Mathematical Aspects of Scalar-Tensor Field Theories

by

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A Thesis

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ABSTRACT

This thesis is based on a study of Lagrange scalar densities which are, in general, concomitants of the metric tensor g_{ij} (and its first and second derivatives) together with a scalar field ϕ (and its first derivative). Three invariance identities relating the "tensorial derivatives" of this Lagrangian are obtained. These identities are used to write the Euler-Lagrange tensors corresponding to our scalar density in a compact form. Furthermore it is shown that the Euler-Lagrange tensor corresponding to variations of the metric tensor is related to the Euler-Lagrange tensor corresponding to variations of the scalar field in a very elementary manner.

The so-called Brans-Dicke scalar-tensor theory of gravitation is a special case of our previous results and the field equations corresponding to this theory are derived and investigated at length. As a result of studying the effects of conformal transformations on the general Lagrange scalar density it is shown that solutions to the Brans-Dicke field equations are conformally related to solutions to a certain system of Einstein field equations. A detailed study of a particular static, spherically symmetric vacuum solution to the Brans-Dicke field equation is then undertaken and compared with the corresponding Einstein case.

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TO MY PARENTS AND BROTHER

TABLE OF CONTENTS

1. Introduction

This thesis is based on a study of Lagrange scalar densities of the form*

$$L = L(g_{ij};\ g_{ij,h};\ g_{ij,hk};\ \phi;\ \phi_{,i}) \quad , \qquad (1.1)$$

where the g_{ij}'s are the symmetric components of the metric tensor of an n-dimensional Riemannian space V_n, and ϕ is a scalar field. With (1.1) we may associate two Euler-Lagrange expressions, viz.,

$$E^{ij}(L) = \frac{\partial}{\partial x^k}\left\{\frac{\partial L}{\partial g_{ij,k}} - \frac{\partial}{\partial x^m}\frac{\partial L}{\partial g_{ij,km}}\right\} - \frac{\partial L}{\partial g_{ij}}, \quad (1.2)$$

and

$$E(L) = \frac{\partial}{\partial x^i}\left(\frac{\partial L}{\partial \phi_{,i}}\right) - \frac{\partial L}{\partial \phi} \quad , \qquad (1.3)$$

where the former is obtained from (1.1) through a variation of the g_{ij}'s regarding ϕ and $\phi_{,i}$ as arbitrary preassigned functions of position, and the latter is obtained from (1.1) through a variation of ϕ regarding the g_{ij}'s and their derivatives as arbitrary preassigned quantities.

Our analysis of (1.1) follows very closely the method outlined by Rund in [22] and [23]. In these two papers Rund makes an extensive study of the properties of Lagrange scalar densities of the form

$$L = L(g_{ij};\ g_{ij,h};\ g_{ij,hk}) \quad , \qquad (1.4)$$

and, more generally

$$L = L(g_{ij};\ g_{ij,h};\ g_{ij,hk};\ \psi_i;\ \psi_{i,j}) \quad , \qquad (1.5)$$

where ψ_i is a covariant vector field.

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*In (1.1) Latin indices run from 1 to n, and a comma is used to denote partial differentiation with respect to the local coordinates, x^i, of our V_n. The summation convention is used throughout this thesis.

Following Rund's method we first construct the various tensors associated with the derivatives of (1.1) with respect to each of its arguments. These tensors make it possible for us to write the Euler-Lagrange expressions $E^{ij}(L)$ and $E(L)$ in manifestly tensorial form. Our next step in the study of (1.1) is devoted to the derivation of the so-called "invariance identities." These identities are obtained by examining the behaviour of (1.1) under essentially arbitrary coordinate transformations. The invariance identities make it possible to greatly simplify the form of the Euler-Lagrange tensors corresponding to (1.1). We conclude section 2 by showing that

$$E^{ij}(L)_{|j} = \tfrac{1}{2} g^{ij} \phi_{,j} E(L) , \qquad (1.6)$$

where the vertical bar is used to denote covariant differentiation. As a consequence of (1.6) we see that whenever the field equations governing the metric tensor are satisfied; i.e.,

$$E^{ij}(L) = 0 , \qquad (1.7)$$

then the field equation for ϕ; viz.,

$$E(L) = 0 , \qquad (1.8)$$

will be satisfied automatically.

In section 3 we apply the results of section 2 to examine four special Lagrange scalar densities of the form (1.1). Three of these examples are then used to discuss the construction of Lagrangians of the form (1.1) that yield field equations which are at most of second order in the derivatives of both g_{ij} and ϕ. It is shown that although the scalar density used by Brans and Dicke to obtain their vacuum field equations satisfies the above condition; i.e., is of the form (1.1) and yields second order field equation, it is <u>not</u> the most general Lagrangian of the form (1.1) which enjoys this property. Furthermore it is shown that even the more general Lagrangian suggested by Bergmann [3] is not the most general scalar density of the form (1.1) which yields second order field equations.

Section 4 is the first of eight sections dealing exclusively with the Brans-Dicke theory. In this section we use the results presented in section 2 to derive the Brans-Dicke field equations from a suitably chosen Lagrange scalar density.

Section 5 is devoted to a study of the behaviour of $E^{ij}(L)$ under a conformal transformation of the form

$$\bar{g}_{ij} = e^{2\sigma} g_{ij} , \tag{1.9}$$

where σ is an arbitrary function of class C^2. It is shown that if L is of the form (1.1) then under (1.9)

$$E^{ij}(L) = e^{2\sigma} E^{ij}(\bar{L}) , \tag{1.10}$$

where $\bar{L}$ denotes the form assumed by L as a result of (1.9).

Using (1.10) we show in section 6 that it is possible to obtain solutions to the Brans-Dicke field equations from solutions of a certain system of Einstein field equations. The theory presented in section 6 is then used to develop a method for generating static solutions to the Brans-Dicke vacuum field equations from static solutions to the Einstein vacuum field equations. To illustrate the use of the above method the Schwarzschild vacuum solution is employed to obtain a static solution to the Brans-Dicke vacuum field equations. The solution so obtained is in fact one of the four possible static, spherically symmetric, isotropic solutions to the Brans-Dicke vacuum field equations. Sections 8 and 9 are essentially devoted to showing that the remaining three (i.e., those which were not obtained from the Schwarzschild vacuum solution) static, spherically symmetric vacuum solutions are, in a certain sense, physically unacceptable.

In section 10 we use the weak field approximate solution to the Brans-Dicke field equations presented in section 8 to identify the constants appearing in the physically acceptable static solution of the Brans-Dicke vacuum field equations. After making this identification we examine other properties of this exact vacuum solution, e.g., it singularities and its geodesics.

We conclude the thesis by showing that whenever the metric tensor of the Brans-Dicke theory is known throughout a matter free region of space (excluding certain pathological cases) then it is possible to express the Brans-Dicke scalar field in terms of geometrical objects; i.e., objects constructed from the metric tensor and its derivatives.

2. Lagrangian Scalar Densities--Invariance Identities

In the Brans Dicke theory we shall be dealing with a Riemannian V_4; however, for the purposes of this section we shall consider an n-dimensional Riemannian space V_n with line element[1]

$$ds^2 = g_{ij}dx^i dx^j;$$

where all Latin indices run from 1 to n. The functions g_{ij} appearing in the above expression are the symmetric components of the metric tensor, and are assumed to be of class C^5. We shall place no restriction upon the signature of g_{ij} but we do demand that

$$g = \det(g_{ij}) \neq 0.$$

In all further calculations we shall make use of the Christoffel symbols of the second kind, Γ^i_{jk}, along with the Riemann curvature tensor, $R_j{}^i{}_{kl}$, the Ricci tensor R_{jk}, and the scalar curvature invariant, R. We define these objects as follows[2]:

$$\Gamma^i_{jk} = \tfrac{1}{2}g^{il}(g_{jl,k} + g_{kl,j} - g_{jk,l}) \ ,$$

$$R_j{}^i{}_{kl} = \Gamma^i_{jk,l} - \Gamma^i_{jl,k} + \Gamma^m_{jk}\Gamma^i_{ml} - \Gamma^m_{jl}\Gamma^i_{mk} \ ,$$

$$R_{jk} = R_{kj} = R_j{}^i{}_{ki} \ , \text{ and } \ R = g^{jk}R_{jk} \ ,$$

where the g^{jk} are characterised by

$$g^{jk}g_{ik} = \delta^j_i \ .$$

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[1]The summation convention will be used throughout this thesis.

[2](,k) denotes partial differentiation with respect to the coordinate x^k, and (|k) denotes covariant differentiation with respect to x^k. Repeated partial differentiation, e.g., $\frac{\partial^2}{\partial x^k \partial x^l}$ is denoted by (,kl) , while repeated covariant differentiation with respect to x^k, x^l,... is denoted by (|kl...).

We concentrate our attention on scalar densities L of the form

$$L = L(g_{ij};\ g_{ij,h};\ g_{ij,hk};\ \phi;\ \phi_{,i})\ , \qquad (2.1)$$

where ϕ represents any scalar field of class C^4, and thus $\phi_{,i}$ is a covariant vector field. In what follows L will be assumed to be of class C^4 in its arguments.

When computing the Euler-Lagrange equations corresponding to the g_{ij}'s of the above Lagrangian, L, we shall regard ϕ as a preassigned function of position. Similar considerations are made with regards to g_{ij} when calculating the Euler-Lagrange equation for ϕ.

It is essential to note that there are certain symmetry properties associated with the first three functions in the argument of L. Thus if one desires to take a partial derivative of L with respect to g_{kl} it is necessary to replace each g_{ij} appearing in L by

$$\tfrac{1}{2}(g_{ij} + g_{ji})\ ,$$

and then to regard the g_{ij}'s as n^2 independent quantities when differentiating L with respect to g_{kl}. Similar considerations have to be made for the quantities $g_{ij,k}$ and $g_{ij,kl}$.

Rund [22] has considered in detail Lagrangian scalar densities of the type

$$L = L(g_{ij};\ g_{ij,k};\ g_{ij,hk})$$

and has obtained results which, in many cases, are directly applicable to the Lagrangian (2.1). In what follows we shall adhere very closely to the presentation of Rund.

We shall first construct tensors which are associated with the various derivatives of the Lagrangian (2.1). In order to simplify the form of the ensuing expressions we shall adopt the following notation:

$$\Phi = \frac{\partial L}{\partial \phi}\ , \quad \Phi^{i} = \frac{\partial L}{\partial \phi_{,i}}\ , \quad \Lambda^{ij} = \frac{\partial L}{\partial g_{ij}}\ , \quad \Lambda^{ij,k} = \frac{\partial L}{\partial g_{ij,k}}\ ,$$

and

$$\Lambda^{ij,kl} = \frac{\partial L}{\partial g_{ij,kl}}\ .$$

Since the essence of tensorial character is contained in transformation properties let us now study the behaviour of L and its derivatives under coordinate transformations of the form

$$\bar{x}^{h} = \bar{x}^{h}(x^{k})\ , \tag{2.2}$$

which are arbitrary except that they be of class C^3 and that

$$\det\left|\frac{\partial \bar{x}^{i}}{\partial x^{j}}\right| > 0.$$

We shall set

$$B^{i}_{j} = \frac{\partial x^{i}}{\partial \bar{x}^{j}}\ ,$$

and by the above we have

$$B = \det(B^{i}_{j}) > 0.$$

The higher derivatives of x^i with respect to $\bar{x}^h$ will be denoted by

$$B^{i}_{jk} = \frac{\partial^2 x^{i}}{\partial \bar{x}^{j}\partial \bar{x}^{k}}\ , \text{ and } \quad B^{i}_{jkl} = \frac{\partial^3 x^{i}}{\partial \bar{x}^{j}\partial \bar{x}^{k}\partial \bar{x}^{l}}\ .$$

Under the coordinate transformation (2.2) we have

$$\bar{\phi}(\bar{x}) = \phi(x), \tag{2.3}$$

and

$$\bar{\phi}_{,i} = \phi_{,j}B^{j}_{i}\ , \tag{2.4}$$

where $\bar{\phi}$ denotes the functional form of ϕ in the new, barred, coordinate system. We also find:

$$\bar{g}_{hk} = g_{ij}B^{i}_{h}B^{j}_{k}\ , \tag{2.5}$$

$$\bar{g}_{hk,l} = g_{ij,p}B^{p}_{l}B^{i}_{h}B^{j}_{k} + g_{ij}B^{i}_{hl}B^{j}_{k} + g_{ij}B^{i}_{h}B^{j}_{kl}\ , \tag{2.6}$$

and

$$\bar{g}_{hk,lm} = g_{ij}(B^i_{hlm}B^j_k + B^i_hB^j_{klm} + B^i_{hl}B^j_{km} + B^i_{hm}B^j_{kl}) + \\ + g_{ij,p}B^p_m(B^i_{hl}B^j_k + B^i_hB^j_{kl}) + g_{ij,p}(B^p_{lm}B^i_hB^j_k + \\ + B^p_lB^i_{hm}B^j_k + B^p_lB^i_hB^j_{km}) + g_{ij,pq}B^p_lB^q_mB^i_hB^j_k \,. \quad (2.7)$$

Since L is a scalar density we have

$$\bar{L}(\bar{g}_{ij};\ \bar{g}_{ij,h};\ \bar{g}_{ij,hk};\ \bar{\phi};\ \bar{\phi}_{,i}) = \\ = B\, L(g_{ij};\ g_{ij,h};\ g_{ij,hk};\ \phi;\ \phi_{,i}). \quad (2.8)$$

Upon differentiating this expression with respect to ϕ and $\phi_{,i}$ we find:

$$B\,\Phi = \bar{\Phi}\,, \quad (2.9)$$

and

$$B\,\Phi^i = \bar{\Phi}^k B^i_k \quad (2.10)$$

respectively. The above equations imply that Φ is a scalar density and Φ^i is a contravariant vector density. Following Rund we see that differentaition of (2.8) with respect to $g_{ij,kl}$, $g_{ij,k}$ and g_{ij} yields:

$$B\,\Lambda^{ij,kl} = \bar{\Lambda}^{rs,tu}\frac{\bar{g}_{rs,tu}}{g_{ij,kl}}\,, \quad (2.11)$$

$$B\,\Lambda^{ij,k} = \bar{\Lambda}^{rs,t}\frac{\bar{g}_{rs,t}}{g_{ij,k}} + \bar{\Lambda}^{rs,tu}\frac{\bar{g}_{rs,tu}}{g_{ij,k}}\,, \quad (2.12)$$

and

$$B\,\Lambda^{ij} = \bar{\Lambda}^{rs}\frac{\bar{g}_{rs}}{g_{ij}} + \bar{\Lambda}^{rs,t}\frac{\bar{g}_{rs,t}}{g_{ij}} + \bar{\Lambda}^{rs,tu}\frac{\bar{g}_{rs,tu}}{g_{ij}} \quad (2.13)$$

respectively.

Quite clearly $\Lambda^{ij,kl}$ is a contravariant tensor density of rank four, whereas $\Lambda^{ij,k}$ and Λ^{ij} are not tensorial quantities. However, Rund has shown that when suitably combined with $\Lambda^{ij,kl}$ and the Christoffel symbols we can construct tensor densities from them. These are denoted by $\Pi^{ij,k}$ and Π^{ij} and are respectively defined by

$$\Pi^{ij,k} = \Lambda^{ij,k} + \Gamma^k_{rs}\Lambda^{ij,rs} + 2\Gamma^i_{rs}\Lambda^{rj,ks} + 2\Gamma^j_{rs}\Lambda^{ir,ks}\,, \quad (2.14)$$

(which is a contravariant tensor density of rank 3),

and

$$\Pi^{ij} = \Lambda^{ij} + \Gamma^{i}_{kl,m}\Lambda^{kj,lm} + \Gamma^{j}_{kl,m}\Lambda^{ki,lm} +$$

$$+ \Gamma^{i}_{kl}(\Pi^{kj,l} - \Gamma^{l}_{mn}\Lambda^{kj,mn} - \Gamma^{k}_{mn}\Lambda^{mj,ln} - \Gamma^{j}_{mn}\Lambda^{mk,ln}) +$$

$$+ \Gamma^{j}_{kl}(\Pi^{ki,l} - \Gamma^{l}_{mn}\Lambda^{ki,mn} - \Gamma^{k}_{mn}\Lambda^{mi,ln} - \Gamma^{i}_{mn}\Lambda^{mk,ln}), \quad (2.15)$$

(which is a symmetric contravariant tensor density of rank 2).

The Euler-Lagrange equations corresponding to (2.1) are

$$E^{ij}(L) = 0 \quad (2.16)$$

and

$$E(L) = 0, \quad (2.17)$$

where

$$E^{ij}(L) = \frac{\partial}{\partial x^k}\left(\frac{\partial L}{\partial g_{ij,k}} - \frac{\partial}{\partial x^m}\left(\frac{\partial L}{\partial g_{ij,km}}\right)\right) - \frac{\partial L}{\partial g_{ij}}, \quad (2.18)$$

and

$$E(L) = \Phi^{i}{}_{,i} - \Phi \, . \quad (2.19)$$

Equation (2.18) was obtained from (2.1) through a variation of the g_{ij}'s regarding ϕ as an arbitrary preassigned function, whereas equation (2.19) was obtained from (2.1) by varying ϕ with the g_{ij}'s being regarded as arbitrary preassigned functions.

From our previous remarks it is obvious that each of the terms in the expression for $E^{ij}(L)$ are non-tensorial. However, Rund has shown that (2.18) may be written in tensor form in terms of $\Lambda^{hk,lm}$, $\Pi^{hk,l}$, and Π^{hk}. More exactly we have

$$E^{ij}(L) = -(\Lambda^{ij,kl}{}_{|kl} - \Pi^{ij,k}{}_{|k} + \Pi^{ij}) \, . \quad (2.20)$$

Of the two components in the expression for E(L) only the first could be non-tensorial. However, since Φ^i is a contravariant vector density we have

$$\Phi^{i}{}_{|i} = \Phi^{i}{}_{,i} + \Phi^{k}\Gamma^{i}_{ki} - \Phi^{i}\Gamma^{k}_{ki} = \Phi^{i}{}_{,i} \quad ,$$

which is a scalar density. Thus each term in (2.19) is tensorial.

We shall now derive the so-called "invariance identities" of our Lagrangian (2.1). To accomplish this we must first substitute into the left hand side of (2.8) the explicit functional representations for the $\bar{g}_{ij}$, $\bar{g}_{ij,k}$, $\bar{g}_{ij,kl}$, $\bar{\phi}$ and $\bar{\phi}_{,i}$, thereby obtaining an identity in B^i_j and its derivatives. If we now compute the three first derivatives of this identity with respect to B^r_{stv}, B^r_{st}, and B^r_s we find that each derivative yields an identity. The computations necessary to obtain these three identities are given below.

We begin by differentiating (2.8) with respect to B^r_{stv}. Since the right hand side of (2.8) is independent of this quantity we obtain

$$\frac{\partial \bar{L}}{\partial B^r_{stv}} = 0 . \tag{2.21}$$

From our expressions relating the barred an unbarred arguments of L we find that B^r_{stv} appears only in the transformation of $\bar{g}_{hk,lm}$. So (2.21) may be written

$$\bar{\Lambda}^{hk,lm} \frac{\partial \bar{g}_{hk,lm}}{\partial B^r_{stv}} = 0. \tag{2.22}$$

After using (2.7) to compute $\dfrac{\partial \bar{g}_{hk,lm}}{\partial B^r_{stv}}$ we find that (2.22) becomes

$$\bar{\Lambda}^{hk,lm} \frac{\partial \bar{g}_{hk,lm}}{\partial B^r_{stv}} = \frac{2}{3}\, g_{rj} B^j_k \left\{ \bar{\Lambda}^{sk,tv} + \bar{\Lambda}^{tk,vs} + \bar{\Lambda}^{vk,st} \right\} = 0 . \tag{2.23}$$

If we now multiply the above expression by $B^{r}_{l}\bar{g}^{-lp}$ we obtain

$$\bar{\Lambda}^{sp,tv} + \bar{\Lambda}^{tp,vs} + \bar{\Lambda}^{vp,st} = 0. \tag{2.24}$$

From our previous work we know that $\Lambda^{sk,tv}$ is tensorial, and thus (2.24) is valid in all coordinate systems; in particular we have

$$\Lambda^{sk,tv} + \Lambda^{tk,vs} + \Lambda^{vk,st} = 0 \qquad (2.25)$$

which is our FIRST INVARIANCE IDENTITY, and is identical with Rund's. By repeated use of (2.25) we obtain

$$\Lambda^{sk,tv} = \Lambda^{tv,sk} \qquad (2.26)$$

where we have used

$$\Lambda^{hi,kj} = \Lambda^{hi,jk} = \Lambda^{ih,kj} .$$

To determine our second invariance identity we differentiate (2.8) with respect to B^r_{st}. Since the right hand side of (2.8) is independent of this quantity we find

$$\frac{\partial \bar{L}}{\partial B^r_{st}} = 0 . \qquad (2.27)$$

The B^r_{st}'s arise in the transformations of $\bar{g}_{hk,l}$ and $\bar{g}_{hk,lm}$. Thus (2.27) may be written

$$\bar{\Lambda}^{hk,l}\frac{\partial \bar{g}_{hk,l}}{\partial B^r_{st}} + \bar{\Lambda}^{hk,lm}\frac{\partial \bar{g}_{hk,lm}}{\partial B^r_{st}} = 0 . \qquad (2.28)$$

Using (2.6) we easily obtain

$$\bar{\Lambda}^{hk,l}\frac{\partial g_{hk,l}}{\partial B^r_{st}} = g_{rj}B^j_k(\bar{\Lambda}^{sk,t} + \bar{\Lambda}^{tk,s}). \qquad (2.29)$$

From (2.7) and our first invariance identity we find that the second term in (2.28) may be written

$$\bar{\Lambda}^{hk,lm}\frac{\partial \bar{g}_{hk,lm}}{\partial B^r_{st}} = -2g_{rj}B^j_{km}\bar{\Lambda}^{mk,st} + g_{ij,r}B^i_hB^j_k\bar{\Lambda}^{hk,st} + -2g_{rj,p}B^p_mB^j_k\bar{\Lambda}^{mk,st} . \qquad (2.30)$$

The above expression may be rewritten as follows:

$$\bar{\Lambda}^{hk,lm}\frac{\partial \bar{g}_{hk,lm}}{\partial B^r_{st}} = -2g_{rj}B^j_{km}\bar{\Lambda}^{mk,st} + -2(g_{uj}\Gamma^u_{rp} + g_{ru}\Gamma^u_{jp})B^p_mB^j_k\bar{\Lambda}^{mk,st} + +(g_{uj}\Gamma^u_{ir} + g_{iu}\Gamma^u_{jr})B^i_hB^j_k\bar{\Lambda}^{hk,st} . \qquad (2.31)$$

The last term appearing in (2.31) may be simplified to yield

$$2g_{uj}\Gamma^u_{rp}B^p_mB^j_k\bar{\Lambda}^{mk,st} ,$$

which cancels with the first part of the second term on the right hand side of (2.31) leaving us with

$$\bar{\Lambda}^{hk,lm}\frac{\partial \bar{g}_{hk,lm}}{\partial B^r_{st}} = -2g_{rj}B^j_{km}\bar{\Lambda}^{mk,st} - 2g_{ru}\Gamma^u_{jp}B^p_m B^j_k\bar{\Lambda}^{mk,st} \,. \quad (2.32)$$

To further reduce the form of the above expression it will be necessary to make use of the well known transformation law of the Christoffel symbols; viz.,

$$\Gamma^u_{jp}B^j_k B^p_m = \bar{\Gamma}^a_{km}B^u_a - B^u_{mk} \,.$$

Upon inserting this expression into (2.32) we obtain

$$\bar{\Lambda}^{hk,lm}\frac{\partial \bar{g}_{hk,lm}}{\partial B^r_{st}} = -2g_{ru}\bar{\Gamma}^a_{km}B^u_a\bar{\Lambda}^{mk,st}. \quad (2.33)$$

Combining (2.33) with (2.29) we find that (2.28) becomes

$$g_{rj}\,B^j_k\,(\bar{\Lambda}^{sk,t} + \bar{\Lambda}^{tk,s} - 2\,\bar{\Gamma}^k_{lm}\,\bar{\Lambda}^{lm,st}) = 0 \,. \quad (2.34)$$

If we now multiply the above quantity by $B^r_l\,\bar{g}^{lp}$ we find

$$\bar{\Lambda}^{sp,t} + \bar{\Lambda}^{tp,s} = 2\bar{\Gamma}^p_{lm}\,\bar{\Lambda}^{lm,st} \,. \quad (2.35)$$

Upon replacing the left hand side of (2.35) with the tensorial quantities introduced in (2.14) we obtain

$$\bar{\Pi}^{sk,t} + \bar{\Pi}^{tk,s} = 0 \,. \quad (2.36)$$

If we now use a little algebra and the tensorial properties of $\Pi^{sk,t}$, noting that $\Pi^{sk,t} = \Pi^{ks,t}$, we find that (2.36) is equivalent to

$$\Pi^{sk,t} = 0, \quad (2.37)$$

which is our SECOND INVARIANCE IDENTITY, and again agrees with Rund's.

An immediate consequence of (2.37) is that if one has a Lagrange density of the type

$$L = L(g_{ij};\; g_{ij,h};\; \phi;\; \phi_{,i}) \quad (2.38)$$

then L is actually independent of $g_{ij,k}$. For if we had an L of the form (2.38) then $\Lambda^{hk,lm} = 0$. Thus equation (2.14) becomes

$$\Pi^{hk,l} = \Lambda^{hk,l} \,.$$

But since $\Pi^{hk,l} = 0$, we have $\Lambda^{hk,l} = 0$, and therefore L must be independent of $g_{ij,k}$.

From the procedure used to obtain the first and second invariance identities it is obvious that similar identities hold for any relative tensor of weight w, contravariant valency p and covariant valency q which is of the form

$$T^{i_1 \ldots i_p}_{j_1 \ldots j_q} = T^{i_1 \ldots i_p}_{j_1 \ldots j_q}(g_{ab};\, g_{ab,v};\, g_{ab,vw};\, \phi;\, \phi_{,a}).$$

In particular we may conclude that if $T^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ is independent of $g_{ab,vw}$, then it is also independent of $g_{ab,v}$.

It should also be noted that due to our second invariance identity the expression for $E^{ij}(L)$, given by (2.20), reduces to

$$E^{ij}(L) = -(\Lambda^{ij,kl}{}_{|kl} + \Pi^{ij}). \qquad (2.39)$$

To determine our final invariance identity we begin by differentiating (2.8) with respect to B^r_s to obtain

$$\bar{\Phi}\frac{\partial \bar{\phi}}{\partial B^r_s} + \bar{\Phi}^h \frac{\partial \bar{\phi}_{,h}}{\partial B^r_s} + \bar{\Lambda}^{hk}\frac{\partial \bar{g}_{hk}}{\partial B^r_s} + \bar{\Lambda}^{hk,l}\frac{\partial \bar{g}_{hk,l}}{\partial B^r_s} + \bar{\Lambda}^{hk,lm}\frac{\partial \bar{g}_{hk,lm}}{\partial B^r_s} = \frac{\partial B}{\partial B^r_s} L = A^s_r BL, \qquad (2.40)$$

where

$$A^s_r = \frac{\partial \bar{x}^s}{\partial x^r},$$

and

$$A^s_r B^r_m = \delta^s_m .$$

The first term on the left hand side of (2.40) does not contribute since $\frac{\partial \bar{\phi}}{\partial B^r_s} = 0$. Rund has found that $\bar{\Lambda}^{hk}\frac{\partial \bar{g}_{hk}}{\partial B^r_s}$, $\bar{\Lambda}^{hk,l}\frac{\partial \bar{g}_{hk,l}}{\partial B^r_s}$ and $\bar{\Lambda}^{hk,lm}\frac{\partial \bar{g}_{hk,lm}}{\partial B^r_s}$ are given by:

$$\bar{\Lambda}^{hk}\frac{\partial \bar{g}_{hk}}{\partial B^r_s} = 2g_{rj}\bar{\Lambda}^{sk}B^j_k ,$$

$$\bar{\Lambda}^{hk,l}\frac{\partial \bar{g}_{hk,l}}{\partial B^r_s} = g_{ij,r}B^i_hB^j_k\bar{\Lambda}^{hk,s} + 2g_{rj}B^j_{hl}\bar{\Lambda}^{hs,l} + + 2g_{rj,p}B^p_lB^j_k\bar{\Lambda}^{sk,l} ,$$

and

$$\bar{\Lambda}^{hk,lm}\frac{\partial \bar{g}_{hk,lm}}{\partial B^r_s} = 2B^i_hB^j_kB^q_m(g_{rj,iq}\bar{\Lambda}^{sk,hm} + g_{ij,rq}\bar{\Lambda}^{hk,sm}) + + 2g_{rj}B^j_{hlm}\bar{\Lambda}^{hs,lm} + 4g_{ij,r}B^i_{hl}B^j_k\bar{\Lambda}^{hk,ls} + + 2g_{rj,p}(B^p_{lm}B^j_k + 2B^j_{km}B^p_l)\bar{\Lambda}^{sk,lm} .$$

Thus $\dfrac{\partial \bar{\phi}_{,h}}{\partial B^r_s}$ remains to be calculated.

Since

$$\bar{\phi}_{,h} = \phi_{,l}B^l_h ,$$

we wee that

$$\frac{\partial \bar{\phi}_{,h}}{\partial B^r_s} = \phi_{,r}\delta^s_h .$$

Upon substituting the above expressions for

$\bar{\Lambda}^{hk}\dfrac{\partial \bar{g}_{hk}}{\partial B^r_s}$, $\bar{\Lambda}^{hk,l}\dfrac{\partial \bar{g}_{hk,l}}{\partial B^r_s}$, $\bar{\Lambda}^{hk,lm}\dfrac{\partial \bar{g}_{hk,lm}}{\partial B^r_s}$ and $\dfrac{\partial \bar{\phi}_{,h}}{\partial B^r_s}$

into (2.40), dividing that result by 2 and multiplying through by B^r_t we obtain

$$g_{rj}B^j_{hlm}B^r_t\bar{\Lambda}^{hs,lm} + g_{rj,iq}B^r_tB^i_hB^j_kB^q_m\bar{\Lambda}^{sk,hm} + + g_{ij,rq}B^i_hB^j_kB^r_tB^q_m\bar{\Lambda}^{hk,sm} + 2g_{ij,r}B^r_tB^i_{hl}B^j_k\bar{\Lambda}^{hk,ls} + + g_{rj,p}\bar{\Lambda}^{sk,lm}B^r_t(B^p_{lm}B^j_k + 2B^j_{km}B^p_l) + \tfrac{1}{2}g_{ij,r}B^r_tB^i_hB^j_k\bar{\Lambda}^{hk,s} + + g_{rj,p}B^r_tB^p_lB^j_k\bar{\Lambda}^{sk,l} + g_{rj}B^r_tB^j_{hl}\bar{\Lambda}^{hs,l} + \bar{g}_{tk}\bar{\Lambda}^{sk} + \tfrac{1}{2}\bar{\Phi}^s\phi_{,r}B^r_t = = \tfrac{1}{2}\delta^s_tBL . \qquad (2.41)$$

Using (2.7) we see that the first three terms appearing on the left hand side of the above expression involve B^j_{hlm}. Rund has shown that the coefficient of B^j_{hlm} appearing in (2.41) is

$$-g_{ji}B^i_k(\bar\Lambda^{sk,hm} + 2\bar\Lambda^{hk,sm}). \qquad (2.42)$$

If we now make use of our first invariance identity we find that[3]

$$-g_{ji}B^i_kB^j_{hlm}(\bar\Lambda^{sk,hm} + 2\bar\Lambda^{hk,sm}) = 0. \qquad (2.43)$$

Due to (2.43) equation (2.41) may be rewritten

$$\bar\Lambda^{hk,sm}\left\{\bar g_{hk,tm} + \bar g_{tm,hk}\right\} + 2\bar\Lambda^{mk,sh}\left\{g_{ij}B^i_{ht}B^j_{km} + g_{ij,p}B^p_mB^j_kB^i_{th}\right\} - \bar\Lambda^{hk,sm}g_{ij,p}B^p_{tm}B^i_hB^j_k + \tfrac{1}{2}g_{ij,r}B^r_tB^i_hB^j_k\bar\Lambda^{hk,s} + g_{rj,p}B^r_tB^p_lB^j_k\bar\Lambda^{sk,l} + g_{rj}B^r_tB^j_{hl}\bar\Lambda^{hs,l} + \bar g_{tk}\bar\Lambda^{sk} + \tfrac{1}{2}\bar\Phi^s\bar\phi_{,t} = \tfrac{1}{2}\delta^s_t BL. \qquad (2.44)$$

The last term on the left hand side of (2.44) was obtained from $\tfrac{1}{2}\bar\Phi^s\phi_{,r}B^r_t$ by using equation (2.4).

After further manipulations similar to those performed by Rund equation (2.44) becomes

$$\bar\Lambda^{hk,sm}\left\{\bar g_{hk,tm} + \bar g_{tm,hk}\right\} + \bar g_{tk,l}\bar\Lambda^{sk,l} + \tfrac{1}{2}\bar g_{hk,t}\bar\Lambda^{hk,s} + \bar g_{tk}\bar\Lambda^{sk} + \tfrac{1}{2}\bar\Phi^s\bar\phi_{,t} - \tfrac{1}{2}\delta^s_t\bar L = -g_{ij}B^j_kB^i_{th}\left\{2\bar\Gamma^k_{mp}\bar\Lambda^{mp,sh} - \bar\Lambda^{hk,s} - \bar\Lambda^{sk,h}\right\}. \qquad (2.45)$$

If we now make use of our second invariance identity we find that the right hand side of the above expression is zero. Since the left hand side of equation (2.45) refers to a single coordinate syatem we shall let that be the unbarred one[4], and so (2.45) becomes

$$\Lambda^{hk,sm}\left\{g_{hk,tm} + g_{tm,hk}\right\} + g_{tk,l}\Lambda^{sk,l} + \tfrac{1}{2}g_{hk,t}\Lambda^{hk,s} + g_{tk}\Lambda^{sk} + \tfrac{1}{2}\Phi^s\phi_{,t} = \tfrac{1}{2}\delta^s_t L. \qquad (2.46)$$

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[3]To obtain this result we make use of the fact that

$$B^j_{hlm} = B^j_{mlh}.$$

[4]This step is in order since the left hand side of (2.45) will be shown to be tensorial.

We desire to express the left hand side of (2.46) in tensorial form. We know that $\Phi^s \phi_{,t}$ is a mixed tensor density, thus we need only consider the remaining terms. However, these are precisely the terms dealt with by Rund. Thus we find

$$g_{tk}\Pi^{sk} + g_{im}R_h{}^i{}_{tk}\Lambda^{hk,sm} - \tfrac{1}{3}g_{tk}R_l{}^s{}_{hm}\Lambda^{hk,lm} + \tfrac{1}{2}\Phi^s\phi_{,t} = \\ = \tfrac{1}{2}\delta^s_t L. \qquad (2.47)$$

Upon multiplying this result by g^{tr} we obtain

$$\Pi^{sr} + R_k{}^r{}_{mh}\Lambda^{hk,sm} - \tfrac{1}{3}R_k{}^s{}_{hm}\Lambda^{hr,km} + \tfrac{1}{2}\Phi^s\phi^{,r} = \tfrac{1}{2}g^{rs}L\ , \quad (2.48)$$

where

$$\phi^{,r} = g^{rt}\phi_{,t}\ .$$

Since Π^{rs} and g^{rs} are symmetric in r and s the remaining group of terms in (2.48) must also be symmetric under the interchange of r and s; i.e.,

$$R_k{}^r{}_{mh}\Lambda^{hk,sm} - \tfrac{1}{3}R_k{}^s{}_{hm}\Lambda^{hr,km} + \tfrac{1}{2}\Phi^s\phi^{,r} = \\ = R_k{}^s{}_{mh}\Lambda^{hk,rm} - \tfrac{1}{3}R_k{}^r{}_{hm}\Lambda^{hs,km} + \tfrac{1}{2}\Phi^r\phi^{,s}\ . \qquad (2.49)$$

Using our first invariance identity we obtain

$$R_k{}^r{}_{mh}\Lambda^{hs,km} = -R_k{}^r{}_{mh}\Lambda^{hk,sm}\ ,$$

and thus equation (2.49) becomes

$$\tfrac{4}{3}R_k{}^r{}_{mh}\Lambda^{hk,sm} + \tfrac{1}{2}\Phi^s\phi^{,r} = \tfrac{4}{3}R_k{}^s{}_{mh}\Lambda^{hk,rm} + \tfrac{1}{2}\Phi^r\phi^{,s}\ . \quad (2.50)$$

Upon rearrangement of equation (2.50) we find

$$R_k{}^s{}_{mh}\Lambda^{hk,rm} = R_k{}^r{}_{mh}\Lambda^{hk,sm} + \tfrac{3}{8}(\Phi^s\phi^{,r} - \Phi^r\phi^{,s})\ . \quad (2.51)$$

The substitution of (2.51) into equation (2.48) yields

$$\Pi^{rs} + \tfrac{2}{3}R_k{}^r{}_{mh}\Lambda^{hk,sm} + \left\{\tfrac{3}{8}\Phi^s\phi^{,r} + \tfrac{1}{8}\Phi^r\phi^{,s}\right\} = \tfrac{1}{2}g^{rs}L\ , \quad (2.52)$$

which is the THIRD INVARIANCE IDENTITY for our Lagrangian (2.1). This result differs from Rund's owing to the presence of the term within the curly brackets. However, (2.52) reduces formally to the form of Rund's result if L is independent of $\phi_{,r}$.

In passing we note that it is possible to obtain precisely the three invariance identities (2.25), (2.37) and (2.52) from the expression obtained by differentiating (2.8) first with respect to B^r_s and subsequently with respect to B^r_{stu} and B^r_{st} respectively. However, in general this technique is not as general as the method we have used.

The next step in the study of our Lagrangian (2.1) is devoted to the divergence of $E^{hk}(L)$, viz.,

$$E^{hk}(L)_{|k} = -(\Lambda^{hk,lm}{}_{|lmk} + \Pi^{hk}{}_{|k}) . \qquad (2.53)$$

Rund has shown that in his case the covariant divergence of $E^{hk}(L)$ is identically zero. In general, this will not be true in our case.

Our calculation of $E^{hk}(L)_{|k}$ is carried out in three steps. The first step is to determine $\Lambda^{hk,lm}{}_{|lmk}$, the second $\Pi^{hk}{}_{|k}$, and then finally $E^{hk}(L)_{|k}$. Firstly we see that

$$\Lambda^{hk,lm}{}_{|lmk} = \frac{1}{3}(p^h + q^h) , \qquad (2.54)$$

where

$$p^h = \Lambda^{hl,km}{}_{|mkl} - \Lambda^{hl,km}{}_{|mlk} \qquad (2.55)$$

and[5]

$$q^h = \Lambda^{hm,kl}{}_{|klm} - \Lambda^{hm,kl}{}_{|mlk} . \qquad (2.56)$$

Equation (2.55) may be rewritten by using Ricci's identity[6] along with our first invariance identity to obtain

$$p^r = \Lambda^{hk,ij}{}_{|i} R_h{}^r{}_{jk} . \qquad (2.57)$$

The modifying of (2.56) is quite a lengthy task resulting in

$$q^r = 2\Lambda^{hk,ij}{}_{|i} R_h{}^r{}_{jk} + \Lambda^{rk,hj}{}_{|i} R_h{}^i{}_{jk} - \Lambda^{hk,ij} R_h{}^r{}_{ki|j} + - \Lambda^{rk,ih} R_h{}^j{}_{ki\,j} . \qquad (2.58)$$

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[5]In Rund's paper π^h is used in place of q^h.

[6][10], page 32.

Equation (2.58) may be further simplified if we make use of equation (2.51), which can be written as

$$R_h{}^r{}_{kj}\Lambda^{hk,ij} = R_h{}^i{}_{kj}\Lambda^{hk,rj} + \tfrac{3}{8}(\Phi^i\phi^{,r} - \Phi^r\phi^{,i}) \,. \quad (2.59)$$

By taking the covariant derivative of each side of equation (2.59) with respect to x^i we obtain

$$R_h{}^r{}_{kj|i}\Lambda^{hk,ij} + R_h{}^r{}_{kj}\Lambda^{hk,ij}{}_{|i} = R_h{}^i{}_{kj|i}\Lambda^{hk,rj} + R_h{}^i{}_{kj}\Lambda^{hk,rj}{}_{|i} + \tfrac{3}{8}(\Phi^i\phi^{,r} - \Phi^r\phi^{,i})_{|i} \,.$$

This expression may be rewritten in the form

$$\Lambda^{hk,ij}{}_{|i}R_h{}^r{}_{kj} - \Lambda^{hk,rj}{}_{|i}R_h{}^i{}_{kj} - \tfrac{3}{8}(\Phi^i\phi^{,r} - \Phi^r\phi^{,i})_{|i} = -(\Lambda^{hk,ij}R_h{}^r{}_{ki|j} + \Lambda^{rk,ih}R_h{}^j{}_{ki|j}) \,,$$

which when substituted into (2.58) yields

$$q^r = \Lambda^{hk,ij}{}_{|i}R_h{}^r{}_{jk} + R_h{}^i{}_{jk}(\Lambda^{rk,hj}{}_{|i} + \Lambda^{hk,rj}{}_{|i}) + - \tfrac{3}{8}(\Phi^i\phi^{,r} - \Phi^r\phi^{,i})_{|i} \,. \quad (2.60)$$

By the use of our first invariance identity and the symmetry properties of the curvature tensor q^r reduces to

$$q^r = R_h{}^r{}_{jk}\Lambda^{hk,ij}{}_{|i} - \tfrac{3}{8}(\Phi^i\phi^{,r} - \Phi^r\phi^{,i})_{|i} \,. \quad (2.61)$$

Combining (2.54), (2.57) and (2.61) we obtain the final expression for $\Lambda^{rk,lm}{}_{|lmk}$, which is

$$\Lambda^{rk,lm}{}_{|lmk} = \tfrac{2}{3}R_h{}^r{}_{jk}\Lambda^{hk,ij}{}_{|i} - \tfrac{1}{8}(\Phi^i\phi^{,r} - \Phi^r\phi^{,i})_{|i} . \quad (2.62)$$

We shall now determine $\Pi^{hk}{}_{|k}$. This may be accomplished through the use of our third invariance identity, which can be rewritten as

$$\Pi^{rs} = -\tfrac{2}{3}R_k{}^r{}_{mh}\Lambda^{hk,sm} - \tfrac{3}{8}\Phi^s\phi^{,r} - \tfrac{1}{8}\Phi^r\phi^{,s} + \tfrac{1}{2}g^{rs}L \,. \quad (2.63)$$

The first step in this calculation is the evaluation of $L_{|r}$.

