SYMMETRIES

by

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Let the time and effort invested in this work be dedicated to the memory of my grandfathers, Bota Sandor and Foldes Mor.

ABSTRACT

Automorphisms of graphs, hypergraphs and digraphs are investigated.

The invariance of the chromatic polynomial in the rotor effect is disproved. New invariance results are obtained.

It is shown that given any integer $k > 2$, almost every finite group acts as the regular full automorphism group of some k-uniform hypergraph.

Permutation groups that can be represented as automorphism groups of digraphs are characterized.

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I would like to thank my supervisor, Professor W.T. Tutte, for his constant help throughout this work. It was he who introduced me to the rotor problem, which appears to me as a source of intuitive ideas and methods. In matters of publications and lectures I found the advice of my supervisor of great value. At a more material and human level, I am indebted to Professor Tutte for his intervention at a moment when the instability of my financial situation threatened the continuation of this research.

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INTRODUCTION

The main concern of this thesis is automorphisms of a class of incidence structures covered by the concept of general graphs and of their sub-structures. General graphs are defined so as to include graphs and hypergraphs, as well as digraphs, without restriction on the cardinality of the vertex set or the multiplicity of the edges. We try to introduce the least possible amount of new definitions and use a standard terminology, as far as such can be extracted from the abundant literature appearing on both sides of the Atlantic and as far as it is compatible with the specific purposes of this work. Chapter 0 provides some basic concepts and definitions.

Asymmetry is one of the most recurrent ideas in the investigation of symmetries. Indeed, the rotationally symmetric rotors must be reflectionally asymmetric if they are to produce a non-trivial graph transformation leaving invariant certain functions. Securing the existence of appropriate asymmetric independent sets, we disprove the invariance in the rotor effect of the second highest coefficient of the chromatic polynomial. The rotor effect occupies the first chapter of this thesis, containing also special results on planar rotors and extensions of some known invariance results.

The second chapter deals with symmetries of a general graph which are realizable when it is embedded into another general graph.

The third chapter is based on our collaboration with Professor Singhi of the Tata Institute of Fundamental Research. It is devoted to representations of regular permutation groups by uniform hypergraphs. The main result here is that given any integer k larger than 2 , all but a finite number of finite groups act as the regular full automorphism group of some k-uniform hypergraph. Again, it presents less difficulty to secure that all translations induce a symmetry than to ensure that no non-trivial symmetry has a fixed point. The analogy with the construction of non-trivial rotors inspired the particular representation of regular cyclic permutation groups by 3-uniform hypergraphs given in section 3, contrasting the fact that these groups have been the earliest examples of groups having no graphical regular representation.

Last, but not least, in the fourth chapter permutation groups that can be represented as automorphism groups of digraphs are characterized. The case of abelian permutation groups is examined extensively, and the complete equivalence of the problem of finding a digraph with a given abelian group and the problem of the validity of a generalized Chinese remainder theorem for the family of stabilizer subgroups is established. Some classes of permutation groups naturally arising in algebra are also examined.

The axiom of choice [H8 ,J1] will be assumed. As a consequence of Zermelo's theorem [Z1], all transfinite cardinal numbers will be alephs.

CHAPTER 0

BASIC CONCEPTS AND DEFINITIONS

0.1. The elements of a <u>set</u> are thought of as occurring without any particular order and without repetitions. Thus for any two objects x and y, distinct or not, $\{x,y\} = \{y,x\}$ and if $x = y$ also $\{x,y\} = \{x\} = \{y\}$. We denote by (x,y) an <u>ordered pair</u>, by (x,y,z) an ordered triplet. These objects are thought of as essentially different from sets. In particular for every set V, the set $P(V)$ of all subsets of V is disjoint from the set of all ordered pairs of elements of V. This disjointness will be needed to make an unambiguous distinction between oriented darts and unoriented lines of a general graph, to be defined later.

Given a set S and a set V, the <u>cartesian product</u> $S\times V$ is defined by

$$S\times V = \{(x,t) \mid x\in S,\ t\in V\}$$

Let F be a subset of $S\times V$ such that for every $t\in V$ there is exactly one $x\in S$ with $(x,t)\in F$. Then F is called a <u>family</u> of elements of S

indexed by V. We also write $x=x_t$ if $(x,t)\in F$ and $F=(x_t)_{t\in V}$. F is said to be a finite family if the set $\{x_t \mid t\in V\}$ is finite, even if V is an infinite indexing set.

0.2. A function, or mapping, from a set V to a set S can be defined as an ordered pair (F,S), where F is a family of elements of S indexed by V. Injectivity, surjectivity and bijectivity are defined in the usual way.

For any ordinal numbers α and β the set of ordinal numbers γ such that $\alpha\leq\gamma$ and $\gamma\leq\beta$ is denoted by $[\alpha,\beta]$. We write $[o,\beta] \setminus \{\beta\}=W(\beta)$. Thus, ω denoting the first transfinite ordinal number, $W(\omega)=\mathbb{N}$, the set of natural numbers. A cardinal number is an ordinal number α such that there exists no bijective mapping from $W(\alpha)$ to any $W(\beta),\beta<\alpha$. The axiom of choice being assumed, for every set S there is a unique cardinal number k such that there exists a bijective mapping from S to W(k). k is then called the cardinality, or size, of S and we write $|S|=k$. The cardinality of a finite set S is the number of elements of S.

The set of integers will be denoted by $\mathbb{Z}$.

0.3. A permutation of a set V is a bijective function from V to itself. The set of all permutations of V is denoted by S_V. The product $\sigma\tau$ of two permutations σ and τ is defined by

$$\sigma\tau(x) = \sigma(\tau(x)),$$

for every $x \in V$. Then S_V is a group under this binary operation, called the symmetric group on V. (We shall always denote a group and the set of its elements by the same symbol.) Any subgroup of the symmetric group on V is a permutation group on V.

0.4. If G is a permutation group on a set V and U is a subset of V such that $\sigma(U) \subseteq U$ for every $\sigma \in G$, then U is a constituent of G. It is then easy to see that $\sigma(U)=U$ for every $\sigma \in G$. In this case , for a permutation $\sigma \in G$ we can define the restriction $\sigma|U$ of σ to U to be the permutation of U given by

$$\sigma|U(x) = \sigma(x),$$

for every $x \in U$. We also write

$$G|U \doteq \{\sigma|U \mid \sigma\in G\}$$

$G|U$ is a permutation group on U and the restriction mapping r: $G \to G|U$ defined by

$$r(\sigma) \doteq \sigma|U,$$

for every $\sigma\in G$, is a surjective group homomorphism from G to $G|U$. If r is injective, the U is called a <u>faithful constituent</u> and the groups G and $G|U$ are isomorphic, $G \approx G|U$ [W6].

0.5. A permutation group G on a set V is called <u>transitive</u> if its only constituents are V and the empty set $\emptyset$. For arbitrary G, a non empty constituent U such that $G|U$ is transitive is called an <u>orbit</u> of G. (In the case when G is cyclic, i.e. generated by a permutation σ of V, $G \doteq (\sigma)$; an orbit U of G is also called an orbit of σ.) Every element $x\in V$ belongs to exactly one orbit, called the orbit of x.

For $x\in V$, the stabilizer G_x of x in G is the subgroup of G defined by

$$G_x = \{\sigma\in G \mid \sigma(x)=x\}$$

If G is transitive and the stabilizer of some element $x \in V$ is trivial, then G is said to be <u>regular</u>. In this case all the stabilizers G_x, $x \in V$, are trivial.

0.6. A <u>general graph</u> G is an ordered triplet (V,E,ψ), where $V=V(G)$ and $E=E(G)$ are sets, called the set of <u>vertices</u> and the set of <u>edges</u>, respectively, and $\psi=\psi_G$ is a function from E to $(\mathcal{P}(V)\setminus\{\emptyset\})\cup V^2$ called the <u>incidence function</u>. If A is an edge and $\psi(A)\in\mathcal{P}(V)$, then A is called a <u>line</u>, if $\psi(A)\in V^2$, then A is called a <u>dart</u>. If A is a dart and $\psi(A)=(x,y)$, then A is said to be a <u>dart from x to y</u>, or a dart with <u>tail</u> x and <u>head</u> y. A <u>loop</u> is a line A such that $|\psi(A)|=1$. A <u>link</u> is a line A such that $|\psi(A)|=2$. G is called strict if ψ is injective. We deviate from the general practice in defining G to be <u>simple</u> if

$$E \subseteq (\mathcal{P}(V)\setminus \{\emptyset\}) \cup V^2$$

and if ψ is the identity function. Thus a simple general graph is always strict.

Given an arbitrary G, we define the <u>underlying simple general graph</u> $s(G)=(\bar{V},\bar{E},\bar{\psi})$ by

$$\bar{V} = V(G)$$

$$\bar{E} = \{\psi(A) \mid A \in E(G)\},$$

$\bar{\psi}$ being the identity function. Generally, to define a simple general graph it will suffice to define its vertices and edges.

G is a <u>finite general graph</u> if both sets V(G) and E(G) are finite.

A <u>hypergraph</u> is a general graph having no darts, a <u>digraph</u> is a general graph having no lines. A <u>graph</u> is a hypergraph with $|\psi(A)| \leq 2$ for every line A. Given a non-zero cardinal number k, a hypergraph is called <u>k-uniform</u> if $|\psi(A)| = k$ for every line A.

0.7. A <u>subsystem</u> of a general graph $G = (V, E, \psi)$ is a general graph $H = (V_1, E_1, \psi_1)$ such that $V_1 \subseteq V$, $E_1 \subseteq E$ and ψ_1 is the restriction of ψ to E_1. H is called an <u>induced subsystem</u> of G if for every edge A of G

$$\psi(A) \in P(V_1) \cup V_1^2 \Longrightarrow A \in E_1.$$

We also write in this case $H = G[V_1]$ and H is said to be the subsystem induced by V_1. A subsystem H of G is a spanning subsystem if $V(H) = V(G)$.

A subsystem of a hypergraph, a graph, or a digraph is called a sub-hypergraph, a subgraph, a sub-digraph, respectively. Induced and spanning sub-hypergraphs, subgraphs and sub-digraphs are defined accordingly.

A vertex x of G and an edge A are incident if

$$A \notin E(G[V(G) \setminus \{x\}]) .$$

The degree $d(x)$ is the cardinality of the set of edges incident with x. $d^+(x)$, respectively $d^-(x)$, denotes the cardinality of the set of darts having x as tail, respectively as head. The neighbourhood $N_G(x) = N(x)$ of x is the set of vertices $y \neq x$ incident with an edge that is incident with x. The elements of $N(x)$ are the neighbours of x. The neighbourhood $N_G(S) = N(S)$ of a subset $S \subseteq V(G)$ is defined by

$$N(S) = (\bigcup_{x \in S} N(x)) \setminus S .$$

For every $x \in V(G)$, $N_G^+(x) = N^+(x)$ denotes the set of those vertices y for which there is a dart from x to y. Dually,

$$N_G^-(x) = N^-(x) = \{ y \in V(G) \mid x \in N^+(y) \} .$$

For every $S \subseteq V(G)$ let

$$N_G^+(S) = N^+(S) = \bigcup_{x \in S} N^+(x)$$

$$N_G^-(S) = N^-(S) = \bigcup_{x \in S} N^-(x) .$$

If S is a subset of $E(G)$, then the subsystem H <u>spanned</u> by S has edge set $E(H) = S$ and vertex set

$$V(H) = \{ x \in V(G) \mid x \text{ is incident with some } A \in S \}.$$

Whenever this does not lead to confusion, to designate a "subsystem induced by a vertex x", or a "subsystem spanned by a singleton $\{A\}$, $A \in E(G)$", we shall simply speak of the "vertex x" or the "edge A".

A subset $S \subseteq V(G)$ is <u>independent</u> if $E(G[S]) = \emptyset$.

A family $(G_i)_{i\in I}$ of general graphs is <u>compatible</u> if for every pair of indices $i, j \in I$ and every $A \in E(G_i) \cap E(G_j)$,

$$\psi_{G_i}(A) = \psi_{G_j}(A) .$$

The union

$$G = \bigcup_{i\in I} G_i$$

can then be defined as the general graph satisfying

(i) $V(G) = \bigcup_{i\in I} V(G_i)$,

(ii) $E(G) = \bigcup_{i\in I} E(G_i)$,

(iii) every G_i , $i \in I$, is a subsystem of G.

0.8. A <u>partition</u> of a set J is a subset π of $P(J)$ such that

(i) the elements of π are pairwise disjoint,

(ii) $\bigcup_{B\in\pi} B = J$,

(iii) $\emptyset \notin \pi$.

The elements of π are called the <u>blocks</u> of the partition. For $x, y \in J$ we write

$$x \equiv y \mod \pi$$

if x and y belong to the same block of π .

A <u>component</u> of a hypergraph H is a non-empty subset C of V(H) such that for every line A of H either $\psi_H(A) \subseteq C$ or $\psi_H(A) \cap C = \emptyset$. C is called a <u>connected component</u> if the only component of H [C] is C. The set π of connected components is a partition of V(H) and we shall write $c(H) = |\pi|$. H is <u>connected</u> if $c(H) \leq 1$.

Let G be a graph. A <u>polygon</u> is a finite subgraph P of G such that

(i) $|V(P)| = |E(P)| > 0$

(ii) no proper subgraph of P has property (i).

A <u>circuit</u> is the edge set of some polygon.

If G is a strict graph, then for every $x, y \in V(G)$, there is at most one line incident with both x and y. If there is such a line A , then we write

$$A = \langle x, y \rangle = \langle y, x \rangle \quad .$$

If we have vertices $v_1, \ldots, v_k$, $k \geq 3$, such that v_i is adjacent to v_{i+1} for every $1 \leq i \leq k-1$ and v_k is adjacent to v_1, then

$$P(v_1, \ldots, v_k)$$

denotes the polygon of G with edge set

$$\{ \langle v_i, v_{i+1} \rangle \mid 1 \leq i \leq k-1 \} \cup \{ \langle v_k, v_1 \rangle \} \quad .$$

If a graph G has a polygon of length at least 3, then the girth of G is the minimum length of such a polygon. Otherwise we say that the girth of G is ∞.

A path in a graph G is defined as a connected subgraph with two vertices of degree 1, all other vertices having degree 2. A path has to be finite.

0.9. An automorphism of a general graph G is a permutation σ of V(G) such that for every $S \subseteq V$

$$|\{ A \in E(G) \mid \psi(A) = S \}| = |\{ A \in E(G) \mid \psi(A) = \sigma(S) \}| \quad ,$$

and for every $x, y \in V$

$$|\{A\in E(G)\mid \psi(A)=(x,y)\}| = |\{A\in E(G)\mid \psi(A)=(\sigma(x),\sigma(y))\}| .$$

The set of all automorphisms of G is denoted by Aut G. This is a permutation group on V(G), called the automorphism group of G.

CHAPTER 1

ON THE ROTOR EFFECT

1.0. All hypergraphs considered in this chapter will be finite.

1.1. Let H be a hypergraph and let J be a subset of V(H). Let π be a partition of J. We define a hypergraph $H(\pi) = (V, E, \psi)$

by

$$V = \{\{x\} \mid x \in V(H) \setminus J\} \cup \pi \quad ,$$

$$E = E(H) \quad ,$$

and

$$\psi(A) = \{ M \in V \mid M \cap \psi_H(A) \neq \emptyset \}$$

for every line $A \in E$.

$H(\pi)$ is thought of as obtained from H by identification of π- congruent vertices.

Let R be a non-null hypergraph, θ an automorphism of R (thought of as a rotation), J an orbit of Θ, v a distinguished vertex in J. Then $\mathcal{R} = (R, \Theta, J, v)$ is called a <u>rotor</u>. J is called the <u>principal orbit</u> and $|J|$ the <u>order</u> of the rotor $\mathcal{R}$. Clearly, if this order is k, then J consists of v, $\Theta(v), \ldots, \Theta^{k-1}(v)$. The automorphism Θ^k is not necessarily the identity, the order of Θ can be a proper multiple of the order of the rotor.

Example. If R is the strict graph depicted in Figure 1, having for vertex set $V(R) = [0, 13]$, then a rotor $\mathcal{R} = (R, \Theta, J, v)$ can be defined by

$$\Theta = (0,1,2)\ (3,4,5)\ (6,10,8,9,7,11)\ (12,13)\ ,$$

$$J = \{0,1,2\}\ ,$$

$$v = 0\ .$$

The order of the rotor $\mathcal{R}$ is 3, while the order of the automorphism Θ of the graph R is 6.

The <u>border</u> is defined as the ordered pair $(R[J], \Theta\,|\,J)$. In the example of the rotor given above, $R[J]$ is the edgeless graph with 3 vertices 0,1,2 and $\Theta\,|\,J$ is the cyclic permutation (0,1,2).

To the rotor (R, Θ, J, v) we associate a mapping $\phi: J \longrightarrow J$, called <u>reflection</u>, given by

$$\phi\ (\Theta^{i}(v)) = \Theta^{-i}\ (v)\ ,$$

for every $0 \leq i < |\ J\ |$. ϕ is an involution.

Let π be a partition of the principal orbit, called a <u>border-partition</u>. We also have the reflected border-partition $\phi\pi$ defined by

$$\phi\pi = \{\ \phi(B)\ |\ B \in \pi\ \}\ .$$

We shall be interested in common properties of $R(\pi)$ and $R(\phi\pi)$.

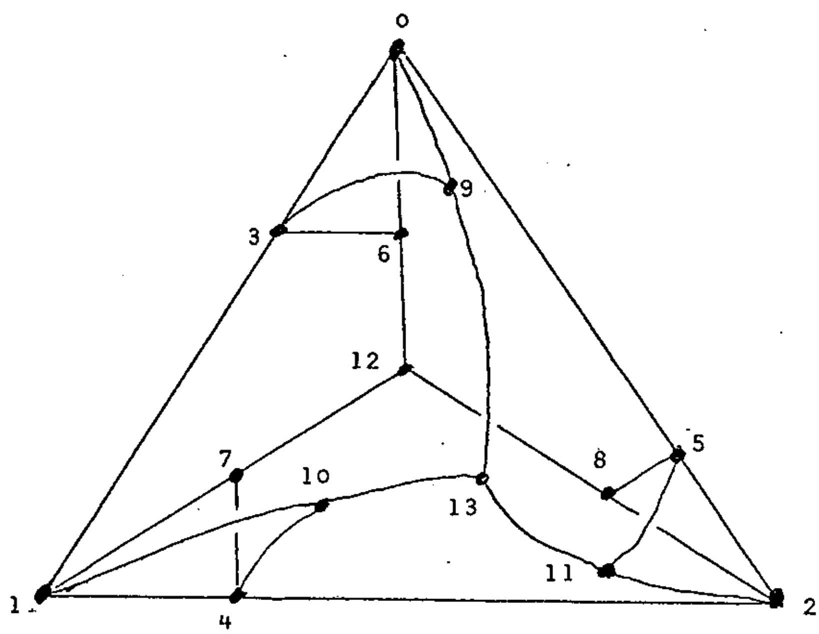


Figure 1

1.2.1. Let H be a hypergraph. Given a positive integer λ , a <u>proper λ-coloring</u> of H is a function $f: V(H) \longrightarrow [1, \lambda]$ such that

$$|f(\psi_H(A))| > 1$$

for every line A of H. The number of proper λ-colorings of H is denoted by $P(H,\lambda)$.

1.2.2. For H as above and $A \in E(H)$, we define a hypergraph $H'_A = (V', E', \psi')$ by

$$V' = V(H),$$

$$E' = E(H) \setminus \{A\},$$

and

$$\psi'(B) = \psi_H(B)$$

for every $B \in E'$.

Also we define a hypergraph $H''_A = (V'', E'', \psi'')$ by

$$V'' = \{\{x\} \mid x \in V(H) \setminus \psi_H(A)\} \cup \{\psi_H(A)\}$$

$$E'' = E(H) \setminus \{A\},$$

and

$$\psi''(B) = \{M \in V'' \mid M \cap \psi_H(B) \neq \emptyset\}$$

for every $B \in E''$.

H'_A can be thought of as obtained from H by deleting the line A and H''_A as obtained from H by contracting A to a single vertex.

As in the case of graphs, for any positive integer λ the set of proper λ-colorings of H is a subset of the set of proper λ-colorings of H'_A . Also, a proper λ-coloring f of H'_A is not a proper λ-coloring of H if and only if

$$|f(\psi_H(A))| = 1 \quad .$$

But it is not difficult to verify that there is a bijective correspondence between proper λ-colorings f of H'_A satisfying this equality and proper λ-colorings of H''_A. Consequently the familiar recursion formula

$$P(H,\lambda) = P(H'_A,\lambda) - P(H''_A,\lambda)$$

holds also for hypergraphs.

It follows that $P(H,\lambda)$ is a polynomial in λ with integer coefficients, having degree $|V|$ unless $|\psi_H(A)| = 1$ for some line A of H, in which case $P(H, \lambda)$ is identically zero. $P(H, \lambda)$ is called the chromatic polynomial of H. It was introduced for graphs by George D. Birkhoff [B 7] , for hypergraphs by C. Benzaken [B 4] as a polynomial whose coefficients count the number of unlabelled colorings, and independently by V. Chvatal [C 6] in the form presented above.

1.2.31. PROPOSITION. Let R and S be two sub-hypergraphs of a hypergraph H suchthat

$$E(R) \cup E(S) = E(H) \quad ,$$

$$E(R) \cap E(S) = \emptyset \quad ,$$

$$V(R) \cup V(S) = V(H) \quad .$$

Let $J=V(R) \cap V(S)$. For a spanning sub-hypergraph T of S, let π_T be the partition of J each block of which is the intersection of J with some connected component of T. Then

$$P(H,\lambda) = \sum_T (-1)^{|E(T)|} \lambda^{c(T)-|\pi_T|} P(R(\pi_T), \lambda) \quad ,$$

the summation ranging over all spanning sub-hypergraphs T of S.

Proof: By induction on $|E(S)|$.

The expansion clearly holds if $|E(S)| = 0$, because in this case the only spanning sub-hypergraph of S is S, $(-1)^0 = 1$, H consists of R and $c(S) - |\pi_S| =$

$|V(S)|-|J|$ isolated vertices, and $P(R(\pi_S),\lambda) = P(R,\lambda)$.

If $|E(S)| \geq 1$, assume that the expansion is valid for lesser values of $|E(S)|$ and let $A \in E(S)$. Since the expansion is valid for $P(H'_A, \lambda)$ and $P(H''_A, \lambda)$, it follows from the recursion formula

$$P(H,\lambda) = P(H'_A,\lambda) - P(H''_A,\lambda)$$

that it holds also for $P(H,\lambda)$.

A similar expansion in the special case of chromatic polynomials of planar maps appears in the paper of George D. Birkhoff and D.C. Lewis [B 8] . Also H.Crapo [C 4] reports an analagous expansion for the more general coboundary polynomial of graphs. Our expansion was motivated by the conjectured invariance of the chromatic polynomial in the rotor effect, to be treated in the sequel, and was obtained independently.

A particular case of proposition 1.2.31 gives the following

1.2.32. PROPOSITION. Let R and S be spanning sub-hypergraphs of a hypergraph H, $E(R) \cup E(S) = E(H)$, $E(R) \cap E(S) = \emptyset$. For every partition π of $V(H)$ and every non-negative integer i let $t_{\pi i}$ denote the number of spanning sub-hypergraphs T of S having i lines and such that $\pi_T = \pi$. Then

$$P(H,\lambda) = \sum_{i \geq 0} \sum_{\pi} (-1)^i \, t_{\pi i} . P(R(\pi),\lambda) \quad ,$$

the double summation varying over all non-negative integers i and all partitions π of $V(H)$.

Proof. For every spanning sub-hypergraph T of S, $c(T) = |\pi_T|$.

1.2.33. COROLLARY. Let H be a hypergraph and let s_{ki} be the number of spanning sub-hypergraphs of H having k connected components and i lines. Then the coefficient of λ^k in $P(H,\lambda)$ is

$$\sum_{i \geq 0} (-1)^i \quad s_{ki}$$

Proof. In proposition 1.2.32 set $S = H$ and let R be the spanning sub-hypergraph of H with $E(R) = \emptyset$. Then for every partition π of $V(H)$

$$P(R(\pi),\lambda) = \lambda^{|\pi|} ,$$

and the result follows.

In the case of planar maps the above corollary is due to George D. Birkhoff [B 7] and in the case of graphs to H.Whitney [W 5].

1.2.4. Let S be a graph. Let $A_1,\ldots, A_m$ be an enumeration of the elements of E(S). A set $C \subseteq E(S)$ is called a <u>broken circuit</u> if there is an integer i, $1 \leq i \leq m$, such that

(i) $C \cup \{A_i\}$ is a circuit ,

(ii) for every $A_j \in C$, $j \geq i$.

In particular all circuits are broken circuits, including loops and digons.

We have the following generalization of Whitney's interpretation of the coefficients of the chromatic polynomial [W 5] :

1.2.41. PROPOSITION. Let R and S be two subgraphs of a graph G such that

$$E(R) \cup E(S) = E(G) ,$$

$$E(R) \cap E(S) = \emptyset \quad ,$$

$$V(R) \cup V(S) = V(G) \quad .$$

Let $J = V(R) \cap V(S)$. For a spanning subgraph T of S, let π_T be the partition of J each block of which is the intersection of J with some connected component of T. Then

$$P(G,\lambda) = \sum_{T} (-1)^{|V(T)| - c(T)} \lambda^{c(T) - |\pi_T|} P(R(\pi_T),\lambda) \quad ,$$

the summation ranging over all spanning subgraphs T of S containing no broken circuit.

Proof: As in Whitney's original theorem, we assume first that S has no loops or multiple links. Considering the sum

$$P(G,\lambda) = \sum_{T} (-1)^{|E(T)|} \lambda^{c(T) - |\pi_T|} P(R(\pi_T),\lambda) \; ,$$

given by proposition 1.2.31 it can be verified, as in [W 5] , that the contributions of those terms

$$(-1)^{|E(T)|} \lambda^{c(T) - |\pi_T|} P(R(\pi_T),\lambda)$$

that correspond to a spanning subgraph T of S containing a broken circuit cancel. Also, if T contains no broken circuit, then T contains no circuit and

$$|E(T)| = |V(T)| - c(T) .$$

In case S has multiple links, the expansion of the proposition is still true. Indeed, if the links between two distinct vertices x and y of S are $A_{j_1}, \ldots, A_{j_k}$, $j_1 < \ldots < j_k$, then every $\{A_{j_\ell}\}$, $1 < \ell \leq k$, is a broken circuit because $\{A_{j_1}, A_{j_\ell}\}$ is a circuit and $j_1 < j_\ell$. Hence in the summation of the proposition we sum only over spanning subgraphs T of S that contain among the links joining two vertices x and y at most the first one (in the fixed enumeration $A_1, \ldots, A_m$). The expansion is therefore reduced to the case where S has no multiple links.

Also, if S has a loop A_j, $P(G,\lambda)$ is identically zero and so is the sum

$$\Sigma(-1)^{|V(T)| - c(T)} \lambda^{c(T) - |\pi_T|} P(R(\pi_T),\lambda) ,$$

which is now taken over the empty set because every spanning subgraph of S contains the broken circuit $\emptyset$, $\{A_j\}$ being a circuit in itself.

By taking $S = G$ and $V(R) = V(G)$, $E(R) = \emptyset$, as in Corollary 1.2.33. , we obtain the original result of Whitney [W 5] saying that for every positive integer k the coefficient of λ^k in $P(G,\lambda)$ is $(-1)^{|V(G)| - k}$ times the number of spanning subgraphs of G having k connected components and containing no broken circuits. As noticed by S.G. Hoggar [H 9] , this yields a much simplified proof of some of G.H.J. Meredith's results on the highest coefficients of the chromatic polynomial [M2] :

1.2.42. COROLLARY (Meredith [M 2]). Let G be a strict loopless graph with n vertices, m edges, finite girth g and p circuits of length g. Let

$$P(G,\lambda) = \sum_{k=1}^{n} (-1)^{n-k} c_k \lambda^k$$

Then

$$c_{n-k} = \binom{m}{k} \qquad \text{for} \quad k = 0, \dots, g-2 ,$$

and

$$c_{n-g+1} = \binom{m}{g-1} - p$$

1.3. Invariance results on the chromatic polynomial.

1.3.1. Let R =(R, Θ,J, v) be a rotor. We say that the <u>chromatic polynomial is partition-invariant</u> with respect to R if for every border-partition π

$$P(R(\pi),\lambda) = P(R(\phi\pi),\lambda) \quad .$$

Let ℓ be a non-negative integer. The <u>lowest ℓ coefficients of the chromatic polynomial are partition-invariant</u> with respect to R if for every border-partition π and every integer k, $0 \le k \le \ell - 1$, the coefficient of λ^k is the same in $P(R(\pi),\lambda)$ and in $P(R(\phi\pi),\lambda)$.

