Stress Intensity Factor for a Crack Embedded in an Infinite Matrix under the Assumption of Plane Micropolar Elasticity

by

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AUTHOR'S DECLARATION

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In last few decades, studies on defining mechanical behaviour of materials with significant microstructure have increased drastically. This is due to the lack of compatibility between the experimental data of mechanical behaviour of such materials and obtained results from classical elasticity theory. Moreover, with growing demand and need of using microstructural materials such as polymers and composites in mechanical, physical, and engineering applications, it has become crucial to have a good insight about their behaviour and how to analyse it.

As mentioned before, the classical theory cannot provide suitable formulae to model the behaviour of microstructural materials. Thus, a new theory had to be developed in order to describe such materials. The basis of a new elasticity theory, which considers the microstructure of materials, was formed during the 1950s. This theory is known as Cosserat theory of elasticity or micropolar elasticity.

An effective way to solve the problems in Micropolar elasticity is to use the boundary integral method. Nevertheless, this method forces some limitations on the properties of boundaries of considerate domains. To be more specific, this method demands more detail to characterize the boundary. By using this method, boundaries can be defined and presented by a twice differentiable curve. As a result, it cannot be applied on domains with reduced boundary smoothness, or the ones containing cracks or cuts. Hence, there is a need of finding methods to define irregular boundaries. There has been some research in this particular area, however this issue has not been completely addressed.

In order to overcome this difficulty of defining irregular boundaries, an advanced mathematical approach can be used. This method includes using the distribution setting in Sobolev spaces to formulate the corresponding boundary value problem. The benefit of using this method is finding the appropriate weak solution in terms of integral potentials, which works perfectly for the aforementioned boundaries.

In this work, boundary integral equation method has been used to find the integral potentials which are the exact analytical solutions, for the corresponding boundary value problems. Moreover, the boundary element method has been used to approximate these exact solutions numerically. Then these solutions can be applied in many practical engineering problems.

As an illustration of the importance of this method, then a crack in a human bone was modeled and solved using these solutions. The bone assumed to follow plane Cosserat elasticity. The stress intensity factor around this crack was calculated and compared to classical analysis. The results approves the high effect of microstructure of the material in stress distribution around the crack.

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I thank my colleague and friend Michael Cohen for his incredible help during my research. And finally, I would like to acknowledge the contributions of many people who have helped me throughout my degree.

Dedication

I dedicate this to my mom, dad, and sister.

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Chapter 1 Introduction

1.1 Background

The classical theory of elasticity is the perfect tool in order to describe the mechanical behaviour of the elastic continuum. This theory focuses on materials with reversible deformations. In this theory, the stress vector is transferring the internal loading through material. For a wide range of materials, such as construction materials, the classical theory of elasticity is very reliable.

Despite the accurate obtained results from the classical theory of elasticity compare to experimental data, this theory cannot provide the appropriate outcomes, when the microstructure of the material influences its behaviour. In so called materials, each element of material will have a microrotation as well as displacement. These microrotations will affect the whole material's deformation.

In order to use the elasticity theory for such materials, a modification and change has to be applied on the classical theory. The new adjustment of theory of elasticity counts aforementioned microrotations as well as displacement for each particle in a material. This new approach, leads to calculating more reliable and exact results for materials such as polymers and granular materials.

1.2 Scope and Objectives of Thesis

The objective of this thesis is to consider a crack problem in a plane Micropolar elasticity, obtain the results for a stress intensity factor and compare it with the results for the stress intensity factor for the same crack but considered under the assumptions of the classical theory. Theoretical background for the computational results presented herein was developed by (Shmoylova, 2006).

1.3 Outline

Next chapter of this thesis (Chapter 2), will focus on literature review. First, a brief history of developing the micropolar elasticity will be provided. As it follows an explanation of granular materials, the fundamentals of cosserat theory, and a brief definition of stress intensity factor will come.

Chapter 3 will give a brief overview of the three-dimensional theory of micropolar elasticity, which comes in detail in a paper published by (Nowacki, 1986). The motivation of this chapter is to introduce the governing equations describing three-dimensional deformations of a linearly elastic Cosserat solid and to formulate the basic constitutive and kinematic relations that will be used for derivation of corresponding relations of the theory of plane Cosserat elasticity in the subsequent chapters.

Chapter 4 is devoted entirely to the plane problems of micropolar elasticity. On the basis of the governing equations and constitutive relations of the three-dimensional Cosserat theory. In this chapter, the governing equations has been derived. Moreover, the fundamental boundary value problems of plane micropolar elasticity has been formulated. The boundary integral equation method that has been used in this chapter, proved the uniqueness and the existence of theorems and obtained the exact solutions to these problems in the form of integral potentials.

Chapter 5 will define the solutions for a crack problem and provides the solutions of these problems. Since the solutions in the form of integral potentials may not be convenient for applications, later in this chapter, the boundary element method has been introduced as a powerful tool to approximate the solutions numerically.

In order to illustrate the importance of study in this area, a brief comparison between the results of analysis with classical and Micropolar elasticity theory for a human bone with a crack will be represented in chapter 6.

Finally, in Chapter 7 several important conclusions and recommendations for future work will be made.

Chapter 2 Literature Review

2.1 Cosserat Media

The mechanical behaviour of a number of materials cannot be completely described using the classical theory of elasticity. The length scales have an effect on the behaviour of such materials, which is known as Cosserat materials. Soils, polycrystalline and composite materials, granular and powder like materials, masonries, cellular or porous media and foams, bones, liquid crystals, as well as electromagnetic and ferromagnetic media are some examples of Cosserat continua. Some of these examples can be seen in Figure 2-1 and Figure 2-2.

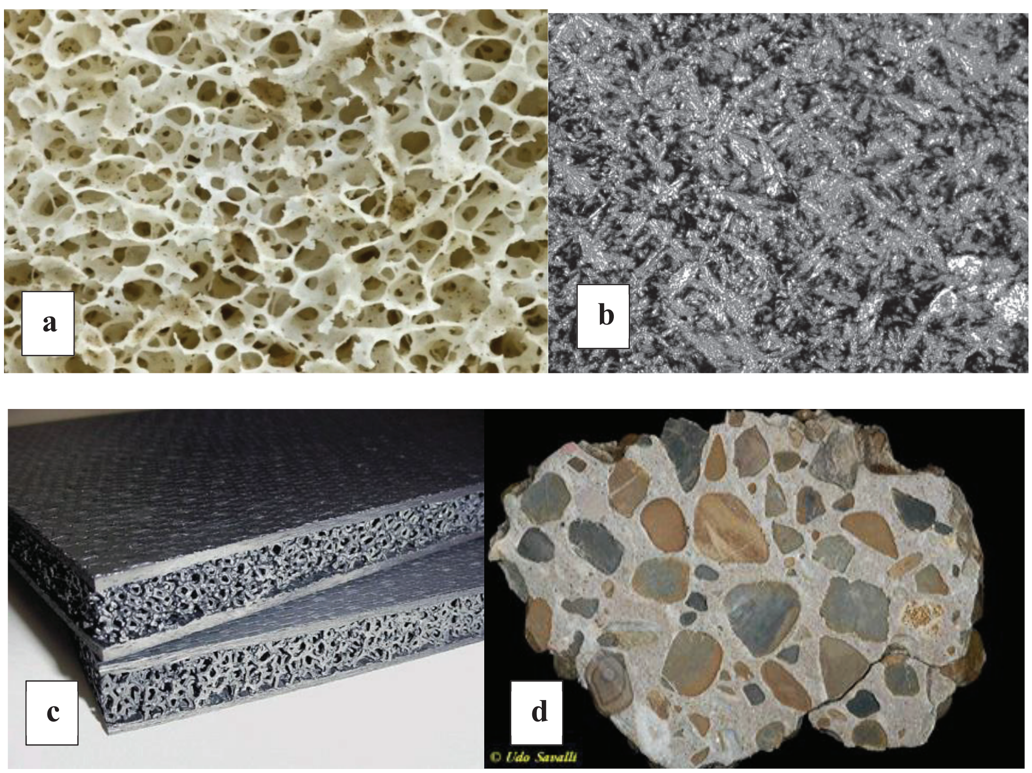

Figure 2-1 Some of the Cosserat Materials examples- (a) Bone texture (adopted from http://ic.steadyhealth.com/causes-of-bone-cance). (b) Liquid crystalline polymer (adopted from (Zheng, et al., 2006)). (c) Ceramic composite (adopted from http://www.globalspec.com/learnmore/materials_chemicals_adhesives/ceramics_glass_materials/cer

amic_matrix_composites). (d) Conglomerate (adopted from http://www.savalli.us/BIO113/Labs/02.Rocks.html).

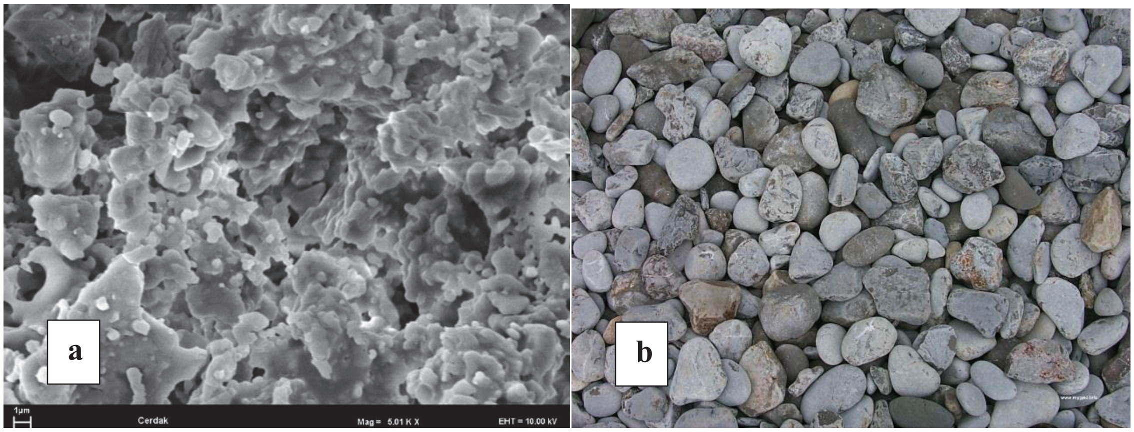

Figure 2-2 Some of the Cosserat Materials examples- (a) Ceramic texture (adopted from http://cerdak.co.za/Technical-Information/). (b) Stones (adopted from http://mail.ipb.ac.rs/~vrhovac/sloba/science/gs.html).

When these materials face a stress, their deformation is influenced by local rotational as well as stress translational behaviour of their particles. A brief overview of the behaviour of such materials under the stress can be seen in Figure 2-3.

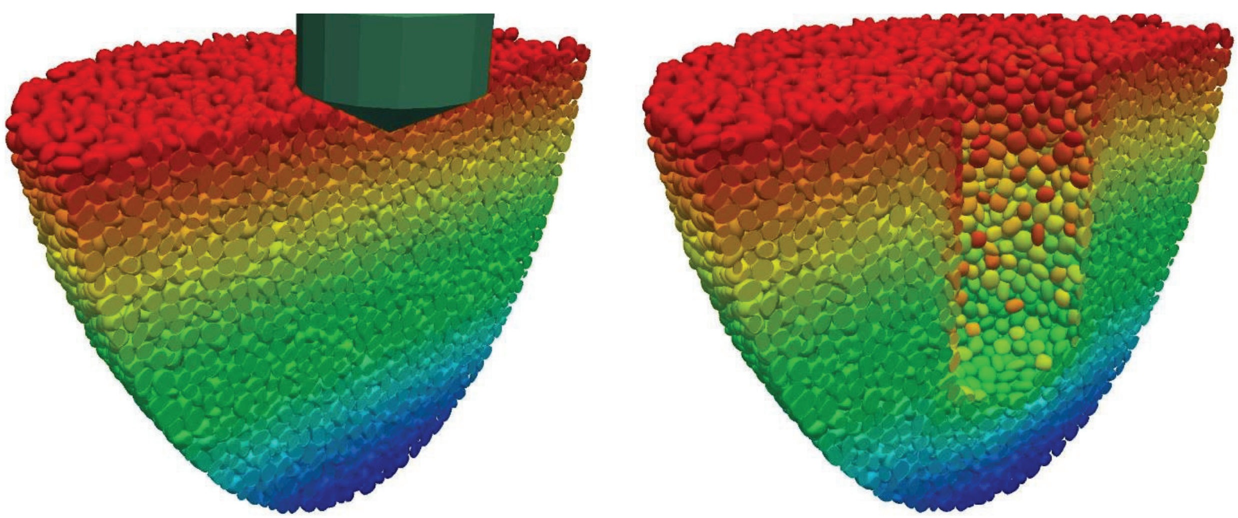

Figure 2-3 Response to stress in Cosserat media (adopted from (Wellmann, et al., 2009))

2.2 Development of Cosserat Theory of Elasticity

The classical theory of elasticity is inspired by the mechanics of the elastic continuum which is, in turn, defined by the transfer of internal loading through the element by only the (force) stress vector. In this theory all the deformations of the body can be described as symmetric strain and stress tensors.

The classical theory of elasticity is reliable for most of the construction materials, such as concrete, steel, or aluminium, as the analytical models obtained from this theory show relatively equivalent results as the experimental results of these materials in their elastic range.

On the other hand, the classical theory of elasticity cannot provide accurate results when the influence of the microstructure of materials on the deformation is particularly evident. As an example, in granular bodies with large molecules (e.g. polymers) or human bones (see, for example, Kruyt, 2003; Rothenburge, et al., 1991; Bathurst, et al., 1988; Lakes, 1982; Lakes, 1995). This inadequacy of the classical theory of elasticity has been more notable in design and manufacture of the modern advanced materials in which the microstructure of the material is playing a vital role in predicting the total mechanical behaviour of it.

In 1886, Voigt, for the first time, considered these small-scale effects of the materials that were absent from the classical theory of elasticity. Voigt assumed that the interactions between two elements of the body transfers through a surface element using both (force) stress vector and an independent moment (couple-stress) vector (Voigt, 1887).

Nonetheless, the first modified and new elasticity theory was established by the Cosserat brothers in 1909 (Cosserat, et al., 1907; Cosserat, et al., 1896; Cosserat, et al., 1909). As a development to Viogt's theory, Cosserat brothers proposed that displacement vector $u(x, v)$, and independent microrotation vector $\varphi(x, v)$ should be used to describe the deformation of a body. In spite of classical theory of elasticity that describes the deformation by a symmetric stress tensor, in Cosserat theory the deformation of the body will be described as asymmetric strain and stress tensors, due to six degree of freedom of a body element in this theory.

Regardless of their whole new ideas, this theory was ignored for a long time. The main downsides of the Cosserat theory caused it to remain unnoticed were, first, it was a non-linear theory, second, the way it was formulated was very unclear and complicated and last and the most important one, the theory was consisted of numerous problems which were exceeded the classical theory of elasticity's framework. Cosserat brothers tried to create a theory that combines mechanics, optics and electrodynamics in which they could address the elasticity problems as well as non-ideal fluid problems, quasi-elastic continuum model of the McCullagh and Kelvin Ether, and some problems related to electrodynamics and magnetism (Schaeffer, 1967).

In the middle of twentieth century, researches and studies in solid mechanics and mechanics of fluids showed that the classical theory of elasticity is dysfunctional in describing certain types of materials and fluids, therefore the Cosserat theory was considered to address these problems.

The simplified Cosserat theory or asymmetric elasticity in so-called Cosserat pseudocontinuum which is sometimes called couple-stress elasticity, was the first area of interest of this theory. There is a possibility of creating asymmetric stresses and couple stresses during a deformation in a Cosserat pseudocontinuum, but the microrotation vector Φ and the displacement vector u are assumed to be dependent like in classical theory of elasticity, thus the whole deformation of the body is describing by displacement field (Lurie, et al., 2005), by means of the following relation:

$$\varphi = \left(\frac{1}{2}\right) \nabla \times u \qquad 22.1$$

where $\nabla \times u$ is the curl of vector u.

One of the first comprehensive works on developing couple-stress theory of elasticity has been done by (Toupin, 1962; Toupin, 1962; Truesdell, et al., 1960), on the linear and non-linear elasticity of Cosserat pseudocontinuum. Grioli, Mindlin and Thiersten continued on expanding the research on this theory (Grioli, 1960; Mindlin, 1963; Mindlin, 1964; Mindlin, 1965; Mindlin, et al., 1963).

It has to be mentioned that, all the papers that were published during that time on simplified Cosserat theory were focusing on general problems of derivation and methods of solutions governing the equations of the couple-stress theory of elasticity and on the applications of the theory, for example calculating the influence of couple stresses on stress concentration factor around holes and rigid inclusions or analysing of bending plates in pseudo-Cosserat media.

Nevertheless, the simplified couple-stress theory, like other simplified theories, cannot determine the deformations of granular media accurately. This inaccuracy was confirmed by a series of experiments (Sternberg, et al., 1965; Sternberg, et al., 1967; Itou, 1977; Ellis, et al., 1968; Gauthier, et al., 1975). As the aim of couple-stress theory was to simplifying the Cosserat theory, there are some similarities between this simplified theory and classical theory of elasticity, for example, the Navier's equations with respect to three unknown equations are the governing equations here as it is in classical theory of elasticity. Because of aforementioned flaws of couple-stress theory, after couple of attempts to use this theory, researchers have come up with general Cosserat theory, which is also more mathematically rigorous.

It was only by the late fifties and early sixties of the twentieth century that Gunther and Schaefer formulated the basis of the Cosserat theory continuum for the first time, in which the microrotations and displacements were independent (Gunther, 1958; Schaefer, 1967; Schaeffer, 1967). Gunther focused on and studied the three-dimensional model of the Cosserat continuum and showed the significant influence of Cosserat theory on dislocation theory, and Schaefer worked on the basis of the Cosserat theory for the plane strain and rediscovered it again. After these authors, constitutive relations and governing equations of the general theory of Cosserat elasticity was presented by (Aero, et al., 1964; Palmov, 1964; Palmov, 1964).

The most systematic development of Cosserat theory was given by Eringen and his coworkers, who named the theory as micropolar or asymmetric elasticity (Eringen, 1966; Eringen, 1967; Eringen, 1999). His main work focused on formulating the general provisions of the theory of micropolar plates, approximating solutions of micropolar plates and shells, and crack growth in a micropolar media (Eringen, 1967). As a more comprehensive reference for the explanation of the theory, a paper by Schaeffer (Schaeffer, 1967) and, a book with an extensive bibliography by Nowacki can be mentioned (Nowacki, 1986).

In the meanwhile of expanding Cosserat theory of elasticity, some developments had been achieved on theory of Cosserat fluids. For more clarification on Cosserat fluids one can use a paper published by (Payne, et al., 1989), and another paper in the same journal, which was published by (Payne, et al., 1989).

As it mentioned before, all of these works were huge steps in developing the Cosserat theory, however, the mathematically rigorous formulation of the boundary value problems appearing in micropolar theory of elasticity and the methods of their solution has never been addressed in any of those papers or monographs. The reason for this problem is the rigorous mathematical analysis of the governing equations and boundary conditions of a micropolar media which is a complicated structure, cannot be addressed by methodology, procedure, and approaches of the classical theory of elasticity (such as, theory of analytical functions, Fredholm's theory of integral equations, theory of one-dimensional singular integral equations). Fortunately, this situation is now changing mostly due to the important work in the area of three-dimensional classical elasticity carried out in the last 40 years.

There are different ways of solving three-dimensional problems of classical theory of elasticity. Some of which can be developed in order to use in the analysis of the boundary value problems of micropolar elasticity. The first possibility is the modern theory of generalized solutions of differential equations (the method of Hilbert spaces, variational methods). The second one is the theory of multidimensional singular potentials and singular integral equations.

The first type of methods which is different than the classical mechanics and built on the state-of-art practical analysis, are addressing general cases with different variable coefficients and numerous different boundaries. As this method covers widespread number of situations, it is the ideal method for demonstrating theorems on existence of non-classical solutions. In order to use it for classical solution, some extra limitations have to be added to this method.

Two books that have been published by (Chudinovich, et al., 2000), and (Constanda, 1990) are including a brief investigation on these topics.

The second type of methods relying on the singular integrals and integral equations, which are growing fast, is an expansion on the theory of potentials and Fredholm equations. These ideas are known as a prevalent approach of the classical mechanics. In spite of the first method, this procedure is not covering so many general cases. This will permit to keep the regulations of the classical mechanics of continua while analyzing the most important cases for the theory and applications thoroughly. The remarkable and initiate research of Muskhelishvili (Muskhelishvili, 1953) on singular integral equations was a breakthrough in this approach. This method then used by Kupradze and his co-workers (Kupradze, 1965), (Kupradze, et al., 1979) to develop the research on the boundary value problems of three-dimensional theory of elasticity. Furthermore, Constanda employed this method to investigate the bending of plates with transverse shear deformation (Constanda, 1990).

Kupradze has established the effective tools for researches on the micropolar theory of elasticity area. One of the remarkable works, using the formulas and tools provided by Kupradze, is Iesan investigation on the three-dimensional problems of micropolar elasticity (Iesan, 1970). He had formulated uniqueness and existences theorems for the boundary-value problems of plain strain of three-dimensional micropolar continuum. Nevertheless, his research cannot be used as a thoroughly solution to the problems, due to the ignorance of considering different possible cases. Schiavone, and Schiavone and Constada, have modified the framework of singular integral equations to form the analytical solutions and analysis of boundary value problems of the theory of micropolar plates, in their research on this topic (Schiavone, et al., 1989; Schiavone, 1989; Schiavone, 1991; Schiavone, 1996). However, none of these works were focused on the cases when the boundary of the domain is not smooth enough or there is a crack in the boundary. In this thesis, a Sobolev setting and distributional approach was used to obtain the exact analytical solutions in the closed form.

One type of deformations in solids is anti-plane shear deformation. The classical anti-plane shear is considered as a compatible method to the more complex classical plane strain. Under the=e assumption of this method, which is more familiar, the deformation will only have two in-plane displacements. Therefore, the displacements will be defined in x_1 and x_2 directions, and a microrotation will be defined on $x_1x_2 - plane$. This scenario can be seen in solids, when their length in x_3 direction is considerably longer than the other dimensions.

As it mentioned before, considering plane strain will reduce the equilibrium equations of Micropolar elasticity from six equations to three partial differential equations. The governing equations and fundamental boundary value problems of a linearly elastic, homogeneous, and isotropic Micropolar continua in plain strain have been formulated and analyzed by some other researchers (Nowacki, 1986; Iesan, 1970; Schiavone, 1996).

The boundary element method has been developed from classical integral equations and finite elements. One of the advantages of this method, same as boundary integral equation, is the fact that the dimension of the problem can be reduced by one and the domain can approach to infinity while having very reliable solutions. Moreover, there is no need to specify the shape functions in this method, to calculate the stresses. However, it makes it possible to distinguish the matrix of fundamental solutions to smoothen the calculation of the stresses and providing more accurate results.

2.3 The Description of Cosserat Theory of Elasticity

The Cosserat theory of elasticity, considers a media as a rigid body in which every small particle has microrotations as well as translations in all directions. The assumption of translation is same as the classical theory of elasticity (Rubin, 2000). This definition has been shown in Figure 2-4.

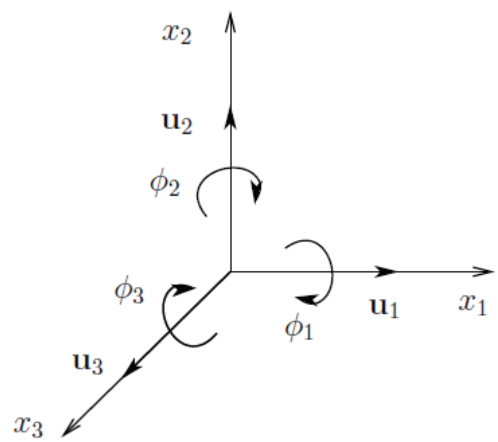

Figure 2-4 The representation of the degrees of freedom in the Cosserat model (adopted from (Kapiturova, 2013))

Another characteristic of this theory is considering a couple stress m (a torque per unit are) as well as force stress σ (force per unit area) (Altenbach, et al., 2013). This has been shown in Figure 2-5, as a brief comparison between the classical representation of a point and micropolar representation of the same point in a body.

