Computing the Residue Class of Partition Numbers

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Combinatorics & Optimization

Waterloo, Ontario, Canada, 2016

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

In 1919, Ramanujan initiated the study of congruence properties of the integer partition function $p(n)$ by showing that

$$p(5n + 4) \equiv 0 \ (mod\ 5)$$

and

$$p(7n + 5) \equiv 0 \ (mod\ 7)$$

hold for all integers n. These results attracted a lot of interest in the mathematical community and inspired other mathematicians to investigate the divisibility of various classes of integer partitions.

The purpose of this thesis is to illustrate the use of generating series in the study of the residue classes of integer partition values. We begin by presenting the work of Mizuhara, Sellers and Swisher in 2015 on the residue classes of restricted plane partitions numbers. Next, we introduce Ramanujan's Conjecture regarding Ramanujan Congruences. Moreover, we use modular forms to present Ahlgren and Boylan's resolution of Ramanujan's Conjecture from 2003. Then, we discuss the open problems surrounding the distribution of the integer partitions values into residue classes and present Judge, Keith and Zanello's work from 2015 on the the distribution of the parity of the partition function. We continue by introducing $m-$ary partitions and provide an account of Andrews, Fraenkel and Sellers' work from 2015 and 2016 which yielded a complete characterization of the congruence classes of $m-$ary partitions with and without gaps. Finally, we present new results regarding the complete characterization of the residue classes of coloured $m-$ary partitions with and without gaps.

Acknowledgements

First, I would like to thank my supervisor Ian Goulden for encouraging me to pursue my interests, providing me with guidance, patience, optimism and the occasional joke and his friendship.

I owe a debt of gratitude to the department of Combinatorics & Optimization for the support and all of the free printing.

Special thanks goes to David Jackson for always knowing the right Welsh proverb to cheer me up, agreeing to read my thesis and guiding me through an exploration of hypermaps and constellations.

I would also like to thank Bruce Richmond for agreeing to read my thesis and providing me with feedback.

I am extremely grateful for the support and guidance I received from Elias Brettler, Ada Chan, Mike Zabrocki and Norman Purzitsky during my time at York University to pursue my interest in mathematics.

To those who made my last two years memorable (in no particular order): Matt, Luis, Nathan, Jason, Randy, Leanne, Rachel, Dan, Mark, Melissa, Carol, Marie, Megan, Gabriel, Kayla, Owen and Djao.

Dedication

To my parents Ilya and Irena, my brother Mark and my best friend Noch.

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Chapter 1

Introduction

1.1 History and Motivation

Addition of natural numbers is arguably one of the most easily understood arithmetic operations. Typically, it is presented as the combination of objects of the same type. In this thesis, we study the ways in which positive integers can add up to another positive integer while satisfying some criteria. These are known as *integer partitions*.

In particular, we shall denote by $p(n)$ the number of distinct number of ways in which we can add positive integers to yield a nonnegative integer n, where the order of the summands is irrelevant. Given the simplicity of the definition of $p(n)$, it should not come as a surprise that it appears in various areas of mathematics such as number theory, combinatorics and representation theory. Before we discuss the intricacies of the theory of integer partitions, we discuss some of their history. The following presentation is based on Andrews work in [And08].

The study of integer partitions dates back to Gottfried Wilhelm Leibniz in 1674 when he wrote a letter to Daniel Bernoulli asking about integer partitions and whether the partition function is prime for $n \geq 2$. This is demonstrably false as $p(7) = 15$.

In the 18th century, Leonard Euler made several significant breakthroughs in the area. He was the first to introduce generating series to the study of integer partitions. In particular, he found the generating series for the set of all integer partitions. Later, he discovered the coefficients of its inverse in what is universally known as The Pentagonal Number Theorem (see Theorem 1.2.13).

In the 19th century, James Joseph Sylvester provided the mathematical community with a new, geometric approach to analyzing integer partitions. He recognized that representing integer partitions as left-aligned rows of dots makes certain partition decompositions become transparent. He named these Ferrers diagrams or Ferrers graphs after the British mathematician Norman Macleod Ferrers, and used them to introduce the conjugates of partitions.

Later, in the 20th century, Srinivasa Ramanujan made profound contributions to the study of divisibility properties of integer partitions [HSAW27]. In particular, he established the following three identities for all integers n:

$$p(5n+4) \equiv 0 \ (mod\ 5)$$
$$p(7n+5) \equiv 0 \ (mod\ 7)$$
$$p(11n+5) \equiv 0 \ (mod\ 11).$$

Ramanujan went on to make the conjecture that $\ell = 5, 7, 11$ are the only primes for which there exists a partition congruence of the type $p(\ell n + \beta) \equiv 0 \ (mod\ \ell)$. The problem of finding congruences of this type for composite numbers has a slightly different flavour and we refer the interested reader to [Ono00], [LO02] and [Atk68]. Since Ramanujan's time, many mathematicians sought to attack problems of similar flavour. In this thesis, we shall examine the divisibility of various classes of integer partitions.

The underlying goal of this thesis is to demonstrate the usefulness of generating series in the study of the congruence properties of integer partitions. The chapters are ordered as follows. In this chapter, we discuss the history of integer partitions and study basic decomposition theorems. We also include, for completeness a few basic facts about generating series. Furthermore, we provide the background for the elementary mathematics we shall use throughout the thesis which pertains to generating series. In Chapter 2, we study the arithmetic properties of restricted plane partitions through the periodicity of their generating series. Moreover, we present an interesting tool used in the verification of plane partition congruences. In Chapter 3, we provide a summary of a proof to Ramanujan's Conjecture using the theory of modular forms. In Chapter 4, we study the distribution of the values of the partition function into congruences classes, the odd-value density of the partition function. Finally, in Chapter 5 we present a complete characterization of two classes of $m-$ary partitions modulo m due to Andrews, Fraenkel, and Sellers and provide a new variation on these objects by introducing colourings.

We note here that Chapter 2 is based on the works of Mizuhara, Sellers, Swisher [MSS15] and Kwong [Kwo89]. The presentation of Chapter 3 is based on the on the works of Ahlgren and Boylan in [AB03]. Chapter 4 borrows from the questions posed by Newman in [New60] and Ono's discussions regarding density and distribution in [Ono00]. Following this, we present a recent investigation of the parity of the partition function due to Judge, Keith and Zanello in [JKZ15]. Chapter 5 is based on the the work of Andrews, Fraenkel and Sellers in [AFS15] and [AFS16]. In Section 5.4, we present new results that extend their work which were discovered in the process of working on this thesis.

1.2 Preliminaries

Throughout this thesis, we shall denote the set of natural numbers $\{1, 2, 3, \ldots\}$ by $\mathbb{N}$ and the set of nonnegative integers by $\mathbb{N}_0$. As we described previously, an integer partition of n is a summation of positive integers which yields n, where we do not care about the order in which we add the positive integers to each other (each of which is called a *part*). In general, we shall write integer partitions as sequences of integers throughout this thesis.

Definition 1.2.1. Fix $n \in \mathbb{N}_0$ an integer. We say that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{N}^k$ is a *partition* of n with $k \in \mathbb{N}_0$ parts if

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$$

and

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_k = n.$$

We let $p(n)$ denote the total number of partitions of n, where we use the convention $p(0) = 1$ and $p(n) = 0$ for n negative. As $\lambda_i \in \mathbb{N}$ for all i, we say that the set of *allowed parts* is the set of positive integers.

We can easily compute all of the partitions of n for small values for n, as we do for $n = 4$ below.

Example 1.2.2. We note that $p(4) = 5$ by enumerating the partitions of 4:

$$\begin{aligned} p(4) &= 4 \\ &= 3 + 1 \\ &= 2 + 2 \\ &= 2 + 1 + 1 \\ &= 1 + 1 + 1 + 1 \end{aligned}$$

We see that there is only one partition with a single part: (4), two partitions with two parts: (3,1) and (2,2), one partition with three parts: (2,1,1), and a single partition with four parts: (1,1,1,1).

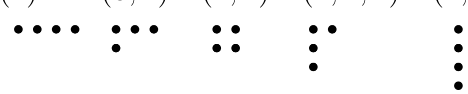
It is often helpful to view partitions geometrically. To do so, we make use of Ferrers diagrams.

Definition 1.2.3. For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, we identify λ with the so-called *Ferrers diagram.* A Ferrers diagram will consist of k rows of dots which lie on top of each other in a left-justified fashion. Row i will have λ_i dots, where we draw our first row at the top.

The convention that we use here is known as the English convention of drawing Ferrers diagrams. Some authors use the French convention where the row sizes increase from top to bottom.

Example 1.2.4. We draw the Ferrers diagrams for the partitions of 4 below.

(4) $(3, 1)$ $(2, 2)$ $(2, 1, 1)$ $(1, 1, 1, 1)$



The graphical representation of partitions is a valuable tool as it allows one to decompose partitions in a more intuitive way. For instance, consider the following:

Proposition 1.2.5. *The number of partitions with at most k parts is equal to the number of partitions where the largest part is at most k.*

Proof. We describe a bijection which relies on Ferrers diagrams. Consider an arbitrary partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ where $r \leq k$. We use λ to construct a partition λ' which has no parts of size larger than k.

We examine the Ferrers diagram of λ and count the number of dots in each column. We set λ' to be the partition where λ'_i is the number of dots in column i of λ. Since λ has r rows, we see that $\lambda'_1 = r \leq k$. As the column lengths decrease from left to right, we see that λ' is a partition with parts bounded by k.

Now, if we apply the same map to λ', it becomes apparent that we obtain λ. Hence, there exists a bijection between the number of partitions with at most k parts and partitions whose parts are bounded by k. ∎

The operation described in Proposition 1.2.5 which sends λ to λ' is known as *conjugation*. It not hard to see that λ' is obtained from λ by reflecting the Ferrers diagram of λ via the line $y = -x$. The partition λ' which is obtained from λ by conjugation is known as the *conjugate partition* of λ. Conjugation is well known in the theory of partitions and appears frequently throughout various decompositions of integer partitions.

We shall often employ generating series in the study of partitions. Generating series allow one to study combinatorial objects by endowing them with unique algebraic structure.

1.2.1 Formal Power Series

This subsection serves as a short review of the basics of generating series. Here, we make use of the terminology found in [Wil13], [GJ04].

Definition 1.2.6. Let $(a_n)_{n\geq 0}$ be a sequence of complex numbers and q an indeterminate. We define the *generating series* $A(q)$ of $(a_n)_{n\geq 0}$ to be the formal sum

$$A(q) := \sum_{n\geq 0} a_n q^n.$$

We call any such summation a *formal power series*. We let $\mathbb{C}[[q]]$ denote the set of all formal power series in q with coefficients in $\mathbb{C}$. Throughout this thesis, we shall write $[q^n]A(q)$ to denote the *extraction* of the $n-$th coefficient of $A(q)$. That is,

$$[q^n]A(q) = a_n$$

for all $n \in \mathbb{Z}$ where we use the convention $a_k = 0$ for $k < 0$. We define addition of two formal power series in the natural way:

$$\sum_{n\geq 0} a_n q^n + \sum_{n\geq 0} b_n q^n = \sum_{n\geq 0} (a_n + b_n) q^n.$$

Moreover, we can extend the usual definition of multiplication of polynomials to formal power series as follows:

$$\left(\sum_{n\geq 0} a_n q^n\right)\left(\sum_{n\geq 0} b_n q^n\right) = \sum_{n\geq 0}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) q^n.$$

With respect to these two operations, the set $\mathbb{C}[[q]]$ forms a ring.

Example 1.2.7. Let $(a_n)_{n\geq 0}$ be the sequence defined by $a_n : n \mapsto 1$ for all $n \in \mathbb{N}_0$. The power series $A(q)$ which corresponds to it is given by

$$A(q) = \sum_{n\geq 0} a_n q^n = \sum_{n\geq 0} q^n = \frac{1}{1-q},$$

and we note that it is easily verified that $\sum_{n\geq 0} q^n$ is the multiplicative inverse of $(1-q)$ in $\mathbb{C}[[q]]$.

In the previous example, $\frac{1}{1-q}$ is an instance of a *closed-form expression* for a generating series. In this thesis, we write *closed-form expression* for a generating series to mean a finite expression in terms of basic arithmetic operations and elementary functions (such as polynomials, trigonometric functions, etc.).

We are often interested in finding closed-form expressions for generating series. Closed-form expressions for generating series often allow one to find a recurrence relation for the sequence, or in some cases an explicit formula for the $n-$th term in the sequence. Additionally, in analytic combinatorics, it allows one to study the asymptotics of a sequence.

Example 1.2.8. If $(b_n)_{n\geq 0}$ is the sequence defined by $n \mapsto (-1)^n$ for all $n \in \mathbb{N}_0$, we see that its generating series is given by

$$B(q) = \sum_{n\geq 0} b_n q^n = \sum_{n\geq 0} a_n (-1)^n q^n = \frac{1}{1+q} = A(-q),$$

which can be obtained via the substitution $q \mapsto -q$ in $A(q)$.

We may use the closed-form expression for $B(q)$ to deduce a recurrence relation for b_n. If we multiply the expression for $B(q)$ by $(1+q)$, we have

$$\begin{aligned} 1 &= (1+q)B(q) \\ &= \sum_{n\geq 0} b_n q^n + \sum_{n\geq 0} b_n q^{n+1}. \end{aligned}$$

Now, comparing coefficients we find

$$\begin{aligned} [q^n]1 &= [q^n]\left(\sum_{n\geq 0} b_n q^n + \sum_{n\geq 0} b_n q^{n+1}\right) \\ &= [q^n]\sum_{n\geq 0} b_n q^n + [q^n]\sum_{n\geq 0} b_n q^{n+1} \\ &= b_n + b_{n-1}. \end{aligned}$$

Since

$$[q^n]1 = \begin{cases} 1, & n = 0 \\ 0, & n \geq 1 \end{cases},$$

it is immediate that

$$b_n + b_{n-1} = \begin{cases} 1, & n = 0 \\ 0, & n \geq 1 \end{cases}.$$

Since $b_{-1} = 0$, we see that $b_0 = 1$ and

$$0 = b_n + b_{n-1}$$

holds for all $n \geq 1$. We summarize the recursive structure of b_n as follows:

$$b_0 = 1; b_n = -b_{n-1}, \text{ for } n \geq 1.$$

1.2.2 Generating Series for Partitions

Now, we investigate the generating series related to integer partitions. An overarching theme in the following chapters is the idea of determining the generating series for a class of partitions and then reducing the series modulo some integer.

The following theorem is due to Euler and is central to the study of integer partitions.

Theorem 1.2.9. *The generating series $P(q)$ for the sequence of integer partitions is given by*

$$P(q) = \sum_{n \geq 0} p(n) q^n = \prod_{i \geq 1} \frac{1}{1 - q^i}.$$

Proof. Fix an integer partition λ of n. As n cannot have parts larger than n, it is evident that

$$p(n) = \sum_{\substack{m_1, m_2, \ldots, m_n \geq 0 \\ 1 \cdot m_1 + 2 \cdot m_2 + \cdots + n \cdot m_n = n}} 1.$$

Therefore, we see that

$$\begin{aligned}
P(q) &= \sum_{n \geq 0} p(n) q^n \\
&= \sum_{n \geq 0} \left(\sum_{\substack{m_1, m_2, \ldots, m_n \geq 0 \\ 1 \cdot m_1 + 2 \cdot m_2 + \cdots + n \cdot m_n = n}} q^{1 \cdot m_1 + 2 \cdot m_2 + \cdots + n \cdot m_n} \right) \\
&= \left(\sum_{m_1 \geq 0} q^{1 \cdot m_1} \right) \left(\sum_{m_2 \geq 0} q^{2 \cdot m_2} \right) \cdots \left(\sum_{m_r \geq 0} q^{r \cdot m_r} \right) \cdots \\
&= \frac{1}{1 - q} \frac{1}{1 - q^2} \cdots \frac{1}{1 - q^r} \cdots \\
&= \prod_{i=1}^{\infty} \frac{1}{1 - q^i}.
\end{aligned}$$

■

We shall encounter partitions where some parts have the same size but are distinguishable from one another. We shall include them in a multiset of parts where we assign an integer label to each part.

Definition 1.2.10. Fix a multiset S of natural numbers. We shall denote by $p(n; S)$ the number of partitions of n where every part is an element of S.

If $k \in \mathbb{N}$ appears in S with multiplicity $f(k)$, we shall distinguish the various parts of size k by assigning them indices from $\{1, 2, \ldots, f(k)\}$. We can identify S with this function f which we call the *multiplicity counter* of S. We shall refer to the multiset S as follows

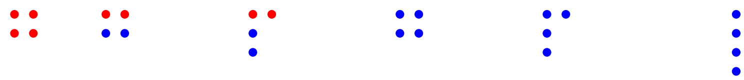
$$S := \{1_1, \ldots, 1_{f(1)}, 2_1, \ldots, 2_{f(2)}, \ldots\} = \{i_j : i \in \mathbb{N}, 1 \leq j \leq f(i)\}.$$

Example 1.2.11. If $S = \mathbb{N}$, then $p(n; S) = p(n)$.
If $S = \{2k - 1 : k \in \mathbb{N}\}$, then $p(n; S)$ is the number of partitions of n into odd parts.
If the multiplicity counter is defined by $f(1) = 1, f(2) = 2$ and $f(k) = 0$ for all $k > 2$, then $S = \{1_1, 2_1, 2_2\}$ and the partitions of 4 are the following

$$(2_2, 2_2), (2_2, 2_1), (2_2, 1_1, 1_1), (2_1, 2_1), (2_1, 1_1, 1_1), (1_1, 1_1, 1_1, 1_1).$$

We have an analogue of Ferrers diagrams for these partitions, where we assign different colours to each part. For instance, we may assign the colour red to the part 2_2 and blue to 2_1 and the color blue to 1_1 as well. This gives us the following Ferrers diagrams for the partitions listed above:

$(2_2, 2_2)$ $\quad (2_2, 2_1)$ $\quad (2_2, 1_1, 1_1)$ $\quad (2_1, 2_1)$ $\quad (2_1, 1_1, 1_1)$ $\quad (1_1, 1_1, 1_1, 1_1)$



Next, we generalize Theorem 1.2.9 to partitions with set of allowed parts given by a multiset S.

