Periodic Adaptive Control for First-Order Discrete-Time Plants

by

Swapnil Kanabar

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In adaptive control the goal is to deal with systems that have unknown and/or time-varying parameters. An adaptive controller typically consists of an LTI compensator together with an identifier or a tuner which is used to adjust the compensator parameters. A common approach to tuning is to invoke the Certainty Equivalence Principle, where at each instance of time the estimated plant parameters are assumed to be correct and the controller gains are updated accordingly.

In this work we consider the first order case. We use the Certainty Equivalence approach to periodically estimate the plant parameters and then update the control action in order to provide stability. The data from first two steps are used to estimate the system parameters for the next two steps; the approach works by using a nominal control law and adding a small perturbation to the gain. The controller is proven to be noise tolerant, and we are able to prove a linear-like bound on the closed-loop behavior.

Acknowledgements

I would like to thank my research supervisors Prof. Daniel Miller and Prof. Andrew Heunis who oversaw my work and provided a lot of help with choosing the research direction and solving the problems I encountered along the way and made this possible. I would also like to thank my Parents and my sister without whom pursuing my graduate studies was just a mere dream.

Dedication

This is dedicated to my beloved parents Mr. and Mrs. Kanabar, who have raised me to be the person I am today. I take this opportunity to thank them for their love, patience and selfless support. I also dedicate this to my sister Karishma and my future wife Shikha for their support to make this dream a reality.

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Chapter 1

Introduction

1.1 The Background

Adaptive control is a systematic technique which provides tools for automatic adjustment of a controller used in order to achieve or to maintain a desired level of control system performance when the parameters of the plant dynamic model are unknown and/or change in time. A classical example of an adaptive controller is a linear time-invariant (LTI) compensator with adjustable parameters. A tuning mechanism is used to adjust the compensator's parameters to appropriately match the plant.

The study of such uncertain linear systems started in the 1950's. The design of autopilots for high-performance aircraft motivated intense research activity in adaptive control. High-performance aircraft undergo drastic changes in their dynamics when they fly and these changes cannot be handled by constant-gain feedback control. A sophisticated controller, such as an adaptive controller, that could learn and accommodate changes in the aircraft dynamics was needed. Model reference adaptive control was suggested by Whitaker et al. [22] to solve the autopilot control problem. The goal in MRACP is to have the output of the plant asymptotically track the output of the stable reference model in response to a piece-wise continuous input. The lack of stability proofs and the lack of understanding of the properties of the proposed adaptive control schemes coupled with a disaster in a flight test caused the interest in adaptive control to diminish.

Due to the initial lack of success to solve this problem, the focus shifted to the easier problem, in which the system parameters are fixed but unknown. In such cases, although the structure of the controller, in general, will not depend on the particular values of the plant model parameters, the correct tuning of the controller parameters cannot be done without some knowledge of the plant parameters. Adaptive control techniques can provide an automatic tuning procedure for the controller parameters. In such cases, in the absence of external disturbances, the effect of the adaptation vanishes as time increases and the system settles down. Some general results which tackled this simplified problem appeared in 1980 [6, 10, 16]. After the initial success, efforts to improve the transient response, noise tolerance and tolerance to unmodelled dynamics led to the introduction and adaptation of two main approaches. The first approach used the classic Certainly Equivalence approach, which used an estimator with an LTI compensator. The plant parameters were first estimated which updated the compensator gains periodically[7, 8]. The other approach was the switching approach, which involved switching between different LTI compensators according to the required performance. Initially prerouted logic-based switching approach [3, 14] were used and then more sophisticated approaches such as supervisory and multi-model switching control were introduced [17, 18, 19]. Specifically, one of the most successful results shown by Morse [17, 18] wherein robust LTI techniques are used to design a family of LTI compensators, and suitable switching techniques are used depending on the performance requirements. These controllers are tolerant to unmodelled dynamics and provided step tracking for a large set of uncertain plants.

In the late 1980s and early 1990s a lot of effort was put into extending the results of the late 1980s to linear time-varying systems. Many of the classical adaptive controllers were suitably modified to tolerate some degree of time variation of the plant parameters. Here is a list of some notable papers and articles on the work done in this area for the last two decades [9, 11, 5, 2, 21]. As far as the author is aware, no work successfully handles plants with non-minimum phase with fast time variations of the plant parameters.[1]

The central idea of adaptive control is to tune the controller in response to the time-varying plant parameters and determine a control action accordingly. So far there were two core ideas discussed namely the Switching approach and the Certainty Equivalence approach, out of which we use the latter approach wherein the key idea is to first estimate the time-varying parameters and then tune the control action accordingly to achieve desired output.

[1] All of this work is in continuous time and does not extend naturally to discrete-time.

1.2 Purpose

The goal of this thesis is to design an Adaptive Controller for a discrete-time first order system with unknown and time-varying parameters. The objective of this is stability and performance with three goals in mind:

- Exponential and BIBO stability for the system with fixed but unknown parameters and no noise.
- Near optimal transient performance for the system with fixed but unknown parameters and no noise.
- Tolerance to time-variations.

We will be using the Certainty Equivalence approach, where the key idea is to periodically estimate the plant parameters and then update the control action in order to provide stability. At every other step, the system parameters are estimated which updates the control action resulting in a controller of period two. The estimation is a basic computation of the data available from the two steps to get the system parameters for the next two steps. The controller runs in a very simple yet sophisticated manner. Roughly speaking, it is designed in such a way that, whenever the system tries to go unstable (the state variable increases), the effect of the noise decreases, giving us correct estimates making the state variable of the closed-loop system close to zero again.

1.3 Organization

The following chapter deals with the required mathematical preliminaries. In Chapter 3 we will define the problems and state the necessary assumptions. Chapter 4 deals with the stability analysis of the systems with fixed but unknown parameters; we will also look at the transient performance of the proposed control action with respect to the performance of an ideal DLQR controller. In Chapter 5 we analyse the case of a time-varying plant. Chapter 6 presents some examples and simulations. And finally, Chapter 7 concludes the thesis by summarizing the main goals achieved and outlining the scope of future work.

Chapter 2

Mathematical Preliminaries

To measure the size of a vector we will use the infinity norm: for $v \in \mathcal{R}^n$, we define $\|v\| := max\{|v_i| : i = 1, 2 \ldots n\}$. With $\mathcal{A} \subset \mathcal{R}$, we let $s(\mathcal{A})$ denote the set of sequences taking values on $\mathcal{A}$; with $\delta > 0$, we let $s(\mathcal{A}, \delta)$ denote those $x \in s(\mathcal{A})$ satisfying $|x[k+1] - x[k]| \le \delta$ for $k \in \mathcal{Z}$.

Let the set $\mathcal{A} \subset \mathcal{R}$ be of the form $\mathcal{A} := [\underline{a}_1, \overline{a}_1] \cup [\underline{a}_2, \overline{a}_2] \cup \cdots \cup [\underline{a}_n, \overline{a}_n]$ with $\underline{a}_1 < \overline{a}_1 < \underline{a}_2 < \overline{a}_2 < \cdots < \underline{a}_n < \overline{a}_n,\ n \in \mathbb{N}$, and define $\delta_{\mathcal{A}} := \frac{1}{2} min\{|\underline{a}_2 - \overline{a}_1|, |\underline{a}_3 - \overline{a}_2| \cdots |\underline{a}_n - \overline{a}_{n-1}|\}$. We define projection function $\Pi_{\mathcal{A}} : \mathcal{R} \to \mathcal{A}$ by

$$\Pi_{\mathcal{A}}(x) := \begin{cases} x & \text{if } x \in \mathcal{A}; \\ \underline{a}_1 & \text{if } x < \underline{a}_1; \\ \overline{a}_j & \text{if } x \in \left(\overline{a}_j,\ \frac{1}{2}\left(\overline{a}_j + \underline{a}_{j+1}\right)\right) \\ & \quad \text{and} \quad j = 1,\ 2,\ \ldots,\ n-1; \\ \underline{a}_{j+1} & \text{if } x \in (\frac{1}{2}(\overline{a}_j + \underline{a}_{j+1}),\ \underline{a}_{j+1}) \\ & \quad \text{and} \quad j = 1,\ 2,\ \ldots,\ n-1; \\ \overline{a}_n & \text{if } x > \overline{a}_n. \end{cases}$$

This projection function will be used in our estimation procedure. In particular, if $a \in \mathcal{A}$ and the estimation error is less than $\delta_{\mathcal{A}}$ from $\mathcal{A}$, then projecting the estimate onto $\mathcal{A}$ reduces the estimation error.

Chapter 3

Problem Statement and Assumptions

3.1 Discrete-Time Linear Systems

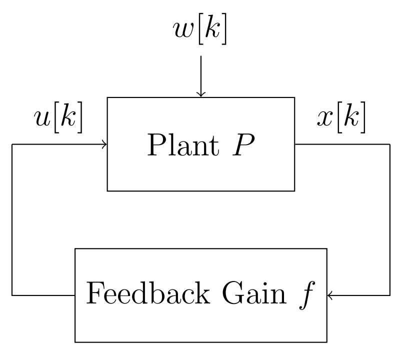


Figure 3.1: Closed-loop system under consideration

In adaptive control of time-varying systems, a first-order Discrete-Time Single-Input Single-Output (SISO) plant can be described by the following state-space equations:

$$x[k+1] = a[k]x[k] + b[k]u[k] + w[k], \; x[0] = x_0, \tag{3.1}$$
$$y[k] = x[k], \tag{3.2}$$

where $x[k] \in \mathcal{R}$ is the system state, $u[k] \in \mathcal{R}$ is the control input, $y[k] \in \mathcal{R}$ is the measured output, $a[k]$ and $b[k]$ are the plant parameters[1] and $w[k] \in \mathcal{R}$ is the external disturbance.

[1]For LTI systems $a[k] = a$ and $a[k] = b$ for every $k \in \mathcal{Z}^+$.

Here, $a[k]$ lies in a compact set $\mathcal{A}$ consisting of a finite number of intervals; there may be additional constraints required to ensure that the control law is well defined. Similarly $b[k]$ lies in a compact set $\mathcal{B}$ of a finite set of intervals not containing zero. This means, in particular, that

$$\underline{a} := min\,\{|a| : a \in \mathcal{A}\} \geq 0,$$

$$\bar{a} := max\,\{|a| : a \in \mathcal{A}\} < \infty,$$

$$\underline{b} := min\,\{|b| : b \in \mathcal{B}\} > 0,$$

$$\bar{b} := max\,\{|b| : b \in \mathcal{B}\} < \infty.$$

Associated with $\mathcal{A}$ and $\mathcal{B}$ are $\delta_{\mathcal{A}}$ and $\delta_{\mathcal{B}}$ (as defined in Chapter 2); we now define

$$\delta := min\,\{\delta_{\mathcal{A}}, \delta_{\mathcal{B}}\}\,.$$

Remark 1*:* The noise is considered in the above way to simplify calculations. That being said, we can have noise at both the input and the output and convert the setup to the above form. To see this, consider a plant with input disturbance $d[k]$ and the output measurement noise $n[k]$:

$$\begin{aligned} x[k+1] &= a[k]x[k] + b[k]u[k] + d[k],\ x[0] = x_0, \\ y[k] &= x[k] + n[k]. \end{aligned}$$

Hence,

$$\begin{aligned} y[k+1] &= x[k+1] + n[k+1] \\ &= a[k]x[k] + b[k]u[k] + d[k] + n[k+1] \\ &= a[k]\,(y[k] - n[k]) + b[k]u[k] + d[k] + n[k+1] \\ &= a[k]y[k] + b[k]u[k] + (d[k] - a[k]n[k] + n[k+1])\,. \end{aligned}$$

If we define

$$w[k] := d[k] - a[k]n[k] + n[k+1],$$

then

$$y[k+1] = a[k]y[k] + b[k]u[k] + w[k],$$

which is of the form (3.1)-(3.2). ∎

Occasionally we will consider the plant (3.1)-(3.2) when there is no time variation, yielding

$$
\begin{aligned}
x[k+1] &= ax[k] + bu[k] + w[k], && (3.3)\\
y[k] &= x[k]. && (3.4)
\end{aligned}
$$

Regardless of whether we consider the time-varying or time-invariant case, the control objective is to provide, at the minimum, some form of stability in the presence of uncertainty in the plant parameters. We plan to achieve this by using a state-feedback control law whose gain depends on $a[k]$ and $b[k]$ (a and b in the linear time-invariant case); since these quantities are not known, we adapt the Certainty Equivalence point of view and use the present estimate.

We allow a very general state-feedback control law of the form

$$u[k] = f\left(a[k], b[k]\right) x[k]$$

with $f : \mathcal{A} \times \mathcal{B} \to \mathcal{R}$. We say that f is *admissible* if:

(i) f is stabilizing in the sense that $a + bf(a,b) \in (-1,1)$ for all $(a,b) \in \mathcal{A} \times \mathcal{B}$, and

(ii) f is globally Lipschitz continuous on $\mathcal{A} \times \mathcal{B}$.

Some examples of $f(a,b)$ are:

(a) Deadbeat Control: $f(a[j], b[j]) = -a[j]/b[j]$.

(b) Pole Placement with $\lambda \in (-1,1)$: $f(a[j], b[j]) = \frac{1}{b[j]}\lambda - \frac{a[j]}{b[j]}$.

(c) Discrete-Time LQR (DLQR): With $r > 0$, for a given initial condition $x[0]$ the control law which minimizes the cost

$$\sum_{k=0}^{\infty} \left[x[k]^2 + ru[k]^2\right]$$

for the LTI plant (3.3)-(3.4) is the state-feedback control law of the from

$$u[k] = f(a,b)x[k],$$

with

$$f(a,b) = \begin{cases} \frac{1-a^2+b^2-\sqrt{(a^2+b^2-1)^2-4b^2}}{2ab} & \text{if } a \neq 0, \\ 0 & \text{if } a = 0. \end{cases}$$

It can be verified that f is globally Lipschitz continuous on $\mathcal{A} \times \mathcal{B}$.

We define the image of f as

$$\mathcal{F} := \{f(a,b) : (a,b) \in \mathcal{A} \times \mathcal{B}\},$$

and

$$\bar{f} := max\,\{|f(a,b)| : f \in \mathcal{F}\} < \infty.$$

Proposition 1*:* If $f : \mathcal{A} \times \mathcal{B} \to \mathcal{R}$ is *admissible* then there exists a $\lambda \in [0,1)$ and $\gamma > 0$ so that

(i) for every $(a,b) \in \mathcal{A} \times \mathcal{B}$,

$$|a + bf(a,b)| \in [-\lambda, \lambda];$$

(ii) for every $(a,b), (\hat{a}, \hat{b}) \in \mathcal{A} \times \mathcal{B}$,

$$|f(a,b) - f(\hat{a}, \hat{b})| \leq \gamma \left\| \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} \right\|.$$

Proof:

Part (i) follows from the continuity of f together with the compactness of $\mathcal{A} \times \mathcal{B}$.

Part (ii) follows from the definition of Lipschitz Continuity.

$\blacksquare$

Even if a and b are time-varying, due to property (i) of f an admissible controller will exponentially stabilize the plant in the absence of noise. The challenge is that a and b are unknown. Hence, we estimate a and b yielding $\hat{a}$ and $\hat{b}$ and we use this to form an adaptive controller. To facilitate this process, we do not estimate the parameters at every step, but rather every other step. Our objective is not only stability but we would also like the adaptive control law to be close to the original state feedback control law.

3.2 The Proposed Approach and Controller Action

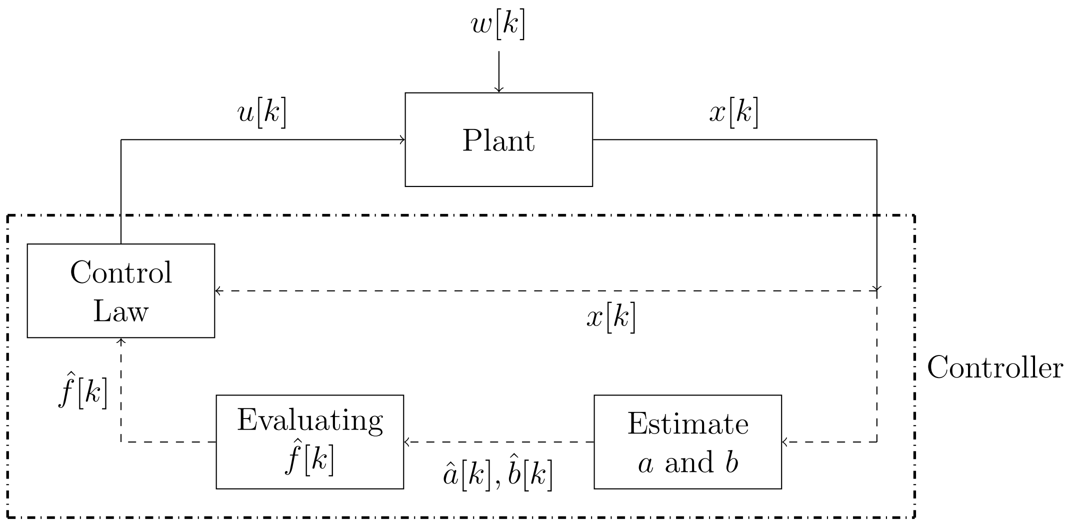


Figure 3.2: The system setup with controller structure

Classical adaptive control uses an estimator which may not converge, and if it does, then it typically converges asymptotically. Here our goal is to obtain a good estimate quickly and so that the controller is close to the original one. Since there are two parameters to estimate, we need at least two equations to solve; this motivates us to:

(i) Estimate a and b every other step.

(ii) We use the nominal control law for two steps, but we add a small perturbation to the gain to facilitate the estimation. [2]

To see how to carry out the estimation, with $\varepsilon > 0$ and $f \in \mathcal{R}$, suppose that

$$\begin{aligned} u[0] &= (f+\varepsilon)x[0], \\ u[1] &= (f-\varepsilon)x[1], \end{aligned}$$

$a[k]$ and $b[k]$ are constant, and the noise is zero. Then

$$\begin{bmatrix} x[1] \\ x[2] \end{bmatrix} = \begin{bmatrix} x[0] & (f+\varepsilon)x[0] \\ x[1] & (f-\varepsilon)x[1] \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}. \tag{3.5}$$

[2]This differs from the classical idea of probing in that here we are perturbing the gain whereas in probing we add an exogenous input.

The determinant of the two by two matrix on the RHS is $-2\varepsilon x[0]x[1]$, so if $x[0]$ and $x[1]$ are non-zero then we can solve for a and b:

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x[0] & (f+\varepsilon)x[0] \\ x[1] & (f-\varepsilon)x[1] \end{bmatrix}^{-1} \begin{bmatrix} x[1] \\ x[2] \end{bmatrix}.$$

With this as motivation we can now define the control law. The goal is to prove that this controller is stabilizing, even in the presence of noise, and ideally provides closed-loop behaviour which is very close to that provided by the original controller.

