W}_3$ algebra. This algebra has two generators: the energy-momentum tensor $T(z)$, and a spin 3 field $W(z)$. A free-field realization of this algebra is obtained from a triplet of free bosons ϕ_i, constrained by $\sum_i \phi_i = 0$. In this realization the currents take the form [60]

$$T = -\frac{1}{2}(\partial\phi_1^2 + \partial\phi_2^2 + \partial\phi_3^2) + Q(\partial^2\phi_3 - \partial^2\phi_1),$$
$$W = i\sqrt{\frac{3}{30Q+8}}\left(\partial\phi_1^3 + \partial\phi_2^3 + \partial\phi_3^3 - Q^2\partial^3\phi_2 - 2QM_{ij}\partial\phi_i\partial^2\phi_j\right) \qquad (2.25)$$

where Q is a parameter of the theory. The matrix M has the nonzero entries $M_{12} = M_{13} = M_{23} = M_{33} = -M_{11} = 1$ Primary fields in this representation satisfy

$$\partial\phi_i(w)V_{\boldsymbol{c}}(z) \sim \frac{c^i}{w-z}V_{\boldsymbol{c}}(z), \qquad (2.26)$$

for $i = 1, 2$. Following the same method as before, we can guess that irregular vectors will

satisfy

$$\partial\phi_i(w)V^{(r)}_{\boldsymbol{c}}(z) \sim \left(\sum_{m=0}^{r}\frac{c^i_m}{(w-z)^{m+1}} + \sum_{m=1}^{r} m\partial_{c^i_m}(w-z)^{m-1} + \mathcal{O}((w-z)^r)\right)V^{(r)}_{\boldsymbol{c}}(z), \tag{2.27}$$

which implies the irregular currents

$$\begin{aligned}
T(w)V^{(r)}_{\boldsymbol{c}}(z) \sim &\left(\sum_{m=0}^{2r}\frac{-\frac{1}{2}\sum_{i,n}c^i_n c^i_{m-n} + (m+1)Q(c^3_m - c^1_m) + \sum_{i,n} n c^i_{n+m}\partial_{c^i_n}}{(w-z)^{m+2}}\right.\\
&\left.+\frac{\partial_z}{(w-z)}\right)V^{(r)}_{\boldsymbol{c}}(z),\\
W(w)V^{(r)}_{\boldsymbol{c}}(z) \sim &\, i\sqrt{\frac{3}{30Q+8}}\left(\sum_{m=r}^{3r}\frac{\sum_{i,n,p}c^i_n c^i_p c^i_{m-n-p} + \sum_{i,n,p}c^i_n c^i_p \partial_{c^i_{n+p-m}}}{(w-z)^{m+3}}\right.\\
&\left.-\frac{2Q\sum_{i,j,n}(n+1)M_{ij}c^i_n c^j_{m-n} + Q^2(m+1)(m+2)c^i_m}{(w-z)^{m+3}} + \mathcal{O}((w-z)^{-r-2})\right)V^{(r)}_{\boldsymbol{c}}(z).
\end{aligned} \tag{2.28}$$

The Ward identities do not fix several singular terms in the OPE of W. This is not a surprise: the Ward identities for a regular vector V of the W-algebra contain descendants such as $W_{-1}V$ and $W_{-2}V$ which cannot be rewritten in terms of derivatives in the parameters of the conformal block. The Ward identities for a rank r irregular vector contain $r+2$ such descendants.

2.3 Correlation functions

Our discussion of the collision limits until this point only concerned the holomorphic (or anti-holomorphic) Ward identities. In the Virasoro case, there is strong evidence [41] that the collision limit is also sensible at the level of full correlation functions for Liouville theory. The Liouville theory correlation functions are assembled from DOZZ three-point functions (see [28, 29, 102]), holomorphic and anti-holomorphic conformal blocks. The pairing is defined on the physical locus for the external Liouville momenta, $\alpha = \frac{Q}{2} + i\mathbb{R}$, and can be analytically continued to the complex α plane by setting the parameter in the

anti-holomorphic conformal block to be $\bar{\alpha} = Q - \alpha$ (or $\bar{\alpha} = \alpha$, the choice is immaterial in conformal blocks). The collision limit is done by sending the Liouville momenta α_k of the colliding puncture to $\pm i\infty$, and involves the cancellations of divergent factors between the DOZZ three-point functions and the conformal blocks.

We expect that an analogous collision limit should be possible in a suitable rational CFT (RCFT) based on the $\widehat{sl}(2)$ current algebra: the so-called H_3^+ WZW model. A standard primary field $\Phi_j(\mu, \bar{\mu}|z, \bar{z})$ in this RCFT carries a spin $j \in -\frac{1}{2} + i\mathbb{R}$. The correlation functions are defined on this physical slice of parameter space (and $\mu^* = \bar{\mu}$), and can be analytically continued away from there, by keeping the spin in the anti-holomorphic conformal blocks $\bar{j} = j$ (the convention $\bar{j} = -j - 1$ is equivalent, but less convenient)[85].

The collision limit gives a generalization of the physical slice $j \in -\frac{1}{2} + i\mathbb{R}$. For a vector of rank r, we require $j_0 \in -\frac{r+1}{2} + i\mathbb{R}$, and $\bar{j}_n = j_n = -j_n^* - (r+1)\delta_{n,0}$.

2.3.1 An example: one-point function of a rank two irregular vector

To show that the above results are consistent, we proceed with an example and calculate the one-point function of a rank two irregular vector, up to a function of j_0. This is the simplest nontrivial case, as rank 1 vectors have a vanishing one-point function by symmetry (see below). We use the fact that the singular modes of the currents $T(w)$ and $J^a(w)$ annihilate the vacuum state. This implies that the differential operators $\mathcal{L}_0$, $\mathcal{L}_{\pm 1}$ and $\mathcal{J}_0^a$ must annihilate the one-point function. In the current case these conditions are sufficient to fix the correlation function, and the generalized KZ equations (derived in appendix 2.B) are not needed. The six equations read

$$\begin{aligned}
\mathcal{J}_0^-\langle\Phi_{\mathbf{j}}^{(2)}(\boldsymbol{\mu}|z)\rangle =& \mu_0\langle\Phi_{\mathbf{j}}^{(2)}(\boldsymbol{\mu}|z)\rangle = 0,\\
\mathcal{J}_0^0\langle\Phi_{\mathbf{j}}^{(2)}(\boldsymbol{\mu}|z)\rangle =& (j_0 - \mu_0\partial_{\mu_0} - \mu_1\partial_{\mu_1} - \mu_2\partial_{\mu_2})\langle\Phi_{\mathbf{j}}^{(2)}(\boldsymbol{\mu}|z)\rangle = 0,\\
\mathcal{J}_0^+\langle\Phi_{\mathbf{j}}^{(2)}(\boldsymbol{\mu}|z)\rangle =& (2j_0\partial_{\mu_0} + 2j_1\partial_{\mu_1} + 2j_2\partial_{\mu_2}\\
&\quad - \mu_0\partial_{\mu_0}^2 - 2\mu_1\partial_{\mu_0}\partial_{\mu_1} - \mu_2(\partial_{\mu_1}^2 + \partial_{\mu_0}\partial_{\mu_2}))\langle\Phi_{\mathbf{j}}^{(2)}(\boldsymbol{\mu}|z)\rangle = 0,\\
\mathcal{L}_{-1}\langle\Phi_{\mathbf{j}}^{(2)}(\boldsymbol{\mu}|z)\rangle =& \partial_z\langle\Phi_{\mathbf{j}}^{(2)}(\boldsymbol{\mu}|z)\rangle = 0,\\
\mathcal{L}_0\langle\Phi_{\mathbf{j}}^{(2)}(\boldsymbol{\mu}|z)\rangle =& (\mu_1\partial_{\mu_1} + 2\mu_2\partial_{\mu_2} + j_1\partial_{j_1} + 2j_2\partial_{j_2} - b^2 j_0(j_0+1))\langle\Phi_{\mathbf{j}}^{(2)}(\boldsymbol{\mu}|z)\rangle = 0,\\
\mathcal{L}_1\langle\Phi_{\mathbf{j}}^{(2)}(\boldsymbol{\mu}|z)\rangle =& (\mu_2\partial_{\mu_1} + j_2\partial_{j_1} - 2b^2 j_1(j_0+1))\langle\Phi_{\mathbf{j}}^{(2)}(\boldsymbol{\mu}|z)\rangle = 0.
\end{aligned} \tag{2.29}$$

This system, together with its anti-holomorphic counterpart, has the solution

$$\begin{aligned}\langle \Phi^{(2)}_{\mathbf{j}}(\boldsymbol{\mu}|z)\rangle =&\delta^{(2)}(\mu_0)|\mu_2|^{2j_0+2}|j_2|^{(j_0+1)(b^2j_0-2)}e^{2ib^2(j_0+1)\mathrm{Im}\left(\frac{j_1^2}{j_2}\right)+\frac{1}{2}i\mathrm{Im}(u^2)}\\ &\times\left(D_{j_0+1}(u),\ D_{-j_0-2}(iu)\right)C(j_0)\left(D_{j_0+1}(\bar{u}),\ D_{-j_0-2}(i\bar{u})\right)^{\mathrm{T}},\end{aligned}\tag{2.30}$$

where $u=i(\mu_2/2)^{-1/2}(j_2(\mu_1/\mu_2)-j_1)$, with $\bar{u}=-iu^*$, $C(j_0)$ is an arbitrary 2×2 matrix, and D_A is the parabolic cylinder function

$$D_A(z)=2^{A/2}\sqrt{\pi}e^{-\frac{1}{4}z^2}\left(\Gamma\left(\tfrac{1-A}{2}\right)^{-1}{}_1F_1\left(-\frac{A}{2};\frac{1}{2};\frac{z^2}{2}\right)-\sqrt{2}\,\Gamma\left(-\tfrac{A}{2}\right)^{-1}z\,{}_1F_1\left(\frac{1-A}{2};\frac{3}{2};\frac{z^2}{2}\right)\right).\tag{2.31}$$

To find the matrix $C(j_0)$ we impose the constraint that the physical correlation function do not grow exponentially in any direction in the u-plane. This property holds for the three-point function whose collision limit defines the correlation function we are after [85]. The parabolic cylinder function has a Stokes phenomenon, so we must make sure the constraint holds for each of the four Stokes sector[3]. This fixes the matrix to $C(j_0)=\mathrm{diag}(1,\,i\,\gamma(j_0+2))$, where $\gamma(A)=\Gamma(A)/\Gamma(1-A)$, up to an overall factor.

The correlation function can be expressed as the double integral

$$\begin{aligned}\langle \Phi^{(2)}_{\mathbf{j}}(\boldsymbol{\mu}|z)\rangle =&\delta^{(2)}(\mu_0)C(j_0)|\mu_2|^{2j_0+2}|j_2|^{(j_0+1)(b^2j_0-2)}e^{2ib^2(j_0+1)\mathrm{Im}\left(\frac{j_1^2}{j_2}\right)}\\ &\times\int_{\mathbb{C}}dxd\bar{x}\ |x|^{-2(j_0+2)}e^{-\frac{1}{2}x^2+\frac{1}{2}\bar{x}^2-ux+u^*\bar{x}}.\end{aligned}\tag{2.32}$$

The integral is calculated in appendix 2.C.3 The integrand is the "square" of the integrand in the contour integral definition of the parabolic cylinder function. The holomorphic and anti-holomorphic parameters are such that the integrand is oscillatory, of modulo 1, and does not blow up exponentially at large u. The double integral can also be interpreted as the Fourier transform of the correlation function in the x-basis, which can be determined independently by the collision limit on the standard three-point function of the WZW model in the x-basis. In appendix 2.D we verify that the same correlation function (2.32) arises from the collision limit (in the μ-basis).

[3]For a review of the Stokes phenomenon, see [100]

2.4 Irregular vectors and the Liouville-H_3^+ duality

2.4.1 The Liouville-H_3^+ duality

We review the duality between WZW and Liouville correlation functions. Primary fields in Liouville theory are parametrized by a Liouville momentum α, and have conformal dimension $\Delta_\alpha = \alpha(Q-\alpha)$. Here $Q = b + b^{-1}$ is a parameter of the theory, related to the central charge by $c = 1 + 6Q^2$. For a review of Liouville theory, see [50, 93].

The Liouville-H_3^+ duality relates the correlation functions of primary fields in the form [85]

$$\left\langle \prod_k^n \Phi_{j^k}(\mu^k|z_k) \right\rangle = \frac{\pi}{2}(-\pi)^{-n} b\ \delta^{(2)}(\sum_k^n \mu_k)\ |\Theta_{n,n-2}|^2 \left\langle (\prod_k^n V_{\alpha_k}(z_k))(\prod_m^{n-2} V_{-\frac{1}{2b}}(y_m)) \right\rangle . \tag{2.33}$$

where the function $\Theta_{n,m}$ is defined as[4]

$$\Theta_{n,m}(z_1,\cdots,z_n|y_1,\cdots,y_m,u) = u^{\sum_k^n \frac{\alpha_k}{b} - \frac{n}{2b^2} - (n-1)} \frac{\prod_{s,k<s}^n (z_k - z_s)^{-\frac{\alpha_k+\alpha_s}{b} + \frac{3}{2b^2} + 2}}{\prod_k^n \prod_p^m (z_k - y_p)^{-\frac{\alpha_k}{b} + \frac{1}{b^2} + 1}} \prod_{p,q<p}^m (y_q - y_p)^{\frac{1}{2b^2}}. \tag{2.34}$$

The WZW variables are related to those on the Liouville side by

$$\alpha_k = b(j^k + 1) + \frac{1}{2b}, \tag{2.35}$$

$$\mu^k = u \frac{\prod_m^{n-2}(z_k - y_m)}{\prod_{s\neq k}^n (z_k - z_s)}. \tag{2.36}$$

The parameters k and b are related under the duality by $b^2 = -(k+2)^{-1}$, and $u = \sum_{k=1}^n \mu^k z_k$.

This somewhat imposing relation can be understood more easily after two simple observations. First, the y_m are essentially the zeroes of $J^-(y)$, or more precisely of

$$\sum_k \frac{\mu^k}{y - z_k}. \tag{2.37}$$

[4]Here the function differs from the one defined in [85] because of the different normalization for the fields.

This statement inverts the relation (2.36). Second, the $\Theta_{n,m}$ prefactor, which allows one to reduce the KZ equations on the WZW side to the BPZ equations satisfied by the degenerate punctures at y_m, can be derived by comparing the various OPE limits of corresponding conformal blocks on the two sides. A similar relation holds at the level of conformal blocks. Each trinion on the WZW side maps to a trinion on the Liouville side with an extra degenerate on one of the three legs. There is a two-dimensional space of three-point junctions on the WZW side, which matches the two-dimensional space of Virasoro three-point junctions with an extra degenerate insertion on one of the three legs.

2.4.2 H_3^+ irregular vectors from Liouville theory

We now turn to the main objective of this section, and look for an equivalent of eq. (2.33) for correlation functions involving irregular vectors. Again, we use the collision limit for this purpose. The duality formula provides an alternative definition of H_3^+ correlation functions involving irregular vectors, in terms of the corresponding Liouville theory correlation functions, which behave as described in section 2.2.1. Most calculations in this section are tedious and are done in the appendix 2.A.2.

We look for the duality involving n punctures of ranks r_k, $k = \{1, \cdots, n\}$, on the WZW side, which can be regular if $r_k = 0$. For this purpose we start from the duality formula with $t = \sum_{k=1}^{n}(r_k + 1)$ H_3^+ regular vectors. This implies on the Liouville side the presence of t regular vectors and $t-2$ degenerate insertions. Then, we do the usual collision limit on the H_3^+ side. Because, by definition, we tune the μ^k so that the $J^-(y)$ Ward identity finite in the limit, the location of the zeros y_m remains generic in the collision limit. On the other hand, the way we tune the j^k to keep the $J^0(y)$ Ward identity finite is compatible with the standard collision limit on the Virasoro side of the duality. Thus the endpoint of the collision limit on the Liouville side involves irregular vectors of rank r_k and $t-2$ degenerate fields.

Thus our final formula includes n fields $\Phi_{\mathbf{j}^k}^{(r_k)}(\boldsymbol{\mu}^k|z_k)$ on the H_3^+ side and the dual fields $V_{\mathbf{c}^k}^{(r_k)}(z_k)$ and $t-2$ degenerate fields $V_{-\frac{1}{2b}}(y_p)$. The parameters are related by

$$c_m^k = bj_m^k + (r+1)(b + \frac{1}{2b})\delta_{m,0}, \tag{2.38}$$

$$\mu_m^k = u\frac{\prod_{p=1}^{t-2}(z_k - y_p)}{\prod_{s\neq k}^{n}(z_k - z_s)^{r_s+1}}W_{r_k-m}^k, \tag{2.39}$$

where $W^k_{r_k-m}$ represents the series expansion

$$\sum_p W^k_p q_i^p = \frac{\prod_m^{t-2}(1+q_i(z_k-y_m)^{-1})}{\prod_{s=r+2}^n (1+q_i(z_k-z_s)^{-1})^{r_s+1}}. \tag{2.40}$$

The first relation follows straightforwardly from (2.35), while the second requires more work and is derived in appendix 2.A.2. Of course, this relation is simply a complicated way to say that the y_m are zeros of the $J^-(y)$ Ward identity.

The duality formula for this field content is also derived in appendix 2.A.2, and takes the form

$$\left\langle \prod_{k=1}^n \Phi^{(r_k)}_{\mathbf{j}^k}(\boldsymbol{\mu}^k|z_k) \right\rangle = \frac{\pi}{2}(-\pi)^{-t} b\ \delta^{(2)}(\sum_{k=1}^n \mu_0^k)\ |\Theta_{\mathbf{r}}|^2 \left\langle (\prod_{k=1}^n V^{(r_k)}_{\mathbf{c}^k}(z_k))(\prod_m^{t-2} V_{-\frac{1}{2b}}(y_m)) \right\rangle. \tag{2.41}$$

The function $\Theta_{\mathbf{r}}$ is the generalization of $\Theta_{n,n-2}$ to irregular vectors and is given by

$$\begin{aligned}
&\Theta_{\mathbf{r}}(z_1,\cdots,z_n|y_1,\cdots,y_{t-2},u) \\
&\quad = u^{\sum_k^n \frac{\alpha_k}{b} - \frac{t}{2b^2} - (t-1)} \prod_{p,q<p}^{t-2} (y_q-y_p)^{\frac{1}{2b^2}} \frac{\prod_{k,s<k}^n (z_k-z_s)^{-\frac{(r_s+1)\alpha_k+(r_k+1)\alpha_s}{b}+(r_k+1)(r_s+1)(2+\frac{3}{2b^2})}}{\prod_k^n \prod_m^{t-2}(z_k-y_m)^{-\frac{\alpha_k}{b}+(r+1)(1+\frac{1}{2b^2})}} \\
&\quad \times \quad \frac{\prod_{k,s<k}^n \exp\left(b^{-1}\sum_{p=1}^{\max(r_k,r_s)}(-1)^p \frac{(r_s+1)c_p^k+(r_k+1)c_p^s}{p(z_k-z_s)^p}\right)}{\prod_k^n \prod_m^{t-2} \exp\left(b^{-1}\sum_{p=1}^r \frac{(-1)^p c_p^k}{p(z_k-y_m)^p}\right)} F^{(r)}(\boldsymbol{j})
\end{aligned} \tag{2.42}$$

with $u = \sum_{k=1}^n (j_0^k z_k + j_1^k)$. The function $F^{(r)}(\boldsymbol{j})$ is the exponent of a rational function of the $\boldsymbol{j}$, which is defined as the solution to the set of differential equations

$$\sum_{n=0}^{r-m} n j_{m+n} \frac{\partial_{j_n} F^{(r)}(\boldsymbol{j})}{F^{(r)}(\boldsymbol{j})} + 2(r-m)(b^2+1)j_m + (b^2+1)r(r+1)\delta_{m0} = 0, \tag{2.43}$$

for $0 \le m \le r$, normalized such that $F(0,\cdots,0,j_r) = j_r^{(b^2+1)(r+1)}$. A similar formula holds at the level of conformal blocks. Because of the structure of the collision limit, we expect each of the irregular junctions described in [41] on the Virasoro side to combine with a degenerate insertion to give an irregular junction in irregular WZW conformal blocks. We leave the details of the dictionary between irregular conformal blocks to future work.

2.5 Semiclassical analysis of irregular vectors

In order to understand better the meaning of irregular vectors in WZW models, we can look at the semiclassical limit of correlation functions. The semiclassical limit is defined as a $k \to \infty$ limit[5], combined with an appropriate rescaling of the parameters at the punctures, in such a way that correlation functions scale as

$$\langle \prod_i \Phi_i(z_i) \rangle \sim e^{\frac{1}{k+2} S_{\rm cl}(z_i)} \tag{2.44}$$

in terms of the classical action for an appropriate solution of the classical WZW equations of motion, determined by the data at the punctures.

Here the action is the WZW action for the gauge group $H_3^+ = SL(2,C)/SU(2)$. One useful parametrization for a group element is

$$h = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\phi} & 0 \\ 0 & e^{\phi} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{\gamma} & 1 \end{pmatrix} = \begin{pmatrix} e^{-\phi} + \gamma\bar{\gamma}e^{\phi} & e^{\phi}\gamma \\ e^{\phi}\bar{\gamma} & e^{\phi} \end{pmatrix}. \tag{2.45}$$

In terms of these parameters, the action reads [26]

$$S_{\rm WZW} = \frac{k}{\pi} \int d^2z {\rm Tr}(\partial h^{-1} \bar{\partial} h) + k\Gamma_{\rm WZ} = \frac{k}{\pi} \int d^2z (\partial\phi\bar{\partial}\phi + e^{2\phi}\partial\bar{\gamma}\bar{\partial}\gamma). \tag{2.46}$$

This also corresponds to a string action in H_3^+, with the metric $ds^2 = d\phi^2 + e^{2\phi} d\gamma d\bar{\gamma}$. The H_3^+ theory is reviewed in [24, 66, 67, 91]

The KZ equations for a spin 1/2 degenerate field is also expected to have a finite limit, and reduce to the equations of motion for the classical WZW solution. More precisely, the KZ differential operator should go to an ordinary differential operator in the semiclassical limit

$$\partial_z - \frac{1}{k+2} J^a \sigma_a \to \partial_z - A(z). \tag{2.47}$$

At regular singularities, $A(z)$ should have a single pole with residue $R = R^a \sigma_a$

$$R^- = m, \qquad R^0 = a - xm, \qquad R^+ = 2ax - x^2 m. \tag{2.48}$$

Here we scaled $j = a(k+2)$, $\mu = m(k+2)$, and $x = \frac{\partial S_{\rm cl}}{\partial \mu}$. This is a rather generic parameterization for a traceless matrix of fixed eigenvalues a and $-a$ in terms of a pair of conjugate variables x and m.

[5] This is equivalent to the standard semiclassical limit $\hbar \to 0$, since k appears as an overall multiplicative factor in the WZW action (2.46)

The group element $G(z, \bar{z})$ which represents a classical solution of the WZW equations of motion should satisfy

$$\partial G = AG, \qquad \bar{\partial} G = G\bar{A}. \tag{2.49}$$

and it should be single-valued on the Riemann surface. The latter constraint is rather strong. We can solve the equations by writing

$$G(z, \bar{z}) = g(z) C \bar{g}(\bar{z}). \tag{2.50}$$

for some holomorphic solution $g(z)$ and constant hermitian matrix C.

