Covering Graphs and Equiangular Tight Frames

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract:

Recently, there has been huge attention paid to equiangular tight frames and their constructions, due to the fact that the relationship between these frames and quantum information theory was established. One of the problems which has been studied is the relationship between equiangular tight frames and covering graphs of complete graphs. In this thesis, we will explain equiangular tight frames and covering graphs of complete graphs and present the results that show the relationship between these two concepts. The latest results about the constructions of equiangular tight frames from projective geometries and Steiner systems also has been presented.

Acknowledgement

I would like to express the deepest appreciation to my supervisor Professor Chris Godsil at The University of Waterloo for his extensive academic, technical and personal support throughout the course of my master.

I also want to thank my friends for their friendship and help.

Finally, I find myself in great debt to my parents for their love and support. For that I will always be grateful.

Dedication

To my dear parents, who have been a constant source and encouragement during the challenges of graduate school and life.

Contents

Introduction

We discuss the relationship between covering graphs of complete graphs and equiangular tight frames.

0.1 Covering Graphs

A distance regular graph G with diameter d is *antipodal* if we can partition the vertices of G into equivalence classes such that each pair of vertices of each class are at distance d from each other; that is, if two vertices are at distance d from a vertex, they are also at distance d from each other. We call each class a *fibre* of G.

Let G be a graph such that we can partition its vertices into cells with the following conditions.

(a) The vertices in each cell are independent; that is, each cell is an independent set.

(b) Either there is not any edge between any pair of cells, or there is a perfect matching between them.

We denote this partition by P and call each cell a *fibre*. Let Q be a graph such that each of its vertices corresponds to a cell of P, and two vertices are adjacent if and only if the edges joining the corresponding cells in P is a matching. We denote Q by G/P and call G, a *covering graph* of G/P. The *covering map* maps each vertex in each fibre of G to the corresponding vertex of that fibre in G/P. In this note, we specifically talk about the covering graphs of complete graphs. The definition of abelian covering graphs of complete graphs is as follows.

Let G be an r-fold cover of K_n with the vertex set $V(G) = V(K_n) \times \{1, ..., r\}$. Then, the vertices of G can be partitioned to n independent sets with r vertices, such that there is a matching

between each pair of these sets. We can show each of these matchings by a permutation of $\{1, ..., r\}$. Thus, we can define an *arc function* of index r over the complete graph K_n as a function f from the set of arcs of K_n to the symmetric group of the permutations of $\{1, ..., r\}$; that is, $Sym(r)$ such that $f(u, v) = f^{-1}(v, u)$ for each pair of vertices u and v of K_n. Therefore, for each pair of vertices u and v, $f(u, v)$ denotes the matching between two independent sets in G corresponding to u and v. We can consider f as the identity permutation if we evaluate it over the edges of a spanning tree. In this situation, we call f a *normalized arc function*.

We denote the permutation group that is generated by the image of the arc function f over the arcs of the complete graph K_n by $\langle f \rangle$. An r fold cover G of the complete graph K_n is an *abelian covering graph* of the complete graph K_n if $\langle f \rangle$ is an abelian group. In Chapter 0.3 we will present a table of feasible parameters of abelian covering graphs of complete graphs.

0.2 Equiangular tight frames

A sequence $F = \{f_1, ..., f_n\}$ is a *frame* for a Hilbert space H if there exist two constants A and B such that for every vector x in H, we have

$$A||x||^2 \leq \sum_{i=1}^{n} |\langle x, f_i \rangle|^2 \leq B||x||^2.$$

A *tight frame* is a frame such that $A = B$. In Chapter 0.11, we will present some characteristics of tight frames. In this note, we specifically discuss equiangular tight frames which are defined as follows.

A set of lines spanned by vectors $v_1, ..., v_n$ in $\mathbb{C}^d$ (or $\mathbb{R}^d$) is a set of complex (or real) *equiangular lines* if there is $\alpha \in \mathbb{R}$ such that, for all i and j with $i \neq j$, we have $|\langle v_i, v_j \rangle| = \alpha$; that is, the angles between each pair of distinct lines are the same.

The set $E = \{v_1, v_2, ..., v_n\}$ of vectors in $\mathbb{R}^d$ is an equiangular tight frame if for each $i \neq j$, $\langle v_i, v_j \rangle = \alpha$ for some $\alpha \in \mathbb{R}$, and there exists a constant A such that,

$$\sum_{i=1}^{n} |\langle x, v_i \rangle|^2 = A||x||^2$$

for each $x \in \mathbb{R}^d$. In Chapter 0.11, we present two bounds for the size of a set of equiangular lines and we show that this set is a tight frame if and only if these bounds hold with equality for it.

To show the relationship between equiangular tight frames and covering graphs of complete graphs we need to present the definition of Seidel matrices which is defined as follows. Suppose $F = \{f_1, ..., f_n\}$ is a frame in H, the Gramian operator for F is the operator $G : \ell^2(n) \to \ell^2(n)$ such that:

$$Gx = (\sum_{i=1}^{n} x_i \langle f_i, f_j \rangle)_{j=1}^{n} = \sum_{i=1}^{n} x_i (\langle f_i, f_j \rangle)_{j=1}^{n},$$

for $x = (x_i)_{i=1}^{n}$ in H. we call G, the Gram matrix of the frame.

Let $E = \{v_1, v_2, ..., v_n\}$ be a set of equiangular lines in $\mathbb{R}^d$, and M be the matrix whose columns are the members of E. Then, the Gram matrix of the vectors of E is as follows

$$G = M^T M.$$

G is a positive semidefinite symmetric matrix which has the same rank as M. Let $E = \{v_1, v_2, ..., v_n\}$ be a set of equiangular lines such that v_i is a unit vector for each $1 \leq i \leq n$ and $v_i^T v_j = \pm\alpha$ for all $1 \leq i, j \leq n$, so we have the following for its Gram matrix G:

$$G = \alpha S + I,$$

such that S is a symmetric matrix whose diagonal entries are zero and off-diagonal entries are ± 1. S can be the nonstandard adjacency matrix of a graph G if -1 shows the adjacency of two vertices and 1 shows the non-adjacency of two vertices of G.

The matrix S is known as the *Seidel matrix* of G.

0.3 Outline of the Thesis

In Chapter 0.3, we will define antipodal distance regular graphs, and antipodal distance regular covering graphs of complete graphs. Secondly, we present the arc function of these graphs and abelian covering graphs of complete graphs. We will demonstrate a table of feasible parameters of abelian covering graphs of complete graphs in this chapter. After that, we will present some constructions of

covering graphs of complete graphs and at last we will present a table of abelian covering graphs of complete graphs.

In Chapter 0.11, we will define frames and present some operators which are related to frames. Secondly, we will present the definition of a set of equiangular lines and present two bounds on the size of these sets. After that, we will demonstrate the equiangular tight frames. At last, we will present the definition of Seidel matrices.

In Chapter 0.18, we will present the new results about constructions of equiangular tight frames from design geometries, Steiner systems, hyperovals, and projective planes.

In Chapter 0.24, we will present two theorems which show the relationship between equiangular tight frames and covering graphs of complete graphs. Secondly, we will present the definition of Steiner systems and show that for each Steiner system there exists an equiangular tight frame corresponding to that system.

Covering graphs

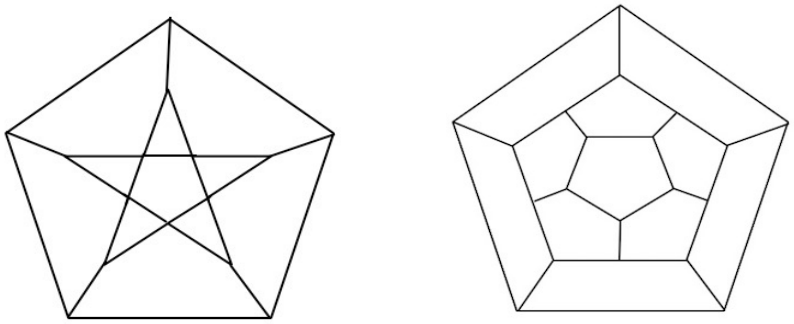
0.4 Introduction

In this chapter, we will talk about covering graphs, specifically antipodal distance regular covering graphs of complete graphs, and discuss their characteristics. Most of the information in this chapter is from Godsil and Hensel [7], which completely describes covering graphs. Firstly, we define antipodal distance regular graphs, and present some characteristics of these graphs. We will discuss the covering graphs, following this we will define covering graphs of complete graphs, and demonstrate some characteristics of these graphs and show their feasibility conditions. After that, we define the arc functions of covering graphs of complete graphs and discuss abelian covering graphs of complete graphs. We also present a table of feasible parameters of abelian covering graphs of complete graphs. At last, we will present some well known constructions of covering graphs of complete graphs.

0.5 Antipodal Distance Regular Graphs

The first part of this section will define distance regular graphs and their properties. A graph G with diameter d is a *distance regular graph* if for each pair of vertices v and u in G and for each $0 \leq i, j \leq d$, the number of vertices which are at distance i from v and j from u is depends just on i and j and the distance between v and u in G and the distance between u and v.

Example. Figure 1 presents two distance-regular graphs. The graph in the figure (a), which is known as the Petersen graph, is a distance regular graph. We can easily see that the diameter of this graph is



(a) The Petersen graph which is a distance regular graph with diameter $d = 2$.

(b) The Dodecahedron graph which is a distance regular graph with diameter $d = 5$.

Figure 1: An example of two distance regular graphs

two. Figure (b) shows a dodecahedron, a distance regular graph with diameter five. If we check the graphs, we can see that for any pair of vertices u and v where $0 \leq i, j \leq 2$, the number of vertices at distance i from u and j from v depends only on i and j.

A distance regular-graph G with diameter d is *antipodal* if we can classify the vertices of G into equivalence classes such that each pair of vertices of each class are at distance d from each other; that is, if two vertices are at distance d from a vertex, they are also at distance d from each other. We call each class a fibre of G.

Let u and v be two vertices of the antipodal distance regular graph G at distance i from each other. We denote the number of neighbours of u which are at distance $i-1$ from v by c_i, the number of neighbours of u which are at distance i from v by a_i and the number of neighbours of u which are at distance $i+1$ from v by b_i.These numbers are called the *intersection numbers* of G.

Since all of the neighbours of u are at distance $i-1$, i, or $i+1$, we can see that $c_i + a_i + b_i$ is equal to the degree of u. Thus, we have a constant $c_i + a_i + b_i$ for each $0 \leq i \leq d$, because G is a regular graph. Therefore, we can associate to each antipodal distance regular graph by its intersection array which is $\{c_1, c_2, ..., c_d; b_0, b_1, ..., b_{d-1}\}$.

Example. The figure 2 shows a graph which is an antipodal distance regular graph with diameter 3 and intersection array $\{1, 1, 2; 2, 1, 1\}$.

The vertices of this graph are partitioned into three independent sets, each has two vertices, and there is a matching between each

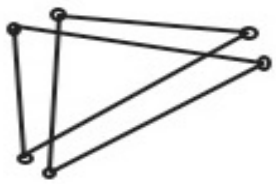


Figure 2: An antipodal distance regular graph eith distance 3

pair of these sets. We can easily see that the distance between the vertices of each set is three which is the diameter of the graph.

The following theorem from [1] and [6] presents two fundamental characteristics of antipodal distance regular graphs.

0.5.1 Theorem. *Let G be an antipodal distance regular graph with intersection array $\{c_1, c_2, ..., c_d; b_0, b_1, ..., b_{a_1}\}$ and diameter d such that $d \geq 2$. Then G has the following characteristics.*

i) If there is an edge between two fibres, then there is a perfect matching between them; that is, each vertex in one fibre has a unique neighbour in the other one.

ii) If the distance between two fibres is i, then for each vertex v in each fibre, there exists a unique vertex u in the other fibre which is at distance i from v, and v is at distance $d - i$ from the vertices in the second fibre other than u. □

We will define another graph which is related to the antipodal distance regular graphs and is an example of a covering graph.