With the initial conditions of $\hat{a}[0] \in \mathcal{A},\ \hat{b}[0] \in \mathcal{B},\ \hat{f}[0] = \hat{f}[1] = f(\hat{a}[0], \hat{b}[0])$ and $\varepsilon > 0$, we propose the adaptive control law

$$u[j] = \left(\hat{f}[j] + (-1)^j \varepsilon\right) x[j],\ j \in \mathcal{Z}^+, \tag{3.6}$$

together with an estimation:

$$\begin{bmatrix} \check{a}[2j+2] \\ \check{b}[2j+2] \end{bmatrix} = \begin{cases} \begin{bmatrix} x[2j] & (\hat{f}[2j]+\varepsilon)x[2j] \\ x[2j+1] & (\hat{f}[2j+1]-\varepsilon)x[2j+1] \end{bmatrix}^{-1} \begin{bmatrix} x[2j+1] \\ x[2j+2] \end{bmatrix}, \\ \qquad \text{if } x[2j]x[2j+1] \neq 0, \\ \begin{bmatrix} \check{a}[2j] \\ \check{b}[2j] \end{bmatrix}, \text{ if } x[2j]x[2j+1] = 0, \end{cases} \tag{3.7}$$

$$\left.\begin{aligned} \hat{a}[2j+2] &= \Pi_{\mathcal{A}}(\check{a}[2j+2]), \\ \hat{b}[2j+2] &= \Pi_{\mathcal{B}}(\check{b}[2j+2]), j \in \mathcal{Z}^+; \end{aligned}\right\} \tag{3.8}$$

using the Certainty Equivalence Principle we then define

$$\left.\begin{aligned} \hat{f}[2j+2] &:= f(\hat{a}[2j], \hat{b}[2j]), \\ \hat{f}[2j+3] &:= f(\hat{a}[2j], \hat{b}[2j]). \end{aligned}\right\} \tag{3.9}$$

At first glance, one might expect that we would need a very small ε even to achieve stability, but this turns out not to be the case. The underlying reason can be seen by considering the non-adaptive linear periodic control law

$$\begin{aligned} u[2j] &= (f+\varepsilon)x[2j], \\ u[2j+1] &= (f-\varepsilon)x[2j+1], \end{aligned}$$

applied to the plant under the assumption that $a + bf \in (-1, 1)$. Assuming that the noise is zero for simplicity we have

$$\begin{aligned} x[2j+1] &= ax[2j] \;+\; bu[2j] \\ &= ax[2j] \;+\; b(f+\varepsilon)x[2j] \\ &= ax[2j] \;+\; bfx[2j] + b\varepsilon x[2j] \\ &= \underbrace{(a+bf)}_{=:a_{cl}} x[2j] + b\varepsilon x[2j] \\ &= (a_{cl} + b\varepsilon)x[2j]. \end{aligned}$$

Similarly,

$$\begin{aligned} x[2j+2] &= (a+bf)x[2j+1] - b\varepsilon x[2j+1] \\ &= (a_{cl})x[2j+1] - b\varepsilon x[2j+1] \\ &= (a_{cl} - b\varepsilon)x[2j+1] \\ &= (a_{cl} - b\varepsilon)(a_{cl} + b\varepsilon)x[2j] \\ &= (a_{cl}^2 - b^2\varepsilon^2)x[2j]. \qquad (3.10) \end{aligned}$$

For the closed-loop system to be stable we need $(a_{cl}^2 - b^2\varepsilon^2) \in (-1, 1)$. If $a_{cl} = a + bf$ is stable, then closed-loop stability is assured if $b^2\varepsilon^2 < 1$, which is the case if $\varepsilon < 1/\bar{b}$. Inspired by this discussion, fix $c_0 \in (0, 1)$ and define, for a given admissible f,

$$\lambda_1 \;=\; max\left\{\, |(a + bf(a,b))^2 - b^2\varepsilon^2| : (a,b) \in \mathcal{A} \times \mathcal{B}, \varepsilon \in [0, c_0/\bar{b}] \right\}, \qquad (3.11)$$

and subsequently define

$$\lambda_2 := \lambda_1^{1/2}; \qquad (3.12)$$

both λ_1 and λ_2 lie in $[0, 1)$. Note that in the deadbeat case we have

$$\lambda_1 = c_0^2 \text{ and } \lambda_2 = c_0.$$

In the DLQR case there is no closed-form description for λ_1 and λ_2, although we do know that they lie in $[0, 1)$.

In Chapter 4 we will analyze the proposed controller (3.6)-(3.9) in the time-invariant case, and in Chapter 5 we analyze it in time-varying case. Before moving to Chapter 4 we present a technical result which will prove useful in both Chapter 4 and Chapter 5.

Lemma 1*:* There exist a constant $c_1 > 0$ so that, for every $a \in s(\mathcal{A})$, $b \in s(\mathcal{B})$, $x(0) \in \mathcal{R}$, $\hat{a}[0] \in \mathcal{A}$, $\hat{b}[0] \in \mathcal{B}$ (yielding $\hat{f}[0] = \hat{f}[1] \in \mathcal{F}$), $w \in l_{\infty}$,, $\varepsilon \in (0, c_0/\bar{b}]$ and $j \in \mathcal{Z}^+$ when the control law (3.6) $-$ (3.9) is applied to the time-varying plant (3.1)-(3.2), if

$$|x[j+1]| > \|w\|_{\infty} \tag{3.13}$$

then

$$\frac{1}{|x[j]|} \leq \frac{c_1}{|x[j+1]| - \|w\|_{\infty}}.$$

Proof: Let $a \in s(\mathcal{A})$, $b \in s(\mathcal{B})$, $x(0) \in \mathcal{R}$, $\hat{a}[0] \in \mathcal{A}$, $\hat{b}[0] \in \mathcal{B}$ (yielding $\hat{f}[0] = \hat{f}[1] \in \mathcal{F}$), $w \in l_{\infty}$ and $\varepsilon \in (0, c_0/\bar{b}]$ applying the control law to plant (3.1)-(3.2) we have:

$$\begin{aligned} x[j+1] &= a[j]x[j] + b[j]u[j] + w[j] \\ &= a[j]x[j] + b[j]x[j](\hat{f}[j] + (-1)^j\varepsilon) + w[j] \end{aligned}$$

From our definitions, since we have bounds on a, b, ε and f in both the time-varying and invariant cases, there exists a constant γ_1 such that

$$|a[j] + b[j](\hat{f}[j] + \varepsilon)| \leq \bar{a} + \bar{b}(\bar{f} + c_0/\bar{b}) =: \gamma_1 \tag{3.14}$$

In the time-invariant case, since a and b are constant, $\bar{a} = a$ and $\bar{b} = b$. Assuming that (3.13) holds we have

$$\begin{aligned} |x[j+1]| &\leq \gamma_1 |x[j]| + \|w\|_{\infty} \\ \Rightarrow |x[j]| &\geq \frac{1}{\gamma_1}(|x[j+1]| - \|w\|_{\infty}), \end{aligned}$$

which means

$$\frac{1}{|x[j]|} \leq \frac{\gamma_1}{(|x[j+1]| - \|w\|_{\infty})}. \tag{3.15}$$

This yields us the desired result.

■

Chapter 4

Linear Time-Invariant System

4.1 The Setup

In this chapter we consider the case of an LTI plant given by

$$\left.\begin{array}{rcl} x[k+1] &=& ax[k]+bu[k]+w[k], \\ y[k] &=& x[k]. \end{array}\right\} \tag{4.1}$$

The controller that we proposed in Chapter 3 is rewritten here for convenience, and with the notation slightly modified. With the original conditions of $\hat{a}[0] \in \mathcal{A}$, $\hat{b}[0] \in \mathcal{B}$, $\hat{f}[0] = \hat{f}[1] = f(\hat{a}[0], \hat{b}[0])$ and $\varepsilon \in \big(0, c_0/\bar{b}\big]$, the proposed adaptive control law, for $k \in \mathcal{Z}^+$ even, is given by:

$$\left.\begin{array}{rcl} u[k] &=& (\hat{f}[k]+\varepsilon)x[k], \\ u[k+1] &=& (\hat{f}[k+1]-\varepsilon)x[k+1], \end{array}\right\} \tag{4.2}$$

$$\left[\begin{array}{c} \check{a}[k+2] \\ \check{b}[k+2] \end{array}\right] = \left\{\begin{array}{l} \left[\begin{array}{cc} x[k] & (\hat{f}[k]+\varepsilon)x[k] \\ x[k+1] & (\hat{f}[k+1]-\varepsilon)x[k+1] \end{array}\right]^{-1} \left[\begin{array}{c} x[k+1] \\ x[k+2] \end{array}\right], \text{if } x[k]x[k+1] \neq 0 \\ \left[\begin{array}{c} \hat{a}[k] \\ \hat{b}[k] \end{array}\right], \text{if } x[k]x[k+1] = 0, \end{array}\right. \tag{4.3}$$

$$\left.\begin{array}{rcl} \hat{a}[k+2] &=& \Pi_{\mathcal{A}}(\check{a}[k+2]), \\ \hat{b}[k+2] &=& \Pi_{\mathcal{B}}(\check{b}[k+2]), \end{array}\right\} \tag{4.4}$$

$$\left.\begin{array}{rcl} \hat{f}[k+2] &=& f(\hat{a}[k], \hat{b}[k]), \\ \hat{f}[k+3] &=& f(\hat{a}[k], \hat{b}[k]). \end{array}\right\} \tag{4.5}$$

We will first prove that the proposed control law is exponentially stabilizing and that the map from the noise to x has a bounded gain; in fact, we also provide a quantitative analysis of the effect of ε on the behavior. Secondly, we analyze how close the behavior of the proposed control law is to the original control law.

The stability of the controller depends on the accuracy of the estimated plant parameters. For this purpose, we will look at the estimation process in detail. With the initial conditions $\hat{f}[0] = \hat{f}[1] \in \mathcal{F}$, let the control law (4.2)-(4.5) be applied to plant (4.1) for even k, yielding the following response:

$$\begin{aligned}
x[k+1] &= ax[k] \ + \ b(\hat{f}[k]+\varepsilon)x[k] \ + w[k], \\
x[k+2] &= ax[k+1] \ + \ b(\hat{f}[k]-\varepsilon)x[k+1] \ + w[k+1], \\
\Rightarrow \underbrace{\begin{bmatrix} x[k+1] \\ x[k+2] \end{bmatrix}}_{(known)} &= \underbrace{\begin{bmatrix} x[k] & (\hat{f}[k]+\varepsilon)x[k] \\ x[k+1] & (\hat{f}[k]-\varepsilon)x[k+1] \end{bmatrix}}_{=:\phi[k+2](known)} \times \begin{bmatrix} a \\ b \end{bmatrix} + \underbrace{\begin{bmatrix} w[k] \\ w[k+1] \end{bmatrix}}_{(unknown)}. \qquad (4.6)
\end{aligned}$$

At this point we would like to get a bound on

$$\Delta[k+2] := \begin{bmatrix} \check{a}[k+2] \\ \check{b}[k+2] \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix};$$

if $x[k]x[k+1] = 0$, then $\phi[k+2]$ is not invertible then we define

$$\Delta[k+2] := \Delta[k].$$

Hence in the following analysis we assume that $x[k]x[k+1] \neq 0$. From (4.3) we have

$$\begin{bmatrix} \check{a}[k+2] \\ \check{b}[k+2] \end{bmatrix} = \phi[k+2]^{-1} \begin{bmatrix} x[k+1] \\ x[k+2] \end{bmatrix}$$

and from (4.6) we have

$$\begin{bmatrix} a \\ b \end{bmatrix} = \phi[k+2]^{-1} \begin{bmatrix} x[k+1] \\ x[k+2] \end{bmatrix} - \phi[k+2]^{-1} \begin{bmatrix} w[k] \\ w[k+1] \end{bmatrix}.$$

So we conclude that

$$\Delta[k+2] = \phi[k+2]^{-1} \begin{bmatrix} w[k] \\ w[k+1] \end{bmatrix}. \qquad (4.7)$$

$$\Rightarrow \|\Delta[k+2]\| \ \leq \ \left\| \phi[k+2]^{-1} \times \begin{bmatrix} w[k] \\ w[k+1] \end{bmatrix} \right\|. \qquad (4.8)$$

This quantity $\|\Delta[k+2]\|$ is of great importance as it reveals the accuracy of the parameter estimation. Analysing (4.7) yields

$$\begin{aligned}\Delta[k+2] &= \phi[k+2]^{-1} \times \begin{bmatrix} w[k] \\ w[k+1] \end{bmatrix} \\ &= \frac{1}{-2\varepsilon x[k]x[k+1]} \times \begin{bmatrix} (\hat{f}[k]-\varepsilon)x[k+1] & -(\hat{f}[k]+\varepsilon)x[k] \\ -x[k+1] & x[k] \end{bmatrix} \begin{bmatrix} w[k] \\ w[k+1] \end{bmatrix}.\end{aligned}$$

So

$$\begin{aligned}\|\Delta[k+2]\| &\leq \frac{1}{2\varepsilon|x[k]||x[k+1]|} \times (1+|\hat{f}[k]|+\varepsilon) \times (|x[k]|+|x[k+1]|) \times \|w\|_\infty \\ &= \frac{1+|\hat{f}[k]|+\varepsilon}{2\varepsilon} \times \left[\frac{|x[k]|+|x[k+1]|}{|x[k]|\;|x[k+1]|}\right] \times \|w\|_\infty. \end{aligned} \tag{4.9}$$

An upper bound on $\hat{f}[k]$ is $\bar{f}$ and an upper bound on ε is $c_0/\bar{b}$, so

$$\begin{aligned}\|\Delta[k+2]\| &\leq \frac{1+\bar{f}+c_0/\bar{b}}{2\varepsilon}\left(\frac{|x[k]|+|x[k+1]|}{|x[k]|\;|x[k+1]|}\right) \times \|w\|_\infty \\ &\leq \frac{1+\bar{f}+c_0/\bar{b}}{2\varepsilon}\left(\frac{1}{|x[k]|}+\frac{1}{|x[k+1]|}\right) \times \|w\|_\infty \end{aligned} \tag{4.10}$$

for all $k \in \mathcal{Z}^+$ even and satisfying

$$x[k]x[k+1] \neq 0.$$

To make $\|\Delta[k+2]\|$ small, we have to make sure that the states are much larger than the noise $\|w\|_\infty$. This motivates the definition of the scaled state variable

$$\bar{x}[k] := \frac{x[k]}{\|w\|_\infty}. \tag{4.11}$$

Lemma 2: There exist strictly positive constants c_2, c_3 and c_4 so that, for every $a \in \mathcal{A}$, $b \in \mathcal{B}$, $x(0) \in \mathcal{R}$, $\hat{a}[0] \in \mathcal{A}$, $\hat{b}[0] \in \mathcal{B}$ (yielding $\hat{f}[0] = \hat{f}[1] \in \mathcal{F}$), $w \in l_\infty$, $\varepsilon \in (0, c_0/\bar{b}]$ and $k \in \mathcal{Z}^+$ even, when the control law (4.2)-(4.5) is applied to the linear time-invariant plant (4.1), if

$$\varepsilon|\bar{x}[k+1]| > c_2 \tag{4.12}$$

then

$$\|\Delta[k+2]\| \leq \frac{1}{\varepsilon}\frac{c_3}{|\bar{x}[k+1]|}$$

and

$$|\tilde{f}[k+2]| := |f(a,b) - f(\hat{a}[k+2], \hat{b}[k+2])| \leq \frac{1}{\varepsilon}\frac{c_4}{|\bar{x}[k+1]|}.$$

Proof: Let $a \in \mathcal{A}$, $b \in \mathcal{B}$, $x(0) \in \mathcal{R}$, $\hat{a}[0] \in \mathcal{A}$, $\hat{b}[0] \in \mathcal{B}$ (yielding $\hat{f}[0] = \hat{f}[1] \in \mathcal{F}$), $w \in l_\infty$, $\varepsilon \in (0, c_0/\bar{b}]$ and $k \in \mathcal{Z}^+$ even with the control law (4.2)-(4.5) is applied to plant (4.1).

First suppose that $x[k]x[k+1] = 0$. If

$$x[k+1] = 0$$

then (4.12) never holds. If

$$x[k] = 0$$

then

$$\begin{aligned} x[k+1] &= ax[k] + bu[k] + w[k] \\ &= w[k], \end{aligned}$$

so

$$|\bar{x}[k+1]| \leq 1.$$

This means that (4.12) will now hold as long as

$$\frac{c_2}{\varepsilon} > 1 \Leftrightarrow c_2 > \varepsilon \Leftrightarrow c_2 > \frac{c_0}{\bar{b}}.$$

Hence, (4.12) holds for the case as long as c_2 is large enough, that is $c_2 > \frac{c_0}{\bar{b}}$.

Now assume that $x[k]x[k+1] \neq 0$, which means that equation (4.10) provides a bound on $\|\Delta[k+2]\|$. With c_1 given by Lemma 1 and

$$\gamma_1 := \frac{1 + \bar{f} + c_0/\bar{b}}{2},$$

as long as $|\bar{x}[k+1]| > 1$, we have

$$\begin{aligned}
\|\Delta[k+2]\| &\leq \frac{\gamma_1}{\varepsilon}\left[\frac{c_1}{(|x[k+1]| - \|w\|_\infty)} + \frac{1}{|x[k+1]|}\right] \times \|w\|_\infty \\
&= \frac{\gamma_1}{\varepsilon}\left[\frac{(c_1+1) - \frac{\|w\|_\infty}{|x[k+1]|}}{|x[k+1]| - \|w\|_\infty}\right] \times \|w\|_\infty \\
&= \frac{\gamma_1}{\varepsilon}\frac{(c_1+1) - \frac{\|w\|_\infty}{|x[k+1]|}}{\frac{|x[k+1]|}{\|w\|_\infty}} \\
&= \frac{\gamma_1}{\varepsilon} \times \frac{c_1 + 1 - \frac{1}{|\bar{x}[k+1]|}}{|\bar{x}[k+1]| - 1} \\
&= \frac{\gamma_1}{\varepsilon} \times \frac{c_1 + 1 - \frac{1}{|\bar{x}[k+1]|}}{1 - \frac{1}{|\bar{x}[k+1]|}} \times \frac{1}{|\bar{x}[k+1]|}. \qquad (4.13)
\end{aligned}$$

The middle term on the R.H.S of (4.13) converges to c_1+1 as $|\bar{x}[k+1]| \to \infty$; hence, there exists a $\gamma_2 > 1$ so that if $|\bar{x}[k+1]| > \gamma_2$, then

$$\|\Delta[k+2]\| \leq \frac{2\gamma_1(c_1+1)}{\varepsilon|\bar{x}[k+1]|}. \qquad (4.14)$$

In order to guarantee that $\begin{bmatrix} \hat{a}[k+2] \\ \hat{b}[k+2] \end{bmatrix}$ is a better estimate of $\begin{bmatrix} a \\ b \end{bmatrix}$ than $\begin{bmatrix} \check{a}[k+2] \\ \check{b}[k+2] \end{bmatrix}$, it is enough that

$$\Delta[k+2] < \delta; \qquad (4.15)$$

this will be the case if

$$\begin{aligned}
\frac{2\gamma_1(c_1+1)}{\varepsilon|\bar{x}[k+1]|} &< \delta. \\
\Leftrightarrow \varepsilon|\bar{x}[k+1]| &> \frac{2\gamma_1(c_1+1)}{\delta}. \qquad (4.16)
\end{aligned}$$

At this point we have to be careful to ensure that $|\bar{x}[k+1]| > \gamma_2$ as well as (4.16) holds. To this end, if we set

$$c_2 := max\left\{\gamma_2\frac{c_0}{\bar{b}}, \frac{2\gamma_1(c_1+1)}{\delta}\right\}, \qquad (4.17)$$

then

$$\varepsilon|\bar{x}[k+1]| > c_2$$

clearly implies that

$$|\bar{x}[k+1]| > \frac{c_2}{\varepsilon} \geq \frac{c_2}{c_0/\bar{b}} \geq \frac{\gamma_2 \cancel{\frac{c_0}{\bar{b}}}}{\cancel{\frac{c_0}{\bar{b}}}} = \gamma_2,$$

which in turn implies that (4.14),(4.15) and (4.16) holds. For such a choice of c_2, if (4.12) holds then

$$\|\Delta[k+2]\| \leq \frac{2\gamma_1\,(c_1+1)}{\varepsilon|\bar{x}[k+1]|} \leq \delta,$$

so we will define

$$c_3 := 2\gamma_1(c_1+1),$$

ensuring that

$$\left\| \begin{bmatrix} \hat{a}[k+2]-a \\ \hat{b}[k+2]-b \end{bmatrix} \right\| \leq \|\Delta[k+2]\|$$

as well. With γ_3 the Lipschitz Constant of f, we can also conclude that

$$\begin{aligned} \underbrace{|f(a,b)-f(\hat{a}[k+2],\hat{b}[k+2])|}_{=:\tilde{f}[k+2]} \leq \gamma_3 \left\| \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} \hat{a}[k+2] \\ \hat{b}[k+2] \end{bmatrix} \right\| &\leq \gamma_3\|\Delta[k+2]\| \\ &\leq \frac{c_3\gamma_3}{\varepsilon|\bar{x}[k+1]|}, \quad (4.18) \end{aligned}$$

so we will define $c_4 = c_3\gamma_3$. ■

We now introduce the theorem that proves that the closed-loop system is stable.