The holomorphic solution $g(z)$ will have monodromy

$$g(e^{2\pi i}(z - z_s) + z_s) = g(z) M_s \tag{2.51}$$

around the regular singularities at the locations z_s of the ordinary vertex operators. Then G will be single-valued if

$$M_s C \bar{M}_s = C, \tag{2.52}$$

i.e. M_a is conjugate to $\bar{M}_s^{-1}$.

This means that the trace of the monodromy along any path is real. This constraint kills half of the degrees of freedom of the system. In principle it fixes, say, the m parameters at all punctures in terms of the x parameters at all punctures. As the relation between A and its monodromies is highly transcendental, it is very hard to describe the constraints on A implied by the constraints on the monodromy matrices M_s. The semiclassical limit of WZW conformal blocks solves this problem. This is analogous to the statement that the semiclassical limit of Virasoro conformal blocks solves the uniformization problem.

This constraints can only be satisfied if the parameter a_s for the regular singularity at z_s is either pure real or pure imaginary: in the first case the eigenvalue $e^{2\pi i a}$ of M is a phase, in the second it is real. If we pick the a_s parameters to be real at all punctures, then we can pick a gauge where all the monodromies are unitary matrices, and $C = 1$. Then G lives in the space of hermitian matrices of unit determinant, which is the same as the hyperbolic space H_3^+. This is the expected target space for the H_3^+ WZW sigma model.

On the other hand, the vertex operators which represent normalizable states in the H_3^+ model have pure imaginary a_s. If we pick the a_s parameters to be imaginary at all punctures, then we can pick a gauge where all the monodromies are real, i.e. belong to $SL(2, R)$. Then we need to pick $C = i\sigma_2$. As a consequence, G lives in the space of hermitian matrices of determinant -1, which is an analytic continuation of H_3^+: it is three-dimensional de-Sitter space dS_3. It may seem strange for the semiclassical saddle points for

correlation functions of normalizable vertex operators to take value in the complexification of the target space, but it becomes less surprising if we look at a much simpler CFT: the theory of a free boson. The semiclassical solutions near a normalizable vertex operator e^{ipX} are imaginary

$$X \sim ip\alpha' \log |z|^2. \tag{2.53}$$

This is also the correct range of parameters for the regular punctures we collide to obtain irregular punctures. Our expectation is based on the Liouville theory analogue setup [41]. It essentially means that in the collision limit, one of the eigenvalues of each of the monodromy matrices M_s involved in the collision should be sent to infinity. We will see momentarily that the natural constraints at irregular singularities which replace the single-valuedness at regular singularities indeed require a C of the form $C = i\sigma_2$.

In order to understand the semiclassical behavior near a regular puncture, it is useful to consider solutions $g_{\pm}^{(s)}(z)$ which behave as $(z - z_s)^{\pm a}$ at the regular singularity. If a_s is real, then the solution must take the diagonal form

$$G = c_s g_+^{(s)} \left(g_+^{(s)}\right)^* + c_s^{-1} g_-^{(s)} \left(g_-^{(s)}\right)^*. \tag{2.54}$$

The coefficient c_s is fixed by the requirement that G should take the diagonal form at all punctures. As we approach the singularity, one of the two solutions blows up, and G goes to the boundary of H_3^+, at a location determined by the x parameter of the regular puncture

$$G \sim \begin{pmatrix} 1 & \bar{x} \\ x & x\bar{x} \end{pmatrix} |z - z_s|^{-2|a_s|}. \tag{2.55}$$

Thus the semiclassical solution is the solution of the equation of motion which reach the boundary at a prescribed set of points.

If a_s is imaginary, then the solution must take the off-diagonal form

$$G = c_s g_+^{(s)} \left(g_-^{(s)}\right)^* + c_s^{-1} g_-^{(s)} \left(g_+^{(s)}\right)^*. \tag{2.56}$$

As we approach the singularity, neither solution diverges. Rather, we get an oscillating approximate solution

$$G \to \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} \bar{m} & 2\bar{a} + \bar{m}\bar{x} \end{pmatrix} |z - z_s|^{2i|a_s|} + \begin{pmatrix} m \\ 2a + mx \end{pmatrix} \begin{pmatrix} 1 & \bar{x} \end{pmatrix} |z - z_s|^{-2i|a_s|}. \tag{2.57}$$

As we approach the regular singularity, the solution winds infinitely many times along a specific circle in dS_3.

At an irregular vertex operator, the differential operator $\partial_z - A$ has an irregular singularity: the matrix A takes the form $A = \sum_{n=0} z^{-n-1} R_n^a \sigma_a$, where

$$R_n^- = m_n, \qquad R_n^0 = a_n - \sum_p m_n x_{p-n}, \qquad R_n^+ = 2\sum_p a_n x_{p-n} - \sum_{p,q} m_p x_q x_{p-q-n+1}. \tag{2.58}$$

To simplify the analysis, we set $x_0 = m_r = 0$ by a H_3^+ transformation, and $z_s = 0$. The solution $g(z)$ will have Stokes phenomena. Given a generic straight ray going into the irregular singularity, we can find a unique solution which decreases exponentially fast along that ray according to a specific asymptotic behavior, which is valid only in an appropriate Stokes sector around the ray. Roughly,

$$g(z) \sim z^{\pm a_0} e^{\mp \sum_{n=1}^r \frac{a_n}{n z^n}} \tag{2.59}$$

with an appropriate vector structure. This procedure identifies $2n$ "small solutions" $g_i^{(s)}$, each decreasing exponentially fast in a sector of width π/n around the irregular singularity.

Pairs of consecutive small solutions $g_i^{(s)}$, $g_{i+1}^{(s)}$ are always linearly independent, and can be normalized so that $\det(g_i^{(s)}, g_{i+1}^{(s)}) = 1$. If we compare $g_{i+1}^{(s)}$ and $g_{i-1}^{(s)}$ in the i-th sector, they will grow at the same rate, and their sum will be proportional to $g_i^{(s)}$:

$$g_{i+1}^{(s)} + g_{i-1}^{(s)} = d_i^{(s)} g_i^{(s)}. \tag{2.60}$$

Imposing this normalization, we will get a periodicity $g_{i+2n}^{(s)} = \eta^{\pm 1} g_i^{(s)}$, where the "formal monodromy" η depends on a_0.

The proportionality coefficients $d_i^{(s)}$ generalize the notion of monodromy around a regular puncture. In particular, if we look at an irregular singularity as the collision of n regular singularities, the coefficients $d_i^{(s)}$ control the behavior of the n monodromy matrices in the limit.

We should ask what is the condition on the $d_i^{(s)}$ which arises from the condition (2.52) in the collision limit. We can take a shortcut: as the solution G did not blow up at the regular singularities, it should also not blow up at an irregular singularity. Thus between each pair of sectors G should take the off-diagonal form proportional to

$$g_i^{(s)} \left(g_{i-1}^{(s)}\right)^* - g_{i-1}^{(s)} \left(g_i^{(s)}\right)^*. \tag{2.61}$$

This is compatible with (2.60) if the $d_i^{(s)}$ are real. The asymptotic solution in all sectors is, to leading order, of the form

$$G(z,\bar{z}) \sim \begin{pmatrix} -2\ \mathrm{Re}(x_1 z S(z,\bar{z})) & S^*(z,\bar{z}) \\ S(z,\bar{z}) & 2\ \mathrm{Re}\left(\frac{m_{r-1}}{2a_r} z S^*(z,\bar{z})\right) \end{pmatrix} + \mathcal{O}(z^2), \tag{2.62}$$

where we defined

$$S(z,\bar{z}) = |z|^{2a_0} e^{-2i\mathrm{Im}\sum_{n=1}\frac{a_n}{nz^n}} = \exp\left[2i\mathrm{Im}\left(a_0 \log|z| - \sum_{n=1}\frac{a_n}{nz^n}\right)\right] = e^{i\psi}. \tag{2.63}$$

At the zeroth order (neglecting the $\mathcal{O}(z)$ terms), the solution is included in a U(1) subgroup of dS_3 parametrized by the angle ψ. For sufficiently small and constant $|z|$ the solution winds quickly in alternating direction, i.e., $\psi'(\theta)$ is separated in $2n$ sectors of opposing sign.

Appendix

In these appendices we derive some of the technica results of this chapter as well as a few extra results, and review some properties of double (complex) integrals. In appendix 2.A, we derive some results by a collision limit, the irregular Virasoro current (2.23) and the H_+^3-Liouville duality formula (2.33). In appendix 2.B we generalize a set of differential equations constraining the correlation functions, the KZ equations [26], to include irregular vertex operators. In appendix 2.C we review some identities for complex integrals and demonstrate them with some examples.In appendix 2.D we rederivenamely the one point-function of a rank two vector, found in section 2.3.1, from the direct collision of the regular three-point function.

2.A Collision limits

2.A.1 The irregular Virasoro current

Here we derive the irregular Virasoro current (2.23) for $\widehat{sl}(2)$ WZW models from a collision limit. As in section 2.2.2 we start with $r+1$ primary fields $\Phi_{j^k}(\mu^k|z+q_k)$, and write the collision in the form

$$\Phi_{\mathbf{j}}^{\prime(r)}(\boldsymbol{\mu}|z,\mathbf{q}) = \prod_{i,j<i} |q_i-q_j|^{2b^2C_{ij}} \prod_{k=1}^{r+1} \Phi_{j^k}(\mu^k|z+q_k),$$
$$\text{with} \quad \tilde{\Phi}_{\mathbf{j}}^{(r)}(\boldsymbol{\mu}|z) = \lim_{\mathbf{q}\to\mathbf{0}} \Phi_{\mathbf{j}}^{\prime(r)}(\boldsymbol{\mu}|z,\mathbf{q}). \tag{2.64}$$

As will be shown in the next section, finiteness of the conformal blocks in the collision limite require $C_{ij} = 2j_ij_j + 2(1+b^{-2})(j_i+j_j+1)$. We look for the collision limit of the

Virasoro Ward identity

$$T(w)\prod_{k=1}^{r+1}\Phi_{j^k}(\mu^k|z+q_k)\sim\sum_{k=1}^{r+1}\left(\frac{\Delta_k}{(w-z-q_k)^2}+\frac{\partial_{z_k}}{(w-z-q_k)}\right)\times\prod_{i,j<i}|q_i-q_j|^{-2b^2C_{ij}}\Phi'^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z,\mathbf{q}),\tag{2.65}$$

with $\Delta_k=-b^2j^k(j^k+1)$. Defining $C_{ji}=C_{ij}$ for $j>i$, this becomes

$$T(w)\Phi'^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z,\mathbf{q})\sim\left(\sum_{m=0}^{\infty}\frac{-(m+1)b^2\sum_i q_i^m j^i(j^i+1)-\sum_{i,j\neq i}^{r+1}C_{ij}b^2q_i^{m+1}(q_i-q_j)^{-1}}{(w-z)^{m+2}}+\sum_{m=-1}^{\infty}\frac{\sum_i^{r+1}q_i^{m+1}\partial_{z_i}}{(w-z)^{m+2}}\right)\Phi'^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z,\mathbf{q}).\tag{2.66}$$

For the moment we neglect the part with the derivative terms, and call the rest $T_j(w)$. The terms containing C_{ij} can be rewritten in the form

$$\sum_{i,j\neq i}^{r+1}C_{ij}q_i^{m+1}(q_i-q_j)^{-1}=\frac{1}{2}\sum_{i,j\neq i}^{r+1}\sum_{n=0}^{m}q_i^nq_j^{m-n-1}C_{ij},\tag{2.67}$$

which implies

$$\begin{aligned}&T_j(w)\Phi'^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z,\mathbf{q})\\&\sim-b^2\sum_{m=0}^{\infty}\frac{(m+1)(j_m+\sum_i q_i^m(j^i)^2)+\sum_{i,j\neq i}^{r+1}\sum_{n=0}^{m}q_i^nq_j^{m-n-1}C_{ij}/2}{(w-z)^{m+2}}\Phi'^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z,\mathbf{q})\\&\sim-\sum_{m=0}^{\infty}\left(\frac{(b^2(2r+1-m)+2(r-m))j_m}{(w-z)^{+2}m}\right.\\&\qquad\left.+\frac{b^2\sum_{n=0}^{m}j_nj_{m-n-1}+(b^2+1)r(r+1)\delta_{m,0}}{(w-z)^{m+2}}\right)\Phi'^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z,\mathbf{q})\end{aligned}\tag{2.68}$$

We now turn to the other part of the Virasoro Ward identity, containing z-derivatives (labeled $T_\partial(w)$). This part cannot easily be treated symmetrically in i, so here we set q_{r+1} to 0. This gives the set of derivatives

$$\partial_{z_{r+1}}=\partial_z-\sum_{i=1}^{r}\partial_{q_i},\qquad\text{and}\qquad\partial_{z_i}=\partial_{q_i},\quad i\leq i\leq r.\tag{2.69}$$

We use this to rewrite

$$T_\partial(w)\Phi'^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z,\mathbf{q}) \sim \left(\frac{\partial_z}{(w-z)} + \sum_{m=0}^{\infty}\frac{\sum_i^r q_i^{m+1}\partial_{q_i}}{(w-z)^{m+2}}\right)\Phi'^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z,\mathbf{q}). \tag{2.70}$$

Using the chain rule to rewrite the ∂_{q_i} as combinations of ∂_{j_n} and ∂_{μ_n}, we find

$$T_\partial(w)\Phi'^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z,\mathbf{q}) \sim \left(\frac{\partial_z}{(w-z)} + \sum_{m=0}^{\infty}\frac{\sum_{n=1}^{r+1} n(\mu_{n+m}\partial_{\mu_n} + j_{n+m}\partial_{j_n})}{(w-z)^{m+2}}\right)\Phi'^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z,\mathbf{q}). \tag{2.71}$$

Assembling the pieces together and taking the collision limit, we get the full Virasoro Ward identity

$$\begin{aligned} T(w)\tilde{\Phi}^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z) \sim \Bigg(&-\sum_{m=0}^{2r}\frac{(b^2(2r+1-m)+2(r-m))j_m}{(w-z)^{m+2}} \\ &-\sum_{m=0}^{2r}\frac{b^2\sum_{n=0}^{m} j_n j_{m-n} + (b^2+1)r(r+1)\delta_{m,0}}{(w-z)^{m+2}} \\ &+\frac{\partial_z}{(w-z)} + \sum_{m=0}^{r-1}\frac{\sum_{n=1}^{r} n(\mu_{n+m}\partial_{\mu_n} + j_{n+m}\partial_{j_n})}{(w-z)^{m+2}}\Bigg)\tilde{\Phi}^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z), \end{aligned} \tag{2.72}$$

This differs slightly from the expected result, however the the difference is only due to a different scaling in j_m. which can be brought to the form (2.23) with the rescaling

$$\Phi^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z) = F^{(r)}(\boldsymbol{j})\tilde{\Phi}^{(r)}_{\mathbf{j}}(\boldsymbol{\mu}|z), \tag{2.73}$$

where $F^{(r)}(\boldsymbol{j})$ is the function introduced in section 2.4.2. By requiring $\Phi^{(r)}$ to satisfy (2.23) we recover the defining set of differential equations (2.43).

2.A.2 The duality formula for irregular vectors

We evaluate the collision limit of eq (2.36) to find the relation between the μ^k_m and the Liouville parameters in the duality. The setup is described in section 2.4.2. Starting with the collisions which do not involve μ^i, we find (assuming the field i collide to the irregular field k)

$$\mu^i = u\frac{\prod_p^{t-2}(z_k+q_i-y_p)}{\prod_{j\neq i}^{r_k+1}(q_i-q_j)\prod_{m\neq k}^{n}(z_k+q_i-z_m)^{r_m+1}}. \tag{2.74}$$

For a regular field $r_k = 0$ there is no factor $(q_i - q_j)$ in the denominator, and we can directly find the result by setting $q_i = 0$, $\mu^i = \mu^k$. For the irregular case we expand in series in q_i

$$\mu^i = u \frac{\prod_p^{t-2}(z_k - y_p)}{\prod_{j\neq i}^{r_k+1}(q_i - q_j)\prod_{m\neq k}^{n}(z_k - z_m)^{r_m+1}} \sum_{p=0}^{\infty} q_i^p \; W_p^k, \tag{2.75}$$

Where the $W_p^k$are the series coefficients defined in (2.40). To compare with the parametrization found in section 2.2.2, we calculate the sums $\sum_i q_i^x \mu^i$ explicitly (where the sum runs over the colliding fields)[6]:

$$\begin{aligned}
\sum_i q_i^m \mu^i =& u \frac{\prod_p^{t-2}(z_k - y_p)}{\prod_{s\neq k}^{n}(z_k - z_s)^{r_s+1}} \left(\sum_{p=0}^{\infty} W_p^k \sum_i \frac{q_i^{m+p}}{\prod_{j\neq i}^{r_k+1}(q_i - q_j)} \right) \\
=& u \frac{\prod_p^{t-2}(z_k - y_p)}{\prod_{s\neq k}^{n}(z_k - z_s)^{r_s+1}} \left(\sum_{p=0}^{\infty} W_p^k \frac{\sum_i (-1)^{r_k+1-i} q_i^{m+p} \prod_{s,t<s,s,t\neq i}(q_s - q_t)}{\prod_{s,t<s}(q_s - q_t)} \right) \\
=& u \frac{\prod_p^{t-2}(z_k - y_p)}{\prod_{s\neq k}^{n}(z_k - z_s)^{r_s+1}} W_{r_k - m} + \mathcal{O}(\mathbf{q}).
\end{aligned} \tag{2.76}$$

Taking the collision limit, we recover the announced result (2.39)

We now turn to the limit of the full duality formula (2.33). Using the results of the previous section, we write

$$\begin{aligned}
&\prod_k \prod_{i,j<i} |q_{ik} - q_{jk}|^{-4b^4 j^{ik} j^{jk} - 4(b^2+1)(j^{ik} + j^{jk} + 1)} |F(\boldsymbol{j})|^2 \left\langle \prod_k \Phi_{\mathbf{j}^k}^{\prime(r_k)}(\boldsymbol{\mu}^k | z_k, \mathbf{q}_k) \right\rangle \\
&\quad = \frac{\pi}{2}(-\pi)^{-t} b \; \delta^{(2)}\Big(\sum_k^n \mu^{ik}\Big) \; |\Theta_{t,t-2}|^2 \prod_k \prod_{i,j<i} |q_{ik} - q_{jk}|^{-4\alpha_{ik}\alpha_{jk}} \left\langle \prod_k V_{\boldsymbol{\alpha}_k}^{\prime(r_k)}(z_k, \mathbf{q}_k) \prod_p^{t-2} V_{-\frac{1}{2b}}(y_p) \right\rangle,
\end{aligned} \tag{2.77}$$

where a pair of indices of the form q_{ik} denotes the field number i in the collision forming

[6] To get the third line we can use the fact that the sum over i is antisymmetric under odd permutations of the q_k, so it has to factor the Vandermonde determinant. For $x + p \neq r$, this fact fixes the sum to 0 or $\mathcal{O}(\mathbf{q})$ times the denominator. For $x + p = r$ we can compare to the Laplace expansion of the Vandermonde determinant

the irregular field k, for example $\mathbf{q}_k = (q_{1k}, \cdots, q_{r_k+1\ k})$. The function $\Theta_{t,t-2}$ goes as

$$F^{(r)}(\boldsymbol{j})\Theta_{t,t-2}(z_1+\mathbf{q}_1,\cdots,z_n+\mathbf{q}_n|y_1,\cdots,y_m,u) = u^{\sum_{ik}\frac{\alpha_{ik}}{b}-\frac{t}{2b^2}-(t-1)}\prod_{p,q<p}^{m}(y_q-y_p)^{\frac{1}{2b^2}}$$

$$\times F^{(r)}(\boldsymbol{j})\prod_k\prod_{i,j<i}(q_{ik}-q_{jk})^{-\frac{\alpha_{ik}+\alpha_{jk}}{b}+\frac{3}{2b^2}+2}\frac{\prod_{s,k<s}\prod_{i,j}(z_k-z_s+q_{ik}-q_{js})^{-\frac{\alpha_{ik}+\alpha_{js}}{b}+\frac{3}{2b^2}+2}}{\prod_{ik}\prod_p(z_k+q_{ik}-y_p)^{-\frac{\alpha_{ik}}{b}+\frac{1}{b^2}+1}}.$$

$$= u^{\sum_k\frac{c_0^k}{b}-\frac{t}{2b^2}-(t-1)}F^{(r)}(\boldsymbol{j})\prod_k\prod_{i,j<i}(q_{ik}-q_{jk})^{-\frac{\alpha_{ik}+\alpha_{jk}}{b}+\frac{3}{2b^2}+2}\prod_{p,q<p}^{m}(y_q-y_p)^{\frac{1}{2b^2}}$$

$$\times\frac{\prod_{s,k<s}(z_k-z_s)^{-\frac{(r_s+1)c_0^k+(r_k+1)c_0^s}{b}+(r_k+1)(r_s+1)\left(\frac{3}{2b^2}+2\right)}\exp\left(\sum_{p=0}^{\infty}\frac{(r_k+1)c_p^s+(-1)^p(r_s+1)c_p^k)}{bp(z_k-z_s)^p}\right)}{\prod_k\prod_p(z_k-y_p)^{-\frac{c_0^k}{b}+(r_k+1)\left(\frac{1}{b^2}+1\right)}\exp\left(\sum_{p=0}^{\infty}\frac{c_p^k}{bp(y_p-z_k)^p}\right)}$$

$$\sim\prod_k\prod_{i,j<i}(q_{ik}-q_{jk})^{-\frac{\alpha_{ik}+\alpha_{jk}}{b}+\frac{3}{2b^2}+2}\Theta_{\mathbf{r}}(z_1,\cdots,z_n|y_1,\cdots,y_{t-2},u). \tag{2.78}$$

The powers of $(q_{ik}-q_{jk})$ cancel with the ones already present in the duality formula. This is the announced validation for the choice of the rescaling for the irregular vector, as if the choice had been different some powers of the $(q_{ik}-q_{jk})$ would remain, leading to an uninteresting limit. Taking the collision limit, we recover the duality formula (2.41).

2.B The generalized KZ equations for irregular vectors

In this appendix we provide the generalization of the KZ equation for $\widehat{sl}(2)$ theories involving irregular vectors, as well as an outline of its derivation. Here we find the generalization from scratch in a way analogous to the regular case [26], but it can also be found by taking the collision limit of the usual KZ equation.

The starting point is the Sugawara construction for the energy-momentum operator in $\widehat{sl}(2)$ WZW models:

$$T(w) = \frac{1}{2(k+2)}(J^aJ^a)(w). \tag{2.79}$$

The KZ equation is obtained by requiring consistency of both sides when inserted in correlation functions. This amounts to imposing the equality of the OPE for singular modes.