1.3.2. Let R=(R, Θ,J, v) be a rotor. Let S be a hypergraph such that

$$V(S) \cap V(R) = \emptyset$$

$$E(S) \cap E(R) = \emptyset \quad .$$

Let w: J $\to$ V(S) be an injective mapping called the <u>attachment function</u>. Then (R, S , w) is called a <u>motor.</u> The hypergraph S is called the <u>stator.</u>

The reflected motor $(\mathcal{R}, S, w\phi)$ is the motor defined to have the same rotor and stator as $(\mathcal{R}, S, w)$, but a different attachment function $w\phi$ given by

$$w\phi(x) = w(\phi(x))$$

for every $x \in J$.

For a motor $(\mathcal{R}, S, w)$, $\mathcal{R} = (R, \Theta, J, v)$, we define the hypergraph $M(\mathcal{R}, S, w) = (V, E, \psi)$ by

$$V = V(R) \cup [V(S) \setminus w(J)],$$

$$E = E(R) \cup E(S),$$

$$\psi(A) = \psi_R(A)$$

for every $A \in E(R)$ and

$$\psi(A) = [\psi_S(A) \setminus w(J)] \cup \{x \in J \mid w(x) \in \psi_S(A)\}$$

for every $A \in E(S)$.

$M(\mathcal{R}, S, w)$ is thought of as obtained from R and S by identifying vertices that correspond under the attachment function.

1.3.3. Let T be a spanning sub-hypergraph of S. We define a sub-hypergraph $\bar{T}$ of M(R , S, w) and a sub-hypergraph $\bar{\bar{T}}$ of M(R,S, wϕ) by

$$V(\bar{T}) = V(\bar{\bar{T}}) = J \cup [V(S) \setminus w(J)] = J \cup [V(S) \setminus w\phi\ (J)],$$
$$E(\bar{T}) = E(\bar{\bar{T}}) = E(T) \quad .$$

(Although $\bar{T}$ and $\bar{\bar{T}}$ have the same vertices and the same lines, in general they do not have the same incidence function because $\bar{T}$ is a sub-hypergraph of M(R,S, w) while $\bar{\bar{T}}$ is a sub-hypergraph of M(R,S,wϕ).)

The above definitions apply in particular for T= S . Then R and $\bar{S}$ are sub-hypergraphs of M(R,S,w),

$$E(R) \cup E(\bar{S}) = E(M\ (\ R,S,w))\ ,$$
$$E(R) \cap E(\bar{S}) = \emptyset\ ,$$
$$V(R) \cup V(\bar{S}) = V(M(R,S,w))\ ,$$
$$V(R) \cap V(\bar{S}) = J \quad .$$

Also R and $\bar{\bar{S}}$ are sub-hypergraphs of the hypergraph M(R,S,w ϕ) obtained from the reflected motor,

$$E(R) \cup E(\bar{\bar{S}}) = E(M(\mathcal{R},S,w\phi)) \quad ,$$

$$E(R) \cap E(\bar{\bar{S}}) = \emptyset \ ,$$

$$V(R) \cup V(\bar{\bar{S}}) = V(M(\mathcal{R},S,w\phi)) \quad ,$$

$$V(R) \cap V(\bar{\bar{S}}) = J \ .$$

These observations can be compared with the hypotheses of proposition 1.2.31.

We also note that the mapping $T \longrightarrow \bar{T}$ (resp. $T \longrightarrow \bar{\bar{T}}$) establishes a bijection between spanning sub-hypergraphs of S and spanning sub-hypergraphs of $\bar{S}$ (resp. spanning sub-hypergraphs of $\bar{\bar{S}}$) .

1.3.4. With respect to a rotor $\mathcal{R}=(R,\Theta,J,v)$ the chromatic polynomial is said to be motor-invariant if for every motor $(\mathcal{R},S,w)$ and the reflected motor $(\mathcal{R},S,w\phi)$ we have

$$P(M(\mathcal{R},S,w),\lambda) = P(M(\mathcal{R},S,w\phi),\lambda) \ .$$

Let ℓ be a non-negative integer. The lowest ℓ coefficients of the chromatic polynomial are motor invariant with respect to $\mathcal{R}$ if for every motor $(\mathcal{R},S,w)$ and every integer k, $0 \le k \le \ell - 1$,the coefficient of λ^k is the same in $P(M(R,S,w),\lambda)$ and in $P(M(R,S,w_\phi),\lambda)$.

W.T. Tutte has shown [T 2] that if the chromatic polynomial is partition-invariant with respect to a given rotor R , then it is motor-invariant with respect to R . (In fact proposition 4.1. of [T 2] contains a more general result about the dichromate.) The argument is valid also for the chromatic polynomial of hypergraphs. We prove the following:

1.3.5. PROPOSITION. Let R be a rotor.

(i) For every non-negative integer ℓ , the lowest ℓ coefficients of the chromatic polynomial are motor-invariant with respect to R if and only if they are partition-invariant.

(ii) The chromatic polynomial is motor-invariant with respect to R if and only if it is partition-invariant.

Proof: The "if" part of (ii) is contained in proposition 4.1. of [T 2] . Also the proof of the "if" part of (i) is essentially the argument used by Tutte to prove proposition 4.1. [T 2] . For the sake of completeness we show how both the "if" and the "only if" parts of (i) can be deduced from proposition 1.2.31 and how (ii) can be viewed as a consequence of (i).

For any polynomial $P(\lambda)$ let $c_k\ P(\lambda)$ denote the coefficient of λ^k in $P(\lambda)$. If k is negative, then $c_k\ P(\lambda) = 0$.

Assume that the lowest ℓ coefficients of the chromatic polynomial are partition-invariant with respect to $R = (R,\Theta,J,v)$. Then for every $0 \le k \le \ell - 1$ and every border-partition π

$$c_k\ P(R(\pi),\lambda) = c_k\ P(R(\phi\pi),\lambda) \quad .$$

Let (R,S,w) be a motor. Let us write

$$M = M(R,S,w) \quad ,$$

$$\phi M = M(R,S,w\,\phi) \quad .$$

By proposition 1.2.31 and the observations made in 1.3.3. we have

$$P(M,\lambda)= \sum_{T} (-1)^{|E(\bar{T})|}\ \lambda^{c(\bar{T})-|\pi_{\bar{T}}|}\ P(R(\pi_{\bar{T}}),\lambda) \quad ,$$

$$P(\phi M,\lambda)= \sum_{T} (-1)^{|E(\bar{\bar{T}})|}\ \lambda^{c(\bar{\bar{T}})-|\pi_{\bar{\bar{T}}}|}\ P(R(\pi_{\bar{\bar{T}}}),\lambda) \quad ,$$

where the summations range over all spanning sub-hypergraphs T of S. But

$$E(\bar{T})= E(\bar{\bar{T}})= E(T) \quad ,$$

$$c(\bar{T})= c(\bar{\bar{T}})= c(T)$$

and

$$|\pi_{\bar{T}}|= |\pi_{\bar{\bar{T}}}|$$

for every spanning sub-hypergraph T of S. We shall also write $p_T= |\pi_{\bar{T}}| = |\pi_{\bar{\bar{T}}}|$ for every such T.
Hence

$$c_k\ P(M,\lambda)= \sum_{T}(-1)^{|E(T)|}\ C_{k-c(T)+p(T)}\ P(R(\pi_{\bar{T}}),\lambda)$$

and

$$c_k\ P(\phi M,\lambda)= \sum_{T} (-1)^{|E(T)|}\ C_{k-c(T)+p(T)}\ P(R(\pi_{\bar{\bar{T}}}),\lambda).$$

We observe now that

$$\pi_{\bar{\bar{T}}} = \phi\pi_{\bar{T}}$$

for every spanning sub-hypergraph T of S.

Also

$$k - c(T) + p(T) \le k \le \ell - 1 ,$$

and by the partition-invariance of the lowest ℓ coefficients

$$c_{k-c(T)+p(T)}\, P(R(\pi_{\overline{T}}),\lambda) = c_{k-c(T)+p(T)}\, P(R(\pi_{\overline{\overline{T}}}),\lambda) ,$$

implying that

$$c_k\, P(M,\lambda) = c_k\, P(\phi\, M,\lambda) .$$

This proves the motor-invariance of the lowest ℓ coefficients.

Conversely, assume that the lowest ℓ coefficients of the chromatic polynomial are motor-invariant with respect to $\mathcal{R}=(R,\Theta,J,\, v)$. Suppose they are not partition-invariant. We show that this leads to a contradiction. In the lattice of partitions of the principal orbit J choose a minimal element π_o for which there is some k, $0 \le k \le \ell - 1$, such that

$$c_k\ P(R(\pi_o),\lambda) \neq c_k\ P(R(\phi\pi_o),\lambda) \quad .$$

(Recall that in the lattice of partitions $\pi_1 \leq \pi_2$ if and only if each block of π_1 is contained in some block of π_2 [S 3].) To form a motor (R,S,w), we define a stator S and an attachment function w as follows. Let V(S) be a set disjoint from V(R) and w a bijection $w: J \rightarrow \underline{V}(S)$. For each non-singleton block B of π_o let L_B be a line of S with $\psi_S(L_B) = B$. Again, let

$$M = M(R,S,w) \quad ,$$

$$\phi M = M(R,S,w\phi) \quad .$$

Since the functions w and $w\phi$ are surjective onto V(S), proposition 1.2.31 gives

$$P(M,\lambda) = \sum_T (-1)^{|E(T)|}\ P(R(\pi_{\bar{T}}),\lambda)$$

and

$$P(\phi M,\lambda) = \sum_T (-1)^{|E(T)|}\ P(R(\pi_{\bar{\bar{T}}}),\lambda) \quad ,$$

the summations ranging over all spanning sub-hypergraphs T of S. As before, $\pi_{\bar{\bar{T}}} = \phi\pi_{\bar{T}}$ for every T. We have $\pi_{\bar{S}} = \pi_o$, but for every $T \neq S$,

$$\pi_{\bar{T}} < \pi_O$$

in the lattice of partitions of J. Thus, by the minimality assumption on π_O ,

$$c_k \; P(R(\pi_{\bar{T}}),\lambda) = \; c_k \; P(R(\pi_{\bar{\bar{T}}}),\lambda)$$

for every $T \neq S$. Also

$$c_k(M,\lambda) = (-1)^{|E(S)|} c_k P(R(\pi_O),\lambda) \; +$$

$$+ \sum_{T \neq S} (-1)^{|E(T)|} c_k P(R(\pi_{\bar{T}}),\lambda) \quad ,$$

$$c_k(\phi M,\lambda) = (-1)^{|E(S)|} \; c_k P(R(\phi\pi_O),\lambda) \; +$$

$$+ \sum_{T \neq S} (-1)^{|E(T)|} c_k P(R(\pi_{\bar{\bar{T}}}),\lambda) \quad ,$$

and

$$c_k \; P(M,\lambda) = \; c_k \; P(\phi \; M,\lambda)$$

by motor-invariance, contradicting

$$c_k \; P(R(\pi_O),\lambda) \neq c_k \; P(R(\phi\pi_O),\lambda) \; .$$

Finally, consider the following four statements, where invariance is always understood with respect to R:

(A) The chromatic polynomial is motor-invariant,

(B) The chromatic polynomial is partition-invariant,

(C) For every $\ell \geq 0$, the lowest ℓ coefficients of the chromatic polynomial are motor-invariant.

(D) For every $\ell \geq 0$, the lowest ℓ coefficients are partition-invariant.

From part (i) of the proposition, proven above, it follows that (C) and (D) are equivalent. On the other hand, (A) is clearly equivalent to (C) and, similarly, (B) is equivalent to (D). Consequently (A) and (B) an equivalent. But this is exactly part (ii) of the proposition, the proof of which is now complete.

In view of proposition 1.3.5., instead of saying that the chromatic polynomial is partition-invariant or motor-invariant with respect to a rotor R, we can say simply that it is invariant with respect to R.

1.3.6. Let π be a border-partition of a rotor $R=(R,\Theta,J, v)$. Given a power Θ^i of Θ, $i\in Z$, we define the border-partition $\Theta^i\pi$ by

$$\Theta^i\pi = \{\Theta^i(B) \mid B\in \pi\}$$

π is said to be bilaterally symmetric if $\phi\pi = \Theta^i\pi$ for some power Θ^i of Θ. ($\phi\pi$ denotes the reflected border-partition.)

Stated originally for graphs, the following results of Tutte [T 2] are clearly true also for hypergraphs:

1.3.61. If π is a bilaterally symmetric border-partition, then $R(\pi)$ and $R(\phi\pi)$ are isomorphic.

1.3.62. Every border-partition of a rotor of order at most 5 is bilaterally symmetric.

1.3.63. With respect to rotors of order at most 5, the chromatic polynomial is invariant.

This last result, immediate consequence of the preceeding,1.3.61 and, 1.3.62, yields a systematic method of constructing non-isomorphic hypergraphs having the same chromatic polynomial, non-trivial examples of which are not abundant (see George D. Birkhoff and D.C. Lewis [B 8] , R.A. Bari [B 3] and L.A. Lee [L 2]) .

Example. Let R be the simple hypergraph with vertex-set V(R) = [0,12] and whose lines are

{ 0, 3, 1 } ,	{ 1, 4, 2 },	{ 2, 5, 0 } ,
{ 0,11,12 } ,	{ 1, 9,12 },	{ 2,10,12 } ,
{ 0, 6, 9 } ,	{ 1, 7,10 },	{ 2, 8,11 } .

A geometric representation of R is given in Figure 2.

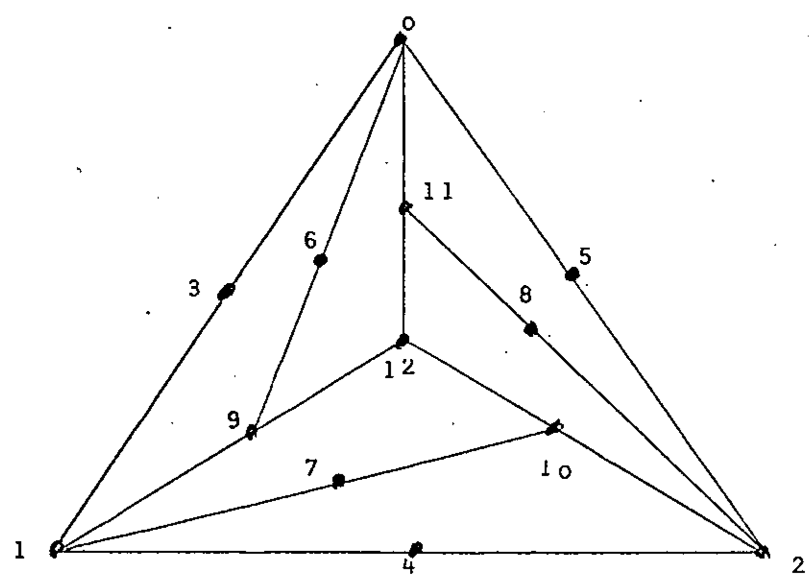


Figure 2

Consider the permutation

$$\theta=(0,\ 1,\ 2)(3,4,5)(6,\ 7,\ 8)(9,\ 10,\ 11)$$

of V(R). Clearly θ is an automorphism of R. Let $J=\{0,\ 1,\ 2\}$ and $v=0$. Then $\mathcal{R}=(R,\theta,J,\ v)$ is a rotor of order 3. Let the stator S be the simple hypergraph given by

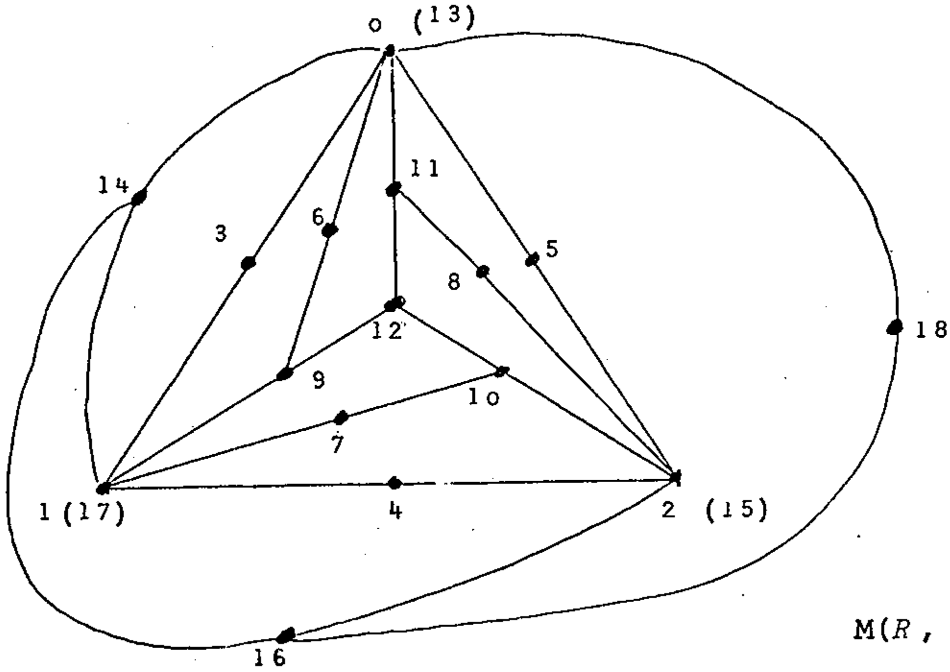
$$V(S)=[13,\ 18]\quad,$$

$$E(S)=\{\{13,\ 14,\ 15\}\ ,\ \ \{14,\ 16,\ 17\}\ ,\ \{16,\ 18,\ 13\}\}.$$

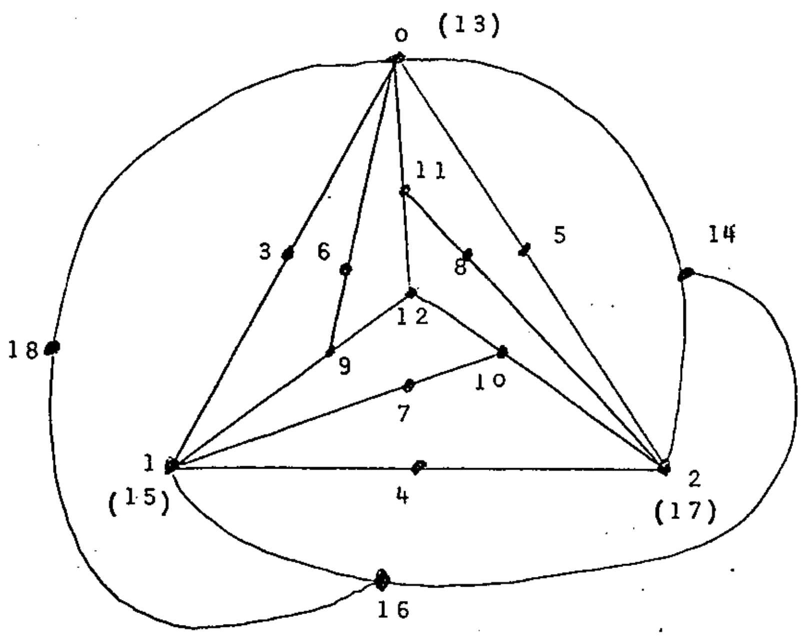
Define the attachment function $w : J \to V(S)$ by

$$w\ (0)=13\quad,\qquad w(1)=15\quad,\quad w(2)=17\quad.$$

According to 1.3.63 the hypergraphs $M(\mathcal{R},S,w)$ and $M(\mathcal{R},S,w\phi)$ have the same chromatic polynomial. A geometric representation of $M(\mathcal{R},S,w)$ and $M(\mathcal{R},S,w\phi)$ is given in Figure 3. There is no difficulty in verifying that these two hypergraphs are not isomorphic.



M(R , S , w)



M(R ,S ,wϕ)

Figure 3

1.3.7. Rotors of order 6 bounded by a hexagon: a simplified proof of a result of Lee.

Let R be a graph and $\mathcal{R} = (R, \theta, J, v)$ a rotor of order 6 such that v is adjecent to $\theta(v)$. For the purposes of this subsection, we call a border-partition π of $\mathcal{R}$ <u>admissible</u> if for every integer i

$$\theta^{i}(v) \not\equiv \theta^{i+1}(v) \mod \pi$$

1.3.71. PROPOSITION. Every admissible border-partition is bilaterally symmetric.

Proof. Let us write, for every integer i, $v^i = \theta^i(v)$. Then

$$J = \{ v^1, v^2, v^3, v^4, v^5, v^6 \} .$$

Consider the following 12 admissible border - partitions:

$$\pi_1 = \{\{v^1, v^3, v^5\}, \{v^2, v^4, v^6\}\} ,$$

$$\pi_2 = \{\{v^1, v^3, v^5\}, \{v^2, v^4\}, \{v^6\}\} ,$$

$$\pi_3 = \{\{v^1, v^4\}, \{v^2, v^5\}, \{v^3, v^6\}\},$$

$$\pi_4 = \{\{v^1, v^4\}, \{v^2, v^6\}, \{v^3, v^5\}\},$$

$$\pi_5 = \{\{v^1, v^3, v^5\}, \{v^2\}, \{v^4\}, \{v^6\}\},$$

$$\pi_6 = \{\{v^1, v^3\}, \{v^2, v^4\}, \{v^5\}, \{v^6\}\},$$

$$\pi_7 = \{\{v^1, v^4\}, \{v^2, v^5\}, \{v^3\}, \{v^6\}\},$$

$$\pi_8 = \{\{v^1, v^4\}, \{v^2, v^6\}, \{v^3\}, \{v^5\}\},$$

$$\pi_9 = \{\{v^1, v^3\}, \{v^2\}, \{v^4\}, \{v^5\}, \{v^6\}\},$$

$$\pi_{10} = \{\{v^1, v^4\}, \{v^2\}, \{v^3\}, \{v^5\}, \{v^6\}\},$$

$$\pi_{11} = \{\{v1\}, \{v^2\}, \{v^3\}, \{v^4\}, \{v^5\}, \{v^6\}\},$$

$$\pi_{12} = \{\{v^1, v^5\}, \{v^2, v^4\}, \{v^3\}, \{v^6\}\} .$$

It is easily seen that for every admissible border-partition π there is some integer i, and exactly one index k, $1 \leq k \leq 12$, such that

$$\pi = \Theta^i (\pi_k)$$

On the other hand, if ρ is a bilaterally symmetric border-partition, then so is $\Theta^i(\rho)$ for every integer i. But it can be verified without difficulty that the border-partitions $\pi_1, \ldots, \pi_{12}$ are bilaterally symmetric. The result follows.

1.3.72. PROPOSITION (Lee [L 2]). Let $\mathcal{R}=(R,\theta,J,v)$ be a rotor of order 6, where R is a graph such that v is adjacent to $\theta(v)$. Then the chromatic polynomial is invariant with respect to $\mathcal{R}$.

Proof. We prove partition-invariance. Let π be a border-partition of $\mathcal{R}$. If π is not admissible, then $R(\pi)$ has a loop and so does $R(\phi\pi)$. In this case

$$P(R(\pi),\lambda) = P(R(\phi\pi),\lambda) = 0 \quad .$$

If π is admissible, then π is bilaterally symmetric by proposition 1.3.71 and $R(\pi)$ is isomorphic to $R(\phi\pi)$ by 1.3.61. We have a fortiori the identity

$$P(R(\pi),\lambda)= P(R(\phi\pi),\lambda) \quad .$$

1.3.8. More on rotors of order 6.

Let $\mathcal{R}=(R,\theta,J,v)$ be a rotor of order 6, where R is a graph such that v is adjacent to $\theta^2(v)$. Throughout this subsection we call a border-partition π <u>admissible</u> if for every integer i

$$\theta^i(v) \neq \theta^{i+2}(v) \mod \pi$$

As in subsection 1.3.7 , we have the following:

1.3.81. PROPOSITION. Every admissible border-partition is bilaterally symmetric.

Proof. Again, let $v^i = \theta^i(v)$ for every integer i. Consider the following 10 admissible border-partitions:

$$\pi_1 = \{\{v^1, v^2\}, \{v^3, v^4\}, \{v^5, v^6\}\},$$
$$\pi_2 = \{\{v^1, v^2\}, \{v^3, v^6\}, \{v^4, v^5\}\},$$
$$\pi_3 = \{\{v^1, v^4\}, \{v^2, v^5\}, \{v^3, v^6\}\},$$
$$\pi_4 = \{\{v^1, v^4\}, \{v^2, v^3\}, \{v^5\}, \{v^6\}\},$$
$$\pi_5 = \{\{v^1, v^2\}, \{v^3, v^4\}, \{v^5\}, \{v^6\}\},$$
$$\pi_6 = \{\{v^1, v^2\}, \{v^4, v^5\}, \{v^3\}, \{v^6\}\},$$
$$\pi_7 = \{\{v^1, v^4\}, \{v^2, v^5\}, \{v^3\}, \{v^6\}\},$$
$$\pi_8 = \{\{v^1, v^2\}, \{v^3\}, \{v^4\}, \{v^5\}, \{v^6\}\},$$
$$\pi_9 = \{\{v^1, v^4\}, \{v^2\}, \{v^3\}, \{v^5\}, \{v^6\}\},$$
$$\pi_{10} = \{\{v^1\}, \{v^2\}, \{v^3\}, \{v^4\}, \{v^5\}, \{v^6\}\}.$$

These border-partitions are bilaterally symmetric. Also every admissible border- partition of R is of the form $\theta^i(\pi_k)$, $i \in Z$, $1 \leq k \leq 10$ (k unique). The result follows as in the proof of proposition 1.3.71 .

1.3.82. PROPOSITION. Let $\mathcal{R} = (R, \Theta, J, v)$ be a rotor of order 6, where R is a graph such that v is adjacent to $\Theta^2(v)$. Then the chromatic polynomial is invariant with respect to $\mathcal{R}$.

Proof. Similar to that of proposition 1.3.72.

1.3.83. Example. Let R be the strict graph with vertex set V(R)= [0, 17] displayed in Figure 4.

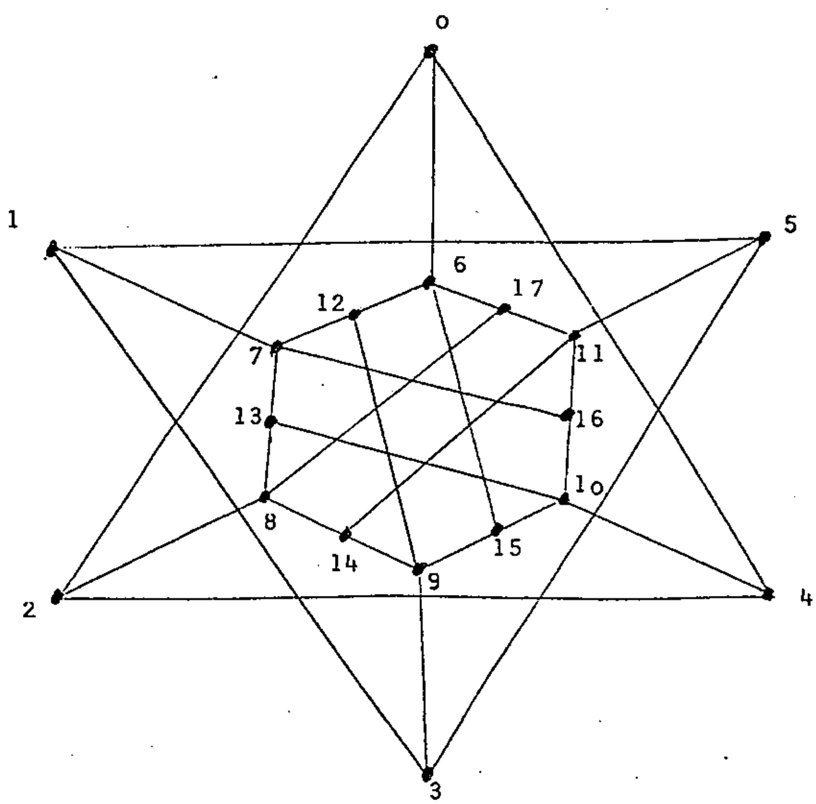


Figure 4

Let

$$\theta = (0,1,2,3,4,5)(6,7,8,9,10,11)(12,13,14,15,16,17),$$

$$J = [0,\ 5]\ , \qquad v = 0\ .$$

Then (R,θ,J,v) is a rotor satisfying the hypotheses of proposition 1.3.82.

1.4. Invariance and non-invariance of the second highest coefficient.

1.4.1. Let G be a graph. The link number εG is defined as the number of links of the underlying simple graph,

$$\varepsilon G = |\{\{x,y\} \mid x,y \in V(G)\ ,\ x \neq y \quad \text{and} \quad x \text{ is adjacent to } y\}|\ .$$

Since the chromatic polynomial of a graph (indeed, of any hypergraph) is identical with that of the underlying simple graph (simple hypergraph), it is an immediate consequence of corollary 1.2.42 , that for any loopless graph G the coefficient of $\lambda^{|V(G)|-1}$ in $P(G,\lambda)$ is $-\varepsilon G$.