Let G be an antipodal distance regular graph with intersection array

$$\{c_1, c_2, ..., c_d; b_0, b_1, ..., b_{d-1}\}$$

and diameter $d \geq 2$. Suppose each fibre of G has size r. Let Q be the graph such that each vertex of this graph corresponds to a fibre of G and two vertices are adjacent if and only if there is a matching between their corresponding fibres in G. This graph is an *antipodal quotient* of G.

The following theorem (see [1, 6, 7]) shows that the antipodal quotient graph Q is a distance regular graph and presents the relationship between the eigenvalues of Q and G.

0.5.2 Theorem. *Let G be an antipodal distance regular graph. If Q is its antipodal quotient, then we have:*

i) Q is a distance regular graph with intersection array $\{c_2, c_3, ..., \alpha c_m; b_0, b_1, ..., b_{m-1}\}$ such that if $d = 2m$, then $\alpha = r$ and if $d = 2m + 1$ then $\alpha = 1$.

ii) The set of eigenvalues of Q is a subset of the set of eigenvalues of G. □

0.6 Covering Graphs

In this section, we will define covering graphs and discuss their properties.

The following definition presents covering graphs. After that, we will discuss the properties of these graphs. Let G be a graph with a permutation of its vertices into the cells, such that the following conditions hold:

(a) The vertices in each cell are independent; that is, each cell is an independent set.

(b) Either there is not any edge between any pair of cells or there is a perfect matching between them.

We denote this partition by P and call each cell a *fibre*. Let Q be a graph such that each of its vertices corresponds to a cell of P, and two vertices are adjacent if and only if there is a matching between their corresponding cells in P. We denote Q by G/P and call G, a *covering graph* of G/P. The *covering map* maps each vertex in each fibre of G to the corresponding vertex of that fibre in G/P.

We can easily see that if G/P is connected, then all of the fibres of G have the same size which is the *index* of the covering. We denote the index of G by r. Thus, we can call G an *r-fold covering graph* of G/P.

0.7 Covering graphs of Complete Graphs

In this section, we present the properties of covering graphs drawn from complete graphs. For this purpose, we discuss the antipodal distance regular graphs of diameter three because they are the covering graphs of complete graphs. At first, we show that we can characterizing many properties of these graphs by three parameters.

Let G be an antipodal distance regular graph with diameter three. Thus, there exist some positive integers r and n such that G is the r-fold cover of a complete graph K_n. Let u be an arbitrary vertex of G, and G_i be the subset of vertices of G which are at distance i from u in G for $1 \leq i \leq 3$.

Let v be a vertex in G_3, so v is at distance three from u and they are in the same fibre. All of the neighbours of u are in other fibres. Since there is a matching between each pair of fibres, the neighbours of u cannot be adjacent to v. Thus, they are at distance two from v. Thus, u has a neighbour in each fibre other than its fibre. Therefore, the degree of u is $n-1$ and $c_3 = n-1$.

Let w represent a vertex at distance two from u. Thus, they are in different fibres and are not adjacent. All of the vertices which are at distance three form u are in the same fibre with u, and just one of them is a neighbour of w. Thus, there exists only one vertex which is a neighbour of w and is at distance three from u. Therefore, $b_2 = 1$.

If we do double counting on the number of edges between G_2 and G_3, we can see that $|G_2| = (n-1)(r-1)$. By double counting the number of edges between G_1 and G_2, we have $(n-1)b_1 = (n-1)(r-1)c_2$. Thus, $b_1 = (r-1)c_2$. Hence, we can describe G with three parameters (n, r, c_2).

Example. The graph which was presented in Example 0.5, the figure 2, is a 2-fold covering graph of the complete graph K_3. This graph is an antpodal distance covering graph with the parameters $(3, 2, 1)$.

The following lemma presents a characteristic of the covering graphs of the complete graphs.

0.7.1 Lemma. *Let G be an r-fold cover of a complete graph with n vertices (K_n). There exists some positive integer $c_2 \geq 1$ such that G is an antipodal distance regular graph with parameters (n, r, c_2) if and only if each pair of non-adjacent vertices, which are from different fibres, have c_2 common neighbours.*

Proof. At first, let G be a covering graph of a complete graph such that any pair of non-adjacent vertices from different fibres have c_2 common neighbours. We prove that G is an antipodal distance regular graph of diameter 3 with parameters (n, r, c_2). Let u and v be two vertices in a fibre f. Since each fibre is an independent set, u and v are not adjacent. If w is a neighbour of u, then w is from another fibre. w can just be adjacent with u in f, thus the distance between u and v is not two. v and w are from different fibres and they are non-adjacent, so they have c_2 common neighbours. Therefore, the distance between w and v is two, and so the distance between u and v is three.

Since G is a cover of a complete graph, each vertex a which is not in f has a neighbour in f like u. As we discussed above, the distance between this vertex and the vertices in $f \setminus u$ is two. Therefore, a has c_2 common neighbours with each of the vertices of $f \setminus u$. Thus, a has $(r-1)c_2$ neighbours which are at distance two from u. Hence, the number of common neighbours of u and a is

$$(n-1) - (r-1)c_2 - 1 = n - 2 - (r-1)c_2.$$

As a result, G is an antipodal distance regular graph with intersection array $\{1, c_2, n-1; n-1, (r-1)c_2, 1\}$.

Now suppose G is an antipodal distance regular graph with parameters (n, r, c_2). Since the distance between two non-adjacent vertices from different fibres is two, They have c_2 common neighbours. □

Now, let u be a vertex of G, and H be the induced subgraph containing the neighbours of u.Thus, H has $n-1$ vertices and is not a complete graph, so $0 \leq a_1 \leq n-3$. Let v be a vertex of H, so it is a neighbour of u. v has c_2 common neighbours with the vertices which are in the same fibre as u in G. Since there are r vertices in each fibre, we can see that $a_1 = (n-2) - (r-1)c_2$.

We know that there are an even number of vertices which have odd degree in a graph. Therefore, if n is even, $n-1$ is odd, and since H is a a_1 regular graph with $n-1$ vertices, a_1 must be even. $n-2$ is even, thus by $a_1 = n-2-(r-1)c_2$ we can conclude that $(r-1)c_2$ must be even. Consequently, if n is even, then $(r-1)c_2$ is also even. In [1], the authors proved that if $\delta := a_1 - c_2$ and $\Delta := \delta^2 + 4(n-1)$,

then the following are the eigenvalues of G:

$$n-1, -1, \theta = \frac{(\delta + \sqrt{\Delta})}{2}, \tau = \frac{(\delta - \sqrt{\Delta})}{2}.$$

G is an $(n-1)$-regular connected graph, so it has an eigenvalue $n-1$ with multiplicity 1. As we said in Theorem 0.5.2, the set of eigenvalues of K_n is a subset of the set of the eigenvalues of G. We also know that the eigenvalues of K_n are $n-1$ with a multiplicity of 1 and -1 with a multiplicity of $n-1$. Thus, -1 is an eigenvalue of G with a multiplicity equals to $n-1$. The multiplicities of θ and τ are

$$m_\theta = \frac{n(r-1)\tau}{\tau - \theta}, m_\tau = \frac{n(r-1)\theta}{\theta - \tau}$$

respectively. Since m_θ and m_τ are the multiplicities of the eigenvalues, they have to be integer numbers.

The following two lemmas, which are from [7], determine the conditions of covering graphs of complete graphs.

0.7.2 Lemma. *If $\delta \neq 0$, then θ and τ are integers, and if $\delta = 0$, then $\theta = -\tau = \sqrt{n-1}$.* □

0.7.3 Lemma. *If n is even, then c_2 is also even.*

Proof.

If n is even, then $n-1$ is odd. We know that $n-1 = \theta\tau$. Thus, either θ and τ are odd integers, or $\delta = 0$. In both cases, $\delta = \theta + \tau$ is even. Therefore, by using $\delta = n - 2 - rc_2$, and since n is even, rc_2 must be even. We have proved that if n is even, then $(r-1)c_2$ is also even. Hence, c_2 is even. □

Now, by using the results that we have proved above, we have the following feasibility conditions for the covering graphs of the complete graphs:

(a) n, r and c_2 are three positive integers such that $1 \leq (r-1)c_2 \leq n-2$.

(b) m_θ and m_τ are positive integers.

(c) If n is even, then c_2 is also even.

The following theorem is a result for covering graphs of complete graphs with $c_2 = 1$, which is from [1].

0.7.4 Theorem. *Let G be a covering graph of a complete graph with parameters $(n, r, 1)$. We have the following:*

- *n is odd,*
- *$(n-r)$ divides $(n-1)$,*
- *$(n-r)(n-r+1)$ divides $rn(n-1)$,*
- *$(n-r)^2 \leq n-1$.*

Proof.

By the feasibility condition (b), we know that if n is even, then c_2 must be even. Therefore, since c_2 is odd, then n must be odd. Now, let v be a vertex of G, and G' be the induced subgraph of G by the neighbours of v. Let u and w be two non-adjacent vertices of G'. They have exactly one common neighbour in G which is v. Therefore, they do not have a common neighbour in G', and their distance is not two. Thus, G' is composed of cliques of size $a_1 + 1$. We know that $a_1 = n - 1 - r$, so the size of each clique is $n - r$. Since G' has $n-1$ vertices, $n-r$ divides $n-1$. The vertex v is in $\frac{(n-1)}{(n-r)}$ cliques which have $n-r+1$ vertices. Therefore, there are

$$(nr)\frac{n-1}{n-r}\frac{1}{n-r+1} = nr(n-1)/(n-r)(n-r+1)$$

of these cliques in G. Hence, $(n-r)(n-r+1)$ divides $rn(n-1)$. Let C be a clique in G' and w be a vertex which is antipodal to v. Let H be an induced subgraph of G by the neighbours of w, and C' be a clique of H. Each vertex of H has a neighbour in G'. Let a and b be two vertices of C and c and d be two vertices of C' such that a is adjacent with c and b is adjacent with d. Since a and b are the vertices of a clique, they are adjacent. In the same way, we can prove that c and d are adjacent. It is a contradiction with $c_2 = 1$, because a and d are two vertices at distance two which have two common neighbours. Thus, each vertex of C has a neighbour in a unique clique in H. Since $|C| = n - r$ and H has $\frac{n-1}{n-r}$ cliques, $n - r \leq \frac{n-1}{n-r}$. Therefore, $(n-r)^2 \leq n-1$. □

Now, we present the following result from [7] which gives us a condition for the multiplicity of an eigenvalue of covering graphs of complete graphs.

0.7.5 Lemma. *Suppose G is an antipodal distance regular graph which is a cover of a complete graph with n vertices. Let λ be an integer eigenvalue of G, not -1 or $n-1$ with multiplicity m. If $n > m - r + 3$, then $\lambda + 1$ divides c_2.* □

The following two theorems, which are the last results of this section, classify the covering graphs of complete graphs depending on the value of $\delta = a_1 - c_2$. These four classes are $\delta = -2, 0, 2$ or none of them. The following theorem has been proved by Godsil and Hensel in [7]

0.7.6 Theorem. *The number of parameter sets for the distance regular covering graphs of the complete graphs with n vertices and constants r and δ is finite, unless $\delta = -2, 0, 2$.*

Proof.