In terms of our notation $L_{|r}$ is given by

$$L_{|r} = \Phi\phi_{,r} + \Phi^{l}\phi_{,lr} + \Lambda^{hk}g_{hk,r} + \Lambda^{hk,l}g_{hk,lr} + \Lambda^{hk,lm}g_{hk,lmr} + - L\Gamma^{j}_{jr} \quad ,$$

and since

$$\phi_{,lr} = \phi_{|lr} + \phi_m\Gamma^{m}_{lr}$$

we may rewrite the above expression for $L_{|r}$ in the form

$$L_{|r} = \Phi\phi_{,r} + \Phi^{l}\phi_{|lr} + \Phi^{l}\phi_{,m}\Gamma^{m}_{lr} + \Lambda^{hk}g_{hk,r} + \Lambda^{hk,l}g_{hk,lr} + + \Lambda^{hk,lm}g_{hk,lmr} - L\Gamma^{j}_{jr} \, . \qquad (2.64)$$

In order to evaluate equation (2.64) we shall use normal coordinates. In this case $g_{hk,j}$ and Γ^{h}_{ij} vanish at the pole P, and from equations (2.14) and (2.37) we see that $\Lambda^{hk,l}$ also vanishes at P. Consequently at P we have

$$L_{|r} = \Lambda^{hk,lm}g_{hk,lmr} + \Phi\phi_{,r} + \Phi^{l}\phi_{|lr} \, .$$

Rund shows that at P

$$\Lambda^{hk,lm}\, g_{hk,lmr} = \tfrac{2}{3}R_{lkhm|r}\Lambda^{hk,lm} \, .$$

Thus at P we find

$$L_{|r} = \tfrac{2}{3}R_{lkhm|r}\Lambda^{hk,lm} + \Phi\phi_{,r} + \Phi^{l}\phi_{|lr} \, . \qquad (2.65)$$

Since both sides of the above expression are tensorial quantities we no longer need restrict ourselves to the pole P of a normal coordinate system, and thus equation (2.65) is valid everywhere in all coordinate systems. By applying the Bianchi identities to the first term on the right hand side of equation (2.65) we finally obtain

$$L_{|r} = \tfrac{4}{3}R_{hrjk|i}\Lambda^{hk,ij} + \Phi\phi_{,r} + \Phi^{l}\phi_{|lr} \, . \qquad (2.66)$$

With this knowledge of $L_{|r}$ at our disposal we may now complete the computation of $\Pi^{rs}{}_{|s}$ by taking the covariant derivative of both sides of (2.63) with respect to x^s. Upon so doing we find:

$$\Pi^{rs}{}_{|s} = -\tfrac{2}{3}R_{k}{}^{r}{}_{mh|s}\Lambda^{hk,sm} - \tfrac{2}{3}R_{k}{}^{r}{}_{mh}\Lambda^{hk,sm}{}_{|s} +$$

$$- \tfrac{1}{8}(3\Phi^{s}\phi^{,r} + \Phi^{r}\phi^{,s})_{|s} + \tfrac{1}{2}g^{rs}L_{|s} .$$

Equation (2.66) allows us to rewrite the above expression in the form

$$\Pi^{rs}{}_{|s} = -\tfrac{2}{3}R_{k}{}^{r}{}_{mh}\Lambda^{hk,sm}{}_{|s} + \tfrac{1}{2}\Phi\phi^{,r} + \tfrac{1}{2}\Phi^{l}\phi_{|l}{}^{|r} + - \tfrac{1}{8}(3\Phi^{s}\phi^{,r} + \Phi^{r}\phi^{,s})_{|s} , \qquad (2.67)$$

where

$$\phi_{|l}{}^{|r} = \phi_{|ls}g^{sr} .$$

Upon combining equations (2.62) and (2.67) with (2.53) we find:

$$E^{rk}(L)_{|k} = -\Big\{ \tfrac{2}{3}R_{h}{}^{r}{}_{jk}\Lambda^{hk,ij}{}_{|i} - \tfrac{1}{8}(\Phi^{i}\phi^{,r} - \Phi^{r}\phi^{,i})_{|i} + - \tfrac{2}{3}R_{k}{}^{r}{}_{mh}\Lambda^{hk,sm}{}_{|s} - \tfrac{1}{8}(3\Phi^{i}\phi^{,r} + \Phi^{r}\phi^{,i})_{|i} + + \tfrac{1}{2}\Phi\phi^{,r} + \tfrac{1}{2}\Phi^{l}\phi_{|l}{}^{|r} \Big\} .$$

This expression may be rewritten in the form

$$E^{rk}(L)_{|k} = \tfrac{1}{2}\Big\{ (\Phi^{k}\phi^{,r})_{|k} - \Phi\phi^{,r} - \Phi^{k}\phi_{|k}{}^{|r} \Big\} , \qquad (2.68)$$

which reduces to

$$E^{rk}(L)_{|k} = \tfrac{1}{2}\phi^{,r}(\Phi^{l}{}_{,l} - \Phi) . \qquad (2.69)$$

Using (2.19) we may rewrite the above expression in its final form; viz.,

$$E^{rk}(L)_{|k} = \tfrac{1}{2}\phi^{,r}E(L) . \qquad (2.70)$$

We may summarize the results of this section with the following

Theorem 2.1: If L is a scalar density of the form

$$L = L(g_{ij};\ g_{ij,h};\ g_{ij,hk};\ \phi;\ \phi_{,i}) \qquad (2.1)$$

then,

(1) L and its associated tensor densities $\Lambda^{hk,lm}$, $\Pi^{hk,l}$, Π^{hk}, Φ^{h}, and Φ(defined by $\frac{\partial L}{\partial g_{hk,lm}}$, (2.14), (2.15), $\frac{\partial L}{\partial \phi_{,h}}$,

<u>and</u> $\frac{\partial L}{\partial \phi}$ <u>respectively), satisfy the following three identities:</u>

(i) $\Lambda^{hk,lm} + \Lambda^{hl,mk} + \Lambda^{hm,kl} = 0$, (2.25)

(ii) $\Pi^{hk,l} = 0$ (2.37)

(iii) $\Pi^{rs} + \frac{2}{3}R_{k}{}^{r}{}_{mh}\Lambda^{hk,sm} + \frac{3}{8}\Phi^{s}\phi^{,r} + \frac{1}{8}\Phi^{r}\phi^{,s} = \frac{1}{2}g^{rs}L$. (2.52)

(2) <u>The Euler-Lagrange tensors corresponding to</u> L <u>are given by</u>

$$E^{ij}(L) = -(\Lambda^{ij,lm}{}_{|lm} + \Pi^{ij}), \qquad (2.39)$$

<u>and</u>

$$E(L) = \Phi^{l}{}_{,l} - \Phi , \qquad (2.19)$$

<u>and these two tensors are related by</u>

$$E^{ij}(L)_{|j} = \tfrac{1}{2}\phi^{,i}E(L). \qquad (2.70)$$

From equation (2.70) we see that <u>when the field equations governing the</u> g_{ij}'s <u>are satisfied</u>; i.e.,

$$E^{rk}(L) = 0 ,$$

<u>then the field equation for</u> ϕ <u>will automatically be satisfied</u>;i.e.,

$$E(L) = 0.$$

However, if

$$E(L) = 0$$

we can use equation (2.70) to conclude that

$$E^{rk}(L)_{|k} = 0 ,$$

but this does <u>not</u> necessarily imply that

$$E^{rk}(L) = 0 .$$

The vacuum field equations of the Brans-Dicke theory serve to exemplify this fact (c.f. equation (4.15) and (4.23)).

In a later section we shall consider a Lagrange scalar density of the form

$$L_T = L(g_{ij};\ g_{ij,h};\ g_{ij,hk};\ \phi;\ \phi_{,i}) + \sqrt{g}\ L_m , \qquad (2.71)$$

where L_m denotes the scalar Lagrangian of matter and may contain vector fields, charge densities, matter densities, etc.

However, L_m will always be independent of both the scalar field ϕ, and the derivatives of g_{ij} greater than the first. We shall let $E^{ij}(L)$ denote the Euler-Lagrange tensor derived from the first term on the right hand side of (2.71) and $E^{ij}(L_T)$ denote the Euler-Lagrange tensor obtained from L_T.

The field equations governing the metric potentials will then be

$$E^{ij}(L_T) = E^{ij}(L) - \frac{\partial}{\partial g_{ij}}(\sqrt{g}\,L_m) + \frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial g_{ij,k}}(\sqrt{g}\,L_m)\right) = 0. \qquad (2.72)$$

The Euler-Lagrange equation for ϕ will still be

$$E(L_T) = E(L) = 0 \ ,$$

where $E(L)$ is given by (2.19). The field equations describing the behaviour of the matter variables contained within $\sqrt{g}\,L_m$ are the usual Euler-Lagrange equations which pertain to a matter Lagrangian. We should also note that

$$E^{ij}(L_T)_{|j} = E^{ij}(L)_{|j} + \left\{\frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial g_{ij,k}}(\sqrt{g}L_m)\right) - \frac{\partial}{\partial g_{ij}}(\sqrt{g}\,L_m)\right\}_{|j} \ ,$$

which due to (2.70) can be written in the form[7]

$$E^{ij}(L_T)_{|j} = \tfrac{1}{2}\phi^{,i}E(L) + \left\{\frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial g_{ij,k}}(\sqrt{g}\,L_m)\right) - \frac{\partial}{\partial g_{ij}}(\sqrt{g}\,L_m)\right\}_{|j} . \qquad (2.73)$$

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[7]The term appearing within the curly brackets on the right hand side of (2.73) is not in general identically zero. As an example consider the matter Lagrangian associated with the electromagnetic field

$$\sqrt{g}\ L_m = \tfrac{1}{2}\sqrt{g}\ F_{hk}F^{hk} \ ,$$

where

$$F_{hk} = A_{h|k} - A_{k|h} \ ,$$

and A_h denotes the electromagnetic vector potentials. In this example

$$\left\{\frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial g_{ij,k}}(\sqrt{g}\,L_m)\right) - \frac{\partial}{\partial g_{ij}}(\sqrt{g}\,L_m)\right\}_{|j} = \sqrt{g}\ F^{ik}F_k{}^j{}_{|j} \ ,$$

which is not identically zero.

3. Lagrange Scalar Densities--Applications[8]

This section will essentially consist of two parts. In the first part we shall consider four examples of possible Lagrange scalar densities of the type (2.1). Three of these examples will be used in the second part of this section where we consider the "uniqueness" of the Brans-Dicke vacuum Lagrange scalar density.

Example (I)

The purpose of our first example is to examine the peculiar form of our third invariance identity (2.52). We consider (2.52) to be peculiar because of the non-symmetrical appearance of $\Phi^r \phi^{,s}$ and $\Phi^s \phi^{,r}$ in this expression. We might have expected the linear combination of $\Phi^r \phi^{,s}$ and $\Phi^s \phi^{,r}$ to be symmetric in r and s. However, if this were the case the symmetries of (2.52) would have implied that

$$R_k{}^r{}_{mh} \Lambda^{hk,sm} = R_k{}^s{}_{mh} \Lambda^{hk,rm} \qquad (3.1)$$

(as is the case[9] for Lagrange scalar densities of the form

$$L = L(g_{ij};\, g_{ij,h};\, g_{ij,hk})).$$

To show that (3.1) is not in general true for scalar densities of the form (2.1) let us consider the following scalar density

$$L_1 = \sqrt{g}\, \phi_{,i} \phi_{,j} R^{ij} . \qquad (3.2)$$

---

[8] Throughout the course of this thesis we shall be dealing with various Lagrange scalar densities. These Lagrangians will be denoted by L_A, where A may represent either a number or a letter. We shall denote the various derivatives and tensors obtainable from L_A by placing an (A) beneath these objects; e.g.

$$\frac{\partial L_A}{\partial g_{rs,tv}} = \underset{(A)}{\Lambda}^{rs,tv}$$

[9] Rund, [22] , [23] .

Using (3.2) we find after a lengthy, but straight forward calculation that $\underset{(1)}{\Lambda}^{hk,sm}$ is given by

$$\underset{(1)}{\Lambda}^{hk,sm} = \tfrac{1}{4}\sqrt{g}\{\phi^{,s}\phi^{,k}g^{hm} + \phi^{,m}\phi^{,k}g^{sh} + \phi^{,h}\phi^{,s}g^{km} + \phi^{,h}\phi^{,m}g^{ks} - 2\phi^{,s}\phi^{,m}g^{hk} - 2\phi^{,h}\phi^{,k}g^{sm}\} \quad (3.3)$$

From (3.3) we find

$$R_k{}^r{}_{mh}\underset{(1)}{\Lambda}^{hk,sm} = \tfrac{3}{4}\sqrt{g}\,(R^{krms}\phi_{,k}\phi_{,m} + R^{rm}\phi_{,m}\phi^{,s}) \ . \quad (3.4)$$

Now since $R^{krms}\phi_{,k}\phi_{,m}$ is symmetric in r and s and $R^{rm}\phi_{,m}\phi^{,s}$ is non-symmetric in r and s we may conclude that

$$R_k{}^r{}_{mh}\underset{(1)}{\Lambda}^{hk,sm} \neq R_k{}^s{}_{mh}\underset{(1)}{\Lambda}^{hk,rm} . \quad (3.5)$$

Thus in general

$$R_k{}^r{}_{mh}\Lambda^{hk,sm} \neq R_k{}^s{}_{mh}\Lambda^{hk,rm} \quad (3.6)$$

when dealing with Lagrangians of the form (2.1).

Example (II)

The following scalar density

$$L_B = \sqrt{g}(f_1(\phi)\,R + f_2(\phi)\phi_{,i}\phi^{,i} + f_3(\phi)), \quad (3.7)$$

where f_1, f_2, and f_3 are arbitrary scalar functions of class C^2 in their arguments, was first investigated by Bergmann[3]. He has suggested that (3.7) is the most general Lagrange scalar density of the form (2.1) which yields field equations of second order in the derivatives of both g_{ij} and ϕ. The merits of Bergmann's choice of Lagrangian will be discussed later in this section. However, we shall now give a trivial example which shows that L_B is not the most general scalar density of the form (2.1) compatible with the above restiction upon the field equations.

Let us consider[10]

$$L = \sqrt{g}\,f(\rho), \quad (3.8)$$

[10] Note that Bergmann's Lagrangian (3.7) does not contain a term of the form (3.8).

where f is an arbitrary scalar function of class C^2 in its argument $\rho = \phi_{,i}\phi^{,i}$. In order to obtain the field equations corresponding to L_2 it will be necessary to compute $\underset{(2)}{\Lambda}^{ij}$ and $\underset{(2)}{\Phi}^{i}$. These are given by

$$\underset{(2)}{\Lambda}^{ij} = \tfrac{1}{2}\sqrt{g}\, g^{ij} f - \sqrt{g}\, f' \phi^{,i}\phi^{,j}, \tag{3.9}$$

and

$$\underset{(2)}{\Phi}^{i} = 2\sqrt{g}\, f' \phi^{,i}, \tag{3.10}$$

where

$$f' = \frac{df}{d\rho} . \tag{3.11}$$

Thus we may use equations (2.18) and (2.19) to conclude that

$$E^{ij}(L_2) = \sqrt{g}\left\{ f'\phi^{,i}\phi^{,j} - \tfrac{1}{2} g^{ij} f \right\}, \tag{3.12}$$

and

$$E(L_2) = 2\sqrt{g}\left\{ \Box\phi\, f' + 2f''\phi^{,p}\phi^{,i}\phi_{|pi} \right\}, \tag{3.13}$$

where

$$\Box\phi = \frac{1}{\sqrt{g}}(\sqrt{g}\, g^{ij}\phi_{,i})_{,j} = g^{ij}\phi_{|ij} . \tag{3.14}$$

Equations (3.12) and (3.13) show that L_2 could be added to L_B without changing the order of the field equations.

<u>Example (III)</u>[11]

The purpose of our present example is to obtain the Euler Lagrange tensors corresponding to the following scalar density

$$L_3 = f_4(\phi)\, L_4 , \tag{3.15}$$

where

$$L_4 = \sqrt{g}(R^2 - 4R_{ij}R^{ij} + R_{ijhk}R^{ijhk}) , \tag{3.16}$$

and $f_4 = f_4(\phi)$ is an arbitrary scalar function of class C^2. The calculations necessary to determine $E^{ij}(L_3)$ will be simplified slightly if we first consider some of the properties of L_4.

We begin by rewriting L_4 in the form

$$L_4 = \tfrac{1}{4}\sqrt{g}\,\delta^{abcd}_{efhi}\, R_{ab}{}^{ef} R_{cd}{}^{hi} , \tag{3.17}$$

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[11] Throughout the rest of this thesis we shall confine our attention to 4 dimensional Riemannian spaces.

where

$$\delta^{abcd}_{efhi} = \det\begin{pmatrix} \delta^a_e & \delta^a_f & \delta^a_h & \delta^a_i \\ \delta^b_e & \delta^b_f & \delta^b_h & \delta^b_i \\ \delta^c_e & \delta^c_f & \delta^c_h & \delta^c_i \\ \delta^d_e & \delta^d_f & \delta^d_h & \delta^d_i \end{pmatrix}. \tag{3.18}$$

Using (3.17) Lovelock [15] has shown that

$$\underset{(4)}{\Lambda}^{rs,tv} = \frac{\sqrt{g}}{4}\,\delta^{abcd}_{efhi}\,(\delta^r_a\delta^s_l + \delta^r_l\delta^s_a)(\delta^t_b\delta^v_k + \delta^t_k\delta^v_b)\; g^{ke}g^{lf}R_{cd}{}^{hi}, \tag{3.19}$$

and that the covariant divergence of the above expression with respect to x^t is identically zero; i.e.,

$$\underset{(4)}{\Lambda}^{rs,tv}{}_{|t} \equiv 0. \tag{3.20}$$

Now since $\underset{(4)}{\Phi}^r$ is identically zero we may use equation (2.52) to deduce that

$$\underset{(4)}{\Pi}^{rs} = \frac{1}{2}\,g^{rs}L_4 - \frac{2}{3}\,R_k{}^r{}_{mh}\underset{(4)}{\Lambda}^{hk,sm}. \tag{3.21}$$

Using (3.19) we find that the second term on the right hand side of (3.21) may be written

$$-\frac{2}{3}\,R_k{}^r{}_{mh}\underset{(4)}{\Lambda}^{hk,sm} = -\frac{1}{6}\sqrt{g}\left\{\left[\delta^{ascd}_{efpq}(R^{fre}{}_a + R_a{}^{ref}) + \right.\right.$$
$$\left.\left. + \delta^{abcd}_{efpq}(R^{fr}{}_{ba} + R_a{}^r{}_b{}^f)g^{se}\right] R_{cd}{}^{pq}\right\}. \tag{3.22}$$

A straightforward computation yields

$$\delta^{abcd}_{efpq}R_{cd}{}^{pq} = 2(\delta^a_e\delta^b_f - \delta^a_f\delta^b_e)R + 4(\delta^a_f R^b_e - \delta^a_e R^b_f) +$$
$$+ 4(\delta^b_e R^a_f - \delta^b_f R^a_e) + 4R^{ab}{}_{ef}. \tag{3.23}$$

Upon combining (3.23) with (3.22) we find

$$-\frac{2}{3}\,R_k{}^r{}_{mh}\underset{(4)}{\Lambda}^{hk,sm} = -2(R\,R^{rs} - 2R^{rf}R^s_f + R_f{}^s{}_{ab}R^{frab} +$$
$$-2R^{af}R_a{}^r{}_f{}^s). \tag{3.24}$$

If we now make use of equations (3.16), (3.21) and (3.24) we see that

$$\underset{(4)}{\Pi}^{rs} = \frac{1}{2}\,g^{rs}(R^2 - 4R^{ab}R_{ab} + R_{abcd}R^{abcd}) - 2(RR^{rs} - 2R^{ra}R^s_a +$$
$$+ R_a{}^s{}_{bc}R^{arbc} - 2R^{ab}R_a{}^r{}_b{}^s), \tag{3.25}$$

which vanishes identically in a four dimensional space as a result of the Bach identity [2].[12]

With the above information with regards to L_4 at our disposal we may now easily obtain $E^{ij}(L_3)$.

To begin we have

$$\underset{(3)}{\Lambda}^{rs,tv} = f_4(\phi)\underset{(4)}{\Lambda}^{rs,tv} , \qquad (3.26)$$

and using (3.20) we find

$$\underset{(3)}{\Lambda}^{rs,tv}{}_{|tv} = f_4(\phi)_{|tv}\underset{(4)}{\Lambda}^{rs,tv} . \qquad (3.27)$$

Since $\underset{(3)}{\Phi}^{r}$ is identically zero we may use our third invariance identity (2.52) to obtain

$$\underset{(3)}{\Pi}^{rs} = \frac{1}{2} g^{rs}L_3 - \frac{2}{3} R_k{}^r{}_{mh}\underset{(3)}{\Lambda}^{hk,sm} . \qquad (3.28)$$

Equations (3.15), (3.21) and (3.26) permit us to rewrite the above expression in the form

$$\underset{(3)}{\Pi}^{rs} = f_4(\phi)\underset{(4)}{\Pi}^{rs} . \qquad (3.29)$$

Due to the fact that $\underset{(4)}{\Pi}^{rs}$ is identically zero we have

$$\underset{(3)}{\Pi}^{rs} = 0 . \qquad (3.30)$$

If we now insert equations (3.27) and (3.30) into (2.39) we find

$$E^{rs}(L_3) = - f_4(\phi)_{|tv}\underset{(4)}{\Lambda}^{rs,tv} , \qquad (3.31)$$

where $\underset{(4)}{\Lambda}^{rs,tv}$ is given by (3.19). (3.31) is clearly of second order in the derivatives of g_{ij} and ϕ.

Using (2.19) we obtain

$$E(L_3) = - \frac{df_4}{d\phi} L_4 , \qquad (3.32)$$

which is of second order in the derivatives of g_{ij} and of zeroth order in the derivatives of ϕ.

In passing we note that if the field equation

$$E^{rs}(L_3) = 0$$

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[12] This identity is usually referred to as the Lanczos identity [12], but since it was first discovered by Bach [2] we shall refer to it as the Bach identity.

is satisfied, and if $\frac{df_4}{d\phi}$ is non-zero then equations (2.70) and (3.32) imply that

$$L_4 = 0 . \tag{3.33}$$

We shall now show that our present example may be used to construct a multitude of scalar densities of the form

$$L = L(g_{ij};\ g_{ij,h};\ g_{ij,hk};\ \phi;\ \phi_{,i};\ \phi_{,ij})$$

which identically satisfy both sets of Euler-Lagrange equations. To see this let us begin by considering

$$\mathcal{L} = g_{rs}E^{rs}(L_3), \tag{3.34}$$

which is given by

$$\mathcal{L} = -\ g_{rs}f_4(\phi)_{|tv}\frac{\sqrt{g}}{4}\,\delta^{abcd}_{efhi}(\delta^r_a\delta^s_l + \delta^r_l\delta^s_a)(\delta^t_b\delta^v_k + \delta^t_k\delta^v_b)g^{ke}g^{lf}R_{cd}{}^{hi}\,, \tag{3.35}$$

Using (3.23) we find that (3.35) simplifies to

$$\mathcal{L} = -\ 4\ f_4(\phi)_{|tv}\sqrt{g}(R^{tv} - \tfrac{1}{2}g^{tv}R)\,. \tag{3.36}$$

Now since

$$(\sqrt{g}\ (\ R^{tv} - \tfrac{1}{2}g^{tv}R))_{|v} \equiv 0$$

we may rewrite (3.36) as follows:

$$\mathcal{L} = -\ 4(f_4(\phi)_{|t}\sqrt{g}(R^{tv} - \tfrac{1}{2}g^{tv}R))_{|v}\,. \tag{3.37}$$

However,

$$-\ 4\sqrt{g}\,f_4(\phi)_{|t}\ (R^{tv} - \tfrac{1}{2}g^{tv}R)$$

is a vector density, implying that the covariant divergence appearing in (3.37) may be replaced by an ordinary divergence. Thus $\mathcal{L}$ is an ordinary divergence. Consequently we may apply Lovelock's [16] result, (with regards to divergences identically satisfying their Euler-Lagrange equations), and conclude that

$$E^{ij}(\mathcal{L}) \equiv 0\ ,$$

and

$$E(\mathcal{L}) \equiv 0\,.$$

The above analysis of $\mathcal{L}$ admits an immediate generalization which assumes the form of the following

Theorem 3.1: Let P^{ij} be a contravariant tensor density of valency two which is a concomitant of g_{ab} and its first p derivatives (where p is any integer ≥ 0) and which enjoys the following property

$$P^{ij}{}_{|j} = 0 ;$$

and let A_i ,

$$A_i = A_i(\phi;\ \phi_{,i_1};\ \ldots;\ \phi_{,i_1 \ldots i_q};\ g_{ab};\ g_{ab,j_1};\ldots;\ g_{ab,j_1\ldots j_r}),$$

(where q and r are any integers ≥ 0) be a covariant vector, then the scalar density

$$L_G = A_{i|j}\, P^{ij}$$

identically satisfies the Euler-Lagrange equations corresponding to both g_{ij} and ϕ.[13]

proof: Due to the fact that $P^{ij}{}_{|j} = 0$ we may write L_G as follows:

$$L_G = (A_i P^{ij})_{|j} .$$

Since P^{ij} is a tensor density $A_i P^{ij}$ is a contravariant vector density and thus

$$L_G = (A_i P^{ij})_{,j} .$$

Thus L_G is an ordinary divergence and we may now apply Lovelock's [16] results to conclude that the Euler-Lagrange tensors corresponding to L_G are identically zero.[14] Q.E.D.

---

[13]This theorem is also valid if

$$A_i = A_i(V_k;\ V_{k,i_1};\ldots;\ V_{k,i_1\ldots i_q};\ g_{ab};\ g_{ab,j_1};\ldots;\ g_{ab,j_1\ldots j_r}),$$

where V_k is a covariant vector.

[14]Note that the proof is actually independent of the dimension of the space under consideration.

In passing we remark that Lovelock [14] has constructed <u>all</u> <u>symmetric</u> divergence free contravariant tensor densities of rank 2 which are concomitants of g_{ij}, and its first two derivatives. Thus by combining any one of Lovelock's divergence free tensors with an A_i of the form

$$A_i = H(\phi)_{|i},$$

where $H(\phi)$ is an arbitrary function of class C^4, we can construct a multitude of scalar densities of the form

$$L = L(g_{ij};\ g_{ij,h};\ g_{ij,hk};\ \phi;\ \phi_{,i};\ \phi_{,ij})$$

which identically satisfy both sets of Euler-Lagrange equations. In particular when $n = 4$ Lovelock has shown that the most general symmetric, divergence free, contravariant tensor density of rank 2 which is a concomitant of g_{ij}, $g_{ij,h}$, and $g_{ij,hk}$ is given by

$$a\sqrt{g}\,(R^{ij} - \tfrac{1}{2}g^{ij}R) + b\sqrt{g}\,g^{ij},$$

where a and b are constants. Thus the Euler Lagrange tensors corresponding to

$$a\sqrt{g}\,H(\phi)_{|ij}\,(R^{ij} - \tfrac{1}{2}g^{ij}R) + b\sqrt{g}\,H(\phi)_{|ij}\,g^{ij},$$

are identically zero. In fact this statement is valid for a space of any dimension.

<u>Example</u> (<u>IV</u>)

As our last example we shall consider the Euler-Lagrange tensors corresponding to

$$L_5 = f_5(\phi)\,L_6, \tag{3.38}$$

where

$$L_6 = \epsilon^{pqhi}R_{hijk}R_{pq}{}^{jk}, \tag{3.39}$$

and ϵ^{pqhi} denotes the Levi-Civita four dimensional permutation symbol which is a tensor density. $f_5 = f_5(\phi)$ is an arbitrary scalar of class C^2.

As in our last example let us begin by examining the various properties and tensors associated with L_6. The first tensor we must consider is $\underset{(6)}{\Lambda}^{rs,tv}$ which is given by

$$\underset{(6)}{\Lambda}^{rs,tv} = \epsilon^{pqhi} g^{lj} g^{mk} \left\{ \frac{\partial R_{hijk}}{\partial g_{rs,tv}} R_{pqlm} + R_{hijk} \frac{\partial R_{pqlm}}{\partial g_{rs,tv}} \right\}.$$

Now since

$$\epsilon^{pqhi} = \epsilon^{hipq},$$

we may rewrite the above expression in the form

$$\underset{(6)}{\Lambda}^{rs,tv} = 2\epsilon^{pqhi} g^{lj} g^{mk} R_{pqlm} \frac{\partial R_{hijk}}{\partial g_{rs,tv}} .$$

After a lengthy calculation we find

$$\epsilon^{pqhi} g^{lj} g^{mk} \frac{\partial R_{hijk}}{\partial g_{rs,tv}} = \frac{1}{4} \left\{ \epsilon^{pqrt} \left[g^{lv} g^{ms} - g^{ls} g^{mv} \right] + \right.$$

$$+ \epsilon^{pqrv} \left[g^{lt} g^{ms} - g^{ls} g^{mt} \right] + \epsilon^{pqst} \left[g^{lv} g^{mr} - g^{lr} g^{mv} \right] + \left. + \epsilon^{pqsv} \left[g^{lt} g^{mr} - g^{lr} g^{mt} \right] \right\}. \qquad (3.40)$$

To obtain $\underset{(6)}{\Lambda}^{rs,tv}$ we must multiply (3.40) by $2R_{pqlm}$ and sum over p, q, l and m. Now since each of the terms appearing within square brackets in (3.40) are antisymmetric in l and m we have

$$\left[g^{la} g^{mb} - g^{lb} g^{ma} \right] R_{pqlm} = 2 g^{la} g^{mb} R_{pqlm} .$$

Thus we find that $\underset{(6)}{\Lambda}^{rs,tv}$ is given by

$$\underset{(6)}{\Lambda}^{rs,tv} = \left\{ \epsilon^{pqrt} g^{lv} g^{ms} + \epsilon^{pqrv} g^{lt} g^{ms} + \epsilon^{pqst} g^{lv} g^{mr} + \right. \left. + \epsilon^{pqsv} g^{lt} g^{mr} \right\} R_{pqlm} . \qquad (3.41)$$

Let us now examine the covariant divergence of (3.41) with respect to x^t, which may be written as follows:

$$\underset{(6)}{\Lambda}^{rs,tv}{}_{|t} = \epsilon^{pqhi} (R_{pqlm|i} \delta^v_j + R_{pqlm|j} \delta^v_i)(\delta^r_h \delta^s_k + \delta^r_k \delta^s_h) g^{lj} g^{mk}. \qquad (3.42)$$

The form of the above expression may be simplified if we note that

$$\epsilon^{pqhi} = \epsilon^{qihp} = \epsilon^{iphq} ,$$

and consequently

$$\epsilon^{pqhi}R_{pqlm|i} = \tfrac{1}{3}\,\epsilon^{pqhi}(R_{pqlm|i} + R_{qilm|p} + R_{iplm|q}) ,$$

which vanishes identically due to the Bianchi identity.

Thus (3.42) can be written in the form

$$\underset{(6)}{\Lambda}^{rs,tv}{}_{|t} = \epsilon^{pqhv}R_{pq\ \ m|j}^{\ \ \ j}\,(\delta^r_h\delta^s_k + \delta^r_k\delta^s_h)g^{mk} . \qquad (3.43)$$

Using the Bianchi identity once again we find that (3.43) may be written as follows:

$$\underset{(6)}{\Lambda}^{rs,tv}{}_{|t} = 2\epsilon^{pqhv}R^k_{q|p}\,(\delta^r_h\delta^s_k + \delta^r_k\delta^s_h) . \qquad (3.44)$$

From (3.44) we find that

$$\underset{(6)}{\Lambda}^{rs,tv}{}_{|tv} = 2\epsilon^{pqhv}R^k_{q|pv}\,(\delta^r_h\delta^s_k + \delta^r_k\delta^s_h) , \qquad (3.45)$$

which may be rewritten

$$\underset{(6)}{\Lambda}^{rs,tv}{}_{|tv} = 2\left\{\epsilon^{pqrv}R^s_{q|pv} + \epsilon^{pqsv}R^r_{q|pv}\right\} . \qquad (3.46)$$

Upon making use of the antisymmetry of ϵ^{pqrv} we see that (3.46) may be put in the following form

$$\underset{(6)}{\Lambda}^{rs,tv}{}_{|tv} = \epsilon^{pqrv}(R^s_{q|pv} - R^s_{q|vp}) + \epsilon^{pqsv}(R^r_{q|pv} - R^r_{q|vp}) . \quad (3.47)$$

The Ricci identity permits us to rewrite the above expression as follows:

$$\underset{(6)}{\Lambda}^{rs,tv}{}_{|tv} = \epsilon^{pqrv}(R^m_qR_m{}^s{}_{pv} + R^s_mR^m{}_{qpv}) +$$

$$+\ \epsilon^{pqsv}(R^m_qR_m{}^r{}_{pv} + R^r_mR^m{}_{qpv}) . \quad (3.48)$$

However, due to the fact that

$$R^m{}_{qpv} + R^m{}_{pvq} + R^m{}_{vqp} = 0 ,$$

we have that

$$\epsilon^{pqrv}R^m{}_{qpv} = 0 ,$$

and thus (3.48) becomes

$$\underset{(6)}{\Lambda}^{rs,tv}{}_{|tv} = \epsilon^{pqrv}R^m_qR_m{}^s{}_{pv} + \epsilon^{pqsv}R^m_qR_m{}^r{}_{pv} . \quad (3.49)$$

Now since $\underset{(6)}{\Lambda}^{rs,tv}{}_{|tv}$ is symmetric in r and s we finally obtain

$$\underset{(6)}{\Lambda}{}^{rs,tv}{}_{|tv} = 2\epsilon^{pqrv}R^m_q R_m{}^s{}_{pv} \, . \qquad (3.50)$$

By combining equation (2.52) and (2.39) we may write the expression for $E^{rs}(L_6)$ as follows:

$$E^{rs}(L_6) = -\underset{(6)}{\Lambda}{}^{rs,tv}{}_{|tv} - \frac{1}{2} g^{rs}L_6 + \frac{2}{3} R_k{}^r{}_{mh}\underset{(6)}{\Lambda}{}^{hk,sm} \, . \qquad (3.51)$$

Using equation (3.41) it is possible to show that

$$\frac{2}{3} R_k{}^r{}_{mh}\underset{(6)}{\Lambda}{}^{hk,sm} = R_k{}^r{}_{mh}(\epsilon^{hmpq}R_{pq}{}^{sk} + \epsilon^{kspq}R_{pq}{}^{mh}). \qquad (3.52)$$

Thus by combining equations (3.39), (3.50) and (3.52) we see that (3.51) becomes

$$E^{rs}(L_6) = -2\epsilon^{pqrv}R^m_q R_m{}^s{}_{pv} - \frac{1}{2} g^{rs}\epsilon^{pqhi}R_{hijk}R_{pq}{}^{jk} + $$
$$+ \; R_k{}^r{}_{mh}(\epsilon^{hmpq}R_{pq}{}^{sk} + \epsilon^{kspq}R_{pq}{}^{mh}). \qquad (3.53)$$

However, the right hand side of the above expression is proportional to

$$- g^{sd}\delta^{pqklr}_{ijtud}\epsilon^{ijhm}R_{hmpq}R_{kl}{}^{tu} \, , \qquad (3.54)$$

which vanishes identically in a four dimensional space, since when $n = 4$ we have*

$$\delta^{pqklr}_{ijtud} \equiv 0 \, .$$

We may now use the above information to obtain $E^{ij}(L_5)$. To begin we have

$$\underset{(5)}{\Lambda}{}^{rs,tv} = f_5(\phi)\underset{(6)}{\Lambda}{}^{rs,tv} \, , \qquad (3.55a)$$

and thus

$$\underset{(5)}{\Lambda}{}^{rs,tv}{}_{|tv} = f_5(\phi)_{|tv}\underset{(6)}{\Lambda}{}^{rs,tv} + 2f_5(\phi)_{|t}\underset{(6)}{\Lambda}{}^{rs,tv}{}_{|v} + $$
$$+ f_5(\phi)\underset{(6)}{\Lambda}{}^{rs,tv}{}_{|tv} \, . \qquad (3.55b)$$

Since L_5 is independent of the derivatives of ϕ we obtain

$$\underset{(5)}{\Phi}{}^r = 0 \, . \qquad (3.56)$$

Upon combining (3.56) with our third invariance identity we find

$$\underset{(5)}{\Pi}{}^{rs} = \frac{1}{2} g^{rs}L_5 - \frac{2}{3} R_k{}^r{}_{mh}\underset{(5)}{\Lambda}{}^{hk,sm} \, , \qquad (3.57)$$

which, due to (3.38) and (3.55a), may be rewritten as follows:

*This type of an approach to the derivation of dimensionally dependent identities is due to D.Lovelock [17].

$$\underset{(5)}{\Pi}^{rs} = f_5(\phi)\left\{\frac{1}{2}g^{rs}L_6 - \frac{2}{3}R_k{}^r{}_{mh}\underset{(6)}{\Lambda}^{hk,sm}\right\} . \qquad (3.58)$$

If we now insert equations (3.56b) and (3.58) into (2.39) we obtain

$$E^{rs}(L_5) = - f_5(\phi)_{|tv}\underset{(6)}{\Lambda}^{rs,tv} - 2f_5(\phi)_{|t}\underset{(6)}{\Lambda}^{rs,tv}{}_{|v} + \\ - f_5(\phi)\left\{\underset{(6)}{\Lambda}^{rs,tv}{}_{|tv} + \frac{1}{2}g^{rs}L_6 - \frac{2}{3}R_k{}^r{}_{mh}\underset{(6)}{\Lambda}^{hk,sm}\right\} . \qquad (3.59)$$

The term within curly brackets in the above expression is simply $-E^{rs}(L_6)$ (c.f.(3.51)) which we know vanishes identically in a four dimensional space. Thus (3.59) reduces to

$$E^{rs}(L_5) = - f_5(\phi)_{|tv}\underset{(6)}{\Lambda}^{rs,tv} - 2f_5(\phi)_{|t}\Lambda^{rs,tv}{}_{|v} , \qquad (3.60a)$$

which, due to equations (3.41) and (3.44), is obviously of second order in the derivatives of ϕ and of third order in the derivatives of g_{ij}.

Due to equation (3.56) we may use equation (2.19) to find

$$E(L_5) = - \frac{df_5}{d\phi} L_6 , \qquad (3.60b)$$

which is of zeroth order in the derivatives of ϕ and of second order in the derivatives of g_{ij}.

As in example (III) we note that if the field equation

$$E^{rs}(L_5) = 0$$

is satisfied, and if $\frac{df_5}{d\phi}$ is non-zero, then equations (2.70) and (3.60b) imply that

$$L_6 = 0 .$$

This completes example (IV).

We shall now proceed to discuss the problem of choosing a Lagrange scalar density of the form (2.1) when working in a Riemannian V_4. In order to limit the field of possible choices let us demand that the Euler-Lagrange tensors corresponding to our scalar density be at most second order in the derivatives of g_{ij} and ϕ.

To begin we shall analyze the approach used by Bergmann [3] to obtain the Lagrangian presented in equation (3.7), viz.,

$$L_B = \sqrt{g}\,(f_1(\phi)\,R + f_2(\phi)\phi_{,i}\phi^{,i} + f_3(\phi)). \quad (3.7)$$

The above scalar density has been suggested by Bergmann as the most general Lagrangian compatible with second order field equations. The argument used by Bergmann to obtain (3.7) is as follows:

(i) The only two scalar densities of the form (2.1) which are independent of ϕ and yield second order field equations in g_{ij} are

$$\sqrt{g}\,R \quad \text{and} \quad \sqrt{g}\ .$$

(ii) Upon introducing ϕ the most general Lagrange scalar density compatible with second order field equarions in both g_{ij} and ϕ may be obtained by considering a linear combination of the above scalar densities and $\sqrt{g}\,\phi_{,i}\phi^{,i}$. The coefficients appearing in this linear combination are to be arbitrary functions of class C^2 in their argument ϕ.

From (i) and (ii) Bergmann obtains (3.7).

Let us now examine Bergmann's reasoning. First of all we note that the statement made in (i) is false: i.e., $\sqrt{g}\,R$ and $\sqrt{g}$ are not the only scalar densities which yield field equations of second order in the derivatives of g_{ij}. This has been pointed out by Bach [2] who was the first to prove that the scalar densities L_4 and L_6, given by (3.16) and (3.39) respectively, satisfy the Euler-Lagrange equations identically, and thus trivially satisfy our demand of second order field equations.[15] In fact it has been shown by Lovelock [13] that the most general scalar density of the form

$$L = L(g_{ij};\ g_{ij,h};\ g_{ij,hk}), \qquad (3.61)$$

which yields field equations of second order when in a four dimensional space is given by

$$L = a\sqrt{g} + b\sqrt{g}\,R + c\sqrt{g}(R^2 - 4R_{ij}R^{ij} + R_{ijhk}R^{ijhk}) + \\ + d\epsilon^{ijrs}R_{rskh}R^{kh}{}_{ij}\,, \qquad (3.62)$$

where a, b, c and d are constants.

Before proceeding to examine the second step of Bergmann's argument let us replace (i) by the following correct conclusion:

(i') The only four scalar densities of the form (3.61) which yield Euler-Lagrange tensors which are at most of second order in g_{ij} are given by

$$a\sqrt{g}\ ,\ b\sqrt{g}\,R\ ,\ c\sqrt{g}(R^2 - 4R_{ij}R^{ij} + R_{ijhk}R^{ijhk})\ ,$$

and

$$d\epsilon^{ijrs}R_{rshk}R^{hk}{}_{ij}\,,$$

where a, b, c and d are constants.[16]

In the second step, (ii), of Bergmann's argument there is an oversight in that he neglects to include terms of the form

$$\sqrt{g}\ f(\phi_{,i}\phi^{,i})$$

in his Lagrangian, where f is an arbitrary scalar function of class C^2 in its argument. From example (II) we know that such terms are in fact compatible with second order field equations and thus must be considered.

Step (ii) also gives rise to another difficulty when we try to apply it to (i'), our corrected version of (i). In this case we note that (ii) would imply that

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[15] This result was also established by Lanczos [12].

[16] This result is valid only in a four dimensional space.

$$L_5 = f_5(\phi)\epsilon^{ijrs}R_{rshk}R^{hk}{}_{ij} ,$$

should yield field equations of second order. However, we have seen in example (IV) that this is not the case, since $E^{ij}(L_5)$ is of third order in the derivatives of g_{ij}.

Due to the above considerations it appears that a more general scalar density of the form (2.1) compatible with the demand for field equations which are at most of second order in the derivatives of g_{ij} and ϕ would be[17]

$$L_H = \sqrt{g}\left\{ h_1(\phi)R + h_2(\phi;\ \phi_{,i};\ g_{ij}) + h_3(\phi)(R^2 - 4R_{ij}R^{ij} + R_{ijhk}R^{ijhk})\right\} + a\epsilon^{ijrs}R_{rshk}R^{hk}{}_{ij} , \qquad (3.63)$$

where h_1, h_2 and h_3 are arbitrary scalar functions of class C^2, and a is a constant.