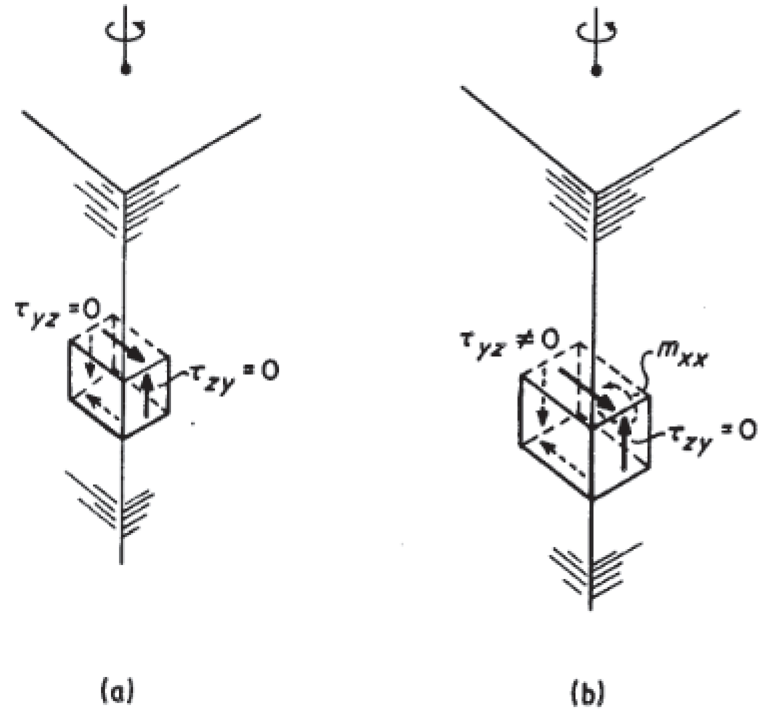

Figure 2-5 Free body diagram of a corner element- (a) Classical theory. (b) Micropolar theory (adopted from (Lakes, et al., 1985)).

In terms of defining the Cosserat theory mathematically, four elastic constants are added to those ones from classical theory. These constants are $(\alpha, \beta, \gamma, \kappa)$ in addition to Lame constants in classical theory of elasticity (μ, λ) (Yang, et al., 1982).

Overall, it can be said that based on the Cosserat theory of elasticity, the description of motion of any particle in a media can be defined by six degree of freedom, three translational (same as classical theory of elasticity) and three others are local rotational motions (Eringen, 1999).

By considering Cosserat elasticity theory in order to describe the behaviour of materials, basically, the deformations of each point and each domain assumed to be like Figure 2-6.

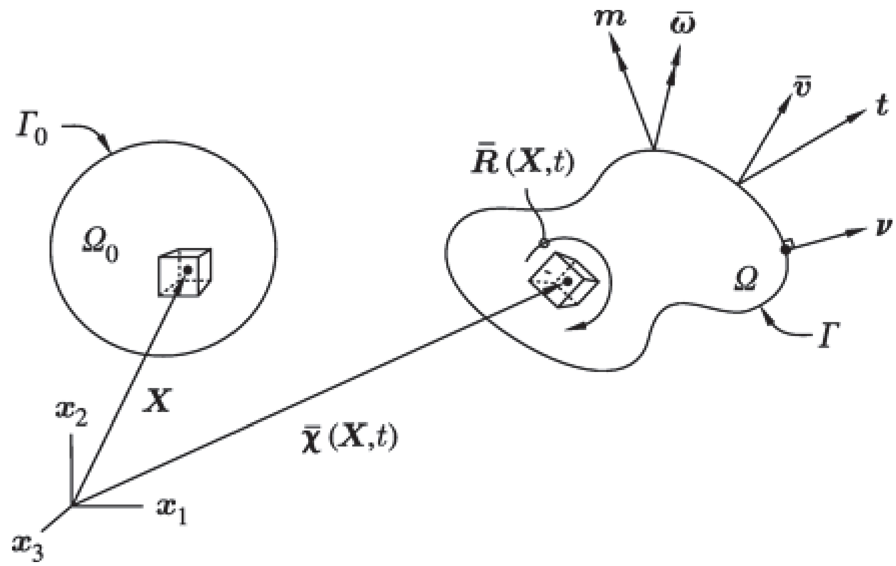

Figure 2-6 Deformation of a point from the reference configuration Ω_0 to the current configuration Ω. Vectors $m, \bar{\omega}, t, \bar{v}$ are representing spatial couple stress, angular velocity, force stress vector, and velocity, respectively (adopted from (Segerstad, et al., 2009).

The importance of this theory will be more illustrated in some practical examples. As a brief explanation, in Figure 2-7 a crack exist in a corner of a body. The crack growth cannot be modeled using the classical theory of elasticity, due to the lack of the existence of any couple stress at the corner in classical theory of elasticity. On the other hand, this growth can be modeled using the Cosserat theory of elasticity.

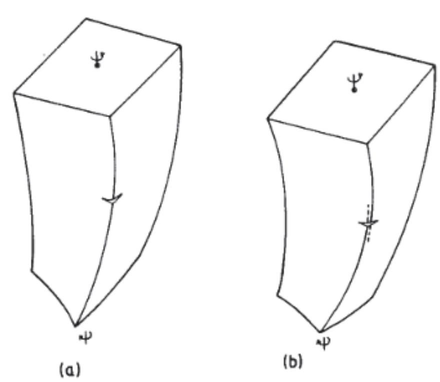

Figure 2-7 Crack growth- (a) Classical theory of elasticity. (b) Cosserat theory of elasticity (adopted from (Lakes, et al., 1985)).

As it is mentioned in section 2.2, the main focus of this work is on plain strain Micropolar elasticity. The importance of such research can be put in two categories. First of all, as it mentioned before, the plain strain has three equation for three unknowns (two displacements and one microrotation), which are relying on two independent variables. This hybrid nature of the problems, will ease finding the analytical solutions. Secondly, In addition to the fact that plane problems are of very interesting nature on their own, a number of practical three dimensional problems often can be reduced to the consideration of plane problems which makes them easier to solve rather than in the case of the formulation using the assumptions of three dimensional theories.

2.4 The Application of Cosserat Theory

Although the experimental data of this theory is still rare, there are some applications of the theory that made it notable and more popular. One of these applications is referring to solids with periodic microstructures. These materials ranges from crystals to composite materials and engineering structures such as infinite-fiber composites, sandwich structures, grid structures, trusses and honeycombs (Pabst, 2005). Some works have calculated the micropolar elastic moduli of different crystals exhibiting polar phenomena (Askar, 1972; Fischer-Hjalmars, 1981; Fischer-Hjalmars, 1982; Pouget, et al., 1986). In other attempt (Bazant, et al., 1972) analysed the steel-concrete grid structures numerically, using Cosserat theory.

Gaining the experimental data from Cosserat materials with random microstructures is very hard and complicated. The reason for this is the need of high precautions in performing the experiment in order to obtain reliable results. Due to this, the information for the experiments on these materials could not be found (Gauthier, 1982).

The other application is very useful for heterogeneous materials. These materials become dispersive under dynamic conditions following the classical elastic theory, due to the same order of magnitude of the wavelength and the size of the heterogeneities. The dispersion relations exist for transverse and rotational waves, which can be stabilized using the micropolar theory of elasticity relations (Bertram, et al., 1998).

Finally, this theory can be used to regulate the methods in the computation of localization phenomena (De Borst, 1991).

2.5 Discontinuities Models

Two types of discontinuities can be defined, first one is the weak discontinuities, and second one is the strong one. In the first type the displacement field in materials remains continuous, and strain field may or may not be continues. These problems are basically continues problems. As an example, material interfaces and inclusions can be named, which has been illustrated in Figure 2-8.

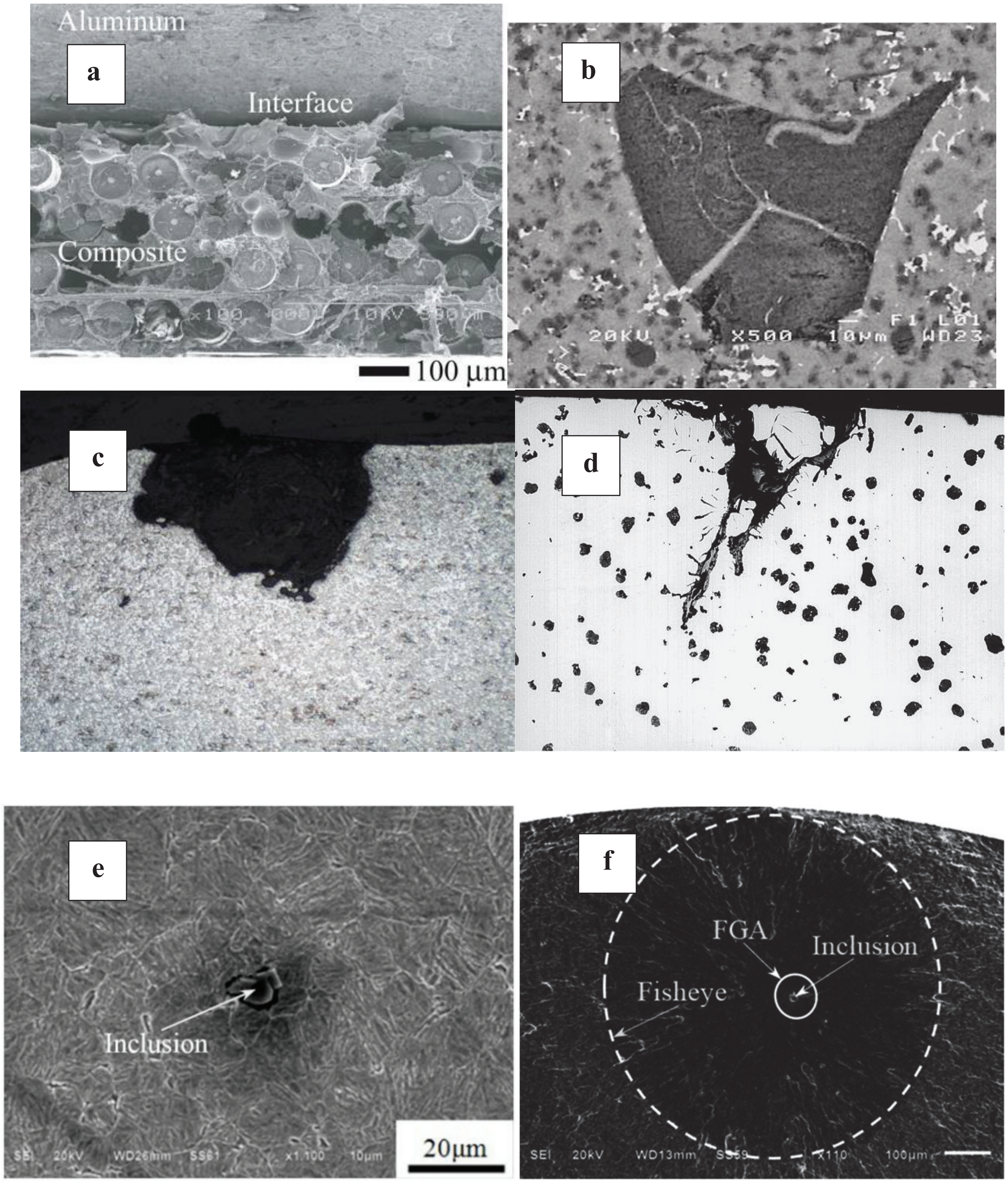

Figure 2-8 Weak discontinuities- (a) Material interface (adopted from (Gan, 2009)). (b) A visible inclusion defect in composite materials (adopted from (BEJGER, et al., 2015)). (c) Inclusion of foreign material (adopted from http://www.prometlab.com/en/index.asp). (d) Oxide inclusion in nodular graphite cast iron (adopted from https://blogs.msdn.microsoft.com/murrays/2015/05/14/equation-numbering-in-office-2016/). (e) Microstructure and inclusion (adopted from (Deng, et al., 2015)). (f) An inclusion in fine granular area (FGA) located at the center of fisheye (adopted from (Deng, et al., 2015)).

However, in strong discontinuities, both displacement field and strain one are discontinuous. Cracks, dislocations, and voids can be some of the examples of this type problems Figure 2-9.

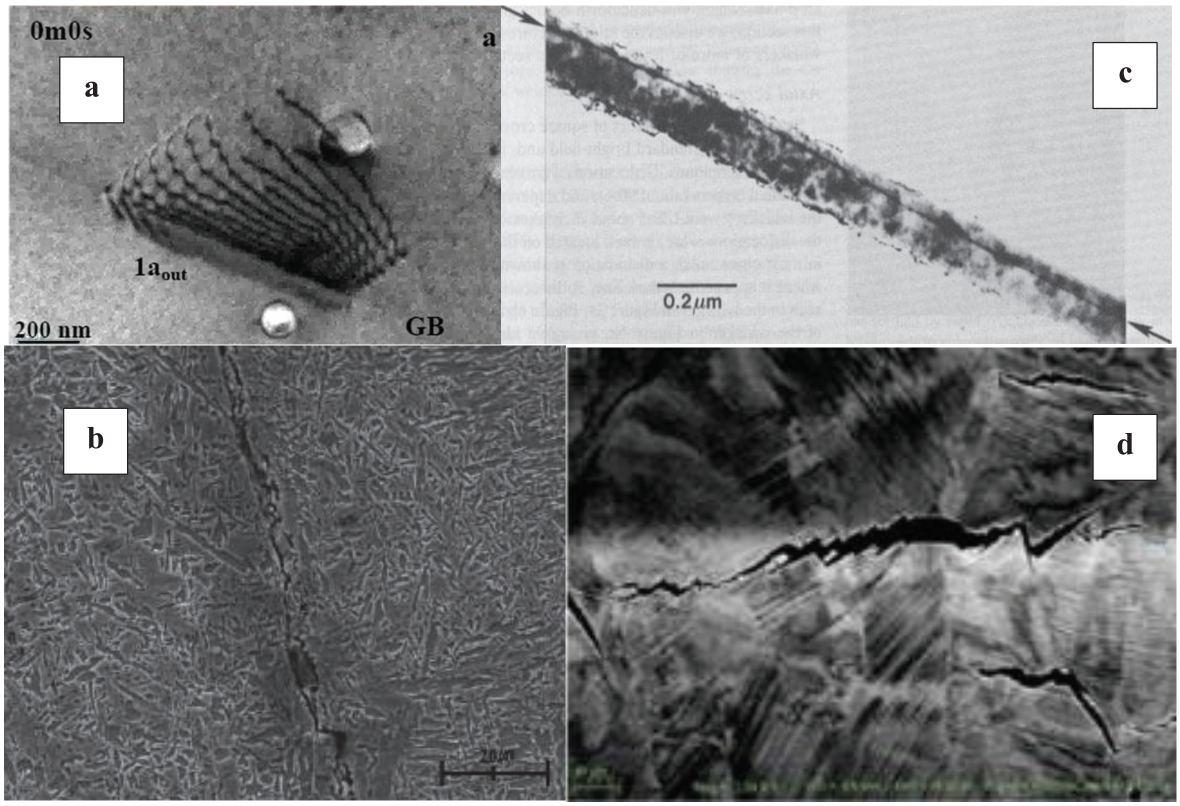

Figure 2-9 Strong discontinuities- (a) Dislocation (adopted from (Robertson) (b) A screw dislocation (adopted from (Velben, et al., 1983)). (c) Branching and blunted crack (adopted from (Yi, et al., 2011)). (d) Crack propagation (adopted from http://www.mpie.de/3173155/Hydrogen-Embrittlement).

As this thesis topic is focused on stress distribution around cracks, different methods of simulating cracks will be represent below.

Some of the most famous models for describing cracks are non-local model, the continuous smeared crack model, and the discrete crack model (Ozbolt, et al., 1996; Willam, et al., 1987; Parsad, et al., 2002).

In non-local models, the fracture behaviour of each point of a material can be influenced by the stress state at that point and some points which are close to that point. One of the most important advantages of these models is the lack of mesh modeling in order to describe crack propagation. Cosserat theory is included in these types of models (Li, et al., 2015). The smeared crack model describes the crack with its stiffness and strength reduction. These models need a crack mesh model and a predefined crack growth path. By which there will be no need to remesh the crack while it's propagating. Finally, the discrete crack model, simulate the discontinuity along finite element edges or the sub-mesh inside the elements. This method needs remeshing the crack, and due to this, it is time consuming and expensive (Rots, et al., 1989).

2.6 Fracture Mechanics

Fractures happens due to many different reasons, from designing inadequacy to defects in materials, which sometimes might lead to failure. In order to prevent the happening of such incidents, fracture mechanics became one of the most important and active fields of research during last decades. Fracture mechanics tries to find a quantitative relation between the crack length, the material's inherent resistance to crack growth, and the stress which makes the crack propagates catastrophically.

2.6.1 Modes of Crack Growth

Three basic modes is defined to describe the crack growth. Mode I, called the opening mode in which crack propagates in the normal direction of its plane. The displacements in this mode are symmetric with respect to x-z and a-y planes. This mode is the most common fracture mode. Mode II, the edge sliding mode, describes crack growth in normal direction to crack front Figure 2-10 (a). The displacements are symmetric with respect to x-y plane and anti-symmetric with respect to x-z plane Figure 2-10 (b). Mode III, the anti-plane shear mode or tearing mode refers to the modes when crack growth parallel to its front. The displacements are asymmetric with respect to x-y and x-z directions Figure 2-10 (c).

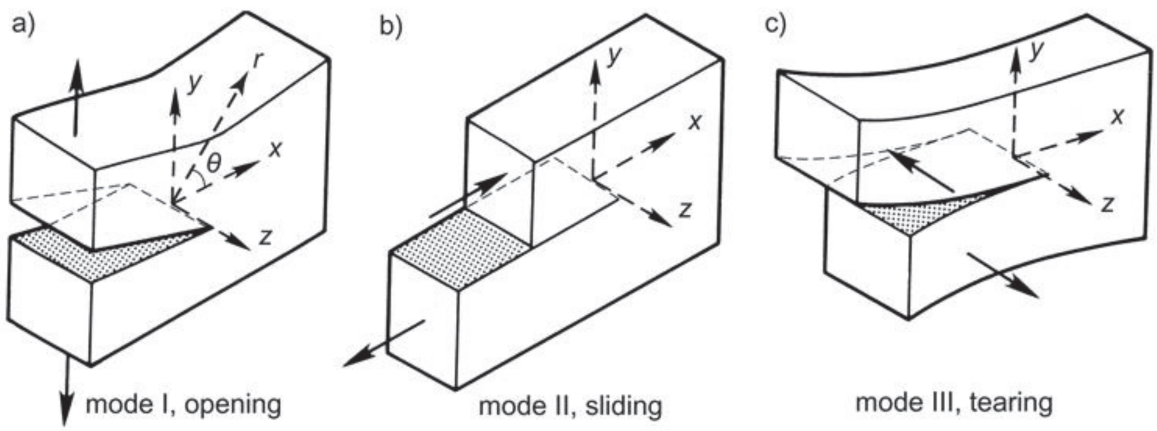

Figure 2-10 Modes of Fracture (adopted from (Kanninen, et al., 1985))

2.6.2 Stress Intensity Factor

Stress intensity factor quantifies the stresses at a crack tip. It defines the happening of fracture only if a critical stress intensity factor reached. In a paper published by (Diegele, et al., 2004) the displacement fields near a crack tip for Mode I opening in a Cosserat medium has been defined and formulated, as it can be seen below.

$$\begin{bmatrix} u_r^{tip} \\ u_\theta^{tip} \end{bmatrix} = \sqrt{\frac{r}{2\pi}} \frac{\widetilde{K}_I}{\mu[\mu+\alpha(3-2\upsilon)]} \begin{bmatrix} \cos\frac{\theta}{2}\ \mu(1-2\upsilon) - \alpha + [\mu + \alpha(7-6\upsilon)]\ \sin^2\frac{\theta}{2} \\ -\sin\frac{\theta}{2}\ [2(\mu+\alpha)(1-\upsilon) - \alpha + [\mu+\alpha(7-6\upsilon)]\ \cos^2\frac{\theta}{2} \end{bmatrix} +$$

$$r\frac{k_I}{2\mu}\begin{bmatrix} 2(\cos^2\theta - \upsilon) \\ -\sin(2\theta) \end{bmatrix} + \begin{bmatrix} O\left(r^{3/2}\right) \\ O\left(r^{3/2}\right) \end{bmatrix} \qquad 2.2$$

$$\phi^{tip} = \sqrt{\frac{2r}{\pi}} \frac{L_I}{\gamma+\delta} \sin\frac{\theta}{2} + O\left(r^{3/2}\right) \qquad 2.3$$

Where $\widetilde{K}_I$ and L_I are the Mode I stress intensity factor for micropolar elasticity, and k_I is the second order stress intensity factor. A notable point is that the couple stresses are singular for Mode I crack, on the other hand these stresses are regular for Mode II. The stress intensity factor can be defined as:

$$\widetilde{K}_I \coloneqq \lim_{r\to 0}\left\{\sqrt{2\pi r}\,T_{\theta\theta}(r,\theta)|\theta = 0\right\} \qquad 2.4$$

$$L_I \coloneqq \lim_{r\to 0}\left\{\sqrt{2\pi r}\,M_{z\theta}(r,\theta)|\theta = 0\right\} \qquad 2.5$$

where $T_{\theta\theta}$ and $M_{z\theta}$ are non-vanishing stress and couple stress, and $\widetilde{K}_I$ and L_I are force stress intensity factor and couple stress intensity factor, respectively (Diegele, et al., 2004; Li, et al., 2009).

The elastic constant α governs the degree of coupling for the rotations and translations. (Diegele, et al., 2004). It has to be mention that for $\alpha = 0$, the problem will become the classical linear elasticity, and for $\alpha \to \infty$ the problem will reduce to couple stress linear elasticity.

In Figure 2-11 and Figure 2-12 the effect of the material parameters α and γ on the stress intensity factor can be seen. It has to be noted that normalized stress intensity factor has been shown in both graphs $\left(\frac{\tilde{K}_I}{K_I}\right)$ (Diegele, et al., 2004).

As it can be seen in figure 2-11 for values of $\frac{\alpha}{E} < 10^{-3}$ the values for $\tilde{K}_I$ will remains the same for different γ parameter values. Then for $10^{-3} < \frac{\alpha}{E} < 10$ materials show different values and patterns for different γ values. Finally, as it is obvious from the graph for $\frac{\alpha}{E}$ greater than 10, $\frac{\tilde{K}_I}{K_I}$ takes a constant value for each γ values.

On the other hand, referring to (Figure 2-12), for $\gamma \rightarrow 0$ and $\gamma \rightarrow \infty$, the graph shows constant values of $\frac{\tilde{K}_I}{K_I}$ for different $\frac{\alpha}{E}$ values. These constant values are below 1 for larger values of γ, and they are greater than 1 for smaller amounts of γ. Another notable point in (Figure 2-12) is for smaller values of $\frac{\alpha}{E}$ (i.e. $\frac{\alpha}{E} < 10^{-3}$) the $\frac{\tilde{K}_I}{K_I}$ graph vs. γ shows almost no change (straight line), which will match the observation in $\frac{\tilde{K}_I}{K_I}$ graph vs. $\frac{\alpha}{E}$.

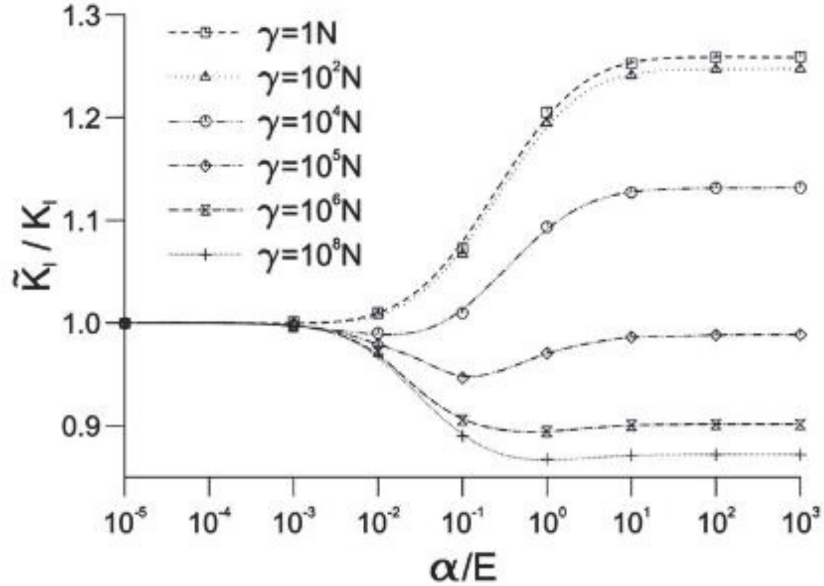

Figure 2-11 Distribution of normalized SIF as a function of $\frac{\alpha}{E}$ against parameter γ for Mode I fracture. *(adopted from (Diegele, et al., 2004).*

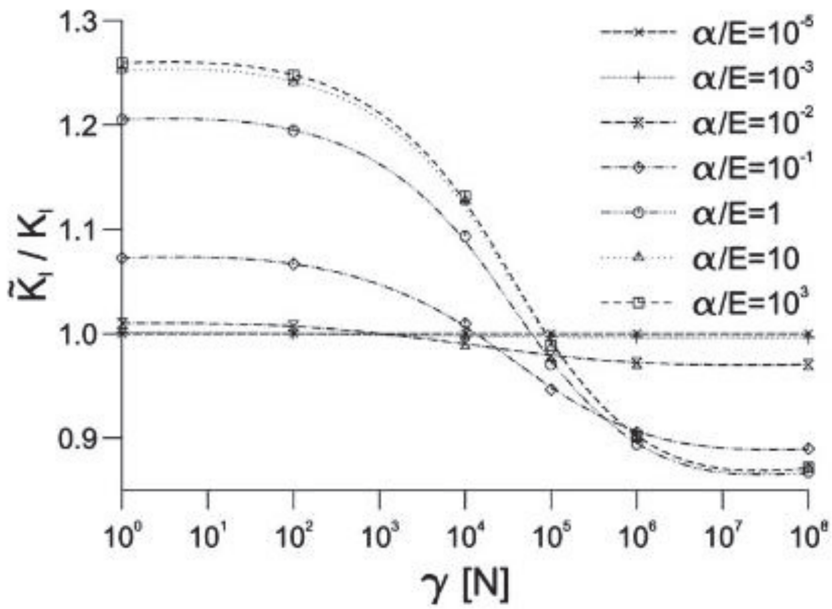

Figure 2-12 Distribution of normalized SIF as a function of γ against $\frac{\alpha}{E}$ for Mode I fracture. *(adopted from (Diegele, et al., 2004).*

Moreover, Figure 2-13 and Figure 2-14 are showing the relation between couple stress intensity factor (L_I) and material parameters. Again in L_I vs. $\frac{\alpha}{E}$ graph, for $\frac{\alpha}{E} < 10^{-3}$ the chart remains at the same value of L_I independently from the γ parameter value. This value is equal to zero. Then for $10^{-3} < \frac{\alpha}{E} < 10$ the L_I values decrease to below one and for values of $\frac{\alpha}{E}$ greater than 10 the graph shows a constant behaviour. On the other hand, the zero same zero value in L_I vs. γ will start at $\gamma = 1$ and ends at $\gamma = 10$, and the graph shows the constant behaviour after $\gamma = 10^7$. It is noticeable that for $\gamma = 1$, the values of L_I remains equal to zero, which matches the observation from the L_I vs. γ graph. It should be noted that for $\frac{\alpha}{E} = 10^{-5}$, same as $\frac{\tilde{K}_I}{K_I}$ vs. $\frac{\alpha}{E}$, L_I graph shows the same behaviour, and has a constant value of zero for different values of γ. Finally, it should be mentioned that $\frac{\tilde{K}_I}{K_I}$ had only positive values but L_I is having zero or negative values.