Let (n, r, c_2) be the feasible parameters of an antipodal distance regular graph G. Suppose, θ and τ are two eigenvalues of G neither -1 nor $n-1$, and m_θ and m_τ are the multiplicities of θ and τ respectively. We have:

$$m_\theta - m_\tau = \frac{n(r-1)(\theta+\tau)}{\theta-\tau} = \frac{n(r-1)\delta}{\sqrt{\Delta}}.$$

We know that m_θ and m_τ are integers, so $m_\theta - m_\tau$ is also integer. If $\delta \neq 0$, then by using Lemma 0.7.2, we can conclude that $\sqrt{\Delta}$ is an integer number. We have:

$$(\sqrt{\Delta} - \delta)^2 = \Delta + \delta^2 - 2\delta\sqrt{\Delta} = 2(\delta^2 + 2(n-1) - \delta\sqrt{\Delta}).$$

Therefore, $(\sqrt{\Delta} - \delta)$ is even. Thus, there is an integer s such that $\sqrt{\Delta} - \delta = 2s$, and so $\Delta = (\delta + 2s)^2$. We know that $\Delta = \delta^2 + 4(n-1)$. Therefore, $4n = ((2s+\delta)^2 + (4-\delta^2))$. We know that $4(m_\theta - m_\tau)$ is an integer number, thus

$$\frac{((2s+\delta)^2 + (4-\delta^2))(r-1)\delta}{2s+\delta} = (2s+\delta)(r-1)\delta + \frac{(4-\delta^2)(r-1)\delta}{2s+\delta}$$

is an integer. Hence, $2s + \delta$ is bounded above by $(4-\delta^2)(r-1)\delta$ unless $\delta = \pm 2$. We know that n is function of δ and s. Thus, n is bounded by a function of δ and s unless $\delta = \pm 2$ or $\delta = 0$. □

0.7.7 Lemma. *If the parameter set (n, r, c_2) satisfies the conditions (a) and (b), then the following hold, depending on the value of δ.*

If $\delta = \pm 2$, then (c) holds if and only if n is a square number.

If $\delta = 0$, then (c) is satisfied.

Proof.

Let (n, r, c_2) be the parameter set which satisfies the conditions (a) and (b).

If $\delta = 0$, then

$$\theta = \sqrt{\Delta}/2 = \sqrt{n-1} \text{ and } \tau = -\sqrt{\Delta}/2 = -\sqrt{n-1}.$$

Thus,

$$m_\theta = \frac{-n(r-1)\sqrt{n-1}}{-2\sqrt{n-1}} = n(r-1)/2$$

and

$$m_\tau = \frac{n(r-1)\sqrt{n-1}}{2\sqrt{n-1}} = n(r-1)/2.$$

We know that $\delta = n - 2 - rc_2 = 0$, so if n is odd then r is also odd. Thus, in either case that n is odd or even, $n(r-1)$ is even, and m_θ and m_τ are integer numbers. Therefore, if $\delta = 0$, then the parameter set satisfies the condition (c).

If $\delta = \pm 2$, then $\Delta = 4+4(n-1) = 4n$. Therefore, $\theta = \frac{(\pm 2+\sqrt{4n})}{2} = \pm 1 + \sqrt{n}$ and $\tau = \frac{(\pm 2-\sqrt{4n})}{2} = \pm 1 - \sqrt{n}$. As the result,

$$m_\theta = \frac{n(r-1)(\pm 1 - \sqrt{n})}{-2\sqrt{n}} = -\sqrt{n}(r-1)(\pm 1 - \sqrt{n})/2$$

and

$$m_\tau = \frac{n(r-1)(\pm 1 + \sqrt{n})}{2\sqrt{n}} = \sqrt{n}(r-1)(\pm 1 + \sqrt{n})/2.$$

Hence, m_θ and m_τ are integer if and only if n is a square. □

In the next section, we will present some constructions of covering graphs, and explain them.

0.8 Abelian Covering Graphs of Complete Graphs

This section will show that we can describe each covering graph of complete graphs with a function, and define the abelian covering

graphs by using this function. Let G be an r-fold cover of K_n with the vertex set $V(K_n) \times \{1, ..., r\}$. Then, the vertices of G can be partitioned into n independent sets with r vertices, such that there is a matching between each pair of these sets. We can show each of theses matchings by a permutation of $\{1, ..., r\}$. Thus, we can define an *arc function* of index r over the complete graph K_n as a function f from the set of arcs of K_n to the symmetric group of the permutations of $\{1, ..., r\}$; that is, $Sym(r)$ such that $f(u, v) = f^{-1}(v, u)$ for each pair of vertices u and v of K_n. Therefore, for each pair of vertices u and v, $f(u, v)$ defines a matching between two independent sets in G corresponding to u and v. We can consider f as the identity permutation if we evaluate it over the edges of a spanning tree. In this situation, we call f a *normalized arc function*.

We use $\langle f \rangle$ to denote the the permutation group that is generated by the image of the arc function f over the arcs of the complete graph.

An r-fold cover G of the complete graph K_n is *regular* if $\langle f \rangle$ is regular. We say G is *cyclic* if $\langle f \rangle$ is cyclic. And G is an *abelian covering graph* of the complete graph K_n if $\langle f \rangle$ is an abelian group. G is regular if and only if $|\langle f \rangle| = r$.

Let f be an arc function for the complete graph K_n. Then, $A(K_n)^f$ is the adjacency matrix of a graph with n vertices, each of its rows and columns are corresponded to a vertex of K_n, and for each pair of distinct vertices u and v of K_n, $A(K_n)^f(u, v) = f(u, v)$, and for all vertices u of K_n, $A(K_n)^f(u, u) = 0$.

Suppose P is a permutation representation of $\langle f \rangle$ with dimension d. We define the matrix $A(K_n)^{P(f)}$ as the matrix which is derived from $A(K_n)^f$ by replacing each entries $A(K_n)^f(u, v)$ by a $d \times d$ matrix which is a permutation matrix corresponding to its image under P for each pair of distinct vertices u and v of K_n, and replacing each entries of the diagonal by a $d \times d$ zero matrix.

Let G be a regular covering graph of K_n which is defined by the arc function f of K_n, and P be the regular representation of $\langle f \rangle$, then we can see that

$$A(K_n)^{P(f)} = A(G)$$

Which is the adjacency matrix of G.

The following theorem, which has been presented in [7] by Godsil and Hensel, shows the relationship between the representations of

n	r	c_2	a_1	θ	τ	m_θ	m_τ
5	3	1	1	2	-2	5	5
6	2	2	2	2.236067977	-2.236067977	3	3
7	5	1	1	2.449489743	-2.449489743	14	14
8	3	2	2	2.645751311	-2.645751311	8	8
9	3	3	1	2	-4	12	6
	5	1	3	4	-2	12	24
	7	1	1	2.828427125	-2.828427125	27	27
10	2	4	4	3	-3	5	5
	4	2	2	3	-3	15	15

Figure 3: The table of feasible parameters of abelian covering graphs of complete graphs for $n \leq 10$

an arc function of a complete graph and the adjacency matrix of its r-fold covering graph.

0.8.1 Theorem. *Suppose f is a normalized arc function corresponding to a connected r-fold covering graph of the complete graph K_n, and $P_1, P_2, ..., P_r$ are the linear characters of $\langle f \rangle$. We have the following results.*

(a) We can write the adjacency matrix of G; that is, $A(G)$ as,

$$\begin{pmatrix} A(K_n)^{P_1(f)} & & & \\ & A(K_n)^{P_2(f)} & & \\ & & \ddots & \\ & & & A(K_n)^{P_r(f)} \end{pmatrix}.$$

(b) The graph G is an antipodal distance regular covering of K_n with parameters (n, r, c_2) if and only if for each $1 \leq i \leq r$, the minimal polynomial of $A(K_n)^{P_i(f)}$ is

$$t^2 - (n - rc_2 - 2)t - (n - 1).$$

We present a table of feasible parameters of abelian covering graphs of complete graphs for $n \leq 10$. We will demonstrate this table for $n \leq 50$ in the appendix.

0.9 Constructions

In this section, we present some constructions of the covering graphs of the complete graphs. The following construction has been presented by Mathon in [1].

Construction 1: To introduce this construction at first we need the following definitions.

A bilinear function $B : V \times V \to GF(q)$ satisfying $S(x, x) = 0$ for all $x \in V$ is a symplectic form on V. A symplectic form S is non-degenerate if $S(x, y) = 0$ results $y = 0$ for all of the elements $x \in V$.

Now suppose q is a prime power and $q = rc_2 + 1$ such that $r > 1$ and if q is odd then c_2 is even. Let V be a two dimensional vector space over the field $GF(q)$ and S be a non-degenerate symplectic form on V. Consider the multiplicative group of $GF(q)$ and let H be the subgroup of index r of this group. Construct a graph with vertex set $\{Hx|x \in V\backslash 0\}$ such two vertices Hx and Hy are adjacent if and only if $S(x, y) \in H$. This graph is a covering graph of the graph K_{q+1} with parameters $(q + 1, r, c_2)$.

The following construction is generated by De Caen and Fon-der-Flaas in [4] and is known by their names.

Construction 2: Suppose V is a vector space over $GF(2^t)$ with dimension d. Let B be a skew product of V that has the following properties:

- $a \to B(a, a)$ is a bijection.
- $B(a, b) = B(b, a)$ if and only if a and b are linearly dependent.

Let $L = (l_{ij})$ such that $i, j \in F$ be a symmetric latin square whose entries are the elements of $GF(2^t)$. Suppose G is a graph whose vertex set is $V \times F \times V$ and each pair of vertices (x, y, z) and (x', y', z') are adjacent if and only if $z + z' = B(x, y) + B(y, x) + l_{ij}(B(x, x) + B(y, y))$. The graph G is an antipodal distance regular covering graph of a complete graph with parameters $(2^{t(d+1)}, 2^{td}, 2^t)$. By its construction, we can see that G is an abelian covering graph and the automorphism group that fixes each fibre is isomorphic to $\mathbb{Z}_2^{td}$.

The following construction was presented first by Thas [25] and then extended by Somma in [23].

Construction 3: Suppose p is a prime number and $q = p^j$ for some j. Let v be a $2i$-dimensional vector space over $GF(q)$ with non-degenerate symplectic form S. Construct a graph with vertex set $\{(x, y)|x \in GF(q), y \in V\}$ such that two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if $S(y_1, y_2) = x_1 - x_2$ and $y_1 \neq y_2$. This graph is a cover of the complete graph $K_{q^{2i}}$ with parameters (q^{2i}, q, q^{2i-1}).

The following construction has been introduced by Godsil and Hensel in [7] which is known as *The Quotient Construction*.

Construction 4: Let p be a prime and $q = p^i$ for some i. Suppose V is a two-dimensional vector space over $GF(q)$ equipped with a non-degenerate symplectic form S. Let A be an additive subgroup of index p^{i-k} in $GF(q)$ such that $0 \leq k < i$. Construct a graph with the vertex set $\{(A+\alpha, u)|\alpha \in GF(q), u \in V\}$ such that two vertices $(A+\alpha, u)$ and $(A+\beta, v)$ are adjacent if and only if $\alpha-\beta-S(u, v) \in A$ and $u \neq v$. This graph is a covering graph of the complete graph $K_{p^{2i}}$ with parameters $(p^{2i}, p^{i-k}, p^{i+k})$.

The following construction was presented by Biggs in [2].

Construction 5: To present this construction we need the following definitions. A projective plane P is a plane whose elements called points and some subsets of points called lines which satisfies the following conditions:

- each pair of points are in exactly one common line.
- each pair of lines have exactly one common point.
- There exists four points such that none of the triple points of it lies in the same line.

A polarity is a one by one function π between points and lines such that for each pair of points x and y, $x \in \pi(y)$ if and only if $y \in \pi(x)$. Now, we have the following definitions.

The point x is *absolute* if $x \in \pi(x)$ and the line l is absolute if $\pi^{-1}(l) \in l$.

Let P be a projective plane with n^2+n+1 (order n) with polarity π having n absolute points. Suppose line l consists of all absolute points. Construct a graph with vertex set consists of the points of P other than the points on the line l and the point $\pi^{-1}(l)$ such that two vertices x and y are adjacent if and only if $x \in \pi(y)$. This graph is a cover for K_n with parameters $(n, n-2, 1)$.

The following construction which is the last construction was demonstrated by Brouwer in [3].