4.2 The Main Result

Theorem 1*:* There exists a constant $c_5 > 0$ such that for every $a \in \mathcal{A}$, $b \in \mathcal{B}$, $x(0) \in \mathcal{R}$, $\hat{a}[0] \in \mathcal{A}$, $\hat{b}[0] \in \mathcal{B}$ (yielding $\hat{f}[0] = \hat{f}[1] \in \mathcal{F}$), $w \in l_\infty$ and $\varepsilon \in (0, c_0/\bar{b}]$, when the control law (4.2)-(4.5) is applied to the linear time-invariant plant (4.1), we have

$$|x[k]| \leq c_5 \lambda_2^k |x[0]| + c_5 \left(1 + \frac{1}{\varepsilon}\right) \|w\|_\infty,\ k \geq 0.$$

Proof: Let $a \in \mathcal{A}$, $b \in \mathcal{B}$, $x(0) \in \mathcal{R}$, $\hat{a}[0] \in \mathcal{A}$, $\hat{b}[0] \in \mathcal{B}$ (yielding $\hat{f}[0] = \hat{f}[1] \in \mathcal{F}$), $w \in l_\infty$, $\varepsilon \in (0, c_0/\bar{b}]$ and $k \in \mathcal{Z}^+$ be even. Since the initial condition $\hat{f}[0]$ will typically be inaccurate, the behaviour over the first two periods will typically differ from the ideal behavior. Hence, we focus much of our analysis on what happens after this. To this end, when the control law (4.2)-(4.5) is applied to the plant (4.1), we have

$$\begin{aligned} x[k+3] &= \Big(a + b(\hat{f}[k+2] + \varepsilon)\Big)\, x[k+2] + w[k+2], && (4.19) \\ x[k+4] &= \Big(a + b(\hat{f}[k+2] - \varepsilon)\Big)\, x[k+3] + w[k+3]. && (4.20) \end{aligned}$$

From this point, we have divided the proof into simple steps which makes it easier to understand.

Step 1: We will first obtain a crude bound showing how fast x increases. We first define γ_1 as defined in (3.14):

$$|a + b(\hat{f}[l] + \varepsilon)| \leq \bar{a} + \bar{b}(\bar{f} + c_0/\bar{b}) =: \gamma_1,\ l \in \mathcal{Z}^+. \qquad (4.21)$$

This can be used to obtain a crude bound on the increase of x over a step:

$$|x[k+l+1]| \leq \gamma_1 |x[k+l]| + \|w\|_\infty,\ l \in \mathcal{Z}^+. \qquad (4.22)$$

Step 2: The next goal is to obtain a bound on $|x[k+4]|$, for which we will use the crude bound derived in step 1. To achieve this we will analyze $x[k+4]$ for two cases, when $\varepsilon|\bar{x}[k+1]|$ is small and when it is large. Using the choice of c_2 asserted to exist by (4.17) in Lemma 2, we consider:

Case 1: $\varepsilon|\bar{x}[k+1]| \leq c_2$. Since $\varepsilon|\bar{x}[k+1]|$ is not large, we can use the crude bound given by (4.22) to get a bound on $|x[k+4]|$. Using (4.22) three times in succession yields

$$\begin{aligned} |x[k+4]| &\leq \gamma_1 |x[k+3]| + \|w\|_\infty \\ &\leq \gamma_1^2 |x[k+2]| + \gamma_1 \|w\|_\infty + \|w\|_\infty \\ &\leq \gamma_1^3 |x[k+1]| + \gamma_1^2 \|w\|_\infty + \gamma_1 \|w\|_\infty + \|w\|_\infty, \end{aligned}$$

which implies that

$$\begin{aligned} |\bar{x}[k+4]| &\leq \gamma_1^3 |\bar{x}[k+1]| + \gamma_1^2 + \gamma_1 + 1 \\ &\leq \frac{\gamma_1^3 c_2}{\varepsilon} + \gamma_1^2 + \gamma_1 + 1. \end{aligned} \tag{4.23}$$

Case 2: $\varepsilon |\bar{x}[k+1]| > c_2$. Combining (4.19) and (4.20) yields

$$\begin{aligned} x[k+4] &= \left[a + b(\hat{f}[k+2] - \varepsilon)\right] \left[(a + b(\hat{f}[k+2] + \varepsilon))x[k+2] + w[k+2]\right] \\ &\quad + w[k+3] \\ &= \left[\left(a + b\hat{f}[k+2]\right)^2 - b^2\varepsilon^2\right] x[k+2] + \left[a + b(\hat{f}[k+2] - \varepsilon)\right] w[k+2] \\ &\quad + w[k+3] \\ &= \underbrace{\left[(a + bf(a,b))^2 - b^2\varepsilon^2\right] x[k+2]}_{\psi_1[k]} \\ &\quad + \underbrace{\left(b^2 \tilde{f}[k+2]^2 - 2\left[a + bf(a,b)\right] b\tilde{f}[k+2]\right) x[k+2]}_{\psi_2[k]} \\ &\quad + \underbrace{\left[a + b(\hat{f}[k+2] - \varepsilon)\right] w[k+2] + w[k+3]}_{\psi_3[k]}. \end{aligned} \tag{4.24}$$

Using the definition of λ_1 given in (3.11) we see that

$$|\psi_1[k]| \leq \lambda_1 |x[k+2]|. \tag{4.25}$$

Using bounds on $a, b, \hat{f}[k+2]$ and ε yields

$$|\psi_3[k]| \leq \left(\bar{a} + \bar{b}\bar{f} + 2\right) \|w\|_\infty. \tag{4.26}$$

Now we turn to $\psi_2[k]$, which is more complicated to analyze. Using (4.22) we convert a bound on $|x[k+2]|$ to one on $|x[k+1]|$ and Lemma 2 to provide a bound on $|\tilde{f}[k+2]|$, we obtain

$$
\begin{aligned}
|\psi_2[k]| &\leq (\bar{b}^2|\tilde{f}[k+2]|^2 + 2\lambda_1\bar{b}|\tilde{f}[k+2]|)\,(\gamma_1|x[k+1]| + \|w\|_\infty) \\
&= |\tilde{f}[k+2]|(\bar{b}^2 \underbrace{|\tilde{f}[k+2]|}_{\leq \frac{c_4}{\varepsilon|\bar{x}[k+1]|} \leq \frac{c_4}{c_2}} + 2\lambda_1\bar{b})\,(\gamma_1|x[k+1]| + \|w\|_\infty) \\
&\leq \frac{c_4}{\varepsilon|\bar{x}[k+1]|}\left(\bar{b}^2\frac{c_4}{c_2} + 2\lambda_1\bar{b}\right)(\gamma_1|x[k+1]| + \|w\|_\infty) \\
&= \bar{b}\left(\frac{\bar{b}c_4}{c_2} + 2\lambda_1\right)\left(c_4\gamma_1\underbrace{\frac{|x[k+1]|}{\varepsilon|\bar{x}[k+1]|}}_{=\frac{\|w\|_\infty}{\varepsilon}} + \underbrace{\frac{c_4}{\varepsilon|\bar{x}[k+1]|}}_{<\frac{c_4}{c_2}}\|w\|_\infty\right) \\
&\leq \bar{b}\left(\frac{\bar{b}c_4}{c_2} + 2\lambda_1\right)\left(\frac{c_4}{c_2} + \frac{c_4\gamma_1}{\varepsilon}\right)\|w\|_\infty. \qquad (4.27)
\end{aligned}
$$

Substituting the bound on $|\psi_1[k]|, |\psi_2[k]|$ and $|\psi_3[k]|$ into (4.24), it follows that there exists a constant γ_2 so that

$$
|x[k+4]| \leq \lambda_1|x[k+2]| + \gamma_2\left(1 + \frac{1}{\varepsilon}\right)\|w\|_\infty.
$$

Equivalently

$$
|\bar{x}[k+4]| \quad \leq \quad \lambda_1|\bar{x}[k+2]| + \gamma_2\left(1 + \frac{1}{\varepsilon}\right). \qquad (4.28)
$$

Step 3: We will now combine the two cases of step 2 to get a bound on $|x[k+4]|$. More specifically, the bounds provided by combining cases 1 and 2 given by (4.23) and (4.28), we conclude that regardless of the value of $|\bar{x}[k+1]|$, we have

$$
|\bar{x}[k+4]| \quad \leq \quad \lambda_1|\bar{x}[k+2]| + (1 + \gamma_1 + \gamma_1^2 + \gamma_2) + \frac{\gamma_1^3 c_2 + \gamma_2}{\varepsilon}.
$$

If we define

$$
\gamma_3 =: max\left\{1 + \gamma_1 + \gamma_1^2 + \gamma_2, \gamma_1^3 c_2 + \gamma_2\right\}
$$

we have

$$
|x[k+4]| \quad \leq \quad \lambda_1|x[k+2]| + \gamma_3(1 + \frac{1}{\varepsilon})\|w\|_\infty, \; k \in \mathcal{Z}^+ \text{ even}. \qquad (4.29)
$$

Step 4: We will now obtain a bound on $|x[l]|$ for $l \in \mathcal{Z}^+$. Observe that (4.29) holds for even $k \in \mathcal{Z}^+$; using the variable substitution with $k = 2(j-1)$ with $j \in \mathbb{N}$, this becomes

$$|x[2j+2]| \quad \leq \lambda_1 |x[2j]| + \gamma_3 (1 + \tfrac{1}{\varepsilon}) \|w\|_\infty, \; j \in \mathbb{N}.$$

Solving iteratively yields

$$|x[2j+2]| \quad \leq \quad \lambda_1^j |x[2]| + \frac{\gamma_3}{1-\lambda_1} \left(1 + \frac{1}{\varepsilon}\right) \|w\|_\infty, \; j \in \mathbb{N}. \tag{4.30}$$

For the odd steps we use (4.21):

$$|x[2j+3]| \quad \leq \quad \gamma_1 |x[2+2j]| + \|w\|_\infty, \; j \in \mathbb{N}.$$

Combining this with (4.30) yields:

$$|x[2j+3]| \quad \leq \quad \gamma_1 \lambda_1^j |x[2]| + \left[1 + \frac{\gamma_1 \gamma_3}{1-\lambda_1} \left(1 + \frac{1}{\varepsilon}\right)\right] \|w\|_\infty, \; j \in \mathbb{N}. \tag{4.31}$$

So for $l \geq 4$ even, (4.30) says that

$$|x[l]| \leq (\lambda_1^{1/2})^{l-2} |x[2]| + \frac{\gamma_3}{1-\lambda_1} \left(1 + \frac{1}{\varepsilon}\right) \|w\|_\infty,$$

and for $l \geq 5$ odd, (4.31) says that

$$|x[l]| \leq \gamma_1 (\lambda_1^{1/2})^{l-3} |x[2]| + \left[1 + \frac{\gamma_1 \gamma_3}{1-\lambda_1} \left(1 + \frac{1}{\varepsilon}\right)\right] \|w\|_\infty.$$

If we define

$$\gamma_4 := max \left\{ \frac{\gamma_3}{1-\lambda_1}, 1 + \frac{\gamma_1 \gamma_3}{1-\lambda_1} \right\},$$

when combined these yields

$$|x[l]| \leq \gamma_1 \lambda_1^{-3/2} \lambda_1^{l/2} |x[2]| + \gamma_4 \left(1 + \frac{1}{\varepsilon}\right) \|w\|_\infty, \; l \geq 4. \tag{4.32}$$

The last step is to analyse the first three steps. With γ_1 defined by (4.21), it is easy to see that

$$|x[1]| \leq \gamma_1 |x[0]| + \|w\|_\infty, \tag{4.33}$$

$$|x[2]| \leq \gamma_1^2 |x[0]| + (\gamma_1 + 1) \|w\|_\infty, \tag{4.34}$$

and

$$|x[3]| \leq \gamma_1^3|x[0]| + (\gamma_1^2 + \gamma_1 + 1)\|w\|_\infty. \tag{4.35}$$

Combining (4.32) - (4.34) yields

$$|x[l]| \leq (\gamma_1^3\lambda_1^{-3/2})(\lambda_1^{1/2})^l|x[0]| + \left(\gamma_1\lambda_1^{-3/2}(\gamma_1 + 1) + \gamma_4 + \frac{\gamma_4}{\varepsilon}\right)\|w\|_\infty,\ l \leq 4. \tag{4.36}$$

From (4.33), (4.34) and (4.35), we also see that

$$|x[l]| \leq max\left\{1, \gamma_1\lambda_1^{-1}, \gamma_1^2\lambda_1^{-2}, \gamma_1^3\lambda_1^{-3}\right\}\lambda_1^l|x[0]| + \left(\gamma_1^2 + \gamma_1 + 1\right)\|w\|_\infty,\ l = 0, 1, 2, 3. \tag{4.37}$$

Using the definition of $\lambda_2 = \lambda_1^{1/2}$, if follows from (5.40) and (5.41) that there exists a constant c_5 so that

$$|x[l]| \leq c_5(\lambda_2)^l|x[0]| + c_5\left(1 + \frac{1}{\varepsilon}\right)\|w\|_\infty, l \in \mathcal{Z}^+.$$

■

4.3 Optimality

The proposed control action yields the closed-loop equation

$$x[k+1] = ax[k] \ + \ b(\hat{f}[k] + (-1)^k \varepsilon)x[k] \ + w[k]. \tag{4.38}$$

On the first two steps, we typically have a poor estimate of f so the closed-loop behavior may be poor. As a consequence, starting from step $k = 2$ we will obtain a bound on the difference between the proposed closed-loop system behavior and the "optimal behavior". The optimal state-feedback control action is defined by

$$\begin{aligned} u^o[k] &= f^o[k]x^o[k], \\ f^o[k] &= f(a,b); \end{aligned}$$

with this control action the behavior of the closed-loop system with optimal controller will be

$$\begin{aligned} x^o[k+1] &= ax^o[k] + bu^o[k] + w[k] \\ &= ax^o[k] + bf^o[k]x^o[k] + w[k], \ k \geq 2, \ x^o[2] = x[2]; \end{aligned}$$

here $x^o[k]$ is the optimal state variable. We also define the difference between $x[k]$ and $x^o[k]$ by:

$$\tilde{x}[k] := x[k] - x^o[k].$$

Since we are comparing the two behaviors starting from step $k = 2$, we have defined the initial condition to be $x^0[2] = x[2]$[1]; since $f^o[k] = f(a,b)$, it means that $a + bf^o[k] = a_{cl}$. This yields

$$x^o[k+1] \ = \ a_{cl}x^o[k] + w[k], \ k \geq 2. \tag{4.39}$$

So for every $\varepsilon \in (0, c_0/\bar{b}]$, from (3.11) and (3.12) we have:

$$\begin{aligned} |a_{cl}^2 - b^2\varepsilon^2| \leq \lambda_1 &= \lambda_2^2 \text{ for all } \varepsilon \in (0, c_0/\bar{b}] \subset [0, 1/\bar{b}] \\ \Rightarrow |a_{cl}^2| &\leq \lambda_2^2 \\ \Rightarrow |a_{cl}| &\leq \lambda_2, \end{aligned} \tag{4.40}$$

and so by (4.40) we obtain a bound on optimal state variable

$$|x^o[k+1]| \leq \lambda_2 |x^o[k]| + \|w\|_\infty. \tag{4.41}$$

[1]Note that $\tilde{x}[2] = 0$.

Theorem 2: There exists constants $c_6 > 0$ and $c_7 > 0$ such that for every $a \in \mathcal{A}$, $b \in \mathcal{B}$, $x(0) \in \mathcal{R}$, $\hat{a}[0] \in \mathcal{A}$, $\hat{b}[0] \in \mathcal{B}$ (yielding $\hat{f}[0] = \hat{f}[1] \in \mathcal{F}$), $w \in l_\infty$ and $\varepsilon \in (0, c_0/\bar{b}]$, when the control law (4.2)-(4.5) is applied to the linear time-invariant plant (4.1), we have

$$|x[k] - x^o[k]| \leq \varepsilon c_6 \left(\frac{\lambda_2 + 1}{2}\right)^k |x[0]| + c_6 \left(1 + \frac{1}{\varepsilon}\right) \|w\|_\infty, \;\; k \geq 2,$$

and

$$|u[k] - u^o[k]| \leq \varepsilon c_7 \left(\frac{\lambda_2 + 1}{2}\right)^k |x[0]| + c_7 \left(1 + \frac{1}{\varepsilon}\right) \|w\|_\infty, \;\; k \geq 2.$$

Proof: let $a \in \mathcal{A}$, $b \in \mathcal{B}$, $x(0) \in \mathcal{R}$, $\hat{a}[0] \in \mathcal{A}$, $\hat{b}[0] \in \mathcal{B}$ (yielding $\hat{f}[0] = \hat{f}[1] \in \mathcal{F}$), $w \in l_\infty$, $\varepsilon \in (0, c_0/\bar{b}]$ and $k \in \mathcal{Z}^+$. Let the control law (4.2)-(4.5) be applied to plant (4.1). We obtain a difference equation for $\tilde{x}[k]$ for $k \geq 2$:

$$\begin{aligned}
\tilde{x}[k+1] &= x[k+1] - x^o[k+1] \\
&= ax[k] \; + \; b[\hat{f}[k] + (-1)^k \varepsilon]x[k] \; + w[k] - (ax^o[k] + bf^o[k]x^o[k] + w[k]) \\
&= ax[k] - ax^o[k] + b\hat{f}[k]x[k] + (-1)^k \varepsilon bx[k] - bf^o[k]x^o[k] \\
&= a\tilde{x}[k] + bf^o[k]x[k] + \left(\hat{f}[k] - f^o[k]\right) x[k] + (-1)^k \varepsilon bx[k] - bf^o[k]x^o[k] \\
&= (a + bf^o[k])\tilde{x}[k] - b(\hat{f}[k] - f^o[k])x[k] - (-1)^k \varepsilon bx[k] \\
|\tilde{x}[k+1]| &\leq \underbrace{|a + bf^o[k]| \; |\tilde{x}[k]|}_{=:g_1[k]} + \underbrace{|b(\hat{f}[k] - f^o[k])x[k]|}_{=:g_2[k]} + \underbrace{|(-1)^k \varepsilon b| \; |x[k]|}_{=:g_3[k]}, \;\; k \geq 2.
\end{aligned} \tag{4.42}$$

We now obtain a bound on each of these three terms.
A bound on $g_1[k]$: We know that

$$a + bf^o[k] = a_{cl},$$

so from (4.40) we have

$$g_1[k] \;\; \leq \;\; \lambda_2 |\tilde{x}[k]|, \;\; k \geq 2. \tag{4.43}$$

A bound on $g_3[k]$: From Theorem 1 we already have an upper bound on $|x[k]|$ for every

$k \geq 0$, a bound on $g_3[k]$ is given by [2]

$$\begin{aligned} g_3[k] &\leq \varepsilon\bar{b}\left[c_5\left(\frac{\lambda_2+1}{2}\right)^k |x[0]| + c_5\left(1+\frac{1}{\varepsilon}\right)\|w\|_\infty\right] \\ \Rightarrow g_3[k] &\leq \varepsilon\bar{b}c_5\left(\frac{\lambda_2+1}{2}\right)^k |x[0]| + c_0c_5\left(1+\frac{1}{\varepsilon}\right)\|w\|_\infty,\ k \geq 0. \end{aligned} \tag{4.44}$$

<u>**A bound on** $g_2[k]$**:**</u> We first obtain a crude bound on the increase of x at every step by defining

$$|a + b(\hat{f}[l] + \varepsilon)| \leq \bar{a} + \bar{b}(\bar{f} + c_0/\bar{b}) =: \gamma_1,\ l \in \mathcal{Z}^+,$$

and then observe that

$$|x[k+1]| \leq \gamma_1|x[k]| + \|w\|_\infty, k \geq 0. \tag{4.45}$$

As discussed earlier, since the estimate of f may be poor for the first two steps, we look at the case of $g_2[k]$ for $k \geq 2$. Equivalently, we obtain a bound on $g_2[k+2]$ for $k \geq 0$:

$$g_2[k+2] \leq \bar{b}|\tilde{f}[k+2]||x[k+2]|,\ k \geq 0. \tag{4.46}$$

We now use Lemma 2 to obtain a bound on $g_2[k+2]$ for $k \in \mathcal{Z}^+$ even and then extend the results to obtain the case when $k \in \mathcal{Z}^+$ odd. Motivated by Lemma 2 and with $j \in \mathcal{Z}^+$, we will analyse $g[2j+2]$ (this is the even case) for the case when $\varepsilon|x[2j+1]| \leq c_2$ and for the case when $\varepsilon|x[2j+1]| > c_2$.
<u>Case 1:</u> $\varepsilon|\bar{x}[2j+1]| \leq c_2$ (equivalently, $|x[2j+1]| \leq \frac{1}{\varepsilon}c_2\|w\|_\infty$). In this case, from (4.45)

$$|x[2j+2]| \leq \gamma_1|x[2j+1]| + \|w\|_\infty.$$

So from (4.42)

$$\begin{aligned} g_2[2j+2] &\leq \bar{b}|\tilde{f}[2j+2]||x[2j+2]| \\ &\leq 2\bar{b}\bar{f}(\gamma_1|x[2j+1]| + \|w\|_\infty). \end{aligned} \tag{4.47}$$

Substituting the upper bound on $|x[2j+1]|$ yields

$$\begin{aligned} g_2[2j+2] &\leq 2\bar{b}\bar{f}(\gamma_1\frac{c_2}{\varepsilon}\|w\|_\infty + \|w\|_\infty) \\ &\leq 2\bar{b}\bar{f}(\gamma_1\frac{c_2}{\varepsilon} + 1)\|w\|_\infty. \end{aligned} \tag{4.48}$$

[2] We use a weaker bound than Theorem 1 provides in order to simplify later calculations.