Theorem 1.2.12. *If S is a multisubset of $\mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}_0$ is its multiplicity counter, then*

$$\sum_{n \geq 0} p(n; S) q^n = \prod_{i \geq 1} \frac{1}{(1 - q^i)^{f(i)}}.$$

Proof. We remark that

$$p(n; S) = \sum_{\substack{m_{1,1},\ldots,m_{1,f(1)},\ldots,m_{n,1},\ldots,m_{n,f(n)} \in \mathbb{N}_0 \\ 1(m_{1,1}+\cdots+m_{1,f(1)})+\cdots+n(m_{n,1}+\cdots+m_{n,f(n)})=n}} 1$$

and from here the proof is analogous to the proof of Theorem 1.2.9. ■

Euler recognized a relationship between the inverse of $P(q)$ and the so-called generalized pentagonal numbers.

Theorem 1.2.13 (Pentagonal Number Theorem). *If $P(q) = \sum_{n \geq 0} p(n) q^n$, then*

$$\frac{1}{P(q)} = \sum_{k \in \mathbb{Z}} (-1)^k q^{\frac{k(3k-1)}{2}} = 1 + \sum_{k \geq 1} (-1)^k \left(q^{\frac{k(3k-1)}{2}} + q^{\frac{k(3k+1)}{2}} \right) = \sum_{n \geq 0} g_n q^n,$$

where

$$g_n := \begin{cases} (-1)^k, & \text{if } n = \dfrac{k(3k \pm 1)}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

■

Euler [EDA12] originally provided an algebraic proof for Theorem 1.2.13 . For a combinatorial proof, we refer the reader to [Gro84].

Theorem 1.2.13 implies the following linear recurrence relation which has no fixed order.

Corollary 1.2.14. *The $n-th$ partition number $p(n)$ is given by*

$$p(n) = \sum_{k\in\mathbb{Z};k\neq 0} (-1)^{k+1} p(n-g_k).$$

Proof. Since Theorem 1.2.13 is equivalent to the statement

$$P(q)\sum_{k\in\mathbb{Z}}(-1)^k q^{\frac{3(3k-1)}{2}} = 1,$$

we expand the left hand side and then equate the coefficients of q^n on both sides. ■

Throughout this thesis, we shall study congruences of various integer partition classes by considering the underlying generating series in some modulus. As we see from the generating series for integer partitions, binomial terms turn up as factors frequently. Fortunately, we have a lemma analogous to the binomial theorem which gives us their reduction modulo a prime.

Lemma 1.2.15. *If ℓ is prime, then for all $j \in \mathbb{N}_0$,*

$$(1-q^j)^\ell \equiv (1-q^{j\ell}) \ (mod\ \ell).$$

Proof. The binomial theorem states that we can expand $(1-q^j)^\ell$ as follows:

$$(1-q^j)^\ell = \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k q^{kj}.$$

Moreover, for $1 \leq k \leq \ell - 1$, we know that

$$\binom{\ell}{k} = \frac{\ell!}{k!(\ell-k)!}$$

is divisible by ℓ, as ℓ appears in the numerator and ℓ is prime. Therefore, we see that

$$\begin{aligned}(1-q^j)^\ell &= \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k q^{kj} \\ &\equiv 1 - q^{\ell j} \ (mod\ \ell)\end{aligned}$$

where we know that $(-1)^\ell \equiv -1$ $(mod\ \ell)$ is true for odd primes and trivially true when ℓ is 2. ■

In some instances, when dealing with a formal power series $A(q) = \sum_{n\geq 0} a_n q^n$, we shall be interested in a related power series where only terms in some congruence class modulo m appear. This motivates the following definition.

Definition 1.2.16. Fix $m \in \mathbb{N}$ and ℓ an integer where $0 \leq \ell \leq m-1$. We say that a sum of the form $\sum_{k\equiv \ell \ (mod\ m)} a_k q^k$ is an *m-th multisection of* $A(q)$ *with* residue ℓ.

In order to find an explicit formulation for the $m-$th multisection of a formal power series, we make use of the following proposition.

Proposition 1.2.17. *If* $w = e^{\frac{2\pi i}{m}}$ *is an* $m-$*th root of unity, then*

$$\sum_{k=0}^{m-1} w^{nk} = \begin{cases} m, & \text{if } m|n \\ 0, & \text{otherwise} \end{cases}.$$

Moreover, if $A(q) = \sum_{r\geq 0} a_r q^r$*, then*

$$\sum_{k\equiv \ell \ (mod\ m)} a_k q^k = \frac{1}{m}\sum_{j=0}^{m-1} A(qw^j)w^{-\ell j}.$$

■

For proof, we refer the interested reader to [GJ04].

Chapter 2

Plane Partitions

2.1 Introduction and History

The study of multi-dimensional partitions dates back to Percy A. MacMahon in the early 20th century [Mac16]. MacMahon was very interested in the so-called plane partitions that are the natural two dimensional generalization of partitions. He was able to prove several interesting results regarding plane partitions. For instance, he showed [MA78] that

$$\sum_{n\geq 0} pl(n)q^n = \prod_{n\geq 1} \frac{1}{(1-q^n)^n}$$

where $pl(n)$ is the number of plane partitions of n. Though one can immediately identify the right hand side as the generating series for coloured partitions, a bijective proof for this result was not obtained until much after MacMahon's time (see [PB04]).

Though some of MacMahon's conjectures regarding higher-dimensional partitions turned out to be false (see [Knu70]), MacMahon had many contributions in the area of plane partitions such as the identification ten different classes of plane partitions subject to different symmetries and several correct conjectures concerning them. For a recent and up-to-date survey, see [Kra15].

Despite the fact that connections between plane partitions and other areas of mathematics were not initially evident, MacMahon's conjectures regarding plane partitions captured the interests of many mathematicians due to their elegance. Over the last century, it became apparent that plane partitions are relevant in many other areas of mathematics. These include the theory of symmetric functions [Vul09], representation theory [Col16], enumeration of matchings in graphs ([Kuo04], [Ciu97]) and statistical mechanics ([GGS16],[Rov16]).

In 2015, Mizuhara, Sellers and Swisher (here after referred to as MSS) [MSS15] were able to make use of Kwong's work on the periodicity of rational polynomial functions [Kwo89] to develop a useful theorem for establishing congruence properties for restricted plane partitions. In this chapter, we will present MSS's theorem for establishing restricted plane partitions and demonstrate its use.

2.2 Preliminaries

A plane partition can be thought of as two-dimensional array of nonnegative integers that increases along rows and columns with the restriction that there are only finitely many nonzero entries.

Definition 2.2.1. For $n \in \mathbb{N}$, we say that an array $\pi = [\pi_{i,j}]_{i,j\in\mathbb{N}}$ is a *plane partition* of n if π is a two-dimensional array of integers satisfying the conditions

1. $\pi_{i,j} \in \mathbb{N}_0$ for all $i, j \in \mathbb{N}$,
2. $\pi_{i,j} \geq \pi_{i,j+1}$ and $\pi_{i,j} \geq \pi_{i+1,j}$ for all i, j,
3. $|\pi| := \sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} \pi_{i,j} = n$.

We note these conditions imply that a plane partition has only finitely many nonzero entries.

Definition 2.2.2. We denote by $pl(n)$ the number of plane partitions of n for $n \in \mathbb{N}$ and we set $pl(0) = 1$. For a triple $(r, s, t) \in \mathbb{N}^3$, we let $\mathcal{R}_{r,s,t}$ denote the number of plane partitions for which the number of nonzero rows is at most r, number of nonzero columns is at most s and the biggest entry is at most t.

We take the convention of displaying our plane partitions in the fourth quadrant as illustrated in the example below.

Example 2.2.3. The plane partitions of 4 are

$$4\ ,\ \begin{matrix}3 & 1\end{matrix}\ ,\ \begin{matrix}3\\1\end{matrix}\ ,\ \begin{matrix}2 & 2\end{matrix}\ ,\ \begin{matrix}2\\2\end{matrix}\ ,\ \begin{matrix}2 & 1 & 1\end{matrix}\ ,\ \begin{matrix}2 & 1\\1 & \end{matrix}\ ,\ \begin{matrix}2\\1\\1\end{matrix}\ ,\ \begin{matrix}1 & 1 & 1 & 1\end{matrix}\ ,\ \begin{matrix}1 & 1 & 1\\1 & & \end{matrix}\ ,\ \begin{matrix}1 & 1\\1 & 1\end{matrix}\ ,\ \begin{matrix}1 & 1\\1 & \\1 & \end{matrix}\ ,\ \begin{matrix}1\\1\\1\\1\end{matrix}\ .$$

Much like their one-dimensional counterparts, plane partitions have a nice graphical representation. For instance, we can represent the plane partition

$$\pi = \begin{matrix}2 & 1\\1 & \end{matrix}\ ,$$

graphically by drawing $\pi_{i,j}$ cubes stacked on top of one another in position (i, j) as depicted in Figure 2.1.

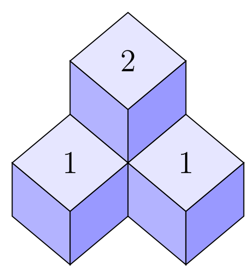


Figure 2.1: A graphical representation of a plane partition of 4.

One of MacMahon's celebrated results [Mac99] is the discovery of a closed-form expression for the generating series of plane partitions in $\mathcal{R}_{r,s,t}$.

Theorem 2.2.4. *The generating series for plane partitions in* $\mathcal{R}_{r,s,t}$ *is given by*

$$\sum_{\pi\in\mathcal{R}_{r,s,t}} q^{|\pi|} = \prod_{i=1}^{r}\prod_{j=1}^{s}\prod_{k=1}^{t}\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}.$$

■

MacMahon used Theorem 2.2.4 to compute the generating series for plane partitions

$$\sum_{n\geq 0} pl(n)q^n = \prod_{n\geq 1}\frac{1}{(1-q^n)^n},$$

by taking a limit as r, s and t tend to infinity.

In the following section, we will be studying the recursive structure of plane partitions with restricted parts.

Definition 2.2.5. If $k \in \mathbb{N}$, then we use $pl_k(n)$ to denote the number of plane partitions of n with parts from $\{1, 2, \ldots, k\}$ and we let $PL_k(q)$ denote the generating series

$$PL_k(q) := \sum_{n \geq 0} pl_k(n) q^n.$$

Set $S_k := \{i_j : 1 \leq i \leq k-1, 1 \leq j \leq i\}$ and recall that $p(n; S)$ denotes the number of integer partitions with parts from S. We will denote the generating series for $p(n; S_k)$ by $F_k(q) = \sum_{n \geq 0} p(n; S_k) q^n$ and observe that

$$F_k(q) = \prod_{i=1}^{k-1} \frac{1}{(1-q^i)^i},$$

by Theorem 1.2.12.

By specializing Theorem 2.2.4, we obtain the following relationship between $PL_k(q)$ and $F_k(q)$:

$$PL_k(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{\min(k,n)}} \tag{2.1}$$

$$= F_k(q) \prod_{n=k}^{\infty} \frac{1}{(1-q^n)^k}. \tag{2.2}$$

2.3 The Periodicity of Partition Functions

In [Kwo89], Kwong showed that given any set of allowed parts S, the generating series $A(q) := \sum_{n \geq 0} p(n; S) q^n$ is periodic modulo ℓ^N for any prime ℓ and $N \in \mathbb{N}$.

Definition 2.3.1. If $A(q) := \sum_{n \geq 0} \alpha_n q^n$ is a formal power series with integer coefficients, then we say that $A(q)$ is *periodic with period* $d \in \mathbb{N}$ and modulo ℓ if the sequence α_n satisfies

$$\alpha_{n+d} \equiv \alpha_n \ (mod\ \ell).$$

We denote by $\pi_m(A(q))$ the smallest period of $A(q)$ in modulo m, if it exists.

In Section 2.4, we will examine MSS's theorem regarding restricted plane partitions which could be paraphrased as follows: If ℓ is a prime, in order to establish a congruence of the form

$$\sum_{i=1}^{s} pl_l(\ell n + a_i) \equiv \sum_{j=1}^{t} pl_\ell(\ell n + b_j) \ (mod\ \ell),$$

it is sufficient to check that it holds for all $n < K(l)$, where $K(\ell)$ is some given integer constant which is a function of our prime ℓ.

We now introduce the terminology necessary to define the constant $K(\ell)$. This is due to a specialization of Kwong's work in [Kwo89].

Definition 2.3.2. For $n \in \mathbb{Z}$ and ℓ prime, we define $ord_\ell(n)$ to be the unique integer such that

$$n = \ell^{ord_\ell(n)} m$$

where $\ell \nmid m$. We say that m is the $\ell-$free part of n.

That is, we think of $ord_\ell(n)$ as the number of occurrences of ℓ in n as a factor.

Definition 2.3.3. Fix a prime ℓ. For a finite multiset of positive integers S, we define $m_\ell(S)$ to be the $\ell-$free part of $lcm(S)$. That is,

$$lcm(S) = \ell^{ord_\ell(lcm(S))} m_\ell(S).$$

We set $b_\ell(S)$ to be the least nonnegative integer such that

$$\ell^{b_\ell(S)} \geq \sum_{n \in S} \ell^{ord_\ell(n)}.$$

Example 2.3.4. We investigate the generating series $F_3(q) := \sum_{n \geq 0} p(n; S_3) q^n$ of partitions with parts from the set $S_3 := \{1_1, 2_1, 2_2\}$, where we recall that a_b denotes a part of size a coloured by b.

We look to compute $m_3(S_3)$ and $b_3(S_3)$. First, we know that

$$m_3(S_3) = \frac{lcm(S_3)}{3^{ord_3(S_3)}}$$

and since $lcm(S_3) = lcm\{1,2\} = 2$, we compute $ord_3(2)$. Clearly, we have

$$2 = 3^0 \cdot 2$$

and hence $ord_3(2) = 0$. We conclude that

$$m_3(S_3) = \frac{2}{3^0} = 2.$$

Next, we compute $ord_3(1)$. Clearly,

$$1 = 3^0 \cdot 1$$

and conclude $ord_3(1) = 0$. This implies that $b_3(S_3)$ must be the least nonnegative integer for which

$$\begin{aligned} 3^{b_3(S_3)} &\geq 3^{ord(1)} + 3^{ord(2)} + 3^{ord(2)} \\ &= 3^0 + 3^0 + 3^0 = 3 \end{aligned}$$

and so it follows that $b_3(S_3) = 1$.

Kwong [Kwo89] proved that partition functions satisfy the following minimal period:

Theorem 2.3.5. *Let S be a multisubset of $\mathbb{N}$, ℓ a prime and $N \in \mathbb{N}$. Then $A(q) = \sum_{n\geq 0} p(n;S)q^n$ is periodic modulo ℓ^N with minimal period*

$$\pi_{\ell^N}(A(q)) = \ell^{N+b(S)-1}m(S).$$

■

MSS [MSS15] specialized Theorem 2.3.5 to the case where $S := S_k$ and obtained the following corollary.

Corollary 2.3.6. *Fix ℓ a prime, $k, N \in \mathbb{N}$. Then the series $F_k(q)$ is periodic modulo ℓ^N with minimal period*

$$\pi_{\ell^N}(F_k(q)) = \ell^{N+b(S_k)-1}m(S_k).$$

■

In particular, Corollary 2.3.6 asserts that for all $n \geq 0$

$$p(n + \pi_{\ell^N}(F_k(q)); S_k) \equiv p(n; S_k) \ \big(mod\ \ell^N\big).$$

In fact, whenever $n \equiv m \ (mod\ \pi_{\ell^N}(F_k))$, it follows from Corollary 2.3.6 that

$$p(n; S_k) \equiv p(m; S_k) \ \left(mod\ \ell^N\right).$$

Next, we specialize Corollary 2.3.6 to the case when $k = \ell$ and evaluate the minimal period.

Proposition 2.3.7. *The minimal periods of $F_\ell(q)$ modulo ℓ^n for any primes ℓ are given by*

$$\pi_{\ell^N}(F_\ell(q)) = \begin{cases} 2^{N-1} & \ell = 2 \\ 3^N \cdot 2 & \ell = 3 \\ \ell^{N+1} \cdot lcm\{1, 2, \ldots, \ell - 1\} & \ell \geq 5 \end{cases}$$

Proof. If $k = \ell$, then by Theorem 2.3.5 we must determine

$$\ell^{N+b(S_\ell)-1} m(S_\ell).$$

Recall that since S_l is given by

$$S_\ell = \{i_j : 1 \leq i \leq \ell - 1, 1 \leq j \leq i\}$$

we know that no integer in S_ℓ is divisible by ℓ and hence $ord_\ell(n) = 0$ for all $n \in S_\ell$. It is then evident that

$$\begin{aligned} \sum_{n \in S_\ell} \ell^{ord_\ell(n)} &= \sum_{n \in S_\ell} \ell^0 \\ &= |S_\ell| \\ &= \frac{\ell(\ell - 1)}{2}. \end{aligned}$$

Therefore, $b_\ell(S_\ell)$ is the minimal nonnegative integer satisfying

$$\ell^{b_\ell(S_\ell)} \geq \frac{\ell(\ell - 1)}{2} = \ell \cdot \frac{(\ell - 1)}{2}.$$

Now, we consider the three cases: $\ell = 2, 3$ or $\ell \geq 5$.

Case 1: If $\ell = 2$, we know that

$$\ell^{b_\ell(S_\ell)} \geq \frac{\ell(\ell - 1)}{2} = 0$$

is satisfied by $b_\ell(S_\ell) = 0$.

Case 2: If $\ell = 3$, we see that

$$\ell^{b_\ell(S_\ell)} \geq \frac{\ell(\ell-1)}{2} = \ell$$

is satisfied by $b_\ell(S_\ell) = 1$.

Case 3: If $\ell \geq 5$, we know that

$$1 < \frac{(\ell-1)}{2} < \ell$$

and so

$$\ell < \ell\frac{(\ell-1)}{2} < \ell^2$$

and it follows that $b_\ell(S_\ell) = 2$.