Construction 5: Suppose G is a strongly regular graph with parameters $(s(t+1), st, 1, t+1)$. Let G have a partition of its vertices into cliques of size $s+1$. Construct a graph by deleting the edges of G which are in these cliques. This graph is a covering graph of the complete graph K_{st+1} with parameters $(st+1, s+1, t-1)$.

Frames

0.10 Introduction

In this chapter, we discuss equiangular tight frames. As we mentioned in Chapter , equiangular tight frames are type of frames and are the most important frames with a finite dimension. One of the problems about equiangular tight frames is the existance of these frames; for example, [10, 20, 24, 22] are about this problem and discuss this problem, or you can find some tables of existence of these frames in [12, 27, 13].

One of the most important problems in this field is constructing equiangular tight frames by combinatorial objects like block designs or graphs. In this thesis, we show the relationship between these frames and covering graphs of complete graphs.

At first, we define frames and present an example of frames and after that, we demonstrate a theorem which has shown an elementary property of frames. Following this, we present the operations which are related to the frames. We will define specifically tight frames, and explain equiangular lines. We demonstrate two bounds for the the size of a set of equiangular lines. These bounds hold if and only if the set of lines is an equiangular tight frame. We will need the Seidel matrices to show the relationship between covering graphs and equiangular tight frames in the next chapter, and so we define these matrices here.

0.11 Frames

As we mentioned, a frame of a vector space is a generalization of the idea of a basis.

A sequence $F = \{f_1, ..., f_n\}$ is a *frame* for a Hilbert space H if

there exist two constants A and B such that for every vector x in H, we have

$$A||x||^2 \leq \sum_{i=1}^{n} |\langle x, f_i\rangle|^2 \leq B||x||^2.$$

We call A and B the frame bounds of F.
To explain furthur, we present an examples of frames. The following example is the famous Mercedes-Benz frame, which is the most well-known frame [21].

Example (Mercedes-Benz Frame). This frame is a set of three vectors

$$F = \{f_1, f_2, f_3\} = \left\{ \left(0, \sqrt{\frac{2}{3}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}\right) \right\}.$$

The set F is a frame for the vector space $\mathbb{R}^2$. Consider the matrix F which has the columns f_1, f_2 and f_3. Thus,

$$F = \begin{pmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ \sqrt{2/3} & -1/\sqrt{6} & -1/\sqrt{6} \end{pmatrix}.$$

Hence,

$$FF^* = \begin{pmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ \sqrt{2/3} & -1/\sqrt{6} & -1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 0 & \sqrt{2/3} \\ -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \end{pmatrix} = I.$$

We have,

$$\sum_{i=1}^{3} |\langle x, f_i\rangle|^2 = FF^*||x||^2 = ||x||^2.$$

Therefore, F is a frame with the frame bounds $A = B = 1$.

There are different kinds of frames. An *equal norm frame* is a frame in which $||f_i|| = ||f_j||$ for all $1 \leq i, j \leq n$. A *uniform norm frame* is an equal norm frame such that for all $1 \leq i \leq n$, $||f_i|| = 1$. A *tight frame* is a frame whose frame bounds are equal; that is, $A = B$. We call a tight frame with frame bounds equal to A, an *A-tight frame*. Tight frames are further described in section 0.14. In the next section, we show the properties of frames.

0.12 Properties of Frames

This section presents some properties of frames. The following theorem shows that a frame is a generalization of a basis, although the vectors of the frame can be linearly dependent.

0.12.1 Theorem. *If a family $F = \{f_i\}_{i\in I}$ of vectors spans the Hilbert space H, then F is a frame of H.*

Proof.

To prove that F is a frame, it is sufficient to find A and B such that for every x in H we have the following inequalities.

$$A||x||^2 \leq \sum\nolimits_{i\in I} |\langle x, f_i\rangle|^2 \leq B||x||^2.$$

To find B, by Cauchy-Schwarz inequality we know that for each $i \in I$,

$$|\langle x, f_i\rangle|^2 \leq \langle x, x\rangle\langle f_i, f_i\rangle = ||x||^2||f_i||^2.$$

Thus, we can write,

$$\sum\nolimits_{i\in I} |\langle x, f_i\rangle|^2 \leq \sum\nolimits_{i\in I} ||x||^2||f_i||^2 = ||x||^2 \sum\nolimits_{i\in I} ||f_i||^2.$$

Therefore, we can consider B as $B = \sum_{i\in I} ||f_i||^2$.
To find A, suppose ϕ is the function from H to $\mathbb{R}$, such that for every x in H,

$$\phi(x) = \sum_{i\in I} |\langle x, f_i\rangle|^2.$$

Consider a unit ball in H. It is bounded and closed, so it is compact. Thus, the set

$$\{\sum\nolimits_{i\in I} |\langle x, f_i\rangle|^2, ||x|| = 1\}$$

has an infimum, and there is a $y \in H$ such that $||y|| = 1$ and,

$$\sum\nolimits_{i\in I} |\langle y, f_i\rangle|^2 = \inf\left\{\sum_{i\in I} |\langle x, f_i\rangle|^2, ||x|| = 1\right\}.$$

Since for each x in H the norm of the vector $\frac{x}{||x||}$ is one, we have:

$$\sum\nolimits_{i\in I} |\langle x, f_i\rangle|^2 = \sum\nolimits_{i\in I} |\left\langle \frac{x}{||x||} f_i \right\rangle|^2 ||x||^2 \geq \sum\nolimits_{i\in I} |\langle y, f_i\rangle|^2.$$

We know that $\sum_{i\in I} |\langle y, f_i\rangle|^2 > 0$, so let $A = \sum_{i\in I} |\langle y, f_i\rangle|^2$, and we have,

$$A||x||^2 \leq \sum\nolimits_{i\in I} |\langle x, f_i\rangle|^2 \leq B||x||^2.$$

Thus, F is a frame. □

0.13 Operations

In this section, we define the frame operations and present the theorems that are related to these operations. The definitions in this section are from [21], Chapter 1. In the following definitions, let H be a real or complex N-dimensional Hilbert space, and L_2 denotes $\ell^2(n)$.

Let $F = \{f_1, ..., f_n\}$ be a frame in H. The associated *analysis operator* $T : H \to L_2$ is

$$Tx := (\langle x, f_i\rangle)_{i=1}^n, x \in H,$$

and the associated *synthesis operator* is defined as

$$T^*y := \sum_{i=1}^n y_i f_i,$$

for $y = (y_i)_{i=1}^n$ in L_2.

By the definition of analysis and synthesis operators, we can show that T^* is the adjoint operator of T. Suppose R is the adjoint of T, then

$$\begin{aligned}\langle Ry, x\rangle = \langle y, Tx\rangle &= \langle (y_i)_{i=1}^n, (\langle x, f_i\rangle)_{i=1}^n\rangle \\ &= \sum_{i=1}^n y_i \overline{\langle x, f_i\rangle} \\ &= \left\langle \sum_{i=1}^n y_i f_i, x \right\rangle.\end{aligned}$$

Therefore we can write R as $\sum\limits_{i=1}^n y_i f_i$, and so $R = T^*$ becomes the adjoint operation of T.

Now, we define the frame operator. Let $F = \{f_1, ...f_n\}$ be a frame with analysis operator T and synthesis operator T^*; the associated *frame operator* S is then defined as follows:

$$Sx := T^*Tx = \sum_{i=1}^{n} \langle x, f_i \rangle f_i,$$

for $x \in H$.

Suppose $F = \{f_1, ..., f_n\}$ is a frame in H with the analysis operator T and synthesis operator T^*; the *Gramian operator* for F is the operator $G : L_2 \to L_2$ such that:

$$Gx = TT^*x = \left(\sum_{i=1}^{n} x_i \langle f_i, f_j \rangle \right)_{j=1}^{n} = \sum_{i=1}^{n} x_i (\langle f_i, f_j \rangle)_{j=1}^{n},$$

for $x = (x_i)_{i=1}^{n}$ in H. we call $G = TT^*$, the Gram matrix of the frame.

0.14 Tight Frames and Equiangular Lines

In Section 0.11 we defined different types of frames, but this thesis specifically discusses tight frames. In this section, at first, we discuss these frames and their properties. After that, we present the definition of special sets of vectors which are called *equiangular lines*, and discuss an example.

In Section 0.17 the relationship between tight frames and equiangular lines will be demonstrated. Tight frames are defined as follows:

A frame $F = \{f_1, ..., f_n\}$ is a *tight frame* if its frame bounds are equal; that is,

$$A||x||^2 \leq \sum_{i=1}^{n} |\langle x, f_i \rangle|^2 \leq A||x||^2.$$

for some constant A. A tight frame F is a *parseval frame*, if its frame bound is equal to one; that is $A = 1$. Next example can clear this definition.

Example. Let $f_1, f_2, ..., f_n$ be an orthonormal basis for H. Then, $F = \{f_1, f_1, f_2, f_2, ..., f_n, f_n\}$ is a tight frame for H with frame bound $A = 2$, since $\{f_1, f_2, ..., f_n\}$ is a parseval frame.

The following definition discusses equiangular lines. after that we present an example about these lines.

A set of lines spanned by vectors $v_1, ..., v_n$ in $\mathbb{C}^d$ (or $\mathbb{R}^d$) is a set of complex (or real) *equiangular lines* if there is $\alpha \in \mathbb{R}$ such that, for all i and j with $i \neq j$, we have $|\langle v_i, v_j \rangle| = \alpha$; that is, the angles between each pair of distinct vectors are the same.
If the angle between each pair of vectors is θ, then for $i \neq j$,

$$\langle v_i, v_j \rangle = v_i^T v_j = \pm \cos(\theta).$$

That is, $\alpha = |\cos(\theta)|$.

Example. Let $E = \{e_1, e_2, ..., e_n\}$ be an orthonormal basis for H, then E is a set of equiangular lines, because for each $1 \leq i < j \leq n$ we have $|\langle e_i, e_j \rangle| = 0$.

In Sections 0.15 and 0.16 we will present two bounds for the size of a set of equiangular lines.

0.15 The Absolute Bound

We explore two bounds for the size of a set of equiangular lines. In [11], the following theorem is demonstrated which provides a bound for a number of equiangular lines in $\mathbb{R}^d$, that is called the *absolute bound*.

0.15.1 Theorem. *(The Absolute Bound) If E is a set of equiangular lines for $\mathbb{R}^d$, then*

$$|E| \leq \binom{d+1}{2}.$$

Proof.

Let $E = \{v_1, v_2, ..., v_n\}$ be a set of n unit vectors in $\mathbb{R}^d$ such that $\langle v_i, v_j \rangle = \alpha$ for each $i \neq j$. Now for each $1 \leq i \leq n$, the matrix $V_i = v_i v_i^T$ is a $d \times d$ matrix in the space of symmetric $d \times d$ matrices. This space has dimension $\binom{d+1}{2}$.

To prove this theorem, we should prove that all V_is are linearly independent in this space. We can easily see that for all $1 \leq i \leq n$, V_i is a symmetric matrix and

$$V_i^2 = V_i V_i = v_i v_i^T v_i v_i^T = v_i v_i^T = V_i.$$

Now we have $V_iV_j = v_iv_i^Tv_jv_j^T$. Thus, $\operatorname{tr}(V_iV_j) = (v_i^Tv_j)^2 = \alpha^2$ for $i \neq j$. Also,

$$\operatorname{tr}(V_i^2) = \operatorname{tr}(V_i) = \operatorname{tr}(v_iv_i^T) = v_i^Tv_i = 1.$$

Now suppose $X = \sum_{i=1}^{n} C_iV_i$ for constants C_i for all $1 \leq i \leq n$. We have

$$\operatorname{tr}(X^2) = \sum_{i,j} C_iC_j \operatorname{tr}(V_iV_j) = \sum_{i=1}^{n} C_i^2 + \sum_{i,j:i\neq j} C_iC_j\alpha^2 =$$

$$\alpha^2 \left(\sum_{i=1}^{n} C_i \right)^2 + (1-\alpha^2) \sum_{i=1}^{n} C_i^2.$$

We know that $1 < \alpha < 0$, so $\operatorname{tr}(X^2) = 0$ if and only if $C_i = 0$ for all $1 \leq i \leq n$. Since V_i is symmetric for all $1 \leq i \leq n$, the matrix X is also symmetric. Thus, $\operatorname{tr}(X^2) \geq 0$ and $\operatorname{tr}(X^2) = 0$ if and only if $X = 0$. Therefore, $X = 0$ if and only if $C_i = 0$ for all $1 \leq i \leq n$. Hence, all V_is are linearly independent. Thus,

$$n \leq \binom{d+1}{2}.$$

□

As an example for absolute bound, we can consider a regular icosahedron centered at 0 in $\mathbb{R}^3$, and the six lines through pairs of opposite vertices. Since $\binom{3+1}{2} = 6$, the absolute bond holds with equality for these lines.