Case 2: $\varepsilon|\bar{x}[2j+1]| > c_2$ (equivalently, $|x[2j+1]| > \frac{1}{\varepsilon}c_2\|w\|_\infty$). From (4.45) we have

$$|x[2j+2]| \le \gamma_1|x[2j+1]| + \|w\|_\infty.$$

So from (4.42)

$$\begin{aligned} g_2[2j+2] &\le \bar{b}|\tilde{f}[2j+2]||x[2j+2]| \\ &\le \bar{b}|\tilde{f}[2j+2]|(\gamma_1|x[2j+1]| + \|w\|_\infty). \end{aligned} \tag{4.49}$$

From Lemma 2 we see that

$$|\tilde{f}[2j+2]| \le \frac{c_4}{\varepsilon|\bar{x}[2j+1]|}.$$

So

$$\begin{aligned} g_2[2j+2] &\le \bar{b}(\gamma_1|x[2j+1]| + \|w\|_\infty)\frac{c_4}{\varepsilon|\bar{x}[2j+1]|} \\ &\le \bar{b}c_4(\frac{\gamma_1}{\varepsilon} + \frac{1}{\varepsilon|\bar{x}[2j+1]|})\|w\|_\infty \\ &\le \bar{b}c_4(\frac{\gamma_1}{\varepsilon} + \frac{1}{c_2})\|w\|_\infty. \end{aligned} \tag{4.50}$$

If we combine Case 1 and Case 2 and define γ_2 by

$$\gamma_2 = max\left\{\bar{b}c_4\gamma_1, 2\bar{b}\bar{f}, 2\bar{b}\bar{f}c_2\gamma_1, b\frac{\bar{c}_4}{c_2}\right\},$$

we end up with

$$g_2[2j+2] \le \gamma_2(1+\frac{1}{\varepsilon})\|w\|_\infty,\ j \in \mathcal{Z}^+. \tag{4.51}$$

Now we consider the case of $k \in \mathcal{Z}^+$ odd. If we set $k = 2j+1$ in (4.46), then we end up with a bound on the odd terms:

$$g_2[2j+3] \le |b||\tilde{f}[2j+3]||x[2j+3]|,\ j \ge 0.$$

Using (4.45) to obtain bound on $|x[2j+3]|$, we get

$$\begin{aligned} g_2[2j+3] &\le |b||\tilde{f}[2j+2]|\,(\gamma_1|x[2j+2]| + \|w\|_\infty) \\ &\le \gamma_1|b\tilde{f}[2j+2]x[2j+2]| + |b||\tilde{f}[2j+2]|\|w\|_\infty \\ &\le \gamma_1 g_2[2j+2] + 2\bar{b}\bar{f}\|w\|_\infty,\ j \ge 0. \end{aligned}$$

Using the bound on $g_2[2j+2]$ by (4.51), we get

$$g_2[2j+3] \leq \left[\gamma_1\gamma_2\left(1+\frac{1}{\varepsilon}\right)+2\bar{b}\bar{f}\right]\|w\|_\infty\ , j \geq 0. \tag{4.52}$$

If we define

$$\gamma_3 := max\left\{2\bar{b}\bar{f}, \gamma_2, \gamma_1\gamma_2\right\},$$

we can combine (4.52) - (4.51) and obtain

$$g_2[k] \ \ \leq \ \ \gamma_3\left(1+\frac{1}{\varepsilon}\right)\|w\|_\infty. \tag{4.53}$$

Using the bounds (4.43), (4.44) and (4.53) in (4.42) we have

$$\begin{aligned} |\tilde{x}[k+1]| \ \ &\leq \ \ \lambda_2|\tilde{x}[k]| + \varepsilon\bar{b}c_5\left(\frac{\lambda_2+1}{2}\right)^k|x[0]| + c_0c_5\left(1+\frac{1}{\varepsilon}\right)\|w\|_\infty \\ &+\gamma_3\left(1+\frac{1}{\varepsilon}\right)\|w\|_\infty,\ \ k \geq 2. \end{aligned}$$

To simplify the above equation we define

$$\gamma_4 = \gamma_3 + c_0c_5$$

so that

$$|\tilde{x}[k+1]| \ \ \leq \ \ \lambda_2|\tilde{x}[k]| + \varepsilon\bar{b}c_5\left(\frac{\lambda_2+1}{2}\right)^k|x[0]| + \gamma_4\left(1+\frac{1}{\varepsilon}\right)\|w\|_\infty \ k \geq 2. \tag{4.54}$$

The last step is to solve this difference inequality. For that purpose, let us define

$$\theta[k] := |\tilde{x}[k]|$$

and

$$\psi[k] := \varepsilon\bar{b}c_5\left(\frac{\lambda_2+1}{2}\right)^k|x[0]| + \gamma_4\left(1+\frac{1}{\varepsilon}\right)\|w\|_\infty.$$

So our difference inequality now looks like

$$\theta[k+1] \leq \lambda_2\theta[k] + \psi[k],\ \ \theta[2] \geq 0,\ \ k \geq 2; \tag{4.55}$$

notice that $\lambda_2 \geq 0$ and that $\theta[k] \geq 0$ and $\psi[k] \geq 0$ for $k \geq 2$. In order to solve this inequality let us define an equation

$$\bar{\theta}[k+1] = \lambda_2 \bar{\theta}[k] + \psi[k],\ \bar{\theta}[2] = \theta[2].$$

Solving iteratively for any k we get

$$\bar{\theta}[k] \quad = \quad \lambda_2^{k-2}\bar{\theta}[2] + \sum_{m=2}^{k-1} \lambda_2^{k-1-m}\psi[m],\ k \geq 2. \tag{4.56}$$

<u>Claim:</u> $\theta[k] \leq \bar{\theta}[k]$ for all $k \geq 2$.

Proof of claim: We will use induction to prove this. Clearly the claim is true for $k = 2$. Now, let us suppose that it is true for $k = 2, 3, 4, \ldots, j$.; we need to prove it for $k = j + 1$. From (4.55) we have

$$\theta[j+1] \leq \lambda_2 \theta[j] + \psi[j]; \tag{4.57}$$

by hypothesis we also know that

$$\lambda_2 \theta[j] + \psi[j] \leq \underbrace{\lambda_2 \bar{\theta}[j] + \psi[j]}_{=\bar{\theta}[j+1]}. \tag{4.58}$$

From (4.57) and (4.58), we conclude that

$$\theta[k+1] \leq \bar{\theta}[k+1].$$

$\blacksquare$

From this claim and (4.56) we get

$$\theta[k] \leq \lambda_2^{k-2}\theta[2] + \sum_{m=2}^{k-1} \lambda_2^{k-1-m}\psi[m],\ k \geq 2;$$

but $\theta[k] = |\tilde{x}[k]|$ and substituting the value of $\psi[k+1]$ yields

$$\begin{aligned}
|\tilde{x}[k]| &\leq \lambda_2^{k-2} \underbrace{|\tilde{x}[2]|}_{=0} + \sum_{m=2}^{k-1} \lambda_2^{k-1-m} \left[\varepsilon \bar{b} c_5 \left(\frac{\lambda_2+1}{2}\right)^m |x[0]| + \gamma_4 \left(1+\frac{1}{\varepsilon}\right) \|w\|_\infty\right] \\
&\leq \varepsilon \bar{b} c_5 |x[0]| \sum_{m=2}^{k-1} \lambda_2^{k-1} \lambda_2^{-m} \left(\frac{\lambda_2+1}{2}\right)^m + \gamma_4 \left(1+\frac{1}{\varepsilon}\right) \|w\|_\infty \sum_{m=2}^{k-1} \lambda_2^{k-1-m} \\
&\leq \varepsilon \bar{b} c_5 \lambda_2^{k-1} |x[0]| \sum_{m=2}^{k-1} \left(\frac{\lambda_2+1}{2\lambda_2}\right)^m + \frac{\gamma_4}{1-\lambda_2} \left(1+\frac{1}{\varepsilon}\right) \|w\|_\infty \\
&\leq \varepsilon \bar{b} c_5 |x[0]| \frac{2\lambda_2^{k-1}\lambda_2}{1-\lambda_2} \left[\left(\frac{\lambda_2+1}{2\lambda_2}\right)^k - 1\right] + \frac{\gamma_4}{1-\lambda_2} \left(1+\frac{1}{\varepsilon}\right) \|w\|_\infty \\
&\leq \varepsilon \bar{b} c_5 \frac{2}{1-\lambda_2} \left[\left(\frac{\lambda_2+1}{2}\right)^k - \lambda_2^k\right] |x[0]| + \frac{\gamma_4}{1-\lambda_2} \left(1+\frac{1}{\varepsilon}\right) \|w\|_\infty \\
&\leq \varepsilon \frac{2\bar{b} c_5}{1-\lambda_2} \left(\frac{\lambda_2+1}{2}\right)^k |x[0]| + \frac{\gamma_4}{1-\lambda_2} \left(1+\frac{1}{\varepsilon}\right) \|w\|_\infty, \ k \geq 2.
\end{aligned}$$

If we define

$$c_6 \ := \ max\left\{\frac{2\bar{b}c_5}{1-\lambda_2}, \frac{\gamma_4}{1-\lambda_2}\right\},$$

then

$$|\tilde{x}[k]| \ \leq \ \varepsilon c_6 \left(\frac{\lambda_2+1}{2}\right)^k |x[0]| + c_6 \left(1+\frac{1}{\varepsilon}\right) \|w\|_\infty, \ k \geq 2. \tag{4.59}$$

For the second part of the theorem, let us look at

$$\begin{aligned}
\tilde{u}[k] &:= u[k] - u^o[k] \\
&= \left(\hat{f}[k] + (-1)^k \varepsilon\right) x[k] - f^o[k] x^o[k] \\
&= \hat{f}[k] x[k] + (-1)^k \varepsilon x[k] - f^o[k] x^o[k] \\
&= \hat{f}[k] x[k] + (-1)^k \varepsilon x[k] - f^o[k] x^o[k] + f^o[k] x[k] - f^o[k] x[k] \\
&= f^o[k] \left(x[k] - x^o[k]\right) + (\hat{f}[k] - f^o[k]) x[k] + (-1)^k \varepsilon x[k]. \\
|\tilde{u}[k]| &\leq \underbrace{|f^o[k]\tilde{x}[k]|}_{h_1[k]} + \underbrace{|\tilde{f}[k] x[k]|}_{h_2[k]} + \underbrace{|\varepsilon x[k]|}_{h_3[k]}, \ k \geq 2.
\end{aligned} \tag{4.60}$$

We now obtain a bound on each of these three terms.
A bound on $h_1[k]$: We know that

$$|f^o[k]| \leq \bar{f};$$

so using (4.59) bound on $h_1[k]$ yields

$$\begin{aligned} h_1[k] &\leq \bar{f}\tilde{x}[k] \\ &\leq \varepsilon \bar{f} c_6 \left(\frac{\lambda_2+1}{2}\right)^k |x[0]| + \bar{f} c_6 \left(1+\frac{1}{\varepsilon}\right) \|w\|_\infty, \ k \geq 2. \end{aligned} \tag{4.61}$$

A bound on $h_3[k]$: From Theorem 1

$$\begin{aligned} h_3[k] &\leq \varepsilon \left[c_5 \left(\frac{\lambda_2+1}{2}\right)^k |x[0]| + c_5 \left(1+\frac{1}{\varepsilon}\right) \|w\|_\infty \right] \\ &\leq \varepsilon c_5 \left(\frac{\lambda_2+1}{2}\right)^k |x[0]| + \frac{c_0 c_5}{\bar{b}} \left(1+\frac{1}{\varepsilon}\right) \|w\|_\infty, \ k \geq 0. \end{aligned} \tag{4.62}$$

A bound on $h_2[k]$: If we look at the bound on $g_2[k]$ and compare it with $h_2[k]$ we get

$$h_2[k] = \frac{g_2[k]}{b},$$

and $g_2[k]$ is bounded by (4.53). Using this bound yields

$$\Rightarrow h_2[k] \leq \frac{\gamma_3}{\underline{b}} \left(1+\frac{1}{\varepsilon}\right) \|w\|_\infty. \tag{4.63}$$

Combining the bounds (4.61), (4.62) and (4.63) will give us

$$\begin{aligned} |\tilde{u}[k]| \leq\ & \varepsilon \bar{f} c_6 \left(\frac{\lambda_2+1}{2}\right)^k |x[0]| + \bar{f} c_6 \left(1+\frac{1}{\varepsilon}\right) \|w\|_\infty + \varepsilon c_5 \left(\frac{\lambda_2+1}{2}\right)^k |x[0]| \\ & + \frac{c_0 c_5}{\underline{b}} \left(1+\frac{1}{\varepsilon}\right) \|w\|_\infty + \frac{\gamma_3}{\underline{b}} \left(1+\frac{1}{\varepsilon}\right) \|w\|_\infty; \end{aligned}$$

if we define

$$c_7 := max\left\{ c_5, \bar{f} c_6, \frac{c_0 c_5}{\underline{b}}, \frac{\gamma_3}{\underline{b}} \right\},$$

then

$$|\tilde{u}[k]| \leq \varepsilon c_7 \left(\frac{\lambda_2+1}{2}\right)^k |x[0]| + c_7 \left(1+\frac{1}{\varepsilon}\right) \|w\|_\infty, \ k \geq 2. \tag{4.64}$$

■

Chapter 5

Linear Time-Varying System

5.1 The Setup

In this chapter we consider the case of linear time-varying plant given by

$$\left.\begin{array}{rcl} x[k+1] &=& a[k]x[k] + b[k]u[k] + w[k], \\ y[k] &=& x[k]. \end{array}\right\} \tag{5.1}$$

With the initial conditions of $\hat{a}[0] \in \mathcal{A}$, $\hat{b}[0] \in \mathcal{B}$, $\hat{f}[0] = \hat{f}[1] = f(\hat{a}[0], \hat{b}[0])$ and $\varepsilon \in \big(0, c_0/\bar{b}\big]$, we again state the proposed adaptive control law; for $k \in \mathcal{Z}^+$ even

$$\left.\begin{array}{rcl} u[k] &=& (\hat{f}[k] + \varepsilon)x[k], \\ u[k+1] &=& (\hat{f}[k+1] - \varepsilon)x[k+1], \end{array}\right\} \tag{5.2}$$

$$\begin{bmatrix} \check{a}[k+2] \\ \check{b}[k+2] \end{bmatrix} = \begin{cases} \begin{bmatrix} x[k] & (\hat{f}[k] + \varepsilon)x[k] \\ x[k+1] & (\hat{f}[k+1] - \varepsilon)x[k+1] \end{bmatrix}^{-1} \begin{bmatrix} x[k+1] \\ x[k+2] \end{bmatrix}, \text{if } x[k]x[k+1] \neq 0 \\ \begin{bmatrix} \hat{a}[k] \\ \hat{b}[k] \end{bmatrix}, \text{if } x[k]x[k+1] = 0, \end{cases} \tag{5.3}$$

$$\left.\begin{array}{rcl} \hat{a}[k+2] &=& \Pi_{\mathcal{A}}(\check{a}[k+2]), \\ \hat{b}[k+2] &=& \Pi_{\mathcal{B}}(\check{b}[k+2]), \end{array}\right\} \tag{5.4}$$

$$\left.\begin{array}{rcl} \hat{f}[k+2] &=& f(\hat{a}[k], \hat{b}[k]), \\ \hat{f}[k+3] &=& f(\hat{a}[k], \hat{b}[k]). \end{array}\right\} \tag{5.5}$$

With some limitations on how fast the plant parameters change, we will prove that the proposed control law is exponentially stabilizing and that the map from the noise to x has a bounded gain.

As discussed in the previous chapter, since the stability of the controller depends on how accurate the estimated plant parameters are, we will look at the estimation process in detail. With the initial conditions $\hat{f}[0] = \hat{f}[1] \in \mathcal{F}$, let the control law (5.2)-(5.5) be applied to plant (5.1) for even k, yielding the following response:

$$\begin{aligned}
x[k+1] &= a[k]x[k] + b[k](\hat{f}[k]+\varepsilon)x[k] + w[k] \\
x[k+2] &= a[k+1]x[k+1] + b[k+1](\hat{f}[k]-\varepsilon)x[k+1] + w[k+1] \\
\Rightarrow \underbrace{\begin{bmatrix} x[k+1] \\ x[k+2] \end{bmatrix}}_{(known)} &= \underbrace{\begin{bmatrix} x[k] & (\hat{f}[k]+\varepsilon)x[k] \\ x[k+1] & (\hat{f}[k]-\varepsilon)x[k+1] \end{bmatrix}}_{=:\phi[k+2](known)} \times \begin{bmatrix} a[k] \\ b[k] \end{bmatrix} + \underbrace{\begin{bmatrix} w[k] \\ w[k+1] \end{bmatrix}}_{(unknown)} \\
&\quad + \underbrace{\begin{bmatrix} 0 & 0 \\ x[k+1] & x[k+1](\hat{f}[k]-\varepsilon) \end{bmatrix}}_{=:\zeta[k](known)} \begin{bmatrix} a[k+1]-a[k] \\ b[k+1]-b[k] \end{bmatrix}. \qquad (5.6)
\end{aligned}$$

At this point we would like to get a bound on

$$\Delta_{ltv}[k+2] := \begin{bmatrix} \check{a}[k+2] \\ \check{b}[k+2] \end{bmatrix} - \begin{bmatrix} a[k] \\ b[k] \end{bmatrix};$$

if $x[k]x[k+1] = 0$, then $\phi[k+2]$ is not invertible and so we can not use (5.6) to obtain a bound on $\|\Delta_{ltv}[k+2]\|$. Hence, in the following analysis we assume that $x[k]x[k+1] \neq 0$. If we compare (5.6) to (4.6) we have an extra term due to the time-varying parameters. We now define

$$\Delta_a[k] := a[k+1] - a[k]$$

and

$$\Delta_b[k] := b[k+1] - b[k].$$

Using these definitions and rearranging (5.6) yields

$$\begin{bmatrix} a[k] \\ b[k] \end{bmatrix} = \phi[k+2]^{-1} \begin{bmatrix} x[k+1] \\ x[k+2] \end{bmatrix} - \phi[k+2]^{-1} \begin{bmatrix} w[k] \\ w[k+1] \end{bmatrix} - \phi[k+2]^{-1}\zeta[k] \begin{bmatrix} \Delta_a[k] \\ \Delta_b[k] \end{bmatrix}$$

and from (5.3) we have

$$\begin{bmatrix} \check{a}[k+2] \\ \check{b}[k+2] \end{bmatrix} = \phi[k+2]^{-1} \begin{bmatrix} x[k+1] \\ x[k+2] \end{bmatrix}.$$

So with this we conclude that

$$\Delta_{ltv}[k+2] = \underbrace{\phi[k+2]^{-1}\left[\begin{array}{c} w[k] \\ w[k+1] \end{array}\right]}_{=:\Delta[k+2]} + \underbrace{\phi[k+2]^{-1}\zeta[k]\left[\begin{array}{c} \Delta_a[k] \\ \Delta_b[k] \end{array}\right]}_{\text{Term 2}}. \tag{5.7}$$

Observing (5.7), we see that we have two error generating terms. We see that the first term, namely $\Delta[k+2]$, is exactly the same as (4.7) (LTI case) and Term 2 is due to the time-varying parameters. Using this yields

$$\Delta_{ltv}[k+2] = \Delta[k+2] + \phi[k+2]^{-1}\zeta[k]\left[\begin{array}{c} \Delta_a[k] \\ \Delta_b[k] \end{array}\right].$$

Similar to Lemma 2, we now introduce an important result which will be helpful in proving exponential stability of the closed-loop LTV system under consideration.

