Contributions to the Study of the Validity of Huygens' Principle for the Non-self-adjoint Scalar Wave Equation on Petrov Type D Spacetimes

by

Kenneth Chung-kan Chu

A thesis
presented to the University of Waterloo
in fulfilment of the
thesis requirement for the degree of
Master of Mathematics
in
Applied Mathematics

Waterloo, Ontario, Canada, 2000

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

This thesis makes contributions to the solution of Hadamard's problem through an examination of the question of the validity of Huygens' principle for the non-self-adjoint scalar wave equation on a Petrov type D spacetime. The problem is split into five further sub-cases based on the alignment of the Maxwell and Weyl principal spinors of the underlying spacetime. Two of these sub-cases are considered, one of which is proved to be incompatible with Huygens' principle, while for the other, it is shown that Huygens' principle implies that the two principal null congruences of the Weyl tensor are geodesic and shear-free. Furthermore, an unpublished result of McLenaghan regarding symmetric spacetimes of Petrov type D is independently verified. This result suggests the possible existence of counter-examples of the Carminati-McLenaghan conjecture.

Acknowledgements

I am very indebted to Dr. Ray McLenaghan for his many years of mentoring since my undergraduate years, and supervision and financial support throughout the course of my Master's program. My sincere thanks also go to Dr. Steve Czapor for his supervision and guidance in graduate study in general and his help with symbolic computation and Gröbner bases in particular.

I thank the Department of Applied Mathematics for financial support. I also wish to specially thank Dr. Josef Paldus and Dr. Pino Tenti for their help and advice, and Ms. Helen Warren, our Graduate Secretary, for her efficient help and accurate information.

I would like to thank Dr. Holger Meißner, Dr. Ulf Nilsson, and Dr. Roman Smirnov, for their friendships and the many discussions we shared on mathematics, physics, history and life.

I also owe many thanks to all my friends for their moral support, the joy and fun they bring me and the great many things I have learned from them. Aaron, Anmar, Corina, Irina, Katrin, Michael, Relu and Valentina: Thank you all tremendously!

Lastly and most importantly, I thank my parents, Erika and Walter — for everything — literally!

This thesis is dedicated, with love and gratitude, to my parents, Erika and Walter Chu.

Contents

1 Introduction **1**

1.1 Historical Notes 1

1.2 Mathematical Preliminaries 4

1.2.1 Spacetimes 4

1.2.2 Normal Neighbourhoods 5

1.2.3 Geodesically Convex Domains 6

1.2.4 Normal Coordinates 8

1.2.5 Causal Domains 9

1.2.6 Hypersurfaces 11

1.2.7 Discussion of Assumptions 11

1.3 Mathematical Formulation of Huygens' Principle 13

1.4 Further Development 18

1.4.1 Hadamard's Criterion 18

1.4.2 Trivial Transformations 19

1.4.3 The Conformally Invariant Scalar Wave Equation . . . 24

1.4.4 Hadamard's Problem & the Carminati-McLenaghan Conjecture 27

1.4.5 Summary of Known Results 29

2 The Necessary Conditions **32**

2.1 The Necessary Conditions in Tensor Form 32

2.2 The Spinor Formalism 36
2.2.1 Spin Structure on a Spacetime 36
2.2.2 Spinor Algebra 37
2.2.3 Petrov Classification of Spacetimes 39
2.2.4 Spinor Analysis 40
2.3 The Necessary Conditions in Spinor Form 45
2.4 Transformation Laws of NP Quantities 48
2.4.1 Dyad Transformations 48
2.4.2 Conformal Transformations 52

3 Gröbner Bases 56
3.1 Solutions of Systems of Algebraic Equations and Ideals in a Polynomial Ring 57
3.2 Multivariate Division Algorithm 59
3.3 Gröbner Bases 72
3.4 Buchberger's Algorithm 77
3.5 Reduced Gröbner Bases 84
3.6 Improving Buchberger's Algorithm by Incorporating Multivariate Factorization: the MAPLE function `gsolve()` 88

4 Reduction of the Problem & Proof of the Inadmissibility of Two Cases 92
4.1 The MAPLE Package `NPspinor` 92
4.2 A Proposition on the General Petrov Type D Spacetime . . . 93
4.3 Alignment of Principal Null Directions between the Maxwell Spinor and the Weyl Spinor 94
4.4 Inadmissibility of Cases 1 and 4 96

5 Results for Symmetric Type D Spacetimes 138
5.1 Symmetric Spacetimes of Petrov Type D 138

5.2 Derivation of Results 140

6 Partial Results for Case 2 **147**

7 Conclusion **152**

A Necessary Conditions in Newman-Penrose Form **155**

A.1 Newman-Penrose Field Equations 155

A.2 Bianchi Identities 156

A.3 The Commutation Relations 158

A.4 Maxwell's Equations (2-index Condition) 158

A.5 Component Equations of the 2-index Condition 159

A.6 Component Equations of the 3-index Condition 161

A.7 Component Equations of the 4-index Condition for Case 4 . . 163

A.8 The $o_{(AB}\iota_{CD)}\overline{o}_{(\dot{A}\dot{B}}\overline{\iota}_{\dot{C}\dot{D})}$ component equation of the 4-index Condition 170

A.9 Component Equations of the 5-index Condition for Case 4 . . 172

B Component Equations for Type D Symmetric Spacetimes **178**

C MAPLE (`NPspinor`) Code **181**

C.1 The 2-index Condition (2.3.13) 181

C.2 The 3-index Condition (2.3.10) 183

C.3 The 4-index Condition (2.3.11) 184

C.3.1 Individual Symmetrized Terms 184

C.3.2 `cV.sum` 186

C.4 The 5-index Condition (2.3.12) 188

C.4.1 `cVI-symm-spinor-basis.txt` 188

C.4.2 `gen-tD-cVI-symmetrize.txt` 189

C.4.3 `gen-tD-cVI-footer.txt` 189

C.4.4 Individual Symmetrized Terms 189

Notation and Conventions

Spacetimes

Spacetime signature: $(+,-,-,-)$

Tensor indices in the natural basis: α, β, $\ldots$; range: $0, \ldots, 3$.

Abstract spinor indices: $A, B, \ldots, \dot{A}, \dot{B}, \ldots$; range: $0, 1$.

Component spinor indices: $a, b, \ldots, \dot{a}, \dot{b}, \ldots$; range: $0, 1$

Symmetrization of indices: $(a_1 \cdots a_m)$

Skew-symmetrization of indices: $(a_1 \cdots a_m)$

Exclusion of indices from (skew-) symmetrization: $|a_k|$

Partial differentiation with respect to x^α: $\frac{\partial}{\partial x^\alpha}$, ∂_α, or $,a$

Covariant derivative: ∇_α, or $_{;\alpha}$

The metric tensor: $g_{\alpha\beta}$

The Riemann curvature tensor:

$$R^{\varepsilon}{}_{\alpha\beta\gamma} X_\epsilon = X_{\alpha;\beta\gamma} - X_{\alpha;\gamma\beta}$$

The Ricci tensor: $R_{\alpha\beta} := R^{\varepsilon}{}_{\alpha\beta\varepsilon}$

The Ricci scalar: $R := R^{\alpha}{}_{\alpha}$

The Weyl tensor:

$$C_{\alpha\beta\gamma\delta} := R_{\alpha\beta\gamma\delta} - g_{\alpha[\gamma}R_{\delta]\beta} - g_{\beta[\delta}R_{\gamma]\alpha} + \frac{1}{3}\, R\, g_{\alpha[\gamma}g_{\delta]\beta}$$

Trace-free symmetrization of a tensor:

$$\mathrm{TS}[\, T_{\alpha_1\cdots\alpha_m}\,] := T_{(\alpha_1\cdots\alpha_m)} - \sum_{k=1}^{[m/2]} g_{(\alpha_1\alpha_2}\cdots g_{\alpha_{2k-1}\alpha_{2k})}\overset{m}{T}_{\alpha_{2k+1}\cdots\alpha_{2m}}$$

where $\overset{m}{T}_{\alpha_{2k+1}\cdots\alpha_{2m}}$ are obtained by solving the $k := [m/2]$ equations which results from contracting both sides of the above equation successively with $g^{\alpha_1\alpha_2}$, $g^{\alpha_3\alpha_4}$, ... , $g^{\alpha_{2k-1}\alpha_{2k}}$.

Spinor Equivalents

- the metric tensor: $g_{\alpha\beta} \longleftrightarrow \epsilon_{AB}\overline{\epsilon}_{\dot{A}\dot{B}}$
- the Riemann tensor[1]

$$\begin{aligned} R_{\alpha\beta\gamma\delta} \quad \longleftrightarrow \quad & -\Psi_{ABCD}\overline{\epsilon}_{\dot{A}\dot{B}}\overline{\epsilon}_{\dot{C}\dot{D}} - \epsilon_{AB}\epsilon_{CD}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}} \\ & -\Phi_{AB\dot{C}\dot{D}}\epsilon_{CD}\overline{\epsilon}_{\dot{A}\dot{B}} - \Phi_{CD\dot{A}\dot{B}}\epsilon_{AB}\overline{\epsilon}_{\dot{C}\dot{D}} \\ & -2\,\Lambda\,(\epsilon_{AC}\epsilon_{BD}\overline{\epsilon}_{\dot{A}\dot{B}}\overline{\epsilon}_{\dot{C}\dot{D}} + \epsilon_{AB}\epsilon_{CD}\overline{\epsilon}_{\dot{A}\dot{D}}\overline{\epsilon}_{\dot{B}\dot{C}}) \end{aligned}$$

- the Ricci scalar: $R = 24\,\Lambda$

[1]The sign convention here is that of Newman and Penrose [36].

- the Ricci tensor: $R_{\alpha\beta} \longleftrightarrow -2\,\Phi_{AB\dot{A}\dot{B}} + 6\,\Lambda\,\epsilon_{AB}\overline{\epsilon}_{\dot{A}\dot{B}}$
- the Weyl tensor: $C_{\alpha\beta\gamma\delta} \longleftrightarrow -\Psi_{ABCD}\overline{\epsilon}_{\dot{A}\dot{B}}\overline{\epsilon}_{\dot{C}\dot{D}} - \epsilon_{AB}\epsilon_{CD}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}$

Gröbner Basis Theory

$\langle S \rangle$: the ideal generated by a subset S of a ring R.

$\mathbb{F}[x_1, \ldots, x_n]$: the ring of polynomials in $x_1, \ldots, x_n$ with coefficients in the field $\mathbb{F}$.

$\mathfrak{M}$: the set of all monomials in $\mathbb{F}[x_1, \ldots, x_n]$.

$\prec$: a monomial ordering on $\mathfrak{M}$:

Let $\prec$ be given monomial ordering on $\mathfrak{M}$; then, any $f \in \mathbb{F}[x_1, \ldots, x_n]$ can be written uniquely as an $\mathbb{F}$-linear combination of monomials, i.e. $f = \sum_{i=1}^{k} c_i m_i$, such that for each $i = 1, \ldots, k$, $c_i \neq 0$, $m_i \in \mathfrak{M}$, and $m_1 \succ \cdots \succ m_k$.

- The support of f: $\mathrm{supp}(f) := \{\{m_i | i = 1, \ldots, k\}$.
- The leading coefficient, $\mathrm{lc}(f)$, of f is c_1.
- The leading monomial, $\mathrm{lm}(f)$, of f is m_1.
- The leading term, $\mathrm{lt}(f)$, of f is $c_1 m_1$.
- Each $c_i m_i$ is called a term of f.

$\mathrm{rem}(f, (g_1, \ldots, g_k))$: the remainder of $f \in \mathbb{F}[x_1, \ldots, x_n]$ with respect to the ordered sequence $(g_1, \ldots, g_k) \subseteq \mathbb{F}[x_1, \ldots, x_n]$ produced by the Multivariate

Division Algorithm.

Lm(S): the leading monomial ideal of the subset $S \subseteq \mathbb{F}[x_1, \ldots, x_n]$ (with respect to some monomial ordering).

lcm(m_1, m_2): the least common multiple of the monomials $m_1, m_2 \in \mathfrak{M}$ (with respect to some monomial ordering). It is defined as follows:

$$\operatorname{lcm}(x_1^{\alpha_1} \cdots x_n^{\alpha_n}, x_1^{\beta_1} \cdots x_n^{\beta_n}) := x_1^{\gamma_1} \cdots x_n^{\gamma_n},$$

where $\gamma_i := \max\{\alpha_i, \beta_i\}$, for each $i = 1, \ldots, n$.

$S(f, g)$: the S-polynomial of $f, g \in \mathbb{F}[x_1, \ldots, x_n]$ (with respect to some monomial ordering).

Chapter 1

Introduction

1.1 Historical Notes

In 1678, the Dutch physicist and mathematician Christiaan Huygens published his *Treatise on Light* [27], in which he presented a theory for the propagation, reflection, and refraction of light.

Huygens' light theory is based on the assumption that light waves traverse a medium of ether particles. Huygens drew two conclusions regarding the propagation of light, both of which were later referred to as Huygens' principle.

In 1923, Jacques Hadamard published his *Lectures on Cauchy's Problem in Linear Partial Differential Equations* [24], in which he mentioned that "... it happened, as is often the case, the question [the formulation of Huygens' principle] under discussion was badly set. Huygens' principle can be taken in several different senses, and these were not sufficiently distinguished."

He then proceeded to present Huygens' principle as the following syllogism:

(A) **Major Premise**

The action of phenomena produced at the instant $t = 0$ on the state of matter at the later time $t = t_0$ takes place by the mediation of every intermediate instant $t = t'$, i.e. (assuming $0 < t' < t_0$), in order to find out what takes place for $t = t_0$, we can deduce from the state at $t = 0$ the state at $t = t'$ and from the latter, the required state at $t = t_0$.

(B) **Minor Premise**

If, at the instant $t = 0$ — or more precisely throughout the short interval $-\epsilon \leq t \leq 0$ — we produce a luminous disturbance localized in the intermediate neighbourhood of O, the effect of it will be, for $t = t'$, localized in the immediate neighbourhood of the surface of the sphere with center O and radius $\omega t'$: that is, will be localized in a very thin spherical shell with centre O including the aforesaid sphere.

(C) **Conclusion**

In order to calculate the effect of our initial luminous phenomenon produced at O at $t = 0$, we may replace it by a proper system of disturbances taking place at $t = t'$ and distributed over the surface of the sphere with centre O and radius $\omega t'$.

The two conclusions regarding the propagation of light drawn by Huygens in [27] were the Minor Premise and Conclusion of Hadamard's syllogism. Despite the fact that Huygens based his theory on ether particles, which is considered incorrect from modern perspectives, Huygens' conclusions nonetheless describe light waves accurately. This success is what led to the continued study of Huygens' principle.

Evidently, Hadamard's Major Premise is an example of the general philosophical belief in cause and effect, whereas the Conclusion is the superposition principle for linear wave phenomena. Modern researchers on Huygens' principle have thus restricted their attention to Hadamard's Minor Premise, and throughout this thesis, Huygens' principle will be taken in this sense.

The settings in which Huygens' principle can be studied are numerous. It has been known for a long time that the ordinary wave equation in three spatial dimensions satisfies Huygens' principle while that in two spatial dimensions does not. (See, for example, [41].) Hadamard proved that in order for Huygens' principle to hold, it is necessary that the total number of dimensions of the underlying spacetime be even and greater than or equal to four.

Huygens' principle can also be extended to "wave" operators on smooth sections of a vector bundle over a Lorentzian manifold of any dimension. (See [23].) Researchers, including Czapor, McLenaghan, and Sasse, have studied the validity of Huygens' principle of Weyl's neutrino equation and Maxwell's equations in spacetimes of certain Petrov types.

In this thesis, however, we will restrict our attention to only scalar wave equations on spacetimes, i.e. 4-dimensional Lorentzian manifolds.

1.2 Mathematical Preliminaries

1.2.1 Spacetimes

Definition 1.2.1 *A* spacetime *is a pair* $(\mathcal{M}, g_{\alpha\beta})$ *where* $\mathcal{M}$ *is a connected, non-compact, oriented, time-oriented 4-dimensional* C^∞ *real manifold and* $g_{\alpha\beta}$ *is a* C^∞ *Lorentzian metric on* $\mathcal{M}$.

Remarks

- We require a manifold to be Hausdorff and second-countable. Proposition 5.5.5 of [1] states that Hausdorff, second-countable and locally compact spaces are paracompact. Thus, manifolds, being locally Euclidean, are automatically paracompact. Paracompactness implies existence of smooth partitions of unity, which in turn implies that an integration process can be defined on manifolds.

- The following result (Proposition 37, Chapter 5, [38]) characterizes Lorentzian manifolds: For a smooth manifold $\mathcal{M}$, the following are equivalent:

  1. $\mathcal{M}$ admits a Lorentz metric.
  2. $\mathcal{M}$ admits a time-orientable Lorentz metric.
  3. $\mathcal{M}$ admits a nowhere vanishing vector fields.
  4. Either $\mathcal{M}$ is non-compact or $\mathcal{M}$ is compact with Euler characteristic zero.

- A Lorentzian manifold is time-orientable if and only if it admits a C^∞ vector field that is everywhere time-like. (Lemma 32, Chapter 5, citeONeill.)

- Let $\mathcal{M}$ be an n-manifold. Then

1. $\mathcal{M}$ is orientable if it admits a volume element (any nowhere-vanishing n-form.) (Theorem 9, §5.3, [43].)
2. $\mathcal{M}$ has a volume element if it is orientable and paracompact. (Theorem 12, §5.3, [43].)

In what follows, $(\mathcal{M}, g_{\alpha\beta})$ will denote a spacetime, ∇_α its Levi-Civita connection.

1.2.2 Normal Neighbourhoods

Theorem 1.2.1 (Proposition 24, Chapter 3, [38]) *For any $p \in \mathcal{M}$ and any $v \in T_p\mathcal{M}$, where $T_p\mathcal{M}$ is the tangent space of $\mathcal{M}$ at p, there exists a unique geodesic $\gamma_v : I \longrightarrow \mathcal{M}$ such that*

1. *$\gamma_v'(0) = v$, and*
2. *If $\sigma : J \longrightarrow \mathcal{M}$ is a geodesic such that $\sigma(0) = p$ and $\sigma'(0) = v$, then $J \subseteq I$ and $\sigma = \gamma_v|_J$.*

In the preceding theorem, γ_v is called the inextendible geodesic of $v \in T_p\mathcal{M}$.

Definition 1.2.2 *For any $p \in \mathcal{M}$, define*

$$\mathcal{D}_p := \{v \in T_p\mathcal{M} \mid \mathrm{Domain}(\gamma_v) \supseteq [0,1]\}.$$

The exponential map *of $\mathcal{M}$ at p is the function*

$$\begin{aligned} \exp_p : \ \mathcal{D}_p &\longrightarrow \mathcal{M} \\ v &\longmapsto \gamma_v(1). \end{aligned}$$

Theorem 1.2.2 (Proposition 30, Chap 3, [38]) *For each point $p \in \mathcal{M}$, there exists a neighbourhood $\tilde{U} \subseteq \mathcal{D}_p$ of $0 \in T_p\mathcal{M}$ such that the exponential map exp_p maps $\tilde{U}$ diffeomorphically onto a neighbourhood U of p in $\mathcal{M}$.*

Definition 1.2.3 *A subset S of a vector space is said to be* starshaped *about $v_0 \in S$ if $v \in S$ implies $v_0 + t(v - v_0) \in S$ for all $0 \leq t \leq 1$.*

Definition 1.2.4 *A neighbourhood U of $p \in \mathcal{M}$ is said to be a* normal neighbourhood *of p if $U \subseteq \exp_p(\mathcal{D}_p)$ and $exp_p^{-1}(U)$ is starshaped about $0 \in T_p\mathcal{M}$.*

Since given any neighbourhood $\tilde{W}$ of $0 \in T_p\mathcal{M}$, there exists a neighbourhood $\tilde{U} \subseteq \tilde{W}$ that is starshaped about $0 \in T_p\mathcal{M}$, it is evident that every $p \in \mathcal{M}$ is contained in a normal neighbourhood.

Theorem 1.2.3 (Proposition 31, Chap 3, [38]) *If U is a normal neighbourhood of $p \in \mathcal{M}$, then for each point $q \in U$, there exists a unique geodesic $\gamma : [0,1] \longrightarrow U$ from p to q. Furthermore, $\gamma'(0) = \exp_p^{-1}(q)$.*

1.2.3 Geodesically Convex Domains

Definition 1.2.5 *An open connected subset $\Omega \subseteq \mathcal{M}$ is said to be* geodesically convex *if any two points $p, q \in \Omega$ can be joined by a unique geodesic in Ω.*

Theorem 1.2.4 (Theorem 1.2.2, [17]) *Every point $p \in \mathcal{M}$ has a normal neighbourhood that is geodesically convex.*

Definition 1.2.6 (Quadratic Geodesic Distance) *Let $\Omega \in \mathcal{M}$ be a geodesically convex domain. Then the* quadratic geodesic distance *between $p, q \in \Omega$*

is defined by

$$\Gamma(p,q) := \begin{cases} \left(\int_\gamma \langle \gamma', \gamma' \rangle^{1/2}\, d\tau\right)^2, & \text{if } p \neq q \\ \\ 0, & \text{if } p = q \end{cases},$$

where γ *is the unique geodesic between* p *and* q *when* $p \neq q$*, and* $\langle \gamma', \gamma' \rangle$ *is the scalar product of the tangent vector of* γ *with itself.*

Theorem 1.2.5 (Theorem 1.2.3, [17]) *Let* $\Omega \subseteq \mathcal{M}$ *be a geodesically convex domain. Then the map*

$$\begin{array}{rccc} \Gamma: & \Omega \times \Omega & \longrightarrow & \mathbb{R} \\ & (p,q) & \longmapsto & \Gamma(p,q) \end{array}$$

has the following properties:

1. *It is* C^∞ *on* $\Omega \times \Omega$ *and symmetric in its two arguments.*

2. *As a function of either argument,* Γ *satisfies*

$$\langle \nabla\Gamma, \nabla\Gamma \rangle = 4\Gamma, \tag{1.2.1}$$

   where ∇ *is the gradient operator on scalar fields.*

3. *Let* q *be fixed, and let* $\gamma : s \longmapsto x^\alpha(s)$ *be a geodesic such that* $x^\alpha(0) = q$*, with* s *being an affine parameter, then*

$$\nabla\Gamma = 2\, s\, \gamma', \tag{1.2.2}$$

   where $\gamma' = \frac{\mathrm{d}x^\alpha}{\mathrm{d}s}$ *is the tangent vector to the geodesic* γ*.*

1.2.4 Normal Coordinates

Definition 1.2.7 (Normal Coordinates) *Let Ω be a normal neighbourhood at $p \in \mathcal{M}$, with exponential map $\exp_p : W \longrightarrow \Omega$, where W is a starshaped neighbourhood about $0 \in T_p\mathcal{M}$.*

Let $\{\mathbf{e}_0, \ldots, \mathbf{e}_3\}$ be an orthonormal basis for $T_p\mathcal{M}$. Define

$$\begin{array}{rrcl} \Xi : & W & \longrightarrow & \mathbb{R}^4 \\ & x^\alpha \mathbf{e}_\alpha & \longmapsto & (x^0, x^1, x^2, x^3) \end{array}$$

Clearly, Ξ is a smooth injection. Let $U := \Xi(W) \subseteq \mathbb{R}^4$. Then Ξ is a diffeomorphism from $W \subseteq T_p\mathcal{M}$ to $U \subseteq \mathbb{R}^4$. Define $\psi := \Xi \circ \exp_p^{-1}$, i.e.,

$$\begin{array}{rrcccl} \psi : & \Omega & \longrightarrow & W & \longrightarrow & U \\ & q & \longmapsto & \exp_p^{-1}(q) = x^\alpha \mathbf{e}_\alpha & \longmapsto & (x^0, x^1, x^2, x^3) \end{array}$$

Then $\psi : \Omega \longrightarrow U$ is a diffeomorphism and (ψ, Ω) a coordinate system containing the point p. (ψ, Ω) is called the normal coordinate system *on Ω centred at p with respect to the orthonormal basis $\{\mathbf{e}_0, \ldots, \mathbf{e}_3\}$ at $T_p\mathcal{M}$.*

Proposition 1.2.1 *Suppose (ψ, Ω) is a normal coordinate system on a normal neighbourhood Ω centred at a point $p \in \mathcal{M}$. For any point $x \in \Omega$, let $(x^0, x^1, x^2, x^3) = \psi(x)$ represent the coordinates of x. Then, the following statements hold:*

1. $g_{\alpha\beta}(p) = \mathrm{diag}(1, -1, -1, -1)$.
2. ${\Gamma^\alpha}_{\beta\gamma}(p) = 0$, *where ${\Gamma^\alpha}_{\beta\gamma}$ are the Christoffel symbols of the Levi-Civita connection.*
3. $g_{\alpha\beta}(x) x^\beta = g_{\alpha\beta}(p) x^\beta$.

4. If, in addition, Ω is geodesically convex, then

$$\Gamma(p,x) = g_{\alpha\beta}(x)x^\alpha x^\beta = g_{\alpha\beta}(p)x^\alpha x^\beta,$$

where $\Gamma(p,x)$ is the quadratic geodesic distance between p and x.

Definition 1.2.7 can be found in [37]. The results stated in Proposition 1.2.1 can be found in Theorem 1.2.3, [17] and Proposition 4.1.18, [37].

1.2.5 Causal Domains

In this section $(\mathcal{M}, g_{\alpha\beta}, X_\alpha)$ is a spacetime which is time-oriented by the everywhere time-like vector field X_α on $\mathcal{M}$.

Definition 1.2.8 *Let $\Omega \subseteq \mathcal{M}$ be open and connected.*

For each $p \in \Omega$, the future of p in Ω, *denoted by $J_+^\Omega(p)$, is defined to be the set of all points $q \in \Omega$ such that there exists a causal, future-pointing piecewise C^1 curve in Ω which starts from p and terminates at q.*

The past of p in Ω, *denoted by $J_-^\Omega(p)$, is defined similarly, but with "future-pointing" replaced by "past-pointing".*

The boundaries of $J_+^\Omega(p)$ and $J_-^\Omega(p)$ are respectively denoted by $C_+^\Omega(p)$ and $C_-^\Omega(p)$. The interiors of $J_+^\Omega(p)$ and $J_-^\Omega(p)$ are respectively denoted by $D_+^\Omega(p)$ and $D_-^\Omega(p)$.

Let $S \subseteq \Omega$ be any subset of Ω. The future of S *and* past of S *in Ω are respectively $J_+^\Omega(S) := \bigcup_{p\in S} J_+^\Omega(p)$ and $J_-^\Omega(S) := \bigcup_{p\in S} J_-^\Omega(p)$.*

Definition 1.2.9 *An open connected $\Omega \subseteq \mathcal{M}$ is said to be* geodesically normal *if it is a normal neighbourhood of each of its points.*

It is trivial to see that every geodesically normal domain is also geodesically convex.

Theorem 1.2.6 (Lemma 1.2, Chap. 1, [23]) *Let $U \subseteq \mathcal{M}$ be open and $p \in \mathcal{M}$. Then there exists a geodesically normal domain Ω such that $p \in \Omega \subseteq U$.*

Theorem 1.2.7 (Lemma 2.4, Chap. 1, [23] and Remarks) *Let $\Omega \subseteq \mathcal{M}$ be a geodesically normal domain, and $p \in \Omega$. Then*

$$\begin{aligned} J_+^\Omega(p) &= \{\, q \in \Omega \mid \Gamma(p,q) \geq 0, X(\Gamma(p,\cdot))|_q \geq 0 \,\}, \\ C_+^\Omega(p) &= \{p\} \cap \{\, q \in \Omega \mid \Gamma(p,q) = 0, X(\Gamma(p,\cdot))|_q > 0 \,\}, \\ J_-^\Omega(p) &= \{\, q \in \Omega \mid \Gamma(p,q) \geq 0, X(\Gamma(p,\cdot))|_q \leq 0 \,\}, \\ C_-^\Omega(p) &= \{p\} \cap \{\, q \in \Omega \mid \Gamma(p,q) = 0, X(\Gamma(p,\cdot))|_q < 0 \,\}, \end{aligned}$$

where Γ is the quadratic geodesic distance function on Ω and X is an everywhere time-like C^∞ vector field which gives the time orientation of $\mathcal{M}$.

Definition 1.2.10 *An open, connected set $\Omega_0 \subseteq \mathcal{M}$ is called a* causal domain *if there exists a geodesically normal domain $\Omega \subseteq \mathcal{M}$ containing Ω_0 such that for every $p, q \in \Omega_0$, $J_+^\Omega(p) \cap J_-^\Omega(q)$ is either a compact subset Ω_0 or it is empty.*

Theorem 1.2.8 (Theorem 4.4.1, [17]) *Every point in a spacetime has a neighbourhood that is a causal domain.*

1.2.6 Hypersurfaces

Definition 1.2.11 *A manifold S is a* submanifold *of $\mathcal{M}$ if*

1. *S is a topological subspace of $\mathcal{M}$.*
2. *The inclusion map $i : S \longrightarrow \mathcal{M}$ is C^∞ and its differential $\mathrm{d}i : TS \to T\mathcal{M} : v \mapsto \mathrm{d}(i)(v)$, where $\mathrm{d}(i)(v)[f] = v[f \circ i]$, $\forall\ v \in TS$, and $\forall\ f \in C^\infty(\mathcal{M})$, is injective.*

Theorem 1.2.9 (Proposition 31, Chap. 1, [38]) *A subset S of an n-manifold $\mathcal{M}$ is an m-dimensional submanifold if and only if at each point $p \in S$, there is a coordinate chart $\varphi : \mathcal{U} \subset \mathcal{M} \longrightarrow U \subset \mathbb{R}^n$, with $p \in \mathcal{U}$, such that $S = \varphi^{-1}(W)$, where $W \subseteq U$ is some some subset of $U \subseteq \mathbb{R}$ for which exactly $n - m$ coordinates on W are constant.*

Definition 1.2.12 *A* hypersurface *of an n-manifold is an $(n-1)$-dimensional submanifold.*

Definition 1.2.13 (Causal Character of Submanifolds) *Let S be a submanifold of a spacetime $\mathcal{M}$. If $T_p(S)$ has the same causal character in $T_p(\mathcal{M})$ for every $p \in S$, then that same causal character is attributed to S.*

Obviously, an arbitrary submanifold need not have a causal character.

1.2.7 Discussion of Assumptions

We have assumed our spacetime to be oriented. Since we will need to perform integration on our spacetime, we need the existence of a volume element on $\mathcal{M}$, which is in general implied by paracompactness and orientability.

We need the time-oriented assumption because we need to be able to distinguish (locally) past and future.

In the preceding sections, we presented a series of rather technical definitions and theorems, culminating in Theorem 1.2.8, which essentially states that every point in a spacetime has a neighbourhood which

1. is diffeomorphic to a domain of Minkowski spacetime that is star-shaped about the origin, and
2. contains the intersection of every pair of half null cones whose vertices belong to the neighbourhood.

We will eventually formulate Huygens' principle within causal domains only. This allows us to make use of the fact that the local topology (in particular, the topology of null cones) within a causal domain is characterised by the quadratic geodesic distance function defined on it, as implied by Theorem 1.2.7.

On the other hand, by working within causal domains, our analysis can only be carried out locally. This restriction results from the fact that Huygens' principle will be formulated as a property of the solution to the Cauchy problem, whose general solution is constructed pointwise using the quadratic geodesic distance and the local causal structure.

1.3 Mathematical Formulation of Huygens' Principle

In this section we will formulate Huygens' principle for scalar wave equations on a causal domain of spacetime, following the treatment of Friedlander [17].

In what follows, $(\mathcal{M}, g_{\alpha\beta})$ represents a spacetime, ∇_α its Levi-Civita connection. The word *smooth* will mean C^∞. $\Omega \subseteq \mathcal{M}$ will represent a causal domain, which has a piecewise smooth boundary $\partial\Omega$.

Definition 1.3.1 *A second-order linear hyperbolic partial differential operator P on $C^2(\Omega)$ is said to have* metric principal part *if*

$$P = \Box + A^\alpha \nabla_\alpha + B, \tag{1.3.1}$$

where ∇_α is the Levi-Civita connection on $(\mathcal{M}, g_{\alpha\beta})$, A^α is any smooth vector field on Ω, B is any smooth scalar field on Ω, and $\Box := g^{\alpha\beta}\nabla_\alpha\nabla_\beta = \frac{1}{|g|}\frac{\partial}{\partial x^\alpha}\left(\sqrt{|g|}\, g^{\alpha\beta}\frac{\partial}{\partial x^\beta}\right)$.

The adjoint of P is the operator:

$$^tP[v] := \Box\, v - \nabla_\alpha(A^\alpha v) + B\, v$$

The operator P is said to be self-adjoint *if $^tP = P$; otherwise, it is said to be* non-self-adjoint.

For brevity, we will refer to differential operators of the form (1.3.1) as *scalar wave operators* on Ω.

It is trivial to see that $P := \Box + A^\alpha\nabla_\alpha + B$ is self-adjoint if and only if A^α is zero, by noting that

$$\begin{aligned} ^tP[v] &:= \Box\, v - \nabla_\alpha(A^\alpha v) + B\, v \\ &= \Box\, v - A^\alpha \nabla_\alpha v + (B - \mathrm{div} A)\, v. \end{aligned}$$

Definition 1.3.2 *A* local Cauchy problem *for the operator* (1.3.1) *on* Ω *is a boundary value problem of the following form:*

$$\begin{cases} Pu &= f, \quad on\ \Omega, \\ u &= g, \quad on\ S, \\ \frac{\partial u}{\partial n} &= h, \quad on\ S, \end{cases} \tag{1.3.2}$$

where $S \subseteq \Omega$ *is a space-like hypersurface in* $\mathcal{M}$, $f \in C^\infty(\Omega)$, $g, h \in C^\infty(S)$ *and* $\frac{\partial u}{\partial n}$ *denotes the normal derivative of* u *on* S.

Definition 1.3.3 *A subset* $S \subseteq \Omega$ *is said to be* past-compact *if* $J_-^\Omega(p) \cap S$ *is either empty or compact for all* $p \in \Omega$. S *is said to be* future-compact *if* $J_+^\Omega(p) \cap S$ *is either empty or compact for all* $p \in \Omega$.

Theorem 1.3.1 (Existence and Uniqueness of the Forward Solution) *Let* $S \subseteq \Omega$ *be a past-compact space-like hypersuface such that* $\partial J_+^\Omega(S) = S$. *Suppose that* $f \in C^\infty(\Omega)$ *and* $g, h \in C^\infty(S)$. *Then the local Cauchy problem* (1.3.2) *has a unique solution* $u \in C^\infty(J_+^\Omega(S))$.

The following theorem, together with the preceding one, give a representation formula for the unique C^∞ solution to the local Cauchy problem.

Theorem 1.3.2 (The Forward Representation Formula) *Let* Ω *be a causal domain and* $P := \Box + A^a\nabla_a + B$ *a scalar wave operator on* $C^2(\Omega)$. *Let* S *be a past-compact space-like hypersurface such that* $\partial J_+^\Omega(S) = S$.

If $u \in C^\infty(\Omega)$, *then for each* $x_0 \in J_+^\Omega(S) \backslash S$, *we have*

$$u(x_0) = u^{(1)}(x_0) + u^{(2)}(x_0) + u^{(3)}(x_0) + u^{(4)}(x_0),$$

where

$$u^{(1)}(x_0) := \frac{1}{2\pi}\int_{C_-^{\Omega}(x_0)\cap J_+^{\Omega}(S)} U P(u)\mu_\Gamma$$
$$u^{(2)}(x_0) := \frac{1}{2\pi}\int_{J_-^{\Omega}(x_0)\cap J_+^{\Omega}(S)} V^+ P(u)\mu$$
$$u^{(3)}(x_0) := \frac{1}{2\pi}\int_{J_-^{\Omega}(x_0)\cap S} * \left(V^+\nabla(u) - u\nabla V^+ + (u\,V^+)A^a\right) \tag{1.3.3}$$
$$u^{(4)}(x_0) := \frac{1}{2\pi}\int_{C_-^{\Omega}(x_0)\cap S} \left(U\,(\,2\langle\xi,\nabla u\rangle + (\theta + \langle a,\xi\rangle)u\,) + V^+\,u\right)\,d\sigma_p.$$

$U \in C^\infty(\Omega\times\Omega)$ *is defined by:*

$$U(x_0,x) := \exp\left\{-\frac{1}{4}\int_0^{s(x)}(\Box\Gamma + A^\alpha\nabla_\alpha\Gamma - 8)\frac{\mathrm{d}t}{t}\right\}, \tag{1.3.4}$$

with the integral taken along the unique geodesic in Ω *starting at* x_0 *and terminating at* x. $V^+(x_0,x) \in C^\infty(\Delta^+)$, *where* $\Delta^+ := \{(p,q)\in\Omega\times\Omega \mid p \in J_+^{\Omega}(q)\}$, *is the solution to the following initial value problem:*

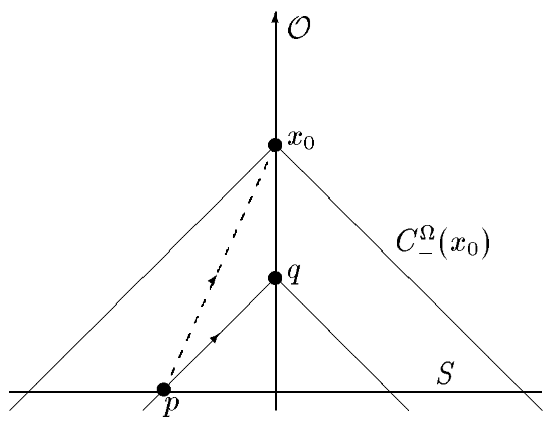
$$\begin{cases} P[V^+] &= 0, & \text{on } \Delta^+, \text{ and} \\ V^+(x_0,x) &= \frac{U(x_0,x)}{s}\int_0^{s(x)}\frac{P[U]}{U}\mathrm{d}t, & \text{when } x \in C_-^{\Omega}(x_0). \end{cases} \tag{1.3.5}$$

In (1.3.4) *and* (1.3.5), *the operator* P *differentiates with respect to* x.

The proof of Theorem 1.3.1 can be found in [17] (Theorem 5.3.2.) In the statement of Theorem 1.3.2, the vector field ξ and scalar field θ are unambiguously determined by the 2-dimensional space-like hypersurface $C_-^{\Omega}(x_0)\cap S$. It is rather involved to establish their definition and existence, and since their actual definitions, as we shall shortly see, play no part in the theory of Huygens' principle, we will omit them in this thesis but refer the interested reader to Chapter 5 of [17]. Also the C^∞ function $V^+(x_0,x)$ of $x \in D_+^{\Omega}(x_0)$ is defined to be the solution to the initial value problem (1.3.5), and the proof of its existence and uniqueness can be found in Section 4.3 of [17].

We next give a diagramatical motivation of the definition of Huygens' principle. Let $\mathcal{O}$ be the worldline of an observer and S a space-like hypersurface. Let x_0 be some point on the worldline of $\mathcal{O}$ and in the future of S. Let p be a point belonging to the intersection of S and the interior of the past null cone of the point x_0. We also suppose that all items mentioned above are contained in some causal domain Ω. This is illustrated in the following diagram.

Illustration of Huygens' Principle



The dash line indicates a "ripple" from the event $p \in S$ to the event x_0.

If at p, a light signal is emitted towards $\mathcal{O}$, it will be received by $\mathcal{O}$ at the event q, travelling along a null geodesic indicated as the solid directed line segment from p to q. If at some later point x_0 on the worldline of $\mathcal{O}$, another signal from p is received, say the one that travels along the dashed line segment in the diagram, then that subsequent signal would be a "ripple". Intutively, this violates Hadamard's Minor Premise, and Huygens' principle should therefore require that the forward solution to the local Cauchy problem at x_0 be independent of the Cauchy data on the interior of $J^{\Omega}_{-}(x_0) \cap S$. Thus we make the following:

Definition 1.3.4 *The operator* $P := \Box + A^\alpha \nabla_\alpha + B$ *on* $C^2(\Omega)$ *is said to be a* forward Huygens operator *in a causal domain* Ω *if for every* $x_0 \in \Omega$, *and every past-compact space-like hypersurface* S, *the support of the solution at* x_0 *of the local Cauchy problem* (1.3.2) *is contained in* $C_-^\Omega(x_0)$.

It is obvious that the "past" counterparts of Theorems 1.3.1 and 1.3.2 also hold, which allows us to define *backward Huygens operator.*

Definition 1.3.5 $P := \Box + A^\alpha \nabla_\alpha + B$ *is a* Huygens *operator on a causal domain* $\Omega \subseteq \mathcal{M}$ *if it is both forward and backward Huygens.*

1.4 Further Development

1.4.1 Hadamard's Criterion

The following necessary and sufficient condition for Huygens' principle is known as Hadamard's Criterion.

Theorem 1.4.1 (Hadamard's Criterion) *The operator* $P := \Box + A^\alpha \nabla_\alpha + B$ *is a forward Huygens operator in a causal domain* Ω *of the spacetime* $(\mathcal{M}, g_{\alpha\beta})$ *if and only if*

$$P[U] = 0, \quad \text{on } C_-^\Omega(x_0), \quad \text{for every } x_0 \in \Omega, \tag{HC}$$

where $U(x_0, x)$ *is the function as defined in Theorem 1.3.2.*

Proof It is obvious from Theorem 1.3.2 that $P := \Box + A^\alpha \nabla_\alpha + B$ is forward Huygens if and only if both $u^{(2)}(x_0)$ and $u^{(3)}(x_0)$ in (1.3.3) vanish for every x_0 in the future of the Cauchy surface, for every local Cauchy problem. It is also obvious from the definition of $u^{(2)}(x_0)$ and $u^{(3)}(x_0)$ that

$$\underbrace{\begin{array}{c} V^+(x_0, x) = 0, \\ \text{for every } x \in J_-^\Omega(x_0), \\ \text{for every } x_0 \in \Omega. \end{array}}_{(\star)} \quad \Longrightarrow \quad \underbrace{\begin{array}{c} u^{(2)}(x_0) = u^{(3)}(x_0) = 0, \\ \text{for every } x_0 \in J_+^\Omega(S), \\ \text{for every local Cauchy problem.} \end{array}}_{(\triangle)}$$

When we take into account that the C^∞ Cauchy surface and Cauchy data are otherwise arbitrary, we see that $(\star)$ is also a necessary condition for $(\triangle)$. Therefore, $(\star)$ is in fact a necessary and sufficient condition for P to be forward Huygens.

We next claim that $(\star)$ is in turn equivalent to (HC). Now, since V^+ is the solution to the initial value problem (1.3.5), it is obvious that (HC) implies ($\star$). Conversely, ($\star$) implies

$$\int_0^{s(x)} \frac{P[U]}{U}\,\mathrm{d}t = 0, \quad \text{on} \;\; C_-^{\Omega}(x_0), \tag{1.4.1}$$

since $U \neq 0$, being an exponential. Since $x \in C_-^{\Omega}(x_0)$ is also arbitrary, we see that (1.4.1) implies that $\frac{P[U]}{U} = 0 \iff P[U] = 0$, i.e. ($\star$) holds. □

1.4.2 Trivial Transformations

Let $P := \Box + A^{\alpha}\nabla_{\alpha} + B$ be a scalar wave operator on a causal domain $\Omega \subseteq (\mathcal{M}, g_{\alpha\beta})$. Then the following two types of transformations on P leave invariant the Huygens nature of P:

(i) $\widetilde{P}[u] = \lambda^{-1} P[\lambda\, u]$, where λ is a nowhere vanishing C^{∞} function on spacetime.

(ii) $\widetilde{P}[u] = e^{-2\phi} P[u]$, where ϕ is a C^{∞} function on spacetime.

Straightforward calculations show that in the case (i), $\widetilde{P}$ is a scalar wave operator on $\Omega \subseteq (\mathcal{M}, g_{\alpha\beta})$. However, in case (ii) $\widetilde{P}$ is a scalar wave operator on $\Omega \subseteq (\mathcal{M}, \bar{g}_{\alpha\beta})$, where $\bar{g}_{\alpha\beta} = e^{2\phi} g_{\alpha\beta}$. $(\mathcal{M}, \bar{g}_{\alpha\beta})$ is a spacetime conformally related to $(\mathcal{M}, g_{\alpha\beta})$.

We state the above claim as a theorem:

Theorem 1.4.2 *Let $P := \Box + A^{\alpha}\nabla_{\alpha} + B$ be a scalar wave operator on some causal domain Ω in $(\mathcal{M}, g_{\alpha\beta})$. Let $\phi, \lambda \in C^{\infty}(\Omega)$ and suppose λ is nowhere vanishing. Then the operator $\widetilde{P}$ defined by:*

$$\widetilde{P}[u] := \lambda^{-1} e^{-2\phi} P[\lambda\, u] \tag{1.4.2}$$

is forward Huygens if and only if P is forward Huygens.

The preceding theorem follows immediately from Hadamard's Criterion and the transformation law of $P[U]$ on $C_-^\Omega(x_0)$, the latter of which we state as the lemma below:

Lemma 1.4.1

$$\left[\widetilde{P}[\widetilde{U}]\right] = \left[\frac{1}{e^{2\phi(x_0)}e^{2\phi}}\,\frac{\lambda(x_0)}{\lambda}\,a_1\,P[U]\right].$$

where the outer square brackets on either side indicates restriction to $C_-^\Omega(x_0)$*, and* a_1 *is a non-vanishing function on* $C_-^\Omega(x_0)$*.*

Outline of proof This proof follows that of McLenaghan [33].

Claim 1 By Theorem 4.5.1, [17],

$$G_{x_0}^+ := \frac{1}{2\pi}(U\delta_+(\Gamma) + V^+) \tag{1.4.3}$$

is a forward fundamental solution of P, i.e. $P[G_{x_0}^+] = \delta_{x_0}$, with support in $J_+^\Omega(x_0)$. By Corollary 5.1.1, [17], it is the only one with pole at x_0 and support in $J_+^\Omega(x_0)$.

Claim 2 $\widetilde{P}$ is the same type of differential operator as P and the fundamental solution of $\widetilde{P}$ is:

$$\widetilde{G}_{x_0}^+ = \frac{\lambda(x_0)}{\lambda}\,\frac{1}{e^{2\phi(x_0)}}\,G_{x_0}^+. \tag{1.4.4}$$

To prove this, we proceed as follows: Since $\widetilde{G}_{x_0}^+$ is the fundamental solution of $\widetilde{P}$, we have

$$\widetilde{P}[\widetilde{G}_{x_0}^+] = \widetilde{\delta}_{x_0}(x),$$

where $\widetilde{\delta}_{x_0}(x) = e^{-4\phi}\,\delta_{x_0}(x)$. This transformation law follows from the defintion of the Dirac delta distribution on a Lorentzian manifold with metric g. It is defined, in terms of any local coordinate system (U,φ), to be $\delta_{x_0}(x) := \frac{\delta(\mathbf{x_0}-\mathbf{x})}{|g|^{1/2}}$ where $x_0, x \in U$, and $\varphi(x_0) = \mathbf{x}_0$, $\varphi(x) = \mathbf{x}$. Taking into account that $\bar{g}^{\alpha\beta} = e^{-2\phi} g^{\alpha\beta} \Longrightarrow |\bar{g}|^{1/2} = e^{4\phi}|g|^{1/2}$, the asserted transformation now clearly follows. Therefore,

$$\frac{1}{\lambda} e^{-2\phi} P[\, \lambda\, \widetilde{G}^{+}_{x_0}\,] = e^{-4\phi}\, \delta_{x_0}(x),$$

and hence,

$$P[\, \lambda\, \widetilde{G}^{+}_{x_0}\,] = \lambda e^{-2\phi} \delta_{x_0}(x) = \lambda_0 e^{-2\phi_0} \delta_{x_0}(x),$$

where $\lambda_0 := \lambda(x_0)$ and $\phi_0 := \phi(x_0)$. Consequently,

$$P\left[\frac{\lambda}{\lambda_0}\, e^{2\phi_0}\, \widetilde{G}^{+}_{x_0} \right] = \delta_{x_0}(x).$$

Now, by uniqueness of the fundamental solution of P, we conclude that

$$\widetilde{G}^{+}_{x_0} = \frac{\lambda_0}{\lambda}\, \frac{1}{e^{2\phi_0}}\, G^{+}_{x_0}.$$

Claim 3 Substituting (1.4.3) into (1.4.4) and equating the regular and singular parts separately, we get:

$$\widetilde{U} = a_1\, \frac{\lambda_0}{\lambda} \frac{1}{e^{2\phi_0}} U \tag{1.4.5}$$

$$\widetilde{V}^{+} = \frac{\lambda_0}{\lambda} \frac{1}{e^{2\phi_0}} V^{+}. \tag{1.4.6}$$

Using the fact that $\delta_{+}(\widetilde{\Gamma}) = \left.\frac{\mathrm{d}\Gamma}{\mathrm{d}\widetilde{\Gamma}}\right|_{\Gamma=0} \delta_{+}(\Gamma) = \frac{1}{a_1}\, \delta_{+}(\Gamma)$, we also have

$$\frac{\widetilde{V}^{+}}{\widetilde{U}} = \frac{1}{a_1}\, \frac{V^{+}}{U}. \tag{1.4.7}$$

Claim 4 For brevity, we write $\Gamma(x_0, x)$ as Γ, and differentiation is performed with respect to $x \in \Omega$. Then, $\widetilde{\Gamma} = 0 \Longleftrightarrow \Gamma = 0$ implies

$$\widetilde{\Gamma} = a_1\Gamma + a_2\Gamma^2 + \cdots\,, \tag{1.4.8}$$

where a_1 is defined to be

$$a_1 = \frac{1}{s}\int_0^{s(x)} e^{2\phi}\mathrm{d}t, \tag{1.4.9}$$

where the integration is carried out along the unique geodesic from x_0 to x with respect to an affine parameter s.

To see this, note that by (1.4.8) and Theorem 1.2.5, we have

$$\begin{aligned}\bar{g}^{\alpha\beta}\widetilde{\Gamma}_{,\alpha}\widetilde{\Gamma}_{,\beta} &= e^{-2\phi}\left(a_1^2\,\overbrace{g^{\alpha\beta}\Gamma_{,\alpha}\Gamma_{,\beta}}^{4\Gamma} + (2\,a_1\,\overbrace{g^{\alpha\beta}a_{1,\alpha}\Gamma_{,\beta}}^{2\frac{\mathrm{d}a_1}{\mathrm{d}s}})\,\Gamma + \mathcal{O}(\Gamma^2)\right)\\ &= 4\,e^{-2\phi}a_1\left(a_1 + s\frac{\mathrm{d}a_1}{\mathrm{d}s}\right)\Gamma + \mathcal{O}(\Gamma^2)\end{aligned} \tag{1.4.10}$$

On the other hand, $\bar{g}^{\alpha\beta}\widetilde{\Gamma}_{,\alpha}\widetilde{\Gamma}_{,\beta} = 4\widetilde{\Gamma}$ (by Theorem 1.2.5 again) implies

$$\bar{g}^{\alpha\beta}\widetilde{\Gamma}_{,\alpha}\widetilde{\Gamma}_{,\beta} = 4\,a_1\Gamma + 4\,a_2\Gamma^2 + \cdots \tag{1.4.11}$$

Thus equating the coefficients of Γ in (1.4.10) and (1.4.11) yields

$$s\frac{\mathrm{d}a_1}{\mathrm{d}s} + a_1 = e^{2\phi},$$

whose obvious solution is given by (1.4.9). **Claim 4** is established.

We now finish the proof of the lemma. By definition,

$$V^{+} = \frac{U}{s} \int_{0}^{s(x)} \frac{P[U]}{U} \mathrm{d}t,$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(s\,\frac{V^{+}}{U} \right) = \left[\frac{P[U]}{U} \right]. \tag{1.4.12}$$

where the outer square brackets on the right hand side indicates restriction to $C_{-}^{\Omega}(x_0)$ only. Similarly, we must have

$$\frac{\mathrm{d}}{\mathrm{d}\widetilde{s}}\left(\widetilde{s}\,\frac{\widetilde{V}^{+}}{\widetilde{U}} \right) = \left[\frac{\widetilde{P}[\widetilde{U}]}{\widetilde{U}} \right], \tag{1.4.13}$$

By (1.4.7), we see that

$$\frac{\mathrm{d}}{\mathrm{d}\widetilde{s}}\left(\widetilde{s}\,\frac{\widetilde{V}^{+}}{\widetilde{U}} \right) = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\widetilde{s}}{a_1}\,\frac{V^{+}}{U} \right) \frac{\mathrm{d}s}{\mathrm{d}\widetilde{s}}. \tag{1.4.14}$$

Claim 5 It can be shown via a straighforward calculation that

$$\widetilde{s} := \int_{0}^{s(x)} e^{2\phi} \mathrm{d}t, \tag{1.4.15}$$

is an affine parameter for the unique null geodesic from x_0 to x with respect to the metric $\bar{g}_{\alpha\beta} = e^{2\phi} g_{\alpha\beta}$. This immediately implies $\frac{\widetilde{s}}{a_1} = s$ and $\frac{\mathrm{d}s}{\mathrm{d}\widetilde{s}} = e^{-2\phi}$. Substituting these into (1.4.14), taking into account (1.4.12), gives

$$\frac{\mathrm{d}}{\mathrm{d}\widetilde{s}}\left(\widetilde{s}\,\frac{\widetilde{V}^{+}}{\widetilde{U}} \right) = \frac{\mathrm{d}}{\mathrm{d}s}\left(s\,\frac{V^{+}}{U} \right) e^{-2\phi} = \left[e^{-2\phi}\,\frac{P[U]}{U} \right]. \tag{1.4.16}$$

Substituting (1.4.5) and (1.4.16) into (1.4.13) yields:

$$\left[e^{-2\phi}\,\frac{P[U]}{U}\right] = \left[\frac{\widetilde{P}[\widetilde{U}]}{a_1\,\frac{\lambda_0}{\lambda}\,\frac{1}{e^{2\phi_0}}U}\right]. \tag{1.4.17}$$

Equivalently,

$$\left[\widetilde{P}[\widetilde{U}]\right] = \left[\frac{1}{e^{2\phi_0}e^{2\phi}}\,\frac{\lambda_0}{\lambda}\,a_1\,P[U]\right]. \tag{1.4.18}$$

$\square$

Definition 1.4.1 *Two scalar wave operators P and $\widetilde{P}$ are said to be* equivalent *if there exist smooth functions λ and ϕ, with λ nowhere vanishing, such that* (1.4.2) *hold.*

1.4.3 The Conformally Invariant Scalar Wave Equation

In §2.1, we will discuss a sequence of necessary conditions for Huygens' principle that have been computed. The first one of these conditions is:

$$B - \frac{1}{2}A^{\alpha}{}_{;\alpha} - \frac{1}{4}A^{\alpha}A_{\alpha} - \frac{1}{6}R = 0,$$

where R is the Ricci scalar of the underlying spacetime. For a self-adjoint equation, it reduces to

$$B = \frac{1}{6}R. \tag{1.4.19}$$

Therefore, we conclude that any self-adjoint scalar wave equation satisfying Huygens' principle must have the form:

$$\Box\, u + \frac{R}{6}\, u = 0. \tag{1.4.20}$$

Equation (1.4.20) is called the *conformally invariant scalar wave equation,* because of the following:

Proposition 1.4.2 *$P := \Box + \frac{R}{6}$ is equivalent to $\bar{P} := \bar{\Box} + \frac{\bar{R}}{6}$ whenever their underlying spacetimes are conformally related.*

Proof First, consider a general scalar wave operator $P := \Box + A^\alpha \nabla_\alpha + B$, and let $\widetilde{P} := \lambda^{-1} e^{-2\phi} P[\lambda\, u]$, as in (1.4.2). It is easy to see that if the principal part of P comes from the metric $g_{\alpha\beta}$, then the principal part of $\widetilde{P}$ comes from $\bar{g}_{\alpha\beta} := e^{2\phi} g_{\alpha\beta}$. Thus, for suitable $\widetilde{A}^\alpha$ and $\widetilde{B}$, we may write

$$\begin{aligned} \widetilde{P}[u] &= \bar{g}^{\alpha\beta}\, u_{\bar{;}\alpha\beta} + \widetilde{A}^\alpha\, u_{,\alpha} + \widetilde{B}\, u \\ &= (\bar{\Box} + \widetilde{A}^\alpha \bar{\nabla}_\alpha + \widetilde{B})\, [u]. \end{aligned}$$

According to McLenaghan [33] and Walton [45], the coefficients of P and $\widetilde{P}$ are related as follows:

$$\bar{g}^{\alpha\beta} = e^{-2\phi}\, g^{\alpha\beta}, \qquad \text{or} \qquad \bar{g}_{\alpha\beta} = e^{2\phi}\, g_{\alpha\beta} \tag{1.4.21}$$

$$\widetilde{A}_\alpha = A_\alpha + 2\nabla_\alpha(\ln\lambda) - 2\,\nabla_\alpha\phi, \quad \text{and} \quad \widetilde{A}^\alpha = \bar{g}^{\alpha\beta}\widetilde{A}_\beta \tag{1.4.22}$$

$$\widetilde{B} = e^{-2\phi}\, \left(B + \lambda^{-1} g^{\alpha\beta}\nabla_\alpha\nabla_\beta\lambda + A^\alpha\nabla_\alpha(\ln\lambda)\right) \tag{1.4.23}$$

Now, suppose $P = \Box + \frac{R}{6}$, and let $(\mathcal{M}, g_{\alpha\beta})$ be the underlying spacetime of P. Let $\phi \in C^\infty(\mathcal{M})$ be given, and define $\lambda := e^\phi$. Let $\widetilde{P}[u] :=$

$\lambda^{-1}e^{-2\phi}P[\lambda\, u] = e^{-3\phi}P[e^{\phi}\, u]$. Then $\widetilde{P}$ is a scalar wave operator on $(\mathcal{M}, \bar{g}_{\alpha\beta})$, where $\bar{g}_{\alpha\beta} = e^{2\phi}g_{\alpha\beta}$. By Theorem 1.4.2, P and $\widetilde{P}$ are equivalent. We shall show that $\widetilde{P} = \bar{\Box} + \frac{\bar{R}}{6} = \bar{P}$.