0.16 The Relative Bound

In the following theorem, Van Lint and Seidel [26] present a bound for the number of equiangular lines in $\mathbb{R}^d$, a bound related to d and α.

0.16.1 Theorem. *Let $E = \{v_1, v_2, ..., v_n\}$ be a set of n equiangular lines in $\mathbb{R}^d$ such that for each $i \neq j$, $\langle v_i, v_j \rangle = \alpha$. If $1/\alpha^2 > d$, then*

$$n \leq \frac{d - d\alpha^2}{1 - d\alpha^2}.$$

Equality holds if and only if $\sum_{i=1}^{n} v_i v_i^T = (n/d)I$.

Proof.

Let $V_i = v_i v_i^T$ and $X = I - \frac{d}{n}\sum_{i=1}^{n} V_i$ where $1 \leq i \leq n$.

Each V_i is symmetric, so X is also symmetric. Thus, $\mathrm{tr}(X^2) \geq 0$ and $\mathrm{tr}(X^2) = 0$ if and only if $X = 0$. We have,

$$X^2 = I^2 - 2\frac{d}{n}\sum_{i=1}^{n} V_i + \frac{d^2}{n^2}(\sum_{i=1}^{n} V_i)^2.$$

Thus,

$$\mathrm{tr}(X^2) = \mathrm{tr}(I) - \frac{2d}{n}\sum_{i=1}^{n}\mathrm{tr}(V_i) + \frac{d^2}{n^2}\,\mathrm{tr}(\sum_{i=1}^{n} V_i)^2.$$

Therefore,

$$d - \frac{2d}{n}n + \frac{d^2}{n^2}\big(\sum_{i=1}^{n}\mathrm{tr}(V_i^2) + \sum_{i\neq j}\mathrm{tr}(V_iV_j)\big) \geq 0.$$

We know that $\mathrm{tr}(V_iV_j) = \alpha^2$ and $\mathrm{tr}(V_i)^2 = 1$. Hence,

$$-d + \frac{d^2}{n^2}(n + n(n-1)\alpha^2) \geq 0.$$

By solving this inequality for n, we have $n \leq \frac{d-d\alpha^2}{1-d\alpha^2}$ when $\frac{1}{\alpha^2} > d$.

The equality holds if and only if $X = 0$. Thus, $I - \frac{d}{n}\sum_{i=1}^{n} V_i = 0$, and so $\sum_{i=1}^{n} V_i = \frac{n}{d}I$.

Therefore, the equality holds if and only if $\sum_{i=1}^{n} v_i v_i^T = \frac{n}{d}I$. □

0.17 Equiangular Tight Frames

As can be seen from the name, the set $E = \{v_1, v_2, ..., v_n\}$ of vectors in $\mathbb{R}^d$ is an equiangular tight frame if there is an $\alpha \in \mathbb{R}$ such that for each $i \neq j$, we have $\langle v_i, v_j\rangle = \alpha$, and there exists a constant A such that,

$$\sum_{i=1}^{n} |\langle x, v_i\rangle|^2 = A||x||^2$$

for each $x \in \mathbb{R}^d$.

In fact, the set E is a tight frame if

$$v_1 v_1^T + v_2 v_2^T + ... + v_n v_n^T = \frac{n}{d} I.$$

Example. Looking at the first example of this chapter, which is about the Mercedes-Benz frame, we can see that this frame is an equiangular tight frame. The Mercedes-Benz frame is

$$F = \{f_1, f_2, f_3\} = \left\{ \left(0, \sqrt{\frac{2}{3}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}\right) \right\}.$$

As we proved in that example, $\sum_{i=1}^{3} |\langle x, f_i \rangle|^2 = ||x||^2$ for each $x \in \mathbb{R}^2$. Thus, F is a tight frame with a frame bound equal to 1. Also, we can see that for each $1 \leq i, j \leq 3$ such that $i \neq j$, $\langle f_i, f_j \rangle = -\frac{1}{3}$. Therefore, F is a set of equiangular lines in $\mathbb{R}^2$, and so F is an equiangular tight frame.

If we go back to Sections 0.15 and 0.16, we can write the theorems in these sections as follows.

0.17.1 Theorem. *If E is a set of unit equiangular lines for $\mathbb{R}^d$, then*

$$|E| \leq \binom{d+1}{2}.$$

If equality holds, then E is a tight frame. □

The following theorem shows that equality holds for the relative bound if the set of lines is a tight frame for $\mathbb{R}^d$.

0.17.2 Theorem. *Let $E = \{v_1, v_2, ..., v_n\}$ be a set of n equiangular lines in $\mathbb{R}^d$ such that if $i \neq j$, then $\langle v_i, v_j \rangle = \alpha$. If $\frac{1}{\alpha^2} > d$, then*

$$n \leq \frac{d - d\alpha^2}{1 - d\alpha^2}.$$

The equality holds if and only if $\sum_{i=1}^{n} v_i v_i^T = \frac{n}{d} I$. Thus, the equality holds if and only if E is a frame for $\mathbb{R}^d$. □

0.18 Seidel matrix

Let $E = \{v_1, v_2, ..., v_n\}$ be a set of equiangular lines in $\mathbb{R}^d$, and M be the matrix whose columns are the members of E. Then, the Gram matrix of the vectors of E is as follows

$$G = M^T M.$$

G is a positive semidefinite symmetric matrix which has the same rank with M. Since, for every $n \times n$ symmetric semidefinite matrix we can find a $d \times n$ matrix M such that $G = M^T M$, We can represent the set of equiangular lines by its Gram matrix. Let $E = \{v_1, v_2, ..., v_n\}$ be a set of equiangular lines such that v_i is a unit vector for each $1 \leq i \leq n$ and $v_i^T v_j = \pm\alpha$ for all $1 \leq i, j \leq n$, so we have the following for its Gram matrix G:

$$G = \alpha S + I,$$

such that S is a symmetric matrix whose diagonal entries are zero and off-diagonal entries are ± 1. S can be regarded as the nonstandard adjacency matrix of a graph G if -1 shows the adjacency of two vertices and 1 shows the non-adjacency of two vertices of G.

The matrix S is known as the *Seidel matrix* of G and the following formula shows the relationship between the Seidel matrix of G and its adjacency matrix $A(G)$,

$$S(G) = J - I - 2A(G).$$

The following theorem, which has been proved by Lemmens and Seidel [11], demonstrates that we can have two sets of equiangular lines from a Seidel matrix which are related to its largest and smallest eigenvalues.

0.18.1 Theorem. *Suppose S is an $n \times n$ Seidel matrix whose least eigenvalue is θ and largest eigenvalue is τ with multiplicities m_θ and m_τ respectively. We have the following Gram matrices of two sets of equiangular lines.*

- *There is a sets of equiangular lines in dimension $n - m_\theta$ with Gram matrix $I - (1/\theta)S$.*
- *There is a set of equiangular lines in dimension $n - m_\tau$ with Gram matrix $I - (1/\tau)S$.*

In addition, a set of equiangular lines $E = \{v_1, v_2, ..., v_n\}$ *with* $v_i^T v_j = \alpha$ *for all* $1 \leq i < j \leq n$ *is an equiangular tight frame if and only if its corresponding Seidel matrix has two distinct eigenvalues* $\theta = -\frac{1}{\alpha}$ *and* $\tau = \frac{n-d}{\alpha d}$ *with multiplicities* $m_\theta = n - d$ *and* $m_\tau = d$ *respectively.* □

New constructions of equiangular tight frames

0.19 Introduction

One of the important problems in frame theory is constructing frames by combinatorial objects. Since tight frames are the most important kinds of frames, discussing the constructions of these frames attracts more attention.

In this chapter we will discuss new results about the constructions of equiangular tight frames. At first we show the constructions which are from design geometries. After that we discuss Steiner equiangular tight frames. Following this, we present equiangular tight frames from hyperovals. We present a theorem to show the relation between equiangular tight frames and projective planes. At last, we demonstrate an abstract of the new results about equiangular tight frames.

0.20 Design geometries

In this section, we present the new results in constructing equiangular tight frames. At first we need some definitions which are useful to show these constructions.

A balanced incomplete block design is the finite set V of n points and any set B of the subsets of V whose elements are called blocks, such that there exist positive integers λ, k and r with the following properties.

(a) The size of each block is exactly k,

(b) every points is in exactly r blocks,

(c) each pair of points is in exactly λ blocks.

We denote these designs by (n, k, λ). They have the following characteristics:

(a) Each point is in exactly $r = (n-1)/(k-1)$ blocks.

(b) The size of B is $|B| = n(n-1)/k(k-1)$.

A Steiner system (t, k, n) is a set A of n points and a collection B of k-subsets of A which are called blocks such that any t points of A are exactly in one sets of B. This Steiner system has the following characteristics:

(a) Each point is in exactly r blocks such that

$$r = \frac{\binom{n-1}{t-1}}{\binom{k-1}{t-1}}.$$

(b) The size of B; that is, the total number of blocks is

$$|B| = \frac{nr}{k}.$$

A Steiner system is *resolvable*, if we can partition its blocks into disjoint subcollections $\{B_r\}_{r\in R}$ such that the blocks in each B_r form a partition for V.

Affine planes are an example of Steiner systems which are defined as follows.

An *affine plane* is a set of points V and a collection B of subsets of V which are called blocks with the following properties.

(a) Each pair of points is exactly in one block,

(b) each block has at least two points,

(c) For any point and block either the point is in block, or there is a unique block which contains a point and does not have a common point with the block,

(d) there are three points which are not in a common block.

The incidence matrix X of a Steiner system is the matrix whose rows are corresponding to the points of the system, and whose columns are corresponding to the blocks of the system such that:

$$X(i,j) = \begin{cases} 1 & \text{if } i \in j, \\ 0 & \text{if } i \notin j. \end{cases}.$$

A *projective plane* of order q is a collection of q^2+q+1 lines and q^2+q+1 points such that:

(a) Every point is on $q+1$ lines,

(b) every line contains $q+1$ points,

(c) any two distinct points lie on exactly one line,

(d) any two distinct lines have exactly one common point.

Therefore, a projective plane is a balanced incomplete block design with $\lambda = 1$.

A *hyperoval* in a projective plane of order q is the set of $q+2$ points such that no three of these points are in a common line.

In [9], Jasper, Mixon and Fickus present a construction from resolvable Steiner systems, providing new families of constant-amplitude equiangular tight frames. We present their construction as follows.