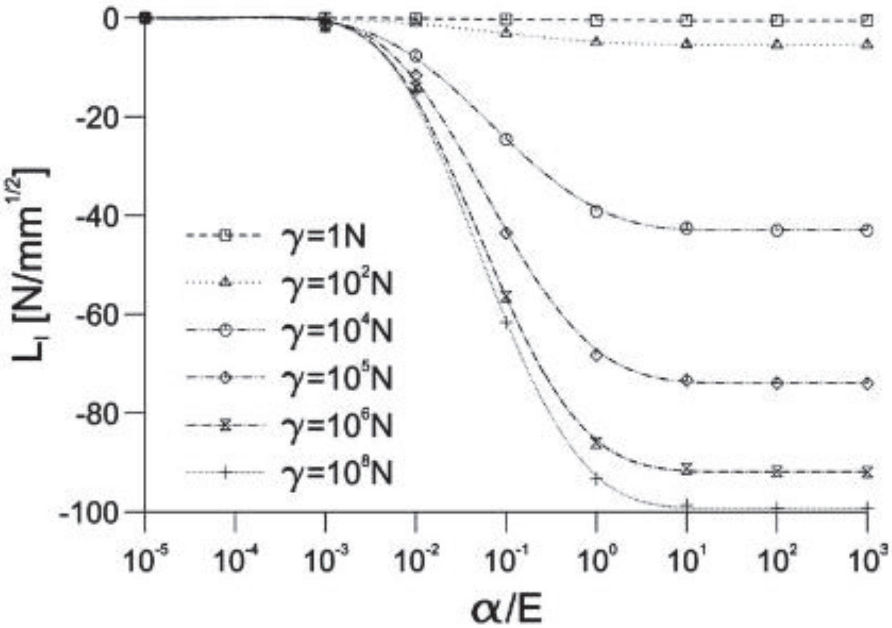

Figure 2-13 Distribution of L_I as a function of $\frac{\alpha}{E}$ against parameter γ for Mode I fracture (adopted from (Diegele, et al., 2004)).

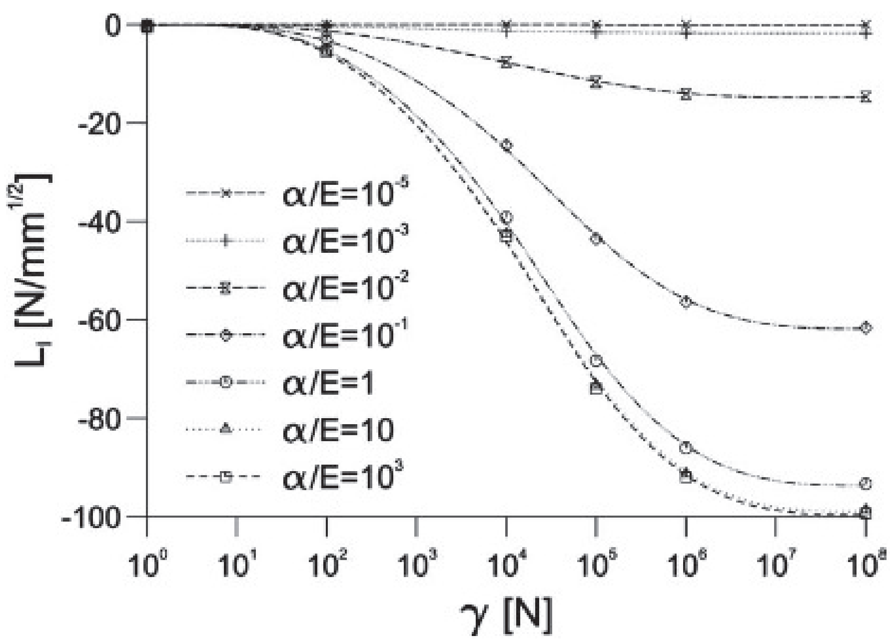

Figure 2-14 Distribution of L_I as a function of γ against $\frac{\alpha}{E}$ for Mode I fracture (adopted from (Diegele, et al., 2004)).

Chapter 3 The Basic Foundations of Theory of Cosserat Elasticity

3.1 Basic Definitions

In this chapter, the provisions of the three-dimensional theory of Cosserat elasticity will be reviewed. As the past works on this topic will be presented here, the detailed procedure of obtaining constitutive equations will be skipped. A detailed description of three-dimensional theory of elasticity can be found in (Nowacki, 1986).

Throughout what follows, Greek and Latin indices take the values 1, and 2 and 1, 2, and 3, respectively, the convention of summation over repeated indices is understood, $\mathcal{M}_{m\times n}$ is the space of $(m \times n)$ matrices, E_n is the identity element in $\mathcal{M}_{n\times n}$, a superscript T indicates matrix transposition and$(\dots)_{,a} = {}^{\partial(\dots)}/_{\partial x_a}$. Also, if X is a space of scalar functions and v a matrix, $v \in X$ means that every component of v belongs to X.

Let an elastic isotropic Cosserat body occupy a domain V in $\mathbb{R}^3$ and be bounded by surface S. Assume that the body undergoes deformation due to the action of external forces $X = (X_1, X_2, X_3)^T$ and external moments $Y = (Y_1, Y_2, Y_3)^T$. The elastic properties of the body can be characterized by elastic constants $\lambda, \mu, \gamma, \beta, \alpha, \kappa$, where λ and μ are usual Lame coefficients as in the classical theory of elasticity and γ, β, α and κ are micropolar elastic constants, representing the contribution of material microstructure to the elastic properties of the body. The state of deformation is characterized by a displacement field

$$u(x) = \big(u_1(x), u_2(x), u_3(x)\big)^T \qquad 3.1$$

and a microrotation field

$$\varphi(x) = \left(\varphi_1(x), \varphi_2(x), \varphi_3(x)\right)^T \qquad 3.2$$

where $x = (x_1, x_2, x_3)$ is a generic point in R^3.

This leads to the description of deformation of the body in terms of asymmetric strain, torsion, and stress and couple-stress tensors (Nowacki, 1986). These tensors can be represented in the form below

$$\varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \qquad \varkappa = \begin{bmatrix} \varkappa_{11} & \varkappa_{12} & \varkappa_{13} \\ \varkappa_{21} & \varkappa_{22} & \varkappa_{23} \\ \varkappa_{31} & \varkappa_{32} & \varkappa_{33} \end{bmatrix} \qquad 3.3$$

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \qquad \varrho = \begin{bmatrix} \varrho_{11} & \varrho_{12} & \varrho_{13} \\ \varrho_{21} & \varrho_{22} & \varrho_{23} \\ \varrho_{31} & \varrho_{32} & \varrho_{33} \end{bmatrix} \qquad 3.4$$

where ε is the strain tensor, $\varkappa$ the torsion tensor, σ the stress tensor, ϱ the couple-stress tensor.

3.2 Equilibrium Equations

The equations of equilibrium can be derived in terms of stresses and couple stresses, following the method provided in some other research (Nowacki, 1986; Eringen, 1966). These equations are:

$$\sigma_{jij} + X_j = 0$$

$$\epsilon_{ijk}\sigma_{jij} + Y_i = 0 \qquad 3.5$$

where ϵ_{ijk} alternating symbol.

It has to be indicate that the equilibrium equations for a micropolar media are formulated in a more complex way compare to classical theory of elasticity. The reason for this complication is the extra equations because of considering couple stresses. This impression defines the behaviour of a Cosserat solid in terms of asymmetric stress and stress-couple tensors. Evidently, if the all couple stresses set equal to zero, the symmetric stress tensor of the classical theory of elasticity will be obtained. Therefore, the asymmetric stress tensor is a result of the existence of couple stress. Using the constitutive relations (Nowacki, 1986)

$$\sigma_{ji} = (\mu + \alpha)\varepsilon_{ji} + (\mu - \alpha)\varepsilon_{ji} + \lambda_{kk}\delta_{ji}$$

$$\varrho_{ji} = (\gamma + \kappa)\varkappa_{ji} + (\gamma - \kappa)\varkappa_{ji} + \beta\varkappa_{kk}\delta_{ji} \qquad 3.6$$

where δ_{ji} is the Kronecker symbol,

and the kinematic relations

$$\varepsilon_{ji} = u_{ji} - \epsilon_{ji}\varphi_k$$

$$\varkappa_{ji} = \varphi_{ij} \qquad 3.7$$

the equilibrium equations can be defined in terms of displacements and microrotations, in the vector forms below:

$$(\mu + \alpha)\Delta u + (\lambda + \mu - \alpha)\nabla . u + 2\alpha(\nabla \times \varphi) + X \ = \ 0,$$

$$[(\gamma + \kappa)\Delta - 4\alpha]\varphi + (\beta + \gamma - \kappa)\nabla . \varphi + 2\alpha(\nabla \times \varphi) + Y \ = \ 0 \qquad 3.8$$

where Δ is the Laplace operator, $\nabla . u$ is the divergence of vector u, and $\nabla \times \varphi$ is the curl of vector φ.

As it mentioned before, the governing equations of micropolar elasticity, the system (3.8) is more complicated than the ones in classical theory of elasticity. The system of governing equations for a Cosserat solid is a system of coupled partial differential equations. This system consists of six unknowns. Three of these unknowns representing the displacements, which are the same in classical theory of elasticity. The other three, present microrotations. As it noted before, these microrotations are independent. If the micropolar elastic constants, $\alpha, \beta, \gamma, \kappa$, set to be equal to zero, Navier's equations, the governing equations in classical theory of elasticity, can be obtained from micropolar governing equations. It shows, the micropolar theory of elasticity is more general than the classical one.

In the studies that has been done before, the boundary integral method had been used to model and formulate the related boundary value problems of (3.5), and (3.8) (Nowacki, 1986). In some other studies, the method of potentials under different sets of boundary conditions was used to integrate the system (3.8) (Nowacki, 1986; Kessel, 1967; Kluge, 1969; Cowin, 1970; Cowin, 1970). In this thesis, the main focus is on studying and solving plane problems of Cosserat Elasticity. In order to reach this goal, the integration of the system (3.8) won't be written here. However, this system will be used to derive the governing equations for plane deformation.

Chapter 4 Plane Deformation of Micropolar Elastic Bodies

4.1 Basic Definitions

In this chapter, the governing equations of plane micropolar elasticity will be derived, using the equations of general three-dimensional theory of Cosserat elasticity (which represented in last chapter). With these equations, the Dirichlet and Neumann boundary value problems will be formed in Sobolev space setting.

Adding to last chapters explanations, along the following calculations, it has been assumed that the convention of summation over repeated indices is understood. Also, the columns of a (3×3)- matrix P are denoted by P^i.

Assuming S be a domain in $\mathbb{R}^2$ that has been occupied by a homogeneous and isotropic elastic Micropolar material. In this domain, the boundary can be represented as ∂S. Also, the notations $\|.\|_{0;S}$ and $\langle .,. \rangle_{0;S}$, are representing the norm and inner product in $L^2(S) \cap \mathcal{M}_{m\times 1}$ for any $m \in \mathbb{N}$. Moreover, when $S = \mathbb{R}^2$, the norm and inner products will get the form of $\|.\|_0$ and $\langle .,. \rangle_0$.

The state of plane Micropolar strain can be characterized by a displacement field

$$u(x') = \big(u_1(x'), u_2(x'), u_3(x')\big)^T \qquad 4.1$$

and a microrotation field

$$\varphi(x') = \big(\varphi_1(x'), \varphi_2(x'), \varphi_3(x')\big)^T \qquad 4.2$$

of the form

$u_a(\mathrm{x}') = u_a(x), \qquad u_3(\mathrm{x}') = 0$

$\varphi_a(\mathrm{x}') = 0, \qquad \varphi_3(\mathrm{x}') = \varphi_3(x)$

where $\mathrm{x}' = (x_1, x_2, x_3)$ and $x = (x_1, x_2)$ are generic points in $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively.

4.2 Plane Micropolar Elasticity

Now the equilibrium equations of plane Micropolar strain which was formulated in terms of displacements and microrotations can be written here (Schiavone, 1996), and (Shmoylova, 2006).

$$L(\partial_x)u(x) + q(x) = 0, \quad x \in S \qquad 4.3$$

where vector $q(x) = (q_1, q_2, q_3)^T$ represents body forces and body couples.

If φ_3 is shown as u_3, then $u(x) = (u_1, u_2, u_3)^T$, the matrix of partial differential operator $L(\partial_x) = L\left(\partial/\partial_x\right)$ can be defined as

$$L(\xi) = L(\xi_a) \qquad 4.4$$

$$= \begin{bmatrix} (\mu+\alpha)\Delta + (\lambda+\mu-\alpha)\xi_1^2 & (\lambda+\mu-\alpha)\xi_1\xi_2 & 2\alpha\xi_2 \\ (\lambda+\mu-\alpha)\xi_1\xi_2 & (\mu+\alpha)\Delta + (\lambda+\mu-\alpha)\xi_2^2 & -2\alpha\xi_1 \\ -2\alpha\xi_2 & 2\alpha\xi_1 & (\gamma+\kappa)\Delta - 4\alpha \end{bmatrix}$$

where $\Delta = \xi_\alpha\xi_\alpha$

Moreover, the boundary stress operator can be defined as

$$T(\xi) = T(\xi_a) \qquad 4.5$$

$$= \begin{bmatrix} (\lambda + 2\mu)\xi_1 n_1 + (\mu + \alpha)\xi_2 n_2 & (\mu - \alpha)\xi_1 n_2 + \lambda\xi_2 n_1 & 2\alpha\xi_2 \\ (\mu - \alpha)\xi_2 n_1 + \lambda\xi_1 n_2 & (\lambda + 2\mu)\xi_2 n_2 + (\mu + \alpha)\xi_1 n_1 & -2\alpha\xi_1 \\ 0 & 0 & (\gamma + \kappa)\xi_\alpha n_\alpha \end{bmatrix}$$

where $n = (n_1, n_2)^T$ is the unit outward normal to ∂S.

In order to assure about the ellipticity of equation (4.1), and the subsequent operators, it has been assumed that:

$$\lambda + \mu > 0, \quad \mu > 0, \quad \gamma + \kappa > 0, \quad \alpha > 0$$

The integral energy density is given by

$$2E(u, v) = 2E_0(u, v) + \mu(u_{1,2} + u_{2,1})(v_{1,2} + v_{2,1}) \qquad 4.6$$

$$+\alpha(u_{1,2} - u_{2,1} + 2u_3)(v_{1,2} - v_{2,1} + 2v_3)$$

$$+(\gamma + \kappa)(u_{3,1}v_{3,1} + u_{3,2}v_{3,2}$$

$$2E_0(u, v) = (\lambda + 2\mu)(u_{1,1}v_{1,1} + u_{2,2}v_{2,2}) + \lambda(u_{1,1}v_{1,1} + u_{2,2}v_{2,2}) \qquad 4.7$$

Clearly $E(u, v)$ is a positive quadratic form.

The space of rigid displacements and microrotations $\mathcal{F}$ is spanned by the columns of the matrix

$$\mathbb{F} = \begin{bmatrix} 1 & 0 & -x_2 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{bmatrix}$$

From which it can be seen that $L\mathbb{F} = 0$ in $\mathbb{R}^2$, $T\mathbb{F} = 0$ on ∂S, and a general rigid displacement can be written as $\mathbb{F}k$, where $k \in \mathcal{M}_{3\times1}$ is constant and arbitrary.

Now let S^+ be a domain in $\mathbb{R}^2$ bounded by a closed curve ∂S, and $S^- = \mathbb{R}^2 \backslash \bar{S}^+$. Using the same technique as in the derivation of the Betti formula, as it shown by (Constanda, 1990), it can be shown that if u is a solution of equation (4.1) in S^+, then for any $v \in C^2(S^+) \cap C^1(\bar{S}^+)$

$$\int_{S^+} v^T q \, dx = -\int_{S^+} v^T Lu \, dx = 2\int_{S^+} E(u,v) \, dx - \int_{\partial S} v^T Tu \, ds \qquad 4.8$$

A Galerkin representation for the solution of equation (4.1) when $q(x) = -\delta(|x-y|)$, where δ is Dirac delta distribution, yields the matrix of fundamental solutions (Schiavone, 1996).

$$D(x,y) = L^*(\partial x)t(x,y) \qquad 4.9$$

where L^* is the adjoint of L,

$$t(x,y) = \frac{a}{8\pi k^4}\{[k^2|x-y|^2 + 4] \ln|x-y| + 4K_0(k|x-y|)\} \qquad 4.10$$

where K_0 is the modified Bessel function of order zero and the constants a, k^2 are defined by

$$a^{-1} = (\gamma + \kappa)(\lambda + 2\mu)(\mu + \alpha) \qquad 4.11$$

$$k^2 = \frac{4\mu\alpha}{(\gamma+\kappa)(\mu+\alpha)} \qquad 4.12$$

In view of equation 4.9 and equation 4.10

$$D(x,y) = D^T(x,y) = D(y,x) \qquad 4.13$$

Along with matrix $D(x,y)$ the matrix of singular solutions can be considered as

$$P(x, y) = \left(T(\partial y)D(y, x)\right)^T \qquad 4.14$$

It is easy to verify $D^{(i)}(x, y)$, and $P^{(i)}(x, y)$ can satisfy equation (4.1) with $q(x) = 0$ at all $x \in \mathbb{R}^2, x \neq y$.

A class $\mathcal{A}$ of vectors $u \in \mathcal{M}_{3\times 1}$ can be introduced, having its components in terms of polar coordinates when $r = |x| \to \infty$, in the form of

$$u_1(r, \theta) = r^{-1}(\beta m_0 \sin\theta + m_1 \cos\theta + m_0 \sin 3\theta + m_2 \cos 3\theta + O(r^{-2})$$

$$u_2(r, \theta) = r^{-1}(m_3 \sin\theta + \beta m_0 \cos\theta + m_3 \sin 3\theta + m_0 \cos 3\theta + O(r^{-2})$$

$$u_1(r, \theta) = r^{-1}(m_5 \sin 2\theta + m_6 \cos 2\theta + O(r^{-3})$$

where

$$\beta = \frac{3\mu + \lambda}{\lambda + \mu}$$

and $m_0, \dots, m_6$ are arbitrary constants.

Also let

$$\mathcal{A}^* = \{u : u = \mathcal{F}c + \sigma^{\mathcal{A}}\}$$

where $c \in \mathcal{M}_{3\times 1}$ is an arbitrary constant and $\sigma^{\mathcal{A}} \in (\mathcal{M}_{3\times 1} \cap \mathcal{A})$.

For the exterior domain the Betti formula, that came in a book published by (Constanda, 1990), can be represented as below

If u is a solution of equation (4.1) in S^-, then for any $v \in C^2(S^-) \cap C^1(\bar{S}^-) \cap \mathcal{A}^*$

$$\int_{S^-} v^T q\, dx = -\int_{S^-} v^T L u\, dx = 2\int_{S^-} E(u, v)\, dx - \int_{\partial S} v^T T u\, ds \qquad 4.15$$

Further the corresponding area, single layer potential, and double layer potential can be represented respectively as below

$$(U\varphi)(x) = \int_{\mathbb{R}^2} D(x,y)\varphi(y)dy \quad 4.16$$

$$(V\varphi)(x) = \int_{\partial S} D(x,y)\varphi(y)ds(y) \quad 4.17$$

$$(W\varphi)(x) = \int_{\partial S} P(x,y)\varphi(y)ds(y) \quad 4.18$$

where $\varphi \in \mathcal{M}_{3\times 1}$ is an unknown density matrix.

It can be concluded that $L(Uq) = q$ in $\mathbb{R}^2$.

The properties of single and double layer potentials can be seen in the following theorem. These properties have been proven in a book published by (Constanda, 1990).

Theorem 4.1

i. If $\varphi \in C(\partial S)$, then $V\varphi$, and $W\varphi$, are analytic and satisfy $L(V\varphi) = L(W\varphi) = 0$ in $S^+ \cup S^-$.

ii. If $\varphi \in C^{0,\alpha}(\partial S)$, and $\alpha \in (0,1)$, then the direct values $V_0\varphi$, and $W_0\varphi$ of $V\varphi$, and $W\varphi$ on ∂S exist (the latter as principle value), the functions $V^+(\varphi) = (V\varphi)|_{\bar{S}^+}$, and $V^-(\varphi) = (V\varphi)|_{\bar{S}^-}$ are of class $C^{1,\alpha}(\bar{S}^+)$, and $C^{1,\alpha}(\bar{S}^-)$ respectively. Also $TV^+(\varphi) = (W_0^* + \frac{1}{2}I)\varphi$, and $TV^-(\varphi) = (W_0^* - \frac{1}{2}I)\varphi$ on ∂S, where W_0^* is the adjoint of W_0 and I is the identity operator.

iii. If $\varphi \in C^{1,\alpha}(S)$, and $\alpha \in (0,1)$, then the function $W^+(\varphi)$ is of class $C^{1,\alpha}(\bar{S}^+)$, and the function $W^-(\varphi)$ is of class $C^{1,\alpha}(\bar{S}^-)$, and $TW^+(\varphi) = TW^-(\varphi)$ on ∂S. Functions $W^+(\varphi)$, and $W^-(\varphi)$ have been represented below

$$W^{+}(\varphi)=\begin{cases}(W\varphi)|_{S^{+}}, & in\ S^{+}\\ \left(W_0-\frac{1}{2}I\right)\varphi, & on\ \partial S\text{'}\end{cases}$$

$$W^{-}(\varphi)=\begin{cases}(W\varphi)|_{S^{-}}, & in\ S^{-}\\ \left(W_0+\frac{1}{2}I\right)\varphi, & on\ \partial S\end{cases}$$

For any $m \in \mathbb{R}$, let $H_m(\mathbb{R}^2)$ be the standard real Sobolev space of three component distributions, equipped with the norm

$$\|u\|_m^2=\int_{\mathbb{R}^2}(1+|\xi|^2)^m|\tilde{u}(\xi)|^2 d\xi$$

where $\tilde{u}$ is the Fourier transform of u.

In the following calculations, the equivalent norms has not been distinguished from each other and has been denoted by the same symbol. Hence, the norm in $H_1(\mathbb{R}^2)$ can be defined by

$$\|u\|_1^2=\|u\|_0^2+\sum_{i=1}^{3}\|\nabla u_i\|_0^2 \qquad 4.19$$

The spaces $H_m(\mathbb{R}^2)$ and $H_{-m}(\mathbb{R}^2)$ are dual with respect to duality induced by $\langle .,. \rangle_0$.

The space $L^2_\omega(\mathbb{R}^2)$ of 3*1-vector functions $u=(\bar{u},u_3)^T$, where $\bar{u}=(u_1,u_2)^T$, such that

$$\|u\|_{0,\omega}^2=\int_{\mathbb{R}^2}\frac{|\bar{u}(x)|^2}{(1+|x|)^2(1+\ln|x|)^2}dx+\int_{\mathbb{R}^2}\frac{|u_3(x)|^2}{(1+|x|)^4(1+\ln|x|)^2}dx<\infty \qquad 4.20$$

The bilinear form $b(u,v)=2\int_{\mathbb{R}^2}E(u,v)\,dx$ has been considered in this analysis. Assuming $H_{1,\omega}(\mathbb{R}^2)$ to be the space of three component distributions on $\mathbb{R}^2$ for which

$$\|u\|_{1,\omega}^2=\|u\|_{0,\omega}^2+b(u,u)<\infty \qquad 4.21$$

$H_{-1,\omega}(\mathbb{R}^2)$ is dual to $H_{1,\omega}(\mathbb{R}^2)$ with respect to duality generated by $\langle .,. \rangle_0$. The norm in $H_{-1,\omega}(\mathbb{R}^2)$ is denoted by $\|.\|_{-1,\omega}$.