1.4.2. Consider a rotor $R=(R,\theta,J,v)$, R being a graph. Clearly for every border-partition π the graphs $R(\pi)$ and $R(\phi\pi)$ have the same number of vertices. We say that the second highest coefficient of the chromatic polynomial is partition-invariant with respect to R if for every border-partition π the coefficient of

$$\lambda^{|V(R(\pi))|-1} = \lambda^{|V(R(\phi\pi))|-1}$$

is the same in $P(R(\pi),\lambda)$ and in $P(R(\phi\pi),\lambda)$.

The chief result of this section will provide an answer to the following question: With respect to a rotor (R,Θ,J,v) , when is it possible to conclude the partition-invariance of the second highest coefficient of the chromatic polynomial by looking at only the border $(R[J], \Theta|J)$? In this respect it should be noted that if B is a non-null graph with k vertices $v_1,\ldots,v_k$ such that the cyclic permutation

$$\sigma = (v_1, \ldots, v_k)$$

is an automorphism of B, then (B,σ) is always the border of some rotor R. E.g. one, rather trivial, rotor with border (B,σ) is $R = (B,\sigma,V(B), v_1)$. In view of this, it is convenient to call the ordered pair (B,σ) a border (of order k), even without reference to any particular rotor.

1.4.3. For every positive integer k we denote by Z_k the cyclic group of integers modulo k. Formally, the elements of Z_k are usually defined as sets of the form $\{ kq + p | q \in Z \}$, where $p \in Z$ is fixed. We write $\bar{p} = \{kq+p | q \in Z \}$, and p is said to be a representative of $\bar{p}$.

Vertices of a border (B,σ) of order k will be indexed by integers modulo k rather than by integers, in such a way that $V(B) = \{v_i \mid i \in Z_k\}$ and

$$\sigma(v_i) = v_{i+\bar{1}}$$

for every $i \in Z_k$. A 3-element subset $\{v_r, v_s, v_t\}$ of V(B) will then be called a <u>scalene 3-set</u> provided that the differences r-s, s-t and t-r are all distinct elements of Z_k ,

$$|\{ r-s, s-t, t-r \}| = 3 \quad .$$

We note that this definition does not depend on the order of the vertices v_r, v_s, v_t. A subset S of V(B), $|S| = n$, is called <u>periodic</u> if there is an index $i \in Z_k$ and a positive integer $p \le \frac{k}{n}$ such that

$$S = \{v_{i+\overline{qp}} \mid q \in [0, n-1]\} \ .$$

The integer p is called a <u>period</u> of S. For $r, s \in Z_k$ we define

$$\vec{k}(r, s) = \min\{ d \in Z \mid d \geq 0 \text{ and } s = r + \bar{d} \}$$

and

$$k(r, s) = \min(\vec{k}(r, s), \vec{k}(s, r)).$$

1.4.4. Let π be a border-partition of a rotor (R, Θ, J, v), R being a graph. Then

$$\varepsilon R(\pi) = \varepsilon R[V(R) \setminus J] + \varepsilon R[J](\pi) + \sum_{A \in \pi} |N_R(A) \cap (V(R) \setminus J)|$$

and

$$\varepsilon R(\phi\pi) = \varepsilon R[V(R) \setminus J] + \varepsilon R[J](\phi\pi) + \sum_{A \in \pi} |N_R(\phi(A)) \cap (V(R) \setminus J)|$$

Although generally the reflection ϕ is not the restriction to J of any automorphism of R, it is clear that $\phi \in \mathrm{Aut}\, R[J]$ and therefore the graphs $R[J](\pi)$ and $R'[J](\phi\pi)$ are isomorphic. Consequently $R(\pi)$ and $R(\phi\pi)$ have the same link number if and only if

$$\sum_{A\in\pi} |N_R(A) \cap (V(R)\setminus J)| = \sum_{A\in\pi} |N_R(\phi(A)) \cap (V(R)\setminus J)|$$

This will always be the case if every block A of π is periodic.

1.4.5. PROPOSITION. Let (B,σ) be a border of order k, where B is a loopless graph, $V(B) = \{v_i \mid i \in Z_k\}$, and $\sigma(v_i) = v_{\overline{i+1}}$ for every $i \in Z_k$. The following two conditions are equivalent:

(i) Whatever the rotor $\mathcal{R} = (R,\Theta,J,v)$, R being a graph , having border $(B,\sigma) = (R[J], \Theta|J)$ may be, the second highest coefficient of the chromatic polynomial is partition-invariant with respect to $\mathcal{R}$.

(ii) (B,σ) has no independent scalene 3-set.

Proof. Assume (ii) and let π be a border-partition of a rotor (R,Θ,J,v) , R being a graph, $R[J] = B$, $\Theta \mid J = \sigma$. If some block A of π is not independent, then $\phi(A)$ is not independent and, both $R(\pi)$ and $R(\phi\pi)$ having loops,

$$P(R(\pi),\lambda) = P(R(\phi\pi),\lambda) = 0$$

Assume therefore that every block of π is independent. According to 1.4.1. and 1.4.4. it will suffice to show that every block A of π is periodic. This is obvious if A is a singleton. Let $|A| \geq 2$. Let p be the smallest positive integer such that A has a periodic subset having period p. Let B be a periodic subset of A having period p and containing the largest possible number of elements, say $|B| = n$,

$$B = \{v_{\overline{i+qp}} \mid q \in [0, n-1]\} .$$

Since every 2- subset of A is periodic, $n \geq 2$. If $A = B$, then A is periodic. If $A \neq B$, let $v_j \in A \setminus B$. We claim that $n = 2$ and

$$k(j, i) = k(j, \overline{i + (n-1)p}) .$$

Indeed, if we had $n > 2$ or

$$k(j, i) > k(j, \overline{i + (n-1)p}) ,$$

then

$$\{ v_{i+\overline{(n-2)p}} , v_{i+\overline{(n-1)p}} , v_j \}$$

would be a scalene 3-subset of A, which contradicts (ii) because A is independent. If we had $n > 2$ or

$$k(j, i) < k(j, i + \overline{(n-1)p}) ,$$

then

$$\{ v_j , v_i , v_{i+\bar{p}} \}$$

would be a scalene 3-subset of A, contradicting again the assumption (ii). If follows now that

$$j = i + \bar{p} + \overline{(\frac{k-p}{2})} ,$$

and

$$A = B \cup \{v_j\} = \{ v_{i+\bar{p}} , v_j , v_i \}$$

is a periodic set with period $\frac{k-p}{2}$. Condition (i) is proved.

Conversely, assume that (ii) is false. Let $\{ v_r, v_s, v_t \}$ be an independent scalene 3-set of the border (B,σ). In order to prove the falsity of (i), we shall construct an $\mathcal{R}= (R,\Theta,J, v)$ having border (B,σ) , and such that the second highest coefficient of the chromatic polynomial is not partition-invariant with respect to $\mathcal{R}$. Let I be a k-set disjoint from $V(B)$, $I= \{x_i \mid i\in Z_k\}$. The graph R is defined by the following conditions:

a.) $V(R) = V(B)\cup I$,

b.) $R[V(B)] = B$,

c.) I is independent ,

d.) $N_R(x_i) = \{v_{r+i}, v_{s+i}, v_{t+i}\}$, for every $i\in Z_k$,

e.) There are no multiple lines between a vertex $v\in V(B)$ and a vertex $x\in I$.

The automorphism Θ of R is defined by

$$\Theta \mid V(B) = \sigma$$

and

$$\Theta(x_i) = x_{i+\bar{1}}$$

for every $i\in Z_k$. Let also

$$J = V(B)$$

and

$$v = v_{\bar{0}}$$

To see that the second highest coefficient of the chromatic polynomial is not partition-invariant with respect to $R=(R,\theta,J,v)$, consider the partition π of J for which

$$C = \{ v_r , v_s , v_t \}$$

is a block and all other blocks are singletons. Note that for every $v_i \in J$

$$|N_R (v_i) \cap I| = |N_R (\phi(v_i)) \cap I| .$$

Hence if we could prove that

$$|N_R (C) \cap I| \neq |N_R (\phi(C) \cap I| ,$$

then it would follow that

$$\sum_{A \in \pi} |N_R (A) \cap I| \neq \sum_{A \in \pi} |N_R (\phi(A)) \cap I| .$$

and, by 1.4.4. , $R(\pi)$ and $R(\phi\pi)$ would have different link numbers. Further since R is loopless by assumption and every block of π, and consequently of $\phi\pi$, is independent, both $R(\pi)$ and $R(\phi\pi)$ are loopless.

It would then follow from 1.4.1. that the respective coefficients of $\lambda^{|V(R(\pi))|-1} = \lambda^{|V(R(\phi\pi))|-1}$ in $P(R(\pi),\lambda)$ and in $P(R(\phi\pi),\lambda)$ are different. This would complete the proof. Let $S = \{r, s, t\}$. We have

$$N_R(C)\cap I = \{x_i \in I \mid \{v_{r+i}, v_{s+i}, v_{t+i}\}\cap\{v_r, v_s, v_t\} \neq \emptyset\},$$

so that

$$|N_R(C)\cap I| = |\{i \in Z_k \mid \{r+i, s+i, t+i\}\cap \{r,s,t\} \neq \emptyset\}| =$$

$$= |\{p - q \mid p, q \in S\}| \quad .$$

On the other hand,

$$N_R(\phi(C))\cap I = \{x_i \in I \mid \{v_{r+i}, v_{s+i}, v_{t+i}\}\cap\{v_{-r}, v_{-s}, v_{-t}\} \neq \emptyset\},$$

so that

$$|N_R(\phi(C))\cap I| = |\{i \in Z_k \mid \{r+i, s+i, t+i\}\cap\{-r,-s,-t\} \neq \emptyset\}| =$$

$$= |\{-p-q \mid p, q \in S\}|$$

$$= |\{p+q \mid p, q \in S\}|$$

We have to distinguish two cases.

Case 1. For every $p, q \in S$, such that $p \neq q$, we have $p - q \neq q - p$. Then it is easy to see that

$$\{r-s,\ s-t,\ t-r\} \cap \{r-t,\ t-s,\ s-r\} = \emptyset$$

and

$$|N_R(C) \cap I| = |\{\ p-q \mid p,q \in S\}| = 7 .$$

But

$$|\{\ p+q \mid p,q \in S\}| \leq 6 ,$$

so that in this case

$$|N_R(C) \cap I| \neq |N_R(\phi(C)) \cap I| .$$

Case 2. For some $p, q \in S$, $p \neq q$, we have $p-q = q-p$. There is no loss of generality in assuming that

$$r - s = s - r$$

i.e.

$$2(r - s) = \overline{0} .$$

Then

$$\{r-s,\ s-t,\ t-r\} \cap \{r-t,\ t-s,\ s-r\} = \{r-s\} \quad ,$$

because otherwise we would have

$$s - t = t - s$$

or

$$t - r = r - t \quad ,$$

i.e. besides

$$2(r - s) = \bar{0} \quad ,$$

also

$$2(s - t) = \bar{0}$$

or

$$2(t - r) = \bar{0} \quad .$$

But, in view of the assumption that $\{v_r, v_s, v_t\}$ is a scalene 3-set, this contradicts the fact that the equation

$$2x = \bar{0}$$

has at most one non-zero solution in Z_k. It follows that

$$|\{p - q \mid p, q \in S\}| = 6 \quad .$$

Also, since $r + r = s + s$,

$$|\{ p+q \mid p,q \in S \}| \leq 5 \quad ,$$

so that

$$|N_R(C) \cap I| \neq |N_R(\phi(C)) \cap I| \quad .$$

The proof of proposition 1.4.5. is now complete.

1.4.6. Clearly a border of order at most 5 cannot have a scalene 3-set, in compliance with Tutte's invariance result for such borders [T 2] . This result is best possible.

1.4.61. PROPOSITION. Let k be any integer ≥ 6. There exists a border (B,σ) of order k, B being a loopless graph, which has an independent scalene 3-set.

Proof. Let

$$V(B) = \{v_i \mid i \in Z_k\} \quad ,$$

$$E(B) = \emptyset$$

and

$$\sigma(v_i) = v_{i+\bar{1}}$$

for every $i \in Z_k$. Then

$$\{ v_{\bar{1}} , v_{\bar{3}} , v_{\bar{K}} \}$$

is an independent scalene 3-set.

In view of proposition 1.4.5 we obtain the following:

1.4.62. COROLLARY (See [F 2]). Let k be any integer ≥ 6. There exists a rotor R of order k such that the second highest coefficient of the chromatic polynomial is not partition-invariant with respect to R .

In particular, following the proofs of propositions 1.4.5 and 1.4.61 , rotors of arbitrary high order $k \geq 6$ can be constructed with respect to which the chromatic polynomial is not invariant. This contrasts with the invariance of the number of spanning trees, proved by R.L.Brooks, C.A.B. Smith, A.H.Stone, W.T.Tutte [B 13] and Tutte [T 4] .

As an example, take $k = 7$. Constructing the rotor $R = (R, \Theta, J, v)$ as in the proofs of propositions 1.4.5 and 1.4.61, R is the Levi graph of the Fano geometry pictured in Figure 5.

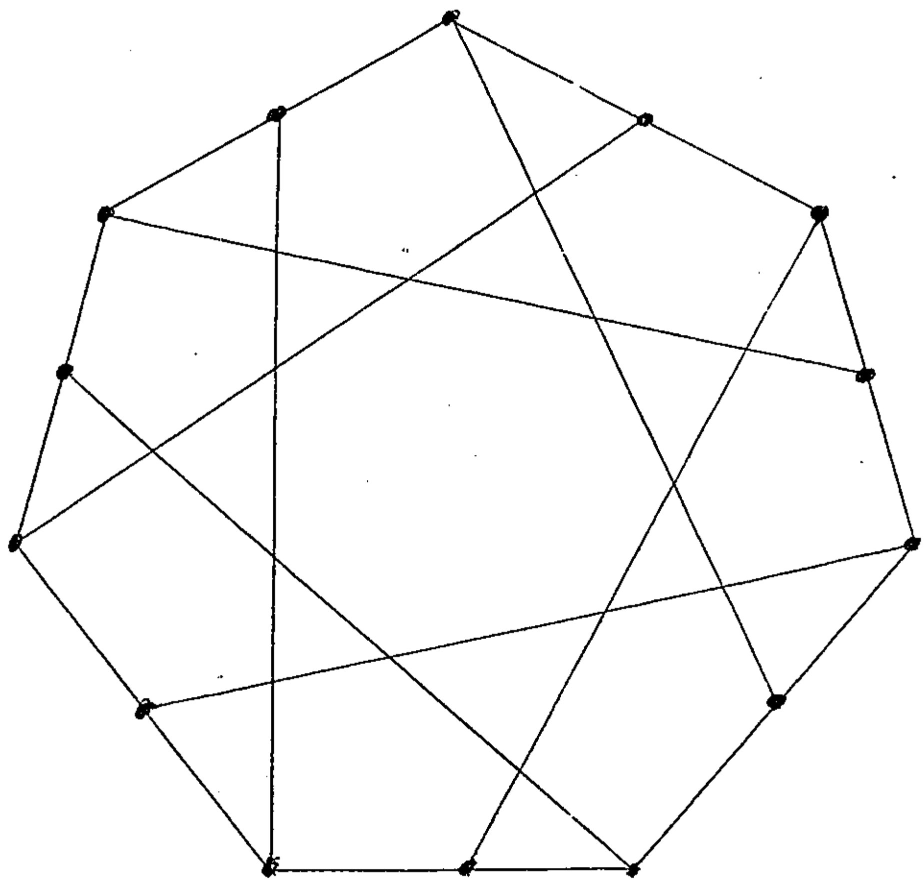


Figure 5

By choosing an appropriate stator, we obtain graphs having the same number of spanning trees but having different chromatic polynomials. Such a pair of graphs is displayed in Figure 6.

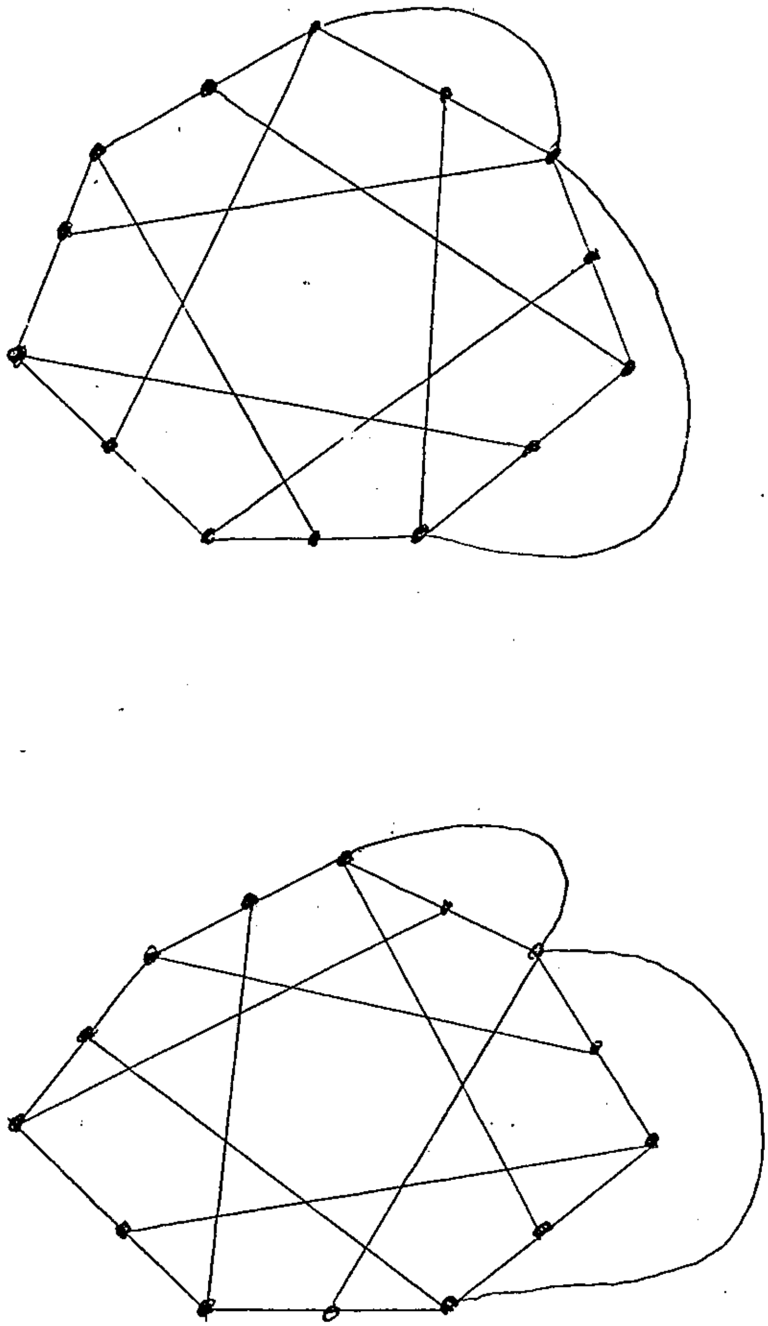


Figure 6

1.5. Planarity of the rotor: the link number

1.5.1. Let π be a border-partition of a rotor (R, Θ, J, v), R being a graph. Then

$$\varepsilon R(\pi) = \varepsilon R\,[V(R) \setminus J] + \varepsilon R\,[J]\,(\pi) \; +$$

$$+ \sum_{x \in V(R) \setminus J} \left| \{ A \in \pi \mid N_R(x) \cap A \neq \emptyset \} \right|$$

and

$$\varepsilon R(\phi\pi) = \varepsilon R\,[V(R) \setminus J] + \varepsilon R\,[J]\,(\phi\pi) \; +$$

$$+ \sum_{x \in V(R) \setminus J} \left| \{ A \in \pi \mid N_R(x) \cap \phi(A) \neq \emptyset \} \right| .$$

By an argument similar to that of 1.4.4 the graphs $R(\pi)$ and $R(\phi\pi)$ have the same link number if and only if

$$\sum_{x \in V(R) \setminus J} \left| \{ A \in \pi \mid N_R(x) \cap A \neq \emptyset \} \right| =$$

$$= \sum_{x \in V(R) \setminus J} \left| \{ A \in \pi \mid N_R(x) \cap \phi(A) \neq \emptyset \} \right| .$$

1.5.2. PROPOSITION. Let the rotor (R,Θ,J,v) and π be as above. Suppose that for every orbit I of Θ, $I \neq J$, there is a permutation ψ of I such that

$$\phi(N_R(x) \cap J) = N_R(\psi(x)) \cap J$$

for every $x \in I$. Then $R(\pi)$ and $R(\phi\pi)$ have the same link number.

Proof. For every orbit I of Θ, $I \neq J$, let us denote by ψ_I the corresponding permutation of I. Define a permutation σ of $V(R)\backslash J$ by the condition that

$$\sigma(x) = \psi_I(x)$$

if $x \in I$. Then clearly

$$\phi(N_R(x) \cap J) = N_R(\sigma(x)) \cap J$$

for every $x \in V(R)\backslash J$.

Since ϕ is a permutation of J, for every $x \in V(R)\backslash J$ and $A \in \pi$

$$\phi(N_R(x) \cap A) = \phi(N_R(x) \cap J \cap A) =$$

$$= N_R(\sigma(x)) \cap J \cap \phi(A) = N_R(\sigma(x)) \cap \phi(A)$$

Consequently

$$\sum_{x \in V(R)\setminus J} |\{A \in \pi \mid N_R(x) \cap A \neq \emptyset\}| =$$

$$= \sum_{x \in V(R)\setminus J} |\{A \in \pi \mid N_R(\sigma^{-1}(x)) \cap A \neq \emptyset\}| =$$

$$= \sum_{x \in V(R)\setminus J} |\{A \in \pi \mid N_R(x) \cap \phi(A) \neq \emptyset\}| .$$

Now the proposition follows from 1.5.1.

1.5.3. Let P be a polygon of a graph G. An equivalence relation $\sim$ is defined on $E(G)\setminus E(P)$ by the condition that for every $A, A' \in E(G)\setminus E(P)$,

$$A \sim A'$$

if and only if there is a path in G whose terminal lines are A and A' and which is internally vertex-disjoint from P. A <u>bridge</u> of P in G is a subgraph B of G spanned by any of the equivalence classes of the relation ~ . The vertices in $V(B) \cap V(P)$, called <u>vertices of attachment</u> of B, partition P into segments. Two distinct bridges B_1 and B_2 of P are said to <u>overlap</u> if the vertices of attachment of B_1 are not confined to a single segment of B_2. (This definition is symmetric in B_1 and B_2.) Two subgraphs H_1 and H_2 of G not contained in P are said to be <u>separated</u> by P if there is some subgraph H of G containing P, P having two overlapping bridges B_1 and B_2 in H such that H_i is a subgraph of B_i for $i = 1,2$. Planar graphs are characterized by the property that no polygon separates every two of three subgraphs H_1, H_2, H_3. Also in any embedding of a planar graph G in the plane, two subgraphs separated by a polygon P lie in different residual domains of P. (See [B 11], [K 4] , [T 3] .)

1.5.4. PROPOSITION. Let $\mathcal{R} = (R,\Theta,J,v)$ be a rotor of order k, R being a planar graph. Assume that the order of Θ is k. Then for every border-partition π of $\mathcal{R}$, $R(\pi)$ and $R(\phi\pi)$ have the same link number.

Proof. Clearly it is enough to prove the proposition for strict R.

We use proposition 1.5.2. We show that for every orbit I of θ, $I \neq J$, there is a permutation ψ of I such that

$$\psi(N_R(x) \cap J) = N_R(\psi(x)) \cap J$$

for every $x \in I$. For every $i \in Z$ and $y \in V(R)$, let $y^i = \theta^i(y)$.

Let therefore I be an orbit of θ, $I \neq J$. Let $\ell = |I|$. If v is not adjacent to any vertex in I, let ψ be the identity permutation of I. If

$$|N_R(v) \cap I| = 1 ,$$

then let x be the unique vertex of I adjacent to v. We have

$$N_R(x) \cap J = \{ v^{i\ell} \mid i \in Z \} .$$

Every vertex in I is of the form x^j for some $j \in Z$.

Clearly

$$N_R\ (x^j) \cap J = \{ v^{j+i\ell} \mid i \in Z \} \quad .$$

We define

$$\psi(x^j) = x^{-j} \quad .$$

for every $j \in Z$.

If v is adjacent to more than 1 vertex of I, then let d be an integer such that for some $x \in I$, $x^d \neq x$ and

$$\{x, x^d\} \subseteq N_R\ (v) \quad .$$

Assume that among all possible d we have chosen one such that the orbit of θ^d containing v has maximum cardinality. Let m be the cardinality of the orbit of θ^d containing x. Obviously $m \geq 2$. For every $i \in Z$

$$P(x^i, v^i, x^{i+d}, v^{i+d}, \ldots, x^{i+(m-1)d}, v^{i+(m-1)d})$$

is a polygon of length 2 m, denoted in the sequel by $P(v^i)$. Throughout this proof polygons of the form $P(v^i)$ will be called <u>μ - polygons</u> .

We have to distinguish several cases, a diagram of which is displayed in Figure 7.

Case 1. d and k are not coprime, $\gcd(d,k) \neq 1$. This means that J breaks up into several orbits under the action of θ^d.

Case 1.1. $N_R(x) \cap J$ intersects several orbits of θ^d. Since $v = v^o \in N_R(x)$, this means that there is a $v^r \in N_R(x)$ such that $r \not\equiv 0 \mod \gcd(d,k)$. Applying θ^{-r} it is easily seen that there is some $x^s \in N_R(v) \cap I$ such that $s \not\equiv 0 \mod \gcd(d,k)$.

Case 1.1.1. We can find r and s as above such that $r \not\equiv s \mod \gcd(d,k)$. This means that the μ - polygons $P(v^r)$ and $P(v^s)$ are vertex-disjoint not only from $P(v)$ but also from each other. Moreover, every $v^{jd} \in V(P(v)) \cap I$ is adjacent to $x^{jd+s} \in V(P(v^s))$ and every $x^{jd} \in V(P(v)) \cap I$ is adjacent to $v^{jd+r} \in V(P(v^r))$. It follows that $P(v^r)$ and $P(v^s)$ are separated by $P(v)$. Since all μ - polygons are similar under some appropriate power of θ, for every μ- polygon P there are two μ - polygons P_1 and P_2 that are separated by P.

In an embedding of R in the plane, let the μ -polygon P be chosen in such a way that no μ- polygon P' vertex-disjoint from P lies in the interior (bounded residual domain) of P. Then both P_1 and P_2 have to lie in the exterior of P, contradicting the fact that they are separated by P.

Case 1.1.2. For every r and s as in Case 1.1., $r \equiv s \bmod \gcd(d,k)$. In particular r and s are unique modulo gcd(d,k). By symmetry under powers of θ, it is seen that for every $i \in Z$, $N_R(x^i) \cap J$ is contained in 2 orbits of θ^d and that the same holds for $N_R(v^i) \cap I$, $i \in Z$. But x^r is adjacent to v^r, v^{r-s} and v^{2r}, and therefore either $2r \equiv r \bmod \gcd(d,k)$ or $2r \equiv r - s \bmod \gcd(d,k)$. The first congruence, equivalent to $r \equiv 0 \bmod \gcd(d,k)$, is false by the definition of r, so that $2r \equiv r - s \equiv 0 \bmod \gcd(d,k)$.

Case 1.1.2.1. $\ell < k$. In this case $v^{r+md} \neq v^r$ and $P(v^r)$ separates the vertex v^{r+md} from $P(v)$. But this is impossible because v^{r+md} is adjacent to $x \in V(P(v))$.

Case 1.1.2.2. $\ell = k$.