Lemma 3: There exits strictly positive constants c_8, c_9, c_{10}, c_{11} and c_{12} so that, for every $\overline{\Delta}_a > 0$, $\overline{\Delta}_b > 0$, $a \in s\left(\mathcal{A}, \overline{\Delta}_a\right)$, $b \in s\left(\mathcal{B}, \overline{\Delta}_b\right)$, $x(0) \in \mathcal{R}$, $\hat{a}[0] \in \mathcal{A}$, $\hat{b}[0] \in \mathcal{B}$ (yielding $\hat{f}[0] = \hat{f}[1] \in \mathcal{F}$), $w \in l_\infty$, $\varepsilon \in (0, c_0/\bar{b}]$ and $k \in \mathcal{Z}^+$ even, when the control law (5.2)-(5.5) is applied to the linear time-varying plant (5.1), if

$$\varepsilon|\bar{x}[k+1]| > c_8 \tag{5.8}$$

and

$$\frac{2c_9}{\varepsilon}(\overline{\Delta}_a + \overline{\Delta}_b) < \delta \tag{5.9}$$

then

$$\|\Delta_{ltv}[k+2]\| \leq \frac{1}{\varepsilon}\frac{c_{10}}{|\bar{x}[k+1]|} + \frac{c_9}{\varepsilon}(\overline{\Delta}_a + \overline{\Delta}_b) \tag{5.10}$$

and

$$|\tilde{f}[k+2]| \leq \frac{1}{\varepsilon}\frac{c_{11}}{|\bar{x}[k+1]|} + \frac{c_{12}}{\varepsilon}(\overline{\Delta}_a + \overline{\Delta}_b).$$

Proof: Let $\overline{\Delta}_a > 0$, $\overline{\Delta}_b > 0$, $a \in s\left(\mathcal{A}, \overline{\Delta}_a\right)$, $b \in s\left(\mathcal{B}, \overline{\Delta}_b\right)$, $x(0) \in \mathcal{R}$, $\hat{a}[0] \in \mathcal{A}$, $\hat{b}[0] \in \mathcal{B}$ (yielding $\hat{f}0] = \hat{f}[1] \in \mathcal{F}$) , $w \in l_\infty$, $\varepsilon \in (0, c_0/\bar{b}]$ and suppose that the control law (5.2)-(5.5) is applied to the linear time-varying plant (5.1); let $k \in \mathcal{Z}^+$ be even.

First suppose that $x[k]x[k+1] = 0$. If

$$x[k+1] = 0$$

then (5.8) never holds. If

$$x[k] = 0$$

then

$$\begin{aligned} x[k+1] &= a[k]x[k] + b[k]u[k] + w[k] \\ &= w[k]. \end{aligned}$$

Using the definition of $\bar{x}[k]$ given in (4.11) yields

$$|\bar{x}[k+1]| \leq 1,$$

which means that (5.8) will not hold as long as

$$\frac{c_8}{\varepsilon} > 1 \Leftrightarrow c_8 > \varepsilon \Leftrightarrow c_8 > \frac{c_0}{\bar{b}}.$$

Now assume that $x[k]x[k+1] \neq 0$. Using the same argument as in the proof of Lemma 2, we can prove that there exists constants γ_1 and γ_2 so that if $|\bar{x}[k+1]| > \gamma_2$ then

$$\|\Delta[k+2]\| \leq \frac{2\gamma_1\,(c_1+1)}{\varepsilon|\bar{x}[k+1]|}.$$

Hence,

$$\begin{aligned} \|\Delta_{ltv}[k+2]\| &\leq \|\Delta[k+2]\| + \left\| \phi[k+2]^{-1}\zeta[k] \begin{bmatrix} \Delta_a[k] \\ \Delta_b[k] \end{bmatrix} \right\| \\ &\leq \frac{2\gamma_1\,(c_1+1)}{\varepsilon|\bar{x}[k+1]|} + \underbrace{\left\| \phi[k+2]^{-1}\zeta[k] \begin{bmatrix} \Delta_a[k] \\ \Delta_b[k] \end{bmatrix} \right\|}_{\text{Term 2}}. \end{aligned} \tag{5.11}$$

Now, let us look at Term 2 on the R.H.S of the above equation in detail. Substituting the value of $\phi[k+2]^{-1}$ and $\zeta[k]$ in $Term$ 2 yields

$$\begin{aligned} \|Term\ 2\| &= \Bigg\| \frac{1}{2\varepsilon x[k]x[k+1]} \begin{bmatrix} (\hat{f}[k]-\varepsilon)x[k+1] & -(\hat{f}[k]+\varepsilon)x[k] \\ -x[k+1] & x[k] \end{bmatrix} \\ &\quad \times \begin{bmatrix} 0 & 0 \\ x[k+1] & x[k+1](\hat{f}[k]-\varepsilon) \end{bmatrix} \begin{bmatrix} \Delta_a[k] \\ \Delta_b[k] \end{bmatrix} \Bigg\| \\ &= \Bigg\| \frac{1}{2\varepsilon x[k]x[k+1]} \times \begin{bmatrix} -(\hat{f}[k]+\varepsilon)x[k]x[k+1] & -(\hat{f}[k]^2-\varepsilon^2)x[k]x[k+1] \\ x[k]x[k+1] & (\hat{f}[k]-\varepsilon)x[k]x[k+1] \end{bmatrix} \\ &\quad \times \begin{bmatrix} \Delta_a[k] \\ \Delta_b[k] \end{bmatrix} \Bigg\| \\ &= \frac{1}{2\varepsilon} \left\| \begin{bmatrix} -(\hat{f}[k]+\varepsilon) & -(\hat{f}[k]-\varepsilon)(\hat{f}[k]+\varepsilon) \\ 1 & (\hat{f}[k]-\varepsilon) \end{bmatrix} \begin{bmatrix} \Delta_a[k] \\ \Delta_b[k] \end{bmatrix} \right\| \\ &\leq \frac{1}{2\varepsilon} \Bigg[\underbrace{(1+\bar{f}+c_0/\bar{b})(1+\bar{f}+c_0/\bar{b})}_{=:2c_9} \Bigg] (\overline{\Delta}_a + \overline{\Delta}_b) \\ &\leq \frac{c_9}{\varepsilon}(\overline{\Delta}_a + \overline{\Delta}_b). \end{aligned} \tag{5.12}$$

Substituting this inequality into (5.11) yields

$$\|\Delta_{ltv}[k+2]\| \leq \frac{2\gamma_1\,(c_1+1)}{\varepsilon|\bar{x}[k+1]|} + \frac{c_9}{\varepsilon}(\overline{\Delta}_a + \overline{\Delta}_b). \tag{5.13}$$

In order to guarantee that $\begin{bmatrix} \hat{a}[k+2] \\ \hat{b}[k+2] \end{bmatrix}$ is a better estimate of $\begin{bmatrix} a \\ b \end{bmatrix}$ than $\begin{bmatrix} \check{a}[k+2] \\ \check{b}[k+2] \end{bmatrix}$, it is enough that

$$\|\Delta_{ltv}[k+2]\| \quad < \quad \delta; \tag{5.14}$$

this will be the case if

$$\frac{2\gamma_1\,(c_1+1)}{\varepsilon|\bar{x}[k+1]|} + \frac{c_9}{\varepsilon}(\overline{\Delta}_a + \overline{\Delta}_b) \quad < \quad \delta,$$

which in turn will be the case if

$$\begin{aligned} \frac{2\gamma_1\,(c_1+1)}{\varepsilon|\bar{x}[k+1]|} \quad &< \quad \frac{\delta}{2} \\ \Leftrightarrow \varepsilon|\bar{x}[k+1]| \quad &> \quad \frac{4\gamma_1\,(c_1+1)}{\delta} \end{aligned} \tag{5.15}$$

and

$$\frac{c_9}{\varepsilon}(\overline{\Delta}_a + \overline{\Delta}_b) < \frac{\delta}{2}. \tag{5.16}$$

At this point we have to be careful to ensure that $|\bar{x}[k+1]| > \gamma_2$ as well as (5.15) holds. To this end, if we set

$$c_8 := max\left\{\gamma_2\frac{c_0}{\overline{b}}, \frac{4\gamma_1\,(c_1+1)}{\delta}\right\}, \tag{5.17}$$

then

$$\varepsilon|\bar{x}[k+1]| > c_8$$

clearly implies that

$$|\bar{x}[k+1]| > \frac{c_8}{\varepsilon} \geq \frac{c_8}{c_0/\overline{b}} \geq \frac{\gamma_2\cancel{\frac{c_0}{\overline{b}}}}{\cancel{\frac{c_0}{\overline{b}}}} = \gamma_2$$

as well as

$$\varepsilon|\bar{x}[k+1]| > \frac{4\gamma_1\,(c_1+1)}{\delta}.$$

For such a choice of c_8, if (5.8) and (5.9) holds then (5.13) and (5.14) are true; if we define

$$c_{10} := 2\gamma_1(c_1+1),$$

then (5.13) becomes (5.10); furthermore

$$\left\| \begin{bmatrix} \hat{a}[k+2]-a \\ \hat{b}[k+2]-b \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} \check{a}[k+2]-a \\ \check{b}[k+2]-b \end{bmatrix} \right\| = \|\Delta_{ltv}[k+2]\|$$

as well. With γ_3 the Lipschitz constant of f, we conclude that

$$\begin{aligned} \tilde{f}[k+2] \leq \gamma_3 \left\| \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} \hat{a}[k+2] \\ \hat{b}[k+2] \end{bmatrix} \right\| &\leq \gamma_3 \|\Delta_{ltv}[k+2]\| \\ &\leq \frac{\gamma_3 c_{10}}{\varepsilon |\bar{x}[k+1]|} + \frac{\gamma_3 c_9}{\varepsilon}(\overline{\Delta}_a + \overline{\Delta}_b), \end{aligned} \tag{5.18}$$

so we will define $c_{11} = \gamma_3 c_{10}$ and $c_{12} = \gamma_3 c_9$. ■

For the stability analysis of the closed-loop system involving the time-varying plant (5.1), we define

$$a_{cl}[k] := a[k] + b[k]f(a[k], b[k]) \in (-1, 1)$$

and using the definition of (3.12) we can see

$$|a_{cl}[k]| \leq \lambda_2;$$

we choose

$$\lambda_3 \in (\lambda_2, 1). \tag{5.19}$$

5.2 The Main Result

Theorem 3: There exists constants $c_{13} > 0$ and $c_{14} > 0$ such that for every $a \in s(\mathcal{A}, c_{13}\varepsilon)$, $b \in s(\mathcal{B}, c_{13}\varepsilon)$, $x(0) \in \mathcal{R}$, $\hat{a}[0] \in \mathcal{A}$, $\hat{b}[0] \in \mathcal{B}$ (yielding $\hat{f}0] = \hat{f}[1] \in \mathcal{F}$), $w \in l_\infty$ and $\varepsilon \in (0, c_0/\bar{b}]$, when the control law (5.2)-(5.5) is applied to the linear time-varying plant (5.1), we have

$$|x[k]| \leq c_{14}\lambda_3^k |x[0]| + c_{14}\left(1 + \frac{1}{\varepsilon}\right)\|w\|_\infty, \; k \geq 0.$$

Proof: Let $a[k] \in s(\mathcal{A}, c_{13}\varepsilon)$, $b[k] \in s(\mathcal{B}, c_{13}\varepsilon)$, $x(0) \in \mathcal{R}$, $\hat{a}[0] \in \mathcal{A}$, $\hat{b}[0] \in \mathcal{B}$ (yielding $\hat{f}0] = \hat{f}[1] \in \mathcal{F}$), $w \in l_\infty, \varepsilon \in (0, c_0/\bar{b}]$ and $k \in \mathcal{Z}^+$ be even. As discussed in previous theorems, since $\hat{f}[0] = \hat{f}[1]$ are typically inaccurate, the behaviour over the first few steps will typically be poor. To this end, when the control law (5.2)-(5.5) is applied to the linear time-varying plant (5.1), we have

$$\begin{aligned} x[k+3] &= \left[a[k+2] + b[k+2](\hat{f}[k+2] + \varepsilon)\right] x[k+2] + w[k+2], \\ x[k+4] &= \left[a[k+3] + b[k+3](\hat{f}[k+2] - \varepsilon)\right] x[k+3] + w[k+3]. \end{aligned} \tag{5.20}$$

We now divide the proof into simple steps which makes it easier to understand.

Step 1: We will first obtain a crude bound showing how fast x increases. We first define γ_1 as defined in (3.14):

$$|a[l] + b[l](\hat{f}[l] + \varepsilon)| \;\; \leq \;\; \bar{a} + \bar{b}(\bar{f} + c_0/\bar{b}) =: \gamma_1, \; l \in \mathcal{Z}^+. \tag{5.21}$$

This can be used to obtain a crude bound on the increase of x over a step:

$$|x[k+l+1]| \leq \gamma_1 |x[k+l]| + \|w\|_\infty, \; l \in \mathcal{Z}^+. \tag{5.22}$$

Step 2: The next goal is to obtain a bound on $|x[k+4]|$, for which we will use the crude bound derived in step 1. To achieve this we will analyze $x[k+4]$ for two cases, when $\varepsilon|\bar{x}[k+1]|$ is small and when it is large. Using the choice of c_8 asserted to exist by Lemma 3, we consider:

Case 1: $\varepsilon|\bar{x}[k+1]| \leq c_8$. Since $\varepsilon|\bar{x}[k+1]|$ is not large, we can use the crude bound given by (5.22) to get a bound on $|x[k+4]|$. Using (5.22) three times in succession yields

$$\begin{aligned} |x[k+4]| &\leq \gamma_1 |x[k+3]| + \|w\|_\infty \\ &\leq \gamma_1^2 |x[k+2]| + \gamma_1\|w\|_\infty + \|w\|_\infty \\ &\leq \gamma_1^3 |x[k+1]| + \gamma_1^2\|w\|_\infty + \gamma_1\|w\|_\infty + \|w\|_\infty \end{aligned}$$

which implies that

$$
\begin{aligned}
|\bar{x}[k+4]| &\le \gamma_1^3|\bar{x}[k+1]| + \gamma_1^2 + \gamma_1 + 1. \\
&\le \frac{\gamma_1^3 c_8}{\varepsilon} + \gamma_1^2 + \gamma_1 + 1.
\end{aligned} \tag{5.23}
$$

Case 2: $\varepsilon|\bar{x}[k+1]| > c_8$. Expanding (5.20) yields

$$
\begin{aligned}
x[k+4] &= \Big[a[k+3] + b[k+3](\hat{f}[k+2] - \varepsilon)\Big] \\
&\quad \times \Big\{\Big[a[k+2] + b[k+2](\hat{f}[k+2] + \varepsilon)\Big]\, x[k+2] + w[k+2]\Big\} + w[k+3] \\
&= \Big[(a[k+2] + \Delta_a[k+2]) + (b[k+2] + \Delta_b[k+2])(\hat{f}[k+2] - \varepsilon)\Big] \\
&\quad \times \Big[a[k+2] + b[k+2](\hat{f}[k+2] + \varepsilon)\Big]\, x[k+2] \\
&\quad + \Big[a[k+3] + b[k+3](\hat{f}[k+2] - \varepsilon)\Big]\, w[k+2] + w[k+3] \\
&= \Big[a[k+2] + b[k+2](\hat{f}[k+2] - \varepsilon) + \Delta_a[k+2] + \Delta_b[k+2](\hat{f}[k+2] - \varepsilon)\Big] \\
&\quad \times \Big[a[k+2] + b[k+2](\hat{f}[k+2] + \varepsilon)\Big]\, x[k+2] \\
&\quad + \Big[a[k+3] + b[k+3](\hat{f}[k+2] - \varepsilon)\Big]\, w[k+2] + w[k+3] \\
&= \Big[a[k+2] + b[k+2](\hat{f}[k+2] - \varepsilon)\Big] \\
&\quad \times \Big[a[k+2] + b[k+2](\hat{f}[k+2] + \varepsilon)\Big]\, x[k+2] \\
&\quad + \Big[\Delta_a[k+2] + \Delta_b[k+2](\hat{f}[k+2] - \varepsilon)\Big] \\
&\quad \times \Big[a[k+2] + b[k+2](\hat{f}[k+2] + \varepsilon)\Big]\, x[k+2] \\
&\quad + \Big[a[k+3] + b[k+3](\hat{f}[k+2] - \varepsilon)\Big]\, w[k+2] + w[k+3],
\end{aligned}
$$

so,

$$
\begin{aligned}
x[k+4] = {} & \underbrace{\left[(a[k+2]+b[k+2]f[k+2])^2 - b[k+2]^2\varepsilon^2\right] x[k+2]}_{=:\psi_1[k]} \\
& + \underbrace{\Big[b[k+2]^2\tilde{f}[k+2]^2 - 2b[k+2]\tilde{f}[k+2]\left(a[k+2]+b[k+2]f[k+2]\right)\Big] x[k+2]}_{=:\psi_2[k]} \\
& + \underbrace{\Big[\Delta_a[k+2]+\Delta_b[k+2](\hat{f}[k+2]-\varepsilon)\Big]\Big[a[k+2]+b[k+2](\hat{f}[k+2]+\varepsilon)\Big] x[k+2]}_{=:\psi_3[k]} \\
& + \underbrace{\Big[a[k+3]+b[k+3](\hat{f}[k+2]-\varepsilon)\Big] w[k+2] + w[k+3]}_{=:\psi_4[k]}. \qquad (5.24)
\end{aligned}
$$

We now obtain bounds on $\psi_1[k], \psi_2[k], \psi_3[k]$ and $\psi_4[k]$ in order of difficulty. From the definition of λ_2 given by (3.12) we have

$$
|\psi_1[k]| \;\leq\; \lambda_2^2|x[k+2]|. \qquad (5.25)
$$

It is easy to see that

$$
\begin{aligned}
|\psi_4[k]| &\leq \left[\bar{a}+\bar{b}(\bar{f}+\varepsilon)+1\right]\|w\|_\infty \\
&\leq (\gamma_1+1)\|w\|_\infty. \qquad (5.26)
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
|\psi_3[k]| &\leq \left[\overline{\Delta}_a+\overline{\Delta}_b\left(\bar{f}+\varepsilon\right)\right]\left[\bar{a}+\bar{b}\left(\bar{f}+\varepsilon\right)\right]|x[k+2]| \\
&\leq \left[\overline{\Delta}_a+\overline{\Delta}_b\left(\bar{f}+\frac{c_0}{\bar{b}}\right)\right]\left[\bar{a}+\bar{b}\bar{f}+c_0\right]|x[k+2]| \\
&\leq \underbrace{\left(\bar{a}+\bar{b}\bar{f}+c_0\right) max\left\{1, \bar{f}+\frac{c_0}{\bar{b}}\right\}}_{=:\gamma_2}\left(\overline{\Delta}_a+\overline{\Delta}_b\right)|x[k+2]|. \qquad (5.27)
\end{aligned}
$$

Now we turn to $\psi_2[k]$, which is the most complicated to analyse one to analyse. Since $\varepsilon|\bar{x}[k+1]| > c_8$ by hypothesis, if

$$
\overline{\Delta}_a+\overline{\Delta}_b < \frac{\delta}{2c_9}\varepsilon \qquad (5.28)
$$

then by Lemma 3 we have

$$
|\tilde{f}[k+2]| \leq \frac{1}{\varepsilon}\frac{c_{11}}{|\bar{x}[k+1]|}+\frac{c_{12}}{\varepsilon}(\overline{\Delta}_a+\overline{\Delta}_b),
$$

which can be used to form a bound on $|\psi_2[k]|$:

$$
\begin{aligned}
|\psi_2[k]| &\leq \left[\bar{b}^2|\tilde{f}[k+2]|^2 + 2\bar{b}|\tilde{f}[k+2]|\right]|x[k+2]| \\
&\leq \bar{b}\left(\bar{b}|\tilde{f}[k+2]| + 2\right)|\tilde{f}[k+2]||x[k+2]| \\
&\leq \underbrace{\bar{b}\left(2\bar{b}\bar{f} + 2\right)}_{=:\gamma_3}|\tilde{f}[k+2]||x[k+2]| \\
&\leq \gamma_3\left[\frac{1}{\varepsilon}\frac{c_{11}}{|\bar{x}[k+1]|} + \frac{c_{12}}{\varepsilon}\left(\overline{\Delta}_a + \overline{\Delta}_b\right)\right]|x[k+2]| \\
&\leq \frac{\gamma_3 c_{11}}{\varepsilon|\bar{x}[k+1]|}\left(\gamma_1|x[k+1]| + \|w\|_\infty\right) + \frac{\gamma_3 c_{12}}{\varepsilon}\left(\overline{\Delta}_a + \overline{\Delta}_b\right)|x[k+2]| \\
&\leq \frac{\gamma_1\gamma_3 c_{11}}{\varepsilon}\|w\|_\infty + \frac{\gamma_3 c_{11}}{c_8}\|w\|_\infty + \frac{\gamma_3 c_{12}}{\varepsilon}\left(\overline{\Delta}_a + \overline{\Delta}_b\right)|x[k+2]|. \qquad (5.29)
\end{aligned}
$$

If we substitute the bounds on $\psi_1[k], \psi_2[k], \psi_3[k]$ and $\psi_4[k]$ onto (5.24) we conclude that if (5.28) holds, then

$$
\begin{aligned}
|x[k+4]| \leq\ & \left[\lambda_2^2 + \left(\gamma_2 + \frac{\gamma_3 c_{12}}{\varepsilon}\right)\left(\overline{\Delta}_a + \overline{\Delta}_b\right)\right]|x[k+2]| \\
& + \left[1 + \gamma_1 + \frac{\gamma_1\gamma_3 c_{11}}{\varepsilon} + \frac{\gamma_3 c_{11}}{c_8}\right]\|w\|_\infty. \qquad (5.30)
\end{aligned}
$$

If (5.28) holds and

$$
\begin{aligned}
\left(\gamma_2 + \frac{\gamma_3 c_{12}}{\varepsilon}\right)\left(\overline{\Delta}_a + \overline{\Delta}_b\right) &\leq \lambda_3^2 - \lambda_2^2 \\
\Leftrightarrow \left(\overline{\Delta}_a + \overline{\Delta}_b\right) &\leq \frac{\varepsilon}{\gamma_2\varepsilon + \gamma_3 c_{12}}\lambda_3^2 - \lambda_2^2 \\
\Leftarrow \left(\overline{\Delta}_a + \overline{\Delta}_b\right) &\leq \underbrace{\frac{\lambda_3^2 - \lambda_2^2}{\gamma_2\varepsilon + \gamma_3 c_{12}}}_{=:\gamma_4}\varepsilon, \qquad (5.31)
\end{aligned}
$$

then (5.30) becomes

$$
|x[k+4]| \leq \lambda_3^2|x[k+2]| + \left[1 + \gamma_1 + \frac{\gamma_1\gamma_3 c_{11}}{\varepsilon} + \frac{\gamma_3 c_{11}}{c_8}\right]\|w\|_\infty;
$$

if we define

$$
\gamma_5 = max\left\{1 + \gamma_1 + \frac{\gamma_8 c_{11}}{c_8}, \gamma_1\gamma_3 c_{11}\right\},
$$

then this becomes

$$|x[k+4]| \leq \lambda_3^2 |x[k+2]| + \gamma_5 \left(1 + \frac{1}{\varepsilon}\right) \|w\|_\infty. \tag{5.32}$$

Now observe that if we define

$$c_{13} := \frac{1}{3} min \left\{ \frac{\delta}{2c_9}, \gamma_4 \right\},$$

then

$$\overline{\Delta}_a \leq c_{13}\varepsilon$$

and

$$\overline{\Delta}_b \leq c_{13}\varepsilon$$

implies that (5.28) and (5.31) both holds, which means that (5.32) holds.