Let $\dot{H}_m(S^+)$ be the subspace of $H_m(\mathbb{R}^2)$ consisting of all u which have a compact support in S^+. $H_m(S^+)$ is the space of the restrictions to S^+ of all $u \in H_m(\mathbb{R}^2)$. The norm of $u \in H_m(S^+)$ will be introduced by $\|u\|_{m;S^+} = inf_{v \in H_m(\mathbb{R}^2):\pi^+ v = u} \|v\|_m$, if the operators of restrictions from $\mathbb{R}^2$ to $S^\pm$, denote by $\pi^\pm$. If $m = 1$, then the norms of $u \in \dot{H}_1(S^+)$ and $u \in H_1(S^+)$ are equivalent to

$$\left\{\|u\|_{0;S^+}^2 + \sum_{i=1}^3 \int_{S^+} |\nabla u_i(x)|^2 dx\right\}^{1/2} \qquad 4.22$$

The spaces $\dot{H}_m(S^+)$ and $H_{-m}(S^+)$ are dual with respect to duality induced by $\langle .,. \rangle_{0;S^+}$.

Let $\dot{H}_{1,\omega}(S^-)$ be the subspace of $H_{1,\omega}(\mathbb{R}^2)$ consisting of all u which have a compact support in S^-. $H_{1,\omega}(S^-)$ is the space of the restrictions to S^- of all $u \in H_{1,\omega}(\mathbb{R}^2)$. The norm of $u \in H_{1,\omega}(S^-)$ will be introduced by

$$\|u\|_{1,\omega;S^-} = inf_{v \in H_{1,\omega}(\mathbb{R}^2):\pi^- v = u} \|v\|_{1,\omega} \qquad 4.23$$

if the operators of restrictions from $\mathbb{R}^2$ to $S^\pm$, denote by $\pi^\pm$.

From the definition it follows that $H_{1,\omega}(S^-)$ is isometric to $\frac{H_{1,\omega}(\mathbb{R}^2)}{\dot{H}_1(S^+)}$. It can be shown that the norm of $u \in H_{1,\omega}(S^-)$ equivalent to

$$\left\{\|u\|_{0,\omega;S^-}^2 + b(u,u)\right\}^{1/2} \qquad 4.24$$

where

$$\|u\|^2_{0,\omega;S^-} = \int_{S^-} \frac{|\bar{u}(x)|^2}{(1+|x|)^2(1+\ln|x|)^2} dx + \int_{S^-} \frac{|u_3(x)|^2}{(1+|x|)^4(1+\ln|x|)^2} dx \qquad 4.25$$

and $b_{\pm}(u,v) = 2\int_{S^{\pm}} E(u,v)\, dx$. This norm is compatible with asymptotic class $\mathcal{A}$.

The dual of $\dot{H}_{1,\omega}(S^-)$ with respect to the duality generated by $\langle .,. \rangle_{0;S^-}$ is the space $H_{-1,\omega}(S^-)$, with norm $\|.\|_{-1,\omega;S^-}$. The dual of $H_{1,\omega}(S^-)$ is $\dot{H}_{-1,\omega}(S^-)$, which can be identified with a subspace of $H_{-1,\omega}(\mathbb{R}^2)$. It can be shown that if $u \in \dot{H}_{-1}(S^-)$ and has compact support in S^-, or if

$$\int_{S^-} |\bar{u}(x)|^2(1+|x|)^2(1+\ln|x|)^2\, dx + \int_{S^-} |u_3(x)|^2(1+|x|)^4(1+\ln|x|)^2\, dx < \infty \qquad 4.26$$

then $u \in \dot{H}_{-1,\omega}(S^-)$.

Let $H_m(\partial S)$ be the standard Sobolev space of distributions on ∂S, with norm $\|.\|_{m;\partial S}$. $H_m(\partial S)$, and $H_{-m}(\partial S)$ are dual with respect to the duality generated by the inner product $\langle .,. \rangle_{0;\partial S}$ in $L^2(\partial S)$.

The trace operators are defined on $C_0^\infty(S^{\pm})$. Then by denoting γ^+, and γ^-, it can be extended by continuity to surjections $\gamma^+ : H_1(S^+) \to H_{\frac{1}{2}}(\partial S)$, and $\gamma^- : H_{1,\omega}(S^-) \to H_{\frac{1}{2}}(\partial S)$. This conclusion is accurate due to the local equivalency of $H_{1,\omega}(S^-)$ and $H_1(S^-)$. Moreover, a continuous extension operators has been considered as $l^+ : H_{\frac{1}{2}}(\partial S) \to H_1(S^+)$, and $l^- : H_{\frac{1}{2}}(\partial S) \to H_1(S^-)$. Since the norm in $H_1(S^-)$ is stronger than the norm in $H_{1,\omega}(S^-)$, the latter operator can also be regarded as a continuous operator from $H_{\frac{1}{2}}(\partial S)$ into $H_{1,\omega}(S^-)$.

To proceed further, the following fact from functional analysis is needed.

Theorem 4.2 *(Lax-Milgram Lemma)* Let H be a Hilbert space and $b(u,v)$ be a bilinear functional defined for every ordinate pair $u,v \in H$, for which there exist two constants h and k such that

$$|b(u,v)| \leq h\|u\|\|v\|, \quad \|u\|^2 \leq k|b(u,v)| \qquad \forall u,v \in H \tag{4.27}$$

In this case it can be said that $b(u,v)$ is coercive. Although the bounded linear functional $\mathcal{L}(v)$ has been assigned on H, there exist one and only one u such that

$$b(u,v) = \mathcal{L}(v), \qquad \forall v \in H, \quad \|u\| \leq c\|\mathcal{L}\| \tag{4.28}$$

where $\|.\|_*$ is the norm on the dual H' of H.

The proof for this lemma can be found in a book published by (Miranda, 1970).

Chapter 5 Crack in Plane Micropolar Elasticity

5.1 Background

When a domain is weakened by a crack, the representation of the boundary conditions across the crack region could be challenging in the boundary integral analysis in a classical setting. Several researches has been done focusing on a crack problem in two-dimensional Cosserat elasticity under the assumption of classical elastic setting, using the finite element method (Lakes, et al., 1990). Another types of attempts to investigate a crack problem in two-dimensional Cosserat elasticity, has been taken under the assumption of a simplified theory of plane Cosserat elasticity, when displacements and microrotations are constrained (couple-stress elasticity) (Mühlhaus, et al., 2002) (Atkinson, et al., 1977). Moreover, it has to be mentioned that some researches has been done to investigate the crack analysis in three-dimensional Cosserat elasticity (De Borst, et al., 1998) (Yavari, et al., 2002) (Diegele, et al., 2004) (Garanjeu, et al., 2003). However, an analysis on the crack problem under plane Cosserat elasticity assumption in the general case is still absent from the literature.

The boundary integral equation method in a weak (Sobolev) space setting has been used to obtain the solutions for several crack problems in a theory of bending of classical elastic plates (Chudinovich, et al., 2000). Although, the method that has been used in the aforementioned research is mathematically complicated, it can be counted as one of the most effective methods in this area that produces very reliable results. Due to this fact, in this work the effectiveness of this method has been studied with a view to the analysis and solution of the plane problems of Cosserat elasticity.

In this chapter, the boundary value problems for both finite and infinite domains that has been weakened by a crack has been formulated. These calculations were taken by consideration of plane micropolar elasticity, when displacements and microrotations or stresses and couple stresses exist on both sides of the crack, in Sobolev spaces. The aim was to find the corresponding weak solutions in terms of integral potentials with distributional densities.

5.2 Basic Definitions

In this research an infinite domain with a crack has been considered. The crack is modelled by an open arc Γ_0, which is a part of a simple closed C^2-curve, Γ. This curve divides $\mathbb{R}^2$ into interior and exterior domains Ω^+ and Ω^-. The superscripts + and – in Ω^+ and Ω^- are denoting the limiting values of functions as $x \to \Gamma$. Also, $\Omega = \mathbb{R}^2 \backslash \bar{\Gamma}_0$ and $\Gamma_1 = \Gamma \backslash \bar{\Gamma}_0$. Regarding definition of Ω, for the norm and inner product in $L^2(\Omega)$, $\|.\|_0$ and $\langle .,. \rangle_0$ can be used respectively.

Let $H_m(\Gamma)$ be the standard Sobolev space of distributions on Γ, with the norm product $\|.\|_{m,\Gamma}$. $H_m(\Gamma)$ and $H_{-m}(\Gamma)$ are dual with respect to the duality generated by the inner product $\langle .,. \rangle_{0,\Gamma}$ in $L^2(\Gamma)$. The subspace of all $f \in H_m(\Gamma)$ with a compact support on Γ_0 will be denoted by $\dot{H}_m(\Gamma_0)$. Also, the space of the restrictions to Γ_0 of all $f \in H_m(\Gamma)$ will be denoted by $H_m(\Gamma_0)$. Moreover, π_0 and π_1 are set to be the operators from Γ to Γ_0 and Γ_1.

The norm of $f \in H_m(\Gamma_0)$ can be defined by $\|f\|_{m,\Gamma_0} = inf_{v \in H_m(\Gamma):\pi_0 v = f} \|v\|_{m,\Gamma}$. For any $m \in \mathbb{R}$, $\dot{H}_m(\Gamma_0)$ and $H_{-m}(\Gamma_0)$ are dual with respect to the duality generated by the inner product $\langle .,. \rangle_{0;\Gamma_0}$ in $L^2(\Gamma_0)$.

Assume γ^{+} and γ^{-} are continuous trace operators from $H_1(\Omega^{+})$ and $H_{1,\omega}(\Omega^{-})$ to $H_{\frac{1}{2}}(\Gamma)$. Also, it has been assumed that $\gamma_i^{\pm} = \pi_i\gamma^{\pm}$, $i = 0{,}1$. For any u defined in Ω or $\mathbb{R}^2$, it can be written that $u = \{u_+, u_-\}$, where $u_{\pm} = \pi^{\pm}u$.

Let $H_{1,\omega}(\Omega)$ be the space of all $u = \{u_+, u_-\}$ such that $u_+ \in H_1(\Omega^{+})$ and $u_- \in H_{1,\omega}(\Omega^{-})$ and $\gamma_1^{+}u_+ = \gamma_1^{-}u_-$. The norm in $H_{1,\omega}(\Omega)$ can then be defined as

$$\|u\|_{1,\omega;\Omega}^2 = \|u_+\|_{1;\Omega^+}^2 + \|u_-\|_{1,\omega;\Omega^-}^2$$

From defining $\dot{H}_{1,\omega}(\Omega)$ as the subspace of $H_{1,\omega}(\Omega)$, $\dot{H}_{1,\omega}(\Omega)$ can be identified with a subspace of $H_{1,\omega}(\mathbb{R}^2)$. It should be mentioned that $H_{1,\omega}(\Omega)$ consists of all u that $\gamma_0^{+}u_+ = \gamma_0^{-}u_- = 0$.

The duals of $\dot{H}_{1,\omega}(\Omega)$ and $H_{1,\omega}(\Omega)$, with respect to the duality induced by $\langle .,. \rangle_0$, can be defined as $H_{-1,\omega}(\Omega)$ and $\dot{H}_{-1,\omega}(\Omega)$. Moreover, the norms in $H_{-1,\omega}(\Omega)$ and $\dot{H}_{-1,\omega}(\Omega)$ are denoted by $\|.\|_{-1,\omega;\Omega}$ and $\|.\|_{-1,\omega}$.

5.3 Boundary Value Problem

As it mentioned in introduction chapter of this thesis, two types of boundary value problems have been considered, Dirichlet and Neumann boundary value problems.

The first boundary value problem consist of seeking $u \in C^2\,(\Omega) \,\cap C\,(\bar{\Omega})$, $u_- \in \mathcal{A}^*$ such that

$$\begin{cases} Lu(x) + q(x) \;=\; 0, \;\; x \in \Omega, \\ u^{+}(x) \;\;=\; f^{+}(x), \;\; u^{-}(x) = f^{-}(x), \;\; x \in \Gamma_0 \end{cases} \qquad 5.1$$

where f^{+} and f^{-} are prescribed on Γ_0.

The second one is focusing on finding $\in C^2(\Omega) \cap C^1(\bar{\Omega})$, $u_- \in \mathcal{A}$ such that

$$\begin{cases} Lu(x) + q(x) = 0, \quad x \in \Omega, \\ (Tu)^+(x) = g^+(x), \quad (Tu)^-(x) = g^-(x), \quad x \in \Gamma_0 \end{cases} \qquad 5.2$$

where g^+ and g^- are prescribed on Γ_0. Asymptotic classes $\mathcal{A}^*$ and $\mathcal{A}$ have been introduced in chapter 4.

In the following the variational formulation of Dirichlet can be seen. Noting that the goal is to find $u \in H_{1,\omega}(\Omega)$

$$b(u,v) = \langle q, v\rangle_0 \qquad \forall v \in \dot{H}_{1,\omega}(\Omega) \qquad 5.3$$

$$\gamma^+ u_+ = f^+, \qquad \gamma^- u_- = f^-$$

where $q \in H_{-1,\omega}(\Omega)$ and $f^+, f^- \in H_{\frac{1}{2}}(\Gamma_0)$ are given.

In the following the variational formulation of Neumann can be seen. Noting that the goal is to find $u \in H_{1,\omega}(\Omega)$

$$b(u,v) = \langle q, v\rangle_0 + \langle g^+, \gamma_0^+ v_+\rangle_{0;\Gamma_0} - \langle g^-, \gamma_0^- v_-\rangle_{0;\Gamma_0} \qquad \forall v \in H_{1,\omega}(\Omega) \qquad 5.4$$

$$\gamma^+ u_+ = f^+, \qquad \gamma^- u_- = f^-$$

where $q \in \dot{H}_{-1,\omega}(\Omega)$ and $g^+, g^- \in H_{\frac{1}{2}}(\Gamma_0)$ are given.

In what follows $\delta f = f^+ - f^-$ and $\delta g = g^+ - g^-$ are representing the jump across the crack.

Theorem 5.1 Problem (5.1) has a unique solution $u \in H_{1,\omega}(\Omega)$ for any $q \in H_{-1,\omega}(\Omega)$ and $f^+, f^- \in H_{\frac{1}{2}}(\Gamma_0)$, such that $\delta f \in \dot{H}_{\frac{1}{2}}(\Gamma_0)$. This solution satisfies

$$\|u\|_{1,\omega;\Omega} \leq c(\|q\|_{-1,\omega;\Omega} + \|f^{+}\|_{1/2;\Gamma_0} + \|\delta f\|_{1/2;\Gamma} \qquad 5.5$$

Proof Assume that $f^{+} = f^{-} = 0$. In order to prove this assumption, it is necessary to verify that $b(u,v)$ is coercive on $\dot{H}_{1,\omega}(\Omega)$. It can be shown that any $u = \{u_{+}, u_{-}\} \in \dot{H}_{1,\omega}(\Omega)$ can satisfy $\|u_{-}\|^{2}_{1,\omega;\Omega^{+}} \leq cb_{+}(u_{+},u_{+})$ and $\|u_{-}\|^{2}_{1,\omega;\Omega^{-}} \leq cb_{-}(u_{-},u_{-})$, where $b_{\pm}(u,v) = 2\int_{\Omega^{\pm}} E_{\pm}(u,v)\,dx$. As a result it can be concluded that

$$\|u\|^{2}_{1,\omega;\Omega} = \|u_{+}\|^{2}_{1,\omega;\Omega^{+}} + \|u_{-}\|^{2}_{1,\omega;\Omega^{-}} \leq c[b_{+}(u_{+},u_{+}) + b_{-}(u_{-},u_{-})] = cb(u,u)$$

By the Lax-Milgram lemma, Dirichlet boundary value problem, with $f^{+} = f^{-} = 0$, has a unique solution $u \in \dot{H}_{1,\omega}(\Omega)$ and

$$\|u\|_{1,\omega} \leq c\|q\|_{-1,\omega;\Omega} \qquad 5.6$$

In the full problem of Dirichlet boundary value, the operator l_0 will be considered from Γ_0 to Γ, which maps $H_{\frac{1}{2}}(\Gamma_0)$ continuously to $H_{\frac{1}{2}}(\Gamma)$. Let $F^{+} = l_0 f^{+}$, and let F^{-} be the extension of f^{-} to Γ such that $\pi_1 F^{+} = \pi_1 F^{-}$. The operators of extension from Γ to $\Omega^{\pm}$ by $l_{\pm}$, which map $H_{\frac{1}{2}}(\Gamma)$ continuously to $H_1(\Omega^{+})$ and $H_{1,\omega}(\Omega^{-})$, respectively. Let $w = l_{+}F^{+} \in H_1(\Omega^{+})$, and $w_{-} = l_{-}F^{-} \in H_1(\Omega^{-})$. It is obvious from this definition that $w = \{w_{+}, w_{-}\} \in \dot{H}_{1,\omega}(\Omega)$. The goal is to find a solution for Dirichlet boundary value problems in the form of $u = u_0 + w$, where $u_0 \in \dot{H}_{1,\omega}(\Omega)$ satisfies

$$b(u_0, v) = \langle q, v\rangle_0 - b(w,u) \qquad \forall v \in \dot{H}_{1,\omega}(\Omega) \qquad 5.7$$

On the other hand, it can be said for all $v \in \dot{H}_{1,\omega}(\Omega)$

$$|b(w,v)| \leq |b_{+}(w_{+},v_{+})| + |b_{-}(w_{-},v_{-})| \leq c\big(\|w_{+}\|_{1;\Omega^{+}} + \|w_{-}\|_{1,\omega;\Omega^{-}}\big)\|v\|_{1,\omega}$$

$$\leq c\left(\|F^+\|_{1/_2;\Gamma} + \|F^-\|_{1/_2;\Gamma}\right)\|v\|_{1,\omega} \leq c\left(\|f^+\|_{1/_2;\Gamma_0} + \|f^-\|_{1/_2;\Gamma_0}\right)\|v\|_{1,\omega}$$

$$\leq c\left(\|f^+\|_{1/_2;\Gamma_0} + \|\delta f\|_{1/_2;\Gamma}\right)\|v\|_{1,\omega}$$

The right hand side of the equation (5.7), $L(v) = \langle q, v\rangle_0 - b(w, u)$, defines the continuous linear functional on $\dot{H}_{1,\omega}(\Omega)$, and $\|L\|_{1,\omega;\Omega} \leq c\left(\|q\|_{-1,\omega;\Omega} + \|f^+\|_{1/_2;\Gamma_0} + \|\delta f\|_{1/_2;\Gamma}\right)$. Hence the equation (5.7) has a unique solution $u_0 \in \dot{H}_{1,\omega}(\Omega)$, and

$$\|u_0\|_{1,\omega;\Omega} \leq c\left(\|q\|_{-1,\omega;\Omega} + \|f^+\|_{1/_2;\Gamma_0} + \|\delta f\|_{1/_2;\Gamma}\right)$$

The theorem now follows from this inequality and the estimate

$$\|w\|_{1,\omega;\Omega} \leq c\left(\|f^+\|_{1/_2;\Gamma_0} + \|\delta f\|_{1/_2;\Gamma}\right)$$

■

For Neumann boundary value problems, it can be clearly understood that, from the properties of rigid displacements point of view, the equation (5.8) is a necessary solvability condition for these boundary value problems.

$$\langle q, z\rangle_0 + \langle g^+, z\rangle_{0;\Gamma_0} - \langle g^-, z\rangle_{0;\Gamma_0} = 0 \qquad \forall z \in \mathcal{F} \qquad 5.8$$

Theorem 5.2 Problem (5.2) is solvable for any $q \in \dot{H}_{-1,\omega}(\Omega)$ and any $g^+, g^- \in H_{-\frac{1}{2}}(\Gamma_0)$, such that $\delta f \in \dot{H}_{-\frac{1}{2}}(\Gamma_0)$ satisfies equation (5.8). Each solution is differ from the other by a rigid displacement. Hence, there is a solution u_0 that satisfies the estimate

$$\|u_0\|_{1,\omega;\Omega} \leq c(\|q\|_{-1,\omega} + \|\delta g\|_{-1/_2;\Gamma} + \|g^-\|_{1/_2;\Gamma_0}$$

Proof Assume the expression $L(v)$ such that

$L(v) = \langle g^+, \gamma_0^+ v_+ \rangle_{0;\Gamma_0} - \langle g^-, \gamma_0^- v_- \rangle_{0;\Gamma_0}$

$= \langle \delta g, \gamma_0^+ v_+ \rangle_{0;\Gamma_0} + \langle g^-, \delta v \rangle_{0;\Gamma_0} \qquad \forall v \in H_{1,\omega}(\Omega)$

where $\delta v = \gamma_0^+ v_+ - \gamma_0^- v_-$, which defines a continuous linear function on $H_{1,\omega}(\Omega)$. Therefore, it can be concluded that $q_1 \in \dot{H}_{1,\omega}(\Omega)$ exists, which will satisfy $L(v) = \langle q_1, v \rangle_0$ for all $v \in H_{1,\omega}(\Omega)$, and

$$\|q_1\|_{-1,\omega} \leq c\left(\|g^-\|_{-\frac{1}{2},\Gamma_0} + \|\delta g\|_{-\frac{1}{2},\Gamma}\right) \qquad 5.9$$

Now by setting $q + q_1 = \tilde{q}$ the equation (5.2) can be rewritten in the form of $b(u,v) = \langle \tilde{q}, v \rangle_0 , v \in H_{1,\omega}(\Omega)$. Also, the factor space has been defined as $\mathbb{H}_{1,\omega}(\Omega) = H_{1,\omega}(\Omega) \backslash \mathcal{F}$. The norm for the factor space is $\|U\|_{\mathbb{H}_{1,\omega}(\Omega)} = inf_{u \in H_{1,\omega}(\Omega), u \in U} \|u\|_{1,\omega;\Omega}$. Moreover, a bilinear form $\mathcal{B}(U,V)$, and a linear functional $\mathcal{L}(V)$, will be defined as

$$\mathcal{B}(U,V) = b(u,v), \qquad \mathcal{L}(V) = L(v) = \langle \tilde{q}, v \rangle_0 \qquad 5.10$$

where u and v are arbitrary representatives of the classes $U, V \in \mathbb{H}_{1,\omega}(\Omega)$. Since $b(z,z) = 0$ and $\langle \tilde{q}, z \rangle_0 = 0$ for any $z \in \mathcal{F}$, the definitions (5.10) are consistent.

The goal of the problem that will be considered now is to find $U \in \mathbb{H}_{1,\omega}(\Omega)$, such that

$$\mathcal{B}(U,V) = \mathcal{L}(V), \qquad \forall V \in \mathbb{H}_{1,\omega}(\Omega) \qquad 5.11$$

It is claimed that equation (5.11) has a unique solution. From equation (5.9) it can be concluded that

$$|\mathcal{L}(V)| \leq c\left(\|q\|_{-1,\omega} + \|g^-\|_{-\frac{1}{2},\Gamma_0} + \|\delta g\|_{-\frac{1}{2},\Gamma}\right)\|v\|_{1,\omega;\Omega} \quad v \in V$$

which will results

$$|\mathcal{L}(V)| \leq c\left(\|q\|_{-1,\omega} + \|g^-\|_{-\frac{1}{2},\Gamma_0} + \|\delta g\|_{-\frac{1}{2},\Gamma}\right)\|V\|_{\mathbb{H}_{1,\omega}(\Omega)}$$

This conclusion means that $\mathcal{L}(V)$ is continuous. The continuity of $\mathcal{B}$ can be clearly understood. In every class U a representative u has been chosen that $\langle \gamma_0^+ u_+, z\rangle_{0;\Gamma_0} = 0$ for all $z \in \mathcal{F}$.

$$\|u\|^2_{1,\omega;\Omega} \leq c\left[b_-(u_-,u_-) + \|\gamma_1^- u_-\|^2_{0,\Gamma_1}\right] \leq c\left[b_-(u_-,u_-) + \|\gamma_1^+ u_+\|^2_{0,\Gamma_1}\right]$$

$$\leq c\left[b_-(u_-,u_-) + \|u_+\|^2_{1,\Omega^+}\right]$$

where $\|.\|_{0,\Gamma_1}$ is the norm in $L^2(\Gamma_1)$, and

$$\|u_+\|^2_{1,\Omega^+} \leq c[b_+(u_+,u_+)]$$

Consequently

$$\|U\|_{\mathbb{H}_{1,\omega}(\Omega)} \leq \|u_+\|^2_{1,\Omega^+} \leq \mathcal{B}(U,V)$$

This proves that $\mathcal{B}$ is coercive on $\mathbb{H}_{1,\omega}(\Omega)$. By the Lax-Milgram lemma, equation (5.10) has a unique solution $U \in \mathbb{H}_{1,\omega}(\Omega)$, and

$$\|U\|_{\mathbb{H}_{1,\omega}(\Omega)} \leq c\left(\|q\|_{-1,\omega} + \|g^-\|_{-\frac{1}{2},\Gamma_0} + \|\delta g\|_{-\frac{1}{2},\Gamma}\right)$$

Clearly any element u in U is a solution of equation (5.2).If u_1 and u_2 are two solutions of equation (5.2), then $w = u_1 - u_2$, which satisfies

$$b(w,w) = 0, \qquad w \in \mathbb{H}_{1,\omega}(\Omega)$$

It can be concluded now that $w \in \mathcal{F}$. By choosing $u_0 \in U$ that will satisfy $\|u_0\|_{1,\omega;\Omega} = \|U\|_{\mathbb{H}_{1,\omega}(\Omega)}$, the proof will be completed. ■

Theorem 5.3 $H_{-1,\omega}(\Omega)$ consists of all $q = (\bar{q}^T, q_3)^T$. Also, $\bar{q}$ is defined as $\bar{q} = (q_1, q_2)^T$. $\bar{q}$ and q_3 satisfy the equation (5.12).

$$\bar{q} = DivP + GradQ, \quad q_3 = divV - 2Q \qquad 5.12$$

where $P \in L^2(R^2) \cap \mathcal{M}_{3\times1}$, $Q \in L^2(R^2) \cap \mathcal{M}_{1\times1}$, and $V \in L^2(R^2) \cap \mathcal{M}_{2\times1}$. Div shows the divergence of a vector and $Grad$ represents the gradient. Moreover, the constants $c_1 > 0$ and $c_2 > 0$ are defined as

$$c_1 \parallel q \parallel_{-1,\omega;\Omega} \leq \parallel P \parallel_0 + \parallel Q \parallel_0 + \parallel V \parallel_0 \leq c_2 \parallel q \parallel_{-1,\omega;\Omega}$$

The proof of this theorem can be find in (Shmoylova, 2006).