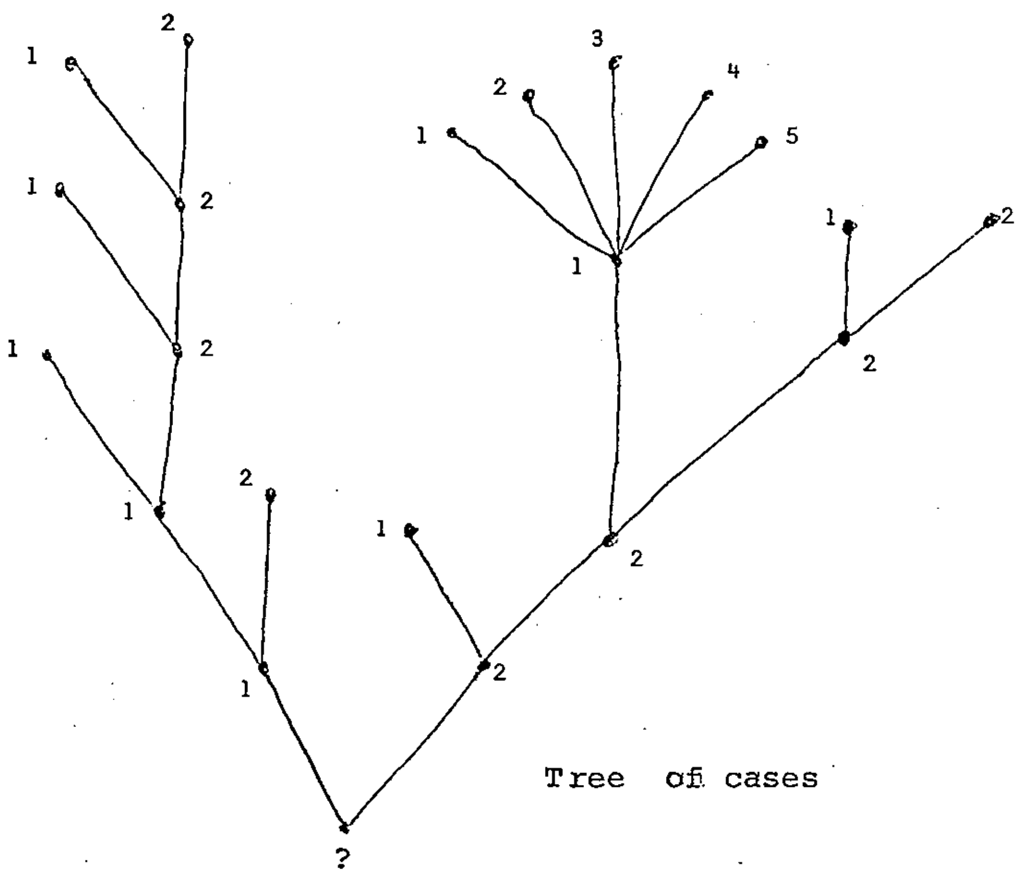
Case 1.1.2.2.1. $m = 2$. Then $x^{2d} = x$, and $x^{2s} = x$ or $x^{2s} = x^d$. If $x^{2s} = x^d$, then $2s \equiv d \bmod k$ and $v^{2s} = v^d$. Clearly the orbit of θ^s containing v has more elements than the orbit of θ^d containing v. Since also $\{x, x^s\} \subseteq N_R(v)$, this contradicts the choice of d. On the other hand, if $x^{2s} = x$, then $2s \equiv 0 \bmod k$. But also $2d \equiv 0 \bmod k$ because $x^{2d} = x$. Since $s \not\equiv 0 \bmod k$ and $d \not\equiv 0 \bmod k$, we must have $s \equiv d \bmod k$, which is again impossible.

Case 1.1.2.2.2. $m \geq 3$. It is not difficult to see that the only possible vertex in $V(P(v)) \cap J$ that can be adjacent to x^r is v^{-d}. Therefore v^{2r} must equal v^{-d}, $2r \equiv -d \bmod k$ and $v^{2(r+d)} = v^d$. Since $\{x, x^{r+d}\} \subseteq N_R(v)$, this, again, leads to a contradiction with the definition of d.

Case 1.2. $N_R(x) \cap J$ is contained in one orbit of θ^d. The existence of ψ can be shown by essentially the same argument as the one used in the following Case 2.

Case 2. $(d, k) = 1$. Then $m = \ell$.

Case 2.1. $x^{2d} = x$. Then $I = \{x, x^{1}\}$ and every vertex of I is adjacent to every vertex of J. Let ψ be the identity permutation of I.



Tree of cases

Figure 7

Case 2.2. $x^{2d} \neq x$.

Case 2.2.1. $k = \ell$.

Case 2.2.1.1. There is an integer r, $r \not\equiv -2d$, $r \not\equiv -d$, $r \not\equiv 0$ and $r \not\equiv d$ mod k, such that v^r is adjacent to x. Then any two of the three lines

$$\langle x, v^r \rangle, \quad \langle x^d, v^{r+d} \rangle, \quad \langle x^{2d}, v^{r+2d} \rangle$$

are separated by the polygon $P(v)$. This case is impossible.

Case 2.2.1.2. v^{-2d} and v^d are both adjacent to x. Then any two of the three lines

$$\langle x, v^d \rangle, \quad \langle x^d, v^{2d} \rangle, \quad \langle x^{2d}, v \rangle$$

are separated by $P(v)$. This is again impossible.

Case 2.2.1.3. $N_R(x) \cap J = \{v, v^{-d}, v^d\}$.

Then

$$N_R(x^i) \cap J = \{v^i, v^{i-d}, v^{i+d}\}$$

for every $i \in Z$. Let

$$\psi(x^i) = x^{-i}$$

Case 2.2.1.4. $N_R(x) \cap J = \{ v, v^{-d}, v^{-2d} \}$.

Then

$$N_R(x^i) \cap J = \{v^i, v^{i-d}, v^{i-2d}\}$$

for every $i \in Z$. Let

$$\psi(x^i) = x^{2d-i}$$

Case 2.2.1.5. $N_R(x) \cap J = \{v, v^{-d}\}$

Then

$$N_R(x^i) \cap J = \{v^i, v^{i-d}\}$$

for every $i \in Z$. Let

$$\psi(x^i) = x^{d-i} .$$

Case 2.2.2. $\ell < k$.

Case 2.2.2.1 $N_R(x) \cap J = \{v^{j\ell}, v^{j\ell-d} \mid j \in Z\}$.

Then

$$N_R(x^i) \cap J \ \{v^{i+j\ell}, v^{i+j\ell-d} \mid j \in Z\}$$

for every $i \in Z$. Let

$$\psi(x^i) = x^{d-i}$$

Case 2.2.2.2. $N_R(x) \cap J \neq \{ v^{j\ell}, v^{j\ell-d} \mid j \in Z\}$.

Let

$$v^r \in N_R(x) \cap J \setminus \{ v^{j\ell}, v^{j\ell-d} \mid j \in Z\} .$$

We can assume that $v^r \in V(P(v))$. Also

$$v^{r+\ell} \in N_R(x) \cap J ,$$

and $P(v)$ separates $v^{r+\ell}$ from each of the lines

$$< x, v^r > , < x^d , v^{r+d} > .$$

But $P(v)$ also separates these lines from each other, contradicting the planarity of R.

All cases have been exhausted and the proof of proposition 1.5.4 is finished.

1.6. Planarity of the rotor: the girth.

1.6.1. Throughout this section $\mathcal{R} = (R, \Theta, J, v)$ will be a rotor of order $k \geq 3$, R being a strict loopless connected planar graph. Assume that the automorphism Θ has also order k. Moreover, assume that R is embedded in the plane so as to realize a planar map M, the boundary of the outer face being the polygon

$$B = P(v, \Theta(v), \ldots, \Theta^{k-1}(v))$$

and the automorphism Θ being induced by some map-automorphism σ of M preserving incidence between vertices, lines and faces, and preserving the natural counterclockwise cyclic order of edges around any vertex as well as around any face. There is no loss of generality in assuming that the sequence

$$v, \Theta(v), \ldots, \Theta^{k-1}(v)$$

is a counter-clockwise description of the boundary polygon B. Then it is clear that the outer face must be fixed by σ. Also the polygon B has no chords, or equivalently:

1.6.11. PROPOSITION. B is an induced subgraph of R.

Proof. Indeed, suppose that A is a line joining two non-consecutive vertices of B. Then A and σ (A) are separated by B, contradicting the assumption that both are drawn in the interior of B.

1.6.2. If x is a vertex of R, let d(B,x) be the minimum length of a path between x and a vertex of B.

1.6.21. PROPOSITION. None of the orbits of Θ has cardinality less than k and greater than 1. Θ has at most 1 fixed vertex.

Proof. We first prove that if Θ has a fixed vertex x, then the cardinality of every orbit I not containing x is k. Then we complete the proof of the proposition by showing that if Θ has an orbit of cardinality less than k, then Θ has a fixed vertex.

Let x be a fixed vertex of Θ such that d(B,x) is smallest possible. Let I be an orbit of Θ, $x \notin I$. Choose a vertex $y \in I$. Let P be a path of length d(B, x) between x and a vertex $\Theta^i(v)$ of B. For every $j \in Z$,

$$P_j = \sigma^{j-i}(P)$$

is path of length $d(B,x)$ from x to $\theta^j(v)$. It follows from the minimality of $d(B,x)$ that any two of the paths

$$P_1, \ldots, P_k$$

have only the vertex x in common. Consequently if y lies on one of the P_j's, $j = 1, \ldots, k$, then

$$|I| = |\{\theta^i(y) \mid i = 1, \ldots, k\}| = k \quad .$$

If this is not the case, then there is some j such that y lies in the interior of the polygon formed by P_j, P_{j+1} and the line $<\theta^j(v), \theta^{j+1}(v)>$. Again

$$\theta(y), \ldots, \theta^k(y)$$

must be all distinct and $|I| = k$.

Assume now that θ has no fixed vertex but has orbits of cardinality less than k. Let x be a vertex such that

$$\theta^{\ell}(x) = x$$

for some integer $0<\ell< k$, and assume that $d(B,x)$ is smallest possible. Since ℓ divides k and $k \geq 3$, we must have

$$\ell + 1 < k \quad .$$

Also $1<\ell$ because otherwise x would be a fixed vertex of Θ . Let P be a path of length $d(B,x)$ between x and a vertex $\Theta^i(v)$ of B. As before, let

$$P_j = \sigma^{j-i} (P)$$

for every $j \in \mathbb{Z}$. The subgraph union of P_0 and P_ℓ must have a common vertex y with the union of P_1 and $P_{\ell+1}$, because otherwise they would be separated by the polygon P. It is easy to see that

$$\Theta^n (y) = y$$

for some integer $0 < n \leq \ell +1$. Since $\ell +1 < k$,

$$d(B, y) < d(B, x)$$

contradicts the minimality of $d(B, x)$.

1.6.3. Let n be a positive integer, $n < k$. If there is a path P joining v to $\theta^n(v)$ such that

(i) P is internally disjoint from B, and

(ii) no fixed vertex of θ lies on P,

then let $p(n)$ be the minimum length of such a path P. If no such path exists between v and $\theta^n(v)$, then we write $p(n) = \infty$.

1.6.31. PROPOSITION. Let $0 < n < k$. Then $p(n) = \infty$ or $p(n) > \min(n, k-n)$.

Proof. Assume that the proposition is false. Among those $n \in [1, k-1]$, $d(n) \neq \infty$, that maximize the value of

$$\min(n, k-n) - p(n) ,$$

choose one for which $\min(n, k-n)$ is smallest possible. Since B is chordless,

$$2 \leq n \leq k-2 .$$

We can also assume without loss of generality that $n \leq k/2$. For this fixed n let the vertices of a shortest path P, internally disjoint from B, joining v to $\theta^n(v)$ be, in consecutive order,

$$v = x_0, x_1, \ldots, x_{p(n)} = \theta^n(v) .$$

Since P and $\sigma(P)$ are not separated by B, they must have an internal vertex in common. Let i be the smallest integer with $x_i \in V(\sigma(P))$, say

$$x_i = \theta(x_j) .$$

By assumption no fixed vertex of θ lies on P and therefore $i \neq j$.

If $i < j$, then the path P' joining v to $\theta^{n+1}(v)$ consisting of

$$P[\{x_0, \ldots, x_i\}]$$

and

$$\sigma(P)[\{\theta(x_j), \ldots, \theta(x_{p(n)})\}]$$

is internally disjoint from B and has length at most $p(n) - 1$. Consequently

$$p(n+1) < p(n) .$$

Also

$$\min(n+1, k-(n+1)) \geq \min(n, k-n) - 1$$

If

$$\min(n+1, k-(n+1)) \geq \min(n, k-n) ,$$

then

$$\min(n+1, k-(n+1)) - p(n+1) > \min(n, k-n) - p(n) ,$$

contradicting the choice of n. If

$$\min(n+1, k-(n+1)) = \min(n, k-n) - 1 ,$$

which happens only if k is even and $n = k/2$, then still

$$\min(n+1, k-(n+1)) - p(n+1) \geq \min(n, k-n) - p(n) ,$$

(in fact equality must hold), so that the minimality of

$$\min(n, k-n)$$

in contradicted.

If $i > j$, then the subgraph of R consisting of

$$\sigma(P)\ [\ \{\ \Theta(x_o)\ ,\ \ldots\ ,\ \Theta(x_j)\}\ ]$$

and

$$P[\ \{x_i\ ,\ \ldots\ ,\ x_{d(n)}\ \}\ ]$$

must contain a path P' joining $\theta(v)$ to $\theta^n(v)$ and internally disjoint from B. Since the length of P' is less than $p(n)$, so is the length of $\sigma^{-1}(P')$. But $\sigma^{-1}(P')$ is a path joining v to $\theta^{n-1}(v)$ and internally disjoint from B. This means that

$$p(n - 1) < p(n) .$$

Since $n \le k/2$, we also have

$$\min (n-1, k-(n-1)) = \min (n,k-n) - 1 ,$$

which implies

$$\min (n-1, k-(n-1)) - p(n-1) \ge \min (n,k-n) - p(n)$$

(in fact equality must hold), and contradicts the minimality of

$$\min (n, k - n) .$$

The proof of proposition 1.6.31 is finished.

1.6.4. Let π be a border-partition of R . Since

$$E (R(\pi)) = E (R) ,$$

for every subgraph M of $R(\pi)$, $E(M) \subseteq E(R)$. The subgraph of R spanned by $E(M)$ is then called the <u>subgraph of R corresponding to M.</u> Also for every subgraph N of R, $E(N) \subseteq E(R(\pi))$, and the subgraph of $R(\pi)$ spanned by $E(N)$ is called the <u>subgraph of $R(\pi)$ corresponding to N.</u>

Let C be a polygon of $R(\pi)$. A <u>maximal segment</u> of C is a subgraph P of C such that

(i) P is a path of length at least 2,

(ii) both terminal vertices of P belong to $V(B(\pi))$,

(iii) P is internally disjoint from $B(\pi)$.

1.6.41. PROPOSITION. Let π be a border-partition of R and let the positive integer g be the girth of $R(\pi)$. Let C be a polygon of $R(\pi)$ having length g. Then exactly one of the following three conditions holds:

1. $|V(C) \cap V(B(\pi))| \leq 1$,
2. C is a subgraph of $B(\pi)$,
3. C has exactly 1 maximal segment.

Proof. Suppose that C has two distinct maximal segments M_1 and M_2. Let N_1 and N_2 be the corresponding subgraphs of R. By proposition 1.6.21 , Θ

has at most one fixed vertex, and therefore one of N_1 and N_2, say N_1 , does not contain any fixed vertex of Θ . Let P be a shortest path in B between the two terminal vertices of the path N_1. By proposition 1.6.31, the length of P is less than the length of N_1 . Replacing in C, M_1 by the subgraph of $R(\pi)$ corresponding to P, we obtain a subgraph that may not be a polygon, but that will necessarily contain a polygon C' such that

(i) M_2 is a subgraph of C' ,
(ii) the length of C' is smaller than that of C.
If follows from (i) that the length of C' is at least 3. But than (ii) contradicts the assumption that the length of C is the girth g of $R(\pi)$.

1.6.5. PROPOSITION. For every border-partition π of R , $R(\pi)$ and $R(\phi\pi)$ have the same girth.

Proof. We have only to prove that the girth of $R(\pi)$ is not less than the girth of $R(\phi\pi)$. Then, by applying the argument again to the border-partition $\phi\pi$, we would obtain the desired result.

Let the positive integer g be the girth of $R(\pi)$ and let C be a polygon of $R(\pi)$ having length g.

We shall find a polygon C' of $R\ (\phi\pi)$ having the same length.

If

$$V(C) \cap V(B(\pi)) = \emptyset \quad ,$$

then C is a polygon of $R(\phi\pi)$ and we let $C' = C$.

If

$$|V(C) \cap V(B(\pi))| = 1 \ ,$$

then the subgraph S of R corresponding to C is either a polygon or a path. If it is a polygon, then the subgraph of $R(\phi\pi)$ corresponding to S is also a polygon C'. If it is a path P with terminal vertices $\theta^i(v)$ and $\theta^j(v)$, then

$$\theta^i(v) \equiv \theta^j(v) \mod \pi$$

and

$$\theta^{-i}(v) \equiv \theta^{-j}(v) \mod \phi\pi \quad ,$$

so that the subgraph of $R(\phi\pi)$ corresponding to σ^{-i-j} (P) is a polygon C', having still the same length as C.

If C is a subgraph of $B(\pi)$ then, since $B(\pi)$ and $B(\phi\pi)$ are isomorphic, $B(\phi\pi)$ must also have a polygon C' of the same length as C.

According to proposition 1.6.41, there remains only the case where C has exactly one maximal segment M. Let the corresponding subgraph of R be the path N with terminal vertices $\theta^{i}(v)$ and $\theta^{j}(v)$. Also there is a path in $B(\pi)$, subgraph of C, joining the vertex of $R(\pi)$ containing $\theta^{i}(v)$ to the vertex containing $\theta^{j}(v)$. (Recall that the vertices of $R(\pi)$ are sets of vertices of R.) Consequently there is a path P in $B(\phi\pi)$ joining the vertex containing $\theta^{-i}(v)$ to the vertex containing $\theta^{-j}(v)$. But P, together with the path in $R(\phi\pi)$ corresponding to $\sigma^{-i-j}(N)$, forms a polygon C' having the same length g as C.

The proof of proposition 1.6.5. is finished.

1.7. Planarity of the stator

1.7.1. Let R be a graph and $\mathcal{R} = (R, \Theta, J, v)$ a rotor of order k such that v is adjacent to $\Theta(v)$. Let $(\mathcal{R}, S, w)$ be a motor. In 1.3.3. the subgraph $\bar{S}$ of $M(\mathcal{R}, S, w)$ was defined by

$$V(\bar{S}) = J \cup [\, V(S) \setminus w(J) \,] ,$$

$$E(\bar{S}) = E(S) .$$

$(\mathcal{R}, S, w)$ is called a <u>planar motor</u> if

(i) the subgraph Q of $M(\mathcal{R}, S, w)$ consisting of $\bar{S}$ and the polygon

$$P = P(v, \Theta(v), \ldots, \Theta^{k-1}(v))$$

is planar

(ii) P do not have overlapping bridges in Q.

The chromatic polynomial is said to be <u>planar motor invariant</u> with respect to $\mathcal{R}$ if for every planar motor $(\mathcal{R}, S, w)$

$$P(M(\mathcal{R}, S, w), \lambda) = P(M(\mathcal{R}, S, w\phi), \lambda)$$

Let ℓ be a non-negative integer. The <u>lowest ℓ coefficients of the chromatic polynomial are planar motor invariant</u> with respect to R if for every planar motor (R,S,w) and every integer j, $0 \le j \le \ell - 1$, the coefficient of λ^j is the same in $P(M(R,S,w),\lambda)$ and in $P(M(R,S,w\phi),\lambda)$.

1.7.2. Let the rotor $R = (R,\theta,U,v)$ be as above. A border-partition π of R is called <u>planar</u> if there are no four integers i_1, i_2, i_3, i_4 with

(i) $0 \le i_1 < i_2 < i_3 < i_4 < k$,

(ii) $\theta^{i}(v) \equiv \theta^{i_3}(v) \mod \pi$,

(iii) $\theta^{i_2}(v) \equiv \theta^{i_4}(v) \mod \pi$,

(iv) $\theta^{i_1}(v) \not\equiv \theta^{i_2}(v) \mod \pi$.

The chromatic polynomial is said to be <u>planar partition-invariant</u> with respect to R if for every planar border-partition π

$$P(R(\pi),\lambda) = P(R(\phi\pi),\lambda) .$$

Let ℓ be a non-negative integer. The <u>lowest ℓ coefficients of the chromatic polynomial are planar partition invariant</u> with respect to R if for every planar border-partition π and every integer j, $0 \leq j \leq \ell - 1$, the coefficient of λ^j is the same in $P(R(\pi),\lambda)$ and in $P(R(\phi\pi),\lambda)$.

1.7.3. PROPOSITION. Let the rotor $R = (R,\Theta,J,v)$ be as above.

(i) For every non-negative integer ℓ the lowest ℓ coefficients of the chromatic polynomial are planar motor invariant with respect to R if and only if they are planar partition invariant.

(ii) The chromatic polynomial is planar motor invariant with respect to R if and only if it is planar partition invariant.

Proof. Analogous to that of proposition 1.3.5.

To prove the "if" part of (i), we observe that if (R,S,w) is a planar motor, then so is $(R,S,\phi w)$ and for every spanning subgraph T of S, the border partitions $\pi_{\overline{T}}$ and $\pi_{\overline{\overline{T}}}$ are planar. Therefore planar partition invariance of the lowest ℓ coefficients implies planar motor invariance as in the proof of proposition 1.3.5. Also a slight modification of the argument used in proving proposition 1.3.5. shows the validity of the "only if" part of (i). Given the border partition π_o, this time planar, we have to construct a planar motor (R, S, w) such that w is surjective onto $V(S)$ and

$$\pi_{\bar{S}} = \pi_0 \quad ,$$

but for every proper spanning subgraph T of S,

$$\pi_{\bar{T}} < \pi_0 \quad .$$

This can be done as follows. Define, as in the general case of hypergraphs, w to be a bijection $w: V(R) \longrightarrow V(S)$, where $V(S)$ is any set disjoint from $V(R)$. For each block B of π_0 ,

$$B = \{ \theta^{i_1}(v) , \ldots , \theta^{i_t}(v) \} ,$$

$$0 \le i_1 < \ldots < i_t < k ,$$

let a connected component of S induce a path with consecutive vertices

$$w(\theta^{i_1}(v)) , \ldots , w(\theta^{i_t}(v)) .$$

Clearly the stator S satisfies all the requirements, and the remaining part of the argument used for general hypergraphs is valid also in this case.

Finally, (ii) follows from (i) exactly as in the proof of proposition 1.3.5.

1.7.4. In view of the above proposition, we call the chromatic polynomial simply planar invariant with respect to $\mathcal{R}$, if it is planar motor invariant or equivalently, planar partition invariant.

1.7.5. Planar rotors of order 7 bounded by a heptagon.

Let $\mathcal{R} = (R, \theta, J, v)$ be a rotor of order 7, where R is a graph in which v is adjacent to $\theta(v)$. As in 1.3.7, we call a border partition π admissible if for every integer i

$$\theta^{i}(v) \not\equiv \theta^{i+1}(v) \mod \pi .$$

We have then the following:

1.7.51. PROPOSITION. Every admissible planar border partition of $\mathcal{R}$ is bilaterally symmetric.

Proof. Let $v^{i} = \theta^{i}(v)$ for every integer i. Consider the following 6 admissible planar border partitions:

$$\pi_1 = \{\{ v^1, v^3, v^6\}, \{v^2\}, \{v^4\}, \{v^5\}, \{v^7\}\} ,$$

$$\pi_2 = \{\{ v^1 \}, \{ v^3\}, \{v^6\}, \{ v^2, v^4\}, \{v^5, v^7\}\} ,$$

$$\pi_3 = \{\{ v^1 \}, \{ v^3\}, \{v^4\}, \{v^5\}, \{v^6\}, \{ v^2, v^7\}\} ,$$

$$\pi_4 = \{\{ v^1 \}, \{ v^4\}, \{v^5\}, \{v^2, v^7\}, \{ v^3, v^6\}\} ,$$

$$\pi_5 = \{\{ v^1 \}, \{ v^2\}, \{v^4\}, \{v^5\}, \{v^7\}, \{ v^3, v^6\}\} ,$$

$$\pi_6 = \{\{ v^1 \}, \{ v^2\}, \{v^3\}, \{v^4\}, \{v^5\}, \{ v^6\}, \{v^7\}\} .$$

These border partitions are bilaterally symmetric. Also every admissible planar border partition of $\mathcal{R}$ is of the form $\theta^i(\pi_k)$, $i \in \mathbb{Z}$, $1 \leq k \leq 6$ (k unique). As in proposition 1.3.71, the result follows.

1.7.52. Remark. Not every admissible border partition of $\mathcal{R}$ is bilaterally symmetric. A counter-example is the partition

$$\pi = \{\{v^1, v^3, v^6\}, \{ v^5, v^7\}, \{ v^2\}, \{ v^4\}\} .$$

1.7.53. PROPOSITION. Let $\mathcal{R} = (R, \theta, J, v)$ be a rotor of order 7, where R is a graph such that v is adjacent to $\theta(v)$. Then the chromatic polynomial is planar invariant with respect to $\mathcal{R}$.

Proof. Analogous to that of proposition 1.3.72.

1.7.54. Example. Let R be the strict graph with vertex set $V(R) = [0, 13]$ depicted in Figure 8.

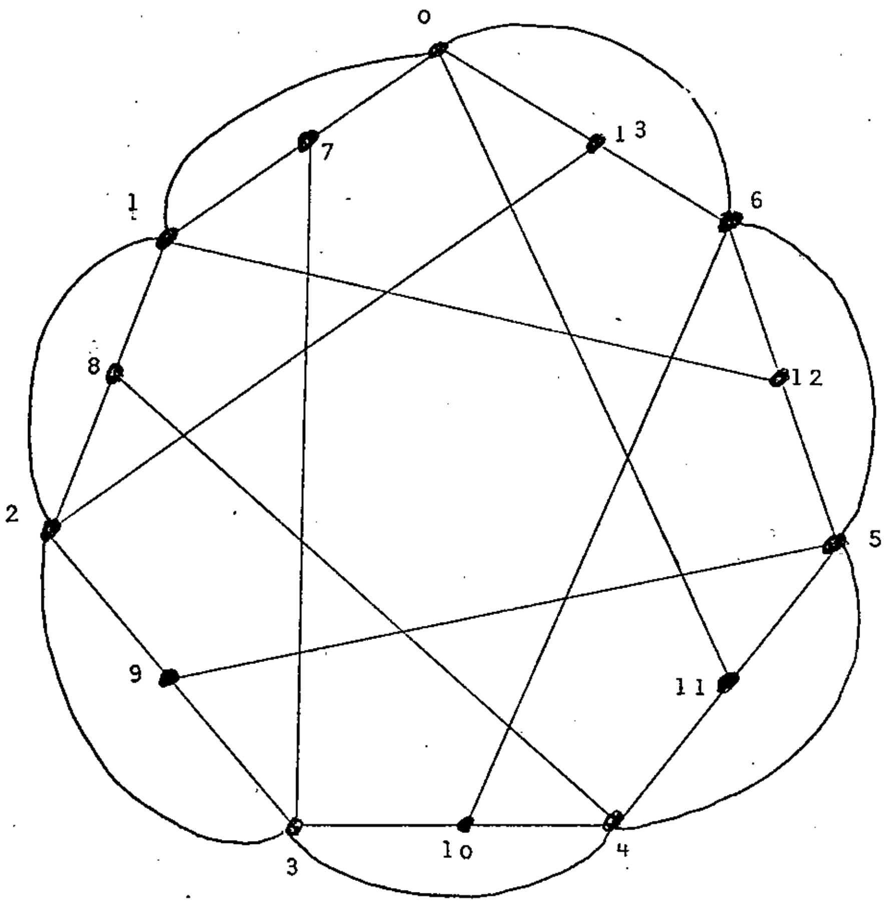


Figure 8

Let

$$\Theta = (0,2,3,4,5,6)\ (7,8,9,10,11,12,13)\ ,$$

$$J = [0,6]\quad ,\qquad v = 0\ .$$

Then $\mathcal{R} = (R,\Theta,J,v)$ is a rotor satisfying the hypotheses of proposition 1.7.53. We note that R is not a planar graph. Figure 9 displays a pair of non-isomorphic graphs, obtained from $\mathcal{R}$, having the same chromatic polynomial.