∎

2.4 A Congruence Characterization Theorem for Restricted Plane Partitions

Equipped with the knowledge of the minimal period of F_ℓ in modulo ℓ, in this section we present MSS's [MSS15] restricted plane partition congruence theorem.

Theorem 2.4.1. *Fix $s, t \in \mathbb{N}$ and $a_i, b_j \in \mathbb{N}$ for $1 \leq i \leq s, 1 \leq j \leq t$. For a prime ℓ, if*

$$\sum_{i=1}^{s} pl_l(\ell n + a_i) \equiv \sum_{j=1}^{t} pl_l(\ell n + b_j) \ (mod\ \ell),$$

holds for all $n < \frac{\pi_\ell(F_\ell(q))}{\ell}$, then it holds for all $n \geq 0$.

Proof. Let ℓ be a prime. Using (2.2) and Lemma 1.2.15, we see that

$$PL_\ell(q) = \sum_{n\geq 0} pl_\ell(n)q^n \equiv \left(\sum_{i\geq 0} p(i; S_\ell)q^i\right)\left(\prod_{j\geq \ell} \frac{1}{1-q^{j\ell}}\right) \ (mod\ \ell) \tag{2.3}$$

$$\equiv \left(\sum_{i\geq 0} \alpha_i q^i\right)\left(\sum_{m\geq 0} \beta_m q^{m\ell}\right) \ (mod\ \ell) \tag{2.4}$$

where for ease of notation we set $\alpha_i := p(i; S_\ell)$ and

$$\prod_{j\geq\ell} \frac{1}{(1-q^{j\ell})} = \sum_{m\geq0} \beta_m q^{m\ell}.$$

Thus, by comparing coefficients we can immediately deduce from (2.3) that

$$pl_\ell(n\ell+k) \equiv \sum_{i=0}^{n} \alpha_{i\ell+k}\beta_{n-i} \ (mod\ \ell). \tag{2.5}$$

We assumed in the hypothesis that

$$\sum_{i=1}^{s} pl_l(\ell n+a_i) \equiv \sum_{j=1}^{t} pl_l(\ell n+b_j) \ (mod\ \ell)$$

holds. We use (2.5) to rewrite our hypothesis as

$$\sum_{i=1}^{s}\sum_{r=0}^{n} \alpha_{r\ell+a_i}\beta_{n-r} \equiv \sum_{j=1}^{t}\sum_{r=0}^{n} \alpha_{r\ell+b_j}\beta_{n-r} \ (mod\ \ell).$$

By rearranging the summations, we can write

$$\sum_{r=0}^{n} \beta_{n-r}\left(\sum_{i=1}^{s} \alpha_{r\ell+a_i}\right) \equiv \sum_{r=0}^{n} \beta_{n-r}\left(\sum_{j=1}^{t} \alpha_{r\ell+b_j}\right) \ (mod\ \ell). \tag{2.6}$$

In particular, (2.6) is equivalent to

$$\sum_{i=1}^{s} \alpha_{n\ell+a_i} + \sum_{r=0}^{n-1} \beta_{n-r}\left(\sum_{i=1}^{s} \alpha_{r\ell+a_i}\right) \equiv \sum_{j=1}^{t} \alpha_{n\ell+b_j} + \sum_{r=0}^{n-1} \beta_{n-r}\left(\sum_{j=1}^{t} \alpha_{r\ell+b_j}\right) \ (mod\ \ell) \tag{2.7}$$

when $n \geq 1$ as $\beta_0 = 1$. Therefore, to show that (2.6) holds for all $n \in \mathbb{N}$, it suffices to show that

$$\sum_{i=1}^{s} \alpha_{n\ell+a_i} \equiv \sum_{j=1}^{t} \alpha_{n\ell+b_j} \ (mod\ \ell)$$

holds for all n. We consider the two cases $\ell = 2$ and $\ell > 2$.

Case 1: If $\ell = 2$, then we know that $\pi_2(F_2(q)) = 1$ by Proposition 2.3.7 and thus the coefficients are all congruent modulo 2.

Case 2: If $\ell > 2$, then we know that $\pi_\ell(F_\ell(q)) = K\ell$, for some $K \in \mathbb{N}$. Let $n \geq K = \frac{\pi_\ell(F_\ell(q))}{\ell}$, where we write n uniquely as

$$n = xK + y$$

for some $x \in \mathbb{N}, 0 \leq y < K$.

Now, for any $1 \leq i \leq s, 1 \leq j \leq t$, we can write

$$n\ell + a_i = xK\ell + (y\ell + a_i)$$
$$n\ell + b_j = xK\ell + (y\ell + b_j)$$

and so

$$n\ell + a_i \equiv y\ell + a_i \ (mod\ \pi_\ell(F_\ell(q)))$$
$$n\ell + b_j \equiv y\ell + b_j \ (mod\ \pi_\ell(F_\ell(q)))$$

holds for all $1 \leq i \leq s, 1 \leq j \leq t$. Since π_ℓ is the minimal period, we have that

$$\alpha_{n\ell+a_i} \equiv \alpha_{y\ell+a_i} \ (mod\ \ell) \tag{2.8}$$

and

$$\alpha_{n\ell+b_j} \equiv \alpha_{y\ell+b_j} \ (mod\ \ell) \tag{2.9}$$

hold. Since $y\ell + a_i, y\ell + b_j < \pi_\ell(F_\ell)$ and $y < \pi_\ell(F_\ell)/\ell$, we see that

$$\begin{aligned}
\sum_{i=1}^{s} \alpha_{n\ell+a_i} &\equiv \sum_{i=1}^{s} \alpha_{y\ell+a_i} \ (mod\ \ell) && \text{by (2.8)} \\
&\equiv \sum_{j=1}^{t} \alpha_{y\ell+b_j} \ (mod\ \ell) && \text{by hypothesis} \\
&\equiv \sum_{j=1}^{t} \alpha_{n\ell+b_j} \ (mod\ \ell) && \text{by (2.9)}
\end{aligned}$$

as was needed to be shown. ■

2.5 Applications

Moreover, MSS [MSS15] were able to prove the following congruences hold using Theorem 2.4.1.

Theorem 2.5.1. *For all $n \geq 0$, $(pl_k(m))_{m\geq 0}$ satisfies the following congruence relations:*

$$pl_2(2n+1) \equiv pl_2(2n) \ (mod\ 2) \tag{2.10}$$

$$pl_3(3n+2) \equiv 0 \ (mod\ 3) \tag{2.11}$$

$$pl_5(5n+2) \equiv pl_5(5n+4) \ (mod\ 5) \tag{2.12}$$

$$pl_5(5n+1) \equiv pl_5(5n+3) \ (mod\ 5). \tag{2.13}$$

■

We demonstrate the power of Theorem 2.4.1 by proving the first two identities as follows.

Corollary 2.5.2. *For all $n \in \mathbb{N}_0$,*

$$pl_3(3n+2) \equiv 0 \ (mod\ 3).$$

Proof. We apply Theorem 2.4.1 by specializing it to $\ell = 3, a_1 = 2$ and $b_1 = 5$. It suffices to show that

$$pl_3(3n+2) \equiv pl_3(3n+5) \ (mod\ 3) \tag{2.14}$$

holds for all $n \geq 0$ and that

$$pl_3(2) \equiv 0 \ (mod\ 3).$$

By listing all plane partitions of 2 with parts in $\{1, 2, 3\}$, we find that $pl_3(2) = 3 \equiv 0 \ (mod\ 3)$. Since

$$\frac{\pi_3(F_3(q))}{3} = \frac{3 \cdot 2}{3} = 2,$$

we need only check that (2.14) holds for all $0 \leq n < 2$. This is easily verified because

$$pl_3(2) = 3 \equiv 0 \ (mod\ 3)$$

and

$$pl_5(2) = 21 \equiv 0 \ (mod\ 3),$$

which completes the proof. ■

Corollary 2.5.3. *For $n \in \mathbb{N}_0$,*

$$pl_2(2n+1) \equiv pl_2(2n) \ (mod\ 2).$$

Proof. We specialize Theorem 2.4.1 to $\ell = 2, a_1 = 1, b_1 = 0$. By Proposition 2.3.7, it suffices to show that

$$pl_2(2n+1) \equiv pl_2(2n) \ (mod\ 2)$$

holds for all $0 \leq n < \frac{\pi_2(F_2(q))}{2} = \frac{2^0}{2} = \frac{1}{2}$. Therefore, we compute $pl_2(2)$ and $pl_2(3)$ and check that they are congruent modulo 2. By listing the relevant plane partitions, we find

$$pl_2(2) = 1 \qquad pl_2(3) = 5.$$

These are congruent modulo 2, and the claim holds. ∎

Chapter 3

Modular Forms

3.1 Introduction

In 1919, Ramanujan [Ram19] initiated the study of partition number congruences by discovering the so-called Ramanujan Congruences . A Ramanujan Congruence for a prime ℓ is the assertion that the integer partition function maps integers which lie in residue class β $(mod\ \ell)$ to 0 $(mod\ \ell)$, for some β.

Ramanujan established the two identities

$$\sum_{n\geq 0} p(5n+4)q^n = 5\frac{\prod_{i\geq 1}(1-q^5)^5}{\prod_{i\geq 1}(1-q)^6} \tag{3.1}$$

and

$$\sum_{n\geq 0} p(7n+6)q^n = 7\frac{\prod_{i\geq 1}(1-q^7)^3}{\prod_{i\geq 1}(1-q)^4} \tag{3.2}$$

in order to prove that for any $n \in \mathbb{Z}$, $p(5n+4)$ is divisible by 5 and $p(7n+6)$ is divisible by 7, respectively.

Subsequently in [Ram21], Ramanujan found another method which allowed him to deduce that $p(11n+6)$ is divisible by 11 for all n. The method he used relied on functional equations he derived from modular forms. Ramanujan conjectured that there are no such congruences holding for primes other than $5, 7$ and 11.

In 2003, Ahlgren and Boylan (here after referred to as AB) [AB03] were able to provide a proof to this conjecture using the theory of modular forms. In this chapter, we introduce modular forms and provide a summary of AB's proof of Ramanujan's Conjecture.

3.2 Ramanujan Congruences

The existence of a Ramanujan congruence for a prime ℓ implies that the partition function maps some linear sequence of integers with growth rate ℓ into the line $y = \ell x$.

Definition 3.2.1. For ℓ a prime and $\beta \in \mathbb{Z}$, we say that

$$p(\ell n + \beta) \equiv 0 \ (mod\ \ell) \tag{3.3}$$

is a *Ramanujan congruence* if (3.3) holds for all $n \in \mathbb{Z}$.

Ramanujan showed the following Ramanujan congruences for $\ell = 5, 7, 11$:

$$p(5n + 4) \equiv 0 \ (mod\ 5) \tag{3.4}$$

$$p(7n + 5) \equiv 0 \ (mod\ 7) \tag{3.5}$$

$$p(11n + 6) \equiv 0 \ (mod\ 11) \tag{3.6}$$

Brendt [Ber07] presented proofs for (3.4), (3.5) and (3.6) relying on the theory of modular forms. Although there are several mathematical statements which can be referred to as *Ramanujan's Conjecture*, in this chapter we will denote the following statement with this name.

Conjecture 3.2.2 (Ramanujan's Conjecture)**.** *Fix ℓ a prime. If*

$$p(\ell n + \beta) \equiv 0 \ (mod\ \ell)$$

is a Ramanujan congruence for some $\beta \in \mathbb{Z}$, then $\ell \in \{5, 7, 11\}$. Equivalently, equations (3.4), (3.5) and (3.6) are the only Ramanujan congruences.

3.3 The Basics of Modular Forms

We begin by studying modular forms. The content and definitions in this section are based on the work of Eric Bucher in [Buc10].

In order to define modular forms, we need to introduce modular groups and Dirichlet characters. However, for our purposes we shall consider only the trivial Dirichlet character.

Definition 3.3.1. We define the *Special Linear Group* $SL_2(\mathbb{Z})$ to be the group of 2×2 matrices over $\mathbb{Z}$ with unit determinant. That is,

$$SL_2(\mathbb{Z}) := \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z} \text{ and } \det(A) = 1 \right\}.$$

We note that $SL_2(\mathbb{Z})$ is generated by two matrices

$$S := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad T := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

We let $SL_2(\mathbb{Z})$ act on the upper complex half-plane $\mathcal{H} := \{z \in \mathbb{C} : Im(z) > 0\}$ by setting

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az + b}{cz + d}$$

for any $z \in \mathcal{H}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$. We remark that in some literature, this action is known as the *Möbius transformation.*

We recall that a complex-valued function $f : S \to \mathbb{C}$ is said to be *analytic* on an open set $D \subseteq S$ if for any $x_0 \in D$, we can write

$$f(x) = \sum_{n \geq 0} a_n (x - x_0)^n$$

where $a_0, a_1, \ldots$ are all complex numbers and the series converges to $f(x)$ for all x in some nontrivial neighbourhood $N \subseteq D$ of x_0.

Definition 3.3.2. We say that a complex-valued function $f : \mathcal{H} \to \mathbb{C}$ is *holomorphic* on $\mathcal{H}$ if it is analytic on $\mathcal{H}$. If f is analytic at ∞ as well, then we say that f is *holomorphic at* ∞.

Dirichlet characters are studied throughout analytic number theory (see [Apo76]). They are used to construct Dirichlet L-Functions which generalize the Riemann Zeta function and hence appear extensively throughout the study of the generalized Riemann hypothesis.

For the sake of being thorough, we provide their definition here despite using only the trivial Dirichlet character for our construction.

Definition 3.3.3. A *Dirichlet character* χ is a function $\chi : \mathbb{Z} \to \mathbb{C}$ that satisfies the following properties:

1. There exists $k \in \mathbb{Z}$ for which
$$\chi(n) = \chi(n+k)$$
for all $n \in \mathbb{Z}$. We call k the *period* of χ and we say that χ is a character to the modulus k.

2. If $\gcd(n, k) > 1$, then
$$\chi(n) = 0.$$
Otherwise, if $gcd(n, k) = 1$, then
$$\chi(n) \neq 0.$$

3. For all $m, n \in \mathbb{Z}$,
$$\chi(mn) = \chi(m)\chi(n).$$

4. $\chi(1) = 1$.

5. If $a \equiv b \ (mod\ k)$, then
$$\chi(a) = \chi(b).$$

6. If $gcd(a, k) = 1$, then $\chi(a)$ is a $\phi(k)^{th}$ complex root of unity, where ϕ is Euler's totient function, which counts the number of positive integers less than some integer n that are relatively prime to n.

We define the *trivial* Dirichlet character χ_1 by
$$\chi_1(n) = 1$$
for all $n \in \mathbb{Z}$ and note that it is periodic with period 1.

Now, we relate these mathematical constructions to define a modular form.

Definition 3.3.4. We say that $f : \mathcal{H} \to \mathbb{C}$ is a *modular form of weight k* on $SL_2(\mathbb{Z})$ with character χ if

1. The mapping f satisfies the functional equation
$$f(\gamma \cdot z) = \chi(d)(cz+d)^k f(z) \tag{3.7}$$
for all $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$ and $z \in \mathcal{H}$,
2. f is holomorphic on $\mathcal{H}$.
3. f is holomorphic at ∞.

The proof of Conjecture 3.2.2 relies on weakly modular forms.

Definition 3.3.5. We say that a modular form f of weight k is a *weakly modular form* of weight k for $SL_2(\mathbb{Z})$ if f satisfies (3.7) with $\chi = \chi_1$. That is,
$$f(\gamma \cdot z) = (cz+d)^k f(z) \tag{3.8}$$
holds for all $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$.

Example 3.3.6. A good example of a modular form for $SL_2(\mathbb{Z})$ is $\Delta(z)$ which is defined by
$$\Delta(z) := q \prod_{n \geq 1} (1-q^n)^{24}. \tag{3.9}$$
One can verify that
$$\Delta\left(-\frac{1}{z}\right) = z^{12}\Delta(z)$$
which in turn tells us that $\Delta(z)$ has weight 12.

Moreover, Δ is related to the so-called Dedekind eta function η, which is defined by
$$\eta(z) := q^{\frac{1}{24}} \prod_{n \geq 1} (1-q^n).$$
We note that $\eta(z)$ has an obvious relationship with the generating series of ordinary partitions, since
$$\frac{1}{\eta(z)} = q^{-\frac{1}{24}} \sum_{n \geq 0} p(n) q^n$$
which is often used in the literature in order to facilitate a connection between modular forms and integer partitions (for instance, see [GO97], [CDJ$^+$08]).

We note that the set of all weakly modular forms of weight k forms a complex vector space which we will denote by M_k.

We recall that a Fourier series is an expansion of a periodic function $f(x)$ as an infinite summation of oscillating functions. In particular, any periodic function has such an expansion. Here, we show that weakly modular functions have such expansions.

Lemma 3.3.7. *If f is a weakly modular form, then f has a Fourier series expansion.*

Proof. Since S and T generate $SL_2(\mathbb{Z})$, we can see that f satisfies (3.8) if and only if f satisfies (3.8) for $z = S, T$. This is equivalent to

$$\begin{aligned} f(S \cdot z) &= f\left(\frac{0 \cdot z - 1}{1 \cdot z + 0}\right) \\ &= f\left(\frac{-1}{z}\right) \\ &= (1 \cdot z + 0)^k f(z) \\ &= z^k f(z) \end{aligned}$$

and

$$\begin{aligned} f(T \cdot z) &= f\left(\frac{1 \cdot z + 1}{0 \cdot z + 1}\right) \\ &= f(z+1) \\ &= (0 \cdot z + 1)^k f(z) \\ &= f(z). \end{aligned}$$

Hence, we can conclude that $f(z)$ satisfies (3.7) if and only if

$$f\left(\frac{-1}{z}\right) = z^k f(z) \tag{3.10}$$

and

$$f(z+1) = f(z). \tag{3.11}$$

In particular, (3.11) tells us that f is periodic and has period 1. Since modular functions are periodic, we know that they have a Fourier series. Additionally, we know that they have a Fourier series expansion at ∞ with the form

$$f(z) = \sum_{n \geq 0} a_n q^n$$

where $q := e^{2\pi i z}$. ■

Before we summarize AB's proof of Conjecture 3.2.2 in the next section, we must first introduce some of the machinery we will use.