We have already mentioned that the principal part of $\widetilde{P}$ is $\bar{\Box}$. By the choice of λ, it is immediate from (1.4.22) that $\widetilde{A}^{\alpha} = 0$, i.e. $\widetilde{P}$ is still self-adjoint. Thus, it remains only to prove that $\widetilde{B} = \bar{R}/6$.

The transformation law of the Ricci scalar under a conformal transformation is as follows:

$$\bar{R} = e^{-2\phi}\left(R + 6\, g^{\alpha\beta}\nabla_{\alpha}\nabla_{\beta}\phi + 6\, g^{\alpha\beta}(\nabla_{\alpha}\phi)(\nabla_{\beta}\phi)\right).$$

The derivation of the above can be found, for example, in §3.4, [15]. Dividing through by 6, we get

$$\begin{aligned}\frac{\bar{R}}{6} &= e^{-2\phi}\left(\frac{R}{6} + g^{\alpha\beta}(\nabla_{\alpha}\nabla_{\beta}\phi + (\nabla_{\alpha}\phi)(\nabla_{\beta}\phi))\right)\\ &= e^{-2\phi}\left(B + e^{-\phi}g^{\alpha\beta}\nabla_{\alpha}\nabla_{\beta}\, e^{\phi}\right),\end{aligned} \tag{1.4.24}$$

where we have used (1.4.19) and $\nabla_{\alpha}\nabla_{\beta}\, e^{\phi} = e^{\phi}\,(\nabla_{\alpha}\nabla_{\beta}\,\phi + \nabla_{\alpha}\phi\,\nabla_{\beta}\phi)$ to obtain the last equality. On the other hand, substituting $\lambda = e^{\phi}$ and $A^{\alpha} = 0$ into (1.4.23), we get

$$\widetilde{B} = e^{-2\phi}\left(B + e^{-\phi}g^{\alpha\beta}\nabla_{\alpha}\nabla_{\beta}\, e^{\phi}\right). \tag{1.4.25}$$

Thus (1.4.24) and (1.4.25) together imply

$$\widetilde{B} = \frac{\bar{R}}{6}.$$

□

1.4.4 Hadamard's Problem & the Carminati-McLenaghan Conjecture

The first working conjecture regarding the validity of Huygens' principle was the following one proposed by Hadamard:

Conjecture 1.4.1 (Hadamard's Conjecture) *Every Huygens' operator is equivalent to the ordinary wave operator in Minkowski spacetime.*

Hadamard's conjecture is now known to be false. We have so far restricted our consideration to spacetimes, i.e. 4-dimensional Lorentzian manifolds. In fact the question of the validity of Huygens' principle was originally posed for Lorentzian manifolds of any finite dimension. Hadamard himself proved that Huygens' principle implies that the dimension of the underlying manifold must be even and greater than or equal to four. Stellmacher [44] constructed counter-examples to Hadamard's conjecture for all even dimensions ≥ 4, i.e. Lorentzian manifolds that are not equivalent to any "Minkowskian" manifold (i.e. with a flat Lorentzian metric) but on which Huygens' principle is satisfied.

In 1965, Günther established further counter-examples: Any exact plane wave spacetime, whose metric has the form (in Ehlers-Kundt coordinates, [16]),

$$\mathrm{d}s^2 = 2\mathrm{d}v\left\{\mathrm{d}u + \left(D(v)z^2 + \bar{D}(v)\bar{z}^2 + e(v)z\bar{z}\right)\mathrm{d}v\right\} - 2\mathrm{d}z\mathrm{d}\bar{z},$$

is a non-conformally flat spacetime on which the conformally invariant scalar wave equation satisfies Huygens' principle. See [22].

For many decades, researchers have endeavoured to solve the following:

Hadamard's Problem

Determine all equivalence classes of Huygens' operators modulo the trivial *transformations on the set of all scalar wave operators on spacetimes.*

The following three facts:

(1) The conformally invariant scalar wave equation satisfies Huygens' principle on any conformally flat spacetime and also on any spacetime conformally equivalent to an exact plane wave spacetime. See [30], [25], [6] and [22].

(2) These are the only known spacetimes in which Huygens' principle is valid for the conformally invariant scalar wave equation.

(3) These are the only conformally empty[1] spacetimes on which Huygens' principle is valid for the conformally invariant scalar wave equation. See [32]

have prompted the proposition of the following conjecture by Carminati and McLenaghan:

Conjecture 1.4.2 (Carminati-McLenaghan)

(1) Every non-conformally flat spacetime on which Huygens' principle is valid for the conformally invariant scalar wave equation is conformally equivalent to an exact plane wave spacetime.

(2) Every non-self-adjoint scalar wave equation that satisfies Huygens' principle is equivalent to a self-adjoint equation on a conformally flat spacetime or on an exact plane wave spacetime.

[1] i.e. conformally related to an empty spacetime. A spacetime is said to be *empty* if its Ricci tensor vanishes identically

	Petrov Type					
Equation	I	II	D	III	N	0
conformally invariant			$\times$	$\times$	e.p.w.	$\surd$
non-self-adjoint			?	$\times$	e.p.w.	$\surd$

Table 1.1: Partial results towards the solution of Hadamard's Problem

1.4.5 Summary of Known Results

The Carminati-McLenaghan conjecture is a "negative" conjecture in the sense that it asserts the invalidity of Huygens' principle except when the underlying spacetime is either conformally flat or conformally equivalent to an exact plane wave spacetime.

In the attempt to establish the Carminati-McLenaghan conjecture, researchers have considered separately the disjoint classes of spacetimes by their Petrov types. Necessary conditions for the validity of Huygens' principle have been derived from Hadamard's Criterion, and they have been used to disprove the validity of Huygens' principle for a particular Petrov type of spacetimes by being shown to lead to contradictions in those spacetimes.

A spinor formalism has been commonly used since it offers a number of computational advantages due to the general algebraic properties of spinors which will be discussed in §2.2. Also, the Petrov classification of spacetimes becomes more transparent in the spinor formalism where it can be determined simply by examining the alignment of the principal spinors of the Weyl spinor.

Table 1.1 summarizes the results that have been obtained so far. The symbol $\surd$ indicates the validity of Huygens' principle has been established, while $\times$ indicates the invalidity of Huygens' principle has been proved. The entry e.p.w. under the column for type N indicates that for Huygens' principle to hold on a Petrov type N spacetime, the spacetime must be conformally equivalent to an exact plane wave spacetime. The empty slots indicate that no results have been obtained for the corresponding cases. The case marked with ? is the case considered in this thesis.

The following is the list of citations of the results mentioned above.

- (Mathisson [30], 1939) The non-self-adjoint scalar wave equation in a conformally flat spacetime satisfies Huygens' principle.

- (Günther [22], 1965) Huygens' principle is valid for the conformally invariant scalar wave equation on any conformally flat spacetime and also on any spacetime conformally related to an exact plane wave spacetime, the metric of which has the form (in Ehlers-Kundt coordinates):

$$\mathrm{d}s^2 = 2\mathrm{d}v \left\{ \mathrm{d}u + \left(D(v)z^2 + \bar{D}(v)\bar{z}^2 + e(v)z\bar{z} \right) \mathrm{d}v \right\} - 2\mathrm{d}z\mathrm{d}\bar{z}$$

- (Carminati & McLenaghan [9], 1986) The conformally invariant scalar wave equation on a Petrov Type N spacetime satisfies Huygens' principle if and only if the spacetime is conformally related to an *exact plane wave* spacetime.

- (McLenaghan & Walton [34], 1988) Any non-self-adjoint scalar wave equation satisfies Huygens' principle on a Petrov Type N spacetime if and only if it is equivalent to an wave equation on an exact plane wave spacetime.

- (McLenaghan & Williams [35], 1990) There are no Petrov Type D spacetimes on which the conformally invariant scalar wave equation satisfies Huygens' principle.

- (Anderson & McLenaghan [4], 1994) Derivation of a sixth necessary condition (the 5-index condition) for Huygens' principle.

- (Anderson, McLenaghan & Sasse [5], 1999) Any non-self-adjoint scalar wave equation satisfying Huygens' principle on a Petrov type III space-time is equivalent to a conformally invariant scalar wave equation.

- (Czapor, McLenaghan & Sasse [14], 1999) There are no Petrov Type III spacetimes on which the scalar wave equation (self-adjoint or not) satisfies Huygens' principle.

Chapter 2

The Necessary Conditions

2.1 The Necessary Conditions in Tensor Form

In this section, we present the first six necessary conditions for Huygens' principle derived from Hadamard's Criterion and describe how they can be obtained. Define

$$\sigma := \frac{P[U]}{U}. \tag{2.1.1}$$

Then, since U, being an exponential, is nowhere vanishing, σ vanishes whenever $P[U]$ does. Thus, it is obvious that

$$[\sigma] = 0 \quad \Longleftrightarrow \quad \sigma = 0 \text{ on } C_-^{\Omega}(x_0), \text{ for every } x_0 \in \Omega, \tag{2.1.2}$$

is another necessary and sufficient condition for Huygens' principle.

McLenaghan [31] proved that (2.1.2) has the following consequence:

Lemma 2.1.2 (Lemma 4.1, [31]) *If $\sigma = 0$ on $C_+^{\Omega}(x_0)$ (correspondingly $C_-^{\Omega}(x_0)$) and if $\overset{\circ}{\sigma}_{;\alpha_1 \cdots \alpha_m}$ denotes the m-th order covariant derivative of σ*

evaluated at x_0 ($m = 0$ corresponds to the undifferentiated σ), then

$$\overset{\circ}{\sigma}_{;\alpha_1\cdots\alpha_m} k^{\alpha_1} \cdots k^{\alpha_m} = 0 \tag{2.1.3}$$

for any choice of a future-pointing (correspondingly past-pointing) null vector $k^\alpha \in T_{x_0}\mathcal{M}$.

McLenaghan then showed that (2.1.3) can be stated without reference to the null vector k^α:

Lemma 2.1.3 (Lemma 4.2, [31] and subsequent remarks)
If $\sigma = 0$ on $C_+^\Omega(x_0)$ or on $C_-^\Omega(x_0)$, then for every integer $m \geq 0$, we have

$$\mathrm{TS}[\,\overset{\circ}{\sigma}_{;\alpha_1\cdots\alpha_m}\,] = 0, \tag{2.1.4}$$

where TS[ ] *is the operation of trace-free symmetrization.*[1]

Note that it follows from Lemma 2.1.3 that the forward and backward Huygens' principle give rise to the same set of necessary conditions. For each non-negative integer m, (2.1.4) thus gives a necessary condition of Huygens' principle. The following nomenclature has been established to refer to these necessary conditions:

m	Condition	Name of Condition
0	$\overset{\circ}{\sigma} = 0$	0-index Condition
1	$\overset{\circ}{\sigma}_{;\alpha} = 0$	1-index Condition
2	$\mathrm{TS}[\,\overset{\circ}{\sigma}_{;\alpha\beta}\,] = 0$	2-index Condition
$\vdots$	$\vdots$	$\vdots$
m	$\mathrm{TS}[\,\overset{\circ}{\sigma}_{;\alpha_1\cdots\alpha_m}\,] = 0$	m-index Condition
$\vdots$	$\vdots$	$\vdots$

[1] *See* Notations and Conventions *of this thesis for the definition of the trace-free symmetrization.*

The above equations are useful because their left hand sides can be expressed in terms of the tensors $g_{\alpha\beta}$, A^α and B as appearing in $P := \Box + A^\alpha \nabla_\alpha + B$, (albeit via very long calculations carried out in normal coordinates) thereby giving conditions on these quantities and hence on P, whenever the validity of Huygens' principle is assumed.

The calculations expressing the 0-index, ... , 5-index conditions in terms of $g_{\alpha\beta}$, A^α and B can be found in [31], [3], [45], and [42]. These calculations are lengthy, and at present, only the first six necessary conditions have been computed for the non-self-adjoint scalar wave equation.

To state the necessary conditions in the desired form, we first make the following definitions:

$$
\begin{aligned}
R_{\alpha\beta} &:= R^{\epsilon}{}_{\alpha\beta\epsilon} && (2.1.5)\\
R &:= R^{\alpha}{}_{\alpha} && (2.1.6)\\
L_{\alpha\beta} &:= -R_{\alpha\beta} + \frac{1}{6} g_{\alpha\beta} R && (2.1.7)\\
S_{\alpha\beta\gamma} &:= L_{\alpha[\beta;\gamma]} && (2.1.8)\\
C_{\alpha\beta\gamma\delta} &:= R_{\alpha\beta\gamma\delta} + 2 g_{[\alpha[\delta} L_{\beta]\gamma]} && (2.1.9)\\
H_{\alpha\beta} &:= A_{[\alpha;\beta]} && (2.1.10)
\end{aligned}
$$

Now, the first six necessary conditions can be stated in terms of these quantities:

$$
\begin{aligned}
0 &= B - \frac{1}{2} A^{\alpha}{}_{;\alpha} - \frac{1}{4} A^\alpha A_\alpha - \frac{1}{6} R && (2.1.11)\\
0 &= H^{\nu}{}_{\alpha;\nu} && (2.1.12)\\
0 &= S_{\alpha\beta\nu;}{}^{\nu} - \frac{1}{2} C^{\mu}{}_{\alpha\beta}{}^{\nu} L_{\mu\nu} + 5 \left(H_{\alpha\mu} H_{\beta}{}^{\mu} - \frac{1}{4} g_{\alpha\beta} H_{\mu\nu} H^{\mu\nu} \right) && (2.1.13)\\
0 &= \mathrm{TS}[\, 3 S_{\alpha\beta\mu} H^{\mu}{}_{\gamma} + C^{\mu}{}_{\alpha\beta}{}^{\nu} H_{\gamma\mu;\nu} \,] && (2.1.14)
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\; & \mathrm{TS}[\, 3C_{\mu\alpha\beta\nu;\lambda}C^{\mu}{}_{\gamma\delta}{}^{\nu\lambda}{}_{;} + 8C^{\mu}{}_{\alpha\beta}{}^{\nu}{}_{;\gamma}S_{\mu\nu\delta} + 40S_{\alpha\beta}{}^{\mu}S_{\gamma\delta\mu} \\
& -8C^{\mu}{}_{\alpha\beta}{}^{\nu}S_{\mu\nu\gamma;\delta} - 24C^{\mu}{}_{\alpha\beta}{}^{\nu}S_{\gamma\delta\mu;\nu} + 4C^{\mu}{}_{\alpha\beta}{}^{\nu}C_{\nu}{}^{\lambda}{}_{\gamma\mu}L_{\delta\lambda} \\
& +12C^{\mu}{}_{\alpha\beta}{}^{\nu}C^{\lambda}{}_{\gamma\delta\nu}L_{\mu\lambda} + 12H_{\mu\alpha;\beta\gamma}H^{\mu}{}_{\delta} - 16H_{\mu\alpha;\beta}H^{\mu}{}_{\gamma\delta} \\
& -84H^{\mu}{}_{\alpha}C_{\mu\beta\gamma\nu}H^{\nu}{}_{\delta} - 18H_{\mu\alpha}H^{\mu}{}_{\beta}L_{\gamma\delta}\,]
\end{aligned}
\qquad (2.1.15)
$$

$$
\begin{aligned}
0 \;=\; & \mathrm{TS}[\, 36C^{\mu}{}_{\alpha\beta}{}^{\nu}C_{\nu\gamma\delta\lambda;\mu}H^{\lambda}{}_{\epsilon} - 6C^{\mu}{}_{\alpha\beta}{}^{\nu}{}_{;\gamma}C_{\nu\delta\epsilon}{}^{\lambda}H_{\mu\lambda} \\
& -138S_{\alpha\beta}{}^{\mu}C_{\mu\gamma\delta\nu}H^{\lambda}{}_{\epsilon} + 6S_{\alpha\beta\mu}H^{\mu}{}_{\gamma;\delta\epsilon} + 6C^{\mu}_{\alpha\beta}{}^{\nu}_{;\gamma}H_{\mu\delta;\nu\epsilon} \\
& -24S_{\alpha\beta\mu;\gamma}H^{\mu}{}_{\delta;\epsilon} + 12C^{\mu}{}_{\alpha\beta}{}^{\nu}L_{\mu\gamma}H_{\nu\delta;\epsilon} - 9C^{\mu}{}_{\alpha\beta}{}^{\nu}{}_{;\gamma}L_{\mu\delta}H_{\nu\epsilon} \\
& -9S_{\alpha\beta\mu}L_{\gamma\delta}H^{\mu}{}_{\epsilon}\,]
\end{aligned}
\qquad (2.1.16)
$$

Equations (2.1.11), … , (2.1.14) were obtained by Günther [21]. Equation (2.1.15) was obtained by McLenaghan [33], and equation (2.1.16) was derived by Anderson and McLenaghan [4].

The anti-symmetric rank 2 tensor $H_{\alpha\beta}$ will be referred to as the (associated) *Maxwell tensor* of $P := \Box + A^{\alpha}\nabla_{\alpha} + B$. This is because if P is a Huygens' operator, then $H_{\alpha\beta}$ satisfies (2.1.12), which has the same form as the source-free Maxwell's equations.

2.2 The Spinor Formalism

This section is a summary of the theory of spinor analysis over a spacetime, which uses concepts from the theory of complex vector bundles. For an account of complex vector bundles, see [37].

2.2.1 Spin Structure on a Spacetime

Definition 2.2.1 *Let $(\mathcal{M}, g_{\alpha\beta})$ be spacetime.*

(1) *A* spinor bundle *over $\mathcal{M}$ is a vector bundle S, with base space $\mathcal{M}$, whose fibre is a 2-dimensional vector space over $\mathbb{C}$.*

(2) *Given a spinor bundle S over $\mathcal{M}$, a* spinor field of type $\binom{r\,s}{t\,u}$ *is a smooth section of the vector bundle $S^{r\,s}_{t\,u}$.*

(3) *$\Gamma(S^{r\,s}_{t\,u})$ denotes the set of all spinor fields of type $\binom{r\,s}{t\,u}$.*

Definition 2.2.2 *Suppose there exists a spinor bundle S over $\mathcal{M}$.*

- *A* Levi-Civita spinor field *on $\mathcal{M}$ is a smooth section ϵ of $S^{2\,0}_{0\,0}$ that is anti-symmetric and nowhere vanishing.*
- *A* van der Waerden-Infeld correspondence *is a map σ from the set of all smooth (contravariant) vector fields on M to the set of all Hermitian spinor fields of type $\binom{1\,1}{0\,0}$ which possesses the following properties:*
  - *Linearity: $\sigma(r_1 V_1 + r_2 V_2) = r_1\,\sigma(V_1) + r_2\,\sigma(V_2)$, for all smooth vector fields V_1, V_2 on $\mathcal{M}$ and for all $r_1, r_2 \in \mathbb{R}$.*
  - *For every smooth vector field V on $\mathcal{M}$, we have*

    $$\det(\sigma(V)) = \frac{1}{2}\,|\det(\epsilon)|\,g(V,V),$$

    with respect to any local basis of the fibre.

Hereinafter, Penrose's abstract index notation will be employed throughout. A Levi-Civita spinor field will then be denoted by ϵ^{AB} and a van der Waerden-Infeld correspondence by $\sigma_\alpha^{\ A\dot{A}}$.

Definition 2.2.3 *A* spin structure *on* $(\mathcal{M}, g_{\alpha\beta})$ *is a triple* $(S, \epsilon^{AB}, \sigma_\alpha^{\ A\dot{A}})$ *where* S *is spinor bundle over* $\mathcal{M}$*,* $\epsilon^{AB} \in S^{\,2\,0}_{\,0\,0}$ *is a Levi-Civita spinor field on* $\mathcal{M}$ *and* $\sigma_\alpha^{\ A\dot{A}}$ *is a van der Waerden-Infeld correspondence between the smooth fields on* $\mathcal{M}$ *and the Hermitian spinor fields in* $S^{\,1\,1}_{\,0\,0}$*.*

2.2.2 Spinor Algebra

We will make use of the following well-known facts from spinor algebra. Note that each statement in this section holds at every point of the spacetime $\mathcal{M}$, i.e. the spinors in this section do not have to be smooth spinor fields on $\mathcal{M}$. For proofs of these results, see [19], [23] and [40].

(1) The van der Waerden-Infeld correspondence is a bijection between the set of vectors and the set of Hermitian $\binom{1\,1}{0\,0}$-spinors. This correspondence extends in an obvious way from tensors to Hermitian spinors. As a result, we call the image of a tensor under the van der Waerden-Infeld correspondence its spinor equivalent. For example, if $T^{\alpha\beta}$ is a tensor, then $\sigma_{\alpha A\dot{A}}\sigma_{\beta B\dot{B}}T^{\alpha\beta}$ is its spinor equivalent.

(2) Every spinor dyad $\{o^A, \iota^A\}$ determines a complex null tetrad on $\mathcal{M}$ as follows:

$$
\begin{aligned}
l^\alpha &:= \sigma^\alpha_{A\dot{A}}\, o^A \overline{o}^{\dot{A}}, & \qquad n^\alpha &:= \sigma^\alpha_{A\dot{A}}\, \iota^A \overline{\iota}^{\dot{A}}, \\
m^\alpha &:= \sigma^\alpha_{A\dot{A}}\, o^A \overline{\iota}^{\dot{A}}, & \qquad \overline{m}^\alpha &:= \sigma^\alpha_{A\dot{A}}\, \iota^A \overline{o}^{\dot{A}}.
\end{aligned}
$$

(3) Every spinor of the form $T_{A_1\cdots A_m}$ is the sum of the totally symmetric spinor $T_{(A_1\cdots A_m)}$ and direct products of ϵ_{AB}'s with totally symmetric

spinors of lower valence.

(4) Every totally symmetric spinor $S_{A_1 \cdots A_m}$ can be written in the form:

$$S_{A_1 \cdots A_m} = \overset{1}{\alpha}_{(A_1} \cdots \overset{1}{\alpha}_{A_m)}, \tag{2.2.1}$$

for some m $\binom{0\,0}{1\,0}$-spinors, $\overset{1}{\alpha}_{A_1}, \ldots, \overset{1}{\alpha}_{A_m}$, which are called the *principal spinors* of $S_{A_1 \cdots A_m}$. Furthermore, the principal spinors of a totally symmetric spinor are unique up to ordering and scaling factors.

(5) If a tensor $R_{\alpha\beta\gamma\delta}$ has the following symmetries:

$$R_{\alpha\beta\gamma\delta} = R_{[\alpha\beta][\gamma\delta]} = R_{[\gamma\delta][\alpha\beta]}, \qquad R_{\alpha[\beta\gamma\delta]} = 0, \tag{2.2.2}$$

then its spinor equivalent has the form:

$$\begin{aligned} -\Psi_{ABCD}\overline{\epsilon}_{\dot{A}\dot{B}}\overline{\epsilon}_{\dot{C}\dot{D}} - \epsilon_{AB}\epsilon_{CD}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}} - \Phi_{AB\dot{C}\dot{D}}\epsilon_{CD}\overline{\epsilon}_{\dot{A}\dot{B}} - \Phi_{CD\dot{A}\dot{B}}\epsilon_{AB}\overline{\epsilon}_{\dot{C}\dot{D}} \\ -2\,\Lambda\,(\epsilon_{AC}\epsilon_{BD}\overline{\epsilon}_{\dot{A}\dot{B}}\overline{\epsilon}_{\dot{C}\dot{D}} + \epsilon_{AB}\epsilon_{CD}\overline{\epsilon}_{\dot{A}\dot{D}}\overline{\epsilon}_{\dot{B}\dot{C}}), \end{aligned} \tag{2.2.3}$$

where Ψ_{ABCD} is a totally symmetric spinor, $\Phi_{AB\dot{A}\dot{B}}$ is a Hermitian spinor symmetric in each pair of its indices, and Λ is a scalar.

(6) The curvature tensor $R_{\alpha\beta\gamma\delta}$ has the symmetries in (2.2.2), and if its spinor equivalent is expressed as in (2.2.3), then the spinor equivalents of the Ricci tensor $R_{\alpha\beta}$, the Ricci scalar R, and the Weyl tensor $C_{\alpha\beta\gamma\delta}$ are as follows:

$$R_{\alpha\beta} \quad \longleftrightarrow \quad -2\,\Phi_{AB\dot{A}\dot{B}} + 6\,\Lambda\,\epsilon_{AB}\overline{\epsilon}_{\dot{A}\dot{B}}, \tag{2.2.4}$$

$$R \quad = \quad 24\,\Lambda, \tag{2.2.5}$$

$$C_{\alpha\beta\gamma\delta} \quad \longleftrightarrow \quad -\Psi_{ABCD}\overline{\epsilon}_{\dot{A}\dot{B}}\overline{\epsilon}_{\dot{C}\dot{D}} - \epsilon_{AB}\epsilon_{CD}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}. \tag{2.2.6}$$

Type	Ψ_{ABCD}	Alignment	Vanishing Components
I	$\alpha_{(A}\,\beta_B\,\gamma_C\,\delta_{D)}$		none
II	$\alpha_{(A}\,\alpha_B\,\beta_C\,\gamma_{D)}$		Ψ_0, Ψ_1
D	$\alpha_{(A}\,\alpha_B\,\beta_C\,\beta_{D)}$		$\Psi_0, \Psi_1, \Psi_3, \Psi_4$
III	$\alpha_{(A}\,\alpha_B\,\alpha_C\,\beta_{D)}$		Ψ_0, Ψ_1, Ψ_2
N	$\alpha_{(A}\,\alpha_B\,\alpha_C\,\alpha_{D)}$		$\Psi_0, \Psi_1, \Psi_2, \Psi_3$
0	0		all

Table 2.1: The Petrov Classification

2.2.3 Petrov Classification of Spacetimes

The spinor Ψ_{ABCD} in (2.2.6) is called the Weyl spinor. By (2.2.1), it can be decomposed into four principal spinors. One method of defining the Petrov classification of spacetimes is by the alignment of the principal spinors of the Weyl spinor. For example, a spacetime whose four Weyl principal spinors are pairwise independent is said to be of Petrov type I. If two of the four Weyl principal spinors are aligned, with the other two independent, then the spacetime is said to be of Petrov type II. The rest of the classification continues analogously. This classification scheme is summarized in Table 2.1.

Since the Weyl spinor Ψ_{ABCD} is totally symmetric, with respect to any spinor dyad $\{\zeta^i_A\}_{i=0,1}$, it can be expressed as follows:

$$\begin{aligned}\Psi_{ABCD} \;=\; & \Psi_{0000}\,\zeta^1_{(A}\zeta^1_B\zeta^1_C\zeta^1_{D)} - 4\Psi_{0001}\,\zeta^1_{(A}\zeta^1_B\zeta^1_C\zeta^0_{D)} + 6\Psi_{0011}\,\zeta^1_{(A}\zeta^1_B\zeta^0_C\zeta^0_{D)} \\ & -4\Psi_{0111}\,\zeta^1_{(A}\zeta^0_B\zeta^0_C\zeta^0_{D)} + \Psi_{1111}\,\zeta^0_{(A}\zeta^0_B\zeta^0_C\zeta^0_{D)},\end{aligned}$$

where Ψ_{0000}, … ,Ψ_{1111} are the spinor components of Ψ_{ABCD} with respect to the dyad $\{\zeta^i_A\}_{i=0,1}$. We introduce the notational convention:

$$\Psi_0 := \Psi_{0000}, \quad \Psi_1 := \Psi_{0001}, \quad \Psi_2 := \Psi_{0011}, \quad \Psi_3 := \Psi_{0111}, \quad \Psi_4 := \Psi_{1111}.$$

Definition 2.2.4 *We say a spinor dyad is* canonical *to* Ψ_{ABCD} *if each of its degenerate principal spinors is aligned with one of the dyad spinors.*

The advantage in tackling the question of the validity of Huygens' principle by considering the distinct Petrov types separately lies in the fact that some of the Weyl spinor components vanish with respect to any dyad that is canonical to the Weyl spinor, thereby simplifying the necessary conditions (in spinor form) for Huygens' principle listed in the next section. The vanishing components (with respect to a canonical dyad) of the Weyl spinor in each Petrov type are shown in the third column in Table 2.1.

2.2.4 Spinor Analysis

We compile a list of the results and notations from spinor analysis that will be useful in the sequel. For proofs of these results, see [19], [23] and [40].

(1) If S is a spinor bundle on $\mathcal{M}$ which admits a spin structure, then there exists a unique linear connection D on S and a unique linear connection $\overline{D}$ on $\overline{S}$ which have the following properties:

(a) $\overline{D}_X(\overline{\xi}^{\dot{A}}) = \overline{D_X(\xi^A)}$ for all vector fields X, and for all $\binom{1\,0}{0\,0}$-spinor fields ξ^A on $\mathcal{M}$,

(b) $D_X(\epsilon^{AB}) = 0$, for any vector field X on $\mathcal{M}$,

(c) $D_X(\sigma(Y)) = \sigma(\nabla_X Y)$,

where ∇ is the Levi-Civita connection on $\mathcal{M}$. It is clear that there is an obvious way of extending D and $\overline{D}$ to spinor fields of higher valences.

In the sequel, we will follow the convention of denoting both D and $\overline{D}$ with ∇.

(2) Define

$$\nabla_{A\dot{A}} := \sigma^{\alpha}_{A\dot{A}} \nabla_{\frac{\partial}{\partial x^{\alpha}}}$$

The fact that the operator $\nabla_{A\dot{A}}$ is independent of the coordinates x^{α} can be seen as follows: Let x^{α} and y^{β} be two different local coordinate systems. Then

$$\begin{aligned} \sigma^{\beta}_{A\dot{A}} \nabla_{\frac{\partial}{\partial y^{\beta}}} &= \sigma^{\beta}_{A\dot{A}} \nabla_{\frac{\partial x^{\alpha}}{\partial y^{\beta}} \frac{\partial}{\partial x^{\alpha}}} \\ &= \sigma^{\beta}_{A\dot{A}} \frac{\partial x^{\alpha}}{\partial y^{\beta}} \nabla_{\frac{\partial}{\partial y^{\beta}}} \\ &= \sigma^{\alpha}_{A\dot{A}} \nabla_{\frac{\partial}{\partial x^{\alpha}}} \end{aligned}$$

This shows that $\nabla_{A\dot{A}}$ is independent of coordinate systems.

(3) Recall that a spinor dyad $\{o^A, \iota^A\}$ determines a complex null tetrad $\{l^{\alpha}, n^{\alpha}, m^{\alpha}, \overline{m}^{\alpha},\}$ on $\mathcal{M}$. Therefore, $\{o^A, \iota^A\}$ — now they need to be smooth spinor fields — also determines four directional derivative operators:

$$\begin{aligned} D &:= l^{\alpha}\nabla_{\partial/\partial x^{\alpha}} & \Delta &:= n^{\alpha}\nabla_{\partial/\partial x^{\alpha}} \\ \delta &:= m^{\alpha}\nabla_{\partial/\partial x^{\alpha}} & \overline{\delta} &:= \overline{m}^{\alpha}\nabla_{\partial/\partial x^{\alpha}} \end{aligned}$$

The operators D, Δ, δ, and $\overline{\delta}$ are called the Pfaffian operators of the spinor dyad $\{o^A, \iota^A\}$.

(4) It is often convenient to use a component notation to denote the Pfaffians. To this end, we proceed as follows: Given any spinor dyad

$\{\zeta_a^A\}_{a=0,1}$, define

$$\partial_{a\dot{a}} := \zeta_a^A \overline{\zeta}_{\dot{a}}^{\dot{A}} \nabla_{A\dot{A}}$$

Then these operators coincide with the Pfaffian operators as follows:

$$\begin{array}{rclcl}
D & := & l^\alpha \nabla_{\partial/\partial x^\alpha} & = & (\sigma^\alpha_{A\dot{A}} \zeta_0^A \overline{\zeta}_{\dot{0}}^{\dot{A}}) \nabla_\alpha = \zeta_0^A \overline{\zeta}_{\dot{0}}^{\dot{A}} \nabla_{A\dot{A}} = \partial_{0\dot{0}} \\
\Delta & := & n^\alpha \nabla_{\partial/\partial x^\alpha} & = & (\sigma^\alpha_{A\dot{A}} \zeta_1^A \overline{\zeta}_{\dot{1}}^{\dot{A}}) \nabla_\alpha = \zeta_1^A \overline{\zeta}_{\dot{1}}^{\dot{A}} \nabla_{A\dot{A}} = \partial_{1\dot{1}} \\
\delta & := & m^\alpha \nabla_{\partial/\partial x^\alpha} & = & (\sigma^\alpha_{A\dot{A}} \zeta_0^A \overline{\zeta}_{\dot{1}}^{\dot{A}}) \nabla_\alpha = \zeta_0^A \overline{\zeta}_{\dot{1}}^{\dot{A}} \nabla_{A\dot{A}} = \partial_{0\dot{1}} \\
\overline{\delta} & := & \overline{m}^\alpha \nabla_{\partial/\partial x^\alpha} & = & (\sigma^\alpha_{A\dot{A}} \zeta_1^A \overline{\zeta}_{\dot{0}}^{\dot{A}}) \nabla_\alpha = \zeta_1^A \overline{\zeta}_{\dot{0}}^{\dot{A}} \nabla_{A\dot{A}} = \partial_{1\dot{0}}
\end{array}$$

(5) Let $\{\zeta_0^A, \zeta_1^A\}$ be any spinor dyad. The spinor coefficients with respect to the ζ_a^A are defined to be:

$$\Gamma_{abc\dot{c}} := \zeta_b^B \, \zeta_c^C \, \overline{\zeta}_{\dot{c}}^{\dot{C}} \, \nabla_{C\dot{C}} \zeta_{aB},$$

where x^α are any local coordinates. The spin coefficients satisfy

$$\Gamma_{abc\dot{c}} = \Gamma_{bac\dot{c}}.$$

Thus, with respect to each given spinor dyad, there are only twelve independent spin coefficients; these are usually denoted by

$$\begin{array}{llll}
\kappa := \Gamma_{000\dot{0}} & \sigma := \Gamma_{000\dot{1}} & \rho := \Gamma_{001\dot{0}}, & \tau := \Gamma_{001\dot{1}} \\
\epsilon := \Gamma_{010\dot{0}} & \beta := \Gamma_{010\dot{1}} & \alpha := \Gamma_{011\dot{0}}, & \gamma := \Gamma_{011\dot{1}} \\
\pi := \Gamma_{110\dot{0}} & \mu := \Gamma_{110\dot{1}} & \lambda := \Gamma_{111\dot{0}}, & \nu := \Gamma_{111\dot{1}}
\end{array}$$

(6) The spin coefficients with respect to any spinor dyad satisfy the following equations:

(a) **Commutation Relations**: For any smooth scalar function ϕ on $\mathcal{M}$, we have

$$[\partial_{a\dot{b}}\partial_{c\dot{d}} - \partial_{c\dot{d}}\partial_{a\dot{b}}]\ \phi = \left[\epsilon^{fe}(\Gamma_{fac\dot{d}}\partial_{fca\dot{b}} - \Gamma_{fca\dot{b}}\partial_{e\dot{d}}) + \bar{\epsilon}^{\dot{f}\dot{e}}(\overline{\Gamma}_{\dot{f}\dot{b}dc}\partial_{a\dot{e}} - \overline{\Gamma}_{\dot{f}\dot{d}\dot{b}a}\partial_{c\dot{e}})\right]\ \phi \qquad (2.2.7)$$

(b) **Newman-Penrose Field Equations**

$$\begin{aligned}\partial_{f\dot{e}}\Gamma_{acd\dot{b}} - \partial_{d\dot{b}}\Gamma_{acf\dot{e}} \quad = \quad & \epsilon^{pq}\left(\Gamma_{apd\dot{b}}\Gamma_{qcf\dot{e}} + \Gamma_{acp\dot{b}}\Gamma_{qdf\dot{e}} - \Gamma_{apf\dot{e}}\Gamma_{qcd\dot{b}} - \Gamma_{acp\dot{e}}\Gamma_{qfd\dot{b}}\right)\\ & +\bar{\epsilon}^{\dot{r}\dot{s}}\left(\Gamma_{acd\dot{r}}\overline{\Gamma}_{\dot{s}\dot{b}\dot{e}f} - \Gamma_{acf\dot{r}}\overline{\Gamma}_{\dot{s}\dot{e}\dot{b}d}\right) + \Psi_{acdf}\bar{\epsilon}_{\dot{e}\dot{b}}\\ & +\Lambda\,\bar{\epsilon}_{\dot{e}\dot{b}}(\epsilon_{cd}\epsilon_{af} + \epsilon_{ad}\epsilon_{cf}) + \Phi_{ac\dot{b}\dot{e}}\epsilon_{fd} \qquad (2.2.8)\end{aligned}$$

The component form of (2.2.7) and (2.2.8) with respect to a spinor dyad are displayed in Appendix A.

(7) **The Bianchi Identities in Spinor Form**

The Bianchi Identities

$$\nabla_{[\mu}R_{\alpha\beta]\gamma\delta} = 0$$

in spinor form have the form:

$$\left\{\begin{aligned}\nabla^{D}{}_{\dot{A}}\,\Phi_{ABCD} \quad &= \quad \nabla_{(C}{}^{\dot{B}}\Phi_{AB)\dot{A}\dot{B}},\\ \nabla^{B\dot{B}}\,\Phi_{AB\dot{A}\dot{B}} \quad &= \quad -3\,\nabla_{A\dot{A}}\Lambda,\end{aligned}\right. \qquad (2.2.9)$$

When expressed with respect to a spinor dyad, (2.2.9) takes the form:

$$\left\{\begin{array}{lcl} \partial^{p}{}_{\dot{d}}\Psi_{abcp} - \partial_{(c}{}^{\dot{t}}\,\Phi_{ab)\dot{d}\dot{t}} & = & \left(3\Psi_{pr(ab}\Gamma_{c)}{}^{pr}{}_{\dot{d}} + \Psi_{abcp}\Gamma^{p}{}_{r}{}^{r}{}_{\dot{d}}\right) - 2\Gamma^{p}{}_{(ab}{}^{\dot{t}}\Phi_{c)p\dot{t}\dot{d}} \\ & & - \left(\overline{\Gamma}_{\dot{t}\dot{d}\dot{p}(a}\Phi_{bc)}{}^{\dot{t}\dot{p}} + \overline{\Gamma}_{\dot{t}}{}^{\dot{p}}{}_{\dot{p}(a}\Phi_{bc)}{}^{\dot{t}}{}_{\dot{d}}\right), \\ & & \\ 3\,\partial_{a\dot{b}}\Lambda + \partial^{p\dot{t}}\Phi_{ap\dot{b}\dot{t}} & = & \epsilon^{\dot{v}\dot{w}}\left(\Phi_{ap\dot{b}}{}^{\dot{t}}\overline{\Gamma}_{\dot{t}\dot{w}\dot{v}}{}^{p} + \Phi_{ap}{}^{\dot{t}}{}_{\dot{w}}\overline{\Gamma}_{\dot{b}\dot{t}\dot{v}}{}^{p}\right) \\ & & - \left(\Phi_{pr\dot{b}}{}^{\dot{t}}\Gamma_{a}{}^{pr}{}_{\dot{t}} + \Phi_{ap\dot{b}}{}^{\dot{t}}\Gamma^{p}{}_{r}{}^{r}{}_{\dot{t}}\right). \end{array}\right. \tag{2.2.10}$$

The individual component equations of (2.2.10) are displayed in Appendix A.

2.3 The Necessary Conditions in Spinor Form

In Chapters 4 to 6, we will make extensive use of the dyad form of the spinor necessary conditions for Huygens' principle. This section lists the 0-index to the 5-index condition in spinor form, which can be obtained via a straightforward conversion process from their tensorial counterparts given in §2.1. The dyad forms of these equations, which can be obtained through contractions with appropriate dyad spinors, are displayed in Appendix A.

The terms defined in (2.1.5), ... , (2.1.10) have the following spinor equivalents:

$$
\begin{aligned}
R_{\alpha\beta} &\longleftrightarrow -2\,\Phi_{AB\dot{A}\dot{B}} + 6\,\Lambda\,\epsilon_{AB}\overline{\epsilon}_{\dot{A}\dot{B}} &\quad (2.3.1)\\
R &\longleftrightarrow 24\,\Lambda &\quad (2.3.2)\\
L_{\alpha\beta} &\longleftrightarrow 2\Phi_{AB\dot{A}\dot{B}} - 2\Lambda\epsilon_{AB}\overline{\epsilon}_{\dot{A}\dot{B}} &\quad (2.3.3)\\
S_{\alpha\beta\gamma} &\longleftrightarrow \Psi^{D}{}_{ABC;D\dot{A}}\epsilon_{\dot{C}\dot{B}} + \overline{\Psi}^{\dot{D}}{}_{\dot{A}\dot{B}\dot{C};\dot{D}A}\epsilon_{CB} &\quad (2.3.4)\\
C_{\alpha\beta\gamma\delta} &\longleftrightarrow -\Psi_{ABCD}\overline{\epsilon}_{\dot{A}\dot{B}}\overline{\epsilon}_{\dot{C}\dot{D}} - \epsilon_{AB}\epsilon_{CD}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}} &\quad (2.3.5)\\
H_{\alpha\beta} &\longleftrightarrow \epsilon_{AB}\overline{\phi}_{\dot{A}\dot{B}} + \overline{\epsilon}_{\dot{A}\dot{B}}\phi_{AB} &\quad (2.3.6)
\end{aligned}
$$

Using the above conversions, the spinor equivalents of the equations, (2.1.11), ... , (2.1.16) — the tensor form of the necessary conditions for Huygens' principle — can be directly computed, and they are as follows:

$$0 = B - \frac{1}{2}A^{\alpha}{}_{;\alpha} - \frac{1}{4}A^{\alpha}A_{\alpha} - \frac{1}{6}R \tag{2.3.7}$$

$$0 = \phi_{AK;}{}^{K}{}_{\dot{A}} \tag{2.3.8}$$

$$\begin{aligned} 0 = {} & \Psi_{ABKL}{}^{K}{}_{\dot{A}}{}^{L}{}_{\dot{B}} + \overline{\Psi}_{\dot{A}\dot{B}\dot{K}\dot{L}}{}^{\dot{K}}{}_{A}{}^{\dot{L}}{}_{B} + \\ & \Psi_{AB}{}^{KL}\Phi_{KL\dot{A}\dot{B}} + \overline{\Psi}_{\dot{A}\dot{B}}{}^{\dot{K}\dot{L}}\Phi_{AB\dot{K}\dot{L}} + 10\phi_{AB}\overline{\phi}_{\dot{A}\dot{B}} \end{aligned} \tag{2.3.9}$$

$$\begin{aligned} 0 = {} & 3\Psi_{ABCK;}{}^{K}{}_{(\dot{A}}\phi_{\dot{B}\dot{C})} + 3\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{K};}{}^{\dot{K}}{}_{(A}\phi_{BC)} \\ & -\Psi_{ABC}{}^{K}\overline{\phi}_{(\dot{A}\dot{B};\dot{C})K} - \overline{\Psi}_{\dot{A}\dot{B}\dot{C}}{}^{\dot{K}}\phi_{(AB;C)\dot{K}} \end{aligned} \tag{2.3.10}$$

$$\begin{aligned} 0 = {} & 3\Psi_{ABCD;K\dot{K}}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D};}{}^{K\dot{K}} - 40\Psi_{(ABC|K|;}{}^{K}{}_{(\dot{A}}\overline{\Psi}_{\dot{B}\dot{C}\dot{D})\dot{K};}{}^{\dot{K}}{}_{D)} \\ & +4\Psi^{K}{}_{(ABC;D)(\dot{A}}\overline{\Psi}_{\dot{B}\dot{C}\dot{D})\dot{L};K}{}^{\dot{L}} + 4\overline{\Psi}^{\dot{K}}{}_{(\dot{A}\dot{B}\dot{C};\dot{D})(A}\Psi_{BCD)L;\dot{K}}{}^{L} \\ & -4\Psi^{K}{}_{(ABC}\overline{\Psi}_{(\dot{A}\dot{B}\dot{C}|\dot{K};K|}{}^{\dot{K}}{}_{\dot{D})D)} - 4\overline{\Psi}^{\dot{K}}{}_{(\dot{A}\dot{B}\dot{C}}\Psi_{(ABC|K;\dot{K}|}{}^{K}{}_{D)\dot{D})} \\ & +12\Psi^{K}{}_{(ABC}\overline{\Psi}_{(\dot{A}\dot{B}\dot{C}|\dot{K}|;D)}{}^{\dot{K}}{}_{K\dot{D})} + 12\overline{\Psi}^{\dot{K}}{}_{(\dot{A}\dot{B}\dot{C}}\Psi_{(ABC|K|;\dot{D})}{}^{K}{}_{\dot{K}D)} \\ & -16\Psi^{K}{}_{(ABC}\Phi_{D)K\dot{K}(\dot{A}}\overline{\Psi}_{\dot{B}\dot{C}\dot{D})}{}^{\dot{K}} - 32\Lambda\Psi_{ABCD}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}} \\ & -6\phi_{(AB;CD)(\dot{C}\dot{D}}\overline{\phi}_{\dot{A}\dot{B})} - 6\overline{\phi}_{(\dot{A}\dot{B};\dot{C}\dot{D})(CD}\phi_{AB)} \\ & -42\phi_{(AB}\phi_{CD)}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}} - 42\overline{\phi}_{(\dot{A}\dot{B}}\overline{\phi}_{\dot{C}\dot{D})}\Psi_{ABCD} \\ & +16\phi_{(AB;C(\dot{C}}\overline{\phi}_{\dot{A}\dot{B};\dot{D})D)} + 36\phi_{(AB}\Phi_{CD)(\dot{C}\dot{D})}\overline{\phi}_{\dot{A}\dot{B}} \end{aligned} \tag{2.3.11}$$

$$\begin{aligned} 0 = {} & \mathrm{S}[-6\Psi^{K}{}_{ABC;K\dot{A}}\overline{\phi}_{\dot{B}\dot{C};D\dot{D}E\dot{E}} + 6\Psi_{ABC}{}^{K}{}_{;D\dot{D}}\overline{\phi}_{\dot{A}\dot{B};K\dot{C}E\dot{E}} \\ & +24\Psi^{K}{}_{ABC;K\dot{A}D\dot{D}}\overline{\phi}_{\dot{B}\dot{C};E\dot{E}} + 24\Psi^{K}{}_{ABC}\Phi_{KD\dot{A}\dot{D}}\overline{\phi}_{\dot{B}\dot{C};E\dot{E}} \\ & -18\Psi^{K}{}_{ABC;E\dot{E}}\Phi_{KD\dot{A}\dot{D}}\overline{\phi}_{\dot{B}\dot{C}} + 18\Psi^{K}{}_{ABC;K\dot{A}}\Phi_{DE\dot{D}\dot{E}}\overline{\phi}_{\dot{B}\dot{C}} \\ & -36\Psi^{K}{}_{ABC}\overline{\Psi}_{\dot{B}\dot{C}\dot{D}\dot{E};K\dot{A}}\phi_{DE} - 138\Psi^{K}{}_{ABC\ K\dot{A}}\overline{\Psi}_{\dot{B}\dot{C}\dot{D}\dot{E}}\phi_{DE} \\ & +6\Psi^{K}{}_{ABC;D\dot{D}}\overline{\Psi}_{\dot{B}\dot{C}\dot{E}\dot{A}}\phi_{KE} + 6\Psi_{ABCD;E\dot{E}}\overline{\Psi}^{\dot{K}}{}_{\dot{A}\dot{B}\dot{C}}\overline{\phi}_{\dot{K}\dot{D}} \\ & + \mathrm{c.c.}] \end{aligned} \tag{2.3.12}$$

The following is a stronger form of equation (2.3.9):

$$0 = \Psi_{ABKL;}{}^{K}{}_{\dot{A}}{}^{L}{}_{\dot{B}} + \Psi_{AB}{}^{KL}\,\Phi_{KL\dot{A}\dot{B}} + 5\,\phi_{AB}\overline{\phi}_{\dot{A}\dot{B}}. \tag{2.3.13}$$

It was obtained by Wünsch [46] and McLenaghan and Williams [35], and it will be used in the sequel instead of (2.3.9).

2.4 Transformation Laws of NP Quantities

Recall that the Huygens' nature of an operator $P := \Box + A^\alpha \nabla_\alpha + B$ is invariant under the trivial transformations. We will exploit this freedom in Chapters 4 and 6 to simplify the necessary conditions. In this section, we shall present the transformation laws of the the spin coefficients, the curvature spinor components, Weyl spinor components and the Maxwell spinor components under a dyad transformation or a conformal transformation. The derivations of these transformation laws can be found in [3] and [45].