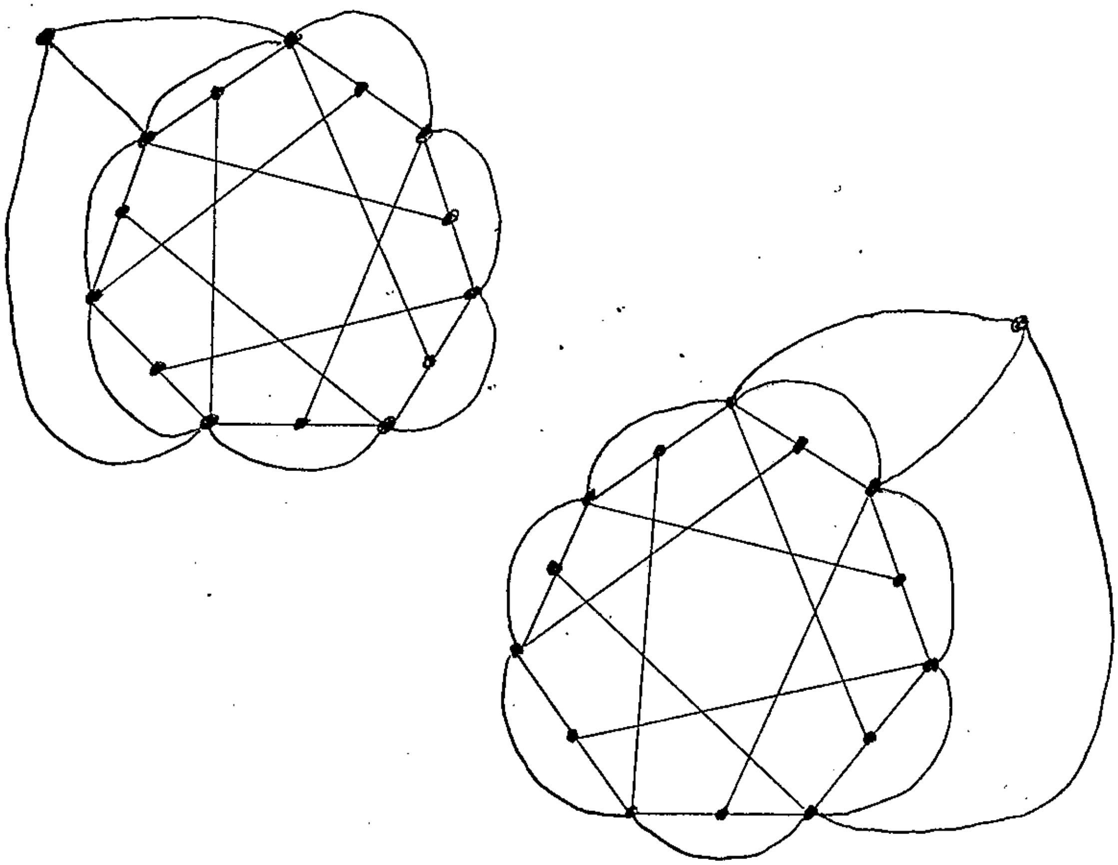


Figure 9

CHAPTER 2

AUTOMORPHISMS OF SUBSYSTEMS

2.1. This section deals with the automorphism group of a general graph G and that of a subsystem H induced by a constituent of Aut G.

Given n groups $A_1, \ldots, A_n$, $n \geq 3$, and n-1 group homomorphisms

$$h_1 : A_1 \to A_2 \;, \; \ldots \;, \; h_{n-1} : A_{n-1} \longrightarrow A_n \quad ,$$

the sequence

$$h_1 \;, \; \ldots \;, \; h_{n-1} \;,$$

represented diagrammatically as

$$A_1 \xrightarrow{h_1} A_2 \qquad \cdots \qquad A_{n-1} \xrightarrow{h_{n-1}} A_n$$

is called an <u>exact sequence</u> (see [L 1]) if for every $1 \leq i \leq n-2$

$$\text{Im } h_i = \text{Ker } h_{i+1} \quad .$$

Let a diagram D represent groups $A_1, \ldots, A_n, B_1, \ldots, B_n$ and group homomorphisms

$$h_1: A_1 \longrightarrow B_1, \ldots, h_n: A_n \longrightarrow B_n .$$

Suppose that for every pair of sequences $h_{i_1}, \ldots, h_{i_k}$ and $h_{j_1}, \ldots, h_{j_m}$ such that

(i) for every $1 \leq t \leq k-1, B_{i_t} = A_{i_{t+1}}$,

(ii) for every $1 \leq t \leq m-1, B_{j_t} = A_{j_{t+1}}$,

(iii) $A_{i_1} = A_{j_1}$, $B_{i_k} = B_{j_m}$,

the compositions $h_{i_k} \ldots h_{i_1}$ and $h_{j_m} \ldots h_{j_1}$ are equal. Then the diagram D is said to be <u>commutative</u>.

We denote the trivial group Z_1 by 0.

Let G be a general graph and let U be a constituent of Aut G. Let

$$\mathrm{Fix}\ (G, G[U]) = \bigcap_{x \in U} (\mathrm{Aut}\ G)_x$$

be the subgroup of those automorphisms of G that fix every vertex of the subsystem G[U] induced by U.

Let r: Aut $G \longrightarrow$ Aut $G[U]$ be the restriction homomorphism,

$$r(\sigma) = \sigma|U$$

for every $\sigma \in$ Aut G.

2.1.1. PROPOSITION. The sequence

$$0 \longrightarrow \text{Fix}(G, G[U]) \longrightarrow \text{Aut } G \xrightarrow{r} \text{Aut } G[U] \quad ,$$

where the unlabelled arrows stand for the canonical group homomorphisms, is exact.

Proof. Follows without difficulty from the definitions.

The main proposition of this section is the following:

2.1.2. PROPOSITION. Given a general graph H and an exact sequence of group homomorphisms

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} \text{Aut } H \quad ,$$

there exists a general graph G such that

(i) H is a subsystem of G induced by a constituent of Aut G ,

(ii) for some group isomorphisms $h\colon \mathrm{Fix}\,(G, H) \longrightarrow A$ and $k\colon \mathrm{Aut}\ G \longrightarrow B$, the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[dr, "g"] & \\
 & & & \mathrm{Aut}\ H \\
0 \arrow[r] & \mathrm{Fix}(G,H) \arrow[r] \arrow[uu, "h"] & \mathrm{Aut}\ G \arrow[uu, "k"'] \arrow[ur] &
\end{tikzcd}$$

is commutative.

Proof. Deferred to 2.1.4.

2.1.3. PROPOSITION. Given a general graph H and a subgroup B of Aut H, there exists a general graph G such that (i) H is a subsystem of G induced by a faithful constituent of Aut G,

(ii) $\mathrm{Aut}\ G\,|V(H) = B$.

Proof. Let α be the cardinality of V(H). α is the first ordinal number with $|W(\alpha)| = |V(H)|$. We can in fact assume that

$$V(H) = W(\alpha) \quad .$$

Assume also that $\alpha \geq 2$.

For each $\sigma \in B$ let v_σ be a new element not belonging to $V(H)$,

$$v_\sigma = v_\tau \Rightarrow \sigma = \tau$$

We write

$$V_B = \{v_\sigma | \sigma \in B\} \quad .$$

For every $(\sigma, \beta) \in B \times V(H)$ let $G(\sigma, \beta)$ be a strict digraph such that

(i) the vertices of $G(\sigma, \beta)$ are in one-to-one correspondence with $[0, \sigma^{-1}(\beta) + 2]$,

$$V(G(\sigma,\beta)) = \{w_\gamma \mid 0 \leq \gamma \leq \sigma^{-1}(\beta) + 2\} \quad ,$$

$$w_0 = v_\sigma \quad , \quad w_{\sigma^{-1}(\beta)+2} = \beta \quad ,$$

(ii) there is a dart from w_γ to w_δ if and only if $\gamma < \delta$.

Moreover, assume that for every σ , $\sigma' \in B$ and β , $\beta' \in V(H)$

$$V(G(\sigma,\beta)) \cap (V_B \cup V(H)) = \{v_\sigma , \beta\}$$

and

$$V(G(\sigma, \beta)) \cap V(G(\sigma', \beta')) \subseteq V_B \cup V(H)$$

if $\sigma \neq \sigma'$ or $\beta \neq \beta'$. Let also

$$E(G(\sigma, \beta)) \cap E(G(\sigma', \beta')) = \emptyset$$

if $\sigma \neq \sigma'$ or $\beta \neq \beta'$, and

$$E(G(\sigma, \beta)) \cap E(H) = \emptyset$$

for every σ and β .

We define G by

$$G = H \cup \bigcup_{\substack{\sigma \in B \\ 0 \leq \beta < \alpha}} G(\sigma, \beta)$$

Obviously H is an induced subsystem of G. We have

$$V(H) = \{x \in V(G) \mid N^+(x) \subseteq \{x\} \quad \text{or } \exists y \in N^-(x) \text{ with } N^+(x) \nsubseteq N^+(y)\} .$$

Also

$$V_B = \{x \in V(G) \mid d^-(x) = 0\} \quad .$$

Consequently $V(H)$ and V_B are constituents of Aut G.

For a $\Theta \in$ Aut G let us write $\Theta_H = \Theta | V(H)$. Let i be the identity element of the group B. Let

$$v_\tau = \Theta \; (v_i) \quad .$$

For every $\beta \in V(H)$, $\Theta(V(G(i,\beta))) = V(G(\tau, \Theta(\beta)))$.
Consequently $G(i,\beta)$ is isomorphic to $G(\tau, \Theta(\beta))$, so that

$$\tau^{-1} \; (\Theta(\beta)) + 2 = \; i^{-1} \; (\beta) \; + 2 \quad ,$$

$$\tau^{-1} \; (\Theta_H(\beta)) \; = \; \beta \quad ,$$

i.e.

$$\Theta_H \; (\beta) \; = \tau(\beta) \quad ,$$

implying that

$$\Theta_H = \tau \; \in \; B \; .$$

Thus

$$\text{Aut } G \; | V(H) \subseteq \; B \; .$$

Further, for every $\tau, \sigma \in B$ and $\beta \in V(H)$ there is a (unique) isomorphism between the digraphs $G(\sigma, \beta)$ and $G(\tau\sigma, \tau(\beta))$. It can be then verified that for every $\tau \in B$ there exists an automorphism Θ_τ of G such that

$$\Theta_\tau(\beta) = \tau(\beta)$$

for every $\beta \in V(H)$ and

$$\Theta_\tau(v_\sigma) = v_{\tau\sigma}$$

for every $v_\sigma \in V_B$. Clearly

$$\Theta_\tau \mid V(H) = \tau .$$

This proves the equality

$$\text{Aut } G \mid V(H) = B .$$

To prove that V(H) is a faithful constituent of Aut G, we have to show that

$$\Theta_H = \Theta'_H$$

holds only if $\Theta = \Theta'$.

Since for $\tau, \sigma \in B$ and $\beta \in V(H)$ the digraph isomorphism between $G(\sigma,\beta)$ and $G(\tau\sigma,\tau(\beta))$ is unique, it suffices to show that for every $\Theta \in \text{Aut } G$, if

$$\Theta_H = \tau \in B \quad ,$$

then

$$\Theta(v_\sigma) = v_{\tau\sigma}$$

for every $v_\sigma \in V_B$. Let indeed

$$\Theta(v_\sigma) = v_\pi \quad .$$

For every $\beta \in V(H)$,

$$\Theta(V(G(\sigma,\beta))) = V(G(\pi,\tau(\beta))) \quad ,$$

so that

$$\sigma^{-1}(\beta) + 2 = \pi^{-1}\tau(\beta) + 2 \ .$$

It follows that

$$\sigma^{-1} = \pi^{-1}\tau \quad ,$$

i.e.

$$\pi = \tau\sigma \quad ,$$

proving our claim. The proof of the proposition is complete.

Remark. In the above construction no vertex of G has degree 0 .

2.1.4. Proof of proposition 2.1.2.

Im g is a subgroup B_o of Aut H. According to proposition 2.1.3. we can construct a general graph G_o with no vertices of degree 0 such that

(i) H is a subsystem of G_o induced by a faithful constituent of Aut G_o,

(ii) Aut $G_o | V(H) = B_o$.

To continue the construction, we assume that B is disjoint from $V(G_o)$, We define a general graph H_1 by

$$V(H_1) = V(G_o) \cup B ,$$

$$E(H_1) = E(G_o)$$

and the requirement that G_o be a subsystem of H_1. Since

$$V(G_o) = \{x \in V(H_1) \mid d(x) \neq 0\} ,$$

$V(G_o)$ and B are constituents of Aut H_1 , and so is V(H).

For every $\sigma \in B_0$, let $\bar{\sigma}$ be the unique automorphism of G_0 whose restriction to V(H) is σ. For every $\alpha \in B$, define an automorphism τ_α of H_1 by

$$\tau_\alpha(\beta) = \alpha \beta .$$

for every $\beta \in B$ and

$$\tau_\alpha(v) = \overline{g(\alpha)} \quad (v)$$

for every $v \in V(G_0)$. Clearly

$$B_1 = \{\tau_\alpha \mid \alpha \in B\}$$

is a subgroup of Aut H_1. Let G be a general graph such that

(i) H_1 is a subsystem of G induced by a faithful constituent of Aut G,

(ii) Aut G $| V(H_1) = B_1$.

H is an induced subsystem of G and V(H) is a constituent of Aut G. Let i denote the identity element of the group B. Since B is a constituent of Aut G, we can define k by

$$k(\sigma) = \sigma(i)$$

for every $\sigma \in$ Aut G. Clearly for every $\sigma \in$ Fix (G, H), $g\,k(\sigma)$ is the identity automorphism of H, i.e.

$$k(\sigma) \in \operatorname{Ker} g = \operatorname{Im} f .$$

On the other hand, since f is injective, $f^{-1}(\gamma)$ is well defined for every $\gamma \in \operatorname{Im} f$. Let

$$h(\sigma) = f^{-1}(k(\sigma))$$

for every $\sigma \in$ Fix (G, H). There is no difficulty in verifying that G, k, h satisfy the requirements of proposition 2.1.2.

Remark. If H is finite. then so is G.

2.2. k- uniform hypergraphs.

2.2.1. For every integer $k \geq 2$ define a simple k - uniform hypergraph A_k by

$$V(A_k) = [1, 2k+2] ,$$

$$E(A_k) = \{[i, i+k-1] \mid 1 \leq i \leq k+2 \}$$

$$\cup \{ [i, i+k-2] \cup \{ 2k+2\} \mid i = 3, k+2\} .$$

A_2 is pictured in Figure 10.

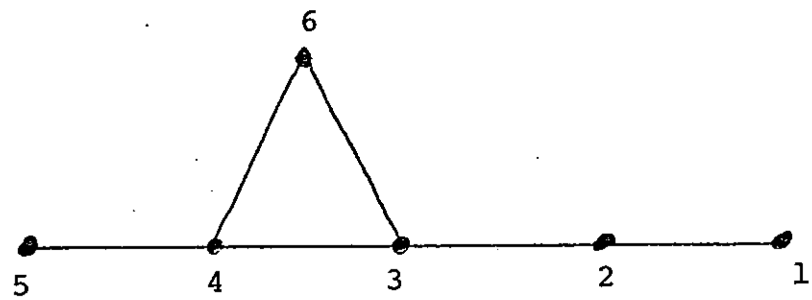


Figure 10

A hypergraph H is called a <u>k-arrow</u> from x to y, where $x, y \in V(H)$, if its vertices can be labelled $v_1, \ldots, v_{2k+2}$ in such a way that

(i) $v_1 = x$ and $v_{2k+1} = y$,

(ii) the mapping $i \longrightarrow v_i$ $(i=1, \ldots, 2k+2)$ is an isomorphism from A_k to H.

2.2.11. LEMMA. The automorphism group of any k-arrow is trivial.

Proof. We show that Aut A_k is trivial for any $k \geq 2$. First we observe that A_k has exactly two vertices of degree one, namely 1 and 2k+1. It has also two vertices of degree two, namely 2 and 2k+2. But 1 is the only vertex of degree one that has a neighbour of degree two. Also 2 is the only vertex of degree two that has a neighbour of degree one. Consequently every automorphism of A_k leaves fixed each of the vertices 1, 2, 2k+1 and 2k+2 . Suppose Aut A_k is not trivial, and let i be the smallest integer, $i \in [1, 2k+2]$, which is not fixed by every automorphism of A_k. Clearly $3 \leq i \leq 2k$. Both $[1, i-1]$ and $[i, 2k+1]$ are constituents of Aut A_k. If $i \leq k+1$, then i and $2k+1$ are the only vertices of degree one in the sub-hypergraph of A_k induced by $[i, 2k+1]$, and since $2k+1$ is fixed by every automorphism of A_k, so is i. Since this contradicts the choice of i, $i > k+1$. Then i has degree one in the sub-hypergraph of A_k induced by

$$[1, i-1] \cup \{i\} = [1, i] \quad ,$$

while any other vertex $j \in [i, 2k+1]$ has degree zero in the sub-hypergraph induced by

$$[1, i-1] \cup \{j\} \quad .$$

It follows that i has to be fixed by every automorphism of A_k, a contradiction proving the lemma.

2.2.2. For every $k \geq 2$ and every positive integer n define a k-uniform simple hypergraph $C_{k,n}$ by

$$V(C_{k,n}) = [1, n] ,$$
$$E(C_{k,n}) = \{[i, i+k-1] \mid 1 \leq i \leq n-k+1\} .$$

$C_{2,5}$ is pictured in Figure 11.

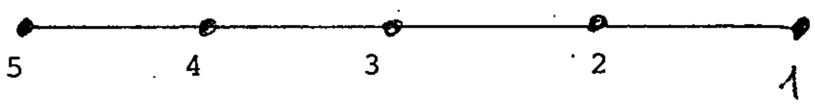


Figure 11

Let H be a hypergraph, $x \in V(H)$. H is called a <u>k-chain</u> of length n with endpoint x if there is an isomorphism $f: H \longrightarrow C_{k,\, n+1}$ such that $f(x) = 1$.

2.2.21. LEMMA. Let H be a k-chain of length at least 2k-2 with endpoint x. Then the stabilizer $(\mathrm{Aut}\ H)_x$ is trivial.

Proof. We show that if $n \geq 2k-2$, then $(\mathrm{Aut}\ C_{k,n+1})_1$ is trivial. Since the only vertices of degree one of $C_{k,\, n+1}$ are 1 and n+1, every automorphism in $(\mathrm{Aut}\ C_{k,\, n+1})_1$ fixes also n+1. Suppose $(\mathrm{Aut}\ C_{k,n+1})_1$ is not trivial and let i be the smallest integer, $i \in [1, n+1]$ which is not fixed by every automorphism in $(\mathrm{Aut}\ C_{k,\, n+1})_1$. Both $[1, i-1]$ and $[i, n+1]$ are constituents of $(\mathrm{Aut}\ C_{k,\, n+1})_1$. Distinguishing the cases $i \leq k - 1$ and $i > k - 1$, contradictions are obtained as in the proof of Lemma 2.2.11.

2.2.3. PROPOSITION. Let k be any integer ≥ 2. Given a k-uniform hypergraph H and an exact sequence of group homomorphisms

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} \mathrm{Aut}\ H ,$$

there exists a k-uniform hypergraph G such that

(i) H is a sub-hypergraph of G induced by a constituent of Aut G ,

(ii) for some group isomorphisms $h\colon \mathrm{Fix}\,(G,H) \longrightarrow A$ and $k\colon \mathrm{Aut}\,G \longrightarrow B$, the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[dr, "g"] & \\
 & & & \mathrm{Aut}\ H \\
0 \arrow[r] & \mathrm{Fix}(G,H) \arrow[uu, "h"] \arrow[r] & \mathrm{Aut}\ G \arrow[uu, "k"] \arrow[ur] &
\end{tikzcd}$$

is commutative.

Proof. Following the proof of proposition 2.1.2, we construct a general graph G_o such that

(i) the hypergraph H is a subsystem of G_o induced by a constituent of Aut G_o,

(ii) there are isomorphisms h_o and k_o making the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[dr, "g"] & \\
 & & & \mathrm{Aut}\ H \\
0 \arrow[r] & \mathrm{Fix}(G_o,H) \arrow[uu, "h_o"] \arrow[r] & \mathrm{Aut}\ G_o \arrow[uu, "k_o"] \arrow[ur] &
\end{tikzcd}$$

commutative,

(iii) every dart is incident with exactly 2 vertices, and no two distinct darts are incident with the same 2 vertices,

(iv) every edge in $E(G_o) \setminus E(H)$ is a dart .

We now construct a k-uniform hypergraph G such that

(i) $V(G_o) \subseteq V(G)$,

(ii) $V(G_o)$ is a faithful constituent of Aut G,

(iii) Aut G $|V(G_o)$ = Aut G_o ,

(iv) $G[V(H)] = G_o[V(H)] = H$.

For each dart of G_o with tail x and head y, let $A(x,y)$ be a k-arrow from x to y. For each $z \in V(G_o)$, and each $i \in [2k-2, 3k]$ let $C(z, i)$ be a k-chain of length i with endpoint z. Assume that

$$\forall\ x,y \quad V(A(x,y)) \cap V(G_o) = \{x, y\} ,$$

$$\forall\ z, i \quad V(C(z, i)) \cap V(G_o) = \{z\} ,$$

$$\forall\ x,y, x', y' \quad V(A(x,y)) \cap V(A(x',y')) \subseteq V(G_o)$$

if $x \neq x'$ or $y \neq y'$,

$$\forall\ z, i, z', i' \quad V(C(z, i)) \cap V(C(z', i')) \subseteq V(G_O)$$

if $z \neq z'$ or $i \neq i'$,

$$\forall\ x, y, z, i \quad V(A(x, y)) \cap V(C(z, i)) \subseteq V(G_O)$$

Suppose also that

$$\forall\ x, y \quad E(A(x, y)) \cap E(G_O) = \emptyset ,$$

$$\forall\ z, i \quad E(C(z, i)) \cap E(G_O) = \emptyset ,$$

$$\forall\ x, y, x', y' \quad E(A(x, y)) \cap E(A(x', y')) = \emptyset$$

if $x \neq x'$ or $y \neq y'$,

$$\forall\ z, i, z', i' \quad E(C(z, i)) \cap E(C(z', i')) = \emptyset$$

if $z \neq z'$ or $i \neq i'$,

$$\forall\ x, y, z, i \quad E(A(x, y)) \cap E(C(z, i)) = \emptyset \ .$$

Let

$$G = H \cup \bigcup_{x,y} A(x, y) \cup \bigcup_{z,i} C(z, i)$$

We have

$$V(G_O) = \{\ x \in V(G) \mid d(x) \geq k+2\ \} \ .$$

Hence $V(G_O)$ is a constituent of Aut G. That it is a faithful constituent, follows from lemmas 2.2.11 and 2.2.21. Also every automorphism of G_O extends to an automorphism of G, so that

$$\mathrm{Aut}\ G \mid V(G_O) = \mathrm{Aut}\ G_O \quad .$$

Finally, it is clear that H is an induced sub-hypergraph of G.

Define k: Aut G $\longrightarrow$ B by

$$k(\sigma) = k_O(\sigma \mid V(G_O))$$

for every $\sigma \in$ Aut G, and define h:Fix $(G, H) \longrightarrow A$ by

$$h(\sigma) = h_O(\sigma \mid V(G_O))$$

for every $\sigma \in$ Fix (G, H). Then G, h, k satisfy all the requirements of the proposition.

As a particular instance, the following result corresponds to proposition 2.1.3. in the case of k-uniform hypergraphs, $k \geq 2$:

2.2.4. PROPOSITION. Let k be any integer ≥ 2. Given a k-uniform hypergraph H and a subgroup B of Aut H, there exists a k-uniform hypergraph G such that

(i) H is a sub-hypergraph of G induced by a faithful constituent of Aut G,

(ii) Aut $G \mid V(H) = B$.

Proof. In proposition 2.2.3, let A be the trivial group, $A = 0$, and let f and g be the canonical embeddings.

For finite H the case $k = 2$ of proposition 2.2.4 was proved by I.Z.Bouwer [B 12] and L.Babai [B 1]. Our results were obtained independently.

2.2.5. Some classical theorems follow from the previous results. They are included in the following corollary, which, for the case of finite groups, was first proved by P.Hell and J.Nesetril [H 4, H 5] .

2.2.51. COROLLARY. Given an integer $k \geq 2$ and a group B, there exists a k-uniform hypergraph G whose automorphism group is isomorphic to B.

Proof. Define first a k-uniform hypergraph H by

$$V(H) = B \quad , \quad E(H) = \emptyset$$

Clearly

$$\text{Aut } H = S_B \ .$$

Define the injection $f: B \longrightarrow \text{Aut } H$ by

$$f(\alpha)(\beta) = \alpha\beta$$

for every $\alpha, \beta \in B$. Let A be the trivial group, $A = 0$. Applying proposition 2.2.3 to the exact sequence

$$0 \longrightarrow A \longrightarrow B \xrightarrow{f} \text{Aut } H,$$

we obtain the desired k-uniform hypergraph G.

The case $k = 2$, B finite, is the well-known theorem of R.Frucht [F 4] . It has been generalized to infinite groups by Frucht himself [F 5] and G. Sabidussi [S 1] .

2.3. $\aleph_\alpha$-uniform hypergraphs.

2.3.1. For every ordinal number α define a simple $\aleph_\alpha$-uniform hypergraph A_α by

$$V(A_\alpha) = [0, \omega_\alpha] ,$$

$$E(A\ ) = \{[\beta, \omega_\alpha] \mid 0 \le \beta < \omega_\alpha\}.$$

A hypergraph H is called an $\aleph_\alpha$-arrow from x to y, where $x, y \in V(H)$, if there is an isomorphism $f: H \to A_\alpha$ such that $f(x) = 0$ and $f(y) = \omega_\alpha$.

2.3.11. LEMMA. The automorphism group of any $\aleph_\alpha$- arrow is trivial.

Proof. We show that Aut A_α is trivial for any ordinal number α . We prove by transfinite induction on β , $0 \le \beta \le \omega_\alpha$, that $[0, \beta]$ is a constituent of Aut A_α . Indeed, let β be such that $[0, \gamma]$ is a constituent of Aut A for every $\gamma < \beta$. Then

$$W(\beta) = \bigcup_{\gamma < \beta} [0, \gamma]$$

is a constituent of Aut A_α and so is

$$[0, \omega_\alpha] \setminus W(\beta) = [\beta , \omega_\alpha] .$$

But β is the only vertex of degree less than 2 in the sub-hypergraph of A_α induced by $[\beta, \omega_\alpha]$. Therefore β is fixed by every automorphism of A_α and

$$[0, \beta] = W(\beta) \cup \{ \beta \}$$

is a constituent of Aut A_α . The Lemma follows.

2.3.2. PROPOSITION. α being any ordinal number, propositions 2.2.3 and 2.2.4 hold also for $\aleph_\alpha$-uniform hypergraphs.

Proof. Let indeed H be an $\aleph_\alpha$-uniform hypergraph and

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} \text{Aut } H$$

an exact sequence. We find a general graph G_o and group homomorphisms h_o, k_o exactly as described in the proof of proposition 2.2.3. It can also be assumed that every vertex of G_o is incident with some dart . Again we construct an $\aleph_\alpha$-uniform hypergraph G such that

(i) $V(G_o) \subseteq V(G)$,

(ii) $V(G_o)$ is a faithful constituent of Aut G,

(iii) Aut $G \mid V(G_o)$ = Aut G_o ,

(iv) $G[V(H)] = G_0[V(H)] = H$.

The construction of G will of course be different. For each dart of G_0 with tail x and head y, let $A(x, y)$ be an $\aleph_\alpha$-arrow from x to y. Assume that

$$\forall\ x, y \quad V(A(x, y)) \cap V(G_0) = \{x, y\},$$

$$\forall\ x, y \quad E(A(x, y)) \cap E(G_0) = \emptyset$$

and

$$\forall\ x, y, x', y' \quad V(A(x, y)) \cap V(A(x', y')) \subseteq V(G_0),$$

$$\forall\ x, y, x', y' \quad E(A(x, y)) \cap E(A(x', y')) = \emptyset$$

if $x \neq x'$ or $y \neq y'$.

Let

$$G = H \cup \bigsqcup_{x,y} A(x, y)$$

$V(G_0)$ is a constituent of Aut G because

$$V(G) \setminus V(G_0) = \{x \in V(G) \mid 1 < d(x) < \aleph_\alpha \text{ and } (\forall\ L, L' \in E(G) \text{ incident with } x, \text{ either } \psi(L) \subseteq \psi(L') \text{ or } \psi(L') \subset \psi(L))\}.$$

Lemma 2.3.11 implies that it is a faithful constituent. The remaining parts of the proofs of propositions 2.2.3 and 2.2.4 apply mutatis mutandis.

Corresponding to Corollary 2.2.51 we have the generalization of Frucht's theorem [F 4] to uniform hypergraph with infinite lines:

2.3.21. COROLLARY. Given any ordinal number α and a group B, there exists an $\aleph_\alpha$-uniform hypergraph G whose automorphism group is isomorphic to B.

2.4. Symmetry blocks in general graphs.

2.4.1. A <u>symmetry block</u> in a general graph G is a subset $U \subseteq V(G)$ such that for every automorphism σ of G, $\sigma(U) \subseteq U$ or $\sigma(U) \cap U = \emptyset$. Actually, if U is a symmetry block then $\sigma(U) \subset U$ is impossible, because this would imply

$$\sigma^{-1} \sigma(U) = U \subset \sigma^{-1}(U) \quad ,$$

despite the fact that $\sigma^{-1} \in$ Aut G. Every constituent of Aut G is trivially a symmetry block.