In [AB03], the authors demonstrated the following striking property of primes ℓ which have a Ramanujan congruence.

Proposition 3.3.8. *For a prime ℓ and $\beta \in \mathbb{Z}$, if*

$$p(\ell n + \beta) \equiv 0 \ (mod\ \ell)$$

holds for all $n \in \mathbb{Z}$, then

$$24\beta \equiv 1 \ (mod\ \ell).$$

■

We can check that Proposition 3.3.8 agrees with the statement of Conjecture 3.2.2. For instance, we can see that for $\ell = 5, \beta = 4$ we have

$$24 \cdot 4 = 96 \equiv 1 \ (mod\ 5),$$

for $\ell = 7, \beta = 5$ we have

$$24 \cdot 5 = 120 \equiv 1 \ (mod\ 7),$$

and when $\ell = 11, \beta = 6$ we have

$$24 \cdot 6 = 144 \equiv 1 \ (mod\ 11),$$

as asserted.

In proving Ramanujan's Conjecture 3.2.2, [AB03] carefully examined the following functions on prime numbers $\ell > 3$.

Definition 3.3.9. For $\ell > 3$ a prime, we define δ_ℓ by

$$\delta_\ell := \frac{\ell^2 - 1}{24}$$

and f_ℓ by

$$f_\ell := \Delta^{\delta_\ell}(z).$$

Clearly, if $\delta_\ell \in \mathbb{Z}$, then δ_ℓ satisfies

$$24\delta_\ell \equiv \ell^2 - 1 \equiv -1 \ (mod\ \ell).$$

From Proposition 3.3.8, we see that

$$24\delta_\ell \equiv -1 \equiv 24\beta \ (mod\ \ell)$$

which allows us to conclude that $\delta_\ell \equiv -\beta \ (mod\ \ell)$ because ℓ is a prime which does not divide 24.

Following [AB03], we now show that δ_ℓ is a nonnegative integer.

Proposition 3.3.10. *If $\ell > 3$ is a prime, then $\delta_\ell \in \mathbb{N}$.*

Proof. Consider $\ell^2 - 1 \ (mod\ 24)$. We seek to show that $24|\ell^2 - 1$ by showing that $8|\ell^2 - 1$ and $3|\ell^2 - 1$:

Since ℓ is an odd prime, we know that either $\ell \equiv 1 \ (mod\ 4)$ or $\ell \equiv 3 \ (mod\ 4)$.

If $\ell \equiv 1 \ (mod\ 4)$, then $\ell - 1 \equiv 0 \ (mod\ 4)$ and $\ell + 1 \equiv 2 \ (mod\ 4)$. This means that $4|\ell - 1$ and $2|\ell + 1$ and so

$$8|(\ell - 1)(\ell + 1) = \ell^2 - 1.$$

Otherwise, if $\ell \equiv 3 \ (mod\ 4)$, we know $\ell - 1 \equiv 2 \ (mod\ 4)$ and $\ell + 1 \equiv 0 \ (mod\ 4)$. In particular, this means that $2|\ell - 1$ and $4|\ell + 1$ and thus allows us to conclude that

$$8|(\ell^2 - 1).$$

Since $\gcd(3, 8) = 1$, we only need to show now that $3|\ell^2 - 1$. Consider the consecutive integers $\ell - 1, \ell, \ell + 1$. Exactly one of these will be divisible by 3. Since ℓ is a prime greater than 3, we know that $3 \nmid \ell$, and hence one of $\ell - 1$ and $\ell + 1$ is divisible by 3, which concludes our proof. ■

In their study of Ramanujan's conjecture, [AB03] made use of several operators on the set of all integer formal power series. We mention one in particular below as it shows up in the study of $m-$ary partitions in Chapter 5 as well.

Definition 3.3.11. Let $U_\ell : \mathbb{Z}[[q]] \to \mathbb{Z}[[q]]$ be the operator defined by

$$U_\ell\left(\sum_{n\geq 0} a_n q^n\right) := \sum_{n\geq 0} a_{\ell n} q^n.$$

For the purposes of our summary, we will use the following nice factorization property that is satisfied by U_ℓ:

Lemma 3.3.12. *If a_n and b_n are integer sequences, then*

$$U_\ell\left[\left(\sum_{n\geq 0} a_n q^{\ell n}\right)\left(\sum_{m\geq 0} b_m q^m\right)\right] = \left(\sum_{n\geq 0} a_n q^n\right)\left(\sum_{m\geq 0} b_{m\ell} q^m\right).$$

Proof. We expand and compare coefficients:

$$\begin{aligned}
U_\ell\left[\left(\sum_{n\geq 0} a_n q^{\ell n}\right)\left(\sum_{m\geq 0} b_m q^m\right)\right] &= U_\ell\left[\sum_{s\geq 0}\left(\sum_{\substack{\ell n+m=s\\ n,m\geq 0}} a_n b_m\right) q^s\right] \\
&= \sum_{s\geq 0}\left(\sum_{\substack{\ell n+m=\ell s\\ n,m\geq 0}} a_n b_m\right) q^s
\end{aligned}$$

Since $\ell n + m = \ell s$ and $n, m \geq 0$ is equivalent to $m = \ell k$, for some $k \geq 0$, we can rewrite our summation as

$$\begin{aligned}
\sum_{s\geq 0}\left(\sum_{\substack{\ell n+m=\ell s\\ n,m\geq 0}} a_n b_m\right) q^s &= \sum_{s\geq 0}\left(\sum_{\substack{\ell n+\ell k=\ell s\\ n,k\geq 0}} a_n b_{\ell k}\right) q^s \\
&= \sum_{s\geq 0}\left(\sum_{\substack{n+k=s\\ n,k\geq 0}} a_n b_{\ell k}\right) q^s \\
&= \left(\sum_{n\geq 0} a_n q^n\right)\left(\sum_{k\geq 0} b_{\ell k} q^k\right).
\end{aligned}$$

■

3.4 Classification of Ramanujan's Congruence

Now, we outline AB's proof of Conjecture 3.2.2. We omit several statements regarding the reduction of modular forms modulo ℓ and instead emphasize the elegant mathematics involved in their proof.

Theorem 3.4.1. *Conjecture 3.2.2 is true.*

Proof. We let ℓ be a prime and we consider the three cases $\ell = 2, 3$ or $\ell \geq 5$.

Case 1: Suppose $\ell = 2$ and fix $\beta \in \mathbb{Z}$. Using the division algorithm, we can write

$$\beta = 2 \cdot n + k$$

where $k \in \{0, 1\}$ and $n \in \mathbb{Z}$. Then

$$p(2(-n) + \beta) = p(k)$$

and since $p(0) = p(1) = 1$, we conclude $p(k) \not\equiv 0 \ (mod\ 2)$.

Case 2: Suppose $\ell = 3$ and $\beta \in \mathbb{Z}$. Once again, using the division algorithm, we find

$$\beta = 3 \cdot n + k$$

for some $k \in \{0, 1, 2\}$ and $n \in \mathbb{Z}$. Thus,

$$p(3(-n) + \beta) = p(k)$$

and because $p(k) \in \{p(0), p(1), p(2)\} = \{1, 2\}$, we see that $p(k) \not\equiv 0 \ (mod\ 3)$.

Case 3: Finally, we suppose that $\ell \geq 5$.

By Proposition 3.3.10, $\delta_\ell \in \mathbb{N}$. Moreover, we see that $\{\ell n + \beta : n \in \mathbb{N}\} = \{\ell n - \delta_\ell : n \in \mathbb{N}\}$ as $\delta_\ell \equiv -\beta \ (mod\ \ell)$.

Therefore, in order to prove Ramanujan's conjecture, it is enough to show that if $\ell \geq 13$ is prime then

$$\sum_{n \geq 0} p(\ell n - \delta_\ell) q^n \not\equiv 0 \ (mod\ \ell).$$

We examine f_ℓ in detail:

$$\begin{aligned} f_\ell(z) &= q^{\delta_\ell} \prod_{n \geq 1} (1 - q^n)^{24\delta_\ell} \\ &\equiv q^{\delta_\ell} \prod_{n \geq 1} (1 - q^n)^{\ell^2 - 1} \ (mod\ \ell) \\ &= q^{\delta_\ell} \prod_{n \geq 1} \frac{(1 - q^n)^{\ell^2}}{(1 - q^n)} \ (mod\ \ell) \end{aligned}$$

as $24\delta_\ell \equiv \ell^2 - 1 \ (mod\ \ell)$. Moreover, by the binomial theorem

$$\begin{aligned} q^{\delta_\ell} \prod_{n\geq 1} \frac{(1-q^n)^{\ell^2}}{(1-q^n)} &\equiv q^{\delta_\ell} \prod_{n\geq 1} (1-q^{\ell n})^\ell \prod_{n\geq 1} \frac{1}{(1-q^n)} \ (mod\ \ell) \\ &= q^{\delta_\ell} \prod_{n\geq 1} (1-q^{\ell n})^\ell \sum_{n\geq 0} p(n) q^n \\ &= \prod_{n\geq 1} (1-q^{\ell n})^\ell \sum_{n\geq 0} p(n-\delta_\ell) q^n \end{aligned}$$

If we apply U_ℓ, then by Lemma 3.3.12 we see that

$$U_\ell(f_\ell(z)) = \prod_{n\geq 1} (1-q^n)^\ell \sum_{n\geq 0} p(\ell n - \delta_\ell) q^n. \tag{3.12}$$

Therefore, it follows that ℓ gives a Ramanujan congruence if and only if

$$U_\ell(f_\ell(z)) \equiv 0 \ (mod\ \ell) \tag{3.13}$$

holds. AB [AB03] demonstrated that (3.13) is equivalent to the assertion that

$$(\delta_\ell + 1)^{\frac{\ell+3}{2}} \equiv 241 \cdot \delta_\ell^{\frac{\ell+3}{2}} \ (mod\ \ell).$$

We rewrite this as

$$\frac{(\delta_\ell + 1)}{\delta_\ell}^{\frac{\ell+3}{2}} \equiv 241 \ (mod\ \ell)$$

and then we simplify the left hand side by recalling that $24\delta_\ell \equiv -1 \ (mod\ \ell)$ to obtain

$$\begin{aligned} \frac{(\delta_\ell + 1)}{\delta_\ell}^{\frac{\ell+3}{2}} &= -24(\delta_\ell + 1) \ (mod\ \ell) \\ &= (-24\delta_\ell - 24)^{\frac{\ell+3}{2}} \\ &\equiv (1-24)^{\frac{\ell+3}{2}} \ (mod\ \ell) \\ &= (-23)^{\frac{\ell+3}{2}} \\ &\equiv 241. \end{aligned}$$

In order to make use of Fermat's Little Theorem, we write

$$\frac{\ell+3}{2} = 2 + \frac{\ell-1}{2}$$

and simplify

$$\begin{aligned}
(-23)^{\frac{\ell+3}{2}} &= (-23)^2(-23)^{\frac{\ell-1}{2}} \\
&\equiv 529 \cdot (\pm 1) \ (mod\ \ell) \quad \text{by Fermat's Little Theorem} \\
&\equiv 241 \ (mod\ \ell).
\end{aligned}$$

Therefore, we need to check for which values of ℓ we have

$$529 \equiv (\pm 241) \ (mod\ \ell)$$

or equivalently, which primes ℓ satisfy either one of

$$529 + 241 = 770 = 2 \cdot 5 \cdot 7 \cdot 11 \equiv 0 \ (mod\ \ell)$$

or

$$529 - 241 = 288 = 2^5 \cdot 3^2 \equiv 0 \ (mod\ \ell)$$

Since we excluded $\ell = 2$ and $\ell = 3$ in case 1 and case 2, we have demonstrated that ℓ must be one of $5, 7$ or 11. ■

Chapter 4

The Distribution of the Partition Function

4.1 Introduction

In the previous chapter, we analyzed the Ramanujan congruences. In essence, they stated that the partition function maps certain arithmetic progressions to 0 $(mod\ \ell)$, for certain primes ℓ. Similar to this notion is the study of the distribution of the values of the partition function modulo an integer m, not necessarily prime.

More specifically, we will be interested in the problem: given positive integer m, r and X satisfying $0 \leq r < m$, how often is $p(n) \equiv r\ (mod\ m)$ for $n \in \{1, 2, \ldots, X\}$?

In this chapter we will present Newman's conjecture [New60] regarding the distribution of the partition function, an elementary proof of the ever-changing parity of the partition function and recent progress pertaining to the odd-value density of the partition function [JKZ15].

4.2 The Distribution of the Partition Function

In the 1960's, Morris Newman [New60] studied distribution of the partition function modulo an integer m. He was motivated by the belief that for all positive integers m, the partition function values took on all residues in modulo m infinitely often, as in the following conjecture.

Conjecture 4.2.1 (Newman's Conjecture)**.** *If $m \in \mathbb{N}$ and $0 \leq r \leq m-1$, then there is an infinite sequence $(a_n^{(r)})_{n\geq 0}$ for which*

$$p(a_n^{(r)}) \equiv r \ (mod\ m)$$

holds for all $n \geq 0$.

Newman showed that Conjecture 4.2.1 holds for $m = 2$ using a elegant algebraic argument. It relies on the following Lemma regarding formal power series.

Lemma 4.2.2. *Fix $m > 1$ an integer and let $(e_n)_{n=0}^{\infty}, (c_n)_{n=0}^{\infty}$ be sequences of integers such that*

$$\lim_{n\to\infty} e_{n+1} - e_n = \infty$$

and for all $n \in \mathbb{N}$,

$$gcd(c_n, c_{n+1}, \ldots) = 1, \quad e_n \geq e_{n-1}.$$

If we let $f(q)$ be the generating series

$$f(q) := \sum_{n\geq 0} c_n q^{e_n},$$

then there are no polynomials $\alpha(q), \beta(q) \in \mathbb{Z}[q]$ for which $\alpha(q) = 1$ and

$$f(q) \equiv \beta(q)/\alpha(q) \ (mod\ m).$$

Proof. Suppose there are $\alpha(q) = \sum_{i=0}^{r} a_i q^i, \beta(q) = \sum_{j=0}^{s} b_j q^j \in \mathbb{Z}[q]$ for which

$$f(q)\alpha(q) \equiv \beta(q) \ (mod\ m),$$

where $\alpha(q) = 1$. Fixing $n \in \mathbb{N}$, we see that

$$\sum_{e_k \leq n} c_k a_{n-e_k} \equiv b_n \ (mod\ m)$$

and thus,

$$\begin{aligned} b_{e_n} &\equiv \sum_{k=1}^{n} c_k a_{e_n - e_k} \ (mod\ m) \\ &= c_n + \sum_{k=1}^{n-1} c_k a_{e_n - e_k} \ (mod\ m) \end{aligned}$$

because $a_0 = 1$. Since

$$\lim_{n\to\infty} e_{n+1} - e_n = \infty,$$

we know that there exists n_0 for which

$$e_n - e_{n-1} > r \text{ and } e_n > s$$

holds for all $n \geq n_0$. However, this in turn implies that for $n \geq n_0$,

$$\begin{aligned} b_{e_n} &= 0 && \text{since } \beta(q) \text{ is a polynomial} \\ &\equiv c_n + \sum_{k=1}^{n-1} c_k a_{e_n - e_k} \ (mod\ m) \\ &= c_n + \sum_{k=1}^{n-1} c_k \cdot 0 \\ &= c_n \end{aligned}$$

as $e_n - e_k \geq e_n - e_{n-1}$ for all $1 \leq k \leq n-1$ and $a_i = 0$ for $i > r$. Therefore, we find that $c_n \equiv 0 \ (mod\ m)$ for all $n \geq n_0$. However, this means that

$$gcd(c_n, c_{n+1}, \ldots,) > 1$$

contradicting our hypothesis. ■

Related to the notion of distribution in the context of Conjecture 4.2.1 is the *ultimate periodicity* of a sequence. That is, the property that a sequence is eventually distributed modulo m with recurrent pattern.

Definition 4.2.3. We say that a sequence $(a_n)_{n=0}^{\infty}$ is *ultimately periodic* modulo m if there are $k, r \in \mathbb{N}$ for which

$$a_{l+r} \equiv a_l \ (mod\ m)$$

holds for all $l \geq k$. We say that r is a period of (a_n) modulo m with constant k. The smallest r satisfying this criteria is known as the *minimal period* of (a_n) modulo m with constant k.

It follows immediately from the definition that if a sequence $(a_n)_{n\geq 0}$ is not ultimately periodic, then it takes on at least two values modulo m infinitely often. Newman [New60] used Lemma 4.2.2 to demonstrate that $p(n)$ is not ultimately periodic and thus takes on at least two values in modulo m infinitely often. In particular, this proved that $p(n)$ is odd infinitely often and even infinitely often.

Theorem 4.2.4. *The sequence $(p(n))_{n\geq 0}$ is not ultimately periodic $\pmod{m}$ and consequently takes on at least two different residue modulo m infinitely often.*
In particular, Conjecture 4.2.1 holds for $m = 2$.

Proof. We know by the Pentagonal Number Theorem 1.2.13 that

$$\prod_{n\geq 0}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{(3n^2+n)/2}.$$

We write

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{(3n^2+n)/2} = \sum_{n\geq 0} g_n q^{e_n},$$

where g_n is as given by

$$g_n = \begin{cases} (-1)^k, & \text{if } n = \dfrac{k(3k\pm 1)}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Since $(3n^2+n)/2 > 0$ for all $n \in \mathbb{Z}$, we see that

$$0 \leq e_0 < e_1 < e_2 < \cdots .$$

Moreover, it is evident that

$$gcd(g_n, g_{n+1}, g_{n+1}, \ldots) = 1$$

as $g_\ell = \pm 1$ infinitely often.