2.4.1 Dyad Transformations

A dyad transformation is of the form:

$$o' = e^{w/2}\, o \qquad \iota' = e^{-w/2}(\iota + q o). \tag{2.4.1}$$

To obtain the component transformation laws for Ψ_{ABCD}, ϕ_{AB}, and $\Phi_{AB\dot{A}\dot{B}}$, we simply need to contract the respective spinors with the transformed dyad. For example,

$$\begin{aligned}
\Psi'_0 &= \Psi_{ABCD}\; o'^A o'^B o'^C o'^D \\
&= \Psi_{ABCD}\; (e^{w/2} o^A)\,(e^{w/2} o^B)\,(e^{w/2} o^C)\,(e^{w/2} o^D) \\
&= (e^{w/2})^4\; \Psi_{ABCD}\; o^A o^B o^C o^D \\
&= e^{2w}\Psi_0.
\end{aligned}$$

The transformation laws of all other components of Ψ_{ABCD}, ϕ_{AB}, and $\Phi_{AB\dot{A}\dot{B}}$ can be derived the same way. For later reference, they are listed below:

$$
\begin{aligned}
\Psi_0' &= \Psi_0 e^{2w} && (2.4.2)\\
\Psi_1' &= e^{w}(\Psi_1 + \Psi_0 q) && (2.4.3)\\
\Psi_2' &= \Psi_2 + 2\Psi_1 q + \Psi_0 q^2 && (2.4.4)\\
\Psi_3' &= e^{-w}(3\Psi_2 q + 3\Psi_1 q^2 + \Psi_0 q^3 + \Psi_3) && (2.4.5)\\
\Psi_4' &= e^{-2w}(4\Psi_3 q + 6\Psi_2 q^2 + 4\Psi_1 q^3 + \Psi_4 + \Psi_0 q^4) && (2.4.6)
\end{aligned}
$$

$$
\begin{aligned}
\phi_0' &= \phi_0 e^{w} && (2.4.7)\\
\phi_1' &= \phi_0 q + \phi_1 && (2.4.8)\\
\phi_2' &= e^{-w}(2\phi_1 q + \phi_2 + \phi_0 q^2) && (2.4.9)
\end{aligned}
$$

$$
\begin{aligned}
\Phi_{00}' &= \Phi_{00} e^{(w+\overline{w})} && (2.4.10)\\
\Phi_{01}' &= e^{w}(\Phi_{01} + \Phi_{00}\overline{q}) && (2.4.11)\\
\Phi_{02}' &= e^{(w-\overline{w})}(2\Phi_{01}\overline{q} + \Phi_{00}\overline{q}^2 + \Phi_{02}) && (2.4.12)\\
\Phi_{11}' &= \Phi_{01} q + \Phi_{11} + \Phi_{00} q\overline{q} + \Phi_{10}\overline{q} && (2.4.13)\\
\Phi_{12}' &= e^{(-\overline{w})}(2\Phi_{01} q\overline{q} + 2\Phi_{11}\overline{q} + \Phi_{00} q\overline{q}^2 + \Phi_{10}\overline{q}^2 \\
&\quad +\Phi_{02} q + \Phi_{12}) && (2.4.14)\\
\Phi_{22}' &= e^{(-w-\overline{w})}(4\Phi_{11} q\overline{q} + 2\Phi_{21}\overline{q} + 2\Phi_{01} q^2\overline{q} + 2\Phi_{10} q\overline{q}^2 \\
&\quad +\Phi_{20}\overline{q}^2 + \Phi_{00} q^2\overline{q}^2 + 2\Phi_{12} q + \Phi_{22} + \Phi_{02} q^2) && (2.4.15)
\end{aligned}
$$

To derive the transformation laws for the spin coefficients under a dyad transformation, first note that if we define

$$
\begin{aligned}
I_{B\dot{B}} &:= \gamma\, o_B \overline{o}_{\dot{B}} - \alpha\, o_B \overline{\iota}_{\dot{B}} - \beta\, \iota_B \overline{o}_{\dot{B}} + \varepsilon\, \iota_B \overline{\iota}_{\dot{B}} \\
II_{B\dot{B}} &:= -\tau\, o_B \overline{o}_{\dot{B}} + \rho\, o_B \overline{\iota}_{\dot{B}} + \sigma\, \iota_B \overline{o}_{\dot{B}} - \kappa\, \iota_B \overline{\iota}_{\dot{B}} \\
III_{B\dot{B}} &:= \nu\, o_B \overline{o}_{\dot{B}} - \lambda\, o_B \overline{\iota}_{\dot{B}} - \mu\, \iota_B \overline{o}_{\dot{B}} + \pi\, \iota_B \overline{\iota}_{\dot{B}},
\end{aligned}
$$

then, we can express $\nabla_{B\dot{B}}\, o_A$ and $\nabla_{B\dot{B}}\, \iota_A$ as follows:

$$
\begin{aligned}
\nabla_{B\dot{B}}\, o_A &= I_{B\dot{B}}\, o_A + II_{B\dot{B}}\, \iota_A \\
\nabla_{B\dot{B}}\, \iota_A &= III_{B\dot{B}}\, o_A - I_{B\dot{B}}\, \iota_A.
\end{aligned}
$$

Of course, the two sets of equations above hold with respect to any dyad. On the other hand, we have

$$
\begin{aligned}
\nabla_{B\dot{B}} o'_A &= \nabla_{B\dot{B}} \left(e^{w/2}\, o_A\right) \\
&= e^{w/2} \left(\nabla_{B\dot{B}} o_A + \frac{1}{2}\, o_A\, \nabla_{B\dot{B}} w\right),
\end{aligned}
$$

and similarly,

$$
\nabla_{B\dot{B}} \iota'_A = e^{-w/2} \left(\nabla_{B\dot{B}} \iota_A + q \nabla_{B\dot{B}} o'_A + o_A \left(\nabla_{B\dot{B}} q - \frac{1}{2} q \nabla_{B\dot{B}} w\right) - \frac{1}{2} \iota_A \nabla_{B\dot{B}} w\right).
$$

We can therefore compute the transformation laws for $I_{B\dot{B}}$, $II_{B\dot{B}}$ and $III_{B\dot{B}}$, and they will turn out to be

$$
\begin{aligned}
I'_{B\dot{B}} &= I_{B\dot{B}} + \frac{1}{2} \nabla_{B\dot{B}} w - q II_{B\dot{B}} && (2.4.16) \\
II'_{B\dot{B}} &= e^{w}\, II_{B\dot{B}} && (2.4.17) \\
III'_{B\dot{B}} &= e^{-w} \left(III_{B\dot{B}} + 2q II_{B\dot{B}} - q^2 II_{B\dot{B}} + \nabla_{B\dot{B}}\, q\right). && (2.4.18)
\end{aligned}
$$

Contracting the preceding equations with the transformed dyad yields the transformation laws for the spin coefficients, which are listed below:

$$\varepsilon' = (\epsilon + \frac{1}{2}D(w) + q\kappa)e^{(\frac{1}{2}w+\frac{1}{2}\overline{w})} \tag{2.4.19}$$

$$\beta' = (\beta + \frac{1}{2}\delta(w) + q\sigma + \overline{q}\epsilon + \frac{1}{2}\overline{q}D(w) + \overline{q}q\kappa)e^{(\frac{1}{2}w-\frac{1}{2}\overline{w})} \tag{2.4.20}$$

$$\alpha' = (q\epsilon + \frac{1}{2}qD(w) + q^2\kappa + \alpha + \frac{1}{2}\overline{\delta}(w) + q\rho)e^{(-\frac{1}{2}w+\frac{1}{2}\overline{w})} \tag{2.4.21}$$

$$\begin{aligned}\gamma' = {} & (q\beta + \frac{1}{2}q\delta(w) + q^2\sigma + \gamma + \frac{1}{2}\Delta(w) + q\tau + q\overline{q}\epsilon \\ & +\frac{1}{2}q\overline{q}D(w) + q^2\overline{q}\kappa + \overline{q}\alpha + \frac{1}{2}\overline{q}\overline{\delta}(w) + \overline{q}q\rho)e^{(-\frac{1}{2}w-\frac{1}{2}\overline{w})}\end{aligned} \tag{2.4.22}$$

$$\kappa' = \kappa e^{(\frac{3}{2}w+\frac{1}{2}\overline{w})} \tag{2.4.23}$$

$$\sigma' = (\sigma + \kappa\overline{q})e^{(\frac{3}{2}w-\frac{1}{2}\overline{w})} \tag{2.4.24}$$

$$\rho' = (q\kappa + \rho)e^{(\frac{1}{2}w+\frac{1}{2}\overline{w})} \tag{2.4.25}$$

$$\tau' = (q\sigma + \tau + \overline{q}q\kappa + \rho\overline{q})e^{(\frac{1}{2}w-\frac{1}{2}\overline{w})} \tag{2.4.26}$$

$$\pi' = e^{(-\frac{1}{2}w+\frac{1}{2}\overline{w})}(\pi + 2q\epsilon + q^2\kappa + D(q)) \tag{2.4.27}$$

$$\begin{aligned}\mu' = {} & (2q\beta + q^2\sigma + \mu + \delta(q) + \overline{q}\pi + 2q\overline{q}\epsilon \\ & +q^2\overline{q}\kappa + \overline{q}D(q))e^{(-\frac{1}{2}w-\frac{1}{2}\overline{w})}\end{aligned} \tag{2.4.28}$$

$$\begin{aligned}\lambda' = {} & (q\pi + 2q^2\epsilon + q^3\kappa + qD(q) + 2q\alpha + q^2\rho \\ & +\lambda + \overline{\delta}(q))e^{(-\frac{3}{2}w+\frac{1}{2}\overline{w})}\end{aligned} \tag{2.4.29}$$

$$\begin{aligned}\nu' = {} & (2q^2\beta + q^3\sigma + q\mu + q\delta(q) + 2q\gamma + \nu + \Delta(q) + q^2\tau \\ & +q\overline{q}\pi + 2q^2\overline{q}\epsilon + q^3\overline{q}\kappa + q\overline{q}D(q) + 2\overline{q}q\alpha \\ & +\overline{q}q^2\rho + \overline{q}\lambda + \overline{q}\overline{\delta}(q))e^{(-\frac{3}{2}w-\frac{1}{2}\overline{w})}\end{aligned} \tag{2.4.30}$$

2.4.2 Conformal Transformations

A conformal transformation can be considered to be the identity map from $(\mathcal{M}, g_{\alpha\beta})$ to $(\mathcal{M}, e^{2\phi} g_{\alpha\beta})$, where ϕ is a C^∞ scalar function on $\mathcal{M}$.

It is somewhat more complicated to obtain the transformation laws for a conformal transformation. We first remark that a conformal transformation does not *a priori* stipulate the dyad and the van der Waerden correspondence on the transformed spacetime. Hence, we will make use of this freedom and choose the following:

$$
\begin{aligned}
\epsilon'^{AB} &= \epsilon^{AB}, & \epsilon'_{AB} &= \epsilon_{AB}, \\
\sigma'_\alpha{}^{A\dot{A}} &= e^{\phi}\, \sigma_\alpha{}^{A\dot{A}}, & \sigma'^\alpha{}_{A\dot{A}} &= e^{-\phi}\, \sigma^\alpha{}_{A\dot{A}}, \\
o'_A &= e^{\frac{r-1}{2}}\, o_A, & \iota'_A &= e^{\frac{1-r}{2}}\, \iota_A,
\end{aligned}
$$

as the van der Waerden correspondence and the dyad on the transformed spacetime respectively. Then the transformation laws for Ψ_{ABCD}, $\Phi_{AB\dot{A}\dot{B}}$, ϕ_{AB}, $I_{B\dot{B}}$, $II_{B\dot{B}}$, $III_{B\dot{B}}$ are as follows[2]:

$$
\begin{aligned}
\Psi'_{ABCD} &= e^{-2\phi}\, \Psi_{ABCD} \\
\Phi'_{AB\dot{A}\dot{B}} &= e^{-2\phi} \left(\Phi_{AB\dot{A}\dot{B}} - \phi_{;(A(\dot{A}B)\dot{B})} + \phi_{;(A(\dot{A}}\phi_{;B)\dot{B})} \right) \\
\phi'_{AB} &= e^{-2\phi}\, \phi_{AB} \\
\Lambda' \epsilon_{AB} \overline{\epsilon}_{\dot{A}\dot{B}} &= e^{-2\phi} \left(\Lambda \epsilon_{AB} \overline{\epsilon}_{\dot{A}\dot{B}} + \frac{1}{2} \epsilon_{AB} \epsilon_{\dot{A}\dot{B}} \phi_{;K\dot{K}} \phi_{;}{}^{K\dot{K}} \right) \\
I'_{B\dot{B}} &= e^{-\phi} \left(I_{B\dot{B}} + \frac{r}{2} \nabla_{B\dot{B}}\phi - o_B \iota^A \nabla_{A\dot{B}}\phi \right) \\
II'_{B\dot{B}} &= e^{(r-2)\phi} \left(II_{B\dot{B}} + o_B o^A \nabla_{A\dot{B}}\phi \right) \\
III'_{B\dot{B}} &= e^{-r\phi} \left(III_{B\dot{B}} + \iota_B \iota^A \nabla_{A\dot{B}}\phi \right).
\end{aligned}
$$

Appropriately contracting the above equations with the transformed dyad

[2]See [45] for their derivations.

yields the desired component transformation laws, listed below:

$$\Psi_0' = \Psi_0 e^{2\phi(-2+r)} \tag{2.4.31}$$

$$\Psi_1' = \Psi_1 e^{\phi(-3+r)} \tag{2.4.32}$$

$$\Psi_2' = \Psi_2 e^{-2\phi} \tag{2.4.33}$$

$$\Psi_3' = \Psi_3 e^{-\phi(1+r)} \tag{2.4.34}$$

$$\Psi_4' = \Psi_4 e^{-2\phi r} \tag{2.4.35}$$

$$\phi_0' = \phi_0 e^{\phi(-3+r)}, \qquad \phi_1' = \phi_1 e^{-2\phi}, \qquad \phi_2' = \phi_2 e^{-\phi(1+r)}. \tag{2.4.36}$$

$$\begin{aligned}
\Lambda' = e^{-2\phi}\Big(&\frac{1}{2}D(\phi)\Delta(\phi) - \frac{1}{2}\bar{\delta}(\phi)\delta(\phi) - \frac{1}{4}(-\epsilon - \bar{\epsilon} + \bar{\rho} + \rho)\Delta(\phi) \\
&-\frac{1}{4}(-\tau + \bar{\pi} + \beta - \bar{\alpha})\bar{\delta}(\phi) - \frac{1}{4}(\pi - \alpha - \bar{\tau} + \bar{\beta})\delta(\phi) \\
&-\frac{1}{4}(-\bar{\mu} + \gamma + \bar{\gamma} - \mu)D(\phi) - \frac{1}{4}\delta(\bar{\delta}(\phi)) + \Lambda + \frac{1}{4}D(\Delta(\phi)) \\
&-\frac{1}{4}\bar{\delta}(\delta(\phi)) + \frac{1}{4}\Delta(D(\phi))\Big)
\end{aligned} \tag{2.4.37}$$

$$\begin{aligned}
\Phi'_{00} &= e^{2(r-2)\phi}\big(\,\Phi_{00} - \bar{\delta}(\phi)\kappa - \delta(\phi)\bar{\kappa} + D(\phi)^2 + (\bar{\epsilon}+\epsilon)D(\phi) \\
&\quad - D(D(\phi))\,\big) &(2.4.38)\\
\Phi'_{01} &= e^{(r-3)\phi}\bigg(\Phi_{01} - \frac{1}{2}\Delta(\phi)\kappa - \frac{1}{2}\bar{\delta}(\phi)\sigma + D(\phi)\delta(\phi) + \frac{1}{2}(-\bar{\epsilon}-\bar{\rho}+\epsilon)\delta(\phi) \\
&\quad + \frac{1}{2}(\bar{\alpha}+\beta+\bar{\pi})D(\phi) - \frac{1}{2}D(\delta(\phi)) - \frac{1}{2}\delta(D(\phi))\bigg) &(2.4.39)\\
\Phi'_{02} &= e^{-2\phi}(\Phi_{02} - \Delta(\phi)\sigma + \delta(\phi)^2 - (-\beta+\bar{\alpha})\delta(\phi) + D(\phi)\bar{\lambda} \\
&\quad - \delta(\delta(\phi))) &(2.4.40)\\
\Phi'_{11} &= e^{-2\phi}\bigg(\Phi_{11} + \frac{1}{2}D(\phi)\Delta(\phi) + \frac{1}{4}(-\bar{\epsilon}-\rho-\bar{\rho}-\epsilon)\Delta(\phi) \\
&\quad + \frac{1}{2}\bar{\delta}(\phi)\delta(\phi) + \frac{1}{4}(\bar{\alpha}+\bar{\pi}-\tau-\beta)\bar{\delta}(\phi) \\
&\quad + \frac{1}{4}(-\bar{\beta}+\pi+\alpha-\bar{\tau})\delta(\phi) + \frac{1}{4}(\bar{\mu}+\gamma+\mu+\bar{\gamma})D(\phi) \\
&\quad - \frac{1}{4}D(\Delta(\phi)) - \frac{1}{4}\delta(\bar{\delta}(\phi)) - \frac{1}{4}\bar{\delta}(\delta(\phi)) - \frac{1}{4}\Delta(D(\phi))\bigg) &(2.4.41)\\
\Phi'_{12} &= e^{-(r+1)\phi}\bigg(\Phi_{12} + \delta(\phi)\Delta(\phi) - \frac{1}{2}(\beta+\bar{\alpha}+\tau)\Delta(\phi) \\
&\quad + \frac{1}{2}\bar{\delta}(\phi)\bar{\lambda} - \frac{1}{2}(-\mu+\bar{\gamma}-\gamma)\delta(\phi) \\
&\quad + \frac{1}{2}D(\phi)\bar{\nu} - \frac{1}{2}\delta(\Delta(\phi)) - \frac{1}{2}\Delta(\delta(\phi))\bigg) &(2.4.42)\\
\Phi'_{22} &= e^{-2r\phi}\big(\,\Phi_{22} + \Delta(\phi)^2 + (-\bar{\gamma}-\gamma)\Delta(\phi) \\
&\quad + \bar{\delta}(\phi)\bar{\nu} + \delta(\phi)\nu - \Delta(\Delta(\phi))\,\big) &(2.4.43)
\end{aligned}$$

$$\varepsilon' = (\epsilon + \frac{1}{2}rD(\phi))e^{\phi(-2+r)} \tag{2.4.44}$$

$$\beta' = (\beta + \frac{1}{2}r\delta(\phi))e^{-\phi} \tag{2.4.45}$$

$$\alpha' = (\alpha + \frac{1}{2}r\bar{\delta}(\phi) - \bar{\delta}(\phi))e^{-\phi} \tag{2.4.46}$$

$$\gamma' = (\gamma + \frac{1}{2}r\Delta(\phi) - \Delta(\phi))e^{-\phi r} \tag{2.4.47}$$

$$\kappa' = \kappa e^{\phi(-3+2r)} \tag{2.4.48}$$

$$\sigma' = \sigma e^{\phi(-2+r)} \tag{2.4.49}$$

$$\rho' = (\rho - D(\phi))e^{\phi(-2+r)} \tag{2.4.50}$$

$$\tau' = (\tau - \delta(\phi))e^{-\phi} \tag{2.4.51}$$

$$\pi' = (\pi + \bar{\delta}(\phi))e^{-\phi} \tag{2.4.52}$$

$$\mu' = (\mu + \Delta(\phi))e^{-\phi r} \tag{2.4.53}$$

$$\lambda' = \lambda e^{-\phi r} \tag{2.4.54}$$

$$\nu' = \nu e^{-\phi(-1+2r)} \tag{2.4.55}$$

Chapter 3

Gröbner Bases

The necessary conditions for Huygens' principle in spinor form are spinor equations. Computationally, one works with the component equations which are algebraic-differential scalar equations involving the spin coefficients, and the components of the curvature spinor, the Weyl spinor and the Maxwell spinor.

In Chapter 4, we will simplify these equations and eventually form a number of purely algebraic systems of polynomial equations involving only three spin coefficients and their complex conjugates. The theory of Gröbner bases is then used to prove that these systems admit only the zero solution, which subsequently will lead to the desired result (to be stated there). In this chapter, we present an account of Gröbner basis theory that will be used in Chapter 4.

The observation given in §3.1 is quoted from [29]. The development of the theory from §3.2 to §3.5 is primarily adapted from [2] and [18]. The theory behind the implementation of the MAPLE function `gsolve()` outlined in §3.6 can be found in [11], [12] and [20].

3.1 Solutions of Systems of Algebraic Equations and Ideals in a Polynomial Ring

Recall that if R is a commutative ring with identity, then the set $R[x_1, \dots, x_n]$ of multivariate polynomials with coefficients in R in the n indeterminates $x_1, \dots, x_n$ itself forms a ring under the usual polynomial addition and multiplication, and the concept of an ideal in $R[x_1, \dots, x_n]$ is thus well-defined.

Definition 3.1.1 *A* system of algebraic equations in n variables *over a commutative ring R with identity is a subset $S \subseteq R[x_1, \dots, x_n]$. A* solution *of S in R (or some super-ring*[1] *$\bar{R} \geq R$) is an n-tuple $(r_1, \dots, r_n) \in R^n$ (or $\bar{R}^n$) such that $s(r_1, \dots, r_n) = 0$, for all $s \in S$.*

Given any subset $S \subseteq R[x_1, \dots, x_n]$, $\langle S \rangle$ is used to denote the set

$$\langle S \rangle := \left\{ \sum_{i=1}^{k} f_i\, s_i \;\middle|\; s_i \in S, f_i \in R[x_1, \dots, x_n] \right\}$$

We remark that $\langle S \rangle$ is the smallest ideal of $R[x_1, \dots, x_n]$ that contains S. It is called the ideal generated by S.

Theorem 3.1.1 *Let $S \subseteq R[x_1, \dots, x_n]$. Then, $(r_1, \dots, r_n) \in \bar{R}^n$, where $\bar{R} \geq R$ is possibly some super-ring of R, is a solution of S if and only if it is a solution of $\langle S \rangle$.*

Proof Since $S \subseteq \langle S \rangle$, if $(r_1, \dots, r_n)$ is a solution of $\langle S \rangle$, it is necessarily a solution of S.

Conversely, suppose $(r_1, \dots, r_n) \in R^n$ is a solution of S, and let $p \in \langle S \rangle$ be given. Then $p = \sum_{i=1}^{k} f_i\, s_i$ for some $f_1, \dots, f_k \in R[x_1, \dots, x_n]$ and

[1] *A ring $\bar{R}$ is said to be a super-ring of the ring R if $\bar{R}$ contains R as a sub-ring.*

$s_1, \ldots, s_k \in S$. Therefore,

$$\begin{aligned} p(r_1, \ldots, r_n) &= \textstyle\sum_{i=1}^{k} f_i(r_1, \ldots, r_n)\, s_i(r_1, \ldots, r_n) \\ &= \textstyle\sum_{i=1}^{k} f_i(r_1, \ldots, r_n)\, 0 \\ &= 0 \end{aligned}$$

Since $p \in \langle S \rangle$ is arbitrary, $(r_1, \ldots, r_n)$ is a solution of $\langle S \rangle$. $\square$

Corollary 3.1.1 *Let $S, T \subseteq R[x_1, \ldots, x_n]$. If $\langle S \rangle = \langle T \rangle$, then S and T have the same solutions.*

Thus, in order to find solutions of given a finite set of polynomials $S = \{s_1, \ldots, s_k\} \subseteq R[x_1, \ldots, x_n]$, one can attempt to look for a set $G = \{g_1, \ldots, g_m\}$, with $\langle G \rangle = \langle S \rangle$, which is "simpler" to solve than S. The corollary above ensures that G and S have precisely the same solutions. One application of Gröbner bases is that it provides an algorithm for constructing the "simplest" generating sets of the ideal of a given system of multivariate polynomials. It is this application of Gröbner bases with which we will be concerned in this chapter.

3.2 Multivariate Division Algorithm

Let $\mathbb{F}$ be a field. We seek to extend the Division Algorithm in the polynomial ring $\mathbb{F}[x]$ over $\mathbb{F}$ in one indeterminate to the multivariate polynomial ring $\mathbb{F}[x_1, \ldots, x_n]$ over $\mathbb{F}$ in n indeterminates.

Definition 3.2.1 *Let $\mathbb{F}$ be a field and $\mathbb{F}[x_1, \ldots, x_n]$ be the polynomial ring in n indeterminates over $\mathbb{F}$. A* monomial *in $\mathbb{F}[x_1, \ldots, x_n]$ is an element in $\mathbb{F}[x_1, \ldots, x_n]$ of the form $x_1^{i_1} \cdots x_n^{i_n}$, where i_k is a non-negative integer for each $k = 1, \ldots, n$. The set of all monomials in $\mathbb{F}[x_1, \ldots, x_n]$ is denoted by $\mathfrak{M}$.*

Definition 3.2.2 *A* monomial ordering *on $\mathfrak{M}$ is a total ordering $\prec$ on $\mathfrak{M}$ that is compatible with the multiplication of monomials in the following sense:*

(1) *For any pair of monomials m, n, exactly one of the following holds: $m \prec n$ or $n \prec m$ or $m = n$.*

(2) $m_1 \prec m_2$ *and* $m_2 \prec m_3 \Longrightarrow m_1 \prec m_3$, $\forall\, m_1, m_2, m_3 \in \mathfrak{M}$.

(3) $1 \prec m$, *for any monomial* $m \neq 1$.

(4) $m_1 \prec m_2 \Longrightarrow m\, m_1 \prec m\, m_2$, $\forall\, m, m_1, m_2 \in \mathfrak{M}$.

Example 3.2.1 (Pure Lexicographical Ordering)
The pure lexicographical ordering *with $x_n \prec x_{n-1} \prec \cdots \prec x_1$, is defined as follows: $x_1^{i_1} \cdots x_n^{i_n} \prec x_1^{j_1} \cdots x_n^{j_n}$ if $i_1 = j_1, \ldots, i_k = j_k, i_{k+1} < j_{k+1}$ for some $k, 1 \leq k < n$.*

It is routine to verify that the pure lexicographical ordering is indeed a monomial ordering and its proof will be omitted. There are many other different monomial orderings, however, in this thesis, the pure lexicographical ordering alone will suffice for our purposes. For other examples of monomial orderings, consult [2], [20] or [18].

We will fix the following notations. Assume a monomial ordering $\prec$ has been given on $\mathfrak{M}$. Then, each $f \in \mathbb{F}[x_1, \ldots, x_n]$ can be written uniquely as an $\mathbb{F}$-linear combination of monomials in $\mathfrak{M}$, i.e. $f = \sum_{i=1}^{k} c_i m_i$, such that for each $i = 1, \ldots, k$, $c_i \neq 0$, $m_i \in \mathfrak{M}$, and $m_1 \succ \cdots \succ m_k$.

(1) The support of f, denoted by $\mathrm{supp}(f)$, is the set $\{m_i | i = 1, \ldots, k\}$.

(2) The leading coefficient, $\mathrm{lc}(f)$, of f is c_1.

(3) The leading monomial, $\mathrm{lm}(f)$, of f is m_1.

(4) The leading term, $\mathrm{lt}(f)$, of f is $c_1 m_1$.

(5) Each $c_i m_i$ is called a term of f.

Definition 3.2.3 *A polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$ is said to be* reduced *with respect to a set of non-zero polynomials $P = \{p_1, \ldots, p_k\}$ if either $f = 0$ or no monomial in* $\mathrm{supp}(f)$ *is divisible by any element of* $\{\mathrm{lm}(p_i) \,|\, i = 1, \ldots, k\}$.

Multivariate Division Algorithm

INPUT:

- a monomial ordering $\prec$ on the set $\mathfrak{M}$ of monomials in $\mathbb{F}[x_1, \ldots, x_n]$
- $f \in \mathbb{F}[x_1, \ldots, x_n]$

- a finite ordered sequence $(g_1, \ldots, g_k)$ with

$$g_i \in \mathbb{F}[x_1, \ldots, x_n] \quad \text{and} \quad g_i \neq 0, \quad \forall\, i = 1, \ldots k.$$

OUTPUT:

- $u_1, \ldots, u_k, r \in \mathbb{F}[x_1, \ldots, x_n]$ such that

$$f = u_1 g_1 + \cdots + u_k g_k + r, \tag{3.2.1}$$

and r is reduced with respect to $(g_1, \ldots, g_k)$. r is called the remainder of f with respect to $(g_1, \ldots, g_k)$; it will be denoted by $\operatorname{rem}(f, (g_1, \ldots, g_k))$.

INITIALIZATION: $u_1 := 0, u_2 := 0, \ldots, u_k := 0, r := 0, h := f$.

WHILE $h \neq 0$ DO
 IF $\{i \in \{1, 2, \ldots, k\} \mid \operatorname{lm}(g_i) \text{ divides } \operatorname{lm}(h)\} \neq \varnothing$ THEN
  $i_0 := \min\{i \in \{1, 2, \ldots, k\} \mid \operatorname{lm}(g_i) \text{ divides } \operatorname{lm}(h)\}$
  $u_{i_0} := u_{i_0} + \frac{\operatorname{lt}(h)}{\operatorname{lt}(g_{i_0})}$
  $h := h - \frac{\operatorname{lt}(h)}{\operatorname{lt}(g_{i_0})} g_{i_0}$
 ELSE
  $r := r + \operatorname{lt}(h)$
  $h := h - \operatorname{lt}(h)$
 ENDIF
ENDWHILE

Of course, we need to prove that the Algorithm indeed terminates after finitely many iterations, which turns out to be rather involved. It relies on the fact that every monomial ordering is a well-ordering on $\mathfrak{M}$, whose proof in turn requires the famous Hilbert's Basis Theorem. To prove that the output of the Algorithm does have the asserted properties, we will make use of a lemma that states that the algorithm maintains a certain "invariant" throughout the iterations.

We state here the termination and correctness theorem of the Multivariate Division Algorithm. We then prove a number of the technical results we have mentioned and return to the proof of the theorem after that.

Theorem 3.2.1 *Given a monomial ordering on the set $\mathfrak{M}$ of monomials in $\mathbb{F}[x_1, \ldots, x_n]$, a finite ordered sequence $G = (g_1, \ldots, g_k)$ of non-zero polynomials in $\mathbb{F}[x_1, \ldots, x_n]$, and $f \in \mathbb{F}[x_1, \ldots, x_n]$, the Multivariate Division Algorithm produces $u_1, \ldots, u_k, r \in \mathbb{F}[x_1, \ldots, x_n]$ such that*

$$f = u_1 g_1 + \cdots + u_k g_k + r \tag{3.2.2}$$

with r reduced with respect to G and

$$\operatorname{lm}(f) = \max\left\{\operatorname{lm}(r), \max_{1 \le i \le k}\{\operatorname{lm}(u_i)\operatorname{lm}(g_i)\}\right\} \tag{3.2.3}$$

We will first prove Hilbert's Basis Theorem, and subsequently a theorem to the effect that every monomial ordering is a well-ordering. It is the well-ordering that will be used to prove the termination of the Multivariate Division Algorithm. We begin with a few definitions.

Definition 3.2.4 *Let R be a commutative ring. An ideal I of R is said to be* finitely generated *if there exist a finite number of elements $r_1, \ldots, r_n \in R$ such that $I = \langle r_1, \ldots, r_n \rangle$.*

Definition 3.2.5 *Let R be a commutative ring. Suppose for every ascending chain of ideals, $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$, of R, there exists $N \in \mathbb{N}$ such that $I_n = I_N$, $\forall\, n \geq N$. Then, R is called a* Noetherian ring, *and we say that the* Ascending Chain Condition on Ideals *is satisfied in R.*

Theorem 3.2.2 *A commutative ring R is Noetherian if and only if every ideal of R is finitely generated.*

Proof Suppose every ideal of R is finitely generated, and let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ be an ascending chain of ideals of R. Since the chain is ascending, it follows that $I = \bigcup_{n=1}^{\infty} I_n$ is closed under addition and multiplication with elements of R, i.e. I is an ideal of R. By hypothesis, $I = \langle r_1, \ldots, r_k \rangle$ for some $r_1, \ldots, r_k \in R$. For each $i = 1, \ldots, k$, $r_i \in I$; hence, there exists $N_i \in \mathbb{N}$ such that $r_i \in I_{N_i}$. Let $N = \max_{1 \leq i \leq k}\{N_i\}$. Then $r_i \in I_N$ for all $i = 1, \ldots, k$ and so $I = \langle r_1, \ldots, r_k \rangle \subseteq I_N$. The Ascending Chain Condition on Ideals is therefore satisfied in R.

Conversely, suppose R is Noetherian. Assume on the contrary that R has an ideal I that is not finitely generated. I must therefore be non-empty. Choose some $r_1 \in I$ and we have $\langle r_1 \rangle \subsetneq I$, since I is by hypothesis not finitely generated. Choose some $r_2 \in I \backslash \langle r_1 \rangle$; then, $\langle r_1 \rangle \subsetneq \langle r_1, r_2 \rangle \subsetneq I$. Since I is not finitely generated, this process can be continuted indefinitely, producing a strictly ascending chain of ideals of R, contradicting the fact that R is Noetherian. □

Theorem 3.2.3 (Hilbert's Basis Theorem) *If R is a Noetherian ring, then so is $R[x]$.*

Proof Let R be a Noetherian ring and J an ideal of $R[x]$. By Theorem 3.2.2, it is sufficient to prove that J is finitely generated.

For each $n \geq 0$, define

$$I_n := \{0\} \cup \{r \in R \mid r = \mathrm{lc}(p), \text{ for some } p \in J \text{ with } \deg(p) = n\}.$$

We claim that each I_n is an ideal of R, and $I_n \subseteq I_{n+1}$. To see this, let $s, t \in I_n$, then there exist $p_s, p_t \in J$, both of degree n, with leading coefficients s and t respectively. Then $p_s + p_t$ is a polynomial in J of degree n with leading coefficient $s + t$; therefore, $s + t \in I_n$. Let $r \in R$. Then rp_s is a polynomial of degree n in J with leading coefficient rs; hence $rs \in I_n$. Thus, I_n is indeed an ideal of R. For the second statement, note that for every $s \in I_n$ with p_s a polynomial in J of degree n and leading coefficient s, xp_s is a polynomial in J of degree $n+1$ with leading coefficient s. Thus, $s \in I_{n+1}$.

Since R is Noetherian, there exists $N \in \mathbb{N}$ such that $I_n = I_N$ for all $n \geq N$. By Theorem 3.2.2, for each $i = 1, \ldots, N$, there exist $r_{i,j} \in I_i$, with $j = 1, \ldots, k_i$, such that $I_i = \langle r_{i,1}, \ldots r_{i,k_i} \rangle$. By construction of the I_n's, for each $i = 1, \ldots, N$ and $j = 1, \ldots, k_i$, there exist $f_{i,j} \in J$ of degree i and with leading coefficient $r_{i,j}$.

Claim 1: $J = \langle f_{i,j} \mid j = 1, \ldots, k_i,\ i = 1, \ldots, N \rangle$.

Let $J^* := \langle f_{i,j} \mid j = 1, \ldots, k_i,\ i = 1, \ldots, N \rangle$. Since each $f_{i,j} \in J$, clearly $J^* \subseteq J$. Let $f \in J$. If $f = 0$, then $f \in I_0 \subseteq J^*$. If $f \neq 0$, let $n := \deg(f)$. If $n = 0$, i.e. f is a constant polynomial, then $f \in I_0 \subseteq J^*$, since f is then the leading coefficient of itself. We now proceed with an induction on n. Assume all polynomials in J of degree less than or equal to $n - 1$ are in J^*. We consider the two disjoint sub-cases: $n \leq N$ or $n > N$ separately.

Suppose $n \leq N$. Let $c := \mathrm{lc}(f)$. Since f is in J, c is in I_n by definition of I_n. But, $I_n = \langle r_{n,1}, \ldots, r_{n,k_n} \rangle$. Therefore, $c = \sum_{i=1}^{k_n} s_i r_{n,i}$, for some

$s_i \in R$. Then $g := \sum_{i=1}^{k_n} s_i f_{n,i}$ is a polynomial of degree n (since each $\deg(f_{n,i}) = n$) with leading coefficient $\sum_{i=1}^{k_n} s_i r_{n,i} = c$. Thus, $f - g$ is a polynomial of degree at most $n-1$ since their leading terms cancel. Since g is an R-linear combination of $f_{n,i}$, g is in J^*. By induction hypothesis, $\deg(f-g) = n-1 < n \Longrightarrow f - g \in J^*$. Thus, $f \in J^*$.

Suppose $n > N$. Let $c := \mathrm{lc}(f)$. Then $f \in J$ and $\deg(f) = n \Longrightarrow c \in I_n$. But, $I_n = I_N = \langle r_{N,1}, \ldots, r_{N,k_N} \rangle$. Therefore, $c = \sum_{i=1}^{k_N} s_i r_{N,i}$, for some $s_i \in R$. Now, $g := \sum_{i=1}^{k_N} s_i f_{N,i}$ is a polynomial of degree N and whose leading coefficient is $\sum_{i=1}^{k_N} s_i r_{N,i} = c$. Hence, $f - x^{n-N} g$ is a polynomial of degree at most $n-1$ since their leading terms cancel, implying $f - x^{n-N} g \in J^*$ by induction hypothesis. As before, $g \in J^*$ since g is an R-linear combination of $f_{N,i}$. This implies $x^{n-N} g \in J^*$, which in turn implies $f \in J^*$. Claim 1 is now proved and so is the theorem. $\square$

We will use the following result to prove that every monomial ordering on $\mathfrak{M}$ is a well-ordering on $\mathfrak{M}$.

Corollary 3.2.2 *If $\mathbb{F}[x_1, \ldots, x_n]$ is the multivariate polynomial ring over the field $\mathbb{F}$, then $\mathbb{F}[x_1, \ldots, x_n]$ is a Noetherian ring. In particular, every ideal in $\mathbb{F}[x_1, \ldots, x_n]$ is finitely generated, and the Ascending Chain Condition on Ideals is satisfied in $\mathbb{F}[x_1, \ldots, x_n]$.*

PROOF Recall that, whenever $\mathbb{F}$ is a field, $\mathbb{F}[x_1]$ is a Euclidean domain, hence a principal ideal domain, and thus trivially Noetherian. That $\mathbb{F}[x_1, \ldots, x_n]$ is Noetherian now follows by a straightforward induction argument. The rest of the statement follows from Theorem 3.2.2. $\square$

Theorem 3.2.4 *Let $\mathfrak{M}$ be the set of all monomials in $\mathbb{F}[x_1, \ldots, x_n]$. Then, every monomial ordering $\prec$ on $\mathfrak{M}$ is a well-ordering on $\mathfrak{M}$; i.e. for every non-empty subset $\mathfrak{A} \subseteq \mathfrak{M}$, there exists $m_0 \in \mathfrak{A}$ such that either $m_0 = m$ or $m_0 \prec m$, for all $m \in \mathfrak{A}$.*

Proof Suppose to the contrary that some monomial ordering $\prec$ is not a well-ordering. Then, there exists a non-empty subset $\mathfrak{A} \subseteq \mathfrak{M}$ that contains a strictly descending sequence of monomials, i.e. there exist $m_1, m_2, \ldots \in \mathfrak{A}$ such that

$$m_1 \succ m_2 \succ m_3 \succ \cdots$$

Claim 1: $\langle m_1, \ldots, m_k \rangle \subsetneq \langle m_1, \ldots, m_{k+1} \rangle$ for each $k \in \mathbb{N}$.
Note that the inclusion part is obvious, so to prove the Claim 1, we only need to disprove the equality. If we had equality, then $m_{k+1} \in \langle m_1, \ldots, m_k \rangle$ and

$$m_{k+1} = \sum_{i=1}^{k} p_i \, m_i, \quad p_i \in \mathbb{F}[x_1, \ldots, x_n]. \tag{3.2.4}$$

Since the LHS of (3.2.4) is m_{k+1}, a monomial, m_{k+1} must equal one of the elements of $\mathfrak{S} = \mathrm{supp}(\sum_{i=1}^{k} p_i \, m_i)$. However, every element of $\mathfrak{S}$ is a multiple of one of $m_1, m_2, \ldots, m_k$. Thus, m_{k+1} must be divisible by m_i for some $i = 1, \ldots, k$. This is equivalent to $m_{k+1} = m \, m_i$ for some $m \in \mathfrak{M}$. By the definition of a monomial ordering, we have $1 \prec m \Longrightarrow m_i = 1 \, m_i \prec m \, m_i = m_{k+1}$. This contradicts the original hypothesis that the m_i's form a strictly descending sequence of monomials. Claim 1 is proved.

However, Claim 1 now implies that we have the following strictly ascending chain of ideals:

$$\langle m_1 \rangle \subsetneq \langle m_1, m_2 \rangle \subsetneq \cdots \subsetneq \langle m_1, \ldots, m_k \rangle \subsetneq \cdots$$

This contradicts Hilbert's Basis Theorem, or more explicitly, Corollary 3.2.2. We conclude that every monomial ordering on $\mathfrak{M}$ is a well-ordering on $\mathfrak{M}$. □

We have developed adequate technical results to prove the termination of the Multivariate Division Algorithm after finitely many iterations. We now state and prove one more technical lemma which will be used to establish the correctness of the Algorithm.

Lemma 3.2.4 *For the Multivariate Division Algorithm,*

$$f - h = \sum_{i=1}^{k} u_i\, g_i + r \tag{3.2.5}$$

holds at the end of every iteration of the Algorithm.

Proof In this proof, the "unprimed" variables denote the values of the variables before a particular iteration of the WHILE loop and the "primed" variables denote the values of the corresponding variables after the that iteration.

We proceed by induction on the number N of iterations that has been performed. Consider the first iteration ($N = 1$). The initial values are:

$$\left\{ \begin{array}{lcl} h & = & f \\ u_i & = & 0,\ \forall\, i \\ r & = & 0 \end{array} \right.$$

If the IF-segment was executed, we have

$$\begin{cases} h' &= f - \frac{\mathrm{lt}(f)}{\mathrm{lt}(g_{i_0})}\, g_{i_0} \\ u'_{i_0} &= \frac{\mathrm{lt}(f)}{\mathrm{lt}(g_{i_0})}\, g_{i_0} \\ u'_i &= 0,\ \forall\, i \neq i_0 \\ r' &= 0 \end{cases}$$

Therefore,

$$\begin{aligned} f - h' &= f - \left(f - \frac{\mathrm{lt}(f)}{\mathrm{lt}(g_{i_0})}\, g_{i_0}\right) \\ &= \frac{\mathrm{lt}(f)}{\mathrm{lt}(g_{i_0})}\, g_{i_0} \\ &= 0\, g_1 + \cdots + 0\, g_{i_0-1} + \frac{\mathrm{lt}(f)}{\mathrm{lt}(g_{i_0})}\, g_{i_0} + 0\, g_{i_0+1} + \cdots + 0\, g_k + 0 \\ &= u'_1\, g_1 + \cdots + u'_{i_0-1}\, g_{i_0-1} + \frac{\mathrm{lt}(f)}{\mathrm{lt}(g_{i_0})}\, g_{i_0} + u'_{i_0+1}\, g_{i_0+1} + \cdots + u'_k\, g_k + r' \end{aligned}$$

Thus (3.2.5) holds for this case. If the ELSE-segment was executed, then

$$\begin{cases} h' &= f - \mathrm{lt}(f) \\ u'_i &= 0,\ \forall\, i \\ r' &= \mathrm{lt}(f) \end{cases}$$

Hence,

$$\begin{aligned} f - h' &= f - (f - \mathrm{lt}(f)) \\ &= \mathrm{lt}(f) \\ &= 0\, g_1 + \cdots + 0\, g_k + \mathrm{lt}(f) \\ &= u'_1\, g_1 + \cdots + u'_k\, g_k + r' \end{aligned}$$

Thus, (3.2.5) holds after the first iteration ($N = 1$) of the WHILE loop — regardless of which segment of the IF-ELSE statement was executed.

Our induction hypothesis states that (3.2.5) holds up to the end of the N-th iteration of the Multivariate Division Algorithm. We now prove that this implies that (3.2.5) still holds after $N+1$ iterations.

Depending on which segment was executed last, we have either one of the following:

$$\text{IF:} \quad \left\{ \begin{array}{rcl} h' & := & h - \frac{\mathrm{lt}(h)}{\mathrm{lt}(g_{i_0})}\, g_{i_0} \\ u'_{i_0} & := & u_{i_0} + \frac{\mathrm{lt}(h)}{\mathrm{lt}(g_{i_0})}\, g_{i_0} \\ u'_i & := & u_i, \forall\, i \neq i_0 \\ r' & := & r \end{array} \right.$$

$$\text{ELSE:} \quad \left\{ \begin{array}{rcl} h' & = & h - \mathrm{lt}(h) \\ u'_i & := & u_i, \forall\, i = 1, \ldots, k \\ r' & := & r + \mathrm{lt}(h) \end{array} \right.$$

Hence,

$$\begin{array}{llcl} \text{IF:} & f - h' & = & f - \left(h - \frac{\mathrm{lt}(h)}{\mathrm{lt}(g_{i_0})}\, g_{i_0}\right) \\ & & = & (f-h) + \frac{\mathrm{lt}(h)}{\mathrm{lt}(g_{i_0})}\, g_{i_0} \\ & & = & \left(\sum_{i=1}^k u_i g_i + r\right) + \frac{\mathrm{lt}(h)}{\mathrm{lt}(g_{i_0})}\, g_{i_0} \\ & & = & \sum_{i \neq i_0} u_i g_i + \left(u_{i_0} + \frac{\mathrm{lt}(h)}{\mathrm{lt}(g_{i_0})}\right) g_{i_0} + r \\ & & = & \sum_{i=1}^k u'_i g_i + r' \end{array}$$

$$\begin{array}{llcl} \text{ELSE:} & f - h' & = & f - (h - \mathrm{lt}(h)) \\ & & = & (f-h) + \mathrm{lt}(h) \\ & & = & \left(\sum_{i=1}^k u_i g_i + r\right) + \mathrm{lt}(h) \\ & & = & \sum_{i=1}^k u_i g_i + (r + \mathrm{lt}(h)) \\ & & = & \sum_{i=1}^k u'_i g_i + r' \end{array}$$

Hence (3.2.5) holds in either case, and the lemma is proved. □

We are finally ready to prove Theorem 3.2.1, and it is now rather straightforward after all the hard work.

Proof of Theorem 3.2.1. We first prove termination. If given some input, the Algorithm does not terminate, then we will have an infinite sequence of non-zero polynomials $h_i \in \mathbb{F}[x_1, \ldots, x_n]$, $i \in \mathbb{N}$, where h_i is the value of the variable h in the Algorithm after i iterations. Observe that the h_i's are constructed by the Algorithm so that the $\mathrm{lm}(h_i)$'s form a strictly descending sequence of monomials. But this contradicts the well-orderedness of $\mathfrak{M}$ by $\prec$ (Theorem 3.2.4). We conclude that the Algorithm must terminate after finitely many iterations.

To prove that (3.2.2) holds, suppose that the Algorithm terminates after N iterations. By Lemma 3.2.4, (3.2.5) holds when the Algorithm terminates, which occurs when $h = 0$. Equation (3.2.2) now easily follows when we substitute 0 into h in (3.2.5).

Also, we note that r is reduced with respect to G because each term of r is not divisible by any of the $\mathrm{lm}(g_i)$.

It remains to prove (3.2.3). First, (3.2.2) immediately implies that

$$\mathrm{lm}(f) \preceq \max\left\{\mathrm{lm}(r), \max_{1\le i\le k}\{\mathrm{lm}(u_i)\mathrm{lm}(g_i)\}\right\}.$$

For the reverse inequality, we argue as follows: Initially $h := f$, and in every iteration, h is modified by subtracting its own leading term (and possibly adding some lower terms). Hence, after every iteration, $\mathrm{lm}(h)$ strictly drops and so $\mathrm{lm}(h) \preceq \mathrm{lm}(f)$ after every iteration. Now for each i, u_i is obtained by adding terms $\frac{\mathrm{lt}(h)}{\mathrm{lt}(g_i)}$, where $\frac{\mathrm{lt}(h)}{\mathrm{lt}(g_i)} g_i$ cancels the leading term of h. Therefore, we conclude that after each iteration,

$$\mathrm{lm}(u_i)\mathrm{lm}(g_i) \preceq \mathrm{lm}(h) \preceq \mathrm{lm}(f), \quad \text{for each } \ i.$$

Similarly, r is obtained by adding leading terms of h, and so $\mathrm{lm}(r) \preceq \mathrm{lm}(f)$. It is now obvious that (3.2.3) follows. $\square$

3.3 Gröbner Bases

The Multivariate Divison Algorithm has a number of "pathologies" in the sense that the following phenomena, which do not occur in the univariate case, do occur when there is more than one indeterminate:

Let $G := \{g_1, \ldots, g\} \subseteq \mathbb{F}[x_1, \ldots, x_n]$. Given $f \in \mathbb{F}[x_1, \ldots, x_n]$, let $\mathrm{rem}(f, (g_1, \ldots, g_k))$ denote the remainder of the f with respect to G generated by the Multivariate Division Algorithm (using a given monomial ordering). Then

- $f \in \langle g_1, \ldots, g_k \rangle$ does not imply $\mathrm{rem}(f, (g_1, \ldots, g_k)) = 0$ (however, the converse is obviously true).
- In general, $\mathrm{rem}(f, (g_1, \ldots, g_k))$ depends on the ordering of the g_i's.
- $\langle g_1, \ldots, g_k \rangle = I = \langle h_1, \ldots, h_r \rangle$ does not imply $\mathrm{rem}(f, (g_1, \ldots, g_k)) = \mathrm{rem}(f, (h_1, \ldots, h_r))$, i.e. the representation of $f + I \in \mathbb{F}[x_1, \ldots, x_n]/I$ may not be unique if an arbitrary generating set for I is used as the divisors in the Multivariate Division Algorithm.

For concrete examples of the above pathologies, consult [2] and [18]. Gröbner bases were introduced to overcome the above difficulties.

Definition 3.3.1 *Let $S \subseteq \mathbb{F}[x_1, \ldots, x_n]$ and let $\prec$ be a monomial ordering on the monomials in $\mathbb{F}[x_1, \ldots, x_n]$. The* leading monomial ideal *of S with respect to $\prec$ is the ideal*

$$\mathrm{Lm}(S) := \langle \mathrm{lm}(f) \mid f \in S \rangle.$$

Definition 3.3.2 *Let $I \subseteq \mathbb{F}[x_1, \ldots, x_n]$ be an ideal of $\mathbb{F}[x_1, \ldots, x_n]$. Let $\prec$ be any monomial ordering on the monomials in $\mathbb{F}[x_1, \ldots, x_n]$. A finite subset*

$\{g_1, \ldots, g_k\} \subseteq I$ *is called a* Gröbner basis *for* I *with respect to* $\prec$ *if*

$$\langle\, \mathrm{lm}(g_1), \ldots, \mathrm{lm}(g_k)\, \rangle = \mathrm{Lm}(I).$$

Theorem 3.3.1 *Let I be an ideal of $\mathbb{F}[x_1, \ldots, x_n]$ and let $\prec$ be any monomial ordering on the set of monomials in $\mathbb{F}[x_1, \ldots, x_n]$. Then I admits a Gröbner basis with respect to $\prec$.*

Proof This essentially follows from Hilbert's Basis theorem. $\mathbb{F}[x_1, \ldots, x_n]$ is a Noetherian ring by Corollary 3.2.2; hence the leading monomial ideal $\mathrm{Lm}(I)$ of I with respect to $\prec$ is finitely generated, i.e. there exist $f_1, \ldots, f_N \in I$ such that

$$\langle \mathrm{lm}(f_1), \ldots, \mathrm{lm}(f_N)\rangle = \mathrm{Lm}(I).$$

Therefore, $\{f_1, \ldots, f_N\}$ is a Gröbner basis for I with respect to the monomial ordering $\prec$. □

We now state and prove a series of results that describe how Gröbner bases remedy the pathologies with the Multivariate Division Algorithm.

Theorem 3.3.2 *If $\{g_1, \ldots, g_k\}$ is a Gröbner basis for an ideal I in $\mathbb{F}[x_1, \ldots, x_n]$ with respect to some monomial ordering, then $\langle\, g_1, \ldots, g_k\,\rangle = I$.*

Proof Clearly, $\langle\, g_1, \ldots, g_k\,\rangle \subseteq I$. Let $f \in I$. Then, $\mathrm{lm}(f) \in \mathrm{Lm}(I) = \langle\, g_1, \ldots, g_k\,\rangle$. This implies $\mathrm{lm}(f) = \sum_{i=1}^{k} p_i \mathrm{lm}(g_i)$, for some polynomials $p_i \in \mathbb{F}[x_1, \ldots, x_n]$. But, since $\mathrm{lm}(f)$ is a monomial, the RHS must collapse to a monomial[2] for equality to hold. However, each term in the RHS is

[2]to lm(f) itself, in fact

divisible by one of the $\mathrm{lm}(g_i)$. This in turn implies that $\mathrm{lm}(f)$ is divisible by $\mathrm{lm}(g_{i_1})$ for some $i_1 = 1, \ldots, k$. Thus, for a suitable monomial $m_1 \in \mathfrak{M}$, f and $m_1 g_{i_1}$ have the same leading term, and thus $\mathrm{lm}(f - m_1 g_{i_1}) \prec \mathrm{lm}(f)$. Since $f - m_1 g_{i_1} \in I$, by the same argument, we can find $m_2 \in \mathfrak{M}$ such that $\mathrm{lm}(f - m_1 g_{i_1} - m_2 g_{i_2}) \prec \mathrm{lm}(f - m_1 g_{i_1})$. Clearly, this process can be continued as long as $f - \sum_{j=1}^{s} m_j g_{i_j} \neq 0$. However, for some finite s, it must be zero, since otherwise $\{\mathrm{lm}(f - \sum_{j=1}^{s} m_j g_{i_j})\}_{s=1}^{\infty}$ forms a strictly descending sequence of non-zero monomials, contradicting the fact that $\prec$ is a well-ordering on $\mathfrak{M}$ Now, $f - \sum_{j=1}^{s} m_j g_{i_j} = 0$ is equivalent to $f \in \langle g_1, \ldots, g_k \rangle$. □

Theorem 3.3.3 *Let $\{g_1, \ldots, g_k\}$ be a Gröbner basis for an ideal I in $\mathbb{F}[x_1, \ldots, x_n]$. Let the set $\mathfrak{M}$ of monomials in $\mathbb{F}[x_1, \ldots, x_n]$ possess a fixed monomial ordering $\prec$. Then* $\mathrm{rem}(f_1, (g_1, \ldots, g_k)) = \mathrm{rem}(f_2, (g_1, \ldots, g_k))$ *if and only if $f_1 - f_2 \in I$. In particular,* $\mathrm{rem}(f, (g_1, \ldots, g_k)) = 0$ *if and only if $f \in I$.*

Proof Suppose $f_1 - f_2 \in I$. For brevity, let G denote the ordered sequence $(g_1, \ldots, g_k)$. Since $f_i - \mathrm{rem}(f_i, G) \in I$ for $i = 1, 2$ and $\mathrm{rem}(f_1, G) - \mathrm{rem}(f_2, G) = (f_2 - \mathrm{rem}(f_2, G)) - (f_1 - \mathrm{rem}(f_1, G)) - (f_2 - f_1)$, it follows that $\mathrm{rem}(f_1, G) - \mathrm{rem}(f_2, G) \in I$. This implies

$$\mathrm{lm}(\mathrm{rem}(f_1, G) - \mathrm{rem}(f_2, G)) \in \mathrm{Lm}(I). \tag{3.3.1}$$

On the other hand, $\mathrm{rem}(f_i, G)$ is an $\mathbb{F}$-linear combination of monomials outside $\mathrm{Lm}(I) = \mathrm{Lm}(g_1, \ldots, g_k)$, since $\mathrm{rem}(f_i, G)$ is reduced with respect to G. In particular, if $\mathrm{rem}(f_1, G) - \mathrm{rem}(f_2, G) \neq 0$, then we must have

$$\mathrm{lm}(\mathrm{rem}(f_1, G) - \mathrm{rem}(f_2, G)) \notin \mathrm{Lm}(I),$$

a contradiction to (3.3.1). □

Theorem 3.3.4 *Let $\mathfrak{M}$ be the set of monomials in $\mathbb{F}[x_1, \ldots, x_n]$ and I an ideal of $\mathbb{F}[x_1, \ldots, x_n]$. Then $\mathbb{F}[x_1, \ldots, x_n]/I$ is a vector space over $\mathbb{F}$. Furthermore, for every monomial ordering $\prec$ on $\mathfrak{M}$, $\{m + I \mid m \in \mathfrak{M} \backslash \mathrm{Lm}(I)\}$ forms an $\mathbb{F}$-basis for $\mathbb{F}[x_1, \ldots, x_n]/I$.*

Proof It is routine to verify that $\mathbb{F}[x_1, \ldots, x_n]/I$ forms a vector space over $\mathbb{F}$.

Let the monomial ordering $\prec$ be given. Let $\mathfrak{S} = \{m + I \mid m \in \mathfrak{M} \backslash \mathrm{Lm}(I)\}$.

We first prove that $\mathfrak{S}$ is an $\mathbb{F}$-linearly independent subset of $\mathbb{F}[x_1, \ldots, x_n]/I$. Suppose $s_1, \ldots, s_N \in \mathfrak{S}$ are distinct and there exist $c_1, \ldots, c_N \in \mathbb{F}$ such that

$$c_1 s_1 + \cdots + c_N s_N \in I \tag{3.3.2}$$

If $c_1 s_1 + \cdots + c_N s_N \neq 0$, (3.3.2) implies $\mathrm{lm}(c_1 s_1 + \cdots + c_N s_N) \neq 0$ and

$$\mathrm{lm}(c_1 s_1 + \cdots + c_N s_N) \in \mathrm{Lm}(I) \tag{3.3.3}$$

Since the s_i's are distinct, $\mathrm{lm}(c_1 s_1 + \cdots + c_N s_N)$ must be equal to one of the s_i. (3.3.3) therefore contradicts the fact that $s_i \notin \mathrm{Lm}(I)$ for every $i = 1, \ldots, N$. Hence, we must have $c_1 s_1 + \cdots + c_N s_N = 0$. Since the s_i's are distinct monomials, $c_i = 0$ for every $i = 1, \ldots, N$. This implies that $\mathfrak{S}$ is $\mathbb{F}$-linearly independent in $\mathbb{F}[x_1, \ldots, x_n]/I$.

It remains to prove that $\mathfrak{S}$ spans $\mathbb{F}[x_1, \ldots, x_n]/I$. Fix a monomial ordering $\prec$ on the monomials in $\mathbb{F}[x_1, \ldots, x_n]$. Then, by Theorem 3.3.1, I admits a Gröbner basis $G = \{g_1, \ldots, g_k\}$ with respect to $\prec$. Let $f + I \in \mathbb{F}[x_1, \ldots, x_n]/I$. Then $f + I = \mathrm{rem}(f, G) + I$ by the Multivariate Division Algorithm. Since $\mathrm{rem}(f, G)$ is reduced with respect to G, every term of it is not

divisible by $\mathrm{lm}(g_i)$ for any $i = 1, \ldots, k$. On the other hand, $\mathrm{Lm}(G) = \mathrm{Lm}(I)$, G being a Gröbner basis for I, implies that for every $p \in I$, $\mathrm{lm}(p)$ is divisible by $\mathrm{lm}(g_i)$ for some $i = 1, \ldots, k$. These two facts together imply every term of $\mathrm{rem}(f, G)$ is not divisible by $\mathrm{lm}(p)$ for any $p \in I$. In particular, every monomial in the support of $\mathrm{rem}(f, G)$ is outside of $\mathrm{Lm}(I)$, i.e. $\mathrm{rem}(f, G)$ is an $\mathbb{F}$-linear combination of monomials in $\mathfrak{S}$. Hence $\mathfrak{S}$ spans $\mathbb{F}[x_1, \ldots, x_n]/I$.
□

Corollary 3.3.3 *Let I be an ideal of $\mathbb{F}[x_1, \ldots, x_n]$ and let $\prec$ be a monomial ordering on the set $\mathfrak{M}$ of monomials of $\mathbb{F}[x_1, \ldots, x_n]$. Then*

$$\mathrm{rem}(f, G) = \mathrm{rem}(f, G')$$

for any two Gröbner bases G, G' for I with respect to $\prec$.