0.20.1 Theorem. *Suppose (V,B) is an resolvable $(2,k,v)$-Steiner system. Assume $\{f_i\}_{i=0}^{R}$ is a unimodular regular simplex in $\mathbb{C}^R$ and $\{h_s\}_{s\in S}$ is a unimodular orthogonal basis for $\mathbb{C}^S$. Let $\{B_r\}_{r\in R}$ be a partition of B such that for any r, the subcollection $B_r = \{b_{r,s}\}_{s\in S}$ is a partition of V.*

Let $M = R\times S$ and $N = V\times\{0,...,R\}$, then $\{\psi_{i,k}\}_{(i,k)\in N}$ is a Steiner equiangular tight frame for the space $\mathbb{C}^M$ as follows:

$$\psi_{i,k}(r,s) := R^{-1/2}\begin{cases} f_i(r) & \text{if } k \in b_{r,s}, \\ 0 & \text{if } k \notin b_{r,s}. \end{cases}$$

In addition, we have the following Kirkman equiangular tight frame

$$\psi_{i,k}(r,s) := B^{-1/2} f_i(r) h_{s(r,k)}(s),$$

such that for each $r\in R$ and $k\in V$, $s(r,k)$ is the unique $s\in S$ where $k\in b_{r,s}$. □

By the above theorem, they demonstrate some constructions of equiangular tight frames as follows. All of the these constructions are from [9].

Affine geometries over finite fields: Let q be a prime power and $j \geq 1$. There is a resolvable $(2, k, n)$-Steiner system with $k = q$ and $n = q^{j+1}$. The points of this design are the vectors in $\mathbb{F}_q^{j+1}$ such that $\mathbb{F}_q$ is the finite field of order q. The blocks of this design are the sets of the form $\{au + v : a \in \mathbb{F}_q\}$ for some initial point $v \in \mathbb{F}_q^{j+1}$ and direction vector $u \in \mathbb{F}_q^{j+1} \setminus \{0\}$. We can partition its blocks into disjoint subcollections $\{B_u\}_{u \in \mathbb{F}_q^{j+1} \setminus \{0\}}$ such that each $\{B_u\}$ is a collection of blocks $\{au + v : a \in \mathbb{F}_q\}$ for a constant u.

For the Kirkman equiangular tight frame with N vectors in a space with dimension M, we have:

$$M = q^j\big(\frac{q^{j+1} - 1}{q - 1}\big), N = q^{j+1}\big(\frac{q^{j+1} - 1}{q - 1} + 1\big).$$

Denniston designs: Let i and j be two positive integers such that $i < j$. There is a resolvable $(2, k, n)$-Steiner system with $k = 2^i$ and $n = 2^{i+j} + 2^i - 2^j$. The resulting Kirkman equiangular tight frame has

$$N = (2^j + 1)(2^j + 1 - 2^{j-i})$$

vectors in a M-dimensional space such that

$$M = 2^i(2^j + 2)(2^j + 1 - 2^{j-i}).$$

Kirkman's Schoolgirl problem: Let $n \equiv 3 \bmod 6$. There is a resolvable $(2, 3, n)$-Steiner system. The resolution of this system is the subcollections $\{B_r\}_{r \in R}$ of its blocks such that each $\{B_r\}$ is a collection of 3-subsets of $\{1, ..., n\}$ which are pairwise disjoint. The resulting Kirkman equiangular tight frame has $\frac{n(n+1)}{2}$ vectors in a $\frac{n(n-1)}{6}$-dimensional space.

Three-dimensional projective geometries: If q is a prime power, then there is a resolvable $(2, q + 1, q^3 + q^2 + q + 1)$-Steiner system. The resulting Kirkman equiangular tight frame has

$$N = (q^2 + 1)(q^2 + q + 1)$$

vectors in M-dimensional space such that

$$M = (q^2 + q + 2)(q^3 + q^2 + q + 1).$$

0.21 Steiner equiangular tight frames

In [18] Fickus, Mixon and Tremain present a construction of equiangular tight frames from Steiner systems and provide some examples of these constructions from different kinds of Steiner systems. After that in [14] Fickus, Mixon and Jasper generalize the construction in [18] to construct a new infinite family of complex equiangular tight frames. We present their construction as the following theorem, and provide their example to show how this construction works. After that, we demonstrate one of their constructions from Steiner system that is a construction from projective geometries.

0.21.1 Theorem. *We can construct an equiangular tight frame from every $(2, k, n)$-Steiner system with $N = n(1 + \frac{n-1}{k-1})$ vectors in $M = \frac{n(n-1)}{k(k-1)}$-dimensional space.*

In addition, if there exists a real Hadamard matrix of size $1+\frac{n-1}{k-1}$, then a $\frac{n(n-1)}{k(k-1)} \times n\big(1 + \frac{n-1}{k-1}\big)$ equiangular tight frame matrix will be constructed as follows:

(a) Suppose A^T is the $\frac{n(n-1)}{k(k-1)} \times n$ transpose of the incidence matrix of $(2, k, n)$-Steiner system.

(b) For each $1 \leq i \leq n$, suppose Hadamard matrix H be any

$$\left(1 + \frac{n-1}{k-1}\right) \times \left(1 + \frac{n-1}{k-1}\right)$$

matrix whose rows are orthogonal and entries are unimodular, such as a possibly complex Hadamard matrix.

(c) For each $1 \leq i \leq n$, suppose F_i is a $\frac{n(n-1)}{k(k-1)} \times (1 + \frac{n-1}{k-1})$ matrix which is generated by replacing each of the one-valued entries of the ith column of A^T with a distinct row of H_i and every zero-valued with a row of zeros.

(d) concatenate and rescale the F_i's to reach the matrix

$$F = \left(\frac{k-1}{n-1}\right)^{1/2} [F_1 F_2 ... F_n].$$

The columns of matrix F form an equiangular tight frame.

Proof. To prove that F is a tight frame we have to prove that the inner product of each pair of distinct rows of F is zero and the rows of F have constant norm. The inner product of each pair of distinct rows is the sum of the inner products of the corresponding rows of the F_i's over all $1 \leq i \leq n$. For any $1 \leq i \leq n$, the inner product of each pair of rows of F_i is zero, because either these rows correspond to the rows of H_i which are orthogonal, or one of these rows is zero and so their inner product is zero. To prove that the rows of F have constant norm, we know that each block of a $(2, k, n)$-Steiner system has k elements in it and so each row of A^T has k ones. For each $1 \leq i \leq n$, the matrix H_i has entries which have the same absolute value, thus the squared-norm of any row of F is the squared-scaling factor $\frac{k-1}{n-1}$ times a sum of $k\big(1 + \frac{k-1}{n-1}\big)$ ones; that is,

$$\frac{M}{N} = \frac{k-1}{n-1}\left(1 + \frac{n-1}{k-1}\right) = k\left(1 + \frac{k-1}{n-1}\right).$$

To prove that F is equiangular we should show that each of the columns of F has unit norm and the inner product of each pair of distinct columns has constant modulus. To prove that each column has unit form, we should find the norm of each column of F_i for each $1 \leq i \leq n$. The entries of H_i have the same absolute values and each column of A^T contains $\frac{n-1}{k-1}$ ones, because each point is in $\frac{n-1}{k-1}$ blocks. The squared norm of each column of F is $\frac{k-1}{n-1}$ times the squared norm of a column of F_i's, and so it is equal to

$$\left(\frac{k-1}{n-1}\right)\left(\frac{n-1}{k-1}\right)1 = 1.$$

Now, we prove that the inner product of each pair of distinct columns of F has constant modulus. We know that each pair of distinct blocks have a unique common point, thus any pair of distinct columns of A^T have a single entry of mutual support, so any pair of distinct columns of F that arise from distinct F_i bocks have a single entry of mutual support. Therefore, the inner product of such columns is $\frac{k-1}{n-1}$ times the product of two unimodular numbers. Thus, the square-magnitude of inner product of each pair of columns is $\frac{N-M}{M(N-1)} = \big(\frac{k-1}{n-1}\big)^2$. For each $1 \leq i \leq n$, the matrix H_i is a scalar multiple of a unitary matrix, thus its columns are orthogonal. The matrix F_i contains all of the rows of the matrix H_i except one of

them, namely one for each of the 1-valued entries of A^T. Therefore, the inner product of the rows in the common part of H_i and F_i is equal to zero minus the contribution from the left over entries. The matrix H_i is unimodular, so the squared-magnitude of such inner product is $\frac{N-M}{M(N-1)} = \left(\frac{k-1}{n-1}\right)^2$. As a result, the inner product of each pair of columns of F has constant modulus. Thus, F is an equiangular tight frame. □

The following example shows how this construction works.

Example. As an example, we will find an equiangular tight frame F for $(2, 2, 4)$-Steiner system. The transpose of incidence matrix of this system is:

$$A^T = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Since there are three 1s in each column of A^T, we need a 4×4 matrix H with unimodular entries and orthogonal rows. We consider H as follows:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Now, we replace each 1-valued entries of A^T with a distinct row of H. For each one entry we choose either the second, third or fourth row of H. As a result, the following matrix F is an equiangular tight frame with $M = 6$ and $N = 16$.

$$F = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

We can easily see that the rows of F are orthogonal and have constant norm. The inner product of each pair of columns which are from the same block is $-\frac{1}{3}$ and from distinct blocks is $\pm\frac{1}{3}$. Thus, F is an equiangular tight frame.

Projective Geometries: Let q be a prime power, then for any $n \geq 2$, there exists a $(2, k, v)$-Steiner system such that $k = q+1$ and $n = \frac{q^{n+1}-1}{q-1}$. These projective geometries generate equiangular tight frames with $\frac{q^{n+1}-1}{q-1}(1 + \frac{q^n-1}{q-1})$ for a $\frac{(q^n-1)(q^{n+1}-1)}{(q+1)(q-1)^2}$-dimensional space.

0.22 Construction from hyperovals

We present the construction of equiangular tight frames which has been demonstrated in [14] as follows. At first, we need some definitions and lemmas to prove the main theorem.

Suppose X is an incidence matrix of a balanced incomplete block design $(n, k, 1)$ with n points and b blocks. For each $i = 1, ..., n$, a corresponding *embedding* $E_i : \mathbb{F}^r \to \mathbb{F}^b$ is an operator which maps $\mathbb{F}^r$ to the subspace of $\mathbb{F}^b$ which consists of vectors supported on

$$\{j : X(j, i) = 1\}.$$

This subspace is an r-dimensional subspace.

For any $r \geq 3$, a corresponding *unimodular cosimplex* is a sequence $\{c_k\}_{k=1}^{r-1}$ in $\mathbb{F}^r$ with the following properties:

(a) For each $1 \leq i \leq r-1$, the vector c_i has unimodular entries.

(b) For each $1 \leq i \leq r-1$, the last two entries of c_i sum to zero.

(c) For all $1 \leq i, i' \leq r-1$, we have $|\langle c_i, c_{i'} \rangle| = 1$.

Let X be a matrix with the following form

$$X = \begin{bmatrix} X_{1,1} & X_{1,2} \\ 0 & X_{2,2} \end{bmatrix}$$

such that $X_{1,1}$ is the incidence matrix of a balanced incomplete block design $(q+2, 2, 1)$, and the matrix $X_{2,2}^T$ is the incidence matrix of a balanced incomplete block design $(\frac{1}{2}q(q-1), \frac{1}{2}q, 1)$ where q is even.

Let Y be a matrix with the following form

$$Y = \begin{bmatrix} Y_{1,1} & Y_{1,2} \\ 0 & Y_{2,2} \end{bmatrix}$$

such that $Y_{1,1}$ is the incidence matrix a balanced incomplete block design $(\frac{1}{2}q(q-1), \frac{1}{2}q, 1)$ and $Y_{2,2}^T$ is the incidence matrix of a balanced incomplete block design $(q+1, 2, 1)$.

The following theorem shows there exists an equiangular tight frame from a projective plane which contains a hyperoval. This theorem has been proved by Fickus, Mixon and Jasper in [14]. First we need the following definition.

Suppose D is a balanced incomplete block design with parameter $(n, k, 1)$. For each $1 \leq i \leq n$, a corresponding embedding is an operator $E_i : \mathbb{F}^r \to \mathbb{F}^s$ which maps the standard basis $\mathbb{F}^r$ to the standard basis of the subspace of $\mathbb{F}^s$ which consists of the vectors supported on $\{k : D(k, i) = 1\}$ and has dimension r.