Thus, $f(q) := 1/P(q)$ satisfies the hypothesis for Lemma 4.2.2 and we can conclude that there are no $\alpha(q), \beta(q) \in \mathbb{Z}[q]$ for which

$$f(q) = \frac{\alpha(q)}{\beta(q)}.$$

But this also implies that there are no $\alpha(q), \beta(q) \in \mathbb{Z}[q]$ for which

$$P(q) = \frac{\alpha(q)}{\beta(q)}.$$

If $(p_n)_{n\geq 0}$ is ultimately periodic modulo m with minimal period r and constant k, then

$$\begin{aligned} P(q) &= \sum_{n\geq 0} p_n q^n \\ &= \sum_{n=0}^{k} p_n q^n + \sum_{n=k+1}^{k+r} p_n q^n + \sum_{n=k+r+1}^{k+2r} p_n q^n + \sum_{n=k+2r+1}^{k+3r} p_n q^n + \cdots \\ &\equiv \sum_{n=0}^{k} p_n q^n + \sum_{n=k+1}^{k+r} p_n q^n + q^r \sum_{n=k+1}^{k+r} p_{n-r} q^{n-r} + q^{2r} \sum_{n=k+2r+1}^{k+3r} p_{n-2r} q^{n-2r} + \cdots \ (mod\ m) \\ &= \sum_{n=0}^{k} p_n q^n + \sum_{n=k+1}^{k+r} p_n q^n \left(\frac{1}{1-q^r}\right). \end{aligned}$$

If we denote by $\alpha(q) = \sum_{n=0}^{k} p_n q^n$ and $\beta(q) = \sum_{n=k+1}^{k+r} p_n q^n$, then

$$\begin{aligned} P(q) &\equiv \alpha(q) + \beta(q)\frac{1}{1-q^r} \ (mod\ m) \\ &= \frac{1}{1-q^r}(\alpha(q)(1-q^r) + \beta(q)), \end{aligned}$$

contradicting the fact that $P(q)$ cannot be expressed as a rational function of two integer polynomials. So, we conclude that $p(n)$ is not ultimately periodic and hence its values take on at least two residue classes infinitely often. ■

For an alternative proof of the fact that $p(n)$ is odd and even infinitely often, see Kolberg's work in [Kol59].

Moreover in [New60], Newman showed that Conjecture 4.2.1 holds for $m = 5$ and $m = 13$ by exploiting congruences which were derived from the theory of elliptic modular forms. Later on, Atkin [Atk68] verified Conjecture 4.2.1 for $m = 7, 11$ and $m = 13$

In [Ono00], K. Ono devised a groundbreaking tool for verifying if Conjecture 4.2.1 holds for any arbitrary *good prime*.

Definition 4.2.5. We say that a prime $m > 3$ is a *good* prime if for all $r \in \{0, 1, \ldots, m-1\}$ there exists $n_r \in \mathbb{N}_0$ such that $mn_r \equiv -1$ (mod 24) and

$$p\left(\frac{mn_r+1}{24}\right) \equiv r \ (mod\ m).$$

Theorem 4.2.6. *If $m \geq 5$ is a good prime, then Conjecture 4.2.1 is true for m. Moreover, for every residue class r modulo m, we have*

$$\#\{0 \leq n \leq X : p(n) \equiv r \ (mod\ m)\} \ll \begin{cases} \sqrt{X}/\log X & \text{if } 1 \leq r \leq m-1 \\ X & \text{if } r = 0 \end{cases}.$$

■

Ono noted that although it appears likely that every prime $m \geq 13$ is good, it is computationally expensive to prove. He was able to use Theorem 4.2.6 in order to assert the Corollary 4.2.7.

Corollary 4.2.7. *Conjecture 4.2.1 holds for all good primes $m < 1000$ with the possible exception of $m = 3$.*

■

4.3 The Density of the Partition Function

When studying the distribution of a mathematical object, it is often natural to ask questions about the density of the distribution. Before we examine the distribution of the values of $p(n)$ modulo an integer m, we introduce some essential terminology.

Definition 4.3.1. If $m \in \mathbb{N}$ and $r \in \{0, 1, \ldots, m-1\}$, we denote by $\delta_r(m, X)$ the ratio

$$\delta_r(m, X) = \frac{\#\{0 \leq n < X : p(n) \equiv r \ (mod\ m)\}}{X}$$

for any $X \in \mathbb{N}$.

Example 4.3.2. For instance, if we consider $m = 2$ and compute the first 12 values of $p(n)$.

We see that $\delta_0(2, 12) = 5/12$ and $\delta_1(2, 12) = 7/12$.

The following conjecture is due to Ahlgren and Ono [AO01]:

Conjecture 4.3.3. *The following are true.*

n	$p(n)$	$p(n)$ (*mod* 2)
0	1	1
1	1	1
2	2	0
3	3	1
4	5	1
5	7	1
6	11	1
7	15	1
8	22	0
9	30	0
10	42	0
11	56	0

Table 4.1: Values of $p(n)$ for $0 \leq n < 12$.

1. *If* $0 \leq r < m$, *then the limit*

$$\lim_{X \to \infty} \delta_r(m, X) = \delta_r(m)$$

exists and is in the interval (0,1).

2. *If* $s \geq 1$ *and* $m = 2^s$, *then for all* $0 \leq i < 2^s$, *we have*

$$\delta_i(2^s) = \frac{1}{2^s}.$$

3. *If* $s \geq 1$ *and* $m = 3^s$, *then for all* $0 \leq i < 3^s$, *we have*

$$\delta_i(3^s) = \frac{1}{3^s}.$$

4. *If there is a prime* $\ell \geq 5$ *for which* $\ell | m$, *then for all* $0 \leq r < m$ *we have*

$$\delta_r(m) \neq \frac{1}{m}.$$

■

The best known results that support Conjecture 4.3.3 may be found in [NRS98] and [Ahl99], though they are still far from affirming the conjecture.

4.3.1 The Density of the Odd Values of the Partition Function

The study of the odd-value density of the partition function dates back to Parkin and Shanks in 1967 [PS67]. Their interest in this mathematical construct stemmed from Ramanujan's simple sufficiency condition for divisibility of the partition function: if $n = 5k + 4$, for some $k \in \mathbb{Z}$ then $5|p(n)$.

Parkin and Shanks [PS67] computed the parity of the partition function $p(n)$ for n up to $n = 2,039,999$ empirically and conjectured that $\delta_1 = \frac{1}{2}$, where we utilize the notation below.

Definition 4.3.4. We define the *odd density* of $p(n)$ to be the quantity δ_1, where

$$\delta_1 := \lim_{x\to\infty} \frac{\#\{n \leq x : \; p(n) \text{ is odd}\}}{n}.$$

Recall that for $t \in \mathbb{N}$, we say that $(\lambda_1, \ldots, \lambda_t)$ is a *t-multipartition* of an integer n if λ_i is an integer partition for all $i = 1, \ldots, t$ and

$$|\lambda_1| + \cdots + |\lambda_t| = n.$$

We use $p_t(n)$ to denote the number of $t-$multipartitions of n. We denote the *odd density* of $p_t(n)$ to be the quantity δ_t where

$$\delta_t := \lim_{x\to\infty} \frac{|\{n \leq x : \; p_t(n) \text{ is odd}\}|}{n}.$$

After conducting extensive computations, [JKZ15] Judge, Keith and Zanello (hereafter referred to as JKZ) were led to believe the following.

Conjecture 4.3.5. *The odd density of the partition function δ_1 exists and is equal to* $1/2$*. Additionally, if $t = 2^k t_0 \in \mathbb{N}$, where t_0 is odd, then δ_t exists and equals*

$$\delta_t = \frac{1}{2^{k+1}}.$$

In particular, they extended part 2 of Conjecture 4.3.3.

JKZ's work [JKZ15] established a connection between δ_1 and δ_t by showing that there is an intricate relationship between $p(n)$ and $p_t(n)$. In particular, consider the following theorem.

Theorem 4.3.6. *If $t \in \{5, 7, 11, 13, 17, 19, 23, 25\}$ and $\delta_t > 0$, then $\delta_1 > 0$. Moreover, if $\delta_t > 0$, then $\delta_r > 0$ holds for the following (t, r) pairs:*

$$(27, 9), (9, 3), (25, 5), (15, 3), (21, 3), (27, 3).$$

In order to prove this connection, the JKZ established the following congruences using modular forms.

Theorem 4.3.7. *For $(a, b, t) \in \{(5, 4, 1), (7, 5, 1), (11, 6, 1), (13, 6, 1), (17, 5, 1), (19, 4, 1), (23, 1, 1), (3, 2, 3), (5, 2, 3), (7, 1, 3), (5, 0, 5), (3, 0, 9)\}$,*

$$q \sum_{n \geq 0} p_t(an + b)q^n \equiv \frac{1}{\prod_{i \geq 1}(1 - q^i)^{at}} + \frac{1}{\prod_{i \geq 1}(1 - q^{ai})^t} \ (mod\ 2) \tag{4.1}$$

and for $(a, b, t) \in \{(3, 8, 3), (5, 24, 1)\}$,

$$q^2 \sum_{n \geq 0} p_t(a^2n + b)q^n \equiv \frac{1}{\prod_{i \geq 1}(1 - q^i)^{a^2t}} + \frac{1}{\prod_{i \geq 1}(1 - q^{ai})^{at}} + \frac{q}{\prod_{i \geq 1}(1 - q^i)^t}. \tag{4.2}$$

■

Theorem 4.3.7 allowed JKZ to prove that Theorem 4.3.6 holds through simple arguments analogous to coefficient comparisons of two sides of a mathematical identitiy. For instance, one can show that $\delta_5 > 0$ implies $\delta_1 > 0$ as follows:

Proof. Suppose that $\delta_5 > 0$ and that Theorem 4.3.7 holds in the case $(5, 4, 1)$ so that

$$q \sum_{n \geq 0} p(5n + 4)q^n \equiv \frac{1}{\prod_{i \geq 1}(1 - q^i)^5} + \frac{1}{\prod_{i \geq 1}(1 - q^{5i})} \ (mod\ 2). \tag{4.3}$$

Since $\delta_5 > 0$, we know that $\frac{1}{\prod_{i \geq 1}(1-q^i)^5}$ has a positive odd density. Therefore, the odd densities of $\frac{1}{\prod_{i \geq 1}(1-q^{5i})}$ and $\sum_{n \geq 0} p(5n + 4)q^n$ cannot both be zero.
We see that the odd density of $P(q^5) = \frac{1}{\prod_{i \geq 1}(1-q^{5i})}$ is equal to the odd density of $P(q)$, and hence it has odd density δ_1. Since $(5n + 4)_{n \geq 0}$ is a subsequence of $(n)_{n \geq 0}$, if it has a positive density, then so must $p(n)$. Thus, we conclude $\delta_1 > 0$. ■

This technique can be generalized to other cases of (4.1) as well by recognizing that $p_t(an + b)$ is a subprogression of of $p_t(n)$, $\frac{1}{\prod_{i \geq 1}(1-q^i)^{at}}$ is $P_{at}(q)$ the generating series for $at-$multipartitions and $\frac{1}{\prod_{i \geq 1}(1-q^{ai})^t}$ has the same density as $P(q)$. We note that a similar argument holds for congruences of the form of (4.2).

Moreover, these congruences allowed JKZ to show how one could test absurd hypotheses in examinations of Conjecture 4.3.5, such as the Proposition 4.3.8 below.

Proposition 4.3.8. *If* $\delta_1 = 1$*, then* $\delta_5 = 4/5$*. In particular, the odd density of* $\sum_{n\geq 0} p_5(5n)q^{5n}$ *is 0 and odd density of 1 among*

$$\sum_{n\geq 0} p_5(5n+1)q^{5n+1}, \sum_{n\geq 0} p_5(5n+2)q^{5n+2}, \sum_{n\geq 0} p_5(5n+3)q^{5n+3}, \sum_{n\geq 0} p_5(5n+4)q^{5n+4}.$$

Proof. If $\delta_1 = 1$, then in (4.3), we find that since $p(5n+4)$ is a subprogression of $p(n)$, it will have odd density 1 as well. Similarly, $P(q^5)$ will also have odd density 1.

This means that the coefficients corresponding to multiples of 5 in $P_5(q)$ must have density 0. In particular, this means that $\{p_5(5n)\}_{n\geq 0}$ has odd density 0 and density 1 on all other linear subprogressions, as needed. ■

Chapter 5

Enumerating $m-$ary Partitions modulo a prime m

5.1 Introduction and History

A binary partition is a partition where every part is a power of 2. In the 1960's, Churchhouse investigated and made several conjectures concerning congruences of binary partitions. In particular, he conjectured that if $l \geq 1$, and $l \equiv 1$ $(mod\ 2)$ then

$$b_2(2^{2k+2}l) \equiv b(2^{2k}l) \ \left(mod\ 2^{2k+2}\right)$$

and

$$b_2(2^{2k+1}l) \equiv b(2^{2k-1}l) \ \left(mod\ 2^{3k}\right)$$

where $b_2(n)$ is the number of binary partitions of n.

Shortly after Churchhouse published his conjectures regarding binary partitions congruences ([Chu69]), other mathematicians proved and generalized his conjectures (see [Gup76], [And71]).

In 2015, Andrews, Fraenkel and Sellers, hereafter referred to as AFS, were able to provide an elementary and ingenious proof to a generalization of Churchhouse's conjectures regarding binary partitions (see [AFS15]). More specifically, they introduced $m-$ary partitons: integer partitions where the only allowed parts are powers of m, and then they found an elegant closed-form expression for the residue class modulo m for each member of this sequence. Their approach relied on simple manipulations of the generating series for $m-$ary partitions and some basic tools from the theory of formal power series.

Their result was remarkable for two reasons: it is a complete characterization of $m-$ary partitions $(mod\ m)$ which only depends on the base m representation of an integer and such characterizations are extremely rare for integer partitions. It motivated other mathematicians to study the distribution of $m-$ary partitions modulo m ([Edg16]). Moreover, Ekhad and Zeilberger [EZ15] were able to generalize AFS's approach to compute $m-$sections of any formal power series satisfying a similar functional equation to that of $m-$ary partitions.

In [AFS16], Andrews, Fraenkel and Sellers were able to find similar congruence properties for a related class of $m-$ary partitions: those for which we use all powers of m as parts up to some m^k. This time, their approach for determining the residue class was to exploit a congruence property derived from a recurrence relation.

In this chapter, we will illustrate AFS's techniques for deriving their characterization theorems. Afterwards, we will extend their results to a certain class of coloured $m-$ary partitions.

5.2 $m-$ary Partitions modulo m

Definition 5.2.1. For $m \geq 2$ an integer, we say that an integer partition is m-ary if all of the parts are powers of m. We denote by $b_m(n)$ the number of $m-$ary partitions of n.

By considering the set of allowed parts for $m-$ary partitions in the context of Theorem 1.2.12, it is evident that the generating series for $m-$ary partitions $B_m(q)$ is

$$B_m(q) := \prod_{j \geq 0} \frac{1}{(1-q^{m^j})}.$$

Moreover, it is obvious that $B_m(q)$ satisfies the functional equation

$$\frac{B_m(q^m)}{(1-q)} = B_m(q). \tag{5.1}$$

Now, we show that the values of $(b_m)_{m \geq 0}$ agree in tuples of m.

Lemma 5.2.2. *For all $r \in \{1, \ldots, m-1\}$ and $l \geq 0$, the coefficients $b_m(n)$ satisfy*

$$b_m(ml+r) = b_m(ml).$$

Proof. Fix an arbitrary $m-$ary partition λ of ml. In order to construct a unique $m-$ary partition of $ml+r$, we may add r parts of size 1 to λ.

Conversely, let γ be an arbitrary partition of $ml+r$ where $r \in \{1, \ldots, m-1\}$ and $\ell \geq 0$. We know that it has s parts of size 1 where $s \equiv r \ (mod\ m)$ since it is the only way for the partition to have size congruent to $ml + r$. In particular, since $s \geq r \geq 1$, we know that we can remove r 1's to obtain a unique $m-$ary partition of ml.

This gives us the desired bijection and concludes the proof. ■

In order to prove their characterization theorem, the approach that AFS took made use of the following elementary tools.

Lemma 5.2.3. *For $m \geq 2$, we have*

$$\frac{1-q^m}{(1-q)^2} \equiv \sum_{k=1}^{m} kq^{k-1} \ (mod\ m).$$

Proof. Formal differentiation of both sides of the geometric series gives us

$$\frac{d}{dq}\left(\frac{1}{1-q}\right) = \frac{d}{dq}\left(\sum_{k\geq 0} q^k\right)$$

which in turn implies that

$$\frac{1}{(1-q)^2} = \sum_{k\geq 1} kq^{k-1}.$$

So, if we multiply both sides by $(1-q^m)$, we find

$$\begin{aligned}
\frac{(1-q^m)}{(1-q)^2} &= (1-q^m)\sum_{k=1}^{\infty} kq^{k-1} \\
&= \sum_{k\geq 1} kq^{k-1} - \sum_{k\geq 1} kq^{k+m-1} \\
&= \sum_{k\geq 1} kq^{k-1} - \sum_{k\geq m+1} (k-m)q^{k-1} \\
&= \sum_{k=1}^{m} kq^{k-1} + \sum_{k\geq m+1} mq^{k-1} \\
&\equiv \sum_{k=1}^{m} kq^{k-1} \ (mod\ m).
\end{aligned}$$

■

Lemma 5.2.4. *If ζ is the $m-$th root of unity given by $e^{2\pi i/m}$, then*

$$\sum_{k=0}^{m-1} \frac{1}{1-\zeta^k q} = m\left(\frac{1}{1-q^m}\right).$$

Proof. We expand and then use an elementary result regarding roots of unity:

$$\begin{aligned}
\sum_{k=0}^{m-1} \frac{1}{1-\zeta^k q} &= \sum_{k=0}^{m-1}\sum_{r\geq 0} \zeta^{kr} q^r \\
&= \sum_{r\geq 0} q^r \sum_{k=0}^{m-1} \zeta^{kr} \\
&= \sum_{r\geq 0} q^r \sum_{k=0}^{m-1} \left(e^{2\pi i/m}\right)^{kr}
\end{aligned}$$

In order to simplify this summation, we analyze what the contribution of the inner sum is relative to the divisibility of r by m.