Proof Theorem 3.3.4 implies that

$$\mathrm{rem}(f, G) + I \;=\; f + I \;=\; \mathrm{rem}(f, G') + I$$

Therefore, $\mathrm{rem}(f, G) - \mathrm{rem}(f, G') \in I$, which in turn implies that $\mathrm{lm}(\mathrm{rem}(f, G) - \mathrm{rem}(f, G')) \in \mathrm{Lm}(I)$. On the other hand, the supports of both $\mathrm{rem}(f, G)$ and $\mathrm{rem}(f, G')$ do not intersect $\mathrm{Lm}(I)$; in particular, suppose to the contrary that $\mathrm{rem}(f, G) - \mathrm{rem}(f, G') \neq 0$; then, we must have $\mathrm{lm}(\mathrm{rem}(f, G) - \mathrm{rem}(f, G')) \notin \mathrm{Lm}(I)$, a contradiction. □

3.4 Buchberger's Algorithm

In the preceding section, we defined Gröbner bases and proved that using them as divisors in the Multivariate Division Algorithm remedies the difficulties that otherwise could occur. We also proved that every ideal of $\mathbb{F}[x_1, \ldots, x_n]$ admits a Gröbner basis, but we did not show how one can actually obtain a Gröbner basis for a given ideal when a generating set is given.

We shall do so in this section by presenting Buchberger's Algorithm. The theoretical basis of Buchberger's Algorithm is a theorem also discovered by Buchberger (Theorem 3.4.2), whose proof requires the following theorem. It gives stronger characterizations of Gröbner bases than what we have developed so far. We will omit its proof. (See Theorem 1.6.2 in [2].)

Theorem 3.4.1 *Let I be a non-zero ideal of $\mathbb{F}[x_1, \ldots, x_n]$. The following statements are equivalent for a set of non-zero polynomials $G = \{g_1, \ldots, g_k\} \subseteq I$.*

(1) *G is a Gröbner basis for I.*

(2) *$f \in I$ if and only if $\mathrm{rem}(f, G) = 0$.*

(3) *$f \in I$ if and only if $f = \sum_i^n h_i g_i$, with $\mathrm{lm}(f) = \max_{1 \le i \le n}\{\mathrm{lm}(h_i)\mathrm{lm}(g_i)\}$.*

(4) *For all $0 \neq f \in I$, $\mathrm{lm}(g_i)$ divides $\mathrm{lm}(f)$ for some $i \in \{1, 2, \ldots, k\}$.*

It turns out that the objects defined below play a pivotal role in constructing Gröbner bases.

Definition 3.4.1 *Let $0 \neq f, g \in \mathbb{F}[x_1, \ldots, x_n]$. Then the S-*polynomial *of f and g is defined to be*

$$S(f, g) := \frac{\mathrm{lcm}(\mathrm{lm}(f), \mathrm{lm}(g))}{\mathrm{lt}(f)} f - \frac{\mathrm{lcm}(\mathrm{lm}(f), \mathrm{lm}(g))}{\mathrm{lt}(g)} g.$$

We will use the following technical lemma to prove Buchberger's theorem.

Lemma 3.4.5 *Let $f_1, \ldots, f_s \in \mathbb{F}[x_1, \ldots, x_n]$ be such that $\mathrm{lm}(f_i) = X \neq 0$, for all $i = 1, \ldots, s$. Let $f = \sum_{i=1}^{s} c_i f_i$, with $c_i \in \mathbb{F}$ for all $i = 1, \ldots, s$. Then*

$$\mathrm{lm}(f) \prec X \quad \Longrightarrow \quad \begin{array}{c} \textit{for each } i, j \textit{ with } 1 \leq i < j \leq s, \exists\ d_{ij} \in \mathbb{F}, \\ \textit{such that } f = \sum_{1 \leq i < j \leq s} d_{ij} S(f_i, f_j). \end{array}$$

Proof Write $f_i = a_i X +$ lower terms. Then by hypothesis, $\sum_{i=1}^{s} c_i a_i = 0$. Since $X = \mathrm{lm}(f_i) = \mathrm{lm}(f_j) \Longrightarrow X = \mathrm{lcm}(\mathrm{lm}(f_i), \mathrm{lm}(f_j))$, we have

$$\begin{aligned} S(f_i, f_j) &= \frac{X}{a_i X} f_i - \frac{X}{a_j X} f_j \\ &= \frac{1}{a_i} f_i - \frac{1}{a_j} f_j. \end{aligned}$$

Therefore,

$$\begin{aligned} f &= c_1 f_1 + \cdots + c_s f_s \\ &= c_1 a_1 (\frac{1}{a_1} f_1) + c_2 a_2 (\frac{1}{a_2} f_2) + \cdots + c_s a_s (\frac{1}{a_s} f_s) \\ &= c_1 a_1 (\frac{1}{a_1} f_1 - \frac{1}{a_2} f_2) + (c_1 a_1 + c_2 a_2)(\frac{1}{a_2} f_2 - \frac{1}{a_3} f_3) + \cdots \\ &\quad + (c_1 a_1 + \cdots + c_{s-1} a_{s-1})(\frac{1}{a_{s-1}} f_{s-1} - \frac{1}{a_s} f_s) + \underbrace{(c_1 a_1 + \cdots + c_s a_s)}_{0} \frac{1}{a_s} f_s \\ &= c_1 a_1 S(f_1, f_2) + (c_1 a_1 + c_2 a_2) S(f_2, f_3) + \cdots \\ &\quad + (c_1 a_1 + \cdots + c_{s-1} a_{s-1}) S(f_{s-1}, f_s). \end{aligned}$$

□

Theorem 3.4.2 (Buchberger) *Let $G = \{g_1, \ldots, g_k\}$ be a set of non-zero polynomials in $\mathbb{F}[x_1, \ldots, x_n]$. Let a monomial ordering $\prec$ be given. Then G*

is a Gröbner basis for the ideal $I := \langle g_1, \ldots, g_k \rangle$ *with respect to* $\prec$ *if and only if for all* $i \neq j$, $\mathrm{rem}(S(g_i, g_j), G) = 0$.

Proof If G is a Gröbner basis for I, then for all $i \neq j$, $\mathrm{rem}(S(g_i, g_j), G) = 0$, by Theorem 3.4.1.

Conversely, suppose $\mathrm{rem}(S(g_i, g_j), G) = 0$ for all $i \neq j$. Let $f \in I$. Then the set

$$\mathcal{X} := \left\{ X \in \mathfrak{M} \;\middle|\; X = \max_{1 \leq i \leq k} \{\mathrm{lm}(h_i)\mathrm{lm}(g_i)\}, \quad \begin{array}{l} \text{for some } \; h_1, \ldots, h_k \in \mathbb{F}[x_1, \ldots, x_n] \\ \text{such that } \; f = \sum_{i=1}^k h_i g_i \end{array} \right\}$$

is a non-empty subet of $\mathfrak{M}$. Since $\prec$ is a well-ordering on $\mathfrak{M}$, $\mathcal{X}$ admits a least element, say X.

If $\mathrm{lm}(f) = X$, then the theorem follows by Theorem 3.4.1. Otherwise, we have $\mathrm{lm}(f) \prec X$, while $X = \max_{1 \leq i \leq k}\{\mathrm{lm}(h_i)\mathrm{lm}(g_i)\}$ for some $h_1, \ldots, h_k \in \mathbb{F}[x_1, \ldots, x_n]$ such that $f = \sum_{i=1}^k h_i g_i$. Let $S := \{i \in \{1, \ldots, k\} | \mathrm{lm}(h_i)\mathrm{lm}(g_i) = X\}$. For each $i \in S$, let $X_i := \mathrm{lm}(h_i)$ and $c_i := \mathrm{lc}(h_i)$.

Then,

$$\begin{aligned} f &= \sum_{i=1}^k h_i g_i \\ &= \sum_{i \in S} h_i g_i + \sum_{j \notin S} h_j g_j \\ &= \sum_{i \in S} (c_i X_i + \text{lower terms of } \; h_i)\, g_i + \sum_{j \notin S} h_j g_j \\ &= \sum_{i \in S} c_i X_i g_i + \sum_{i \in S} (\text{lower terms of } \; h_i)\, g_i + \sum_{j \notin S} h_j g_j. \end{aligned}$$

We will show that the assumption $\mathrm{lm}(f) \prec X$ implies that $g := \sum_{i \in S} c_i X_i g_i$ admits a representation $g = \sum_{\nu=1}^k h'_\nu g_\nu$ such that $\max_{1 \leq \nu \leq k}\{\mathrm{lm}(h'_\nu)\mathrm{lm}(g_\nu)\} \prec$

X. This in turn leads to a new representation for $f = \sum_{\nu=1}^{k} h''_\nu g_\nu$, with

$$\max_{1\le\nu\le k}\{\mathrm{lm}(h''_\nu)\mathrm{lm}(g_\nu)\} \prec X,$$

contradicting the minimality of X.

Now, $g = \sum_{i\in S} c_i X_i g_i$ such that $\mathrm{lm}(X_i g_i) = X$ for all $i \in S$, but $\mathrm{lm}(g) \prec X$. By Lemma 3.4.5, there exist $d_{ij} \in \mathbb{F}$ such that

$$g = \sum_{i,j\in S, i\ne j} d_{ij} S(g_i, g_j).$$

On the other hand, since $X = \mathrm{lm}(X_i g_i)$ for all $i \in S$, we have $X = \mathrm{lcm}(\mathrm{lm}(X_i g_i), \mathrm{lm}(X_j g_j))$ for all $i, j \in S$ with $i \ne j$. Therefore,

$$\begin{aligned} S(X_i g_i, X_j g_j) &= \frac{X}{\mathrm{lt}(X_i g_i)} X_i g_i - \frac{X}{\mathrm{lt}(X_j g_j)} X_j g_j \\ &= \frac{X}{\mathrm{lt}(g_i)} g_i - \frac{X}{\mathrm{lt}(g_j)} g_j \\ &= \frac{X}{X_{ij}} S(g_i, g_j), \end{aligned} \tag{3.4.1}$$

where $X_{ij} := \mathrm{lcm}(\mathrm{lm}(g_i), \mathrm{lm}(g_j))$. By hypothesis, $\mathrm{rem}(S(g_i, g_j), G) = 0$, which implies that $\mathrm{rem}(S(X_i g_i, X_j g_j), G) = 0$. This in turn implies that for all $i, j \in S$, $i \ne j$, there exist $h_{ij\nu} \in \mathbb{F}[x_1, \ldots, x_n]$, $\nu = 1, \ldots, k$, such that

$$S(X_i g_i, X_j g_j) = \sum_{\nu=1}^{k} h_{ij\nu} g_\nu.$$

By Theorem 3.2.1, this implies

$$\mathrm{lm}(S(X_i g_i, X_j g_j)) = \max_{1\le\nu\le k}\{\mathrm{lm}(h_{ij\nu})\mathrm{lm}(g_\nu)\}.$$

Since for all $i, j \in S$,

$$\mathrm{lm}\left(\frac{X}{\mathrm{lt}(X_i g_i)} X_i g_i\right) = X = \mathrm{lm}\left(\frac{X}{\mathrm{lt}(X_j g_j)} X_j g_j\right),$$

we see from (3.4.1) that for all $i, j \in S$, $i \neq j$,

$$\mathrm{lm}(S(X_i g_i, X_j g_j)) \prec X.$$

Therefore, for all $i, j \in S$, $i \neq j$,

$$\max_{1 \leq \nu \leq k} \{\mathrm{lm}(h_{ij\nu})\mathrm{lm}(g_\nu)\} \prec X.$$

Now,

$$\begin{aligned}
f &= g + \sum_{i \in S} (\text{lower terms of } h_i)\, g_i + \sum_{j \notin S} h_j g_j \\
&= \sum_{i,j \in S, i \neq j} d_{ij} S(X_i g_i, X_j g_j) + \sum_{i \in S} (\text{lower terms of } h_i)\, g_i + \sum_{j \notin S} h_j g_j \\
&= \sum_{i,j \in S, i \neq j} d_{ij} \sum_{\nu=1}^{k} h_{ij\nu} g_{nu} + \sum_{i \in S} (\text{lower terms of } h_i)\, g_i + \sum_{j \notin S} h_j g_j \\
&= \sum_{\nu=1}^{k} \left(\sum_{i,j \in S, i \neq j} d_{ij} h_{ij\nu} \right) g_\nu + \sum_{i \in S} (\text{lower terms of } h_i)\, g_i + \sum_{j \notin S} h_j g_j \\
&= \sum_{\nu=1}^{k} h''_\nu g_\nu,
\end{aligned}$$

where h''_ν are obtained simply by collecting terms and by the preceding arguments, they satisfy (3.4), completing our proof. $\square$

We are finally ready to present Buchberger's algorithm for constructing Gröbner bases.

Buchberger's Algorithm

INPUT: $F = \{f_1, \dots, f_s\} \subseteq \mathbb{F}[x_1, \dots, x_n]$ with $f_i \neq 0$, for $i = 1, \dots, s$.
OUTPUT: $G = \{g_1, \dots, g_k\}$, a Gröbner basis for $\langle f_1, \dots, f_s \rangle$.
INITIALIZATION: $G := F$, $\mathcal{G} := \{\{f_i, f_j\} \mid f_i \neq f_j \in G\}$.

WHILE $\mathcal{G} \neq \emptyset$ **DO**
  Choose any $\{f, g\} \in \mathcal{G}$.
  $\mathcal{G} := \mathcal{G} \backslash \{\{f, g\}\}$.
  $h := \mathrm{rem}(S(f, g), G)$
  IF $h \neq 0$ **THEN**
    $\mathcal{G} := \mathcal{G} \cup \{\{u, h\} \mid \text{for all } u \in G\}$
    $G := G \cup \{h\}$
  ENDIF
ENDWHILE

The following is the termination and correctness theorem for Buchberger's Algorithm.

Theorem 3.4.3 *Given $F = \{f_1, \dots, f_s\}$ with $f_i \neq 0$ for all $i = 1, \dots, s$, Buchberger's Algorithm will produce a Gröbner basis for the ideal $I = \langle f_1, \dots, f_s \rangle$ in finitely many iterations.*

Proof If Buchberger's Algorithm does not terminate, then it would generate a strictly increasing sequence $\{G_i\}$ of sets, where each G_i is the set G in the i-th iteration of the Algorithm.

Each G_i is obtained from G_{i-1} by adding some $h \in I$ to G_{i-1} where h is the non-zero remainder, with respect to G_{i-1}, of an S-polynomial of two elements of G_{i-1}. Thus $\operatorname{lm}(h) \notin \operatorname{Lm}(G_{i-1})$. Thus, we get

$$\operatorname{Lm}(G_1) \subsetneq \operatorname{Lm}(G_2) \subsetneq \operatorname{Lm}(G_3) \subsetneq \cdots$$

which is a strictly ascending chain of ideals. This contradicts Hilbert's Basis Theorem. Thus we conclude that Buchberger's Algorithm must terminate.

Now, $F \subseteq G \subseteq I$ implies that $I := \langle f_1, \ldots, f_s \rangle \subseteq \langle g_1, \ldots, g_k \rangle \subseteq I$. Therefore, G is a generating set of I. Moreover, by construction of G, $\operatorname{rem}(S(g_i, g_j), G) = 0$, for all $g_i, g_j \in G$. Therefore, G is a Gröbner basis for I by Theorem 3.4.2. $\square$

3.5 Reduced Gröbner Bases

An arbitrary ideal of $\mathbb{F}[x_1, \ldots, x_n]$ generally admits more than one Gröbner basis. In this section, we introduce *reduced Gröbner bases*, a special kind of Gröbner bases which is unique for every given ideal of $\mathbb{F}[x_1, \ldots, x_n]$.

Definition 3.5.1 *A Gröbner basis $G = \{g_1, \ldots, g_k\}$ for an ideal I of $\mathbb{F}[x_1, \ldots, x_n]$ with respect to a fixed monomial ordering is said to be* minimal *if*

(1) every element of G is monic, i.e. $\mathrm{lc}(g_i) = 1$ for all $i = 1, \ldots, k$, and

(2) $\mathrm{lm}(g_i)$ does not divide $\mathrm{lm}(g_j)$ for all $i \neq j$.

Lemma 3.5.6 *Let $G = \{g_1, \ldots, g_k\}$ be a Gröbner basis for an ideal I of $\mathbb{F}[x_1, \ldots, x_n]$ with respect to a monomial ordering $\prec$. If $\mathrm{lm}(g_2)$ divides $\mathrm{lm}(g_1)$, then $\{g_2, \ldots, g_k\}$ is also a Gröbner basis for I with respect to the monomial ordering $\prec$.*

Proof By hypothesis, $\mathrm{lm}(g_1) = m\,\mathrm{lm}(g_2)$ for some monomial m. Let $f \in \mathrm{Lm}(g_1, \ldots, g_k)$. Then,

$$\begin{aligned} f &= \textstyle\sum_{i=1}^k p_i \,\mathrm{lm}(g_i) \\ &= p_1 \,\mathrm{lm}(g_1) + \textstyle\sum_{i=2}^k p_i \,\mathrm{lm}(g_i) \\ &= p_1\, m \,\mathrm{lm}(g_2) + \textstyle\sum_{i=2}^k p_i \,\mathrm{lm}(g_i), \end{aligned}$$

for some $p_1, \ldots, p_k \in \mathbb{F}[x_1, \ldots, x_n]$. Therefore, $f \in \mathrm{Lm}(g_2, \ldots, g_k)$, and we have

$$\mathrm{Lm}(g_2, \ldots, g_k) \subseteq \mathrm{Lm}(g_1, \ldots, g_k) \subseteq \mathrm{Lm}(g_2, \ldots, g_k).$$

$\{g_2, \ldots, g_k\}$ is thus a Gröbner basis for I with respect to $\prec$. $\square$

Corollary 3.5.4 *Given a Gröbner basis $G = \{g_1, \ldots, g_k\}$ for an ideal I of $\mathbb{F}[x_1, \ldots, x_n]$, a minimal Gröbner basis can be obtained as follows:*

(1) *Eliminate all g_i for which there exists $j \neq i$ such that $\mathrm{lm}(g_j)$ divides $\mathrm{lm}(g_i)$.*

(2) *Divide all remaining g_i by $\mathrm{lc}(g_i)$.*

Lemma 3.5.7 *If $G = \{g_1, \ldots, g_k\}$ and $F = \{f_1, \ldots, f_s\}$ are minimal Gröbner bases for an ideal I of $\mathbb{F}[x_1, \ldots, x_n]$ with respect to a common monomial ordering, then $k = s$ and, after renumbering if necessary, $\mathrm{lm}(g_i) = \mathrm{lm}(f_i)$ for every $i = 1, \ldots, k$.*

Proof Since f_1 is in I and G is Gröbner basis for I, there exists g_i such that $\mathrm{lm}(g_i)$ divides $\mathrm{lm}(f_1)$. After renumbering if necessary, we may assume $\mathrm{lm}(g_1)$ divides $\mathrm{lm}(f_1)$. Now since g_1 is itself in I, there exists some f_j such that $\mathrm{lm}(f_j)$ divides $\mathrm{lm}(g_1)$. This implies $\mathrm{lm}(f_j)$ divides $\mathrm{lm}(f_1)$. Since F is a minimal Gröbner basis, we have $j = 1$. Therefore, $\mathrm{lm}(f_1) = \mathrm{lm}(g_1)$ since they divide each other.

Now, f_2 is in I, so there exists $i = 1, \ldots, k$ such that $\mathrm{lm}(g_i)$ divides $\mathrm{lm}(f_2)$. Minimality of F implies that $i \neq 1$ since $\mathrm{lm}(g_1) = \mathrm{lm}(f_1)$. As before, we may assume, after renumbering if necessary, $\mathrm{lm}(g_2) = \mathrm{lm}(f_2)$. This matching process continues until one of G or F is exhausted.

It remains to show that $k = s$. Suppose to the contrary that $k > s$. In the preceding paragraph we proved that, after renumbering if necessary, $\mathrm{lm}(g_i) = \mathrm{lm}(f_i)$, for $i = 1, \ldots, s$. By minimality of G, and the fact that F is a Gröbner basis for I,

$$\mathrm{lm}(g_{s+1}) \notin \langle\, \mathrm{lm}(g_1), \ldots, \mathrm{lm}(g_s)\,\rangle = \langle\, \mathrm{lm}(f_1), \ldots, \mathrm{lm}(f_s)\,\rangle = \mathrm{Lm}(I),$$

which contradicts the fact that $g_{s+1} \in I$. Similarly, $s > k$ leads to a contradiction and we must have $k = s$. $\square$

Definition 3.5.2 *A Gröbner basis $G = \{g_1, \ldots, g_k\}$ for an ideal I of $\mathbb{F}[x_1, \ldots, x_n]$ with respect to some monomial ordering is said to be* reduced *if*

(1) *G is minimal, and*

(2) *each g_i is reduced with respect to $G \setminus \{g_i\}$, or equivalently, no non-zero term in g_i is divisible by any $\mathrm{lm}(g_j)$, for every $j \neq i$.*

Theorem 3.5.1 *Let I be an ideal of $\mathbb{F}[x_1, \ldots, x_n]$ and let a monomial ordering $\prec$ on $\mathbb{F}[x_1, \ldots, x_n]$ be given. Then I admits a unique reduced Gröbner basis with respect to $\prec$.*

Proof By Theorem 3.3.1, I admits a Gröbner basis $G = \{g_1, \ldots, g_k\}$ with respect to $\prec$. By Corollary 3.5.4, a minimal Gröbner basis can be constructed from G. So, without loss of generality, we assume G is a minimal Gröbner basis.

Let $G' := \{\mathrm{rem}(g_i, G \backslash \{g_i\}) \mid i = 1, \ldots, k\}$. Note that $\mathrm{lm}(g_i) = \mathrm{lm}(\mathrm{rem}(g_i, G \backslash \{g_i\}))$, for each $i = 1, \ldots, k$ by minimality of G. Therefore, G' is still a Gröbner basis for I with respect to $\prec$. By construction, G' is reduced. We have proved the existence of a reduced Gröbner basis for I.

To prove uniqueness, observe first that by Lemma 3.5.7, any two reduced Gröbner bases must have the same number of elements, since both are minimal. Assume $G = \{g_1, \ldots, g_k\}$ and $F = \{f_1, \ldots, f_k\}$ are two reduced Gröbner bases for I with respect to $\prec$. By Lemma 3.5.7 again, we furthermore have $\mathrm{lm}(f_i) = \mathrm{lm}(g_i)$, for all $i = 1, \ldots, k$. Now, suppose to the contrary that for some $i = 1, \ldots, k$, $f_i \neq g_i$. Then,

$$\begin{aligned} f_i, g_i \in I &\implies 0 \neq f_i - g_i \in I \\ &\implies 0 \neq \mathrm{lm}(f_i - g_i) \in \mathrm{Lm}(I). \end{aligned}$$

Since G is a Gröbner basis, there exists some $j = 1, \ldots, k$ such that $\mathrm{lm}(g_j)$ divides $\mathrm{lm}(f_i - g_i)$. Now, $j \neq i$ because $\mathrm{lm}(f_i - g_i) \prec lm(g_i) = \mathrm{lm}(f_i)$. Now $\mathrm{lm}(f_i - g_i)$ is a non-zero monomial in $\mathrm{supp}(f_i)$ or $\mathrm{supp}(g_i)$. In either case, it is divisible by $\mathrm{lm}(f_j) = \mathrm{lm}(g_j)$, with $j \neq i$, which contradicts the hypothesis that both G and F are reduced Gröbner bases. $\square$

Note that the proof of the preceding theorem also provides a algorithm for constructing the reduced Gröbner basis for a given ideal of $\mathbb{F}[x_1, \ldots, x_n]$.

3.6 Improving Buchberger's Algorithm by Incorporating Multivariate Factorization: the MAPLE function `gsolve()`

In Chapter 4, we will need to determine the roots of a number of multivariate systems of polynomials. As the preceding sections have indicated, one way of achieving this is to consider the given system as generators of an ideal I in some polynomial ring, and compute a Gröbner basis for I with respect to some (suitable) monomial ordering.

We presented Buchberger's algorithm for obtaining Gröbner bases in §3.4. In practice, however, a straightforward implementation of Buchberger's algorithm may be unfeasibly inefficient. To determine the roots of the systems of polynomials that appear in Chapter 4, we will use instead the function `gsolve()` of the `Groebner` package of the symbolic algebra system MAPLE.

`gsolve()` implements a variant of Buchberger's algorithm. Rather than returning the reduced Gröbner basis of the original system given to it, `gsolve()` computes the Gröbner bases of "sub-systems" of the original system such that the union of the roots of these sub-systems coincides with the roots of the whole system. So, even though the reduced Gröbner basis of the original system may not be easily recovered from the output of `gsolve()` , the eventual purpose of determining the roots of the original system is served.

In the remainder of this section, we will demonstrate the functionality of `gsolve()` by working through a concrete example. We will, however, omit the actual theory behind the implementation of `gsolve()` , which combines multvariate factorization with Buchberger's algorithm. For an account of that theory, see Czapor [11] and [12]. Czapor is the author of the original `Groebner` package in the MAPLE library, which includes the function `gsolve()` .

Consider the following set of polynomials:

$$\left\{ \begin{array}{lcl} f_1 & := & 4\,x^2 + xy^2 - z + \frac{1}{4} \\ f_2 & := & 2\,x + y^2 z + \frac{1}{2} \\ f_3 & := & x^2 z - \frac{1}{2}\,x - y^2 \end{array} \right. . \tag{3.6.1}$$

The reduced Gröbner basis of the system (3.6.1) with respect to the pure lexicographical ordering with $x \succ y \succ z$ is

$$\left\{ \begin{array}{c} -342z^2 + 75z^3 + 266z - 60 + 52z^4 + z^5 - 8z^6 + 16z^7, \\ 1988y^2 - 481837z^2 + 1407741z - 595666 - 4197z^4 - 251555z^3 + 1272z^5 - 76752z^6, \\ 3976x + 37104z^6 - 600z^5 + 2111z^4 + 122062z^3 + 232833z^2 - 680336z + 288814 \end{array} \right\} \tag{3.6.2}$$

Note that the first polynomial in the reduced Gröbner basis (3.6.2) factors as follows:

$$\begin{array}{cl} & -342z^2 + 75z^3 + 266z - 60 + 52z^4 + z^5 - 8z^6 + 16z^7 \\ = & (z-1)\,(16z^6 + 8z^5 + 9z^4 + 61z^3 + 136z^2 - 206z + 60). \end{array} \tag{3.6.3}$$

Thus the solution to (3.6.1) is obviously the union of the solutions to the following two "sub-systems":

$$\left\{ \begin{array}{c} z - 1, \\ 1988y^2 - 481837z^2 + 1407741z - 595666 - 4197z^4 - 251555z^3 + 1272z^5 - 76752z^6, \\ 3976x + 37104z^6 - 600z^5 + 2111z^4 + 122062z^3 + 232833z^2 - 680336z + 288814 \end{array} \right. \tag{3.6.4}$$

$$\left\{ \begin{array}{c} 16z^6 + 8z^5 + 9z^4 + 61z^3 + 136z^2 - 206z + 60, \\ 1988y^2 - 481837z^2 + 1407741z - 595666 - 4197z^4 - 251555z^3 + 1272z^5 - 76752z^6, \\ 3976x + 37104z^6 - 600z^5 + 2111z^4 + 122062z^3 + 232833z^2 - 680336z + 288814 \end{array} \right. \tag{3.6.5}$$

Now, the reduced Gröbner basis of (3.6.4), with respect to the pure lexicographical ordering with $x \succ y \succ z$, is

$$\{\ 2x+1, z-1, 2y^2-1\ \}, \tag{3.6.6}$$

whereas that of (3.6.5) is

$$\left\{ \begin{array}{c} 16z^6+8z^5+9z^4+61z^3+136z^2-206z+60, \\ 284y^2+5664z^5+5568z^4+5866z^3+24365z^2+59937z-43978, \\ 568x-2736z^5-2680z^4-2771z^3-11793z^2-28946z+21382 \end{array} \right\} \tag{3.6.7}$$

We shall say that the Gröbner basis (3.6.2) *decomposes* into the *components* (3.6.6) and (3.6.7). It is precisely the fact that factorization of the Gröbner basis polynomials leads to decomposition of the basis as illustrated above that `gsolve()` exploits in order to enhance the efficiency of Buchberger's algorithm.

We are now ready to discuss the output of `gsolve()` . We invoke the MAPLE function `gsolve()` , with the system (3.6.1) as input, using the following commands:

```
> with(Groebner):
> gsolve({4*x∧2+x*y∧2-z+1/4,2*x+y∧2*z+1/2,x∧2*z-x/2-y∧2},{x,y,z});
```

`gsolve()` then returns:

$\{[[16z^6+8z^5+9z^4+61z^3+136z^2-206z+60, 284y^2+5664z^5+5568z^4+5866z^3+24365z^2+59937z-43978, 568x-2736z^5-2680z^4-2771z^3-11793z^2-28946z+21382], \mathrm{plex}(x,y,z), \{z-1\}], [[z-1, 2y^2-1, 2x+1], \mathrm{plex}(x,y,z), \{\}]\}$

This output is a `set` of two `list`'s, each of which is of the form

$$[\ G, T, \{S\}\ ],$$

where G is a Gröbner basis of a sub-system of the reduced Gröbner basis (3.6.2) of the original system (3.6.1), T is the monomial ordering with respect to which G is constructed, and S is a set of polynomial(s) such that G is indeed a sub-system only under the assumption that the polynomial(s) in S are non-zero.

The first entry of first list is the Gröbner basis (3.6.7) of the sub-system (3.6.5) of the reduced Gröbner basis (3.6.2) of the original system (3.6.1). Similarly, the first entry of the second list is the Gröbner basis (3.6.6) of the other sub-system (3.6.4).

Note, incidentally, that the sub-systems (3.6.4) and (3.6.5) are not explicitly returned by `gsolve()` . We have identified them by first determining the reduced Gröbner basis of the original system (3.6.1) and then realizing that one of the basis polynomial factors (see (3.6.3)). We computed them in order to illustrate how to interpret the output of `gsolve()` . In general, constructing the reduced Gröbner basis from the output of `gsolve()` is neither easy nor necessary (as far as obtaining solutions to the given system is concerned).

In this example, $\{z-1, 2y^2-1, 2x+1\}$ is a basis simple enough to allow us to read off *some* of the solutions of (3.6.1); however, since the other basis is rather complicated, in this case, we are unable to determine *all* the solutions of (3.6.1) by inspection. Fortunately, in the systems that appear in Chapter 4, the decompositions happen to be sufficiently simple that we can read off *all* the solutions to those systems.

The second entries, $\text{plex}(x, y, z)$, indicate that the Gröbner bases are with respect to the pure lexicographical ordering with $x \succ y \succ z$. The third entry of the first list, $\{z-1\}$, indicates that sub-system applies under the assumptions that $z - 1 \neq 0$. Again, this is obvious from the factorization (3.6.3).

Chapter 4

Reduction of the Problem & Proof of the Inadmissibility of Two Cases

4.1 The MAPLE Package `NPspinor`

The component equations of the spinor equations (2.3.7), ... , (2.3.13) (see § 2.3) are the primary tools we will use in Chapters 4 to 6.

Note that a totally symmetric spinor equation with n dotted and n undotted indices has $(n+1)^2$ (symmetrized) components. Now, each of (2.3.7), ... , (2.3.12) is symmetrized in the dotted and undotted indices separately, and hence the 0-index condition has one component, the 1-index condition has four components, ... , the 5-index condition has thirty-six components, etc.

The total number of component equations of (2.3.7), ... , (2.3.12) is thus $(0+1)^2 + (1+1)^2 + \cdots + (5+1)^2 = 91$, and a considerable number of them involve hundreds of terms. (See Appendices A and B.) In addition, the spacetime must also satisfy the Newman-Penrose field equations, the Bianchi identities, which together give another twenty-nine equations. (See Appendix A.)

Due to the large number and sizes of the equations to work with, the use of a symbolic algebra system is necessary. The computations in Chapters 4 to 6 are carried out using the symbolic algebra system MAPLE . The expansion of the spinor equations (2.3.10), ... , (2.3.13) into their component equations is performed using the MAPLE package `NPspinor` [13], developed primarily by Czapor. The actual MAPLE code used for the expansion of (2.3.10), ... , (2.3.13) can be found in Appendix C.

4.2 A Proposition on the General Petrov Type D Spacetime

Proposition 4.2.1 *The validity of Huygens' principle for any non-self-adjoint scalar wave equation on a Petrov type D spacetime implies that with respect to any canonical spinor dyad (one in which the only non-vanishing component of the Weyl spinor is Ψ_2), the following equation holds:*

$$\kappa\bar{\sigma}\bar{\lambda}\nu - \bar{\kappa}\sigma\lambda\bar{\nu} = 0 \tag{4.2.1}$$

Proof Let $\mathcal{M}$ be a Petrov type D spacetime[1] on which there exists a non-self-adjoint scalar wave equation that satisfies Huygens' principle. Suppose on the contrary that (4.2.1) does not hold with respect to some canonical spinor dyad on $\mathcal{M}$. Note, first of all, that this implies none of κ, σ, λ, and ν can vanish with respect to this dyad.

Since the dyad is canonical, all components of the Weyl spinor vanish except Ψ_2. Equations (A.6.6), (A.6.11), (A.6.15) and (A.6.16) then lead to

[1] $\mathcal{M}$ could be just a geodesically convex domain of a larger spacetime.

the following homogeneous linear system of equations in ϕ_0, ϕ_2 and their conjugates:

$$\begin{pmatrix} \overline{\Psi}_2\bar{\lambda} & 0 & 0 & \Psi_2\sigma \\ 0 & \Psi_2\lambda & \overline{\Psi}_2\bar{\sigma} & 0 \\ \overline{\Psi}_2\bar{\kappa} & \Psi_2\kappa & 0 & 0 \\ 0 & 0 & \overline{\Psi}_2\bar{\nu} & \Psi_2\nu \end{pmatrix} \begin{pmatrix} \phi_0 \\ \bar{\phi}_0 \\ \phi_2 \\ \bar{\phi}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.2.2}$$

The determinant of the coefficient matrix in (4.2.2) is $-\Psi_2^2\overline{\Psi}_2^2(\kappa\bar{\sigma}\bar{\lambda}\nu - \bar{\kappa}\sigma\lambda\bar{\nu})$. Since Ψ_2 does not vanish on $\mathcal{M}$, if (4.2.1) does not hold, then both ϕ_0, and ϕ_2 must vanish (and so must their conjugates). This in turn implies $\phi_1 \neq 0$ and $\bar{\phi}_1 \neq 0$, since by hypothesis the scalar wave equation is non-self-adjoint.

With the conditions that $\phi_0 = \bar{\phi}_0 = \phi_2 = \bar{\phi}_2 = 0$, (A.6.7) and (A.6.12) lead to the following homogeneous linear system of equations in ϕ_1 and $\bar{\phi}_1$:

$$\begin{pmatrix} 2\overline{\Psi}_2\kappa & 6\Psi_2\kappa \\ -6\overline{\Psi}_2\bar{\kappa} & -2\Psi_2\bar{\kappa} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \bar{\phi}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.2.3}$$

Since $\phi_1 \neq 0$ and $\bar{\phi}_1 \neq 0$, the determinant of the coefficient matrix in (4.2.3) must vanish. This determinant is $32\Psi_2\overline{\Psi}_2\kappa\bar{\kappa}$, implying that κ vanishes; this contradicts our original assumption that (4.2.1) does not hold. □

4.3 Alignment of Principal Null Directions between the Maxwell Spinor and the Weyl Spinor

We decompose the question of the validity of Huygens' principle on a Petrov type D spacetime into sub-cases according to the alignment of the two principal null directions of the Maxwell spinor with the two doubly degenerate principal null directions of the Weyl spinor.

We fix a canonical spinor dyad, $\{o_A, \iota_B\}$, for the underlying Petrov type D spacetime, i.e. a spinor dyad with respect to which the only non-vanishing component of the Weyl spinor is Ψ_2. Being totally symmetric, by equation (2.2.1), the Maxwell spinor ϕ_{AB} takes the form:

$$\phi_{AB} = \xi_{(A}\zeta_{B)}, \tag{4.3.1}$$

where ξ_A and ζ_A are the principal spinors of the Maxwell spinor.

The alignment of the Maxwell principal spinors with the Weyl principal spinors determines which of the Maxwell spinor components vanish with respect to the chosen canonical spinor dyad. This is shown in Table 4.1, where 0 and N represent the vanishing and non-vanishing of the corresponding Maxwell spinor component respectively. Note that

(1) Case 0 is the self-adjoint case.

(2) Case 1 and Case 4 are equivalent; so are Case 3 and Case 6. We can see this equivalence by interchanging o_A and ι_B. So, there are in fact only five geometrically distinct sub-cases that are non-self-adjoint.

We now give the geometric meaning of these sub-cases. First we express ϕ_{AB} with respect to the chosen canonical spinor dyad $\{o_A, \iota_B\}$ as follows:

$$\phi_{AB} = \phi_0\, \iota_A \otimes \iota_B - 2\,\phi_1\, o_{(A} \otimes \iota_{B)} + \phi_2\, o_A \otimes o_B, \tag{4.3.2}$$

If both Maxwell principal spinors are aligned with, say ι_A, then we see from (4.3.2) that $\phi_1 = \phi_2 = 0$. This corresponds to Case 4 shown in Table 4.1. If both Maxwell principal spinors are aligned with o_A instead, then again by (4.3.2), we must have $\phi_0 = \phi_1 = 0$. This corresponds to Case 1 in Table 4.1. Also, if one of the Maxwell principal spinors is aligned with o_A and the

Table 4.1: Possible alignments between the Maxwell and Weyl principal spinors

Case	$\phi_{AB} \propto$	ϕ_0	ϕ_1	ϕ_2
0	0	0	0	0
1	$o_A o_B$	0	0	N
2	$o_{(A}\iota_{B)}$	0	N	0
3		0	N	N
4	$\iota_A \iota_B$	N	0	0
5		N	0	N
6		N	N	0
7		N	N	N

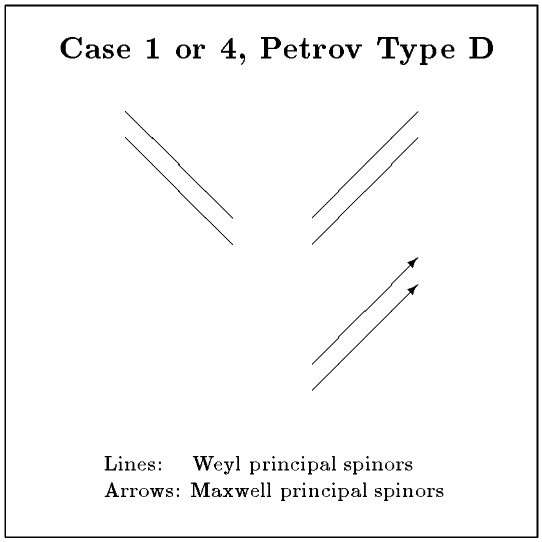
other with ι_A, then clearly $\phi_0 = \phi_2 = 0$. This corresponds to Case 2. The geometric interpretations of the other cases are similar.

4.4 Inadmissibility of Cases 1 and 4

This following proposition is the main result of this thesis. It states that Case 1 and Case 4 as discussed in §4.3 are inadmissible if Huygens' principle is to hold on a Petrov type D spacetime.

Proposition 4.4.1 *Let $P := \Box + A^a\nabla_a + C$ be a non-self-adjoint scalar wave operator on a Petrov type D spacetime. If the Maxwell spinor (or tensor) associated to A^a is algebraically special and its degenerate principal null direction coincides with one of the doubly degenerate principal null directions of the Weyl spinor (or tensor) of the underlying spacetime, then P is not a Huygens' operator.*

Outline of proof Restated in terms of spinor components, this proposition states precisely that, with respect to a suitable canonical spinor dyad of the underlying Petrov type D spacetime, Case 1 and Case 4 described in §4.3 are inadmissible for any Huygens' operator. The alignment of the principal spinors between the Maxwell and Weyl spinors in Case 1 or 4 is depicted in the following diagram:



We shall make extensive use of the necessary conditions for Huygens' principle listed in Appendix A, with ϕ_1 and ϕ_2 set to zero. In other words, we shall establish explicitly the inadmissibility of Case 4. The inadmissibility of Case 1 then follows trivially since the two cases are equivalent geometrically. The proof will proceed in the following steps:

(1) We simplify the necessary conditions by eliminating sequentially four sets of variables[2] and seven sets of Pfaffian derivatives.

(2) Due to factorization of a number of the simplified necessary conditions, the analysis splits into a number of further sub-cases.

(3) In each such sub-case, Ψ_2 is solved for as a polynomial expression of α, τ, π and their conjugates. Subsequently, each sub-case is shown to

[2] spin coefficients and components of the curvature spinor

lead to a system of multivariate polynomial equations in α, τ, π and their conjugates.

(4) Gröbner basis methods are used to prove that all these polynomial systems of equations admit only trivial solutions, i.e. each implies that all of α, τ, π and their conjugates must vanish.

(5) Substitution of $\alpha = \bar{\alpha} = \tau = \bar{\tau} = \pi = \bar{\pi} = 0$ into the expression for Ψ_2 obtained in Step (3) then implies that Ψ_2 vanishes, contrary to the original assumption that the chosen spinor dyad is canonical in the underlying Petrov type D spacetime. $\square$

Proof of Proposition 4.4.1 Suppose Proposition 4.4.1 is false, i.e. there exists a Petrov type D spacetime $(\mathcal{M}, g_{\alpha\beta})$ and a Huygens' scalar wave operator $P := \Box + A^\alpha \nabla_\alpha + B$ on $\mathcal{M}$ such that the Maxwell spinor ϕ_{AB} associated to A^α has the form $\phi_{AB} = \phi_0 \iota_A \iota_B$, with $\phi_0 \neq 0$, where $\{o_A, \iota_A\}$ is a spinor dyad canonical to the Weyl spinor of $(\mathcal{M}, g_{\alpha\beta})$.

Let w be the smooth function defined on $\mathcal{M}$ by

$$e^w = \frac{\Psi_2}{\phi_0}.$$

Then under the dyad transformation

$$o'_A = e^{w/2}\, o_A \qquad \iota'_A = e^{-w/2}\, \iota_A,$$

the Maxwell spinor component ϕ_0 transform as follows:

$$\phi'_0 = e^w\, \phi_0 = \frac{\Psi_2}{\phi_0}\phi_0 = \Psi_2 = \Psi'_2,$$

while ϕ'_1, and ϕ'_2 remain zero. The above assertions follow from the transformation laws of the respective spinor components listed in §2.4.

Next, let ϕ be the smooth function on $\mathcal{M}$ defined by $e^{4\phi} = \Psi_2' \overline{\Psi}_2'$. Under a conformal transformation to $(\mathcal{M}, e^{2\phi} g_{\alpha\beta})$, and the following choices for the van der Waerden correspondence and spinor dyad:

$$
\begin{aligned}
\epsilon''^{AB} &= \epsilon'^{AB}, & \qquad \epsilon''_{AB} &= \epsilon'_{AB}, \\
\sigma''_{\alpha}{}^{A\dot{A}} &= e^{\phi}\, \sigma'_{\alpha}{}^{A\dot{A}}, & \qquad \sigma''^{\alpha}{}_{A\dot{A}} &= e^{-\phi}\, \sigma'^{\alpha}{}_{A\dot{A}}, \\
o''_A &= e^{\frac{r-1}{2}}\, o'_A, & \qquad \iota''_A &= e^{\frac{1-r}{2}}\, \iota'_A,
\end{aligned}
$$

Ψ_2' transforms (see §2.4) as follows

$$\Psi_2'' = e^{-2\phi}\, \Psi_2'.$$

As a result,

$$\Psi_2''\, \overline{\Psi}_2'' = (e^{-2\phi}\, \Psi_2')(e^{-2\phi}\, \overline{\Psi}_2') = e^{-4\phi} \Psi_2 \overline{\Psi}_2 = e^{-4\phi} e^{4\phi} \equiv 1,$$

i.e. Ψ_2'' is identically unimodular. If we choose $r = 1$, then ϕ_0' transforms (see §2.4) as follows

$$\phi_0'' = e^{(r-3)\phi}\, \phi_0' = e^{(r-3)\phi}\, \Psi_2' = e^{-2\phi}\, \Psi_2' = \Psi_2''.$$

Therefore, under the above conformal and dyad transformations, we have made

$$\Psi_2'' \overline{\Psi}_2'' \equiv 1 \;\text{ and }\; \phi_0'' \equiv \Psi_2''. \tag{4.4.1}$$

Since the Huygens' nature of P is preserved under trivial transformations, P'', the transformed operator of P under the above transformations, is also Huygens'. In what follows, we assume that these transformations have been made, hence (4.4.1) holds, and will drop the double prime.

Without further mention, all references to the component necessary conditions assume that the substitutions $\phi_0 = \Psi_2$ as well as the Case 4 assumptions ($\phi_0 \neq 0$, $\phi_1 = \phi_2 = 0$) have been made. Then,

$$\left\{ \begin{array}{rcl} (\mathrm{A}.4.2) & \Longrightarrow & \boxed{\begin{array}{c} \lambda = \bar{\lambda} = 0 \\ \nu = \bar{\nu} = 0 \\ \bar{\kappa} = -\kappa \end{array}} \\ (\mathrm{A}.4.4) & \Longrightarrow & \\ (\mathrm{A}.6.11) & \Longrightarrow & \end{array} \right. \qquad (\mathcal{V}1)$$

Making the substitutions ($\mathcal{V}1$) into the following equations give:

$$\left\{ \begin{array}{rcl} (\mathrm{A}.1.7) & \Longrightarrow & \bar{\delta}(\pi) = -\bar{\sigma}\mu - \pi^2 - \pi\alpha + \pi\bar{\beta} - \Phi_{20} \\ (\mathrm{A}.1.9) & \Longrightarrow & \Delta(\pi) = -\mu\bar{\tau} - \mu\pi - \pi\gamma + \pi\bar{\gamma} - \Phi_{21} \\ (\mathrm{A}.1.13) & \Longrightarrow & \bar{\delta}(\mu) = -\mu\pi + \pi\bar{\mu} - \mu\alpha - \mu\bar{\beta} - \Phi_{21} \\ (\mathrm{A}.1.14) & \Longrightarrow & \Delta(\mu) = -\mu^2 - \mu\gamma - \mu\bar{\gamma} - \Phi_{22} \\ (\mathrm{A}.4.1) & \Longrightarrow & \bar{\delta}(\Psi_2) = -\pi\Psi_2 + 2\Psi_2\alpha \\ (\mathrm{A}.4.3) & \Longrightarrow & \Delta(\Psi_2) = -\mu\Psi_2 + 2\Psi_2\gamma \\ (\mathrm{A}.6.7) & \Longrightarrow & D(\Psi_2) = \frac{1}{8}\Psi_2(27\rho + 6\bar{\epsilon} - 9\bar{\sigma} + 3\sigma - 2\epsilon - 9\bar{\rho}) \end{array} \right. \qquad (\mathcal{D}1)$$

If we now make the substitution ($\mathcal{V}1$) into (A.6.5), and then make the substitution for $\Delta\Psi_2$ and its conjugates as in ($\mathcal{D}1$) into the resulting equation, we get:

$$\begin{aligned} (\mathrm{A}.6.5) \overset{(\mathcal{V}1)}{\Longrightarrow}\ & 2\Psi_2\bar{\Psi}_2\bar{\mu} - \Psi_2\Delta(\bar{\Psi}_2) + 2\Psi_2\bar{\Psi}_2\bar{\gamma} - 9\bar{\Psi}_2\Psi_2\mu - 3\bar{\Psi}_2\Delta(\Psi_2) = 0 \\ \overset{(\mathcal{D}1)}{\Longrightarrow}\ & -3\Psi_2\bar{\Psi}_2(-\bar{\mu} + 2\mu + 2\gamma) = 0 \end{aligned} \qquad (4.4.2)$$

(4.4.2) and its conjugate equation then imply:

$$\boxed{\gamma = -\mu + \tfrac{\bar{\mu}}{2}, \quad \text{and} \quad \bar{\gamma} = -\bar{\mu} + \tfrac{\mu}{2}.} \qquad (\mathcal{V}2')$$

Next, we show that $\mu = 0$. Employing an "implication chart"[3] as in (4.4.2), we have:

$$
\begin{aligned}
\text{(A.7.21)} \quad \overset{(\mathcal{V}1)(\mathcal{V}2')(\mathcal{D}1)}{\Longrightarrow} \quad & 72\bar{\Psi}_2\Psi_2\Delta(\mu) + 72\Psi_2\bar{\Psi}_2\Delta(\bar{\mu}) + 72\Psi_2\bar{\mu}\Delta(\bar{\Psi}_2) \\
& +72\bar{\Psi}_2\mu\Delta(\Psi_2) - 1440\Psi_2\mu\bar{\Psi}_2\bar{\mu} - 1152\bar{\Psi}_2\Psi_2\mu\bar{\gamma} \\
& +648\bar{\Psi}_2\Psi_2\mu^2 + 648\Psi_2\bar{\Psi}_2\bar{\mu}^2 + 288\Psi_2\Phi_{22}\bar{\Psi}_2 \\
& -1152\Psi_2\bar{\Psi}_2\bar{\mu}\gamma - 144\bar{\Psi}_2\Psi_2\Delta(\gamma) - 144\bar{\Psi}_2\Delta(\Psi_2)\gamma \\
& -144\Psi_2\bar{\Psi}_2\Delta(\bar{\gamma}) - 144\Psi_2\Delta(\bar{\Psi}_2)\bar{\gamma} - 1728\Psi_2\gamma\bar{\Psi}_2\bar{\gamma} = 0 \\[1em]
\overset{(\mathcal{V}1)(\mathcal{V}2')(\mathcal{D}1)}{\Longrightarrow} \quad & -72\Psi_2\bar{\Psi}_2(-\Delta(\mu) - \Delta(\bar{\mu}) + 16\mu\bar{\mu} - 5\mu\gamma + 3\mu\bar{\gamma} \\
& -9\mu^2 - 9\bar{\mu}^2 - 2\Phi_{22} - 5\bar{\mu}\bar{\gamma} + 3\bar{\mu}\gamma) = 0 \\[1em]
\overset{(\mathcal{V}1)(\mathcal{V}2')(\mathcal{D}1)}{\Longrightarrow} \quad & 144\Psi_2\bar{\Psi}_2(\mu - \bar{\mu})^2 = 0 \\[1em]
\Longrightarrow \quad & \mu - \bar{\mu} = 0 \qquad (4.4.3)
\end{aligned}
$$

On the other hand, from the unimodularity of Ψ_2, we have:

$$
\begin{aligned}
\Delta(\Psi_2\overline{\Psi}_2 = 1) \quad & \Longleftrightarrow \quad \Psi_2\Delta(\bar{\Psi}_2) + \bar{\Psi}_2\Delta(\Psi_2) = 0 \\
& \overset{(\mathcal{D}1)}{\Longrightarrow} \quad \Psi_2\bar{\Psi}_2(2\gamma - \mu + 2\bar{\gamma} - \bar{\mu}) = 0 \\
& \overset{(\mathcal{V}2')}{\Longrightarrow} \quad -2\Psi_2\bar{\Psi}_2(\mu + \bar{\mu}) = 0 \qquad (4.4.4)
\end{aligned}
$$

$$\text{Hence,} \quad (4.4.3) \quad \text{and} \quad (4.4.4) \quad \Longrightarrow \quad \boxed{\mu = \bar{\mu} = 0} \qquad (\mathcal{V}3)$$

[3]This technique of repeated substitutions will be routinely used in this proof and some computational details will be omitted in later arguments.

$$\text{And,} \quad (\mathcal{V}2') \quad \text{and} \quad (\mathcal{V}3) \implies \boxed{\gamma = \bar{\gamma} = 0} \tag{$\mathcal{V}2$}$$

Using the unimodularity of Ψ_2 again, we can obtain expressions for $\delta\Psi_2$ and its conjugate; they are the missing Pfaffian derivatives of Ψ_2 not solved for in $(\mathcal{D}1)$.

$$\begin{aligned}
\delta(\Psi_2\overline{\Psi}_2 = 1) &\iff & \Psi_2\delta(\bar{\Psi}_2) + \bar{\Psi}_2\delta(\Psi_2) = 0 \\
&\overset{(\mathcal{D}1)}{\implies} & -\bar{\Psi}_2(\Psi_2\bar{\pi} - 2\Psi_2\bar{\alpha} - \delta(\Psi_2)) = 0 \\
&\implies & \boxed{\delta(\Psi_2) = \Psi_2\bar{\pi} - 2\Psi_2\bar{\alpha}}
\end{aligned} \tag{$\mathcal{D}2$}$$

We continue with our simplifications:

$$\begin{array}{llcl}
(\text{A.1.13}) & & \overset{(\mathcal{V}1)..(\mathcal{V}3)}{\implies} & \\
(\text{A.1.14}) & & \overset{(\mathcal{V}1)..(\mathcal{V}3)}{\implies} & \\
(\text{A.6.2}) & \overset{(\mathcal{D}1)(\mathcal{D}2)}{\implies} & 3\Psi_2\bar{\Psi}_2(7\pi + 4\alpha - 2\bar{\beta}) = 0 & \implies \\
(\text{A.6.13}) & \overset{(\mathcal{D}1)(\mathcal{D}2)}{\implies} & 3\Psi_2\bar{\Psi}_2(7\bar{\pi} + 4\bar{\alpha} - 2\beta) = 0 & \implies
\end{array}
\boxed{\begin{array}{c}
\Phi_{21} = \Phi_{12} = 0 \\
\Phi_{22} = 0 \\
\beta = \frac{7}{2}\bar{\pi} + 2\bar{\alpha} \\
\bar{\beta} = \frac{7}{2}\pi + 2\alpha
\end{array}} \tag{$\mathcal{V}4$}$$

The third set of solved Pfaffians is obtained by repeatedly substituting all of $(\mathcal{V}1)$, ... , $(\mathcal{V}4)$, $(\mathcal{D}1)$, and $(\mathcal{D}2)$ into:

$$\begin{array}{rcl} (A.1.6) & \Longrightarrow & \Delta(\epsilon) = -\alpha\tau - \alpha\bar{\pi} - \frac{7}{2}\bar{\pi}\bar{\tau} - 2\bar{\alpha}\bar{\tau} - \frac{7}{2}\pi\bar{\pi} \\ & & \quad -2\pi\bar{\alpha} - \tau\pi - \Psi_2 + \Lambda - \Phi_{11} \\ (A.1.8) & \Longrightarrow & \delta(\pi) = -9/2\pi\bar{\pi} - \pi\bar{\alpha} - \Psi_2 - 2\Lambda \\ (A.1.15) & \Longrightarrow & \Delta(\bar{\alpha}) = 0 \\ (A.2.6) & \Longrightarrow & \Delta(\Phi_{11}) = 0 \\ (A.2.7) & \Longrightarrow & \Delta(\Phi_{20}) = 0 \\ (A.5.9) & \Longrightarrow & \bar{\delta}(\alpha) = -\frac{1}{2}\pi\alpha - \alpha^2 + \frac{1}{2}\Phi_{20} - \pi^2 \\ (A.7.17) & \Longrightarrow & \delta(\sigma) = -\frac{3}{2}\bar{\pi}\sigma - 3\bar{\alpha}\sigma + 3\tau\sigma \end{array} \qquad (\mathcal{D}3)$$