For regular vectors, the positive modes annihilate the states, and the equation for the L_0 mode is trivially realized, so there is only one equation coming from L_{-1}. This equation reads [26]

$$\left(\partial_z + \frac{1}{k+2}\sum_{i=1}^{n}\frac{\mathcal{D}^a\mathcal{D}^a_i}{z-z_i}\right)\langle\Phi_j(\mu|z)\Phi_{j_1}(\mu_1|z_1)\cdots\Phi_{j_n}(\mu_n|z_n)\rangle = 0. \tag{2.80}$$

For irregular vectors, some positive modes act nontrivially, so we have to generalize the consistency condition. The mode expansion of eq. (2.79) reads

$$L_n = \frac{1}{2(k+2)}\sum_m :J^a_{m-n}J^a_m: = \begin{cases} \frac{1}{2}J^a_{n/2}J^a_{n/2} + \sum_{m=1} J^a_{n/2-m}J^a_{n/2+m}, & n \text{ even}, \\ \sum_{m=0} J^a_{(n-1)/2-m}J^a_{(n+1)/2+m}, & n \text{ odd}. \end{cases} \tag{2.81}$$

We insert this equality inside a correlation function by applying it to a field $\Phi^{(r)}_{\mathbf{B}}(\mathbf{A}|z)$, in the presence of k other fields $\Phi^{(r_i)}_{\mathbf{B_i}}(\mathbf{A_i}|z_i)$. This forces the equality

$$\left\langle\left(\left(L_n - \frac{1}{2(k+2)}\sum_m J^a_{m-n}J^a_m\right)\Phi^{(r)}_{\mathbf{B}}(\mathbf{A}|z)\right)(z)\prod_{i=1}^{k}\Phi^{(r_i)}_{\mathbf{B_i}}(\mathbf{A_i}|z_i)\right\rangle = 0, \quad n \geq -1. \tag{2.82}$$

The effect of the irregular modes of $T(w)$ and $J^a(w)$ is known from eq. (2.17) and (2.23). However, there are also contributions from the non-singular part of $J^a(w)$. These can be expressed in terms of the other fields of the correlation function using the residue theorem ($n > 0$):

$$\begin{aligned}\left\langle(J^a_{-n}\Phi^{(r)})(z)\prod_{i=1}^{k}\Phi^{(r_i)}_{j_i}(\mu_i|z_i)\right\rangle &= \oint\frac{dw}{(w-z)^n}\left\langle(J^a\Phi^{(r)})(z)\prod_{i=1}^{k}\Phi^{(r_i)}_{j_i}(\mu_i|z_i)\right\rangle \\ &= -\sum_{s=1}^{k}\oint\frac{dw}{(w-z)^n}\left\langle\Phi^{(r)}(z)(J^a\Phi^{(r_s)}_{j_s})(\mu_s|z_s)\prod_{i=1,i\neq s}^{k}\Phi^{(r_i)}_{j_i}(\mu_i|z_i)\right\rangle \\ &= -\sum_{s=1}^{k}\sum_{m=0}^{r_s}\frac{(-1)^m(n+m-1)!\mathcal{J}^a_{m(s)}}{(n-1)!(z_s-z)^{n+m}}\left\langle\Phi^{(r)}(z)\prod_{i=1}^{k}\Phi^{(r_i)}_{j_i}(\mu_i|z_i)\right\rangle\end{aligned} \tag{2.83}$$

where $\mathcal{J}^a_{m(s)}$ is the differential operator representing the action of J^a_m on the field s. For the special case of a regular vector, we have only the term $m = 0$, with $\mathcal{J}^a_{0(s)} = \mathcal{D}^a_s$. For

even n the equation becomes

$$0 = \left(\mathcal{L}_{2n} - \frac{1}{k+2} \left(\frac{1}{2} \mathcal{J}^a_n \mathcal{J}^a_n + \sum_{m=1}^{n} \mathcal{J}^a_{n-m} \mathcal{J}^a_{n+m} - \sum_{m=n+1}^{r-n} \sum_{s=1}^{k} \sum_{p=0}^{r_s} \frac{(-1)^p (m-n+p-1)! \mathcal{J}^a_{p(s)} \mathcal{J}^a_{n+m}}{(m-n-1)! (z_s - z)^{m-n+p}} \right) \right)$$
$$\times \left\langle \Phi^{(r)}(z) \prod_{i=1}^{k} \Phi^{(r_i)}_{j_i}(\mu_i | z_i) \right\rangle, \qquad n \geq 0. \tag{2.84}$$

Similarly, for odd n,

$$0 = \left(\mathcal{L}_{2n+1} - \frac{1}{k+2} \left(\sum_{m=1}^{n} \mathcal{J}^a_{n+1-m} \mathcal{J}^a_{n+m} - \sum_{m=n}^{r-n} \sum_{s=1}^{k} \sum_{p=0}^{r_s} \frac{(-1)^p (m-n+p-2)! \mathcal{J}^a_{p(s)} \mathcal{J}^a_{n+m}}{(m-n-2)! (z_s - z)^{m-n+1+p}} \right) \right)$$
$$\times \left\langle \Phi^{(r)}(z) \prod_{i=1}^{k} \Phi^{(r_i)}_{j_i}(\mu_i | z_i) \right\rangle, \qquad n \geq -1. \tag{2.85}$$

This is the generalized form of the KZ equation for $\widehat{sl}(2)$ theories[7]. The differential operators are given by (2.17), (2.23).

2.C Review of double integrals

We discuss some integral identities, all of which are variations on the theme of the Riemann bilinear identity, which relates an integral over a Riemann surface to a bilinear of contour integrals over a basis of 1-cycles on the surface

$$\int_\Sigma \omega \wedge \tilde{\omega} = \sum_i \oint_{\alpha_i} \omega \oint_{\beta_i} \tilde{\omega} - \oint_{\beta_i} \omega \oint_{\alpha_i} \tilde{\omega} \tag{2.86}$$

Here ω is a holomorphic $(1,0)$ form, $\tilde{\omega}$ is an anti-holomorphic $(0,1)$ form, and the basis of cycles α_i, β_j is chosen as usual so that the intersection matrix is $\langle \alpha_i, \alpha_j \rangle = \langle \beta_i, \beta_j \rangle = 0$, $\langle \alpha_i, \beta_j \rangle = \delta_{ij}$. The formula has obvious extensions to higher dimensional manifolds.

A standard strategy to prove this relation is to look at this integral as a contour integral in $\Sigma \times \bar{\Sigma}$, with local coordinates z and $\tilde{z}$ and integration contour $\mathcal{I} : \tilde{z} = \bar{z}$ Then we can

[7] Actually, this formula is completely general and holds for any sort of field in any WZW model (provided we replace the factor $(k+2)$ by $(k+g)$). What differs between theories is the actual form of the differential operators and the algebra in which the index a is valued.

simply decompose the integration contour into a basis for the second homology of $\Sigma \times \bar{\Sigma}$, which can be taken to consist of cycles of the form $\alpha_i \times \bar{\alpha}_j$, $\alpha_i \times \bar{\beta}_j$, etc. The coefficient of a basis element in the decomposition is simply the intersection number of $\mathcal{I}$ with a dual basis element. For example, the coefficient of $\alpha_i \times \bar{\beta}_j$ is the intersection number of $\mathcal{I}$ with $-\beta_i \times \bar{\alpha}_j$, which is equal to the intersection of α_j and β_i (some orientation sleight of hand here...), etcetera. This immediately leads to the bilinear identity. It is also useful to write the identity in terms of a generic basis of cycles γ_a with intersection matrix I_{ab}:

$$\int_\Sigma \omega \wedge \tilde{\omega} = -\oint_{\gamma_a} \omega I^{-1}_{ab} \oint_{\gamma_b} \tilde{\omega} \tag{2.87}$$

It is useful to give a trivial example of the bilinear identity. Consider the area of a torus,

$$\int_{E_\tau} dz d\bar{z} = -2i\mathrm{Im}\tau \tag{2.88}$$

The period of dz on α is 1, and on β it is τ.

2.C.1 Multivaluedness

We will need two simple generalizations of this strategy. The first is to consider situations where $\omega \wedge \tilde{\omega}$ is single-valued, but ω is not. The second is to consider non-compact situations where ω may diverge at infinity, while $\omega \wedge \tilde{\omega}$ is integrable.

Consider a situation where ω is a section of some line bundle, i.e. it is a multi-valued holomorphic $(1,0)$ form with constant Abelian monodromies $\omega \to \mu_p \omega$ when transported along some closed path p. We take the μ_p to be monomials in a certain set of n generators μ_s. Suppose that $\tilde{\omega}$ has opposite monodromies, so that $\omega \wedge \tilde{\omega}$ is single-valued, and the integral

$$\int_\Sigma \omega \wedge \tilde{\omega} \tag{2.89}$$

can still be considered as a contour integral on $\mathcal{I}$. Now the contour integrals for ω do not belong to the homology of Σ. We can consider a cover Σ_μ of Σ on which ω is single-valued, and work with the homology of that cover. We can take the cover to have fiber $\mathbb{Z}^n$, gluing it together by the map $p \to \mu_p$.

As we only really care about integration cycles for ω, we can naturally represent the images of a cycle γ under $\mathbb{Z}^n$ deck transformations as $\prod_s \mu_s^{n_s} \gamma$, so that the period of ω on $\prod_s \mu_s^{n_s} \gamma$ is $\prod_s \mu_s^{n_s}$ times the period on γ. Once we work with a homology whose coefficients

are rational functions of the μ_s, we can usually find a basis of cycles γ_a, and use the Riemann bilinear identity, with an intersection matrix which will depend on the μ_s.

As an example, consider the following integral, which leads to the Virasoro-Shapiro amplitude

$$\int_{\mathbb{C}} dz d\bar{z} |z|^{2A} |1-z|^{2B} \tag{2.90}$$

This integral converges as long as the real parts of A and B are larger than -1, and their sum smaller than -1. The form ω is now $z^A(1-z)^B dz$, and has monodromies by $\mu_A = e^{2\pi i A}$ and $\mu_B = e^{2\pi i B}$ around 0 and 1 respectively. These monodromies combine to a monodromy $\mu_A^{-1}\mu_B^{-1}$ around infinity. The three ramification points 0, 1 and ∞ are really on the same footing, and we could move them to generic positions:

$$\begin{aligned}&\int_{\mathbb{C}} dz d\bar{z} |z-z_1|^{2A} |z-z_2|^{2B} |z-z_3|^{-2A-2B-4} \\ &= |z_1-z_2|^{2A+2B+2} |z_2-z_3|^{-2A-2} |z_3-z_1|^{-2B-2} \int_{\mathbb{C}} dz d\bar{z} |z|^{2A} |1-z|^{2B}\end{aligned} \tag{2.91}$$

There is a single homology generator, which takes the form of a Pochhammer contour γ, depicted in figure 2.C.1, together with its $\mu_A^{n_A}\mu_B^{n_B}$ images. The intersection matrix takes the nice, symmetric form

$$I = -(1-\mu_A)(1-\mu_B)(1-\mu_A^{-1}\mu_B^{-1}) \tag{2.92}$$

The next example,

$$\int_{\mathbb{C}} dz d\bar{z} |z|^{2A} |1-z|^{2B} |1-tz|^{2C} \tag{2.93}$$

or more generically

$$\begin{aligned}&\int_{\mathbb{C}} dz d\bar{z} |z-z_1|^{2A} |z-z_2|^{2B} |z-z_3|^{2C} |z-z_4|^{-2A-2B-2C-4} \\ &= |z_1-z_2|^{2A+2B+2} |z_1-z_3|^{2C} |z_4-z_2|^{-2A-2} |z_1-z_4|^{-2B-2C-2} \int_{\mathbb{C}} dz d\bar{z} |z|^{2A} |1-z|^{2B} |1-tz|^{2C}\end{aligned} \tag{2.94}$$

where $t = \frac{(z_3-z_4)(z_1-z_2)}{(z_3-z_1)(z_4-z_2)}$.

The corresponding one form ω has monodromies $\mu_A = e^{2\pi i A}$ around 0, $\mu_B = e^{2\pi i B}$ around 1, $\mu_C = e^{2\pi i C}$ around $1/t$ and thus $\mu_A^{-1}\mu_B^{-1}\mu_C^{-1}$ around infinity. The appropriate

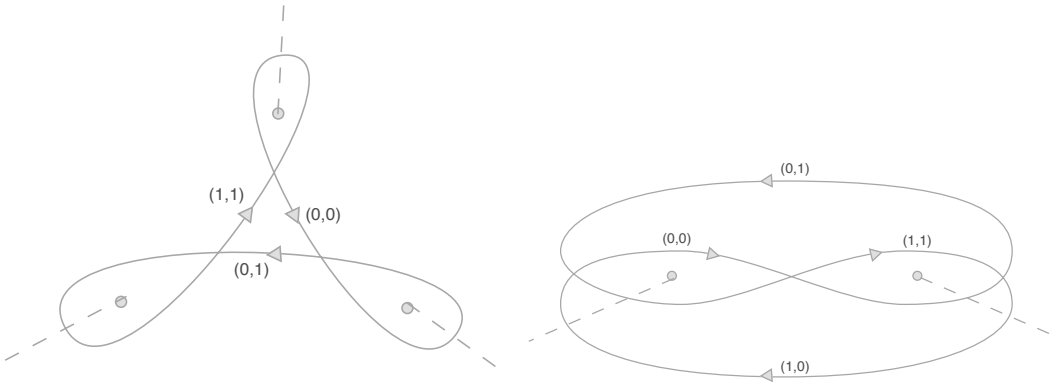

Figure 2.C.1: Two views of the Pochhammer cycle. On the left: a symmetric presentation. The three punctures are 0 on the left, 1 on the right, ∞ at the top. We indicate on which sheets the contour runs. On the right, a more traditional presentation, as a cycle wrapping several times around 0 and 1. This presentation makes more manifest the relation to an open integration cycle from 0 to 1. A positive intersection point between sheets which differ by (n_A, n_B) units contributes $\mu_A^{n_A}\mu_B^{n_B} - \mu_A^{-n_A}\mu_B^{-n_B}$ to the intersection matrix.

homology of integration contours has two generators. We can take them to be a Pochhammer cycle around 0 and 1, and a Pochhammer cycle around $1/t$ and ∞. This choice is convenient, as they do not intersect, and we already know their self-intersection.

Furthermore, we can use the well-known integral

$$\int_0^1 z^A(1-z)^B(1-tz)^C dz = \frac{\Gamma(A+1)\Gamma(B+1)}{\Gamma(A+B+2)} {}_2F_1(-C, A+1, A+B+2, t) \tag{2.95}$$

or more generally

$$\int_{z_1}^{z_2} (z-z_1)^A(z-z_2)^B(z-z_3)^C(z-z_4)^{-A-B-C-2}dz = (z_1-z_2)^{A+B+1}(z_1-z_3)^C(z_1-z_4)^{-B-C-1}$$
$$(z_4-z_2)^{-A-1}\frac{\Gamma(A+1)\Gamma(B+1)}{\Gamma(A+B+2)} {}_2F_1(-C, A+1, A+B+2, \frac{(z_3-z_4)(z_1-z_2)}{(z_3-z_1)(z_4-z_2)}) \tag{2.96}$$

which we can specialize to

$$\int_\infty^{1/t} z^A(1-z)^B(1-tz)^C dz = t^{-A-B-1}(t-1)^{B+C+1}\frac{\Gamma(C+1)\Gamma(-B-C-A-1)}{\Gamma(-B-A)}$$
$$\times \quad {}_2F_1(-A, C+1, -B-C, t) \tag{2.97}$$

Thus

$$\begin{aligned}\int_{\mathbb{C}} dzd\bar{z}|z|^{2A}|1-z|^{2B}|1-tz|^{2C} \\ = &-2\pi i\frac{\Gamma(A+1)\Gamma(B+1)\Gamma(-A-B-1)}{\Gamma(-A)\Gamma(-B)\Gamma(A+B+2)}|_2F_1(-C,A+1,A+B+2,t)|^2 \\ &-2\pi i\frac{\Gamma(C+1)\Gamma(-A-B-C-1)\Gamma(A+B+1)}{\Gamma(-C)\Gamma(A+B+C+2)\Gamma(-A-B)}|_2F_1(-A,C+1,-B-C,t)|^2\end{aligned} \tag{2.98}$$

2.C.2 Exponential growth

The next generalization involves an integral of the form

$$\int_\Sigma \omega\wedge\tilde{\omega}e^{W(z)-\bar{W}(\bar{z})} \tag{2.99}$$

where W grows polynomially at infinity. The rapid oscillation makes the integral barely convergent. After the usual analytic continuation, we can improve the behavior at infinity, pushing the boundary of the integration cycle towards region where the real part of $W(z)$ grows arbitrarily large and negative, and the real part of $\bar{W}(\tilde{z})$ grows arbitrarily large and positive. The integral is then exponentially convergent.

Next, we can try to decompose the integration contour in a convenient basis of appropriate homology of integration cycles for $\omega e^{W(z)}$ and for $\tilde{\omega}e^{-\bar{W}(\tilde{z})}$. The former includes both closed contours and contours which are allowed to end at infinity, in regions where the real part of $W(z)$ grows arbitrarily large and negative. The latter includes both closed contours and contours which are allowed to end at infinity, in regions where the real part of $\bar{W}(\tilde{z})$ grows arbitrarily large and positive.

It is important to observe that there is no well-defined notion of mutual intersection for contours which are allowed to end at infinity, in regions where $\mathrm{Re}W(z)\ll 0$. Luckily, we do not need that. Rather, there is a well-defined intersection pairing between contours which are allowed to end at infinity, in regions where $\mathrm{Re}W(z)\ll 0$, and contours which are allowed to end at infinity, in regions where $\mathrm{Re}W(z)\gg 0$ If we pick a set of contours in α_a in the first class, and $\bar{\beta}_b$ in the second, with intersection I_{ij}, we can write as usual

$$\int_\Sigma \omega\wedge\tilde{\omega}e^{W(z)-\bar{W}(\bar{z})} = -\oint_{\alpha_a}\omega e^{W(z)}I^{-1}_{ab}\oint_{\beta_b}\tilde{\omega}e^{-\bar{W}(\bar{z})} \tag{2.100}$$

The first example is a version of the Airy integral

$$\int_{\mathbb{C}} dz d\bar{z} e^{\frac{z^3}{3} - tz - \frac{\bar{z}^3}{3} + \bar{t}\bar{z}} \tag{2.101}$$

There are three regions where the contours of integration can end, which are centered around rays of phase $\pi/3$, π, $-\pi/3$. We can denote the integration contours which join consecutive regions counterclockwise as C_1, C_2, C_3, with $\sum_i C_i = 0$. Dual contours D_1, D_2, D_3 join regions centered around rays of phase $-2\pi/3$, 0, $2\pi/3$. The intersections numbers are basically $I_{i,i+1} = -I_{i,i-1} = 1$. Thus the integral can be written in the rough form $Ai(t)\bar{A}i'(\bar{t}) - Ai'(t)\bar{A}i(\bar{t})$ where Ai and Ai' are two of the contour integrals.

Another useful example is

$$\int dz d\bar{z} |z|^{2A} e^{z - \bar{z}} \tag{2.102}$$

which is well-defined before analytic continuation if the real part of A is -1. We can improve the behavior at infinity by deforming the contour at large $|z|$ to something like $\tilde{z} = \bar{z} + \epsilon|z|$, after which we can allow the real part of A to be bigger than -1.

There is a unique basic integration contour, comes from negative real infinity, goes around the origin counterclockwise, and goes back to negative real infinity. A dual contour runs from positive real infinity and back. The contours have intersection $1 - \mu_A$. We can write

$$\oint_\alpha dz (-z)^A e^z = (1 - \mu_A) \int_0^\infty t^A e^{-t} dt = (1 - \mu_A)\Gamma(A + 1) \tag{2.103}$$

and the integral is proportional to $(1 - \mu_A)\Gamma(A + 1)^2 \sim \frac{\Gamma(A+1)}{\Gamma(-A)}$

In a similar way, one can express the integral

$$\int dz d\bar{z} |z|^{2A} |1 - tz|^{2B} e^{z - \bar{z}} \tag{2.104}$$

in terms of confluent hypergeometric functions, and

$$\int dz d\bar{z} |1 - tz|^{2B} e^{z^2 - \bar{z}^2} \tag{2.105}$$

in terms of parabolic cylinder functions.

2.C.3 The parabolic cylinder double integral

Here we calculate the parabolic cylinder double integral

$$\int_{\mathbb{C}} dzd\bar{z}\ |z|^{-2(A+1)} e^{-\frac{1}{2}z^2+\frac{1}{2}\bar{z}^2-tz+\bar{t}\bar{z}}. \tag{2.106}$$

A convenient basis of contours is made of a path from zero to positive infinity and another from negative infinity to zero. A dual basis is a pair of paths from $-i\infty \pm \epsilon$ to $+i\infty \pm \epsilon$. The branch cut is on the negative imaginary axis, such that it splits the dual contours. All the contours can be reduced to the basic integral

$$\int_0^\infty dz\ z^{-A-1} e^{-\frac{1}{2}z^2-tz} = e^{\frac{1}{4}t^2}\Gamma(-A)D_A(t), \tag{2.107}$$

which leads to the result

$$\begin{aligned}\int_{\mathbb{C}} dzd\bar{z}\ |z|^{-2(A+1)} e^{-\frac{1}{2}z^2+\frac{1}{2}\bar{z}^2-tz+\bar{t}\bar{z}} &= e^{\frac{1}{4}(t-\bar{t})}\Gamma(-A)^2 e^{i\frac{\pi}{2}A}\left[\left(D_A(t)\left(D_A(\bar{t})-e^{-i\pi A}D_A(-i\bar{t})\right)\right.\right.\\ &\quad\left.\left.-e^{-i\pi A}D_A(-t)\left(D_A(i\bar{t})-e^{i\pi A}D_A(-i\bar{t})\right)\right]\right.\\ &= -2i\pi e^{\frac{1}{4}(t-\bar{t})}\gamma(-A)e^{i\frac{\pi}{2}A}\left(D_A(t)D_A(-i\bar{t})+i\ \gamma(A+1)D_{-A-1}(it)D_{-A-1}(\bar{t})\right).\end{aligned} \tag{2.108}$$

This agrees with the results of section 2.3.1.

2.D Direct collision limit of the regular three-point function

We rederive the result of section 2.3.1, namely the one point-function of a rank two vector, from the direct collision of the regular three-point function.