In the following, it will be shown that equation (5.1) and equation (5.2), by using the area potential, can be reduced to similar problems for the homogeneous equilibrium.

In order to show this for equation (5.1), using theorem (5.3), it has been assumed that any $q \in H_{-1,\omega}(\Omega)$ can be represented in the form (5.12), where the range of the equality is defined on $S'(\Omega)$. Let $q \in H_{-1,\omega}(R^2)$ be defined by the same formula (5.12), in which the range of the equality is defined on $S'(R^2)$. Now, the solution of equation (5.1) can be represented in the form $u = U(-\hat{q}) + w$.

Since

$$b(U(-\hat{q}), v) = \langle \hat{q}, v \rangle_0 = \langle q, v \rangle_0$$

for

$$v \in H_{1,\omega}(\Omega)$$

it will be concluded that $w \in H_{1,\omega}(\Omega)$, which satisfies

$$b(w,v) = 0 \quad \forall v \in H_{1,\omega}(\Omega)$$

$$\gamma_0{}^+ w_+ = f^+ - \gamma_0{}^+(U(-\hat{q}))_+,\ \gamma_0{}^- w_- = f^- - \gamma_0{}^-(U(-\hat{q}))_-$$

Let γ_0 be the trace operator defined on $H_{1,\omega}(\Omega)$ by $\gamma_0 v = \{\gamma_0{}^+ v_+, \gamma_0{}^+ v_+ - \gamma_0{}^- v_-\}$. It can clearly be understood that γ_0 is continuous from $H_{1,\omega}(\Omega)$ to $H_{1/2}(\Gamma_0) \times \dot{H}_{1/2}$. In the following, the goal is finding $u \in H_{1,\omega}(\Omega)$ as a solution of problem (5.1), without the loss of generality, which will satisfy

$$b(u,v) = 0 \ \ \forall v \in H_{1,\omega}(\Omega), \quad \gamma_0 u = \{f^+, \delta f\} \qquad 5.13$$

On the other hand, in problem (5.2), the aim is to seek $u \in H_{1,\omega}(\Omega)$ such that

$$b(u,v) = \langle \tilde{q}, v\rangle_0, \ \ \forall v \in H_{1,\omega}(\Omega) \qquad 5.14$$

where $\tilde{q} \in H_{-1,\omega}(\Omega)$ was defined in Theorem (5.2) and satisfies

$$\langle \tilde{q}, z\rangle_0 = 0 \quad \forall z \in \mathcal{F} \qquad 5.15$$

Since $H_{1,\omega}(R^2)$ is a subspace of $H_{1,\omega}(\Omega)$, it is possible to consider $\tilde{q}$ which belongs to $H_{-1,\omega}(R^2)$. Moreover, from (5.14) it can be understood that $q \in H_{-1,\omega}(R^2)$. So, the solution of equation (5.14) can be represented in the form $u = U\tilde{q} + w$. Then equation (5.14) becomes

$$b(w,v) = \langle \tilde{q}, v\rangle_0 - b(U\tilde{q}, v) \quad \forall v \in H_{1,\omega}(\Omega)$$

Lemma 5.1 For all $\tilde{q} \in \dot{H}_{-1,\omega}(\Omega)$ satisfying equation (5.15), the expression

$$\mathcal{L}(\gamma_0 v) = \langle \tilde{q}, v\rangle_0 - b(U\tilde{q}, v), \ v \in H_{1,\omega}(\Omega) \qquad 5.16$$

defines a continuous linear functional on $H_{1/2}(\Gamma_0) \times \dot{H}_{1/2}$. Hence, $\mathcal{L}(\gamma_0 v)$ can be written in the form

$$\langle \tilde{q}, v \rangle_0 - b(U\tilde{q}, v) = \langle \delta g, \gamma_0{}^{+} v_{+} \rangle_{0;\Gamma_0} + \langle g^{-}, \delta v \rangle_{0;\Gamma_0}, \ \ v \in H_{1,\omega}(\Omega)$$

where $\{\delta g, g^{-}\} \in \dot{H}_{-1/2}(\Gamma_0) \times H_{-1/2}(\Gamma_0)$

Proof Let $v_1, v_2 \in H_{1,\omega}(\Omega)$ such that $\gamma_0 v_1 = \gamma_0 v_2$. The difference $(v^1 - v^2) \in H_{1,\omega}(\Omega) \subset H_{1,\omega}(R^2)$. Also, it is understandable that $b\big(U\tilde{q}, (v^1 - v^2)\big) = \langle q, (v^1 - v^2) \rangle_0$ Due to these definitions, the following equation can be concluded

$$\mathcal{L}(\gamma_0 v_1) = \mathcal{L}(\gamma_0 v_2)$$

This means that definition (5.16) of $\mathcal{L}$ on $H_{1/2}(\Gamma_0) \times \dot{H}_{1/2}$ is consistent.

Let $\{f^{+}, \delta f\} \in H_{1/2}(\Gamma_0) \times \dot{H}_{1/2}$. Repeating the proof of Theorem (5.1), this time by choosing $v \in H_{1,\omega}(\Omega)$ so that $\gamma_0 v = \{f^{+}, \delta f\}$ and $\| v \|_{1,\omega;\Omega} \leq c(\| f^{+} \|_{1/2;\Gamma_0} + \| \delta f \|_{1/2;\Gamma})$. It then will be concluded that

$$|\mathcal{L}(\{f^{+}, \delta f\})| \leq c \| \tilde{q} \|_{-1,\omega} \| v \|_{1,\omega;\Omega} \leq c \| \tilde{q} \|_{-1,\omega} (\| f^{+} \|_{1/2;\Gamma_0} + \| \delta f \|_{1/2;\Gamma}),$$

which shows that $\mathcal{L}$ is continuous on $H_{1/2}(\Gamma_0) \times \dot{H}_{1/2}(\Gamma_0)$. By definition, $H_{-1/2}(\Gamma_0) \times \dot{H}_{-1/2}(\Gamma_0)$ is the dual of $H_{1/2}(\Gamma_0) \times \dot{H}_{1/2}$, so the proof is completed.■

Lemma (5.1) implies that, without loss of generality, it is possible to consider equation 5.2 only for the homogeneous equilibrium equation. So, if the goal has been defined as seeking $u \in H_{1,\omega}(\Omega)$ such that

$$b(u, v) = \langle \delta g, \gamma_0{}^{+} v_{+} \rangle_{0;\Gamma_0} + \langle g^{-}, \delta v \rangle_{0;\Gamma_0} \quad \forall v \in H_{1,\omega}(\Omega) \qquad 5.17$$

equation (5.17) is solvable only if

$$\langle z, \delta g\rangle_{0;\Gamma_0} = 0 \quad \forall z \in \mathcal{F} \tag{5.18}$$

5.4 Poincaré--Steklov operator for the crack problem

For $F = \{f^+, \delta f\} \in H_{1/2}(\Gamma_0) \times \dot{H}_{1/2}(\Gamma_0)$ and $G = \{\delta g, g^-\} \in H_{-1/2}(\Gamma_0) \times \dot{H}_{-1/2}(\Gamma_0)$, the following notation has been used

$$[F, G]_{0;\Gamma_0} = \langle f^+, \delta g\rangle_{0;\Gamma_0} + \langle \delta f, g^-\rangle_{0;\Gamma_0}$$

Now the Poincaré-Steklov operator $\mathcal{T}$ can be defined on $H_{1/2}(\Gamma_0) \times \dot{H}_{1/2}(\Gamma_0)$ by

$$[\mathcal{T}F, \Psi]_{0;\Gamma_0} = b(u, v) \quad \forall \Psi \in H_{1/2}(\Gamma_0) \times \dot{H}_{1/2}(\Gamma_0) \tag{5.19}$$

$$F \in H_{1/2}(\Gamma_0) \times \dot{H}_{1/2}(\Gamma_0)$$

where u is the solution of equation (5.13) and v is any element in $H_{1,\omega}(\Omega)$ such that $\gamma_0 v = \Psi = \{\psi^+, \delta\psi\}$. The definition is independent of the choice of v. In particular, v might be taken as $v = l\Psi$, where l is an operator of extension from Γ_0 to Ω which maps $H_{1/2}(\Gamma_0) \times \dot{H}_{1/2}(\Gamma_0)$ continuously to $H_{1,\omega}(\Omega)$.

To continue, $\mathcal{F}$ will be identified with the subspace of $H_{1/2}(\Gamma_0) \times \dot{H}_{1/2}(\Gamma_0)$ consisting of all $Z = \{z, 0\}, z \in \mathcal{F}$. Also, the spaces $\widehat{\mathcal{H}}_{1/2}(\Gamma_0)$ and $\widehat{\mathcal{H}}_{-1/2}(\Gamma_0)$ will be introduced as

$$\widehat{\mathcal{H}}_{1/2}(\Gamma_0) = \{F \in H_{1/2}(\Gamma_0) \times \dot{H}_{1/2}(\Gamma_0) \colon \langle f^+, z\rangle_{0;\Gamma_0} = 0 \quad \forall z \in \mathcal{F}\},$$

$$\widehat{\mathcal{H}}_{-1/2}(\Gamma_0) = \{G \in H_{-1/2}(\Gamma_0) \times \dot{H}_{-1/2}(\Gamma_0) \colon \langle \delta g, z\rangle_{0;\Gamma_0} = 0 \quad \forall z \in \mathcal{F}\},$$

Theorem 5.4

i. $\mathcal{T}: H_{1/2}(\Gamma_0) \times \dot{H}_{1/2}(\Gamma_0) \to H_{-1/2}(\Gamma_0) \times \dot{H}_{-1/2}(\Gamma_0)$ is self-adjoint and continuous.

ii. The kernel of $\mathcal{T}$ coincides with $\mathcal{F}$.

iii. The range of $\mathcal{T}$ coincides with $\widehat{\mathcal{H}}_{-1/2}(\Gamma_0)$.

iv. The restriction $\mathcal{N}$ of $\mathcal{T}$ to $\widehat{\mathcal{H}}_{1/2}(\Gamma_0)$ is a homeomorphism from $\widehat{\mathcal{H}}_{1/2}(\Gamma_0)$ to $\widehat{\mathcal{H}}_{-1/2}(\Gamma_0)$.

Proof

i. If u is the solution of equation 5.13, and $v = l\Psi$, then, by the definition of $\mathcal{T}$, for $F, \Psi \in H_{1/2}(\Gamma_0) \times \dot{H}_{1/2}(\Gamma^0)$

$$|[\mathcal{T}F, \Psi]|^2 = |b(u,v)|^2 \leq b(u,u)b(v,v) \leq cb(u,u) \parallel \Psi \parallel^2_{H_{1/2}(\Gamma_0)\times \dot{H}_{1/2}(\Gamma^0)}$$

Consequently

$$\mathcal{T}F \in H_{-1/2}(\Gamma_0) \times \dot{H}_{-1/2}(\Gamma_0)$$

and

$$\|\mathcal{T}f\|^2_{H_{-1/2}(\Gamma_0)\times \dot{H}_{-1/2}(\Gamma_0)} \leq cb(u,u) = c[\mathcal{T}F, F]_{0;\Gamma_0} \qquad 5.20$$

$$\leq c\|\mathcal{T}f\|_{H_{-1/2}(\Gamma_0)\times \dot{H}_{-1/2}(\Gamma_0)} \|F\|_{H_{1/2}(\Gamma_0)\times \dot{H}_{1/2}(\Gamma_0)}$$

From equation (5.20) it follows that

$$\|\mathcal{T}F\|_{H_{-1/2}(\Gamma_0)\times \dot{H}_{-1/2}(\Gamma_0)} \leq c\|F\|_{H_{-1/2}(\Gamma_0)\times \dot{H}_{-1/2}(\Gamma_0)} \qquad 5.21$$

which proves the continuity of $\mathcal{T}$. The definition of $\mathcal{T}$ shows that it is self-adjoint in the sense that

$[\mathcal{T}F,\Psi]_{0;\Gamma_0} = [\Psi,\mathcal{T}F]_{0;\Gamma_0} \quad \forall F,\Psi \in H_{1/2}(\Gamma_0) \times \dot{H}_{1/2}(\Gamma_0)$

ii. It is clear that $\mathcal{T}Z = 0$ for $Z \in \mathcal{F}$. If $F \in H_{1/2}(\Gamma_0) \times \dot{H}_{1/2}(\Gamma_0)$, then $\mathcal{T}F = 0$. Also, u is the solution of equation (5.13) As a result, it can be concluded that $b(u,u) = 0$. Hence, $u \in \mathcal{F}$, which implies that $F = \gamma_0 u \in \mathcal{F}$. This also proves that $\mathcal{N}$ is injective.

iii. By equation (5.21), it can be understood that the range of $\mathcal{T}$ is a subset of $\widehat{\mathcal{H}}_{-1/2}(\Gamma_0)$. Let $\{\tilde{z}^{(i)}\}_{i=1}^{3}$ be an $L^2(\Gamma_0)$-orthonormal basis for$\mathcal{F}$. From the previous explanations it follows that any $u \in H_{1,\omega}(\Omega)$ satisfies

$$\|u\|_{1,\omega;\Omega}^2 \le c\left[b(u,u) + \textstyle\sum_{i=1}^{3}\langle \gamma^{0+}u^+, z^{(i)}\rangle_{0;\Gamma^0}^2\right] \qquad 5.22$$

Let $F \in \widehat{\mathcal{H}}_{-1/2}(\Gamma_0)$. By the trace theorem from (Chudinovich, et al., 2000), and equation (5.22)

$$\|F\|_{H_{1/2}(\Gamma_0)\times\dot{H}_{1/2}(\Gamma_0)}^2 \le c\|u\|_{1,\omega;\Omega}^2 \le cb(u,u) = c[\mathcal{T}F,F]_{0;\Gamma_0}$$

hence,

$$\|F\|_{H_{1/2}(\Gamma_0)\times\dot{H}_{1/2}(\Gamma_0)} \le c\|\mathcal{T}F\|_{H_{-1/2}(\Gamma_0)\times\dot{H}_{-1/2}(\Gamma_0)}$$

which shows that $\mathcal{N}^{-1}$ is continuous.

When the range of $\mathcal{T}$ is not dense in $\widehat{\mathcal{H}}_{-1/2}(\Gamma_0)$, then there is a nonzero $\hat{F}$ in the dual $\left[H_{1/2}(\Gamma^0) \times \dot{H}_{1/2}(\Gamma^0)\right]\backslash\mathcal{F}$ of $\widehat{\mathcal{H}}_{-1/2}(\Gamma_0)$ such that

$$\langle \mathcal{T}F,\Psi\rangle_{0;\Gamma_0} = 0$$

for all representatives F of the class $\hat{F}$ and all $\Psi \in H_{1/2}(\Gamma_0) \times \dot{H}_{1/2}(\Gamma_0)$.

Now by taking $F \in \widehat{\mathcal{H}}_{1/2}(\Gamma_0)$ and $\Psi = F$, it will be found that

$$[\mathcal{T}F,F]_{0;\Gamma_0} = 0$$

therefore, $F \in \mathcal{F}$ and $\hat{F} = 0$. This contradiction proves the third statement.

iv. This claim has been proved in the las ones.■

5.5 Boundary equations

Here the single and double layer potentials on the crack will be defined by

$$(V\varphi)(x) = \int_{\Gamma_0} \int D(x,y)\varphi(y)\, ds(y)$$

$$(W\varphi)(x) = \int_{\Gamma_0} \int P(x,y)\varphi(y)\, ds(y)$$

Let $\dot{\mathcal{H}}_{-1/2}(\Gamma_0)$ be the subspace of $\dot{H}_{-1/2}(\Gamma_0)$ of all g such that $\langle g, z\rangle_{0;\Gamma_0} = 0$ for all $z \in \mathcal{F}$.

The modified single layer potential V of density $\varphi \in \dot{\mathcal{H}}_{-1/2}(\Gamma_0)$ can be defined by

$$(\mathcal{V}\varphi)(x) = (V\varphi)(x) - \langle (V\varphi)_0, z^{\sim(i)}\rangle_{0;\Gamma_0} z^{\sim(i)}(x),\ \ x \in R^2$$

where $V\varphi$ is the single layer potential, and V_0 is the boundary operator defined by

$$(V\varphi)_0 = \gamma_0^{\pm}\pi^{\pm}V\varphi$$

Let $\mathcal{V}_0\varphi$ be the operator defined on $\dot{\mathcal{H}}_{-1/2}(\Gamma_0)$ by

$$\varphi \to (\mathcal{V}\varphi)^0 = \gamma_0^{\pm}\pi^{\pm}\mathcal{V}\varphi$$

From the results established in (Shmoylova, 2006), $\mathcal{V}_0$ is continuous from $\dot{\mathcal{H}}_{-1/2}(\Gamma_0)$ to the subspace $\mathcal{H}_{-1/2}(\Gamma_0)$ of all $f^{+} \in H\_\{1/2\}(\Gamma_0)$ such that $< f^{+}, z > \_\{0; \Gamma_0\} = 0$ for all $z \in F$. Let $\tilde{\mathcal{V}}$ be the continuous operator from $\dot{\mathcal{H}}_{-1/2}(\Gamma_0)$ to $\widehat{\mathcal{H}}_{-1/2}(\Gamma_0)$ defined by $\tilde{\mathcal{V}}\varphi = \{\mathcal{V}_0\varphi, 0\}$.

Theorem 5.5 The operator $\mathcal{V}_0$ is a homeomorphism from $\dot{\mathcal{H}}_{-1/2}(\Gamma_0)$ to $\mathcal{H}_{-1/2}(\Gamma_0)$.

Proof The continuity of $\mathcal{V}_0$ has been proved in (Shmoylova, 2006). From the jump formula for the normal boundary stresses and couple stresses of the single layer potential it follows that the first component of $\mathcal{N}\tilde{\mathcal{V}}\varphi \in \hat{\mathcal{H}}_{-1/2}(\Gamma_0)$ is φ. By Theorem (5.4),

$$\|\varphi\|_{-1/2;\Gamma_0} \leq \left\|\mathcal{N}\tilde{\mathcal{V}}\varphi\right\|_{H_{-1/2}(\Gamma_0)\times\dot{H}_{-1/2}(\Gamma_0)}$$

$$\leq c\left\|\tilde{\mathcal{V}}\varphi\right\|_{H_{1/2}(\Gamma_0)\times\dot{H}_{1/2}(\Gamma_0)} = c\|\mathcal{V}_0\varphi\|_{1/2;\Gamma_0}$$

which shows that ${\mathcal{V}_0}^{-1}$ is continuous.

Now, it can be claimed that the range of $\mathcal{V}_0$ is $\mathcal{H}_{1/2}(\Gamma_0)$. Let $f^+ \in \mathcal{H}_{1/2}(\Gamma_0)$ and $F = \{f^+, 0\} \in \hat{\mathcal{H}}_{-1/2}(\Gamma_0)$, and let $u \in H_{1,\omega}(\Omega)$ be the solution of equation (5.13) with $\delta f = 0$. The G and φ will be taken as

$$G = \{\delta g, g^-\} = \mathcal{N}F \in \hat{\mathcal{H}}_{-1/2}(\Gamma_0)$$

$$\varphi = \delta g \in \dot{\mathcal{H}}_{-1/2}(\Gamma_0)$$

On the other hand, there is a w which $w = u - \mathcal{V}_0\varphi$ and satisfies

$$\gamma_0 w = \{f^+ - \mathcal{V}_0\varphi, 0\} = \Psi$$

By the jump formula, the first component of $\mathcal{N}\Psi$ is zero. It can be resulted that $b(w,w) = [\mathcal{N}\Psi, \Psi]_{0;\Gamma_0} = 0$. This conclusion means that $w \in \mathcal{F}$, so ${\gamma_0}^+ w_+$ is a rigid displacement on Γ_0. Since ${\gamma_0}^+ w_+ = f^+ - \mathcal{V}_0\varphi \in \mathcal{H}_{1/2}(\Gamma_0)$, it can be concluded that, $f^+ = \mathcal{V}_0\varphi$, and the assertion is proved.■

For the double layer potential, the modified double layer potential can be introduce by $\mathcal{W}$ of density $\psi \in \dot{H}_{1/2}(\Gamma_0)$, which will satisfy

$$(\mathcal{W}\psi)(x) = (W\psi)(x) - \langle \pi_0 W^{+}\psi, z\text{^}\{(i)\} \rangle_{0;\Gamma_0} z^{\sim(i)}(x),\ x \in \Omega$$

Clearly, if $\psi \in \dot{H}_{1/2}(\Gamma_0)$, then $W\psi \in H_{1,\omega}(\Omega)$ and $\|W\psi\|_{1,\omega;\Omega} \leq c\|\psi\|_{1/2;\Gamma}$. Hence, for $\psi \in \dot{H}_{1/2}(\Gamma_0)$, the operators $\mathcal{W}^{\pm}$ has been defined. These operators are of the limiting values of the modified double layer potential on Γ from within $\Omega^{\pm}$, which can be shown by writing $\mathcal{W}^{\pm}\psi = \gamma^{\pm}\pi^{\pm}\mathcal{W}\psi$. It is obvious that all $\mathcal{W}^{\pm}$ are continuous from $\dot{H}_{1/2}(\Gamma_0)$ to $H_{1/2}(\Gamma)$ and satisfy the jump formula

$$\mathcal{W}^{+}\psi - \mathcal{W}^{-}\psi = -\psi \qquad 5.23$$

For $\psi \in \dot{H}_{1/2}(\Gamma_0)$, the operator $\mathcal{W}_0$ of the limiting values of the modified double layer potential on Γ_0 from within Ω, can be defined by writing

$$\mathcal{W}_0\psi = \{\pi_0\mathcal{W}^{+}\psi, \pi_0(\mathcal{W}^{+}\psi - \mathcal{W}^{-}\psi)\} = \{\pi_0\mathcal{W}^{+}\psi, -\psi\}$$

Clearly, $\mathcal{W}_0$ is continuous from $\dot{H}_{1/2}(\Gamma_0)$ to $\widehat{\mathcal{H}}_{1/2}(\Gamma_0)$.

Let $\tilde{\mathcal{G}} = \mathcal{N}\mathcal{W}_0$. From the jump formula for the normal boundary stresses and couple stresses of the double layer potential it follows that the first component of $\tilde{\mathcal{G}}\psi$ is zero for any $\psi \in \dot{H}_{1/2}(\Gamma_0)$. Hence, as a result, it can be written that $\tilde{\mathcal{G}}\psi = \{0, \mathcal{G}\psi\}$ for all $\psi \in \dot{H}_{1/2}(\Gamma_0)$.

Theorem 5.6 $\mathcal{G}$ is a homeomorphism from $\dot{H}_{1/2}(\Gamma_0)$ to $H_{-1/2}(\Gamma_0)$.