Next, we note that:

$$\begin{array}{lcl} (A.1.12) & \overset{(\mathcal{V}1)\ldots(\mathcal{V}4)}{\Longrightarrow} & 4\delta(\alpha) - 14\bar{\delta}(\bar{\pi}) - 8\bar{\delta}(\bar{\alpha}) - 4\alpha\bar{\alpha} - 49\pi\bar{\pi} \\ & & -28\pi\bar{\alpha} + 4\Psi_2 - 4\Lambda - 4\Phi_{11} = 0 \\ & \overset{(\mathcal{D}3)}{\Longrightarrow} & 2\delta(\alpha) + 7\pi\bar{\pi} + 7\alpha\bar{\pi} + 7\bar{\Psi}_2 + 12\Lambda - 4\bar{\delta}(\bar{\alpha}) \\ & & -2\alpha\bar{\alpha} - 14\pi\bar{\alpha} + 2\Psi_2 - 2\Phi_{11} = 0 \\ & \Longrightarrow & \delta(\alpha) = \frac{7}{2}\pi\bar{\pi} - \frac{7}{2}\alpha\bar{\pi} + \frac{11}{6}\bar{\Psi}_2 + 6\Lambda - \alpha\bar{\alpha} + \frac{8}{3}\Psi_2 - \Phi_{11} \end{array}$$

$$\begin{array}{lcl} (A.2.11) & \overset{(\mathcal{V}1)\ldots(\mathcal{V}4)}{\Longrightarrow} & \Delta(\Phi_{11}) + 3\Delta(\Lambda) = 0 \\ & \overset{(\mathcal{D}3)}{\Longrightarrow} & \Delta(\Lambda) = 0 \end{array}$$

$$\begin{array}{lcl} (A.5.3) & \overset{(\mathcal{V}1)\ldots(\mathcal{V}4)}{\Longrightarrow} & -10\Delta(\Psi_2)\tau + 6\Delta(\Psi_2)\bar{\alpha} + 2\Delta(\delta(\Psi_2)) \\ & & -6\Psi_2\Delta(\tau) + 2\delta(\Delta(\Psi_2)) + 7\Delta(\Psi_2)\bar{\pi} = 0 \\ & \overset{(\mathcal{V}4)(\mathcal{D}1)\ldots(\mathcal{D}3)}{\Longrightarrow} & -\Psi_2\Delta(\tau) = 0 \\ & \Longrightarrow & \Delta(\tau) = 0 \end{array}$$

Therefore, the fourth set of solved Pfaffian derivatives is:

$$\boxed{\begin{array}{c} \Delta(\Lambda) = \Delta(\tau) = 0 \\ \\ \delta(\alpha) = \tfrac{7}{2}\pi\bar{\pi} - \tfrac{7}{2}\alpha\bar{\pi} + \tfrac{11}{6}\bar{\Psi}_2 + 6\Lambda - \alpha\bar{\alpha} + \tfrac{8}{3}\Psi_2 - \Phi_{11} \end{array}} \qquad (\mathcal{D}4)$$

Now substitute all of $(\mathcal{V}1) \ldots (\mathcal{V}4)$, and $(\mathcal{D}1) \ldots (\mathcal{D}4)$ into to the following three equations to obtain an (over-determined) system of equations linear in the two Pfaffian derivatives $\Delta(\sigma)$ and $\delta(\tau)$:

$$\text{(A.1.16)} \implies \begin{array}{l} 2\delta(\tau) - 2\Delta(\sigma) - 2\tau^2 \\ -2\tau\bar{\alpha} - 7\bar{\pi}\tau - 2\Phi_{02} = 0 \end{array} \tag{4.4.5}$$

$$\text{(A.5.6)} \implies \begin{array}{l} 2\Phi_{02} - 4\bar{\pi}^2 - 8\bar{\pi}\bar{\alpha} - 16\bar{\alpha}^2 - 18\tau^2 \\ -9\bar{\pi}\tau - 30\tau\bar{\alpha} + 6\Delta(\sigma) + 6\delta(\tau) = 0 \end{array} \tag{4.4.6}$$

$$\text{(A.7.12)} \implies \begin{array}{l} -6\tau^2 + 8\bar{\pi}^2 - 32\bar{\alpha}^2 - 6\delta(\tau) - 47\bar{\pi}\tau \\ -50\tau\bar{\alpha} - 24\bar{\pi}\bar{\alpha} + 2\Delta(\sigma) - 2\Phi_{02} = 0 \end{array} \tag{4.4.7}$$

Solving (4.4.5) and (4.4.7) yields:

$$\delta(\tau) = \frac{7}{6}\bar{\pi}^2 - \frac{10}{3}\bar{\alpha}^2 - \frac{11}{2}\bar{\pi}\tau - 5\tau\bar{\alpha} - \frac{8}{3}\bar{\pi}\bar{\alpha} - \frac{1}{3}\Phi_{02} \tag{4.4.8}$$

$$\Delta(\sigma) = 4\bar{\pi}\bar{\alpha} + 6\bar{\alpha}^2 - \frac{1}{2}\bar{\pi}^2 + 3\tau^2 + 7\bar{\pi}\tau + 10\tau\bar{\alpha} \tag{4.4.9}$$

Substituting (4.4.8) and (4.4.9) into (4.4.5), we get

$$\Phi_{02} = \frac{5}{4}\bar{\pi}^2 - 7\bar{\alpha}^2 - 12\bar{\pi}\tau - 12\tau\bar{\alpha} - 5\bar{\pi}\bar{\alpha} - 3\tau^2 \tag{4.4.10}$$

Therefore,

$$(4.4.8) - \frac{4}{3} \times (4.4.10) \Longrightarrow$$
$$\delta(\tau) = -\frac{1}{2}\bar{\pi}^2 + 6\bar{\alpha}^2 + \frac{21}{2}\bar{\pi}\tau + 11\tau\bar{\alpha} + 4\bar{\pi}\bar{\alpha} + \Phi_{02} + 4\tau^2 \quad (4.4.11)$$

(4.4.9) and (4.4.11) together give the fifth set of solved Pfaffians:

$$\boxed{\begin{array}{rcl} \Delta(\sigma) & = & 4\bar{\pi}\bar{\alpha} + 6\bar{\alpha}^2 - \frac{1}{2}\bar{\pi}^2 + 3\tau^2 + 7\bar{\pi}\tau + 10\tau\bar{\alpha} \\ \delta(\tau) & = & -\frac{1}{2}\bar{\pi}^2 + 6\bar{\alpha}^2 + \frac{21}{2}\bar{\pi}\tau + 11\tau\bar{\alpha} + 4\bar{\pi}\bar{\alpha} + \Phi_{02} + 4\tau^2 \end{array}} \quad (\mathcal{D}5)$$

If we apply the commutator relation (A.3.1) on Ψ_2 and then make the substitutions $(\mathcal{V}1)$ ... $(\mathcal{V}4)$, and $(\mathcal{D}1)$ ... $(\mathcal{D}5)$, we get:

$$\begin{array}{l} -\Psi_2(-24\bar{\pi}\bar{\alpha} - 54\Delta(\rho) + 18\Delta(\bar{\rho}) + 54\bar{\tau}^2 - 4\Psi_2 + 12\bar{\Psi}_2 + 8\Phi_{11} \\ +36\bar{\alpha}\bar{\tau} - 18\bar{\pi}\bar{\tau} + 54\tau\pi + 28\pi\bar{\pi} - 8\Lambda - 9\pi^2 + 126\bar{\tau}\pi + 36\pi\bar{\alpha} \\ -18\tau^2 - 60\tau\bar{\alpha} - 36\bar{\alpha}^2 + 108\alpha^2 + 3\bar{\pi}^2 - 12\alpha\tau - 12\alpha\bar{\pi} + 72\pi\alpha \\ -42\bar{\pi}\tau + 180\alpha\bar{\tau}) = 0 \end{array} \quad (4.4.12)$$

(4.4.12) and its conjugate allow us to solve for $\Delta(\rho)$ (and $\Delta(\bar{\rho})$), in particular, they give:

$$\begin{array}{rcl} \Delta(\rho) & = & \bar{\tau}^2 + \frac{2}{9}\bar{\Psi}_2 + \frac{2}{9}\Phi_{11} + \frac{2}{3}\bar{\alpha}\bar{\tau} + \tau\pi + \frac{7}{9}\pi\bar{\pi} - \frac{2}{9}\Lambda \\ & & -\frac{1}{6}\pi^2 + \frac{7}{3}\bar{\tau}\pi + \frac{2}{3}\pi\bar{\alpha} + 2\alpha^2 + \frac{4}{3}\pi\alpha + \frac{10}{3}\alpha\bar{\tau} \end{array} \quad (4.4.13)$$

We next substitute $(\mathcal{V}1)$ ... $(\mathcal{V}4)$, and $(\mathcal{D}1)$ ... $(\mathcal{D}5)$ and (4.4.13) into:

$$\begin{array}{lcl} & & 8\bar{\tau}^2 + 4\bar{\Psi}_2 + 4\Phi_{11} + 12\bar{\alpha}\bar{\tau} - 45\tau\pi + 14\pi\bar{\pi} + 32\Lambda \\ (\mathrm{A.1.17}) & \Longrightarrow & -3\pi^2 + 42\bar{\tau}\pi + 12\pi\bar{\alpha} + 36\alpha^2 + 24\pi\alpha + 60\alpha\bar{\tau} - 18\bar{\delta}(\tau) \\ & & +18\tau\bar{\tau} - 18\alpha\tau + 18\Psi_2 = 0 \\ & & \\ & & 8\bar{\delta}(\tau) + 18\bar{\tau}^2 + 2\Psi_2 + 20\bar{\Psi}_2 - 20\Phi_{11} + 12\bar{\alpha}\bar{\tau} + 117\tau\pi \\ (\mathrm{A.5.2}) & \Longrightarrow & +26\pi\bar{\pi} + 32\Lambda - 3\pi^2 + 48\alpha\bar{\alpha} + 42\bar{\tau}\pi + 36\pi\bar{\alpha} + 18\tau\bar{\tau} \\ & & +36\alpha^2 + 90\alpha\tau + 24\pi\alpha + 60\alpha\bar{\tau} = 0 \end{array} \tag{4.4.14}$$

Solving (4.4.14) as a linear system for the two unknowns Λ and $\bar{\delta}(\tau)$ yields:

$$\bar{\delta}(\tau) = \frac{4}{9}\Psi_2 - \frac{4}{9}\bar{\Psi}_2 + \frac{2}{3}\Phi_{11} - \frac{9}{2}\tau\pi - \frac{1}{3}\pi\bar{\pi} - \frac{4}{3}\alpha\bar{\alpha} - \frac{2}{3}\pi\bar{\alpha} - 3\alpha\tau \tag{4.4.15}$$

$$\begin{aligned} \Lambda = & -\frac{9}{16}\bar{\tau}^2 - \frac{5}{16}\Psi_2 - \frac{3}{8}\bar{\Psi}_2 + \frac{1}{4}\Phi_{11} - \frac{3}{8}\bar{\alpha}\bar{\tau} - \frac{9}{8}\tau\pi - \frac{5}{8}\pi\bar{\pi} \\ & + \frac{3}{32}\pi^2 - \frac{3}{4}\alpha\bar{\alpha} - \frac{21}{16}\bar{\tau}\pi - \frac{3}{4}\pi\bar{\alpha} - \frac{9}{16}\tau\bar{\tau} - \frac{9}{8}\alpha^2 - \frac{9}{8}\alpha\tau \\ & - \frac{3}{4}\pi\alpha - \frac{15}{8}\alpha\bar{\tau} \end{aligned} \tag{4.4.16}$$

The sixth set of solved Pfaffians is given by (4.4.13) and (4.4.15):

$$\boxed{\begin{aligned} \bar{\delta}(\tau) &= \tfrac{4}{9}\Psi_2 - \tfrac{4}{9}\bar{\Psi}_2 + \tfrac{2}{3}\Phi_{11} - \tfrac{9}{2}\tau\pi - \tfrac{1}{3}\pi\bar{\pi} - \tfrac{4}{3}\alpha\bar{\alpha} - \tfrac{2}{3}\pi\bar{\alpha} - 3\alpha\tau \\ \Delta(\rho) &= \bar{\tau}^2 + \tfrac{2}{9}\bar{\Psi}_2 + \tfrac{2}{9}\Phi_{11} + \tfrac{2}{3}\bar{\alpha}\bar{\tau} + \tau\pi + \tfrac{7}{9}\pi\bar{\pi} - \tfrac{2}{9}\Lambda \\ &\quad -\tfrac{1}{6}\pi^2 + \tfrac{7}{3}\bar{\tau}\pi + \tfrac{2}{3}\pi\bar{\alpha} + 2\alpha^2 + \tfrac{4}{3}\pi\alpha + \tfrac{10}{3}\alpha\bar{\tau} \end{aligned}} \tag{$\mathcal{D}6$}$$

The seventh set of Pfaffians will be obtained by solving simultaneously eighteen equations linear in eighteen unknown Pfaffians.[4] To build this system, as usual we make the substitutions $(\mathcal{V}1)\ \ldots\ (\mathcal{V}4)$, and $(\mathcal{D}1)\ \ldots\ (\mathcal{D}6)$ into:

$$
\begin{aligned}
(\text{A.1.3}) \quad &\Longrightarrow \quad D(\tau) - \Delta(\kappa) - \rho\tau - \rho\bar{\pi} - \sigma\bar{\tau} - \sigma\pi - \tau\epsilon \\
&\qquad +\tau\bar{\epsilon} - \Phi_{01} = 0 \\[1em]
(\text{A.1.4}) \quad &\Longrightarrow \quad 2D(\alpha) - 2\bar{\delta}(\epsilon) - 2\alpha\rho - 2\alpha\bar{\epsilon} + 8\alpha\epsilon - 7\bar{\pi}\bar{\sigma} - 4\bar{\alpha}\bar{\sigma} \\
&\qquad +5\pi\epsilon - 2\pi\rho - 2\Phi_{10} = 0 \\[1em]
(\text{A.1.5}) \quad &\Longrightarrow \quad 7D(\bar{\pi}) + 4D(\bar{\alpha}) - 2\delta(\epsilon) - 2\sigma\alpha - 2\sigma\pi - 7\bar{\pi}\bar{\rho} - 4\bar{\alpha}\bar{\rho} \\
&\qquad +7\bar{\pi}\bar{\epsilon} + 4\bar{\alpha}\bar{\epsilon} - 2\epsilon\bar{\pi} + 2\epsilon\bar{\alpha} = 0 \\[1em]
(\text{A.1.11}) \quad &\Longrightarrow \quad 2\delta(\rho) - 2\bar{\delta}(\sigma) - 6\rho\bar{\alpha} - 7\rho\bar{\pi} + 2\sigma\alpha - 7\sigma\pi - 2\rho\tau \\
&\qquad +2\tau\bar{\rho} - 2\Phi_{01} = 0
\end{aligned}
\tag{4.4.17}
$$

[4]It has been verified that every sub-system of this 18-equation linear system involves more Pfaffians than equations. Hence, this large system has no smaller solvable subsystems for the Pfaffians.

$$(\mathrm{A}.5.7) \implies \begin{aligned}&-16D(\bar{\pi})+48D(\tau)+32D(\bar{\alpha})+48\Delta(\kappa)+4\delta(\epsilon)-12\delta(\bar{\epsilon})-6\delta(\rho)\\&+18\delta(\bar{\rho})+18\delta(\bar{\sigma})+48\bar{\delta}(\sigma)-60\tau\bar{\rho}-32\Phi_{01}+18\tau\sigma-61\bar{\pi}\bar{\rho}\\&-45\bar{\pi}\bar{\sigma}+24\bar{\pi}\sigma+14\bar{\pi}\bar{\epsilon}-94\bar{\alpha}\bar{\rho}+116\bar{\alpha}\bar{\epsilon}-126\bar{\alpha}\bar{\sigma}+60\bar{\alpha}\sigma\\&-108\tau\bar{\sigma}+42\rho\bar{\alpha}+36\rho\tau+15\rho\bar{\pi}+48\sigma\bar{\tau}+232\sigma\pi-72\tau\epsilon\\&+120\tau\bar{\epsilon}+112\sigma\alpha+6\epsilon\bar{\pi}-60\epsilon\bar{\alpha}=0\end{aligned}$$

$$(\mathrm{A}.5.8) \implies \begin{aligned}&32D(\pi)+32D(\alpha)-4\bar{\delta}(\epsilon)+12\bar{\delta}(\bar{\epsilon})+6\bar{\delta}(\rho)-18\bar{\delta}(\bar{\rho})+6\bar{\delta}(\sigma)\\&+32\Phi_{10}+16\bar{\pi}\bar{\sigma}-32\bar{\alpha}\bar{\sigma}-48\tau\bar{\sigma}-38\pi\bar{\epsilon}+9\pi\bar{\rho}+36\pi\bar{\sigma}\\&-18\alpha\bar{\rho}+36\alpha\bar{\sigma}-3\sigma\pi+38\alpha\rho-20\alpha\bar{\epsilon}+28\alpha\epsilon+34\pi\epsilon\\&+29\pi\rho+6\sigma\alpha-54\bar{\tau}\bar{\sigma}=0\end{aligned}$$

$$(\mathrm{A}.7.6) \implies \begin{aligned}&-96D(\bar{\tau})-96D(\alpha)+32\Delta(\kappa)+32\delta(\bar{\sigma})-28\bar{\delta}(\epsilon)+20\bar{\delta}(\bar{\epsilon})+42\bar{\delta}(\rho)\\&-30\bar{\delta}(\bar{\rho})+42\bar{\delta}(\sigma)+224\bar{\pi}\bar{\sigma}-64\bar{\alpha}\bar{\sigma}-272\tau\bar{\sigma}-10\pi\bar{\epsilon}+31\pi\bar{\rho}\\&+60\pi\bar{\sigma}-14\alpha\bar{\rho}+300\alpha\bar{\sigma}-128\bar{\tau}\epsilon-144\bar{\tau}\bar{\rho}+144\rho\bar{\tau}-64\bar{\tau}\bar{\epsilon}\\&+48\sigma\bar{\tau}+123\sigma\pi-6\alpha\rho-44\alpha\bar{\epsilon}-92\alpha\epsilon-82\pi\epsilon-117\pi\rho\\&-6\sigma\alpha+150\bar{\tau}\bar{\sigma}=0\end{aligned} \tag{4.4.18}$$

$$
\begin{aligned}
(A.7.9) \quad \Longrightarrow \quad & 32D(\kappa) - 24D(\bar{\pi}) - 16D(\bar{\alpha}) - 4\delta(\epsilon) - 4\delta(\bar{\epsilon}) + 6\delta(\rho) + 6\delta(\bar{\rho}) \\
& -2\delta(\bar{\sigma}) - 32\Phi_{01} - 32\kappa\bar{\epsilon} - 96\rho\kappa + 32\epsilon\kappa - 352\kappa\sigma + 64\kappa\bar{\rho} \\
& -6\tau\sigma + 51\bar{\pi}\bar{\rho} + 31\bar{\pi}\bar{\sigma} - 22\bar{\pi}\sigma - 74\bar{\pi}\bar{\epsilon} + 146\bar{\alpha}\bar{\rho} - 124\bar{\alpha}\bar{\epsilon} \\
& +58\bar{\alpha}\bar{\sigma} - 48\bar{\alpha}\sigma - 174\rho\bar{\alpha} - 93\rho\bar{\pi} - 24\sigma\pi - 16\sigma\alpha + 86\epsilon\bar{\pi} \\
& +132\epsilon\bar{\alpha} = 0 \\[2ex]
(A.7.18) \quad \Longrightarrow \quad & -96D(\tau) - 96D(\bar{\alpha}) - 32\Delta(\kappa) + 20\delta(\epsilon) - 28\delta(\bar{\epsilon}) - 30\delta(\rho) + 42\delta(\bar{\rho}) \\
& +42\delta(\bar{\sigma}) + 32\bar{\delta}(\sigma) + 144\tau\bar{\rho} + 150\tau\sigma - 117\bar{\pi}\bar{\rho} + 123\bar{\pi}\bar{\sigma} + 60\bar{\pi}\sigma \\
& -82\bar{\pi}\bar{\epsilon} - 6\bar{\alpha}\bar{\rho} - 92\bar{\alpha}\bar{\epsilon} - 6\bar{\alpha}\bar{\sigma} + 300\bar{\alpha}\sigma + 48\tau\bar{\sigma} - 14\rho\bar{\alpha} \\
& -144\rho\tau + 31\rho\bar{\pi} - 272\sigma\bar{\tau} + 224\sigma\pi - 64\tau\epsilon - 128\tau\bar{\epsilon} - 64\sigma\alpha \\
& -10\epsilon\bar{\pi} - 44\epsilon\bar{\alpha} = 0 \\[2ex]
(A.7.23) \quad \Longrightarrow \quad & -32D(\kappa) - 24D(\pi) - 16D(\alpha) - 4\bar{\delta}(\epsilon) - 4\bar{\delta}(\bar{\epsilon}) + 6\bar{\delta}(\rho) + 6\bar{\delta}(\bar{\rho}) \\
& -2\bar{\delta}(\sigma) - 32\Phi_{10} - 32\kappa\bar{\epsilon} - 64\rho\kappa + 32\epsilon\kappa + 96\kappa\bar{\rho} - 24\bar{\pi}\bar{\sigma} \\
& -16\bar{\alpha}\bar{\sigma} + 86\pi\bar{\epsilon} - 93\pi\bar{\rho} - 22\pi\bar{\sigma} - 174\alpha\bar{\rho} - 48\alpha\bar{\sigma} + 31\sigma\pi \\
& +146\alpha\rho + 132\alpha\bar{\epsilon} - 124\alpha\epsilon - 74\pi\epsilon + 51\pi\rho + 58\sigma\alpha + 352\bar{\sigma}\kappa \\
& -6\bar{\tau}\bar{\sigma} = 0
\end{aligned}
\tag{4.4.19}
$$

$$
\begin{aligned}
\text{Conjugate of (A.1.3)} \quad &\Longrightarrow \quad D(\bar{\tau}) + \Delta(\kappa) - \bar{\tau}\bar{\rho} - \pi\bar{\rho} - \tau\bar{\sigma} - \bar{\pi}\bar{\sigma} - \bar{\tau}\bar{\epsilon} \\
&\qquad +\bar{\tau}\epsilon - \Phi_{10} = 0 \\[1em]
\text{Conjugate of (A.1.4)} \quad &\Longrightarrow \quad 2D(\bar{\alpha}) - 2\delta(\bar{\epsilon}) - 2\bar{\alpha}\bar{\rho} - 2\epsilon\bar{\alpha} + 8\bar{\alpha}\bar{\epsilon} - 7\sigma\pi - 4\sigma\alpha \\
&\qquad +5\bar{\pi}\bar{\epsilon} - 2\bar{\pi}\bar{\rho} - 2\Phi_{01} = 0 \\[1em]
\text{Conjugate of (A.1.5)} \quad &\Longrightarrow \quad 7D(\pi) + 4D(\alpha) - 2\bar{\delta}(\bar{\epsilon}) - 2\bar{\alpha}\bar{\sigma} - 2\bar{\pi}\bar{\sigma} - 7\pi\rho - 4\alpha\rho \\
&\qquad +7\pi\epsilon + 4\alpha\epsilon - 2\pi\bar{\epsilon} + 2\alpha\bar{\epsilon} = 0 \\[1em]
\text{Conjugate of (A.1.11)} \quad &\Longrightarrow \quad 2\bar{\delta}(\bar{\rho}) - 2\delta(\bar{\sigma}) - 6\alpha\bar{\rho} - 7\pi\bar{\rho} + 2\bar{\alpha}\bar{\sigma} - 7\bar{\pi}\bar{\sigma} - 2\bar{\tau}\bar{\rho} \\
&\qquad +2\rho\bar{\tau} - 2\Phi_{10} = 0
\end{aligned}
\tag{4.4.20}
$$

$$[\delta, D]\,\Psi_2 \quad\Longrightarrow\quad \begin{aligned}&-24\bar\pi\sigma + 32D(\bar\alpha) - 36\bar\alpha\sigma + 54\delta(\rho) + 12\delta(\bar\epsilon) - 18\delta(\bar\sigma) - 4\delta(\epsilon)\\&-18\delta(\bar\rho) - 16D(\bar\pi) + 18\tau\sigma - 162\rho\bar\alpha - 4\bar\alpha\bar\epsilon + 54\bar\alpha\bar\sigma - 20\epsilon\bar\alpha\\&+22\bar\alpha\bar\rho - 135\rho\bar\pi - 46\bar\pi\bar\epsilon + 45\bar\pi\bar\sigma + 26\epsilon\bar\pi + 61\bar\pi\bar\rho - 16\sigma\pi\\&+32\sigma\alpha = 0\end{aligned}$$

$$[\delta, D]\,\overline{\Psi}_2 \quad\Longrightarrow\quad \begin{aligned}&-18\delta(\rho) - 32D(\bar\alpha) - 4\delta(\bar\epsilon) + 6\delta(\bar\sigma) + 16D(\bar\pi) + 54\delta(\bar\rho) + 12\delta(\epsilon)\\&+72\bar\pi\sigma + 108\bar\alpha\sigma - 54\tau\sigma - 130\bar\alpha\bar\rho - 4\epsilon\bar\alpha - 18\bar\alpha\bar\sigma - 20\bar\alpha\bar\epsilon\\&+54\rho\bar\alpha - 151\bar\pi\bar\rho - 46\epsilon\bar\pi - 15\bar\pi\bar\sigma + 26\bar\pi\bar\epsilon + 45\rho\bar\pi + 16\sigma\pi\\&-32\sigma\alpha = 0\end{aligned}$$

$$\left[\bar\delta, D\right]\,\Psi_2 \quad\Longrightarrow\quad \begin{aligned}&16D(\pi) + 54\bar\delta(\rho) + 12\bar\delta(\bar\epsilon) + 6\bar\delta(\sigma) - 4\bar\delta(\epsilon) - 18\bar\delta(\bar\rho) - 32D(\alpha)\\&+108\alpha\bar\sigma + 72\pi\bar\sigma - 54\bar\tau\bar\sigma - 130\alpha\rho - 4\alpha\bar\epsilon - 18\sigma\alpha - 20\alpha\epsilon\\&+54\alpha\bar\rho - 151\pi\rho - 46\pi\bar\epsilon - 15\sigma\pi + 26\pi\epsilon + 45\pi\bar\rho + 16\bar\pi\bar\sigma\\&-32\bar\alpha\bar\sigma = 0\end{aligned}$$

$$\left[\bar\delta, D\right]\,\overline{\Psi}_2 \quad\Longrightarrow\quad \begin{aligned}&-24\pi\bar\sigma - 36\alpha\bar\sigma + 12\bar\delta(\epsilon) + 54\bar\delta(\bar\rho) - 18\bar\delta(\sigma) - 18\bar\delta(\rho) - 4\bar\delta(\bar\epsilon)\\&+32D(\alpha) - 16D(\pi) + 18\bar\tau\bar\sigma - 162\alpha\bar\rho - 4\alpha\epsilon + 54\sigma\alpha - 20\alpha\bar\epsilon\\&+22\alpha\rho - 135\pi\bar\rho - 46\pi\epsilon + 45\sigma\pi + 26\pi\bar\epsilon + 61\pi\rho - 16\bar\pi\bar\sigma\\&+32\bar\alpha\bar\sigma = 0\end{aligned}$$

$$(4.4.21)$$

The eighteen equations in (4.4.17), (4.4.18), (4.4.19), (4.4.20), (4.4.21) form a closed system for these eighteen Pfaffians: $D(\kappa)$, $D(\pi)$, $D(\bar\pi)$, $D(\tau)$, $D(\bar\tau)$, $D(\alpha)$, $D(\bar\alpha)$, $\Delta(\kappa)$, $\delta(\epsilon)$, $\delta(\bar\epsilon)$, $\delta(\rho)$, $\delta(\bar\rho)$, $\delta(\bar\sigma)$, $\bar\delta(\epsilon)$, $\bar\delta(\bar\epsilon)$, $\bar\delta(\rho)$, $\bar\delta(\bar\rho)$, $\bar\delta(\sigma)$.

The solution[5] to this 18-equation system for the above Pfaffian derivatives is:

$$\begin{aligned}
\bar{\delta}(\bar{\epsilon}) \quad = \quad & \tfrac{9}{32}\tau\bar{\rho} - \tfrac{1}{10}\Phi_{10} - \tfrac{3}{5}\Phi_{01} + \tfrac{3}{5}\kappa\bar{\epsilon} + \tfrac{3}{2}\rho\kappa - \tfrac{3}{5}\epsilon\kappa + \tfrac{33}{10}\kappa\sigma \\
& - \tfrac{3}{2}\kappa\bar{\rho} + \tfrac{261}{80}\tau\sigma - \tfrac{603}{640}\bar{\pi}\bar{\rho} - \tfrac{1307}{128}\bar{\pi}\bar{\sigma} + \tfrac{201}{640}\bar{\pi}\sigma - \tfrac{219}{320}\bar{\pi}\bar{\epsilon} - \tfrac{501}{320}\bar{\alpha}\bar{\rho} \\
& + \tfrac{87}{32}\bar{\alpha}\bar{\epsilon} - \tfrac{2013}{320}\bar{\alpha}\bar{\sigma} + \tfrac{1107}{320}\bar{\alpha}\sigma + \tfrac{63}{8}\tau\bar{\sigma} + \tfrac{111}{64}\rho\bar{\alpha} - \tfrac{121}{320}\pi\bar{\epsilon} + \tfrac{3639}{640}\pi\bar{\rho} \\
& - \tfrac{869}{640}\pi\bar{\sigma} + \tfrac{525}{64}\alpha\bar{\rho} - \tfrac{2103}{320}\alpha\bar{\sigma} + \tfrac{69}{80}\bar{\tau}\epsilon + \tfrac{909}{160}\bar{\tau}\bar{\rho} - \tfrac{99}{32}\rho\bar{\tau} + \tfrac{69}{16}\bar{\tau}\bar{\epsilon} \\
& + \tfrac{9}{160}\rho\tau - \tfrac{51}{640}\rho\bar{\pi} + \tfrac{3}{16}\sigma\bar{\tau} + \tfrac{735}{128}\sigma\pi - \tfrac{39}{16}\tau\epsilon + \tfrac{249}{80}\tau\bar{\epsilon} - \tfrac{2191}{320}\alpha\rho \\
& + \tfrac{249}{160}\alpha\bar{\epsilon} + \tfrac{133}{32}\alpha\epsilon + \tfrac{1631}{320}\pi\epsilon - \tfrac{2993}{640}\pi\rho + \tfrac{1737}{320}\sigma\alpha - \tfrac{51}{320}\epsilon\bar{\pi} - \tfrac{381}{160}\epsilon\bar{\alpha} \\
& - \tfrac{33}{10}\bar{\sigma}\kappa - \tfrac{51}{20}\bar{\tau}\bar{\sigma}
\end{aligned}$$

$$\begin{aligned}
\bar{\delta}(\rho) \quad = \quad & -\tfrac{3}{16}\tau\bar{\rho} - \tfrac{3}{5}\Phi_{10} + \tfrac{2}{5}\Phi_{01} - \tfrac{2}{5}\kappa\bar{\epsilon} - \rho\kappa + \tfrac{2}{5}\epsilon\kappa - \tfrac{11}{5}\kappa\sigma \\
& + \kappa\bar{\rho} - \tfrac{87}{40}\tau\sigma + \tfrac{201}{320}\bar{\pi}\bar{\rho} + \tfrac{393}{64}\bar{\pi}\bar{\sigma} - \tfrac{67}{320}\bar{\pi}\sigma + \tfrac{73}{160}\bar{\pi}\bar{\epsilon} + \tfrac{167}{160}\bar{\alpha}\bar{\rho} \\
& - \tfrac{29}{16}\bar{\alpha}\bar{\epsilon} + \tfrac{2333}{480}\bar{\alpha}\bar{\sigma} - \tfrac{369}{160}\bar{\alpha}\sigma - \tfrac{17}{4}\tau\bar{\sigma} - \tfrac{37}{32}\rho\bar{\alpha} + \tfrac{227}{160}\pi\bar{\epsilon} - \tfrac{973}{320}\pi\bar{\rho} \\
& - \tfrac{971}{960}\pi\bar{\sigma} - \tfrac{127}{32}\alpha\bar{\rho} + \tfrac{301}{160}\alpha\bar{\sigma} - \tfrac{23}{40}\bar{\tau}\epsilon - \tfrac{303}{80}\bar{\tau}\bar{\rho} + \tfrac{33}{16}\rho\bar{\tau} - \tfrac{23}{8}\bar{\tau}\bar{\epsilon} \\
& - \tfrac{3}{80}\rho\tau + \tfrac{17}{320}\rho\bar{\pi} - \tfrac{1}{8}\sigma\bar{\tau} - \tfrac{261}{64}\sigma\pi + \tfrac{13}{8}\tau\epsilon - \tfrac{83}{40}\tau\bar{\epsilon} + \tfrac{917}{160}\alpha\rho \\
& - \tfrac{3}{80}\alpha\bar{\epsilon} - \tfrac{39}{16}\alpha\epsilon - \tfrac{517}{160}\pi\epsilon + \tfrac{1291}{320}\pi\rho - \tfrac{659}{160}\sigma\alpha + \tfrac{17}{160}\epsilon\bar{\pi} + \tfrac{127}{80}\epsilon\bar{\alpha} \\
& + \tfrac{11}{5}\bar{\sigma}\kappa + \tfrac{37}{10}\bar{\tau}\bar{\sigma}
\end{aligned}$$

[5]MAPLE V Release 5.1 was used here.

$$
\begin{aligned}
D(\tau) \quad = \quad & -\tfrac{1}{32}\tau\bar{\rho} - \tfrac{1}{15}\Phi_{10} + \tfrac{19}{15}\Phi_{01} - \tfrac{4}{15}\kappa\bar{\epsilon} - \tfrac{2}{3}\rho\kappa + \tfrac{4}{15}\epsilon\kappa - \tfrac{22}{15}\kappa\sigma \\
& + \tfrac{2}{3}\kappa\bar{\rho} - \tfrac{43}{40}\tau\sigma + \tfrac{253}{640}\bar{\pi}\bar{\rho} + \tfrac{509}{128}\bar{\pi}\bar{\sigma} - \tfrac{253}{1920}\bar{\pi}\sigma + \tfrac{7}{960}\bar{\pi}\bar{\epsilon} + \tfrac{713}{960}\bar{\alpha}\bar{\rho} \\
& - \tfrac{131}{96}\bar{\alpha}\bar{\epsilon} + \tfrac{2969}{960}\bar{\alpha}\bar{\sigma} - \tfrac{357}{320}\bar{\alpha}\sigma - \tfrac{5}{24}\tau\bar{\sigma} - \tfrac{187}{192}\rho\bar{\alpha} + \tfrac{503}{960}\pi\bar{\epsilon} - \tfrac{1417}{1920}\pi\bar{\rho} \\
& - \tfrac{53}{1920}\pi\bar{\sigma} - \tfrac{283}{192}\alpha\bar{\rho} + \tfrac{363}{320}\alpha\bar{\sigma} - \tfrac{47}{240}\bar{\tau}\epsilon - \tfrac{149}{160}\bar{\tau}\bar{\rho} + \tfrac{11}{32}\rho\bar{\tau} - \tfrac{47}{48}\bar{\tau}\bar{\epsilon} \\
& + \tfrac{111}{160}\rho\tau + \tfrac{1823}{1920}\rho\bar{\pi} - \tfrac{1}{12}\sigma\bar{\tau} - \tfrac{219}{128}\sigma\pi + \tfrac{85}{48}\tau\epsilon - \tfrac{587}{240}\tau\bar{\epsilon} + \tfrac{953}{960}\alpha\rho \\
& + \tfrac{11}{160}\alpha\bar{\epsilon} - \tfrac{33}{32}\alpha\epsilon - \tfrac{1153}{960}\pi\epsilon + \tfrac{213}{640}\pi\rho - \tfrac{2951}{960}\sigma\alpha + \tfrac{143}{960}\epsilon\bar{\pi} + \tfrac{433}{480}\epsilon\bar{\alpha} \\
& + \tfrac{22}{15}\bar{\sigma}\kappa + \tfrac{21}{20}\bar{\tau}\bar{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
D(\kappa) \quad = \quad & -\tfrac{1}{16}\tau\bar{\rho} - \tfrac{8}{15}\Phi_{10} + \tfrac{2}{15}\Phi_{01} - \tfrac{17}{15}\kappa\bar{\epsilon} - \tfrac{7}{3}\rho\kappa + \tfrac{17}{15}\epsilon\kappa - \tfrac{11}{15}\kappa\sigma \\
& + \tfrac{10}{3}\kappa\bar{\rho} - \tfrac{29}{40}\tau\sigma + \tfrac{67}{320}\bar{\pi}\bar{\rho} + \tfrac{131}{64}\bar{\pi}\bar{\sigma} - \tfrac{67}{960}\bar{\pi}\sigma + \tfrac{73}{480}\bar{\pi}\bar{\epsilon} + \tfrac{167}{480}\bar{\alpha}\bar{\rho} \\
& - \tfrac{29}{48}\bar{\alpha}\bar{\epsilon} + \tfrac{431}{480}\bar{\alpha}\bar{\sigma} - \tfrac{123}{160}\bar{\alpha}\sigma - \tfrac{13}{6}\tau\bar{\sigma} - \tfrac{37}{96}\rho\bar{\alpha} + \tfrac{1187}{480}\pi\bar{\epsilon} - \tfrac{3853}{960}\pi\bar{\rho} \\
& - \tfrac{377}{960}\pi\bar{\sigma} - \tfrac{703}{96}\alpha\bar{\rho} - \tfrac{33}{160}\alpha\bar{\sigma} - \tfrac{23}{120}\bar{\tau}\epsilon - \tfrac{101}{80}\bar{\tau}\bar{\rho} + \tfrac{11}{16}\rho\bar{\tau} - \tfrac{23}{24}\bar{\tau}\bar{\epsilon} \\
& - \tfrac{1}{80}\rho\tau + \tfrac{17}{960}\rho\bar{\pi} - \tfrac{1}{24}\sigma\bar{\tau} - \tfrac{23}{64}\sigma\pi + \tfrac{13}{24}\tau\epsilon - \tfrac{83}{120}\tau\bar{\epsilon} + \tfrac{3317}{480}\alpha\rho \\
& + \tfrac{319}{80}\alpha\bar{\epsilon} - \tfrac{77}{16}\alpha\epsilon - \tfrac{1477}{480}\pi\epsilon + \tfrac{1017}{320}\pi\rho + \tfrac{301}{480}\sigma\alpha + \tfrac{17}{480}\epsilon\bar{\pi} + \tfrac{127}{240}\epsilon\bar{\alpha} \\
& + \tfrac{176}{15}\bar{\sigma}\kappa + \tfrac{3}{20}\bar{\tau}\bar{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
D(\pi) \quad = \quad & \tfrac{3}{16}\tau\bar{\rho} + \tfrac{3}{5}\Phi_{10} - \tfrac{2}{5}\Phi_{01} + \tfrac{2}{5}\kappa\bar{\epsilon} + \rho\kappa - \tfrac{2}{5}\epsilon\kappa + \tfrac{11}{5}\kappa\sigma \\
& - \kappa\bar{\rho} + \tfrac{87}{40}\tau\sigma - \tfrac{201}{320}\bar{\pi}\bar{\rho} - \tfrac{393}{64}\bar{\pi}\bar{\sigma} + \tfrac{67}{320}\bar{\pi}\sigma - \tfrac{73}{160}\bar{\pi}\bar{\epsilon} - \tfrac{167}{160}\bar{\alpha}\bar{\rho} \\
& + \tfrac{29}{16}\bar{\alpha}\bar{\epsilon} - \tfrac{671}{160}\bar{\alpha}\bar{\sigma} + \tfrac{369}{160}\bar{\alpha}\sigma + \tfrac{17}{4}\tau\bar{\sigma} + \tfrac{37}{32}\rho\bar{\alpha} - \tfrac{67}{160}\pi\bar{\epsilon} + \tfrac{973}{320}\pi\bar{\rho} \\
& - \tfrac{103}{320}\pi\bar{\sigma} + \tfrac{127}{32}\alpha\bar{\rho} - \tfrac{621}{160}\alpha\bar{\sigma} + \tfrac{23}{40}\bar{\tau}\epsilon + \tfrac{303}{80}\bar{\tau}\bar{\rho} - \tfrac{33}{16}\rho\bar{\tau} + \tfrac{23}{8}\bar{\tau}\bar{\epsilon} \\
& + \tfrac{3}{80}\rho\tau - \tfrac{17}{320}\rho\bar{\pi} + \tfrac{1}{8}\sigma\bar{\tau} + \tfrac{261}{64}\sigma\pi - \tfrac{13}{8}\tau\epsilon + \tfrac{83}{40}\tau\bar{\epsilon} - \tfrac{437}{160}\alpha\rho \\
& + \tfrac{3}{80}\alpha\bar{\epsilon} + \tfrac{39}{16}\alpha\epsilon + \tfrac{357}{160}\pi\epsilon - \tfrac{171}{320}\pi\rho + \tfrac{659}{160}\sigma\alpha - \tfrac{17}{160}\epsilon\bar{\pi} - \tfrac{127}{80}\epsilon\bar{\alpha} \\
& - \tfrac{11}{5}\bar{\sigma}\kappa - \tfrac{27}{10}\bar{\tau}\bar{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
D(\bar{\pi}) \;=\;& -\tfrac{21}{16}\tau\bar{\rho} - \tfrac{2}{5}\Phi_{10} + \tfrac{3}{5}\Phi_{01} - \tfrac{8}{5}\kappa\bar{\epsilon} - 4\rho\kappa + \tfrac{8}{5}\epsilon\kappa - \tfrac{44}{5}\kappa\sigma \\
& +4\kappa\bar{\rho} - \tfrac{123}{40}\tau\sigma + \tfrac{759}{320}\bar{\pi}\bar{\rho} + \tfrac{423}{64}\bar{\pi}\bar{\sigma} - \tfrac{253}{320}\bar{\pi}\sigma - \tfrac{153}{160}\bar{\pi}\bar{\epsilon} + \tfrac{593}{160}\bar{\alpha}\bar{\rho} \\
& -\tfrac{59}{16}\bar{\alpha}\bar{\epsilon} + \tfrac{809}{160}\bar{\alpha}\bar{\sigma} - \tfrac{711}{160}\bar{\alpha}\sigma - \tfrac{11}{4}\tau\bar{\sigma} - \tfrac{115}{32}\rho\bar{\alpha} + \tfrac{353}{160}\pi\bar{\epsilon} - \tfrac{1087}{320}\pi\bar{\rho} \\
& -\tfrac{83}{320}\pi\bar{\sigma} - \tfrac{205}{32}\alpha\bar{\rho} + \tfrac{279}{160}\alpha\bar{\sigma} - \tfrac{17}{40}\bar{\tau}\epsilon - \tfrac{177}{80}\bar{\tau}\bar{\rho} + \tfrac{15}{16}\rho\bar{\tau} - \tfrac{17}{8}\bar{\tau}\bar{\epsilon} \\
& +\tfrac{123}{80}\rho\tau - \tfrac{97}{320}\rho\bar{\pi} + \tfrac{11}{8}\sigma\bar{\tau} - \tfrac{231}{64}\sigma\pi + \tfrac{19}{8}\tau\epsilon - \tfrac{77}{40}\tau\bar{\epsilon} + \tfrac{863}{160}\alpha\rho \\
& +\tfrac{183}{80}\alpha\bar{\epsilon} - \tfrac{69}{16}\alpha\epsilon - \tfrac{583}{160}\pi\epsilon + \tfrac{729}{320}\pi\rho - \tfrac{521}{160}\sigma\alpha + \tfrac{303}{160}\epsilon\bar{\pi} + \tfrac{313}{80}\epsilon\bar{\alpha} \\
& +\tfrac{44}{5}\bar{\sigma}\kappa + \tfrac{9}{5}\bar{\tau}\bar{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
D(\bar{\tau}) \;=\;& \tfrac{1}{32}\tau\bar{\rho} + \tfrac{16}{15}\Phi_{10} - \tfrac{4}{15}\Phi_{01} + \tfrac{4}{15}\kappa\bar{\epsilon} + \tfrac{2}{3}\rho\kappa - \tfrac{4}{15}\epsilon\kappa + \tfrac{22}{15}\kappa\sigma \\
& -\tfrac{2}{3}\kappa\bar{\rho} + \tfrac{43}{40}\tau\sigma - \tfrac{253}{640}\bar{\pi}\bar{\rho} - \tfrac{381}{128}\bar{\pi}\bar{\sigma} + \tfrac{253}{1920}\bar{\pi}\sigma - \tfrac{7}{960}\bar{\pi}\bar{\epsilon} - \tfrac{713}{960}\bar{\alpha}\bar{\rho} \\
& +\tfrac{131}{96}\bar{\alpha}\bar{\epsilon} - \tfrac{2969}{960}\bar{\alpha}\bar{\sigma} + \tfrac{357}{320}\bar{\alpha}\sigma + \tfrac{29}{24}\tau\bar{\sigma} + \tfrac{187}{192}\rho\bar{\alpha} - \tfrac{503}{960}\pi\bar{\epsilon} + \tfrac{3337}{1920}\pi\bar{\rho} \\
& +\tfrac{53}{1920}\pi\bar{\sigma} + \tfrac{283}{192}\alpha\bar{\rho} - \tfrac{363}{320}\alpha\bar{\sigma} - \tfrac{193}{240}\bar{\tau}\epsilon + \tfrac{309}{160}\bar{\tau}\bar{\rho} - \tfrac{11}{32}\rho\bar{\tau} + \tfrac{95}{48}\bar{\tau}\bar{\epsilon} \\
& +\tfrac{49}{160}\rho\tau + \tfrac{97}{1920}\rho\bar{\pi} + \tfrac{13}{12}\sigma\bar{\tau} + \tfrac{347}{128}\sigma\pi - \tfrac{37}{48}\tau\epsilon + \tfrac{347}{240}\tau\bar{\epsilon} - \tfrac{953}{960}\alpha\rho \\
& -\tfrac{11}{160}\alpha\bar{\epsilon} + \tfrac{33}{32}\alpha\epsilon + \tfrac{1153}{960}\pi\epsilon - \tfrac{213}{640}\pi\rho + \tfrac{2951}{960}\sigma\alpha - \tfrac{143}{960}\epsilon\bar{\pi} - \tfrac{433}{480}\epsilon\bar{\alpha} \\
& -\tfrac{22}{15}\bar{\sigma}\kappa - \tfrac{21}{20}\bar{\tau}\bar{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
D(\alpha) \;=\;& -\tfrac{3}{16}\tau\bar{\rho} - \tfrac{11}{10}\Phi_{10} + \tfrac{2}{5}\Phi_{01} - \tfrac{2}{5}\kappa\bar{\epsilon} - \rho\kappa + \tfrac{2}{5}\epsilon\kappa - \tfrac{11}{5}\kappa\sigma \\
& +\kappa\bar{\rho} - \tfrac{87}{40}\tau\sigma + \tfrac{201}{320}\bar{\pi}\bar{\rho} + \tfrac{393}{64}\bar{\pi}\bar{\sigma} - \tfrac{67}{320}\bar{\pi}\sigma + \tfrac{73}{160}\bar{\pi}\bar{\epsilon} + \tfrac{167}{160}\bar{\alpha}\bar{\rho} \\
& -\tfrac{29}{16}\bar{\alpha}\bar{\epsilon} + \tfrac{751}{160}\bar{\alpha}\bar{\sigma} - \tfrac{369}{160}\bar{\alpha}\sigma - \tfrac{7}{2}\tau\bar{\sigma} - \tfrac{37}{32}\rho\bar{\alpha} + \tfrac{167}{160}\pi\bar{\epsilon} - \tfrac{793}{320}\pi\bar{\rho} \\
& -\tfrac{37}{320}\pi\bar{\sigma} - \tfrac{91}{32}\alpha\bar{\rho} + \tfrac{561}{160}\alpha\bar{\sigma} - \tfrac{23}{40}\bar{\tau}\epsilon - \tfrac{303}{80}\bar{\tau}\bar{\rho} + \tfrac{33}{16}\rho\bar{\tau} - \tfrac{23}{8}\bar{\tau}\bar{\epsilon} \\
& -\tfrac{3}{80}\rho\tau + \tfrac{17}{320}\rho\bar{\pi} - \tfrac{1}{8}\sigma\bar{\tau} - \tfrac{273}{64}\sigma\pi + \tfrac{13}{8}\tau\epsilon - \tfrac{83}{40}\tau\bar{\epsilon} + \tfrac{377}{160}\alpha\rho \\
& +\tfrac{17}{80}\alpha\bar{\epsilon} - \tfrac{51}{16}\alpha\epsilon - \tfrac{497}{160}\pi\epsilon + \tfrac{111}{320}\pi\rho - \tfrac{719}{160}\sigma\alpha + \tfrac{17}{160}\epsilon\bar{\pi} + \tfrac{127}{80}\epsilon\bar{\alpha} \\
& +\tfrac{11}{5}\bar{\sigma}\kappa + \tfrac{69}{20}\bar{\tau}\bar{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
D(\bar{\alpha}) \;=\;& \tfrac{27}{16}\tau\bar{\rho} - \tfrac{3}{2}\Phi_{01} + \tfrac{27}{8}\tau\sigma - \tfrac{9}{64}\bar{\pi}\bar{\rho} - \tfrac{381}{64}\bar{\pi}\bar{\sigma} + \tfrac{3}{64}\bar{\pi}\sigma - \tfrac{57}{32}\bar{\pi}\bar{\epsilon} \\
& + \tfrac{33}{32}\bar{\alpha}\bar{\rho} - \tfrac{7}{16}\bar{\alpha}\bar{\epsilon} - \tfrac{135}{32}\bar{\alpha}\bar{\sigma} + \tfrac{105}{32}\bar{\alpha}\sigma + \tfrac{3}{2}\tau\bar{\sigma} - \tfrac{15}{32}\rho\bar{\alpha} - \tfrac{15}{32}\pi\bar{\epsilon} \\
& + \tfrac{33}{64}\pi\bar{\rho} - \tfrac{3}{64}\pi\bar{\sigma} + \tfrac{39}{32}\alpha\bar{\rho} - \tfrac{81}{32}\alpha\bar{\sigma} + \tfrac{3}{8}\bar{\tau}\epsilon + \tfrac{27}{16}\bar{\tau}\bar{\rho} - \tfrac{9}{16}\rho\bar{\tau} \\
& + \tfrac{15}{8}\bar{\tau}\bar{\epsilon} - \tfrac{33}{16}\rho\tau - \tfrac{129}{64}\rho\bar{\pi} - \tfrac{15}{8}\sigma\bar{\tau} + \tfrac{285}{64}\sigma\pi - \tfrac{21}{8}\tau\epsilon + \tfrac{15}{8}\tau\bar{\epsilon} \\
& - \tfrac{9}{32}\alpha\rho + \tfrac{15}{16}\alpha\bar{\epsilon} + \tfrac{15}{16}\alpha\epsilon + \tfrac{57}{32}\pi\epsilon + \tfrac{9}{64}\pi\rho + \tfrac{159}{32}\sigma\alpha + \tfrac{15}{32}\epsilon\bar{\pi} \\
& - \tfrac{7}{16}\epsilon\bar{\alpha} - \tfrac{9}{4}\bar{\tau}\bar{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
\Delta(\kappa) \;=\;& -\tfrac{1}{32}\tau\bar{\rho} - \tfrac{1}{15}\Phi_{10} + \tfrac{4}{15}\Phi_{01} - \tfrac{4}{15}\kappa\bar{\epsilon} - \tfrac{2}{3}\rho\kappa + \tfrac{4}{15}\epsilon\kappa - \tfrac{22}{15}\kappa\sigma \\
& + \tfrac{2}{3}\kappa\bar{\rho} - \tfrac{43}{40}\tau\sigma + \tfrac{253}{640}\bar{\pi}\bar{\rho} + \tfrac{509}{128}\bar{\pi}\bar{\sigma} - \tfrac{253}{1920}\bar{\pi}\sigma + \tfrac{7}{960}\bar{\pi}\bar{\epsilon} + \tfrac{713}{960}\bar{\alpha}\bar{\rho} \\
& - \tfrac{131}{96}\bar{\alpha}\bar{\epsilon} + \tfrac{2969}{960}\bar{\alpha}\bar{\sigma} - \tfrac{357}{320}\bar{\alpha}\sigma - \tfrac{5}{24}\tau\bar{\sigma} - \tfrac{187}{192}\rho\bar{\alpha} + \tfrac{503}{960}\pi\bar{\epsilon} - \tfrac{1417}{1920}\pi\bar{\rho} \\
& - \tfrac{53}{1920}\pi\bar{\sigma} - \tfrac{283}{192}\alpha\bar{\rho} + \tfrac{363}{320}\alpha\bar{\sigma} - \tfrac{47}{240}\bar{\tau}\epsilon - \tfrac{149}{160}\bar{\tau}\bar{\rho} + \tfrac{11}{32}\rho\bar{\tau} - \tfrac{47}{48}\bar{\tau}\bar{\epsilon} \\
& - \tfrac{49}{160}\rho\tau - \tfrac{97}{1920}\rho\bar{\pi} - \tfrac{13}{12}\sigma\bar{\tau} - \tfrac{347}{128}\sigma\pi + \tfrac{37}{48}\tau\epsilon - \tfrac{347}{240}\tau\bar{\epsilon} + \tfrac{953}{960}\alpha\rho \\
& + \tfrac{11}{160}\alpha\bar{\epsilon} - \tfrac{33}{32}\alpha\epsilon - \tfrac{1153}{960}\pi\epsilon + \tfrac{213}{640}\pi\rho - \tfrac{2951}{960}\sigma\alpha + \tfrac{143}{960}\epsilon\bar{\pi} + \tfrac{433}{480}\epsilon\bar{\alpha} \\
& + \tfrac{22}{15}\bar{\sigma}\kappa + \tfrac{21}{20}\bar{\tau}\bar{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
\delta(\epsilon) \;=\;& -\tfrac{39}{32}\tau\bar{\rho} - \tfrac{7}{5}\Phi_{10} - \tfrac{9}{10}\Phi_{01} - \tfrac{28}{5}\kappa\bar{\epsilon} - 14\rho\kappa + \tfrac{28}{5}\epsilon\kappa - \tfrac{154}{5}\kappa\sigma \\
& + 14\kappa\bar{\rho} - \tfrac{321}{80}\tau\sigma + \tfrac{2893}{640}\bar{\pi}\bar{\rho} + \tfrac{1437}{128}\bar{\pi}\bar{\sigma} - \tfrac{1711}{640}\bar{\pi}\sigma - \tfrac{1091}{320}\bar{\pi}\bar{\epsilon} + \tfrac{4171}{320}\bar{\alpha}\bar{\rho} \\
& - \tfrac{377}{32}\bar{\alpha}\bar{\epsilon} + \tfrac{2963}{320}\bar{\alpha}\bar{\sigma} - \tfrac{2877}{320}\bar{\alpha}\sigma - \tfrac{53}{8}\tau\bar{\sigma} - \tfrac{865}{64}\rho\bar{\alpha} + \tfrac{2171}{320}\pi\bar{\epsilon} - \tfrac{6949}{640}\pi\bar{\rho} \\
& - \tfrac{641}{640}\pi\bar{\sigma} - \tfrac{1279}{64}\alpha\bar{\rho} + \tfrac{333}{320}\alpha\bar{\sigma} - \tfrac{59}{80}\bar{\tau}\epsilon - \tfrac{699}{160}\bar{\tau}\bar{\rho} + \tfrac{69}{32}\rho\bar{\tau} - \tfrac{59}{16}\bar{\tau}\bar{\epsilon} \\
& + \tfrac{201}{160}\rho\tau - \tfrac{3259}{640}\rho\bar{\pi} + \tfrac{17}{16}\sigma\bar{\tau} - \tfrac{605}{128}\sigma\pi + \tfrac{49}{16}\tau\epsilon - \tfrac{239}{80}\tau\bar{\epsilon} + \tfrac{5861}{320}\alpha\rho \\
& + \tfrac{1581}{160}\alpha\bar{\epsilon} - \tfrac{423}{32}\alpha\epsilon - \tfrac{2941}{320}\pi\epsilon + \tfrac{5283}{640}\pi\rho - \tfrac{787}{320}\sigma\alpha + \tfrac{2101}{320}\epsilon\bar{\pi} + \tfrac{2211}{160}\epsilon\bar{\alpha} \\
& + \tfrac{154}{5}\bar{\sigma}\kappa + \tfrac{9}{5}\bar{\tau}\bar{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
\delta(\bar{\epsilon}) \quad = \quad & \tfrac{27}{16}\tau\bar{\rho} - \tfrac{5}{2}\Phi_{01} + \tfrac{27}{8}\tau\sigma - \tfrac{73}{64}\bar{\pi}\bar{\rho} - \tfrac{381}{64}\bar{\pi}\bar{\sigma} + \tfrac{3}{64}\bar{\pi}\sigma + \tfrac{23}{32}\bar{\pi}\bar{\epsilon} \\
& + \tfrac{1}{32}\bar{\alpha}\bar{\rho} + \tfrac{57}{16}\bar{\alpha}\bar{\epsilon} - \tfrac{135}{32}\bar{\alpha}\bar{\sigma} + \tfrac{105}{32}\bar{\alpha}\sigma + \tfrac{3}{2}\tau\bar{\sigma} - \tfrac{15}{32}\rho\bar{\alpha} - \tfrac{15}{32}\pi\bar{\epsilon} \\
& + \tfrac{33}{64}\pi\bar{\rho} - \tfrac{3}{64}\pi\bar{\sigma} + \tfrac{39}{32}\alpha\bar{\rho} - \tfrac{81}{32}\alpha\bar{\sigma} + \tfrac{3}{8}\bar{\tau}\epsilon + \tfrac{27}{16}\bar{\tau}\bar{\rho} - \tfrac{9}{16}\rho\bar{\tau} \\
& + \tfrac{15}{8}\bar{\tau}\bar{\epsilon} - \tfrac{33}{16}\rho\tau - \tfrac{129}{64}\rho\bar{\pi} - \tfrac{15}{8}\sigma\bar{\tau} + \tfrac{61}{64}\sigma\pi - \tfrac{21}{8}\tau\epsilon + \tfrac{15}{8}\tau\bar{\epsilon} \\
& - \tfrac{9}{32}\alpha\rho + \tfrac{15}{16}\alpha\bar{\epsilon} + \tfrac{15}{16}\alpha\epsilon + \tfrac{57}{32}\pi\epsilon + \tfrac{9}{64}\pi\rho + \tfrac{95}{32}\sigma\alpha + \tfrac{15}{32}\epsilon\bar{\pi} \\
& - \tfrac{23}{16}\epsilon\bar{\alpha} - \tfrac{9}{4}\bar{\tau}\bar{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
\delta(\rho) \quad = \quad & -\tfrac{45}{32}\tau\bar{\rho} - \tfrac{1}{15}\Phi_{10} + \tfrac{19}{15}\Phi_{01} - \tfrac{4}{15}\kappa\bar{\epsilon} - \tfrac{2}{3}\rho\kappa + \tfrac{4}{15}\epsilon\kappa - \tfrac{22}{15}\kappa\sigma \\
& + \tfrac{2}{3}\kappa\bar{\rho} - \tfrac{103}{40}\tau\sigma + \tfrac{313}{640}\bar{\pi}\bar{\rho} + \tfrac{281}{128}\bar{\pi}\bar{\sigma} - \tfrac{313}{1920}\bar{\pi}\sigma + \tfrac{1147}{960}\bar{\pi}\bar{\epsilon} + \tfrac{533}{960}\bar{\alpha}\bar{\rho} \\
& - \tfrac{71}{96}\bar{\alpha}\bar{\epsilon} + \tfrac{2069}{960}\bar{\alpha}\bar{\sigma} - \tfrac{897}{320}\bar{\alpha}\sigma - \tfrac{7}{12}\tau\bar{\sigma} + \tfrac{545}{192}\rho\bar{\alpha} + \tfrac{503}{960}\pi\bar{\epsilon} - \tfrac{1417}{1920}\pi\bar{\rho} \\
& - \tfrac{53}{1920}\pi\bar{\sigma} - \tfrac{283}{192}\alpha\bar{\rho} + \tfrac{363}{320}\alpha\bar{\sigma} - \tfrac{47}{240}\bar{\tau}\epsilon - \tfrac{149}{160}\bar{\tau}\bar{\rho} + \tfrac{11}{32}\rho\bar{\tau} - \tfrac{47}{48}\bar{\tau}\bar{\epsilon} \\
& + \tfrac{291}{160}\rho\tau + \tfrac{7283}{1920}\rho\bar{\pi} + \tfrac{43}{24}\sigma\bar{\tau} - \tfrac{315}{128}\sigma\pi + \tfrac{97}{48}\tau\epsilon - \tfrac{287}{240}\tau\bar{\epsilon} + \tfrac{953}{960}\alpha\rho \\
& + \tfrac{11}{160}\alpha\bar{\epsilon} - \tfrac{33}{32}\alpha\epsilon - \tfrac{1153}{960}\pi\epsilon + \tfrac{213}{640}\pi\rho - \tfrac{3191}{960}\sigma\alpha - \tfrac{157}{960}\epsilon\bar{\pi} + \tfrac{733}{480}\epsilon\bar{\alpha} \\
& + \tfrac{22}{15}\bar{\sigma}\kappa + \tfrac{21}{20}\bar{\tau}\bar{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
\delta(\bar{\rho}) \quad = \quad & \tfrac{21}{16}\tau\bar{\rho} + \tfrac{2}{5}\Phi_{10} - \tfrac{3}{5}\Phi_{01} + \tfrac{8}{5}\kappa\bar{\epsilon} + 4\rho\kappa - \tfrac{8}{5}\epsilon\kappa + \tfrac{44}{5}\kappa\sigma \\
& - 4\kappa\bar{\rho} + \tfrac{163}{40}\tau\sigma + \tfrac{361}{320}\bar{\pi}\bar{\rho} - \tfrac{423}{64}\bar{\pi}\bar{\sigma} - \tfrac{521}{960}\bar{\pi}\sigma - \tfrac{7}{160}\bar{\pi}\bar{\epsilon} - \tfrac{113}{160}\bar{\alpha}\bar{\rho} \\
& + \tfrac{59}{16}\bar{\alpha}\bar{\epsilon} - \tfrac{809}{160}\bar{\alpha}\bar{\sigma} + \tfrac{391}{160}\bar{\alpha}\sigma + \tfrac{11}{4}\tau\bar{\sigma} + \tfrac{115}{32}\rho\bar{\alpha} - \tfrac{353}{160}\pi\bar{\epsilon} + \tfrac{1087}{320}\pi\bar{\rho} \\
& + \tfrac{83}{320}\pi\bar{\sigma} + \tfrac{205}{32}\alpha\bar{\rho} - \tfrac{279}{160}\alpha\bar{\sigma} + \tfrac{17}{40}\bar{\tau}\epsilon + \tfrac{177}{80}\bar{\tau}\bar{\rho} - \tfrac{15}{16}\rho\bar{\tau} + \tfrac{17}{8}\bar{\tau}\bar{\epsilon} \\
& - \tfrac{123}{80}\rho\tau + \tfrac{97}{320}\rho\bar{\pi} - \tfrac{11}{8}\sigma\bar{\tau} + \tfrac{231}{64}\sigma\pi - \tfrac{19}{8}\tau\epsilon + \tfrac{77}{40}\tau\bar{\epsilon} - \tfrac{863}{160}\alpha\rho \\
& - \tfrac{183}{80}\alpha\bar{\epsilon} + \tfrac{69}{16}\alpha\epsilon + \tfrac{583}{160}\pi\epsilon - \tfrac{729}{320}\pi\rho + \tfrac{1883}{480}\sigma\alpha - \tfrac{143}{160}\epsilon\bar{\pi} - \tfrac{313}{80}\epsilon\bar{\alpha} \\
& - \tfrac{44}{5}\bar{\sigma}\kappa - \tfrac{9}{5}\bar{\tau}\bar{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
\delta(\bar\sigma) \;=\;& \tfrac{1}{32}\tau\bar\rho + \tfrac{1}{15}\Phi_{10} - \tfrac{4}{15}\Phi_{01} + \tfrac{4}{15}\kappa\bar\epsilon + \tfrac{2}{3}\rho\kappa - \tfrac{4}{15}\epsilon\kappa + \tfrac{22}{15}\kappa\sigma \\
& -\tfrac{2}{3}\kappa\bar\rho + \tfrac{43}{40}\tau\sigma - \tfrac{253}{640}\bar\pi\bar\rho - \tfrac{925}{128}\bar\pi\bar\sigma + \tfrac{253}{1920}\bar\pi\sigma - \tfrac{7}{960}\bar\pi\bar\epsilon - \tfrac{713}{960}\bar\alpha\bar\rho \\
& +\tfrac{131}{96}\bar\alpha\bar\epsilon - \tfrac{2249}{960}\bar\alpha\bar\sigma + \tfrac{357}{320}\bar\alpha\sigma + \tfrac{37}{12}\tau\bar\sigma + \tfrac{187}{192}\rho\bar\alpha - \tfrac{803}{960}\pi\bar\epsilon + \tfrac{2077}{1920}\pi\bar\rho \\
& -\tfrac{7}{1920}\pi\bar\sigma + \tfrac{439}{192}\alpha\bar\rho - \tfrac{903}{320}\alpha\bar\sigma + \tfrac{107}{240}\bar\tau\epsilon + \tfrac{329}{160}\bar\tau\bar\rho - \tfrac{23}{32}\rho\bar\tau + \tfrac{107}{48}\bar\tau\bar\epsilon \\
& +\tfrac{49}{160}\rho\tau + \tfrac{97}{1920}\rho\bar\pi + \tfrac{17}{24}\sigma\bar\tau + \tfrac{119}{128}\sigma\pi - \tfrac{37}{48}\tau\epsilon + \tfrac{347}{240}\tau\bar\epsilon - \tfrac{1133}{960}\alpha\rho \\
& +\tfrac{89}{160}\alpha\bar\epsilon + \tfrac{53}{32}\alpha\epsilon + \tfrac{2293}{960}\pi\epsilon - \tfrac{153}{640}\pi\rho + \tfrac{2051}{960}\sigma\alpha - \tfrac{143}{960}\epsilon\bar\pi - \tfrac{433}{480}\epsilon\bar\alpha \\
& -\tfrac{22}{15}\bar\sigma\kappa - \tfrac{51}{20}\bar\tau\bar\sigma
\end{aligned}
$$