As previously, we assume that the j^i are in the slice $j \in -\frac{1}{2}+i\mathbb{R}$. The three-point function of primary fields reads [85]

$$\begin{aligned}\langle \Phi_{j^1}(\mu^1|z_1)\Phi_{j^2}(\mu^2|z_2)\Phi_{j^3}(\mu^3|z_3)\rangle =& |z_3-z_2|^{-2\Delta^1_{23}}|z_3-z_1|^{-2\Delta^2_{13}}|z_2-z_1|^{-2\Delta^3_{12}}\\ &\times \delta^{(2)}(\mu^1+\mu^2+\mu^3)D^H\begin{bmatrix} j^1 & j^2 & j^3\\ \mu^1 & \mu^2 & \mu^3\end{bmatrix} C^H(j^1,j^2,j^3),\end{aligned} \tag{2.109}$$

where

$$C^H(j^1,j^2,j^3) = -\frac{1}{2\pi^3 b}\left[\frac{\gamma(b^2)b^{2-2b^2}}{\pi}\right]^{-2-j_{123}}\frac{\Upsilon_b'(0)}{\Upsilon_b(-b(j_{123}+1))} \\ \times\frac{\Upsilon_b(-b(2j^1+1))\Upsilon_b(-b(2j^2+1))\Upsilon_b(-b(2j^3+1))}{\Upsilon_b(-bj^3_{12})\Upsilon_b(-bj^1_{23})\Upsilon_b(-bj^2_{31})},$$

$$D^H\left[\begin{array}{ccc} j^1 & j^2 & j^3 \\ \mu^1 & \mu^2 & \mu^3 \end{array}\right] = \pi\frac{|\mu^1|^{4j^1+2}|\mu^2|^{-2j^2_{13}-2}|\mu^3|^{4j^3+2}}{\gamma(-j_{123}-1)}\int_{\mathbb{C}} dxd\bar{x}|x|^{2j^1_{23}}|x+1|^{2j^3_{12}}\left|x-\frac{\mu^1}{\mu^2}\right|^{-2j_{123}-4}. \tag{2.110}$$

We use the notation $j_{123} = j^1+j^2+j^3$ and $j^3_{12} = j^1+j^2-j^3$, and the function $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. The function Υ_b is defined by the integral ($Q = b+b^{-1}$, domain $0 < \mathrm{Re}(x) < Q$)

$$\log\Upsilon_b(x) = \int_0^\infty \frac{dt}{t}\left[\left(\frac{Q}{2}-x\right)^2 e^{-t} - \frac{\sinh^2[(\frac{Q}{2}-x)\frac{t}{2}]}{\sinh\frac{bt}{2}\sinh\frac{t}{2b}}\right]. \tag{2.111}$$

For the collision of C^H we use the asymptotics ($\tilde{\Delta}_x = x(Q-x)$)

$$\Upsilon_b(x) \sim \tilde{\Delta}_x^{-\frac{1}{2}\tilde{\Delta}_x+\frac{1}{12}(1+Q^2)}e^{\frac{3}{2}\tilde{\Delta}_x}, \tag{2.112}$$

valid when x has a large imaginary part, with $0 < \mathrm{Re}(x) < Q$. This leads to the limit

$$C^H(j^1,j^2,j^3) \sim -2^{(j_0+2)(b^2(j_0+3)+1)}\pi^{j_0-1}b^{(j_0+2)(b^2(j_0+5)-1)}\frac{\gamma(b^2)^{-(j_0+2)}\Upsilon_b'(0)}{\Upsilon_b(-b(j_0+1))} \\ \times j_2^{(j_1+3)(b^2(j_0+2)+1)}\left(q_{12}^{-j^3_{12}-1}q_{13}^{-j^2_{13}-1}q_{23}^{-j^1_{23}-1}e^{\frac{j_1^2}{j_2}}\right)^{2(b^2(j_0+2)+1)} \\ \sim -2^{(j_0+2)(b^2(j_0+3)+1)}\pi^{j_0-1}b^{(j_0+2)(b^2(j_0+5)-1)}\frac{\gamma(b^2)^{-(j_0+2)}\Upsilon_b'(0)}{\Upsilon_b(-b(j_0+1))} \\ \times\left(|q_{12}|^{-2(j^3_{12}+1)}|q_{13}|^{-2(j^2_{13}+1)}|q_{23}|^{-2(j^1_{23}+1)}|j_2|^{(j_0+3)}e^{\frac{j_1^2}{j_2}+\frac{\bar{j}_1^2}{\bar{j}_2}}\right)^{(b^2(j_0+2)+1)}, \tag{2.113}$$

where we used the equality $j = \bar{j}$ to relate q_{ij} to $|q_{ij}|$. In D^H, the limit of the integral can be evaluated using the change of variable

$$x = \frac{\mu^1}{\mu^2}\left(1+i\frac{q_{12}\eta}{\sqrt{2j_2}}\right), \tag{2.114}$$

and its anti-holomorphic counterpart. This leads to

$$D^H \begin{bmatrix} j^1 & j^2 & j^3 \\ \mu^1 & \mu^2 & \mu^3 \end{bmatrix} \sim |q_{12}q_{13}q_{23}|^{-2j_0-2} 2^{j_0+1}\pi |\mu_2|^{2j_0+2}|j_2|^{j_0+1}\gamma(j_0+2) \\ \times \int_{\mathcal{C}} d\eta d\bar{\eta}\ |\eta|^{-2(j_0+2)} e^{-\frac{1}{2}\eta^2+\frac{1}{2}\bar{\eta}^2-u\eta+u^*\bar{\eta}}, \tag{2.115}$$

with u is the same parameter as defined in section 2.3.1. Inserting these results back in eq. (2.109), the powers of q_{ij} cancel and we find the limit

$$\langle \tilde{\Phi}^{(2)}_{\boldsymbol{j}}(\boldsymbol{\mu}|z)\rangle = -\, 2^{(j_0+2)(b^2(j_0+3)-1)}\pi^{j_0} b^{(j_0+2)(b^2(j_0+3)-1)} |\mu_2|^{2(j_0+1)} |j_2|^{(j_0+2)(b^2(j_0+3)+2)} \\ \times \delta^{(2)}(\mu_0)\gamma(j_0+2)\frac{\gamma(b^2)^{-(j_0+2)}\Upsilon_b'(0)}{\Upsilon_b(-b(j_0+1))} e^{2i(b^2(j_0+2)+1)\mathrm{Im}\left(\frac{j_1^2}{j_2}\right)} \\ \times \int_{\mathcal{C}} d\eta d\bar{\eta}\ |\eta|^{-2(j_0+2)} e^{-\frac{1}{2}\eta^2+\frac{1}{2}\bar{\eta}^2-u\eta+u^*\bar{\eta}} \tag{2.116}$$

Using the function $F^{(2)} = j_2^{-(b^2+1)(4j_0+6)} e^{-(b^2+1)j_1^2/j_2}$ to find the one-point function of $\Phi^{(2)}$, we recover the result (2.32) found directly for the irregular vector, and fix the overall factor $C(j_0)$.

Chapter 3

Localization of a supersymmetric gauge theory in the presence of a surface defect

3.1 Introduction

Supersymmetry, despite being so far of little use for realistic physical models, is an interesting framework for studying nonperturbative phenomena in quantum field theory. Indeed, many physical quantities invariant under some supersymmetry can be computed exactly using various tools. One such tool is localization [98], which allows to compute some partition functions and supersymmetric quantities by deforming the action in a suitable way. In recent years, localization allowed to compute partition functions and Wilson loops of supersymmetric gauge theories on S^4 and its deformations [45, 48, 81], on S^2 [12, 31], and on various other spacetimes. In two dimensions, the method has also proven successful in computing various other physical and mathematical quantities [9, 13, 30, 44, 56, 59].

In this chapter, we explore another direction in which the localization program can be expanded, that of supersymmetric gauge theories with surface defects. Surface defects play an important role in quantum theory, and therefore are interesting objects to study [32, 37]. Our goal is to localize a $\mathcal{N} = 2$ supersymmetric gauge theory on S^4 interacting with some extra GLSM degrees of freedom on a S^2 surface[1]. The surface is embedded as a great

[1]It would also be possible include twisted multiplets and perform localization as in [30], but we do not explore this possibility here

sphere of S^4 (i.e. it contains antipodal points). The presence of the defect breaks half the supercharges, leaving a $\mathcal{N} = (2, 2)$ supersymmetry group. We also focus on the partition function of that theory. In addition to being an interesting result by itself, the partition function plays an important role in the AGT duality [1, 3], and thus its computation allows for an additional check of the conjecture. The computation also sets the ground for that of other physical quantities (compatible with localization) in the same setup. For most of the paper, we specialize the computation to the case of a pure supersymmetric gauge theory on S^4 interacting with chiral multiplets on the defect. The simpler computation already shows all the important features of the general case, allowing for a straightforward generalization. In particular, charging a chiral multiplet under a 2d representation of a 4d vector multiplet gives the only coupling between the bulk and the defect relevant for the computation. Superpotential interactions are also possible, but they do not directly affect the path integral.

We compute the partition using supersymmetric localization in the Coulomb branch. The result is an integral over the coulomb branches of both the 4d and 2d multiplets, of the form:

$$\begin{aligned} Z_{2d-4d} = &\int_{4d \text{ Coulomb}} da\ e^{-S^{4d}_{cl}(a)} Z^{4d}_{1-\text{loop}}(a) \\ &\times \sum_B \int_{2d \text{ Coulomb}} d\tilde{a}\ e^{-S^{2d}_{cl}(\tilde{a},B)} |Z^{2d-4d}_{\text{inst}}(\tilde{a}, B; a)|^2 Z^{\text{chiral}}_{1-\text{loop}}(\tilde{a}, B; a) Z^{2d\ \text{vector}}_{1-\text{loop}}(\tilde{a}). \end{aligned} \tag{3.1}$$

(See eq. (3.25) for the exact formula.) The one-loop determinants are the same as in the partition function of the isolated 2d and 4d theories, except for the chiral multiplets which see the 4d vector multiplets as background vector fields. The instanton partition function corresponds to a modified version of Nekrasov's partition function, where the gauge theory in the Ω-background now contains a $\mathbb{R}^2$ defect. It is expected to be computable using the methods of [38], but its derivation is left for future work.

This chapter is organized as follows. In section 3.2, we describe the theory of a 2d chiral multiplet and a 4d vector multiplet and its relevant properties. The relevant computations and some technical details are left for the appendices. The localization is performed in section 3.3, where we find the appropriate setup for the localization and assemble the components of the partition function. Most of the components can be taken from the literature with little change, and we review the computation of the one-loop determinant for the chiral multiplet in appendix 3.E. In section 3.4, we show how the results generalize in the presence of matter on S^4 and vector multiplets on the defect.

3.2 The theory

In this section we write down the action and some basic properties of the theory of a vector multiplet on S^4 coupled to a chiral multiplet on a S^2 defect. The separate uncoupled theories on S^2 and S^4 have been described previously [12, 31, 81]. Our goal is to find the coupling between the defect and the four dimensional vector multiplet. For this purpose, we express the vector multiplet in a "two dimensional" language, where only the unbroken symmetries of the theory are manifest. We split the tangent space near the S^2 subspace in a parallel and a perpendicular part, and we write the spinors as two dimensional Dirac spinors (relative to the parallel part of the tangent space). Then the restriction of the multiplet to the S^2 subspace can be seen as a set of field on that subspace, hence it can interact locally with the matter on the defect. In particular, the derivatives of the fields in the orthogonal direction appear as an extra set of fields. In the following, we call this process the "restriction" of the vector multiplet from S^4 to S^2.

When restricted to S^2, the four dimensional vector multiplet appears as a tower of fields, which consists of the vector multiplet on the defect and its transverse derivatives. Each level appears as a two dimensional vector multiplet together with a chiral multiplet. In principle we can couple the defect to any of this fields, but renormalizability forbids any supersymmetric coupling to the derivative fields (see appendix 3.D).

The resulting chiral multiplet is coupled to the vector multiplet in an unusual way, which encodes its variation in the directions transverse to the defect. However, the (restricted) vector multiplet itself appears exactly as a $\mathcal{N} = (2, 2)$ vector multiplet, with the correct supersymmetry transformations. Therefore we can couple the 2d matter to the restricted vector multiplet as if it was a vector multiplet on S^2, and the result is guaranteed to be supersymmetric[2]. As shown in appendix 3.D, this is the most general renormalizable coupling between the 2d and 4d fields compatible with the symmetries of the theory.

[2] Alternatively, we can interpret S^4 in some neighborhood of S^2 as a bundle $S^2 \times I$,over S^2, where I is some interval, and encode the restricted dimensions in a larger gauge group $I \otimes G$ living on that bundle. In this framework, a 4d vector multiplets split into 2d chiral and vector multiplets, and the "unusual coupling" is encoded in the representation of the chiral multiplet under the new gauge group. The restricted fields are obtained by considering the section of the bundle corresponding to S^2

3.2.1 Field content and action

We now proceed to describe the action for theory. It separates into a two dimensional part and a four dimensional one, in the generic form

$$S = S_{S^4} + S_{S^2} = \int_{S^4} d^4x \sqrt{g_{S^4}} \mathcal{L}_{S^4} + \int_{S^2} d^2x \sqrt{g_{S^2}} \mathcal{L}_{S^2}. \tag{3.2}$$

The four-sphere has a radius r, and we describe it using the stereographic coordinates x^μ, $\mu = 1, \cdots, 4$, centered at the north pole. The metric is conformally flat, $g_{\mu\nu} = e^{2\Omega}\delta_{\mu\nu}$, with conformal factor $e^{-\Omega} = 1 + \frac{x^\mu x^\mu}{r^2}$. The two-sphere is taken to be the subspace $x^3 = x^4 = 0$, and we use the coordinate system x^i, $i = 1, \cdots, 2$, induced by inclusion. Our conventions for spinors are described in appendix 3.A.

We begin with the description[3] of the vector multiplet on S^4. The multiplet consists of a gauge field A_μ, a $SU(2)_R$ R-symmetry doublets of Weyl fermions λ_A, $\bar{\lambda}_A$, a pair of scalars ϕ, $\bar{\phi}$, and a R-symmetry triplet of auxiliary fields $D_{AB} = D_{BA}$. All fields are in the adjoint representation of the gauge group G, with Lie algebra $\mathfrak{g}$. The $SU(2)_R$ indices $\{A, B = 1, 2\}$ are raised and lowered by the antisymmetric matrices C and $\tilde{C}$, defined by $C_{21} = \tilde{C}^{12} = 1$, in the form $\lambda_A = C_{AB}\lambda^B$, $\lambda^A = \tilde{C}^{AB}\lambda_B$. The Lagrangian for the multiplet is [48]

$$\begin{aligned}\mathcal{L}_{S^4} = \tfrac{1}{g^2}\mathrm{Tr}\Big(&\tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} - 4\mathcal{D}_\mu\bar{\phi}\mathcal{D}^\mu\phi - \tfrac{8}{r^2}\bar{\phi}\phi - 2i\lambda^A\sigma^\mu\mathcal{D}_\mu\bar{\lambda}_A - \tfrac{1}{2}D^{AB}D_{AB} \\ &- 2\lambda^A[\bar{\phi}, \lambda_A] + 2\bar{\lambda}^A[\phi, \bar{\lambda}_A] + 4[\bar{\phi}, \phi]^2\Big).\end{aligned} \tag{3.3}$$

Here $F_{\mu\nu}$ is the field strength for A_μ, and $\mathcal{D}_\mu = \nabla_\mu - iA_\mu$ is the gauge covariant derivative.

On the two dimensional side, the defect consists in a set of chiral multiplets, or equivalently a single chiral multiplet in a representation $R = \bigoplus_I R_I$ of the gauge group G , decomposing into a direct sum of irreducible representations (flavors) R_I. A chiral multiplet in a representation R consists of a scalar χ, a Dirac fermion ψ, and an auxiliary scalar F (the corresponding fields are $\bar{\chi}$, $\bar{\psi}$, $\bar{F}$ for the antichiral multiplet). The multiplet is characterized by a mass m and a R-charge q, where m, q are matrix valued in the flavor space and take constant values m_I, q_I in each irreducible representation. As discussed before, the coupling to the S^4 vector multiplet takes the same form as the coupling to a S^2 vector multiplet. Therefore we can write the Lagrangian as [31]

$$\begin{aligned}\mathcal{L}_{S^2} = \bar{\chi}\left(-\mathcal{D}_i^2 + \sigma_1^2 + \sigma_2^2 + iD + M^2 + (2M - \tfrac{i}{r})\sigma_2\right)\chi - i\bar{\psi}\left(\gamma^i\mathcal{D}_i - \sigma_1 - i\sigma_2\gamma^3 - iM\gamma^3\right)\psi \\ + \bar{F}F + i\bar{\psi}\lambda\chi - i\bar{\chi}\bar{\lambda}\psi + \mathcal{L}_W,\end{aligned} \tag{3.4}$$

[3] Our conventions on S^4 mostly follow [48], and those on S^2 mostly follow [31]

where $M = m + \frac{i}{2r}q$ and $\mathcal{D}_i = \nabla_i - iA_i$. The fields σ_1, σ_2, D, λ, $\bar{\lambda}$, and A_i and form a vector multiplet representation of $\mathcal{N} = (2,2)$ supersymmetry. Their exact expression is obtained by decomposing the vector multiplet in terms of representations of the $\mathcal{N} = (2,2)$ supersymmetry algebra, which is done by comparing the supersymmetry transformations of the multiplets (see appendix 3.C). The identification of the vector fields is trivial, and the scalars are given by

$$\begin{aligned} \sigma_1 &= i(\phi + \bar{\phi}), \quad \sigma_2 = \phi - \bar{\phi}, \\ D &= -D_{12} - \tfrac{1}{r}(\phi - \bar{\phi}) - iF_{34}. \end{aligned} \tag{3.5}$$

The term $\mathcal{L}_W$ is a superpotential term, which we will ignore for the purpose of this work since it doesn't affect the partition function. (This is due to the fact that the superpotential term id $\mathcal{Q}$-exact, and by the principle of localization described in section 3.3 it has no effect on the path integral.)

3.2.2 Supersymmetry transformations

Supersymmetry transformations in curved space are most naturally obtained as a subset of the superconformal transformations. Such transformations are parametrized by a set of conformal Killing spinors. Each of these spinors can be expressed as a linear combination of a basis of conformal Killing spinors, each associated to one of the supercharges. In this paper, we describe supersymmetry and superconformal symmetry through conformal Killing spinors, i.e. through the realization of the algebra on the fields.

For generic conformal Killing spinors, the commutator of superconformal transformations contains conformal transformations through conformal Killing vectors. Supersymmetry is obtained by restricting to a maximal subset of the conformal Killing spinors generating only Killing vectors, i.e. isometries.

On S^4, the $\mathcal{N} = 2$ superconformal transformations are given in terms of $SU(2)_R$ doublets of conformal Killing spinors ϵ^A, $\bar{\epsilon}^A$. The fields of the vector multiplet transform

as

$$
\begin{aligned}
(\delta_\epsilon+\delta_{\bar\epsilon})A_\mu &= i\epsilon^A\sigma_{g\mu}\bar\lambda_A - i\bar\epsilon^A\bar\sigma_{g\mu}\lambda_A,\\
(\delta_\epsilon+\delta_{\bar\epsilon})\phi &= -i\epsilon^A\lambda_A,\\
(\delta_\epsilon+\delta_{\bar\epsilon})\bar\phi &= i\bar\epsilon^A\bar\lambda_A,\\
(\delta_\epsilon+\delta_{\bar\epsilon})\lambda_A &= \frac{1}{2}\sigma_g^{\mu\nu}\epsilon_A F_{\mu\nu} + 2\sigma_g^\mu\bar\epsilon_A\mathcal{D}_\mu\phi + \sigma_g^\mu\nabla_\mu\bar\epsilon_A\phi + 2i\epsilon_A[\phi,\bar\phi] + D_{AB}\epsilon^B,\\
(\delta_\epsilon+\delta_{\bar\epsilon})\bar\lambda_A &= \frac{1}{2}\bar\sigma_g^{\mu\nu}\bar\epsilon_A F_{\mu\nu} + 2\bar\sigma_g^\mu\epsilon_A\mathcal{D}_\mu\bar\phi + \bar\sigma_g^\mu\nabla_\mu\epsilon_A\bar\phi - 2i\bar\epsilon_A[\phi,\bar\phi] + D_{AB}\bar\epsilon^B,\\
(\delta_\epsilon+\delta_{\bar\epsilon})D_{AB} &= -2i\bar\epsilon_{(B}\bar\sigma_g^\mu\mathcal{D}_\mu\lambda_{A)} + 2i\epsilon_{(A}\sigma_g^\mu\mathcal{D}_\mu\bar\lambda_{B)} - 4[\phi,\bar\epsilon_{(A}\bar\lambda_{B)}] + 4[\bar\phi,\epsilon_{(A}\lambda_{B)}].
\end{aligned}
\tag{3.6}
$$

For $\mathcal{N}=2$ supersymmetry, the allowed set of spinors is given in terms of constant spinors ϵ_0^A, $\bar\epsilon_0^A$, by

$$
\epsilon^A = e^{\frac{1}{2}\Omega}\left(\epsilon_0^A + \tfrac{1}{2r}x^\mu\sigma^\mu(\tau^3)^A{}_B\bar\epsilon_0^B\right), \qquad \bar\epsilon^A = e^{\frac{1}{2}\Omega}\left(\bar\epsilon_0^A - \tfrac{1}{2r}x^\mu\bar\sigma^\mu(\tau^3)^A{}_B\epsilon_0^B\right). \tag{3.7}
$$

(See appendix 3.B.2.) The four constant spinors correspond to the eight supercharges of the theory. In the presence of a S^2 defect, half of the supersymmetries are broken, and the unbroken ones correspond to spinors of definite "two dimensional chirality". The restriction takes the form:

$$
\begin{aligned}
(-i\sigma^{12})\epsilon_0^1 &= +\epsilon_0^1, \qquad & (i\bar\sigma^{12})\bar\epsilon_0^1 &= -\bar\epsilon_0^1,\\
(-i\sigma^{12})\epsilon_0^2 &= -\epsilon_0^2, \qquad & (i\bar\sigma^{12})\bar\epsilon_0^2 &= +\bar\epsilon_0^2.
\end{aligned}
\tag{3.8}
$$

On S^2, the $\mathcal{N}=(2,2)$ superconformal transformations are generated by a pair of conformal Killing spinors ε, $\bar\varepsilon$. The fields of the chiral (and antichiral) multiplet transform as

$$
\begin{aligned}
(\delta_\varepsilon+\delta_{\bar\varepsilon})\chi &= \bar\varepsilon\psi,\\
(\delta_\varepsilon+\delta_{\bar\varepsilon})\bar\chi &= \varepsilon\bar\psi,\\
(\delta_\varepsilon+\delta_{\bar\varepsilon})\psi &= i\left(\gamma_g^i\mathcal{D}_i\chi + (\sigma_1 - i(\sigma_2+m)\gamma^3)\chi + \tfrac{q}{2}\chi\gamma_g^i\nabla_i\right)\varepsilon + F\bar\varepsilon\\
(\delta_\varepsilon+\delta_{\bar\varepsilon})\bar\psi &= i\left(\gamma^i\mathcal{D}_i\bar\chi + (\sigma_1 + i(\sigma_2+m)\gamma^3)\bar\chi + \tfrac{q}{2}\bar\chi\gamma_g^i\nabla_i\right)\bar\varepsilon + \bar F\varepsilon\\
(\delta_\varepsilon+\delta_{\bar\varepsilon})F &= -i\left(\mathcal{D}_i\psi\gamma_g^i + \sigma_1\psi - i(\sigma_2+m)\psi\gamma^3 + \lambda\chi + \tfrac{q}{2}\psi\gamma_g^i\nabla_i\right)\varepsilon,\\
(\delta_\varepsilon+\delta_{\bar\varepsilon})\bar F &= -i\left(\mathcal{D}_i\bar\psi\gamma_g^i + \bar\psi\sigma_1 + i\bar\psi(\sigma_2+m)\gamma^3 + \bar\chi\bar\lambda + \tfrac{q}{2}\bar\psi\gamma_g^i\nabla_i\right)\bar\varepsilon.
\end{aligned}
\tag{3.9}
$$

For supersymmetry, the set of allowed spinors is given in terms of a pair of constant spinors ε_0, $\bar\varepsilon_0$, by

$$
\varepsilon = e^{\frac{1}{2}\Omega}\left(\varepsilon_0 + \tfrac{1}{2r}x^i\gamma^i\gamma^3\varepsilon_0\right), \qquad \bar\varepsilon = e^{\frac{1}{2}\Omega}\left(\bar\varepsilon_0 - \tfrac{1}{2r}x^i\gamma^i\gamma^3\bar\varepsilon_0\right). \tag{3.10}
$$

(See appendix 3.B.3.) For the theory being considered, the supercharges on S^2 and S^4 are the same, so the allowed conformal Killing spinors are related. In components, the relation is

$$(\epsilon^1)_1 = \tfrac{1}{\sqrt{2}}\varepsilon_1, \quad (\epsilon^2)_2 = \tfrac{1}{\sqrt{2}}\bar{\varepsilon}_2, \quad (\bar{\epsilon}^1)^2 = -\tfrac{i}{\sqrt{2}}\varepsilon_2, \quad (\bar{\epsilon}^2)^1 = \tfrac{i}{\sqrt{2}}\bar{\varepsilon}_1. \tag{3.11}$$

(See appendix 3.B.4.)