If $m|r$, then we write $r = mj$ and

$$\begin{aligned}
\sum_{k=0}^{m-1} \left(e^{\frac{2\pi i}{m}}\right)^{kr} &= \sum_{k=0}^{m-1} e^{2\pi ikj} \\
&= m
\end{aligned}$$

as $e^{2\pi i} = 1$.

Otherwise, if $m \nmid r$, we write $r = ms + j$ where $0 < j < m$ and hence

$$\begin{aligned}
\sum_{k=0}^{m-1} \left(e^{\frac{2\pi i}{m}}\right)^{kr} &= \sum_{k=0}^{m-1} e^{2\pi isk + \frac{2\pi ikj}{m}} \\
&= \sum_{k=0}^{m-1} e^{\frac{2\pi ikj}{m}} \\
&= 0
\end{aligned}$$

as the sum of all the $m-$th roots of unity is 0.

Therefore, when r is divisible by m, we find that the inner sum contributes m and otherwise, it contributes 0. We can write this generating series in the form

$$m\sum_{n\geq 0} q^{nm} = m\left(\frac{1}{1-q^m}\right)$$

as needed. ■

Next, we observe that Lemma 5.2.2 implies that in order to find $(b_m(n))_{n=0}^{\infty}$ $(mod\ m)$ it is sufficient to know $(b_m(mn))_{n=0}^{\infty}$ $(mod\ m)$. Thus, it is natural to investigate the generating series for $b_m(mn)_{n\geq 0}$.

Lemma 5.2.5. *If $T_m(q) := \sum_{n\geq 0} b_m(mn)q^n$, then*

$$T_m(q) = \frac{1}{1-q} B_m(q).$$

Proof. Let $\zeta = e^{2\pi i/m}$. Then

$$\begin{aligned}
T_m(q^m) &= \sum_{n\geq 0} b_m(mn)q^{mn} \\
&= \frac{1}{m}\left(B_m(q) + B_m(\zeta q) + \cdots + B_m(\zeta^{m-1}q)\right) \quad \text{by Proposition 1.2.17} \\
&= \frac{1}{m}\left(\prod_{j\geq 0}\frac{1}{1-q^{m^j}} + \prod_{j\geq 0}\frac{1}{1-\zeta^{m^j}q^{m^j}} + \cdots + \prod_{j\geq 0}\frac{1}{1-\zeta^{(m-1)m^j}q^{m^j}}\right) \\
&= \frac{1}{m}\left(\prod_{j\geq 1}\frac{1}{1-q^{m^j}}\right)\left(\sum_{k=0}^{m-1}\frac{1}{1-\zeta^k q}\right)
\end{aligned}$$

since $\zeta^{k\cdot m^j} = 1$ holds for all $j \geq 1$. Moreover, by Lemma 5.2.4

$$\begin{aligned}
\frac{1}{m}\left(\prod_{j\geq 1}\frac{1}{1-q^{m^j}}\right)\left(\sum_{k=0}^{m-1}\frac{1}{1-\zeta^k q}\right) &= \frac{1}{m}\left(m\frac{1}{1-q^m}\prod_{j\geq 1}\frac{1}{1-q^{m^j}}\right) \\
&= B_m(q)
\end{aligned}$$

as needed to be shown. ∎

Finally, in their construction, AFS showed that the generating series $T_m(q)$ is congruent to a generating series they denoted by $U_m(q)$. This generating series made their characterization of $b_m(mn)$ clear.

Lemma 5.2.6. *If $U_m(q)$ is the generating series*

$$U_m(q) := \prod_{j=0}^{\infty}\left(1 + 2q^{m^j} + 3q^{2m^j} + \cdots + mq^{(m-1)m^j}\right),$$

then

$$T_m(q) \equiv U_m(q)\ (mod\ m).$$

Proof. We show algebraically that

$$\frac{1}{T_m(q)} U_m(q) \equiv 1 \ (mod\ m)$$

By Lemma 5.2.5, we know that

$$\begin{aligned}\frac{1}{T_m(q)} U_m(q) &= (1-q)^2 \left[\prod_{j\geq 1}(1-q^{m^j})\right] U_m(q) \\ &\equiv (1-q)^2 \left[\prod_{j\geq 1}(1-q^{m^j})\right] \prod_{i\geq 0} \frac{1-q^{m^{j+1}}}{(1-q^{m^j})^2}\end{aligned}$$

using Lemma 5.2.3. Finally, we can simplify

$$\begin{aligned}(1-q)^2 \left[\prod_{j\geq 1}(1-q^{m^j})\right] \prod_{i\geq 0} \frac{1-q^{m^{j+1}}}{(1-q^{m^j})^2} &= \prod_{j\geq 0}(1-q^{m^{j+1}}) \prod_{j\geq 1} \frac{1}{1-q^{m^j}} \\ &= 1\end{aligned}$$

as needed. ■

Now, we have the tools available to prove the following characterization theorem.

Theorem 5.2.7. *If* $n = a_0 + a_1 m + \cdots + a_j m^j$ *in base* m*, then*

$$b_m(mn) \equiv \prod_{i=0}^{j}(a_i + 1) \ (mod\ m).$$

Proof. Lemma 5.2.6 tells us that

$$[q^n]T_m(q) \equiv [q^n]U_m(q) \ (mod\ m).$$

In order to determine the coefficient

$$[q^{a_0+a_1m+\cdots+a_jm^j}] \prod_{i\geq 0} \left(1 + 2q^{m^j} + 3q^{2m^j} + \cdots + mq^{(m-1)m^j}\right)$$

we note that by construction, it is enough to compute

$$\begin{aligned}&[q^{a_0+a_1m+\cdots+a_jm^j}] \prod_{i\geq 0}^{j} \left(1 + 2q^{m^j} + 3q^{2m^j} + \cdots + mq^{(m-1)m^j}\right) \\ &= \prod_{i=0}^{j} [q^{a_i m^i}] \left(1 + 2q^{m^j} + 3q^{2m^j} + \cdots + mq^{(m-1)m^j}\right) \\ &= \prod_{i=0}^{j}(a_i + 1)\end{aligned}$$

as needed. ■

We offer an alternative proof for Theorem 5.2.7 by exploiting a congruence recurrence relation that is satisfied by $b_m(mn)$ (see Theorem 5.4.8 in the case when $k = 1$).

5.3 $m-$ary Partitions Without Gaps

AFS were also able to give a characterization of a family of $m-$ary partitions satisfying the property of not having gaps. The analysis still uses generating series heavily but with a slightly different flavour. Here, the authors were able to exploit a recurrence relation to prove their complete characterization of these partition numbers modulo m.

Definition 5.3.1. Fix λ an $m-$ary partition of $n \in \mathbb{N}$. We say that λ is an $m-$ary partition without gaps if λ satisfies the property

$$\text{If } m^i \text{ is a part in } \lambda, \text{ then so is } m^j \text{ for all } 0 \leq j \leq i.$$

Example 5.3.2. For instance, when $m = 2$, the only two binary partitions without gaps of 4 are $(1, 1, 1, 1)$ and $(2, 1, 1)$. Note that (4) is not an binary partition without gaps of 4 as 2 and 1 do not appear as parts.

Lemma 5.3.3. *The generating series $C_m(q)$ for $c_m(n)$ is given by*

$$C_m(q) = 1 + \sum_{n \geq 0} \frac{q^{1+m+m^2+\cdots+m^n}}{(1-q)(1-q^m)\cdots(1-q^{m^2})}.$$

Proof. We decompose the set $\mathcal{C}_m$ of all $m-$ary partitions without gaps uniquely below.

Fix λ an $m-$ary partition without gaps. Let n be the unique maximal integer such that m^n is a part in λ. Since λ is without gaps, we know that $\{1, m, \ldots, m^n\}$ are all parts in λ and hence

$$|\lambda| \geq 1 + m + \cdots + m^n.$$

As n is maximal, it must be that λ only has parts from $\{1, m, \ldots, m^n\}$ and the number of parts from each power of m in this set uniquely determines λ. ■

Now, we show that that $c_m(n)$ also agree in $m-$tuples. Here, we show this combinatorially but later, we offer an algebraic proof via the specialization of Lemma 5.4.15 to the case $k = 1$.

Lemma 5.3.4. *For all $r \in \{0, 1, \ldots, m\}$ and $n \geq 1$,*

$$c_m(mn) = c_m(mn-1) = \cdots = c_m(mn-r).$$

Proof. Fix $0 \leq r \leq m-1$ and let λ be an arbitrary $m-$ary partition of mn without gaps where $n \geq 1$. Since the number of 1's must be congruent to 0 and we have at least one part of size 1, we know that λ has mk parts of size 1 for some $k \geq 1$. So, if we remove r 1's from λ, we produce a unique partition of $mn - r$ which is still $m-$ary and has no gaps.

Conversely, starting with an $m-$ary partition α of $mn - r$ which has no gaps, if we add r parts of size 1 then we construct a unique partition $m-$ary partition of mn without gaps. ■

This means that we can reduce the problem of computing $c_m(l)$ $(mod\ m)$ to that of computing $c_m(rm)$ $(mod\ m)$ where rm is the rounding up of l to the nearest multiple of m.

Next, we use $C_m(q)$ to compute the generating series for $c_m(mn)_{n\geq 0}$.

Lemma 5.3.5. *The generating series for $(c_m(mn))_{n\geq 0}$ is given by*

$$\sum_{n\geq 0} c_m(mn)q^n = 1 + \frac{q}{1-q}C_m(q).$$

Proof. By Lemma 5.3.3, we have

$$\begin{aligned} C_m(q) &= 1 + \sum_{n\geq 0} \frac{q}{1-q} \frac{q^{m+m^2+\cdots+m^n}}{(1-q^m)\cdots(1-q^{m^n})} \\ &= 1 + \frac{q}{1-q} + \sum_{n\geq 1} \frac{q}{1-q} \frac{q^{m+m^2+\cdots+m^n}}{(1-q^m)\cdots(1-q^{m^n})} \end{aligned}$$

where we interpret the three summands as the terms counting the trivial partition of 0, trivial partitions where only parts are of size 1 and nontrivial partitions, respectively. Now, if we were to only consider partitions where the sum of the parts is a multiple of m, we see that

$$\sum_{n\geq 0} c_m(mn)q^{mn} = 1 + \frac{q^m}{1-q^m} + \sum_{n\geq 1} \frac{q^m}{1-q^m} \frac{q^{m+m^2+\cdots+m^n}}{(1-q^m)\cdots(1-q^{m^n})}$$

as the number of 1's must be a multiple of m. This can be expressed as

$$\begin{aligned} \sum_{n\geq 0} c_m(mn)q^{mn} &= \frac{1}{1-q^m} + \frac{q^m}{1-q^m}(C_m(q^m) - 1) \\ &= \frac{1}{1-q^m} - \frac{q^m}{1-q^m} + \frac{q^m}{1-q^m}C_m(q^m). \end{aligned}$$

Now, the claim follows by replacing q^m by q and simplifying. ■

The relationship between $\sum_{n\geq 0} c_m(mn)q^{mn}$ and $C_m(q)$ allowed AFS to establish the following recurrence.

Lemma 5.3.6. *For $n \geq 1$,*

$$c_m(mn) = c_m(0) + c_m(1) + \cdots + c_m(n-1).$$

Proof. We extract coefficients using the series we found in Lemma 5.3.5

$$\begin{aligned}
[q^n]\sum_{n\geq 0} c_m(mn)q^{mn} &= [q^n]1 + \frac{q}{1-q}C_m(q) \\
&= [q^{n-1}]\frac{C_m(q)}{1-q} \\
&= [q^{n-1}]\sum_{r\geq 0} q^r \sum_{k\geq 0} c_m(k)q^k \\
&= c_m(0) + c_m(1) + \cdots + c_m(n-1).
\end{aligned}$$

■

We use this recurrence to obtain a recurrence for $c_m(mn)$ modulo m.

Lemma 5.3.7. *If $n \equiv k \ (mod\ m)$, where $1 \leq k \leq m$ then*

$$c_m(mn) \equiv 1 + (k-1)c_m(n) \ (mod\ m).$$

Proof. By Lemma 5.3.6, we have

$$\begin{aligned}
c_m(mn) &= c_m(0) + c_m(1) + \cdots + c_m(n-1) \\
&= c_m(0) + c_m(1) + \cdots + c_m(m) + c_m(m+1) + \cdots + c_m(2m) \\
&+ \cdots + c_m((j-1)m+1) + \cdots + c_m((j-1)m+m) \\
&+ c_m(mj+1) + \cdots + c_m(mj+k-1).
\end{aligned}$$

Taking into consideration Lemma 5.3.4, it becomes evident that reducing the above equality modulo m gives to some cancellation. In particular, in $\mathbb{Z}_m$ we know that $m \cdot i \equiv 0 \ (mod\ m)$ for all $i \in \mathbb{Z}_m$ and thus, the non vanishing terms from this reduction are

$$\begin{aligned}
c_m(mn) &\equiv c_m(0) + c_m(mj+1) + \cdots + c_m(mj+k-1) \ (mod\ m) \\
&\equiv 1 + (k-1)c_m(mj) \ (mod\ m) \\
&\equiv 1 + (k-1)c_m(n) \ (mod\ m)
\end{aligned}$$

as needed. ■

Next, we prove that the residue class of $c_m(k)$ is invariant with respect to multiplication of k by m^2.

Lemma 5.3.8. *For all $n \geq 0$,*

$$c_m(m^3 n) \equiv c_m(mn) \ (mod\ m).$$

Proof. We apply Lemma 5.3.7 several times:

$$\begin{aligned}
c_m(m^3 n) &= c_m(m(m^2 n)) \\
&\equiv 1 + (m-1)c_m(m^2 n) \ (mod\ m) \\
&\equiv 1 + (m-1)c_m(m(mn)) \ (mod\ m) \\
&\equiv 1 + (m-1)(1 + (m-1)c_m(mn)) \ (mod\ m) \\
&= 1 + (m-1) + (m-1)^2 c_m(mn) \\
&= m + (m^2 - 2m + 1)c_m(mn) \ (mod\ m) \\
&\equiv c_m(mn) \ (mod\ m).
\end{aligned}$$

∎

Lemma 5.3.8 is instrumental in the case analysis that is considered when proving AFS's characterization theorem, which is carried out below.

Theorem 5.3.9. *If $n = \sum_{i \geq j} \alpha_i m^i$ in base m, then*

$$c_m(mn) \equiv \begin{cases} \alpha_j + (\alpha_j - 1)\sum_{i \geq j+1} \alpha_{j+1} \cdots \alpha_i \ (mod\ m) & j \ even \\ 1 - \alpha_j - (\alpha_j - 1)\sum_{i \geq j+1} \alpha_{j+1} \cdots \alpha_i \ (mod\ m) & j \ odd \end{cases}$$

Proof. Since we assume $mn = \sum_{i \geq j} \alpha_i m^{j+1}$, we are able to divide the argument of $c_m(mn)$ by m^2 iteratively while $j + 1 \geq 3$ using Lemma 5.3.8. This means that we have two cases to consider, based on the parity of j.

Case 1: j is even.
Without loss of generality, $j = 0$. Therefore, we can write

$$n = \alpha_0 + a_1 m + \cdots + \alpha_k m^k \tag{5.2}$$

and hence $n \equiv a_0 \ (mod\ m)$. By Lemma 5.3.7:

$$c_m(mn) \equiv 1 + (\alpha_0 - 1)c_m(\alpha_0 + \alpha_1 m + \cdots + \alpha_k m^k) \ (mod\ m).$$

As $m > \alpha_0 \geq 1$, we can replace α_0 by m due to Lemma 5.3.4:

$$\begin{aligned}
&\equiv 1 + (\alpha_0 - 1)c_m((\alpha_1 + 1)m + \alpha_2 m^2 + \cdots + \alpha_k m^k) \\
&\equiv 1 + (\alpha_0 - 1)c_m(m(\alpha_1 + 1 + \alpha_2 m + \cdots + \alpha_k m^{k-1})) \ (mod\ m) \\
&\equiv 1 + (\alpha_0 - 1)(1 + \alpha_1 c_m((a_1 + 1) + \alpha_2 m + \cdots + a_k m^{k-1})) \ (mod\ m)
\end{aligned}$$

if we repeat the process of applying Lemma 5.3.4 and Lemma 5.3.7 and simplifying until $\alpha_i = 0$ for i minimal, we obtain

$$c_m(mn) \equiv \alpha_0 + (\alpha_0 - 1) \sum_{i \geq j} \alpha_1 \alpha_2 \cdots \alpha_i \ (mod\ m).$$

This is because when $\alpha_i = 0$, we see that by Lemma 5.3.7

$$\begin{aligned}
c_m(m(\alpha_i + 1 + \alpha_{i+1} m + \cdots + \alpha_k m^{k-i})) &\equiv 1 + (1-1)c_m((\alpha_i + 1 + \alpha_{i+1} m + \cdots + \alpha_k m^{k-i})) \ (mod\ m) \\
&\equiv 1 \ (mod\ m)
\end{aligned}$$

Case 2: j is odd.

Without loss of generality, we can then assume $j = 1$. Therefore, we know that

$$n \equiv m \ (mod\ m)$$

and hence

$$\begin{aligned}
c_m(mn) &= 1 - c_m(n) \ (mod\ m) \\
&= 1 - c_m(m \sum_{j \geq 0}^{k} \alpha_{j+1} m^j)
\end{aligned}$$

and then we apply our analysis from the first case $j = 0$ to

$$n' = \sum_{j \geq 0} \alpha_{j+1} m^j$$

and we get the desired result. ■

5.4 Colouring $m-$ary Partitions

5.4.1 Allowing Gaps

In this section, we provide a partial generalization to the work of AFS in [AFS15] by allowing the colouring of the parts of $m-$ary partitions with respect to two rules. This work is due to the author of the thesis and his supervisor Ian Goulden. We first suppose that m is a prime and $k \in \mathbb{N}$ where $k < m$. We give a complete characterization of the number of $m-$ary partitions where the parts of size m^i for $i \geq 1$ are coloured by any one of k fixed colours. Later, we will introduce a new variable k_1 which denotes the number of colours we have available for the units and consider $m-$ary partitions where the units are coloured by k_1 colours and the other powers of m are coloured by k colours.