$$
\begin{aligned}
\bar\delta(\epsilon) \;=\;& -\tfrac{3}{16}\tau\bar\rho - \tfrac{21}{10}\Phi_{10} + \tfrac{2}{5}\Phi_{01} - \tfrac{2}{5}\kappa\bar\epsilon - \rho\kappa + \tfrac{2}{5}\epsilon\kappa - \tfrac{11}{5}\kappa\sigma \\
& +\kappa\bar\rho - \tfrac{87}{40}\tau\sigma + \tfrac{201}{320}\bar\pi\bar\rho + \tfrac{169}{64}\bar\pi\bar\sigma - \tfrac{67}{320}\bar\pi\sigma + \tfrac{73}{160}\bar\pi\bar\epsilon + \tfrac{167}{160}\bar\alpha\bar\rho \\
& -\tfrac{29}{16}\bar\alpha\bar\epsilon + \tfrac{431}{160}\bar\alpha\bar\sigma - \tfrac{369}{160}\bar\alpha\sigma - \tfrac{7}{2}\tau\bar\sigma - \tfrac{37}{32}\rho\bar\alpha + \tfrac{167}{160}\pi\bar\epsilon - \tfrac{793}{320}\pi\bar\rho \\
& -\tfrac{37}{320}\pi\bar\sigma - \tfrac{91}{32}\alpha\bar\rho + \tfrac{561}{160}\alpha\bar\sigma - \tfrac{23}{40}\bar\tau\epsilon - \tfrac{303}{80}\bar\tau\bar\rho + \tfrac{33}{16}\rho\bar\tau - \tfrac{23}{8}\bar\tau\bar\epsilon \\
& -\tfrac{3}{80}\rho\tau + \tfrac{17}{320}\rho\bar\pi - \tfrac{1}{8}\sigma\bar\tau - \tfrac{273}{64}\sigma\pi + \tfrac{13}{8}\tau\epsilon - \tfrac{83}{40}\tau\bar\epsilon + \tfrac{217}{160}\alpha\rho \\
& -\tfrac{63}{80}\alpha\bar\epsilon + \tfrac{13}{16}\alpha\epsilon - \tfrac{97}{160}\pi\epsilon - \tfrac{209}{320}\pi\rho - \tfrac{719}{160}\sigma\alpha + \tfrac{17}{160}\epsilon\bar\pi + \tfrac{127}{80}\epsilon\bar\alpha \\
& +\tfrac{11}{5}\bar\sigma\kappa + \tfrac{69}{20}\bar\tau\bar\sigma
\end{aligned}
$$

$$
\begin{aligned}
\bar\delta(\bar\rho) \;=\;& \tfrac{1}{32}\tau\bar\rho + \tfrac{16}{15}\Phi_{10} - \tfrac{4}{15}\Phi_{01} + \tfrac{4}{15}\kappa\bar\epsilon + \tfrac{2}{3}\rho\kappa - \tfrac{4}{15}\epsilon\kappa + \tfrac{22}{15}\kappa\sigma \\
& -\tfrac{2}{3}\kappa\bar\rho + \tfrac{43}{40}\tau\sigma - \tfrac{253}{640}\bar\pi\bar\rho - \tfrac{477}{128}\bar\pi\bar\sigma + \tfrac{253}{1920}\bar\pi\sigma - \tfrac{7}{960}\bar\pi\bar\epsilon - \tfrac{713}{960}\bar\alpha\bar\rho \\
& +\tfrac{131}{96}\bar\alpha\bar\epsilon - \tfrac{3209}{960}\bar\alpha\bar\sigma + \tfrac{357}{320}\bar\alpha\sigma + \tfrac{37}{12}\tau\bar\sigma + \tfrac{187}{192}\rho\bar\alpha - \tfrac{803}{960}\pi\bar\epsilon + \tfrac{8797}{1920}\pi\bar\rho \\
& -\tfrac{7}{1920}\pi\bar\sigma + \tfrac{1015}{192}\alpha\bar\rho - \tfrac{903}{320}\alpha\bar\sigma + \tfrac{107}{240}\bar\tau\epsilon + \tfrac{489}{160}\bar\tau\bar\rho - \tfrac{55}{32}\rho\bar\tau + \tfrac{107}{48}\bar\tau\bar\epsilon \\
& +\tfrac{49}{160}\rho\tau + \tfrac{97}{1920}\rho\bar\pi + \tfrac{17}{24}\sigma\bar\tau + \tfrac{119}{128}\sigma\pi - \tfrac{37}{48}\tau\epsilon + \tfrac{347}{240}\tau\bar\epsilon - \tfrac{1133}{960}\alpha\rho \\
& +\tfrac{89}{160}\alpha\bar\epsilon + \tfrac{53}{32}\alpha\epsilon + \tfrac{2293}{960}\pi\epsilon - \tfrac{153}{640}\pi\rho + \tfrac{2051}{960}\sigma\alpha - \tfrac{143}{960}\epsilon\bar\pi - \tfrac{433}{480}\epsilon\bar\alpha \\
& -\tfrac{22}{15}\bar\sigma\kappa - \tfrac{51}{20}\bar\tau\bar\sigma
\end{aligned}
$$

$$
\begin{aligned}
\bar{\delta}(\sigma) \quad = \quad & -\tfrac{13}{32}\tau\bar{\rho} - \tfrac{1}{15}\Phi_{10} + \tfrac{4}{15}\Phi_{01} - \tfrac{4}{15}\kappa\bar{\epsilon} - \tfrac{2}{3}\rho\kappa + \tfrac{4}{15}\epsilon\kappa - \tfrac{22}{15}\kappa\sigma \\
& + \tfrac{2}{3}\kappa\bar{\rho} - \tfrac{103}{40}\tau\sigma + \tfrac{313}{640}\bar{\pi}\bar{\rho} + \tfrac{281}{128}\bar{\pi}\bar{\sigma} - \tfrac{313}{1920}\bar{\pi}\sigma + \tfrac{1147}{960}\bar{\pi}\bar{\epsilon} + \tfrac{533}{960}\bar{\alpha}\bar{\rho} \\
& - \tfrac{71}{96}\bar{\alpha}\bar{\epsilon} + \tfrac{2069}{960}\bar{\alpha}\bar{\sigma} - \tfrac{897}{320}\bar{\alpha}\sigma - \tfrac{7}{12}\tau\bar{\sigma} - \tfrac{31}{192}\rho\bar{\alpha} + \tfrac{503}{960}\pi\bar{\epsilon} - \tfrac{1417}{1920}\pi\bar{\rho} \\
& - \tfrac{53}{1920}\pi\bar{\sigma} - \tfrac{283}{192}\alpha\bar{\rho} + \tfrac{363}{320}\alpha\bar{\sigma} - \tfrac{47}{240}\bar{\tau}\epsilon - \tfrac{149}{160}\bar{\tau}\bar{\rho} + \tfrac{11}{32}\rho\bar{\tau} - \tfrac{47}{48}\bar{\tau}\bar{\epsilon} \\
& + \tfrac{131}{160}\rho\tau + \tfrac{563}{1920}\rho\bar{\pi} + \tfrac{43}{24}\sigma\bar{\tau} - \tfrac{763}{128}\sigma\pi + \tfrac{97}{48}\tau\epsilon - \tfrac{287}{240}\tau\bar{\epsilon} + \tfrac{953}{960}\alpha\rho \\
& + \tfrac{11}{160}\alpha\bar{\epsilon} - \tfrac{33}{32}\alpha\epsilon - \tfrac{1153}{960}\pi\epsilon + \tfrac{213}{640}\pi\rho - \tfrac{2231}{960}\sigma\alpha - \tfrac{157}{960}\epsilon\bar{\pi} + \tfrac{733}{480}\epsilon\bar{\alpha} \\
& + \tfrac{22}{15}\bar{\sigma}\kappa + \tfrac{21}{20}\bar{\tau}\bar{\sigma}
\end{aligned}
\tag{$\mathcal{D}7$}
$$

Note that ($\mathcal{D}7$) refers to the entire group of the eighteen Pfaffian derivatives just determined. If we now make the substitutions ($\mathcal{V}1$) … ($\mathcal{V}4$), and ($\mathcal{D}1$) … ($\mathcal{D}7$) into (A.9.3), remarkably, we obtain:

$$
\begin{aligned}
& -\Psi_2\bar{\Psi}_2\bar{\sigma}(12\alpha^2 + 10\pi\alpha - 21\bar{\tau}\pi - 18\alpha\bar{\tau} + 6\pi^2 \\
& +44\bar{\Psi}_2 - 12\alpha\bar{\alpha} - 21\tau\pi - 18\alpha\tau - 14\pi\bar{\alpha}) = 0
\end{aligned}
\tag{4.4.22}
$$

(4.4.22) implies that we have either

Scenario 1 $\qquad \sigma = \bar{\sigma} = 0, \quad$ or

Scenario 2 $\qquad \begin{aligned} & 12\alpha^2 + 10\pi\alpha - 21\bar{\tau}\pi - 18\alpha\bar{\tau} + 6\pi^2 + 44\bar{\Psi}_2 \\ & -12\alpha\bar{\alpha} - 21\tau\pi - 18\alpha\tau - 14\pi\bar{\alpha} = 0. \end{aligned}$

Scenario 1 We assume in this scenario that

$$
\boxed{\sigma = \bar{\sigma} = 0} \tag{$\mathcal{S}1$}
$$

Substituting ($\mathcal{S}1$), and ($\mathcal{V}4$) into (A.1.2) gives:

$$
\delta(\kappa) = \kappa\tau + \frac{19}{2}\kappa\bar{\pi} + 7\kappa\bar{\alpha} \tag{4.4.23}
$$

Substituting ($\mathcal{S}1$), ($\mathcal{V}4$) and the conjugate of (4.4.23) into (A.1.1) yield:

$$D(\rho) = \bar{\tau}\kappa + 7\kappa\pi + 2\kappa\alpha + \rho^2 + \rho\epsilon + \rho\bar{\epsilon} - \bar{\kappa}\tau + \Phi_{00} \tag{4.4.24}$$

Substituting ($\mathcal{V}1$) ... ($\mathcal{V}4$), ($\mathcal{D}1$) ... ($\mathcal{D}7$) into (A.7.19) now gives:

$$\begin{aligned}
(A.7.19) \quad &\Longrightarrow \quad -\Psi_2\bar{\Psi}_2(8\bar{\pi}^2 + 32\bar{\alpha}^2 + 32\bar{\pi}\bar{\alpha} + 15\sigma\bar{\sigma} - 25\sigma\bar{\rho} + 48\kappa\bar{\pi} + 3\sigma\rho - 48\kappa\tau \\
&\qquad +32\kappa\bar{\alpha} - 18\sigma\epsilon + 6\sigma\bar{\epsilon} + 28\Psi_2 - 45\sigma^2 + 16D(\sigma) - 16\delta(\kappa)) = 0 \\
&\overset{(\mathcal{S}1)}{\Longrightarrow} \quad 2\bar{\pi}^2 + 8\bar{\alpha}^2 + 8\bar{\pi}\bar{\alpha} + 12\kappa\bar{\pi} - 12\kappa\tau + 8\kappa\bar{\alpha} + 7\Psi_2 - 4\delta(\kappa) = 0 \\
&\overset{(4.4.23)}{\Longrightarrow} \quad \Psi_2 = -\tfrac{2}{7}\bar{\pi}^2 - \tfrac{8}{7}\bar{\alpha}^2 - \tfrac{8}{7}\bar{\pi}\bar{\alpha} + \tfrac{26}{7}\kappa\bar{\pi} + \tfrac{16}{7}\kappa\tau + \tfrac{20}{7}\kappa\bar{\alpha} \qquad (4.4.25)
\end{aligned}$$

We next solve for Φ_{11} and Λ in terms of the spin coefficents κ, α, τ, π and their conjugates:

$$\begin{aligned}
[\bar{\delta}, \delta]\,\Psi_2 \quad &\Longleftrightarrow \quad \bar{\delta}(\delta(\Psi_2)) - \delta(\bar{\delta}(\Psi_2)) = (-\mu + \bar{\mu})D(\Psi_2) + (-\rho + \bar{\rho})\Delta(\Psi_2) \\
&\qquad +(\alpha - \bar{\beta})\delta(\Psi_2) + (-\bar{\alpha} + \beta)\bar{\delta}(\Psi_2) \\
&\overset{\substack{(\mathcal{V}1)\ldots(\mathcal{V}4) \\ (\mathcal{D}1)\ldots(\mathcal{D}4)}}{\Longrightarrow} \quad 5\bar{\Psi}_2 + 8\pi\bar{\pi} - 2\Phi_{11} + 5\Psi_2 + 14\Lambda = 0 \qquad (4.4.26)
\end{aligned}$$

$$\begin{aligned}
(A.1.17) \quad \overset{\substack{(\mathcal{V}1)\ldots(\mathcal{V}4) \\ (\mathcal{D}1)\ldots(\mathcal{D}4)}}{\Longrightarrow} \quad &18\bar{\tau}^2 + 10\Psi_2 + 12\bar{\Psi}_2 - 8\Phi_{11} + 12\bar{\alpha}\bar{\tau} + 36\tau\pi \\
&+20\pi\bar{\pi} + 32\Lambda - 3\pi^2 + 24\alpha\bar{\alpha} + 42\bar{\tau}\pi + 24\pi\bar{\alpha} \qquad (4.4.27) \\
&+18\tau\bar{\tau} + 36\alpha^2 + 36\alpha\tau + 24\pi\alpha + 60\alpha\bar{\tau} = 0
\end{aligned}$$

Solving (4.4.26) and (4.4.27) for Λ and Φ_{11} gives:

$$\begin{cases} \Phi_{11} = \frac{1}{6}\bar{\Psi}_2 + \frac{1}{2}\pi\bar{\pi} + \frac{21}{4}\bar{\tau}^2 - \frac{5}{12}\Psi_2 + \frac{7}{2}\bar{\alpha}\bar{\tau} + \frac{21}{2}\tau\pi - \frac{7}{8}\pi^2 + 7\alpha\bar{\alpha} \\ \quad + \frac{49}{4}\bar{\tau}\pi + 7\pi\bar{\alpha} + \frac{21}{4}\tau\bar{\tau} + \frac{21}{2}\alpha^2 + \frac{21}{2}\alpha\tau + 7\pi\alpha + \frac{35}{2}\alpha\bar{\tau} \\ \\ \Lambda = \frac{3}{4}\bar{\tau}^2 - \frac{5}{12}\Psi_2 - \frac{1}{3}\bar{\Psi}_2 - \frac{1}{2}\pi\bar{\pi} + \frac{1}{2}\bar{\alpha}\bar{\tau} + \frac{3}{2}\tau\pi - \frac{1}{8}\pi^2 + \alpha\bar{\alpha} \\ \quad + \frac{7}{4}\bar{\tau}\pi + \pi\bar{\alpha} + \frac{3}{4}\tau\bar{\tau} + \frac{3}{2}\alpha^2 + \frac{3}{2}\alpha\tau + \pi\alpha + \frac{5}{2}\alpha\bar{\tau} \end{cases} \tag{4.4.28}$$

Eliminating Ψ_2 in (4.4.28) using (4.4.25), we have

$$\begin{cases} \Phi_{11} = \frac{10}{21}\bar{\pi}\bar{\alpha} + \frac{21}{4}\bar{\tau}^2 + \frac{7}{2}\bar{\alpha}\bar{\tau} + \frac{21}{2}\tau\pi + \frac{1}{2}\pi\bar{\pi} - \frac{155}{168}\pi^2 + 7\alpha\bar{\alpha} + \frac{49}{4}\bar{\tau}\pi \\ \quad + 7\pi\bar{\alpha} + \frac{21}{4}\tau\bar{\tau} + \frac{10}{21}\bar{\alpha}^2 + \frac{433}{42}\alpha^2 + \frac{5}{42}\bar{\pi}^2 - \frac{20}{21}\kappa\tau - \frac{10}{21}\kappa\alpha \\ \quad - \frac{13}{21}\kappa\pi - \frac{65}{42}\kappa\bar{\pi} - \frac{25}{21}\kappa\bar{\alpha} + \frac{21}{2}\alpha\tau + \frac{143}{21}\pi\alpha - \frac{8}{21}\bar{\tau}\kappa + \frac{35}{2}\alpha\bar{\tau} \\ \\ \Lambda = \frac{10}{21}\bar{\pi}\bar{\alpha} + \frac{3}{4}\bar{\tau}^2 + \frac{1}{2}\bar{\alpha}\bar{\tau} + \frac{3}{2}\tau\pi - \frac{1}{2}\pi\bar{\pi} - \frac{5}{168}\pi^2 + \alpha\bar{\alpha} + \frac{7}{4}\bar{\tau}\pi \\ \quad + \pi\bar{\alpha} + \frac{3}{4}\tau\bar{\tau} + \frac{10}{21}\bar{\alpha}^2 + \frac{79}{42}\alpha^2 + \frac{5}{42}\bar{\pi}^2 - \frac{20}{21}\kappa\tau + \frac{20}{21}\kappa\alpha + \frac{26}{21}\kappa\pi \\ \quad - \frac{65}{42}\kappa\bar{\pi} - \frac{25}{21}\kappa\bar{\alpha} + \frac{3}{2}\alpha\tau + \frac{29}{21}\pi\alpha + \frac{16}{21}\bar{\tau}\kappa + \frac{5}{2}\alpha\bar{\tau} \end{cases} \tag{4.4.29}$$

We have proved that $(\mathcal{S}1)$ implies both (4.4.25) and (4.4.29). We next list six polynomial equations in α, τ, π, $\bar{\alpha}$, $\bar{\tau}$, $\bar{\pi}$ and κ which hold in **Scenario 1** and which will be used repeatedly to establish the inadmissibility of **Scenario 1**:

$$\text{(A.6.9)} \overset{(\mathcal{D}1),(\mathcal{D}2)}{\Longrightarrow} 2\alpha + 3\bar{\tau} + 2\bar{\alpha} + 3\tau = 0 \tag{4.4.30}$$

$$\text{(4.4.9)} \overset{(\mathcal{S}1)}{\Longrightarrow} \begin{array}{l} -8\bar{\pi}\bar{\alpha} - 12\bar{\alpha}^2 + \bar{\pi}^2 - 6\tau^2 \\ -14\bar{\pi}\tau - 20\tau\bar{\alpha} = 0 \end{array} \tag{4.4.31}$$

$$\text{Conjugate of (4.4.9)} \Longrightarrow \begin{array}{l} -8\pi\alpha - 12\alpha^2 + \pi^2 - 6\bar{\tau}^2 \\ -14\bar{\tau}\pi - 20\alpha\bar{\tau} = 0 \end{array} \tag{4.4.32}$$

$$
\begin{aligned}
&&& -24\bar{\pi}\bar{\alpha} - 18\bar{\tau}^2 + 14\Psi_2 + 14\bar{\Psi}_2 + 16\Phi_{11} - 138\bar{\alpha}\bar{\tau} \\
&&& -63\bar{\pi}\bar{\tau} - 63\tau\pi - 121\pi\bar{\pi} + 32\Lambda + 3\pi^2 - 84\alpha\bar{\alpha} \\
&(A.7.24) & \overset{(\mathcal{V}1)\cdots(\mathcal{V}4)}{\underset{}{\overset{(\mathcal{D}1)\cdots(\mathcal{D}7)}{\Longrightarrow}}} \quad & -42\bar{\tau}\pi - 96\pi\bar{\alpha} - 198\tau\bar{\tau} - 18\tau^2 - 60\tau\bar{\alpha} - 36\bar{\alpha}^2 \\
&&& -36\alpha^2 + 3\bar{\pi}^2 - 138\alpha\tau - 96\alpha\bar{\pi} - 24\pi\alpha - 42\bar{\pi}\tau \\
&&& -60\alpha\bar{\tau} = 0
\end{aligned}
$$

$$
\begin{aligned}
&& -40\bar{\pi}\bar{\alpha} + 210\bar{\tau}^2 - 154\bar{\alpha}\bar{\tau} - 147\bar{\pi}\bar{\tau} + 357\tau\pi \\
&& -301\pi\bar{\pi} - 39\pi^2 + 140\alpha\bar{\alpha} + 490\bar{\tau}\pi + 112\pi\bar{\alpha} \\
& \overset{(4.4.25)}{\overset{(4.4.29)}{\Longrightarrow}} \quad & -210\tau\bar{\tau} - 42\tau^2 - 140\tau\bar{\alpha} - 68\bar{\alpha}^2 + 404\alpha^2 \\
&& +11\bar{\pi}^2 - 32\kappa\tau - 40\kappa\alpha - 52\kappa\pi - 52\kappa\bar{\pi} \\
&& -40\kappa\bar{\alpha} + 182\alpha\tau - 224\alpha\bar{\pi} + 264\pi\alpha - 32\bar{\tau}\kappa \\
&& -98\bar{\pi}\tau + 700\alpha\bar{\tau} = 0
\end{aligned}
\tag{4.4.33}
$$

The real and imaginary parts of (4.4.33) respectively are:

$$
\begin{aligned}
& 16\bar{\pi}\bar{\alpha} + 12\bar{\tau}^2 + 2\bar{\alpha}\bar{\tau} + 15\bar{\pi}\bar{\tau} + 15\tau\pi - 43\pi\bar{\pi} - 2\pi^2 + 20\alpha\bar{\alpha} \\
& +28\bar{\tau}\pi - 8\pi\bar{\alpha} - 30\tau\bar{\tau} + 12\tau^2 + 40\tau\bar{\alpha} + 24\bar{\alpha}^2 + 24\alpha^2 - 2\bar{\pi}^2 \\
& +2\alpha\tau - 8\alpha\bar{\pi} + 16\pi\alpha + 28\bar{\pi}\tau + 40\alpha\bar{\tau} = 0
\end{aligned}
\tag{4.4.34}
$$

$$
\begin{aligned}
& -152\bar{\pi}\bar{\alpha} + 126\bar{\tau}^2 - 168\bar{\alpha}\bar{\tau} - 252\bar{\pi}\bar{\tau} + 252\tau\pi - 25\pi^2 + 294\bar{\tau}\pi \\
& +168\pi\bar{\alpha} - 126\tau^2 - 420\tau\bar{\alpha} - 236\bar{\alpha}^2 + 236\alpha^2 + 25\bar{\pi}^2 - 32\kappa\tau \\
& -40\kappa\alpha - 52\kappa\pi - 52\kappa\bar{\pi} - 40\kappa\bar{\alpha} + 168\alpha\tau - 168\alpha\bar{\pi} + 152\pi\alpha \\
& -32\bar{\tau}\kappa - 294\bar{\pi}\tau + 420\alpha\bar{\tau} = 0
\end{aligned}
\tag{4.4.35}
$$

We remind the reader that equations (4.4.30) to (4.4.35) hold in **Scenario 1**. Next, we split **Scenario 1** into two subcases. Recall that $\delta(\Psi_2) = \Psi_2\bar{\pi} - 2\Psi_2\bar{\alpha}$

as shown in ($\mathcal{D}2$). If we substitute (4.4.25) into it, we will get:

$$\delta(\Psi_2) = \Psi_2\bar{\pi} - 2\Psi_2\bar{\alpha} \overset{(4.4.25)}{\Longrightarrow} \begin{array}{l} -2\bar{\pi}\delta(\bar{\pi}) - 8\bar{\alpha}\delta(\bar{\alpha}) - 4\bar{\pi}\delta(\bar{\alpha}) - 4\bar{\alpha}\delta(\bar{\pi}) + 13\kappa\delta(\bar{\pi}) \\ +13\bar{\pi}\delta(\kappa) + 8\kappa\delta(\tau) + 8\tau\delta(\kappa) + 10\kappa\delta(\bar{\alpha}) + 10\bar{\alpha}\delta(\kappa) \\ +\bar{\pi}^3 - 4\bar{\pi}\bar{\alpha}^2 + 2\bar{\pi}^2\bar{\alpha} - 13\kappa\bar{\pi}^2 - 8\bar{\pi}\kappa\tau + 16\bar{\pi}\kappa\bar{\alpha} \\ -8\bar{\alpha}^3 + 16\bar{\alpha}\kappa\tau + 20\kappa\bar{\alpha}^2 = 0 \end{array}$$

$$\overset{(\mathcal{V}1)\ldots(\mathcal{V}4)}{\underset{}{\overset{(\mathcal{D}1),(\mathcal{D}3)\ldots(\mathcal{D}4)}{\Longrightarrow}}} \qquad \kappa(129\bar{\pi}^2 + 170\tau\bar{\alpha} + 242\bar{\pi}\bar{\alpha} + 40\tau^2 + 165\bar{\pi}\tau + 128\bar{\alpha}^2) = 0 \tag{4.4.36}$$

Equation (4.4.36) implies **Scenario 1**, i.e. ($\mathcal{S}1$), further implies that either one of the following must hold:

$$\textbf{Scenario 1A} \qquad \kappa = 0, \quad \text{or} \tag{4.4.37}$$

$$\textbf{Scenario 1B} \qquad \begin{array}{l} 129\bar{\pi}^2 + 170\tau\bar{\alpha} + 242\bar{\pi}\bar{\alpha} \\ +40\tau^2 + 165\bar{\pi}\tau + 128\bar{\alpha}^2 = 0. \end{array} \tag{4.4.38}$$

<u>**Scenario 1A**</u> Recall from ($\mathcal{D}1$) that $\bar{\delta}(\Psi_2) = -\pi\Psi_2 + 2\Psi_2\alpha$. The following series of substitutions, quite remarkably, leads to:

$$\bar{\delta}(\Psi_2) = -\pi\Psi_2 + 2\Psi_2\alpha \overset{(4.4.25)}{\Longrightarrow} \begin{array}{l} -2\bar{\pi}\bar{\delta}(\bar{\pi}) - 8\bar{\alpha}\bar{\delta}(\bar{\alpha}) - 4\bar{\pi}\bar{\delta}(\bar{\alpha}) - 4\bar{\alpha}\bar{\delta}(\bar{\pi}) + 13\kappa\bar{\delta}(\bar{\pi}) \\ +13\bar{\pi}\bar{\delta}(\kappa) + 8\kappa\bar{\delta}(\tau) + 8\tau\bar{\delta}(\kappa) + 10\kappa\bar{\delta}(\bar{\alpha}) + 10\bar{\alpha}\bar{\delta}(\kappa) \\ -\pi\bar{\pi}^2 - 4\pi\bar{\alpha}^2 - 4\pi\bar{\pi}\bar{\alpha} + 13\pi\kappa\bar{\pi} + 8\pi\kappa\tau + 10\pi\kappa\bar{\alpha} \\ +2\alpha\bar{\pi}^2 + 8\alpha\bar{\alpha}^2 + 8\alpha\bar{\pi}\bar{\alpha} - 26\alpha\kappa\bar{\pi} - 16\alpha\kappa\tau - 20\alpha\kappa\bar{\alpha} = 0 \end{array}$$

$$\overset{(\mathcal{V}1)\ldots(\mathcal{V}4)}{\underset{(\mathcal{D}2)\ldots(\mathcal{D}7)}{\Longrightarrow}} \quad \begin{aligned} &72\bar{\alpha}\Phi_{11} - 360\bar{\alpha}\Lambda - 180\bar{\pi}\Lambda + 197\kappa\Psi_2 - 42\kappa\Phi_{11} - 66\Psi_2\bar{\pi} \\ &-132\Psi_2\bar{\alpha} + 36\Phi_{11}\bar{\pi} - 156\bar{\Psi}_2\bar{\alpha} - 78\bar{\pi}\bar{\Psi}_2 + 468\alpha\kappa\bar{\pi} \\ &+144\alpha\kappa\tau + 264\alpha\kappa\bar{\alpha} + 144\alpha\bar{\pi}\bar{\alpha} + 993\pi\kappa\bar{\pi} + 432\pi\kappa\tau \\ &+582\pi\kappa\bar{\alpha} - 54\pi\bar{\pi}^2 + 144\alpha\bar{\alpha}^2 + 36\alpha\bar{\pi}^2 + 216\pi\bar{\alpha}^2 \\ &+90\bar{\alpha}\bar{\tau}\kappa + 72\tau\bar{\tau}\kappa + 117\bar{\pi}\bar{\tau}\kappa + 91\kappa\bar{\Psi}_2 + 306\kappa\Lambda = 0 \end{aligned}$$

$$\overset{(4.4.37)}{\underset{}{\overset{(4.4.25)\,(4.4.29)}{\Longrightarrow}}} \quad \begin{aligned} &(\bar{\pi} + 2\bar{\alpha})(4\bar{\pi}^2 + 16\bar{\pi}\bar{\alpha} + 126\pi\bar{\pi} + 84\alpha\bar{\pi} \\ &+126\bar{\tau}^2 + 200\pi\alpha + 252\alpha\tau + 126\tau\bar{\tau} + 84\bar{\alpha}\bar{\tau} \\ &+420\alpha\bar{\tau} + 420\pi\bar{\alpha} + 336\alpha\bar{\alpha} + 284\alpha^2 \\ &+294\bar{\tau}\pi - 13\pi^2 + 16\bar{\alpha}^2 + 252\tau\pi) = 0 \end{aligned} \tag{4.4.39}$$

Equation (4.4.39) thus again implies **Scenario 1A** itself splits into two further sub-cases:

$$\textbf{Scenario 1A.1} \qquad \alpha = -\tfrac{1}{2}\pi, \quad \bar{\alpha} = -\tfrac{1}{2}\bar{\pi}, \qquad \text{or} \tag{4.4.40}$$

$$\textbf{Scenario 1A.2} \qquad \begin{aligned} &4\bar{\pi}^2 + 16\bar{\pi}\bar{\alpha} + 126\pi\bar{\pi} + 84\alpha\bar{\pi} + 126\bar{\tau}^2 \\ &+200\pi\alpha + 252\alpha\tau + 126\tau\bar{\tau} + 84\bar{\alpha}\bar{\tau} \\ &+420\alpha\bar{\tau} + 420\pi\bar{\alpha} + 336\alpha\bar{\alpha} + 284\alpha^2 \\ &+294\bar{\tau}\pi - 13\pi^2 + 16\bar{\alpha}^2 + 252\tau\pi = 0 \end{aligned} \tag{4.4.41}$$

Scenario 1A.1, or equivalently (4.4.40), leads to a contradiction, as we presently show. Under the **Scenario 1A.1** assumption (i.e. (4.4.40)), the equation (4.4.34) becomes:

$$-2\bar{\pi}^2 + 6\bar{\tau}^2 + 7\bar{\pi}\bar{\tau} + 7\tau\pi - 15\pi\bar{\pi} - 2\pi^2 + 4\bar{\tau}\pi - 15\tau\bar{\tau} + 6\tau^2 + 4\bar{\pi}\tau = 0 \tag{4.4.42}$$

Now, we have solved for $\Delta\sigma$ earlier. Hence, the **Scenario 1** assumption that $\sigma = \bar{\sigma} = 0$ gives us the following equation:

$$
\begin{aligned}
(4.4.9) \quad &\overset{(S1)}{\Longrightarrow} \quad -8\bar{\pi}\bar{\alpha} - 12\bar{\alpha}^2 + \bar{\pi}^2 - 6\tau^2 - 14\bar{\pi}\tau - 20\tau\bar{\alpha} = 0 \\
&\overset{(4.4.40)}{\Longrightarrow} \quad -(\tau + \bar{\pi})(3\tau - \bar{\pi}) = 0 \\
&\Longrightarrow \quad \text{either} \quad \begin{cases} \pi = -\bar{\tau} \\ \bar{\pi} = -\tau \end{cases} \quad \text{or} \quad \begin{cases} \pi = 3\bar{\tau} \\ \bar{\pi} = 3\tau \end{cases}
\end{aligned}
\tag{4.4.43}
$$

Substituting the two possibilities in (4.4.43) into (4.4.42) yields respectively:

$$
\text{either} \qquad 44\tau\bar{\tau} = 0, \qquad \text{or} \qquad 108\tau\bar{\tau} = 0. \tag{4.4.44}
$$

Equation (4.4.44) of course implies that $\tau = \bar{\tau} = 0$. Equations (4.4.43) and (4.4.40) then imply $\pi = \bar{\pi} = 0$ and $\alpha = \bar{\alpha} = 0$ respectively. Therefore, (4.4.25) implies that $\Psi_2 = \overline{\Psi}_2 = 0$, contrary to the fact that Ψ_2 is the non-vanishing component of the Weyl spinor with respect to the canonical spinor dyad we have chosen. We have proved that (4.4.40) leads to a contradiction, and hence **Scenario 1A.1** is inadmissible.

We next consider **Scenario 1A.2**. In this scenario, (4.4.41) holds and the real and imaginary parts of (4.4.41) respectively are:

$$
\begin{aligned}
&-72\bar{\pi}\bar{\alpha} - 42\bar{\tau}^2 - 112\bar{\alpha}\bar{\tau} - 84\bar{\pi}\bar{\tau} - 84\tau\pi - 84\pi\bar{\pi} + 3\pi^2 - 224\alpha\bar{\alpha} \\
&-98\bar{\tau}\pi - 168\pi\bar{\alpha} - 84\tau\bar{\tau} - 42\tau^2 - 140\tau\bar{\alpha} - 100\bar{\alpha}^2 - 100\alpha^2 + 3\bar{\pi}^2 \\
&-112\alpha\tau - 168\alpha\bar{\pi} - 72\pi\alpha - 98\bar{\pi}\tau - 140\alpha\bar{\tau} = 0
\end{aligned}
\tag{4.4.45}
$$

$$
\begin{aligned}
&184\bar{\pi}\bar{\alpha} - 126\bar{\tau}^2 + 168\bar{\alpha}\bar{\tau} + 252\bar{\pi}\bar{\tau} - 252\tau\pi + 17\pi^2 - 294\bar{\tau}\pi \\
&-336\pi\bar{\alpha} + 126\tau^2 + 420\tau\bar{\alpha} + 268\bar{\alpha}^2 - 268\alpha^2 - 17\bar{\pi}^2 - 168\alpha\tau \\
&+336\alpha\bar{\pi} - 184\pi\alpha + 294\bar{\pi}\tau - 420\alpha\bar{\tau} = 0
\end{aligned}
\tag{4.4.46}
$$

Also, substituting the **Scenario 1A.2** assumption that $\kappa = 0$ into (4.4.34) yields:

$$\begin{aligned}&-152\bar{\pi}\bar{\alpha} + 126\bar{\tau}^2 - 168\bar{\alpha}\bar{\tau} - 252\bar{\pi}\bar{\tau} + 252\tau\pi - 25\pi^2 + 294\bar{\tau}\pi\\&+168\pi\bar{\alpha} - 126\tau^2 - 420\tau\bar{\alpha} - 236\bar{\alpha}^2 + 236\alpha^2 + 25\bar{\pi}^2 + 168\alpha\tau\\&-168\alpha\bar{\pi} + 152\pi\alpha - 294\bar{\pi}\tau + 420\alpha\bar{\tau} = 0\end{aligned} \tag{4.4.47}$$

Therefore, the equations (4.4.30), (4.4.31), (4.4.32), (4.4.34), (4.4.45), (4.4.46) and (4.4.47) form a system of seven multivariate polynomial equations in the 6 indeterminates α, $\bar{\alpha}$, τ, $\bar{\tau}$, π, $\bar{\pi}$. We will call this system **System 1A.2**, since it arises in **Scenario 1A.2**. We claim that this system has only the trivial solution, i.e. **System 1A.2** implies all of α, $\bar{\alpha}$, τ, $\bar{\tau}$, π, $\bar{\pi}$ must vanish. This can be proved using the following lemma:

Lemma 4.4.8 *The Gröbner basis for the ideal (in $\mathbb{C}[\tau, \alpha, \pi, \bar{\pi}, \bar{\alpha}, \bar{\tau}]$) generated by the left-hand-sides*[6] *of the equations in* **System 1A.2** *with respect to the pure lexicographical ordering with $\tau \succ \alpha \succ \pi \succ \bar{\pi} \succ \bar{\alpha} \succ \bar{\tau}$ decomposes into the components:*

$$\{\alpha, \bar{\alpha}, \tau, \bar{\tau}, \pi, \bar{\pi}\}, \quad \{\bar{\tau}, \bar{\pi} + 2\bar{\alpha}, \pi, \alpha, 3\tau + 2\bar{\alpha}\}, \quad \{\bar{\alpha}, \bar{\pi}, -3\bar{\tau} + \pi, 2\alpha + 3\bar{\tau}, \tau\}. \tag{4.4.48}$$

The decomposition asserted in Lemma 4.4.8 can be computed using the MAPLE function `gsolve()` . It is clear that each component in (4.4.48) leads to the trivial solution when we take into account that $\{\alpha, \bar{\alpha}\}$ $\{\tau, \bar{\tau}\}$, $\{\pi, \bar{\pi}\}$ are conjugate pairs. This in turn implies that **Scenario 1A.2**, or (4.4.41), implies that all of α, $\bar{\alpha}$, τ, $\bar{\tau}$, π, $\bar{\pi}$ must vanish. Hence, by (4.4.25) again, Ψ_2 vanishes, contrary to the fact that Ψ_2 is the non-vanishing component of the Weyl spinor with respect to the canonical spinor dyad we have chosen. We

[6] considered, in the present context, as multivariate polynomials in the indeterminates α, $\bar{\alpha}$, τ, $\bar{\tau}$, π, $\bar{\pi}$.

have therefore proved that **Scenario 1A.2** is also inadmissible.

Since both **Scenario 1A.1** and **Scenario 1A.2** are inadmissible, **Scenario 1A** is inadmissible.

Scenario 1B (4.4.38) and its conjugate are:

$$40\tau^2 + 242\bar{\pi}\bar{\alpha} + 165\bar{\pi}\tau + 129\bar{\pi}^2 + 170\tau\bar{\alpha} + 128\bar{\alpha}^2 = 0 \tag{4.4.49}$$

$$40\bar{\tau}^2 + 242\pi\alpha + 165\bar{\tau}\pi + 129\pi^2 + 170\alpha\bar{\tau} + 128\alpha^2 = 0 \tag{4.4.50}$$

Equations (4.4.30), (4.4.31), (4.4.32), (4.4.34), (4.4.49) and (4.4.50) form a system of 6 multivariate polynomial equations in the 6 indeterminates α, τ, π, $\bar{\alpha}$, $\bar{\tau}$ and $\bar{\pi}$. We call this system **System 1B**. We claim that the following lemma is true:

Lemma 4.4.9 *The ideal in $\mathbb{C}[\tau, \alpha, \pi, \bar{\pi}, \bar{\alpha}, \bar{\tau}]$ generated by the left-hand-sides of the equations in* **System 1B** *has the Gröbner basis $\{\alpha, \bar{\alpha}, \tau, \bar{\tau}, \pi, \bar{\pi}\}$ with respect to the pure lexicographical ordering with $\tau \succ \alpha \succ \pi \succ \bar{\pi} \succ \bar{\alpha} \succ \bar{\tau}$.*

Again, Lemma 4.4.9 can be proved via a computation using `gsolve()` . Obviously, the Gröbner basis in Lemma 4.4.9 leads only to the trivial solution. By the same argument as in the two earlier scenarios, (4.4.25) implies that Ψ_2 vanishes, a contradiction. Thus, **Scenario 1B** is also inadmissible. We conclude here that **Scenario 1** is inadmissible. It remains to treat **Scenario 2**.

Scenario 2 In this scenario, the following equation holds:

$$12\alpha^2 + 10\pi\alpha - 21\bar{\tau}\pi - 18\alpha\bar{\tau} + 6\pi^2 + 44\bar{\Psi}_2 - 12\alpha\bar{\alpha} - 21\tau\pi - 18\alpha\tau - 14\pi\bar{\alpha} = 0 \tag{$\mathcal{S}2$}$$

We first obtain expressions for Ψ_2, Λ, Φ_{11}, Φ_{02} in terms of α, τ, π and their conjugates. To begin, note that one of the Newman-Penrose field equations simplifies to the following:

$$(\text{A.1.17}) \quad \overset{(\mathcal{V}1)\dots(\mathcal{V}4)}{\underset{(\mathcal{D}1)\dots(\mathcal{D}7)}{\Longrightarrow}} \quad \begin{aligned} &18\bar{\tau}^2 + 10\Psi_2 + 12\bar{\Psi}_2 - 8\Phi_{11} + 12\bar{\alpha}\bar{\tau} + 36\tau\pi \\ &+20\pi\bar{\pi} + 32\Lambda - 3\pi^2 + 24\alpha\bar{\alpha} + 42\bar{\tau}\pi + 24\pi\bar{\alpha} \\ &+18\tau\bar{\tau} + 36\alpha^2 + 36\alpha\tau + 24\pi\alpha + 60\alpha\bar{\tau} = 0 \end{aligned} \tag{4.4.51}$$

Equation (4.4.51) and its conjugate form a system of equations linear in Ψ_2 and $\overline{\Psi}_2$. If we solve this system, we get the following expression[7] for Ψ_2 (the expression for $\overline{\Psi}_2$ is not shown):

$$\begin{aligned} \Psi_2 \quad = \quad & -\frac{72}{11}\bar{\pi}\bar{\alpha} + \frac{45}{11}\bar{\tau}^2 + \frac{4}{11}\Phi_{11} - \frac{78}{11}\bar{\alpha}\bar{\tau} - \frac{108}{11}\bar{\pi}\bar{\tau} + \frac{90}{11}\tau\pi \\ & -\frac{10}{11}\pi\bar{\pi} - \frac{16}{11}\Lambda - \frac{15}{22}\pi^2 - \frac{12}{11}\alpha\bar{\alpha} + \frac{105}{11}\bar{\tau}\pi + \frac{60}{11}\pi\bar{\alpha} \\ & -\frac{9}{11}\tau\bar{\tau} - \frac{54}{11}\tau^2 - \frac{180}{11}\tau\bar{\alpha} - \frac{108}{11}\bar{\alpha}^2 + \frac{90}{11}\alpha^2 + \frac{9}{11}\bar{\pi}^2 \\ & +\frac{54}{11}\alpha\tau - \frac{72}{11}\alpha\bar{\pi} + \frac{60}{11}\pi\alpha - \frac{126}{11}\bar{\pi}\tau + \frac{150}{11}\alpha\bar{\tau} \end{aligned} \tag{4.4.52}$$

[7]Note that (4.4.52) is different from (4.4.25); and they are valid in **Scenario 2** and **Scenario 1**, respectively.