Proof The continuity of $\mathcal{G}$ follows from the properties of $\mathcal{W}_0$ and $\mathcal{N}$. The claim is that $\mathcal{G}^{-1}$ is continuous. Let $\psi \in \dot{H}_{1/2}(\Gamma_0)$. By equation (5.23) and the trace theorem (Chudinovich, et al., 2000), it can be concluded

$$\|\psi\|^2_{1/2;\Gamma} = \|\mathcal{W}^{+}\psi - \mathcal{W}^{-}\psi\|^2_{1/2;\Gamma} \leq c\|\mathcal{W}\psi\|^2_{1,\omega;\Omega}$$

$\leq\ cb(\mathcal{W}\psi, \mathcal{W}\psi) = -c\langle \mathcal{G}\psi, \psi\rangle_{0;\Gamma_0}$

$\leq\ c\|\mathcal{G}\psi\|_{-1/2;\Gamma_0}\|\psi\|_{1/2;\Gamma}$

consequently, $\|\psi\|_{1/2;\Gamma} \leq c\|\mathcal{G}\psi\|_{-1/2;\Gamma_0}$. If the range of $\mathcal{G}$ is not dense in $H_{-1/2}(\Gamma_0)$, then there is a nonzero ψ in the dual $\dot{H}_{1/2}(\Gamma_0)$ such that $\langle \psi, \mathcal{G}\xi\rangle_{0;\Gamma_0} = 0$ for all $\xi \in \dot{H}_{1/2}(\Gamma_0)$. So it is assumed that $\xi = \psi$, hence, it will be obtained that $\langle \psi, \mathcal{G}\psi\rangle_{0;\Gamma_0} = 0$. This result means that $\mathcal{W}\psi \in \mathcal{F}$. Therefore, $\psi = \mathcal{W}^{-}\psi - \mathcal{W}^{+}\psi = 0$. And it can be said that this contradiction completes proof.■

Now the solution of equation (5.13) will be represented in the form

$$u = (\mathcal{V}\varphi)_\Omega + W\psi + z \qquad 5.24$$

where $\varphi \in \dot{\mathcal{H}}_{-1/2}(\Gamma_0)$ and $\psi \in \dot{H}_{1/2}(\Gamma_0)$ are unknown densities, $(\mathcal{V}\varphi)_\Omega$ is the restriction of $\mathcal{V}\phi$ to Ω and

$$z = \langle f^{+} - \pi_0 W^{+}\psi, z^{(i)}\rangle_{0;\Gamma_0} z^{\sim(i)}$$

Representation of equation (5.24) leads to the system of boundary equations

$$\{\mathcal{V}_0\phi + \pi_0 W^{+}\psi + \gamma_0{}^{+}z, -\psi\} = \{f^{+}, \delta f\} \qquad 5.25$$

Theorem 5.7 For any $\{f^{+}, \delta f\} \in H_{1/2}(\Gamma_0) \times \dot{H}_{1/2}(\Gamma_0)$, system (5.25) has a unique solution

$$\{\varphi, \psi\} \in \dot{\mathcal{H}}_{-1/2}(\Gamma_0) \times \dot{H}_{1/2}(\Gamma_0)$$

respectively, and

$$\|\{\varphi, \psi\}\|_{\dot{H}_{-1/2}(\Gamma_0)\times \dot{H}_{1/2}(\Gamma_0)} \leq c\|\{f^{+}, \delta f\}\|_{H_{1/2}(\Gamma_0)\times \dot{H}_{1/2}(\Gamma_0)}$$

In this case, equation (5.24) is the solution of problem (5.13).

Proof From system (5.25) $\psi = -\delta f \in \dot{H}_{1/2}(\Gamma_0)$, consequently the equation for φ becomes

$$\mathcal{V}_0\varphi = f^+ + \pi_0 W^+\delta f - \langle f^+ + \pi_0 W^+\delta f, z^{\sim(i)}\rangle_{0;\Gamma_0} z^{\sim(i)}\} \qquad 5.26$$

The right-hand side in equation (5.26) belongs to $\mathcal{H}_{1/2}(\Gamma_0)$. By referring to Theorem (5.6), equation (5.26) has a unique solution $\varphi \in \dot{\mathcal{H}}_{1/2}(\Gamma_0)$ and

$$\|\varphi\|_{-1/2;\Gamma} \le c\left(\|f^+\|_{\frac{1}{2};\Gamma_0} + \|\pi_0 W^+\delta f\|_{\frac{1}{2};\Gamma_0}\right)$$

$$\le c\left(\|f^+\|_{\frac{1}{2};\Gamma_0} + \|\delta f\|_{\frac{1}{2};\Gamma}\right) = c\|\{f^+,\delta f\}\|_{H_{1/2}(\Gamma_0)\times\dot{H}_{1/2}(\Gamma_0)}$$

■

In the next attempt, the solution of problem (5.17) will be represented in the form

$$u = (\mathcal{V}\varphi)\_\{\Omega\} + \mathcal{W}\psi + z \qquad 5.27$$

where $\varphi \in \dot{\mathcal{H}}_{1/2}(\Gamma_0)$ and $\psi \in \dot{H}_{1/2}(\Gamma_0)$ are unknown densities, and $z \in \mathcal{F}$ is arbitrary. Representation of equation (5.27) leads to the systems of boundary equations

$$\mathcal{N}\tilde{\mathcal{V}}\varphi + \tilde{\mathcal{G}}\psi = \{\delta g, g^-\} \qquad 5.28$$

Theorem 5.8 For any $\{\delta g, g^-\} \in H_{-1/2}(\Gamma_0) \times \dot{H}_{-1/2}(\Gamma_0)$ satisfying equation (5.18), system (5.28) has a unique solution $\{\varphi,\psi\} \in \dot{\mathcal{H}}_{1/2}(\Gamma_0) \times \dot{H}_{1/2}(\Gamma_0)$ and

$$\|\{\varphi,\psi\}\|_{\dot{H}_{-1/2}(\Gamma_0)\times\dot{H}_{1/2}(\Gamma_0)} \le c\|\{\delta g, g^-\}\|_{H_{-1/2}(\Gamma_0)\times\dot{H}_{-1/2}(\Gamma_0)}$$

In this case, equation (5.27) is the solution of problem (5.17).

Proof Comparing first components on both sides of equation (5.28), it can be seen that $\varphi = \delta g$. Hence, equation (5.28) takes form

$$\mathcal{G}\psi = g^{-} - (\mathcal{N}\tilde{\mathcal{V}}\delta g)^{-} \qquad 5.29$$

where $\left(\mathcal{N}\tilde{\mathcal{V}}\delta g\right)^{-}$ is the second component of $\mathcal{N}\tilde{\mathcal{V}}\delta g$. By Theorems (5.6), and (5.7), equation (5.28) has a unique solution $\psi \in \dot{H}_{1/2}(\Gamma_0)$ and

$$\|\psi\|_{1/2;\Gamma} \le c\left(\|g^{-}\|_{-\frac{1}{2};\Gamma_0} + \|\delta g\|_{-\frac{1}{2};\Gamma}\right)$$

■

5.6 The boundary equations for a finite domain

Assuming ∂S is a simple closed C^2-curve that divides R^2 into interior and exterior domains S^+ and S^-. It has been undertaken that S^+ holds inside an auxiliary simple closed C^2-curve $\Gamma = \Gamma_0 \cup \Gamma_1$, where Γ_0 is an open arc modeling crack. Then it can be written that $\Omega = S^+ \backslash \Gamma_0$. Let Ω^+ be the interior domain bounded by Γ, and $\Omega^- = S^+ \backslash \Omega^+$.

If u is defined in Ω, then its restrictions to Ω^+ and Ω^- will be represented by u_+ and u_-, respectively. So it can be shown as $u = \{u_+, u_-\}$. The spaces $H_1(\Omega^{\pm})$ are introduced in the usual way. The traces of the elements $u_{\pm} \in H_1(\Omega^{\pm})$ on Γ are denoted by $\gamma^+ u_+$ and $\gamma^- u_-$.

Also, the operators of restrictions from Γ to Γ_i will be represented by $\pi_i, i = 0,1$. So

$$\gamma_i^{\pm} = \pi_i \gamma^{\pm} \quad , i = 0,1$$

The space $H_1(\Omega)$ consists of all $u = \{u_+, u_-\}$ defined in Ω and such that

$$u_+ \in H_1(\Omega^+), \qquad u_- \in H_1(\Omega^-)$$

and

$$\gamma_1{}^{+} u_+ = \gamma_1{}^{-} u_-$$

The norm in $H_1(\Omega)$ is defined by

$\|u\|^2_{1;\Omega} = \|u_+\|^2_{1;\Omega^+} + \|u_-\|^2_{1;\Omega^-}$

Let γ_0 be the trace operator that acts on $u \in H_1(\Omega)$ according to the formula $\gamma_0 u = \{\gamma_0{}^+u_+, \gamma_0{}^+u_+ - \gamma_0{}^-u_-\}$. Clearly, γ_0 is continuous from $H_1(\Omega)$ to $H_{1/2}(\Gamma_0) \times \dot{H}_{1/2}(\Gamma_0)$. The trace of $u \in H_1(\Omega)$ on ∂S is denoted by $\gamma^+_{\partial S}u$. $\dot{H}_1(\Omega)$ is the subspace of $H_1(\Omega)$ consisting of all $u \in H_1(\Omega)$ such that $\gamma_0 u = \{0,0\}$ and $\gamma^+_{\partial S}u = 0$.

Let $\Gamma = \Gamma_0 \cup \partial S$. In what follows the spaces $H_{1/2}(\hat{\Gamma}) = H_{1/2}(\Gamma_0) \times \dot{H}_{1/2}(\Gamma_0) \times H_{1/2}(\partial S)$ of all $\hat{F} = \{F, f_{\partial S}\}$ will be used. In the aforementioned spaces $F = \{f^+, \delta f\}$, and $H_{-1/2}(\hat{\Gamma}) = H_{-1/2}(\Gamma_0) \times \dot{H}_{-1/2}(\Gamma_0) \times H_{-1/2}(\partial S)$ of all $\hat{G} = \{G, g_{\partial S}\}$, where $G = \{\delta g, g^-\}$. It is clear that these spaces are dual with respect to the duality $[F,G]_{0;\Gamma} = [F,G]_{0;\Gamma_0} + \langle f_{\partial S}, g_{\partial S}\rangle_{0;\partial S}$. This duality is generated by the inner product $[\cdot,\cdot]_{0;\Gamma}$ in $L^2(\Gamma) = L^2(\Gamma_0) \times L^2(\Gamma_0) \times L^2(\partial S)$.

the following boundary value problems has been considered.

Assuming $F = \{F, f_{\partial S}\} \in H_{1/2}(\hat{\Gamma})$, the aim is to seek $u \in H_1(\Omega)$ such that

$$b_\Omega(u,v) = 0 \quad \forall v \in \dot{H}_1(\Omega), \quad \gamma_0 u = F,\ \gamma^+_{\partial S}u = f_{\partial S} \qquad 5.30$$

where

$$b_\Omega(u,v) = \int_\Omega E(u,v)dx.$$

Assuming $\hat{G} = \{G, g_{\partial S}\} \in H_{1/2}(\hat{\Gamma})$, the aim is to seek $u \in H_1(\Omega)$ such that

$$b_\Omega(u,v) = [G, \gamma_0 v]_{0;\Gamma_0} + \langle g_{\partial S}, \gamma_{\partial S} v\rangle_{0;\partial S}, \quad \forall v \in H_1(\Omega) \qquad 5.31$$

Clearly, equation (5.31) is solvable only if

$\langle \delta g, z\rangle_{0;\Gamma_0} + \langle g_{\partial S}, z\rangle_{0;\partial S} = 0, \ \ \forall z \in \mathcal{F}$ *5.32*

In what follows it has been assumed that equation (5.32) holds. The proofs of the unique solvability of equations (5.30) and (5.31) repeat those of Theorems (5.1) and (5.2) with the obvious changes, so they will be omitted.

The Poincaré-Steklov operator $\tilde{\mathcal{T}}$ by $[\tilde{\mathcal{T}}\hat{F}, \hat{\Psi}]_{0;\Gamma} = b_{\Omega}(u, v)$, where $\hat{F}, \hat{\Psi} \in H_{1/2}(\hat{\Gamma})$ are arbitrary. $v \in H_1(\Omega)$ is any extension of $\hat{\Psi}$ to Ω. Let $\mathcal{F}(\hat{\Gamma})$ be the space of all $\hat{Z} = \{Z, z\}$, $Z = \{z, 0\}$, where $z \in \mathcal{F}$ is arbitrary. The spaces below then defined as

$$\mathcal{H}_{1/2}(\hat{\Gamma}) = \{\hat{F} \in H_{1/2}(\hat{\Gamma}) : [F, Z]_{0;\Gamma} = 0 \ \ \forall Z \in \mathcal{F}(\hat{\Gamma})\}$$

$$\mathcal{H}_{-1/2}(\hat{\Gamma}) = \{G \in H_{-1/2}(\hat{\Gamma}) : [G, Z]_{0;\Gamma} = 0 \ \ \forall Z \in \mathcal{F}(\hat{\Gamma})\}$$

Theorem 5.9

i. $\tilde{\mathcal{T}}$ is self-adjoint and continuous from $H_{\frac{1}{2}}(\hat{\Gamma})$ to $H_{-\frac{1}{2}}(\hat{\Gamma})$.

ii. The kernel of T coincides with $\mathcal{F}(\hat{\Gamma})$.

iii. The range of $\tilde{\mathcal{T}}$coincides with $\mathcal{H}_{-1/2}(\hat{\Gamma})$.

iv. The restriction $\hat{\mathcal{N}}$ of $\tilde{\mathcal{T}}$ from $H_{\frac{1}{2}}(\hat{\Gamma})$ to $\mathcal{H}_{1/2}(\hat{\Gamma})$ is a homeomorphism from $\mathcal{H}_{1/2}(\hat{\Gamma})$ to $\mathcal{H}_{-1/2}(\hat{\Gamma})$.

Proof the proof of this theorem is same at theorem 5.5. ■

Let $\dot{\mathcal{H}}_{-1/2}(\hat{\Gamma})$ be the subspace of $\dot{H}_{-1/2}(\Gamma_0) \times H_{-1/2}(\partial S)$ of all $\varphi = \{\varphi_0, \varphi_{\partial S}\}$ such that $\langle \varphi_0, z\rangle_{0;\Gamma_0} + \langle \varphi_{\partial S}, z\rangle_{0;\partial S} = 0$ for all $z \in \mathcal{F}$. $\hat{\mathcal{H}}_{1/2}(\hat{\Gamma})$ is the subspace of $H_{1/2}(\Gamma_0) \times H_{1/2}(\partial S)$ consisting of all $f = \{f^{+}, f_{\partial S}\}$ such that $\langle f^{+}, z\rangle_{0;\Gamma_0} + \langle f_{\partial S}, z\rangle_{0;\partial S} = 0$ for all $z \in \mathcal{F}$.

The single layer potential of density $\varphi \in \dot{\mathcal{H}}_{-1/2}(\hat{\Gamma})$ has been defined by

$$(V\varphi)(x) = (V_0\varphi_0)(x) + (V_{\partial S}\varphi_{\partial S})(x), \quad x \in R^2$$

where $V_0\varphi_0$ and $V_{\partial S}\varphi_{\partial S}$ are the single layer potentials defined on Γ_0 and ∂S, respectively. Let $\{\hat{Z}^{(i)}\}_{i=1}^3$ be an $L^2(\hat{\Gamma})$-orthonormal basis for $\mathcal{F}(\hat{\Gamma})$, where $\hat{Z}^{(i)} = \{Z^{(i)}, z^{(i)}\}$ and $Z^{(i)} = \{z^{(i)}, 0\}$. The rigid displacements $z^{(i)}$ satisfies equation (5.30) with boundary data $F = \hat{Z}^{(i)}, f_{\partial S} = z^{(i)}$. The modified single layer potential can be introduced as

$$(\mathcal{V}\varphi)(x) = (V\varphi)(x) - \left[\langle (V\varphi)_0, z^{(i)}\rangle_{0;\Gamma_0} + \langle (V\varphi)_{\partial S}, z^{(i)}\rangle_{0;\partial S}\right] z^{(i)}(x), \; x \in R^2$$

where $(V\varphi)_0$ and $(V\varphi)_{\partial S}$ are the restrictions of $V\varphi$ to Γ_0 and ∂S. The corresponding boundary operator $\mathcal{V}_{\hat{\Gamma}}$ is defined by $\mathcal{V}_{\hat{\Gamma}}\varphi = \{\gamma_0^+(\mathcal{V}\varphi)_+, \gamma_{\partial S}^+(\mathcal{V}\varphi)_\Omega\}$, where $(\mathcal{V}\varphi)_\pm$ are the restrictions of $V\phi \; to \; \Omega^\pm$. Moreover, a boundary operator $\overset{\frown\frown}{\mathcal{V}}$ has been defined which satisfies

$$\hat{\mathcal{V}}\varphi = \{\gamma_0^+(\mathcal{V}\varphi)_+, \gamma_{\partial S}^+(\mathcal{V}\varphi)_\Omega\}.$$

Theorem 5.10 $\mathcal{V}_{\hat{\Gamma}}$ is a homeomorphism from $\dot{\mathcal{H}}_{-1/2}(\hat{\Gamma})$ to $\hat{\mathcal{H}}_{1/2}(\hat{\Gamma})$.

Proof From the properties of the single layer potential it follows that

$$(\mathcal{V}\varphi)_\Omega \in H_1(\Omega), \qquad (\mathcal{V}\varphi)_{S^-} \in H_{1,\omega}(S^-)$$

and

$$\|(\mathcal{V}\varphi)_\Omega\|_{1;\Omega}^2 \le c b_\Omega((\mathcal{V}\varphi)_\Omega, (\mathcal{V}\varphi)_\Omega) \qquad 5.33$$

Here $(\mathcal{V}\varphi)_{S^-}$ is the restriction of $\mathcal{V}\varphi$ to S^-. By using the properties of the Poincaré-Steklov operator $\mathcal{T}^-$, and knowing that for any $\varphi = \{\varphi_0, \varphi_{\partial S}\} \in \dot{\mathcal{H}}_{-1/2}(\hat{\Gamma})$, the jump formula for normal boundary stresses and couple stresses can be written as

$$\varphi_0 = (\widehat{\mathcal{N}}\widehat{\mathcal{V}}\varphi)_1, \quad \varphi_{\partial S} = (\widehat{\mathcal{N}}\widehat{\mathcal{V}}\varphi)_3 - \mathcal{T}^-(\mathcal{V}_{\hat{\Gamma}}\varphi)_2 \qquad 5.34$$

where $(\widehat{\mathcal{N}}\widehat{\mathcal{V}}\varphi)_i$ are the components of $\widehat{\mathcal{N}}\widehat{\mathcal{V}}\varphi$ and $(\mathcal{V}_{\hat{\Gamma}}\varphi)_\alpha$ are the components of $\mathcal{V}_{\hat{\Gamma}}\varphi$. From equation (5.34) it follows that

$$b_\Omega((\mathcal{V}\varphi)_\Omega, (\mathcal{V}\varphi)_\Omega) + b_{S^-}((\mathcal{V}\varphi)_{S^-}, (\mathcal{V}\varphi)_{S^-}) = \langle(\mathcal{V}_{\hat{\Gamma}}\varphi)_1, \varphi_0\rangle_{0;\Gamma_0} + \langle(\mathcal{V}_{\hat{\Gamma}}\varphi)_2, \varphi_{\partial S}\rangle_{0;\partial S} \qquad 5.35$$

It can be claimed that $\mathcal{V}_{\hat{\Gamma}}$ is continuous.

Let $\varphi = \{\varphi_0, \varphi_{\partial S}\} \in \dot{\mathcal{H}}_{-1/2}(\hat{\Gamma})$. By the trace theorem (Chudinovich, et al., 2000), $\|\mathcal{V}_{\hat{\Gamma}}\varphi\|^2_{H_{1/2}(\Gamma_0)\times H_{1/2}(\partial S)} \leq c\|(\mathcal{V}\varphi)_\Omega\|_{1;\Omega}$. By equation (5.33) and equation (5.35)

$$\|\mathcal{V}_{\hat{\Gamma}}\varphi\|^2_{H_{1/2}(\Gamma_0)\times H_{1/2}(\partial S)} \leq c[\langle(\mathcal{V}_{\hat{\Gamma}}\varphi)_1, \varphi_0\rangle_{0;\Gamma_0} + \langle(\mathcal{V}_{\hat{\Gamma}}\varphi)_2, \varphi_{\partial S}\rangle_{0;\partial S}]$$

$$\leq c\|\mathcal{V}_{\hat{\Gamma}}\varphi\|_{H_{1/2}(\Gamma_0)\times H_{1/2}(\partial S)}\|\varphi\|_{\dot{H}_{-1/2}(\Gamma_0)\times H_{-1/2}(\partial S)}$$

consequently,

$$\|\mathcal{V}_{\hat{\Gamma}}\varphi\|_{H_{1/2}(\Gamma_0)\times H_{1/2}(\partial S)} \leq c\|\varphi\|_{\dot{H}_{-1/2}(\Gamma_0)\times H_{-1/2}(\partial S)}$$

which proves the continuity of $\mathcal{V}_{\hat{\Gamma}}$.

If $\mathcal{V}_{\hat{\Gamma}}\varphi = 0$, then $\widehat{\mathcal{V}}\varphi = 0$ also, and equation (5.34) gives that $\varphi = 0$; therefore, $\mathcal{V}_{\hat{\Gamma}}$ is injective. By equation (5.34)

$$\|\varphi\|_{\dot{H}_{-1/2}(\Gamma_0)\times H_{-1/2}(\partial S)} \leq c\|\mathcal{V}_{\hat{\Gamma}}\varphi\|_{H_{1/2}(\Gamma_0)\times H_{1/2}(\partial S)}$$

which means that $\mathcal{V}_{\hat{\Gamma}}^{-1}$ is continuous.

To complete the proof, it sufficient to show that the range of $\mathcal{V}_{\hat{\Gamma}}$ is $\widehat{\mathcal{H}}_{1/2}(\hat{\Gamma})$.

Let

$f = \{f^{+}, f_{\partial S}\} \in \hat{\mathcal{H}}_{1/2}(\hat{\Gamma})$

and

$F = \{f^{+}, 0\} \in \dot{H}_{1/2}(\Gamma_0) \times H_{1/2}(\Gamma_0)$

and let

$\hat{F} = \{F, f_{\partial S}\} \in H_{1/2}(\hat{\Gamma})$

the solution of equation (5.30) will be represented by $u_{\Omega} \in H_1(\Omega)$ and the solution of problem below will be denoted by $u_{S^-} \in H_{1,\omega}(S^-)$

$b_{S^-}(u_{S^-}, v_{S^-}) = 0 \quad \forall v_{S^-} \in \dot{H}_{1,\omega}(S^-),\ \ \gamma_{S^-}^{-} u_{S^-} = f_{\partial S}$

Let

$\tilde{\mathcal{T}}\hat{\mathcal{V}} = \hat{G} = \{\delta g, g^{-}, g_{\partial S}^{+}\}$

and let

$\mathcal{T}^{-} f_{\partial S} = g_{\partial S}^{-}$

By taking $\varphi_0 = \delta g$, $\varphi_{\partial S} = g_{\partial S}^{+} - g_{\partial S}^{-}$, and $\varphi = \{\varphi_0, \varphi_{\partial S}\}$, it is possible to write

$w_{\Omega} = u_{\Omega} - (\mathcal{V}\varphi)_{\Omega} \in H_1(\Omega)$

and

$w_{S^-} = u_{S^-} - (\mathcal{V}\varphi)_{S^-} \in H_{1,\omega}(S^-)$

Then

$\gamma_0 w_{\Omega} = \{f^{+} - \gamma_0{}^{+}(\mathcal{V}\varphi)_{+}, 0\}$

$$\gamma_{\partial S}^{+} w_{\Omega} \ = \ f_{\partial S} - \gamma_{\partial S}^{+}(\mathcal{V}\varphi)_{\Omega}$$

$$\gamma_{\partial S}^{-} w_{\partial S} \ = \ f_{\partial S} - \gamma_{\partial S}^{-}(V\varphi)_{S^{-}} = f_{\partial S} - \gamma_{\partial S}^{+}(\mathcal{V}\varphi)_{\Omega}$$

From the jump formulae and the definition of φ it follows that

$$b_{\Omega}(w_{\Omega}, w_{\Omega}) + b_{S^{-}}(w_{S^{-}}, w_{S^{-}})$$

$$= \langle g_{\partial S}^{+} - (\widehat{\mathcal{N}}\widehat{\mathcal{V}}\varphi)_{3}, f_{\partial S} - \gamma_{\partial S}^{+}(\mathcal{V}\varphi)_{\Omega}\rangle_{0;\partial S}$$

$$-\langle g_{\partial S}^{-} - \mathcal{T}^{-}(\mathcal{V}_{\hat{\Gamma}}\varphi)_{2}, f_{\partial S} - \gamma_{\partial S}^{+}(V\varphi)_{S^{-}}\rangle_{0;\partial S}$$

$$= \langle g_{\partial S}^{+} - g_{\partial S}^{-} - \varphi_{\partial S}, f_{\partial S} - \gamma_{\partial S}^{+}(\mathcal{V}\varphi)_{\Omega}\rangle_{0;\partial S} = 0$$

hence, $\widehat{\Psi} = \{f^{+} - \gamma_{0}{}^{+}(V\varphi)_{+}, 0, f_{\partial S}\} - \gamma_{\partial S}^{+}(\mathcal{V}\varphi)_{\Omega}\} \in \mathcal{F}(\hat{\Gamma})$. Since $\widehat{\Psi} \in \mathcal{H}_{1/2}(\hat{\Gamma})$, it can be concluded that $\widehat{\Psi} = 0$, which completes the proof.■

Let $\mathcal{H}_{1/2}(\partial S)$ be the subspace of $H_{1/2}(\partial S)$ consisting of all f such that $\langle f, z\rangle_{0;\partial S} = 0$ for all $z \in \mathcal{F}. \mathcal{H}_{-1/2}(\partial S)$ is the subspace of $H_{-1/2}(\partial S)$ of all g such that $\langle g, z\rangle_{0;\partial S} = 0$ for all $z \in F$.