3.3 Localization

The goal of this paper is to compute the partition function

$$Z = \int \mathcal{D}\Phi_g \mathcal{D}\Phi e^{-S_{2d}[\Phi]-S_{4d}[\Phi]-S_g[\Phi_g,\Phi]}, \tag{3.12}$$

where Φ denotes the set of fields of the chiral and vector multiplets, Φ_g is a set of ghosts, $S = S_{2d} + S_{4d}$ is the action defined in the previous section, and S_g is a gauge fixing action. We perform the path integral over a contour in which the bosons satisfy the reality conditions

$$A_\mu^\dagger = A_\mu, \quad \phi^\dagger = -\bar{\phi}, \quad D_{AB}^\dagger = -D^{AB}, \quad \chi^\dagger = \bar{\chi}, \quad F^\dagger = \bar{F}. \tag{3.13}$$

We compute the partition function using supersymmetric localization [81, 98]. The method relies on the fact that given a supercharge $\mathcal{Q}$, only the $\mathcal{Q}$-invariant field configurations contribute to the path integral. Indeed, if the orbit of $\mathcal{Q}$ is non-trivial, then the path integral over that orbit vanishes (see section 1.6). In practice, we can simplify the computation by deforming the action by a $\mathcal{Q}$-exact term in the form $S \to S + t\mathcal{Q}\cdot V$. Under certain assumptions (V must be $\mathcal{Q}^2$-invariant, and the deformation must not modify the asymptotics of the action), the path integral is not affected by such deformation. If the deformation term is non-negative, we can take the limit $t \to \infty$. In that limit, the saddle-point approximation becomes exact, and thus can be used to compute the partition function for the original theory exactly. Schematically, the path integral reduces to a sum (or integral) over the set F of zeros of $\mathcal{Q}\cdot V$, and a Gaussian integral, in the form

$$Z = \sum_{\Phi_0 \in F} e^{-S[\Phi_0]} \int \mathcal{D}\Phi e^{-(\mathcal{Q}\cdot V)[\Phi+\Phi_0]_{\text{Quad}}} = \sum_{\Phi_0 \in F} e^{-S[\Phi_0]} Z_{1-\text{loop}}[\Phi_0]. \tag{3.14}$$

In the $t \to \infty$ limit, the only smooth classical configurations correspond to a set F_0 of configurations with zero instanton number, which in the case of interest will correspond

to the Coulomb branch of the theory. However we also need to consider the contribution of singular instanton configurations, arrising from the $t \to \infty$ limit of smooth instanton configurations. (One can show by a careful computation of the limit that these need to be included.) We can package the instanton contributions at each pole into an "instanton partition function" $Z_{1-\text{loop}}[\Phi_0]$, giving a full partition function of the form

$$Z = \sum_{\Phi_0 \in F_0} e^{-S[\Phi_0]} Z_{1-\text{loop}}[\Phi_0] \left| Z_{\text{inst}}[\Phi_0] \right|^2 . \tag{3.15}$$

The one-loop partition function (for smooth field configurations) $Z_{1-\text{loop}}[\Phi_0]$ is a ratio of functional determinants, and is greatly simplified by supersymmetry. Indeed, as mentioned before, only the supersymmetric configurations contribute, the others canceling pairwise. We compute it using the Atiyah-Singer index theorem (see appendix 3.E).

3.3.1 The supercharge and deformation terms

To perform the localization, we choose a supercharge compatible with the localization for both the two and four dimensional fields, i.e. a supercharge of the unbroken supersymmetry. We pick the supercharge $\mathcal{Q}$ such that the poles are left invariant by $\mathcal{Q}^2$. In that case $\mathcal{Q}^2$ generates a $U(1)$ group, which consists of a combination of $SO(2) \subset SO(3)$ and $SO(2)_\perp$ rotations, where $SO(2)_\perp$ corresponds to rotation in the plane perpendicular to the defect. In terms of conformal Killing spinors in the two dimensional formalism, the constraint is $\varepsilon_0 \gamma^i \bar{\chi}_0 + \bar{\varepsilon}_0 \gamma^i \chi_0 = 0$ (i.e. the coefficient of M_{i5} must vanish in the Killing vector (3.50)), and is satisfied by imposing $\gamma^3 \varepsilon_0 = +\varepsilon_0$, $\gamma^3 \bar{\varepsilon}_0 = -\bar{\varepsilon}_0$. This leaves two independent spinors, and we choose the combination defined by $\varepsilon_0 = i\gamma^1 \bar{\varepsilon}_0$. In components, we write $(\varepsilon_0)_1 = i(\bar{\varepsilon}_0)_2 = \varepsilon_{\mathcal{Q}}$, where $\varepsilon_{\mathcal{Q}}$ is the parameter for the transformation $\delta_{\mathcal{Q}} = r^{-\frac{1}{2}} \varepsilon_{\mathcal{Q}} \mathcal{Q}$. The square of the supercharge $\mathcal{Q}$ is realized as

$$\begin{aligned} \mathcal{Q}^2 &= M_{12} + M_{34} + \tfrac{1}{2} R + \mathcal{G}[\Lambda] - irm, \\ \Lambda &= -iv \cdot A + r(f(x)\sigma_1 - i\sigma_2), \\ f(x) &= e^{\Omega}(1 - \tfrac{x^\mu x^\mu}{4r^2}) = \cos(\theta), \end{aligned} \tag{3.16}$$

where v is the Killing vector associated with $M_{12} M_{34}$, θ measures the angle on S^4 relative to the north pole, and m is the mass of the multiplet.

This choice of supercharge is compatible with previous computations on S^2 and S^4, hence we can pick a similar deformation term, and the computations follow a similar pattern. In particular, the classical and one-loop computations are almost the same, the

main difference being in the classical field configuration appearing in the two-dimensional one-loop determinant. We review these computations and adapt them to the present case, mostly following the conventions of [31] on S^2, and those of [48] on S^4. However, the instanton partition function for the four dimensional theory is affected by the presence of the defect, and must be computed separately.

We deform the action by a non-negative $\mathcal{Q}$-exact term $\mathcal{Q}V = \mathcal{Q}V_{4d} + \mathcal{Q}V_{2d}$. The two-dimensional part $\mathcal{Q}V_{2d}$ can be taken to be the action S_{S^2} itself, as it is $\mathcal{Q}$-exact. On the four-dimensional side, we take

$$V = \mathrm{Tr}[(\mathcal{Q}\lambda_A)^\dagger \lambda_A + (\mathcal{Q}\bar{\lambda}_A)^\dagger \bar{\lambda}_A]. \tag{3.17}$$

The resulting deformation term $\mathcal{Q}V_{4d}$ manifestly satisfies the required properties.

3.3.2 Classical configurations and one-loop determinants

For the vector multiplet, the classical configurations are given by the zeros of the deformation term $\mathcal{Q}V_{4d}$, for which the bosonic part is

$$\mathcal{Q}V_{4d}^{\mathrm{bos}} = \mathrm{Tr}[(\mathcal{Q}\lambda_A)^\dagger (\mathcal{Q}\lambda_A) + (\mathcal{Q}\bar{\lambda}_A)^\dagger (\mathcal{Q}\bar{\lambda}_A)]. \tag{3.18}$$

The classical configurations thus coincide with the supersymmetric configurations. Up to a gauge transformation, the smooth solutions to $\mathcal{Q}V_{4d}^{\mathrm{bos}} = 0$ are given in terms of a constant $\mathfrak{g}$-valued parameter a, by [81]

$$A_\mu = 0, \quad \phi = -\bar{\phi} = \tfrac{1}{2r} a, \quad D_{12} = -\tfrac{1}{r^2} a, \quad D_{11} = D_{22} = 0. \tag{3.19}$$

For the chiral multiplet, the equation of motion for D_{12} and the above configuration fix $\chi = 0$, and F must vanish by its own equation of motion. The action (3.2) for such field configuration is $S_{cl}(a) = \frac{8\pi^2}{g^2}\mathrm{Tr}(a_0^2)$.

The one-loop determinants for the chiral and vector multiplets are computed independently from each other, although they both depend on the background of the vector multiplet. For a generic vector multiplet background, the one-loop determinant for the chiral multiplet is a product over the weights of R:

$$\tilde{Z}^{2d}_{1-\mathrm{loop}}(\Lambda) = \prod_{w \in R} \frac{\Gamma(\omega \cdot \Lambda_N - irM)}{\Gamma(1 - \omega \cdot \Lambda_S + irM)} \tag{3.20}$$

where N, S denote the values at the north and south poles. This result is computed in appendix 3.E. For the above background, the formula reduces to

$$Z^{2d}_{1-\text{loop}}(a) = \prod_{w\in R} \frac{\Gamma(-i\omega\cdot a - irM)}{\Gamma(1+i\omega\cdot a + irM)} \tag{3.21}$$

The one-loop determinant for the vector multiplet is (see [81])

$$\tilde{Z}^{4d}_{1-\text{loop}}(a) = \prod_{\alpha\in\Delta} G(1+ia\cdot\alpha)G(1-ia\cdot\alpha), \tag{3.22}$$

where Δ is the set of roots of $\mathfrak{g}$, and $G(z)$ is the Barnes G-function [11].

3.3.3 The instanton partition function

In [81], it was shown that the instanton contribution to the partition function at each pole in the absence of defect is given by Nekrasov's instanton partition function [76] of the theory in the Ω-background on $\mathbb{R}^4$. The argument given in that paper is still valid here, however the presence of the defect modifies the instanton partition function, which is now that of a similar theory with a $\mathbb{R}^2$ defect, i.e. the $\mathbb{R}^4$ version of the theory considered in this paper in the Ω-background. We leave the computation of the exact instanton partition function $Z^{2d-4d}_{\text{inst}}(a)$ for future work.

Summing up the previous computations, we write the complete partition function as

$$Z_{2d-4d} = \int_{\mathfrak{h}} da e^{-\frac{8\pi^2}{g^2}\text{Tr}(a^2)} Z^{2d}_{1-\text{loop}}(a) Z^{4d}_{1-\text{loop}}(a) |Z^{2d-4d}_{\text{inst}}(a)|^2, \tag{3.23}$$

where we reduced the integral over a to the Cartan subalgebra $\mathfrak{h}$ at the price of a Jacobian factor $\prod_{\alpha\in\Delta}(\alpha\cdot a)$, inserted in the modified one-loop determinant

$$Z^{4d}_{1-\text{loop}}(a) = \prod_{\alpha\in\Delta} (\alpha\cdot a) G(1+ia\cdot\alpha)G(1-ia\cdot\alpha), \tag{3.24}$$

3.4 Generalizations

In this section we consider the generalization of the above results, where we allow additional types of multiplets. Namely, we consider the addition of hypermultiplets on S^4 and

vector multiplets on the defect. As in the previous case (see appendix 3.D) symmetry and renormalizability imposes heavy constraints on the possible couplings between the defect and the bulk. A vector multiplet on the defect cannot be coupled consistently with a 4d field, and a hypermultiplet can only couple to the defect through a (heavily constrained) superpotential. However, a superpotential term does not affect the partition function, so the localization procedure involves no more couplings between the bulk and the defect than the one considered previously.

A vector multiplet on the defect cannot be coupled consistently with a 4d field, but a hypermultiplet can couple to the defect in two different ways. The first one is through a superpotential: the restricted hypermultiplet appears as two series of chiral multiplets, which can appear in the superpotential. By renormalizability the superpotential must be at most linear in the 4d fields, and only the top chiral multiplet in each series is allowed (i.e. without transverse derivative), and gauge invariance constrains the superpotential further. In any case, a superpotential term does not affect the partition function, so it is irrelevant for the present computation. The other possible coupling is through a four dimensional vector multiplet frozen to its vacuum expectation value [43]: if a Lagrangian is invariant under some flavor symmetry, we can weakly gauge it by introducing a vector multiplet, then freezing it.

The localization procedure for the generalized theory is unchanged from the one described previously. Here we consider vector multiplets V_{4d} and V_{2d}, with gauge groups G_{4d} and G_{2d}. V_{2d} is also associated to a Fayet-Iliopoulos parameter ξ and a topological angle θ, appearing in the combination $\tau = \frac{\theta}{2\pi} + i\xi$. The matter on S^4 forms a hypermultiplet H with mass m_H, in a representation R_H of G_{4d}[4]. The chiral multiplet Ψ on the defect has mass and R-charge $M_\Psi = m_\Psi + \frac{i}{2r} q_\Psi$, and is in a representation R_Ψ of $G_{4d} \times G_{2d}$. Localization works exactly as before, and we obtain the formula

$$Z_{2d-4d} = \sum_B \int_{\mathfrak{h}_{2d}} d\tilde{a} e^{-4\pi i r \mathrm{Im}\mathrm{Tr}[\tau(\tilde{a}+\frac{i}{2r}B)]} \int_{\mathfrak{h}_{4d}} da e^{-\frac{8\pi^2}{g^2}\mathrm{Tr}(a^2)} Z_{1-\mathrm{loop}}(a,\tilde{a},B) |Z^{2d-4d}_{\mathrm{inst}}(a,\tilde{a},B)|^2, \tag{3.25}$$

$$\text{where} \quad Z_{1-\mathrm{loop}}(a,\tilde{a},B) = Z^{V_{2d}}_{1-\mathrm{loop}}(\tilde{a},B) Z^{V_{4d}}_{1-\mathrm{loop}}(a) Z^{H}_{1-\mathrm{loop}}(a) Z^{\Psi}_{1-\mathrm{loop}}(a,\tilde{a},B), \tag{3.26}$$

[4]As in the 2d case, hypermultiplet masses are obtained through a four dimensional vector multiplet frozen to its vacuum expectation value. In principle, one could also couple such vector multiplet to chiral fields on the defect, but the effect is equivalent to giving (equal) twisted masses to the 2d fields [43]. Namely, one can obtain such coupling from others simply by constraining masses, so we do not need to consider it here.

$$\text{and} \quad Z^{\Psi}_{1-\text{loop}}(a,\tilde{a},B) = \prod_{w\in R} \frac{\Gamma(-i\omega\cdot(a,\tilde{a}+\frac{i}{2r}B)-irM)}{\Gamma(1+i\omega\cdot(a,\tilde{a}+\frac{i}{2r}B)+irM)}. \tag{3.27}$$

The other one-loop determinants are unchanged from the separate expressions on S^2 and S^4 [12, 31, 81], and as before we leave the computation of the instanton partition function for future work.

Appendix

In these appendices we review some of the conventions used in this chapter, and derive some technical results. In appendix 3.A we review our conventions concerning the coordinate systems and spinors. In appendix 3.B we describe the supersymmetry algebras on S^4 and S^2, and we relate the two algebras by describing the embedding of the 2d algebra inside the 4d algebra. In appendix 3.C we decompose the $\mathcal{N}=2$ vector multiplet into representations of the $\mathcal{N}=(2,2)$ superconformal algebra, focusing on the $\mathcal{N}=(2,2)$ vector multiplet. In appendix 3.D we describe the restrivtion process from a superspace point of view, and use it to find the most generic couplings allowed by symmetry and renormalizability. In appendix 3.E we compute the one-loop determinant of the chiral multiplet using the equivariant index theorem for transversally elliptic operators.

3.A Coordinates and spinors

In this section we review some basic conventions used in this chapter concerning the coordinate systems and spinors.

We use the stereographic coordinates on S^4. The coordinates are labeled x^μ, $\mu = 1, \cdots, 4$, and the metric is conformally flat:

$$g_{\mu\nu} = e^{2\Omega}\delta_{\mu\nu}, \qquad e^{-\Omega} = 1 + \frac{x^\mu x^\mu}{4r^2}, \tag{3.28}$$

where r is the radius of the sphere. By $x^\mu x^\mu$, it is understood that the contraction is performed using the flat space metric, $x^\mu x^\mu = \delta_{\mu\nu}x^\mu x^\nu$. The S^2 subspace is taken to be along the 1-2 plane. When needed, we split the coordinates and indices into parallel and orthogonal pairs, and write the indices as $i, j = 1, 2$, and $\tilde{i}, \tilde{j} = 3, 4$. The coordinates x^i on S^2 are given by inclusion. The induced metric is $g_{ij} = e^{2\Omega}\delta_{ij}$, with conformal factor $e^{-\Omega} = 1 + \frac{x^i x^i}{4r^2}$.

On S^4, we use the Weyl spinor formalism. The matrices $(\sigma_g^\mu)_{\alpha\dot\alpha}$, $(\bar\sigma_g^\mu)^{\dot\alpha\alpha}$ satisfy $\{\sigma_g^\mu, \bar\sigma_g^\nu\} = 2g^{\mu\nu}$. Since the space is conformally flat, they are simply related to the flat space matrices σ^μ, $\bar\sigma^\mu$ (anticommuting to $\delta_{\mu\nu}$), by $\sigma_g^\mu = e^{-\Omega}\sigma^\mu$, $\bar\sigma_g^\mu = e^{-\Omega}\bar\sigma^\mu$. We take a basis in which the flat space matrices are given by $\sigma^4 = \bar\sigma^4 = 1$ and $\sigma^m = -\bar\sigma^m = -i\tau^m$, $m = 1, 2, 3$, where τ^m are the Pauli matrices. Spinor indices are raised and lowered by the charge conjugation matrices C and $\tilde C$ in the form $\lambda_\alpha = C_{\alpha\beta}\lambda^\beta$, $\lambda^\alpha = \tilde C^{\alpha\beta}\lambda_\beta$ (and similarly for right-handed indices). The charge conjugation matrices are antisymmetric and satisfy $C_{\alpha\gamma}\tilde C^{\gamma\beta} = \delta_\alpha^\beta$. By convention we take $C_{21} = \tilde C^{12} = -C_{12} = -\tilde C^{21} = 1$.

The two dimensional spinors are taken to be Dirac spinors, and the Dirac matrices $(\gamma_g^i)_a{}^b$ satisfy the Clifford algebra $\{\gamma_g^i, \gamma_g^j\} = 2g^{ij}$. As in the four dimensional case, they can be expressed in terms of the flat space Dirac matrices γ^i as $\gamma_g^i = e^{-\Omega}\gamma^i$. The chirality matrix is $\gamma^3 = -i\gamma^1\gamma^2$. We take a basis in which γ^m ($m = 1, 2, 3$) are numerically equal to the Pauli matrices τ^m. Spinor indices are raised and lowered as four dimensional spinors.

3.B Conformal Killing spinors and supersymmetry

In this section we describe the supersymmetry algebras on S^4 and S^2, using conformal Killing spinors. We also relate the two algebras by describing the embedding of the 2d algebra inside the 4d algebra.

3.B.1 Conformal Killing spinors on S^4 and S^2

Conformal Killing spinor ϵ, $\bar\epsilon$ in four dimensions are solutions of the equations

$$\begin{aligned} \nabla_\mu\epsilon &= \sigma_{g\mu}\bar\epsilon', \qquad & \nabla_\mu\bar\epsilon' &= -\tfrac{1}{4r^2}\bar\sigma_{g\mu}\epsilon, \\ \nabla_\mu\bar\epsilon &= \bar\sigma_{g\mu}\epsilon', \qquad & \nabla_\mu\epsilon' &= -\tfrac{1}{4r^2}\sigma_{g\mu}\bar\epsilon, \end{aligned} \tag{3.29}$$

where ϵ', $\bar\epsilon'$ are some auxiliary spinors. The solutions to these equations are

$$\epsilon = e^{\frac{1}{2}\Omega}(\epsilon_0 + x^\mu\sigma^\mu\bar\epsilon_1), \qquad \bar\epsilon = e^{\frac{1}{2}\Omega}(\bar\epsilon_0 + x^\mu\bar\sigma^\mu\epsilon_1), \tag{3.30}$$

where ϵ_0, ϵ_1, $\bar\epsilon_0$, $\bar\epsilon_1$ are arbitrary constant spinors, and can be obtained using the flat space ($r \to \infty$) solution together with Weyl covariance.

Conformal Killing spinors in two dimensions work exactly as in four dimension. In this case, the equations are

$$\nabla_i\varepsilon = \gamma_{gi}\varepsilon', \qquad \nabla_i\varepsilon' = -\tfrac{1}{4r^2}\gamma_{gi}\varepsilon, \tag{3.31}$$

and are solved by

$$\varepsilon = e^{\frac{1}{2}\Omega}(\varepsilon_0 + x^i\gamma^i\varepsilon_1) \tag{3.32}$$

3.B.2 $\mathcal{N} = 2$ supersymmetry on S^4

The supersymmetry algebra on S^4 is most conveniently obtained as a subalgebra of the superconformal algebra, in which the spacetime transformations are restricted to the isometries. In the following we describe this process through the realization of the algebra on fields and spacetime.