We next assemble a system of equations linear in Λ and Φ_{11}:

$$[\bar{\delta}, \delta]\,\Psi_2 \overset{(\mathcal{V}1)\ldots(\mathcal{V}4)}{\underset{(\mathcal{D}1)\ldots(\mathcal{D}7)}{\Longrightarrow}} -5\bar{\Psi}_2 - 8\pi\bar{\pi} + 2\Phi_{11} - 5\Psi_2 - 14\Lambda = 0$$

$$\overset{(4.4.52)}{\Longrightarrow} \begin{aligned} &40\bar{\pi}\bar{\alpha} + 30\bar{\tau}^2 - 12\Phi_{11} + 80\bar{\alpha}\bar{\tau} + 60\bar{\pi}\bar{\tau} + 60\tau\pi \\ &+8\pi\bar{\pi} + 4\Lambda - 5\pi^2 + 80\alpha\bar{\alpha} + 70\bar{\tau}\pi + 40\pi\bar{\alpha} \\ &+60\tau\bar{\tau} + 30\tau^2 + 100\tau\bar{\alpha} + 60\bar{\alpha}^2 + 60\alpha^2 \\ &-5\bar{\pi}^2 + 80\alpha\tau + 40\alpha\bar{\pi} + 40\pi\alpha + 70\bar{\pi}\tau + 100\alpha\bar{\tau} = 0 \end{aligned} \tag{4.4.53}$$

$$(\mathcal{S}2) \overset{(4.4.52)}{\Longrightarrow} \begin{aligned} &-240\bar{\pi}\bar{\alpha} + 216\bar{\tau}^2 - 16\Phi_{11} - 216\bar{\alpha}\bar{\tau} - 360\bar{\pi}\bar{\tau} \\ &+453\tau\pi + 40\pi\bar{\pi} + 64\Lambda - 42\pi^2 + 60\alpha\bar{\alpha} + 525\bar{\tau}\pi \\ &+302\pi\bar{\alpha} + 36\tau\bar{\tau} - 180\tau^2 - 600\tau\bar{\alpha} - 360\bar{\alpha}^2 \\ &+420\alpha^2 + 30\bar{\pi}^2 + 330\alpha\tau - 240\alpha\bar{\pi} + 278\pi\alpha \\ &-420\bar{\pi}\tau + 738\alpha\bar{\tau} = 0 \end{aligned} \tag{4.4.54}$$

Solving (4.4.53) and (4.4.54) for Λ and Φ_{11} yields:

$$\begin{aligned} \Lambda \;=\; & 5\bar{\pi}\bar{\alpha} - 3\bar{\tau}^2 + \frac{11}{2}\bar{\alpha}\bar{\tau} + \frac{15}{2}\bar{\pi}\bar{\tau} - \frac{1119}{176}\tau\pi - \frac{1}{2}\pi\bar{\pi} + \frac{53}{88}\pi^2 \\ & + \frac{35}{44}\alpha\bar{\alpha} - \frac{1295}{176}\bar{\tau}\pi - \frac{373}{88}\pi\bar{\alpha} + \frac{3}{4}\tau\bar{\tau} + \frac{15}{4}\tau^2 + \frac{25}{2}\tau\bar{\alpha} \\ & + \frac{15}{2}\bar{\alpha}^2 - \frac{255}{44}\alpha^2 - \frac{5}{8}\bar{\pi}^2 - \frac{335}{88}\alpha\tau + 5\alpha\bar{\pi} - \frac{337}{88}\pi\alpha \\ & + \frac{35}{4}\bar{\pi}\tau - \frac{907}{88}\alpha\bar{\tau} \end{aligned} \tag{4.4.55}$$

$$\begin{aligned} \Phi_{11} \;=\; & 5\bar{\pi}\bar{\alpha} + \frac{3}{2}\bar{\tau}^2 + \frac{17}{2}\bar{\alpha}\bar{\tau} + \frac{15}{2}\bar{\pi}\bar{\tau} + \frac{507}{176}\tau\pi + \frac{1}{2}\pi\bar{\pi} - \frac{19}{88}\pi^2 \\ & + \frac{305}{44}\alpha\bar{\alpha} + \frac{595}{176}\bar{\tau}\pi + \frac{169}{88}\pi\bar{\alpha} + \frac{21}{4}\tau\bar{\tau} + \frac{15}{4}\tau^2 + \frac{25}{2}\tau\bar{\alpha} \\ & + \frac{15}{2}\bar{\alpha}^2 + \frac{135}{44}\alpha^2 - \frac{5}{8}\bar{\pi}^2 + \frac{475}{88}\alpha\tau + 5\alpha\bar{\pi} + \frac{181}{88}\pi\alpha \\ & + \frac{35}{4}\bar{\pi}\tau + \frac{431}{88}\alpha\bar{\tau} \end{aligned} \tag{4.4.56}$$

To get Φ_{02}, we note:

$$(\text{A.5.6}) \overset{(\mathcal{V}1)\ldots(\mathcal{V}4)}{\underset{}{\overset{(\mathcal{D}1)\ldots(\mathcal{D}7)}{\Longrightarrow}}} \quad -20\bar{\pi}\bar{\alpha} - 28\bar{\alpha}^2 - 4\Phi_{02} + 5\bar{\pi}^2 - 12\tau^2 - 48\bar{\pi}\tau - 48\tau\bar{\alpha} = 0$$

$$\Longrightarrow \quad \Phi_{02} = \frac{5}{4}\bar{\pi}^2 - 7\bar{\alpha}^2 - 12\bar{\pi}\tau - 12\tau\bar{\alpha} - 5\bar{\pi}\bar{\alpha} - 3\tau^2 \qquad (4.4.57)$$

We are now ready to build a system of seven multivariate polynomial equations in the indeterminates α, $\bar{\alpha}$, τ, $\bar{\tau}$, π, $\bar{\pi}$.

$$(\text{A.6.9}) \overset{(\mathcal{V}1)\ldots(\mathcal{V}4)}{\overset{(\mathcal{D}1)\ldots(\mathcal{D}7)}{\Longrightarrow}} \quad 2\alpha + 3\bar{\tau} + 2\bar{\alpha} + 3\tau = 0 \qquad (4.4.58)$$

$$(\text{A.7.24}) \overset{(\mathcal{V}1)\ldots(\mathcal{V}4)}{\overset{(\mathcal{D}1)\ldots(\mathcal{D}7)}{\Longrightarrow}} \quad \begin{aligned} &-24\bar{\pi}\bar{\alpha} - 18\bar{\tau}^2 + 14\Psi_2 + 14\bar{\Psi}_2 + 16\Phi_{11} - 138\bar{\alpha}\bar{\tau} \\ &-63\bar{\pi}\bar{\tau} - 63\tau\pi - 121\pi\bar{\pi} + 32\Lambda + 3\pi^2 - 84\alpha\bar{\alpha} \\ &-42\bar{\tau}\pi - 96\pi\bar{\alpha} - 198\tau\bar{\tau} - 18\tau^2 - 60\tau\bar{\alpha} - 36\bar{\alpha}^2 \\ &-36\alpha^2 + 3\bar{\pi}^2 - 138\alpha\tau - 96\alpha\bar{\pi} - 24\pi\alpha - 42\bar{\pi}\tau \\ &-60\alpha\bar{\tau} = 0 \end{aligned}$$

$$\overset{(4.4.52)}{\overset{(4.4.55)\,(4.4.56)}{\Longrightarrow}} \quad \begin{aligned} &16\bar{\pi}\bar{\alpha} + 12\bar{\tau}^2 + 2\bar{\alpha}\bar{\tau} + 15\bar{\pi}\bar{\tau} + 15\tau\pi - 43\pi\bar{\pi} \\ &-2\pi^2 + 20\alpha\bar{\alpha} + 28\bar{\tau}\pi - 8\pi\bar{\alpha} - 30\tau\bar{\tau} + 12\tau^2 \\ &+40\tau\bar{\alpha} + 24\bar{\alpha}^2 + 24\alpha^2 - 2\bar{\pi}^2 + 2\alpha\tau - 8\alpha\bar{\pi} \\ &+16\pi\alpha + 28\bar{\pi}\tau + 40\alpha\bar{\tau} = 0 \end{aligned} \qquad (4.4.59)$$

$$
\text{(A.9.1)} \quad \overset{\substack{(\mathcal{V}1)\ldots(\mathcal{V}4)\\(\mathcal{D}1)\ldots(\mathcal{D}7)}}{\Longrightarrow} \quad
\begin{aligned}
&105\tau\Psi_2 + 60\tau\bar{\alpha}^2 + 33\bar{\pi}^2\bar{\alpha} + 54\bar{\pi}\tau^2 + 18\tau^2\bar{\alpha}\\
&+102\tau\bar{\pi}^2 + 72\bar{\pi}\bar{\alpha}^2 + 132\bar{\alpha}\Lambda + 102\bar{\pi}\Lambda - 24\bar{\pi}^2\alpha\\
&+36\bar{\tau}\bar{\alpha}^2 + 24\bar{\alpha}^2\alpha - 12\Phi_{11}\bar{\alpha} - 36\bar{\tau}\bar{\pi}^2 + 12\bar{\pi}\Phi_{02}\\
&+54\pi\bar{\pi}^2 + 32\Psi_2\bar{\pi} + 112\Psi_2\bar{\alpha} - 6\Phi_{11}\bar{\pi} + 150\tau\bar{\alpha}\bar{\pi}\\
&+50\bar{\Psi}_2\bar{\alpha} + 40\bar{\pi}\bar{\Psi}_2 + 36\bar{\alpha}^3 - 12\bar{\pi}^3 + 72\pi\bar{\alpha}\bar{\pi}\\
&+12\tau\bar{\pi}\alpha + 12\tau\bar{\alpha}\alpha + 18\tau\bar{\pi}\bar{\tau} + 18\tau\bar{\alpha}\bar{\tau} = 0
\end{aligned}
$$

$$
\overset{\substack{(4.4.52)\,(4.4.57)\\(4.4.55)\,(4.4.56)}}{\Longrightarrow} \quad
\begin{aligned}
&36856\bar{\alpha}\tau\pi + 29436\bar{\alpha}\alpha\bar{\tau} + 11396\bar{\alpha}\pi\alpha\\
&+20790\bar{\alpha}\bar{\tau}\pi + 46830\tau\alpha\bar{\tau} + 18130\tau\pi\alpha\\
&+33075\tau\bar{\tau}\pi - 54120\tau\bar{\alpha}^2 + 2376\bar{\pi}^2\bar{\alpha}\\
&-31020\bar{\pi}\tau^2 - 54120\tau^2\bar{\alpha} + 4158\tau\bar{\pi}^2\\
&-8976\bar{\pi}\bar{\alpha}^2 + 1056\bar{\pi}^2\alpha - 10032\bar{\tau}\bar{\alpha}^2\\
&+1320\bar{\alpha}^2\alpha + 1584\bar{\tau}\bar{\pi}^2 - 33968\tau\bar{\alpha}\bar{\pi}\\
&-15840\bar{\alpha}^3 - 132\bar{\pi}^3 + 28455\tau^2\pi\\
&-2520\tau\pi^2 - 542\pi\bar{\alpha}\bar{\pi} - 18410\tau\bar{\pi}\alpha\\
&+12960\tau\bar{\alpha}\alpha - 26796\tau\bar{\pi}\bar{\tau} - 17688\tau\bar{\alpha}\bar{\tau}\\
&+27300\tau\alpha^2 + 19110\alpha\tau^2 + 13860\tau\bar{\tau}^2\\
&+8712\bar{\alpha}\bar{\tau}^2 - 1584\bar{\alpha}\pi^2 + 11924\pi\bar{\alpha}^2 - 813\bar{\pi}\tau\pi\\
&-780\bar{\pi}\alpha^2 - 1338\bar{\pi}\alpha\bar{\tau} - 396\bar{\pi}\bar{\tau}^2 + 72\bar{\pi}\pi^2\\
&+17160\bar{\alpha}\alpha^2 - 13860\tau^3 - 10044\bar{\pi}\alpha\bar{\alpha}\\
&-945\bar{\pi}\bar{\tau}\pi - 518\bar{\pi}\pi\alpha - 14520\bar{\pi}\bar{\alpha}\bar{\tau} = 0
\end{aligned}
\qquad (4.4.60)
$$

$$
\begin{aligned}
(\mathrm{A}.9.2) \overset{\substack{(\mathcal{V}1)\ldots(\mathcal{V}4)\\(\mathcal{D}1)\ldots(\mathcal{D}7)}}{\Longrightarrow} \quad & 252\bar{\alpha}\tau\pi + 8\Lambda\tau + 192\bar{\alpha}\alpha\bar{\tau} + 72\bar{\alpha}\pi\alpha + 276\bar{\alpha}\bar{\tau}\pi \\
& -8\Phi_{11}\pi - 8\Phi_{11}\bar{\tau} + 48\Phi_{20}\bar{\pi} + 48\Phi_{20}\bar{\alpha} + 214\pi\Psi_2 \\
& +12\tau\pi\alpha + 18\tau\bar{\tau}\pi - 48\Lambda\bar{\alpha} + 152\Lambda\bar{\pi} + 41\pi\bar{\pi}^2 + 80\Psi_2\alpha \\
& -36\bar{\pi}^2\alpha + 48\bar{\alpha}^2\alpha + 60\bar{\Psi}_2\pi - 20\bar{\Psi}_2\alpha + 60\Psi_2\bar{\pi} \\
& -20\Psi_2\bar{\alpha} - 8\Phi_{11}\tau - 8\Phi_{11}\bar{\pi} + 48\Phi_{02}\alpha + 48\Phi_{02}\pi \\
& +80\bar{\Psi}_2\bar{\alpha} + 214\bar{\pi}\bar{\Psi}_2 + 162\tau^2\pi + 24\pi\bar{\alpha}\bar{\pi} \\
& +276\tau\bar{\pi}\alpha + 192\tau\bar{\alpha}\alpha + 18\tau\bar{\pi}\bar{\tau} + 144\alpha\tau^2 + 52\bar{\Psi}_2\tau \\
& +152\Lambda\pi + 8\bar{\tau}\Lambda - 48\alpha\Lambda + 144\bar{\alpha}\bar{\tau}^2 - 36\bar{\alpha}\pi^2 + 84\pi\bar{\alpha}^2 \\
& +254\bar{\pi}\tau\pi + 84\bar{\pi}\alpha^2 + 252\bar{\pi}\alpha\bar{\tau} + 162\bar{\pi}\bar{\tau}^2 + 41\bar{\pi}\pi^2 \\
& +48\bar{\alpha}\alpha^2 + 72\bar{\pi}\alpha\bar{\alpha} + 254\bar{\pi}\bar{\tau}\pi + 24\bar{\pi}\pi\alpha \\
& +12\bar{\pi}\bar{\alpha}\bar{\tau} + 52\Psi_2\bar{\tau} = 0
\end{aligned}
$$

$$
\begin{aligned}
\overset{\substack{(4.4.52)\,(4.4.57)\\(4.4.55)\,(4.4.56)}}{\Longrightarrow} \quad & -36688\bar{\alpha}\tau\pi - 12640\bar{\alpha}\alpha\bar{\tau} + 7956\bar{\alpha}\pi\alpha \\
& -11344\bar{\alpha}\bar{\tau}\pi + 3192\tau\alpha\bar{\tau} + 27758\tau\pi\alpha \\
& +6716\tau\bar{\tau}\pi + 55390\bar{\tau}\pi\alpha + 1848\pi\bar{\pi}^2 \\
& +1276\bar{\pi}^2\alpha - 7040\bar{\tau}\bar{\alpha}^2 - 15472\bar{\alpha}^2\alpha \\
& +572\bar{\tau}\bar{\pi}^2 + 12480\alpha^3 + 12834\alpha\pi^2 \\
& +29348\pi\alpha^2 + 24939\bar{\tau}\pi^2 + 18252\bar{\tau}^2\pi \\
& -3696\bar{\tau}\tau^2 + 27648\bar{\tau}\alpha^2 + 17040\alpha\bar{\tau}^2 \\
& -10392\tau^2\pi + 21947\tau\pi^2 - 13878\pi\bar{\alpha}\bar{\pi} \\
& -16058\tau\bar{\pi}\alpha - 25312\tau\bar{\alpha}\alpha - 7348\tau\bar{\pi}\bar{\tau} \\
& -11880\tau\bar{\alpha}\bar{\tau} + 7936\tau\alpha^2 - 6984\alpha\tau^2 \\
& -528\tau\bar{\tau}^2 - 1944\pi^3 - 5016\bar{\alpha}\bar{\tau}^2 \\
& +14758\bar{\alpha}\pi^2 - 22304\pi\bar{\alpha}^2 - 24513\bar{\pi}\tau\pi \\
& -9916\bar{\pi}\alpha^2 - 17906\bar{\pi}\alpha\bar{\tau} - 6204\bar{\pi}\bar{\tau}^2 \\
& -96\bar{\pi}\pi^2 - 2992\bar{\alpha}\alpha^2 - 9004\bar{\pi}\alpha\bar{\alpha} \\
& -20949\bar{\pi}\bar{\tau}\pi - 14526\bar{\pi}\pi\alpha - 4136\bar{\pi}\bar{\alpha}\bar{\tau} \\
& +3168\bar{\tau}^3 = 0
\end{aligned}
\tag{4.4.61}
$$

$$
\begin{aligned}
(\text{A.9.4}) \quad \overset{(\mathcal{V}1)\dots(\mathcal{V}4)}{\underset{}{\overset{(\mathcal{D}1)\dots(\mathcal{D}7)}{\Longrightarrow}}} \quad & 12\bar{\alpha}\alpha\bar{\tau} + 150\alpha\bar{\tau}\pi + 12\bar{\alpha}\bar{\tau}\pi - 6\Phi_{11}\pi + 40\pi\Psi_2 + 18\tau\alpha\bar{\tau} \\
& +18\tau\bar{\tau}\pi + 132\alpha\Lambda + 102\Lambda\pi + 50\Psi_2\alpha + 32\bar{\Psi}_2\pi + 112\bar{\Psi}_2\alpha \\
& +54\pi\bar{\tau}^2 + 72\pi\alpha^2 + 60\alpha^2\bar{\tau} + 102\bar{\tau}\pi^2 + 18\alpha\bar{\tau}^2 + 33\alpha\pi^2 \\
& -36\tau\pi^2 + 36\tau\alpha^2 - 12\pi^3 + 36\alpha^3 - 24\bar{\alpha}\pi^2 + 54\bar{\pi}\pi^2 + 24\bar{\alpha}\alpha^2 \\
& +72\bar{\pi}\pi\alpha - 12\alpha\Phi_{11} + 12\Phi_{20}\pi + 105\bar{\Psi}_2\bar{\tau} = 0
\end{aligned}
$$

$$
\begin{aligned}
\overset{(4.4.52)(4.4.57)}{\overset{(4.4.55)(4.4.56)}{\Longrightarrow}} \quad & 948\bar{\alpha}\alpha\bar{\tau} + 3614\alpha\bar{\tau}\pi + 1880\bar{\alpha}\pi\alpha + 1106\bar{\alpha}\bar{\tau}\pi \\
& +1422\tau\alpha\bar{\tau} + 2820\tau\pi\alpha + 1659\tau\bar{\tau}\pi + 1659\pi\bar{\tau}^2 \\
& +1640\pi\alpha^2 + 2616\alpha^2\bar{\tau} + 693\bar{\tau}\pi^2 + 1422\alpha\bar{\tau}^2 \\
& +274\alpha\pi^2 + 771\tau\pi^2 + 1980\tau\alpha^2 - 60\pi^3 + 1320\alpha^3 \\
& +514\bar{\alpha}\pi^2 + 1320\bar{\alpha}\alpha^2 = 0
\end{aligned}
\tag{4.4.62}
$$

$$(\text{A.4.1}) \quad \Longrightarrow \quad \bar{\delta}(\Psi_2) + \pi\Psi_2 - 2\Psi_2\alpha = 0$$

$$\begin{aligned}
& 360\bar{\alpha}\tau\pi - 312\bar{\alpha}\alpha\bar{\tau} + 120\alpha\bar{\tau}\pi + 264\bar{\alpha}\pi\alpha \\
& +156\bar{\alpha}\bar{\tau}\pi - 8\bar{\delta}(\Phi_{11}) + 32\bar{\delta}(\Lambda) - 8\Phi_{11}\pi \\
& -36\tau\alpha\bar{\tau} + 252\tau\pi\alpha + 18\tau\bar{\tau}\pi - 18\pi\bar{\pi}^2 \\
& +36\bar{\pi}^2\alpha - 432\bar{\alpha}^2\alpha - 90\pi\bar{\tau}^2 - 210\pi\bar{\delta}(\bar{\tau}) \\
& +60\pi\alpha^2 + 600\alpha^2\bar{\tau} - 210\bar{\tau}\pi^2 - 210\bar{\tau}\bar{\delta}(\pi) \\
& +180\alpha\bar{\tau}^2 - 150\alpha\pi^2 + 24\bar{\alpha}\bar{\delta}(\alpha) + 108\tau^2\pi \\
& -180\tau\pi^2 + 144\pi\bar{\alpha}\bar{\pi} - 504\tau\bar{\pi}\alpha - 720\tau\bar{\alpha}\alpha \\
& +216\tau\alpha^2 - 216\alpha\tau^2 + 15\pi^3 + 360\alpha^3 \\
& +24\alpha\bar{\delta}(\bar{\alpha}) + 16\alpha\Phi_{11} + 30\pi\bar{\delta}(\pi) - 64\alpha\Lambda \\
\overset{(4.4.52)}{\Longrightarrow} \quad & -120\bar{\alpha}\pi^2 + 20\pi\bar{\delta}(\bar{\pi}) + 216\pi\bar{\alpha}^2 + 20\bar{\pi}\bar{\delta}(\pi) \\
& +252\bar{\pi}\tau\pi - 288\bar{\pi}\alpha^2 + 32\pi\Lambda - 432\bar{\pi}\alpha\bar{\tau} \\
& +20\bar{\pi}\pi^2 - 48\bar{\alpha}\alpha^2 - 180\pi\bar{\delta}(\tau) + 216\bar{\pi}\bar{\delta}(\bar{\tau}) \\
& -288\bar{\pi}\alpha\bar{\alpha} + 216\bar{\tau}\bar{\delta}(\bar{\pi}) + 216\bar{\pi}\bar{\tau}\pi - 180\tau\bar{\delta}(\pi) \\
& +156\bar{\alpha}\bar{\delta}(\bar{\tau}) + 156\bar{\tau}\bar{\delta}(\bar{\alpha}) + 104\bar{\pi}\pi\alpha - 180\bar{\tau}\bar{\delta}(\bar{\tau}) \\
& +144\bar{\pi}\bar{\delta}(\bar{\alpha}) - 360\alpha\bar{\delta}(\alpha) - 36\bar{\pi}\bar{\delta}(\bar{\pi}) - 108\alpha\bar{\delta}(\tau) \\
& -108\tau\bar{\delta}(\alpha) + 144\alpha\bar{\delta}(\bar{\pi}) + 144\bar{\pi}\bar{\delta}(\alpha) - 120\pi\bar{\delta}(\alpha) \\
& -120\alpha\bar{\delta}(\pi) + 252\bar{\pi}\bar{\delta}(\tau) + 252\tau\bar{\delta}(\bar{\pi}) - 300\alpha\bar{\delta}(\bar{\tau}) \\
& -300\bar{\tau}\bar{\delta}(\alpha) - 120\pi\bar{\delta}(\bar{\alpha}) - 120\bar{\alpha}\bar{\delta}(\pi) + 18\tau\bar{\delta}(\bar{\tau}) \\
& +18\bar{\tau}\bar{\delta}(\tau) + 216\tau\bar{\delta}(\tau) + 360\tau\bar{\delta}(\bar{\alpha}) + 360\bar{\alpha}\bar{\delta}(\tau) \\
& +432\bar{\alpha}\bar{\delta}(\bar{\alpha}) + 144\bar{\alpha}\bar{\delta}(\bar{\pi}) = 0
\end{aligned}$$

$$
\begin{aligned}
& 8076\bar{\alpha}\tau\pi + 368\bar{\alpha}\alpha\bar{\tau} + 7868\alpha\bar{\tau}\pi + 3440\bar{\alpha}\pi\alpha \\
& +6136\bar{\alpha}\bar{\tau}\pi + 12960\tau\alpha\bar{\tau} + 6216\tau\pi\alpha + 10920\tau\bar{\tau}\pi \\
& -18216\tau\bar{\alpha}^2 + 1452\bar{\pi}^2\bar{\alpha} - 11220\bar{\pi}\tau^2 - 19536\tau^2\bar{\alpha} \\
& +396\pi\bar{\pi}^2 + 2706\tau\bar{\pi}^2 - 1584\bar{\pi}\bar{\alpha}^2 + 1848\bar{\pi}^2\alpha \\
& -10912\bar{\tau}\bar{\alpha}^2 - 7624\bar{\alpha}^2\alpha + 2376\bar{\tau}\bar{\pi}^2 + 2949\pi\bar{\tau}^2 \\
& -9768\tau\bar{\alpha}\bar{\pi} + 3124\pi\alpha^2 + 15272\alpha^2\bar{\tau} - 2256\bar{\tau}\pi^2 \\
\overset{(4.4.52)\,(4.4.57)}{\underset{(\mathcal{V}1)\ldots(\mathcal{V}4)}{\overset{(4.4.55)\,(4.4.56)}{\underset{(\mathcal{D}1)\ldots(\mathcal{D}7)}{\Longrightarrow}}}} \quad & -4356\bar{\tau}\tau^2 + 8970\alpha\bar{\tau}^2 - 4752\bar{\alpha}^3 - 132\bar{\pi}^3 \\
& -1744\alpha\pi^2 + 8235\tau^2\pi - 3858\tau\pi^2 - 820\pi\bar{\alpha}\bar{\pi} \\
& -14820\tau\bar{\pi}\alpha - 10200\tau\bar{\alpha}\alpha - 18348\tau\bar{\pi}\bar{\tau} - 19800\tau\bar{\alpha}\bar{\tau} \\
& +9384\tau\alpha^2 + 1590\alpha\tau^2 + 1548\bar{\tau}^3 + 144\pi^3 \\
& +7144\alpha^3 + 2340\tau\bar{\tau}^2 - 3456\bar{\alpha}\bar{\tau}^2 - 2528\bar{\alpha}\pi^2 \\
& +1196\pi\bar{\alpha}^2 - 2022\bar{\pi}\tau\pi - 4464\bar{\pi}\alpha^2 - 11564\bar{\pi}\alpha\bar{\tau} \\
& -7260\bar{\pi}\bar{\tau}^2 + 34\bar{\pi}\pi^2 - 5148\tau^3 + 2912\bar{\alpha}\alpha^2 \\
& -6712\bar{\pi}\alpha\bar{\alpha} - 1142\bar{\pi}\bar{\tau}\pi - 684\bar{\pi}\pi\alpha - 8008\bar{\pi}\bar{\alpha}\bar{\tau} = 0
\end{aligned}
\tag{4.4.63}
$$

$$
\text{Conjugate of } (\mathcal{S}2) \quad \Longrightarrow \quad
\begin{aligned}
& -12\bar{\alpha}^2 - 10\bar{\pi}\bar{\alpha} + 21\bar{\pi}\tau + 18\tau\bar{\alpha} - 6\bar{\pi}^2 - 44\Psi_2 \\
& +12\alpha\bar{\alpha} + 21\bar{\pi}\bar{\tau} + 18\bar{\alpha}\bar{\tau} + 14\alpha\bar{\pi} = 0
\end{aligned}
$$

$$
\begin{aligned}
& 518\bar{\pi}\bar{\alpha} - 396\bar{\tau}^2 + 546\bar{\alpha}\bar{\tau} + 813\bar{\pi}\bar{\tau} \\
& -813\tau\pi + 72\pi^2 - 945\bar{\tau}\pi - 542\pi\bar{\alpha} \\
\overset{(4.4.52)\,(4.4.57)}{\underset{(\mathcal{V}1)\ldots(\mathcal{V}4)}{\overset{(4.4.55)\,(4.4.56)}{\underset{(\mathcal{D}1)\ldots(\mathcal{D}7)}{\Longrightarrow}}}} \quad & +396\tau^2 + 1338\tau\bar{\alpha} + 780\bar{\alpha}^2 - 780\alpha^2 \\
& -72\bar{\pi}^2 - 546\alpha\tau + 542\alpha\bar{\pi} - 518\pi\alpha \\
& +945\bar{\pi}\tau - 1338\alpha\bar{\tau} = 0
\end{aligned}
\tag{4.4.64}
$$

Equations (4.4.58), (4.4.59), (4.4.60), (4.4.61), (4.4.62), (4.4.63) and (4.4.64) form a system of seven multivariate polynomial equations in the 6 indeterminates α, $\bar{\alpha}$, τ, $\bar{\tau}$, π, and $\bar{\pi}$. We will call this **System 2**. As before, we may use `gsolve()` to establish the following

Lemma 4.4.10 *The ideal in* $\mathbb{C}[\tau, \alpha, \pi, \bar{\pi}, \bar{\alpha}, \bar{\tau}]$ *generated by the left-hand-sides of the equations in* **System 2** *admits, with respect to the pure lexicographical ordering with* $\bar{\pi} \succ \alpha \succ \tau \succ \pi \succ \bar{\alpha} \succ \bar{\tau}$*, the Gröbner basis:*

$$\{27\bar{\tau}^2 + 12\bar{\alpha}\bar{\tau} + 20\pi\bar{\alpha} + \pi^2 - 12\bar{\tau}\pi, 3\bar{\tau} + 2\bar{\alpha} + 3\tau - \pi, 2\alpha + \pi, \bar{\pi} + 2\bar{\alpha}\}. \tag{4.4.65}$$

According to the last two members of the Gröbner basis in Lemma 4.4.10, we have the substitution: $\pi = -2\alpha$, $\bar{\pi} = -2\bar{\alpha}$. Making this substitution into the first two members of the basis yields the following system:

$$2\alpha + 3\bar{\tau} + 2\bar{\alpha} + 3\tau = 0 \tag{4.4.66}$$

$$27\bar{\tau}^2 + 12\bar{\alpha}\bar{\tau} + 4\alpha^2 - 40\alpha\bar{\alpha} + 24\alpha\bar{\tau} = 0 \tag{4.4.67}$$

Differentiating (4.4.67) with δ and $\bar{\delta}$ respectively yields:

$$\begin{aligned}
\delta\,(4.4.67) \quad \Longrightarrow \quad & 27\bar{\tau}\delta(\bar{\tau}) + 6\bar{\alpha}\delta(\bar{\tau}) + 6\bar{\tau}\delta(\bar{\alpha}) + 10\pi\delta(\bar{\alpha}) \\
& +10\bar{\alpha}\delta(\pi) + \pi\delta(\pi) - 6\bar{\tau}\delta(\pi) - 6\pi\delta(\bar{\tau}) = 0 \\
& -216\bar{\alpha}\alpha\bar{\tau} + 48\bar{\alpha}\pi\alpha + 144\bar{\alpha}\bar{\tau}\pi - 24\Phi_{11}\pi \\
& +108\Phi_{11}\bar{\tau} + 10\pi\Psi_2 - 120\bar{\alpha}\Lambda - 60\pi\bar{\pi}^2 \\
& -144\bar{\tau}\bar{\alpha}^2 - 48\bar{\alpha}^2\alpha - 36\bar{\tau}\bar{\pi}^2 - 16\bar{\Psi}_2\pi \\
\overset{(\mathcal{V}1)\ldots(\mathcal{V}4)}{\underset{(\mathcal{D}1)\ldots(\mathcal{D}7)}{\Longrightarrow}} \quad & -76\Psi_2\bar{\alpha} + 30\Phi_{02}\pi + 18\Phi_{02}\bar{\tau} + 16\bar{\Psi}_2\bar{\alpha} \\
& -312\pi\bar{\alpha}\bar{\pi} + 24\bar{\alpha}\Phi_{11} - 486\bar{\alpha}\bar{\tau}^2 - 6\bar{\alpha}\pi^2 \\
& -120\pi\bar{\alpha}^2 - 108\bar{\pi}\alpha\bar{\tau} - 729\bar{\pi}\bar{\tau}^2 - 15\bar{\pi}\pi^2 \\
& -24\bar{\pi}\alpha\bar{\alpha} + 270\bar{\pi}\bar{\tau}\pi + 24\bar{\pi}\pi\alpha - 180\bar{\pi}\bar{\alpha}\bar{\tau} \\
& -12\pi\Lambda + 72\bar{\tau}\Lambda + 72\bar{\tau}\bar{\Psi}_2 - 36\bar{\tau}\Psi_2 = 0
\end{aligned}$$

$$\overset{(4.4.52)(4.4.57)}{\underset{(4.4.55)(4.4.56)}{\Longrightarrow}}\ \substack{(\mathcal{V}1)\dots(\mathcal{V}4)\\(\mathcal{D}1)\dots(\mathcal{D}7)} \qquad \begin{aligned}&-2188\bar{\alpha}\alpha\bar{\tau}+6294\tau\alpha\bar{\tau}-1584\tau\bar{\alpha}^2\\&+1188\tau^2\bar{\alpha}-7172\bar{\tau}\bar{\alpha}^2-5048\bar{\alpha}^2\alpha\\&+2824\alpha^2\bar{\tau}+3465\bar{\tau}\tau^2+1278\alpha\bar{\tau}^2\\&-1584\bar{\alpha}^3-1984\tau\bar{\alpha}\alpha-3960\tau\bar{\alpha}\bar{\tau}\\&+2456\tau\alpha^2+2310\alpha\tau^2-1386\bar{\tau}^3\\&+904\alpha^3+2277\tau\bar{\tau}^2-5742\bar{\alpha}\bar{\tau}^2\\&+608\bar{\alpha}\alpha^2=0\end{aligned} \tag{4.4.68}$$

$$\bar{\delta}\,(4.4.67) \quad \Longrightarrow \quad \begin{aligned}&27\bar{\tau}\bar{\delta}(\bar{\tau})+6\bar{\alpha}\bar{\delta}(\bar{\tau})+6\bar{\tau}\bar{\delta}(\bar{\alpha})+10\pi\bar{\delta}(\bar{\alpha})\\&+10\bar{\alpha}\bar{\delta}(\pi)+\pi\bar{\delta}(\pi)-6\bar{\tau}\bar{\delta}(\pi)-6\pi\bar{\delta}(\bar{\tau})=0\end{aligned}$$

$$\overset{(\mathcal{V}1)\dots(\mathcal{V}4)}{\underset{}{\overset{(\mathcal{D}1)\dots(\mathcal{D}7)}{\Longrightarrow}}} \qquad \begin{aligned}&360\bar{\alpha}\alpha\bar{\tau}+216\alpha\bar{\tau}\pi+144\bar{\alpha}\pi\alpha+252\bar{\alpha}\bar{\tau}\pi\\&-60\Phi_{11}\pi-36\Phi_{11}\bar{\tau}-24\Phi_{20}\bar{\alpha}+110\pi\Psi_2\\&+160\bar{\Psi}_2\pi+1557\pi\bar{\tau}^2-216\pi\alpha^2+972\alpha^2\bar{\tau}\\&-549\bar{\tau}\pi^2+1782\alpha\bar{\tau}^2-138\alpha\pi^2-42\pi\Phi_{20}\\&+648\bar{\tau}^3+33\pi^3+198\bar{\tau}\Phi_{20}+144\bar{\alpha}\bar{\tau}^2\\&-78\bar{\alpha}\pi^2+210\bar{\pi}\pi^2+216\bar{\alpha}\alpha^2+126\bar{\pi}\bar{\tau}\pi\\&+360\pi\Lambda+216\bar{\tau}\Lambda+96\bar{\tau}\bar{\Psi}_2+66\bar{\tau}\Psi_2=0\end{aligned}$$

$$\overset{(4.4.52)(4.4.57)}{\underset{(4.4.55)(4.4.56)}{\Longrightarrow}}\ \substack{(\mathcal{V}1)\dots(\mathcal{V}4)\\(\mathcal{D}1)\dots(\mathcal{D}7)} \qquad \begin{aligned}&1396\bar{\alpha}\alpha\bar{\tau}+1302\tau\alpha\bar{\tau}-396\bar{\tau}\bar{\alpha}^2\\&+1320\bar{\alpha}^2\alpha+2248\alpha^2\bar{\tau}+297\bar{\tau}\tau^2\\&+4734\alpha\bar{\tau}^2+1320\tau\bar{\alpha}\alpha-396\tau\bar{\alpha}\bar{\tau}\\&-3240\tau\alpha^2-990\alpha\tau^2-198\bar{\tau}^3\\&-664\alpha^3-99\tau\bar{\tau}^2-66\bar{\alpha}\bar{\tau}^2\\&-2512\bar{\alpha}\alpha^2=0\end{aligned} \tag{4.4.69}$$

Now, equations (4.4.66), (4.4.67), (4.4.68) and (4.4.69) form another multivariate polynomial system of equations in α, $\bar{\alpha}$, τ and $\bar{\tau}$. We call this system **System 2′**. The following lemma can again be proved using `gsolve()` :

Lemma 4.4.11 *The Gröbner basis of the ideal (in $\mathbb{C}[\tau, \alpha, \bar{\alpha}, \bar{\tau}]$), with respect to the pure lexicographical ordering with $\tau \succ \alpha \succ \bar{\alpha} \succ \bar{\tau}$, generated by the left-hand-sides of the equations in* **System 2$'$** *decomposes into the following components*

$$\{\bar{\alpha}, 2\alpha + 3\bar{\tau}, \tau\}, \qquad \{\bar{\tau}, \alpha, 3\tau + 2\bar{\alpha}\}. \tag{4.4.70}$$

Obviously, Lemma 4.4.11 implies $\alpha = \bar{\alpha} = \tau = \bar{\tau} = 0$, once we take into account that $\{\alpha, \bar{\alpha}\}$ and $\{\tau, \bar{\tau}\}$ are conjugate pairs. This in turn implies that $\pi = \bar{\pi} = 0$ by Lemma 4.4.10. By (4.4.55), (4.4.56) and (4.4.57), $\Lambda = \Phi_{11} = \Phi_{02} = 0$. Lastly, by (4.4.52), $\Psi_2 = 0$, the same contradiction as in the earlier cases. This proves that **Scenario 2** is inadmissible, and the proof of Proposition 4.4.1 is now complete. $\square$

Chapter 5

Results for Symmetric Type D Spacetimes

This chapter contains an independent confirmation of an unpublished result of McLenaghan that any symmetric Petrov type D spacetime admits non-self-adjoint scalar wave operators that satisfy all the necessary conditions that have been worked out for the non-self-adjoint scalar wave equation. The concluding result is stated in a different form from the original one obtained by McLenaghan. The two forms have been verified to be equivalent.

5.1 Symmetric Spacetimes of Petrov Type D

Definition 5.1.1 *A spacetime* $(\mathcal{M}, g_{\alpha\beta})$ *is a said to be* symmetric[1] *if its curvature tensor* $R_{\alpha\beta\gamma\delta}$ *satisfies* $R_{\alpha\beta\gamma\delta;\epsilon} = 0$.

A symmetric spacetime can only be of either Petrov type 0, N, or D. (See Cahen & McLenaghan [8].) As we have seen in Chapter 1, the question

[1] *Some authors call this property* local symmetry *instead and reserve the term* symmetry *for another related property which is of a global nature. For a discussion of these two related concepts, see for example Chapter 8, [38].*

of the validity of Huygens' principle in conformally flat (type 0) and type N spacetimes has been solved. It has also been shown that the conformally invariant scalar wave equation does not satisfy Huygens' principle in any type D spacetime. (See Carminati & McLenaghan [10].) The question, however, remains open for the non-self-adjoint scalar wave equation on a general Petrov type D spacetime.

Symmetric spacetimes of Petrov type D exist, (see §31.2, [28]), and according to Cahen & McLenaghan [8], any such spacetime admits coordinates with respect to which its metric has the form:

$$\mathrm{d}s^2 = \frac{2\mathrm{d}u\mathrm{d}v}{\left(1 - \frac{(R+\beta)uv}{8}\right)^2} - \frac{2\mathrm{d}\zeta\mathrm{d}\overline{\zeta}}{\left(1 + \frac{(R-\beta)\zeta\overline{\zeta}}{8}\right)^2}, \tag{5.1.1}$$

where R, the curvature scalar, and β are constant. The above coordinates are called the Robinson-Bertotti coordinates.

According to Cahen & Defrise [7], any symmetric type D spacetime admits a 6-parameter transitive isometry group due to the fact that its metric (5.1.1) decomposes into the product of two metrics of constant curvature. Thus, the local geometry at any point in a symmetric type D spacetime is identical to that at any other point.

The Carminati-McLenaghan conjecture states that type D spacetimes do not admit any Huygens scalar wave operator. However, the 0- to 5-index necessary conditions for symmetric *and* type D spacetimes not only fail to lead to contradiction but in fact admit "constant" solutions. This, to some extent, suggests that symmetric type D spacetimes may allow counter-examples of the Carminati-McLenaghan conjecture. This chapter presents the details of how to construct scalar wave operators on a symmetric type D spacetime that satisfy the 0- to 5-index neccessary conditions.

5.2 Derivation of Results

By (2.2.3), the Ricci spinor, Weyl spinor and Ricci scalar of a symmetric spacetime satisfy the following:

$$\Phi_{AB\dot{A}\dot{B};C\dot{C}} = 0, \tag{5.2.2}$$

$$\Psi_{ABCD;E\dot{E}} = 0, \tag{5.2.3}$$

$$\Lambda_{;A\dot{A}} = 0. \tag{5.2.4}$$

In a Petrov type D symmetric spacetime, with respect to any canonical dyad of the Weyl spinor, the component equations of the above spinor equations take the form listed in Appendix B.

Equations (B.0.1),..., (B.0.4), and (B.0.13),..., (B.0.16) immediately show that Λ and Ψ_2 are constant, whereas equations (B.0.5),..., (B.0.12) implies that

$$\tau = \pi = \sigma = \kappa = \nu = \lambda = \mu = \pi = 0, \tag{5.2.5}$$

since $\Psi_2 \neq 0$ in a type D spacetime. The vanishing of these spin coefficients, when substituted into the Newman-Penrose field equations, gives:

$$\text{(A.1.1)} \implies \Phi_{00} = 0, \tag{5.2.6}$$

$$\text{(A.1.3)} \implies \Phi_{01} = 0, \tag{5.2.7}$$

$$\text{(A.1.7)} \implies \Phi_{20} = 0, \tag{5.2.8}$$

$$\text{(A.1.8)} \implies \Lambda = \frac{1}{2}\Psi_2, \tag{5.2.9}$$

$$\text{(A.1.9)} \implies \Phi_{21} = 0, \tag{5.2.10}$$

$$\text{(A.1.14)} \implies \Phi_{22} = 0. \tag{5.2.11}$$

Equation (5.2.9) further implies:

$$\overline{\Psi}_2 = \Psi_2. \tag{5.2.12}$$

When we substitute (B.0.13), (5.2.5) and (5.2.6) into (A.5.4), we get

$$\phi_0\bar{\phi}_0 = 0 \quad \Longrightarrow \quad \phi_0 = \bar{\phi}_0 = 0. \tag{5.2.13}$$

Similarly, substituting (B.0.14), (5.2.5) and (5.2.11) into (A.5.5), we get

$$\phi_2\bar{\phi}_2 = 0 \quad \Longrightarrow \quad \phi_2 = \bar{\phi}_2 = 0. \tag{5.2.14}$$

Now substitute (5.2.5), (5.2.13) and (5.2.14) into (A.4.1), (A.4.2), (A.4.3), (A.4.4) yields

$$\text{(A.4.1)} \quad \Longrightarrow \quad D(\phi_1) = 0, \tag{5.2.15}$$

$$\text{(A.4.2)} \quad \Longrightarrow \quad \Delta(\phi_1) = 0, \tag{5.2.16}$$

$$\text{(A.4.3)} \quad \Longrightarrow \quad \delta(\phi_1) = 0, \tag{5.2.17}$$

$$\text{(A.4.4)} \quad \Longrightarrow \quad \bar{\delta}(\phi_1) = 0, \tag{5.2.18}$$

that is, ϕ_1 is constant. Taking into account all the preceding results, namely, (5.2.5), ... , (5.2.18), it can be routinely verified that *all* the component necessary conditions for Huygens' principle are identically satisfied except two: (A.5.2) and (A.8.1). These two remaining equations under the above substitutions simplify to

$$\text{(A.5.2)} \quad \Longrightarrow \quad 0 = 5\phi_1\bar{\phi}_1 - 2\Psi_2\Phi_{11}, \tag{5.2.19}$$

$$\text{(A.8.1)} \quad \Longrightarrow \quad 0 = 4\Psi_2^3 - 7\phi_1^2\Psi_2 - 7\bar{\phi}_1^2\Psi_2 + 4\phi_1\Phi_{11}\bar{\phi}_1. \tag{5.2.20}$$

Solving (5.2.19) for Φ_{11} gives

$$\Phi_{11} = \frac{5}{2}\frac{\phi_1\bar{\phi}_1}{\Psi_2}, \tag{5.2.21}$$

which implies that Φ_{11} is also constant, since ϕ_1 and Ψ_2 have been shown to be constant. Furthermore, substituting (5.2.21) into (5.2.20) gives

$$4\,(\Psi_2)^4 - 7\,(\phi_1^2 + \bar{\phi}_1^2)\,(\Psi_2)^2 + 10\,\phi_1^2\bar{\phi}_1^2 = 0, \tag{5.2.22}$$

which is quadratic in $(\Psi_2)^2$.

If we write the complex constant ϕ_1 in standard form

$$\phi_1 = U + \mathrm{i}V,$$

and write $A := (\Psi_2)^2$, then (5.2.22) becomes

$$4\,A^2 - 14\,(U^2 - V^2)\,A + 10\,(U^2 + V^2)^2 = 0, \tag{5.2.23}$$

a quadratic equation in A with real coefficients. Since $A = (\Psi_2)^2$, where Ψ_2 is real, we seek real and non-negative solutions for A. Now the solution for A (in $\mathbb{C}$) of (5.2.23) is

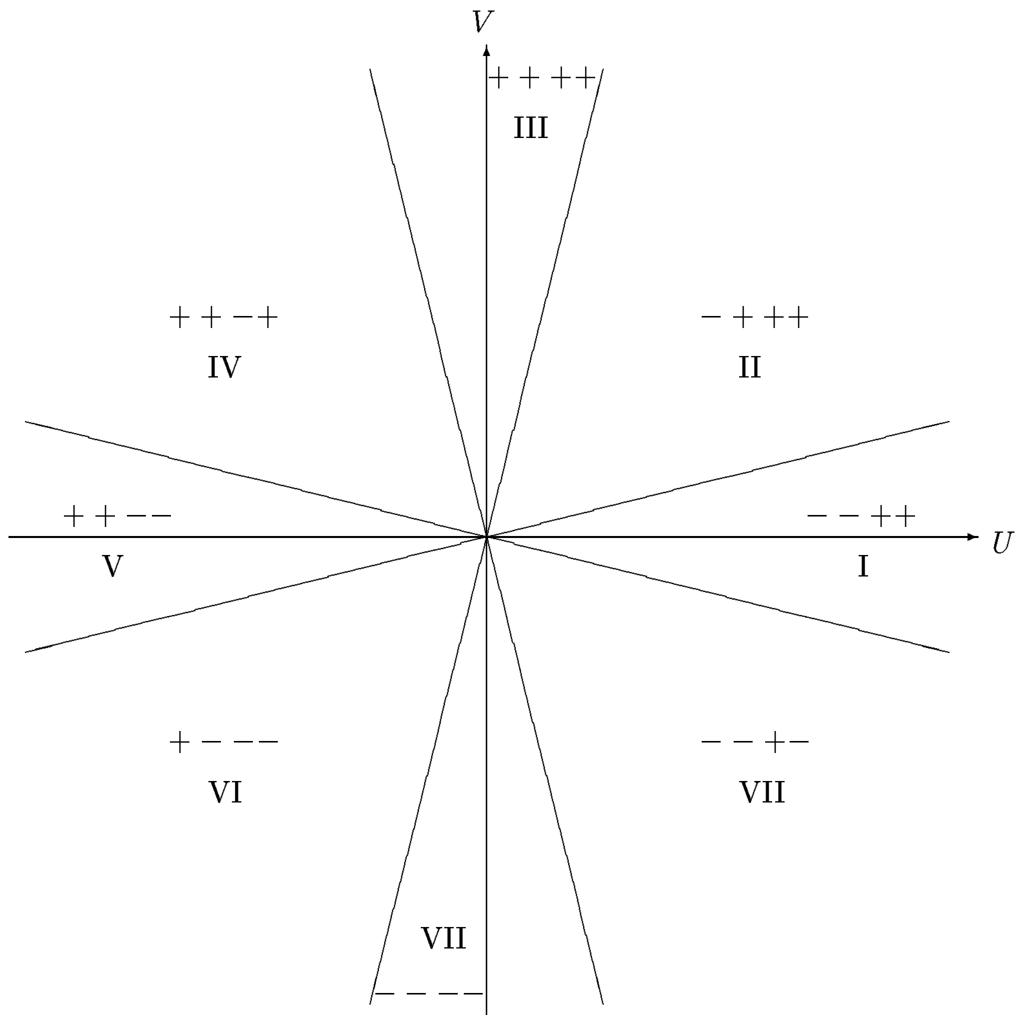
$$A(U,V) = \begin{cases} \frac{7}{4}\,U^2 - \frac{7}{4}\,V^2 + \frac{1}{4}\,\sqrt{9U^4 - 178U^2V^2 + 9V^4}, & \text{or} \\ \frac{7}{4}\,U^2 - \frac{7}{4}\,V^2 - \frac{1}{4}\sqrt{9U^4 - 178U^2V^2 + 9V^4} & \end{cases} \tag{5.2.24}$$

and the discriminant of (5.2.23) is $4\,(9\,U^4 - 178\,U^2V^2 + 9V^4)$, which factors to

$$\frac{4}{9}\,(3U - 7V + 2V\sqrt{10})\,(3U - 7V - 2V\sqrt{10})\,(3U + 7V - 2V\sqrt{10})\,(3U + 7V + 2V\sqrt{10}). \tag{5.2.25}$$

We want to determine the set of $(U, V) \in \mathbb{R}^2$ such that both the discriminant and the value for $A(U, V)$ given in (5.2.24) is non-negative.

The following diagram shows, on the U-V plane, the lines along which one of the factors of the discriminant (5.2.25) vanishes. It also shows the sign of each factor (in the respective order) in the 8 different regions into which the U-V plane is divided by these lines.



Clearly, regions I, III, V and VII are where the discriminant (5.2.25) is non-negative and hence $A(U,V)$ given by in (5.2.24) is real-valued in those 4 regions (and only there).

Next, we observe that

$$\left(\frac{7}{4}U^2 - \frac{7}{4}V^2\right)^2 = \frac{49}{16}\,U^4 - \frac{49}{8}\,U^2V^2 + \frac{49}{16}\,V^4,$$

which implies

$$\left(\frac{7}{4}U^2 - \frac{7}{4}V^2\right)^2 - \left(\frac{1}{4}\sqrt{9U^4 - 178U^2V^2 + 9V^4}\right)^2 = \frac{5}{2}\,\left(U^2 + V^2\right)^2 \geq 0. \tag{5.2.26}$$

Therefore, we have

$$\left|\frac{7}{4}U^2 - \frac{7}{4}V^2\right| \geq \left|\frac{1}{4}\sqrt{9U^4 - 178U^2V^2 + 9V^4}\right|. \tag{5.2.27}$$

This implies that each possible root for $A(U,V)$ in (5.2.24) has the same sign as $\frac{7}{4}U^2 - \frac{7}{4}V^2$. It is now clear that in regions I and V, both possible roots are non-negative, whereas in regions III and VII, both are strictly negative (except at the origin). Thus we conclude that regions I and V are the desired subset of $\mathbb{R}^2$ (or $\mathbb{C}$) on which $A(U,V)$ as in (5.2.24) is non-negative.

To summarize, we state the following

Proposition 5.2.1 *If $P := \Box + A^\alpha \nabla_\alpha + B$ is a non-self-adjoint scalar wave operator on a symmetric Petrov type D spacetime, then, with respect to any spinor dyad canonical to the Weyl spinor, the* 1- *to* 5-*index necessary conditions for Huygens' principle simplify (or, are equivalent) to the following:*

(1) $\tau = \rho = \sigma = \kappa = \nu = \lambda = \mu = \pi = 0.$

$\Phi_{00} = \Phi_{01} = \Phi_{20} = \Phi_{21} = \Phi_{22} = \phi_0 = \phi_2 = 0.$

(2) ϕ_1 *is constant, and it is contained in the following subset of* $\mathbb{C}$*:*

$$\left\{ U + \mathrm{i}V \in \mathbb{C} \;\middle|\; \begin{array}{c} U > 0, \\ 3U - 7V - 2V\sqrt{10} \leq 0, \\ 3U + 7V + 2V\sqrt{10} \geq 0 \end{array} \right\} \bigcup \left\{ U + \mathrm{i}V \in \mathbb{C} \;\middle|\; \begin{array}{c} U < 0, \\ 3U - 7V - 2V\sqrt{10} \geq 0, \\ 3U + 7V + 2V\sqrt{10} \leq 0 \end{array} \right\}$$

(3) $\Lambda = \frac{1}{2}\Psi_2$, *hence* Ψ_2 *is real, and* $\Phi_{11} = \frac{5}{2}\frac{\phi_1\bar{\phi}_1}{\Psi_2}$.

(4)

$$(\Psi_2)^2 = \begin{cases} \frac{7}{4}U^2 - \frac{7}{4}V^2 + \frac{1}{4}\sqrt{9U^4 - 178U^2V^2 + 9V^4}, & \text{or} \\ \frac{7}{4}U^2 - \frac{7}{4}V^2 - \frac{1}{4}\sqrt{9U^4 - 178U^2V^2 + 9V^4}, \end{cases}$$

where U *and* V *are the real and imaginary parts of* ϕ_1 *respectively. Thus,* Ψ_2, Λ, *and* Φ_{11} *are all real constants.*

The significance of this proposition is that it shows that the six necessary conditions that have been computed for the non-self-adjoint equation actually admit "constant" solutions in symmetric spacetimes of Petrov type D. This can be demonstrated as follows: Choose any constant $\phi_1 \in \mathbb{C}$ in the admissible region as described in the proposition. This (together with $\phi_0 = \phi_2 = 0$) determines ϕ_{AB} via (4.3.2), which in turn determines the vector A^α via $A_{[\alpha;\beta]} \longleftrightarrow \phi_{AB}\overline{\epsilon}_{\dot{A}\dot{B}} + \epsilon_{AB}\overline{\phi}_{\dot{A}\dot{B}}$. Statement (4) of the proposition determines Ψ_2 (up to sign), and statement (3) then determines Λ and Φ_{11}, which subsequently determine the metric (5.1.1) of the underlying symmetric type D spacetime by the following two facts:

$$\Psi_2 = \frac{R}{12}, \qquad \text{and} \qquad \Phi_{11} = -\frac{\beta}{8}.$$

Lastly, to complete the construction of $P := \Box + A^\alpha\nabla_\alpha + B$ that will satisfy all the known necessary conditions, we need only use the 0-index condition to determine B.

However, whether these scalar wave operators in fact satisfy Hadamard's Criterion (thus equivalently Huygens' principle) is not known, but the proposition does suggest that they could be counter-examples of the Carminati-McLenaghan conjecture.

To determine whether these operators are indeed (forward) Huygens by directly examining whether Hadamard's Criterion is fulfilled, one needs only examine the Criterion at any one point due to the presence of the transitive isometry group on the underlying symmetric type D spacetime. For example, one may choose to examine Hadamard's Criterion at the origin of the Robinson-Bertotti coordinate system.

Chapter 6

Partial Results for Case 2

This chapter contains some partial results for Case 2 stated as the following

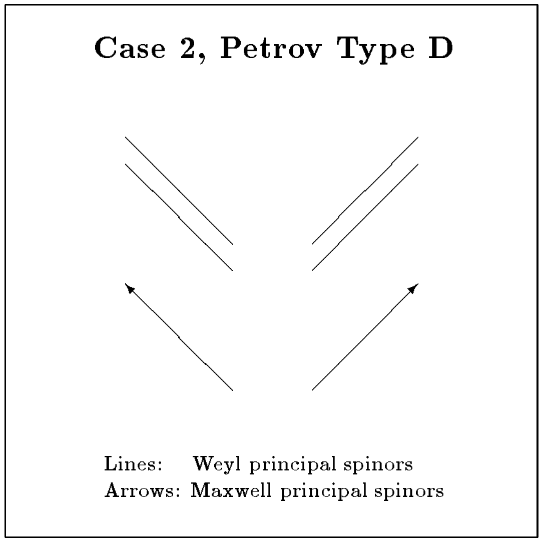
Proposition 6.0.1 *Let* $P := \Box + A^{\alpha}\nabla_{\alpha} + B$ *be a non-self-adjoint Huygens' scalar wave operator on a Petrov type D spacetime such that each Weyl principal spinor of the underlying spacetime is aligned with one of the principal spinors of the Maxwell spinor* ϕ_{AB} *associated to* $A_{[\alpha;\beta]}$. *Then,*

(1) *the principal null congruences of the Weyl spinor (tensor) are geodesic and shear-free, and*

(2) *there exists a conformal gauge in which the principal null congruences of the Weyl spinor (tensor) are expansion-free, and the following equality holds:*

$$t = \bar{\pi}. \tag{6.0.1}$$

Note that the hypothesis on the alignment between the Maxwell and Weyl principal spinors is just the Case 2 assumption discussed in §4.3. The alignment between the Maxwell and Weyl principal spinors in Case 2 is depicted in the following diagram:



A null congruence is said to be *geodesic* if the curves it contains are all geodesics. For the definitions of the shear, expansion (and the motivation behind these definitions) of a geodesic null congruence, see, for example, §8.5 of [15].