The Brans and Dicke vacuum field equations are derived from the following scalar density:

$$L_{BD} = \sqrt{g}\left\{\phi R - \frac{\omega\phi_{,i}\phi^{,i}}{\phi}\right\} , \qquad (3.64)$$

where ω is a dimensionless constant.[18] From the form of (3.64) it is obvious that L_{BD} could not be the most general scalar density of the form (2.1) which yields second order field equations. However, we shall now show that L_{BD} and L_B are closely related by means of a conformal transformation.[19]

To see the relationship between L_{BD} and L_B let us begin by considering a conformal transformation of the form

$$\bar{g}_{ij} = e^{2\sigma}g_{ij} , \qquad (3.65)$$

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[17] We are not claiming (3.63) to be the most general scalar density of the form (2.1) with the aforementioned properties.

[18] In section 4 we shall replace the g appearing in (3.64) by (-g), since it is customary to assume that g<0 in general relativity.

[19] The conformal relationship between L_{BD} and L_B has been pointed out by Bergmann in [3].

where σ is a scalar function of class C^2 in its argument. Under the above transformation we find[20]

$$\sqrt{g} = e^{-4\sigma}\sqrt{\bar{g}} \,, \qquad (3.66)$$

and

$$R = e^{2\sigma}\bar{R} + \frac{6e^{4\sigma}}{\sqrt{\bar{g}}}(e^{-2\sigma}\sqrt{\bar{g}}\,\bar{g}^{ij}\sigma_{,i})_{,j} + 6\bar{g}^{ij}e^{2\sigma}\sigma_{,i}\sigma_{,j} \,. \qquad (3.67)$$

Equations (3.65), (3.66) and (3.67) permit us to rewrite L_B as follows:

$$L_B = e^{-4\sigma}\sqrt{\bar{g}}\Big\{f_1\,e^{2\sigma}\bar{R} + 6e^{4\sigma}f_1\frac{1}{\sqrt{\bar{g}}}(e^{-2\sigma}\sqrt{\bar{g}}\,\bar{g}^{ij}\sigma_{,i})_{,j} + 6\bar{g}^{ij}f_1\,e^{2\sigma}\sigma_{,i}\sigma_{,j} + f_2\phi_{,i}\phi_{,j}\,e^{2\sigma}\bar{g}^{ij} + f_3\Big\} \,. \qquad (3.68)$$

We shall now choose σ so that the coefficient of $\sqrt{\bar{g}}\,\bar{R}$ appearing in the above expression is ϕ. Thus σ is given by

$$\sigma = \frac{1}{2}\ln\left(\frac{f_1}{\phi}\right) . \qquad (3.69)$$

Using (3.69) we find that (3.68) may be written as follows:

$$L_B = \phi\sqrt{\bar{g}}\,\bar{R} + 3\Big\{\sqrt{\bar{g}}\,\phi\,\bar{g}^{ij}\frac{1}{f_1}\Big(f_1' - \frac{f_1}{\phi}\Big)\phi_{,i}\Big\}_{,j} + - 3\sqrt{\bar{g}}\,\bar{g}^{ij}\Big\{\phi\Big(\frac{f_1'}{f_1}\Big)^2 - \frac{f_1'}{f_1}\Big\}\phi_{,i}\phi_{,j} + \frac{3}{2}\sqrt{\bar{g}}\,\bar{g}^{ij}\Big\{\phi\Big(\frac{f_1'}{f_1}\Big)^2 - 2\frac{f_1'}{f_1} + \frac{1}{\phi}\Big\}\phi_{,i}\phi_{,j} + + \phi\frac{f_2}{f_1}\sqrt{\bar{g}}\,\bar{g}^{ij}\phi_{,i}\phi_{,j} + f_3\Big(\frac{\phi}{f_1}\Big)^2\sqrt{\bar{g}} \,, \qquad (3.70)$$

where

$$f_1' = \frac{df_1}{d\phi} \,.$$

Upon combining similar terms the above expression becomes

$$L_B = \phi\sqrt{\bar{g}}\,\bar{R} + 3\Big\{\sqrt{\bar{g}}\,\phi\,\bar{g}^{ij}\frac{1}{f_1}\Big(f_1' - \frac{f_1}{\phi}\Big)\phi_{,i}\Big\}_{,j} + + \sqrt{\bar{g}}\,\bar{g}^{ij}\phi_{,i}\phi_{,j}\Big\{\frac{3}{2\phi}\Big[1 - \Big(\frac{f_1'\phi}{f_1}\Big)^2\Big] + \frac{\phi f_2}{f_1}\Big\} + f_3\Big(\frac{\phi}{f_1}\Big)^2\sqrt{\bar{g}} \,. \qquad (3.71)$$

The second term appearing on the right hand side of (3.71) is a divergence of the type considered by Lovelock in [16]. Drawing upon Lovelock's results we may conclude that

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[20] These results were obtained from Eisenhart [10], pages 89-90. Note that Eisenhart's Ricci tensor is the negative of ours.

the Euler-Lagrange tensors corresponding to this second term are identically zero. So upon neglecting this term (3.71) becomes[21]

$$L_B = \sqrt{\bar{g}}\left\{ \phi \bar{R} + F_1(\phi)\phi_{,i}\phi_{,j}\bar{g}^{ij} + F_2(\phi)\right\} \quad , \qquad (3.72)$$

where we have set

$$F_1(\phi) = \frac{3}{2\phi}\left[1 - \left(\frac{f_1'\phi}{f_1}\right)^2\right] + \phi\frac{f_2}{f_1} \quad , \qquad (3.73)$$

and

$$F_2(\phi) = f_3\left(\frac{\phi}{f_1}\right)^2 . \qquad (3.74)$$

If we now set $F_2(\phi) = 0$ equation (3.72) reduces to

$$L_B = \sqrt{\bar{g}}\left\{\phi\bar{R} + F_1(\phi)\phi_{,i}\phi_{,j}\bar{g}^{ij}\right\} . \qquad (3.75)$$

Setting $F_2(\phi) = 0$ is not a severe restriction, since it is equivalent to setting $f_3(\phi) = 0$; i.e., in assuming that the cosmological term is negligible.

It is now apparent that L_{BD} can be obtained from (3.75) if we set

$$F_2(\phi) = -\frac{\omega}{\phi} \qquad (3.76)$$

which is quite a severe restriction upon the functions $f_1(\phi)$ and $f_2(\phi)$.

To summarize the above results we have the following

<u>Theorem 3.2</u>: <u>Under the conformal transformation</u>

$$\bar{g}_{ij} = \frac{f_1(\phi)}{\phi}\, g_{ij}$$

<u>the Bergmann Lagrangian</u>

$$L_B = \sqrt{g}\left\{ f_1(\phi)R + f_2(\phi)\phi_{,i}\phi_{,j}g^{ij} + f_3(\phi)\right\}$$

<u>becomes</u>

$$L_B = \sqrt{\bar{g}}\left\{\phi\bar{R} + F_1(\phi)\phi_{,i}\phi_{,j}\bar{g}^{ij} + F_2(\phi)\right\} ,$$

(<u>where</u> $F_1(\phi)$ <u>and</u> $F_2(\phi)$ <u>are given by equations</u> (3.73) <u>and</u> (3.74) <u>respectively</u>) <u>which assumes the form of the Brans-</u>

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[21] In order to compute the Euler-Lagrange tensors corresponding to (3.72) we would have to assume that $f_1(\phi)$ is of class C^3.

Dicke Lagrangian

$$L_{BD} = \sqrt{\bar{g}}\left\{\phi\bar{R} - \frac{\omega}{\phi}\phi_{,i}\phi_{,j}\bar{g}^{ij}\right\},$$

when the cosmological term $F_2(\phi)$ is neglected and

$$F_1(\phi) = -\frac{\omega}{\phi}.$$

Now it should be noted that Theorem 3.2 tells us nothing about the relationship between the Euler-Lagrange tensors corresponding to L_B and L_{BD}. The problem of relating the Euler-Lagrange tensors of conformally related scalar densities will be dealt with in section 5.

4. The Field Equations of the Brans-Dicke Theory

The Lagrange scalar density from which the field equations are to be obtained is[22]

$$L = L_{BD} + \frac{16\pi}{c^4}\sqrt{-g}\,L_m \,, \qquad (4.1)$$

where

$$L_{BD} = \sqrt{-g}\left\{\phi R - \frac{\omega}{\phi}\phi_{,i}\,\phi_{,j}\,g^{ij}\right\}, \qquad (4.2)$$

and ω is a dimensionless constant. In (4.1) $\frac{16\pi}{c^4}L_m$ plays the role of ths scalar Lagrangian of matter mentioned at the conclusion of section 2.[23]

The Euler-Lagrange equations for the metric potentials are given by

$$E^{ij}(L) = 0 \,, \qquad (4.3)$$

where

$$E^{ij}(L) = E^{ij}(L_{BD}) + \frac{16\pi}{c^4}\left\{\frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial g_{ij,k}}(\sqrt{-g}\,L_m)\right) - \frac{\partial}{\partial g_{ij}}(\sqrt{-g}\,L_m)\right\}. \qquad (4.4)$$

We shall now proceed to determine the explicit functional form of (4.4).

Using the fact that du Plessis has shown[24]

$$\frac{\partial(\sqrt{-g}\,R)}{\partial g_{ij,kl}} = -\sqrt{-g}\left\{g^{ij}g^{kl} - \tfrac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk})\right\}, \qquad (4.5)$$

we easily find that

$$\underset{(BD)}{\Lambda}^{ij,kl} = -\phi\sqrt{-g}\left\{g^{ij}g^{kl} - \tfrac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk})\right\}, \qquad (4.6)$$

and thus

$$\underset{(BD)}{\Lambda}^{ij,kl}{}_{|kl} = -\phi_{|kl}\sqrt{-g}\left\{g^{ij}g^{kl} - \tfrac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk})\right\}. \qquad (4.7)$$

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[22] Throughout the rest of this thesis Latin indices will run from 0 to 3 and Greek indices will run from 1 to 3. We shall also assume that the signature of g_{ij} is $(-1, 1, 1, 1)$.

[23] Brans and Dicke([4], page 929) assume that L_m is independent of $g_{ij,k}$; however, we shall assume that L_m contains both g_{ij} and its first derivative.

[24] Reference [22], page 169, or [23], page 258 .

The above expression may be written

$$\underset{(BD)}{\Lambda}{}^{ij,kl}{}_{|kl} = -\sqrt{-g}(g^{ij}\Box\phi - \phi^{|ij}) , \qquad (4.8)$$

where

$$\Box\phi = g^{kl}\phi_{|kl} , \qquad (4.9)$$

and

$$\phi^{|ij} = \phi_{|kl}\, g^{ki} g^{jl}. \qquad (4.10)$$

In obtaining (4.8) we made use of the fact that since ϕ is a scalar

$$\phi_{|kl} = \phi_{|lk} .$$

In order to determine $\underset{(BD)}{\Pi}{}^{ij}$ we shall use our third invariance identity (2.52); viz.,

$$\underset{(BD)}{\Pi}{}^{ij} = \frac{1}{2} g^{ij} L_{BD} - \frac{2}{3} R_k{}^i{}_{mh} \underset{(BD)}{\Lambda}{}^{hk,jm} - \frac{3}{8} \underset{(BD)}{\Phi}{}^j \phi^{,i} - \frac{1}{8} \underset{(BD)}{\Phi}{}^i \phi^{,j} . \quad (4.11)$$

From (4.2) we readily obtain

$$\underset{(BD)}{\Phi}{}^i = -2\sqrt{-g}\,\frac{\omega\phi^{,i}}{\phi} . \qquad (4.12)$$

Using (4.6) we find

$$R_k{}^i{}_{mh} \underset{(BD)}{\Lambda}{}^{hk,jm} = -\phi\sqrt{-g}\, R_k{}^i{}_{mh} \left\{ g^{hk} g^{jm} - \frac{1}{2}(g^{hj} g^{km} + g^{hm} g^{jk}) \right\} ,$$

which simplifies to

$$R_k{}^i{}_{mh} \underset{(BD)}{\Lambda}{}^{hk,jm} = \frac{3}{2}\phi\sqrt{-g}\, R^{ij} . \qquad (4.13)$$

Upon inserting (4.2), (4.12) and (4.13) into (4.11) we obtain

$$\underset{(BD)}{\Pi}{}^{ij} = \frac{1}{2} g^{ij}\sqrt{-g}(\phi R - \frac{\omega}{\phi}\phi_{,i}\phi^{,i}) - \phi\sqrt{-g}\, R^{ij} + \frac{\omega}{\phi}\sqrt{-g}\,\phi^{,i}\phi^{,j} ,$$

which may be written in the form

$$\underset{(BD)}{\Pi}{}^{ij} = -\phi\sqrt{-g}\left\{ R^{ij} - \frac{1}{2} g^{ij} R \right\} - \frac{\omega}{\phi}\sqrt{-g}\left\{ \frac{g^{ij}}{2}\phi_{,k}\phi^{,k} - \phi^{,i}\phi^{,j} \right\} . \quad (4.14)$$

If we now combine equations (2.39), (4.8) and (4.14) we find

$$E^{ij}(L_{BD}) = \phi\sqrt{-g}\left\{ R^{ij} - \frac{1}{2} g^{ij} R \right\} + \frac{\omega}{\phi}\sqrt{-g}\left\{ \frac{g^{ij}}{2}\phi_{,k}\phi^{,k} - \phi^{,i}\phi^{,j} \right\} + $$

$$+ \sqrt{-g}\,(g^{ij}\Box\phi - \phi^{|ij}) , \qquad (4.15)$$

which is the Euler-Lagrange tensor corresponding to a variation of the g_{ij}'s within the Brans-Dicke vacuum Lagrangian.

Thus by combining equations (4.4) and (4.15) we find that our first set of field equations (4.3) may be written

$$R^{ij} - \frac{1}{2}g^{ij}R = \frac{8\pi}{\phi c^4}T^{ij} + \frac{\omega}{\phi^2}(\phi^{,i}\phi^{,j} - \frac{1}{2}g^{ij}\phi_{,k}\phi^{,k}) + \\ + \frac{1}{\phi}(\phi^{|ij} - g^{ij}\Box\phi), \qquad (4.16)$$

where we have set

$$T^{ij} = \frac{2}{\sqrt{-g}}\left\{\frac{\partial}{\partial g_{ij}}(\sqrt{-g}\,L_m) - \frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial g_{ij,k}}(\sqrt{-g}\,L_m)\right)\right\}. \quad (4.17)$$

In the presence of matter equation (4.16) governs the behaviour of the metric potentials in the Brans-Dicke theory of gravitation.

The form of equation (4.16) is not completely unfamiliar to us, since many of the elements in this expression appear in the field equations of Einstein's theory. On the left hand side of this equation we have the Einstein tensor, and the first term on the right is the usual energy momentum tensor of matter, however, the gravitational constant has been replaced by the variable coupling parameter $\frac{1}{\phi}$. The second term on the right hand side of (4.16) is the conventional energy momentum tensor of a zero rest mass scalar field, however, once again we have $\frac{1}{\phi}$ replacing the gravitational "constant". The last term in equation (4.16) is peculiar to the Brans-Dicke theory.

We shall now proceed to obtain the Euler-Lagrange tensor for ϕ. At the close of section 2 we remarked that for a Lagrangian scalar density of the form

$$L = L'(g_{ij};\ g_{ij,h};\ g_{ij,hk};\ \phi;\ \phi_{,i}) + \sqrt{-g}\,L_m$$

we have

$$E(L) = E(L').$$

Thus equation (2.70) may be used to determine E(L) where, in our case, L is given by (4.1) and $L' = L_{BD}$.

Taking the covariant derivative of (4.15) with respect to x^j gives us

$$E^{ij}(L_{BD})_{|j} = \phi_{|j}\sqrt{-g}(R^{ij} - \tfrac{1}{2}g^{ij}R) + \frac{\omega}{2}\sqrt{-g}\,g^{ij}\left(\frac{\phi_{,k}\phi_{,l}\,g^{kl}}{\phi}\right)_{|j} +$$
$$- \omega\sqrt{-g}\left(\frac{\phi_{,h}\phi_{,k}}{\phi}\,g^{hi}g^{kj}\right)_{|j} + \sqrt{-g}(g^{ij}g^{hk}\phi_{|hkj} - g^{ih}g^{jk}\phi_{|hkj}). \quad (4.18)$$

Using the Ricci identity we may rewrite the last term appearing in (4.18) as follows:

$$\sqrt{-g}(g^{ij}g^{hk}\phi_{|hkj} - g^{ih}g^{jk}\phi_{|hkj}) =$$
$$= \sqrt{-g}(g^{ij}g^{hk}(\phi_{|hjk} + \phi_{|m}R^{m}{}_{hkj}) - g^{ih}g^{jk}\phi_{|hkj}). \quad (4.19)$$

Since $\phi_{|hk} = \phi_{|kh}$ the above expression reduces to

$$\sqrt{-g}(g^{ij}g^{hk}\phi_{|hkj} - g^{ih}g^{jk}\phi_{|hkj}) = -\sqrt{-g}\,R^{im}\phi_{|m}. \quad (4.20)$$

Let us now examine the second and third terms on the right hand side of (4.18) which we denote by A^i; viz.,

$$A^i = \frac{\omega}{2}\sqrt{-g}\,g^{ij}\left(\frac{\phi_{|k}\phi_{|l}\,g^{kl}}{\phi}\right)_{|j} - \omega\sqrt{-g}\left(\frac{\phi_{|h}\phi_{|k}\,g^{hi}g^{kj}}{\phi}\right)_{|j}.$$

Upon making use of Ricci's Lemma we find that the above expression for A^i reduces to

$$A^i = \frac{\omega}{2\phi}\sqrt{-g}\left\{\frac{\phi^{,i}}{\phi}\phi_{,k}\phi^{,k} - 2\phi^{,i}\Box\phi\right\}. \quad (4.21)$$

If we now insert equations (4.20) and (4.21) into (4.18) we find

$$E^{ij}(L_{BD})_{|j} = -\tfrac{1}{2}\phi^{,i}\sqrt{-g}\,R + \frac{\omega}{2\phi}\sqrt{-g}\,\phi^{,i}\left\{\frac{\phi_{,k}\phi^{,k}}{\phi} - 2\Box\phi\right\}. \quad (4.22)$$

By applying (2.70) we can conclude that

$$E(L) = E(L_{BD}) = -\sqrt{-g}\left\{R - \frac{\omega}{\phi^2}\phi_{,k}\phi^{,k} + \frac{2\omega\Box\phi}{\phi}\right\}. \quad (4.23)$$

Thus the field equation governing the behaviour of the scalar field in the Brans-Dicke theory of gravitation is given by

$$R - \frac{\omega}{\phi^2}\phi_{,k}\phi^{,k} + \frac{2\omega\Box\phi}{\phi} = 0 . \qquad (4.24)$$

From our previous work in section 2 we have

$$E^{ij}(L)_{|j} = \frac{1}{2}\phi^{,i}E(L) - \frac{8\pi}{c^4}\sqrt{-g}\, T^{ij}{}_{|j} . \qquad (4.25)$$

We shall now assume that our energy momentum tensor, T^{ij}, has been constructed so that $T^{ij}{}_{|j}$ vanishes when the field equations governing the matter variables have been satisfied.[25] Thus when this is the case we may conclude (as we did for Lagrange scalar densities of the form (2.1)) <u>that the field equation for ϕ is superfluous</u>, and perhaps may more properly be considered as a consistency equation. For when the field equations governing the metric potentials, (4.16), and the field equations for the matter variables are satisfied we have

$$E^{ij}(L)_{|j} = 0 \qquad (4.26)$$

and

$$T^{ij}{}_{|j} = 0 . \qquad (4.27)$$

Equation (4.25) then implies that E(L) must automatically vanish.

However, it should be noted that when E(L) ($\equiv E(L_{BD})$) vanishes neither $E^{ij}(L)$ nor $E^{ij}(L_{BD})$ will vanish in general.

In passing we note that when

$$E^{ij}(L_{BD}) = E(L_{BD}) = 0 \qquad (4.28)$$

then

$$L_{BD} = 0 . \qquad (4.29)$$

To see this let us examine $g_{ij}E^{ij}(L_{BD})$ and $\phi E(L_{BD})$. From equations (4.2), (4.15) and (4.23) we easily find that

$$g_{ij}E^{ij}(L_{BD}) = - L_{BD} + 3\sqrt{-g}\Box\phi , \qquad (4.30)$$

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[25]In conventional general relativity T^{ij} is usually constructed so as to be compatible with this demand.

and

$$\phi E(L_{BD}) = - L_{BD} - 2\omega\sqrt{-g}\,\Box\phi \ , \qquad (4.31)$$

where

$$\sqrt{-g}\,\Box\phi = (\sqrt{-g}\ g^{ij}\phi_{,i})_{,j} \ . \qquad (4.32)$$

Using the above expressions we obtain

$$3\phi E(L_{BD}) + 2\omega g_{ij}\, E^{ij}(L_{BD}) = - (2\omega + 3) L_{BD} \ . \ (4.33)$$

Thus when $E^{ij}(L_{BD})$ and $E(L_{BD})$ vanish so does L_{BD} (provided we assume that $\omega \neq -\frac{3}{2}$).

We shall now proceed to use equation (4.16) to rewrite our expression for E(L) which is given by (4.23). We begin by multiplying (4.16) by g_{ij} to obtain

$$R = \frac{-8\pi T}{c^4\phi} + \frac{\omega}{\phi^2}\phi_{,k}\phi^{,k} + \frac{3}{\phi}\Box\phi \ .^{26} \qquad (4.34)$$

Upon inserting (4.34) into (4.23) we find

$$E(L) = -\sqrt{-g}\left\{\frac{(2\omega + 3)}{\phi}\Box\phi - \frac{8\pi T}{c^4\phi}\right\} , \qquad (4.35)$$

and so the field equation for ϕ becomes

$$\Box\phi = \frac{8\pi T}{(2\omega + 3)c^4} \ . \qquad (4.36)$$

Thus we see that ϕ obeys a scalar wave equation in which the source of the scalar wave is the contracted energy momentum tensor of matter.

Let us now return to equation (4.16) and ask the following question: Is it ever possible for the first term on the right hand side of (4.16) to be so "large" as to completely dominate that side of the equation and thereby yield the approximate field equation

$$R^{ij} - \frac{1}{2}g^{ij}R \cong \frac{8\pi T^{ij}}{c^4\phi} \ ? \qquad (4.37)$$

Brans and Dicke ([4], page 930) assert that the answer to this

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[26] T is given by $g_{ij}T^{ij}$.

question is in the affirmative. However, (4.36) permits us to replace the

$$-\frac{g^{ij}\Box\phi}{\phi}$$

term appearing in (4.16) by

$$\frac{-8\pi g^{ij}T}{(2\omega+3)c^4\phi} .$$

Thus when T^{ij} is "large" T may also be "large" and hence the

$$\frac{8\pi T^{ij}}{c^4\phi}$$

term need not necessarily dominate the right hand side of (4.16).[27] Therefore we must disagree with Brans and Dicke and conclude that (4.37) is not a generally valid approximation to (4.16) in the limit of "large" T^{ij}.

Given below (for the ease of later refrence) are the Brans-Dicke field equations of gravitation:

$$R^{ij} - \frac{1}{2}g^{ij}R = \frac{8\pi T^{ij}}{\phi c^4} + \frac{\omega}{\phi^2}(\phi^{,i}\phi^{,j} - \frac{1}{2}g^{ij}\phi_{,k}\phi^{,k}) + \\ + \frac{1}{\phi}(\phi^{|ij} - g^{ij}\Box\phi) , \tag{4.16}$$

and

$$R - \frac{\omega}{\phi^2}\phi_{,k}\phi^{,k} + \frac{2\omega\Box\phi}{\phi} = 0 , \tag{4.24}$$

or equivalently

$$\Box\phi = \frac{8\pi T}{(2\omega+3)c^4} , \tag{4.36}$$

where

$$T^{ij} = \frac{2}{\sqrt{-g}}\left\{\frac{\partial}{\partial g_{ij}}(\sqrt{-g}L_m) - \frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial g_{ij,k}}(\sqrt{-g}L_m)\right)\right\} , \tag{4.17}$$

and

$$T = g_{ij}T^{ij}. \tag{4.38}$$

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[27] When T^{ij} is "large" T need not be large, as is the case for the electromagnetic field tensor, $T \equiv 0$.

5. The Effects of Conformal Transformations Upon Scalar Densities of the Form

$$L = L(g_{ij};\ g_{ij,h};\ g_{ij,hk};\ \phi;\ \phi_{,i})$$

Recall that in section 3 we considered how Bergmann's Lagrangian (3.7) transformed under a conformal transformation of the form

$$\bar{g}_{ij} = e^{2\sigma} g_{ij}\,, \tag{5.1}$$

where σ is a scalar function of class C^2. However, we have thus far neglected to consider the relationship between the Euler-Lagrange tensors corresponding to conformally related scalar densities of the form (2.1). This section will be devoted to examining this relationship.

To begin let us consider the following

Example:

When

$$L = \sqrt{-g}\ \phi_{,i}\phi_{,j}g^{ij} \tag{5.2}$$

it is easily shown that

$$E(L) = 2\Box\phi\,. \tag{5.3}$$

If we now perform the following conformal transformation:

$$\bar{g}_{ij} = \frac{1}{\phi}\, g_{ij}\,, \tag{5.4}$$

we find that in terms of the barred metric L becomes

$$\bar{L} = \phi\sqrt{-\bar{g}}\,\bar{g}^{ij}\,\phi_{,i}\phi_{,j}\,, \tag{5.5}$$

from which it can be shown that[28]

$$E(\bar{L}) = 2\phi(\sqrt{-\bar{g}}\,\bar{g}^{kj}\,\phi_{,k})_{,j} + \sqrt{-\bar{g}}\,\bar{g}^{kj}\,\phi_{,k}\phi_{,j}\,. \tag{5.6}$$

Using (5.4) to rewrite (5.3) in terms of the barred metric we find that E(L) becomes

$$E(L) = 2\phi(\sqrt{-\bar{g}}\,\bar{g}^{kj}\,\phi_{,k})_{,j} + 2\sqrt{-\bar{g}}\,\bar{g}^{kj}\phi_{,k}\phi_{,j}\,. \tag{5.7}$$

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[28] To obtain (5.6) from (5.5) we have performed a variation of ϕ regarding the $\bar{g}_{ij}$ as arbitrary preassigned functions of position.

Thus it is apparent that a solution to

$$E(\bar{L}) = 0$$

need not yield a solution to

$$E(L) = 0 ,$$

even though L and $\bar{L}$ are related by a conformal transformation.

From the above example it is apparent that E(L) and $E(\bar{L})$ will not, in general, be related although L and $\bar{L}$ are related by the conformal transformation (5.1). We shall now proceed to show that $E^{ij}(L)$ and $E^{ij}(\bar{L})$ are closely connected, when L is of the form (2.1). We shall then exploit this connection to show that even though E(L) and $E(\bar{L})$ are not related (in general) this difficulty can be overcome through the use of equation (2.70).

Let us now assume that we have two Lagrange scalar densities L_1 and L_2 of the form[29]

$$L_1 = L_1(g_{ij};\ g_{ij,h};\ g_{ij,hk};\ \phi;\ \phi_{,i}) \tag{5.8}$$

and

$$L_2 = L_2(\bar{g}_{ij};\ \bar{g}_{ij,h};\ \bar{g}_{ij,hk};\ \phi;\ \phi_{,i};\ \sigma;\ \sigma_{,i};\ \sigma_{,ij}) , \tag{5.9}$$

where L_1 can be transformed into the form of L_2 (up to the addition of a divergence) by the conformal transformation (5.1). We shall now consider the problem of relating $E^{ab}(L_1)$ to $E^{ab}(L_2)$ where

$$E^{ab}(L_1) = \frac{\partial}{\partial x^c}\left(\frac{\partial L_1}{\partial g_{ab,c}} - \frac{\partial}{\partial x^d}\frac{\partial L_1}{\partial g_{ab,cd}}\right) - \frac{\partial L_1}{\partial g_{ab}} , \tag{5.10}$$

and

$$E^{ab}(L_2) = \frac{\partial}{\partial x^c}\left(\frac{\partial L_2}{\partial \bar{g}_{ab,c}} - \frac{\partial}{\partial x^d}\frac{\partial L_2}{\partial \bar{g}_{ab,cd}}\right) - \frac{\partial L_2}{\partial \bar{g}_{ab}} . \tag{5.11}$$

It will be shown that under (5.1) $E^{ab}(L_1)$ transforms to $e^{2\sigma}E^{ab}(L_2)$, and this transformation is exact; i.e., there

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[29] L_1 and L_2 should <u>not</u> be confused with the L_1 and L_2 introduced in section 3.

is no divergence dropped from consideration as in the case of transforming L_1 to L_2.

In order to establish this relationship between $E^{ab}(L_1)$ and $E^{ab}(L_2)$ we begin by examining how the derivatives of $\bar{g}_{ij}$ are related to g_{ij} and its derivatives. From equation (5.1) we find:

$$\bar{g}_{ij,k} = 2\sigma_{,k}e^{2\sigma}g_{ij} + e^{2\sigma}g_{ij,k} \,, \tag{5.12}$$

and

$$\bar{g}_{ij,kl} = 2\sigma_{,kl}e^{2\sigma}g_{ij} + 4\sigma_{,k}\sigma_{,l}e^{2\sigma}g_{ij} + 2\sigma_{,k}e^{2\sigma}g_{ij,l} + 2\sigma_{,l}e^{2\sigma}g_{ij,k} + e^{2\sigma}g_{ij,kl} \,. \tag{5.13}$$

Upon combining equation (5.10) with the above results we may conclude that under (5.1)

$$E^{ab}(L_1) = \frac{\partial}{\partial x^c}\left[\left(\frac{\partial L_2}{\partial \bar{g}_{ij,k}}\frac{\partial \bar{g}_{ij,k}}{\partial g_{ab,c}} + \frac{\partial L_2}{\partial \bar{g}_{ij,kl}}\frac{\partial \bar{g}_{ij,kl}}{\partial g_{ab,c}}\right) - \frac{\partial}{\partial x^d}\left(\frac{\partial L_2}{\partial \bar{g}_{ij,kl}}\frac{\partial \bar{g}_{ij,kl}}{\partial g_{ab,cd}}\right)\right] + - \left\{\frac{\partial L_2}{\partial \bar{g}_{ij}}\frac{\partial \bar{g}_{ij}}{\partial g_{ab}} + \frac{\partial L_2}{\partial \bar{g}_{ij,k}}\frac{\partial \bar{g}_{ij,k}}{\partial g_{ab}} + \frac{\partial L_2}{\partial \bar{g}_{ij,kl}}\frac{\partial \bar{g}_{ij,kl}}{\partial g_{ab}}\right\} . \tag{5.14}$$

In order to simplify the form of equation (5.14) we must compute the following derivatives:

$$\frac{\partial \bar{g}_{ij,kl}}{\partial g_{ab,cd}} \,,\; \frac{\partial \bar{g}_{ij,kl}}{\partial g_{ab,c}} \,,\; \frac{\partial \bar{g}_{ij,kl}}{\partial g_{ab}} \,,\; \frac{\partial \bar{g}_{ij,k}}{\partial g_{ab,c}} \,,\; \frac{\partial \bar{g}_{ij,k}}{\partial g_{ab}} \,,\; \frac{\partial \bar{g}_{ij}}{\partial g_{ab}} \,. \tag{5.15}$$

From equations (5.1), (5.12) and (5.13) we easily obtain:

$$\frac{\partial \bar{g}_{ij,kl}}{\partial g_{ab,cd}} = \frac{e^{2\sigma}}{4}\left\{\delta^a_i\delta^b_j\delta^c_k\delta^d_l + \delta^a_j\delta^b_i\delta^c_k\delta^d_l + \delta^a_i\delta^b_j\delta^c_l\delta^d_k + \delta^a_j\delta^b_i\delta^c_l\delta^d_k\right\} , \tag{5.16}$$

$$\frac{\partial \bar{g}_{ij,kl}}{\partial g_{ab,c}} = \sigma_{,k}e^{2\sigma}(\delta^a_i\delta^b_j\delta^c_l + \delta^a_j\delta^b_i\delta^c_l) + \sigma_{,l}e^{2\sigma}(\delta^a_i\delta^b_j\delta^c_k + \delta^a_j\delta^b_i\delta^c_k), \tag{5.17}$$

$$\frac{\partial \bar{g}_{ij,kl}}{\partial g_{ab}} = \sigma_{,kl}e^{2\sigma}(\delta^a_i\delta^b_j + \delta^a_j\delta^b_i) + 2\sigma_{,k}\sigma_{,l}e^{2\sigma}(\delta^a_i\delta^b_j + \delta^a_j\delta^b_i) \,, \tag{5.18}$$

$$\frac{\partial \bar{g}_{ij,k}}{\partial g_{ab,c}} = \frac{e^{2\sigma}}{2}(\delta^a_i \delta^b_j \delta^c_k + \delta^a_j \delta^b_i \delta^c_k), \tag{5.19}$$

$$\frac{\partial \bar{g}_{ij,k}}{\partial g_{ab}} = \sigma_{,k} e^{2\sigma}(\delta^a_i \delta^b_j + \delta^a_j \delta^b_i) , \tag{5.20}$$

and

$$\frac{\partial \bar{g}_{ij}}{\partial g_{ab}} = \frac{1}{2} e^{2\sigma}(\delta^a_i \delta^b_j + \delta^a_j \delta^b_i). \tag{5.21}$$

We shall now proceed to use the above results to put equation (5.14) into a more familiar form.

Using equations (5.17) and (5.19) we find

$$\frac{\partial L_2}{\partial \bar{g}_{ij,k}} \frac{\partial \bar{g}_{ij,k}}{\partial g_{ab,c}} + \frac{\partial L_2}{\partial \bar{g}_{ij,kl}} \frac{\partial \bar{g}_{ij,kl}}{\partial g_{ab,c}} = e^{2\sigma} \frac{\partial L_2}{\partial \bar{g}_{ab,c}} + $$

$$+ 2e^{2\sigma}\sigma_{,k} \frac{\partial L_2}{\partial \bar{g}_{ab,kc}} + 2e^{2\sigma}\sigma_{,l} \frac{\partial L_2}{\partial \bar{g}_{ab,cl}} , \tag{5.22}$$

and from (5.16) we obtain

$$\frac{\partial L_2}{\partial \bar{g}_{ij,kl}} \frac{\partial \bar{g}_{ij,kl}}{\partial g_{ab,cd}} = \frac{\partial L_2}{\partial \bar{g}_{ab,cd}} e^{2\sigma} . \tag{5.23}$$

Upon differentiating equation (5.23) with respect to x^d we obtain

$$\frac{\partial}{\partial x^d}\left(\frac{\partial L_2}{\partial \bar{g}_{ij,kl}} \frac{\partial \bar{g}_{ij,kl}}{\partial g_{ab,cd}}\right) = 2\sigma_{,d} e^{2\sigma} \frac{\partial L_2}{\partial \bar{g}_{ab,cd}} + e^{2\sigma} \frac{\partial}{\partial x^d}\left(\frac{\partial L_2}{\partial \bar{g}_{ab,cd}}\right). \tag{5.24}$$

By combining equations (5.22) and (5.24) we find that the term within square brackets appearing in equation (5.14) is given by

$$e^{2\sigma} \frac{\partial L_2}{\partial \bar{g}_{ab,c}} + 2e^{2\sigma}\sigma_{,d} \frac{\partial L_2}{\partial \bar{g}_{ab,cd}} - e^{2\sigma} \frac{\partial}{\partial x^d} \frac{\partial L_2}{\partial \bar{g}_{ab,cd}} . \tag{5.25}$$

If we now differentiate the above expression with respect to x^c we find that under (5.1)

$$\frac{\partial}{\partial x^c}\left(\frac{\partial L_1}{\partial g_{ab,c}} - \frac{\partial}{\partial x^d} \frac{\partial L_1}{\partial g_{ab,cd}}\right)$$

becomes

$$e^{2\sigma}\frac{\partial}{\partial x^c}\left(\frac{\partial L_2}{\partial \bar{g}_{ab,c}} - \frac{\partial}{\partial x^d}\frac{\partial L_2}{\partial \bar{g}_{ab,cd}}\right) + 2\sigma_{,c}e^{2\sigma}\frac{\partial L_2}{\partial \bar{g}_{ab,c}} +$$
$$+ 4\sigma_{,c}\sigma_{,d}e^{2\sigma}\frac{\partial L_2}{\partial \bar{g}_{ab,cd}} + 2\sigma_{,cd}e^{2\sigma}\frac{\partial L_2}{\partial \bar{g}_{ab,cd}} \ . \qquad (5.26)$$

Using equations (5.18), (5.20) and (5.21) we find that the term appearing within curly brackets in equation (5.14) is given by

$$\frac{\partial L_2}{\partial \bar{g}_{ab}}e^{2\sigma} + 2\sigma_{,k}e^{2\sigma}\frac{\partial L_2}{\partial \bar{g}_{ab,k}} + 2\sigma_{,kl}e^{2\sigma}\frac{\partial L_2}{\partial \bar{g}_{ab,kl}} +$$
$$+ 4\sigma_{,k}\sigma_{,l}e^{2\sigma}\frac{\partial L_2}{\partial \bar{g}_{ab,kl}} \ . \qquad (5.27)$$

From equations (5.26) and (5.27) it is now apparent that under (5.1)

$$E^{ab}(L_1) = e^{2\sigma}E^{ab}(L_2) \ . \qquad (5.28)$$

Similarly, under

$$g_{ij} = e^{-2\sigma}\bar{g}_{ij} \qquad (5.29)$$

we have

$$E^{ab}(L_2) = e^{-2\sigma}E^{ab}(L_1) \ . \qquad (5.30)$$

Let us now assume that we have found a solution to

$$E^{ab}(L_2) = 0 \ ; \qquad (5.31)$$

that is, we have determined $\bar{g}_{ij}$, ϕ and σ which are such that $E^{ab}(L_2)$, as given by equation (5.11), is zero. We shall now consider the transformation (5.29) where $\bar{g}_{ij}$, and σ are such that (5.31) is valid. It is obvious that under (5.29) zero is transformed to zero. Thus we may use (5.30) to conclude that in the present case

$$E^{ab}(L_1) = 0 \ , \qquad (5.32)$$

when g_{ij} is given by (5.29). Furthermore it should be noted that the same scalar function ϕ will satisfy both(5.31) and (5.32).

Now equation (2.70) tells us that

$$E^{ab}(L_1)_{|b} = \frac{1}{2}\,\phi^{,a}E(L_1)\,, \qquad (5.33)$$

and thus when (5.32) holds

$$E(L_1) = 0. \qquad (5.34)$$

Thus the vanishing of $E^{ab}(L_2)$ implies that <u>both</u> $E^{ab}(L_1)$ and $E(L_1)$ vanish. This is true independently of the vanishing or non-vanishing of $E(L_2)$, and hence serves to emphasize the fact that $E(L_1)$ and $E(L_2)$ are quite unrelated (in general).

It should also be noted that since L_2 is not (in general) of the form (2.1) we <u>cannot</u> say that

$$E^{ab}(L_2)_{|b} = \frac{1}{2}\,\phi^{,a}E(L_2)\,. \qquad (5.35)$$

However when the above relationship does hold[30] the vanishing of $E^{ab}(L_2)$ implies that $E^{ab}(L_1)$, $E(L_1)$ and $E(L_2)$ all vanish.

To summarize the above results we have the following

<u>Theorem 5.1</u>: <u>If L_1 and L_2 are scalar densities of the form</u>

$$L_1 = L_1(g_{ij};\ g_{ij,h};\ g_{ij,hk};\ \phi;\ \phi_{,i})$$

<u>and</u>

$$L_2 = L_2(\bar{g}_{ij};\ \bar{g}_{ij,h};\ \bar{g}_{ij,hk};\ \phi;\ \phi_{,i};\ \sigma;\ \sigma_{,i};\ \sigma_{,ij})$$

<u>and if L_1 can be transformed into the form of L_2 (up to the addition of a divergence) through a conformal transformation of the form</u>

$$g_{ij} = e^{-2\sigma}\,\bar{g}_{ij}\,,$$

<u>where σ is the scalar function of class C^2 appearing in L_2,</u>

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[30] L_2 will be of the form (2.1) whenever σ is a function of ϕ, and the $\sigma_{,ij}$ terms appearing in L_2 can all be put into the form of a divergence. This type of situation arose when we considered conformal transformations of the Bergmann Lagrangian (3.7) (c.f. equation (3.71)).

then whenever

$$\frac{\partial}{\partial x^c}\left(\frac{\partial L_2}{\partial \bar{g}_{ab,c}} - \frac{\partial}{\partial x^d}\left(\frac{\partial L_2}{\partial \bar{g}_{ab,cd}}\right)\right) - \frac{\partial L_2}{\partial \bar{g}_{ab}} = 0 ,$$

we have both

$$\frac{\partial}{\partial x^c}\left(\frac{\partial L_1}{\partial g_{ab,c}} - \frac{\partial}{\partial x^d}\left(\frac{\partial L_1}{\partial g_{ab,cd}}\right)\right) - \frac{\partial L_1}{\partial g_{ab}} = 0 ,$$

and

$$\frac{\partial}{\partial x^c}\left(\frac{\partial L_1}{\partial \phi_{,c}}\right) - \frac{\partial L_1}{\partial \phi} = 0 .$$

6. Using a Conformal Transformation to Relate the Brans-Dicke and Einstein Field Equations[31]

We have previously seen that Brans and Dicke's field equations are more complex than Einstein's field equations. This can be attributed to the scalar field ϕ, whose reciprocal appears to play the role of a variable gravitational "constant". The purpose of this section is to show that the solution to a certain system of Einsteinian field equations can be used to obtain a solution to the Brans-Dicke field equations.

We begin with the Brans-Dicke Lagrangian in the presence of matter, viz.,

$$L = \sqrt{-g}\left\{\phi R - \frac{\omega}{\phi}\phi_{,i}\phi^{,i} + \frac{16\pi}{c^4}L_m\right\} . \qquad (6.1)$$

Let $\bar{L}$ denote the form assumed by (6.1) after we have performed the conformal transformation

$$\bar{g}_{ij} = e^{2\sigma}g_{ij} . \qquad (6.2)$$

In the previous section we saw that if L_1 and L_2, as given by equations (5.8) and (5.9) respectively, are related by a conformal transformation of the type (6.2) then under (6.2) we have

$$E^{ij}(L_1) = e^{2\sigma}E^{ij}(L_2) . \qquad (6.3)$$

The above result can obviously be applied to L and $\bar{L}$ to conclude that under (6.2)[32]

$$E^{ij}(L) = e^{2\sigma}E^{ij}(\bar{L}) . \qquad (6.4)$$

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[31] The material found in Dicke's paper [9] serves to motivate this section. However, the approach given below is not due to Dicke.