$\mathcal{N} = 2$ superconformal transformations are realized through $SU(2)$ doublets of conformal Killing spinors ϵ^A, $\bar{\epsilon}^A$. For the vector multiplet, the superconformal transformations are given by (3.6), and the algebra is realized on the vector multiplet as

$$[\delta_{\bar{\epsilon}} + \delta_{\bar{\epsilon}}, \delta_\eta + \delta_{\bar{\eta}}] = \mathcal{L}_v + \mathcal{G}(-iv\cdot A + \Phi) + \omega\Omega + \tilde{\Theta}\tilde{R} + \Theta_{AB}R^{AB}, \tag{3.33}$$

where $\mathcal{L}_v$ is a lie derivative,$\mathcal{G}$ is a gauge transformation, Ω is a Weyl transformation, and $\tilde{R}$, R^{AB} are $U(1)$ and $SU(2)$ R-symmetry transformations. The various parameters are given by

$$\begin{aligned}
v^\mu &= 2i\epsilon^A\sigma^\mu_g\bar{\eta}_A - (\epsilon \leftrightarrow \eta), \qquad \omega = \tfrac{1}{4}\nabla_\mu v^\mu,\\
\tilde{\Theta} &= \tfrac{i}{4}(\epsilon^A\sigma^\mu_g\nabla_\mu\bar{\eta}_A - \nabla_\mu\epsilon^A\sigma^\mu_g\bar{\eta}_A) - (\epsilon \leftrightarrow \eta),\\
\Theta_{AB} &= i(\epsilon_{(A}\sigma^\mu_g\nabla_\mu\bar{\eta}_{B)} - \nabla_\mu\epsilon_{(A}\sigma^\mu_g\bar{\eta}_{B)}) - (\epsilon \leftrightarrow \eta),\\
\Phi &= -4\epsilon^A\eta_A\bar{\phi} + 4\bar{\epsilon}^A\bar{\eta}_A\phi
\end{aligned} \tag{3.34}$$

In particular, the spacetime transformations are generated by the conformal Killing vector v. To restrict to supersymmetry, we restrict the set of allowed conformal Killing spinors in such a way that v is a Killing vector, i.e. it generates only isometries of the sphere. We expand

$$\begin{aligned}
v &= 2i\epsilon_0^A\sigma^\mu\bar{\eta}_{0A}\partial_\mu + 2i(\epsilon_0^A\eta_{1A} - \bar{\epsilon}_1^A\bar{\eta}_{0A})x^\mu\partial_\mu - 2i(\epsilon_0^A\sigma^{\mu\nu}\eta_{1A} + \bar{\epsilon}_1^A\bar{\sigma}^{\mu\nu}\bar{\eta}_{0A})x_\mu\partial_\nu\\
&\quad + 2i\bar{\epsilon}_1^A\bar{\sigma}^\mu\eta_{1A}(x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu) - (\epsilon \leftrightarrow \eta)\\
&= -2\epsilon_0^A\sigma^\mu\bar{\eta}_{0A}P_\mu + 2(\epsilon_0^A\eta_{1A} - \bar{\epsilon}_1^A\bar{\eta}_{0A})D + (\epsilon_0^A\sigma^{\mu\nu}\eta_{1A} + \bar{\epsilon}_1^A\bar{\sigma}^{\mu\nu}\bar{\eta}_{0A})M_{\mu\nu}\\
&\quad - 2\bar{\epsilon}_1^A\bar{\sigma}^\mu\eta_{1A}K_\mu - (\epsilon \leftrightarrow \eta),
\end{aligned} \tag{3.35}$$

where P_μ, D, $M_{\mu\nu}$, K_μ are the generators of the conformal transformations in the scalar representation. The $SO(5,1)$ symmetry can be made manifest by defining

$$M_{\mu 5} = nP_\mu - mK_\mu, \qquad M_{\mu 6} = nP_\mu + mK_\mu, \qquad M_{56} = D, \tag{3.36}$$

where $n = \cosh\alpha$, $m = \sinh\alpha$ for some hyperbolic angle α.

$$\begin{aligned} v = &- \left[\tfrac{1}{n}\epsilon_0^A \sigma^\mu \bar\eta_{0A} + \tfrac{1}{m}\bar\epsilon_1^A \bar\sigma^\mu \eta_{1A}\right] M_{\mu 5} - \left[\tfrac{1}{n}\epsilon_0^A \sigma^\mu \bar\eta_{0A} - \tfrac{1}{m}\bar\epsilon_1^A \bar\sigma^\mu \eta_{1A}\right] M_{\mu 6} \\ &+ (\epsilon_0^A \sigma^{\mu\nu}\eta_{1A} + \bar\epsilon_1^A \bar\sigma^{\mu\nu}\bar\eta_{0A}) M_{\mu\nu} + 2(\epsilon_0^A \eta_{1A} - \bar\epsilon_1^A \bar\eta_{0A}) M_{56} - (\epsilon \leftrightarrow \eta) \end{aligned} \tag{3.37}$$

For the sphere, the isometry group is $SO(5)$, which is made manifest as a subgroup of $SO(5,1)$ by setting $\tanh\alpha = \frac{m}{n} = \frac{1}{4r^2}$. For this choice of angle, the generators $M_{\mu 6}$ and M_{56} are not allowed. This imposes the conditions

$$\begin{aligned} (\epsilon_1^A \sigma^\mu \bar\eta_{1A} - \eta_1^A \sigma^\mu \bar\epsilon_{1A}) &= -\tfrac{1}{4r^2}(\epsilon_0^A \sigma^\mu \bar\eta_{0A} - \eta_0^A \sigma^\mu \bar\epsilon_{0A}), \\ \epsilon_0^A \eta_{1A} + \epsilon_1^A \eta_{0A} &= \bar\epsilon_1^A \bar\eta_{0A} + \bar\epsilon_0^A \bar\eta_{1A}, \end{aligned} \tag{3.38}$$

which are satisfied by

$$\begin{aligned} \epsilon_1^A = \tfrac{1}{2r}(\tau^3)^A{}_B \epsilon_0^B, \qquad \bar\epsilon_1^A = -\tfrac{1}{2r}(\tau^3)^A{}_B \bar\epsilon_0^B, \\ \text{or} \qquad \nabla_\mu \epsilon^A = -\tfrac{1}{2r}\sigma_{g\mu}(\tau^3)^A{}_B \bar\epsilon^B, \qquad \nabla_\mu \bar\epsilon^A = \tfrac{1}{2r}\bar\sigma_{g\mu}(\tau^3)^A{}_B \epsilon^B. \end{aligned} \tag{3.39}$$

(and similarly for η^A, $\bar\eta^A$). After imposing these conditions, we obtain $\mathcal{N} = 2$ supersymmetry, and the Killing vector is given by

$$v = -\tfrac{2}{n}(\epsilon_0^A \sigma^\mu \bar\eta_{0A} - \eta_0^A \sigma^\mu \bar\epsilon_{0A}) M_{\mu 5} - \tfrac{1}{r}(\tau^3)^B{}_A (\epsilon_0^A \sigma^{\mu\nu}\eta_{0B} + \bar\epsilon_0^A \bar\sigma^{\mu\nu}\bar\eta_{0B}) M_{\mu\nu}. \tag{3.40}$$

The other parameters in (3.34) reduce to

$$\begin{aligned} \omega = \tilde\Theta = \Theta_{11} = \Theta_{22} = 0, \\ \Theta_{12} = 2\Theta = -\tfrac{2i}{r}(\epsilon^A \eta_A - \bar\epsilon^A \bar\eta_A), \\ \Phi = -4\epsilon^A \eta_A \bar\phi + 4\bar\epsilon^A \bar\eta_A \phi, \end{aligned} \tag{3.41}$$

and the algebra simplifies to

$$[\delta_{\bar\epsilon} + \delta_{\bar\epsilon}, \delta_\eta + \delta_{\bar\eta}] = \mathcal{L}_v + \mathcal{G}(-iv\cdot A + \Phi) + \Theta R, \tag{3.42}$$

where $R = 4R^{12}$ is the unbroken R-charge.

The coupling of the theory to matter on a S^2 surface breaks the spacetime symmetry of the theory to $SO(3) \times SO(2)_\perp$. Here $SO(3)$ represents the isometries of S^2, i.e. rotations

and translations on the 1-2 plane, and $SO(2)_\perp$ represents the rotations on the 3-4 plane. Hence to obtain the set of unbroken supersymmetries, we need a subset of the Killing spinors which does not generate $M_{\tilde{i}5}$ or $M_{i\tilde{j}}$. This requirement imposes the conditions

$$\epsilon_0^A \sigma^{\tilde{i}} \bar{\eta}_{0A} - \eta_0^A \sigma^{\tilde{i}} \bar{\epsilon}_{0A} = 0, \qquad \epsilon_0^A \sigma^{i\tilde{j}} \eta_{0B} + \bar{\epsilon}_0^A \bar{\sigma}^{i\tilde{j}} \bar{\eta}_{0B} = 0, \tag{3.43}$$

which can be satisfied by requiring the spinors to have a definite "two dimensional chirality". We choose them to satisfy

$$\begin{aligned} (-i\sigma^{12})\epsilon_0^1 = +\epsilon_0^1, \qquad (-i\bar{\sigma}^{12})\bar{\epsilon}_0^1 = -\bar{\epsilon}_0^1, \\ (-i\sigma^{12})\epsilon_0^2 = -\epsilon_0^2, \qquad (-i\bar{\sigma}^{12})\bar{\epsilon}_0^2 = +\bar{\epsilon}_0^2 \end{aligned} \tag{3.44}$$

(and similarly for η^A, $\bar{\eta}^A$). The Killing spinor then simplifies to

$$v = \tfrac{4}{n}(\epsilon_0^{[1}\sigma^i\bar{\eta}_0^{2]} - \eta_0^{[1}\sigma^i\bar{\epsilon}_0^{2]})M_{i5} + \tfrac{4i}{r}\left(\epsilon_0^{[1}\eta_0^{2]} - \bar{\epsilon}_0^{[1}\bar{\eta}_0^{2]}\right)M_{12} + \tfrac{4i}{r}\left(\epsilon_0^{[1}\eta_0^{2]} + \bar{\epsilon}_0^{[1}\bar{\eta}_0^{2]}\right)M_{34} \tag{3.45}$$

3.B.3 $\mathcal{N} = (2,2)$ supersymmetry on S^2

For supersymmetry on S^2, we proceed exactly as in the S^4 case. $\mathcal{N} = (2,2)$ supersymmetry are realized through a pair of real conformal Killing spinors ε, $\bar{\varepsilon}$, and the algebra is realized as

$$[\delta_{\bar{\varepsilon}} + \delta_{\bar{\varepsilon}}, \delta_\chi + \delta_{\bar{\chi}}] = \mathcal{L}_v + \mathcal{G}(-iv\cdot A + \Phi) + \omega\Omega + \Theta R + \tilde{\Theta}\mathcal{A} + \alpha m, \tag{3.46}$$

where R and $\mathcal{A}$ are R-symmetry transformations, m is the mass of the multiplet, and the parameters are

$$\begin{aligned} v^\mu &= -i\varepsilon\gamma_g^i\bar{\eta} - (\varepsilon \leftrightarrow \chi), \qquad \omega = \tfrac{1}{4}\nabla_\mu v^\mu, \\ \Theta &= -\tfrac{i}{4}(\nabla_i\varepsilon\gamma_g^i\bar{\chi} - \varepsilon\gamma_g^i\nabla_i\bar{\chi}) - (\varepsilon \leftrightarrow \chi) \\ \tilde{\Theta} &= \tfrac{i}{4}(\nabla_i\varepsilon\gamma^3\gamma_g^i\bar{\chi} - \varepsilon\gamma^3\gamma_g^i\nabla_i\bar{\chi}) - (\varepsilon \leftrightarrow \chi), \\ \Phi &= i\varepsilon\bar{\chi}\sigma_1 - \varepsilon\gamma_g^3\bar{\chi}\sigma_2 - (\varepsilon \leftrightarrow \chi), \quad \alpha = -\varepsilon\gamma^3\bar{\chi} - (\varepsilon \leftrightarrow \chi). \end{aligned} \tag{3.47}$$

The conformal Killing vector v expands as

$$\begin{aligned} v &= \varepsilon_0\gamma^i\bar{\chi}_0 P_i - (\varepsilon_0\bar{\chi}_1 - \varepsilon_1\bar{\chi}_0)D - \tfrac{1}{2}(\varepsilon_0\gamma^{ij}\bar{\chi}_1 + \varepsilon_1\gamma^{ij}\bar{\chi}_0)M_{ij} + \varepsilon_1\gamma^i\bar{\chi}_1 K_i - (\varepsilon \leftrightarrow \chi) \\ &= \tfrac{1}{2}\left[\tfrac{1}{n}\varepsilon_0\gamma^i\bar{\chi}_0 - \tfrac{1}{m}\varepsilon_1\gamma^i\bar{\chi}_1\right]M_{i5} + \tfrac{1}{2}\left[\tfrac{1}{n}\varepsilon_0\gamma^i\bar{\chi}_0 + \tfrac{1}{m}\varepsilon_1\gamma^i\bar{\chi}_1\right]M_{i6} \\ &\quad - \tfrac{1}{2}(\varepsilon_0\gamma^{ij}\bar{\chi}_1 + \varepsilon_1\gamma^{ij}\bar{\chi}_0)M_{ij} - (\varepsilon_0\bar{\chi}_1 - \varepsilon_1\bar{\chi}_0)M_{56} - (\varepsilon \leftrightarrow \chi), \end{aligned} \tag{3.48}$$

where we used the restriction of (3.36) to make the $SO(3,1)$ symmetry manifest. We reduce the symmetry group to $SO(3)$ by imposing

$$\begin{aligned} & \varepsilon_1 = \tfrac{1}{2r}\gamma^3\varepsilon_0, \qquad \bar{\varepsilon}_1 = -\tfrac{1}{2r}\gamma^3\bar{\varepsilon}_0, \\ \text{or} \qquad & \nabla_i\varepsilon = +\tfrac{1}{2r}\gamma_{gi}\gamma^3\varepsilon, \qquad \nabla_i\bar{\varepsilon} = -\tfrac{1}{2r}\gamma_{gi}\gamma^3\bar{\varepsilon}. \end{aligned} \tag{3.49}$$

Under these constraints, the Killing vector is

$$\begin{aligned} v &= \tfrac{1}{n}(\varepsilon_0\gamma^i\bar{\chi}_0 + \bar{\varepsilon}_0\gamma^i\chi_0)M_{i5} + \tfrac{1}{2r}(\varepsilon_0\gamma^{ij}\gamma^3\bar{\chi}_0 - \bar{\varepsilon}_0\gamma^{ij}\gamma^3\chi_0)M_{ij} \\ &= \tfrac{1}{n}(\varepsilon_0\gamma^i\bar{\chi}_0 + \bar{\varepsilon}_0\gamma^i\chi_0)M_{i5} + \tfrac{i}{r}(\varepsilon_0\bar{\chi}_0 - \bar{\varepsilon}_0\chi_0)M_{12}, \end{aligned} \tag{3.50}$$

and the other parameters of the algebra simplify to

$$\omega = \tilde{\Theta} = 0, \quad \Theta = \tfrac{i}{2r}\alpha \tag{3.51}$$

3.B.4 Relating the spinor formalisms

We now proceed to relate the two spinor formalisms introduced for $\mathcal{N} = (2,2)$ supersymmetry. This amounts to matching the conformal Killing spinors on both sides. We recall that on one side we have four four dimensional Weyl spinors ϵ_0^A, $\bar{\epsilon}_0^A$ of definite two dimensional chirality, and on the other we have two unconstrained two dimensional Dirac spinors ε_0, $\bar{\varepsilon}_0$. To proceed, we write the spinors in components in chiral bases, which gives four elementary spinors on both sides:

$$(\epsilon_0^1)_1, (\epsilon_0^2)_2, (\bar{\epsilon}_0^1)^2, (\bar{\epsilon}_0^2)^1, \quad \text{and} \quad (\varepsilon_0)_1, (\varepsilon_0)_2, (\bar{\varepsilon}_0)_1, (\bar{\varepsilon}_0)_2. \tag{3.52}$$

The matching must respect the supersymmetry algebra, i.e. the Killing vector v^μ must be preserved on $\mathcal{S}^2$. This can be achieved by setting

$$(\epsilon_0^1)_1 = \tfrac{1}{\sqrt{2}}(\varepsilon_0)_1, \quad (\epsilon_0^2)_2 = \tfrac{1}{\sqrt{2}}(\bar{\varepsilon}_0)_2, \quad (\bar{\epsilon}_0^1)^2 = -\tfrac{i}{\sqrt{2}}(\varepsilon_0)_2, \quad (\bar{\epsilon}_0^2)^1 = \tfrac{i}{\sqrt{2}}(\bar{\varepsilon}_0)_1. \tag{3.53}$$

This form allows to relate the nonzero components of the conformal Killing spinors by

$$(\epsilon^1)_1 = \tfrac{1}{\sqrt{2}}\varepsilon_1, \quad (\epsilon^2)_2 = \tfrac{1}{\sqrt{2}}\bar{\varepsilon}_2, \quad (\bar{\epsilon}^1)^2 = -\tfrac{i}{\sqrt{2}}\varepsilon_2, \quad (\bar{\epsilon}^2)^1 = \tfrac{i}{\sqrt{2}}\bar{\varepsilon}_1. \tag{3.54}$$

3.C Decomposition of the vector multiplet on S^4

In this section we decompose the $\mathcal{N} = 2$ vector multiplet into representations of the $\mathcal{N} = (2,2)$ superconformal algebra. We focus on the $\mathcal{N} = (2,2)$ vector multiplet, for which

an exact expression is needed to the coupling to the defect. Our goal is to find a set of fields $(A_i, \sigma_1, \sigma_2, D, \lambda, \bar{\lambda})$ among these which transform as a $\mathcal{N} = (2,2)$ vector multiplet, in the form

$$\begin{aligned}
\delta_\varepsilon A_i &= -\tfrac{i}{2}(\bar{\varepsilon}\gamma_{gi}\lambda + \varepsilon\gamma_{gi}\bar{\lambda}),\\
\delta_\varepsilon \sigma_1 &= \tfrac{1}{2}(\bar{\varepsilon}\lambda - \varepsilon\bar{\lambda}),\\
\delta_\varepsilon \sigma_2 &= -\tfrac{i}{2}(\bar{\varepsilon}\gamma^3\lambda + \varepsilon\gamma^3\bar{\lambda}),\\
\delta_\varepsilon \lambda &= (\tfrac{1}{2}\gamma_g^{ij}F_{ij} - \gamma^3\gamma_g^i\mathcal{D}_i\sigma_2 + i\gamma_g^i\mathcal{D}_i\sigma_1 - \gamma^3[\sigma_1,\sigma_2] - D)\varepsilon + i(\sigma_1 + i\sigma_2\gamma^3)\gamma_g^i\nabla_i\varepsilon,\\
\delta_\varepsilon \bar{\lambda} &= (\tfrac{1}{2}\gamma_g^{ij}F_{ij} - \gamma^3\gamma_g^i\mathcal{D}_i\sigma_2 - i\gamma_g^i\mathcal{D}_i\sigma_1 - \gamma^3[\sigma_1,\sigma_2] + D)\bar{\varepsilon} - i(\sigma_1 - i\sigma_2\gamma^3)\gamma_g^i\nabla_i\varepsilon,\\
\delta_\varepsilon D &= \tfrac{i}{2}\mathcal{D}_i(\varepsilon\gamma_g^i\bar{\lambda} - \bar{\varepsilon}\gamma_g^i\lambda) - \tfrac{i}{2}\left[\sigma_1, \varepsilon\bar{\lambda} + \bar{\varepsilon}\lambda\right] + \tfrac{1}{2}\left[\sigma_2, \varepsilon\gamma^3\bar{\lambda} - \bar{\varepsilon}\gamma^3\lambda\right].
\end{aligned} \tag{3.55}$$

For the gauge field, the identification $A_i^{\mathcal{N}=(2,2)} = A_i^{\mathcal{N}=2}$ is obvious. We evaluate

$$\begin{aligned}
\delta_\epsilon A_i &= i\epsilon^A\sigma_\mu\bar{\lambda}_A - i\bar{\epsilon}^A\bar{\sigma}_\mu\lambda_A\\
&= \tfrac{i}{\sqrt{2}}\varepsilon^1(\gamma_{gi})_1{}^2(\lambda_1)_2 + \tfrac{1}{\sqrt{2}}\varepsilon^2(\gamma_{gi})_2{}^1(\bar{\lambda}_1)^1 + \tfrac{1}{\sqrt{2}}\bar{\varepsilon}^1(\gamma_{gi})_1{}^2(\bar{\lambda}_2)^2 - \tfrac{i}{\sqrt{2}}\bar{\varepsilon}^2(\gamma_{gi})_2{}^1(\lambda_2)_1,
\end{aligned} \tag{3.56}$$

which implies the identifications

$$(\lambda_1)_2 = -\tfrac{1}{\sqrt{2}}\bar{\lambda}_2, \quad (\lambda_2)_1 = \tfrac{1}{\sqrt{2}}\lambda_1, \quad (\bar{\lambda}_1)^1 = -\tfrac{i}{\sqrt{2}}\bar{\lambda}_1, \quad (\bar{\lambda}_2)^2 = -\tfrac{i}{\sqrt{2}}\lambda_2. \tag{3.57}$$

Under these identifications, the scalars ϕ, $\bar{\phi}$ transform as

$$\begin{aligned}
\delta_\epsilon\phi &= -i\epsilon^A\lambda_A = \tfrac{i}{2}\varepsilon^2\bar{\lambda}_2 - \tfrac{i}{2}\bar{\varepsilon}^1\lambda_1,\\
\delta_\epsilon\bar{\phi} &= i\bar{\epsilon}^A\bar{\lambda}_A = \tfrac{i}{2}\varepsilon^1\bar{\lambda}_1 - \tfrac{i}{2}\bar{\varepsilon}^2\lambda_2,
\end{aligned} \tag{3.58}$$

implying $\sigma_1 = i(\phi + \bar{\phi})$, $\sigma_2 = \phi - \bar{\phi}$. The auxiliary field D_{12} transforms as

$$\begin{aligned}
\delta_\epsilon D_{12} &= -\tfrac{i}{2}(\varepsilon\gamma_g^i\mathcal{D}_i\bar{\lambda} - \bar{\varepsilon}\gamma_g^i\mathcal{D}_i\lambda) + \tfrac{i}{2}\left[\sigma_1, \varepsilon\bar{\lambda} + \bar{\varepsilon}\lambda\right] - \tfrac{1}{2}\left[\sigma_2, \varepsilon\gamma^3\bar{\lambda} - \bar{\varepsilon}\gamma^3\lambda\right] + i\delta(F_{34})\\
&= -\delta(D + \tfrac{1}{r}\sigma_2 + iF_{34}),
\end{aligned} \tag{3.59}$$

implying the identification

$$D = -D_{12} - \tfrac{1}{r}(\phi - \bar{\phi}) - iF_{34}. \tag{3.60}$$

A similar computation can be done to show that the supersymmetry transformations of the fermions are consistent with the above identifications.

3.D Restriction in the superfield formalism and supersymmetric couplings

In this section we present an alternative description of the theory, based on the superfield formalism in flat space. This description has the advantage of making explicit the possible supersymmetric couplings between the 2d and 4d multiplets. It is equivalent to the $S^2 \subset S^4$ case by classical conformal invariance, which allows to relate the Lagrangians and supersymmetry transformations in the two cases.

We first write down the vector multiplet on $\mathbb{R}^4$ in terms of representations of $\mathcal{N} = (2,2)$ supersymmetry. It takes the form of a chiral multiplet $\Phi = \Phi(x^i, x^{\tilde{i}}, \theta, \bar{\theta})$ coupled to a vector multiplet $V = V(x^i, x^{\tilde{i}}, \theta, \bar{\theta})$. The symmetry in the directions orthogonal to the defect is encoded in the gauge transformations $\delta_\Lambda V = i(\Lambda - \Lambda^\dagger)$, $\delta_\Lambda \Phi = 2\sqrt{2}\partial_+ \Lambda$, where $\Lambda = \Lambda(x^i, x^{\tilde{i}}, \theta, \bar{\theta})$ is a chiral field, and $\partial_\pm = \frac{1}{2}(\partial_3 \pm i\partial_4)$. The four dimensional Lagrangian

$$\mathcal{L}_{\mathbb{R}^4} = \int d^4\theta \left((\Phi^\dagger - 2\sqrt{2}i\partial_- V)(\Phi + 2\sqrt{2}i\partial_+ V) - 2\partial_+ V \partial_- V \right) + \int d^4\theta W^2 + \int d^4\bar{\theta} \bar{W}^2 \tag{3.61}$$

is gauge invariant, and can be shown to reproduce that of a $\mathcal{N} = 2$ vector multiplet on $\mathbb{R}^4$. The restriction to $\mathbb{R}^2$ contains the fundamental fields Φ and V at $x^{\tilde{i}} = 0$, but also their derivatives on the defect. Therefore the set of restricted fields consists of an infinite tower of vector multiplets $V_R^{(m,n)} = (\partial_+^m \partial_-^n V)(x^{\tilde{i}} = 0)$, and a tower of chiral multiplets $\Phi_R^{(m,n)} = (\partial_+^m \partial_-^n \Phi)(x^{\tilde{i}} = 0)$. In particular, the defect sees infinitely many copies of the gauge group G, which we label by $G^{(m,n)}$. Also in the Wess-Zumino gauge the different chiral multiplets mix under supersymmetry.