Definition 5.4.1. Fix m a prime number and $k < m$ a positive integer. Let $b_m^{(k)}(n)$ denote the number of $m-$ary partitions of n where the set of parts allowed is

$$\{1, m_{(j)}^i : j \in \{1, \ldots, k\} \text{ and } i \geq 1\}$$

where $m_{(j)}^i$ is a part of size m^i coloured by j.

We denote by $B_m^{(k)}(q)$ the generating series

$$B_m^{(k)}(q) := \sum_{n \geq 0} b_m^{(k)}(n) q^n,$$

and note that

$$B_m^{(k)}(q) := \frac{1}{1-q} \prod_{i \geq 1} \frac{1}{(1-q^{m^i})^k}$$

by Theorem 1.2.12. These coefficients agree in $m-$tuples, in the same fashion as their $b_m(n)$ counterparts (see Lemma 5.2.2).

Lemma 5.4.2. *For all $r \in \{1, \ldots, m-1\}$ and $l \geq 0$, the coefficients $b_m^{(k)}(n)$ satisfy*

$$b_m^{(k)}(ml + r) = b_m^{(k)}(ml).$$

Proof. We provide a combinatorial proof. Given any partition counted by $b_m^{(k)}(ml)$, we can add r parts of size 1 in order to construct a unique partition of $b_m^{(k)}(ml + r)$.

Now, since the 1 is the only part in our set of parts allowed which is not a multiple of m, we know that any $m-$ary partition of $ml + r$ will have number of $1's$ congruent to r $(mod\ m)$. ■

Lemma 5.4.2 tells us that the problem of classifying $b_m^{(k)}(l)$ $(mod\ m)$ is equivalent to the problem: For any fixed $n \geq 0$, what is $b_m^{(k)}(mn)$ $(mod\ m)$?

We attack this problem by studying the generating series $\Phi_m^{(k)}(q)$ defined by

$$\Phi_m^{(k)}(q) := \sum_{n\geq 0} b_m^{(k)}(mn)q^n.$$

First, we establish the following relationship between $\Phi_m^{(k)}(q)$ and $B_m^{(k)}(q)$.

Lemma 5.4.3. *For all $m, k \in \mathbb{N}$, $\Phi_m^{(k)}(q)$ satisfies*

$$\Phi_m^{(k)}(q) = \frac{B_m^{(k)}(q)}{(1-q)^k}.$$

Proof. We observe that every $m-$ary partition of mn must have number of 1's which is a multiple of m. As this is the only restriction on the parts of a partition enumerated by $\Phi_m^{(k)}(q^m)$, we conclude

$$\begin{aligned}\Phi_m^{(k)}(q^m) &= \frac{1}{1-q^m}\frac{1}{(1-q^m)^k}\frac{1}{(1-q^{m^2})^k}\cdots \\ &= \frac{1}{1-q^m}\prod_{i\geq 1}\frac{1}{(1-q^{m^i})^k} \\ &= \frac{B_m^{(k)}(q^m)}{(1-q^m)^k}.\end{aligned}$$

and therefore the result follows by replacing q^m by q. ■

Now, we can use Lemma 5.4.3 to deduce the following recurrence relation.

Lemma 5.4.4. *For all $n \in \mathbb{N}$, $b_m^{(k)}(mn)$ satisfies the recurrence*

$$b_m^{(k)}(mn) = \sum_{l=0}^{n}\binom{k+l-1}{k-1} b_m^{(k)}(n-l)$$

Proof. Suppose $n = jm + p$, where$p \in \mathbb{Z}_m$. Then by Lemma 5.4.3

$$\begin{aligned}b_m^{(k)}(mn) = [q^n]\Phi_m^{(k)}(q) &= [q^n]\frac{B_m^{(k)}(q)}{(1-q)^k} \\ &= [q^n]\sum_{l\geq 0}\binom{k+l-1}{k-1}q^l\sum_{r\geq 0} b_m^{(k)}(r)q^r \\ &= \sum_{l=0}^{n}\binom{k+l-1}{k-1} b_m^{(k)}(n-l).\end{aligned}$$

■

In order to simplify this recurrence in $\mathbb{Z}_m$, we establish two elementary lemmas.

Lemma 5.4.5. *Given any $n \in \mathbb{N}$, if m is a prime and $k < m$ then we have*

$$\binom{n+m}{k} \equiv \binom{n}{k} \ (mod\ m).$$

Proof. We can write the binomial coefficients as follows

$$\binom{n+m}{k} = \frac{(n+m)(n+m-1)\cdots(n+m+1-k)}{k!}.$$

Since m is prime and $k < m$, we know that $1/k!$ exists and the problem becomes equivalent to showing

$$(n+m)(n+m-1)\cdots(n+m+1-k) \equiv n(n-1)\cdots(n+1-k) \ (mod\ m).$$

This follows immediately from the fact that $n+j \equiv n+m+j \ (mod\ m)$ holds for all j. ■

Lemma 5.4.6. *For any $i, j, n \in \mathbb{N}$,*

$$\sum_{j=0}^{n} \binom{i+j}{i} = \binom{n+i+1}{i+1}.$$

Proof. Through repeated application's of Pascal's Identity, we have

$$\begin{aligned}
\sum_{j=0}^{n} \binom{i+j}{i} &= \binom{i}{i} + \binom{i+1}{i} + \cdots + \binom{n-1+i}{i} + \binom{n+i}{i} \\
&= \binom{i+1}{i+1} + \binom{i+1}{i} + \cdots + \binom{n+i}{i} \\
&= \binom{i+2}{i+1} + \binom{i+2}{i} + \cdots + \binom{n+i}{i} \\
&\vdots \\
&= \binom{n+i}{i+1} + \binom{n+i}{i} \\
&= \binom{n+i+1}{i+1}
\end{aligned}$$

■

Now, we have the tools necessary to find the recurrence that $b_m^{(k)}(mn)$ satisfies $(mod\ m)$.

Lemma 5.4.7. *If $n = jm + p \geq 0$, where $p \in \mathbb{Z}_m$, then*

$$b_m^{(k)}(mn) \equiv \binom{p+k}{k} b_m^{(k)}(n) \ (mod\ m).$$

Proof. From Lemma 5.4.4, we find that

$$\begin{aligned} b_m^{(k)}(mn) &= \sum_{l=0}^{n} \binom{k+l-1}{k-1} b_m^{(k)}(n-l) \\ &\equiv \sum_{i=0}^{j-1} b_m^{(k)}(im) \left[\sum_{s=n-m+1}^{n} \binom{s+k-1}{k-1} \right] + b_m^{(k)}(jm) \sum_{s=0}^{p} \binom{s+k-1}{k-1} \ (mod\ m) \end{aligned}$$

by Lemma 5.4.5. Moreover, using Lemma 5.4.6, we see that we can express the binomial summation in the left summand as a difference of binomial coefficients to get

$$b_m^{(k)}(mn) \equiv \sum_{i=0}^{j-1} b_m^{(k)}(im) \left[\binom{n+k}{k} - \binom{n-m+k}{k} \right] + b_m^{(k)}(jm) \binom{p+k}{k} \ (mod\ m)$$

and finally using Lemma 5.4.5 and Lemma 5.4.2, we may conclude

$$b_m^{(k)}(mn) \equiv b_m^{(k)}(n) \binom{p+k}{k} \ (mod\ m).$$

■

This allows us to prove the following characterization of the number of $m-$ary partitions where every part after the first can utilize any one of k colours.

Theorem 5.4.8. *If $n = a_0 + a_1 m + \cdots + a_j m^j$ in base m, then*

$$b_m^{(k)}(mn) \equiv \prod_{i=0}^{j} \binom{a_i + k}{k} \ (mod\ m).$$

Proof. We provide an inductive proof on the parameter j.
Base case: If $n = a_0$, then

$$\begin{aligned} b_m^{(k)}(ma_0) &\equiv \binom{a_0+k}{k} b_m^{(k)}(a_0) \ (mod\ m) \\ &= \binom{a_0+k}{k} \end{aligned}$$

since $b_m^{(k)}(a_0) = 1$ for all $a_0 \in \mathbb{Z}_m$.

Now, suppose that the result holds for some j fixed and all $a_0, a_1, \ldots, a_j \in \{0, 1, \ldots, m-1\}$. If $a_{j+1} \in \{0, 1, 2, \ldots, m-1\}$ is fixed arbitrarily, then by Lemma 5.4.7, we have

$$\begin{aligned} b_m^{(k)}(mn) &\equiv \binom{a_0+k}{k} b_m^{(k)}(a_0 + a_1 m + \cdots + a_{j+1}m^{j+1}) \ (mod\ m) \\ &= \binom{a_0+k}{k} b_m^{(k)}(a_1 m + \cdots + a_{j+1}m^{j+1}) \quad \text{Lemma 5.4.2} \\ &\equiv \binom{a_0+k}{k} \prod_{i=1}^{j+1} \binom{a_i+k}{k} \ (mod\ m) \quad \text{inductive hypothesis} \end{aligned}$$

as needed. ■

If we let $s = m^k n$ where n is the $m-$free part of s, Theorem 5.4.8 tells us that $b_m^{(k)}(ms) \equiv b_m^{(k)}(mn)\ (mod\ m)$.

Corollary 5.4.9. *: For all $r \in \mathbb{N}$,*

$$b_m^{(k)}(m^r n) \equiv b_m^{(k)}(mn)\ (mod\ m).$$

Proof. Suppose $n = a_0 + a_1 m + \cdots + a_j m^j$ in base m. By Theorem 5.4.8,

$$b_m^{(k)}(mn) \equiv \prod_{i=0}^{j} \binom{a_i+k}{k} \ (mod\ m).$$

Now, we inspect the base m representation of $m^{r-1}n$ for a fixed $r \in \mathbb{N}$ and find

$$m^{r-1}n = 0 + 0 \cdot m + \cdots + 0 \cdot m^{r-2} + a_0 m^{r-1} + a_1 m^r + \cdots + a_j m^{j+r-1}.$$

Next, we use Theorem 5.4.8 again to compute $b_m^{(k)}(m^{r-1}n)$ and find

$$\begin{aligned} b(m^r n) &\equiv \left[\prod_{s=0}^{r} \binom{0+k}{k}\right] \left[\prod_{i=0}^{j} \binom{a_i+k}{k}\right] \ (mod\ m) \\ &\equiv \left[\prod_{i=0}^{j} \binom{a_i+k}{k}\right] \ (mod\ m) \\ &\equiv b_m^{(k)}(mn)\ (mod\ m), \end{aligned}$$

as needed. ■

Now, we generalize our result further by allowing the colouring of the parts of size 1 as well.

Definition 5.4.10. Let $\widetilde{b_m^{(k_1,k)}}(\ell)$ be the number of $m-$ary partitions of l where we have k_1 colours available for parts of size 1 and k colours available for the rest of the parts.

Using our results regarding $b_m^{(k)}(mn)$, we will give a complete characterization of $\widetilde{b_m^{(k_1,k)}}(\ell)$ modulo m for any $\ell \in \mathbb{N}$. In particular, our characterization depends on two variables: the residue class of ℓ modulo m and the base m representation of the quotient in the division of ℓ by m.

With this in mind, we suppose $\ell = mn + a_0 = a_0 + a_1 m + \cdots + a_{j+1} m^{j+1}$ in base m. Let $\Phi_m^{(k_1,k)}(q)$ denote the generating series

$$\Phi_m^{(k_1,k)}(q) := \sum_{n \geq 0} \widetilde{b_m^{(k_1,k)}}(mn + a_0) q^{mn+a_0}.$$

Since the number of parts in a partition enumerated by $\Phi_m^{(k_1,k)}(q)$ is congruent to a_0 modulo m, it follows that

$$\Phi_m^{(k_1,k)}(q) = \sum_{n \geq 0} \binom{k_1 + nm + a_0 - 1}{k_1 - 1} q^{mn+a_0} \prod_{i \geq 1} \frac{1}{(1 - q^{m^i})^k}. \tag{5.3}$$

Now, we introduce and prove our $\widetilde{b_m^{(k_1,k)}}$ characterization.

Corollary 5.4.11. *For any* $0 \leq a_0 \leq m - 1$, *if* $l = mn + a_0 = a_0 + a_1 m + \cdots + a_{j+1} m^{j+1}$ *then*

$$\widetilde{b_m^{(k_1,k)}}(l) \equiv \binom{k_1 + a_0 - 1}{k_1 - 1} \prod_{i=1}^{j+1} \binom{a_i + k}{k} \ (mod\ m).$$

Proof. We apply Lemma 5.4.5 to conclude

$$\sum_{n \geq 0} \binom{k_1 + nm + a_0 - 1}{k_1 - 1} q^{mn+a_0} \equiv \sum_{n \geq 0} \binom{k_1 + a_0 - 1}{k_1 - 1} q^{mn+a_0} \ (mod\ m).$$

So, (5.3) modulo m is

$$\begin{aligned} \Phi_m^{(k_1,k)}(q) &\equiv \sum_{n \geq 0} \binom{k_1 + a_0 - 1}{k_1 - 1} q^{mn+a_0} \prod_{i \geq 1} \frac{1}{(1 - q^{m^i})^k} \ (mod\ m) \\ &= \binom{k_1 + a_0 - 1}{k_1 - 1} \frac{q^{a_0}}{(1 - q^m)} \prod_{i \geq 1} \frac{1}{(1 - q^{m^i})^k} \end{aligned}$$

If we compare coefficients, we find that

$$\begin{aligned}
\widetilde{b_m^{(k_1,k)}}(mn+a_0) &= [q^{mn+a_0}]\Phi_m^{(k_1,k)}(q) \\
&\equiv [q^{mn+a_0}]\binom{k_1+a_0-1}{k_1-1}\frac{q^{a_0}}{(1-q^m)}\prod_{i\geq 1}\frac{1}{(1-q^{m^i})^k} \ (mod\ m) \\
&= \binom{k_1+a_0-1}{k_1-1}[q^{mn}]\frac{1}{1-q^m}\prod_{i\geq 1}\frac{1}{(1-q^{m^i})^k} \\
&= \binom{k_1+a_0-1}{k_1-1}b_m^{(k)}(mn) \\
&\equiv \binom{k_1+a_0-1}{k_1-1}\prod_{i=1}^{j+1}\binom{a_i+k}{k} \ (mod\ m),
\end{aligned}$$

by Theorem 5.4.8. ■

5.4.2 Coloured m-ary Partitions with No Gaps Allowed

In this section, we extend the results of AFS regarding $m-$ary partitions without gaps by introducing colours to the parts.

Definition 5.4.12. Fix m a prime and $k < m$ a positive integer. Let $r_m^{(k)}(l)$ denote the number of $m-$ary partitions λ of l satisfying the two conditions

1. If m^i is a part appearing in λ, then m^k is a part in λ for every $0 \leq k \leq i-1$.
2. With the exception of the 1's, every part is coloured using one of k colours.

We denote by $R_m^{(k)}(q) := \sum_{l\geq 0} r_m^{(k)}(l)q^l$ the generating series for $r_m^{(k)}(l)$ and see that

$$R_m^{(k)}(q) = 1 + \frac{q}{1-q} + \sum_{n\geq 1}\frac{q}{1-q}\left[\frac{1}{(1-q^m)^k}-1\right]\cdots\left[\frac{1}{(1-q^{m^n})^k}-1\right]. \tag{5.4}$$

Moreover, one sees that the generating series for the multisection $\Psi_m^{(k)}(q^m) := \sum_{l\geq 0} r_m^{(k)}(ml)q^{ml}$ of $R_m^{(k)}(q)$ is given by

$$\Psi_m^{(k)}(q^m) = 1 + \frac{q^m}{1-q^m} + \sum_{n\geq 1}\frac{q^m}{1-q^m}\left[\frac{1}{(1-q^m)^k}-1\right]\cdots\left[\frac{1}{(1-q^{m^n})^k}-1\right]. \tag{5.5}$$

Now, we express $\Psi_m^{(k)}(q)$ in terms of $R_m^{(k)}(q)$.

Lemma 5.4.13. $\Psi_m^{(k)}(q)$ *satisfies the functional equation*

$$\Psi_m^{(k)}(q) = \frac{1}{1-q} + \left[R_m^{(k)}(q) - 1\right]\left[\frac{1}{(1-q)^k} - 1\right].$$

Proof. Replacing q^m by q in (5.5) gives us

$$\begin{aligned}\Psi_m^{(k)}(q) &= \frac{1}{1-q} + \frac{q}{1-q}\sum_{n\geq 1}\prod_{j=1}^{n}\left[\frac{1}{(1-q^{m^{j-1}})^k} - 1\right] \\ &= \frac{1}{1-q} + \frac{q}{1-q}\sum_{n\geq 1}\prod_{j=0}^{n-1}\left[\frac{1}{(1-q^{m^j})^k} - 1\right].\end{aligned}$$

Hence, we have

$$\begin{aligned}\left(\Psi_m^{(k)}(q) - \frac{1}{1-q}\right) &= \frac{q}{1-q}\left[\frac{1}{(1-q)^k} - 1\right]\left\{1 + \sum_{n\geq 1}\prod_{j=1}^{n}\left[\frac{1}{(1-q^{m^j})^k} - 1\right]\right\} \\ &= \left[\frac{1}{(1-q)^k} - 1\right]\left\{\frac{q}{1-q}\sum_{n\geq 1}\prod_{j=1}^{n}\left[\frac{1}{(1-q^{m^j})^k} - 1\right] + \frac{q}{1-q}\right\}.\end{aligned}$$

By (5.4), we know that

$$R_m^{(k)}(q) - \frac{1}{1-q} = \frac{q}{1-q}\sum_{n\geq 1}\prod_{j=1}^{n}\left[\frac{1}{(1-q^{m^j})^k} - 1\right]$$

which by substitution gives us

$$\begin{aligned}\left(\Psi_m^{(k)}(q) - \frac{1}{1-q}\right) &= \left[\frac{1}{(1-q)^k} - 1\right]\left(R_m^{(k)}(q) - \frac{1}{1-q} + \frac{q}{1-q}\right) \\ &= \left[\frac{1}{(1-q)^k} - 1\right]\left(R_m^{(k)}(q) - 1\right).\end{aligned}$$

■

Next, we make use of Lemma 5.4.13 to derive a recurrence for $r_m^{(k)}(mn)$.