[32] That (6.3) can be applied to L, as given by (6.1) is due to the fact that the derivation of (6.3) depends only upon L_1's being a concomitant of g_{ij}, and its first and second derivatives. Since L_m contains no derivatives of g_{ij} greater than the first (6.4) follows quite readily from (6.3).

Thus a solution to

$$E^{ij}(\bar{L}) = 0 \tag{6.5}$$

can be used to obtain a solution to

$$E^{ij}(L) = 0 . \tag{6.6}$$

Now after applying (6.2) to (6.1) it is quite possible that

$$\frac{16\pi}{c^4}\sqrt{-g}\,L_m \longrightarrow \frac{16\pi}{c^4}e^{-4\sigma}\sqrt{-\bar{g}}\,\bar{L}_m \neq \frac{16\pi}{c^4}\sqrt{-\bar{g}}\,L_m^* , \tag{6.7}$$

where L_m^* is obtained from L_m by simply replacing g_{ij} and $g_{ij,k}$ appearing in L_m by $\bar{g}_{ij}$ and $\bar{g}_{ij,k}$ respectively, and $\bar{L}_m$ is the form assumed by L_m after we have performed the conformal transformation (6.2). (For example, if

$$L_m = F_{hk}F_{ab}g^{ah}g^{bk} , \tag{6.8}$$

then L_m^* would be given by

$$L_m^* = F_{hk}F_{ab}\bar{g}^{ah}\bar{g}^{bk} , \tag{6.9}$$

whereas $\bar{L}_m$ would be given by

$$\bar{L}_m = e^{4\sigma}F_{hk}F_{ab}\bar{g}^{ah}\bar{g}^{bk} .) \tag{6.10}$$

Whenever (6.7) holds we <u>cannot</u> conclude (in general) that when a matter variable satisfies the matter field equations obtained from $\bar{L}$ it will also satisfy the matter field equations obtained from L. In order to overcome this difficulty we shall henceforth assume that under (6.2)

$$\sqrt{-g}\;L_m \longrightarrow e^{-4\sigma}\sqrt{-\bar{g}}\,\bar{L}_m = \sqrt{-\bar{g}}\;L_m^* . \tag{6.11}$$

This implies that

$$e^{-4\sigma}\bar{L}_m = L_m^* , \tag{6.12}$$

and thus L_m must be homogenuous of degree -2 in g_{ij}. (6.8) serves as an example of such a Lagrangian.

Let T^{ij} correspond to the energy momentum tensor obtained from $\sqrt{-g}\,L_m$; viz.,

$$T^{ij} = \frac{2}{\sqrt{-g}}\left\{\frac{\partial}{\partial g_{ij}}(\sqrt{-g}\,L_m) - \frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial g_{ij,k}}(\sqrt{-g}\,L_m)\right)\right\} . \quad (6.13)$$

We shall assume that T^{ij} has been constructed so that when the field equations governing the matter variables are satisfied

$$T^{ij}{}_{|j} = 0 . \quad (6.14)$$

From equation (4.25) we have that

$$E^{ij}(L)_{|j} = \frac{1}{2}\,\phi^{,i}E(L) - \frac{8\pi}{c^4}\sqrt{-g}\;T^{ij}{}_{|j} , \quad (6.15)$$

and thus when the field equations governing the matter variables are satisfied we find

$$E^{ij}(L)_{|j} = \frac{1}{2}\,\phi^{,i}E(L) . \quad (6.15a)$$

If we now assume that a solution has been found to

$$E^{ij}(\bar{L}) = 0 ,$$

then we may use equations (6.4) and (6.15a) to conclude that both

$$E^{ij}(L) = 0 ,$$

and

$$E(L) = 0.$$

In summary we have shown that if under (6.2) $\sqrt{-g}\,L_m$ transforms as follows:

$$\sqrt{-g}\;L_m \longrightarrow \sqrt{-\bar{g}}\;L^*_m$$

and if T^{ij} has been constructed so that

$$T^{ij}{}_{|j} = 0$$

when the matter field equations are satisfied, then when

$$E^{ij}(\bar{L}) = 0 ,$$

and the matter field equations obtained from

$$\frac{16}{c^4}\sqrt{-\bar{g}}\,L^*_m ,$$

are satisfied we shall find that the field equations

$$E^{ij}(L) = 0 ,$$
$$E(L) = 0 ,$$

and the matter field equations obtained from

$$\frac{16\pi\sqrt{-g}\,L_m}{c^4}$$

are all satisfied.

Let us now consider the following conformal transformation

$$\bar{g}_{ij} = k\phi\, g_{ij}\,, \tag{6.16}$$

where k is a constant. Dicke motivates this conformal transformation by a discussion of units. Our reason for introducing (6.16) is that it transforms our original Lagrangian, (6.1), into the form of an Einstein type Lagrangian, and consequently leads to a system of Einstein field equations.

In section 3 (c.f. equations (3.7) and (3.68)) it was shown that under the conformal transformation (6.2) a scalar density of the type

$$\sqrt{-g}\left\{f_1(\phi)\,R + f_2(\phi)\phi_{,i}\phi_{,j}g^{ij} + f_3(\phi)\right\} \tag{6.17a}$$

becomes

$$e^{-4\sigma}\sqrt{-\bar{g}}\left\{f_1(\phi)e^{2\sigma}\bar{R} + \frac{6e^{4\sigma}}{\sqrt{-\bar{g}}}f_1(\phi)\left(e^{-2\sigma}\sqrt{-\bar{g}}\,\bar{g}^{ij}\sigma_{,i}\right)_{,j} + \right.$$

$$\left. + 6\bar{g}^{ij}f_1(\phi)e^{2\sigma}\sigma_{,i}\sigma_{,j} + f_2(\phi)\phi_{,i}\phi_{,j}e^{2\sigma}\bar{g}^{ij} + f_3(\phi)\right\}. \tag{6.17b}$$

In order to use (6.17a) and (6.17b) to assist in transforming (6.1) into the barred metric we make the following identifications:

$$f_1(\phi) = \phi\,, \tag{6.18}$$

$$f_2(\phi) = -\frac{\omega}{\phi}\,, \tag{6.19}$$

and

$$f_3(\phi) = 0\,. \tag{6.20}$$

Using equations (6.2) and (6.16) we see that

$$e^{2\sigma} = k\,\phi \tag{6.21}$$

and consequently

$$\sigma = \frac{1}{2}\ln(k\,\phi)\,. \tag{6.22}$$

Thus by combining equations (6.1) and (6.11) in conjunction with equations (6.16)-(6.22) we find that $\bar{L}$ is given by

$$\bar{L} = \sqrt{-\bar{g}}\,\frac{1}{k}\left\{\bar{R} + \frac{3\phi}{\sqrt{-\bar{g}}}\left[\frac{1}{\phi^2}\sqrt{-\bar{g}}\,\bar{g}^{ij}\phi_{,i}\right]_{,j} + \left(\frac{3}{2} - \omega\right)\frac{\phi_{,i}\phi_{,j}}{\phi^2}\,\bar{g}^{ij}\right\} + \frac{16\pi}{c^4}\sqrt{-\bar{g}}\,L^*_m\,. \qquad (6.23)$$

The second term appearing within the curly brackets of the above expression may be rewritten as follows:

$$\frac{3\phi}{\sqrt{-\bar{g}}}\left[\frac{1}{\phi^2}\sqrt{-\bar{g}}\,\bar{g}^{ij}\phi_{,i}\right]_{,j} = \frac{3}{\sqrt{-\bar{g}}}\left[\frac{1}{\phi}\sqrt{-\bar{g}}\,\bar{g}^{ij}\phi_{,i}\right]_{,j} - \frac{3}{\phi^2}\,\bar{g}^{ij}\phi_{,i}\phi_{,j}\,. \qquad (6.24)$$

Upon inserting (6.24) into (6.23) we find that $\bar{L}$ may be written

$$\bar{L} = \sqrt{-\bar{g}}\,\frac{1}{k}\left\{\bar{R} - \frac{1}{2}(2\omega + 3)\frac{1}{\phi^2}\,\phi_{,i}\phi_{,j}\,\bar{g}^{ij} + \frac{16\pi k}{c^4}\,L^*_m\right\} + \frac{3}{k}\left[\frac{1}{\phi}\sqrt{-\bar{g}}\,\bar{g}^{ij}\phi_{,i}\right]_{,j}\,. \qquad (6.25)$$

Since k is a constant the last term appearing in (6.25) is a divergence. Hence we may use Lovelock's result [16] to conclude that from the point of view of the calculus of variations the following Lagrangian, $\bar{L}'$, is equivalent to $\bar{L}$

$$\bar{L}' = \sqrt{-\bar{g}}\left\{\bar{R} - \frac{1}{2\phi^2}(2\omega + 3)\phi_{,i}\phi_{,j}\bar{g}^{ij} + \frac{16\pi k}{c^4}\,L^*_m\right\}, \qquad (6.26)$$

where we have dropped the constant factor $\frac{1}{k}$ which appears in (6.25). If we now set

$$\bar{L}_\phi = \frac{-(3 + 2\omega)c^4}{32\pi k\,\phi^2}\,\phi_{,i}\phi_{,j}\bar{g}^{ij}\,, \qquad (6.27)$$

we may rewrite $\bar{L}'$ in its final form; viz.,

$$\bar{L}' = \sqrt{-\bar{g}}\left\{\bar{R} + \frac{16\pi k}{c^4}\left(\bar{L}_\phi + L^*_m\right)\right\}, \qquad (6.28)$$

which is identical in form to the Lagrangian used to obtain the field equations of Einstein's theory of gravitation when matter and a zero rest mass scalar field, ϕ, are present.

We shall now use (6.28) to obtain the field equations governing the barred metric potentials $\bar{g}_{ij}$, and the scalar field ϕ. Since (6.28) is of the form of Einstein's Lagrangian the Euler-Lagrange tensor corresponding to a variation of the $\bar{g}_{ij}$'s, regarding ϕ and all other matter variables as arbitrary preassigned quantities, is given by

$$E^{ij}(\bar{L}') = \sqrt{-\bar{g}}\left\{\bar{R}^{ij} - \frac{1}{2}\bar{g}^{ij}\bar{R} - \frac{8\pi k}{c^4}\underset{(\phi)}{\bar{T}^{ij}}\right\} + \\ - \sqrt{-\bar{g}}\,\frac{8\pi k}{c^4}\underset{(m)}{\bar{T}^{ij}}, \qquad (6.29)$$

where

$$\underset{(m)}{\bar{T}^{ij}} = \frac{2}{\sqrt{-\bar{g}}}\left\{\frac{\partial}{\partial\bar{g}_{ij}}(\sqrt{-\bar{g}}\,L_m^*) - \frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial\bar{g}_{ij,k}}(\sqrt{-\bar{g}}\,L_m^*)\right)\right\}, \qquad (6.30)$$

and[33]

$$\underset{(\phi)}{\bar{T}^{ij}} = \frac{2}{\sqrt{-\bar{g}}}\left\{\frac{\partial}{\partial\bar{g}_{ij}}(\sqrt{-\bar{g}}\,\bar{L}_\phi)\right\}. \qquad (6.31)$$

Thus the field equations governing the barred metric potentials are given by

$$\bar{R}^{ij} - \frac{1}{2}\bar{g}^{ij}\bar{R} = \frac{8\pi k}{c^4}\left\{\underset{(m)}{\bar{T}^{ij}} + \underset{(\phi)}{\bar{T}^{ij}}\right\}. \qquad (6.32)$$

To obtain the field equation governing ϕ from (6.28) we may apply the results of section 2. This is so because $\bar{L}'$, as given by (6.28), is of the form

$$\bar{L}' = L'(\bar{g}_{ij};\ \bar{g}_{ij,h};\ \bar{g}_{ij,hk};\ \phi;\ \phi_{,i}) + \frac{16\pi k}{c^4}\sqrt{-\bar{g}}\,L_m^*, \qquad (6.33)$$

where

$$L' = \sqrt{-\bar{g}}\left\{\bar{R} + \frac{16\pi k}{c^4}\bar{L}_\phi\right\}. \qquad (6.34)$$

Since we have assumed that under (6.2)

$$\sqrt{-g}\,L_m \longrightarrow \sqrt{-\bar{g}}\,L_m^* \qquad (6.11)$$

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[33]Since $\bar{L}_\phi$ is independent of the derivatives of $\bar{g}_{ij}$, (6.31) need not involve a derivative with respect to $\bar{g}_{ij,k}$.

we are guaranteed that $\sqrt{-\bar{g}}\,L^*_m$ is independent of ϕ and consequently $\bar{L}'$ is formally identical to the Lagrangian given in (2.71). Thus we may conclude that

$$E(\bar{L}') = E(L') \, , \tag{6.35}$$

and since $E(L')$ and $E^{ij}(L')$ are related by[34]

$$E^{ij}(L')_{|j} = \frac{1}{2}\,\phi^{,i}E(L') \, , \tag{6.36}$$

we have

$$E^{ij}(L')_{|j} = \frac{1}{2}\,\phi^{,i}E(\bar{L}') \, . \tag{6.37}$$

Now by comparing (6.28) and (6.33) we readily deduce that $E^{ij}(L')$ is given by the term appearing within the curly brackets of equation (6.29), viz.,

$$E^{ij}(L') = \sqrt{-\bar{g}}\left\{\bar{R}^{ij} - \frac{1}{2}\,\bar{g}^{ij}\bar{R} - \frac{8\pi k}{c^4}\,\underset{(\phi)}{\bar{T}^{ij}}\right\} . \tag{6.38}$$

Thus $E^{ij}(L')_{|j}$ is found to be

$$E^{ij}(L')_{|j} = -\,\frac{8\pi k}{c^4}\sqrt{-\bar{g}}\;\underset{(\phi)}{\bar{T}^{ij}}{}_{|j} \quad . \tag{6.39}$$

Using equations (6.27) and (6.31) we easily find

$$-\,\frac{8\pi k}{c^4}\sqrt{-\bar{g}}\;\underset{(\phi)}{\bar{T}^{ij}} = \frac{(3+2\omega)}{2\phi^2}\sqrt{-\bar{g}}\left\{\frac{\bar{g}^{ij}\bar{g}^{rs}}{2}\phi_{|r}\phi_{|s} - \phi^{|i}\phi^{|j}\right\} . \tag{6.40}$$

Upon inserting (6.40) into (6.39) we obtain

$$E^{ij}(L')_{|j} = \frac{(3+2\omega)}{2}\sqrt{-\bar{g}}\left\{\frac{\bar{g}^{ij}\bar{g}^{rs}}{2\phi^2}(\phi_{|rj}\,\phi_{|s} + \phi_{|r}\,\phi_{|sj}) + {} - \frac{(\phi^{|i}{}_{|j}\,\phi^{|j} + \phi^{|i}\,\phi^{|j}{}_{|j})}{\phi^2} - \frac{2\phi_{|j}}{\phi^3}\left(\frac{\bar{g}^{ij}\bar{g}^{rs}}{2}\phi_{|r}\,\phi_{|s} - \phi^{|i}\,\phi^{|j}\right)\right\} , \tag{6.41}$$

which simplifies to

$$E^{ij}(L')_{|j} = -\,\phi^{,i}\frac{(3+2\omega)}{2\phi}\sqrt{-\bar{g}}\,\bar{\Box}\ln\phi \, , \tag{6.42}$$

where

$$\bar{\Box}\ln\phi = \frac{1}{\sqrt{-\bar{g}}}\left(\frac{\sqrt{-\bar{g}}\,\bar{g}^{ij}}{\phi}\,\phi_{|i}\right)_{|j} \quad . \tag{6.43}$$

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[34]The covariant derivatives appearing in equations (6.36) and (6.37) are taken with respect to the barred metric tensor.

Thus we may use (6.37) to conclude that

$$E(\bar{L}') = -\frac{(3+2\omega)}{\phi}\sqrt{-\bar{g}}\,\bar{\Box}\ln\phi\ , \qquad (6.44)$$

and hence the field equation governing ϕ, in terms of the barred metric tensor is

$$\bar{\Box}\ln\phi = 0\ , \qquad (6.45)$$

provided we assume that $\omega \neq -\frac{3}{2}$.

Equation (6.45) does <u>not</u> agree with Dicke's field equation for ϕ.[35] This is so because, as previously mentioned, we are not following Dicke's procedure. Dicke's considerations are based upon a transormation of the system of units being used, from which he motivates the conformal transformation

$$\bar{g}_{ij} = k\phi\, g_{ij}\ .$$

However, Dicke's transformation of units also effects the matter variables within L_m , and for this reason

$$\frac{\partial L_m^*}{\partial \phi} \neq 0$$

in his case. Thus Dicke's field equation for ϕ is given by

$$\bar{\Box}\ln\phi = \frac{-16\pi k\phi}{(2\omega+3)c^4}\frac{\partial L_m^*}{\partial\phi}\ .$$

In passing we must point out that in our case the field equation for ϕ is superflous. This is so because we've assumed that $\bar{T}_{(m)}^{ij}$ has been constructed so that $\bar{T}_{(m)}^{ij}{}_{|j}$ vanishes when the matter field equations have been solved. Consequently once we've found a solution to (6.32) and a solution to the field equations governing the matter variables we can use equation (4.25); viz.,

$$E^{ij}(\bar{L})_{|j} = \frac{1}{2}\phi^{,i}E(\bar{L}) - \frac{8\pi k}{c^4}\sqrt{-\bar{g}}\,\bar{T}_{(m)}^{ij}{}_{|j}$$

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[35]In [9]Dicke sets $\phi = \frac{\lambda}{k}$.

to conclude that the field equation governing ϕ, (6.45), will also be satisfied.

We shall now summarize what we have shown. We began with the Brans-Dicke matter Lagrangian

$$L = \sqrt{-g}\left\{\phi R - \frac{\omega}{\phi}\phi_{,i}\phi_{,j}g^{ij} + \frac{16\pi}{c^4}L_m\right\}, \qquad (6.1)$$

and assumed that under the conformal transformation

$$\bar{g}_{ij} = k\phi g_{ij}, \qquad (6.16)$$

we have

$$\sqrt{-g}\,L_m \longrightarrow \sqrt{-\bar{g}}\,L_m^* . \qquad (6.11)$$

We then saw that under this conformal transformation L becomes

$$\bar{L}' = \sqrt{-\bar{g}}\left\{\bar{R} + \frac{16\pi k}{c^4}(\bar{L}_\phi + L_m^*)\right\}, \qquad (6.28)$$

where $\bar{L}_\phi$ is given by (6.27). Thus under (6.16) our original Brans-Dicke Lagrangian takes on the form of an Einstein type Lagrangian. We then used (6.28) to find the field equations governing $\bar{g}_{ij}$ and ϕ; viz.,

$$\bar{R}^{ij} - \frac{1}{2}\bar{g}^{ij}\bar{R} = \frac{8\pi k}{c^4}\bar{T}^{ij}_{(m)} + \frac{(3+2\omega)}{2\phi^2}\left\{\phi^{|i}\phi^{|j} - \frac{1}{2}\bar{g}^{ij}\bar{g}^{rs}\phi_{|r}\phi_{|s}\right\} \qquad (6.46)$$

and

$$\bar{\Box}\ln\phi = 0, \qquad (6.45)$$

respectively; where we have used (6.40) to rewrite (6.32), and $\bar{T}^{ij}_{(m)}$ is given by (6.30). Thus according to the remarks made earlier in this section we may conclude that when we have a solution to (6.46) and when the field equations governing the matter variables (in terms of the barred metric) have been satisfied then, our original Brans-Dicke field equations (4.16) and (4.36) shall also be satisfied, along with the field equations governing the matter variables (in terms of the unbarred metric) provided we set

$$g_{ij} = \frac{1}{k\phi}\bar{g}_{ij}.$$

7. A Method for Obtaining Exact Solutions to the Brans-Dicke Vacuum Field Equations[36]

In the previous section it was shown how a solution to a certain system of Einstein field equations can be used to obtain a solution to the Brans-Dicke field equations. In particular a solution to the following Einstein equations

$$\bar{G}_{ij} = \frac{(3+2\omega)}{2\phi^2}\left\{\phi_{,i}\phi_{,j} - \tfrac{1}{2}\bar{g}_{ij}\bar{g}^{rs}\phi_{,r}\phi_{,s}\right\}, \qquad (7.1)$$

and

$$\bar{\Box}\ln\phi = 0, \qquad (7.2)$$

can be used to obtain a solution to the Brans-Dicke vacuum field equations

$$G_{ij} = \frac{\omega}{\phi^2}(\phi_{,i}\phi_{,j} - \tfrac{1}{2}g_{ij}g^{rs}\phi_{,r}\phi_{,s}) + \frac{1}{\phi}(\phi_{|ij} - g_{ij}\Box\phi), \qquad (7.3)$$

and

$$\Box\phi = 0, \qquad (7.4)$$

where

$$G_{ij} = R_{ij} - \tfrac{1}{2}g_{ij}R. \qquad (7.5)$$

Consequently let us begin our search for exact solutions to equations (7.3) and (7.4) by studying the following system of Einstein field equations

$$\bar{G}_{ij} = \frac{8\pi k}{c^4}\bar{T}_{ij}, \qquad (7.6)$$

and

$$\bar{\Box}\Psi = 0, \qquad (7.7)$$

where

$$T_{ij} = \Psi_{,i}\Psi_{,j} - \tfrac{1}{2}\bar{g}_{ij}\bar{g}^{km}\Psi_{,k}\Psi_{,m}, \qquad (7.8)$$

[36]The results appearing in this section were obtained from a paper by A.I.Janis, D.C.Robinson and J.Winicour [11]. However, the author has recently discovered that many of the results presented in this section also appear in a paper by N.De [8], which was published prior to [11].

k is a constant and Ψ is a scalar field of class C^2. It should be noted that equations (7.6) and (7.7) become identical with equations (7.1) and (7.2) when we set

$$\Psi = p \ln k\phi$$

where

$$p = \left\{\frac{c^4(3+2\omega)}{16\pi k}\right\}^{1/2} .$$

We shall restrict our considerations to static fields in which case we can write the line element in the form[37]

$$ds^2 = -e^{2U}(dx^0)^2 + e^{-2U}h_{\alpha\beta}dx^\alpha dx^\beta , \qquad (7.9)$$

where

$$U = U(x^\alpha) \quad \text{and} \quad h_{\alpha\beta} = h_{\alpha\beta}(x^\gamma)$$

are functions of class C^2 and

$$x^0 = c\,t.$$

Using this line element it will be shown that our field equations (7.6) and (7.7) can be rewritten as follows:

$$\Psi_{\|\alpha}{}^{\|\alpha} = 0 , \qquad (7.10a)$$

$$U_{\|\alpha}{}^{\|\alpha} = 0 , \qquad (7.10b)$$

and

$$H_{\alpha\beta} - 2U_{,\alpha}U_{,\beta} = \frac{8\pi k}{c^4}\Psi_{,\alpha}\Psi_{,\beta} , \qquad (7.10c)$$

where $H_{\alpha\beta}$ is the Ricci tensor for the auxilliary metric tensor $h_{\alpha\beta}$ and ($\|$) denotes covariant differentiation with respect to the $h_{\alpha\beta}$'s.

We shall now proceed to establish the results appearing in (7.10a)-(7.10c). We begin by multiplying (7.6) by $\bar{g}^{ij}$ to obtain

$$-\bar{R} = -\frac{8\pi k}{c^4}\Psi_{,i}\Psi_{,j}\bar{g}^{ij} .$$

Thus the following equation is equivalent to (7.6)

$$\bar{R}_{ij} = \frac{8\pi k}{c^4}\Psi_{,i}\Psi_{,j} . \qquad (7.11)$$

[37]Recall that Greek indices run from 1 to 3.

Our line element (7.9) can be written in the form

$$ds^2 = e^{-2U}(-e^{-4U}(dx^0)^2 + h_{\alpha\beta}\, dx^\alpha dx^\beta) \;, \qquad (7.12)$$

and when expressed as such we see that calculations may be simplified if we make the conformal transformation

$$p_{ij} = e^{2U}\bar{g}_{ij} \,. \qquad (7.13)$$

The above transformation implies that

$$p_{\alpha\beta} = h_{\alpha\beta} \;,\; p_{\alpha 0} = 0 \;\text{ and }\; p_{00} = -e^{4U} . \qquad (7.14)$$

The Christoffel symbols corresponding to each of the two metrics, p_{ij} and $\bar{g}_{ij}$, are related by[38]

$$\bar{\Gamma}^q_{ij} = \left\{ {q \atop ij} \right\} - \delta^q_i U_{,j} - \delta^q_j U_{,i} + p_{ij}p^{qm}U_{,m} \;; \qquad (7.15)$$

where $\left\{ {q \atop ij} \right\}$ denotes the Christoffel symbols constructed from the p_{ij}'s and $\bar{\Gamma}^q_{ij}$ denotes the Christoffel symbols constructed from the $\bar{g}_{ij}$'s.

By examining the matrix representing p_{ij} it is apparent that

$$p^{00} = -e^{-4U} \,,\; p^{0\alpha} = 0 \text{ and } p^{\alpha\beta} = h^{\alpha\beta} \;, \qquad (7.16)$$

where $h^{\alpha\beta}$ is characterized by

$$h^{\alpha\beta} h_{\gamma\beta} = \delta^\alpha_\gamma . \qquad (7.17)$$

Since $\left\{ {\alpha \atop \beta\gamma} \right\}$ is defined by

$$\left\{ {\alpha \atop \beta\gamma} \right\} = \frac{1}{2} p^{\alpha i}(p_{\beta i,\gamma} + p_{\gamma i,\beta} - p_{\beta\gamma,i}) \;, \qquad (7.18)$$

and all quantities are assumed to be time independent we have

$$\left\{ {\alpha \atop \beta\gamma} \right\} = \frac{h^{\alpha\mu}}{2}(h_{\beta\mu,\gamma} + h_{\gamma\mu,\beta} - h_{\beta\gamma,\mu}) . \qquad (7.19)$$

The components of $\left\{ {q \atop ij} \right\}$ which contain the index 0 are easily found to be:

$$\left\{ {0 \atop 00} \right\} = \left\{ {\alpha \atop 0\beta} \right\} = 0 \,, \qquad (7,20a)$$

$$\left\{ {\alpha \atop 00} \right\} = 2h^{\alpha\beta} U_{,\beta}\, e^{4U} \,, \qquad (7.20b)$$

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[38]Eisenhart [10], page 89.

and

$$\left\{ {0 \atop 0\alpha} \right\} = 2U_{,\alpha} . \qquad (7.20c)$$

With this information at our disposal we may now proceed to determine $\bar{\Box}\Psi$ and $\bar{R}_{ij}$ in terms of the $h_{\alpha\beta}$'s and U.

From (7.13) and the definition of $\bar{\Box}\Psi$ we find

$$\bar{\Box}\Psi = e^{2U}p^{ij}(\Psi_{,ij} - \Psi_{,m}\bar{\Gamma}^m_{ij}) ,$$

which, due to time independence, may be rewritten as follows:

$$\bar{\Box}\Psi = e^{2U}(-p^{00}\Psi_{,\alpha}\bar{\Gamma}^{\alpha}_{00} + p^{\alpha\beta}(\Psi_{,\alpha\beta} - \Psi_{,\mu}\bar{\Gamma}^{\mu}_{\alpha\beta})).$$

Using (7.15) we may rewrite the above expression in the form

$$\bar{\Box}\Psi = e^{2U}\Big\{e^{-4U}\Psi_{,\alpha}\left(\left\{ {\alpha \atop 00} \right\} - \delta^{\alpha}_{0}U_{,0} - \delta^{\alpha}_{0}U_{,0} + p_{00}p^{\alpha\beta}U_{,\beta}\right) +$$
$$+ h^{\alpha\beta}\left(\Psi_{,\alpha\beta} - \Psi_{,\mu}\left[\left\{ {\mu \atop \alpha\beta} \right\} - \delta^{\mu}_{\alpha}U_{,\beta} - \delta^{\mu}_{\beta}U_{,\alpha} + p_{\alpha\beta}p^{\mu\gamma}U_{,\gamma}\right]\right)\Big\} . \qquad (7.21)$$

Using (7.20b) we find that (7.21) becomes

$$\bar{\Box}\Psi = e^{2U}\Big\{h^{\alpha\beta}\left(\Psi_{,\alpha\beta} - \Psi_{,\gamma}\left\{ {\gamma \atop \alpha\beta} \right\}\right) + e^{-4U}(2\Psi_{,\alpha}U_{,\beta}e^{4U}h^{\alpha\beta} +$$
$$- e^{4U}\Psi_{,\alpha}U_{,\beta}h^{\alpha\beta}) + 2U_{,\alpha}U_{,\beta}h^{\alpha\beta} - 3U_{,\alpha}U_{,\beta}h^{\alpha\beta}\Big\} .$$

Thus we see that $\bar{\Box}\Psi$ is given by

$$\bar{\Box}\Psi = h^{\alpha\beta}\left(\Psi_{,\alpha\beta} - \Psi_{,\gamma}\left\{ {\gamma \atop \alpha\beta} \right\}\right)e^{2U} . \qquad (7.22)$$

Since Ψ is governed by

$$\bar{\Box}\Psi = 0 , \qquad (7.7)$$

Ψ must also satisfy

$$h^{\alpha\beta}\left(\Psi_{,\alpha\beta} - \Psi_{,\gamma}\left\{ {\gamma \atop \alpha\beta} \right\}\right) = \Psi_{\|\alpha}{}^{\|\alpha} = 0 , \qquad (7.23)$$

which establishes (7.10a).

One should note that (7.22) is valid if Ψ is replaced by any other time independent scalar function such as U; i.e.,

$$\bar{\Box}U = \bar{g}^{hk}U_{|hk} = h^{\alpha\beta}\left(U_{,\alpha\beta} - U_{,\gamma}\left\{ {\gamma \atop \alpha\beta} \right\}\right)e^{2U} . \qquad (7.24)$$

We shall now proceed to use equation (7.11) to establish (7.10b) and (7.10c). By examining (7.11) when i=j=0 we shall arrive at (7.10b). Similarly by considering (7.11) when

$i=\alpha$ and $j=\beta$ it will be shown that (7.10c) is valid. In order to establish each of these results it will be necessary to construct the Ricci tensor in terms of the $\left\{ {q \atop ij} \right\}$ affinities. This Ricci tensor will be denoted by K_{ij} and is defined by

$$K_{ij} = \frac{\partial}{\partial x^k}\left\{ {k \atop ij} \right\} - \frac{\partial^2}{\partial x^i \partial x^j}(\ln\sqrt{-p}\,) - \left\{ {m \atop ik} \right\}\left\{ {k \atop mj} \right\} + \left\{ {m \atop ij} \right\}\left\{ {k \atop mk} \right\}, \quad (7.25)$$

where

$$p = \det(p_{ij}) \;. \quad (7.26)$$

From Eisenhart[39] we find that K_{ij} and $\bar{R}_{ij}$ are related by

$$\bar{R}_{ij} = K_{ij} + 2(U_{|ij} - U_{|i}\,U_{|j}\,) + \bar{g}_{ij}\bar{g}^{hk}U_{|hk} +$$
$$+ 2\bar{g}_{ij}\bar{g}^{hk}U_{|h}\,U_{|k} \;. \quad (7.27)$$

Let us consider equation (7.11) when $i=j=0$. Due to the fact that Ψ is independent of time (7.11) becomes in this case

$$\bar{R}_{00} = 0 \;. \quad (7.28)$$

We shall now use (7.27) to rewrite the above expression in terms of K_{00} and U.

From (7.25) we have

$$K_{00} = \frac{\partial}{\partial x^k}\left\{ {k \atop 00} \right\} - \frac{\partial^2}{\partial (x^0)^2}(\ln\sqrt{-p}) + \left\{ {m \atop 00} \right\}\left\{ {k \atop mk} \right\} - \left\{ {m \atop 0k} \right\}\left\{ {k \atop m0} \right\} \;,$$

which may be rewritten as follows:

$$K_{00} = \frac{\partial}{\partial x^\alpha}\left\{ {\alpha \atop 00} \right\} + \left\{ {m \atop 00} \right\}\left\{ {0 \atop m0} \right\} + \left\{ {m \atop 00} \right\}\left\{ {\alpha \atop m\alpha} \right\} - \left\{ {0 \atop 00} \right\}\left\{ {0 \atop 00} \right\} - 2\left\{ {\alpha \atop 00} \right\}\left\{ {0 \atop \alpha 0} \right\} +$$
$$- \left\{ {\alpha \atop 0\beta} \right\}\left\{ {\beta \atop \alpha 0} \right\} .$$

Using (7.20a), (7.20b) and (7.20c) we find that the above expression becomes

$$K_{00} = 2\frac{\partial}{\partial x^\alpha}(h^{\alpha\beta}U_{,\beta}e^{4U}) - 4U_{,\alpha}U_{,\beta}h^{\alpha\beta}e^{4U} + 2h^{\alpha\beta}U_{,\alpha}\left\{ {\mu \atop \beta\mu} \right\} .$$

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[39] Eisenhart, [10] page 90. Recall that Eisenhart's Ricci tensor is the negative of ours.

After performing the indicated differentiation we obtain

$$K_{00} = 2e^{4U}\left\{\frac{\partial h^{\alpha\beta}}{\partial x^{\alpha}} U_{,\beta} + h^{\alpha\beta} U_{,\alpha\beta} + 4U_{,\alpha} U_{,\beta} h^{\alpha\beta}\right\} + \\ - 4U_{,\alpha} U_{,\beta} h^{\alpha\beta} e^{4U} + 2h^{\alpha\beta} U_{,\alpha}\left\{{\mu \atop \beta\mu}\right\},$$

which can be rewritten in the form

$$K_{00} = 2e^{4U}\left\{ - (h^{\mu\beta}\left\{{\alpha \atop \mu\alpha}\right\} + h^{\alpha\mu}\left\{{\beta \atop \mu\alpha}\right\}) U_{,\beta} + h^{\alpha\beta} U_{,\alpha\beta} + \\ + 4U_{,\alpha} U_{,\beta} h^{\alpha\beta} e^{4U} + 2h^{\alpha\beta} U_{,\alpha}\left\{{\mu \atop \beta\mu}\right\}\right\}.$$

The above equation simplifies to our final expression for K_{00} which is

$$K_{00} = 2e^{4U} h^{\alpha\beta} (U_{,\alpha\beta} - U_{,\gamma}\left\{{\gamma \atop \alpha\beta}\right\}) + 4e^{4U} U_{,\alpha} U_{,\beta} h^{\alpha\beta} . \quad (7.29)$$

From (7.27) we obtain

$$\bar{R}_{00} = K_{00} + 2(U_{|00} - U_{,0}U_{,0}) + \bar{g}_{00}\bar{g}^{hk}U_{|hk} + \\ + 2\bar{g}_{00}\bar{g}^{hk}U_{,h}U_{,k} . \quad (7.30)$$

Using (7.15), (7.16) and (7.20b) we find that $U_{|00}$ is given by

$$U_{|00} = - U_{,\alpha} U_{,\beta} h^{\alpha\beta} e^{4U}. \quad (7.31)$$

Equations (7.13), (7.14), (7.16), (7.24), (7.29), (7.30) and (7.31) allow us to write $\bar{R}_{00}$ as follows:

$$\bar{R}_{00} = 2e^{4U} h^{\alpha\beta} (U_{,\alpha\beta} - U_{,\gamma}\left\{{\gamma \atop \alpha\beta}\right\}) + 4e^{4U} U_{,\alpha} U_{,\beta} h^{\alpha\beta} + \\ - 2U_{,\alpha} U_{,\beta} h^{\alpha\beta} e^{4U} - e^{4U} h^{\alpha\beta} (U_{,\alpha\beta} - U_{,\gamma}\left\{{\gamma \atop \alpha\beta}\right\}) + \\ - 2e^{4U} h^{\alpha\beta} U_{,\alpha} U_{,\beta} .$$

After obvious cancellations the above expression reduces to

$$\bar{R}_{00} = e^{4U} U_{\|\alpha}{}^{\|\alpha}. \quad (7.32)$$

Thus we can conclude that the field equation (7.28) will be valid only if

$$U_{\|\alpha}{}^{\|\alpha} = 0, \quad (7.33)$$

which establishes (7.10b).

In order to obtain (7.10c) it will be necessary to examine $K_{\alpha\beta}$.

From equation (7.14) it is apparent that

$$-p = e^{4U}h \tag{7.34}$$

where

$$h = \det(h_{\alpha\beta}) . \tag{7.35}$$

Consequently we have

$$\ln\sqrt{-p} = \ln\sqrt{h} + 2U . \tag{7.36}$$

Using (7.25) we find that $K_{\alpha\beta}$ may be written in the form

$$K_{\alpha\beta} = \frac{\partial}{\partial x^{\mu}}\left\{ {\mu \atop \alpha\beta} \right\} - \frac{\partial^2 \ln\sqrt{h}}{\partial x^{\alpha}\partial x^{\beta}} - 2U_{,\alpha\beta} - \left\{ {0 \atop \alpha 0} \right\}\left\{ {0 \atop 0\beta} \right\} - \left\{ {0 \atop \alpha\mu} \right\}\left\{ {\mu \atop 0\beta} \right\} + \\ - \left\{ {\mu \atop 0\alpha} \right\}\left\{ {0 \atop \mu\beta} \right\} - \left\{ {\mu \atop \alpha\gamma} \right\}\left\{ {\gamma \atop \mu\beta} \right\} + \left\{ {\mu \atop \alpha\beta} \right\}\frac{\partial}{\partial x^{\mu}}(\ln\sqrt{h}) + 2U_{,\mu}\left\{ {\mu \atop \alpha\beta} \right\} . \tag{7.37}$$

Upon applying (7.20a) and (7.20c) we find that (7.37) becomes

$$K_{\alpha\beta} = H_{\alpha\beta} - 2U_{\|\alpha\beta} - 4U_{,\alpha}U_{,\beta} , \tag{7.38}$$

where $H_{\alpha\beta}$ is the Ricci tensor for our auxilliary metric $h_{\alpha\beta}$; viz.,

$$H_{\alpha\beta} = \left\{ {\gamma \atop \alpha\beta} \right\}_{,\gamma} - (\ln\sqrt{h})_{,\alpha\beta} + \left\{ {\gamma \atop \alpha\beta} \right\}(\ln\sqrt{h})_{,\gamma} - \left\{ {\gamma \atop \alpha\epsilon} \right\}\left\{ {\epsilon \atop \gamma\beta} \right\} . \tag{7.39}$$

In order to determine $\bar{R}_{\alpha\beta}$ from (7.27) we shall examine

$$2(U_{|\alpha\beta} - U_{,\alpha}U_{,\beta}) + \bar{g}_{\alpha\beta}\bar{g}^{hk}U_{|hk} + 2\bar{g}_{\alpha\beta}\bar{g}^{hk}U_{,h}U_{,k} . \tag{7.40}$$

(7.15) permits us to write $U_{|\alpha\beta}$ as follows:

$$U_{|\alpha\beta} = U_{,\alpha\beta} - U_{,\mu}(\left\{ {\mu \atop \alpha\beta} \right\} - \delta^{\mu}_{\alpha}U_{,\beta} - \delta^{\mu}_{\beta}U_{,\alpha} + p_{\alpha\beta}p^{\gamma\mu}U_{,\gamma}) ,$$

which simplifies to

$$U_{|\alpha\beta} = U_{\|\alpha\beta} + 2U_{,\alpha}U_{,\beta} - p_{\alpha\beta}p^{\gamma\mu}U_{,\gamma}U_{,\mu} . \tag{7.41}$$

Thus the first term appearing in (7.40) becomes

$$2(U_{|\alpha\beta} - U_{,\alpha}U_{,\beta}) = 2(U_{\|\alpha\beta} - p_{\alpha\beta}p^{\gamma\mu}U_{,\gamma}U_{,\mu} + U_{,\alpha}U_{,\beta}) . \tag{7.42}$$

By combining (7.13), (7.24), (7.33) and (7.42) we find that (7.40) reduces to

$$2U_{\|\alpha\beta} + 2U_{,\alpha}U_{,\beta} . \tag{7.43}$$

Using (7.27), (7.38) and (7.43) we find

$$\bar{R}_{\alpha\beta} = H_{\alpha\beta} - 2U_{\|\alpha\beta} - 4U_{,\alpha}U_{,\beta} + 2U_{\|\alpha\beta} + 2U_{,\alpha}U_{,\beta} ,$$

which simplifies to

$$\bar{R}_{\alpha\beta} = H_{\alpha\beta} - 2U_{,\alpha}U_{,\beta} . \tag{7.44}$$

Upon inserting (7.44) into (7.11) we obtain

$$H_{\alpha\beta} - 2U_{,\alpha}U_{,\beta} = \frac{8\pi k}{c^4}\Psi_{,\alpha}\Psi_{,\beta} \quad , \qquad (7.45)$$

which establishes (7.10c).

For the sake of completeness we remark that by using the above approach it is easy to establish that $\overline{R}_{0\alpha} \equiv 0$ for the line element (7.9).

We shall now apply the above results. Let us begin by assuming that we have found a static line element

$$ds^2 = -e^{2V}(dx^0)^2 + e^{-2V}h_{\alpha\beta}dx^\alpha dx^\beta \quad , \qquad (7.46)$$

which yields a solution to the Einstein vacuum field equations

$$\overline{R}_{ij} = 0 \; . \qquad (7.47)$$

Due to our previous work we know that we can write $\overline{R}_{ij}$ as follows:

$$\overline{R}_{00} = e^{4V}V_{\|\alpha}{}^{\|\alpha} \quad , \qquad (7.48)$$

$$\overline{R}_{0\alpha} \equiv 0 \quad , \qquad (7.49)$$

and

$$\overline{R}_{\alpha\beta} = H_{\alpha\beta} - 2V_{,\alpha}V_{,\beta} \; . \qquad (7.50)$$

Since (7.46) is a static solution of (7.47) we can now conclude that

$$e^{4V}\,V_{\|\alpha}{}^{\|\alpha} = 0 \qquad (7.51)$$

and

$$H_{\alpha\beta} - 2V_{,\alpha}V_{,\beta} = 0 \; . \qquad (7.52)$$

Using (7.24) we find that (7.51) may be written in the form

$$\overline{\Box}V = 0 \; . \qquad (7.53)$$

We shall now define two new variables U and Ψ by the following equations:

$$V = \left(1 + \left(\frac{8\pi k}{c^4}\right)\frac{A^2}{2}\right)^{1/2} U \quad , \qquad (7.54)$$

and

$$\Psi = A\,U \, , \qquad (7.55)$$

where A is a non-zero constant. Upon substituting (7.54) and (7.55) into equations (7.52) and (7.53) we obtain

$$H_{\alpha\beta} - 2U_{,\alpha}U_{,\beta} = \frac{8\pi k}{c^4}\Psi_{,\alpha}\Psi_{,\beta} \qquad (7.56)$$

and

$$\bar{\Box}\Psi = 0 \ . \qquad (7.57)$$

It is now apparent that a solution to the equations

$$\bar{G}_{ij} = \frac{8\pi k}{c^4}\left(\Psi_{,i}\Psi_{,j} - \tfrac{1}{2}\bar{g}_{ij}\bar{g}^{qm}\Psi_{,q}\Psi_{,m}\right), \qquad (7.6)$$

and

$$\bar{\Box}\Psi = 0 \ , \qquad (7.7)$$

is given by the metric of the line element (7.9) and Ψ when U and Ψ are given by (7.54) and (7.55) respectively.