Step 3: We will now combine the two Cases of Step 2 to get a bound on $|x[k+4]|$. More specifically, the bounds provided by combining Case 1 given by (5.23) and Case 2 given by (5.32), we conclude that regardless of the value of $|\bar{x}[k+1]|$, we have

$$|\bar{x}[k+4]| \quad \leq \quad \lambda_3^2 |\bar{x}[k+2]| + (1 + \gamma_1 + \gamma_1^2 + \gamma_5) + \frac{\gamma_1^3 c_8 + \gamma_5}{\varepsilon}.$$

If we define

$$\gamma_6 := max \left\{ 1 + \gamma_1 + \gamma_1^2 + \gamma_5, \gamma_1^3 c_8 + \gamma_5 \right\}$$

we have

$$|x[k+4]| \quad \leq \quad \lambda_3^2 |x[k+2]| + \gamma_6 (1 + \frac{1}{\varepsilon}) \|w\|_\infty, \; k \in \mathcal{Z}^+ \text{ even.} \tag{5.33}$$

Step 4: We will now obtain a bound on $|x[l]|$ for $l \in \mathcal{Z}^+$. Observe that (5.33) holds for even $k \in \mathcal{Z}^+$; using the variable substitution $k = 2(j-1)$ with $j \in \mathbb{N}$, this becomes

$$|x[2j+2]| \quad \leq \lambda_3^2 |x[2j]| + \gamma_6 (1 + \tfrac{1}{\varepsilon}) \|w\|_\infty, \; j \in \mathbb{N}.$$

Solving iteratively yields

$$|x[2j+2]| \quad \leq \quad \lambda_3^{2j} |x[2]| + \frac{\gamma_6}{1 - \lambda_3^2} \left(1 + \frac{1}{\varepsilon}\right) \|w\|_\infty, \; j \in \mathbb{N}. \tag{5.34}$$

For the odd steps we use (5.21):

$$|x[2j+3]| \quad \le \quad \gamma_1 |x[2+2j]| + \|w\|_\infty, \;\; j \in \mathbb{N}.$$

Combining this with (5.34) yields

$$|x[2j+3]| \quad \le \quad \gamma_1 \lambda_3^{2j} |x[2]| + \left[1 + \frac{\gamma_1\gamma_6}{1-\lambda_3^2}\left(1+\frac{1}{\varepsilon}\right)\right] \|w\|_\infty, \;\; j \in \mathbb{N}. \tag{5.35}$$

So for $l \ge 4$ even, (5.34) says that

$$|x[l]| \le \lambda_3^{l-2}|x[2]| + \frac{\gamma_6}{1-\lambda_3^2}\left(1+\frac{1}{\varepsilon}\right)\|w\|_\infty,$$

and for $l \ge 5$ odd, (5.35) says that

$$|x[l]| \le \gamma_1 \lambda_3^{l-3}|x[2]| + \left[1 + \frac{\gamma_1\gamma_6}{1-\lambda_3^2}\left(1+\frac{1}{\varepsilon}\right)\right]\|w\|_\infty.$$

If we define

$$\gamma_7 := max\left\{\frac{\gamma_6}{1-\lambda_3^2}, 1 + \frac{\gamma_1\gamma_6}{1-\lambda_3^2}\right\},$$

when combined these yield

$$|x[l]| \le \gamma_1 \lambda_3^{-3}\lambda_3^l |x[2]| + \gamma_7\left(1+\frac{1}{\varepsilon}\right)\|w\|_\infty, \;\; l \ge 4. \tag{5.36}$$

The last step is to analyse the first three steps. With γ_1 defined by (5.21), it is easy to see that

$$|x[1]| \le \gamma_1 |x[0]| + \|w\|_\infty, \tag{5.37}$$

$$|x[2]| \le \gamma_1^2 |x[0]| + (\gamma_1 + 1)\|w\|_\infty, \tag{5.38}$$

and

$$|x[3]| \le \gamma_1^3 |x[0]| + (\gamma_1^2 + \gamma_1 + 1)\|w\|_\infty, \tag{5.39}$$

Combining (5.36) and (5.38) yields

$$|x[l]| \le (\gamma_1^3 \lambda_3^{-3})\lambda_3^l |x[0]| + \left(\gamma_1 \lambda_3^{-3}(\gamma_1+1) + \gamma_7 + \frac{\gamma_7}{\varepsilon}\right)\|w\|_\infty, \;\; l \le 4. \tag{5.40}$$

From (5.37),(5.38) and (5.39), we also see that

$$|x[l]| \le max\left\{1, \gamma_1\lambda_3^{-1}, \gamma_1^2\lambda_3^{-2}, \gamma_1^3\lambda_3^{-3}\right\}\lambda_3^l|x[0]| + \left(\gamma_1^2+\gamma_1+1\right)\|w\|_\infty, \;\; l = 0,1,2,3. \tag{5.41}$$

It follows from (5.40) and (5.41) that there exists a constant c_{12} so that

$$|x[l]| \leq c_{12}(\lambda_3)^l |x[0]| + c_{12}\left(1 + \frac{1}{\varepsilon}\right)\|w\|_\infty, \; l \in \mathcal{Z}^+.$$

■

Chapter 6

Examples and Simulations

For all the examples in this Chapter we consider the following setup:

- $a[k]$ takes values in $\mathcal{A} = [0, 5]$.
- $b[k]$ takes value in set $\mathcal{B} = [-5, -1] \cup [1, 5]$, yielding $\bar{b} = 5$.
- f is an LQR optimal gain for $r = 1$ given by

$$f(a,b) = \begin{cases} \frac{1-a^2+b^2-\sqrt{(a^2+b^2-1)^2-4b^2}}{2ab} & \text{if } a \neq 0, \\ 0 & \text{if } a = 0. \end{cases}$$

We construct a controller as defined in Chapter 4 and Chapter 5 satisfying the following equations: with $c_0 = 1$ yielding $\varepsilon = \left(0, \frac{1}{5}\right)$, for $k \in \mathcal{Z}^+$, even

$$\left.\begin{array}{rcl} u[k] &=& (\hat{f}[k] + \varepsilon)x[k], \\ u[k+1] &=& (\hat{f}[k+1] - \varepsilon)x[k+1], \end{array}\right\} \tag{6.1}$$

$$\begin{bmatrix} \check{a}[k+2] \\ \check{b}[k+2] \end{bmatrix} = \begin{cases} \begin{bmatrix} x[k] & (\hat{f}[k]+\varepsilon)x[k] \\ x[k+1] & (\hat{f}[k+1]-\varepsilon)x[k+1] \end{bmatrix}^{-1} \begin{bmatrix} x[k+1] \\ x[k+2] \end{bmatrix}, \text{if } x[k]x[k+1] \neq 0 \\ \begin{bmatrix} \hat{a}[k] \\ \hat{b}[k] \end{bmatrix}, \text{if } x[k]x[k+1] = 0, \end{cases} \tag{6.2}$$

$$\left.\begin{array}{rcl} \hat{a}[k+2] &=& \Pi_{\mathcal{A}}(\check{a}[k+2]), \\ \hat{b}[k+2] &=& \Pi_{\mathcal{B}}(\check{b}[k+2]), \end{array}\right\} \tag{6.3}$$

$$\left.\begin{array}{rcl} \hat{f}[k+2] &=& f(\hat{a}[k], \hat{b}[k]), \\ \hat{f}[k+3] &=& f(\hat{a}[k], \hat{b}[k]). \end{array}\right\} \tag{6.4}$$

6.1 LTI with No Noise

Example 1*:* Consider the following system:

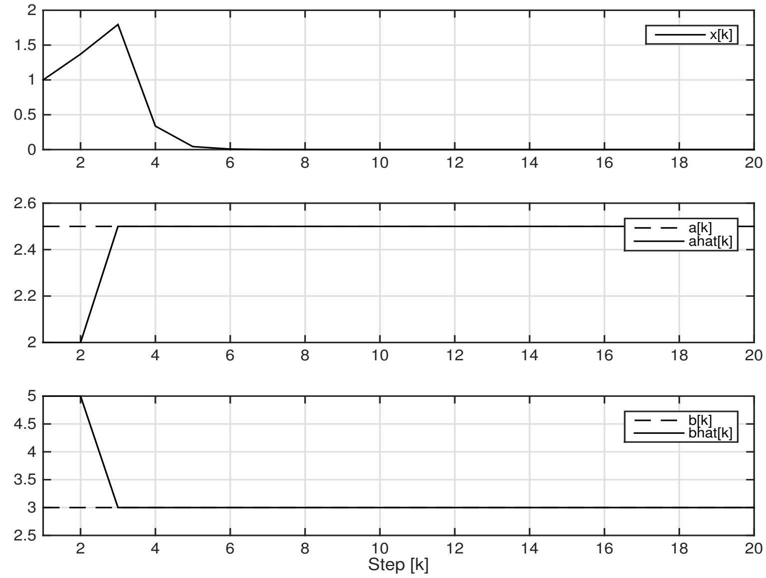
$$\begin{aligned} x[k+1] &= 2.5x[k] + 3u[k], \\ y[k] &= x[k]; \end{aligned}$$

here $\hat{a}[0] = 2$ and $\hat{b}[0] = 5$, yielding $\hat{f}[0] = \hat{f}[1] = f(\hat{a}[0], \hat{b}[0]) = -0.3866$ and $x[0] = 1$.

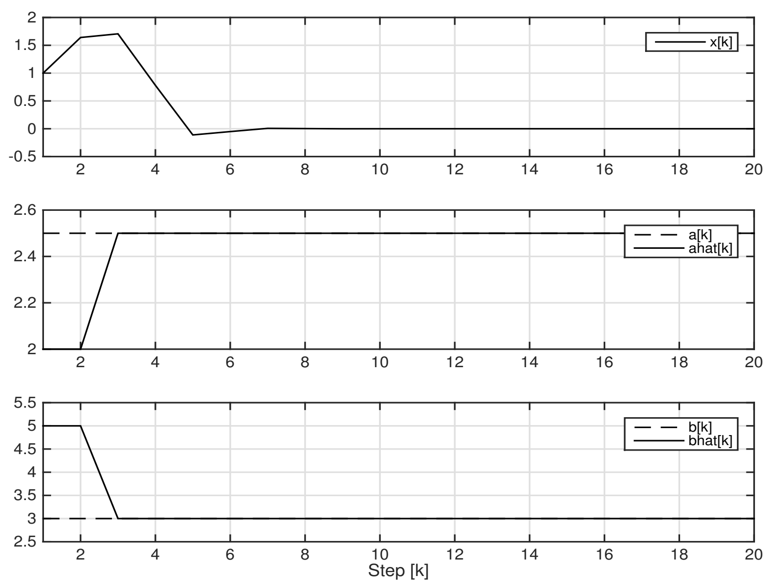
If we apply control law (6.1)-(6.4) with $\varepsilon = 0.01$, as expected the closed-loop system accurately estimates the plant parameters from the very first estimation which can be seen in Figure 6.1a. In absence of noise, the effect of the initial condition diminishes and the state variable $x[k]$ approaches zero, validating the results of Theorem 1 which states that

$$x[k] \leq c_5 \lambda_2^k |x[0]| + c_5 \left(1 + \frac{1}{\varepsilon}\right) \|w\|_\infty, \; k \geq 0.$$

We now vary ε to observe its effect on the plant. We repeat this simulation with $\varepsilon = 0.1$ and $\varepsilon = 0.19$ and the simulation results are shown in Figure 6.1b and Figure 6.1c respectively. These simulations indicate that as we increase ε and approach $\frac{1}{b} = 0.2$, the state variable $x[k]$ ripples before settling down to zero and these ripples increase as $\varepsilon \to 0.2$.

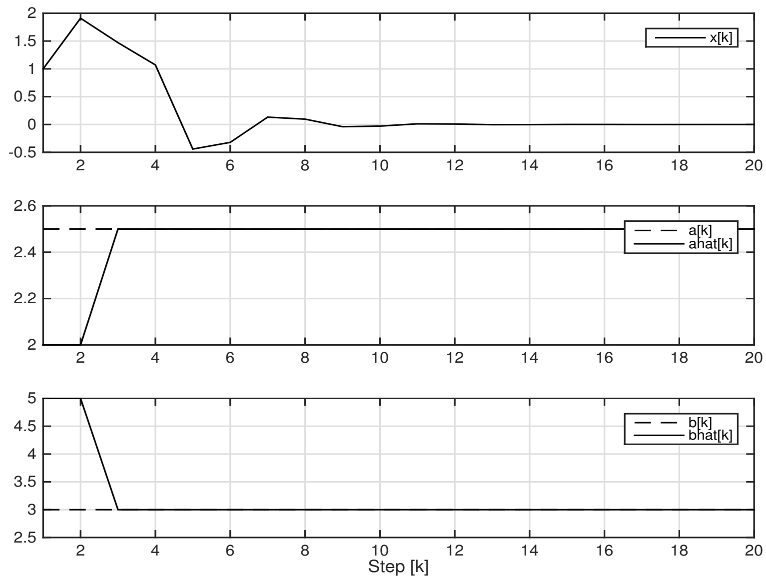


(a) $\varepsilon = 0.01$



(b) $\varepsilon = 0.1$

Figure 6.1: Example 1: Response of closed-loop LTI plant with no noise.



(c) $\varepsilon = 0.19$

Figure 6.1: Example 1: Response of closed-loop LTI plant with no noise.

6.2 LTI with Noise

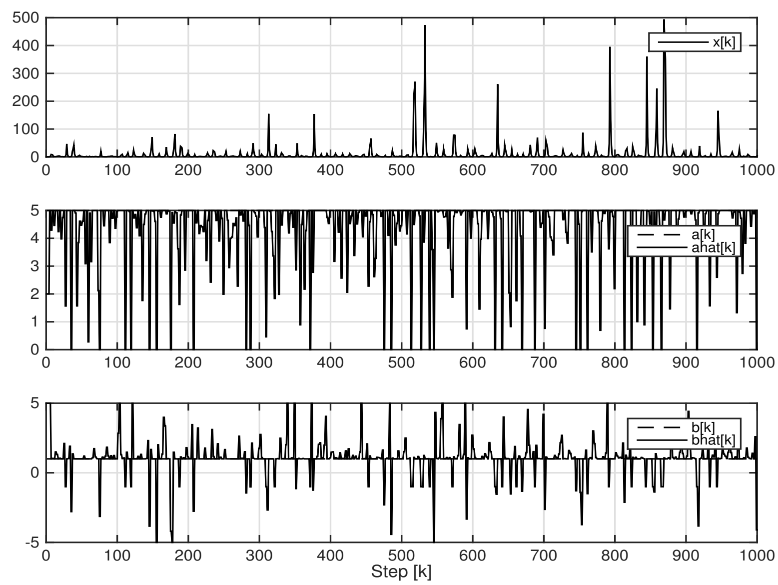
For this case we choose the most unstable value of a i.e. $a = 5$.

Example 2*:* Consider the following system:

$$\begin{aligned} x[k+1] &= 5x[k] + 1u[k] + w[k], \\ y[k] &= x[k]. \end{aligned}$$

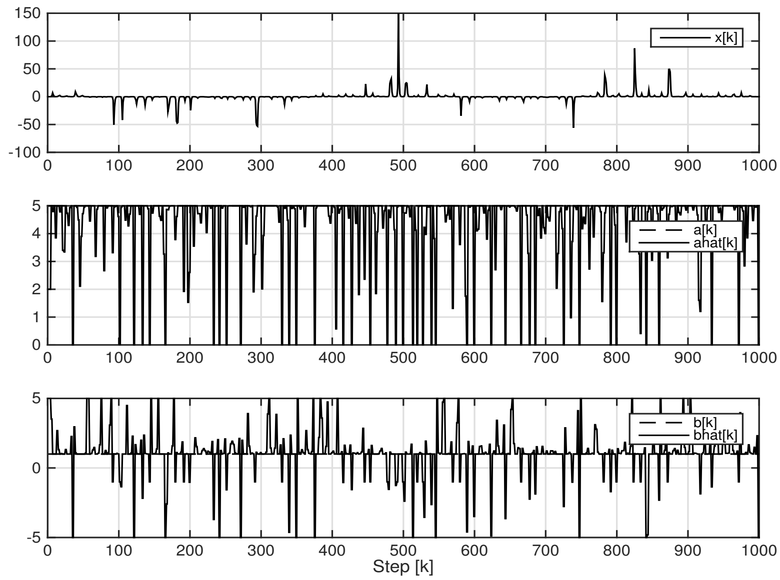
Here $\hat{a}[0] = 2$ and $\hat{b}[0] = 5$, yielding $\hat{f}[0] = \hat{f}[1] = f(\hat{a}[0], \hat{b}[0]) = -0.3866$; we choose $x[0] = 0$ so that we can focus on the effect of the noise $w[k]$ which is a random function generating values in range $[-0.01, 0.01]$.

We apply control law (6.1)-(6.4) to the plant considered in this example with $\varepsilon = 0.01$. For testing the control law we run the simulations for 10,000 steps. Figure 6.2a shows a part of the simulation. It indicates that the closed-loop system is stable and does not blow up, at least for the 1000 steps shown. As a matter of fact, the maximum value of $|x[k]|$ reached for the entire simulation is 1152.8 giving us a noise gain of order 10^5. In order to improve the noise gain, if we increase ε to 0.05 and simulate with the same data set of noise as in the previous simulation, we end up with a better noise gain, as shown in Figure 6.2b. Motivated by this, we repeat the simulations with 3 more test cases $\varepsilon = 0.1$, $\varepsilon = 0.15$ and $\varepsilon = 0.19$; their simulation results are shown in Figures 6.2c, 6.2d and 6.2e respectively. Table 6.1 provides a summary of the effect of the variation of ε on noise gain. One very important thing to note in all these simulations is that even though the parameter estimates are not that accurate, the system is stabilizing because of the adaptive nature of the control law.