0.22.1 Theorem. *Consider a projective plane of order q which contains a hyperoval. Suppose $\{E_i\}_{i=1}^{q^2}$ are embeddings arising from an affine plane of the form of the matrix Y. Assume $\{s_l\}_{l=1}^{q+2}$ and $\{c_l\}_{l=1}^{q}$ respectively are a unimodular simplex and cosimplex for $\mathbb{F}^{q+1}$. Therefore if $P = \{E_i s_l\}$ for $1 \leq i \leq \frac{1}{2}q(q-1)$ and $1 \leq l \leq q+2$, and $Q = \{E_i c_l\}$ for $\frac{1}{2}q(q-1)+1 \leq i \leq q^2$ and $1 \leq l \leq q$,then the set of vectors*

$$P \cup Q$$

is an equiangular tight frame with $q(q^2+q-1)$ vectors in the (q^2+q-1)-dimensional subspace of $\mathbb{F}^{q(q+1)}$ which consists of the vectors whose last $q+1$ entries sum to zero.

In addition, let $\{E_i\}_{i=1}^{q^2+q+1}$ be embeddings arising from the projective plane of the form of the matrix X. Assume $P' = \{E_i s_l\}$ for $1 \leq i \leq \frac{1}{2}q(q-1)$ and $1 \leq l \leq q+2$, and $Q' = \{E_i c_l\}$ for $\frac{1}{2}q(q-1)+1 \leq i \leq q^2+q+1$ and $1 \leq l \leq q$. Thus, the set of vectors

$$P' \cup Q'$$

is an equiangular tight frame with $q^2(q+2)$ vectors for the subspace of $\mathbb{F}^{q^2+q+1}$ with dimension $q(q+1)$ and consists of the vectors whose last $q+2$ entries sum to zero. □

0.23 Projective planes

In [17] Fickus, Jasper, Mixon, Peterson and Watson present the following theorem which shows that there is an equiangular tight

frame from some projective planes with $\lambda = 1$. We present this theorem as follows.

0.23.1 Theorem. *Let ϕ be a matrix whose entrywise squared-modulus $|\phi|^2$ is the incidence matrix of a balanced incomplete block design $(v, k, 1)$ with v points and b blocks. The columns of $|\phi|$ form an equiangular tight frame for their span if and only if*

$$\sum_{k=1}^{b}\sum_{l=1}^{v}\phi(i,l)\overline{\phi(k,l)}\phi(k,j) = 0 \text{ for all } i \text{ and } j \text{ such that } \phi(i,j) = 0. \tag{0.23.1}$$

Proof. Let $r = \frac{v-1}{k-1}$. By using Lemma **??** we can see that the columns of ϕ generate an equiangular tight frames for their span if and only if $\phi\phi^*\phi = a\phi$ such that

$$a = \frac{rn}{d} = r\frac{v}{d} = r + k - 1.$$

Thus, the columns of ϕ generate an equiangular tight frame for their span if and only if the following holds.

$$\begin{aligned}&(r+k-1)\phi(i,j) = (\phi\phi^*\phi)(i,j) = \\ &\sum_{k=1}^{b}\sum_{l=1}^{v}\phi(i,l)\overline{\phi(k,l)}\phi(k,j) \text{ for all } i \text{ and } j.\end{aligned} \tag{0.23.2}$$

Let $|\phi(i,j)| = 1$. The sum in (0.23.2) is nonzero if and only if the following holds.

$$|\phi(i,j)| = |\phi(i,l)| = |\phi(k,l)| = |\phi(k,j)| = 1. \tag{0.23.3}$$

Thus, the sum in (0.23.2) is nonzero if and only if ith and kth points are in both jth and lth blocks. We know that two distinct points are exactly in one block, and the intersection of two distinct blocks is exactly one point. Therefore, the equation (0.23.3) holds if either $i = k$ or $j = l$.

Hence, if $|\phi(i,j)| = 1$, then the sum in (0.23.2) is as follows:

$$\begin{aligned}&\phi(i,j)\overline{\phi(i,j)}\phi(i,j) + \sum_{l=1,l\neq j}^{v}\phi(i,l)\overline{\phi(i,l)}\phi(i,j) \\ &+ \sum_{k=1,k\neq i}^{b}\phi(i,j)\overline{\phi(k,j)}\phi(k,j) = (r+k-1)\phi(i,j).\end{aligned} \tag{0.23.4}$$

Therefore, the columns of ϕ generate an equiangular tight frame if and only if the Equation (0.23.1) holds. □

0.24 More results in equiangular tight frames

In [15] Fickus, Mixon and Tremain presented a new method for constructing equiangular tight frames, that is provided by a tensor-like combination of a Steiner system and a regular simplex. This construction is valid in both the real and complex settings.

In [19] Fickus, Mixon, Peterson and Jasper revisited Steiner equiangular tight frames. They showed that we can construct an equiangular tight frame from the synthesis operator of a Steiner equiangular tight frame. Their proof is more formal and symbolic than the proof which has been demonstrated in [18]

In [16] Fickus, Jasper, Mixon and Peterson provided a new infinite family of complex equiangular tight frames which are called Tremain equiangular tight frames. In some cases, the construction lead to a new strongly regular graphs or a new finite family of distance regular antipodal covering graphs of complete graphs.

Covers and equiangular tight frames

0.25 Introduction

Recently, there has been huge attention to equiangular tight frames, due to the fact that the relationship between these frames and quantum information theory was established. For example, in [18] Fickus, Mixon and Tremain have constructed euiangular tight frames from Steiner systems, and in [13] Fickus and Mixon provide a survey of constructions of equiangular tight frames.

In this chapter, we present the relationship between covering graphs of complete graphs and equiangular tight frames. At first, we show that if there exists an antipodal distance regular covering graph of a complete graph, then there is an equiangular tight frame corresponding to the graph.

After that, conversely we present a theorem which shows that we can construct an abelian covering graph of a complete graph from every equiangular tight frame. To prove this theorem we need the definition of two-graphs which has been explained in this chapter. All of the proofs in this chapter are from Coutinho, Godsil, Shirazi and Zhan [5].

0.26 Equiangular Tight Frames from Covering Graphs

In this section we present a theorem which shows us how we can build an equiangular tight frame from an abelian covering graph of the complete graph.

0.26.1 Theorem. *Suppose G is an abelian antipodal distance regular graph which is a covering graph of a complete graph with parameters (n, r, c_2) and symmetric arc function f. Let $n-1$, -1, θ and τ be the eigenvalues of G with multiplicities 1, $n-1$, m_θ and m_τ, and let ψ be a non-trivial character of $\langle f \rangle$. Then, we have the following:*

(a) There is an equiangular tight frame with n lines in dimension $\big(n - \frac{m_\theta}{r-1}\big)$,

(b) There is an equiangular tight frame with n lines in dimension $\big(n - \frac{m_\tau}{r-1}\big)$.

Proof. Suppose $\psi_1, \psi_2, ..., \psi_r$ are linear characters of $\langle f \rangle$ such that ψ_1 is the trivial character of $\langle f \rangle$. Thus, $A(K_n)^{\psi_1(f)}$ is equal to the adjacency matrix of K_n, and its eigenvalues are $(n-1)$ and (-1) with multiplicities 1 and $(n-1)$. By using Theorem 0.8.1, we know that $A(G)$ is similar to the following matrix:

$$\begin{pmatrix} A(K_n)^{\psi_1(f)} & & & \\ & A(K_n)^{\psi_2(f)} & & \\ & & \ddots & \\ & & & A(K_n)^{\psi_r(f)} \end{pmatrix}.$$

Thus, the eigenvalues of the matrices $A(K_n)^{\psi_i(f)}$ for $i = 2, ..., r$ are the multiple copies of θ and τ. We know that the trace of each of the matrices $A(K_n)^{\psi_i(f)}$ for $i = 2, ..., r$ is zero. Therefore, the multiplicities of their eigenvalues are not dependent on i, and so the multiplicity of θ is the same for all of these matrices. By the same discussion, we know that the multiplicity of τ is also the same for all of these matrices. Hence, the multiplicities of θ and τ for each of the matrices $A(K_n)^{\psi_i(f)}$ for all $i = 2, ..., r$ are $\frac{m_\theta}{(r-1)}$ and $\frac{m_\tau}{(r-1)}$ respectively. By using Theorem 0.18.1, we can conclude that there are n equiangular lines in dimension $n - \frac{m_\theta}{(r-1)}$, and there are n equiangular lines in dimension $n - \frac{m_\tau}{(r-1)}$. □

0.27 Two-graphs

In this section, we will define two-graph which is a combinatorics object. All of the definitions in this section are from Godsil and Royle [8]. At first, we present the definition of switching class.

Let $E = \{v_1, v_2, .., v_n\}$ be a set of n equiangular lines. We can also replace each of the vectors of E by its negative; that is, we can replace v_i by $-v_i$ for each $1 \leq i \leq n$. Therefore, there are 2^n possibilities for E. As we demonstrated in Section 0.18, each of these possibilities has its own Gram matrix, and we can derive a graph from these matrices which are different from each other. Now, suppose P is a subset of $\{1, ..., n\}$, and replace each of the vectors v_i for $i \in P$ by $-v_i$ to reach the set E'. Thus, the Gram matrix of E' is constructed by multiplying -1 to all of the columns and rows of Gram matrix of E which are corresponding to the elements of P. Let G be the graph which is derived from the matrix of E and G' be the graph that is derived from the Gram matrix of E' by the method which was explained in Section 0.18. We can easily see that G' can be constructed from G by removing the edges of G which are between P and $G \setminus P$ and put the edges between every pair of non-adjacent vertices, such that one of them is in P and the other one is in $G \setminus P$. This operation is known as *switching* on the subset P.

Let G be a graph and $P \subset V(G)$, and suppose that we apply the switching operation on P. The result graph is denoted by G^P. Let Δ be the symmetric difference operator. Then, for each pair of subsets P and Q of $V(G)$ we have the following:

(a) $G^P = G^{V(G)\setminus P}$,

(b) $(G^P)^Q = (G^Q)^P = G^{P\Delta Q}$.

The *switching class* of G is the collection of all graphs which are reached by switching on every subset of the set of vertices of G. Another name for switching class is *two-graph*.

In the next section, we will demonstrate how we can construct a two-graph from a set of equiangular lines.

0.28 Covering Graphs from Equiangular Tight Frames

In this section we show that we can build a covering graph from an equiangular tight frame which was proved by Coutinho, Godsil, Shirazi, and Zhan [5].

The following lemma demonstrates that we can construct a two-graph from a set of equiangular lines.

0.28.1 Lemma. *Let $E = \{v_1, v_2, ..., v_n\}$ be a set of n equiangular lines with Seidel matrix S. There exists a graph G which is a 2-fold cover of K_n whose adjacency matrix is derived from S. If E is an equiangular tight frame, then S has only two distinct eigenvalues, and G is an abelian antipodal distance regular graph which is a covering graph of the complete graph K_n.* □

To prove the above lemma, it is sufficient to replace each of its off-diagonal entries by a 2×2 matrix as follows:

replace each -1 by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and replace each 1 by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We know that the diagonal entries of S are zero and its off-diagonal entries are ± 1.

We can see easily that the new matrix is a $2n \times 2n$ matrix which is the adjacency matrix of a 2-fold covering graph of K_n; That is, the graph G in the lemma. These graphs are equivalent to regular two-graphs.

0.28.2 Theorem. *Let $E = \{v_1, v_2, ..., v_n\}$ be an equiangular tight frame in $\mathbb{C}^d$ with Gram matrix G, such that $\langle v_i, v_j \rangle = \alpha$ for each pair of $1 \leq i, j \leq n$ with $i \neq j$ and a constant α. Suppose $S = \frac{1}{\alpha}(G - I)$. If all of the off-diagonal entries of S are r-th roots of unity for some prime r, then there is a cyclic antipodal distance regular graph with parameters (n, r, c_2) which is a covering graph of K_n and*

$$c_2 = \frac{1}{r}\left((n-2) + \frac{(2d-n)}{d\alpha}\right).$$

Proof. Suppose M_r is the multiplicative group of the r-th roots of unity, and ψ is a representation of M_r with degree k. Let S^ψ be a matrix wich is constructed by replacing each of the off-diagonal entries of S by its image under ψ and replacing each of the diagonal entries of S by a $k \times k$ block of 0s.