Corresponding to the above solution of equations (7.6) and (7.7) we have a solution to the Brans-Dicke vacuum field equations (7.3) and (7.4). To establish this result it will be necessary to make use of the results of the previous section.

Recall that we have shown that if functions $\bar{g}_{ij}$ and ϕ have been found which satisfy equations (7.1) and (7.2) then the same scalar function ϕ and the function g_{ij}, defined by

$$g_{ij} = \frac{1}{k\phi}\,\bar{g}_{ij} \qquad (7.58)$$

will satisfy the Brans-Dicke vacuum field equations (7.3) and (7.4). Thus if we can show that solutions to (7.6) and (7.7) can be used to construct solutions to (7.1) and (7.2) we are essentially finished. However, this problem is easily solved because if $\bar{g}_{ij}$ and Ψ satisfy (7.6) and (7.7) then $\bar{g}_{ij}$ and ϕ, where ϕ is defined by

$$\phi = \frac{1}{k}\exp\left(\frac{\Psi}{p}\right), \qquad (7.59)$$

with

$$p = \left(\frac{c^4(3 + 2\omega)}{16\pi k}\right)^{1/2}, \qquad (7.60)$$

will be a solution to equations (7.1) and (7.2). In summary we have shown that a static solution to

$$\bar{R}_{ij} = 0 \qquad (7.47)$$

generates a static solution to the Brans-Dicke vacuum field equations (7.3) and (7.4).

To illustrate the procedure involved we shall now use the Schwarzschild solution to the free space Einstein field equations to obtain a solution to the Brans-Dicke vacuum field equations. The solution which we shall obtain corresponds to the first of Brans's vacuum solution presented in[5].

The isotropic form of the Schwarzschild line element can be written in the form

$$ds^2 = -\left(\frac{\rho - B}{\rho + B}\right)^2 c^2 dT^2 + \frac{(\rho + B)^4}{\rho^4}\left(d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta \, d\epsilon^2\right), \qquad (7.61)$$

where B is a constant and ϵ denotes the azimuthal angle.

To begin we make the following coordinate transformation

$$t = T, \quad r = B\left(\frac{\rho}{B} + \frac{B}{\rho}\right), \quad \theta' = \theta, \text{ and } \epsilon' = \epsilon . \qquad (7.62)$$

Under the above coordinate transformation we find:

$$\left(\frac{\rho - B}{\rho + B}\right)^2 = \frac{r - 2B}{r + 2B}, \qquad (7.63)$$

$$\left(\frac{\rho + B}{\rho}\right)^4 d\rho^2 = \left(\frac{r + 2B}{r - 2B}\right) dr^2 , \qquad (7.64)$$

and

$$\frac{(\rho + B)^4}{\rho^2} = \left(\frac{r + 2B}{r - 2B}\right)(r^2 - 4B^2) . \qquad (7.65)$$

If we set

$$m = 2B \qquad (7.66)$$

and then substitute equations (7.63)-(7.65) into (7.61) we obtain

$$ds^2 = -\left(\frac{r - m}{r + m}\right) c^2 dt^2 + \left(\frac{r - m}{r + m}\right)^{-1} \left\{ dr^2 + (r^2 - m^2) d\Omega^2 \right\} \qquad (7.67)$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\epsilon^2 .$$

It is apparent that the above line element can be brought into the form of (7.46) if we set

$$\left.\begin{aligned} e^{2V} &= \frac{r-m}{r+m}\ , \quad h_{11} = 1\ , \quad h_{22} = (r^2 - m^2)\ , \\ h_{33} &= (r^2 - m^2)\sin^2\theta \quad \text{and} \quad h_{\alpha\beta} = 0 \text{ if } \alpha \neq \beta, \end{aligned}\right\} \tag{7.68}$$

To obtain a static spherically symmetric solution to equations (7.6) and (7.7) we use (7.54) and (7.55) to find

$$U = \frac{1}{2\mu} \ln\left(\frac{r-m}{r+m}\right) \ , \tag{7.69}$$

and

$$\Psi = \frac{A}{2\mu} \ln\left(\frac{r-m}{r+m}\right) \ , \tag{7.70}$$

where

$$\mu = \left(1 + \left(\frac{8\pi k}{c^4}\right)\frac{A^2}{2}\right)^{1/2} \ , \tag{7.71}$$

and A is a <u>non</u>-<u>zero</u> constant. Thus the line element corresponding to our solution of equations (7.6) and (7.7) is given by

$$ds^2 = -\left(\frac{r-m}{r+m}\right)^{1/\mu} c^2 dt^2 + \left(\frac{r+m}{r-m}\right)^{1/\mu}\left\{dr^2 + (r^2 - m^2)d\Omega^2\right\} . \tag{7.72}$$

In order for Ψ and U to be real valued functions we must demand A to be real. This condition implies that

$$\mu = \left(1 + \left(\frac{8\pi k}{c^4}\right)\frac{A^2}{2}\right)^{1/2} > 1 \ , \tag{7.73}$$

(recall that A must be <u>non</u>-<u>zero</u>).

To obtain a solution to the Brans-Dicke vacuum field equations (7.3) and (7.4) we define

$$g_{ij} = \frac{1}{k\phi}\,\bar{g}_{ij} \quad \text{and} \quad \phi = \frac{1}{k}\exp\left(\frac{\Psi}{p}\right) \ ,$$

where in the present case the $\bar{g}_{ij}$'s correspond to the metric coefficients appearing in (7.72). Thus we obtain:

$$\phi = \frac{1}{k}\left(\frac{r-m}{r+m}\right)^{\frac{A}{2\mu p}} , \tag{7.74}$$

$$g_{00} = \left(\frac{r+m}{r-m}\right)^{\left(\frac{A-2p}{2\mu p}\right)} \ , \tag{7.75}$$

$$g_{11} = g_{22} = g_{33} = \left(\frac{r + m}{r - m}\right)^{\left(\frac{A+2p}{2\mu p}\right)}, \tag{7.76}$$

and

$$g_{ij} = 0 \text{ if } i \neq j .$$

The line element corresponding to the above g_{ij}'s has the form

$$ds^2 = -\left(\frac{r + m}{r - m}\right)^{\left(\frac{A-2p}{2\mu p}\right)} c^2 dt^2 + \left(\frac{r + m}{r - m}\right)^{\left(\frac{A+2p}{2\mu p}\right)} \left\{ dr^2 + (r^2 - m^2) d\Omega^2 \right\} . \tag{7.77}$$

In order to put (7.77) into the isotropic form used by Brans we shall make the following transformations:

$$t = \tilde{t}; \quad r = B\left(\frac{\tilde{r}}{B} + \frac{B}{\tilde{r}}\right); \quad \theta = \tilde{\theta}; \quad \epsilon = \tilde{\epsilon}; \quad m = 2B; \tag{7.78}$$

$$\frac{D}{\lambda} = \frac{A}{\mu p}; \quad \frac{A}{\mu p} - \frac{2}{\mu} = -\frac{2}{\lambda},$$

where B, D and λ are constants.

The above transformations yield:

$$\left(\frac{r - m}{r + m}\right) = \left(\frac{\tilde{r} - B}{\tilde{r} + B}\right)^2, \tag{7.79}$$

$$r^2 - m^2 = B^2\left(\frac{\tilde{r}}{B} + \frac{B}{\tilde{r}}\right)^2, \tag{7.80}$$

$$dr^2 = B^2\left(\frac{B}{\tilde{r}^2} - \frac{1}{B}\right)^2 d\tilde{r}^2, \tag{7.81}$$

$$\frac{A - 2p}{2\mu p} = -\frac{1}{\lambda} \quad \text{and} \quad \frac{A + 2p}{2\mu p} = \frac{D}{\lambda} + \frac{1}{\lambda} . \tag{7.82}$$

Inserting equations (7.79)-(7.82) into (7.74) and (7.77) gives us

$$\phi = \phi_0 \left(\frac{\tilde{r} - B}{\tilde{r} + B}\right)^{D/\lambda}, \tag{7.83a}$$

and

$$ds^2 = -\left(\frac{\tilde{r} + B}{\tilde{r} - B}\right)^{-2/\lambda} c^2 d\tilde{t}^2 + \left(\frac{\tilde{r} + B}{\tilde{r} - B}\right)^{2(D+1-\lambda)/\lambda} \left(1 + \frac{B}{\tilde{r}}\right)^4 (d\tilde{r}^2 + \tilde{r}^2 d\Omega^2), \tag{7.83b}$$

where

$$\phi_0 = \frac{1}{k} .$$

The functions appearing in (7.83a) and (7.83b) are just those which appear in Brans's "physical" vacuum solution, [5]. We shall return to this solution in section 9.

The only task remaining before us is to confirm that the range of λ in terms of D agrees with the range determined by Brans. To accomplish this we shall examine our definitions of μ, A, D and λ. From these definitions we have

$$\mu^2 = 1 + \left(\frac{8\pi k}{c^4}\right)\frac{A^2}{2} \qquad (7.84)$$

and

$$\frac{1}{\mu} = \frac{1}{2}\left(\frac{D}{\lambda} + \frac{2}{\lambda}\right) . \qquad (7.85)$$

The second of the above expressions may be rewritten as follows:

$$\mu = \frac{2\lambda}{D+2} . \qquad (7.86)$$

From the last expression appearing in (7.78) we see that

$$A = 2p(1 - \frac{\mu}{\lambda}) ,$$

and thus (7.84) can be rewritten in the form

$$\mu^2 = 1 + \frac{16\pi k\, p^2}{c^4}(1 - \frac{2\mu}{\lambda} + \frac{\mu^2}{\lambda^2}) . \qquad (7.87)$$

If we now make use of equations (7.60) and (7.86) we find that (7.87) becomes

$$\lambda^2 = (D+1)^2 - D(1 - \frac{\omega D}{2}) . \qquad (7.88)$$

For ϕ and the components of our metric tensor to be finite and real λ must be non-zero and real. Therefore we must demand that

$$\lambda^2 = (D+1)^2 - D(1 - \frac{\omega D}{2}) > 0 , \qquad (7.89)$$

which agrees with Brans.

In section 9 we shall see that Brans has found exactly four static spherically symmetric solutions of equations

(7.3) and (7.4) corresponding to the isotropic line element

$$ds^2 = -e^{2\alpha}c^2dt^2 + e^{2\beta}(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\epsilon^2),$$

where α and β are functions of r. We have just seen how to obtain the first of Brans's four solutions. However, at present, we are not sure whether the above procedure can be used to generate the remaining three.

8. The Weak Field Approximation[40]

This section is devoted to finding an approximate solution the the Brans-Dicke field equations; viz.,

$$R_{ij} - \frac{1}{2}g_{ij}R = \frac{8\pi T_{ij}}{\phi c^4} + \frac{\omega}{\phi^2}(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,k}\phi^{,k}) + \\ + \frac{1}{\phi}(\phi_{|ij} - g_{ij}\Box\phi) , \tag{8.1}$$

and

$$\Box\phi = \frac{8\pi T}{(2\omega + 3)c^4} . \tag{8.2}$$

We shall find this approximate solution to be of great value when we attempt to identify the constants which appear in the exact solutions to the above system of equations. It will also be convenient to have a weak field solution available when we begin our study of the scalar field ϕ in the next section.

We begin by assuming that g_{ij} and ϕ may be approximated as follows:[41]

$$g_{ij} = \eta_{ij} + h_{ij} , \tag{8.3}$$

and

$$\phi = \phi_0 + \xi , \tag{8.4}$$

where the η_{ij}'s are the components of the Minkowski tensor; viz.,

$$\eta_{00} = -1 , \eta_{11} = \eta_{22} = \eta_{33} = 1 \text{ and } \eta_{ij} = 0 \text{ if } i \neq j. \tag{8.5}$$

Our task is now to determine the constant ϕ_0, along with the functions h_{ij} and ξ, which we assume to be of class C^2.

In the calculations which follow it will be assumed that

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[40]The material used in this section is based on a paper by C.Brans and R.H.Dicke [4].

[41]The functions h_{ij} given in equation (8.3) should not confused with the functions $h_{\alpha\beta}$ introduced in equation (7.9).

all quantities which are not linear in h_{ij} and ξ are negligible compared with h_{ij} and ξ; e.g., terms like

$$\xi_{,i}h_{kl}\ ,\ \xi_{,i}\xi_{,jk}\ ,\ h_{ij}h_{kp,mn}\ \text{and}\ \xi h_{ij}\ ,$$

will be neglected in this approximation. As a result of this assumption it is easily seen that

$$g^{ij} = \eta^{ij} - \eta^{ik}\eta^{jm}h_{km}\ , \qquad (8.6)$$

and

$$\frac{1}{\phi} = \frac{1}{\phi_0}\left(1 - \frac{\xi}{\phi_0}\right). \qquad (8.7)$$

The first equation to be considered is the scalar wave equation (8.2). Upon expanding the left hand side of that equation we find

$$\Box\phi = (g^{ij}{}_{,i}\phi_{,j} + g^{ij}\phi_{,ij}) + \frac{\sqrt{-g}_{,i}}{\sqrt{-g}}\,g^{ij}\phi_{,j}\ . \qquad (8.8)$$

In the above expression we may replace $g^{ij}{}_{,i}\phi_{,j}$ by

$$-(g^{kj}\Gamma^{i}_{ki} + g^{ik}\Gamma^{j}_{ik})\phi_{,j}\ .$$

However, Γ^{k}_{ij} contains terms of the form $g_{pm,j}$ and thus we see that $g^{ij}{}_{,i}\phi_{,j}$ is of second order and may be dropped from $\Box\phi$. Similarly, since

$$\frac{\sqrt{-g}_{,i}}{\sqrt{-g}}\,g^{ij}\phi_{,j} = \Gamma^{k}_{ik}\,g^{ij}\,\phi_{,j}$$

the last term appearing on the right hand side of (8.8) may be neglected, leaving us with

$$\Box\phi \approx g^{ij}\phi_{,ij}\ . \qquad (8.9)$$

Due to equation (8.6) we find that our expression for $\Box\phi$ becomes

$$\Box\phi \approx \nabla^2\xi - \frac{1}{c^2}\frac{\partial^2\xi}{\partial t^2}\ ,$$

where ∇^2 is the Laplacian operator for a three dimensional flat space. Consequently under our present approximation

(8.2) assumes the following form:

$$\nabla^2\xi - \frac{1}{c^2}\frac{\partial^2\xi}{\partial t^2} = \frac{8\pi T}{(2\omega+3)c^4} \quad . \tag{8.10}$$

A retarded time "solution" to (8.10) is given by[42]

$$\xi = \frac{-2}{(2\omega+3)c^4}\int_V \frac{g^{kq}T_{kq}(\vec{r}\,',\ t-\frac{1}{c}|\vec{r}-\vec{r}\,'|)\,d^3x'}{|\vec{r}-\vec{r}\,'|} \quad , \tag{8.11}$$

where $\vec{r}$ denotes a position vector to the field point of observation, and $\vec{r}\,'$ is a vector whose end point denotes the source points. The integral appearing in (8.11) is performed over the spatial volume V in which T is different from zero, and this volume is generally a function of time.

It should be noted that at present the intergral appearing on the right hand side of (8.11) <u>cannot</u> be evaluated. This is so because the integrand involves g^{kq} and hence contains the unknown functions h_{ij}. Now we know that T_{kq} is independent of ϕ, and consequently can be a first order quantity only if it contains h_{kq}. If such were the case we could set

$$g^{kq}T_{kq} \approx \eta^{kq}T_{kq} \; .$$

However, is would still be impossible (in general) to evaluate the integral appearing in (8.11) because T_{kq} would still involve the unknown h_{ij}'s. Consequently (8.11) should <u>not</u> be regarded as a solution to (8.2) but rather as an integral equation. Later in this section it will be shown that for a special choice of T_{kq} the integral equation (8.11) can be solved quite simply.

[42] We are using a retarded time solution because we do not plan to investigate the problem of radiative reactions; i.e., we are not interested in the behaviour of the scalar field in the immediate vicinity of the source. For a study of radiative reaction in electromagnetic theory see[24].

We shall now rewrite equation (8.1) in accordance with our present approximation. To begin let us examine R_{ij} which is defined by

$$R_{ij} = \Gamma^{k}_{ij,k} - \Gamma^{k}_{ik,j} + \Gamma^{m}_{ij}\Gamma^{k}_{mk} - \Gamma^{m}_{ik}\Gamma^{k}_{mj} . \qquad (8.12)$$

The last two terms appearing in the above expression involve a product of the derivatives of g_{ij}. Thus these terms are of second order and hence may be neglected. In order to complete our rewriting of (8.12) it will be necessary to examine the form assumed by the derivatives of the Christoffel symbols in our approximation.

Using (8.3) and (8.6) we easily find that

$$\Gamma^{q}_{ij} = \frac{\eta^{qk}}{2}(h_{ik,j} + h_{jk,i} - h_{ij,k}) , \qquad (8.12a)$$

and consequently $\Gamma^{q}_{ij,q}$ and $\Gamma^{q}_{iq,j}$ are given by

$$\Gamma^{q}_{ij,q} = \tfrac{1}{2}\eta^{qk}(h_{ik,jq} + h_{jk,iq} - h_{ij,kq}) ,$$

$$\Gamma^{q}_{iq,j} = \tfrac{1}{2}\eta^{qk}(h_{ik,qj} + h_{kq,ij} - h_{iq,kj}) .$$

From the above expressions we find that, in our approximation, the Ricci tensor is given by

$$R_{ij} = -\frac{\eta^{qk}}{2}(h_{kq,ij} + h_{ij,kq} - h_{iq,kj} - h_{jk,iq}) ,$$

and as a consequence of (8.6) the curvature invariant R assumes the following form

$$R = -\frac{\eta^{ij}\eta^{qk}}{2}(h_{kq,ij} + h_{ij,kq} - h_{iq,kj} - h_{jk,iq}).$$

Combining these two equations we find, as usual, the linearized Einstein tensor to be

$$G_{ij} = R_{ij} - \tfrac{1}{2}g_{ij}R = -\tfrac{1}{2}\Big\{ h,_{ij} + \eta^{kq}(h_{ij,kq} - h_{iq,kj} + - h_{jk,iq}) - \tfrac{1}{2}\eta_{ij}h,_{mn}\eta^{mn} - \tfrac{1}{2}\eta_{ij}(h,_{kq}\eta^{kq} - \eta^{kq}\eta^{mn}h_{mq,kn} + - \eta^{kq}\eta^{mn}h_{kn,mq}) \qquad , \quad (8.13)$$

where

$$h = \eta^{kq} h_{kq} \, .$$

(8.13) may be rewritten in a simpler form if we introduce two new quantities γ_{ij} , and σ_i , defined by

$$\gamma_{ij} = h_{ij} - \tfrac{1}{2}\eta_{ij} h \tag{8.14}$$

and

$$\sigma_i = \gamma_{ij,k}\eta^{jk} = h_{ij,k}\eta^{jk} - \tfrac{1}{2}h_{,i} \, . \tag{8.15}$$

Using (8.14) and (8.15) we find that (8.13) may be rewritten as follows:

$$G_{ij} = -\tfrac{1}{2}(\Box\gamma_{ij} - \sigma_{i,j} - \sigma_{j,i} + \eta_{ij}\sigma_{k,q}\eta^{kq}) . \tag{8.16}$$

Now that we have linerarized the Einstein tensor let us devote our attention to the right hand side of equation (8.1). The first thing we observe is that the term

$$\frac{\omega}{\phi^2}(\phi_{,i}\phi_{,j} - \tfrac{1}{2}g_{ij}g^{kq}\phi_{,k}\phi_{,q})$$

is second order in $\xi_{,i}$ and thus may be neglected. The remaining terms on the right hand side of equation (8.1) may be written as follows:

$$\frac{1}{\phi_o}\left\{\xi_{,ij} - \eta_{ij}\Box\xi\right\} + \frac{8\pi}{\phi_o c^4}(1 - \frac{\xi}{\phi_o})T_{ij} \, . \tag{8.17}$$

In obtaining (8.17) we made use of equation (8.7).

At this point Brans and Dicke choose to drop

$$-\frac{8\pi\xi T_{ij}}{\phi_o^2 c^4} \tag{8.18}$$

from consideration. This would be permissible if we knew that this term was of second or higher order. However, at present we have no grounds for believing that this is the case. Furthermore, later in this section we shall choose T_{ij} to be of zeroth order and consequently it is impossible for us to neglect (8.18) at this stage.

If we now equate (8.16) to (8.17) we obtain the linearized field equations governing h_{ij}; viz.,

$$-\tfrac{1}{2}\left\{\Box\gamma_{ij} - \sigma_{i,j} - \sigma_{j,i} + \eta_{ij}\sigma_{k,q}\eta^{kq}\right\} = \frac{1}{\phi_0}\left\{\zeta_{,ij} - \eta_{ij}\Box\zeta\right\} + \frac{8\pi}{\phi_0}\left(1 - \frac{\zeta}{\phi_0}\right)T_{ij} . \qquad (8.19)$$

At this point we shall find it useful to impose a coordinate condition. In order to motivate our particular choice of a coordinate condition we shall make use of the results of section 6. Recall that in that section the conformal transformation

$$\bar{g}_{ij} = k\phi\, g_{ij} , \qquad (8.20)$$

made it possible for us to obtain solutions to the Brans-Dicke field equations from solutions to the Einstein field equations. At that time we wrote the Einstein field equations in terms of the barred metric tensor and the Brans-Dicke field equations were written in terms of the unbarred metric. Now when working with the Einstein field equations it is frequently found convenient to impose the following coordinate condition

$$\bar{g}^{jk}\bar{\Gamma}^{m}_{jk} = 0 , \qquad (8.21)$$

which is usually referred to as the harmonic coordinate condition. Thus the most natural coordinate condition to impose when dealing with the Brans-Dicke field equations would be (8.21) written in terms of the unbarred metric. Let us now use (8.20) to rewrite (8.21) in terms of the unbarred metric.

From Eisenhart ([10], page 89) we find that under (8.20) the barred Christoffel symbols transform as follows:

$$\bar{\Gamma}^m_{ij} = \Gamma^m_{ij} + \tfrac{1}{2}\delta^m_i(\ln k\phi)_{,j} + \tfrac{1}{2}\delta^m_j(\ln k\phi)_{,i} - \tfrac{1}{2}g_{ij}g^{qm}(\ln k\phi)_{,q}. \tag{8.22}$$

Thus we see that in terms of the unbarred metric the harmonic coordinate condition, (8.21), becomes

$$\frac{1}{k\phi}\left\{g^{jk}\Gamma^m_{jk} - g^{qm}(\ln k\phi)_{,q}\right\} = 0 . \tag{8.23}$$

We shall now proceed to rewrite the above expression in accordance with our present approximation.

Using equation (8.12a) we find that

$$g^{jk}\Gamma^m_{jk} \approx \tfrac{1}{2}\eta^{jk}\eta^{mp}(h_{jp,k} + h_{kp,j} - h_{jk,p}) . \tag{8.24}$$

By combining equation (8.6) with (8.7) we find

$$g^{pm}(\ln k\phi)_{,p} \approx \eta^{pm}\frac{\xi_{,p}}{\phi_0} . \tag{8.25}$$

Upon inserting (8.24) and (8.25) into (8.23) we obtain

$$\tfrac{1}{2}\eta^{jk}\eta^{mp}(h_{jp,k} + h_{kp,j} - h_{jk,p}) - \eta^{pm}\frac{\xi_{,p}}{\phi_0} = 0 ,$$

which after multiplying through by η_{mi} becomes

$$\eta^{jk}h_{ji,k} - \tfrac{1}{2}h_{,i} = \frac{\xi_{,i}}{\phi_0} . \tag{8.27}$$

Due to equation (8.15) the above expression may be written as follows:

$$\sigma_i = \frac{\xi_{,i}}{\phi_0} , \tag{8.28}$$

which is precisely the coordinate condition exployed by Brans and Dicke. We shall henceforth assume that our coordinates have been chosen so as to be compatible with (8.28).

Returning now to equation (8.19) we find that our coordinate condition (8.28) permits us to rewrite that expression as follows:

$$-\tfrac{1}{2}\left(\Box\gamma_{ij} - 2\frac{\xi_{,ij}}{\phi_o} + \frac{\eta_{ij}}{\phi_o}\eta^{kq}\xi_{,kq}\right) =$$

$$= \frac{1}{\phi_o}\left(\xi_{,ij} - \eta_{ij}\eta^{kq}\xi_{,kq}\right) + \frac{8\pi}{c^4\phi_o}\left(1 - \frac{\xi}{\phi_o}\right)T_{ij} \; . \quad (8.29)$$

This expression assumes a simpler appearance if we let

$$\alpha_{ij} = \gamma_{ij} - \eta_{ij}\frac{\xi}{\phi_o} = h_{ij} - \eta_{ij}\left(\frac{h}{2} + \frac{\xi}{\phi_o}\right) \; . \quad (8.30)$$

Using (8.30) we find that (8.29) becomes

$$\Box\alpha_{ij} = -\frac{16\pi}{c^4\phi_o}\left(1 - \frac{\xi}{\phi_o}\right)T_{ij} \; . \quad (8.31)$$

A "formal retarded time solution"(c.f. footnote 42) to the above partial differential equation is given by

$$\alpha_{ij} = \frac{4}{\phi_o c^4}\int_V \left\{1 - \frac{\xi(\vec{r}\,', t - \frac{1}{c}|\vec{r} - \vec{r}\,'|)}{\phi_o}\right\} \frac{T_{ij}(\vec{r}\,', t - \frac{1}{c}|\vec{r} - \vec{r}\,'|)d^3x'}{|\vec{r} - \vec{r}\,'|} \; . \quad (8.32)$$

We refer to this solution as a "formal solution" because (c.f. remarks following equation (8.11)) the functions which appear within the integrand on the right hand side of (8.32) have not as yet been determined. However, we shall show that (8.32) can be used to obtain an integral equation for h_{ij}.

We begin by showing that h_{ij} may be expressed in terms of α_{ij} and ξ. From (8.30) we know that

$$\alpha_{ij} = h_{ij} - \tfrac{1}{2}\eta_{ij}h - \eta_{ij}\frac{\xi}{\phi_o} \; . \quad (8.33)$$

Upon multiplying this equation by η^{ij} we obtain

$$\alpha = -h - \frac{4\xi}{\phi_o} \; , \quad (8.34)$$

where we have set

$$\alpha = \eta^{ij}\alpha_{ij} \; . \quad (8.35)$$

If we solve (8.34) for h and insert that result into (8.33) we find that

$$h_{ij} = \alpha_{ij} - \tfrac{1}{2}\eta_{ij}\alpha - \eta_{ij}\frac{\xi}{\phi_o} . \qquad (8.36)$$

Combining equation (8.32) with our definition of α gives us

$$\alpha = \frac{4}{\phi_o c^4}\int_V \left\{1 - \frac{\xi}{\phi_o}\right\}\frac{\eta^{ij}T_{ij}}{|\vec{r} - \vec{r}'|}\, d^3x' , \qquad (8.37)$$

where it is to be understood that all quantities appearing within the above integrand are evaluated at the retarded time (the same convention applies to the integrals given below).

Thus equations (8.11), (8.32), (8.36) and (8.37) permit us to conclude that in our approximation h_{ij} satisfies the following integral equation:

$$h_{ij} = \frac{4}{\phi_o c^4}\int_V (1 - \frac{\xi}{\phi_o})\frac{T_{ij}\, d^3x'}{|\vec{r} - \vec{r}'|} - \frac{2\eta_{ij}}{\phi_o c^4}\int_V (1 - \frac{\xi}{\phi_o})\frac{\eta^{kq}T_{kq}}{|\vec{r} - \vec{r}'|}\, d^3x' +$$

$$+ \frac{2\eta_{ij}}{\phi_o(2\omega + 3)c^4}\int_V \frac{(\eta^{kq} - \eta^{ki}\eta^{qj}h_{ij})T_{kq}}{|\vec{r} - \vec{r}'|}\, d^3x' , \qquad (8.38)$$

where the function ξ is governed by

$$\xi = \frac{-2}{(2\omega + 3)c^4}\int_V \frac{(\eta^{kq} - \eta^{ki}\eta^{qj}h_{ij})T_{kq}}{|\vec{r} - \vec{r}'|}\, d^3x' . \qquad (8.11)$$

In general the coupled system of integral equations represented by (8.11) and (8.38) is quite complicated. However, we shall only be interested in solving this system when T_{kq} corresponds to the energy momentum tensor of a static point mass M situated at the origin of our coordinate system. In this case it is quite simple to determine a solution for ϕ_0, ξ and h_{ij} which is in accordance with our approximation. We shall regard the resultant solution as our weak field solution.

We begin by assuming that the energy momentum tensor T_{jk} corresponding to a static point mass M at the origin may be represented by

$$T_{jk} = \begin{cases} Mc^2\delta(\vec{r}) & \text{if } j=k=0\,, \\ 0, & \text{otherwise}\,, \end{cases} \tag{8.39}$$

where $\delta(\vec{r})$ is the three dimensional Dirac delta function.

Upon inserting (8.39) into (8.11) we find that ξ is given by

$$\xi = \frac{-2}{(2\omega+3)c^4}\int_V (\eta^{00} - \eta^{0i}\eta^{0j}h_{ij})\frac{Mc^2\delta(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d^3x'\;,$$

which due to (8.5) becomes

$$\xi = \frac{2M}{(2\omega+3)c^2}\int_V (1 + h_{00}(|\vec{r}-\vec{r}\,'|))\frac{\delta(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d^3x'\;. \tag{8.40}$$

Upon performing the above integration we find that

$$\xi = \frac{2M}{(2\omega+3)c^2}\left\{\frac{1}{r} + \frac{h_{00}(r)}{r}\right\}\;, \tag{8.41}$$

where

$$r = |\vec{r}|\;. \tag{8.42}$$

Now recall that in our approximation ξh_{00} and $(h_{00})^2$ are considered to be negligible in comparison with ξ and h_{00}. Thus we may use (8.41) to conclude that

$$\frac{h_{00}(r)}{r} \approx 0\;, \tag{8.43}$$

and hence in our approximation ξ is given by

$$\xi = \frac{2M}{(2\omega+3)c^2 r}\;. \tag{8.44}$$

Using (8.39) and (8.44) we find that our integral equation for h_{ij}, (8.38), may be written as follows:

$$h_{ij} = \frac{4}{\phi_o c^4}\int_V\left\{1 - \frac{2M}{\phi_o(2\omega+3)c^2|\vec{r}-\vec{r}'|}\right\}\frac{Mc^2\delta_i^0\delta_j^0\,\delta(\vec{r}')d^3x'}{|\vec{r}-\vec{r}'|} +$$

$$- \frac{2\eta_{ij}}{\phi_o c^4}\int_V\left\{1 - \frac{2M}{\phi_o(2\omega+3)c^2|\vec{r}-\vec{r}'|}\right\}\frac{(-Mc^2)\delta(\vec{r}')\,d^3x'}{|\vec{r}-\vec{r}'|} +$$

$$+ \frac{2\eta_{ij}}{\phi_o(2\omega+3)c^4}\int_V(-Mc^2)(1 + h_{00}(|\vec{r}-\vec{r}'|))\frac{\delta(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d^3x'. \qquad (8.45)$$

Upon evaluating the above integrals we find that h_{ij} is given by:

$$h_{ij} = \frac{4M}{\phi_o c^2}\left\{1 - \frac{2M}{\phi_o(2\omega+3)c^2 r}\right\}\frac{\delta_i^0\delta_j^0}{r} + \qquad (8.46)$$

$$+ \frac{2M\eta_{ij}}{\phi_o c^2 r}\left\{1 - \frac{2M}{\phi_o(2\omega+3)c^2 r}\right\} - \frac{2M\eta_{ij}}{\phi_o(2\omega+3)c^2}\left\{\frac{1}{r} + \frac{h_{00}(r)}{r}\right\},$$

which due to (8.43) may be rewritten as follows:

$$h_{ij} = \frac{4M}{\phi_o c^2 r}\left\{\delta_i^0\delta_j^0 + \eta_{ij}\frac{(1+\omega)}{(2\omega+3)}\right\} + $$

$$- \frac{4M^2}{\phi_o^2 c^4(2\omega+3)r^2}(2\delta_i^0\delta_j^0 + \eta_{ij}). \qquad (8.47)$$

Since $\frac{h_{00}}{r}$ has been shown to be a higher order term we can use (8.47) to conclude that

$$\frac{-4M^2}{\phi_o^2 c^4(2\omega+3)r^2}\left\{2\delta_i^0\delta_j^0 + \eta_{ij}\right\}$$

is also of higher order and hence can be discarded from the above expression for h_{ij}. Thus we find that h_{ij} is given by

$$h_{ij} = \frac{4M}{\phi_o c^2 r}\left\{\delta_i^0\delta_j^0 + \eta_{ij}\left(\frac{1+\omega}{2\omega+3}\right)\right\}. \qquad (8.48)$$

It is now a trivial task to show that when T_{jk} is given by (8.39) then in our approximation ϕ and g_{ij} are given by:

$$\phi = \phi_0 + \frac{2M}{(2\omega + 3)c^2 r}, \tag{8.49}$$

$$g_{00} = -1 + \frac{4M}{\phi_0 c^2 r}\left(\frac{2 + \omega}{2\omega + 3}\right), \tag{8.50}$$

$$g_{11} = g_{22} = g_{33} = 1 + \frac{4M}{\phi_0 c^2 r}\left(\frac{1 + \omega}{2\omega + 3}\right), \tag{8.51}$$

and

$$g_{ij} = 0 \quad \text{if } i \neq j . \tag{8.52}$$

It should now be noted that in the limit as ω goes to infinity equations (8.49)-(8.52) go over to the corresponding weak field solution of the linearized Einstein field equations,[43] provided we set the gravitational coupling constant, k, equal to $\frac{1}{\phi_0}$. Thus we shall henceforth regard ϕ_0 as being the reciprocal of the conventional gravitational constant.

We shall now list the assumptions which have been made in order to obtain the above weak field solution to equations (8.1) and (8.2):

(i) The metric tensor g_{ij} and the scalar field ϕ can be approximated by

$$g_{ij} = \eta_{ij} + h_{ij}, \tag{8.3}$$

and

$$\phi = \phi_0 + \xi, \tag{8.4}$$

where ϕ_0 is a constant, h_{ij} and ξ are functions of class C^2, and η_{ij} is the Minkowski tensor.

(ii) All quantities which are not linear in h_{ij} and ξ can be regarded as negligible in comparison with h_{ij} and ξ.

(iii) The coordinates can be chosen so that

$$\eta^{jk}(h_{ji,k} - \tfrac{1}{2}h_{jk,i}) = \frac{\xi_{,i}}{\phi_0}. \tag{8.27}$$

[43] The Einstein weak field solutions can be found in [1], page 242.

and

(iv) The energy momentum tensor corresponding to a static point mass M situated at the origin can be represented by

$$T_{jk} = \begin{cases} Mc^2 \delta(\vec{r}) & \text{if } j = k = 0 , \\ 0 , & \text{otherwise} , \end{cases} \tag{8.39}$$

where $\delta(\vec{r})$ is the three dimensional Dirac delta function.

9. The Boundary Condition Governing the Scalar Field ϕ in the Region Outside a Static Spherically Symmetric Mass Shell[44]

The purpose of this section is to examine the behaviour of the scalar field ϕ in the region outside a static spherically symmetric mass shell. Our mass shell will be located between $r = R_1$ and $r = R_2$, where $R_1 < R_2$, and at the center of this mass shell we shall place a small mass m. The following picture depicts a crossection of the physical situation we have in mind.

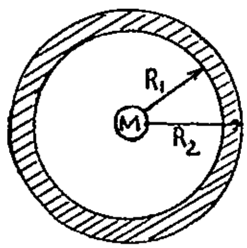

Figure 1

By analyzing the above example we shall obtain a boundary condition for ϕ which is valid in the region outside of (i.e., for $r > R_2$) our spherically symmetric static mass shell.

For convenience we shall denote the three regions of interest in figure 1 as follows:

$$I = \{r : 0 \leq r < R_1\} \quad , \tag{9.1}$$

$$II = \{r : R_1 \leq r \leq R_2\} , \tag{9.2}$$

and

$$III = \{r : R_2 < r\} \quad . \tag{9.3}$$

We shall also denote differentiation with respect to r by a prime.

---

[44] The material in this section is based on a paper by C.Brans, [5].

To begin we make three assumptions:

(i) In region I, II and III the line element assumes the following isotropic form:

$$ds^2 = -e^{2\alpha}c^2dt^2 + e^{2\beta}(dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\epsilon^2) , \quad (9.4)$$

where α and β are functions of r and are of class C^2 in regions I, II and III.

(ii) The functions α, β and the scalar field ϕ are of class C^1 across the boundarys separating region I from region II, and region II from region III.

(iii) Region II makes no contribution to the gravitational field within region I, and the gravitational field in region I can be approximated by our weak field solution.

Due to the above assumptions we may use our weak field solution, equations (8.49)-(8.52), (which is valid in region I) to obtain the sign of α', β', ϕ' and ϕ at $r = R_1$. From equations (8.49)-(8.51) we find that in region I $e^{2\alpha}$, $e^{2\beta}$, and ϕ are given by:

$$e^{2\alpha} = 1 - \frac{2A}{r} \; , \quad e^{2\beta} = 1 + \frac{2(\omega+1)A}{(\omega+2)r} \; , \quad \phi = \frac{1}{k}\left(1 + \frac{A}{r(\omega+2)}\right) ; \quad (9.5)$$

where

$$k = \frac{1}{\phi_o} > 0 \; , \; A = G_o\frac{m}{c^2} \; , \quad m > 0 \; , \; G_o = k\left(\frac{2\omega+4}{2\omega+3}\right) ,$$

and

$$\left|\frac{A}{R_1}\right| \ll 1 .$$

From (9.5) we readily obtain the following expressions for α', β', ϕ' and $\alpha' + \beta'$:

$$\alpha' = \frac{1}{\left(1 - \frac{2A}{r}\right)} \frac{A}{r^2} , \quad (9.6)$$

$$\beta' = \frac{-1}{\left(1 + \frac{2A}{r}\left(\frac{\omega+1}{\omega+2}\right)\right)}\left(\frac{A(\omega+1)}{r^2(\omega+2)}\right), \tag{9.7}$$

$$\phi' = \frac{-2m}{c^2(2\omega+3)r^2}, \tag{9.8}$$

and

$$\alpha' + \beta' = \frac{\left(\frac{2km}{c^2r^2}\right)\frac{1}{(2\omega+3)}\left(1 + \frac{4(\omega+1)A}{r}\right)}{\left(1 - \frac{2A}{r}\right)\left(1 + \frac{2(\omega+1)A}{(\omega+2)r}\right)}. \tag{9.9}$$

Brans states that equations (9.5)-(9.9) "justify the assumption that at $r = R_1$ the following inequalities are valid,"

$$\phi > 0,\ \alpha' \geq 0,\ \beta' \leq 0,\ (2\omega+3)\phi' \leq 0 \text{ and } (2\omega+3)(\alpha'+\beta') \geq 0 \tag{9.10}$$

However, the validity of these inequalities cannot be accepted unconditionally. For it is easily seen that when

$$-2 < \omega < -3/2$$

the value of α' at $r = R_1$ is negative, while when

$$-3/2 < \omega < -1$$

the value of β' at $r = R_1$ is positive.

In order to assure the validity of the inequalities presented in equation (9.10) it is sufficient to assume that

$$|\omega| > 2. \tag{9.11}$$

In a future section it will be shown that ω must be greater than 5 if the Brans-Dicke theory is to correspond reasonably well with the general relativistic experiments performed to date. This fact is used by Brans to justify the imposition of (9.11) on purely physical grounds.

Following Brans let us choose the energy momentum tensor

$T_i^{\ j}$ for region II to be a diagonal fluid type matter tensor in which all components are functions only of r and are such that

$$T_{ij} \geq 0 \quad \text{and} \quad T = T_i^{\ i} \leq 0 \ . \tag{9.12}$$

An example of such a matter tensor is

$$T_1^{\ 1} = T_2^{\ 2} = T_3^{\ 3} = P \ , \ T_0^{\ 0} = -e \ , \ T_i^{\ j} = 0 \text{ for } i \neq j \ ,$$

where P=P(r) is the pressure and e=e(r) is the energy density. In this example we see that assumption (9.12) is equivalent to requiring that the pressures and densities be non-negative, and that the sum of the pressures in all three directions does not exceed the energy density.

We shall now establish that the condition that ϕ vanish anywhere in region III is not consistent with (9.11) and (9.12) when the signs of α', β', ϕ' and ϕ at R_1 are determined by (9.10). In order to establish this result it will be necessary to use the explicit functional form of the Brans-Dicke field equations (4.16) and (4.36). Under our present assumptions the Brans-Dicke field equations are given by:[45]

$$(\beta')^2 + 2\alpha'\beta' + \frac{2}{r}(\alpha' + \beta') - \frac{\phi''}{\phi} + \frac{\beta'\phi'}{\phi} - \frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2$$

$$= \frac{8\pi}{c^4\phi} e^{2\beta}\left(T_1^1 - \frac{T}{2\omega + 3}\right) , \tag{9.13}$$

$$\alpha'' + \beta'' + (\alpha')^2 + \frac{(\alpha' + \beta')}{r} - \frac{\beta'\phi'}{\phi} - \frac{\phi'}{\phi r} + \frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2$$

$$= \frac{8\pi}{c^4\phi} e^{2\beta}\left(T_2^{\ 2} - \frac{T}{2\omega + 3}\right) , \tag{9.14}$$

---

[45] In Brans's paper [5] the first field equation (9.13) is written incorrectly. The error occurs in the fifth term on the left hand side which in [5] is written as $\frac{\beta\phi'}{\phi}$, instead of $\frac{\beta'\phi'}{\phi}$.

$$\alpha'' + \beta'' + (\alpha')^2 + \frac{(\alpha' + \beta')}{r} - \frac{\beta'\phi'}{\phi} - \frac{\phi'}{\phi r} + \frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2$$
$$= \frac{8\pi}{c^4\phi} e^{2\beta}\left(T_3{}^3 - \frac{T}{2\omega + 3}\right) , \qquad (9.15)$$

$$2\beta'' + (\beta')^2 + \frac{4\beta'}{r} - \frac{\alpha'\phi'}{\phi} + \frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2$$
$$= \frac{8\pi}{c^4\phi} e^{2\beta}\left(T_0{}^0 - \frac{T}{2\omega + 3}\right) , \qquad (9.16)$$

and
$$\phi'' + \frac{2\phi'}{r} + (\alpha' + \beta')\phi' = \frac{8\pi}{c^4} e^{2\beta} \frac{T}{2\omega + 3} \quad . \qquad (9.17)$$

The vanishing of the divergence of the energy momentum tensor leads to

$$(T_1{}^1)' + \alpha'(T_1{}^1 - T_0{}^0) + (\beta' + \frac{1}{r})(2T_1{}^1 - T_2{}^2 - T_3{}^3) = 0. \qquad (9.18)$$

As written equations (9.13)-(9.18) apply to region II. To obtain the field equations which are valid in region III we must set $T_i{}^j = 0$, thereby ovtaining the Brans-Dicke vacuum field equations corresponding to the isotropic line element (9.4). Brans [6] has solved this problem; i.e., Brans has found all the exact vacuum solutions to the system of equations (9.13)-(9.18). These solutions are summarized below.