Given a chiral multiplet Ψ, the most general supersymmetric (and gauge invariant) coupling to the vector multiplet is obtained by picking a representation $R^{(m,n)}$ for each of the gauge groups. The Lagrangian may also include a superpotential $\mathcal{W} = \mathcal{W}(\Psi)$ (The superpotential cannot depend on $\Phi_R^{(m,n)}$ by gauge invariance), and twisted masses. Twisted masses are obtained by coupling the chiral multiplet to a (2d) vector multiplet frozen to its vacuum expectation value $\tilde{V}$. In this paper we are interested in a local, renormalizable interaction term, which preserves the $SO(2)_\perp$ symmetry between the transverse directions. Locality requires the interaction to involve only a finite number of multiplets, and by $SO(2)_\perp$ symmetry we must restrict to those with $m = n$. By renormalizability all the representations other than $R^{(0,0)}$ must be trivial. Therefore the most general allowed two

dimensional Lagrangian is of the form

$$\mathcal{L}_{\mathbb{R}^2} = \int d^4\theta\ \Psi^\dagger \exp(V_R^{(0,0)} + \tilde{V})\Psi + \int d^2\theta\ \mathcal{W}(\Psi) + c.c. \tag{3.62}$$

3.E One-loop determinants of the chiral multiplet from index theorem

We compute the one-loop determinant of the chiral multiplet using the equivariant index theorem for transversally elliptic operators . For our purpose, the statement of the index theorem is as follow [10, 81]. Let E_0, E_1 be a pair of vector bundles over a manifold M, and G a compact Lie group acting on the bundles and the manifold. Let $D : \Gamma(E_0) \to \Gamma(E_1)$ map sections of the bundles and commute with the action of G. We also require D to be transversally elliptic[5]. The index of D is defined as

$$\mathrm{ind}D(\hat{t}) = \mathrm{tr}_{\mathrm{Ker}D}\hat{t} - \mathrm{tr}_{\mathrm{Coker}D}\hat{t}, \tag{3.63}$$

where $\hat{t}$ is an element of the maximal torus of G. For a transversally elliptic operator, the Kernel and Cokernel are both infinite dimensional, but can be decomposed as a sum of irreducible representations of G, each appearing with a finite multiplicity. Therefore we can expand the index in formal series. For $G = U(1)$, $\hat{t} = e^{i\epsilon\hat{g}} = t^g$, where $\epsilon \in \mathbb{R}$, and g is the generator of the Lie algebra $u(1)$, and we can expand $\mathrm{ind}D(\hat{t}) = \sum_i c_i t^{w_i}$. However, the expansion is not unique, and some care must be taken in choosing the appropriate expansion. Assuming that G has a discrete set of fixed points F, the index theorem gives the index of D as a sum over F:

$$\mathrm{ind}D(\hat{t}) = \sum_{p\in F} \frac{\mathrm{tr}_{E_0(p)}\hat{t} - \mathrm{tr}_{E_1(p)}\hat{t}}{\det_{T_pM}(1-\hat{t})}. \tag{3.64}$$

To use the index theorem, we first build a $\mathcal{Q}$-complex from the fields. We write the fields of each multiplet as a pair of (sets of) bosons $\{\Phi, \tilde{\Phi}\}$ and a pair of (sets of) fermions $\{\Psi, \tilde{\Psi}\}$, such that $\tilde{\Psi} = \mathcal{Q}\Phi$, $\tilde{\Phi} = \mathcal{Q}\Psi$. The fields are sections of vector bundles, which we write

[5] A differential operator D is said to be transversally elliptic if its symbol is invertible for sections of the cotangent bundle T^*M transversal to the G-orbit. In a local coordinate frame, the symbol is obtained from the highest order part of D by replacing partial derivatives at each point x by momenta, $\partial_i \to ip_i$, where $\{p_i\}$ is set of coordinates for a point p on T^*_xM. For transversal ellipticity, we require the symbol to be invertible for all p in the subspace of T^*_xM transversal to the G-orbit at each point $x \in M$ (in any coordinate frame).

$\{E_\Phi, E_{\tilde{\Phi}}\}$, $\{E_\Psi, E_{\tilde{\Psi}}\}$. Then $\mathcal{Q}^2$ maps each bundle to itself, and we can take G to be the $U(1)$ Lie group generated by $\mathcal{Q}^2$. The quadratic part of the deformation term $\mathcal{V}_q$ in the form

$$\mathcal{V}_q = \begin{pmatrix} \tilde{\Psi} & \Psi \end{pmatrix} \cdot \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \cdot \begin{pmatrix} \Phi \\ \tilde{\Phi} \end{pmatrix} = \mathbf{\Psi}^\dagger \cdot \mathbf{D} \cdot \mathbf{\Phi}. \tag{3.65}$$

This construction gives a smooth linear map $D_{10} : \Gamma(E_\Phi) \to \Gamma(E_\Psi)$. In the following, we assume that D_{10} commutes with the action of $\mathcal{Q}$, allowing us to use the index theorem. The quadratic part of the Lagrangian can then be written as

$$\mathcal{Q}\mathcal{V}_q = (\mathcal{Q}\mathbf{\Psi}) \cdot \mathbf{D} \cdot \mathbf{\Phi} + \mathbf{\Psi} \cdot \mathbf{D} \cdot (\mathcal{Q}\mathbf{\Phi}) = \mathbf{\Phi} \cdot \boldsymbol{\mathcal{Q}}_\mathbf{R}^\mathbf{2} \cdot \mathbf{D} \cdot \mathbf{\Phi} + \mathbf{\Psi} \cdot \mathbf{D} \cdot \boldsymbol{\mathcal{Q}}^\mathbf{2} \cdot \mathbf{\Psi}, \tag{3.66}$$

where $\boldsymbol{\mathcal{Q}}^2 = \mathrm{diag}(1, \mathcal{Q}^2)$, $\boldsymbol{\mathcal{Q}}_R^2 = \mathrm{diag}(\mathcal{Q}_R^2, 1)$, and $\mathcal{Q}_R$ is the supercharge $\mathcal{Q}$ acting on the right. Then for real fields, the one-loop determinant can be written as

$$Z_{1-\mathrm{loop}} = \sqrt{\frac{\det(\mathbf{D} \cdot \boldsymbol{\mathcal{Q}}^\mathbf{2})}{\det(\boldsymbol{\mathcal{Q}}_R^2 \cdot \mathbf{D})}} = \sqrt{\frac{\det_{\mathbf{\Psi}} \mathcal{Q}^2}{\det_{\mathbf{\Phi}} \mathcal{Q}^2}} = \sqrt{\frac{\det_{\mathrm{Coker} D_{10}} \mathcal{Q}^2}{\det_{\mathrm{Ker} D_{10}} \mathcal{Q}^2}}, \tag{3.67}$$

where the last equality follows from the assumption that D_{10} commutes with $\mathcal{Q}$, therefore it relates the action of $\mathcal{Q}$ on both bundles outside its kernel and cokernel. For complex fields, square root is absent. The ratio of determinant can be obtained from the equivariant index through

$$\mathrm{ind} D_{10}(t) = \sum_i c_i t^{w_i} \iff \frac{\det_{\mathrm{Coker} D_{10}} \mathcal{Q}^2}{\det_{\mathrm{Ker} D_{10}} \mathcal{Q}^2} = \prod_i w_i^{-c_i}. \tag{3.68}$$

3.E.1 The chiral multiplet

We now proceed to compute the one-loop determinant of the chiral multiplet using the above method, as done in [12]. Here we assume that the operator D_{10} is transversally elliptic, and avoid computing its exact form. We take the $\mathcal{Q}$-complex defined by $\Phi = \phi$, $\varepsilon_\mathcal{Q}\Psi = \varepsilon\gamma^3\psi$. The two fixed points are the north and south poles. Near the north pole $\mathcal{Q}^2$ acts in complex coordinates as $\mathcal{Q}^2(z, \bar{z}) = (z, -\bar{z})$, so the denominator is $(1-t)(1-t^{-1})$, and similarly at the south pole. The numerator is obtained from the action of $\mathcal{Q}^2$ at the poles, obtained from

$$\begin{aligned} \mathcal{Q}^2\phi_N &= [\Lambda_N - irM]\phi_N, & \mathcal{Q}^2\phi_S &= [\Lambda_S - irM]\phi_S, \\ \mathcal{Q}^2\varepsilon\psi_N &= [\Lambda_N - irM - 1]\varepsilon\psi_N, & \mathcal{Q}^2\varepsilon\psi_S &= [\Lambda_S - irM - 1]\varepsilon\psi_S, \end{aligned} \tag{3.69}$$

where $\Lambda = -iv\cdot A + r(f(x)\sigma_1 - i\sigma_2)$, $M = m + \frac{i}{2r}q$. Here the fields of the vector multiplet are treated as background fields. More formally, the bundles E_Φ, E_Ψ are isomorphic to $K_{q,m}\otimes\mathcal{R}$ and $S\otimes K_{q-1,m}\otimes\mathcal{R}$, where $K_{q,m}$ is a one-dimensional vector bundle encoding a R-charge q and a mass m, $\mathcal{R}$ is a vector bundle transforming in the representation R of G, and S is a (one-dimensional) subbundle of a Spin-$\frac{1}{2}$ bundle interpolating between definite angular momenta $-\frac{1}{2}$ at the north pole and $+\frac{1}{2}$ at the south pole. The index evaluates to a sum over the roots of R:

$$\mathrm{ind}D_{10}(t) = \left[\sum_{w\in R}\frac{t^{\omega\cdot\Lambda_N - irM}}{1-t}\right]_N + \left[\sum_{w\in R}\frac{t^{\omega\cdot\Lambda_S - irM}}{1-t}\right]_S \tag{3.70}$$

The series expansion at each pole is dictated by the symbol of D_{10} (see [81] for details). Here we need to expand the north pole contribution in powers of t, and the south pole contribution in powers of t^{-1}, giving

$$\mathrm{ind}D_{10}(t) = \sum_{w\in R}\sum_{n=0}^{\infty}\left(t^{n+\omega\cdot\Lambda_N - irM} - (t^{-1})^{n+1-\omega\cdot\Lambda_S + irM}\right) \tag{3.71}$$

From this index we deduce the one-loop determinant (up to an irrelevant phase)

$$Z_{1-\mathrm{loop}} = \prod_{w\in R}\frac{\prod_{n=0}^{\infty}(1+n-\omega\cdot\Lambda_S + irM)}{\prod_{n=0}^{\infty}(n+\omega\cdot\Lambda_N - irM)}. \tag{3.72}$$

We regularize the products according to the formula

$$\prod_{n=0}^{\infty}(n+m) \to \frac{1}{\Gamma(m)}, \tag{3.73}$$

giving the result

$$Z_{1-\mathrm{loop}} = \prod_{w\in R}\frac{\Gamma(\omega\cdot\Lambda_N - irM)}{\Gamma(1-\omega\cdot\Lambda_S + irM)}. \tag{3.74}$$

Chapter 4

Dirac zero modes for Abelian BPS multimonopoles

4.1 Introduction

In the construction of BPS monopoles one usually consider a related Dirac operator $\mathcal{D}^\dagger$ and its normalizable zero modes [74]. In particular the zero modes play an important role in the Nahm transform [71, 73] used for computing the monopoles, where they provide a solution to the Nahm equations. However, while in many cases the form of the monopoles is known, the zero modes themselves remain hard to compute.

In this paper we aim to develop a method for finding the Dirac zero modes, by considering the case of Abelian (singular) BPS monopoles in $\mathbb{R}^3$. These monopoles have a relatively simple field configuration, but their zero modes remain difficult to compute. Only a few results are known, concerning single monopoles [22] (see also [68]; some older references on electron-monopole boundstates include [17, 42, 49, 61]) and certain cases of double monopoles [95]. In this paper we present a general formula for arbitrary (finite) monopole configurations. The formula is explicit (in terms of solutions to a finite set of linear equations) for monopoles of charge ± 1 at generic positions, while solutions for the other cases can be recovered by a limiting process. The formula takes the form of a sum of residues at a certain set of poles, where the location of the poles corresponds to the zeros of algebraic functions.

Our method is based on the fact that a large class of (generally non-normalizable) solutions to the equation $\mathcal{D}^\dagger\Psi = 0$ can be built from flat sections of a Lax pair of connections.

We conjecture that the zero modes can be written as superpositions of such solutions. The conjecture can also be stated in the following way: the zero modes are of the form $\mathcal{D}\chi$, where $\mathcal{D}$ is the adjoint of $\mathcal{D}^\dagger$. In the Abelian case, the flat sections are easily found, so the conjecture reduces the problem to finding which superpositions give the correct zero modes, i.e. which ones are normalizable. It is worth noting that while the conjecture itself appears to be new, some similar ideas can be found in the literature. In [74] the zero modes for smooth monopoles are related to flat sections of the Lax pair in a different way, in terms of a differential equation involving normalizable flat sections. Also the dipole solution in [95] is found in the form $\mathcal{D}\chi$.

Our main motivation for computing Dirac zero modes is to obtain the BPS spectrum of two-dimensional gauged linear sigma models (GLSM), and by extension that of many non-linear sigma models (NLSM). As described in the introduction (section 1.7), the BPS spectrum of GLSM is encoded in a cylindrical geometry, through a topological-anti-topological metric [20] and a supersymmetric index [18] (see also [21]). The BPS spectrum is independent of the radius of the cylinder, and for a small radius we obtain an effective one-dimensional theory, corresponding to the quantum mechanics of vector multiplets in the presence of a background of periodic monopoles. In particular the supersymmetric index of [18] can be computed in terms of the ground states of the resulting theory, which correspond precisely to the Dirac zero modes for periodic monopoles. While in this paper we only consider non-periodic monopoles, we hope that our results can be generalized to the periodic case.

This chapter is organized as follow. In section 4.2 we briefly review some background material on Abelian BPS monopoles and the associated Dirac operator. In section 4.3 we review the basic properties of the Lax pair for the Bogomoln'yi equations, and show how it gives an ansatz for the Dirac zero modes. We continue in section 4.4 with the simplest example, a single monopole of unit charge, and use the ansatz to find the correct zero mode. The result takes the form of a residue at a special value of a spectral parameter. In the following section we show that the formula for a generic monopole configuration with positively charged monopoles can also be expressed as a residue formula. We find the exact formula for monopoles of unit charge at generic positions, and the generalization to other cases is straightforward. We leave the details of the computation for appendix 4.A, and we generalize the results for negative charges in appendix 4.B. In section 4.6 we review some of the implications of our results for the Nahm transform, and give an expression for the solutions of the relevant Nahm equations in terms of an integral over an algebraic variety. We conclude in section 4.7 with some open questions concerning the results of this paper and their possible generalizations.

4.2 BPS monopoles and Dirac zero modes

A BPS monopole configuration on $\mathbb{R}^3$ with gauge group G consists of a vector field $\mathbf{A}(\mathbf{x})$ and a Higgs field $\Phi(\mathbf{x})$, satisfying the Bogomoln'yi equation

$$\mathbf{D}\Phi = *\mathbf{F}, \tag{4.1}$$

where $\mathbf{D} = \mathbf{d} - i\mathbf{A}$ is the connection defined by $\mathbf{A}$, and $\mathbf{F}$ is the curvature of $\mathbf{A}$. The configuration must also be sufficiently regular at infinity. In this paper we restrict the gauge group to $G = U(1)$. In this case a monopole configuration corresponds to a set S of monopoles, described by their charge $q \in \mathbb{Z}^*$ and their location $\mathbf{a} \in \mathbb{R}^3$. We can choose a gauge such that the fields are given by [22, 63]

$$\Phi(\mathbf{x}) = \sum_{m\in S} \frac{q_m}{2r_m}, \qquad \mathbf{A}(\mathbf{x}) = i\sum_{m\in S} \frac{z_m \mathbf{d}\bar{z} - \bar{z}_m \mathbf{d}z}{4r_m(r_m - x_m)}. \tag{4.2}$$

Here we work in the coordinates $x = x^3$, $z = x^1 + ix^2$, and we write $\mathbf{x}_m = \mathbf{x} - \mathbf{a}_m$, $r_m = ||\mathbf{x}_m||$.

In this paper we are interested in the Dirac operators

$$\mathcal{D}^\dagger = \boldsymbol{\sigma}\cdot\mathbf{D} + \Phi - t, \qquad \mathcal{D} = \boldsymbol{\sigma}\cdot\mathbf{D} - \Phi + t, \tag{4.3}$$

where $t \in \mathbb{R}$. Specifically we are looking for zero modes of $\mathcal{D}^\dagger$ in the Hilbert space $\mathcal{H}$ of spinors Ψ satisfying $\mathcal{D}^\dagger\Psi = 0$, which are non-singular on $\mathbb{R}^3\backslash\{\mathbf{a}_m\}$ (up to gauge-dependent phase singularities) and square-integrable:

$$\int_{\mathbb{R}^3} d^3x \bar{\Psi}\Psi < \infty. \tag{4.4}$$

It is expected (and proved in [68]) that the number of zero modes is $N_+ = \sum_{m\in S_+} q_m$ for $t \in \mathbb{R}_+$, and $N_- = \sum_{m\in S_-} |q_m|$ for $t \in \mathbb{R}_-$, where $S_\pm = \{i | q_i \in \mathbb{Z}_\pm\}$. Indeed, Abelian monopoles can be obtained from a large Higgs field limit of smooth $SU(2)$ monopoles, for which a similar result holds by an index theorem [53, 75]. The above claim is also predicted by string theoretical constructions of the Nahm transform [23, 27].

4.3 Twistor space, Lax pair and the Dirac equation

In the following we will consider the space T of oriented straight lines in $\mathbb{R}^3$, which is identical to the holomorphic tangent space $T\mathbb{P}^1$ of the (complex) projective line $\mathbb{P}^1$ [51]. A

line in T can be described (non-uniquely) by a base point $\mathbf{x}$ and a direction $\mathbf{v}$. It is useful to view the direction as a point in $\mathbb{P}^1$, represented by the coordinate

$$\zeta(\mathbf{v}) = \frac{||\mathbf{v}|| + v^3}{v^{\bar{z}}} \tag{4.5}$$

For any ζ, there is a convenient choice of coordinates for $\mathbf{x}$:

$$u(\zeta;\mathbf{x}) = \frac{z}{2\zeta} + \frac{\bar{z}\zeta}{2}, \qquad v(\zeta;\mathbf{x}) = -\frac{z}{2\zeta} + \frac{\bar{z}\zeta}{2} + x, \qquad y(\zeta;\mathbf{x}) = \frac{z}{2\zeta} - \frac{\bar{z}\zeta}{2} + x. \tag{4.6}$$

The space of lines passing through a point $\mathbf{x}$ forms a real section of $T\mathbb{P}^1$, noted $P_{\mathbf{x}}$. Given two distinct points $\mathbf{x}$, $\mathbf{y}$, we denote by $C_{\mathbf{x}} \cdot C_{\mathbf{y}}$ the line passing through both points, passing through $\mathbf{x}$ first.

In the study of singular monopoles, it is useful to consider spaces of half-lines in $\mathbb{R}^3$ [63]. We define T^+ as the space of oriented half-lines pointing away from their endpoint, and similarly we define T^- from half-lines pointing towards their endpoint[1]. Given a point $\mathbf{x}$, we write the spaces of lines ending at $\mathbf{x}$ as $C^{\pm}_{\mathbf{x}}$.

Given $\zeta \in \mathbb{C}$, one has special pair of connections [74],

$$\begin{aligned} \nabla_\zeta &= \zeta D_z + \zeta^{-1} D_{\bar{z}} - \Phi + t \equiv D_{u(\zeta)} - \Phi + t, \\ \tilde{\nabla}_\zeta &= \zeta D_z - \zeta^{-1} D_{\bar{z}} + D_3 \equiv 2D_{v(\zeta)}, \end{aligned} \tag{4.7}$$

forming a Lax pair for the Bogonolnyi equation. Indeed, the flatness condition for $(\nabla_\zeta, \tilde{\nabla}_\zeta)$ is equivalent to the Bogonolnyi equation. Here we are interested in the flat sections $\chi(\mathbf{x}, t; \zeta)$ of the Lax pair, satisfying

$$\nabla_\zeta \chi(\mathbf{x}, t; \zeta) = 0, \qquad \tilde{\nabla}_\zeta \chi(\mathbf{x}, t; \zeta) = 0. \tag{4.8}$$

The connections (4.7) can be related to the Dirac operators $\mathcal{D}$ and $\mathcal{D}^\dagger$ as follow. We make the ansatz

$$\tilde{\Psi}(\mathbf{x}, t; \zeta) = \chi(\mathbf{x}, t; \zeta) \begin{pmatrix} 1 \\ \zeta \end{pmatrix}. \tag{4.9}$$

[1] Alternatively, T^+ is built from linear maps $(0, \infty) \to \mathbb{R}^3$, while T^- comes from maps $(-\infty, 0) \to \mathbb{R}^3$

Then the equation $\mathcal{D}\tilde{\Psi} = 0$ reduces to[2]

$$0 = \mathcal{D}\tilde{\Psi} = \begin{pmatrix} D_3 - \Phi + t & 2D_z \\ 2D_{\bar{z}} & -D_3 - \Phi + t \end{pmatrix} \chi(\mathbf{x}, t; \zeta) \begin{pmatrix} 1 \\ \zeta \end{pmatrix}$$
$$= \begin{pmatrix} 2\zeta D_z + D_3 - \Phi + t \\ \zeta(2\zeta^{-1} D_{\bar{z}} - D_3 - \Phi + t) \end{pmatrix} \chi(\mathbf{x}, t; \zeta), \tag{4.10}$$

which is equivalent to eq. (4.8). Thus given a set of solution to eq. (4.8), one can obtain solutions to $\mathcal{D}\tilde{\Psi} = 0$ by integrating over ζ against arbitrary functions $F(\zeta)$. (This corresponds to an integral over $C_{\mathbf{x}}$.) However, here we are interested in the zero modes of $\mathcal{D}^\dagger$. We can actually build a large class of solutions to $\mathcal{D}^\dagger\Psi = 0$ from those of eq. (4.8), of the form

$$\Psi(\mathbf{x}, t; \zeta) = \mathcal{D} \begin{pmatrix} \chi(\mathbf{x}, t; \zeta) \\ 0 \end{pmatrix}. \tag{4.11}$$

Since $\mathcal{D}^\dagger\mathcal{D} = (\mathbf{D}^2 - (\Phi - t)^2) \otimes \mathbb{I}_2$ is diagonal, it follows (by $\mathcal{D}^\dagger D\tilde{\Psi}(\mathbf{x}, t; \zeta) = 0$) that $\mathcal{D}^\dagger\Psi(\mathbf{x}, t; \zeta) = 0$. We now claim that all the zero modes in $\mathcal{H}$ can be obtained as combinations of such solutions. In the rest of this paper we prove this statement for $U(1)$ BPS monopoles in $\mathbb{R}^3$ by finding an explicit expression for $\chi(\mathbf{x}, t; \zeta)$.

The approach of this section is in many points similar to that of [74], although in that paper only smooth monopoles are considered. However in that paper the zero modes of $\mathcal{D}^\dagger$ are obtained in a different way: given a square integrable function η satisfying $\mathcal{D}\eta = 0$, a zero mode Ψ is built from the equation $\boldsymbol{\nabla} \cdot (\Psi^\dagger \boldsymbol{\sigma} \eta) = 0$. It would be interesting to relate the two approaches.