Lemma 5.4.14. *For all* $n \geq 1$,

$$r_m^{(k)}(mn) = 1 + \sum_{l=1}^{n-1} r_m^{(k)}(l)\binom{n-l+k-1}{k-1}.$$

Proof. By extracting coefficients of Lemma 5.4.13,

$$\begin{aligned}
[q^n]\Psi_m^{(k)}(q) &= [q^n]\frac{1}{1-q} + [q^n]\left[R_m^{(k)}(q) - 1\right]\left[\frac{1}{(1-q)^k} - 1\right] \\
&= 1 + [q^n](R_m^{(k)}(q) - 1)\sum_{s\geq 1}\binom{s+k-1}{k-1}q^s \\
&= 1 + [q^n]\sum_{\substack{l\geq 1\\ s\geq 1}} r_m^{(k)}(l)\binom{s+k-1}{k-1}q^{s+l} \\
&= 1 + \sum_{l=1}^{n-1} r_m^{(k)}(l)\binom{n-l+k-1}{k-1}
\end{aligned}$$

■

Next, we show that the sequence $r_m^{(k)}(l)$ comes in $m-$tuples for $\ell \geq 1$. To do so algebraically, we look to express $R_m^{(k)}(q)$ in terms of $\Psi_m^{(k)}(q^m)$.

Lemma 5.4.15. *The relationship between $R_m^{(k)}(q)$ and $\Psi_m^{(k)}(q^m)$ is given by*

$$R_m^{(k)}(q) = \frac{q}{(1-q)}\frac{(1-q^m)}{q^m}\Psi_m^{(k)}(q^m) - (q^{1-m} + q^{2-m} + \cdots + q^{-1}).$$

Proof. Using (5.4) and (5.5), we find that

$$\begin{aligned}
\left(R_m^{(k)}(q) - \frac{1}{(1-q)}\right)\frac{(1-q)}{q} &= \left(\Psi_m^{(k)}(q^m) - \frac{1}{(1-q^m)}\right)\frac{(1-q^m)}{q^m} \\
R_m^{(k)}(q) - \frac{1}{(1-q)} &= \Psi_m^{(k)}(q^m)\frac{(1-q^m)}{(1-q)}q^{1-m} - \frac{q^{1-m}}{(1-q)} \\
R_m^{(k)}(q) &= \Psi_m^{(k)}(q^m)\frac{(1-q^m)}{(1-q)}q^{1-m} + \frac{(1-q^{1-m})}{(1-q)},
\end{aligned}$$

as needed. ■

Lemma 5.4.16. *For all $n \geq 1$,*

$$r_m^{(k)}(mn) = r_m^{(k)}(mn-1) = r_m^{(k)}(mn-2) = \cdots = r_m^{(k)}(mn-(m-1)).$$

Proof. If $0 \le r \le m-1$, then by Lemma 5.4.15 we have

$$\begin{aligned}
r_m^{(k)}(mn-r) &= [q^{mn-r}]R_m^{(k)}(q) \\
&= [q^{mn-r}]\frac{q}{(1-q)}\frac{(1-q^m)}{q^m}\Psi_m^{(k)}(q^m) - (q^{1-m}+q^{2-m}+\cdots+q^{-1}) \\
&= [q^{mn-r}]q^{1-m}(1+q+q^2+\cdots+q^{m-1})\sum_{n\ge 0} r_m^{(k)}(mn)q^{mn} - (q^{1-m}+q^{2-m}+\cdots+q^{-1}) \\
&= [q^{mn-r}](1+q^{-1}+q^{-2}+\cdots+q^{1-m})\sum_{n\ge 0} r_m^{(k)}(mn)q^{mn} \\
&= r_m^{(k)}(mn)
\end{aligned}$$

■

Now, we utilize Lemma 5.4.16 to derive a recurrence for $r_m^{(k)}(mn)$ modulo m.

Lemma 5.4.17. *If $n = jm + p$ and $0 \le p \le m-1$, then*

$$r_m^{(k)}(mn) \equiv 1 + r_m^{(k)}(n)\left[\binom{p+k-1}{k} - 1\right] \ (mod\ m).$$

Proof. By Lemma 5.4.15, we know that

$$\begin{aligned}
r_m^{(k)}(mn) &= 1 + \sum_{l=1}^{n-1} r_m^{(k)}(l)\binom{n+k-1-l}{k-1} \\
&= r_m^{(k)}(0) + r_m^{(k)}(m)\left[\sum_{s=1}^{m}\binom{n+k-1-s}{k-1}\right] + \cdots \\
&+ r_m^{(k)}(jm)\left[\sum_{s=(j-1)m+1}^{jm+p-1}\binom{n+k-1-s}{k-1}\right] \\
&= 1 + r_m^{(k)}(m)\left[\binom{n+k-1}{k} - \binom{n-m+k-1}{k-1}\right] + \\
&\cdots + r_m^{(k)}(jm)\left[\binom{n+k-1-jm}{k} - \binom{n+k-1-jm-(p-1)}{k}\right]
\end{aligned}$$

by Lemma 5.4.6. Now, by Lemma 5.4.5, we have

$$\begin{aligned}
r_m^{(k)}(mn) &\equiv 1 + r_m^{(k)}(m)\cdot 0 + \cdots + r_m^{(k)}(jm)\left[\binom{p+k-1}{k} - \binom{k}{k}\right] \ (mod\ m) \\
&= 1 + r_m^{(k)}(n)\left[\binom{p+k-1}{k} - 1\right].
\end{aligned}$$

■

Next, we show that $r_m^{(k)}$ satisfies an analogous reduction as in the case when $k = 1$ in Lemma 5.3.8.

Lemma 5.4.18. *If $n \geq 1$, then*

$$r_m^{(k)}(m^3 n) \equiv r_m^{(k)}(mn) \ (mod\ m).$$

Proof.

$$\begin{aligned}
r_m^{(k)}(m^3 n) &= r_m^{(k)}(m(m^2 n)) \\
&\equiv 1 + r_m^{(k)}(m^2 n)\left[\binom{k-1}{k} - 1\right] \ (mod\ m) && \text{Lemma 5.4.17} \\
&\equiv 1 - r_m^{(k)}(m^2 n) \ (mod\ m) \\
&\equiv 1 - \left(1 + r_m^{(k)}(mn)\left[\binom{0+k-1}{k} - 1\right]\right) \ (mod\ m) && \text{Lemma 5.4.17} \\
&\equiv 1 - (1 - r_m^{(k)}(mn)) \ (mod\ m) \\
&= r_m^{(k)}(mn) \ (mod\ m)
\end{aligned}$$

■

Lemma 5.4.18 is essential to our characterization of $r_m^{(k)}(mn)$ $(mod\ m)$ as it simplifies the proof to the analysis of two simple cases.

Theorem 5.4.19. *If $n = \sum_{i \geq t} \alpha_i m^i$ in base m where $\alpha_t \neq 0$ and s is smallest such that $\alpha_s = 0$, we have*

$$r_m^{(k)}(mn) \equiv \begin{cases} \binom{\alpha_t + k - 1}{k} + \left[\binom{\alpha_t + k - 1}{k} - 1\right] \sum_{i=t+1}^{t+s-1} \prod_{j=1}^{i} \left[\binom{\alpha_{t+j} + k}{k} - 1\right] & t \ even \\ 1 - \binom{\alpha_t + k - 1}{k} - \left[\binom{\alpha_t + k - 1}{k} - 1\right] \sum_{i=t+1}^{t+s-1} \prod_{j=1}^{i} \left[\binom{\alpha_{t+j} + k}{k} - 1\right] & t \ odd \end{cases}$$

Proof. Suppose $n = \alpha_0 + \alpha_1 m + \cdots + \alpha_l m^l$ where $\alpha_0 \neq 0$. Using Lemma 5.4.16, we can compute $r_m^{(k)}(mn)$ as follows:

$$\begin{aligned}
r_m^{(k)}(mn) &\equiv 1 + \left[\binom{\alpha_0 + k - 1}{k} - 1\right] r_m^{(k)}(\alpha_0 + \alpha_1 m + \cdots + \alpha_l m^l) \ (mod\ m) \\
&\equiv 1 + \left[\binom{\alpha_0 + k - 1}{k} - 1\right] r_m^{(k)}(m(1 + \alpha_1 + \cdots + \alpha_l m^{l-1})) \ (mod\ m)
\end{aligned}$$

by replacing α_0 by m and applying Lemma 5.4.16. Now, we have

$$r_m^{(k)}(mn) \equiv 1 + \left[\binom{\alpha_0 + k - 1}{k} - 1\right]\left(1 + \left[\binom{\alpha_1 + 1 + k - 1}{k} - 1\right] r_m^{(k)}(1 + \alpha_1 + \cdots + \alpha_l m^{l-1})\right)$$

by Lemma 5.4.17. So, we see that

$$r_m^{(k)}(mn) \equiv \binom{\alpha_0 + k - 1}{k} + \left[\binom{\alpha_0 + k - 1}{k} - 1\right]\left[\binom{\alpha_1 + k}{k} - 1\right] r_m^{(k)}(m(1 + \alpha_2 + \cdots + \alpha_l m^{l-2})).$$

We repeat this process iteratively and find that $r_m^{(k)}(mn)$ $(mod\ m)$ has an expansion of the form

$$\begin{aligned} r_m^{(k)}(mn) \equiv \binom{\alpha_0 + k - 1}{k} &+ \left[\binom{\alpha_0 + k - 1}{k} - 1\right] \sum_{i=1}^{s-2} \prod_{j=1}^{i} \left[\binom{\alpha_j + k}{k} - 1\right] \\ &+ \left(\prod_{j=1}^{s-1} \left[\binom{\alpha_j + k}{k} - 1\right]\right) r_m^{(k)}(m(1 + \alpha_s + \alpha_{s+1} m + \cdots + \alpha_l m^{l-s})). \end{aligned}$$

If s is smallest index for which $\alpha_s = 0$, then from Lemma 5.4.17

$$\begin{aligned} r_m^{(k)}(m(1 + \alpha_s + \cdots + \alpha_l m^{l-s})) &\equiv 1 + \left[\binom{1 + \alpha_s + k - 1}{k} - 1\right] r_m^{(k)}(1 + \alpha_s + \cdots + \alpha_l m^{l-s}) \\ &\equiv 1 \end{aligned}$$

and hence our computation stops and we may conclude

$$r_m^{(k)}(mn) \equiv \binom{\alpha_0 + k - 1}{k} + \left[\binom{\alpha_0 + k - 1}{k} - 1\right] \sum_{i=1}^{s-1} \prod_{j=1}^{i} \left[\binom{\alpha_j + k}{k} - 1\right] \ (mod\ m).$$

We suppose n has the base m representation

$$n = \alpha_t m^t + \alpha_{t+1} m^{t+1} + \cdots + \alpha_l m^l,$$

where $\alpha_t \neq 0$. Then

$$\begin{aligned} n &= \alpha_t m^t + \alpha_{t+1} m^{t+1} + \cdots + \alpha_l m^l \\ &= m^t(\alpha_t + \alpha_{t+1} m + \cdots + \alpha_l m^{l-t}) \end{aligned}$$

and so

$$mn = m^{t+1}(\alpha_t + \alpha_{t+1} m + \cdots + \alpha_l m^{l-t}).$$

We can use Lemma 5.4.18 to break this down into the case analysis of the parity of t.

Case 1: Suppose t is even. In this case, we know that

$$\begin{aligned} r_m^{(k)}(m(\alpha_t + \alpha_{t+1}m + \cdots + \alpha_l m^{l-t})) &\equiv r_m^{(k)}(m^2 \cdot m(\alpha_t + \alpha_{t+1}m + \cdots + \alpha_l m^{l-t})) \\ &\equiv r_m^{(k)}(m^4 \cdot m(\alpha_t + \alpha_{t+1}m + \cdots + \alpha_l m^{l-t})) \\ &\vdots \\ &\equiv r_m^{(k)}(m^t \cdot m(\alpha_t + \alpha_{t+1}m + \cdots + \alpha_l m^{l-t})) \\ &= r_m^{(k)}(mn), \end{aligned}$$

and therefore, we can use our prior analysis to conclude

$$\begin{aligned} r_m^{(k)}(mn) &\equiv r_m^{(k)}(m(\alpha_t + \alpha_{t+1}m + \cdots + \alpha_l m^{l-t})) \\ &\equiv \binom{\alpha_t + k - 1}{k} + \left[\binom{\alpha_t + k - 1}{k} - 1\right] \sum_{i=t+1}^{t+s-1} \prod_{j=1}^{i} \left[\binom{\alpha_{t+j} + k}{k} - 1\right] \end{aligned}$$

where $t + s$ is the first index for which $\alpha_{t+s} = 0$.

Case 2: Suppose t is odd. Then without loss of generality, by Lemma 5.4.18 we assume that $t = 1$. Thus, we see that

$$\begin{aligned} r_m^{(k)}(mn) &\equiv r_m^{(k)}(m(0 + \alpha_t m + \alpha_{t+1}m^2 + \cdots + \alpha_l m^{l-t+1})) \\ &\equiv 1 + \left[\binom{k-1}{k} - 1\right] r_m^{(k)}(m(\alpha_t + \alpha_{t+1}m^2 + \cdots + \alpha_l m^{l-t})) \\ &\equiv 1 - r_m^{(k)}(m(\alpha_t + \alpha_{t+1}m^2 + \cdots + \alpha_l m^{l-t})) \\ &\equiv 1 - \binom{\alpha_t + k - 1}{k} - \left[\binom{\alpha_t + k - 1}{k} - 1\right] \sum_{i=t+1}^{t+s-1} \prod_{j=1}^{i} \left[\binom{\alpha_{t+j} + k}{k} - 1\right] \end{aligned}$$

where again, we suppose $t + s$ is the first index for which $\alpha_{t+s} = 0$.

■

This result is notable because it states we can determine $r_m^{(k)}(mn)\ (mod\ m)$ solely from the base m representation of n. In fact, we need only know the parity of the first block of zeros in the representation of n and the first nonzero block of n in order to compute $r_m^{(k)}(mn)\ (mod\ m)$.

Next, we generalize to the case where parts of size 1 have k_1 colours available.

Definition 5.4.20. Let $\widetilde{r_m^{(k_1,k)}}(mn+p)$ is the number of $m-$ary partitions without gaps where 1 has k_1 colours available and k colours are available for the rest.

Corollary 5.4.21. *If $n \geq 1$ and $p \in \mathbb{Z}_m$, then*

$$\widetilde{r_m^{(k_1,k)}}(mn+p) \equiv \binom{k_1+p-1}{k_1-1} r_m^{(k)}(m(n-1)) \ (mod\ m).$$

Proof. We investigate $\sum_{n\geq 0} \widetilde{r_m^{(k_1,k)}}(mn+p)q^{mn+p}$. We know that

$$\sum_{n\geq 0} \widetilde{r_m^{(k_1,k)}}(mn+p)q^{mn+p} = \left[\sum_{s\geq 0}\binom{k_1+sm+p-1}{k_1-1}q^{ms+p}\right]\left(1+\sum_{n\geq 1}\prod_{i=1}^{n}\left[\frac{1}{(1-q^{m^i})^k}-1\right]\right)$$

since the number of $1's$ must be congruent to p $(mod\ m)$. Next, we reduce the generating series for the number of 1's modulo m

$$\begin{aligned}\sum_{s\geq 0}\binom{k_1+sm+p-1}{k_1-1}q^{ms+p} &\equiv \sum_{s\geq 0}\binom{k_1+p-1}{k_1-1}q^{ms+p} \ (mod\ m)\\ &\equiv \binom{k_1+p-1}{k_1-1}\frac{q^p}{1-q^m}\end{aligned}$$

by Lemma 5.4.5. Therefore,

$$\begin{aligned}\sum_{n\geq 0} \widetilde{r_m^{(k_1,k)}}(mn+p)q^{mn+p} &\equiv \binom{k_1+p-1}{k_1-1}\frac{q^p}{1-q^m}\left[1+\sum_{n\geq 1}\prod_{i=1}^{n}\left[\frac{1}{(1-q^{m^i})^k}-1\right]\right]\\ &\equiv \binom{k_1+p-1}{k_1-1}q^p\left(\frac{1}{1-q^m}+\frac{1}{1-q^m}\sum_{n\geq 1}\prod_{j=1}^{n}\left[\frac{1}{(1-q^{m^j})^k}-1\right]\right).\end{aligned}$$

A straightforward computation shows us that

$$\frac{1}{1-q^m}+\frac{1}{1-q^m}\sum_{n\geq 1}\prod_{j=1}^{n}\left[\frac{1}{(1-q^{m^j})^k-1}\right] = \frac{1}{1-q^m}+\frac{\Psi_m^{(k)}(q^m)}{q^m}-\frac{1}{q^m(1-q^m)} \tag{5.6}$$

from which can see that

$$\begin{aligned}\widetilde{r_m^{(k_1,k)}}(mn+p) &\equiv [q^{mn+p}]\binom{k_1+p-1}{k_1-1}q^p\left(\frac{1}{1-q^m}+\frac{1}{1-q^m}\sum_{n\geq 1}\prod_{j=1}^{n}\left[\frac{1}{(1-q^{m^j})^k}-1\right]\right)\\ &= [q^{mn}]\binom{k_1+p-1}{k_1-1}\left(\frac{1}{1-q^m}+\frac{1}{1-q^m}\sum_{n\geq 1}\prod_{j=1}^{n}\left[\frac{1}{(1-q^{m^j})^k}-1\right]\right)\\ &= \binom{k_1+p-1}{k_1-1}r_m^{(k)}(m(n-1))\end{aligned}$$

by (5.4). ■

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