The double layer potential of density $\psi = \{\psi_{0}, \psi_{\partial S}\} \in \dot{H}_{1/2}(\Gamma_{0}) \times \mathcal{H}_{1/2}(\partial S)$ can be defined as

$$(W\psi)(x) = (W_{0}\psi_{0})(x) + (W_{\partial S}\psi_{\partial S})(x), \quad x \in \Omega, x \in S^{-}$$

where $W_{0}\psi_{0}$ and $W_{\partial S}\psi_{\partial S}$ are the double layer potentials defined on Γ_{0} and ∂S, respectively. The modified double layer potential can be introduced

$$(\mathcal{W}\psi)(x) \ = \ (W\psi)(x) - [\langle (W\psi)_{0}^{+}, z^{(i)}\rangle_{0;\Gamma_{0}} + \langle (W\psi)_{\partial S}^{+}, z^{(i)}\rangle_{0;\Gamma_{0}}] z^{(i)}(x),$$

$$x \ \in \ \Omega, \ x \in S^{-}$$

where $(W\psi)_0^+$ and $(W\psi)_{\partial S}^+$ are the limiting values of $W\psi$ on Γ_0 and ∂S from within Ω^+ and S^+. Also the limiting values $\mathcal{W}^\pm$ of the modified double layer potential on Γ from within $\Omega^\pm$ will be defined by writing $\mathcal{W}^\pm\psi = \gamma^\pm\pi^\pm\mathcal{W}\psi$. The corresponding boundary operator $\widehat{\mathcal{W}}\psi = \{\pi_0(\mathcal{W}^+\psi), \pi_0(\mathcal{W}^+\psi - \mathcal{W}^-\psi), \gamma_{\partial S}^+(\mathcal{W}\psi)_\Omega\} = \{\gamma_0{}^+\pi^+(\mathcal{W}\psi), -\psi_0, \gamma_{\partial S}^+(\mathcal{W}\psi)_\Omega\}$.

Let $\hat{\mathcal{G}} = \widehat{\mathcal{N}}\widehat{\mathcal{W}}$. From the jump formula for the normal boundary stresses and couple stresses of the double layer potential it follows that the first component of $\hat{\mathcal{G}}\psi$ is zero for any $\psi \in \dot{H}_{1/2}(\Gamma_0) \times \mathcal{H}_{1/2}(\partial S)$. As a result, it can be written $\hat{\mathcal{G}}\psi = \{0, (\hat{\mathcal{G}}\psi)^-, (\hat{\mathcal{G}}\psi)_{\partial S}\}$ for all $\psi \in \dot{H}_{1/2}(\Gamma_0) \times \mathcal{H}_{1/2}(\partial S)$. Moreover, the boundary operator $\mathcal{G}_{\hat{\Gamma}}\psi = \{(\hat{\mathcal{G}}\psi)^-, (\hat{\mathcal{G}}\psi)_{\partial S}\}$ has been defined from $\dot{H}_{1/2}(\Gamma_0) \times \mathcal{H}_{1/2}(\partial S)$ to $H_{-1/2}(\Gamma_0) \times \mathcal{H}_{-1/2}(\partial S)$.

Theorem 5.11 $\mathcal{G}_{\hat{\Gamma}}$ is a homeomorphism from $\dot{H}_{1/2}(\Gamma_0) \times \mathcal{H}_{1/2}(\partial S)$ to $H_{-1/2}(\Gamma_0) \times \mathcal{H}_{-1/2}(\partial S)$.

Proof From the properties of the double layer potential it can be understood that $(\mathcal{W}\psi)_\Omega \in H_1(\Omega), (\mathcal{W}\psi)_{S^-} \in H_{1,\omega}(S^-)$, and

$$\|(\mathcal{W}\psi)_\Omega\|_{1;\Omega}^2 \le c b_\Omega((\mathcal{W}\psi)_\Omega, (\mathcal{W}\psi)_\Omega) \qquad 5.36$$

Here $(\mathcal{W}\psi)_{S^-}$ is the restriction of $\mathcal{W}\psi$ to S^-. For any $\psi = \{\psi_0, \psi_{\partial S}\} \in \dot{H}_{1/2}(\Gamma_0) \times \mathcal{H}_{1/2}(\partial S)$. The jump formula for double layer potential can be written as

$$\psi_0 = -(\mathcal{W}\psi)_2, \;\; \psi_{\partial S} = -((\mathcal{W}\psi)_3 - \gamma_{\partial S}^-(\mathcal{W}\psi)_{S^-}) \qquad 5.37$$

where $(W\psi)_i$ are the components of $\mathcal{W}\psi$. From equation (5.37) it results that

$$b_\Omega((\mathcal{W}\psi)_\Omega, (\mathcal{W}\psi)_\Omega) + b_{S^-}((\mathcal{W}\psi)_{S^-}, (\mathcal{W}\psi)_{S^-}) \qquad 5.38$$

$$= -\langle(\mathcal{G}_{\hat{\Gamma}}\psi)_1, \psi_0\rangle_{0;\Gamma_0} - \langle(\mathcal{G}_{\hat{\Gamma}}\psi)_2, \psi_{\partial S}\rangle_{0;\partial S}$$

It can be claimed that $\mathcal{G}_{\hat{\Gamma}}$ is continuous. Let $\psi = \{\psi_0, \psi_{\partial S}\} \in \dot{H}_{1/2}(\Gamma_0) \times \mathcal{H}_{1/2}(\partial S)$. By using the proof of Theorem (5.9), equation (5.36) and equation (5.38)

$$\|\mathcal{G}_{\hat{\Gamma}}\psi\|^2_{H_{-\frac{1}{2}}(\Gamma_0)\times\mathcal{H}_{-\frac{1}{2}}(\partial S)} \leq c\|\widehat{\mathcal{W}}\psi\|^2_{H_{-1/2}(\hat{\Gamma})} \leq c\|(\mathcal{W}\psi)_\Omega\|^2_{1;\Omega}$$

$$\leq c\{b_\Omega((\mathcal{W}\psi)_\Omega, (\mathcal{W}\psi)_\Omega) + b_{S^-}((\mathcal{W}\psi)_{S^-}, (\mathcal{W}\psi)_{S^-})\}$$

$$\leq c\{\left|\langle(\mathcal{G}_{\hat{\Gamma}}\psi)^1, \psi^0\rangle_{0;\Gamma^0}\right| + \left|\langle(\mathcal{G}_{\hat{\Gamma}}\psi)_2, \psi_{\partial S}\rangle_{0;\partial S}\right|\}$$

$$\leq c\|\mathcal{G}_{\hat{\Gamma}}\psi\|_{H_{-1/2}(\Gamma_0)\times H_{1/2}(\partial S)}\|\psi\|_{\dot{H}_{\frac{1}{2}}(\Gamma_0)\times H_{\frac{1}{2}}(\partial S)}$$

consequently,

$\|\mathcal{G}_{\hat{\Gamma}}\psi\|_{H_{-1/2}(\Gamma_0)\times H_{1/2}(\partial S)} \leq c\|\psi\|_{\dot{H}_{\frac{1}{2}}(\Gamma_0)\times H_{\frac{1}{2}}(\partial S)}$, which proves the continuity of $\mathcal{G}_{\hat{\Gamma}}$.

If $\mathcal{G}_{\hat{\Gamma}}\psi = 0$, then from the properties of $\widehat{\mathcal{N}}$, and $\widehat{\mathcal{W}}\psi = 0$, also noting that $\mathcal{T}^-\gamma^-_{\partial S}(W\psi)_{S^-} = (\mathcal{G}_{\hat{\Gamma}}\psi)_2 = 0$, from equation (5.37) it will be obtained that $\psi = 0$. Hence, $\mathcal{G}_{\hat{\Gamma}}$ is injective. By using equation (5.37) and Theorem (5.9),

$$\|\psi\|_{\dot{H}_{\frac{1}{2}}(\Gamma_0)\times H_{\frac{1}{2}}(\partial S)} \leq c\|\mathcal{G}_{\hat{\Gamma}}\psi\|_{H_{-\frac{1}{2}}(\Gamma_0)\times H_{\frac{1}{2}}(\partial S)}$$

which means that $\mathcal{G}_{\hat{\Gamma}}{}^{-1}$ is continuous.

To complete the proof, the verification of density of the range, $\mathcal{G}_{\hat{\Gamma}}$, is needed. So, it can be said that the range of $\mathcal{G}_{\hat{\Gamma}}$ is dense in $H_{-1/2}(\Gamma_0) \times \mathcal{H}_{-1/2}(\partial S)$. If the range of $\mathcal{G}_{\hat{\Gamma}}$ is not dense in $H_{-1/2}(\Gamma_0) \times \mathcal{H}_{-1/2}(\partial S)$, then there is a nonzero ψ in the dual $\dot{H}_{\frac{1}{2}}(\Gamma_0) \times \mathcal{H}_{\frac{1}{2}}(\partial S)$ such that $\langle\psi_0, (\mathcal{G}_{\hat{\Gamma}}\beta)_1\rangle_{0;\Gamma_0} + \langle\psi_{\partial S}, (\mathcal{G}_{\hat{\Gamma}}\beta)_2\rangle_{0;\partial S} = 0$ for all $\beta \in \dot{H}_{\frac{1}{2}}(\Gamma_0) \times \mathcal{H}_{\frac{1}{2}}(\partial S)$. it can be easily understood that $\beta = \psi$ and it is possible to obtain $\langle\psi_0, (\mathcal{G}_{\hat{\Gamma}}\psi)_1\rangle_{0;\Gamma_0} + \langle\psi_{\partial S}, (\mathcal{G}_{\hat{\Gamma}}\psi)_2\rangle_{0;\partial S} =$

$[\psi, \hat{\mathcal{G}}\psi]_{0;\hat{\Gamma}} = 0$, which means that $\widehat{\mathcal{W}}\psi \in \mathcal{F}(\hat{\Gamma})$. Therefore, $\psi_0 = -(\pi_0 \mathcal{W}^+\psi - \pi_0 \mathcal{W}^-\psi) = 0$, $\psi_{\partial S} = -(\gamma_{\partial S}^+(W\psi)_\Omega - \gamma_{\partial S}^-(W\psi)_{S^-}) = 0$. This contradiction completes the proof.■

the solution of problem (5.30) can be represented in the form

$$u = (\mathcal{V}\varphi)\_\{\Omega\} + W_0\psi + z \qquad 5.39$$

where $\varphi \in \dot{\mathcal{H}}_{-1/2}(\hat{\Gamma})$. $W_0\psi$ is the double layer potential of density $\psi \in \dot{H}_{1/2}(\Gamma_0)$,

$$z = [\langle f^+ + \gamma_0{}^+(W_0\delta f)_+ z^{(i)}\rangle_{0;\Gamma} + \langle f_{\partial S} + \gamma_{\partial S}^+(W_0\delta f)_\Omega, z^{(i)}\rangle_{0;\partial S}]z^{(i)} \qquad 5.40$$

and $(W_0\delta f)_+$ and $(W_0\delta f)_\Omega$ are the restrictions of $W_0\delta f$ to Ω^+ and Ω. This representation yields the system of boundary equations

$$\hat{\mathcal{V}}\varphi + \{\gamma_0{}^+(W_0\psi)_+, -\psi, \gamma_{\partial S}^+(W_0\delta f)_\Omega\} = \hat{F} - \{z, 0, z\} \qquad 5.41$$

Theorem 5.12 For any $\hat{F} \in H_{1/2}(\hat{\Gamma})$, system (5.41) has a unique solution $\{\phi, \psi\} \in \dot{\mathcal{H}}_{1/2}(\Gamma_0) \times \dot{H}_{1/2}(\Gamma_0)$, which satisfies the estimate

$$\| \{\varphi, \psi\} \|_{\dot{H}_{-\frac{1}{2}}(\Gamma_0) \times H_{\frac{1}{2}}(\partial S) \times \dot{H}_{-\frac{1}{2}}(\Gamma_0)} \leq c \| \hat{F} \|_{H_{1/2}(\hat{\Gamma})}$$

In this case, u that has been defined by equation (5.39) is a solution of problem (5.30).

Proof Assuming $\psi = -\delta f$. Also, the system (5.40) will be reduces

$$\mathcal{V}_{\hat{\Gamma}}\varphi = \{f^+, f_{\partial S}\} + \{\gamma_0{}^+(W_0\delta f)_+, \gamma_{\partial S}^+(W_0\delta f)_\Omega\} - \{z, z\} \qquad 5.42$$

By equation (5.40), the right-hand side in equation (5.42) belongs to $\hat{\mathcal{H}}_{1/2}(\hat{\Gamma})$. The rest of the proof is similar to the proof of Theorem (5.10).■

The solution of (5.31) will now be represented in the form

$$u = \mathcal{V}_{\Gamma_0}\varphi + (W\psi)_\Omega + z \tag{5.43}$$

where $\mathcal{V}_{\Gamma_0}\varphi$ is the modified single layer potential of density$\varphi \in \dot{H}_{-1/2}(\Gamma_0)$, φ and $\psi \in \dot{H}_{1/2}(\Gamma_0) \times \mathcal{H}_{1/2}(\partial S)$ are unknown densities, and $z \in \mathcal{F}$ is arbitrary. This representation yields the system of boundary equations

$$\widehat{\mathcal{N}}\{\gamma_0{}^+(\mathcal{V}_{\Gamma_0}\varphi), 0, \gamma_{\partial S}^+(\mathcal{V}_{\Gamma_0}\varphi)_\Omega\} + \hat{\mathcal{G}}\psi = \hat{G} \tag{5.44}$$

Theorem 5.13 For any $\hat{G} \in H_{-1/2}(\hat{\Gamma})$, system (5.44) has a unique solution $\{\varphi, \psi\} \in \mathcal{H}_{-1/2}(\Gamma_0) \times \dot{H}_{1/2}(\Gamma_0) \times \mathcal{H}_{1/2}(\partial S)$, which satisfies the estimate

$$\| \{\varphi, \psi\} \|_{\dot{H}_{-1/2}(\Gamma_0) \times \dot{H}_{1/2}(\Gamma_0) \times H_{1/2}(\partial S)} \leq c \| \hat{G} \|_{H_{-1/2}(\hat{\Gamma})}$$

In this case, u that has been defined by equation (5.43) is a solution of problem (5.31)

Proof From the jump formula for normal boundary stresses and couple stresses of the single layer potential the first component of

$$\widehat{\mathcal{N}}\{\gamma_0{}^+(\mathcal{V}_{\Gamma_0}\varphi), 0, \gamma_{\partial S}^+(\mathcal{V}_{\Gamma_0}\varphi)_\Omega\}$$

is equal to φ. Comparing the first components on the both sides of equation (5.44) it can be seen that $\varphi = \delta g$. The rest of the proof can be found in proving previous theorems. ■

Remark *Till now Γ and ∂S are assumed to be C^2-curves. However, the results will remain the same for hybrid, and smooth $C^{0,1}$-curves that consist of finitely many C^2-arcs* (Maz'ya, 1985).

5.7 Boundary Element Method

5.7.1 Background

In section 5.6 a rigorous analysis of Dirichlet and Neumann boundary value problem for a domain weakened by a crack in Cosserat continuum has been performed. As a result, the corresponding solution in the form of modified integral potentials has been constructed, with unknown distributional densities. However, it is impossible to find these densities analytically. In order to overcome this drawback, it is necessary to find a numerical technique which will provides a procedure to approximate the solutions numerically. One of the most effective approaches to achieve this goal is the boundary element method (Brebbia, 1978). Later, it has been became a very popular approximation technique in different research areas, including fracture mechanics (Aliabadi, et al., 1993).

The origin of the boundary element method goes back to classical integral equation and finite elements, which inherited the advantages of both techniques. From one point of view, this method allows the reduction of the dimension of a problem by one. Also, it defines domains extending to infinity with a high degree of accuracy. Both of aforementioned benefits are the same as boundary integral equation. On the other hand, in this method there is no need to differentiate the shape functions, which is a major requirement in finite element method in order to find the stresses. Here, the method allows to differentiate the matrix of fundamental solutions, which will ease the way of the calculations of the stresses and provides more accurate results.

In what follows, the boundary element method has been used to find the solution for an infinite domain weakened by a crack in Cosserat continuum, when stresses and couple stresses are prescribed along both sides of the crack (Neumann boundary value problem), and discuss its convergence.

5.7.2 Boundary Element Method

As it has been shown in section (5.6), the solution to problem (5.17) can be represented in the form of equation (5.27), as a reminder, this representation will come below

$$u = (V\varphi)_\Omega + W\psi + z$$

and the corresponding boundary integral equations are uniquely solvable with respect to distributional densities φ and ψ. As it mentioned in the background section of this chapter, these densities cannot be found analytically. In order to approximate these entities numerically, the boundary element method has been used (Gaul, et al., 2003).

Lemma 5.2 *(Somigliana formula)* Using classical techniques, we can prove that if $u \in H_{1,\omega}(\Omega)$ is a solution of $Lu = 0 \; in \; \Omega$, then

$$\int_{\Gamma_0} [D(x,y)\delta(T(\partial_y)u(y)) - P(x,y)\delta u(y)]\, ds(y) = (\tfrac{1}{2})\delta u(x), \;\; x \in \Gamma_0 \qquad 5.45$$

where $\delta(T(\partial_y)u(y))$ denotes the jump of $T(\partial_y)u(y)$ on the crack.

From Theorem (5.9), the density of the modified single layer potential is

$$\varphi = \delta(T(\partial_y)u(y)) = \delta g$$

The next step is to find the density of the modified double layer potential $\psi = -\delta u$. To achieve this goal, Γ_0 has been divided into n elements $\Gamma_0^{(k)}$, each of which possesses one node $\xi^{(k)}$ located in the middle of the element. The values of δg and δu are constant throughout the element and correspond to the values at the node $\delta g(\xi^{(k)})$ and $\delta u(\xi^{(k)})$. Then equation (5.45) becomes

$$\sum_{k=1}^{n}\int_{\Gamma_0^{(k)}}[D(x,y)\delta g(\xi^{(k)})-P(x,y)\delta u(\xi^{(k)})]\,ds(y)=(\tfrac{1}{2})\delta u(x),\quad x\in\Gamma_0$$

By putting x sequentially at all nodes, the linear algebraic system of equations can be obtained

$$\sum_{k=1}^{n}(\int_{\Gamma_0^{(k)}}\big(D\big(\xi^{(i)}\big),y\big)ds(y))\delta u(\xi^{(k)})-\sum_{k=1}^{n}(\int_{\Gamma_0^{(k)}}\big(P\big(\xi^{(i)}\big),y\big)ds(y))\delta u(\xi^{(k)})$$

$$=(\tfrac{1}{2})\delta u(\xi^{(i)}),\qquad i,k=1,n \qquad 5.46$$

with respect to $\delta u(\xi^{(i)})$.

It should be noted that $\int_{\Gamma_0^{(k)}}\big(D\big(\xi^{(i)}\big),y\big)ds(y)$ is defined for any i and k (Schiavone, 1996).

The approximation to ψ has been constructed by solving problem (5.46). If the shape function $\Phi_k(x)$ has been introduced by

$$\Phi_k(x)=\begin{cases}1 & x\in\Gamma_0^{(k)}\\ 0 & x\in\Gamma_0\backslash\Gamma_0^{(k)}\end{cases}$$

then the approximated densities can be represented as

$$\varphi^{(n)}(x)=\sum_{k=1}^{n}\Phi_k(x)\delta g\big(\xi^{(k)}\big)$$

$$\psi^{(n)}(x)=-\sum_{k=1}^{n}\Phi_k(x)\delta u\big(\xi^{(k)}\big)$$

Consequently, the approximate solution is

$$u^{(n)}=(V\phi^{(n)})_{\Omega}+W\psi^{(n)}+z$$

where z is arbitrary.

Now it has to be proven that the approximate numerical solution $u^{(\mathrm{n})}$ will converge to exact analytical solution u when $n \to \infty$.

Theorem 5.14 $u^{(\mathrm{n})} \to u$ as $n \to \infty$.

Proof Since the Neumann problem has been considered, rigid displacement terms are not determined. Hence, it only necessary to show that $V\varphi^{(\mathrm{n})} \to V\varphi$ and $W\psi^{(\mathrm{n})} \to W\psi$ as $n \to \infty$.

In order to prove the firs claim, for $x \in \Omega$

$$\left|V\varphi(x) - V\varphi^{(\mathrm{n})}(x)\right|$$

$$\leq \sum_{i=1}^{3}\left|\int_{\Gamma_0} D^{(i)}(x,y)\varphi(y)\,ds(y) - \sum_{k=1}^{n}\left(\int_{\Gamma_0^{(k)}} D^{(i)}(x,y)ds(y)\right)\varphi(\xi^{(k)})\right|$$

$$= \sum_{i=1}^{3}\left|\sum_{k=1}^{n}\int_{\Gamma_0^{(k)}}\left[D^{(i)}(x,y)\varphi(y) - D^{(i)}(x,y)\varphi(\xi^{(k)})\right]ds(y)\right|$$

$$= \sum_{i=1}^{3}\left|\sum_{k=1}^{n}\int_{\Gamma_0^{(k)}} D^{(i)}(x,y)\left[\varphi(y) - \varphi\left(\xi^{(k)}\right)\right]ds(y)\right|$$

$$\leq \sum_{i=1}^{3}\sum_{k=1}^{n}\left\|D^{(i)}(x,0)\right\|_{1,\omega;\Omega}\left|\varphi(y) - \varphi\left(\xi^{(k)}\right)\right|h_k$$

where h_k is the length of the k^{th} element. Assuming the elements are all equal it can be concluded, $h_k = h = (\frac{L}{n})$, where L is the length of Γ_0.

Also

$$\left|\varphi(y) - \varphi\left(\xi^{(k)}\right)\right|$$

$$\leq \sum_{i=1}^{3}\left|\varphi_i(y) - \varphi_i\left(\xi^{(k)}\right)\right|$$

$$= \sum_{i=1}^{3} \sum_{\alpha=1}^{2} \left| \partial_\alpha \varphi_i \left(\xi^{(k)} \right) \right| h + O(h^2)$$

Then by denoting M_1 as

$$M_1 = max_{\alpha=1,2; i=1,3; k=1,n} \left| \partial_\alpha \varphi_i \left(\xi^{(k)} \right) \right|$$

Since $\left\| D^{(i)}(x,\cdot) \right\|_{1,\omega;\Omega}$ are uniformly bounded (Iesan, 1970), it can obviously be concluded that there exists $M_2 > 0$ such that

$$\left\| D^{(i)}(x,\cdot) \right\|_{1,\omega;\Omega} \leq M_2 \quad \forall x \in \Omega$$

Now it may be written

$$|V\varphi(x) - V\varphi^{(n)}(x)| \leq \left(\frac{(18M^1M^2L^2)}{n} \right) \to 0 \text{ as } n \to \infty.$$

Repeating the proof for $W\psi^{(n)}$, it will be concluded that $u^{(n)} \to u$.

■

Chapter 6 Stress Intensity Factor for a Crack in Human Bone

6.1 Crack in Human Bone

For comparing the results of this calculation and classical analysis and showing the effectiveness of this method, a crack in human bone has been modeled as an open arc of the circle. The circle can be described by equation 6.1. Plane micropolar elasticity was the other assumption in this problem, which will make the problem as a Neumann problem described before. A brief illustration of the problem can be found in Figure 6-1. Changing radius r, will change the crack length. The elastic constants for human bone are having below values:

$\alpha = 4000\ MPa$ $\gamma = 193.6\ N$

$\kappa = 3047\ N$ $\lambda = 5332\ GPa$

$\mu = 4000\ MPa$

Moreover, the crack lengths is taken from the experimental values and put equal to 0.26 mm, 0.52 mm, 0.75 mm, and 10 mm. Another assumption in the example was the assumption of force, which is equal to $P = 2\ MPa$.

In what follows, n is the elements of Γ_0, and ρ is the distance from the crack tip. Also it has to be noted that $0 < \theta < \frac{\pi}{6}$.