(1)**
$$\left.\begin{aligned} \alpha &= \alpha_0 + \frac{1}{\lambda} \ln\left(\frac{r - B}{r + B}\right) , \\ \beta &= \beta_0 + \left(\frac{\lambda - D - 1}{\lambda}\right) \ln\left(\frac{r - B}{r + B}\right) + 2 \ln\left(\frac{r + B}{r}\right) , \\ \phi &= \phi_0 \left(\frac{r - B}{r + B}\right)^{D/\lambda} , \\ \lambda^2 &= (D + 1)^2 - D(1 - \frac{\omega D}{2}) > 0 , \quad \omega > -3/2 \end{aligned}\right\} \qquad (9.19)$$

** Brans implicitly assumes in [5] that solution (1) is valid only for $\omega \geq -3/2$. However, it is easily seen that for all real values of ω real values of D can be found such that $(D + 1)^2 - D(1 - \frac{1}{2}\omega D) > 0$. We shall choose to consider solution (1) for $\omega > -\frac{3}{2}$, for in this case solution (1) is valid for arbitrary real values of D.

(2) $$\left.\begin{aligned} \alpha &= \alpha_0 + \frac{2}{\Lambda}\,\mathrm{Tan}^{-1}\!\left(\frac{r}{B}\right), \\ \beta &= \beta_0 - \frac{2(D+1)}{\Lambda}\,\mathrm{Tan}^{-1}\!\left(\frac{r}{B}\right) - \ln\!\left(\frac{r^2}{r^2+B^2}\right), \\ \phi &= \phi_0 \exp\!\left(\frac{2D}{\Lambda}\,\mathrm{Tan}^{-1}\!\left(\frac{r}{B}\right)\right), \\ \Lambda^2 &= D\left(1-\frac{\omega D}{2}\right) - (D+1)^2 > 0 \;, \quad \omega < -\tfrac{3}{2}. \end{aligned}\right\} \tag{9.20}$$

(3) $$\left.\begin{aligned} &\alpha = \alpha_0 - \frac{r}{B} \;, \beta = \beta_0 - 2\ln\!\left(\frac{r}{B}\right) + (D+1)\left(\frac{r}{B}\right), \\ &\phi = \phi_0 \exp\!\left(\frac{-Dr}{B}\right), \quad D = \frac{-1 \pm \sqrt{-2\omega-3}}{\omega+2} \;, \omega \le -3/2 . \end{aligned}\right\} \tag{9.21}$$

(4) $$\left.\begin{aligned} &\alpha = \alpha_0 - \frac{1}{Br} \;, \beta = \beta_0 + \frac{(D+1)}{Br} \;, \\ &\phi = \phi_0 \exp\!\left(\frac{-D}{Br}\right), \quad D = \frac{-1 \pm (-2\omega-3)^{1/2}}{\omega+2} \;, \omega \le -3/2 . \end{aligned}\right\} \tag{9.22}$$

In each of these four solutions α_0, β_0, ϕ_0, and B are arbitrary constants, and D is a constant whose range is governed by ω.

It should immediately be noted that solution (1) has already been obtained in section 7, and that solutions (2)-(4) are the three other solutions previously referred to. The derivation of all four of these solutions will be found in the appendix to this thesis (page 151).

The solutions (1) through (4), given above, are mathematically valid solutions to the Brans-Dicke field equations in region III, corresponding to our isotropic line element (9.4). By examining the behaviour of the scalar field, ϕ, in region III we shall show that solution (1) is the most physically acceptable of the four solutions.

Let us begin by demanding that ϕ vanish somewhere in region III. This demand will be satisfied if either

(i) $\phi \longrightarrow 0$ as $r \longrightarrow \infty$, (9.23a)

or

(ii) $\phi \longrightarrow 0$ as $r \longrightarrow a$, where $R_2 < a$ $(\neq \infty)$. (9.23b)

We shall now proceed to consider each of the above cases in turn.

Case (i). $\phi \longrightarrow 0$ as $r \longrightarrow \infty$. (9.23a)

Of Brans's four vacuum solutions only the third could be compatible with the present demand on ϕ. Solution (3) will be valid provided

$$2\omega + 3 \leq 0 . \qquad (9.24)$$

The implications of equation (9.24) are quite far reaching. to see this let us examine the field equation governing ϕ in region II.

When in region II the scalar field ϕ satisfies the following field equation:

$$\Box\phi = \frac{8\pi T}{(2\omega+3)c^4} , \qquad (9.25)$$

where

$$\Box\phi = (-g)^{-1/2}((-g)^{1/2} g^{ij}\phi_{,i})_{,j} .$$

Since we are assuming that the line element in region II is of the form (9.4) we easily find that $\Box\phi$ may be written as follows:

$$\Box\phi = (-g)^{-1/2}\frac{d}{dr}((-g)^{1/2}e^{-2/3}\phi') .$$

Thus equation (9.25) becomes

$$\frac{d}{dr}((-g)^{1/2}e^{-2/3}\phi') = \frac{8\pi(-g)^{1/2}T}{(2\omega+3)c^4} , \qquad (9.26)$$

where according to our previous assumptions

$$T = T(r) .$$

Equation (9.26) may be immediately integrated to yield

$$(-g)^{1/2} e^{-2\beta} \phi' = \frac{8\pi}{(2\omega + 3)c^4} \int_{R_1}^{r} (-g(p))^{1/2} T(p)\,dp + + Q , \qquad (9.27)$$

where Q is an integration constant and r lies in region II. To determine Q we set $r = R_1$ obtaining

$$Q = (-g(R_1))^{1/2} e^{-2\beta(R_1)} \phi'(R_1) . \qquad (9.28)$$

Thus equation (9.27) may be written

$$(2\omega + 3)(-g(r))^{1/2} \exp(-2\beta(r))\phi'(r) =$$

$$= \frac{8\pi}{c^4} \int_{R_1}^{r} (-g(p))^{1/2} T(p)\,dp + (2\omega + 3)(-g(R_1))^{1/2} \exp(-2\beta(R_1))\phi'(R_1). \qquad (9.29)$$

According to our assumption (9.12), $T<0$, and consequently the integrand appearing in (9.29) is negative, implying that in region II

$$(2\omega + 3)(-g(r))^{1/2} \exp(-2\beta(r))\phi'(r) \leq (2\omega + 3)(-g(R_1))^{1/2} \exp(-2\beta(R_1))\phi'(R_1) . \qquad (9.30)$$

Since $(-g(r))^{1/2} \exp(-2\beta(r))$ is a positive quantity (9.30) may be rewritten as follows:

$$(2\omega + 3)\phi'(r) \leq (2\omega + 3)\phi'(R_1) \frac{(-g(R_1))^{1/2}}{(-g(r))^{1/2}} \exp(-2\beta(R_1) + 2\beta(r)) . \qquad (9.31)$$

From (9.10) we know that when $r = R_1$

$$(2\omega + 3)\phi'(R_1) \leq 0 ,$$

which when combined with (9.31) implies that

$$(2\omega+3)\phi'(r)\leq 0 \ , \tag{9.32}$$

for all r in region II.

Now we have previously seen that when working in region III the demand that $\phi \longrightarrow 0$ as $r \longrightarrow \infty$ implies that $2\omega+3\leq 0$. If we combine this fact with equations (9.11) and (9.32) we can conclude that

$$\phi'(r)\geq 0 \ , \tag{9.33}$$

when in region II. However, equation (9.10) tells us that

$$\phi(R_1) > 0 \ . \tag{9.34}$$

Combining this with (9.33) permits us to conclude that

$$\phi(R_2) > 0 \ . \tag{9.35}$$

In the present case the scalar field in region III is given by

$$\phi = \phi_0 \exp(-\frac{Dr}{B}) \ . \tag{9.36}$$

Now if ϕ is to vanish as $r \longrightarrow \infty$ $\frac{D}{B}$ must be positive. Due to equation (9.35) we can say that ϕ_0 must be positive and consequently

$$\phi' = -\frac{D}{B}\phi_0 \exp(-\frac{Dr}{B}) \ , \tag{9.37}$$

is negative throughout region III. In particular, $\phi'(R_2)$ is negative which contradicts (9.33).

Thus we see that the demand that $\phi \longrightarrow 0$ as $r \longrightarrow \infty$ is incompatible with our assumptions (9.10), (9.11) and (9.12). So rather than revise these assumptions we shall assert that the demand that $\phi \longrightarrow 0$ as $r \longrightarrow \infty$ is physically unreasonable.

<u>Case (ii).</u> $\phi \longrightarrow 0$ as $r \longrightarrow a$, $R_2 < a\ (\neq \infty)$. (9.23b)

We shall now examine the possibility that ϕ vanish at some point $r = a(\neq \infty)$ lying in region III. In this case solution (1) is the only possible vacuum solution we may consider, and it will be valid provided

$$(D+1)^2 - D(1 - \frac{\omega D}{2}) > 0 \, . \qquad (9.38)$$

It will always be possible to determine real values of D which are compatible with (9.38) provided

$$\omega \geq -3/2 .$$

However, since we are assuming that $|\omega| > 2$ we shall confine our considerations of solution (1) to

$$\omega > 2 \, . \qquad (9.39)$$

Consequently we may use equation (9.32) to conclude that throughout region II

$$\phi'(r) \leq 0 \, . \qquad (9.40)$$

From equation (9.10) we know that

$$\phi(R_1) > 0 \, ,$$

so it now appears that ϕ may possibly vanish somewhere outside of $r = R_1$. However, it will be shown that <u>provided</u> ϕ <u>remains positive in region</u> II the vanishing of ϕ outside the shell contradicts atleast one of our assumptions (9.10), (9.11) and (9.12). In order to establish this result we shall examine the restrictions imposed upon the signs of α', β', ϕ' and $\alpha' + \beta'$ at $r = R_2$ by (9.10), (9.12) and the field equations (9.13)-(9.18) which govern region II.

We begin by adding (9.13) to (9.14) to obtain

$$\phi\left\{z' + z^2 + \frac{3z}{r}\right\} = \left(\phi'' + \frac{\phi'}{r}\right) + \frac{8\pi}{c^4}(T_1{}^1 + T_2{}^2)e^{2\beta} +$$

$$- \frac{8\pi e^{2\beta}T}{c^4(2\omega+3)} - \frac{8\pi e^{2\beta}T}{c^4(2\omega+3)} , \qquad (9.41)$$

where we have set

$$z = \alpha' + \beta' . \qquad (9.42)$$

Using equation (9.17) to replace the last term on the right hand side of (9.41) we find that the above expression may be written as follows:

$$z' + z^2 + \frac{3z}{r} + \frac{z\phi'}{\phi} = \frac{8\pi e^{2\beta}}{c^4\phi}(T_1{}^1 + T_2{}^2) - \frac{8\pi e^{2\beta}T}{\phi c^4(2\omega+3)} - \frac{\phi'}{r\phi} . \quad (9.43)$$

Since we are assuming that ϕ is positive in region II we may use (9.40) to conclude that in the present case

$$-\frac{\phi'}{\phi r} \geq 0 , \qquad (9.44)$$

throughout region II. Upon combining (9.44) with our assumption (9.12) we can deduce that the right hand side of equation (9.43) is non-negative. Thus whenever z vanishes at some point in region II z' is non-negative at that point and consequently z cannot decrease from positive to negative values. Since equations (9.39) and (9.10) imply that

$$z(R_1) \geq 0 , \qquad (9.45)$$

we may conclude that z is non-negative throughout region II.

We shall now turn our attention to equation (9.16) which may be written as follows:

$$x' + \frac{2x}{r} + \frac{x^2}{2} + \frac{x\phi'}{2\phi} = \frac{e^{2\beta}}{2\phi}\left\{\frac{8\pi}{c^4}T_0{}^0 - \frac{8\pi T}{(2\omega+3)c^4} + \right.$$

$$\left. + e^{-2\beta}\left(\phi' z - \frac{\omega(\phi')^2}{2\phi}\right)\right\} , \qquad (9.46)$$

where

$$x = \beta' \quad . \tag{9.47}$$

Due to our previous assumptions and the results we have established with regard to ϕ' and z we can conclude that the right hand side of (9,46) is non-positive throughout region II. Thus whenever x vanishes at some point in region II, x' is non-positive at that point, and consequently x cannot increase from negative to positive values. From equation (9.10) we find that

$$x(R_1) = \beta'(R_1) \leq 0 \ ,$$

and consequently β' is non-positive throughout region II.

As an immediate consequence of the above results we may conclude that in the present case

$$\alpha' \geq 0$$

throughout region II.

To recapitulate we have shown that in the present case

$$\phi' \leq 0 \ , \tag{9.40}$$

$$\alpha' \geq 0 \ , \tag{9.48}$$

$$\beta' \leq 0 \tag{9.49}$$

and

$$\alpha' + \beta' \geq 0 \tag{9.50}$$

throughout region II.

We shall now examine whether solution (1) is compatible with equations (9.39), (9.40) and (9.48)-(9.50).

For solution (1) we find that ϕ is given by

$$\phi = \phi_0 \left(\frac{r - B}{r + B}\right)^{D/\lambda} \ , \tag{9.51}$$

where

$$\lambda^2 = (D + 1)^2 - D(1 - \frac{\omega D}{2}) > 0 \ , \tag{9.52}$$

and ϕ_0 is a non-zero constant. From (9.51) it is obvious that ϕ will vanish in region III on only the following two occasions:

(i) $B > R_2$ and $D/\lambda > 0$;

(ii) $|B| > R_2$, $B < 0$ and $D/\lambda < 0$.

Conditions (i) and (ii) may be written in the following abridged form

$$R_2 < |B| \quad \text{and} \quad \frac{BD}{\lambda} > 0 \, . \tag{9.53}$$

We shall presently show that (9.53) is inconsistent with equations (9.39), (9.40) and (9.48)-(9.50).

To begin we may use equation (9.51) to find

$$\phi' = \phi_0 \frac{D}{\lambda}\left(\frac{r-B}{r+B}\right)^{D/\lambda} \frac{2B}{(r^2-B^2)} \, , \tag{9.54}$$

and thus $\frac{\phi'}{\phi}$ is given by

$$\frac{\phi'}{\phi} = \frac{2BD}{\lambda(r^2-B^2)} \, . \tag{9.55}$$

The values of α and β corresponding to solution (1) are given by:

$$\alpha = \alpha_0 + \frac{1}{\lambda} \ln\left(\frac{r-B}{r+B}\right) , \tag{9.56}$$

$$\beta = \beta_0 + 2 \ln (r+B) - 2 \ln r + \left(\frac{\lambda - D - 1}{\lambda}\right) \ln\left(\frac{r-B}{r+B}\right) . \tag{9.57}$$

Thus we find that α' and β' are given by

$$\alpha' = \frac{2B}{\lambda(r^2-B^2)} \, , \tag{9.58a}$$

and

$$\beta' = \frac{2}{r+B} - \frac{2}{r} + \left(\frac{\lambda - D - 1}{\lambda}\right) \frac{2B}{r^2-B^2} \, . \tag{9.58b}$$

Due to equation (9.58a) we see that (9.55) may be rewritten as follows:

$$\frac{\phi'}{\phi} = D\alpha' \; . \tag{9.59}$$

Since we are assuming that $\phi > 0$ in region II we may use equation (9.40) to conclude that

$$\frac{\phi'(R_2)}{\phi(R_2)} \leq 0 \; . \tag{9.60}$$

Due to equation (9.48) we find that

$$\alpha'(R_2) \geq 0 \; . \tag{9.61}$$

Thus by combining equations (9.59)-(9.61) we may conclude that

$$D \leq 0 \; . \tag{9.62}$$

Upon inserting this result into (9.53) we find that

$$\frac{B}{\lambda} < 0 \; , \tag{9.63}$$

if ϕ is to vanish in region III.

Equation (9.50) can be used to tell us that

$$0 \leq \alpha'(R_2) + \beta'(R_2) \quad . \tag{9.64}$$

Thus by adding equation (9.58a) to (9.58b) we find that when $r = R_2$

$$0 \leq \frac{2}{R_2 + B} - \frac{2}{R_2} + \frac{2B}{(R_2)^2 - B^2} - \frac{2BD}{\lambda((R_2)^2 - B^2)} \; ,$$

which may be rewritten as follows:

$$\left(1 - \frac{D}{\lambda}\right) \frac{B}{(R_2)^2 - B^2} \geq \frac{B}{R_2(R_2 + B)} \quad . \tag{9.65}$$

In order for B and λ to satisfy equation (9.63) we must have either:

(a) $B > 0$ and $\lambda < 0$,

or

(b) $B < 0$ and $\lambda > 0$.

We shall now proceed to examine each of these cases separately.

When B and λ satisfy (a) we find that (9.65) becomes

$$\left(1 - \frac{D}{\lambda}\right)\frac{1}{R_2 - B} \geq \frac{1}{R_2} > 0 \quad . \tag{9.66}$$

Upon applying (9.53) to (9.66) we may conclude that

$$1 - \frac{D}{\lambda} < 0 \quad ,$$

or

$$D^2 > \lambda^2 . \tag{9.67}$$

We shall now assume that B and λ satisfy condition (b). Using (9.53) we can deduce that

$$R_2 + B < 0 ,$$

and consequently (9.65) reduces to

$$\left(1 - \frac{D}{\lambda}\right) \frac{1}{R_2 - B} \geq \frac{1}{R_2} > 0 . \tag{9.68}$$

Since $B < 0$, $R_2 - B$ will be positive and hence the above expression implies that

$$1 - \frac{D}{\lambda} \geq 1 - \frac{B}{R_2} . \tag{9.69}$$

Using (9.69) we may conclude that

$$\frac{D}{\lambda} \leq \frac{B}{R_2} < -1 , \tag{9.70}$$

which in turn implies that

$$D^2 > \lambda^2 . \tag{9.71}$$

In summary we see that for both cases (a) and (b)

$$D^2 > \lambda^2 .$$

Now λ^2 is defined by

$$\lambda^2 = D^2 + \frac{\omega D^2}{2} + D + 1 ,$$

and consequently we may use (9.71) to conclude that

$$0 > \tfrac{1}{2}\omega D^2 + D + 1 . \tag{9.72}$$

Equation (9.72) implies that the range of real D corresponds to the following three intervals:

(i) $\frac{-1-(1-2\omega)^{1/2}}{\omega} < D < \frac{-1+(1-2\omega)^{1/2}}{\omega}$, when $0<\omega<1/2$;

(ii) $D<-1$, when $\omega=0$;

and

(iii) $D< \frac{-1+(1-2\omega)^{1/2}}{\omega}$, or $\frac{-1-(1-2\omega)^{1/2}}{\omega} < D$, when $\omega<0$.

Thus a real value of D compatible with equation (9.72) can always be found provided

$$\omega<1/2 . \qquad (9.73)$$

However, it was shown earlier that in order for the function ϕ, corresponding to solution (1), to vanish in region III and also be compatible with assumption (9.11) ω must be greater than 2. Consequently (9.73) is inconsistent with our previous work.

Thus we see that the following five assumptions imply that in the vacuum surrounding a static spherically symmetric mass shell the scalar field ϕ cannot vanish.

(i) In regions I, II and III the line element assumes the following isotropic form:

$$ds^2 = -e^{2\alpha}c^2dt^2 + e^{2\beta}(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\epsilon^2) ,$$

with α, β and the scalar field ϕ being functions of r which are of class C^2 in regions I, II and III.

(ii) The functions α, β and ϕ are of class C^1 across the boundarys separating region I from region II and region II from region III.

(iii) Region II makes no contribution to the gravitational field in region I, and the gravitational field in region I can be approximated by our weak field solution.

(iv) $|\omega| > 2$.

(v) In region II the matter tensor $T_i^{\ j}$ is diagonal, $\phi > 0$, $T_{ij} \geq 0$ and $T_i^{\ i} \leq 0$.

Thus we may conclude that in the vacuum surrounding our static spherically symmetric mass shell the scalar field ϕ will always be greater than some positive constant N. We shall henceforth take this to be the boundary condition satisfied by ϕ in region III.

Our boundary condition on ϕ eliminates the third of Brans's four vacuum solutions (provided we assume that the scalar field ϕ must remain finite as $r \rightarrow \infty$). In order to reduce the choice even further we have to resort to Brans's interpretation of the gravitational "constant." Out of such a study Brans [5] has shown that ω must be greater than -2 if the Brans-Dicke theory is to conform with Brans's interpretation of Mach's principle. We shall not discuss how Brans has obtained this result, but we shall assume that it is valid. Consequently (due to our assumption that $|\omega| > 2$) we can disregard solutions on the grounds that they are physically unacceptable if they are only valid for values of $\omega \leq 2$. For this reason we may dismiss solutions (2) and (4) as being physically unacceptable.[46] Thus we see that solution (1) is the only static spherically symmetric vacuum solution of the four presented by Brans which is compatible with our demand on ω and which is such that it can have its arbitrary

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[46]It should be noted that solution (3) could also be eliminated on these grounds.

constants chosen to be compatible with our boundary condition on ϕ. We shall henceforth choose to call solution (1) the "physical" vacuum solution to the Brans-Dicke field equations corresponding to the isotropic line element (9.4).

10. Further Properties of Brans's "Physical" Vacuum Solution[47]

Recall that in the previous section it was shown that the most physically acceptable of Brans's four static vacuum solutions, corresponding to a spherically symmetric particle at rest at the origin was given by the isotropic line element

$$ds^2 = -e^{2\alpha}c^2dt^2 + e^{2\beta}(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\epsilon^2) \quad , \qquad (10.1)$$

where

$$e^{2\alpha} = e^{2\alpha_0}\left(\frac{r-B}{r+B}\right)^{2/\lambda} \quad , \qquad (10.2)$$

$$e^{2\beta} = e^{2\beta_0}\left(1+\frac{B}{r}\right)^4\left(\frac{r-B}{r+B}\right)^{2(\lambda-D-1)/\lambda} \quad , \qquad (10.3)$$

$$\phi = \phi_0\left(\frac{r-B}{r+B}\right)^{D/\lambda} \quad , \qquad (10.4)$$

and

$$\lambda^2 = (D+1)^2 - D(1-\frac{\omega D}{2}) > 0 \quad . \qquad (10.5)$$

We shall now proceed to identify the constants appearing in the above solution. After doing so the resultant line element will then be written in non-isotropic form and compared with the Schwarzschild non-isotropic line element.

To determine the constants α_0, β_0, ϕ_0, B and D appearing in equations (10.2)-(10.5) we shall compare these expressions with our weak field solutions (8.49)-(8.51). In order to perform this comparison it will be necessary to expand the above expressions for $e^{2\alpha}$, $e^{2\beta}$ and ϕ in Maclaurin series expansions. To begin let us write $e^{2\alpha}$, $e^{2\beta}$ and ϕ in the

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[47]The material found in this section is based upon H.Nariai's paper, reference [20].

following form:

$$e^{2\alpha} = e^{2\alpha_0}\left(\frac{1-x}{1+x}\right)^{2/\lambda} , \quad (10.6)$$

$$e^{2\beta} = e^{2\beta_0}(1+x)^4\left(\frac{1-x}{1+x}\right)^{2(\lambda-D-1)/\lambda} , \quad (10.7)$$

and

$$\phi = \phi_0\left(\frac{1-x}{1+x}\right)^{-D/\lambda} \quad (10.8)$$

where

$$x = \frac{B}{r} .$$

Using a Maclaurin series to expand equations (10.6)-(10.8) we find that to first order in x

$$e^{2\alpha} \approx e^{2\alpha_0}\left(1 - \frac{4x}{\lambda}\right) , \quad (10.9)$$

$$e^{2\beta} \approx e^{2\beta_0}\left(1 + \frac{4(D+1)x}{\lambda}\right) , \quad (10.10)$$

and

$$\phi \approx \phi_0\left(1 - \frac{2Dx}{\lambda}\right) . \quad (10.11)$$

From our weak field approximation we have

$$e^{2\alpha} \approx 1 - \frac{2MG_0}{rc^2} , \quad (10.12)$$

$$e^{2\beta} \approx 1 + \frac{2MG_0}{rc^2}\left(\frac{1+\omega}{2+\omega}\right) , \quad (10.13)$$

and

$$\phi \approx \frac{1}{k}\left(1 + \frac{MG_0}{rc^2(\omega+2)}\right) , \quad (10.14)$$

where

$$G_0 = k\left(\frac{2\omega+4}{2\omega+3}\right) ,$$

and M denotes the mass of the source of our gravitational field.

Recall that by examining equations (8.50)-(8.52) in the limit as $\omega \longrightarrow \infty$ the constant ϕ_0, appearing in our weak field solution, was shown to be the reciprocal of the Newtonian gravitational constant, k. However, this does not imply that the (locally measured) gravitational constant in the Brans-Dicke theory is $\frac{1}{\phi_o}$. In fact it has been shown by Brans in [5] that G_o represents the (locally measured) gravitational constant in the vacuum surrounding our mass M. One should note that in the limit as $\omega \longrightarrow \infty$, $G_o \longrightarrow k = \frac{1}{\phi_o}$.

Upon comparing (10.9), (10.10) and (10.11) with (10.12), (10.13) and (10.14) we can thus make the following identifications:

$$e^{2\alpha_o} \approx 1 \ , \ e^{2\beta_o} \approx 1 \ , \ \phi_o \approx \frac{1}{k} \ , \tag{10.15}$$

$$\frac{B}{\lambda} \approx \frac{1}{2}\frac{MG_o}{c^2} \ , \tag{10.16}$$

and

$$\frac{(D+1)B}{\lambda} \approx \frac{1}{2}\frac{MG_o}{c^2}\frac{(\omega+1)}{(\omega+2)} \ . \tag{10.17}$$

If we now divide (10.17) by (10.16) we find

$$D + 1 \approx \frac{\omega+1}{\omega+2}$$

which implies that

$$D \approx \frac{-1}{\omega+2} \ . \tag{10.18}$$

Equations (10.5) and (10.18) permit us to conclude that

$$\lambda \approx \left(\frac{2\omega+3}{2\omega+4}\right)^{1/2} . \tag{10.19}$$

Subsituting (10.19) into (10.16) gives us

$$B \approx \frac{MG_o}{2c^2}\left(\frac{2\omega+3}{2\omega+4}\right)^{1/2} \ . \tag{10.20}$$

In summary we have found that the constants appearing in (10.2)-(10.5) are given by

$$\alpha_0 \approx \beta_0 \approx 0 \; ; \quad \phi_0 \approx \frac{1}{k} \; ; \quad D \approx \frac{-1}{\omega + 2} \; ; \quad B \approx \frac{MG_0}{2c^2}\left(\frac{2\omega+3}{2\omega+4}\right)^{1/2} ;$$

and

$$\lambda \approx \left(\frac{2\omega+3}{2\omega+4}\right)^{1/2} .$$

Using the above results we find that the line element corresponding to Brans's "physical" vacuum solution is given by

$$ds^2 = -\xi^{2q}(r)c^2dt^2 + (1 + \frac{B}{r})^4 \xi^{2Q}(r)(dr^2 + r^2 d\Omega^2) \; , \quad (10.21)$$

where

$$q = \left(\frac{2\omega+4}{2\omega+3}\right)^{1/2} , \quad (10.22)$$

$$Q = \frac{(q-1)(q+2)}{q} , \quad (10.23)$$

$$\xi(r) = \frac{r-B}{r+B} , \quad (10.24)$$

and

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\epsilon^2 . \quad (10.25)$$

The associated scalar field ϕ may be written in the form

$$\phi = \frac{1}{k}\,\xi^P(r) , \quad (10.26)$$

where

$$P = -\frac{2(q^2-1)}{q} . \quad (10.27)$$

In passing we note that in general ξ will have to be strictly positive (i.e., $r^2 > B^2$) for the line element (10.21) and the scalar field (10.26) to be real.

In section 8 it was shown that in the limit as ω goes to infinity the Brans-Dicke weak field solutions (8.49)-(8.51) assume the form of the Einstein weak field solutions provided we set $\phi_0 = \frac{1}{k}$. Simailarly we note that in the limit as ω goes to infinity (or equivalently as q goes to 1) the line element given in (10.21) reduces to the isotropic form of the Schwarzschild line element; viz.,

$$ds^2 = -\left[\frac{1 - \frac{kM}{2c^2r}}{1 + \frac{kM}{2c^2r}}\right]^2 c^2dt^2 + \left(1 + \frac{kM}{2c^2r}\right)^4(dr^2 + r^2d\Omega^2)\ , \quad (10.28)$$

and the scalar field given in (10.26) becomes $\frac{1}{k}$. One should note that if $kM>0$ then the line element (10.28) has only one singular point, viz., $r = 0$.

For the purposes of comparison with the non-isotropic Schwarzschild metric of general relativity, namely,

$$ds^2 = -\left(1 - \frac{2kM}{c^2\bar{r}}\right)c^2d\bar{t}^2 + \frac{d\bar{r}^2}{\left(1 - \frac{2kM}{c^2\bar{r}}\right)} + \bar{r}^2(d\bar{\theta}^2 + \sin^2\bar{\theta}d\bar{\epsilon}^2)\ , \quad (10.29)$$

we wish to express the isotropic form of Brans's line element (10.21) in the form

$$ds^2 = -\ e^{\nu(\bar{r})}c^2d\bar{t}^2 + e^{\lambda(\bar{r})}d\bar{r}^2 + \bar{r}^2(d\bar{\theta}^2 + \sin^2\bar{\theta}d\bar{\epsilon}^2)\ . \quad (10.30)$$

In order to rewrite (10.21) in the form of (10.30) we shall perform the following coordinate transformation:

$$t = \bar{t}\ , \quad (10.31)$$

$$\left(1 + \frac{B}{r}\right)^4 r^2 \xi^{2Q}(r) = \bar{r}^2\ , \quad (10.32)$$

$$\theta = \bar{\theta}\ , \quad (10.33)$$

and

$$\epsilon = \bar{\epsilon}\ . \quad (10.34)$$

To assist in rewriting (10.21) in terms of the barred coordinates we shall use

$$\bar{g}_{ij} = g_{hk}B^h_iB^k_j\ , \quad (10.35)$$

where

$$B^h_i = \frac{\partial x^h}{\partial \bar{x}^i}\ . \quad (10.36)$$

Due to the fact that our line element (10.21) is diagonal we can use (10.35) to conclude that:

$$\bar{g}_{ij} = 0 \text{ if } i \neq j \,, \tag{10.37}$$

$$e^{\nu(\bar{r})} = \bar{g}_{00} = g_{00}\left(\frac{dt}{d\bar{t}}\right)^2 \,, \tag{10.38}$$

$$e^{\lambda(\bar{r})} = \bar{g}_{11} = g_{11}\left(\frac{dr}{d\bar{r}}\right)^2 \,, \tag{10.39}$$

$$\bar{g}_{22} = g_{22}\left(\frac{d\theta}{d\bar{\theta}}\right)^2 \,, \tag{10.40}$$

and

$$\bar{g}_{33} = g_{33}\left(\frac{d\epsilon}{d\bar{\epsilon}}\right)^2 \,. \tag{10.41}$$

Since

$$\frac{dt}{d\bar{t}} = \frac{d\theta}{d\bar{\theta}} = \frac{d\epsilon}{d\bar{\epsilon}} = 1 \,, \tag{10.42}$$

we can use equations (10.21), (10.32), (10.38), (10.40) and (10.41) to conclude that

$$e^{\nu(\bar{r})} = \bar{g}_{00}(\bar{r}) = \xi^{2q}(r) \,, \tag{10.43}$$

and

$$\bar{g}_{22} = \bar{g}_{33} = \bar{r}^2. \tag{10.44}$$

Using equation (10.39) we find that

$$e^{-\frac{1}{2}\lambda(\bar{r})} = (g_{11})^{-\frac{1}{2}}\frac{d\bar{r}}{dr} \,, \tag{10.45}$$

which due to equation (10.21) may be rewritten as follows:

$$e^{-\frac{1}{2}\lambda(\bar{r})} = \frac{\frac{d\bar{r}}{dr}}{(1 + \frac{B}{r})^2 \xi^{Q}(r)} \,. \tag{10.46}$$

From (10.32) we find

$$\frac{d\bar{r}}{dr} = 2\left(\frac{r - B}{r + B}\right)^{(Q-1)}\left\{\frac{B(Q-1)}{r} + \frac{B^2}{r^2}\right\} + \left(1 + \frac{B}{r}\right)^2\left(\frac{r - B}{r + B}\right)^Q \,,$$

which can be written in the form

$$\frac{d\bar{r}}{dr} = \left(\frac{r - B}{r + B}\right)^{(Q-1)}\left\{1 + \frac{B^2}{r^2} + \frac{2B(Q-1)}{r}\right\} \,. \tag{10.47}$$

Upon inserting (10.47) into (10.46) we find after simplification that

$$e^{-\frac{1}{2}\lambda(\bar{r})} = \frac{r - B}{r + B} + \frac{2BQr}{(r - B)^2} \; . \qquad (10.48)$$

Making use of our definition of ξ, (10.24), we find that the above expression may be rewritten as follows:

$$e^{-\frac{1}{2}\lambda(\bar{r})} = \xi + \frac{Q}{2}\left(\frac{1 - \xi^2}{\xi}\right) . \qquad (10.49)$$

To summarize we now have our original Brans-Dicke vacuum line element (10.21) in the form

$$ds^2 = -e^{\nu(\bar{r})}c^2 d\bar{t}^2 + e^{\lambda(\bar{r})}d\bar{r}^2 + \bar{r}^2(d\bar{\theta}^2 + \sin^2\bar{\theta}\, d\bar{\epsilon}^2) \; , \qquad (10.30)$$

where $e^{-\frac{1}{2}\lambda}$ is given by (10.49)

$$e^{\frac{1}{2}\nu} = \xi^q \; , \qquad (10.43)$$

and

$$\bar{r} = \frac{2G_0 M}{c^2 q}\left(\frac{\xi^Q}{1 - \xi^2}\right) . \qquad (10.50)$$

This last expression for $\bar{r}$ can easily be obtained from equations (10.24) and (10.32). Associated with the above line element is our scalar field ϕ which in the present case is given by

$$\phi = \frac{1}{k}\,\xi^P \; , \qquad (10.51)$$

where

$$P = -\frac{2(q^2 - 1)}{q} \; . \qquad (10.27)$$

Clearly equations (10.43), (10.49), (10.50) and (10.51) can also be obtained directly by solving the Brans-Dicke free space field equations (which are obtained from equations (4.16) and (4.36) by setting $T_{ij} = 0$ and $T = 0$ respectively) for a line element of the form (10.30). Henceforth we shall work solely with the new form of our line element (10.30).

It should be noted that in the limit as q goes to 1 the line element (10.30) obviously goes over to the Schwarzschild non-isotropic line element given in equation (10.29).

The usual non-isotropic form of the Schwarzschild line element is given by (10.29); viz.,

$$ds^2 = -\left(1 - \frac{2kM}{c^2\bar{r}}\right)c^2d\bar{t}^2 + \frac{d\bar{r}^2}{\left(1 - \frac{2kM}{c^2\bar{r}}\right)} + \bar{r}^2(d\bar{\theta}^2 + \sin^2\theta d\bar{\epsilon}^2) \quad . (10.29)$$

At $\bar{r} = R_g = \frac{2kM}{c^2}$, (10.29) experiences a singularity, and when $\bar{r} < R_g$ (10.29) can no longer be interpreted in the same sense as it was when $\bar{r} > R_g$. This results from the fact that as $\bar{r}$ passes through R_g from above the signature of (10.29) changes from (-1, 1, 1, 1) to (+1, -1, 1, 1). (One should note that the the signature of the isotropic Schwarzschild line element (10.28) never changes, even though g_{00} goes to zero for $r = \frac{kM}{2c^2}$.)

For the non-isotropic form of the Brans-Dicke vacuum line element (10.30) to experience a singularity at $\bar{r} = R_s$ we must have either $e^{-\frac{1}{2}\lambda} \longrightarrow 0$ or $e^{\frac{1}{2}\nu} \longrightarrow \infty$ as $\bar{r} \longrightarrow R_s$. We shall now examine each of these possibilities in turn.

If $e^{-\frac{1}{2}\lambda} \longrightarrow 0$ as $\bar{r} \longrightarrow R_s$ then from (10.49) we see that

$$\xi^2 \longrightarrow \frac{Q}{Q-2}, \text{ as } \bar{r} \longrightarrow R_s, \qquad (10.52)$$

where Q is given by equation (10.23). It will now be shown that (10.52) is not valid. To see this it will be necessary to rewrite $\frac{Q}{Q-2}$ in terms of ω. Using equations (10.22) and (10.23) we find that

$$\frac{Q}{Q-2} = \frac{-2\left(\frac{\omega+1}{2\omega+3}\right) + \left(\frac{2\omega+4}{2\omega+3}\right)^{1/2}}{-2\left(\frac{\omega+1}{2\omega+3}\right) - \left(\frac{2\omega+4}{2\omega+3}\right)^{1/2}} \quad . \tag{10.53}$$

Since we are dealing with Brans's "physical" vacuum solution we know from our previous experience that it is a valid solution when $\omega > -\frac{3}{2}$. However, to be consistent with the assumption that $|\omega| > 2$, which was made in section 9, we shall examine equation (10.53) for $\omega > 2$. In this case it is apparent that

$$\frac{Q}{Q-2} \leq 0 \quad . \tag{10.54}$$

Combining the above result with equations (10.52) and (10.50) permits us to conclude that ξ and hence R_s will be complex except when $\frac{Q}{Q-2} = 0$. However, the case $\frac{Q}{Q-2} = 0$ corresponds to the limiting case $\omega = \infty$. Now as previously mentioned when $\omega \longrightarrow \infty$ (10.30) becomes the non-isotropic Schwarzschild line element (10.29) and consequently we expect (10.30) to experience a singularity for $R_s = \frac{2kM}{c^2}$.

In summary we have shown that for finite values of $\omega > 2$ there exists no positive real value of $\bar{r}$ for which $e^{-\frac{1}{2}\lambda} \longrightarrow 0$.

We shall now examine the possibility that $e^{\frac{1}{2}\nu} \longrightarrow \infty$ as $\bar{r} \longrightarrow R_s$ for $\omega > 2$. Using equations (10.22) and (10.23) it is easily seen that when $\omega > 2$, Q and q lie in the following intervals

$$\left[0, \frac{(8/7)^{1/2} - 6/7}{(8/7)^{1/2}}\right) \quad ,$$

and

$$\left[1, \left(8/7\right)^{1/2}\right),$$

respectively. Now since $e^{\frac{1}{2}\nu} = \xi^q$ we see that $e^{\frac{1}{2}\nu} \longrightarrow \infty$ as $\bar{r} \longrightarrow R_s$ if and only if $\xi \longrightarrow \infty$. However, from (10.50) we see that $\xi \longrightarrow \infty$ implies that $R_s = 0$. Since we have been working under the implicit presupposition that there is a singularity in the metric at the origin we can conclude that $e^{\frac{1}{2}\nu} \longrightarrow \infty$ as $\bar{r} \longrightarrow R_s$ leads to no new singularities.

Thus we have shown that for finite values of $\omega > 2$ the only singularity in the non-isotropic Brans-Dicke line element (10.30) occurs at the origin.

11. Equations of Motion in the Brans-Dicke Theory

In this section we shall study the geodesics corresponding to the "most physically acceptable" of Brans's four vacuum solutions. It should be recalled that this solution has been identified with the gravitational field outside a spherically symmetric static particle of mass M which is at rest at the origin. The line element corresponding to this solution is given by[48]

$$ds^2 = -e^{2\alpha}c^2dt^2 + e^{2\beta}(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)) \ , \qquad (11.1)$$

where

$$e^{2\alpha} = \left\{\frac{1 - \frac{B}{r}}{1 + \frac{B}{r}}\right\}^{2q} , \qquad (11.2)$$

$$e^{2\beta} = \left[1 + \frac{B}{r}\right]^4 \left\{\frac{1 - \frac{B}{r}}{1 + \frac{B}{r}}\right\}^{2Q} , \qquad (11.3)$$

$$q = \left(\frac{2\omega + 4}{2\omega + 3}\right)^{\frac{1}{2}} , \qquad (11.4)$$

$$Q = \frac{(q-1)(q+2)}{q} , \qquad (11.5)$$

$$B = \frac{MG_o}{2c^2q} , \qquad (11.6a)$$

and

$$G_o = k\,q^2 . \qquad (11.6b)$$

Recall that G_o, and not k, represents the (locally measured) gravitational constant (throughout the vacuum surrounding a single mass) in the Brans-Dicke theory of gravitation (c.f. remarks following equation (10.14).)

It is well known (e.g., R.Adler, M.Bazin and M.Schiffer, [1]) that the non-null geodesics corresponding

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[48] Throughout this section ϕ will be used to denote the azimuthal angle and not the scalar field.

to an isotropic line element of the form (11.1) are identical to the Euler-Lagrange equations corresponding to the single particle Lagrangian

$$L^2 = -e^{2\alpha}c^2\dot{t}^2 + e^{2\beta}(\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)) , \qquad (11.7)$$

where a dot is used to denote differentiation with respect to the arc length s. In writing (11.7) it has been assumed that the mass of the corresponding particle is constant, and thus does not appear in L^2 since its presence would add nothing to the equations of motion.

The Euler-Lagrange equations corresponding to the Lagrangian (11.7) are

$$\frac{d}{ds}\frac{\partial L^2}{\partial \dot{x}^i} - \frac{\partial L^2}{\partial x^i} = 0 . \qquad (11.8)$$

These differential equations are to be solved for the unknown coordinates, $x^i = x^i(s)$. We shall now proceed to determine the functional form of the Euler-Lagrange equations corresponding to L^2.