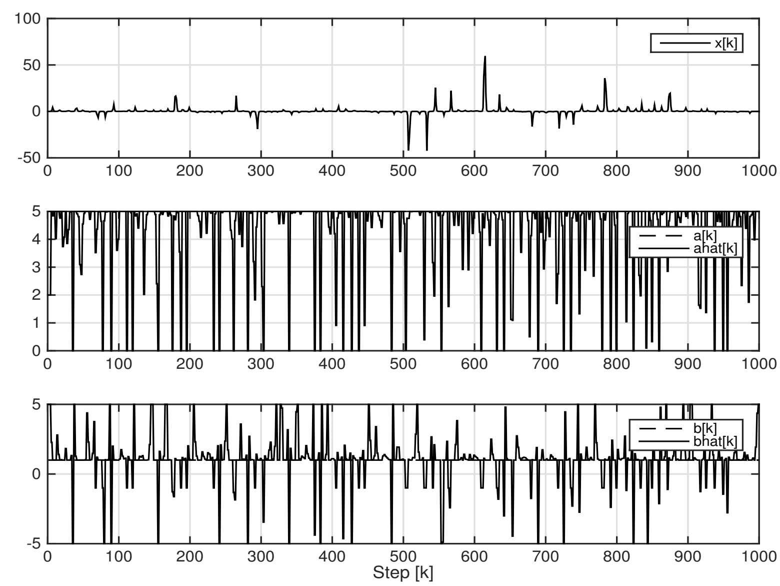


(a) $\varepsilon = 0.01$

Figure 6.2: Example 2: Response of closed-loop LTI plant with noise.

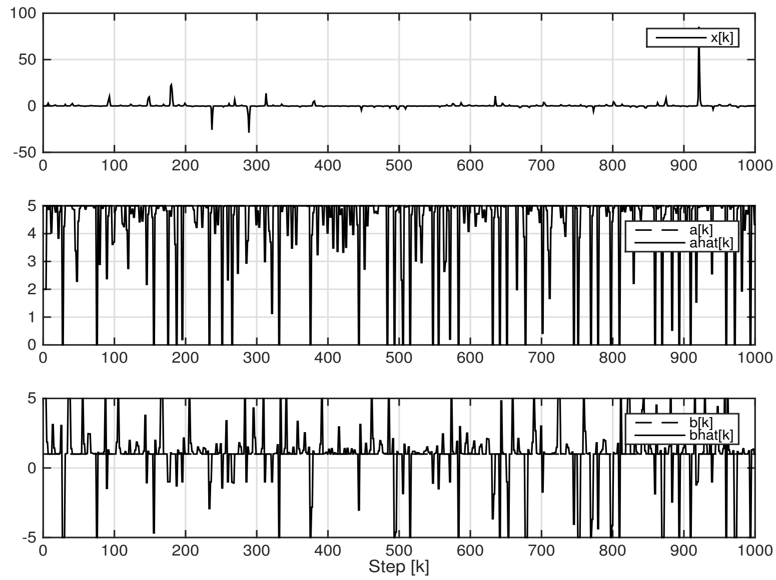


(b) $\varepsilon = 0.05$

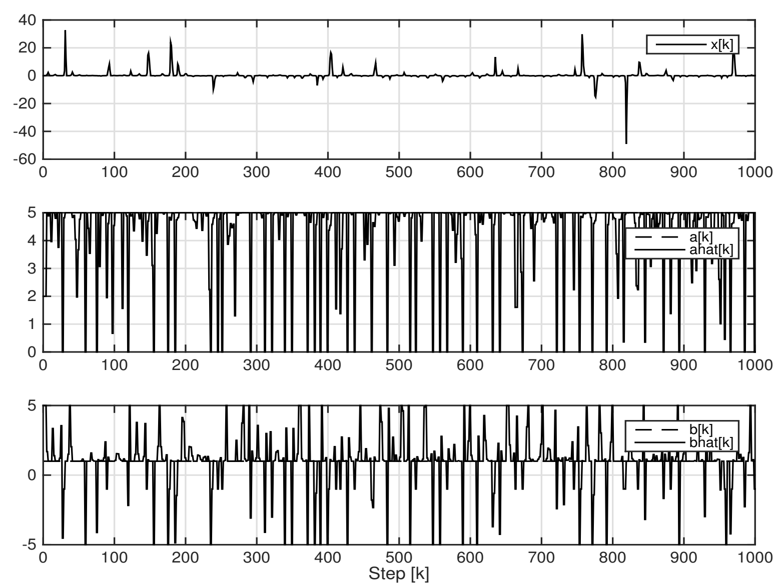


(c) $\varepsilon = 0.10$

Figure 6.2: Example 2: Response of closed-loop LTI plant with noise.



(d) $\varepsilon = 0.15$



(e) $\varepsilon = 0.19$

Figure 6.2: Example 2: Response of closed-loop LTI plant with noise.

ε	Maximum $\lvert x[k] \rvert$ for 10,000 steps	Estimate of Noise Gain
0.01	1152.8	1.15×10^5
0.05	255.3	2.55×10^4
0.10	172.8	1.73×10^4
0.15	120.4	1.20×10^4
0.19	49.14	4.91×10^3

Table 6.1: Summary of the effect of ε on Noise Gain.

If we now focus on the estimation and the estimated parameters, Lemma 2 states that if $\varepsilon|\bar{x}[k+1]|$ is large enough, our estimated $\hat{f}[k]$ is very accurate. To illustrate this, Figure 6.3 shows part of simulation for the test case of $\varepsilon = 0.19$; we see that at the step 757 the value of $x[757]$ is 29.71 (large), so the very next estimate (at step 759), the estimates $\hat{a}[759]$ and $\hat{b}[759]$ are very accurate, which results in $x[k]$ becoming close to 0 again.

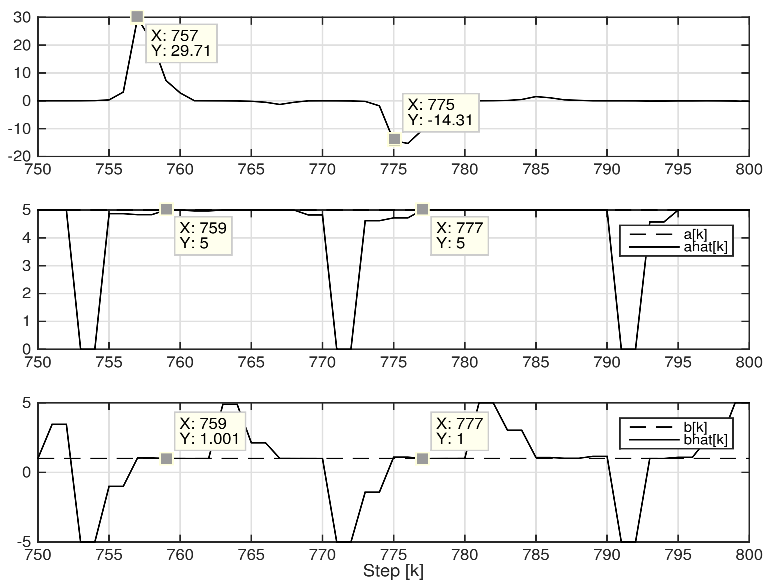


Figure 6.3: Example 2: Simulation for case $\varepsilon = 0.19$ for steps 750 to 800.

6.3 Comparison of the Proposed Control Law to the Ideal Control Law

Example 3*:*

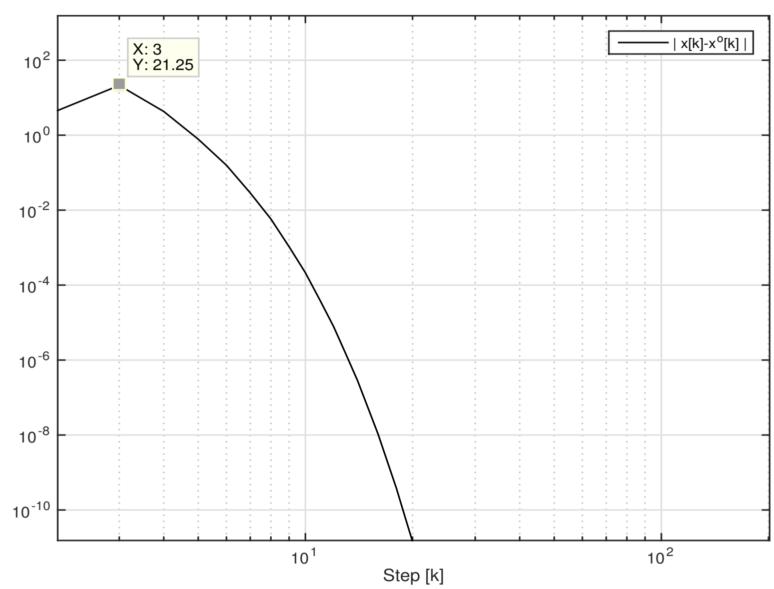
$$\begin{aligned} x[k+1] &= 5x[k] + 1u[k], \\ y[k] &= x[k]; \end{aligned}$$

here, $\hat{a}[0] = 2$ and $\hat{b}[0] = 5$, yielding $\hat{f}[0] = \hat{f}[1] = f(\hat{a}[0], \hat{b}[0]) = -0.3866$; we set $x[0] = 1$.

We simulate the plant with the same control law given by equations (6.1)-(6.4) and compare it with the ideal control action given by

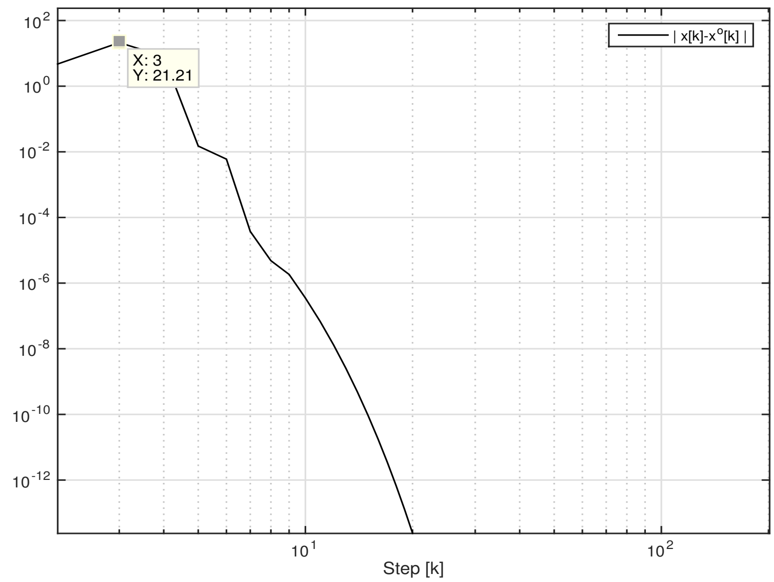
$$u[k] = f[k]x[k].$$

Figure 6.4a shows the the difference $|x[k] - x^o[k]|$ on a log scale for the test case of $\varepsilon = 0.01$, and Figure 6.4b shows for the case when $\varepsilon = 0.19$. Here we observe that the difference quickly goes to zero in both the cases, and the maximum difference in $|x[k] - x^o[k]|$ is almost the same in both test cases.



(a) $\varepsilon = 0.01$

Figure 6.4: Example 3: Difference from optimality ($|x[k] - x^o[k]|$) without noise.



(b) $\varepsilon = 0.19$

Figure 6.4: Example 3: Difference from optimality ($|x[k] - x^o[k]|$) without noise.

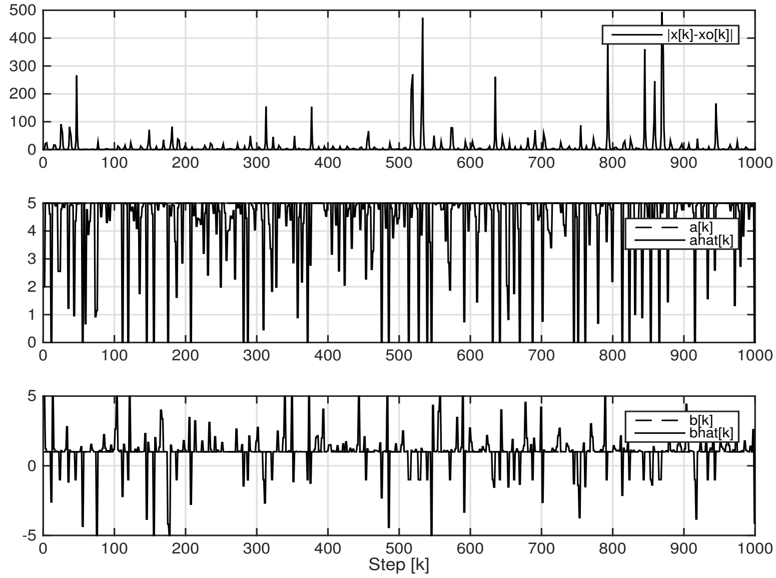
Example 4*:* Now, consider the following system with noise identical to Example 2:

$$\begin{aligned} x[k+1] &= 5x[k] + 1u[k] + w[k], \\ y[k] &= x[k]; \end{aligned}$$

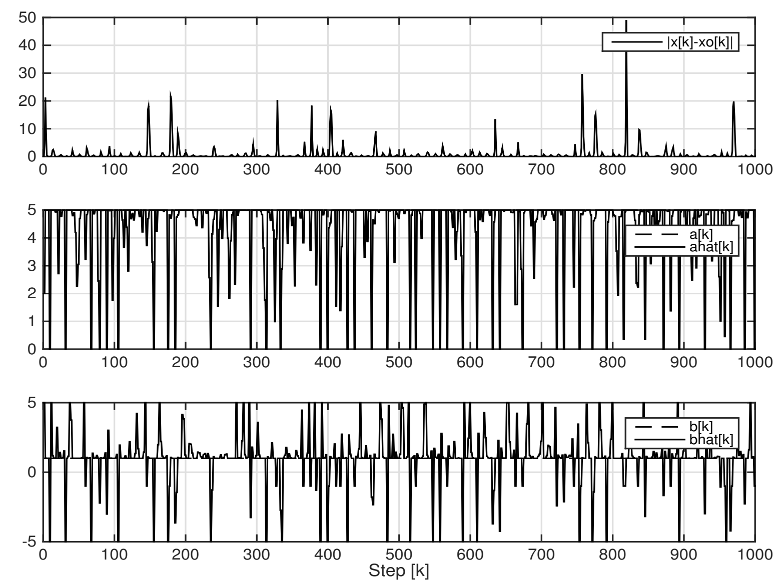
here, $\hat{a}[0] = 2$ and $\hat{b}[0] = 5$, yielding $\hat{f}[0] = \hat{f}[1] = f(\hat{a}[0], \hat{b}[0]) = -0.3866$, we set $x[0] = 0$ and the noise $w[k]$ is a random function generating values in range $[-0.01, 0.01]$.

We apply the Control Law (6.1)-(6.4) and we again compare the state variable of the proposed coontrol law $x[k]$ to the ideal controller state variable in presence of noise. For $\varepsilon = 0.01$ we plot the simulation run for 10,000 steps and the results are shown in Figure 6.5a. We see that the difference is of order 100. If we increase ε to 0.19 and simulate the closed-loop system we see in Figure 6.5b that the difference is of order 10. This is consistent with Theorem 2 which states that

$$|x[k] - x^o[k]| \leq \varepsilon c_6 \left(\frac{\lambda_2 + 1}{2}\right)^k |x[0]| + c_6 \left(1 + \frac{1}{\varepsilon}\right) \|w\|_\infty . \; k \geq 2.$$



(a) $\varepsilon = 0.01$



(b) $\varepsilon = 0.19$

Figure 6.5: Example 4: Difference from optimality $|x[k] - x^o[k]|$ with noise.

Since the maximum value of $|x[k]-x^o[k]|$ is almost the same size as that of the maximum value of $x[k]$, this bound is not useful unless the noise is very small with respect to the initial conditions.

6.4 Linear Time-Variant Plant

For the time-varying plant, we will look at two different examples.

Example 5*:*

$$\begin{aligned} x[k+1] &= a[k]x[k] + b[k]u[k] + w[k], \\ y[k] &= x[k]; \end{aligned}$$

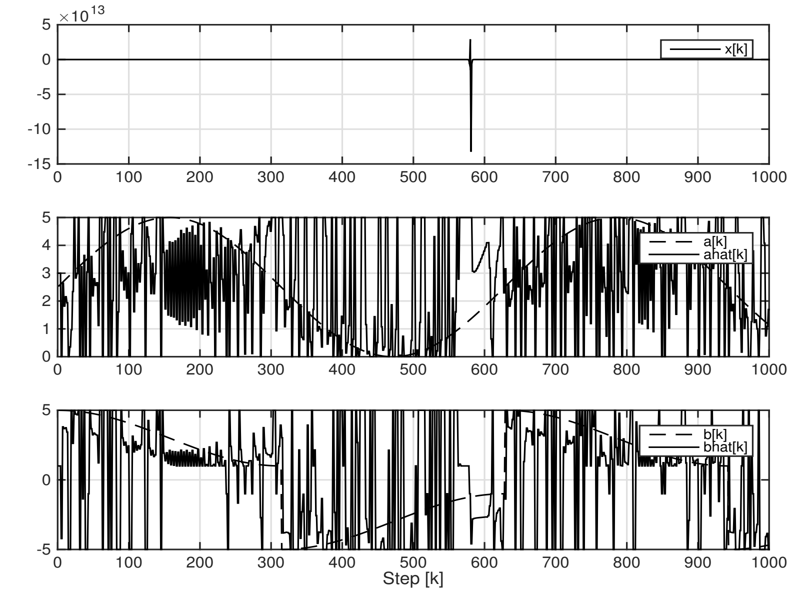
here, $\hat{a}[0] = 3$ and $\hat{b}[0] = 1$, yielding $\hat{f}[0] = \hat{f}[1] = f(\hat{a}[0], \hat{b}[0]) = -2.7033$; we set $x[0] = 0$ so that we can see the effect of noise which is a random number in range $[-0.01, 0.01]$. The value of $a[k]$ is sinusoidal taking values in $\mathcal{A}$:

$$a[k] = 2.5 + 2.5sin(0.01 \times k)$$

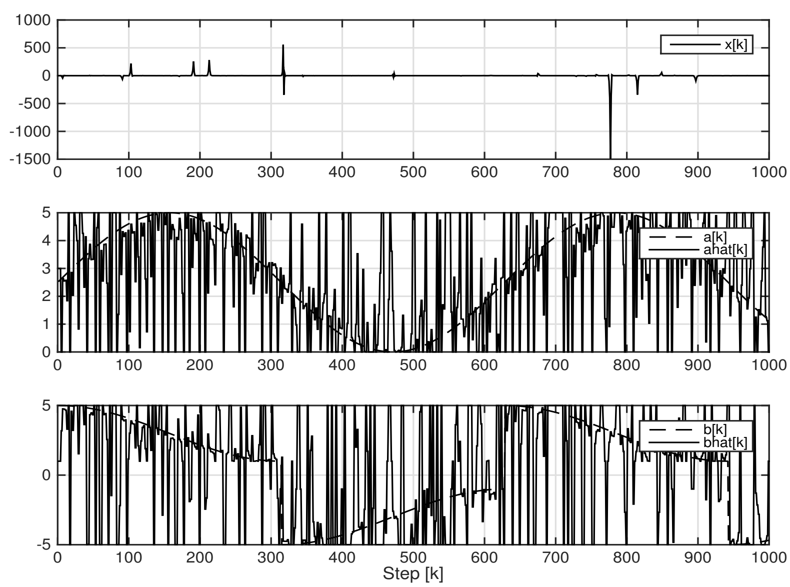
and $b[k]$ is also sinusoidal taking values in $\mathcal{B}$ and switching signs ocassionally:

$$b[k] = 2cos(0.01 \times k) + 3sign(sin(0.01 \times k)).$$

For simulation purposes, we will carry the simulation for 10,000 steps. We start our simulation with $\varepsilon = 0.01$ and the results are shown in Figure 6.6a which show the results for the first 1000 steps. We see that the state variable $x[k]$ gets very large for such a small value of ε. Motivated by the observations of the previous simulations we now increase ε to 0.05 and redo the simulations; we end up with a much smaller maximum value of $x[k]$, as shown in Figure 6.6b. We again increase ε to values $\varepsilon = 0.1$, $\varepsilon = 0.15$ and $\varepsilon = 0.19$, respectively; the simulation results are shown in Figures 6.6c, 6.6d and 6.6d respectively. We observe a constant decrease in noise gain and increasingly accurate estimates $\hat{a}[k]$ and $\hat{b}[k]$.