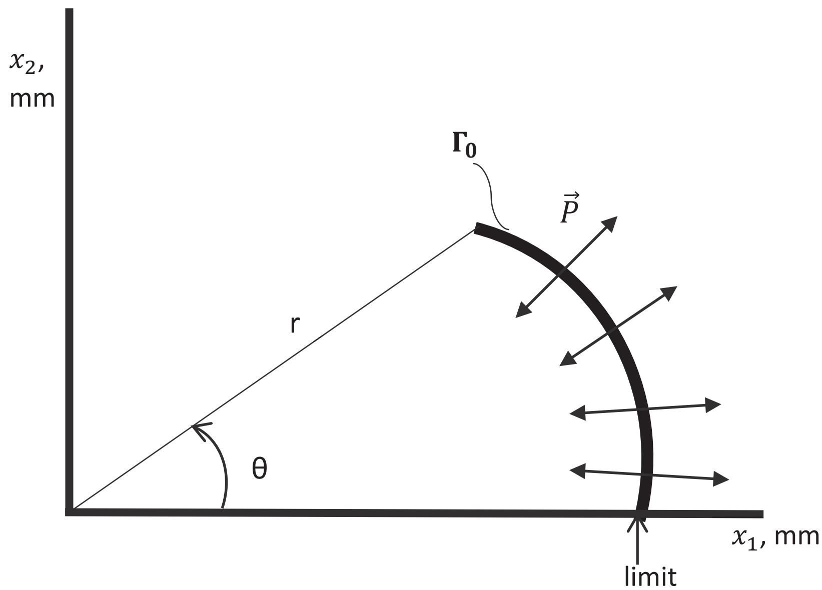

Figure 6-1 Illustration of the thesis problem

$$x_1 = r\cos\theta$$

$$x_2 = r\sin\theta \qquad 6.1$$

Recalling the definition of normalized stress intensity factor:

$$K_{normalized} = K_n = \frac{\widetilde{K_I}}{\sqrt{2\pi r}P} = \frac{T_{\theta\theta}}{P} = \frac{T_n}{P}$$

In (Table 6-1), the approximated solutions for normalized stress intensity factor (K_n), tangential traction (T_s), and moment about z or x_3-axis (M_3) for a crack of length 0.52 mm can be seen.

Table 6.1 Approximate solutions for a crack with length of 0.52 mm

ρ (mm)		0.1	0.5	0.7	1
n=4	K_n	0.491628	0.0543215	0.0411755	0.028641
	T_s (Mpa)	1.175489	0.538926	0.213505	0.156437
	M_3 (N.m)	162.5682	91.24553	79.45362	70.64523
n=10	K_n	0.2846805	0.0343675	0.026339	0.016268
	T_s (Mpa)	0.634936	0.264282	0.091475	0.045343
	M_3 (N.m)	84.24634	49.86301	44.26856	36.09357
n=30	K_n	0.2137745	0.030931	0.019877	0.010915
	T_s (Mpa)	0.506874	0.184756	0.075982	0.033547
	M_3 (N.m)	69.24764	34.25447	30.25879	24.62579
n=50	K_n	0.199231	0.026959	0.015014	0.007622
	T_s (Mpa)	0.454906	0.108063	0.061039	0.026688
	M_3 (N.m)	61.11549	29.69127	22.59611	15.80458
n=52	K_n	0.199228	0.026955	0.0150105	0.0076195
	T_s (Mpa)	0.454899	0.108054	0.061033	0.026678
	M_3 (N.m)	61.11542	29.69119	22.59604	15.80451

In Figure 6-2 to Figure 6-4 the results from Table 6.1 have been plotted. In Figure 6-2 the effect of the distance from crack on the stress intensity factor can be seen. As it is obvious from the graph, for distance values greater than 0.7 mm, all the calculation (with different n) show an almost constant value, which is equal to zero. In Figure 6-3 and Figure 6-4, however, the effect of n is obvious. The graph with n=4 in both cases is having a distance from the other plots. This difference in Figure 6-3, after the distance almost equal to 0.7, for plots with n greater than 4 is vanished. Nevertheless, the effect of n, can be obviously seen in graph 6-4.

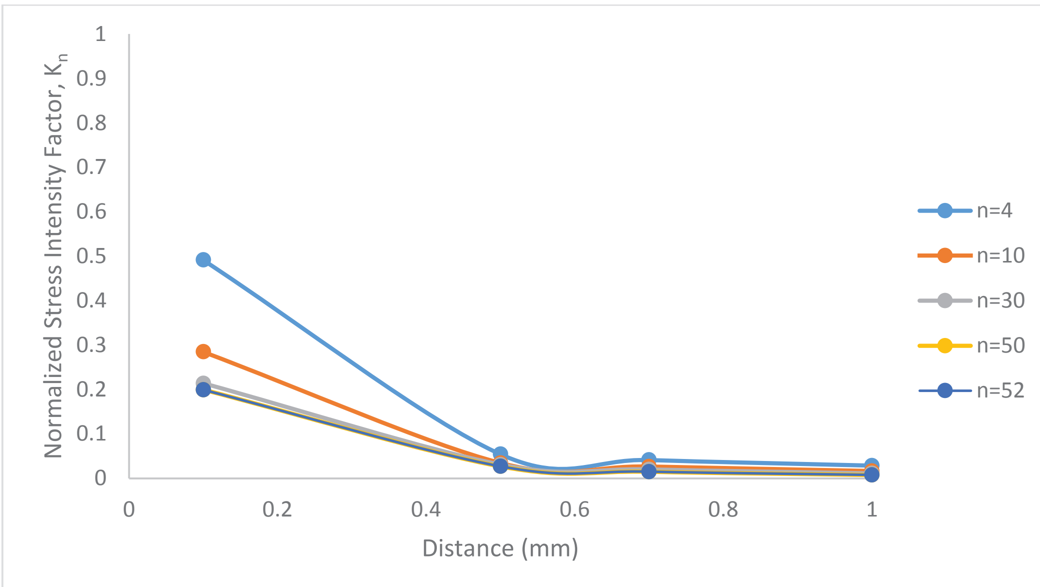

Figure 6-2 Plotting normalized stress intensity factor vs. distance from crack for different elements on the boundary

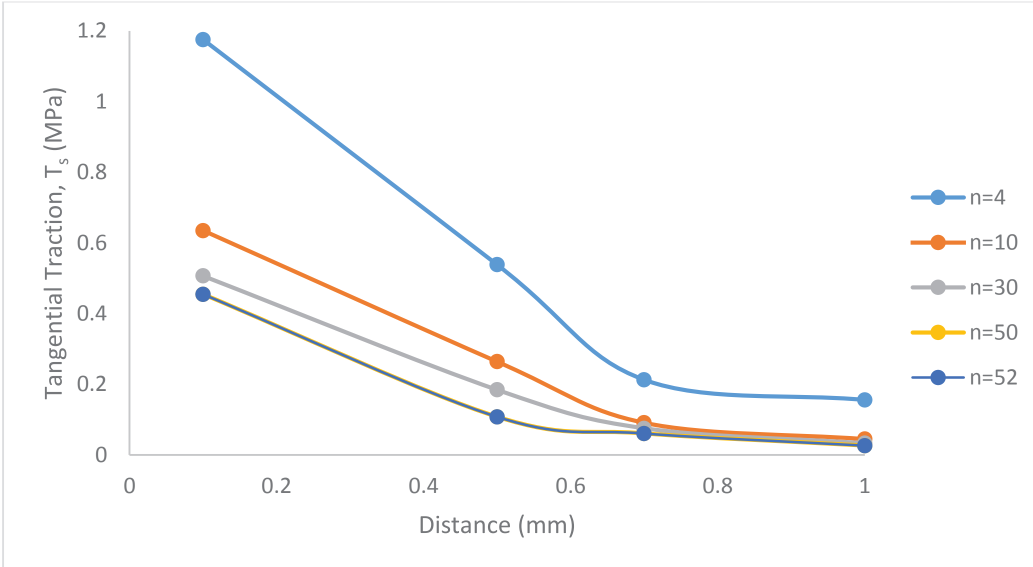

Figure 6-3 Plotting tangential traction vs. distance from crack for different elements on the boundary

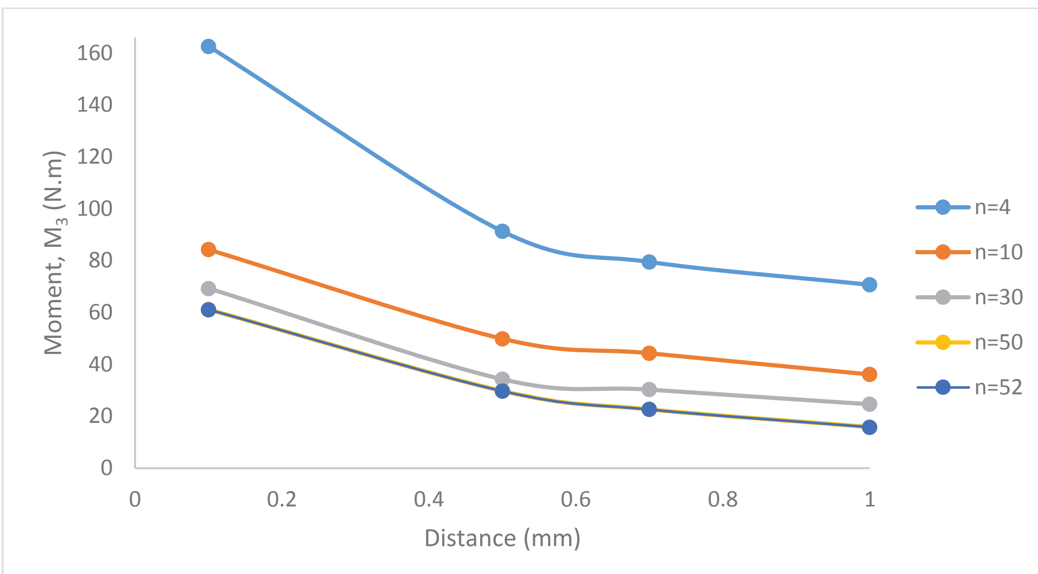

Figure 6-4 Plotting moment vs. distance from crack for different elements on the boundary

As a comparison between the approximate solution from micropolar elasticity and classical elasticity data analysis, Table 6-2 to Table 6-5 and Figure 6-5 to Figure 6-8 have been plotted below. It has to be noted that the numbers that represented in all of the following tables are matching the experimental results of another research in this field (Lakes, et al., 1990). Nevertheless, the assumption in that experiment was slightly different than the assumptions here. In previous experiment, crack assumed to be a squashed ellipse. However, crack here is a piece of a curve, so its shape will affect the stress intensity factor and stress distribution. The main finding in the aforementioned experiment was finding and proving the differences between the classical tractions and micropolar ones are the largest, when the crack length is comparable to the characteristic lengths (Lakes, et al., 1990).

In all of the graphs, as it expected, the stress intensity factor decreases by increasing the distance from the crack tip. It is notable from Figure 6-5 to Figure 6-7, that the stress intensity factor from micropolar calculations is considerably higher than the one from classical. However this difference is decrease when the crack length is equal to 10 mm (figure 6-8). It is interesting to mention that when crack length is equal to 0.26 mm, the classical method finds higher SIFs than micropolar, for far distances from crack tip Figure 6-5. The data in Table 6-2 to Table 6-5 demonstrate these observations.

Table 6.2 Approximate solution for Normalized SIF for a crack with length of 0.26 mm (r= 0.5 mm)

Point (mm)	ρ (mm)	Micropolar $K_{normalized}$	Classical $K_{normalized}$	Difference (%)
(0.5, -0.05)	0.05	0.204210	0.165991	-18.72
(0.5, -0.15)	0.15	0.057154	0.05581	-2.35
(0.5, -0.25)	0.25	0.024529	0.029394	19.84
(0.5, -0.30)	0.30	0.017189	0.022937	33.44
(0.5, -0.35)	0.35	0.012381	0.018456	49.06

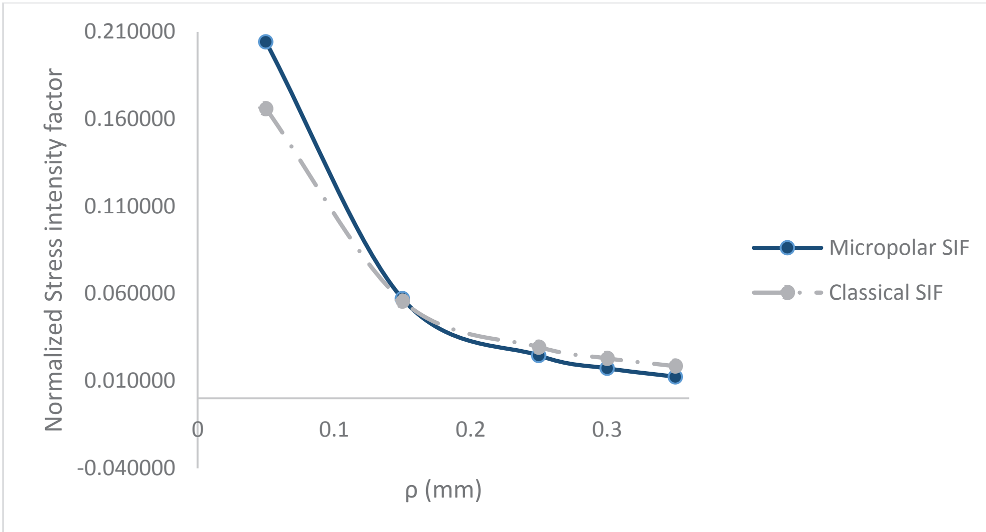

Figure 6-5 Plotting the normalized SIF in micropolar and classical theory for a crack with length of 0.26 mm (r= 0.5 mm)

Table 6.3 Approximate solution for Normalized SIF for a crack with length of 0.52 mm (r= 1 mm)

Point (mm)	ρ (mm)	Micropolar $K_{normalized}$	Classical $K_{normalized}$	Difference (%)
(1, -0.1)	0.1	0.199228	0.151801	-23.81
(1, -0.5)	0.5	0.026955	0.026851	-0.39
(1, -0.7)	0.7	0.015011	0.01686	12.32
(1, -0.8)	0.8	0.011717	0.013889	18.53
(1, -1)	1	0.00762	0.009933	30.36

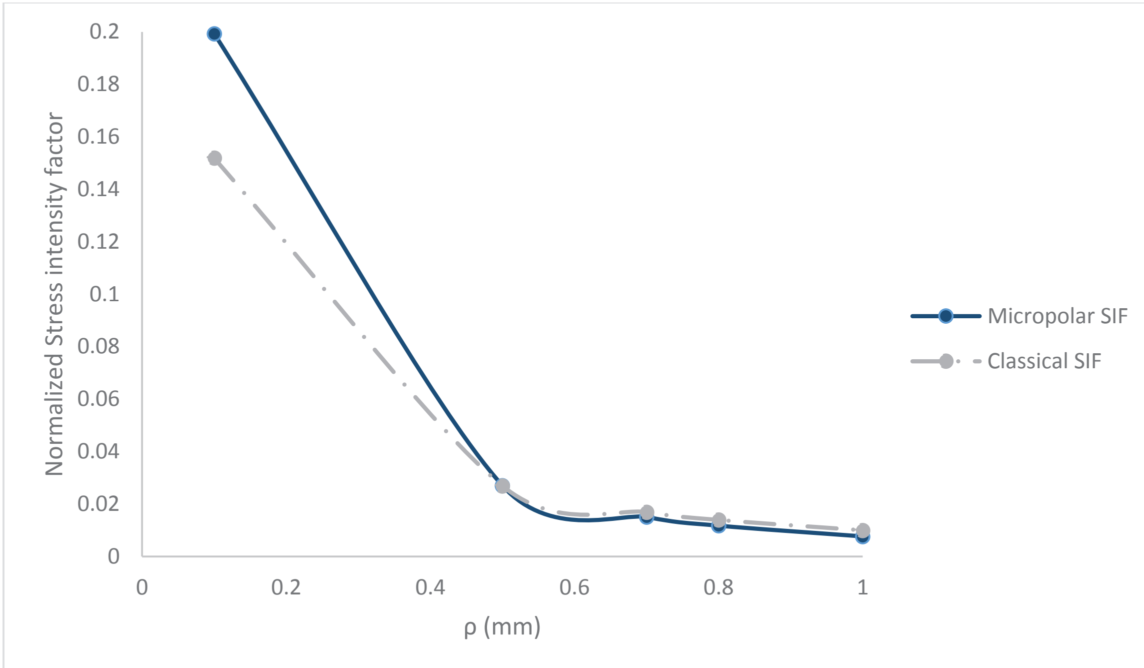

Figure 6-6 Plotting the normalized SIF in micropolar and classical theory for a crack with length of 0.52 mm (r= 1 mm)

Table 6.4 Approximate solution for Normalized SIF for a crack with length of 0.75 mm (r= 1.42 mm)

Point (mm)	ρ (mm)	Micropolar $K_{normalized}$	Classical $K_{normalized}$	Difference (%)
(1.42, -0.142)	0.142	0.197486	0.144589	-26.79
(1.42, -0.71)	0.71	0.028042	0.025559	-8.85
(1.42, -0.994)	0.994	0.016024	0.016049	0.16
(1.42, -1.136)	1.136	0.012669	0.013221	4.36
(1.42, -1.42)	1.42	0.008445	0.009456	11.98

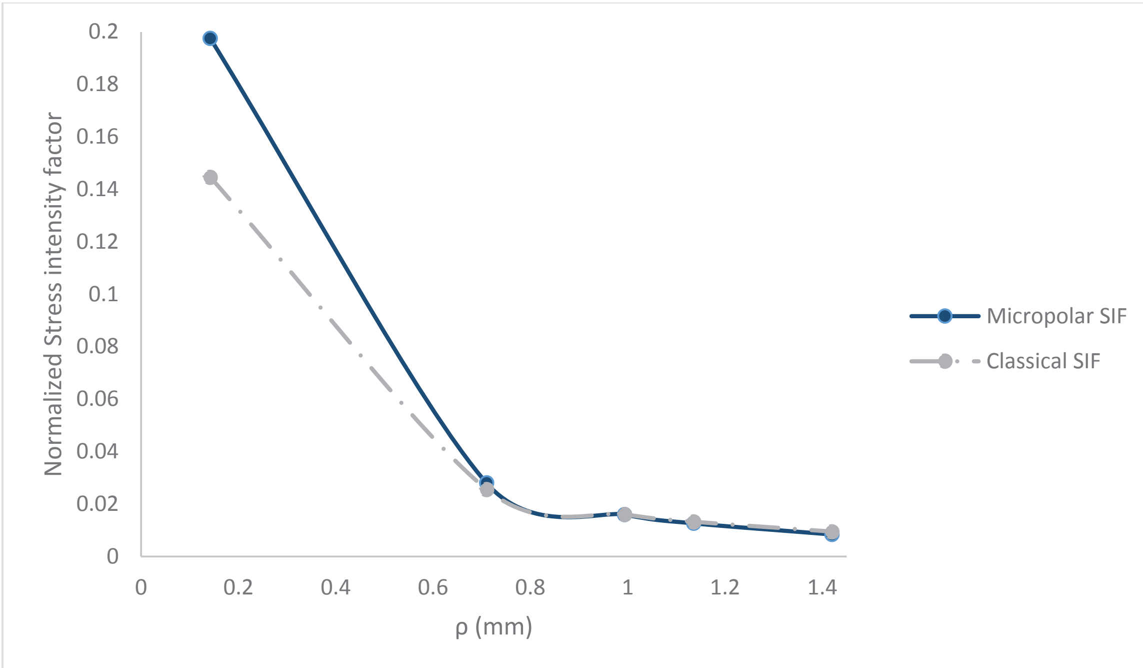

Figure 6-7 Plotting the normalized SIF in micropolar and classical theory for a crack with length of 0.75 mm (r= 1.42 mm)

Table 6.5 Approximate solution for Normalized SIF for a crack with length of 10 mm (r= 20 mm)

Point (mm)	ρ (mm)	Micropolar $K_{normalized}$	Classical $K_{normalized}$	Difference (%)
(20, -2)	2	0.104093	0.090403	-13.15
(20, -10)	10	0.016129	0.015849	-1.74
(20, -14)	14	0.009964	0.009954	-0.11
(20, -16)	16	0.008162	0.008203	0.50
(20, -20)	20	0.00579	0.005872	1.42

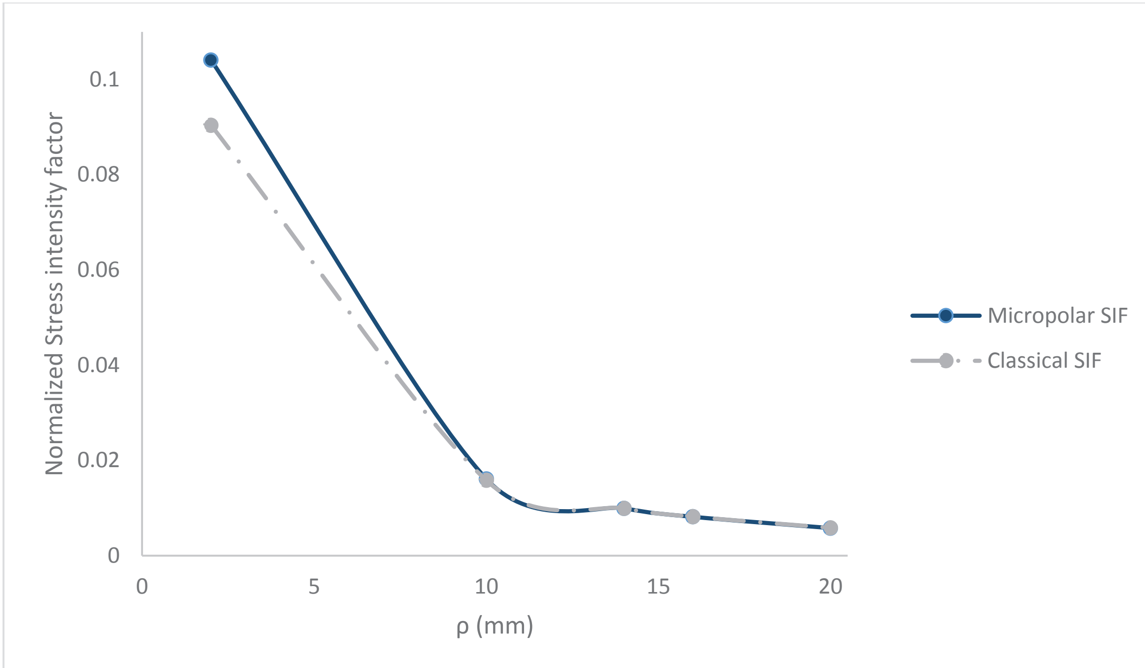

Figure 6-8 Plotting the normalized SIF in micropolar and classical theory for a crack with length of 10 mm (r= 20 mm)

Chapter 7 Conclusion and Future Work

7.1 Conclusion

This thesis focused on the boundary value problems of plane micropolar elasticity. From this work, it can be concluded that the microstructure of materials have a notable influence on the distribution of the stress around a crack. This effect depends on the length of the crack and the effect becomes more prominent moving towards the crack tip.

Another result obtained from this work shows that the formulated interior and exterior Dirichlet and Neumann boundary value problems of plane micropolar elasticity were clearly presented and solved using boundary integral equation method. The analytical solutions have obtained in the form of the corresponding integral potentials with distributional densities are the exact solutions. However, in order to approximate these solutions numerically, considering another method was necessary. Overall, it has been shown that the boundary element method, is a very effective method for approximating the solutions numerically.

In last chapter, it has been shown that the stress intensity factor will have a higher value using the micropolar elasticity assumption. This demonstrates the significance of microstructures of materials on their mechanical behaviour. Moreover, the micropolar and classical theories of elasticity show different results for different crack sizes. It has been shown that this difference is more noticeable for smaller cracks. This conclusion is consistent with the results from previous research (Savin, 1965; Mindlin, 1963; Weitsmann, 1965; Hartranft, et al., 1965; Potapenko, 2005).

7.2 Future Work

The method that has been used in this thesis can be extended to find the solutions of two-dimensional and three-dimensional boundary value problems. The other possibility of using this method is to find the solutions for problems dealing with structures in classical and micropolar elasticity.

Another area of work can be focus on formulating mixed boundary value problems of plane micropolar elasticity. Imposing Dirchlet conditions for two displacements and Neumann condition for microrotation, can be named as an example in this field of work. Considering the crack problem under such assumption reduces the problem to displacement discontinuity problem, which has many applications in geomechanics.

The next important field of work is about finding solutions for boundary value problems of anti-plane micropolar elasticity in Sobolev spaces. (Potapenko, 2005) has formulated the boundary value problems of anti-plane Cosserat elasticity for twice differentiable boundaries using boundary integral equation method. However, this work can be extended to solve the torsion problem of micropolar beams with complicated cross-sections (e.g. rectangular, or square cross-section).

For the other direction of future work, the integration of thermoelastic components in the model can be named. In this area, the fundamental solutions of boundary value problems of three-dimensional thermoelasticity has been found by using boundary integral equations. (Kupradze, et al., 1979). These solutions can be used to find the solutions for thermoelastic deformations problems in weak setting, which will lead to a more general domain.

All of these fields of work in this area will help to have a better understanding of the material's microstructure influence on their mechanical behaviour. This will have many practical applications in structural mechanics and modern day advanced composite materials.

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