For a symmetry block U of G, let

$$N(G,G[U]) = \{\sigma \in \text{Aut } G \mid \sigma(U) = U\} \ .$$

If G is the graph pictured in Figure 12, $V(G)=[1,9]$, then $U=[1,3]$ is a symmetry block and $N(G,G[U]) \simeq Z_2$.

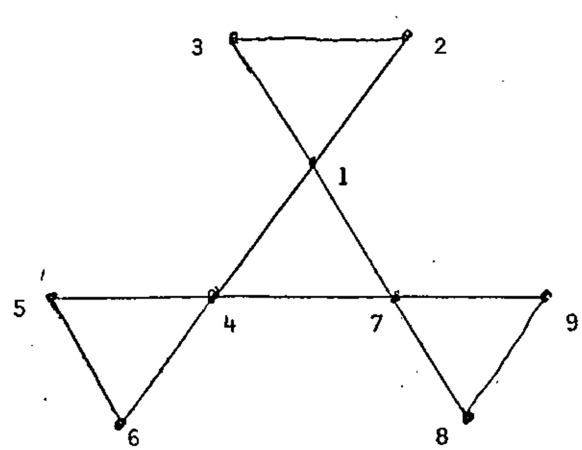


Figure 12

N(G ,G [U]) is the largest subgroup of Aut G of which U is a constituent. Let also

$$\mathrm{Fix}\ (G,\ G\ [U]\ ) = \{\ \sigma \in \mathrm{Aut}\ G\ |\ \forall\ x \in U\ \ \sigma(x) = x\},$$

a definition compatible with the one given in section 2.1 for constituents . Fix (G, G [U]) is a normal subgroup of N(G, G [U]) .

2.4.2 PROPOSITION. Let C be a group, B a subgroup of C and A a normal subgroup of B. There exists a general graph G and an induced subsystem H of G such that

(i) V(H) is a symmetry block of G,

(ii) for some isomorphisms g, h, k the diagram

$$\begin{tikzcd} A \arrow[r] & B \arrow[r] & C \\ \mathrm{Fix}(G,H) \arrow[u, "g"] \arrow[r] & N(G,H) \arrow[u, "h"] \arrow[r] & \mathrm{Aut}\ G \arrow[u, "k"] \end{tikzcd}$$

is commutative (unlabelled arrows standing for the canonical mappings) .

Proof. First we define a general graph H_0. Let

$$C/A = \{\gamma A \mid \gamma \in C\ \}$$

be the set of left cosets of A in C, and assume that $C \cap C/A = \emptyset$. Let

$$V(H_O) = C \cup C/A,$$
$$E(H_O) = \emptyset$$

For each $\delta \in C$ define $\tau_\delta \in \mathrm{Aut}\ H_O$ by

$$\tau_\delta(\gamma) = \delta\gamma$$

for every $\gamma \in C$ and

$$\tau_\delta(\gamma A) = \delta\gamma A$$

for every $\gamma A \in C/A$. Then

$$T = \{\tau_\delta \mid \delta \in C\}$$

is a subgroup of Aut H_O. Let G be a general graph such that

(i) H_O is a subsystem of G induced by a faithful constituent of Aut G,

(ii) $\mathrm{Aut}\ G \mid V(H_O) = T$.

Let

$$U = \{\beta A \mid \beta \in B\} \subset V(H_O)$$

be the set of (left) cosets of A in B. Define

$$H = G[U] .$$

i denoting the identity element of C, define

$$k(\sigma) = \sigma(i)$$

for every $\sigma \in \mathrm{Aut}\, G$, and let h and g be the restrictions of k to N(G, H) and Fix (G, H), respectively. The general graphs G, H and the group isomorphisms g, h, k satisfy the requirements of the proposition.

Remark. It is clear from the previous sections that in the above proposition "general graph" can be replaced by "k-uniform hypergraph" or "$\aleph_\alpha$-uniform hypergraph", for any integer $k \geq 2$ or any ordinal number α .

CHAPTER 3

REGULAR REPRESENTATION OF FINITE GROUPS BY HYPERGRAPHS

3.0. All hypergraphs considered in this chapter will be finite and simple.

3.1.1. A general problem is the characterization of all the permutation groups that are automorphism groups of some k-uniform hypergraph. For k= 2, the problem was raised by Frucht [F 4] . Examples given by Frucht [F 4] and I.N. Kagno [K 1] have shown that the solution might be difficult: although every group is isomorphic to the automorphism group of some hypergraph, not every permutation group is the automorphism group of a hypergraph. The simplest counter example is a regular permutation group of order 3.

For k= 2, the case of regular permutation groups has been extensively studied. If A is a regular abelian permutation group, then A is the automorphism group of some graph if and only if A is isomorphic to Z_2^n for some $n \neq 2, 3, 4$ (C.Y.Chao [C 4] , W.Imrich [I 1, I 2] , M.H. McAndrew [M 1], G. Sabidussi [S2]). The problem is more complicated for non-abelian groups. Using the theorem of W. Feit and J.G. Thompson [F 1] on the solvability of groups of odd order, L.A. Nowitz and M.E. Watkins have shown that if A is a regular, non-abelian permutation group of order coprime to 6, then it is the automorphism group of some graph [N 3] . Imrich extended this result to the case of $|A|$ odd and sufficiently large [I3, I4] . Miscellaneous other classes of groups have also been examined by Watkins [W1, W2, W3, W7] .

3.1.2. Let A be an arbitrary group. A <u>left translation</u> in A is a permutation $\tau \in S_A$ such that

$$\tau(x)\ x^{-1}$$

is the same for every $x \in A$. Left translations form a subgroup L_A of S_A . For every $y \in A$, the mapping $\tau_y\colon A \to A$ given by

$$\tau_y(x) = yx$$

for every $x \in A$, is a left translation. The mapping

$$y \to \tau_y$$

is an isomorphism from A to L_A , a fact well known as Cayley's theorem [C 2] . Every regular permutation group B can be viewed as the group of left translations L_A in some abstract group A, isomorphic to B. In view of this, we shall say that a group A has a <u>regular representation</u> by a k-uniform hypergraph, if

$$L_A = \mathrm{Aut}\ H$$

for some k-uniform hypergraph H.

3.2. Determination of regular cyclic automorphism groups of 3-uniform hypergraphs.

3.2.0. In this section we use integer symbols to denote the elements of Z_n they represent.

3.2.1. Among the cyclic groups Z_n only Z_1 and Z_2 have a regular representation by a graph and indeed they both have a regular representation by a k-uniform hypergraph for every $k \geq 2$: the edgeless hypergraphs on vertex set Z_1 or Z_2 are trivially k-uniform for every k.

3.2.2. If a group B of order n has a regular representation by a k-uniform hypergraph H, $k \leq n$, then it must also have a regular representation by an (n-k) - uniform hypergraph $\bar{H}$. Indeed, $\bar{H}$ can be defined on the vertex set $V(\bar{H}) = V(H) = B$ by

$$E(\bar{H}) = \{B \setminus A \mid A \in E(H)\} \quad .$$

It follows without difficulty that the groups Z_3, Z_4 and Z_5 do not have a regular representation by any 3-uniform hypergraph.

3.2.3. To prove that a group B has a regular representation by a uniform hypergraph H, the general argument consists of two steps. We first define the hypergraph H with vertex set $V(H) = B$ usually in such a way that the relation

$$L_B \subseteq \text{Aut } H$$

becomes obvious. Then, since L_B is transitive on B, in order to prove the equality

$$L_B = \text{Aut } H$$

it suffices to show that the stabilizer in Aut H of the identity element e of B, $(\text{Aut } H)_e$, is trivial.

3.2.4. Let $n \geq 9$. Define a 3-uniform hypergraph H on the vertex set $V(H) = Z_n$ by

$$E(H) = \{\{ i, i+1, i+3\} \mid i \in Z_n\} .$$

Clearly every left translation of Z_n is an automorphism of H. We show that the stabilizer $(\text{Aut } H)_0$ of $0 \in Z_n$ is trivial. The neighbourhood of 0

$$N(0) = \{ -3, -2, -1, 1, 2, 3 \}$$

is a constituent of $(\text{Aut } H)_0$ and

$$(\text{Aut } H)_0 | N(0) \subseteq \text{Aut } (H [N(0)])\ .$$

But the only line of $H [N(0)]$ is

$$\{ -2, -1, 1 \}\ ,$$

which must then be a constituent of $(\text{Aut } H)_0$. On the other hand, define a graph G by

$$V(G) = \{-3, -2, -1, 0, 1, 2, 3\}\ ,$$

$$E(G) = \{\{x, y\} \mid x \neq y \text{ and } \exists A \in E(H) \text{ such that } \{x, y\} \subseteq A\}\ .$$

It is clear that

$$(\text{Aut } H)_0 \mid V(G) \subseteq \text{Aut } G\ .$$

If $n = 9$, then G is the graph displayed in Figure 13. For $n > 9$ it is depicted in Figure 14.

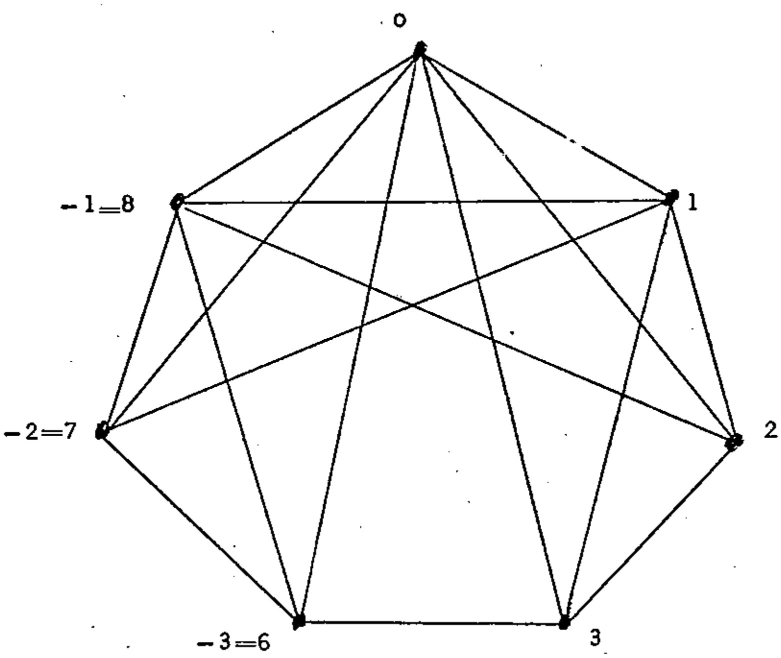


Figure 13

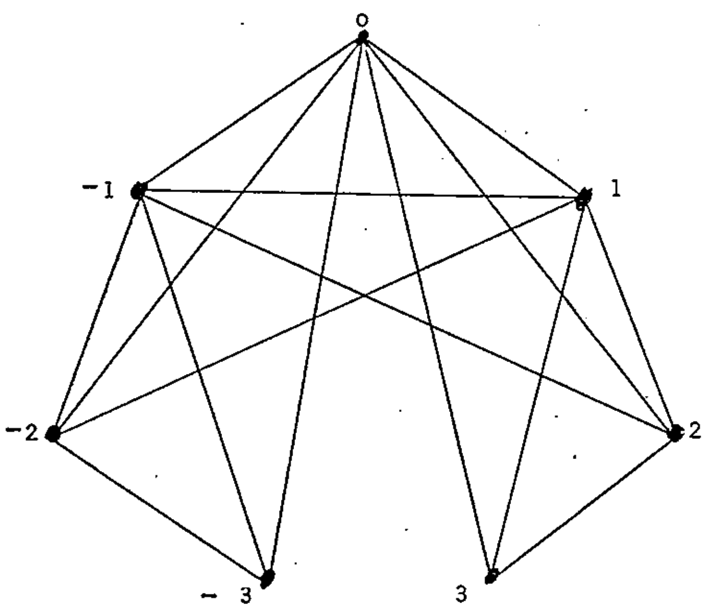


Figure 14

In any case

$$d_G(-2) = 4, \quad d_G(-1) = 5, \quad d_G(1) = 5 \quad ,$$

so that $\{-1, 1\}$ is a constituent of $(\mathrm{Aut}\ H)_0$. Also

$$[N_G(-1) \cap N_G(1)] \setminus \{-2, -1, 0, 1\} = \{2\} \quad ,$$

must be a constituent of $(\mathrm{Aut}\ H)_0$. Suppose an automorphism $\sigma \in (\mathrm{Aut}\ H)_0$ exchanges -1 and 1. Then

$$\sigma(\{-1, 0, 2\}) = \{1, 0, 2\}$$

would be a line of H, which is impossible. Therefore $\sigma(1) = 1$ for every $\sigma \in (\mathrm{Aut}\ H)_0$, i.e.

$$(\mathrm{Aut}\ H)_0 \subseteq (\mathrm{Aut}\ H)_1 \quad .$$

By repeated application of the above argument we obtain

$$(\mathrm{Aut}\ H)_0 \subseteq (\mathrm{Aut}\ H)_1 \subseteq (\mathrm{Aut}\ H)_2 \subseteq \ldots \subseteq (\mathrm{Aut}\ H)_{n-1} \subseteq$$

$$\subseteq (\mathrm{Aut}\ H)_0 \quad ,$$

implying that $(\mathrm{Aut}\ H)_0$ is trivial. Consequently

$$\mathrm{Aut}\ H = L_{Z_n}$$

and Z_n has a regular representation by a 3-uniform hypergraph for every $n \geq 9$.

3.2.5. Let $n \in [6, 8]$. Define a 3-uniform hypergraph H by

$$V(H) = Z_n ,$$

$$E(H) = \{\{i, i+1, i+3\} \mid i \in Z_n\} \cup \{\{i, i+1, i+2\} \mid i \in Z_n\} .$$

Every left translation of Z_n is obviously an automorphism of H. Consider the graph G given by

$$V(G) = V(H) = Z_n ,$$

$$E(G) = \{\{x, y\} \mid x \neq y \text{ and } x \text{ lies together with } y \text{ on 3 different lines of } H\} .$$

We necessarily have

$$\mathrm{Aut}\, H \subseteq \mathrm{Aut}\, G$$

and hence

$$(\mathrm{Aut}\, H)_o \subseteq (\mathrm{Aut}\, G)_o .$$

The stabilizer $(\mathrm{Aut}\, G)_o$ consists of, besides the identity automorphism, the reflection $\phi: Z_n \longrightarrow Z_n$ given by

$$\phi(i) = -i$$

for every $i \in Z_n$. But

$$\phi(\{0, 1, 3\}) \notin E(H) ,$$

so that

$$\phi \notin (\text{Aut } H)_o$$

and $(\text{Aut } H)_o$ can contain only the identity permutation. This proves that each of Z_6, Z_7 and Z_8 have a regular representation by a 3-uniform hypergraph.

We summarize the preceeding results in the following proposition:

3.2.6. PROPOSITION. The cyclic group Z_n has a regular representation by a 3-uniform hypergraph if and only if $n \neq 3, 4, 5$.

3.2.7. Consider the group Z_7. Let H be a 3-uniform hypergraph such that $V(H) = Z_7$ and

$$\text{Aut } H = L_{Z_7} .$$

H must have a line $L = \{h, j, k\}$ such that

$$\{-h, -j, -k\}$$

is not a line, because otherwise $(\mathrm{Aut}\ H)_0$ would contain the non-trivial reflection $i \longrightarrow -i$. It is then easy to see that

$$\{i,\ i+1\} \subset L$$

for some $i \in Z_7$. In fact we can assume that

$$0,\ 1 \in L\ .$$

The third vertex of L must be 3 or 5. In both cases the spanning sub-hypergraph F of H given by

$$E(F) = \{\{i+h\ ,\ i+j\ ,\ i+k\}\ |\ i \in Z_7\}\ .$$

is a projective plane that must be isomorphic to the Fano geometry.

3.3. Groups of exponent > 2 .

3.3.1. Let B be any group. A set D of elements of B is called <u>sum free</u> if

$$\{ x y \mid x, y \in D \} \cap D = \emptyset$$

This is equivalent to the condition

$$x^{-1} y \notin D ,$$

for any $x, y \in D$. Clearly D cannot contain the identity element e of B. A sum free set D is called a <u>δ- set</u> if the following additional conditions are fulfilled:

(i) for every $x \in D$, $x^{-1} \in D$ only if $x^{-1} = x$,

(ii) D has two distinct elements a and b such that

$$a^2 \neq e \quad , \quad b^2 \neq e \quad .$$

It is clear that the elements a and b of condition (ii) must also satisfy

$$a^2 \neq b \ , \ b^2 \neq a \ , \ ab \neq e \ .$$

For every $x \in B$, let (x) denote the subgroup of B generated by x.

3.3.2. An elementary abelian 2-group B (a group of exponent 2, i.e. such that $x^2 = e$ for every $x \in B$) cannot have a δ-set.

3.3.3. PROPOSITION. Let a finite group B of exponent > 2 have order at least 18. Then B has two distinct elements a and b such that

$$a^2 \neq e, \; a^2 \neq b, \; b^2 \neq e, \; b^2 \neq a, \; ab \neq e$$

Proof. Assume that the proposition is false for some group B of exponent > 2, $|B| \geq 18$. Let a be an element of B having largest possible order. Then

$$x^2 \neq a$$

for every $x \notin (a)$, because otherwise we would have

$$(x) \supset (a) \quad ,$$

contradicting the choice of a. Indeed we must have

$$x^2 = e$$

for every $x \notin (a)$, because otherwise letting

$$b = x \quad ,$$

the pair a, b would satisfy the requirements of the proposition. This shows also that

$$(x\ a)^2 = e$$

for every $x \notin (a)$, i.e.

$$x\ a\ x = a^{-1} ,$$

$$x\ a = a^{-1}\ x .$$

Further, if we had

$$|(a)| \geq 7 ,$$

then setting

$$b = a^3$$

the pair a,b would satisfy the requirements of the proposition. Therefore

$$|(a)| \leq 6$$

and, in view of $|B| \geq 18$, we can choose $x, y \in B \setminus (a)$ such that the product

$$x\ y \notin (a) .$$

Then

$$x\ a\ x = a^{-1}\ x\ x = a^{-1}$$

and

$$y\,a\,y = a^{-1}\,y y = a^{-1} \quad ,$$

so that

$$x\,a\,x = y\,a\,y \quad ,$$

$$x\,y\,a = a\,x\,y \quad .$$

But also, since $x\,y \notin (a)$,

$$x\,y\,a = a^{-1}\,x\,y \quad ,$$

and hence

$$a\,x\,y = a^{-1}\,x\,y \quad ,$$

$$a = a^{-1} \quad ,$$

contradicting the choice of a .

3.3.4. PROPOSITION. Let d be an integer ≥ 2 and B a finite group. If

$$|B| \geq 3\,d^3 + 6\,d^2 \quad ,$$

then B has a generating δ-set of size at least d.

Proof. According to proposition 3.3.3, B has a δ-set $\{a, b\}$ containing two elements.

Let D be a maximum size δ-set of B, $|D| = n$,

$$D = \{ x_1, \ldots , x_n\}$$

and assume that the sum

$$\sum_{i=1}^{n} |(x_i)|$$

is largest possible. In order to prove that D generates B, we shall show that the set

$$\bar{D} = \bigcup_{i=1}^{n} (x_i) \cup \{x^{-1}y,\ xy^{-1},\ xy,\ |\ x,y \in D \}$$

is the entire group B. For otherwise let z be any element of $B \setminus \bar{D}$. If

$$z^2 \notin D \quad ,$$

then

$$D \cup \{ z \}$$

is a δ-set, a contradiction with the maximality of D. On the other hand, if

$$z^2 = x_i \in D \quad ,$$

then

$$D' = (D \setminus \{x_i\}) \cup \{ z \}$$

is a δ-set of maximum size $n = |D|$. But $(z) \supset (x_i)$, so that

$$\sum_{x \in D'} |(x)| > \sum_{x \in D} |(x)| ,$$

contradicting the maximality of the latter sum.

There remainsto prove that D contains at least d elements. This again will be a consequence of the equality

$$\bar{D} = B ,$$

Suppose that

$$|D| = n < d .$$

We shall obtain a contradiction. If we had

$$|(x_i)| \leq 3 d^2 + 3 d$$

for every $x_i \in D$, then

$$|B| = |\bar{D}| \leq n (3 d^2 + 3 d) + 3 n^2 ,$$

$$|B| < d (3 d^2 + 3 d) + 3 d^2 = 3 d^3 + 6 d^2 ,$$

a contradiction with the initial assumption on the order of B. Therefore

$$|(x_i)| > 3 d^2 + 3 d$$

for some $x_i \in D$. Keeping this subscript i fixed, observe that for each $x_j \in D$, the equation

$$z^2 = x_j$$

has at most two solutions $z \in (x_i)$. Consequently there are at most $2n$ elements in (x_i) the square of which belongs to D. On the other hand, we have the inequality

$$|\{x^{-1} y, x y^{-1}, x y \mid x, y \in D\}| \le 3 n^2 ,$$

so that it is possible to find an element

$$z \in (x_i) \setminus \{ x^{-1} y, x y^{-1}, x y \mid x, y \in D \}$$

such that

$$z \notin D , \quad z^{-1} \notin D , \quad z^2 \notin D .$$

Then

$$D \cup \{ z \}$$

is a δ-set strictly larger than D, which contradicts the choice of D.

3.3.5. A hypergraph H is called <u>bipartite</u> if $V(H)$ can be partitioned into two independent sets.

A vertex x of a connected graph G is a <u>cut vertex</u> if

$$G[\, V(G) \setminus \{ x \}\,]$$

is not connected.

3.3.6. Let k and n be integers, $k \geq 2$, $n \geq 2k+3$. Let $G_{k,n}$ be the k-uniform hypergraph defined by

$$V(G_{k,n}) = [\, 1, n\,]$$

$$E(G_{k,n}) = \{[\, i, i+k-1] \mid 1 \leq i \leq n-k-1\}$$

$$\cup \{[i, i+k-2\,] \cup \{n-1\} \mid i = 2, k+1\}$$

$$\cup \{ S \cup \{n\} \mid S \subseteq [1, n-1],\ |S| = k-1\}$$

The graph $G_{2,10}$ is pictured in Figure 15 .

A hypergraph G is called a <u>(k , n) - arc</u> if it is isomorphic to $G_{k,n}$.

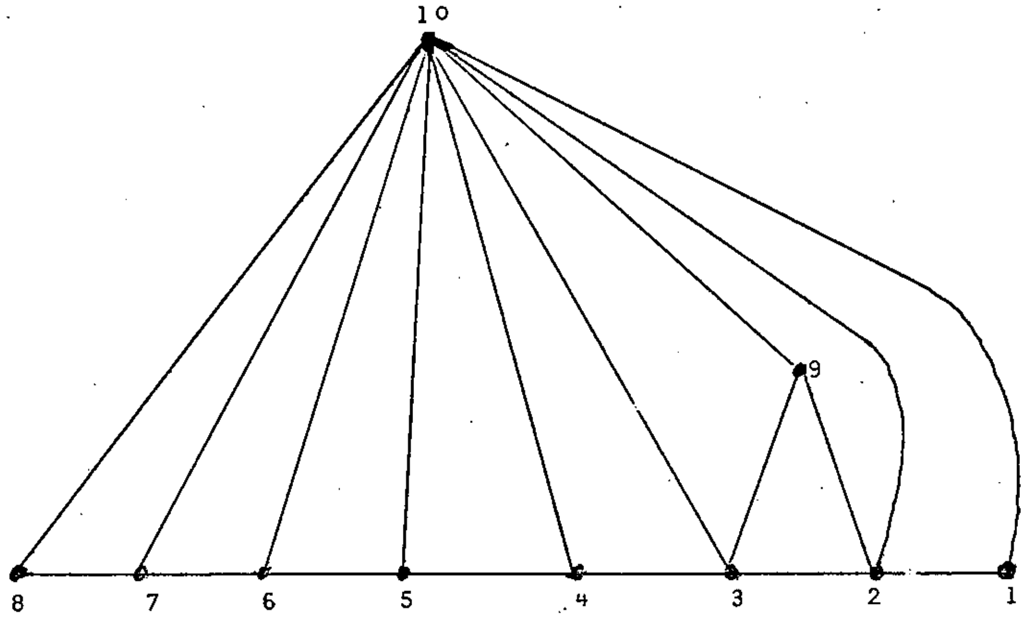


Figure 15

3.3.61. LEMMA. For every $x \in V(G_{k,n})$, $x \neq n$, we have

$$d(x) < d(n) .$$

Proof. It is clear from the definition of $G_{k,n}$ that

$$d(n) = \binom{n-1}{k-1} .$$

Also every $x \in [1, n-1]$ lies together with n in some line exactly $\binom{n-2}{k-2}$ times, and lies in at most $k+1$ lines not containing n , so that

$$d(x) \le k+1 + \binom{n-2}{k-2} .$$

Using the inequalities

$$k + 1 < \binom{2k+1}{k-1} \le \binom{n-2}{k-1} ,$$

we get

$$d(x) \le k+1 + \binom{n-2}{k-2} < \binom{n-2}{k-1} + \binom{n-2}{k-2} = \binom{n-1}{k-1} = d(n) .$$

3.3.62. The vertex of largest degree of any (k,n) - arc G is called the <u>distinguished vertex</u> of G.

3.3.63. LEMMA. The automorphism group of $G_{k,n}$ is trivial.

Proof. Obviously

$$V(G_{k,n}) \setminus \{ n \} = [1, n-1]$$

is a constituent of Aut $G_{k,n}$. By an argument similar to the proof of Lemma 2.2.11 it can be shown that

$$\text{Aut } G_{k,n} [[1, n-1]]$$

is trivial, and consequently so is Aut $G_{k,n}$.

3.3.64. LEMMA. $G_{k,n}$ is not bipartite if $n \geq k^2 - k + 2$.

Proof. Suppose that

$$V(G_{k,n}) = [1, n] = V_1 \cup V_2 ,$$

where V_1 and V_2 are independent sets. Assuming that $n \in V_1$, we must have

$$|V_1 \cap [1, n-2]| \leq k - 2 .$$

But also for every

$$i \in V_1 \cap [1,(n - 2) - k]$$

we must have

$$V_1 \cap [i+1 , i+k] \neq \emptyset ,$$

because otherwise $[i+1, i+k]$ would be a line of $G_{k,n}$ contained in V_2. For similar reasons,

$$V_1 \cap [1, k] \neq \emptyset .$$

It follows that

$$n-2 = |[1, n-2]| \leq (k-1) + |V_1 \cap [1, n-2]| \cdot k \leq$$

$$\leq (k-1) + (k-2)k = k^2 - k - 1 ,$$

a contradiction.

3.3.65. LEMMA. If $n \geq 2k + 6$, then

$$2\, d(x) < d(n)$$

for every $x \in V(G_{k,n})$, $x \neq n$.

Proof. We have already seen in the proof of Lemma 3.3.61 that

$$d(x) \leq k+1 + \binom{n-2}{k-2} .$$

Consequently

$$2\, d(x) \leq 2k+2 + 2\binom{n-2}{k-2} \leq n-4 + 2\binom{n-2}{k-2} .$$

But the assumption $n \geq 2k + 6$ implies also that

$$n - 3 + \binom{n-2}{k-2} \leq \binom{n-2}{k-1}$$

and

$$2\,d(x) < \binom{n-2}{k-1} + \binom{n-2}{k-2} = \binom{n-1}{k-1} = d(n)\ .$$

3.3.7. PROPOSITION. Let k be any integer ≥ 3. Let a finite group B have a generating δ-set D containing at least $k^2 + 4$ elements. Then B has a regular representation by a k-uniform hypergraph H.

Proof.I. Let n be the cardinality of D. Let G be a $(k-1, n)$ - arc with vertex set

$$V(G) = D\ ,$$

and assume that the distinguished vertex of G is an element a of D having order larger than 2.

Let H be defined by

$$V(H) = B\ ,$$

$$E(H) = \{\{t, t x_1, \ldots, t x_{k-1}\} \mid \{x_1, \ldots, x_{k-1}\} \in E(G),\ t \in B\}\ .$$

Obviously every left translation of B is an automorphism of H. We have to prove that the stabilizer

$(\mathrm{Aut}\ H)_e$ is trivial.