4.4 Simple example: single monopole of unit charge

We now proceed to a simple example of the method outlined above and write a known zero mode in terms of flat sections of the Lax connection. We consider a single monopole of unit charge centered at the origin. Assuming $t > 0$, the unique zero mode [22] of $\mathcal{D}^\dagger$ in $\mathcal{H}$ is

$$\Psi(\mathbf{x}, t) = \frac{e^{-rt}}{r} \begin{pmatrix} \sqrt{r - x} \\ -e^{i\phi}\sqrt{r + x} \end{pmatrix} = \mathcal{D} \frac{e^{-rt} + ce^{-xt}}{t\sqrt{r - x}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{4.12}$$

[2]Here and in the following we relax the assumption that the spinors lie in the Hilbert space $\mathcal{H}$, and allow for more singular functions. Hence "solving" the equation $\mathcal{D}\tilde{\Psi} = 0$ makes sense here even though none of the solutions lie in $\mathcal{H}$. The assumption will be restored later, as a condition for the zero modes of $\mathcal{D}^\dagger$ to lie in $\mathcal{H}$.

where c is an arbitrary constant which we set to zero. The flat sections are given by

$$\chi(\mathbf{x}, t; \zeta; F) = \frac{(r-x)^{1/2}}{\bar{z}\zeta + r - x} e^{tu(\zeta)} F(\zeta, y(\zeta)), \tag{4.13}$$

where F is an arbitrary function. In this simple case we can recover the zero mode (4.12) by guesswork: letting $F(\zeta, y) = \zeta^{-1}$ and taking the residue at $\zeta_0 = \zeta(-\mathbf{x}) = -(r-x)/\bar{z}$, we find

$$\frac{(r-x)^{1/2}}{2\pi i \bar{z}} \oint_{\zeta_0} \frac{d\zeta}{\zeta(\zeta - \zeta_0)} e^{tu(\zeta)} = \frac{i(r-x)^{1/2}}{\bar{z}\zeta_0} e^{-rt} = -\frac{e^{-rt}}{\sqrt{r-x}}, \tag{4.14}$$

i.e.

$$\Psi(\mathbf{x}, t) = -\frac{1}{2\pi i t} \mathcal{D} \oint_{\zeta_0} \chi(\mathbf{x}, t; \zeta; \zeta^{-1}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{4.15}$$

4.5 The general formula for Abelian BPS monopoles on $\mathbb{R}^3$

We now show that the residue formula is quite general, and in fact can be generalized to any set of $U(1)$ BPS monopoles. Here we find an exact formula for monopoles of unit charge at generic positions, and as outlined at the end of this section, special cases and generic positive charges can be obtained in a similar way. In appendix 4.B we explore the case where negative charges are also included. In this section, we assume $t > 0$.

We consider a set of N monopoles of unit charges $q_m = 1$, at positions $\mathbf{a}_m$ ($m = 1, \cdots, N$). We assume that the positions are generic, which requires the following:

- No pair of monopoles are separated only by a translation in the x^3 direction, i.e. $\mathbf{a}_m - \mathbf{a}_n$ is not proportional to $\hat{\mathbf{e}}^3$ for any $m \neq n$ (this prevents issues with the choice of gauge).
- No triplet of positions lie on the same line.

In particular, all the positions are different. We now make the following definitions:

$$\begin{aligned} \mathbf{x}_m &= \mathbf{x} - \mathbf{a}_m, \qquad \text{(similarly for } x_m,\ z_m,\ \bar{z}_m,\ r_m,\ u_m,\ v_m,\ y_m\text{)}, \\ \mathbf{a}_{mn} &= \mathbf{a}_m - \mathbf{a}_n, \qquad a_{mn} = ||\mathbf{a}_{mn}|| \end{aligned} \tag{4.16}$$

Flat sections of the Lax connection take the form

$$\chi(\mathbf{x},t;\zeta;F)=\left(\prod_m \chi_0(\mathbf{x}_m)\right)e^{tu(\zeta)}F(\zeta,y(\zeta)),$$
$$\chi_0(\mathbf{x})=\frac{(r-x)^{1/2}}{\bar{z}(\zeta-\zeta(-\mathbf{x}))}=\left(\frac{-\zeta(-\mathbf{x})}{\bar{z}}\right)^{1/2}(\zeta-\zeta(-\mathbf{x}))^{-1}. \tag{4.17}$$

To simplify the twistor notation, we define

$$\zeta_m=\zeta(-\mathbf{x}_m)=-\frac{r_m-x_m}{\bar{z}_m},\qquad \zeta'_m=\zeta(\mathbf{x}_m)=\frac{r_m+x_m}{\bar{z}_m},\qquad \zeta_{mn}=\zeta(\mathbf{a}_{nm}). \tag{4.18}$$

Note that the coordinates $\zeta_m(\mathbf{x})$ and $\zeta'_m(\mathbf{x})$ correspond to the zeros of $y(\zeta;\mathbf{x}_m)$:

$$y(\zeta;\mathbf{x}_m)=-\frac{\bar{z}_m}{2\zeta}(\zeta-\zeta_m)(\zeta-\zeta'_m). \tag{4.19}$$

Inspired by the results of the previous section, we seek a residue formulation for the zero modes. We pick a residue at each of the ζ_m, corresponding to the direction of the monopoles[3], and define

$$X_n[F]\equiv\frac{1}{2\pi i}\oint_{\zeta_n}d\zeta\left(\prod_m\chi_0(\mathbf{x}_m)\right)e^{tu(\zeta)}\zeta^{-1}F(\zeta)$$
$$=\left(\prod_m\frac{-\zeta_m}{\bar{z}_m}\right)^{1/2}\frac{e^{-r_nt}\zeta_n^{-1}F(\zeta_n)}{\prod_{m\neq n}(\zeta_n-\zeta_m)},$$
$$X[\{F_n\}]=\sum_n X_n[F_n]=\left(\prod_m\frac{-\zeta_m}{\bar{z}_m}\right)^{1/2}\sum_n\frac{e^{-r_nt}\zeta_n^{-1}F_n(\zeta_n)}{\prod_{m\neq n}(\zeta_n-\zeta_m)}. \tag{4.20}$$

Note that we dropped the second argument of F, since $y(\zeta_n;\mathbf{x})$ is actually a function of ζ_n only (hence the two argument of F are redundent). We now look for zero modes in $\mathcal{H}$ of the form

$$\Psi[\{F_n\}]=\mathcal{D}\begin{pmatrix}X[\{F_n\}]\\0\end{pmatrix}. \tag{4.21}$$

[3] Here one could try using more complicated residues, but these are the only choices allowing square-integrability at infinity in $\mathbf{x}$. For example if we divide by a power of $y(\zeta;\mathbf{x}_m)$ to obtain a higher order pole at ζ_m, we also create a pole at ζ'_m. That additional pole must then also lie inside the integration contour, as the integration contour cannot depend explicitly on $\mathbf{x}$ (and a contour depending solely on $y(\zeta)$ cannot distinguish the two poles). However $e^{tu(\zeta'_m)}=e^{tr_m}$ grows exponentially at infinity, so the result is not square-integrable. One could also try including in $F(\zeta,y)$ a pole at some other location and integrating around it, however this can be ruled out by a similar argument.

This can be achieved by sets of functions $\{F_n\}$ of the form

$$F_p(\zeta) = \prod_{m\neq p}(\zeta - \zeta_{pm})A_p + \sum_{n\neq p} e^{-a_{pn}t}F_n(\zeta_{pn}) \prod_{m\neq n,p} \frac{\zeta - \zeta_{pm}}{\zeta_{pn} - \zeta_{pm}}, \tag{4.22}$$

for any set of constants A_p. The values $F_n(\zeta_{pn})$ can be fixed by the consistency conditions

$$F_p(\zeta_{qp}) = \prod_{m\neq p}(\zeta_{qp} - \zeta_{pm})A_p + \sum_{n\neq p} e^{-a_{pn}t}F_n(\zeta_{pn}) \prod_{m\neq n,p} \frac{\zeta_{qp} - \zeta_{pm}}{\zeta_{pn} - \zeta_{pm}}. \tag{4.23}$$

This leaves N free parameters A_p, or N zero modes[4]. In appendix 4.A we derive eq. (4.22) and prove that it generates the correct zero modes.

The solutions in the non-generic cases can be obtained by taking a careful limit of the above equations (or by adapting the analysis of this section or appendix 4.A). In particular, monopoles of higher charges can be obtained from monopoles at coincident position. Note that in that case the integrands in the residue formula have multiple poles.

4.6 Relation to Nahm's formulation

The zero modes found in this paper allow to compute solutions to the Nahm equations and the Nahm zero modes via the Nahm transform. In this section we briefly review the procedure, following [73, 74]. In this section we relate the results of this paper to Nahm's approach to monopoles.

Given a set $\{\Psi_i\}$ of zero modes of $\mathcal{D}^\dagger$, one forms the matrices

$$\begin{aligned} \mathbf{T}(t)_{ij} &= -i\int_{\mathbb{R}^3} d^3x \Psi_i(\mathbf{x},t)^\dagger \mathbf{x} \Psi_j(\mathbf{x},t), \\ T^0(t)_{ij} &= -i\int_{\mathbb{R}^3} d^3x \Psi_i(\mathbf{x},t)^\dagger \partial_t \Psi_j(\mathbf{x},t). \end{aligned} \tag{4.24}$$

We can always set $T^0 = 0$ by a suitable gauge transformation. Assuming this, the other three matrices satisfy the Nahm equations

$$\partial_t \mathbf{T} = \mathbf{T} \times \mathbf{T}. \tag{4.25}$$

[4]Here we assumed without proof that the set of equations (4.23) is linearly independent. While this can be justified by the expected number of zero modes, it would be interesting to give a concrete proof of the statement.

Associated to these matrices is a pair of operators

$$D^\dagger = -i\partial_t + \boldsymbol{\sigma}\cdot(\mathbf{T}+\mathbf{x}), \qquad D = i\partial_t + \boldsymbol{\sigma}\cdot(\mathbf{T}+\mathbf{x}), \tag{4.26}$$

playing a role similar to $\mathcal{D}^\dagger$, $\mathcal{D}$. The zero modes $v_i(\mathbf{x},t)$ of $D^\dagger$ can be related to those of $\mathcal{D}^\dagger$ by the formulas

$$\Psi_i(\mathbf{x},t) = -\frac{1}{2\sqrt{2\pi}}\epsilon\mathcal{D}^\dagger\mathcal{D}v_i(\mathbf{x},t)^\dagger, \qquad v_i(\mathbf{x},t) = -\frac{\sqrt{2\pi}}{2}\epsilon(D^\dagger D)_{ij}\Psi_j(\mathbf{x},t), \tag{4.27}$$

where ϵ is the two-dimensional antisymmetric tensor.

4.6.1 An integral formulation

The results of this paper allow to write solutions of the Nahm equation via eq. (4.24), although the integral is in general difficult to compute. We can however use the residue formula to write the solutions as sets of integrals over algebraic varieties. In the following we explore the formulation of these integrals, but some additional work is needed in order to understand their meaning and relevance.

Schematically the integrals take the form[5]

$$\begin{aligned}
\mathbf{T}(t)_{ij} &= -i\sum_{m,n}\int_{\mathbb{R}^3} d^3x \oint_{\zeta_m^*} d\tilde{\zeta} \oint_{\zeta_n} d\zeta \bar{\chi}_i(\mathbf{x},t,\tilde{\zeta})^\dagger \mathbf{x}\chi_j(\mathbf{x},t,\zeta) \\
&= -i\sum_{m,n}\int d^3x d\tilde{\zeta} d\zeta \bar{\chi}_i(\mathbf{x},t,\tilde{\zeta})^\dagger \mathbf{x}\chi_j(\mathbf{x},t,\zeta)\delta(\tilde{\zeta}-\zeta_m^*)\delta(\zeta-\zeta_n) \\
&= -i\sum_{m,n}\int d^3x d\tilde{\zeta} d\zeta \bar{\chi}_i(\mathbf{x},t,\tilde{\zeta})^\dagger \mathbf{x}\chi_j(\mathbf{x},t,\zeta)\bar{y}_m(x,\tilde{\zeta})y_n(x,\zeta)\delta(\bar{y}_m(x,\tilde{\zeta}))\delta(y_n(x,\zeta)),
\end{aligned} \tag{4.28}$$

for some spinors $\chi_i(\mathbf{x},t,\zeta)$ (which include the action of the Dirac operator $\mathcal{D}$). Therefore we have an integral over the algebraic varieties defined by $\bar{y}_m(x,\tilde{\zeta}) = y_n(x,\zeta) = 0$ in the space spanned by $(\mathbf{x}, \tilde{\zeta}, \zeta)$, and the integral is nontrivial only on the components defined by $\tilde{\zeta} = \zeta_m^*$, $\zeta = \zeta_n$. For $n = m$ we can identify $\tilde{\zeta} = \zeta^*$, and the algebraic variety is the tautological line bundle over $C_{\mathbf{a}_m}$, and the contributing "component" is a half-line bundle over $C^-_{\mathbf{a}_m}$. The story is similar for $n \neq m$, but in that case we need to consider a complexification of the space of lines. While this formulation does not simplify the computation of the integral, it gives an algebraic interpretation of the integration contour.

[5]Here the "Dirac delta functions" are just schematic, as we do not define the integration contour properly.

4.7 Conclusion and open questions

In this paper we found the Dirac zero modes for Abelian BPS monopoles on $\mathbb{R}^3$. Our method relies on the following ideas and results:

- A complete set of flat sections for the Lax pair of connections $(\nabla_\zeta, \tilde{\nabla}_\zeta)$, providing a large set of solutions to the Dirac equation $\mathcal{D}^\dagger\Psi = 0$.
- The assumption that the zero modes belong to the above set of solutions.
- An integration contour, which gives a set of residue at specific points in $\mathbb{P}^1$ corresponding to the direction of each of the monopoles, and described by (one of the roots of) the algebraic equations $y(\zeta, \mathbf{x}_m) = 0$. The function $y(\zeta, \mathbf{x}_m)$ describe the algebraic space $C_{\mathbf{a}_m}$, the space of lines passing through the monopole.
- An explicit evaluation of the singularities for the solutions of the Dirac equations obtained from the contour integrals, and a resulting simple set of algebraic conditions for the integrand of the contour integral.

As we look for generalizations of the results of this paper, some of the above ideas are bound to fail. For instance the algebraic varieties $C_{\mathbf{a}_m}$ considered in this paper are specific to singular monopoles, hence a different integration contour is needed for non-Abelian monopoles. A natural guess is the spectral curve [51, 52]. For periodic monopoles the flat section for the Lax pair have an essential singularity near the monopoles, and its cancellation is highly nontrivial. We hope that these difficulties can be overcome.

Appendix

In these appendices we prove our main result by deriving eq. (4.22), and generalize it in the presence of negatively charged monopoles.

4.A Proof of the general formula

In this appendix we prove that the zero modes described in section 4.5 do belong to the Hilbert space $\mathcal{H}$. We first derive equation (4.22) by imposing square integrability near the monopoles, then we show that the singularities at $\zeta_p = \zeta_q$ all cancel. The "Dirac string" type singularities also cancel, however the proof is tedious but straightforward and we do not write it here. Also, it is clear that the solution is regular at infinity if and only if $t > 0$.

4.A.1 Near the monopoles

In the first step we impose square integrability near the monopoles. This requires $\Psi[\{F_n\}]$ to be less divergent that $r_n^{-3/2}$. In components, we have

$$\begin{aligned}
\Psi_1[\{F_n\}] &= \left(\prod_m \frac{-\zeta_m}{\bar{z}_m}\right)^{1/2} \sum_n \frac{e^{-r_n t}\zeta_n^{-1}F(\zeta_n)}{\prod_{m\neq n}(\zeta_n-\zeta_m)} \\
&\quad\times\left(\sum_{m\neq n}\zeta_m^{-1}\partial_3\zeta_m + \frac{F_n'(\zeta_n)}{F_n(\zeta_n)}\partial_3\zeta_n - \sum_{m\neq n}\frac{\partial_3\zeta_n-\partial_3\zeta_m}{\zeta_n-\zeta_m} + t\left(1-\frac{x_n}{r_n}\right)\right), \\
\Psi_2[\{F_n\}] &= \left(\prod_m \frac{-\zeta_m}{\bar{z}_m}\right)^{1/2} \sum_n \frac{e^{-r_n t}\zeta_n^{-1}F(\zeta_n)}{\prod_{m\neq n}(\zeta_n-\zeta_m)} \\
&\quad\times\left(\sum_{m\neq n}\zeta_m^{-1}\bar{\partial}\zeta_m + \frac{F_n'(\zeta_n)}{F_n(\zeta_n)}\bar{\partial}\zeta_n - \sum_{m\neq n}\frac{\bar{\partial}\zeta_n-\bar{\partial}\zeta_m}{\zeta_n-\zeta_m} - t\frac{z_n}{2r_n}\right).
\end{aligned} \tag{4.29}$$

Near a monopole, say $r_p \to 0$, only the terms proportional to $\partial_3\zeta_p$ or $\bar{\partial}\zeta_p$ contribute to the leading singularity, so we have

$$\begin{aligned}
\Psi_1[\{F_n\}] &\sim \left(\prod_{m\neq p}\frac{-\zeta_{pm}}{(a_{pm})_{\bar{z}}}\right)^{1/2}\left(\frac{-\zeta_p}{\bar{z}_p}\right)^{1/2}\zeta_p^{-1}\partial_3\zeta_p \\
&\quad\times\frac{\partial}{\partial\zeta_p}\left(\frac{F_p(\zeta_p)}{\prod_{m\neq p}(\zeta_p-\zeta_{pm})} - \sum_{n\neq p}\frac{e^{-a_{pn}t}F_p(\zeta_{pn})}{(\zeta_p-\zeta_{pn})\prod_{m\neq n,p}(\zeta_{pn}-\zeta_{pm})}\right), \\
\Psi_2[\{F_n\}] &\sim \left(\prod_{m\neq p}\frac{-\zeta_{pm}}{(a_{pm})_{\bar{z}}}\right)^{1/2}\left(\frac{-\zeta_p}{\bar{z}_p}\right)^{1/2}\zeta_p^{-1}\bar{\partial}\zeta_p \\
&\quad\times\frac{\partial}{\partial\zeta_p}\left(\frac{F_p(\zeta_p)}{\prod_{m\neq p}(\zeta_p-\zeta_{pm})} - \sum_{n\neq p}\frac{e^{-a_{pn}t}F_p(\zeta_{pn})}{(\zeta_p-\zeta_{pn})\prod_{m\neq n,p}(\zeta_{pn}-\zeta_{pm})}\right).
\end{aligned} \tag{4.30}$$

Since the leading singularity is of order $r_p^{-3/2}$, it must cancel, leading to the differential equations

$$\frac{\partial}{\partial\zeta_p}\left(\frac{F_p(\zeta_p)}{\prod_{m\neq p}(\zeta_p-\zeta_{pm})} - \sum_{n\neq p}\frac{e^{-a_{pn}t}F_p(\zeta_{pn})}{(\zeta_p-\zeta_{pn})\prod_{m\neq n,p}(\zeta_{pn}-\zeta_{pm})}\right) = 0, \tag{4.31}$$

solved by (4.22).

4.A.2 Cancellation of singularities: $\zeta_p = \zeta_q$

We first consider the case where $\zeta_p = \zeta_q$ for some p, q. This happens on a subset of the line connecting $\mathbf{a}_p$ and $\mathbf{a}_n$, on the two half lines between $\mathbf{a}_p$ and $\mathbf{a}_q$, and ∞ in the opposite direction (i.e. outside the interval between the two points). Consider the case of the (open) half line between $\mathbf{a}_p$ and ∞, where $\zeta_p = \zeta_q = \zeta_{pq}$, the other case being similar. We want to show that $\Psi[\{F_n\}]$ is regular there for any solution of eq. (4.22). In fact $X[\{F_n\}]$ is also regular there. To show this, we write

$$X[\{F_n\}] = \left(\prod_m \frac{-\zeta_m}{\bar{z}_m}\right)^{1/2} \left(\frac{1}{\zeta_p - \zeta_q}\left[\frac{e^{-r_p t}F_p(\zeta_p)}{\prod_{m\neq p,q}(\zeta_p - \zeta_m)} - \frac{e^{-r_q t}F_q(\zeta_q)}{\prod_{m\neq p,q}(\zeta_q - \zeta_m)}\right] + \cdots\right), \tag{4.32}$$

where $\cdots$ is regular on the half line, while the term inside the square brackets evaluates to

$$\frac{e^{-r_p t}F_p(\zeta_{pq}) - e^{-r_q t}F_q(\zeta_{pq})}{\prod_{m\neq p,q}(\zeta_{pq} - \zeta_m)} = \frac{e^{-r_p t}(F_p(\zeta_{pq}) - e^{-a_{pq}t}F_q(\zeta_{pq}))}{\prod_{m\neq p,q}(\zeta_{pq} - \zeta_m)}. \tag{4.33}$$

Meanwhile, evaluating eq. (4.22) at $\zeta = c_{pq}$ gives $F_p(c_{pq}) = e^{-a_{pq}t}F_q(c_{pq})$, so $X[\{F_n\}]$ is regular on the half line. Since the Dirac operator $\mathcal{D}$ cannot add a singularity there, it implies $\Psi[\{F_n\}]$ is also regular there.

4.B Negative charges

In this appendix we generalize the residue formula for a configuration including both monopoles of charge $+1$ and -1 (and by appropriate limits any combination any set of integer charges).

We consider a configuration containing N_+ monopoles of charge $+1$ at positions $\mathbf{a}_m$, $m \in S_+$, and N_- monopoles of charge -1 at $\mathbf{a}_{\hat{m}}$, $\hat{m} \in S_-$. We write the contribution of the negatively charged monopoles to the flat section of the Lax connection in terms of

$$\chi_0'(\mathbf{x};\zeta) = (-2\chi_0 y(\zeta))^{-1} = (-\zeta(-\mathbf{x})\bar{z})^{-1/2}\,(\zeta - \zeta(\mathbf{x}))^{-1} \tag{4.34}$$

We define the residues

$$X_n[F] \equiv \oint_{\zeta_n} d\zeta \left(\prod_m \chi_0(\mathbf{x}_m)\right)\left(\prod_{\hat{m}} \chi_0'(\mathbf{x}_{\hat{m}})\right) e^{tu(\zeta)}\zeta^{-1}F(\zeta),$$
$$X_{\hat{n}}[F] \equiv \oint_{\zeta_{\hat{n}}} d\zeta \left(\prod_m \chi_0(\mathbf{x}_m)\right)\left(\prod_{\hat{m}} \chi_0'(\mathbf{x}_{\hat{m}})\right) e^{tu(\zeta)}\zeta^{-1}F(\zeta),$$
$$X[\{F_n\}] = \sum_n X_n[F_n], \qquad \hat{X}[\{F_{\hat{n}}\}] = \sum_{\hat{n}} X_{\hat{n}}[F_{\hat{n}}], \tag{4.35}$$

Note that X and $\hat{X}$ cannot be mixed, since regularity at infinity requires $t > 0$ for X and $t < 0$ for $\hat{X}$. Proceeding as before, we impose square-integrability for X near the monopoles, and find the set of equations

$$0 = \frac{\partial}{\partial\zeta_p}\left[\frac{1}{\prod_{\hat{m}}(\zeta_p - \zeta_{\hat{m}p})}\left(\frac{F_p(\zeta_p)}{\prod_{m\neq p}(\zeta_p-\zeta_{pm})} - \sum_{n\neq p}\frac{e^{-a_{pn}t}F_p(\zeta_{pn})}{(\zeta_p-\zeta_{pn})\prod_{m\neq n,p}(\zeta_{pn}-\zeta_{pm})}\right)\right],$$
$$0 = F_p(\zeta_{\hat{m}p}). \tag{4.36}$$

The first equation is solved by

$$F_p(\zeta) = \left(\prod_{\hat{m}}(\zeta_p - \zeta_{\hat{m}p})\right)\left(\prod_{m\neq p}(\zeta-\zeta_{pm})A_p + \sum_{n\neq p} e^{-a_{pn}t}F_n(\zeta_{pn})\prod_{m\neq n,p}\frac{\zeta-\zeta_{pm}}{\zeta_{pn}-\zeta_{pm}}\right), \tag{4.37}$$

which trivially satisfies the second. A similar analysis can be performed for $\hat{X}$, and as before we can show that the formula gives valid zero modes. Also the number of zero modes is N_+ for $t > 0$, and N_- for $t < 0$, as expected.

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