If ψ is a regular representation of M_r, then S^ψ is the adjacency matrix of some graph H. We will show that H is an antipodal distance regular covering graph of K_n for some n.

Suppose $\psi_1, \psi_2, ..., \psi_r$ are the linear characters of M_r, such that :

$$\psi_k(e^{2\pi i/r}) = e^{(k-1)2\pi i/r}.$$

We have $A(K_n) = S^{\psi_1}$ and $S = S^{\psi_2}$. Now, we have the following claim.

Claim: The minimal polynomials of the matrices S^{ψ_l} are all equal for all $2 \leq l \leq r$.

Suppose $p_r(t)$ is the r-th cyclotomic polynomial. Since r is a prime number, we have

$$\phi(t) = t^{r-1} + t^{r-2} + ... + t + 1.$$

We know

$$\mathbb{Q}(e^{2\pi i/r}) \cong \mathbb{Q}(t)/\langle \phi_r(t) \rangle.$$

Thus, S can be written as a matrix such that its off-diagonal entries are the indeterminate t subject to $p(t) = 0$.

For matrix A, let $m(A)$ be the minimal polynomial of A. Since for any j, we know $\phi_j(e^{2\pi i/r})$ is a root of the minimal polynomial $P_r(t)$, then $m(S^{\phi_j})$ will vanish. Since the set of lines is equiangular tight frame, it meets the relative bound, and so

$$\alpha^2 = \frac{n-d}{(n-1)d}.$$

By using Theorem 0.18.1, we can conclude that $m(S)$ has degree two, and for all js, the matrix S^{ϕ_j} is not a multiple of the identity matrix. Therefore, the claim holds.

We can see that the trace of all of the matrices S^{ψ_l} is zero. Hence, all of these matrices are cospectral. The matrix S^{ψ} is a block diagonal matrix whose blocks are the matrices S^{ψ_l} for $1 \leq l \leq r$, due to the fact that the eigenvalues of the regular representation of an abelian group are its characters. By using the above facts and using the Theorem 0.18.1 for the expression for the eigenvalues of S, it has been proved that the graph H is an r-fold covering graph of the complete graph with the eigenvalues $n-1$, $\frac{n-d}{\alpha d}$, -1 and $\frac{-1}{\alpha}$ with multiplicities 1, $d(r-1)$, $n-1$ and $(r-1)(n-d)$ respectively. Since r is prime, the graph H is connected, this has been proved by Godsil and Hensel in [7]. By using the above facts and using the Theorem 0.8.1, we can conclude that H is an antipodal distance regular covering graph of the complete graph K_n with parameters (n, r, c_2). □

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Appendix

n	r	c_2	a_1	θ	τ	m_θ	m_τ
5	3	1	1	2	-2	5	5
6	2	2	2	2.236067977	-2.236067977	3	3
7	5	1	1	2.449489743	-2.449489743	14	14
8	3	2	2	2.645751311	-2.645751311	8	8
9	3	3	1	2	-4	12	6
	5	1	3	4	-2	12	24
	7	1	1	2.828427125	-2.828427125	27	27
10	2	4	4	3	-3	5	5
	4	2	2	3	-3	15	15
11	3	3	3	3.16227766	-3.16227766	11	11
	9	1	1	3.16227766	-3.16227766	44	44
12	5	2	2	3.31662479	-3.31662479	24	24
13	11	1	1	3.464101615	-3.464101615	65	65
14	2	6	6	3.605551275	-3.605551275	7	7
	3	4	4	3.605551275	-3.605551275	14	14
	6	2	2	3.605551275	-3.605551275	35	35
15	13	1	1	3.741657387	-3.741657387	90	90
16	2	6	8	5	-3	6	10
	2	8	6	3	-5	10	6
	3	4	6	5	-3	12	20
	4	4	2	3	-5	30	18
	6	2	4	5	-3	30	50
	8	2	0	3	-5	70	42
17	3	5	5	4	-4	17	17
	5	3	3	4	-4	34	34
	15	1	1	4	-4	119	119

Figure 4: The table of feasible parameters of abelian covering graphs of complete graphs for $n \leq 17$

n	r	c_2	a_1	θ	τ	m_θ	m_τ
18	2	8	8	4.123105626	-4.123105626	9	9
	4	4	4	4.123105626	-4.123105626	27	27
	8	2	2	4.123105626	-4.123105626	63	63
19	4	5	2	3	-6	38	19
	7	2	5	6	-3	38	76
	10	2	-1	3	-6	114	57
	17	1	1	4.242640687	-4.242640687	152	152
20	3	6	6	4.358898944	-4.358898944	20	20
	9	2	2	4.358898944	-4.358898944	80	80
21	4	5	4	4	-5	35	28
	10	2	1	4	-5	105	84
	19	1	1	4.472135955	-4.472135955	189	189
22	2	10	10	4.582575695	-4.582575695	11	11
	5	4	4	4.582575695	-4.582575695	44	44
	6	4	0	3	-7	77	33
	10	2	2	4.582575695	-4.582575695	99	99
23	3	7	7	4.69041576	-4.69041576	23	23
	7	3	3	4.69041576	-4.69041576	69	69
	21	1	1	4.69041576	-4.69041576	230	230
25	3	7	9	6	-4	20	30
	5	5	3	4	-6	60	40
	7	3	5	6	-4	60	90
	21	1	3	6	-4	200	300
	23	1	1	4.898979486	-4.898979486	275	275
26	2	12	12	5	-5	13	13
	3	8	8	5	-5	26	26
	4	6	6	5	-5	39	39
	6	4	4	5	-5	65	65
	12	2	2	5	-5	143	143
27	25	1	1	5.099019514	-5.099019514	324	324
28	2	10	16	9	-3	7	21
	2	16	10	3	-9	21	7
	4	8	2	3	-9	63	21
	5	4	10	9	-3	28	84
	8	4	-2	3	-9	147	49
	10	2	8	9	-3	63	189
	13	2	2	5.196152423	-5.196152423	168	168
	16	2	-4	3	-9	315	105

Figure 5: The table of feasible parameters of abelian covering graphs of complete graphs for $18 \leq n \leq 28$

n	r	c_2	a_1	θ	τ	m_θ	m_τ
29	3	9	9	5.291502622	-5.291502622	29	29
	9	3	3	5.291502622	-5.291502622	116	116
	12	2	5	7	-4	116	203
	27	1	1	5.291502622	-5.291502622	377	377
30	2	14	14	5.385164807	-5.385164807	15	15
	7	4	4	5.385164807	-5.385164807	90	90
	14	2	2	5.385164807	-5.385164807	195	195
31	29	1	1	5.477225575	-5.477225575	434	434
32	3	10	10	5.567764363	-5.567764363	32	32
	5	6	6	5.567764363	-5.567764363	64	64
33	3	9	13	8	-4	22	44
	5	7	3	4	-8	88	44
	7	5	1	4	-8	132	66
	9	3	7	8	-4	88	176
	27	1	5	8	-4	286	572
	31	1	1	5.656854249	-5.656854249	495	495
34	2	16	16	5.744562647	-5.744562647	17	17
	4	8	8	5.744562647	-5.744562647	51	51
	8	4	4	5.744562647	-5.744562647	119	119
	16	2	2	5.744562647	-5.744562647	255	255
35	3	11	11	5.830951895	-5.830951895	35	35
	11	3	3	5.830951895	-5.830951895	175	175
	33	1	1	5.830951895	-5.830951895	560	560
36	2	16	18	7	-5	15	21
	2	18	16	5	-7	21	15
	3	12	10	5	-7	42	30
	4	8	10	7	-5	45	63
	6	6	4	5	-7	105	75
	8	4	6	7	-5	105	147
	9	4	2	5	-7	168	120
	16	2	4	7	-5	225	315
	17	2	2	5.916079783	-5.916079783	288	288
	18	2	0	5	-7	357	255

Figure 6: The table of feasible parameters of abelian covering graphs of complete graphs for $29 \leq n \leq 36$

n	r	c_2	a_1	θ	τ	m_θ	m_τ
37	5	7	7	6	-6	74	74
	7	5	5	6	-6	111	111
	11	4	-5	3	-12	296	74
	35	1	1	6	-6	629	629
38	2	18	18	6.08276253	-6.08276253	19	19
	3	12	12	6.08276253	-6.08276253	38	38
	6	6	6	6.08276253	-6.08276253	95	95
	9	4	4	6.08276253	-6.08276253	152	152
	18	2	2	6.08276253	-6.08276253	323	323
39	37	1	1	6.164414003	-6.164414003	702	702
40	3	16	6	3	-13	65	15
	7	4	14	13	-3	45	195
	19	2	2	6.244997998	-6.244997998	360	360
41	3	13	13	6.32455532	-6.32455532	41	41
	13	3	3	6.32455532	-6.32455532	246	246
	14	3	0	5	-8	328	205
	15	3	-3	4	-10	410	164
	39	1	1	6.32455532	-6.32455532	779	779
42	2	20	20	6.403124237	-6.403124237	21	21
	4	10	10	6.403124237	-6.403124237	63	63
	5	8	8	6.403124237	-6.403124237	84	84
	10	4	4	6.403124237	-6.403124237	189	189
43	14	3	2	6	-7	301	258
	40	1	2	7	-6	774	903
	41	1	1	6.480740698	-6.480740698	860	860
44	3	14	14	6.557438524	-6.557438524	44	44
	7	6	6	6.557438524	-6.557438524	132	132
45	2	18	25	11	-4	12	33
	2	25	18	4	-11	33	12
	3	12	19	11	-4	24	66
	4	9	16	11	-4	36	99
	5	10	3	4	-11	132	48
	6	6	13	11	-4	60	165
	9	4	11	11	-4	96	264
	10	5	-2	4	-11	297	108
	12	3	10	11	-4	132	363
	18	2	9	11	-4	204	561
	25	2	-5	4	-11	792	288
	36	1	8	11	-4	420	1155
	43	1	1	6.633249581	-6.633249581	945	945

Figure 7: The table of feasible parameters of abelian covering graphs of complete graphs for $37 \leq n \leq 45$

n	r	c_2	a_1	θ	τ	m_θ	m_τ
46	2	22	22	6.708203932	-6.708203932	23	23
	7	8	-4	3	-15	230	46
	8	6	2	5	-9	207	115
	11	4	4	6.708203932	-6.708203932	230	230
	22	2	2	6.708203932	-6.708203932	483	483
47	3	15	15	6.782329983	-6.782329983	47	47
	5	9	9	6.782329983	-6.782329983	94	94
	9	5	5	6.782329983	-6.782329983	188	188
	45	1	1	6.782329983	-6.782329983	1034	1034
48	23	2	2	6.8556546	-6.8556546	528	528
49	3	15	17	8	-6	42	56
	5	9	11	8	-6	84	112
	5	11	3	4	-12	147	49
	7	7	5	6	-8	168	126
	9	5	7	8	-6	168	224
	13	3	11	12	-4	147	441
	15	3	5	8	-6	294	392
	45	1	3	8	-6	924	1232
	47	1	1	6.92820323	-6.92820323	1127	1127
50	2	24	24	7	-7	25	25
	3	16	16	7	-7	50	50
	4	12	12	7	-7	75	75
	6	8	8	7	-7	125	125
	8	6	6	7	-7	175	175
	12	4	4	7	-7	275	275
	24	2	2	7	-7	575	575

Figure 8: The table of feasible parameters of abelian covering graphs of complete graphs for $46 \leq n \leq 50$

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