Clearly $N_H(e)$ is a constituent of $(\mathrm{Aut}\ H)_e$. Define a $(k-1)$-uniform hypergraph G_1 by

$$V(G_1) = N_H(e) \quad ,$$

$$E(G_1) = \{\{x_1, \ldots, x_{k-1}\} \mid \{e, x_1, \ldots, x_{k-1}\} \in E(H)\}.$$

Then

$$(\mathrm{Aut}\ H)_e \mid N_H(e) \subseteq \mathrm{Aut}\ G_1$$

II. Let

$$E_0 = E(G) \quad ,$$

$$E_1 = \{\{x, xy_1, \ldots, xy_{k-2}\} \mid \{x, y_1, \ldots, y_{k-2}\} \in E(G) \quad ,$$
$$x^2 = e\} \quad ,$$

$$E_2 = \{\{x^{-1}, x^{-1}y_1, \ldots, x^{-1}y_{k-2} \mid \{x, y_1, \ldots, y_{k-2}\} \in E(G)$$
$$x^2 \neq e\} \quad .$$

It follows from the axioms of a δ-set that E_0, E_1 and E_2 are pairwise disjoint. Moreover, for every $A_0 \in E_0$ and $A_2 \in E_2$,

$$A_o \cap A_2 = \emptyset$$

We claim that

$$E(G_1) = E_o \cup E_1 \cup E_2 \quad .$$

The inclusion

$$E_o \cup E_1 \cup E_2 \subseteq E(G_1)$$

is readily verified. On the other hand, let

$$A = \{ z_1, \ldots, z_{k-1}\} \in E(G_1) \quad .$$

By definition

$$\{e, z_1, \ldots, z_{k-1}\} = \{t, tx_1, \ldots, tx_{k-1}\}$$

for some $t \in B$ and

$$\{x_1, \ldots, x_{k-1}\} \in E(G)$$

If $e = t$, then $A \in E_o$. Otherwise e is one of the tx_i, $i = 1, \ldots, k-1$, and there is no loss of generality in assuming that $e = tx_1$. In this case

$$\{e, z_1, \ldots, z_{k-1}\} = \{x_1^{-1}, e, x_1^{-1} x_2, \ldots, x_1^{-1} x_{k-1}\},$$

$$A = \{x_1^{-1}, x_1^{-1}x_2, \ldots, x_1^{-1} x_{k-1}\} \in E_1 \cup E_2$$

III. Let us denote

$$D^{-1} = \{ x^{-1} \mid x \in D \},$$

$$F = \{ x^{-1} y \mid x, y \in D \}.$$

From the axioms of a δ-set it is clear that

$$F \cap D = \emptyset, \quad F \cap D^{-1} = \emptyset.$$

It follows from part II. that

$$V(G_1) \subseteq D \cup D^{-1} \cup F,$$

and also that

$$G_1 [D] = G .$$

Let K be the connected component of G_1 that contains the distinguished vertex a of G. Since, according to Lemma 3.3.64, $G = G_1 [D]$ is not bipartite, it follows that

$$G_1 \ [\, K \,]$$

is not bipartite. For every other connected component $K' \neq K$ of G_1, if there is any, we have

$$K' \cap D = \emptyset$$

$$K' = (D^{-1} \cap K') \cup (F \cap K') \quad .$$

But

$$D^{-1} \cap K'$$

being disjoint from D, is independent in G_1, and so is $F \cap K'$. Hence

$$G_1 \ [K']$$

is bipartite and K is a constituent of Aut G_1 and also of $(\text{Aut } H)_e$. We have

$$\text{Aut } G_1 \mid K \subseteq \text{Aut } G_1 [\, K \,]$$

and consequently

$$(\text{Aut } H)_e \mid K \subseteq \text{Aut } G_1 [\, K \,]$$

It will follow from the subsequent parts IV-VI, that the vertex a of K is fixed by every automorphism of $G_1[K]$. For every $x \in K$, we shall write

$$d(x) = d_{G_1[K]}(x) \quad ,$$

$$N(x) = N_{G_1[K]}(x) \quad .$$

IV. It follows from part II that every $x \in D \setminus D^{-1}$ is incident in G_1 only with lines of G. In view of Lemma 3.3.61,

$$d(x) < d(a)$$

for every $x \in D \setminus \bar{D}$, $x \neq a$.

If $x \in D \cap D^{-1}$, then

$$\{x, y_1, \ldots, y_{k-2}\} \to \{x, xy_1, \ldots, xy_{k-2}\}$$

is a bijection from the set of lines of G incident with x to the set

$$\{ A \in E(G_1) \setminus E(G) \mid x \in A \} .$$

Consequently, in view of Lemma 3.3.65, we have

$$d(x) = 2\, d_G(x) < d(a)$$

for every $x \in D \cap D^{-1}$.

For every $x \in K \cap (D^{-1} \setminus D)$

$$N(x) \subseteq F \quad ,$$

so that $N(x)$ is independent in G_1 , while

$$N(a) = D \setminus \{a\}$$

is not independent in G_1 .

If $x \in K \cap F$, then we have to examine separately the cases $k = 3$ and $k \geq 4$.

V. Let $k = 3$. Since $G_1 [K]$ is a simple loopless graph , we have

$$d(x) = |N(x)|$$

for every $x \in K$. Also, since $x \in F$,

$$N(x) \subseteq D \cup D^{-1} \quad .$$

If $N(x) \subseteq D^{-1} \setminus D$, then $N(x)$ is independent in G_1 while $N(a)$ is not.

If

$$N(x) \subseteq D \quad ,$$

then every element of $N(x)$ has order 2, so that

$$|N(x)| \leq |D| - 2 < n - 1 \quad ,$$

$$d(x) < d(a) \quad .$$

If

$$N(x) \nsubseteq D \text{ and } N(x) \nsubseteq D^{-1} \setminus D$$

then, since no vertex in D is adjacent in G_1 to a vertex in $D^{-1} \setminus D$, x is a cut vertex of

$$G_1[\, N(x) \cup \{ x \} \,] \, .$$

On the contrary, a is not a cut vertex of

$$G_1 \, [\, N(a) \cup \{ a \} \,] = G_1 \, [\, D \,] = G \quad .$$

VI. Let $k \geq 4$.

If

$$|\, N(x) \cap (D \cup D^{-1}) \,| \geq 2 \quad ,$$

then let S be any subset of $N(x)$ such that

$$|S| = k-2 ,$$

$$|S \cap (D \cup D^{-1})| \geq 2 .$$

Clearly

$$S \cup \{x\} \notin E(G_1) .$$

On the contrary, for every subset S of N(a) containing k - 2 elements,

$$S \cup \{a\} \in E(G_1)$$

If

$$|N(x) \cap (D \cup D^{-1})| = 1 ,$$

then every line of G_1 incident with x is incident with the unique element y of $N(x) \cap (D \cup D^{-1})$. But it is easy to find two lines of G, and hence of G_1, the intersection of which contains only a and no other vertex.

VII. The different properties of a and of the other vertices $x \neq a$ of $G_1[K]$, discussed in the preceeding parts IV-VI, show that every automorphism of $G_1[K]$ must fix a. Consequently

$$D = N(a) \cup \{a\} = N_{G_1}(a) \cup \{a\}$$

is a constituent of Aut G_1 $[K]$, hence of Aut G_1, and finally of $(\mathrm{Aut}\ H)_e$. Therefore

$$(\mathrm{Aut}\ H)_e | D \subseteq \mathrm{Aut}\ G_1\ [D] = \mathrm{Aut}\ G.$$

But, according to Lemma 3.3.63, Aut G is trivial. Consequently

$$(\mathrm{Aut}\ H)_e \ | \ D$$

is trivial.

VIII. Since D generates B, every $x \in B$ can be written as a product of elements of D. Let $\ell(x)$ be the minimum number of factors in such an expression of x. We have, e.g.

$$\ell(x) = 0$$

if and only if $x = e$, and

$$\ell(x) = 1$$

if and only if $x \in D$.

We prove by induction on $\ell(x)$ that every $\sigma \in (\mathrm{Aut}\ H)_e$ fixes x. This is true by definition if $\ell(e) = 0$. For $\ell(x) = 1$ this is exactly the triviality of

$$(\mathrm{Aut}\ H)_e \mid D \quad ,$$

proved in VII.

If the claim is false, let $x \in B$ such that

$$\sigma(x) \neq x$$

for some

$$\sigma \in (\mathrm{Aut}\ H)_e \quad ,$$

and assume that $\ell(x) = g$ is smallest possible. Then

$$x = y_1 \cdots y_g$$

with

$$y_i \in D$$

for $1 \leq i \leq g$. Now

$$\ell(x\, y_g^{-1}) = \ell(x) - 1$$

and hence, by the induction hypothesis,

$$\sigma(x\, y_g^{-1}) = x\, y_g^{-1} \quad .$$

Consider the automorphism τ of H given by

$$\tau(z) = x\, y_g^{-1}\, z$$

for every $z \in B$. We have

$$\tau^{-1}\, \sigma\, \tau \in (\mathrm{Aut}\ H)_e \quad ,$$

and consequently

$$\tau^{-1}\, \sigma\, \tau(y_g) = y_g \quad ,$$

$$\sigma\, \tau(y_g) = \tau(y_g) \quad ,$$

$$\sigma(x) = x \quad .$$

3.4. Groups of exponent 2.

3.4.1. We recall that if every non-identity element of a group B has order 2, then B is necessarily isomorphic to some Z_2^n , Although the term elementary abelian 2-group is often used and might be more informative to designate such groups, in the sequel we shall consistently call them groups of exponent 2.

The notation will be kept multiplicative.

3.4.2. LEMMA. For every integer $n \geq 6$ there exists a graph G_n having n vertices, each of them of degree at least 2, and such that Aut G_n is trivial.

Proof. Let

$$V(G_n) = [1, n] \quad ,$$

$$E(G_n)=\{\{ i , i+1\} \mid i\in [1, n - 1] \} \cup$$

$$\cup \{\{ 1, n \} , \{ 1, n - 1\} , \{1, n - 2\}\} .$$

The graph G_6 is pictured in Figure 16.

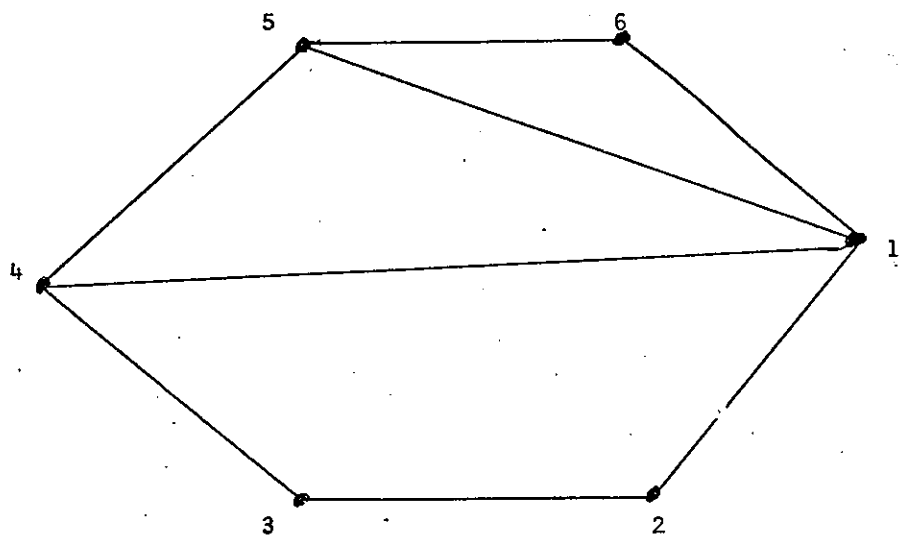


Figure 16

Remark. Every graph having less than 6 and at least 2 vertices has non-trivial automorphism group.

3.4.3. PROPOSITION . Every finite group B of exponent 2 and having order at least 2^6 has a regular representation by a 3-uniform hypergraph H.

Proof. Let D be a minimal set of generators for B. (D is a basis of B if this is viewed as a vector space over the two-element field.) Certainly

$$|D| = \log_2 \ |B| \geq 6 \ .$$

According to Lemma 3.4.2, there is a graph G such that

(i) $V(G) = D$,

(ii) Aut G is trivial ,

(iii) every vertex of G has degree at least 2.

Let H be defined by

$$V(H) = B \ ,$$

$$E(H) = \{\{t, tx, ty\} \mid \{x,y\} \in E(G), t \in B\} \ .$$

Every left translation of B is an automorphism of H. We shall prove that $(\mathrm{Aut}\ H)_e$ is trivial .

Clearly $N_H(e)$ is a constituent of $(\mathrm{Aut}\ H)_e$. Define a graph G_1 by

$$V(G_1) = N_H(e)$$

$$E(G_1) = \{\{x, y\} \mid \{e, x, y\} \in E(H)\} \ .$$

Defining again

$$F = \{x\, y \mid x, y \in D\} \ ,$$

we have

$$V(G_1) \subseteq D \cup F , \quad D \cap F = \emptyset .$$

Also

$$N_{G_1}(x\,y) = \{ x, y \}$$

for every $x\,y \in F \cap V(G_1)$, and

$$G_1 [D] = G .$$

Clearly

$$d_{G_1} (x) = 2\, d_G (x) \geq 4$$

for every $x \in D$, while

$$d_{G_1}(x) = 2$$

for every $x \in F \cap V(G_1)$ consequently D is a constituent of Aut G_1 and hence of $(\text{Aut } H)_e$, so that

$$(\text{Aut } H)_e \mid D \subseteq \text{Aut } G_1 [D] = \text{Aut } G .$$

But Aut G is trivial, so that every $\sigma \in (\text{Aut } H)_e$ fixes every element of D. To prove that every $\sigma \in (\text{Aut } H)_e$ fixes every $x \in B$, i.e. that $(\text{Aut } H)_e$ is trivial, we apply mutatis mutandis the argument of part VIII in the proof of proposition 3.3.7.

3.:.4. PROPOSITION. Let k be any integer ≥ 4 and B a finite group of exponent 2. If $|B| \geq 4k+2$, then B has a regular representation by a, k-uniform hypergraph.

Proof. I. Let

$$|B| = 2^n$$

and let

$$\{x_1, \ldots, x_n\}$$

be a minimal set of generators for B. Let

$$D = \{ \prod_{i \in I} x_i \mid I \subseteq [1, n], |I| \text{ odd} \}.$$

D is a sum free set and

$$|D| = 2^{n-1} \geq 2k+1 .$$

Let G be a $(k - 1, 2^{n-1})$ - arc (see 3.3.6) with

$$V(G) = D .$$

As before, let H be defined by

$$V(H) = B \ ,$$

$$E(H) = \{\{t, tx_1, \ldots, tx_{k-1}\} \mid \{x_1, \ldots, x_{k-1}\} \in E(G) ,$$

$$t \in B\} \ .$$

Let G_1 be the $(k-1)$ - uniform hypergraph defined by

$$V(G_1) = N_H(e) \quad ,$$

$$E(G_1) = \{\{x_1, \ldots, x_{k-1}\} \mid \{e, x_1, \ldots, x_{k-1}\} \in E(H) \quad .$$

To prove that $(\text{Aut } H)_e$ is trivial, it will suffice to show, as in the proof of propositions 3.3.7 and 3.4.3, that every $(\text{Aut } H)_e$ fixes every $x \in D$.

II. Let

$$E_1 = \{\{x, xy_1, \ldots, xy_{k-2}\} \mid \{x, y_1, \ldots, y_{k-2}\} \in E(G)\} \ .$$

Since D is a sum free set,

$$E(G) \cap E_1 = \emptyset \quad .$$

An argument similar to that of part II in the proof of proposition 3.3.7 can show that

$$E(G_1) = E(G) \cup E_1 .$$

Also defining again

$$F = \{x\,y \mid x, y \in D\} ,$$

we see that

$$V(G_1) = D \cup F , \quad D \cap F = \emptyset$$

and

$$G_1 [D] = G .$$

Let $a \in D$ be the distinguished vertex of the $(k-1, 2^{n-1})$ - arc G.

III. For every $x \in D$, the correspondence

$$\{x, y_1, \ldots, y_{k-2}\} \longrightarrow \{x, xy_1, \ldots, xy_{k-2}\}$$

is a bijection from

$$\{A \in E(G) \mid x \in A\}$$

to

$$\{A \in E_1 \mid x \in A\} .$$

It follows from Lemma 3.3.61 that for every $x \in D$, $x \neq a$,

$$d_{G_1}(x) = 2\, d_G(x) < 2\, d_G(a) = d_{G_1}(a) \quad .$$

IV. Setting

$$N_1 = D \setminus \{ a \} ,$$

$$N_2 = \{ ax \mid x \in D \} ,$$

we have

$$N_{G_1}(a) = N_1 \cup N_2 ,$$

$$N_1 \cap N_2 = \emptyset \quad .$$

Moreover, for every subset S of N_1 or of N_2 containing k - 2 elements,

$$S \cup \{a\} \in E(G_1) \quad .$$

On the contrary, assume that for a vertex $x \in F$, $N_{G_1}(x)$ is the union of two disjoint sets

$$N_{G_1}(x) = M_1 \cup M_2$$

such that for every subset S of M_1 or of M_2 containing k - 2 elements

$$S \cup \{x\} \in E(G_1) \quad .$$

Since we can see without difficulty that

$$D \subseteq N_{G_1}(x) \quad ,$$

it is clear that one of the sets M_1 or M_2 , say M_1, has to contain at least $k - 2$ elements of D. Let

$$S \subseteq M_1 \cap D \, , \quad |S| = k - 2 \quad .$$

We should have

$$S \cup \{x\} \in E(G_1)$$

which, in view of $k - 2 \geq 2$, is impossible.

V. It follows from III and IV that every $\sigma \in \mathrm{Aut}\, G_1$ fixes the distinguished vertex a. Therefore $N_{G_1}(a)$ is a constituent of Aut G_1 . But it is easy to see that

$$\Pi = \{ N_1 \, , N_2 \}$$

as defined in IV, is the only partition Π of $N_{G_1}(a)$ into two blocks such that for each block C of Π and every subset S of C containing $k - 2$ elements

$$S \cup \{a\} \in E(G_1).$$

Also N_2 is independent in G_1, while N_1 is not. Consequently

$$D = N_1 \dot{\cup} \{a\}$$

is a constituent of $\mathrm{Aut}\ G_1$ and hence of $(\mathrm{Aut}\ H)_e$. But

$$\mathrm{Aut}\ G_1\ [D] = \mathrm{Aut}\ G$$

is trivial, implying that every $\sigma \in (\mathrm{Aut}\ H)_e$ fixes every $x \in D$.

The proof is finished.

The following proposition summarizes the results of sections 3 and 4:

3.4.5 PROPOSITION. There exists a polynomial $p(x)$ with the property that for every integer $k \geq 3$, every group of order at least $p(k)$ has a regular representation by a k-uniform hypergraph.

Proof. Let

$$p(x) = 3(x^2 + 4)^3 + 6 \quad (x^2 + 4)^2 ,$$

a polynomial of degree 6. The result follows from propositions 3.3.4, 3.3.7, 3.4.3, 3.4.4 and the inequalities

$$p(3) > 2^6$$

and

$$p(k) > 4k + 2$$

for every $k \geq 4$.

Remark. Recently F. Hoffman has shown [H 7] that the theorem of Feit and Thompson on the solvability of groups of odd order, together with a result contained in [F 3] , implies that every finite group of odd order $n \geq 5^7$ has a regular representation by a 3-uniform hypergraph.

CHAPTER 4

SYMMETRIES OF DIGRAPHS

4.1.1. Let P be a class of general graphs having the same vertex set V and such that every spanning subsystem of a member of P also belongs to P. Assume also that if G and G' are two general graphs with

$$V(G) = V(G') = V \ ,$$

having the same underlying simple general graph

$$s(G) = s(G')$$

and if G belongs to P, then G' also belongs to P. We say that P is a distinguished class if either

(i) V is finite and the union of every finite compatible family of members of P belongs to P,

or

(ii) V is infinite and the union of every compatible family of members of P belongs to P.

Given any set V, the following classes of general graphs with vertex set V are examples of distinguished classes:

general graphs,
digraphs,
k-uniform hypergraphs (k being any fixed cardinal number),
hypergraphs in which every line is incident with an infinite number of vertices,
hypergraphs,
graphs.

Given any non-empty set V, the following classes of general graphs with vertex-set V are not distinguished:

strict digraphs,
acyclic digraphs,
connected hypergraphs.

4.1.2. Galois connections.

Let A and B be two sets and let $R \subseteq A \times B$. For every $X \subseteq A$, let

$$X^{\Delta} = \{b \in B \mid \forall\ a \in X \quad (a, b) \in R\}.$$

For every $Y \subseteq B$, let

$$Y^{\nabla} = \{a \in A \mid \forall\ b \in Y \quad (a, b) \in R\}.$$

Let

$$C(A) = \{(X^{\Delta})^{\nabla} \mid X \in \mathcal{P}(A)\}.$$

The mappings $X \longrightarrow X^{\Delta}$ and $Y \longrightarrow Y^{\nabla}$ are often said to form a Galois connection between the lattices $\mathcal{P}(A)$ and $\mathcal{P}(B)$. According to Theorem 19, chapter V of [B 9], $C(A)$ is closed under intersection.

Let now V be any set and B any set of general graphs with vertex set V. Let $A = S_V$, the set of all permutations of V. Let

$$R = \{(\sigma, G) \in A \times B \mid \sigma \in \text{Aut } G\}.$$

Every element of $C(A)$ is a subgroup of S_V of the form

$$\bigcap_{i \in I} \text{Aut } G_i \quad ,$$

where $(G_i)_{i \in I}$ is a family of elements of B, and conversely, every subgroup of S_V of this form belongs to $C(A)$.

We shall need the following Lemma.

4.1.31. LEMMA. Let $i_1 < \ldots < i_k$ and $j_1 < \ldots < j_h$ be two increasing sequences of positive integers. If

$$\sum_{t=1}^{k} 3^{i_t} = \sum_{t=1}^{h} 3^{j_t} \qquad ,$$

then $k = h$ and

$$i_1 = j_1 \; , \; \ldots \; , \; i_k = j_k \quad .$$

Proof. First, we must have $i_1 = j_1$, because if, say

$$i_1 < j_1 \quad ,$$

then 3 does not divide

$$\sum_{t=1}^{k} \frac{3^{i_t}}{3^{i_1}} = \sum_{t=1}^{k} 3^{i_t - i_1}$$

while
$$3 \mid \sum_{t=1}^{h} \frac{3^{j_t}}{3^{i_1}}$$

By induction on t it is then easily proved that

$$i_t = j_t \quad ,$$

yielding the desired result.

4.1.32. PROPOSITION. Let P be a distinguished class of general graphs with given vertex set V. Consider the set B of simple general graphs with vertex set V that are members of P. B has at most

$$2^{2^{|V|} + |V|^2 - 1}$$

elements and for every subgroup H of S_V the following two conditions are equivalent:

(i) $H = \text{Aut } G$ for some member G of P ,

(ii) $H = \bigcap_{i \in I} \text{Aut } G_i$ for some family $(G_i)_{i \in I}$ of elements of B.

Proof. The bound

$$|B| \leq 2^{2^{|V|} + |V|^2 - 1}$$

follows from the observation that the right hand side of the inequality is the cardinal number of all simple general graphs with vertex set V. Of course

$$2^{2^{|V|}+|V|^2-1} = 2^{2^{|V|}}$$

if V is infinite.

(i) $\Longrightarrow$ (ii). Let

$$H = \mathrm{Aut}\ G$$

for some member G of the class P. For every cardinal number k let G_k be the simple general graph defined by

$$V(G_k) = V ,$$

$$E(G_k) = \{\psi_G(A) \mid A \in E(G),\ |\psi_G^{-1}(\psi_G(A))| = k\}.$$

Since P is a distinguished class,

$$G_k \in B$$

for every cardinal number k. It is also clear that there is some cardinal number n with the property that for every $k \geq n$

$$E(G_k) = \emptyset$$

We have

$$H = \bigcap_{i \leq n} \text{Aut } G_i \quad .$$

(ii) $\Longrightarrow$ (i) Let

$$H = \bigcap_{i \in I} \text{Aut } G_i$$

for some family $(G_i)_{i \in I}$ of elements of B. We have to distinguish two cases.

Case 1. The family $(G_i)_{i \in I}$ is finite. We can assume that I is a finite set and

$$I = [1, n]$$

for some positive integer n. For every $i \in [1, n]$ let K_i be a general graph such that

(i) $s(K_i) = s(G_i)$,

(ii) $|\psi_{K_i}^{-1} (\psi_{K_i} (A))| = 3^i$

for every $A \in E(K_i)$.

Moreover, assume that if $i, j \in [1, n]$, $i \neq j$, then

$$E(K_i) \cap E(K_j) = \emptyset .$$

The family

$$(K_i)_{1 \leq i \leq n}$$

is compatible. The union

$$G = \bigcup_{i=1}^{n} K_i$$

is a member of the distinguished class P. Using Lemma 4.1.31., there is no difficulty in verifying that

$$\mathrm{Aut}\ G = H .$$

Case 2. The family $(G_i)_{i \in I}$ is infinite. Then I must be infinite. Let

$$|\ P(I)\ | = \aleph_\alpha$$

and let

$$f: P(I) \longrightarrow W(\omega_\alpha)$$

be a bijection.

For every

$$A \in (P(V) \cup V^2) \setminus \{ \emptyset \}$$

let

$$g(A) = \{ i \in I \mid A \in E(G_i) \} .$$

Let G_A be the general graph such that

(i) $V(G_A) = V$,

(ii) $E(s(G_A)) = \{ A \}$,

(iii) $|\psi_{G_A}^{-1}(A)| = \aleph_{f(g(A))}$.

Moreover, assume that if

$$A, C \in (P(V) \cup V^2) \setminus \{\emptyset\}$$

and $A \neq C$, then

$$E(G_A) \cap E(G_C) = \emptyset .$$

The family $(G_A)_{A \in (P(V) \cup V^2) \setminus \{\emptyset\}}$ is compatible and the union

$$G = \bigcup_A G_A$$

is a member of the distinguished class P. Using the bijectivity of f, it can be shown that

$$\mathrm{Aut}\ G = H .$$

4.1.33. COROLLARY. Let P be a distinguished class of general graphs with vertex set V. The set of subgroups of S_V that are automorphism groups of some member of P is closed under intersection.

4.2.1. Given a set V, the class of digraphs with vertex set V is a distinguished class. The set of permutation groups on V that are automorphism groups of some digraph is closed under intersection. This is not true for strict digraphs. It can indeed be seen that the group of left translations of the Klein group $Z_2 \times Z_2$ is not the automorphism group of any strict digraph, while it is the automorphism group of a digraph isomorphic to the one represented in Figure 17 and hence it is the intersection of automorphism groups of simple digraphs.

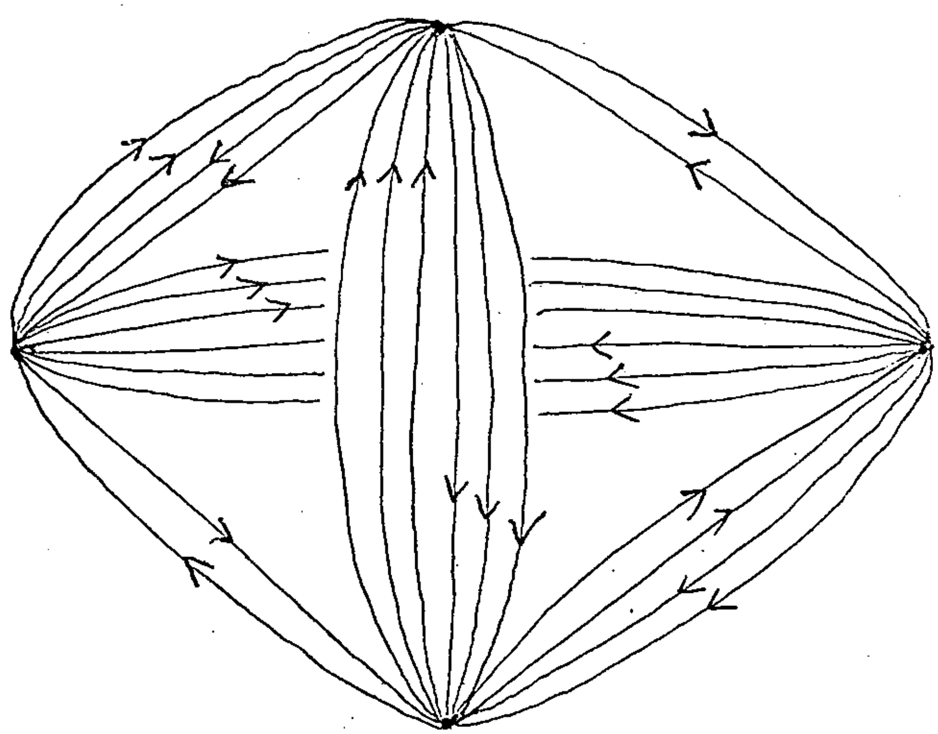


Figure 17

4.2.2. Let V be any set, B a permutation group on V and $\sigma \in S_V$. We say that σ <u>implies</u> B,

$$\sigma \Rrightarrow B ,$$

if every orbit of the permutation σ is contained in some orbit of the group B. In the lattice of partitions of V, this means that the partition into orbits of σ is less than or equal to the partition into orbits of B. If partitions are viewed as equivalence relations, then clearly the above defined implication is to be taken in the ordinary sense of implication between relations.