For the 0, or t coordinate we have

$$\frac{\partial L^2}{\partial \dot{t}} = -2c^2e^{2\alpha}\dot{t} ,$$

and

$$\frac{\partial L^2}{\partial t} = 0 .$$

Thus in this case (11.8) takes on the form

$$\frac{d}{ds}(e^{2\alpha}\dot{t}) = 0 . \qquad (11.9)$$

For the 1, or r coordinate we find

$$\frac{\partial L^2}{\partial \dot{r}} = 2e^{2\beta}\dot{r} ,$$

and

$$\frac{\partial L^2}{\partial r} = -2\frac{d\alpha}{dr}\, e^{2\alpha}c^2\dot{t}^2 + 2\frac{d\beta}{dr}\, e^{2\beta}\dot{r}^2 + 2r\left(1 + r\frac{d\beta}{dr}\right) e^{2\beta}\dot{\theta}^2 +$$

$$+ 2r\left(1 + r\frac{d\beta}{dr}\right)e^{2\beta}\sin^2\theta\,\dot{\phi}^2 .$$

The above two expressions yield the following Euler-Lagrange equation for r :

$$\frac{d}{ds}\left(e^{2\beta}\dot{r}\right) = -\frac{d\alpha}{dr}\, e^{2\alpha}c^2\dot{t}^2 + \frac{d\beta}{dr}\, e^{2\beta}\dot{r}^2 + re^{2\beta}\left(1 + r\frac{d\beta}{dr}\right)\dot{\theta}^2 +$$

$$+ re^{2\beta}\left(1 + r\frac{d\beta}{dr}\right)\sin^2\theta\,\dot{\phi}^2. \qquad (11.10)$$

For the 2, or θ coordinate we obtain

$$\frac{\partial L^2}{\partial\dot{\theta}} = 2r^2e^{2\beta}\dot{\theta} \quad ,$$

and

$$\frac{\partial L^2}{\partial\theta} = 2r^2e^{2\beta}\sin\theta\cos\theta\,\dot{\phi}^2 .$$

Thus the Euler-Lagrange equation for θ is

$$\frac{d}{ds}\left(r^2e^{2\beta}\dot{\theta}\right) = r^2e^{2\beta}\sin\theta\cos\theta\,\dot{\phi}^2. \qquad (11.11)$$

Lastly, for the 3 or ϕ coordinate we have

$$\frac{\partial L^2}{\partial\dot{\phi}} = 2r^2e^{2\beta}\sin^2\theta\,\dot{\phi} ,$$

and

$$\frac{\partial L^2}{\partial\phi} = 0 .$$

So we find the Euler-Lagrange equation for ϕ to be

$$\frac{d}{ds}\left(r^2e^{2\beta}\sin^2\theta\,\dot{\phi}\right) = 0. \qquad (11.12)$$

We shall now show that all of the above Euler-Lagrange equations may be integrated atleast once without appealing to the explicit functional form of α and β.

To begin let us orientate the axes of our coordinate system so that when $s = 0$ our particle lies in the plane $\theta = \frac{\pi}{2}$

with $\dot{\theta} = 0$. In this case we can use equation (11.11) to conclude that when $s = 0$

$$r^2 e^{2\beta} \ddot{\theta} = 0 , \tag{11.13}$$

and hence

$$\ddot{\theta} = 0 . \tag{11.14}$$

We shall now assume that $\theta(s)$ admits a Maclaurin series expansion; i.e.,

$$\theta(s) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\left. \frac{d^n \theta}{ds^n} \right|_{s=0} \right) s^n . \tag{11.15}$$

The expansion coefficients

$$\left. \frac{d^n \theta}{ds^n} \right|_{s=0} , \tag{11.16}$$

may easily be determined by successively differentiating equation (11.11) with respect to s. Due to (11.14) and our choice of coordinates we find that our process of successive differentiation yields

$$\left. \frac{d^n \theta}{ds^n} \right|_{s=0} = 0 , \tag{11.17}$$

for all $n \geq 1$. Thus we may use equations (11.15) and (11.17) to conclude that

$$\theta = \frac{\pi}{2} , \tag{11.18}$$

for all $s \geq 0$.

Upon inserting (11.18) into (11.12) we obtain

$$\frac{d}{ds}(r^2 e^{2\beta} \dot{\phi}) = 0 ,$$

which may be immediately integrated to yield

$$r^2 e^{2\beta} \dot{\phi} = h , \tag{11.19}$$

where h is a constant.

Equation (11.9) may also be integrated to give

$$e^{2\alpha}\dot{t} = d \,, \tag{11.20}$$

where d is a constant.

Since we are presently dealing with the non-null geodesics of (11.1) we can use (11.1) to obtain the following first integral of (11.10):

$$1 = -e^{2\alpha} c^2\dot{t}^2 + e^{2\beta}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) \,. \tag{11.21}$$

Making use of (11.18), (11.19) and (11.20) we find that (11.21) may be written

$$1 = -\, c^2d^2e^{-2\alpha} + e^{2\beta}(\dot{r}^2 + \frac{h^2e^{-4\beta}}{r^2}) \,. \tag{11.22}$$

In summary we have the following first integrals of our Euler-Lagrange equations:

$$\theta = \frac{\pi}{2} \quad , \tag{11.18}$$

$$\dot{\phi} = \frac{h}{r^2e^{2\beta}} \quad , \tag{11.23}$$

$$\dot{t} = de^{-2\alpha} \quad , \tag{11.24}$$

and

$$1 = -c^2d^2e^{-2\alpha} + e^{2\beta}\dot{r}^2 + \frac{h^2e^{-2\beta}}{r^2} \,. \tag{11.22}$$

We shall now concentrate upon rewriting (11.22). To begin let us assume that

$$r = r(\phi(s)) \,.$$

Thus we have

$$\frac{dr}{ds} = r'\dot{\phi} \quad ; \tag{11.25}$$

where a prime denotes differentiation with respect to ϕ. Using (11.23) we find that (11.25) may be written

$$\dot{r} = r'\frac{h}{r^2e^{2\beta}} \,. \tag{11.26}$$

From (11.26) we find that (11.22) can be written in the form

$$1 = -c^2 d^2 e^{-2\alpha} + e^{-2\beta} \frac{(r')^2 h^2}{r^4} + \frac{h^2 e^{-2\beta}}{r^2} . \quad (11.27)$$

Let us now perform the usual change of variables by setting

$$u = \frac{1}{r} . \quad (11.28)$$

This gives us

$$r' = \frac{-1}{u^2} u' . \quad (11.29)$$

Upon combining (11.28) with (11.29) we find that (11.27) becomes

$$1 = -c^2 d^2 e^{-2\alpha} + e^{-2\beta}(u')^2 h^2 + h^2 e^{-2\beta} u^2 . \quad (11.30)$$

It is now formally possible to solve (11.30) to obtain $\phi = \phi(u)$. This can be seen from the fact that (11.30) may be written in the form

$$\left(\frac{du}{d\phi}\right)^2 = \frac{e^{2\beta}}{h^2} + \frac{e^{2(\beta-\alpha)} c^2 d^2}{h^2} - u^2 . \quad (11.31)$$

However, we shall be interested in treating the problem of perihelion rotation and consequently we shall need $u = u(\phi)$.

In order to proceed further it will be necessary for us to differentiate (11.30) with respect to ϕ. However, before taking the desired derivative we shall multiply (11.30) by $e^{2\alpha}$, and thus upon differentiating the resultant expression we obtain an equation independent of the constant d. Upon carrying out the above two operations we find

$$u' \frac{d\alpha}{du} = u' u'' h^2 e^{-2\beta} + h^2 u' (u')^2 \left(\frac{d\alpha}{du} - \frac{d\beta}{du}\right) e^{-2\beta} +$$
$$+ u' h^2 \left(\frac{d\alpha}{du} - \frac{d\beta}{du}\right) u^2 e^{-2\beta} + h^2 u' u e^{-2\beta} . \quad (11.32)$$

One obvious solution of (11.32) is simply

$$u' = 0 , \quad (11.33)$$

which yields

$$r = \text{constant} \;, \tag{11.34}$$

i.e., circular motion. If $u' \neq 0$ then (11.32) becomes

$$e^{2\beta}\frac{d\alpha}{du} = h^2(u'' + u) + h^2\left(\frac{d\alpha}{du} - \frac{d\beta}{du}\right)(u')^2 + $$

$$+ h^2\left(\frac{d\alpha}{du} - \frac{d\beta}{du}\right)u^2 \;, \tag{11.35}$$

which can be rewritten in the form

$$u'' + u = \frac{1}{h^2}e^{2\beta}\frac{d\alpha}{du} + \left(\frac{d\beta}{du} - \frac{d\alpha}{du}\right)((u')^2 + u^2) \;. \tag{11.36}$$

Equations (11.18), (11.23), (11.24) and (11.36) are our final equations governing the non-null geodesics of a general isotropic line element of the form (11.1). One should note that no approximations were used to obtain these expressions.

We shall now write out the differential equations governing the non-null geodesics of the Brans-Dicke theory. In order to accomplish this it will be necessary to determine $\frac{d\beta}{du} - \frac{d\alpha}{du}$ and $\frac{d\alpha}{du}e^{2\beta}$.

From equations (11.2) and (11.3) we find

$$\alpha = q\left\{\ln(1 - Bu) - \ln(1 + Bu)\right\} \;, \tag{11.37}$$

and

$$\beta = 2\ln(1 + Bu) + Q\left\{\ln(1 - Bu) - \ln(1 + Bu)\right\} \;. \tag{11.38}$$

Using the above expressions we easily find

$$\frac{d\alpha}{du} = \frac{-2qB}{1 - B^2u^2} \;, \tag{11.39}$$

and

$$\frac{d\beta}{du} = \frac{2B(1 - Bu - Q)}{1 - B^2u^2} \;. \tag{11.40}$$

Upon subtracting (11.39) from (11.40) we find:

$$\frac{d\beta}{du} - \frac{d\alpha}{du} = \frac{2B(1 - Bu - Q + q)}{(1 - B^2u^2)} \;. \tag{11.41}$$

If we now combine equation (11.3) with (11.39) we find

$$e^{2\beta}\frac{d\alpha}{du} = \frac{-2qB}{1 - B^2u^2}\,(1 + Bu)^4\left(\frac{1 - Bu}{1 + Bu}\right)^{2Q}. \tag{11.42}$$

Substituting (11.2), (11.3), (11.41) and (11.42) into (11.23), (11.24) and (11.36) gives us:

$$\dot{\phi} = hu^2(1 + Bu)^{-4}\left(\frac{1 + Bu}{1 - Bu}\right)^{2Q}, \tag{11.43}$$

$$\dot{t} = d\left(\frac{1 + Bu}{1 - Bu}\right)^{2q}, \tag{11.44}$$

$$u'' + u = -\frac{2qB(1 + Bu)^3}{h^2\,(1 - Bu)}\left(\frac{1 - Bu}{1 + Bu}\right)^{2Q} + \frac{2B(1 - Bu - Q + q)}{(1 - B^2u^2)}((u')^2 + u^2), \tag{11.45}$$

respectively.

These are the exact differential equations which have to be satisfied by the "physical" non-null geodesics of the Brans-Dicke theory. We shall now use these geodesics differential equations to treat the "classical" problems of perihelion rotation and light deflection. Equation (11.45) will suffice to handle the former problem. For the latter problem we shall find it necessary to modify the above equations since they apply to the non-null geodesics whereas light is assumed to follow the null geodesics of our line element.

Before proceeding to consider the above "classical" problems let us compare the non-null Brans-Dicke geodesic differential equations (11.43)-(11.45) with the corresponding differential equations of Newton's and Einstein's theories of gravitation.

In Newtonian theory u and ϕ are governed by (c.f. [1], page 183)

$$u'' + u = \frac{kM}{H^2} \quad , \tag{11.46}$$

and

$$\frac{d\phi}{dt} = H u^2 \, , \tag{11.47}$$

where H is a constant. To begin we note that equations (11.45) and (11.46) are quite different. However, upon dividing (11.43) by (11.44) we find that in the Brans-Dicke theory

$$\frac{d\phi}{dt} = \frac{h}{d} u^2 (1 + Bu)^{-4} \left(\frac{1 + Bu}{1 - Bu}\right)^{2(Q - q)} \quad . \tag{11.48}$$

Thus we see that if $|Bu| \ll 1$, then equations (11.47) and (11.48) are quite similar.

Using the results presented in chapter 6 of [1] we can show that the differential equations governing the geodesics of the isotropic Schwarzschild line element are:

$$\dot{\phi} = \frac{H u^2}{(1 + Gu)^4} \quad , \tag{11.49}$$

$$\dot{t} = D\left(\frac{1 + Gu}{1 - Gu}\right)^2 \quad , \tag{11.50}$$

and

$$u'' + u = \frac{-2G(1 + Gu)^3}{H^2(1 - Gu)} + \frac{2G(2 - Gu)}{1 - G^2u^2}((u')^2 + u^2) \quad , \tag{11.51}$$

where H, D and G are constants. Upon comparing equations (11.43)-(11.45) with equations (11.49)-(11.51) we find that the Brans-Dicke and Einstein expressions for $\dot{\phi}$, $\dot{t}$, and u are quite similar and in fact become identical (up to the choice of constants) when $q = 1$ and $Q = 0$ (i.e., when $\omega \longrightarrow \infty$).

We shall now proceed to use (11.45) to examine the problem of perihelion rotation. In order to treat this topic we shall find it necessary to replace (11.45) by an approximate differential equation.

Assuming that

$$|Bu|<1$$

we may use a binomial series expansion to rewrite (11.45) as follows:

$$u'' + u = -\frac{2qB}{h^2}(1+3x)(1+x)(1-2Qx)^2 + \\ + 2B(1-Q+q-x)((u')^2+(u)^2) + O(x^2) \;, \qquad (11.52)$$

where

$$x = Bu \;. \qquad (11.53)$$

Upon multiplying out the terms appearing in (11.52) we find

$$u'' + u = -\frac{2qB}{h^2}(1+4(1-Q)x) + 2B(1-Q+q-x)((u')^2+u^2) + \\ + O(x^2) \quad . \qquad (11.54)$$

From equation (11.5) we see that

$$1 - Q + q = \frac{2}{q} \;,$$

and consequently (11.54) may be written in the form

$$u'' + u = -\frac{2qB}{h^2}(1+4(1-Q)x) + 2B(\frac{2}{q}-x)((u')^2+u^2) + O(x^2) . \qquad (11.55)$$

Before we can procced any further with (11.55) we have to return to (11.19); viz.,

$$\dot{\phi}\, r^2 e^{2\beta} = h \;, \qquad (11.19)$$

in order to determine the value of the constant h.

When dealing with the classical central force problem of Newtonian physics we find that

$$\frac{d\phi}{dt} r^2 = H \;, \qquad (11.56)$$

where H denotes twice the constant areal velocity of our particle. Using (11.56) we may rewrite (11.19) in the form

$$h^2 = e^{4\beta}\left(\frac{dt}{ds}\right)^2 H^2 . \qquad (11.57)$$

Upon multiplying equation (11.21) by $\left(\frac{ds}{dt}\right)^2$ (recall that $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$) we find

$$\left(\frac{ds}{dt}\right)^2 = -e^{2\alpha}c^2 + e^{2\beta}\left\{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2\right\} . \qquad (11.58)$$

In the above expression the term within the curly brackets denotes the square of the particles velocity as it moves along the geodesic. From (11.2) and (11.3) we see that for fairly large values of r, $e^{2\alpha}$ and $e^{2\beta}$ are approximately 1. Thus for slow moving particles we can use (11.58) to conclude that

$$\left(\frac{ds}{dt}\right)^2 \approx -c^2 , \qquad (11.59)$$

and consequently (11.57) becomes

$$h^2 \approx -\frac{H^2}{c^2} , \qquad (11.60)$$

where we have replaced $e^{4\beta}$ by its limit as r goes to infinity.

Equation (11.60) permits us to rewrite (11.55) in the form

$$u'' + u = \frac{2qBc^2}{H^2}(1 + 4(1 - Q)x) + 2B\left(\frac{2}{q} - x\right)((u')^2 + u^2) + O(x^2). \qquad (11.61)$$

In order to further simplify this equation we shall demand that terms of order greater than first in $\frac{1}{c^2}$ be neglected.

Since B is given by

$$B = \frac{MG_o}{2c^2q} ,$$

we can keep terms proportional to c^2B and c^2B^2, but must neglect all other terms. Thus (11.61) becomes

$$u'' + u = \frac{2qc^2B}{H^2} + \frac{8qc^2B^2}{H^2}(1 - Q)u + \frac{4B}{q}((u')^2 + u^2) \quad . \qquad (11.62)$$

It is fairly apparent that the first term on the right hand side of (11.62) dominates that side of the above differential equation. This is so because the other terms on the right hand side of (11.62) are of first order in $\frac{1}{c^2}$, whereas $\frac{2qc^2B}{H^2}$ is of zeroth order in $\frac{1}{c^2}$. Consequently (11.62) differs only slightly from the differential equation which arises in the classical central force problem; viz.,

$$u'' + u = A \ ,$$

where A is a constant. Hence we shall use a classical perturbation approach to study (11.62).

To begin let us set

$$A = \frac{2qc^2B}{H^2} \ , \qquad (11.63)$$

$$\frac{\epsilon}{A} = \frac{4B}{q} \ , \qquad (11.64)$$

and

$$\epsilon = \frac{8c^2B^2}{H^2} \ . \qquad (11.65)$$

Using (11.63)-(11.65) we may rewrite (11.62) in the form

$$u'' + u = A + \epsilon q\,(1 - Q)u + \frac{\epsilon}{A}((u')^2 + u^2) \ . \qquad (11.66)$$

To obtain an approximate solution to (11.66) we shall assume a solution of the form

$$u = u_o + \epsilon v + O(\epsilon^2) \ . \qquad (11.67)$$

Upon inserting (11.67) into (11.66) we obtain

$$u_o'' + \epsilon v'' + u_o + \epsilon v = A + \epsilon q(1 - Q)u_o + \frac{\epsilon}{A}((u_o')^2 + u_o^2) + \\ + O(\epsilon^2) \ . \qquad (11.68)$$

Equating the zeroth order terms in ϵ we find

$$u_0'' + u_0 = A \, . \qquad (11.69)$$

The general solution to this differential equation is

$$u_0 = A + K\cos(\phi + \delta) \, ,$$

where K and δ are constants. However, by a judicious choice of axes we may make $\delta = 0$, and consequently our solution to (11.69) becomes

$$u_0 = A + K\cos\phi \, . \qquad (11.70)$$

It should immediately be noted that (11.70) is the equation of a conic section with one focus at the origin and eccentricity $\frac{K}{A}$.

If we now equate the first order terms in ϵ which appear in (11.68) we obtain

$$v'' + v = q(1-Q)u_0 + \frac{1}{A}((u_0')^2 + u_0{}^2) \, . \qquad (11.71)$$

Due to (11.70) the above equation becomes

$$v'' + v = \left\{q(1-Q)A + \frac{K^2}{A} + A\right\} + \left\{2 + q(1-Q)\right\}K\cos\phi \, . \qquad (11.72)$$

The exact solution of (11.72) is easily found to be

$$v = \left\{q(1-Q)A + \frac{K^2}{A} + A\right\} + W\cos\phi + $$
$$+ \frac{K(2+q(1-Q))}{2}\phi\sin\phi \, , \qquad (11.73)$$

where W is a constant. However, since our zeroth order solution already contains a term proportional to $\cos\phi$ we can set $W = 0$.

So to $O(\epsilon^2)$ we find that an approximate solution to (11.66) is given by

$$u = A + K\cos\phi + \epsilon(q(1-Q)A + A + \frac{K^2}{A}) +$$
$$+ \frac{\epsilon}{2}(2 + q(1-Q))K\phi\sin\phi \, . \qquad (11.74)$$

In order to put (11.74) into a more useful form we note that to first order in ϵ

$$\cos(\phi - \frac{\epsilon}{2}(2 + q(1 - Q))\phi) \approx \cos\phi + \frac{\epsilon}{2}(2 + q(1 - Q))\phi \sin\phi \quad .$$

Using this (11.74) becomes

$$u = \left\{A + \epsilon(q(1 - Q)A + A + \frac{K^2}{A})\right\} + K\cos(\phi - \frac{\epsilon}{2}(2 + q(1 - Q))\phi) . \quad (11.75)$$

Thus we see that our radial coordinate $r = \frac{1}{u}$ "almost" traces out a conic section of eccentricity ξ given by

$$\xi = \frac{K}{A + \epsilon(q(1 - Q)A + A + \frac{K^2}{A})} \quad . \qquad (11.76)$$

However, just as in the corresponding case in general relativity, we find the appearance of a non-periodic term in the expression for u.

Let us now assume that ξ lies in the range (0,1), implying that our geodesic will have an elliptical appearance. This assumption permits us to consider the problem of perihelion rotation.

The perihelion of a bound orbit occurs when r is at a minimum or, correspondingly, u is at a maximum. From (11.75) it is apparent that u attains its maximal value when

$$\phi(1 - \frac{\epsilon}{2}(2 + q(1 - Q))) = 2\pi n , \qquad (11.77)$$

where $n = 0, 1, 2, \ldots$. Using a binomial expansion we find that to first order in ϵ

$$\phi \approx 2\pi n (1 + \frac{\epsilon}{2}(2 + q(1 - Q))) . \qquad (11.78)$$

Consequently the successive perihelia will be found to occur when

$$\Delta\phi = 2\pi(1 + \frac{\epsilon}{2}(2 + q(1 - Q))) . \qquad (11.79)$$

Thus from the above expression it is obvious that the perihelion shift per revolution is

$$\delta\phi = \pi\epsilon(2 + q(1 - Q)) . \tag{11.80}$$

Using (11.5) we find that $(1 - Q)$ is given by

$$1 - Q = \frac{2 - q^2}{q} ,$$

and thus (11.80) becomes

$$\delta\phi = \pi\epsilon(4 - q^2) . \tag{11.81}$$

From (11.4) we obtain

$$4 - q^2 = \frac{8 + 6\omega}{3 + 2\omega} . \tag{11.82}$$

Upon combining (11.6a), (11.65), (11.81) and (11.82) we find that the perihelion shift per revolution in the Brans-Dicke theory, $\delta\phi_{BD}$, is given by

$$\delta\phi_{BD} = \frac{2\pi}{H^2}\frac{M^2 G_o^{\ 2}}{c^2}\left(\frac{3\omega + 4}{\omega + 2}\right) . \tag{11.83}$$

From conventional theory we know (c.f. [1], page 187) that the perihelion shift per revolution is

$$\delta\phi_E = \frac{2\pi}{H^2}\ \frac{3M^2 G_o^{\ 2}}{c^2} , \tag{11.84}$$

and thus

$$\delta\phi_{BD} = \left(\frac{3\omega + 4}{3\omega + 6}\right)\delta\phi_E , \tag{11.85}$$

which agrees with Brans and Dicke (c.f. [4], page 931).

The observations made of the perihelion shift of Mercury's orbit about the sun seem to be quite good. Thus the observations of Mercury can be used in conjunction with $\delta\phi_E$ to determine a range for ω.

As of present the observed perihelion rotation of Mercury's geodesic (after subtracting planetary perturbations and other effects presumed to be known) is 42.6"± .9" per century (c.f.[4], page 931). We shall now assume (as is done by Brans and Dicke in [4]) that the above result can be reduced by as much as 8%. The cause of such a reduction might be poor experimental technique or perhaps the discovery of other perturbation effects, such as a quadrupole moment of the sun. Thus one would desire the predicted value of the perihelion rotation of Mercury to fall in the range

$$(38.4'', 43.5'') \text{ per century}.$$

Now $\delta\phi_E$ for the perihelion rotation of Mercury is (c.f.[1], page 187)

$$\delta\phi_E = 42.89'' \text{ per century}. \tag{11.86}$$

Thus if

$$\omega \geq 5 \tag{11.87}$$

we see that (11.85) gives us

$$38.4'' \text{ per century} < \left(\frac{3\omega+4}{3\omega+6}\right)\delta\phi_E < 43.5'' \text{ per century}. \tag{11.88}$$

Consequently we must choose $\omega \geq 5$ if we desire the Brans-Dicke theory to compare favorably with the perihelion rotation of Mercury. Brans and Dicke have not fixed their choice of ω at any specific value of $\omega \geq 5$. However, if we choose a certain experiment to fix ω then we must stick to this choice of ω when comparing the Brans-Dicke theory with other experiments.

We shall now consider the null geodesics of our line element (11.1). In this case $ds^2 = 0$, and consequently we shall choose to parameterize the coordinates of our null geodesics by an arbitrary parameter Υ. The differential equations governing the null geodesics of (11.1) are the Euler-Lagrange equations corresponding to the following Lagrangian

$$L_o^2 = -e^{2\alpha}c^2\left(\frac{dt}{d\Upsilon}\right)^2 + e^{2\beta}\left\{\left(\frac{dr}{d\Upsilon}\right)^2 + r^2\left[\left(\frac{d\theta}{d\Upsilon}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\Upsilon}\right)^2\right]\right\}. \quad (11.89)$$

Using our previous results we find that the Euler-Lagrange equations corresponding to (11.89) are:

$$\frac{d}{d\Upsilon}\left(e^{2\alpha}\frac{dt}{d\Upsilon}\right) = 0, \quad (11.90)$$

$$\frac{d}{d\Upsilon}\left(e^{2\beta}\frac{dr}{d\Upsilon}\right) = -\frac{d\alpha}{dr}e^{2\alpha}c^2\left(\frac{dt}{d\Upsilon}\right)^2 + \frac{d\beta}{dr}e^{2\beta}\left(\frac{dr}{d\Upsilon}\right)^2 + e^{2\beta}r\left(1 + r\frac{d\beta}{dr}\right)\left(\frac{d\theta}{d\Upsilon}\right)^2 +$$

$$+ re^{2\beta}\left(1 + r\frac{d\beta}{dr}\right)\sin^2\theta\left(\frac{d\phi}{d\Upsilon}\right)^2, \quad (11.91)$$

$$\frac{d}{d\Upsilon}\left(r^2e^{2\beta}\frac{d\theta}{d\Upsilon}\right) = r^2e^{2\beta}\sin\theta\cos\theta\left(\frac{d\phi}{d\Upsilon}\right)^2, \quad (11.92)$$

and

$$\frac{d}{d\Upsilon}\left(r^2e^{2\beta}\sin^2\theta\frac{d\phi}{d\Upsilon}\right) = 0. \quad (11.93)$$

We may obtain a fifth, non-independent, differential equation from (11.1) by setting $ds^2 = 0$. This differential equation is

$$-e^{2\alpha}c^2\left(\frac{dt}{d\Upsilon}\right)^2 + e^{2\beta}\left\{\left(\frac{dr}{d\Upsilon}\right)^2 + r^2\left[\left(\frac{d\theta}{d\Upsilon}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\Upsilon}\right)^2\right]\right\} = 0. \quad (11.94)$$

Reasoning similar to that used to obtain equations (11.18), (11.22), (11.23) and (11.36) can be applied to equations (11.90)-(11.94) to give us:

$$\theta = \frac{\pi}{2}, \quad (11.95)$$

$$\frac{d\phi}{d\Upsilon} = au^2e^{-2\beta}, \quad (11.96)$$

$$\frac{dt}{d\Upsilon} = be^{-2\alpha}, \quad (11.97)$$

and

$$u'' + u = \left(\frac{d\beta}{du} - \frac{d\alpha}{du}\right)((u')^2 + u^2) \ , \qquad (11.98)$$

where a and b are constants,

$$u = \frac{1}{r} \ ,$$

and a prime is used to denote differentiation with respect to ϕ.

Equations (11.95)-(11.98) represent the exact differential equations governing the null geodesics corresponding to an arbitrary line element of the form (11.1).

Using equations (11.2), (11.3) and (11.41) we find that the exact differential equations governing the null geodesics of Brans's "physical" vacuum solution are:

$$\frac{d\phi}{d\gamma} = \frac{a u^2}{(1 + Bu)^4}\left(\frac{1 + Bu}{1 - Bu}\right)^{2Q} \ , \qquad (11.99)$$

$$\frac{dt}{d\gamma} = b\left(\frac{1 + Bu}{1 - Bu}\right)^{2q} \ , \qquad (11.100)$$

and

$$u'' + u = \frac{2B(1 - Q + q - Bu)}{1 - B^2u^2}((u')^2 + u^2) \ . \qquad (11.101)$$

As in our previous treatment of the non-null geodesics corresponding to (11.1) we shall replace (11.101) by an approximate differential equation. Using a binomial series expansion we find that to first order in $\frac{1}{c^2}$, (11.101) becomes

$$u'' + u = 2B(1 - Q + q)((u')^2 + u^2) \ . \qquad (11.102)$$

Using equation (11.5) it is easily shown that

$$1 - Q + q = \frac{2}{q} \ ,$$

and consequently (11.102) may be written as follows:

$$u'' + u = \frac{4B}{q}((u')^2 + u^2) \ . \qquad (11.103)$$

The differential equation corresponding to (11.103) in Einstein's theory of gravitation is

$$u'' + u = 4Bq\,((u')^2 + u^2)\ . \qquad (11.104)$$

Thus we see that there is only a slight difference between equations (11.103) and (11.104). In the limit as $\omega \longrightarrow \infty$ $q \longrightarrow 1$ and hence in that case the two equations become identical.

In order to obtain an approximate solution to (11.103) we shall make use of a perturbation approach. Such an approach will be permissible since the right hand side of (11.103) is of first order in $\frac{1}{c^2}$ while the left hand side is of zeroth order in $\frac{1}{c^2}$.

To begin let us set

$$\epsilon = \frac{4B}{q}\ . \qquad (11.105)$$

Using (11.105) we may rewrite (11.103) as follows:

$$u'' + u = \epsilon((u')^2 + u^2)\ . \qquad (11.106)$$

We shall assume that an approximate solution to (11.106) is given by

$$u = u_o + \epsilon v + O(\epsilon^2)\ . \qquad (11.107)$$

If we now repeat the procedure used to obtain an approximate solution to (11.66) we find that to first order in ϵ u is given by

$$u = \frac{1}{r_o}\cos\phi + \frac{\epsilon}{r_o^{\,2}}\ , \qquad (11.108)$$

where r_o represents the distance of closest approach to the origin. Upon replacing u by $\frac{1}{r}$ we find that (11.108) becomes

$$\frac{1}{r} = \frac{1}{r_o}\cos\phi + \frac{\epsilon}{r_o^{\,2}}\ . \qquad (11.109)$$

The above expression is the equation of a conic section with eccentricity and latus rectum given by $\frac{r_o}{\epsilon}$ and $\frac{2r_o^2}{\epsilon}$ respectively. Since ϵ is quite small in comparison to r_o (in most cases of interest) we shall have an eccentricity much greater than one. This implies that the trajectory of our light ray is virtually rectilinear and only "slightly" hyperbolic.

To determine the asymptotes of (11.109) we let r go to infinity in that expression to obtain

$$\cos\phi = -\frac{\epsilon}{r_o} . \tag{11.110}$$

Since $\frac{\epsilon}{r_o}$ is quite small we may conclude that the angles corresponding to the asymptotes are close to $\pm\frac{\pi}{2}$. Thus by setting

$$\phi = \pm\frac{\pi}{2} + \delta$$

we find that (11.110) becomes

$$-\sin(\pm\frac{\pi}{2})\sin\delta = -\frac{\epsilon}{r_o} ,$$

which may be written

$$\mp\sin\delta = -\frac{\epsilon}{r_o} .$$

We expect δ to be small, thus we may approximate $\sin\delta$ by δ in the above expression to obtain

$$\delta = \pm\frac{\epsilon}{r_o} . \tag{11.111}$$

Thus the asymptotes corresponding to equation (11.109) are given by

$$\phi = \frac{\pi}{2} + \frac{\epsilon}{r_o} \tag{11.112}$$

and

$$\phi = -\frac{\pi}{2} - \frac{\epsilon}{r_o} . \tag{11.113}$$

The above pair of equations permits us to conclude that the total angular deflection, Δ_{BD}, of our light ray from a straight line trajectory is

$$\Delta_{BD} = \frac{2\epsilon}{r_o} . \tag{11.114}$$

Upon combining equations (11.4), (11.6a), (11.105) and (11.114) we find that Δ_{BD} is given by

$$\Delta_{BD} = \frac{4MG_o}{r_o c^2}\left(\frac{2\omega+3}{2\omega+4}\right) . \tag{11.115}$$

The Schwarzschild solution to Einstein's vacuum field equations predicts that the angular deflection of light, Δ_E, in Einstein's theory of gravitation should approximately be

$$\Delta_E = \frac{4MG_o}{r_o c^2} . \tag{11.116}$$

Consequently we find that

$$\Delta_{BD} = \left(\frac{2\omega+3}{2\omega+4}\right)\Delta_E , \tag{11.117}$$

which agrees with Brans and Dicke (c.f. [4], page 931).

As a result of (11.117) we see that for all finite values of $\omega \geq 5$, Δ_{BD} will always be less than Δ_E.

Conclusion

Our analysis of the geodesics corresponding to Brans's "most physical" line element has shown that the parameter ω can be chosen so as to bring the "classical" predictions of the Brans-Dicke theory into close agreement with the predictions of Einstein's theory. However, we have seen that for all finite values of ω (≥ 5) the predictions of the Brans-Dicke theory, with regards to perihelion rotation and light deflection, will always be less than the corresponding predictions made by Einstein's theory.

The "classical" tests of general relativistic theories performed to date do not have the precision necessary to choose between the Brans-Dicke and Einstein theories of gravitation.[49] In an attempt to choose between these two theories R.E.Morganstern and H.Y.Chiu [19] have devised a test of the Brans-Dicke theory. This test is based upon the radiation of scalar waves which, unlike pure gravitational waves, can be radiated from a spherically symmetric, radially pulsating star. However, Morganstern and Chiu's experiment requires very precise observational techniques which, unfortunately, have not as yet been developed.

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[49] Recently the Pasadena Jet Propulsion Laboratory has succeeded in obtaining a fairly accurate measurement of the time of flight for a radio wave sent from the earth past the sun to a spacecraft near Mars. The results obtained in this experiment seem to favor Einstein's theory over Brans and Dicke's theory of gravitation. An account of this experiment can be found in Time Magazine (November 23, 1970, page 52).

12. The Geometrization of the Brans-Dicke Scalar Field[50]

The Einstein theory of general relativity is said to have geometerized the gravitational field in the sense that all gravitational interactions can be described by the geometry of the space time rather than by field variables independent of the geometry. In this same sense it appears that the Brans-Dicke theory is not completely geometrical since gravitational phenomenon are described by both a scalar field ϕ and the components of the metric tensor. In this section we shall show that ϕ is not independent of the g_{ij}'s.

In order to establish the relationship between ϕ and the components of the metric tensor we confine our attention to source free regions of space, as is similarly done in the geometrization of the electromagnetic fields in the Rainich, Misner and Wheeler theory.

The vacuum field equations governing the metric potentials of the Brans-Dicke theory are given by equation (7.3); viz.,

$$R_{ij} - \tfrac{1}{2}g_{ij}R = \frac{\omega}{\phi^2}(\phi_{,i}\phi_{,j} - \tfrac{1}{2}g_{ij}\phi_{,k}\phi^{,k}) + \frac{1}{\phi}(\phi_{|ij} - g_{ij}\Box\phi) \,. \qquad (12.1)$$

Upon multiplying this equation through by g^{ij} we find

$$-R = -\frac{\omega}{\phi^2}(\phi_{,k}\phi^{,k}) + \frac{3}{\phi}\Box\phi \,.$$

When this expression for $-R$ is substituted into (12.1) we obtain

$$R_{ij} = \frac{\omega}{\phi^2}\phi_{,i}\phi_{,j} + \frac{1}{\phi}\phi_{|ij} + \tfrac{1}{2}g_{ij}\frac{\Box\phi}{\phi} \,. \qquad (12.2)$$

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[50] The material presented in this section is based upon a paper by P.C.Peters, [21].

We shall now replace ϕ by

$$\Psi = \ln \phi \, . \tag{12.3}$$

From (12.3) we find:

$$\frac{\Box\phi}{\phi} = \Box\Psi + \Psi_{,k}\Psi^{,k} \, , \tag{12.4}$$

and

$$\frac{\phi_{|ij}}{\phi} = \Psi_{|ij} + \Psi_{|i}\Psi_{|j} \, . \tag{12.5}$$

The result of substituting (12.3), (12.4) and (12.5) into (12.2) is

$$R_{ij} = (\omega + 1)\Psi_{|i}\Psi_{|j} + \Psi_{|ij} + \tfrac{1}{2}g_{ij}(\Box\Psi + \Psi_{|k}\Psi^{|k}) \, . \tag{12.6}$$

The field equation for ϕ, (4.24), is given by

$$\frac{\Box\phi}{\phi} - \frac{1}{2\phi^2}\phi_{,k}\phi^{,k} + \frac{R}{2\omega} = 0 \, . \tag{12.6a}$$

In terms of our new notation the above expression may be rewritten

$$\Box\Psi + \tfrac{1}{2}\Psi_{|k}\Psi^{|k} + \frac{R}{2\omega} = 0 \, . \tag{12.7}$$

Upon multiplying equation (12.6) by g^{ij} we obtain,

$$R = (\omega + 3)\Psi_{|k}\Psi^{|k} + 3\Box\Psi . \tag{12.8}$$

If we now insert equation (12.8) into (12.7) we find that the field equation for ϕ becomes

$$(\omega + \tfrac{3}{2})(\Box\Psi + \Psi_{|k}\Psi^{|k}) = 0 \, . \tag{12.9}$$

One should recall that in section 2 it was shown that the Euler-Lagrange tensor corresponding to ϕ could be obtained from the divergence of the Euler-Lagrange tensor corresponding to the g_{ij}'s. In the present case this implies that a solution to (12.1) will also be a solution to (12.6a). Consequently we may conclude that any solution to (12.6) will also satisfy (12.9).

We shall now assume that the field equations for the metric potentials, (12.6), are satisfied and proceed to show that it is possible to solve for $\Psi_{|i}$ in terms of geometrical quantities.

To begin let us use (12.6) to calculate

$$C^{kijm}R_{ij|m},$$

where C^{kijm} is the Weyl conformal curvature tensor, and is defined as follows:

$$C_{kijm} = R_{kijm} + g_{k[m}R_{j]i} + g_{i[j}R_{m]k} + \tfrac{1}{3}Rg_{k[j}g_{m]i} \quad , \quad (12.10)$$

where

$$[m\,j] = \tfrac{1}{2}(\,m\,j - j\,m\,).$$

The well known symmetry properties of the Weyl tensor are:

$$C_{kijm} = -C_{ikjm} = -C_{kimj} \quad \text{and} \quad C_{k\ \,ji}^{\ \,i} = 0\ .$$

Using (12.6) and (12.10) we find

$$C^{kijm}R_{ij|m} = C^{kijm}\Big\{(\omega+1)(\Psi_{|im}\Psi_{|j} + \Psi_{|i}\Psi_{|jm}) + \Psi_{|ijm} + \\ + \tfrac{1}{2}g_{ij}(\Box\Psi + \Psi_{|k}\Psi^{|k})_{|m}\Big\}\ ,$$

which simplifies to

$$C^{kijm}R_{ij|m} = (\omega+1)C^{kijm}\Psi_{|im}\Psi_{|j} + C^{kijm}\Psi_{|ijm}\ . \quad (12.11)$$

The last term in (12.11) may be rewritten by using Ricci's identity in conjunction with the fact that $\Psi_{|ijm} = \Psi_{|jim}$ to obtain

$$C^{kijm}\Psi_{|ijm} = \Psi_{|p}R^{p}_{\ jim}C^{kijm}\ . \quad (12.12)$$

The first term on the right hand side of (12.11) may be simplified by considering $C^{kijm}R_{ij}\Psi_{|m}$. From (12.6) we find

$$C^{kijm}R_{ij}\Psi_{|m} = C^{kijm}\Psi_{|m}\Big\{(\omega+1)\Psi_{|i}\Psi_{|j} + \Psi_{|ij} + \tfrac{1}{2}g_{ij}(\Box\Psi + \Psi_{|k}\Psi^{|k})\Big\}\ ,$$

which simplifies to

$$C^{kijm}R_{ij}\Psi_{|m} = -\,C^{kijm}\Psi_{|im}\Psi_{|j}\ . \quad (12.13)$$

Inserting (12.12) and (12.13) into (12.11) gives us

$$C^{kijm}R_{ij|m} = -(\omega+1)C^{kijm}\Psi_{|m}R_{ij} + \Psi_{|p}R^{p}_{\ jim}C^{kijm}\ . \quad (12.14)$$

By exploiting a property of the Riemann curvature tensor we have

$$C^{kijm}(R^p{}_{jim} + R^p{}_{imj} + R^p{}_{mji}) = 0 ,$$

which may be rewritten

$$C^{kijm}R^p{}_{jim} - C^{kijm}R^p{}_{ijm} + C^{kimj}R^p{}_{mij} = 0 .$$

Thus we obtain

$$C^{kijm}R^p{}_{jim} = \tfrac{1}{2}C^{kijm}R^p{}_{ijm}$$

which permits us to write equation (12.14) in the form

$$C^{kijm}R_{ij|m} = -(\omega + 1)\, C^{kijm}R_{ij}\psi_{|m} + \tfrac{1}{2}\psi_{|p}R^p{}_{ijm}C^{kijm} . \quad (12.15)$$

From the definition of the Weyl tensor given in (12.10) we can express the Riemann tensor in terms of the conformal curvature tensor and the Ricci tensor. Doing so gives us

$$R^p{}_{ijm} = C^p{}_{ijm} - \tfrac{1}{2}(\delta^p_m R_{ij} - \delta^p_j R_{im}) - \tfrac{1}{2}(g_{ij}R^p_m - g_{im}R^p_j) + - \frac{R}{6}(\delta^p_j g_{mi} - \delta^p_m g_{ij}) . \quad (12.16)$$

Using the above expression we find

$$C^{kijm}R^p{}_{ijm} = C^{kijm}C^p{}_{ijm} - \tfrac{1}{2}(C^{kijp}R_{ij} - C^{kipm}R_{im}) ,$$

which reduces to

$$C^{kijm}R^p{}_{ijm} = C^{kijm}C^p{}_{ijm} - C^{kijp}R_{ij} . \quad (12.17)$$

This expression may be further simplified if we make use of the Bach [2] identity; viz.,

$$C^{kijm}C^p{}_{ijm} = \frac{g^{kp}}{4}C_{ijmq}C^{ijmq} . \quad (12.18)$$

So we see that (12.17) may be written

$$C^{kijm}R^p{}_{ijm} = \frac{g^{kp}}{4}C_{ijmq}C^{ijmq} - C^{kijp}R_{ij} , \quad (12.19)$$

and thus (12.15) becomes

$$C^{kijm}R_{ij|m} = \frac{1}{8}\psi^{|k}\, C_{ijmq}C^{ijmq} - (\omega + \tfrac{3}{2})C^{kijp}R_{ij}\psi_{|p} . \quad (12.20)$$

When $\omega = -\frac{3}{2}$ equation (12.20) reduces to

$$\psi^{|k} = \frac{8C^{kijm}R_{ij|m}}{C_{ijmq}C^{ijmq}} , \qquad (12.21)$$

and thus we have found an expression for $\psi^{|k}$ in terms of g_{ij} and its derivatives.