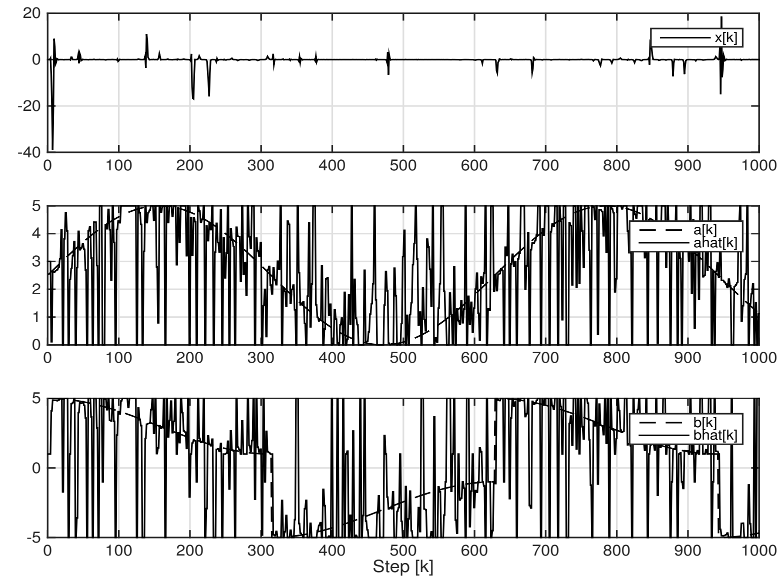


(a) $\varepsilon = 0.01$: Maximum $|x[k]| = 1.0686 \times 10^{15}$ for 10,000 steps.

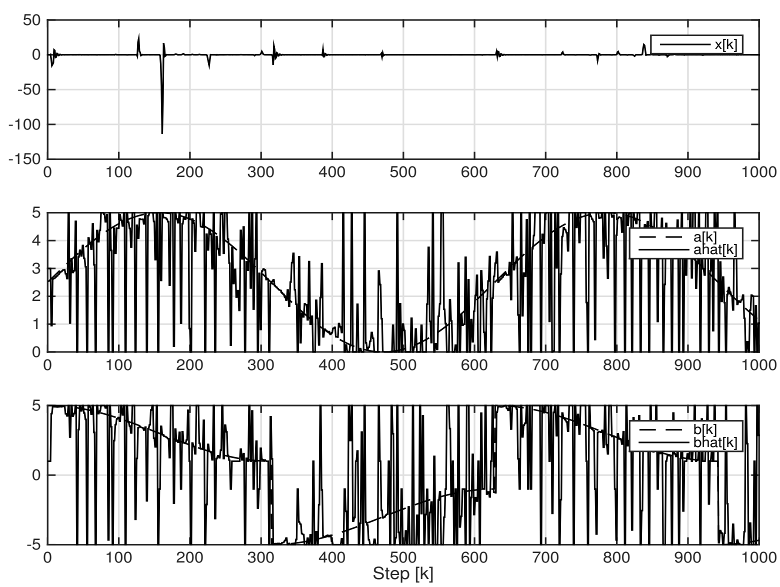


(b) $\varepsilon = 0.05$: Maximum $|x[k]| = 1.1867 \times 10^{4}$ for 10,000 steps.

Figure 6.6: Example 5: Linear Time-Varying Plant with noise.

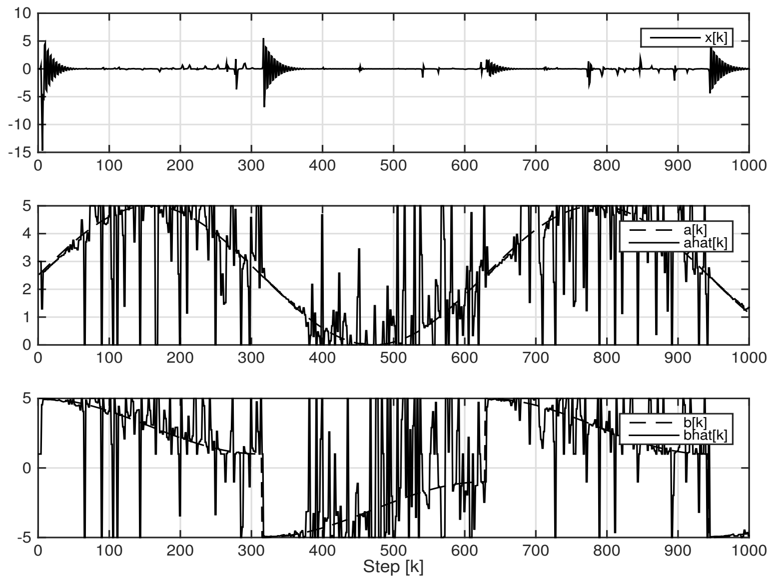


(c) $\varepsilon = 0.1$: Maximum $|x[k]| = 258.93$ for 10,000 steps.



(d) $\varepsilon = 0.15$: Maximum $|x[k]| = 545.30$ for 10,000 steps.

Figure 6.6: Example 5: Linear Time-Varying Plant with noise.



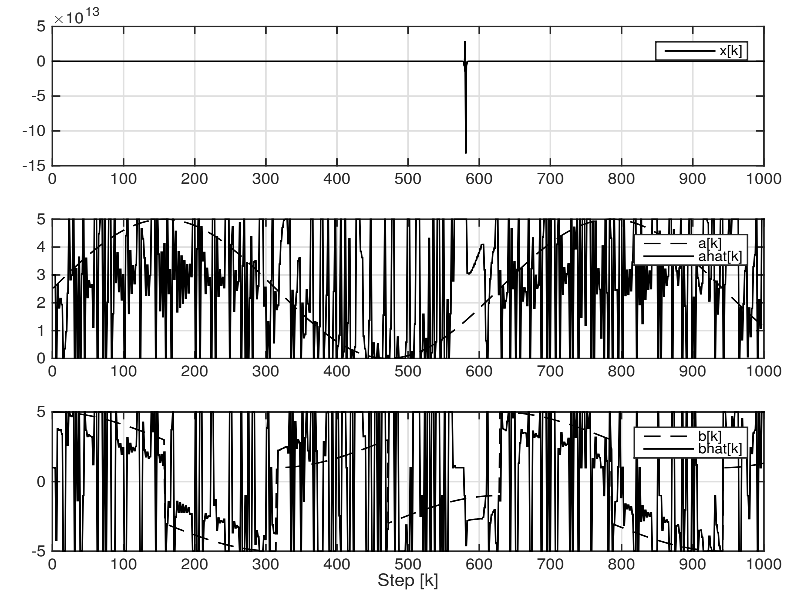
(e) $\varepsilon = 0.19$: Maximum $|x[k]| = 545.30$ for 10,000 steps.

Figure 6.6: Example 5: Linear Time-Varying Plant with noise.

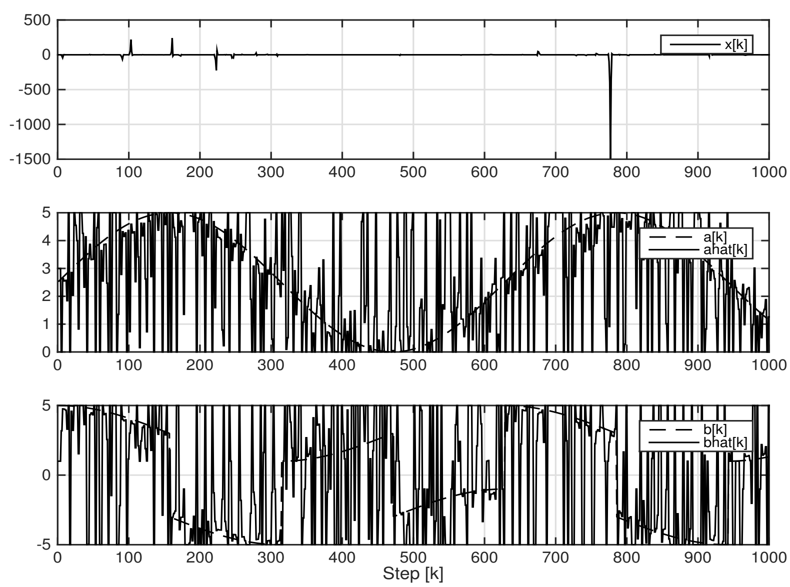
Example 6*:* Here we re-examine the previous example, but with $b[k]$ switching signs twice as often:

$$b[k] = 2cos(0.01 \times k) + 3sign(sin(0.02 \times k)).$$

Following a similar simulation procedure, we do 5 simulations with 5 increasing values of ε starting from 0.01 up to 0.19. The simulation results are shown in Figure 6.7a - 6.7e.

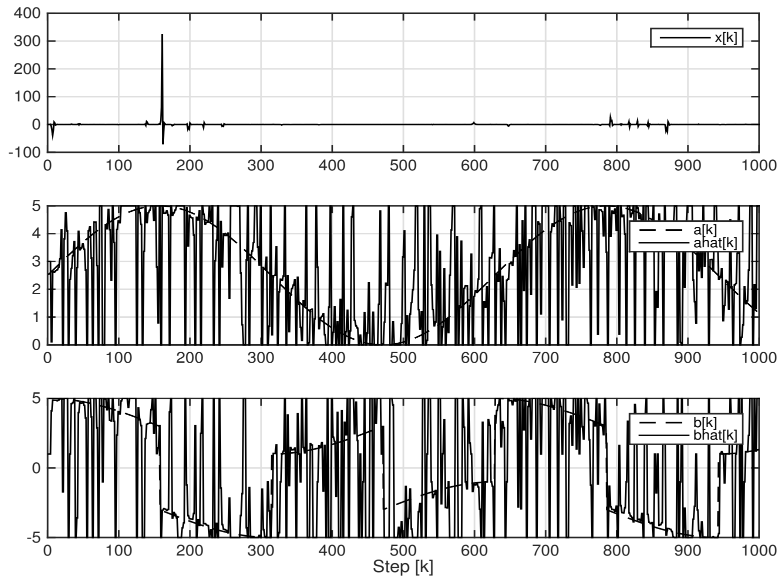


(a) $\varepsilon = 0.01$: Maximum $|x[k]| = 1.0686 \times 10^{15}$ for 10,000 steps.

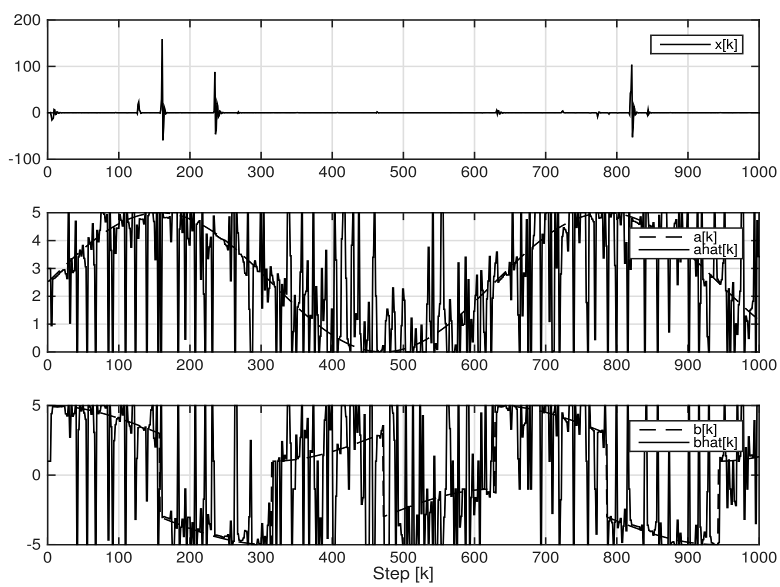


(b) $\varepsilon = 0.05$: Maximum $|x[k]| = 1.1867 \times 10^{4}$ for 10,000 steps.

Figure 6.7: Example 6: Linear Time-Varying Plant with noise (fast switching $b[k]$).

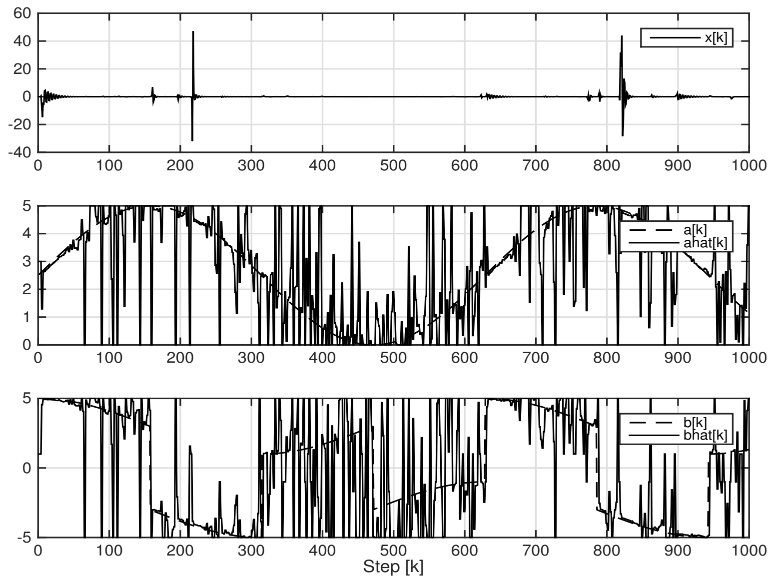


(c) $\varepsilon = 0.1$: Maximum $|x[k]| = 3.5636 \times 10^3$ for 10,000 steps.



(d) $\varepsilon = 0.15$: Maximum $|x[k]| = 2.02 \times 10^3$ for 10,000 steps.

Figure 6.7: Example 6: Linear Time-Varying Plant with noise (fast switching $b[k]$).



(e) $\varepsilon = 0.19$: Maximum $|x[k]| = 1.37 \times 10^3$ for 10,000 steps.

Figure 6.7: Example 6: Linear Time-Varying Plant with noise (fast switching $b[k]$).

We see that the noise gains are now higher than before. For large value of ε we see that the noise gain is increased by a factor of 10, probably because $b[k]$ is switching signs faster. If we look at Figure 6.7e, we see that the estimations are still very accurate when the state gets large because of the adaptive nature of the controller. These results are consistent with Theorem 3, which states that

$$|x[k]| \leq c_{14}\lambda_3^k|x[0]| + c_{14}\left(1 + \frac{1}{\varepsilon}\right)\|w\|_\infty, \;\; k \geq 0.$$

These simulations clearly indicate that the proposed control law provides exponential stability for even a highly unstable plant where $a[k]$ and $b[k]$ lie in such a huge range.

As an observation we perform another simulation, but not with a totally different setup. We simulate for a stable plant with noise. All parameters are same as in previous examples except

- $a[k]$ takes values in $\mathcal{A} = [-0.75, 0.75]$.

Example 7:

$$\begin{aligned} x[k+1] &= a[k]x[k] + b[k]u[k] + w[k], \\ y[k] &= x[k]; \end{aligned}$$

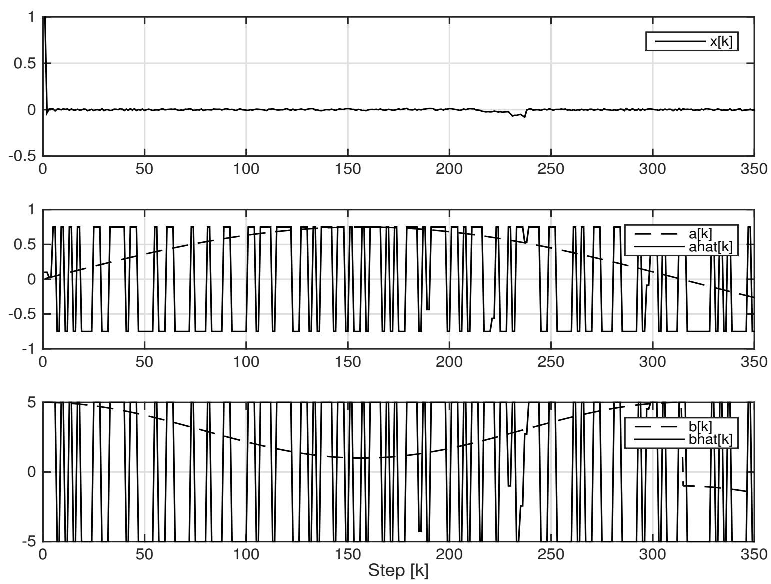
here, $\hat{a}[0] = 0.1$ and $\hat{b}[0] = 5$, yielding $\hat{f}[0] = \hat{f}[1] = f(\hat{a}[0], \hat{b}[0]) = -0.0192$; we set $x[0] = 1$ and noise is a random number in range $[-0.01, 0.01]$. The value of $a[k]$ is sinusoidal taking values in $\mathcal{A}$:

$$a[k] = 0.75 \sin(0.01 \times k)$$

and $b[k]$ is also sinusoidal taking values in $\mathcal{B}$ and switching signs ocassionally:

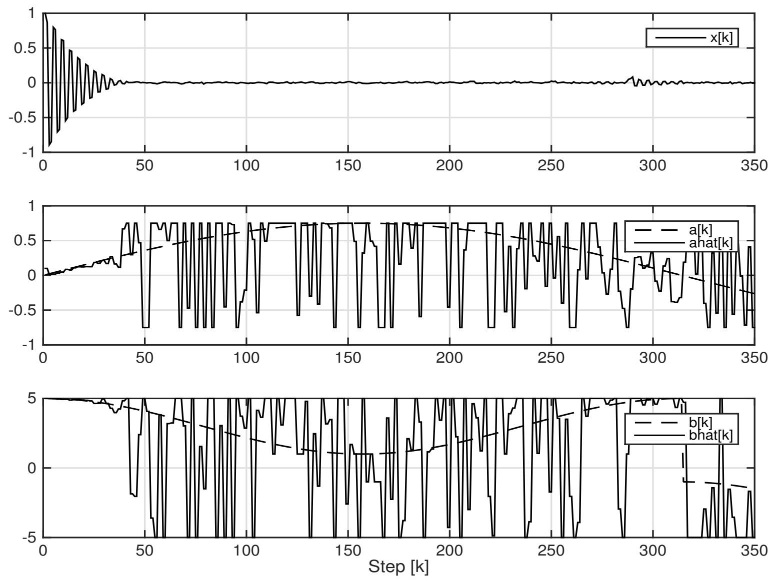
$$b[k] = 2cos(0.01 \times k) + 3sign(sin(0.02 \times k)).$$

We perform simulations for the above plant setup for two different cases one when $\varepsilon = 0.01$ and the other when $\varepsilon = 0.19$. The simulations are shows in Figure 6.8a and Figure 6.8b



(a) $\varepsilon = 0.01$: Maximum $|x[k]| = 1$ for 10,000 steps.

Figure 6.8: Example 7: Linear Time-Varying Stable Plant with noise.



(b) $\varepsilon = 0.19$: Maximum $|x[k]| = 1$ for 10,000 steps.

Figure 6.8: Example 7: Linear Time-Varying Stable Plant with noise.

We observe that initially when the state variable is large we have accurate estimates, which is clearly seen in Figure 6.8b. Once the state variable goes small, the estimates are not accurate, but the stability is maintained.

Chapter 7

Conclusions and Future Work

In this thesis, we considered the problem of adaptively stabilizing a first-order Discrete-Time SISO plant for both Time-Invariant and Time-Varying cases. Here we have looked at a class of such systems and constructed a linear adaptive controller that provides exponential stability and a linear like bound on the closed-loop behavior.

We used the Certainty Equivalence Approach and made the controller adaptive by estimating the plant parameters at every other step. The estimation process was dependent on an estimation parameter ε. With the upper bound of $\varepsilon = \frac{c_0}{b}$, as $\varepsilon \to \frac{c_0}{b}$ the estimated parameters became more accurate. With a given value of ε, when the magnitude of the state variable was small, the estimated plant parameters were not at all accurate, which made the states go large; and with such large state the very next estimated parameter became accurate, which resulted in the state becoming close to 0 again. This property is leveraged to prove the fact that the closed-loop system is exponentially stable and the transient behavior (with no noise) is near optimal, with the error improving as $\varepsilon \to 0$.

Chapters 4 and 5 provided a mathematical analysis of the plant transient behavior when applied with the proposed control law. In spite of conservative bounds, our simulations in Chapter 6 suggests the existence of much stronger bounds. We observed that the effect of noise was also reduced with the increase in ε. In the LTV case, the simulation results clearly shows that we can allow a lot of time variations in $a[k]$ and $b[k]$ than the constants defined in Lemma 3 and Theorem 2, which were intended in showing the existence of the bound. The examples tested the most extreme plant parameter for sets $\mathcal{A}$ and $\mathcal{B}$ and the simulations show that the closed-loop noise gains can be relatively acceptable. If we test the controller with a comparatively nicer plant, we will end up with better noise gains.

With some work, these results should also be extendable to higher order plants as long as the state is measurable. It may also be possible to extend the work to yield step tracking.

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