Equivalently, $\sigma \Rrightarrow B$ means that

$$\sigma(C) = C$$

for every orbit, and hence for every constituent, C of B.

Every element of B implies B, but the converse is generally false. Indeed, every transitive permutation group on V is implied by any permutation of V.

4.2.3. For every permutation group B on V, the following statement (S) is trivially true for every $\sigma \in B$:

(S) For every $x \in V$ there is some $\theta \in B$ such that $\theta \sigma \Rrightarrow B_x$.

Indeed, we can take

$$\theta = \sigma^{-1}$$

for every $x \in V$, where e denotes the identity element of B.

We shall call B <u>closed</u> if the statement (S) does not hold for any

$$\sigma \in S_V \setminus B .$$

Obviously the full symmetric group S_V is closed. It is easy to see that the trivial subgroup $\{ e \}$ of S_V is closed. We also have the following.

4.2.4. PROPOSITION. The intersection of any family of closed permutation groups on V is closed.

Proof. Let $(B_i)_{i \in I}$ be a family of closed subgroups of S_V. Let

$$B = \bigcap_{i \in I} B_i$$

Let $\sigma \in S_V$. Assume that for every $x \in V$ there is some

$$\theta_x \in B$$

such that

$$\theta_x \sigma \equiv\!\!> B_x \quad .$$

For every $i \in I$, every θ_x belongs to B_i. Also every orbit of B_x is contained in some orbit of $(B_i)_x$. Consequently

$$\theta_x \sigma \equiv\!\!> (B_i)_x$$

and by assumption

$$\sigma \in B_i$$

for every $i \in I$, i.e.

$$\sigma \in B \ .$$

4.2.5. PROPOSITION. Let V be any set and B a permutation group on V. The following two conditions are equivalent:

(i) B is the automorphism group of some digraph,

(ii) B is a closed permutation group.

Proof. (i) $\Rightarrow$ (ii) If B is the automorphism group of some digraph, then according to proposition 4.1.32, there exists a family $(D_i)_{i \in I}$ of simple digraphs such that

$$B = \bigcap_{i \in I} \text{Aut } D_i$$

Therefore, in view of proposition 4.2.4, it will suffice to show that the automorphism group Aut D of any simple digraph D with vertex set V is closed.

Let $\sigma \in S_V$ and assume that for every $x \in V$ there is some $\theta_x \in \text{Aut } D$ with

$$\theta_x \sigma \Rrightarrow (\text{Aut } D)_x \ .$$

Clearly $\{ x \}$ is an orbit of the stabilizer $(\text{Aut } D)_x$.

We must have

$$\theta_x \sigma (x) = x \quad ,$$

$$\theta_x \sigma (N^+(x)) = N^+(x) \quad ,$$

$$\theta_x \sigma (V \setminus N^+(x)) = V \setminus N^+(x)$$

for any $x \in V$. We are now able to show that σ is an automorphism of D.

Let

$$(x, y) \in E(D) \quad .$$

Then

$$y \in N^+(x)$$

and

$$\theta_x \sigma (y) \in N^+(x) \quad ,$$

i.e.

$$(x, \theta_x \sigma(y)) \in E(D) \quad .$$

Applying the automorphism θ_x^{-1} of D, we get

$$(\theta_x^{-1} (x) \ , \ \theta_x^{-1} \ \theta_x \ \sigma(y)) \in E(D) \quad .$$

But

$$\theta_x^{-1} (x) = \theta_x^{-1} \ \theta_x \ \sigma(x) = \sigma(x) \quad ,$$

so that

$$(\sigma(x) , \sigma(y)) \in E(D) .$$

Similarly, if

$$(x, y) \in V^2 \setminus E(D) ,$$

i.e.

$$y \in V \setminus N^+(x) ,$$

we can prove that

$$(\sigma(x) , \sigma(y)) \notin E(D) .$$

It follows that

$$\sigma \in \text{Aut } D ,$$

and Aut D is closed.

(ii) $\Longrightarrow$ (i) Assume that B is closed. According to Corollary 4.1.33 it will suffice to show that

$$B = \bigcap_{i \in I} \text{Aut } D_i$$

for some family $(D_i)_{i \in I}$ of simple digraphs.

For every stabilizer B_X let $O(B_X)$ denote the set of orbits of B_x . Let

$$I = \bigcup_{x \in V} \{x\} \times O(B_x) \quad .$$

Let $i \in I$. Then

$$i = (x, C)$$

for some $x \in V$, $C \in O(B_x)$. Define the simple digraph D_i by

$$V(D_i) = V \ ,$$

$$E(D_i) = \bigcup_{\theta \in B} \{ \theta (x)\} \times \theta (C) \ .$$

Clearly

$$B \subseteq \bigcap_{i \in I} \text{Aut } D_i \quad .$$

To prove that equality holds, it is enough to show that for every

$$\sigma \in \bigcap_{i \in I} \text{Aut } D_i \ , \quad x \in V \ ,$$

there is some $\theta_x \in B$ with

$$\theta_x \sigma \Rrightarrow B_x \quad .$$

Since B is closed, the result will follow.

Let σ be as above. For $x \in V$, the orbit of B containing x is

$$\{ y \in V \mid (y, y) \in E(D_{(x, \{x\})}) \} .$$

Since

$$\sigma \in \operatorname{Aut} D_{(x, \{x\})}$$

for every $x \in V$, we must have

$$\sigma \Rrightarrow B .$$

For every $x \in V$ there is some $\theta_x \in B$ with

$$\theta_x (\sigma (x)) = x .$$

If now C is an orbit of the stabilizer B_x,

$$C \in O(B_x) ,$$

then

$$C = \{ y \in V \mid (x, y) \in E(D_{(x, C)}) \} .$$

Consequently C is a constituent of the stabilizer

$$(\operatorname{Aut} D_{(x, C)})_x ,$$

and since $\theta_x \sigma$ belongs to this stabilizer,

$$\theta_x \sigma (C) = C .$$

It follows that $\theta_x \sigma$ implies B_x , as claimed. The proof is finished.

Remark. The underlying principle of the above proof is a generalization of the Cayley color graph construction [C 3, O 3] . Analogous methods were used also by B. Jonsson in the description of automorphism groups of universal algebras [J 2] .

4.3.1. PROPOSITION. A permutation group B having at least one trivial stabilizer B_y is the automorphism group of some digraph.

Proof. According to proposition 4.2.5. we have to show that B is closed. Let $\sigma \in S_V$ and assume that for every $x \in V$ there is some $\theta_x \in B$ with

$$\theta_x \sigma \Rrightarrow B_x \quad .$$

Let $x = y$. Then $\theta_x \sigma$ implies the trivial permutation group on V, hence $\theta_x\sigma$ must be the identity permutation and

$$\sigma = \theta_x^{-1} \in B .$$

4.3.11. COROLLARY. Every regular permutation group is the automorphism group of some digraph.

4.3.12. According to proposition 4.1.32 and corollary 4.3.11, every regular permutation group B is of the form

$$B = \bigcap_{i \in I} \text{Aut } D_i \quad ,$$

where the D_i are strict digraphs. Clearly it can be required that

$$E(D_i) \cap E(D_j) = \emptyset$$

if $i \neq j$. If we then think of the different D_i as represented in the same diagram, the darts of each D_i being distinguished from the other darts by the assignment of some "color i", then we have essentially a redundant Cayley color graph [C 3,0 3].

4.3.13. COROLLARY. If at least one stabilizer B_y of a permutation group B is trivial, then every subgroup of B is the automorphism group of some digraph.

Proof. For any subgroup A of B, A_y is trivial and proposition 4.3.1 applies to A as it does to B itself.

4.3.2. Galois groups.

Let an algebraic extension E of a field F have a primitive element y [L 1] . Let $G(E|F)$ be the group of automorphisms of E over F. Every element of E is a polynomial expression in y, with coefficients in F. Consequently

$$G(E \mid F)_y$$

is trivial and according to Corollary 4.3.13 every subgroup of $G(E \mid F)$ is the automorphism group of some digraph.

4.3.3. Linear groups.

Let F be a field and n a non-zero cardinal number. Consider an n-dimensional vector space V over F. Let $GL(n, F)$ be the group of its invertible linear transformations (i.e. vector space automorphisms). We shall determine when $GL(n, F)$ is the automorphism group of some digraph.

Case 1. $n = 1$. Then the stabilizer

$$GL(1, F)_x$$

of any non-zero element x of V is trivial and by proposition 4.3.1 $GL(1, F)$ is the automorphism group of some digraph.

Case 2. $n > 1$ and $|F| \neq 2$. Then the stabilizer $GL(n, F)_x$ of any $x \in V$ has singleton orbits of the form

$$\{ \lambda x \} ,$$

$\lambda \in F$, and also one non-singleton orbit

$$V \setminus \{ \lambda x \mid \lambda \in F \}$$

Choose a field element

$$\alpha \in F \setminus \{ 0 , 1 \} ,$$

a non-zero vector $v \in V$, and define a permutation $\sigma \in S_V$ as follows:

$$\sigma (\lambda v) = \alpha \lambda v$$

for every $\lambda \in F$ and

$$\sigma (x) = x$$

if

$$x \notin \{\lambda v \mid \lambda \in F \} .$$

We claim that for every $y \in V$ there is some $\theta_y \in GL(n, F)$ such that

$$\theta_y \quad \sigma \Rrightarrow GL(n, F)_y \quad .$$

Indeed, if

$$y = \lambda v$$

for some $\lambda \in F \setminus \{0\}$, then let θ_y be given by

$$\theta_y (z) = \frac{1}{\alpha} z$$

for every $z \in V$. Otherwise let θ_y be the identity transformation.

However, it is clear that

$$\sigma \notin GL(n , F) \quad .$$

Consequently $GL(n , F)$ is not closed and according to proposition 4.2.5 it is not the automorphism group of any digraph.

Case 3. $n = 2$ and $|F| = 2$. Since any permutation of V fixing the zero vector is a linear transformation, it is easy to see that $GL(2 , F)$ is the automorphism group of some digraph.

Case 4. $n > 2$ and $|F| = 2$. For every $x \in V$, the 3 orbits of $GL(n, F)_x$ are

$$\{0\} \ , \ \{ x \} \ , \ V \setminus \{ 0 , x \} \ .$$

If σ is a permutation of V fixing the zero vector and $x \in V$, then we can find some $\theta_x \in GL(n , F)$ such that

$$\theta_x \ \sigma \Rrightarrow GL(n , F)_x \ .$$

Indeed, θ_x can be any linear transformation such that

$$\theta_x \ (\sigma \ (x)) = x \ .$$

On the other hand, it is possible to find a permutation σ of V fixing the zero vector that is not a linear transformation. Let

$$v_1 \ , \ v_2 \ , \ v_3$$

be three linearly independent vectors. Let σ be the permutation of V having the unique non-trivial cycle

$$(v_1 + v_2 \ , v_2 + v_3 \ , \ v_3 + v_1) \ .$$

Clearly

$$\sigma \in (S_V)_o \ \backslash \ GL(n, F) \ ,$$

so that GL(n, F) is not closed and according to proposition 4.2.5 it is not the automorphism group of any digraph.

4.4. Digraphs with abelian group.

4.4.0. The notation is kept multiplicative except in the group Z of integers.

4.4.1. If S_1 and S_2 are subgroups of an abelian group A, then let

$$S_1 S_2 = \{xy \mid x \in S_1 , \quad y \in S_2 \} \quad .$$

Clearly $S_1 S_2$ is the intersection of all subgroups of A that contain simultaneously S_1 and S_2.
If $A = Z$ and

$$S_1 = (m_1) \quad , \quad S_2 = (m_2) \quad ,$$

then

$$S_1 S_2 = \quad (\gcd\ (m_1, m_2)) \quad .$$

Given any subgroup S of A, we say that two elements x and y of A are <u>congruent modulo S</u> ,

$$x \equiv y \mod S \ ,$$

if

$$x\, y^{-1} \in S \quad .$$

If $A = Z$ and

$$S = (m) \quad ,$$

then congruence modulo S is just the usual concept of congruence modulo the integer m generating S.

4.4.2. Let $(S_k)_{k \in K}$ be a family of subgroups of an abelian group A. Clearly, for every family $(x_k)_{k \in K}$ of elements of A, the condition

(i) $\exists\, x \in A \quad \forall\, k \in K \quad x \equiv x_k \mod S_k$

implies

(ii) $\forall\, k, h \in K \quad x_k \equiv x_h \mod S_k\, S_h$.

We say that the <u>Chinese remainder theorem holds</u> for the family $(S_k)_{k \in K}$ if, for every family $(x_k)_{k \in K}$ of elements of A condition (i) is equivalent to (ii).

It is well known that the Chinese remainder theorem holds for every finite family of subgroups of Z (see e.g. [0 2]).

If $(S_k)_{k \in K}$ and $(S_h)_{h \in H}$ are two families of subgroups of an abelian group such that

$$\{ S_k \mid k \in K \} = \{ S_h \mid h \in H \} ,$$

then the Chinese remainder theorem holds for $(S_k)_{k\in K}$ if and only if it holds for $(S_k)_{h\in H}$.

4.4.3. Let A be an abelian permutation group on a set V and let $O(A)$ denote the set of orbits of A. For every $\gamma \in A$ and every orbit $i \in O(A)$ define the permutation (γ , i) of V by

$$(\gamma , i) (x) = \gamma (x)$$

if $x \in i$, and

$$(\gamma , i) (x) = x$$

if $x \notin i$. We observe that for every $i \in O(A)$

$$\gamma \longrightarrow (\gamma , i)$$

is homomorphism from A to S_V .

For every orbit $i \in O(A)$ and any two elements $x, y \in i$, the stabilizers A_x and A_y are conjugate in A, and hence they are identical. Let A_i denote the common stabilizer of all the elements of i . The family $(A_i)_{i\in O(A)}$ will be called the <u>family of stabilizers</u> of A.

4.4.4. Let π be a partition of a set V. Let $(\gamma_i)_{i\in\pi}$ be a family of permutations of V such that for every block i of π we have $\gamma_i(i) = i$. We define the permutation $\gamma = \prod_{i\in\pi} \gamma_i$ of V by

$$\gamma(x) = \gamma_i(x)$$

if $x \in i$.

For every element γ of an abelian permutation group A we have

$$\gamma = \prod_{i\in O(A)} (\gamma, i)$$

4.4.5. PROPOSITION. Let A be an abelian permutation group on a set V. For every permutation σ of V the following three conditions are equivalent:

(i) $\forall x \in V \ \exists \theta_x \in A \quad \theta_x \sigma \equiv\!\!> A_x$,

(ii) $\forall i \in O(A) \ \exists \theta_i \in A \quad \theta_i^{-1} \sigma \equiv\!\!> A_i$,

(iii) there is a family $(\theta_i)_{i \in O(A)}$ of elements of A indexed by O(A), such that

$$\sigma = \prod_{i \in O(A)} (\theta_i , i)$$

and

$$\theta_i \equiv \theta_j \mod A_i A_j$$

for every $i, j \in O(A)$.

Proof. The equivalence of (i) and (ii) is trivial.

(ii) $\Longrightarrow$ (iii). Assume (ii). For every $i \in O(A)$,

$$\theta_i^{-1} \sigma \equiv\!> A_i$$

implies a fortiori that

$$\sigma (x) = \theta_i (x)$$

for every $x \in i$. Consequently we have

$$\sigma = \prod_{i \in O(A)} (\theta_i , i)$$

Also

$$\theta_i^{-1} \prod_{j \in O(A)} (\theta_j , j) \Rrightarrow A_i$$

implies that

$$(\theta_i^{-1} , j) \ (\theta_j , j) \Rrightarrow A_i$$

for every $j \in O(A)$. But

$$(\theta_i^{-1} , j) \ (\theta_j , j) = (\theta_i^{-1} \theta_j , j) \ .$$

Let us abbreviate

$$\theta_i^{-1} \ \theta_j = \alpha_{ij}$$

for every $i, j \in O(A)$. Let x be any element of the orbit j. From

$$(\alpha_{ij} , j) \Rrightarrow A_i$$

it follows that there exists a

$$\beta_{ij} \in A_i$$

with

$$\beta_{ij}(x) = (\alpha_{ij}, j)(x) .$$

But

$$(\alpha_{ij}, j)(x) = \alpha_{ij}(x)$$

and consequently

$$\beta_{ij}^{-1} \alpha_{ij} (x) = x ,$$

i.e.

$$\alpha_{ij} \equiv \beta_{ij} \mod A_j .$$

Hence

$$\theta_i \equiv \theta_j \qquad \mod A_i A_j .$$

The latter congruence clearly holds for every $i, j \in O(A)$, proving condition (iii).

(iii) $\Longrightarrow$ (ii). Assume (iii). We shall prove that

$$\theta_i^{-1} \sigma \Rrightarrow A_i$$

for every $i \in O(A)$. Let i be fixed. We have to show that for every $x \in V$

$$\theta_i^{-1} \sigma(x) = \beta(x)$$

for some $\beta \in A_i$. Let j be the orbit of A that contains x. Since

$$\theta_i \equiv \theta_j \mod A_i A_j$$

holds, there are some

$$\beta_{ij} \in A_i$$

and

$$\gamma_{ij} \in A_j$$

such that

$$\theta_i^{-1} \theta_j = \beta_{ij} \gamma_{ij} .$$

Then

$$\theta_i^{-1} \sigma(x) = \theta_i^{-1} \theta_j(x) =$$

$$= \beta_{ij} \gamma_{ij}(x) = \beta_{ij}(x) ,$$

and we can take

$$\beta = \beta_{ij} .$$

This completes the proof.

4.4.6. PROPOSITION , Let $(\theta_i)_{i\in O(A)}$ be a family of elements of an abelian permutation group A indexed by O(A). We have

$$\prod_{i\in O(A)} (\theta_i, i) \in A$$

if and only if there is some $\theta\in A$ such that

$$\forall\ i\in O(A) \quad \theta \equiv \theta_i \mod A_i \ .$$

Proof. If

$$\prod_{i\in O(A)} (\theta_i, i) \in A ,$$

then

$$\forall\ i\in O(A) \quad \prod_{i\in O(A)} (\theta_i, i) \equiv \theta_i \mod A_i \ .$$

On the other hand, if $\theta \in A$ is a solution of the congruence system

$$\forall\ i\in O(A) \quad \theta \equiv \theta_i \mod A_i \ ,$$

then we must have

$$(\theta, i) = (\theta_i, i)$$

for every $i \in O(A)$, and

$$\prod_{i \in O(A)} (\theta_i, i) = \theta \quad .$$

4.4.7. PROPOSITION. An abelian permutation group is the automorphism group of some digraph if and only if the Chinese remainder theorem holds for the family of stabilizers.

Proof. Propositions 4.2.4, 4.4.5, 4.4.6.

4.5. On the Chinese remainder theorem.

4.5.1. For any abelian group A, let L(A) denote the lattice of subgroups of A. P. Camion , C.S. Levy and H.B. Mann have proved [C 1] that the Chinese remainder theorem holds for a given finite family $(S_i)_{i \in I}$ of subgroups of A if the S_i generate a distributive sublattice of L(A). In particular the Chinese remainder theorem holds for every two subgroups of A. Here we prove the following:

4.5.21. PROPOSITION. Let L_o be a sublattice of the lattice L(A) of all subgroups of an abelian group A. The following two conditions are equivalent:

(i) L_o is a distributive lattice,

(ii) the Chinese remainder theorem holds for every finite family of subgroups belonging to L_o .

Proof. (i) implies (ii) according to the mentioned result of Camion , Levy and Mann [C 1].

(ii) $\Longrightarrow$ (i) We shall in fact show that if

$$S_1 , \ldots , S_n$$

and

$$R_1 , \ldots , R_m$$

are subgroups of A such that

$$\left(\bigcap_{i=1}^{n} S_i\right) \left(\bigcap_{j=1}^{m} R_j\right) \neq \bigcap_{i,j} (S_i R_j) ,$$

then the Chinese remainder theorem fails to hold for the family

$$(S_1 , \ldots , S_n, R_1 , \ldots , R_m) .$$

Indeed,

$$\left(\bigcap_{i=1}^{n} S_i\right) \left(\bigcap_{j=1}^{m} R_j\right)$$

is strictly contained in $\bigcap_{i,j} (S_i R_j)$. Let

$$t \in \bigcap_{i,j} (S_i R_j) \setminus \left(\bigcap_{i} S_i\right) \left(\bigcap_{j} R_j\right) .$$

Then, 0 denoting the identity element of A, the congruence system

$$
\begin{aligned}
x &\equiv 0 \quad \bmod S_1 \\
&\vdots \\
x &\equiv 0 \quad \bmod S_n \\
x &\equiv t \quad \bmod R_1 \\
&\vdots \\
x &\equiv t \quad \bmod R_m
\end{aligned}
$$

does not have any solution $x \in A$.

4.5.22. COROLLARY. The Chinese remainder theorem holds for every finite family of subgroups of an abelian group A if and only if every finitely generated subgroup of A is cyclic.

Proof. Indeed, Ore has proved [O 1] that the lattice L(A) is distributive if and only if every finitely generated subgroup of A is cyclic.

4.5.23. In view of the above corollary, it can be said that the classical Chinese remainder theorem is due to the fact that every subgroup of $\mathbb{Z}$, the group of integers, is cyclic. In the next subsection we shall consider the Chinese remainder theorem for infinite families of subgroups of $\mathbb{Z}$.

4.5.31. The Chinese remainder theorem trivially holds for a given family $(S_i)_{i \in I}$ of subgroups of an abelian group A if one of the S_i is the trivial subgroup.

4.5.32. PROPOSITION. Let $(S_i)_{i \in I}$ be an infinite family of subgroups of the group $\mathbb{Z}$ of integers. The Chinese remainder theorem holds for the family $(S_i)_{i \in I}$ only if one of the S_i is trivial.

Proof. The notation in this proof will be additive and the congruences will be written modulo integers rather than modulo subgroups.

Suppose that none of the S_i is trivial. We can assume that all the S_i are distinct and that I is the set of positive integers. Let

$$S_i = (m_i)$$

for every $i \in I$ and assume that

$$0 < m_1 < m_2 < \ldots < m_i < m_{i+1} < \ldots$$

We shall define a sequence

$$n_1 \ , n_2 \ , \ \ldots \ , n_i \ , \ \ldots$$

of integers and a surjective function

$$f : I \longrightarrow Z \quad ,$$

such that for every $i, j \in I$,

$$n_i \equiv n_j \bmod \gcd(m_i, m_j) \quad ,$$

but for every $i \in I$, no element

$$x \in f([1, i])$$

is a solution of the system

$$\left.\begin{array}{llll} x & \equiv n_1 & \bmod & m_1 \\ & \vdots & & \\ x & \equiv n_i & \bmod & m_i \end{array}\right\}$$

This will clearly prove that the Chinese remainder theorem fails to hold for the family $(S_i)_{i \in I}$.

We define n_i and $f(i)$ by induction on i. Let n_1 be any integer not divisible by m_1 ,

$$m_1 \nmid n_1 \quad ,$$

and let

$$f(1) = 0 \; .$$

Suppose that n_1 , ... , n_i and $f(1)$, ... , $f(i)$ have already been defined. There exists an integer $k > i$ such that

$$m_k \nmid \operatorname*{lcm}_{j<k} m_j \quad .$$

Take the smallest possible k. Let $\bar{x}$ be any solution of the system

$$\begin{aligned} \bar{x} &\equiv n_1 \mod m_1 \\ &\vdots \\ \bar{x} &\equiv n_i \mod m_i \quad . \end{aligned}$$

For every

$$j \in [i+1 , k] \setminus \{k\}$$

define

$$n_j = \bar{x} \quad , \quad f(j) = f(i) \quad .$$

Define $f(k)$ to be an element of

$$Z \setminus f([\,1,i\,])$$

having minimal absolute value. We claim that there is an integer p not divisible by m_k such that

$$f(k) + p \equiv n_j \mod \gcd(m_j, m_k)$$

for every $j \in [1, k-1]$. Then we define

$$n_k = f(k) + p.$$

The inductive step will then be accomplished and the proof finished.

In order to prove our claim, we have to find a solution p to the system

$$\forall\; j \in [\,1, k-1\,] \quad p \equiv n_j - f(k) \mod \gcd(m_j, m_k)\;.$$

This is possible because for every $j_1, j_2 \in [1, k-1]$

$$n_{j_1} \equiv n_{j_2} \mod \gcd(m_{j_1}, m_{j_2})$$

and hence

$$n_{j_1} - f(k) \equiv n_{j_2} - f(k) \mod \gcd(\gcd(m_{j_1}, m_k),$$
$$\gcd(m_{j_2}, m_k)) .$$

If p_o is a particular solution not divisible by m_k, then let

$$p = p_o .$$

If

$$m_k \mid p_o ,$$

then let

$$p_1 = p_o + \operatorname{lcm}_{j<k} (\gcd(m_j, m_k)) ,$$

which is another particular solution. But

$$\operatorname{lcm}_{j<k} (\gcd(m_j, m_k)) =$$
$$\gcd(m_k, \operatorname{lcm}_{j<k} m_j) ,$$

and since

$$m_k \nmid \operatorname{lcm}_{j<k} m_j ,$$

we have

$$m_k \nmid \operatorname*{lcm}_{j<k} (\gcd (m_j , m_k))$$

and

$$m_k \nmid p_1 .$$

Let

$$p = p_1 .$$

The proof is now complete.

Remark. Although the infinite congruence system

$$x \equiv n_i \mod m_i \quad i = 1, 2, \ldots$$

does not have a solution x, every finite subsystem of it has a solution.

4.6. Consequences for the representability of abelian permutation groups by digraphs.

4.6.1. PROPOSITION. Every abelian permutation group with at most two orbits is the automorphism group of some digraph.

Proof. As noticed in 4.5.1, the Chinese remainder theorem holds for any two subgroups of an abelian group.

4.6.2. PROPOSITION. For every abelian group B the following three conditions are equivalent:

(i) $L(B)$ is a distributive lattice,

(ii) every permutation group A abstractly isomorphic to B and having only a finite number of orbits is the automorphism group of a digraph,

(iii) every permutation group A abstractly isomorphic to B and having exactly 3 orbits is the automorphism group of a digraph.

Proof. The implications (i) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (iii) are clear.

To prove that (iii) $\Longrightarrow$ (i), suppose that $L(B)$ is not distributive. We can find three subgroups S_1, S_2, S_3 of B such that

$$S_1\,(S_2 \cap S_3) \neq S_1 S_2 \cap S_1 S_3 .$$

Moreover, it can be seen without difficulty that S_1, S_2, S_3 can be found subject to the additional requirement that the intersection

$$S_1 \cap S_2 \cap S_3$$

be the trivial subgroup of B. For every $x \in B$, and $i = 1, 2, 3$, let

$$x S_i = \{ x y \mid y \in S_i \}$$

and let

$$B / S_i = \{ x S_i \mid x \in B \}$$

be the quotient group by S_i. Let

$$V = B / S_1 \cup B / S_2 \cup B / S_3 .$$

For every $b \in B$, define a permutation τ_b of V by

$$\tau_b (x S_i) = (bx) S_i$$

for every $x S_i \epsilon V$. Let

$$A = \{\tau_b \mid b \epsilon B \} .$$

The permutation group A has 3 orbits

$$B / S_1 , B / S_2 , B / S_3$$

and

$$b \longrightarrow \tau_b$$

is an isomorphism from B to A under which the subgroups S_1 , S_2, S_3 correspond to the stabilizers of A. Since according to the proof of proposition 4.5.21 the Chinese remainder theorem does not hold for S_1, S_2, S_3 , it also fails to hold for the family of stabilizers of A. Hence A is not the automorphism group of any digraph and (iii) fails.

4.6.3. PROPOSITION. A cyclic permutation group A is the automorphism group of some digraph if and only if A is of finite order or has an infinite orbit.

Proof. If A is of finite order, then every family of subgroups of A is finite. It follows from proposition 4.4.7 and corollary 4.5.22 that A is the automorphism group of some digraph.

If A is infinite and has an infinite orbit I, then the stabilizer of any element of I is trivial. According to proposition 4.4.7 and the observation made in 4.5.31, A is the automorphism group of some digraph. (This could also be inferred directly from corollary 4.3.13.)

If A is infinite but has no infinite orbit, then none of the stabilizers is trivial and the family of stabilizers is infinite. According to propositions 4.4.7 and 4.5.32, A is not the automorphism group of any digraph.

Example. Let V be the set of integers strictly larger than 2 and define a permutation σ of V as follows:

$$\sigma(k) = k + 1$$

if k is not a power of 2,

$$\sigma(k) = \frac{k}{2} + 1$$

if k is a power of 2.

Let A be the subgroup of S_V generated by σ . A has infinite order and its orbits are the sets

$$[\ 2^n + 1\ ,\ 2^{n+1}\ ]\ ,$$

$n = 1, 2, \ldots$ According to proposition 4.5.32 A is not the automorphism group of any digraph.

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