For the purpose of further considerations we shall assume that $\omega \neq -\frac{3}{2}$. With this assumption we find from equation (12.9) that

$$\Psi_{|k}\Psi^{|k} + \Box\Psi = 0 ,$$

and thus equation (12.6) becomes

$$R_{ij} = (\omega + 1)\Psi_{|i}\Psi_{|j} + \Psi_{|ij} . \qquad (12.22)$$

Equation (12.7) now tells us that

$$\frac{R}{\omega} = \Psi_{|k}\Psi^{|k} = -\Box\Psi. \qquad (12.23)$$

We shall use (12.22) and (12.23) to express the second term on the right of equation (12.20) in a form in which $\Psi^{|p}$ appears with the free index k.

To accomplish this we shall first consider $R_{i[j|m]}$. Using (12.22) we find

$$R_{ij|m} = (\omega + 1)(\Psi_{|im}\Psi_{|j} + \Psi_{|i}\Psi_{|jm}) + \Psi_{|ijm} ,$$

and

$$R_{im|j} = (\omega + 1)(\Psi_{|ij}\Psi_{|m} + \Psi_{|i}\Psi_{|mj}) + \Psi_{|imj} .$$

Thus we have

$$R_{i[j|m]} = \tfrac{1}{2}(\omega + 1)(\Psi_{|im}\Psi_{|j} - \Psi_{|ij}\Psi_{|m}) + \tfrac{1}{2}(\Psi_{|ijm} - \Psi_{|imj}) ,$$

which simplifies to

$$R_{i[j|m]} = (\omega + 1)\Psi_{|i[m}\Psi_{|j]} + \tfrac{1}{2}\Psi_{|p} R^{p}{}_{ijm} . \qquad (12.24)$$

From (12.22) we find

$$R_{i[m}\Psi_{|j]} = \tfrac{1}{2}(R_{im}\Psi_{|j} - R_{ij}\Psi_{|m}) = \Psi_{|i[m}\Psi_{|j]} . \qquad (12.25)$$

Using (12.25) we find that (12.24) now assumes the following form

$$R_{i[j|m]} = (\omega+1)R_{i[m}\psi_{|j]} + \tfrac{1}{2}\psi_{|p}\,R^{p}{}_{ijm} \quad ,$$

which may be written

$$\psi_{|p}\,R^{p}{}_{ijm} = 2R_{i[j|m]} - 2(\omega+1)R_{i[m}\psi_{|j]} \;. \qquad (12.26)$$

Upon multiplying (12.26) by g^{im} we find

$$\psi_{|m}R^{m}_{j} = R^{m}_{j|m} - R_{|j} + (\omega+1)(R^{m}_{j}\psi_{|m} - R\psi_{|j}) \;.$$

From the Bianchi identity we have

$$R^{m}_{j|m} = \tfrac{1}{2}R_{|j} \;,$$

and so the above expression becomes

$$\psi_{|m}R^{m}_{j} = \left(\frac{\omega+1}{\omega}\right)R\,\psi_{|j} + \frac{1}{2\omega}\,R_{|j} \;. \qquad (12.27)$$

In order to take advantage of equations (12.26) and (12.27) we must rewrite $C^{kijp}R_{ij}\psi_{|p}$ in a manner in which terms of the form $\psi_{|p}R^{p}{}_{ijm}$ and $\psi_{|m}R^{m}_{j}$ appear. To accomplish this we shall make use of the Weyl tensor.

Using the contravariant version of the Weyl tensor we find that we can write the second term appearing on the right hand side of (12.20) in the form

$$C^{kijp}R_{ij}\psi_{|p} = R^{kijp}R_{ij}\psi_{|p} + \tfrac{1}{2}(g^{kp}R^{ji} - g^{kj}R^{pi})R_{ij}\psi_{|p} +$$

$$+ \tfrac{1}{2}(g^{ij}R^{pk} - g^{ip}R^{jk})\,R_{ij}\psi_{|p} + \frac{R}{6}(g^{kj}g^{pi} - g^{kp}g^{ij})R_{ij}\psi_{|p} \;.$$

This expression simplifies to

$$C^{kijp}R_{ij}\psi_{|p} = R^{kijp}R_{ij}\psi_{|p} + \psi^{|k}\left(\tfrac{1}{2}R^{ij}R_{ij} - \frac{R^{2}}{6}\right) +$$

$$- R^{kj}R^{p}_{j}\psi_{|p} + \frac{2}{3}RR^{ki}\psi_{|i} \;. \qquad (12.28)$$

Equations (12.26) and (12.27) permit us to rewrite the first, fourth and last terms on the right hand side of equation (12.28) as :

$$R^{kijp}R_{ij}\psi_{|p} = R_{ij}(R^{ij|k} - R^{jk|i}) - \frac{(\omega+1)}{2\omega}R^{jk}R_{|j} + - \frac{(\omega+1)^2}{2\omega^2}RR^{|k} + \left\{(\omega+1)R_{ij}R^{ij} - \frac{(\omega+1)^3R^2}{\omega^2}\right\}\psi^{|k} ; \quad (12.29)$$

$$R^{kj}R^{p}_{j}\psi_{|p} = \frac{1}{2\omega}R^{kj}R_{|j} + \frac{(\omega+1)^2R^2\psi^{|k}}{\omega^2} + \frac{(\omega+1)}{2\omega^2}RR^{|k} ; \quad (12.30)$$

and

$$\frac{2}{3}RR^{ki}\psi_{|i} = \frac{2}{3}R^2\frac{(\omega+1)\psi^{|k}}{\omega} + \frac{R}{3\omega}R^{|k}. \quad (12.31)$$

Upon inserting (12.29), (12.30) and (12.31) into (12.28) we find after a little algebra

$$C^{kijp}R_{ij}\psi_{|p} = (\omega+\tfrac{3}{2})\left\{R_{ij}R^{ij} - \frac{R^2(\omega^2+2\omega+\tfrac{4}{3})}{\omega^2}\right\}\psi^{|k} + + R^{ij}(R_{ij}{}^{|k} - R_{i}{}^{k}{}_{|j}) - \left(\frac{\omega+2}{2\omega}\right)R^{jk}R_{|j} + - \frac{(\omega^2+\tfrac{7}{3}\omega+2)}{2\omega^2}RR^{|k}. \quad (12.32)$$

Inserting (12.32) into (12.20) gives us

$$\psi_{|k} = \alpha_k \quad , \quad (12.33)$$

where α_k is defined by

$$\alpha_k = \frac{1}{P}\left\{C_k{}^{ijm}R_{ij|m} + (\omega+\tfrac{3}{2})\left[R^{ij}(R_{ij|k} - R_{ik|j}) + - \frac{(\omega+2)}{2\omega}R_{jk}R^{|j} - \frac{(\omega^2+\tfrac{7}{3}\omega+2)}{2\omega^2}RR_{|k}\right]\right\} , \quad (12.34a)$$

with

$$P = \frac{1}{8}C_{ijmq}C^{ijmq} - (\omega+\tfrac{3}{2})^2\left[R_{ij}R^{ij} - R^2\frac{(\omega^2+2\omega+\tfrac{4}{3})}{2\omega^2}\right] \quad (12.34b)$$

From (12.6) we see that the geometrical equations which necessarily must be satisfied if the Brans-Dicke scalar field is present are

$$R_{ij} = (\omega+1)\alpha_i\alpha_j + \alpha_{i|j} + \tfrac{1}{2}g_{ij}(\alpha_k{}^{|k} + \alpha_k\alpha^k) , \quad (12.35a)$$

where α_k is given by equation (12.34a) and satisfies

$$\alpha_{[i|j]} = 0 . \qquad (12.35b)$$

Due to (12.35b) we may use (12.34a) to obtain Ψ by employing the technique used by Misner and Wheeler [18] to obtain the complexion of the electromagnetic field in their "already unified field theory." This procedure yield

$$\Psi = \int \alpha_i dx^i .$$

Since $\Psi = \ln \phi$ we have

$$\phi = \phi_o \exp \int_0^x \alpha_i dx^i \quad , \qquad (12.36)$$

where ϕ_o is a constant which serves to indicate the value of ϕ at some initial point. It is not possible to obtain the value of ϕ_o from our original field equations (12.2) and (12.6a), since these equations are invariant under a constant change in the scale of ϕ. One should note that since we desire ϕ to be determined by α_i we must avoid those regions of space for which P, as given by (12.34b), is zero.

We summarize the above results with the following

Theorem 12.1: When the metric tensor of the Brans-Dicke theory of gravitation is known throughout a matter free region of space and is such that α_i, as given by equation (12.34a), is well defined throughout this region then the scalar field ϕ in the region under consideration is given by

$$\phi = \phi_o \exp \int_0^x \alpha_i dx^i \quad ,$$

where ϕ_o is a constant.

At this point we must emphasize that the above procedure can be used to obtain the scalar field ϕ only when the g_{ij}'s are known. However, we have not shown how to obtain the g_{ij}'s; i.e., we have not given a system of equations which govern the g_{ij}'s and which are independent of the scalar field ϕ. In this sense our geometrization of the scalar field falls short of the Rainich, Misner and Wheeler geometrization of the electromagnetic field.

The case $\omega = -\frac{3}{2}$ is quite interesting in itself for reasons unrelated to the Brans-Dicke theory. To see this let us consider the form of the Ricci tensor when we perform the conformal transformation $\bar{g}_{ij} = \phi g_{ij}$. For a conformal transformation of the type

$$\bar{g}_{ij} = e^{2\sigma} g_{ij}$$

we find

$$\bar{R}_{ij} = R_{ij} - 2(\sigma_{|ij} - \sigma_{|i}\sigma_{|j}) - g_{ij}(g^{km}\sigma_{|km} + 2\sigma_{|k}\sigma^{|k})$$

when the dimension of the space is four. In our case

$$\phi = e^{2\sigma} \quad \text{or} \quad \tfrac{1}{2}\ln\phi = \frac{\Psi}{2} = \sigma .$$

Thus we obtain

$$\bar{R}_{ij} = R_{ij} - (\Psi_{|ij} - \tfrac{1}{2}\Psi_{|i}\Psi_{|j}) - \tfrac{1}{2}g_{ij}(\Box\Psi + \Psi_{|k}\Psi^{|k}) \quad .(12.37)$$

If our original field equations (12.6) are satisfied for $\omega = -\frac{3}{2}$ we find that (12.37) reduces to

$$\bar{R}_{ij} = 0.$$

Consequently we see that a solution to (12.6) for $\omega = -\frac{3}{2}$ can be used to obtain a solution to the Einstein vaccum field equations $\bar{R}_{ij} = 0$.

Now let us assume that we have a Riemannian V_4, and we want to know if this space is conformal to another Riemannian space, $\overline{V}_4$, whose geometry is a solution of the Einstein vacuum field equations[51]

$$\overline{R}_{ij} = 0 .$$

To solve this problem we would construct α_k for $\omega = -3/2$, which we shall denote by β_k. Using equation (12.34a) we find that β_k is given by

$$\beta_k = \frac{8C_k{}^{ijm}R_{ij|m}}{C_{ijmq}C^{ijmq}} . \qquad (12.38)$$

One then constructs the following tensor from β_k,

$$W_{ij} = -\tfrac{1}{2}\beta_i\beta_j + \beta_{i|j} + \tfrac{1}{2}g_{ij}(\beta_k{}^{|k} + \beta_k\beta^k) . \qquad (12.39)$$

If W_{ij} equals the Ricci tensor for the original V_4, then that V_4 is conformal to a $\overline{V}_4$ with the property that

$$\overline{R}_{ij} = 0 .$$

Furthermore the function ϕ needed to perform this conformal transformation may be obtained from equation (12.36) by replacing α_i by β_i.

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[51]For an alternative approach to this problem see the paper by H.W.Brinkmann [7].

Appendix

The purpose of this appendix is to determine all solutions to the Brans-Dicke vacuum field equations corresponding to the following isotropic line element:

$$ds^2 = -e^{2\alpha}c^2dt^2 + e^{2\beta}(dr^2 + r^2d\theta^2 + \sin^2\theta\, d\epsilon^2), \quad (A.1)$$

where α, β and ϕ are functions of r. In this case the Brans-Dicke field equations assume the following form:

$$(\beta')^2 + 2\alpha'\beta' + \frac{2(\alpha' + \beta')}{r} - \frac{\phi''}{\phi} + \frac{\beta'\phi'}{\phi} - \frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2 = 0, \quad (A.2)$$

$$\alpha'' + \beta'' + (\alpha')^2 + \frac{(\alpha' + \beta')}{r} - \frac{\beta'\phi'}{\phi} - \frac{\phi'}{\phi r} + \frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2 = 0, \quad (A.3)$$

$$2\beta'' + (\beta')^2 + 4\frac{\beta'}{r} - \frac{\alpha'\phi'}{\phi} + \frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2 = 0, \quad (A.4)$$

$$\phi'' + 2\frac{\phi'}{r} + (\alpha' + \beta')\phi' = 0, \quad (A.5)$$

where a prime is used to denote differentiation with respect to r.

To begin we set

$$\gamma = \alpha + \beta \quad (A.6)$$

which in turn permits us to rewrite equations (A.2), (A.4) and (A.5) as follows:

$$\left(2\gamma'\beta' - (\beta')^2 + 2\frac{\gamma'}{r}\right) = \frac{\phi''}{\phi} - \frac{\beta'\phi'}{\phi} + \frac{\omega}{2}\frac{(\phi')^2}{\phi^2}, \quad (A.7)$$

$$\left(2\beta'' + (\beta')^2 + 4\frac{\beta'}{r}\right) = \frac{\gamma'\phi'}{\phi} - \frac{\phi'\beta'}{\phi} - \frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2, \quad (A.8)$$

and

$$\phi'' + 2\frac{\phi}{r} + \gamma'\phi' = 0, \quad (A.9)$$

respectively. Upon adding (A.2) to (A.3) we find that we may put the resultant expression into the form

$$\phi\left(\gamma'' + (\gamma')^2 + 3\frac{\gamma'}{r}\right) = \phi'' + \frac{\phi'}{r}. \quad (A.10)$$

(It should be noted that the system of equations (A.7)-(A.10) is equivalent to our original system (A.2)-(A.5).)

Equation (A.9) can be immediately integrated to yield

$$\gamma = -\ln\phi' - 2\ln r + \ln K , \tag{A.11}$$

where K is a constant. Using either (A.9) or (A.11) we easily find that γ' and γ'' are given by

$$\gamma' = -\frac{\phi''}{\phi'} - \frac{2}{r} , \tag{A.12}$$

and

$$\gamma'' = -\frac{\phi'''}{\phi'} + \frac{(\phi'')^2}{(\phi')^2} + \frac{2}{r^2} . \tag{A.13}$$

Upon inserting equations (A.12) and (A.13) into (A.10) we obtain the following differential equation for ϕ

$$-\phi'''\phi'\phi + 2\phi(\phi'')^2 + \frac{\phi''\phi'\phi}{r} = (\phi')^2\phi'' + \frac{(\phi')^3}{r} . \tag{A.14}$$

We shall now concentrate upon determining all solutions to the above equation for ϕ.

We begin our search for solutions to (A.14) by setting

$$\phi = Ee^y , \tag{A.15}$$

where E is a constant and

$$y = y(r) . \tag{A.16}$$

Using (A.15) we find that ϕ', ϕ'' and ϕ''' are given by

$$\phi' = Ey'e^y , \tag{A.17}$$

$$\phi'' = E(y'' + (y')^2)e^y , \tag{A.18}$$

and

$$\phi''' = E(y''' + 3y'y'' + (y')^3)e^y . \tag{A.19}$$

If we now insert equations (A.15) and (A.17)-(A.19) into (A.14) we find that y must satisfy

$$-y'''y' + 2(y'')^2 + \frac{y'y''}{r} = 0 . \tag{A.20}$$

In order to simplify the form of the above equation let us set

$$y' = u \;, \tag{A.21}$$

where

$$u = u(r) \;.$$

Using (A.21) we find that (A.20) becomes

$$- u''u + 2(u')^2 + \frac{uu'}{r} = 0 \;. \tag{A.22}$$

To reduce the form of equation (A.22) even further we shall make the following substitution

$$u = H\, e^x \;, \tag{A.23}$$

where H is a constant and

$$x = x(r) \;.$$

Using (A.23) we find that

$$u' = H x' e^x \;, \tag{A.24}$$

and

$$u'' = H(x'' + (x')^2)\, e^x \;, \tag{A.25}$$

and hence equation (A.22) becomes

$$- x'' + (x')^2 + \frac{x'}{r} = 0 \;. \tag{A.26}$$

If we now set

$$x' = v \;, \tag{A.27}$$

we find that (A.26) may be written

$$- v' + v^2 + \frac{v}{r} = 0 \;. \tag{A.28}$$

An immediate solution to (A.28) is simply $v = 0$. To obtain other solutions to (A.28) let us assume that $v \neq 0$. In this case the function w, defined by

$$w = \frac{2r}{v} + r^2 \;, \tag{A.29}$$

is a constant, when v obeys equation (A.28). To see this let us examine w' which may be written as follows:

$$w' = \frac{- 2rv' + 2rv^2 + 2v}{v^2} \;. \tag{A.30}$$

Due to equation (A.28) w' vanishes, and thus w must be a constant which we shall denote by Q. Consequently we have determined a second, non-trivial, solution to (A.28); viz.,

$$v = \frac{2r}{Q - r^2} \quad . \tag{A.31}$$

In order to obtain the solutions to our original differential equation for ϕ, (A.14), we shall now reverse the order of the steps which lead from (A.15) to (A.28). We begin by examining the case $v = 0$ first, and then proceed to examine (A.31) for each of the three cases, $Q<0$, $Q = 0$ and $Q>0$.

<u>Case (i).</u> $v = 0$.

When $v = 0$ we may use equation (A.27), (A.23) and (A.21) to conclude that

$$y = -A\,r + B \ , \tag{A.32}$$

where A and B are constants. Combining (A.32) in conjunction with (A.15) leads to the following solution for ϕ

$$\phi = \phi_o\, e^{-Ar} \ , \tag{A.33}$$

where ϕ_o is a constant.

<u>Case (ii).</u> $v = \dfrac{2r}{Q - r^2}$, $Q < 0$.

In order to handle this case we shall set

$$Q = -\,B^2 \ , \tag{A.34}$$

where B is a non-zero real number. Thus v can be written as follows:

$$v = \frac{-2r}{B^2 + r^2} \quad . \tag{A.35}$$

Using (A.27) we find that x is given by

$$x = -\,\ln(B^2 + r^2) + G \ , \tag{A.36}$$

where G is a constant. Thus we can use (A.23) to conclude that

$$u = \frac{H}{B^2 + r^2} , \qquad (A.37)$$

where H is a constant. From equation (A.21) we may deduce that

$$y = \frac{H}{B} \operatorname{Tan}^{-1}\left(\frac{r}{B}\right) + J , \qquad (A.38)$$

and consequently (A.15) permits us to conclude that in the present case

$$\phi = \phi_o \exp\left(\frac{H}{B} \operatorname{Tan}^{-1}\left(\frac{r}{B}\right)\right) , \qquad (A.39)$$

where J and ϕ_o are constants.

Case (iii). $v = \frac{2r}{Q - r^2}$, $Q = 0$.

In this case our expression for v reduces to

$$v = -\frac{2}{r} . \qquad (A.40)$$

Upon inserting (A.40) into (A.27) we find that

$$x = -2 \ln r + G , \qquad (A.41)$$

where G is a constant. Using equations (A.41), (A.23) and (A.21) we find that

$$y = -\frac{H}{r} + J , \qquad (A.42)$$

and hence equation (A.15) permits us to conclude that

$$\phi = \phi_o \exp\left(-\frac{H}{r}\right) , \qquad (A.43)$$

where H, J and ϕ_o are constants.

Case (iv). $v = \frac{2r}{Q - r^2}$, $Q > 0$.

In order to handle this case let us set

$$Q = B^2 ,$$

where B is a real, non-zero, constant. This permits us to write v as follows:

$$v = \frac{2r}{B^2 - r^2} . \tag{A.44}$$

Upon inserting (A.44) into (A.27) we find

$$x = -\ln(B^2 - r^2) + G , \tag{A.45}$$

where G is a constant. Equations (A.45), (A.23) and (A.21) permit us to conclude that

$$y = -\frac{H}{2B}\ln\left(\frac{r - B}{r + B}\right) + F , \tag{A.46}$$

where H and F are constants. If (A.46) is now substituted into (A.15) we obtain

$$\phi = \phi_o\left(\frac{r - B}{r + B}\right)^{-H/2B} , \tag{A.47}$$

where ϕ_o is a constant.

We shall now proceed to determine the functions α and β which correspond to the function ϕ determined in each of the above cases.

<u>Case (i).</u> $\phi = \phi_o e^{-Ar}$. (A.33)

Upon adding equation (A.7) to (A.8) we find that β satisfies the following linear second order differential equation:

$$2\beta'' + \left(\frac{4}{r} + 2\gamma' + \frac{2\phi'}{\phi}\right)\beta' = \frac{\phi''}{\phi} + \frac{\gamma'\phi'}{\phi} - \frac{2\gamma'}{r} , \tag{A.48}$$

where, as we have previously shown, γ is given by

$$\gamma = -\ln\phi' - 2\ln r + \ln K . \tag{A.11}$$

Using equation (A.33) we find that

$$\phi' = -A\phi_o e^{-Ar} , \tag{A.49}$$

$$\phi'' = A^2\phi_o e^{-Ar} , \tag{A.50}$$

and

$$\gamma = Ar - 2\ln r + \ln\left(\frac{-K}{A\phi_o}\right) . \tag{A.51}$$

Equations (A.22) and (A.49)-(A.51) permit us to rewrite (A.48) as follows (in our case):

$$\beta'' = \frac{2}{r^2} . \qquad (A.52)$$

Consequently β is given by

$$\beta = -2 \ln r + G r + \ln P , \qquad (A.53)$$

where G and P are constants.

Since

$$\gamma = \alpha + \beta \quad ,$$

we may use equations (A.51) and (A.53) to conclude that in the present case

$$\alpha = (A - G) r + \ln \left(\frac{-K}{A \phi_o P} \right) . \qquad (A.54)$$

Let us now redefine our constants as follows:

$$P = e^{A} B^2 , \alpha_o = \ln \left(\frac{-K}{A \phi_o P} \right) , \quad A = \frac{D}{B} \text{ and } G = \left(\frac{D + 1}{B} \right) . \qquad (A.55)$$

For this choice of constant we find that

$$\alpha = \alpha_o - \frac{r}{B} , \qquad (A.56)$$

$$\beta = \beta_o - 2 \ln \left(\frac{r}{B} \right) + \left(\frac{D + 1}{B} \right) r , \qquad (A.57)$$

and

$$\phi = \phi_o \exp \left(-\frac{Dr}{B} \right) . \qquad (A.58)$$

The constant D appearing in the above expressions is not arbitrary. To see this we simply have to insert (A.56)-(A.58) into either equation (A.7) or (A.8). Upon doing so we easily find that

$$\frac{D^2(\omega + 2)}{2} + D + 1$$

must vanish if the above expressions for α, β and ϕ are to yield a solution to equations (A.7)-(A.10). Thus we may conclude that a solution to equations (A.2)-(A.5) is furnished by α, β and ϕ, as given by (A.56)-(A.58) respectively, provided

$$D = \frac{-1 \pm \sqrt{-2\omega - 3}}{\omega + 2} . \tag{A.59}$$

This corresponds to Brans's third solution (9.21).

Case (ii). $\phi = \phi_0 \exp\left(\frac{H}{B} \mathrm{Tan}^{-1}\left(\frac{r}{B}\right)\right)$. (A.39)

Using (A.39) we find that

$$\phi' = \frac{H\phi}{B^2 + r^2} , \tag{A.60}$$

$$\phi'' = \frac{-2Hr\phi}{(B^2 + r^2)^2} + \frac{H^2\phi}{(B^2 + r^2)^2} , \tag{A.61}$$

and consequently

$$\frac{\phi'}{\phi} = \frac{H}{B^2 + r^2} , \quad \frac{\phi''}{\phi} = \frac{-2Hr + H^2}{(B^2 + r^2)^2} . \tag{A.62}$$

Upon inserting (A.60) into (A.11) we find that γ is given by

$$\gamma = -\ln(\phi_0 H) + \ln(B^2 + r^2) - \frac{H}{B} \mathrm{Tan}^{-1} \frac{r}{B} - 2\ln r + \ln K . \tag{A.63}$$

From (A.63) we find

$$\gamma' = \frac{2r}{B^2 + r^2} - \frac{H}{B^2 + r^2} - \frac{2}{r} . \tag{A.64}$$

If we now insert (A.62) and (A.64) into (A.48) we see that β satisfies the following differential equation:

$$\beta'' + \frac{2r}{B^2 + r^2}\beta' = \frac{-2}{B^2 + r^2} + \frac{2}{r^2} . \tag{A.65}$$

To solve the above differential equation we set

$$W = \beta' , \tag{A.66}$$

and thus obtain

$$W' + \frac{2rW}{B^2 + r^2} = \frac{-2}{B^2 + r^2} + \frac{2}{r^2} . \tag{A.67}$$

Using the standard techniques for handling a linear first order differential equation we find

$$W = \frac{-2B^2}{r(B^2 + r^2)} + \frac{G}{B^2 + r^2} , \qquad (A.68)$$

where G is a constant. Upon combining (A.66) with (A.68) we find

$$\beta = \ln\left(\frac{B^2 + r^2}{r^2}\right) + \frac{G}{B} \operatorname{Tan}^{-1}\left(\frac{r}{B}\right) + \ln P , \qquad (A.69)$$

where P is a constant.

Since

$$\gamma = \alpha + \beta$$

we may use equations (A.63) and (A.69) to conclude that in the present case

$$\alpha = -\ln\left(\frac{\phi_o HP}{K}\right) - \frac{(H+G)}{B} \operatorname{Tan}^{-1}\left(\frac{r}{B}\right) . \qquad (A.70)$$

We shall now choose to redefine our constants as follows:

$$\alpha_o = -\ln\left(\frac{\phi_o HP}{K}\right) ; \ \beta_o = \ln P ; \ H = \frac{2BD}{\Lambda} \text{ and } G = -\frac{2B(D+1)}{\Lambda} . \qquad (A.71)$$

Thus we now find that α, β and ϕ may be written in the following form:

$$\alpha = \alpha_o + \frac{2}{\Lambda} \operatorname{Tan}^{-1}\left(\frac{r}{B}\right) ; \qquad (A.72)$$

$$\beta = \beta_o - \frac{2(D+1)}{\Lambda} \operatorname{Tan}^{-1} \frac{r}{B} - \ln\left(\frac{r^2}{r^2 + B^2}\right) ; \qquad (A.73)$$

and

$$\phi = \phi_o \exp\left(\frac{2D}{\Lambda} \operatorname{Tan}^{-1}\left(\frac{r}{B}\right)\right) . \qquad (A.74)$$

As in the previous case our field equations impose a constraint upon our choice of constants. In the present case we shall show that Λ^2 must be given by

$$\Lambda^2 = D(1 - \frac{\omega D}{2}) - (D+1)^2 . \qquad (A.75)$$

Using equations (A.62), (A.64) and (A.71) we find that

$$\frac{\phi'}{\phi} = \frac{2BD}{\Lambda(r^2 + B^2)} , \qquad (A.76)$$

and

$$\gamma' = \frac{-2BD}{\Lambda(r^2 + B^2)} - \frac{2}{r} + \frac{2r}{r^2 + B^2} . \qquad (A.77)$$

From equation (A.73) we find that

$$\beta' = \frac{-2(D+1)B}{\Lambda(r^2 + B^2)} - \frac{2}{r} + \frac{2r}{r^2 + B^2} , \qquad (A.78)$$

and

$$\beta'' = \frac{4(D+1)Br}{\Lambda(r^2 + B^2)^2} + \frac{2}{r^2} + \frac{2}{r^2 + B^2} - \frac{4r^2}{r^2 + B^2} . \qquad (A.79)$$

If we now insert equations (A.76)-(A.79) into (A.8) we find:

$$\frac{4}{r^2 + B^2} - \frac{4r^2}{(r^2 + B^2)^2} + \frac{4B^2(D+1)^2}{\Lambda^2(r^2 + B^2)^2} =$$

$$= \frac{4B^2D}{\Lambda^2(r^2 + B^2)^2} - \frac{2\omega B^2D^2}{\Lambda^2(r^2 + B^2)^2} . \qquad (A.80)$$

Upon multiplying (A.80) through by $\Lambda^2(r^2 + B^2)^2$ we obtain

$$4\Lambda^2(r^2 + B^2) - 4r^2\Lambda^2 + 4B^2(D+1)^2 = 4B^2D - 2\omega B^2D^2, \qquad (A.81)$$

which reduces to

$$\Lambda^2 = -\frac{D^2}{2}(\omega + 2) - D - 1 . \qquad (A.82)$$

(A.82) may be rewritten in the following form

$$\Lambda^2 = D(1 - \frac{D}{2}) - (D+1)^2 , \qquad (A.83)$$

which agrees with (A.75). (If we had used equation (A.7) rather than (A.8) we would have still obtained (A.75).)

Thus we have shown that the functions α, β and ϕ given by equations (A.72), (A.73) and (A.74) respectively will provide a solution to equations (A.2)-(A.5) provided that the constants Λ and D are related as in equation (A.75) and $\Lambda^2 > 0$. The latter demand must be made upon Λ in order to guarantee that our expressions for α, β and ϕ are finite and real.

The above solution corresponds to Brans's second solution (9.20).

Case (iii). $\phi = \phi_0 \exp(-\frac{H}{r})$. (A.43)

Using (A.43) we find that

$$\phi' = \frac{H\phi}{r^2}, \quad (A.84a)$$

$$\phi'' = -\frac{2H\phi}{r^3} + \frac{H^2}{r^4}\phi, \quad (A.84b)$$

and consequently

$$\frac{\phi'}{\phi} = \frac{H}{r^2}, \quad (A.85a)$$

and

$$\frac{\phi''}{\phi} = -\frac{2H}{r^3} + \frac{H^2}{r^4}. \quad (A.85b)$$

Upon inserting (A.84a) into (A.11) we find that γ is given by

$$\gamma = \ln\left(\frac{K}{H\phi_0}\right) + \frac{H}{r}, \quad (A.86a)$$

and thus

$$\gamma' = -\frac{H}{r^2}. \quad (A.86b)$$

If we now substitute (A.85a), (A.85b) and (A.86b) into (A.48) we find that, in the present case, β is governed by the following differential equation

$$\beta'' + \frac{2\beta'}{r} = 0. \quad (A.87)$$

To solve (A.87) we first set

$$W = \beta', \quad (A.88a)$$

and as a result (A.87) becomes

$$W' = -\frac{2W}{r} .$$

A simple integration then yields

$$W = \frac{G}{r^2} , \qquad \text{(A.88b)}$$

and consequently β is given by

$$\beta = -\frac{G}{r} + \ln P , \qquad \text{(A.89)}$$

where G and P are constants.

Due to the fact that

$$\gamma = \alpha + \beta$$

we may use equations (A.86a) and (A.89) to conclude that

$$\alpha = \frac{(H+G)}{r} + \ln\left(\frac{K}{H\phi_o P}\right) . \qquad \text{(A.90)}$$

Let us now redefine our constants as follows:

$$\alpha_o = \ln\left(\frac{K}{H\phi_o P}\right) ; \beta_o = \ln P \;\; ; \; G = -\frac{(D+1)}{B} \text{ and } H = \frac{D}{B} \; . \qquad \text{(A.91)}$$

Using (A.91) we find that in the present case α, β and ϕ may be written in the form

$$\alpha = \alpha_o - \frac{1}{Br} , \qquad \text{(A.92)}$$

$$\beta = \beta_o + \frac{(D+1)}{Br} , \qquad \text{(A.93)}$$

and

$$\phi = \phi_o \exp\left(\frac{-D}{Br}\right) . \qquad \text{(A.94)}$$

Once again we have to check and see if the above expressions for α, β and ϕ are compatible with equations (A.7) and (A.8). It will be shown that they will be consistent with these equations provided

$$D = \frac{-1 \pm \sqrt{-2\omega - 3}}{\omega + 2} . \qquad \text{(A.95)}$$

In order to establish (A.95) we shall need expressions for β', γ', $\frac{\phi'}{\phi}$ and $\frac{\phi''}{\phi}$. Using equations (A.85a), (A.85b), (A.86b), (A.88a), (A.88b) and (A.91) we find:

$$\beta' = -\frac{(D+1)}{Br^2}, \quad \gamma' = \frac{-D}{Br^2}, \quad \frac{\phi'}{\phi} = \frac{D}{Br^2},$$

and (A.96)

$$\frac{\phi''}{\phi} = \frac{-2D}{Br^3} + \frac{D^2}{B^2r^4}.$$

Upon substituting (A.96) into (A.7) we find:

$$D^2(1+\frac{\omega}{2}) + D + 1 = 0. \tag{A.97}$$

Equation (A.97) implies that D must be given by

$$D = \frac{-1 \pm \sqrt{-2\omega - 3}}{\omega + 2}, \tag{A.95}$$

if α, β and ϕ, as given by (A.92)-(A.94) respectively, are to yield a solution to equations (A.2)-(A.5). (The same constraint upon D would have been found if we had used equation (A.8) rather than (A.7).)

The solution which we have obtained in this case corresponds to Brans's fourth solution (9.22).

<u>Case (iv).</u> $\phi = \phi_0\left(\frac{r-B}{r+B}\right)^{-H/2B}$. (A.47)

As in our three previous cases we begin by determining ϕ' and ϕ'' which we easily find to be

$$\phi' = -H\phi_0\left(\frac{r-B}{r+B}\right)^{-(\frac{H}{2B}+1)}\frac{1}{(B+r)^2}; \tag{A.98}$$

and

$$\phi'' = \phi_0 H(H+2B)\left(\frac{r-B}{r+B}\right)^{-(\frac{H}{2B}+2)}\frac{1}{(r+B)^4} +$$

$$+ \frac{2\phi_0 H}{(r+B)^3}\left(\frac{r-B}{r+B}\right)^{-(\frac{H}{2B}+1)}. \tag{A.99}$$

Upon combining equations (A.47), (A.98) and (A.99) we find

$$\frac{\phi'}{\phi} = \frac{-H}{r^2 - B^2} , \qquad (A.100)$$

and

$$\frac{\phi''}{\phi} = \frac{H(H+2B)}{(r^2-B^2)^2} + \frac{2H}{(r+B)(r^2-B^2)} . \qquad (A.101)$$

Using equations (A.11) and (A.98) we see that in the present case

$$\gamma = -\ln\left(\frac{-\phi_o H}{K}\right) + \left(\frac{H}{2B} + 1\right)\ln\left(\frac{r-B}{r+B}\right) + 2\ln(B+r) - 2\ln r , \qquad (A.102)$$

and consequently γ' is given by

$$\gamma' = \frac{(H+2B)}{r^2 - B^2} + \frac{2}{B+r} - \frac{2}{r} . \qquad (A.103)$$

Upon inserting (A.100), (A.101) and (A.103) into (A.48) we find:

$$2\beta'' + \left(\frac{4}{r} + \frac{2(H+2B)}{r^2-B^2} + \frac{4}{B+r} - \frac{4}{r} - \frac{2H}{r^2-B^2}\right)\beta' =$$

$$= \frac{H(H+2B)}{(r^2-B^2)^2} + \frac{2H}{(r+B)(r^2-B^2)} - \frac{H(H+2B)}{(r^2-B^2)^2} +$$

$$- \frac{2H}{(r+B)(r^2-B^2)} + \frac{2H}{r(r^2-B^2)} - \frac{2(H+2B)}{r(r^2-B^2)} +$$

$$- \frac{4}{r(B+r)} + \frac{4}{r^2} ,$$

which simplifies to

$$\beta'' + \frac{2r}{r^2-B^2}\beta' = \frac{-2B^2}{r^2(r^2-B^2)} . \qquad (A.104)$$

In order to solve (A.104), we begin by setting

$$W = \beta' , \qquad (A.105)$$

and consequently (A.104) becomes

$$W' + \frac{2r}{r^2-B^2} W = \frac{-2B^2}{r^2(r^2-B^2)} . \qquad (A.106)$$

Using the standard techniques for dealing with a linear first order differential equation we find that W is given by

$$W = \frac{2B^2}{r(r^2 - B^2)} + \frac{G}{r^2 - B^2}, \qquad (A.107)$$

where G is a constant. From equations (A.105) and (A.107) we obtain, through a straightforward integration,

$$\beta = \ln P + \frac{G}{2B} \ln\left(\frac{r - B}{r + B}\right) + \ln\left(\frac{r^2 - B^2}{r^2}\right), \qquad (A.108)$$

where P is a constant.

Due to the fact that

$$\gamma = \alpha + \beta$$

we may use equations (A.102) and (A.108) to find

$$\alpha = -\ln\left(\frac{-\phi_o HP}{K}\right) + \left(\frac{H - G}{2B} + 2\right)\ln\left(\frac{r - B}{r + B}\right). \qquad (A.109)$$

We shall now choose to redefine our constants as follows:

$$\alpha_o = -\ln\left(\frac{-\phi_o HP}{K}\right); \ \beta_o = \ln P; \ \frac{1}{\lambda} = \frac{H - G}{2B} + 2; \text{ and } \frac{-H}{2B} = \frac{D}{\lambda}. \qquad (A.110)$$

Using (A.110) we find that in the present case α, β and ϕ are given by

$$\alpha = \alpha_o + \frac{1}{\lambda} \ln\left(\frac{r - B}{r + B}\right), \qquad (A.111)$$

$$\beta = \beta_o + \left(\frac{\lambda - D - 1}{\lambda}\right)\ln\left(\frac{r - B}{r + B}\right) + 2\ln\left(\frac{r + B}{r}\right), \qquad (A.112)$$

and

$$\phi = \phi_o \left(\frac{r - B}{r + B}\right)^{D/\lambda}. \qquad (A.113)$$

As in our three previous cases (A.111)-(A.113) will not yield a solution to equations (A.7)-(A.10) for an arbitrary choice of constants. In the present case we shall show that λ and D must be related by

$$\lambda^2 = (D + 1)^2 - D(1 - \frac{\omega D}{2}). \qquad (A.114)$$

In order to establish (A.114) we shall need expressions for β', β'', γ' and $\frac{\phi'}{\phi}$. Using our previous results we find that:

$$\beta' = \left(\frac{\lambda - D - 1}{\lambda}\right)\frac{2B}{r^2 - B^2} - \frac{2B}{r(r+B)} , \qquad (A.115)$$

$$\beta'' = \left(\frac{\lambda - D - 1}{\lambda}\right)\frac{-4rB}{(r^2 - B^2)^2} - \frac{2}{(r+B)^2} + \frac{2}{r^2} , \qquad (A.116)$$

$$\gamma' = \left(\frac{\lambda - D}{\lambda}\right)\frac{2B}{r^2 - B^2} + \frac{2}{r+B} - \frac{2}{r} , \qquad (A.117)$$

and

$$\frac{\phi'}{\phi} = \frac{2BD}{(r^2 - B^2)} . \qquad (A.118)$$

Upon inserting (A.115)-(A.118) into equation (A.8) and then multiplying the resultant expression by

$$r^2(r+B)^2(r-B)^2$$

we find after simplification that

$$-4B^2r^2 + \left(\frac{D^2}{\lambda^2} + \frac{1}{\lambda^2} + \frac{2D}{\lambda^2}\right)4B^2r^2 = \frac{-4B^2r^2D^2}{\lambda^2} + \left(\frac{D}{\lambda^2} + \frac{1}{\lambda^2}\right)4B^2r^2D - \frac{2\omega B^2D^2r^2}{\lambda^2} . \qquad (A.119)$$

The above expression reduces to

$$\lambda^2 = D^2(1 + \frac{\omega}{2}) + D + 1 , \qquad (A.120)$$

which may be rewritten in the following form

$$\lambda^2 = (D+1)^2 - D(1 - \frac{\omega D}{2}) . \qquad (A.114)$$

(This same result can be obtained using equation (A.7) rather than (A.8).)

In order to guarantee that our solutions for α, β and ϕ be finite and real we shall demand that

$$\lambda^2 > 0 . \qquad (A.121)$$

In summary we have shown that in the present case when

α, β and ϕ are given by equations (A.111)-(A.113) respectively, and when λ and D are related by (A.114) with

$$\lambda^2 > 0$$

then we have found a solution to equations (A.2)-(A.5). This solution corresponds to Brans's first solution (9.19).

Thus we have constructed all of the exact solutions to the system of differential equations represented by (A.2)-(A.5). We list our solutions below.

Case (i).

$$\phi = \phi_0 \exp\left(-\frac{Dr}{B}\right) , \qquad (A.58)$$

$$\alpha = \alpha_0 - \frac{r}{B} , \qquad (A.56)$$

$$\beta = \beta_0 - 2\ln\left(\frac{r}{B}\right) + \left(\frac{D+1}{B}\right)r, \qquad (A.57)$$

where

$$D = \frac{-1 \pm \sqrt{-2\omega - 3}}{\omega + 2} . \qquad (A.59)$$

Case (ii).

$$\phi = \phi_0 \exp\left(\frac{2D}{\Lambda} \operatorname{Tan}^{-1}\left(\frac{r}{B}\right)\right) , \qquad (A.74)$$

$$\alpha = \alpha_0 + \frac{2}{\Lambda} \operatorname{Tan}^{-1}\left(\frac{r}{B}\right) , \qquad (A.72)$$

$$\beta = \beta_0 - \frac{2(D+1)}{\Lambda} \operatorname{Tan}^{-1}\left(\frac{r}{B}\right) - \ln\left(\frac{r^2}{r^2 + B^2}\right) , \qquad (A.73)$$

where

$$\Lambda^2 = D\left(1 - \frac{\omega D}{2}\right) - (D+1)^2 > 0 . \qquad (A.75)$$

Case (iii).

$$\phi = \phi_o \exp\left(-\frac{D}{Br}\right) , \qquad (A.94)$$

$$\alpha = \alpha_o - \frac{1}{Br} , \qquad (A.92)$$

$$\beta = \beta_o + \frac{D+1}{Br} , \qquad (A.93)$$

where

$$D = \frac{-1 \pm \sqrt{-2\omega - 3}}{\omega + 2} . \qquad (A.95)$$

Case (iv).

$$\phi = \phi_o \left(\frac{r-B}{r+B}\right)^{D/\lambda} , \qquad (A.113)$$

$$\alpha = \alpha_o + \frac{1}{\lambda} \ln\left(\frac{r-B}{r+B}\right) , \qquad (A.111)$$

$$\beta = \beta_o + \left(\frac{\lambda - D - 1}{\lambda}\right) \ln\left(\frac{r-B}{r+B}\right) + 2 \ln\left(\frac{r+B}{r}\right) , \qquad (A.112)$$

where

$$\lambda^2 = (D+1)^2 - D\left(1 - \frac{\omega D}{2}\right) > 0 . \qquad (A.114)$$

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