Proof

(1) By the hypothesis on the alignment of the Maxwell and Weyl principal spinors, we have, with respect to any canonical spinor dyad,

$$\phi_1 \neq 0, \qquad \phi_0 = \phi_2 = 0. \tag{6.0.2}$$

Equations (A.6.1) and (A.6.10), under the assumption 6.0.2, form the following linear system:

$$\begin{pmatrix} 6\lambda\overline{\Psi}_2 & -18\lambda\Psi_2 \\ -18\bar{\lambda}\overline{\Psi}_2 & 6\bar{\lambda}\Psi_2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \bar{\phi}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{6.0.3}$$

Since $\phi_1 \neq 0$, the determinant of the coefficient matrix, which is $-288\Psi_2\overline{\Psi}_2\lambda\bar{\lambda}$, must vanish. Since $\Psi_2 \neq 0$, we therefore must have

$$\lambda = 0. \tag{6.0.4}$$

Again, under the assumption 6.0.2, the following pairs of equations — (A.6.2) and (A.6.13), (A.6.7) and (A.6.12), (A.6.3) and (A.6.14) — lead to three other homogeneous linear systems for $\begin{pmatrix} \phi_1 \\ \bar{\phi}_1 \end{pmatrix}$ similar to (6.0.3). The determinants of these systems are, respectively, $-288\Psi_2\overline{\Psi}_2\sigma\bar{\sigma}$, $-288\Psi_2\overline{\Psi}_2\kappa\bar{\kappa}$, and $-288\Psi_2\overline{\Psi}_2\nu\bar{\nu}$. By the same argument as before, we conclude

$$\sigma = \kappa = \nu = 0. \tag{6.0.5}$$

Recall that the vanishing of κ and ν is equivalent to the fact that the principal null congruences of the Weyl tensor are geodesic, which is due to the following fact (see [19] for a proof):

$$\begin{aligned} l^{\alpha}{}_{;\beta}\, l^{\beta} &= D\, l^{\alpha} &&= (\varepsilon + \bar{\varepsilon})\, l^{\alpha} - \bar{\kappa}\, m^{\alpha} - \kappa\, \overline{m}^{\alpha}, \\ n^{\alpha}{}_{;\beta}\, n^{\beta} &= \Delta\, n^{\alpha} &&= -(\gamma + \bar{\gamma})\, n^{\alpha} + \nu\, m^{\alpha} + \bar{\nu}\, \overline{m}^{\alpha}. \end{aligned}$$

The vanishing of σ and λ is equivalent to shear-freeness of these null congruences. (See §8.5, [15].) The first assertion is proved.

(2) Let ϕ be the smooth function on the underlying spacetime determined by $e^{4\phi} = \phi_1\bar{\phi}_1$. Then, under the gauge transformation $g'_{\alpha\beta} = e^{2\phi} g_{\alpha\beta}$, ϕ_1 transforms as follows (See §2.4):

$$\phi'_1 = e^{-2\phi}\, \phi_1. \tag{6.0.6}$$

Hence,

$$\phi_1'\,\bar{\phi}_1' = (e^{-2\phi}\phi_1)(e^{-2\phi}\bar{\phi}_1) = e^{-4\phi}\,\phi_1\bar{\phi}_1 = e^{-4\phi}\,e^{4\phi} \equiv 1. \tag{6.0.7}$$

Therefore ϕ_1' is identically unimodular. An examination on the transformation laws of ϕ_0 and ϕ_2 shows that they remain zero under the conformal gauge, i.e. the Case 2 assumption (6.0.2) still holds, which in turn implies both (6.0.4) and (6.0.5) also hold under the conformal gauge. We shall now drop the prime.

Substituting $\phi_1 = 0$ into (A.4.1) gives

$$D(\phi_1) = 2\rho\phi_1. \tag{6.0.8}$$

Differentiating $\phi_1\,\bar{\phi}_1 \equiv 1$ with the Pfaffian operator D and substituting with (6.0.8) yields

$$\begin{aligned} D(\phi_1\,\bar{\phi}_1 \equiv 1) \quad &\Longrightarrow \quad \phi_1\,D(\bar{\phi}_1) + \bar{\phi}_1\,D(\phi_1) = 0 \\ &\Longrightarrow \quad 2\,\phi_1\bar{\phi}_1\,(\rho + \bar{\rho}) = 0, \end{aligned}$$

which immediately gives $\rho + \bar{\rho} = 0$.

Substituting $\phi_2 = 0$ into (A.4.4) gives

$$\Delta(\phi_1) = -2\mu\phi_1. \tag{6.0.9}$$

Differentiating $\phi_1\,\bar{\phi}_1 \equiv 1$ with the Pfaffian operator Δ and substituting with (6.0.9) yields

$$\begin{aligned} \Delta(\phi_1\,\bar{\phi}_1 \equiv 1) \quad &\Longrightarrow \quad \phi_1\,\Delta(\bar{\phi}_1) + \bar{\phi}_1\,\Delta(\phi_1) = 0 \\ &\Longrightarrow \quad -2\,\phi_1\bar{\phi}_1\,(\mu + \bar{\mu}) = 0, \end{aligned}$$

which gives $\mu + \bar{\mu} = 0$. Now, recall that the vanishing of $\rho + \bar{\rho}$ and $\mu + \bar{\mu}$ is equivalent to the fact that both null congruences of the Weyl spinor (tensor) are expansion-free. (See §8.5, [15].)

Substituting $\phi_0 = 0$ into (A.4.3) gives

$$\delta(\phi_1) = 2\tau\phi_1. \tag{6.0.10}$$

Substituting $\phi_2 = 0$ into (A.4.2) gives

$$\bar{\delta}(\phi_1) = -2\pi\phi_1. \tag{6.0.11}$$

Differentiating $\phi_1\,\bar{\phi}_1 \equiv 1$ with the Pfaffian operator δ and substituting with (6.0.10) and the conjugate of (6.0.11) yields

$$\begin{aligned} \delta(\phi_1\,\bar{\phi}_1 \equiv 1) \quad &\Longrightarrow \quad \phi_1\,\delta(\bar{\phi}_1) + \bar{\phi}_1\,\delta(\phi_1) = 0 \\ &\Longrightarrow \quad 2\,\phi_1\bar{\phi}_1\,(-\bar{\pi} + \tau) = 0, \end{aligned}$$

which gives $\tau = \bar{\pi}$. $\square$

Chapter 7

Conclusion

A scheme was outlined in §4.3 that can be followed in further study of Huygens' principle for the non-self-adjoint scalar wave equation on a Petrov type D spacetime. In particular, the type D problem was split into 5 geometrically distinct sub-cases based on the alignment of the Maxwell and Weyl principal spinors.

The main result of this thesis was Proposition 4.4.1, which states that Case 4 (hence also the geometrically equivalent Case 1) is incompatible with Huygens' principle.

For Case 2, it was shown that the two principal null congruences of the Weyl tensor are geodesic and shear-free. Significant simplifications of the component equations for this case have also been obtained (not included in this thesis). It has been observed that a number of the component equations of the 5-index condition factor after sufficient simplifications, and systematic exploitation of these factorizations may eventually yield the solution to this sub-case.

The result established in Proposition 4.2.1 was not used elsewhere in this thesis. It is a result that holds in a general Petrov type D spacetime, regardless of the alignment between the Weyl and Maxwell principal spinors.

It may turn out to be helpful in the remaining cases (other than Cases 1 and 4) mentioned in §4.3.

Chapter 5 shows that on a symmetric Petrov type D spacetime, there exist scalar wave operators that satisfy all the six available necessary conditions for Huygens' principle, which suggests the existence of counter-examples of the Carminati-McLenaghan conjecture on these spacetimes. One may return to an examination of Hadamard's Criterion in the attempt to directly determine whether these spacetimes are indeed counter-examples. The fact that the underlying spacetime admits a 6-parameter transitive isometry group reduces this problem to determining whether Hadamard's Criterion is fulfilled at any one point inside a given causal domain. An intelligent choice of the point of examination should further simplify this problem.

It was also proved in Chapter 5 that any Huygens' scalar wave operator on a symmetric type D spacetime must satisfy the Case 2 assumption. It has been conjectured by McLenaghan that all other sub-cases except Case 2 are incompatible with Huygens' principle, while Case 2 admits only the complex recurrent[1] spacetimes whenever Huygens' principle is to hold.

The symbolic algebra system MAPLE , and in particular, the MAPLE package `NPspinor`, were essential computational tools used throughout this thesis. The expansion of the spinor equations (2.3.13), ... , (2.3.12) have been independently verified by Czapor (private communication). Furthermore, `NPspinor` was also used to successfully confirm the unpublished result of McLenaghan on symmetric type D spacetimes obtained by hand-calculations. This serves as an additional verification for both McLenaghan's hand-calculations and for the dyad expansion by `NPspinor`. These computations should also be checked by a comparison of results with a symbolic

[1]A spacetime is said to be *complex recurrent* if its Weyl spinor Ψ_{ABCD} satisfies $\Psi_{ABCD;E\dot{E}} = K_{E\dot{E}}\,\Psi_{ABCD}$ for some smooth spinor field $K_{E\dot{E}}$. Obviously, symmetric spacetimes are special cases of complex recurrent spacetimes.

algebra system other than MAPLE .

Appendix A

Necessary Conditions in Newman-Penrose Form

A.1 Newman-Penrose Field Equations

$$D(\rho) - \bar{\delta}(\kappa) = \rho^2 + \sigma\bar{\sigma} + (\epsilon + \bar{\epsilon})\rho - \bar{\kappa}\tau - (3\alpha + \bar{\beta} - \pi)\kappa + \Phi_{00} \tag{A.1.1}$$

$$D(\sigma) - \delta(\kappa) = (\rho + \bar{\rho})\sigma + (3\epsilon - \bar{\epsilon})\sigma - (\tau - \bar{\pi} + \bar{\alpha} + 3\beta)\kappa + \Psi_0 \tag{A.1.2}$$

$$D(\tau) - \Delta(\kappa) = (\tau + \bar{\pi})\rho + (\bar{\tau} + \pi)\sigma + (\epsilon - \bar{\epsilon})\tau - (3\gamma + \bar{\gamma})\kappa + \Psi_1 + \Phi_{01} \tag{A.1.3}$$

$$D(\alpha) - \bar{\delta}(\epsilon) = (\rho + \bar{\epsilon} - 2\epsilon)\alpha + \beta\bar{\sigma} - \bar{\beta}\epsilon - \kappa\lambda - \bar{\kappa}\gamma + (\epsilon + \rho)\pi + \Phi_{10} \tag{A.1.4}$$

$$D(\beta) - \delta(\epsilon) = (\alpha + \pi)\sigma + (\bar{\rho} - \bar{\epsilon})\beta - (\mu + \gamma)\kappa + (\bar{\pi} - \bar{\alpha})\epsilon + \Psi_1 \tag{A.1.5}$$

$$D(\gamma) - \Delta(\epsilon) = (\tau + \bar{\pi})\alpha + (\bar{\tau} + \pi)\beta - (\epsilon + \bar{\epsilon})\gamma - (\gamma + \bar{\gamma})\epsilon + \tau\pi - \nu\kappa + \Psi_2 - \Lambda + \Phi_{11} \tag{A.1.6}$$

$$D(\lambda) - \bar{\delta}(\pi) = \rho\lambda + \bar{\sigma}\mu + \pi^2 + (\alpha - \bar{\beta})\pi - \nu\bar{\kappa} + (\bar{\epsilon} - 3\epsilon)\lambda + \Phi_{20} \tag{A.1.7}$$

$$D(\mu) - \delta(\pi) = \bar{\rho}\mu + \sigma\lambda + \pi\bar{\pi} - (\epsilon + \bar{\epsilon})\mu - (\bar{\alpha} - \beta)\pi - \nu\kappa + \Psi_2 + 2\Lambda \tag{A.1.8}$$

$$D(\nu) - \Delta(\pi) = (\bar{\tau} + \pi)\mu + (\tau + \bar{\pi})\lambda + (\gamma - \bar{\gamma})\pi - (3\epsilon + \bar{\epsilon})\nu + \Psi_3 + \Phi_{21} \tag{A.1.9}$$

$$\Delta(\lambda) - \bar{\delta}(\nu) = (\bar{\gamma} - 3\gamma - \mu - \bar{\mu})\lambda + (3\alpha + \bar{\beta} + \pi - \bar{\tau})\nu - \Psi_4 \tag{A.1.10}$$

$$\delta(\rho) - \bar{\delta}(\sigma) = (\bar{\alpha} + \beta)\rho - (3\alpha - \bar{\beta})\sigma + (\rho - \bar{\rho})\tau + (\mu - \bar{\mu})\kappa - \Psi_1 + \Phi_{01} \tag{A.1.11}$$

$$\delta(\alpha) - \bar{\delta}(\beta) = \mu\rho - \sigma\lambda + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + (\rho - \bar{\rho})\gamma + (\mu - \bar{\mu})\epsilon - \Psi_2 + \Lambda + \Phi_{11} \tag{A.1.12}$$

$$\delta(\lambda) - \bar{\delta}(\mu) = (\rho - \bar{\rho})\nu + (\mu - \bar{\mu})\pi + (\alpha + \bar{\beta})\mu + (\bar{\alpha} - 3\beta)\lambda - \Psi_3 + \Phi_{21} \tag{A.1.13}$$

$$\delta(\nu) - \Delta(\mu) = \mu^2 + \lambda\bar{\lambda} + (\gamma + \bar{\gamma})\mu - \bar{\nu}\pi + (\tau - \bar{\alpha} - 3\beta)\nu + \Phi_{22} \tag{A.1.14}$$

$$\delta(\gamma) - \Delta(\beta) = (\tau - \bar{\alpha} - \beta)\gamma + \mu\tau - \sigma\nu - \epsilon\bar{\nu} - (\gamma - \bar{\gamma} - \mu)\beta + \alpha\bar{\lambda} + \Phi_{12} \tag{A.1.15}$$

$$\delta(\tau) - \Delta(\sigma) = \mu\sigma + \bar{\lambda}\rho + (\tau - \bar{\alpha} + \beta)\tau - (3\gamma - \bar{\gamma})\sigma - \kappa\bar{\nu} + \Phi_{02} \tag{A.1.16}$$

$$\Delta(\rho) - \bar{\delta}(\tau) = -\rho\bar{\mu} - \sigma\lambda + (\gamma + \bar{\gamma})\rho - (\bar{\tau} + \alpha - \bar{\beta})\tau + \nu\kappa - \Psi_2 - 2\Lambda \tag{A.1.17}$$

$$\Delta(\alpha) - \bar{\delta}(\gamma) = (\epsilon + \rho)\nu - (\tau + \beta)\lambda + (\bar{\gamma} - \bar{\mu})\alpha + (\bar{\beta} - \bar{\tau})\gamma - \Psi_3 \tag{A.1.18}$$

A.2 Bianchi Identities

$$\begin{aligned}\bar{\delta}(\Psi_0) - D(\Psi_1) + D(\Phi_{01}) - \delta(\Phi_{00}) \quad = \quad & (4\alpha - \pi)\Psi_0 - 2(2\rho + \epsilon)\Psi_1 + 3\kappa\Psi_2 + (\bar{\pi} - 2\bar{\alpha} - 2\beta)\Phi_{00} \\ & +2(\epsilon + \bar{\rho})\Phi_{01} + 2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \bar{\kappa}\Phi_{02}\end{aligned} \tag{A.2.1}$$

$$\begin{aligned}\Delta(\Psi_0) - \delta(\Psi_1) + D(\Phi_{02}) - \delta(\Phi_{01}) \quad = \quad & (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 + 3\sigma\Psi_2 - \bar{\lambda}\Phi_{00} + 2(\bar{\pi} - \beta)\Phi_{01} \\ & +2\sigma\Phi_{11} + (2\epsilon - 2\bar{\epsilon} + \bar{\rho})\Phi_{02} - 2\kappa\Phi_{12}\end{aligned} \tag{A.2.2}$$

$$\begin{aligned}& 3\bar{\delta}(\Psi_1) - 3D(\Psi_2) + 2D(\Phi_{11}) - 2\delta(\Phi_{10}) + \bar{\delta}(\Phi_{01}) - \Delta(\Phi_{00}) = \\ & \quad 3\lambda\Psi_0 - 9\rho\Psi_2 + 6(\alpha - \pi)\Psi_1 + 6\kappa\Psi_3 + (\bar{\mu} - 2\mu - 2\gamma - 2\bar{\gamma})\Phi_{00} + (2\alpha + 2\pi + 2\bar{\tau})\Phi_{01} \\ & \quad\quad + 2(\tau - 2\bar{\alpha} + \bar{\pi})\Phi_{10} + 2(2\bar{\rho} - \rho)\Phi_{11} + 2\sigma\Phi_{20} - \bar{\sigma}\Phi_{02} - 2\bar{\kappa}\Phi_{12} - 2\kappa\Phi_{21}\end{aligned} \tag{A.2.3}$$

$$3\Delta(\Psi_1) - 3\delta(\Psi_2) + 2D(\Phi_{12}) - 2\delta(\Phi_{11}) + \bar{\delta}(\Phi_{02}) - \Delta(\Phi_{01}) = \\ 3\nu\Psi_0 + 6(\gamma-\mu)\Psi_1 - 9\tau\Psi_2 + 6\sigma\Psi_3 - \bar{\nu}\Phi_{00} + 2(\bar{\mu}-\mu-\gamma)\Phi_{01} - 2\bar{\lambda}\Phi_{10} + 2(\tau+2\bar{\pi})\Phi_{11} \\ + (2\alpha+2\pi+\bar{\tau}-2\bar{\beta})\Phi_{02} + (2\bar{\rho}-2\rho-4\bar{\epsilon})\Phi_{12} + 2\sigma\Phi_{21} - 2\kappa\Phi_{22} \quad (A.2.4)$$

$$3\bar{\delta}(\Psi_2) - 3D(\Psi_3) + D(\Phi_{21}) - \delta(\Phi_{20}) + 2\bar{\delta}(\Phi_{11}) - 2\Delta(\Phi_{10}) = \\ 6\lambda\Psi_1 - 9\pi\Psi_2 + 6(\epsilon-\rho)\Psi_3 + 3\kappa\Psi_4 - 2\nu\Phi_{00} + 2(\bar{\mu}-\mu-2\bar{\gamma})\Phi_{10} + (2\pi+4\bar{\tau})\Phi_{11} \\ + (2\beta+2\tau+\bar{\pi}-2\bar{\alpha})\Phi_{20} - 2\bar{\sigma}\Phi_{12} + 2(\bar{\rho}-\rho-\epsilon)\Phi_{21} - \bar{\kappa}\Phi_{22} + 2\lambda\Phi_{01} \quad (A.2.5)$$

$$3\Delta(\Psi_2) - 3\delta(\Psi_3) + D(\Phi_{22}) - \delta(\Phi_{21}) + 2\bar{\delta}(\Phi_{12}) - 2\Delta(\Phi_{11}) = \\ 6\nu\Psi_1 - 9\mu\Psi_2 + 6(\beta-\tau)\Psi_3 + 3\sigma\Psi_4 - 2\nu\Phi_{01} - 2\bar{\nu}\Phi_{10} + 2(2\bar{\mu}-\mu)\Phi_{11} + 2\lambda\Phi_{02} - \bar{\lambda}\Phi_{20} \\ + 2(\pi+\bar{\tau}-2\bar{\beta})\Phi_{12} + 2(\beta+\tau+\bar{\pi})\Phi_{21} + (\bar{\rho}-2\epsilon-2\bar{\epsilon}-2\rho)\Phi_{22} \quad (A.2.6)$$

$$\begin{aligned} \bar{\delta}(\Psi_3) - D(\Psi_4) + \bar{\delta}(\Phi_{21}) - \Delta(\Phi_{20}) \quad = \quad & 3\lambda\Psi_2 - 2(\alpha+2\pi)\Psi_3 + (4\epsilon-\rho)\Psi_4 - 2\nu\Phi_{10} + 2\lambda\Phi_{11} \\ & +(2\gamma-2\bar{\gamma}+\bar{\mu})\Phi_{20} + 2(\bar{\tau}-\alpha)\Phi_{21} - \bar{\sigma}\Phi_{22} \end{aligned} \quad (A.2.7)$$

$$\begin{aligned} \Delta(\Psi_3) - \delta(\Psi_4) + \bar{\delta}(\Phi_{22}) - \Delta(\Phi_{21}) \quad = \quad & 3\nu\Psi_2 - 2(\gamma+2\mu)\Psi_3 + (4\beta-\tau)\Psi_4 - 2\nu\Phi_{11} - \bar{\nu}\Phi_{20} \\ & +2\lambda\Phi_{12} + 2(\gamma+\bar{\mu})\Phi_{21} + (\bar{\tau}-2\bar{\beta}-2\alpha)\Phi_{22} \end{aligned} \quad (A.2.8)$$

$$D(\Phi_{11}) - \delta(\Phi_{10}) - \bar{\delta}(\Phi_{01}) + \Delta(\Phi_{00}) + 3D(\Lambda) = \\ (2\gamma-\mu+2\bar{\gamma}-\bar{\mu})\Phi_{00} + (\pi-2\alpha-2\bar{\tau})\Phi_{01} + (\bar{\pi}-2\bar{\alpha}-2\tau)\Phi_{10} \\ + 2(\rho+\bar{\rho})\Phi_{11} + \bar{\sigma}\Phi_{02} + \sigma\Phi_{20} - \bar{\kappa}\Phi_{12} - \kappa\Phi_{21} \quad (A.2.9)$$

$$D(\Phi_{12}) - \delta(\Phi_{11}) - \bar{\delta}(\Phi_{02}) + \Delta(\Phi_{01}) + 3\delta(\Lambda) = \\ (2\gamma-\mu-2\bar{\mu})\Phi_{01} + \bar{\nu}\Phi_{00} - \bar{\lambda}\Phi_{10} + 2(\bar{\pi}-\tau)\Phi_{11} + (\pi+2\bar{\beta}-2\alpha-\bar{\tau})\Phi_{02} \\ + (2\rho+\bar{\rho}-2\bar{\epsilon})\Phi_{12} + \sigma\Phi_{21} - \kappa\Phi_{22} \quad (A.2.10)$$

$$D(\Phi_{22}) - \delta(\Phi_{21}) - \bar{\delta}(\Phi_{12}) + \Delta(\Phi_{11}) + 3\Delta(\Lambda) = \\ \nu\Phi_{01} + \bar{\nu}\Phi_{10} - 2(\mu+\bar{\mu})\Phi_{11} - \lambda\Phi_{02} - \bar{\lambda}\Phi_{20} + (2\pi-\bar{\tau}+2\bar{\beta})\Phi_{12} \\ + (2\beta-\tau+2\bar{\pi})\Phi_{21} + (\rho+\bar{\rho}-2\epsilon-2\bar{\epsilon})\Phi_{22} \quad (A.2.11)$$

A.3 The Commutation Relations

$$(\Delta D - D\Delta)\,\xi = [\,(\gamma+\bar{\gamma})D + (\epsilon+\bar{\epsilon})\Delta - (\bar{\tau}+\pi)\delta - (\tau+\bar{\pi})\bar{\delta}\,]\,\xi \tag{A.3.1}$$

$$(\delta D - D\delta)\,\xi = [\,(\bar{\alpha}+\beta-\bar{\pi}) + \kappa\Delta - (\bar{\rho}+\epsilon-\bar{\epsilon})\delta - \sigma\bar{\delta}\,]\,\xi \tag{A.3.2}$$

$$(\bar{\delta} D - D\bar{\delta})\,\xi = [\,(\alpha+\bar{\beta}-\pi)D + \bar{\kappa}\Delta - \bar{\sigma}\delta - (\rho-\epsilon+\bar{\epsilon})\bar{\delta}\,]\,\xi \tag{A.3.3}$$

$$(\bar{\delta}\delta - \delta\bar{\delta})\,\xi = [\,(-\mu+\bar{\mu})D + (-\rho+\bar{\rho})\Delta + (\alpha-\bar{\beta})\delta + (-\bar{\alpha}+\beta)\bar{\delta}\,]\,\xi \tag{A.3.4}$$

A.4 Maxwell's Equations (2-index Condition)

$$D(\phi_1) - \bar{\delta}(\phi_0) = (\pi - 2\alpha)\phi_0 + 2\rho\phi_1 - \kappa\phi_2 \tag{A.4.1}$$

$$D(\phi_2) - \bar{\delta}(\phi_1) = -\lambda\phi_0 + 2\pi\phi_1 + (\rho - 2\epsilon)\phi_2 \tag{A.4.2}$$

$$\delta(\phi_1) - \Delta(\phi_0) = (\mu - 2\gamma)\phi_0 + 2\tau\phi_1 - \sigma\phi_2 \tag{A.4.3}$$

$$\delta(\phi_2) - \Delta(\phi_1) = -\nu\phi_0 + 2\mu\phi_1 + (\tau - 2\beta)\phi_2 \tag{A.4.4}$$

A.5 Component Equations of the 2-index Condition

$$o_{AB}\bar{o}_{(\dot{A}}\bar{\iota}_{\dot{B})} \qquad \begin{aligned}&-9\Psi_2\lambda\beta-3\Psi_2\mu\bar{\beta}-3\Psi_2\mu\bar{\tau}-3\Psi_2\pi\gamma\\&-3\Psi_2\mu\alpha+3\Psi_2\lambda\bar{\pi}-9\Psi_2\nu\epsilon+9\Psi_2\tau\lambda\\&+3\Psi_2\pi\bar{\gamma}+3\Psi_2\lambda\bar{\alpha}-3\Psi_2\nu\bar{\rho}-18\Psi_2\mu\pi\\&+3\Psi_2\pi\bar{\mu}+9\Psi_2\nu\rho-3\Psi_2\nu\bar{\epsilon}-2\Psi_2\Phi_{21}\\&-10\phi_2\bar{\phi}_1+\bar{\delta}(\Psi_2)\bar{\mu}-5\delta(\Psi_2)\lambda-6\Delta(\Psi_2)\pi\\&-\Delta(\Psi_2)\alpha-\Delta(\Psi_2)\bar{\beta}-6\mu\bar{\delta}(\Psi_2)-5\nu D(\Psi_2)\\&-\bar{\delta}(\Psi_2)\gamma-\Delta(\Psi_2)\bar{\tau}+\bar{\delta}(\Psi_2)\bar{\gamma}-3\Psi_2\Delta(\pi)\\&-3\Psi_2\bar{\delta}(\mu)-3\Psi_2D(\nu)-3\Psi_2\delta(\lambda)-\bar{\delta}(\Delta(\Psi_2))\\&-\Delta(\bar{\delta}(\Psi_2))=0\end{aligned} \tag{A.5.1}$$

$$o_{(A}\iota_{B)}\bar{o}_{(\dot{A}}\bar{\iota}_{\dot{B})} \qquad \begin{aligned}&12\Psi_2\nu\kappa-6\Psi_2\rho\bar{\mu}+6\Psi_2\tau\bar{\tau}-2\Delta(D(\Psi_2))\\&-2\bar{\delta}(\delta(\Psi_2))-2D(\Delta(\Psi_2))-2\delta(\bar{\delta}(\Psi_2))-2\Delta(\Psi_2)\bar{\epsilon}\\&-2\delta(\Psi_2)\bar{\beta}+10\tau\bar{\delta}(\Psi_2)+2\delta(\Psi_2)\alpha-10\mu D(\Psi_2)\\&+2\bar{\delta}(\Psi_2)\bar{\pi}+2D(\Psi_2)\bar{\mu}-2\bar{\delta}(\Psi_2)\beta+2D(\Psi_2)\bar{\gamma}\\&-2\Delta(\Psi_2)\bar{\rho}+2D(\Psi_2)\gamma+2\bar{\delta}(\Psi_2)\bar{\alpha}-2\delta(\Psi_2)\bar{\tau}\\&+6\Psi_2\bar{\delta}(\tau)-6\Psi_2D(\mu)+6\Psi_2\Delta(\rho)-6\Psi_2\delta(\pi)\\&+20\phi_1\bar{\phi}_1-8\Psi_2\Phi_{11}+10\Delta(\Psi_2)\rho-2\Delta(\Psi_2)\epsilon\\&-10\delta(\Psi_2)\pi+6\Psi_2\tau\bar{\beta}-6\Psi_2\tau\alpha+12\Psi_2\sigma\lambda\\&-6\Psi_2\mu\epsilon+6\Psi_2\pi\bar{\pi}-6\Psi_2\mu\bar{\epsilon}+24\Psi_2\tau\pi\\&+24\Psi_2\mu\rho-6\Psi_2\rho\gamma-6\Psi_2\rho\bar{\gamma}+6\Psi_2\pi\bar{\alpha}\\&-6\Psi_2\pi\beta-6\Psi_2\mu\bar{\rho}=0\end{aligned} \tag{A.5.2}$$

$$o_{(A}\iota_{B)}\bar{o}_{\dot{A}\dot{B}} \qquad \begin{aligned}&4\Psi_2\Phi_{12}-10\Delta(\Psi_2)\tau-2D(\Psi_2)\bar{\nu}+6\Psi_2\mu\beta\\&+6\Psi_2\mu\bar{\alpha}+6\Psi_2\delta(\mu)-6\Psi_2\pi\bar{\lambda}+2\Delta(\Psi_2)\bar{\alpha}\\&+2\delta(\Psi_2)\bar{\gamma}-2\delta(\Psi_2)\gamma+10\delta(\Psi_2)\mu-12\Psi_2\nu\sigma\\&-24\Psi_2\tau\mu-2\bar{\delta}(\Psi_2)\bar{\lambda}+2\Delta(\delta(\Psi_2))-6\Psi_2\tau\bar{\gamma}\\&+6\Psi_2\rho\bar{\nu}-6\Psi_2\Delta(\tau)+6\Psi_2\tau\gamma+2\delta(\Delta(\Psi_2))\\&+2\Delta(\Psi_2)\beta-10\phi_1\bar{\phi}_2=0\end{aligned} \tag{A.5.3}$$

$$\iota_{AB}\bar{\iota}_{\dot{A}\dot{B}} \qquad \begin{aligned}&\Psi_2\Phi_{00}+5\phi_0\bar{\phi}_0+\delta(\Psi_2)\bar{\kappa}+9\Psi_2\kappa\alpha\\&-3\Psi_2\bar{\delta}(\kappa)+3\Psi_2\kappa\bar{\beta}-3\Psi_2\sigma\bar{\sigma}-5\kappa\bar{\delta}(\Psi_2)\\&+9\Psi_2\rho^2-6D(\Psi_2)\rho+3\Psi_2\rho\epsilon-3\Psi_2\tau\bar{\kappa}\\&-3\Psi_2D(\rho)+3\Psi_2\rho\bar{\epsilon}-D(\Psi_2)\bar{\epsilon}-9\Psi_2\kappa\pi\\&+D(D(\Psi_2))-D(\Psi_2)\epsilon=0\end{aligned} \tag{A.5.4}$$

$$o_{AB}\bar{o}_{\dot{A}\dot{B}} \qquad \begin{aligned}&9\Psi_2\nu\beta+3\Psi_2\delta(\nu)-3\Psi_2\lambda\bar{\lambda}+3\Psi_2\nu\bar{\alpha}\\&+\Delta(\Delta(\Psi_2))+6\mu\Delta(\Psi_2)+9\Psi_2\mu^2+5\delta(\Psi_2)\nu\\&-\bar{\delta}(\Psi_2)\bar{\nu}+\Delta(\Psi_2)\gamma+\Delta(\Psi_2)\bar{\gamma}-9\Psi_2\tau\nu\\&+3\Psi_2\mu\gamma+3\Psi_2\Delta(\mu)-3\Psi_2\pi\bar{\nu}+3\Psi_2\mu\bar{\gamma}\\&+\Psi_2\Phi_{22}+5\phi_2\bar{\phi}_2=0\end{aligned} \tag{A.5.5}$$

$$\iota_{AB}\bar{o}_{\dot{A}\dot{B}} \qquad \begin{aligned} &-5\sigma\Delta(\Psi_2) + \delta(\delta(\Psi_2)) + \delta(\Psi_2)\bar{\alpha} - 9\Psi_2\sigma\mu \\ &+9\Psi_2\tau^2 - 6\delta(\Psi_2)\tau - \delta(\Psi_2)\beta + 9\Psi_2\sigma\gamma \\ &+3\Psi_2\kappa\bar{\nu} - 3\Psi_2\Delta(\sigma) - 3\Psi_2\sigma\bar{\gamma} - D(\Psi_2)\bar{\lambda} \\ &-3\Psi_2\tau\bar{\alpha} + 3\Psi_2\tau\beta - 3\Psi_2\delta(\tau) + 3\Psi_2\rho\bar{\lambda} \\ &+\Psi_2\Phi_{02} + 5\phi_0\bar{\phi}_2 = 0 \end{aligned} \tag{A.5.6}$$

$$\iota_{AB}\bar{o}_{(\dot{A}}\bar{\iota}_{\dot{B})} \qquad \begin{aligned} &-3\Psi_2\tau\epsilon - 3\Psi_2\rho\beta - 9\Psi_2\sigma\alpha - 18\Psi_2\tau\rho \\ &+3\Psi_2\tau\bar{\epsilon} + 9\Psi_2\sigma\pi - 3\Psi_2\rho\bar{\alpha} - 3\Psi_2\rho\bar{\pi} \\ &-3\Psi_2\kappa\bar{\mu} + 3\Psi_2\sigma\bar{\tau} - 3\Psi_2\kappa\bar{\gamma} + 3\Psi_2\sigma\bar{\beta} \\ &-9\Psi_2\kappa\gamma + 9\Psi_2\mu\kappa + 3\Psi_2\tau\bar{\rho} - D(\delta(\Psi_2)) \\ &-\delta(D(\Psi_2)) - 2\Psi_2\Phi_{01} - 10\phi_0\bar{\phi}_1 + \delta(\Psi_2)\epsilon \\ &+6\delta(\Psi_2)\rho + D(\Psi_2)\bar{\pi} + 5\sigma\bar{\delta}(\Psi_2) + 6\tau D(\Psi_2) \\ &-\delta(\Psi_2)\bar{\epsilon} + 5\Delta(\Psi_2)\kappa + D(\Psi_2)\beta - \delta(\Psi_2)\bar{\rho} \\ &+D(\Psi_2)\bar{\alpha} + 3\Psi_2\bar{\delta}(\sigma) + 3\Psi_2 D(\tau) + 3\Psi_2\delta(\rho) \\ &+3\Psi_2\Delta(\kappa) = 0 \end{aligned} \tag{A.5.7}$$

$$o_{(A}\iota_{B)}\bar{\iota}_{\dot{A}\dot{B}} \qquad \begin{aligned} &4\Psi_2\Phi_{10} - 10\phi_1\bar{\phi}_0 + 10\pi D(\Psi_2) - 12\Psi_2\lambda\kappa \\ &-24\Psi_2\rho\pi + 2\delta(\Psi_2)\bar{\sigma} - 10\bar{\delta}(\Psi_2)\rho + 2\bar{\delta}(\Psi_2)\epsilon \\ &-2D(\Psi_2)\bar{\beta} + 2D(\bar{\delta}(\Psi_2)) - 6\Psi_2\tau\bar{\sigma} + 6\Psi_2\rho\bar{\beta} \\ &-6\Psi_2\bar{\delta}(\rho) + 6\Psi_2\rho\alpha + 2\Delta(\Psi_2)\bar{\kappa} + 6\Psi_2\pi\epsilon \\ &+6\Psi_2 D(\pi) - 6\Psi_2\pi\bar{\epsilon} + 6\Psi_2\mu\bar{\kappa} - 2D(\Psi_2)\alpha \\ &-2\bar{\delta}(\Psi_2)\bar{\epsilon} + 2\bar{\delta}(D(\Psi_2)) = 0 \end{aligned} \tag{A.5.8}$$

$$o_{AB}\bar{\iota}_{\dot{A}\dot{B}} \qquad \begin{aligned} &5\phi_2\bar{\phi}_0 + 5D(\Psi_2)\lambda + \bar{\delta}(\Psi_2)\alpha + 9\Psi_2\pi^2 \\ &-9\Psi_2\rho\lambda + 3\Psi_2\pi\alpha - 3\Psi_2\pi\bar{\beta} + 3\Psi_2\bar{\delta}(\pi) \\ &+3\Psi_2\mu\bar{\sigma} - \bar{\delta}(\Psi_2)\bar{\beta} + 9\Psi_2\lambda\epsilon - 3\Psi_2\lambda\bar{\epsilon} \\ &+3\Psi_2 D(\lambda) + 3\Psi_2\nu\bar{\kappa} + \Delta(\Psi_2)\bar{\sigma} + 6\pi\bar{\delta}(\Psi_2) \\ &+\bar{\delta}(\bar{\delta}(\Psi_2)) + \Psi_2\Phi_{20} = 0 \end{aligned} \tag{A.5.9}$$

A.6 Component Equations of the 3-index Condition

$$o_{ABC}\bar{o}_{(\dot{A}}\bar{\iota}_{\dot{B}\dot{C})} \qquad \begin{aligned}&-18\Psi_2\lambda\bar{\phi}_1 - 3\bar{\Psi}_2\bar{\delta}(\phi_2) - 6\bar{\Psi}_2\phi_2\alpha + 6\bar{\Psi}_2\phi_1\lambda\\&-9\phi_2\bar{\delta}(\bar{\Psi}_2) + 27\phi_2\bar{\Psi}_2\bar{\tau} - 9\Psi_2\nu\bar{\phi}_0 = 0\end{aligned} \tag{A.6.1}$$

$$o_{(AB}\iota_{C)}\bar{\iota}_{\dot{A}\dot{B}\dot{C}} \qquad \begin{aligned}&3\Psi_2\bar{\delta}(\bar{\phi}_0) + 6\Psi_2\bar{\phi}_1\bar{\sigma} - 6\Psi_2\bar{\phi}_0\bar{\beta} + 27\bar{\phi}_0\Psi_2\pi\\&+9\bar{\phi}_0\bar{\delta}(\Psi_2) - 18\bar{\Psi}_2\bar{\sigma}\phi_1 - 9\bar{\Psi}_2\bar{\kappa}\phi_2 = 0\end{aligned} \tag{A.6.2}$$

$$o_{ABC}\bar{o}_{(\dot{A}\dot{B}}\bar{\iota}_{\dot{C})} \qquad \begin{aligned}&18\Psi_2\nu\bar{\phi}_1 - 3\bar{\Psi}_2\Delta(\phi_2) - 6\bar{\Psi}_2\phi_2\gamma + 6\bar{\Psi}_2\phi_1\nu\\&+9\Psi_2\lambda\bar{\phi}_2 - 27\phi_2\bar{\Psi}_2\bar{\mu} - 9\phi_2\Delta(\bar{\Psi}_2) = 0\end{aligned} \tag{A.6.3}$$

$$o_{(AB}\iota_{C)}\bar{o}_{(\dot{A}\dot{B}}\bar{\iota}_{\dot{C})} \qquad \begin{aligned}&-6\bar{\Psi}_2\phi_0\nu + 6\bar{\Psi}_2\Delta(\phi_1) + 6\bar{\Psi}_2\phi_2\tau + 27\bar{\phi}_2\Psi_2\pi\\&+9\bar{\phi}_2\bar{\delta}(\Psi_2) + 27\phi_2\bar{\Psi}_2\bar{\pi} + 9\phi_2\delta(\bar{\Psi}_2) + 54\phi_1\bar{\Psi}_2\bar{\mu}\\&+18\phi_1\Delta(\bar{\Psi}_2) - 6\Psi_2\bar{\phi}_0\bar{\nu} + 6\Psi_2\Delta(\bar{\phi}_1) + 6\Psi_2\bar{\phi}_2\bar{\tau}\\&-6\Psi_2\bar{\phi}_1\bar{\mu} + 3\Psi_2\bar{\delta}(\bar{\phi}_2) + 6\Psi_2\bar{\phi}_2\bar{\beta} + 54\bar{\phi}_1\Psi_2\mu\\&+18\bar{\phi}_1\Delta(\Psi_2) + 6\bar{\Psi}_2\phi_2\beta - 6\bar{\Psi}_2\phi_1\mu + 3\bar{\Psi}_2\delta(\phi_2) = 0\end{aligned} \tag{A.6.4}$$

$$o_{(AB}\iota_{C)}\bar{o}_{(\dot{A}}\bar{\iota}_{\dot{B}\dot{C})} \qquad \begin{aligned}&-54\bar{\phi}_1\Psi_2\pi - 18\bar{\phi}_1\bar{\delta}(\Psi_2) - 6\Psi_2\bar{\delta}(\bar{\phi}_1) - 6\Psi_2\bar{\phi}_2\bar{\sigma}\\&+6\Psi_2\bar{\phi}_0\bar{\mu} + 18\phi_1\bar{\delta}(\bar{\Psi}_2) - 54\phi_1\bar{\Psi}_2\bar{\tau} - 6\Psi_2\bar{\phi}_1\bar{\tau}\\&-3\Psi_2\Delta(\bar{\phi}_0) + 6\Psi_2\bar{\phi}_0\bar{\gamma} + 6\bar{\Psi}_2\phi_2\epsilon - 6\bar{\Psi}_2\phi_1\pi\\&+3\bar{\Psi}_2D(\phi_2) - 27\bar{\phi}_0\Psi_2\mu - 9\bar{\phi}_0\Delta(\Psi_2) - 6\bar{\Psi}_2\phi_0\lambda\\&+6\bar{\Psi}_2\bar{\delta}(\phi_1) + 6\bar{\Psi}_2\phi_2\rho + 9\phi_2D(\bar{\Psi}_2) - 27\phi_2\bar{\Psi}_2\bar{\rho} = 0\end{aligned} \tag{A.6.5}$$

$$o_{ABC}\bar{\iota}_{\dot{A}\dot{B}\dot{C}} \qquad 9\Psi_2\lambda\bar{\phi}_0 + 9\bar{\Psi}_2\bar{\sigma}\phi_2 = 0 \tag{A.6.6}$$

$$o_{(A}\iota_{BC)}\bar{\iota}_{\dot{A}\dot{B}\dot{C}} \qquad \begin{aligned}&9\bar{\phi}_0D(\Psi_2) - 27\bar{\phi}_0\Psi_2\rho + 6\Psi_2\bar{\phi}_1\bar{\kappa} + 3\Psi_2D(\bar{\phi}_0)\\&-6\Psi_2\bar{\phi}_0\bar{\epsilon} + 18\bar{\Psi}_2\bar{\kappa}\phi_1 + 9\bar{\Psi}_2\bar{\sigma}\phi_0 = 0\end{aligned} \tag{A.6.7}$$

$$o_{(A}\iota_{BC)}\bar{o}_{(\dot{A}\dot{B}}\bar{\iota}_{\dot{C})} \qquad \begin{aligned}&6\Psi_2\delta(\bar{\phi}_1) + 6\Psi_2\bar{\phi}_2\bar{\rho} - 6\Psi_2\bar{\phi}_0\bar{\lambda} + 9\bar{\phi}_2D(\Psi_2)\\&-27\bar{\phi}_2\Psi_2\rho + 6\bar{\Psi}_2\phi_0\mu - 6\bar{\Psi}_2\delta(\phi_1) - 6\bar{\Psi}_2\phi_2\sigma\\&+18\bar{\phi}_1\delta(\Psi_2) - 54\bar{\phi}_1\Psi_2\tau - 54\phi_1\bar{\Psi}_2\bar{\pi} - 18\phi_1\delta(\bar{\Psi}_2)\\&-6\bar{\Psi}_2\phi_1\tau - 3\bar{\Psi}_2\Delta(\phi_0) + 6\bar{\Psi}_2\phi_0\gamma - 27\phi_0\bar{\Psi}_2\bar{\mu}\\&-9\phi_0\Delta(\bar{\Psi}_2) - 6\Psi_2\bar{\phi}_1\bar{\pi} + 3\Psi_2D(\bar{\phi}_2) + 6\Psi_2\bar{\phi}_2\bar{\epsilon} = 0\end{aligned} \tag{A.6.8}$$

$o_{(A}\iota_{BC)}\bar{o}_{(\dot{A}}\bar{\iota}_{\dot{B}\dot{C})}$

$$\begin{aligned}
&-6\Psi_2 D(\bar{\phi}_1) - 6\Psi_2\bar{\phi}_2\bar{\kappa} + 6\Psi_2\bar{\phi}_0\bar{\pi} - 9\phi_0\bar{\delta}(\bar{\Psi}_2) \\
&+27\phi_0\bar{\Psi}_2\bar{\tau} - 6\bar{\Psi}_2\phi_1\rho - 3\bar{\Psi}_2\bar{\delta}(\phi_0) + 6\bar{\Psi}_2\phi_0\alpha \\
&-6\Psi_2\bar{\phi}_1\bar{\rho} - 3\Psi_2\delta(\bar{\phi}_0) + 6\Psi_2\bar{\phi}_0\bar{\alpha} - 18\bar{\phi}_1 D(\Psi_2) \\
&+54\bar{\phi}_1\Psi_2\rho + 6\bar{\Psi}_2\phi_0\pi - 6\bar{\Psi}_2 D(\phi_1) - 6\bar{\Psi}_2\phi_2\kappa \\
&-9\bar{\phi}_0\delta(\Psi_2) + 27\bar{\phi}_0\Psi_2\tau - 18\phi_1 D(\bar{\Psi}_2) + 54\phi_1\bar{\Psi}_2\bar{\rho} = 0
\end{aligned} \tag{A.6.9}$$

$o_{(A}\iota_{BC)}\bar{o}_{\dot{A}\dot{B}\dot{C}}$

$$\begin{aligned}
&-6\Psi_2\bar{\phi}_2\bar{\alpha} + 6\Psi_2\bar{\phi}_1\bar{\lambda} - 3\Psi_2\delta(\bar{\phi}_2) - 9\bar{\Psi}_2\bar{\nu}\phi_0 \\
&-9\bar{\phi}_2\delta(\Psi_2) + 27\bar{\phi}_2\Psi_2\tau - 18\bar{\Psi}_2\bar{\lambda}\phi_1 = 0
\end{aligned} \tag{A.6.10}$$

$\iota_{ABC}\bar{\iota}_{\dot{A}\dot{B}\dot{C}}$

$$-9\Psi_2\kappa\bar{\phi}_0 - 9\bar{\Psi}_2\bar{\kappa}\phi_0 = 0 \tag{A.6.11}$$

$\iota_{ABC}\bar{o}_{(\dot{A}}\bar{\iota}_{\dot{B}\dot{C})}$

$$\begin{aligned}
&9\Psi_2\sigma\bar{\phi}_0 + 18\Psi_2\kappa\bar{\phi}_1 + 6\bar{\Psi}_2\phi_1\kappa + 3\bar{\Psi}_2 D(\phi_0) \\
&-6\bar{\Psi}_2\phi_0\epsilon + 9\phi_0 D(\bar{\Psi}_2) - 27\phi_0\bar{\Psi}_2\bar{\rho} = 0
\end{aligned} \tag{A.6.12}$$

$\iota_{ABC}\bar{o}_{(\dot{A}\dot{B}}\bar{\iota}_{\dot{C})}$

$$\begin{aligned}
&-18\Psi_2\sigma\bar{\phi}_1 + 27\phi_0\bar{\Psi}_2\bar{\pi} + 9\phi_0\delta(\bar{\Psi}_2) + 6\bar{\Psi}_2\phi_1\sigma \\
&+3\bar{\Psi}_2\delta(\phi_0) - 6\bar{\Psi}_2\phi_0\beta - 9\Psi_2\kappa\bar{\phi}_2 = 0
\end{aligned} \tag{A.6.13}$$

$o_{(AB}\iota_{C)}\bar{o}_{\dot{A}\dot{B}\dot{C}}$

$$\begin{aligned}
&-27\bar{\phi}_2\Psi_2\mu - 9\bar{\phi}_2\Delta(\Psi_2) - 6\Psi_2\bar{\phi}_2\bar{\gamma} + 6\Psi_2\bar{\phi}_1\bar{\nu} \\
&-3\Psi_2\Delta(\bar{\phi}_2) + 9\bar{\Psi}_2\bar{\lambda}\phi_2 + 18\bar{\Psi}_2\bar{\nu}\phi_1 = 0
\end{aligned} \tag{A.6.14}$$

$\iota_{ABC}\bar{o}_{\dot{A}\dot{B}\dot{C}}$

$$9\Psi_2\sigma\bar{\phi}_2 + 9\bar{\Psi}_2\bar{\lambda}\phi_0 = 0 \tag{A.6.15}$$

$o_{ABC}\bar{o}_{\dot{A}\dot{B}\dot{C}}$

$$-9\Psi_2\nu\bar{\phi}_2 - 9\bar{\Psi}_2\bar{\nu}\phi_2 = 0 \tag{A.6.16}$$

A.7 Component Equations of the 4-index Condition for Case 4

In the following equations, ϕ_1,$\bar{\phi}_1$,ϕ_2, and $\bar{\phi}_1$ have been set to zero[1]. These assumptions correspond to the Case 4 assumptions as discussed in §4.3.

$$\iota_{ABCD}\overline{o}_{\dot{A}\dot{B}\dot{C}\dot{D}} \quad -432\Psi_2\sigma\bar{\Psi}_2\bar{\lambda} - 12\phi_0\bar{\phi}_0\bar{\lambda}^2 = 0 \tag{A.7.1}$$

$$o_{(A}\iota_{BCD)}\overline{o}_{\dot{A}\dot{B}\dot{C}\dot{D}} \quad \begin{aligned} &360\Psi_2\sigma\bar{\Psi}_2\bar{\nu} - 1152\bar{\Psi}_2\bar{\lambda}\Psi_2\tau + 216\Psi_2\bar{\Psi}_2\bar{\pi}\bar{\lambda} + 216\Psi_2\bar{\Psi}_2\bar{\alpha}\bar{\lambda} \\ &-72\Psi_2\bar{\Psi}_2\bar{\lambda}\beta + 24\phi_0\bar{\phi}_0\bar{\lambda}\bar{\nu} + 432\delta(\Psi_2)\bar{\Psi}_2\bar{\lambda} + 72\Psi_2\bar{\Psi}_2\delta(\bar{\lambda}) \\ &+144\Psi_2\bar{\lambda}\delta(\bar{\Psi}_2) = 0 \end{aligned} \tag{A.7.2}$$

$$o_{(A}\iota_{BCD)}\overline{o}_{(\dot{A}}\overline{\iota}_{\dot{A}\dot{B}\dot{C})} \quad \begin{aligned} &-26\phi_0\mu D(\bar{\phi}_0) - 26\bar{\delta}(\phi_0)\bar{\phi}_0\bar{\pi} - 26D(\phi_0)\bar{\phi}_0\bar{\mu} - 26\phi_0\gamma D(\bar{\phi}_0) \\ &-26\phi_0\pi\delta(\bar{\phi}_0) - 38\Delta(\phi_0)\bar{\phi}_0\bar{\epsilon} - 3456\Psi_2\rho\bar{\Psi}_2\bar{\rho} + 1368\Psi_2\kappa\bar{\Psi}_2\bar{\tau} \\ &+1368\Psi_2\tau\bar{\Psi}_2\bar{\kappa} - 720\Psi_2\sigma\bar{\Psi}_2\bar{\sigma} + 72\Psi_2\bar{\Psi}_2\beta\bar{\kappa} + 216\Psi_2\bar{\Psi}_2\bar{\kappa}\bar{\alpha} \\ &+72\bar{\Psi}_2\Psi_2\bar{\beta}\kappa + 216\bar{\Psi}_2\Psi_2\kappa\alpha + 40\phi_0\pi\bar{\phi}_0\bar{\alpha} + 64\phi_0\gamma\bar{\phi}_0\bar{\epsilon} \\ &+40\phi_0\alpha\bar{\phi}_0\bar{\pi} + 64\phi_0\mu\bar{\phi}_0\bar{\epsilon} - 216\Psi_2\bar{\Psi}_2\bar{\rho}\epsilon - 216\Psi_2\bar{\Psi}_2\bar{\rho}\bar{\epsilon} \\ &+648\Psi_2\bar{\Psi}_2\bar{\pi}\bar{\kappa} + 64\phi_0\epsilon\bar{\phi}_0\bar{\mu} - 6\phi_0\delta(\bar{\delta}(\bar{\phi}_0)) + 16\Delta(\phi_0)D(\bar{\phi}_0) \\ &-432D(\Psi_2)D(\bar{\Psi}_2) - 6\phi_0\bar{\delta}(\delta(\bar{\phi}_0)) - 6\bar{\phi}_0\Delta(D(\phi_0)) - 6\bar{\phi}_0\delta(\bar{\delta}(\phi_0)) \\ &-6\phi_0\Delta(D(\bar{\phi}_0)) - 72\Psi_2 D(D(\bar{\Psi}_2)) + 16\bar{\delta}(\phi_0)\delta(\bar{\phi}_0) + 16D(\phi_0)\Delta(\bar{\phi}_0) \\ &+16\delta(\phi_0)\bar{\delta}(\bar{\phi}_0) - 6\bar{\phi}_0\bar{\delta}(\delta(\phi_0)) - 72\bar{\Psi}_2 D(D(\Psi_2)) - 6\phi_0 D(\Delta(\bar{\phi}_0)) \\ &-6\bar{\phi}_0 D(\Delta(\phi_0)) + 144\bar{\Psi}_2\rho D(\Psi_2) + 72\bar{\Psi}_2\bar{\delta}(\Psi_2)\kappa - 72\bar{\Psi}_2\Psi_2\bar{\delta}(\kappa) \\ &+144\Psi_2\bar{\rho}D(\bar{\Psi}_2) + 72\Psi_2\delta(\bar{\Psi}_2)\bar{\kappa} - 72\Psi_2\bar{\Psi}_2\delta(\bar{\kappa}) + 216\Psi_2\bar{\Psi}_2 D(\bar{\rho}) \\ &+72\Psi_2 D(\bar{\Psi}_2)\bar{\epsilon} + 72\Psi_2 D(\bar{\Psi}_2)\epsilon - 38\phi_0\beta\bar{\delta}(\bar{\phi}_0) - 38\delta(\phi_0)\bar{\phi}_0\bar{\beta} \\ &-38\phi_0\epsilon\Delta(\bar{\phi}_0) - 26D(\phi_0)\bar{\phi}_0\bar{\gamma} - 26\phi_0\alpha\delta(\bar{\phi}_0) - 26\bar{\delta}(\phi_0)\bar{\phi}_0\bar{\alpha} \\ &+1152\Psi_2\rho D(\bar{\Psi}_2) + 216\bar{\Psi}_2\Psi_2 D(\rho) + 72\bar{\Psi}_2 D(\Psi_2)\bar{\epsilon} + 72\bar{\Psi}_2 D(\Psi_2)\epsilon \\ &+12\phi_0\bar{\phi}_0 D(\bar{\gamma}) + 30\phi_0\bar{\delta}(\bar{\phi}_0)\bar{\pi} + 30\phi_0 D(\bar{\phi}_0)\bar{\mu} + 12\phi_0\bar{\phi}_0 D(\bar{\mu}) \\ &+30\phi_0 D(\bar{\phi}_0)\bar{\gamma} + 18\phi_0\bar{\epsilon}\Delta(\bar{\phi}_0) + 30\bar{\phi}_0 D(\phi_0)\mu + 12\bar{\phi}_0\phi_0\delta(\pi) \\ &+30\bar{\phi}_0\delta(\phi_0)\pi + 12\bar{\phi}_0\phi_0 D(\mu) + 30\bar{\phi}_0 D(\phi_0)\gamma - 6\phi_0\Delta(\bar{\phi}_0)\bar{\rho} \\ &+30\phi_0\bar{\delta}(\bar{\phi}_0)\bar{\alpha} + 12\bar{\phi}_0\phi_0\Delta(\epsilon) - 6\bar{\phi}_0\bar{\delta}(\phi_0)\tau - 6\phi_0\Delta(\bar{\phi}_0)\rho \\ &+12\phi_0\bar{\phi}_0\bar{\delta}(\bar{\alpha}) + 12\phi_0\bar{\phi}_0\delta(\bar{\beta}) + 18\phi_0\bar{\beta}\delta(\bar{\phi}_0) - 6\phi_0\bar{\delta}(\bar{\phi}_0)\tau \\ &+12\phi_0\bar{\phi}_0\Delta(\bar{\epsilon}) - 6\phi_0\delta(\bar{\phi}_0)\bar{\tau} + 12\phi_0\bar{\phi}_0\bar{\delta}(\bar{\pi}) - 6\bar{\phi}_0\delta(\phi_0)\bar{\tau} \\ &+30\bar{\phi}_0\delta(\phi_0)\alpha - 6\bar{\phi}_0\Delta(\phi_0)\bar{\rho} + 18\bar{\phi}_0\beta\bar{\delta}(\phi_0) + 12\bar{\phi}_0\phi_0\delta(\alpha) \\ &+12\bar{\phi}_0\phi_0\bar{\delta}(\beta) - 6\bar{\phi}_0\Delta(\phi_0)\rho - 360\Psi_2\kappa\bar{\delta}(\bar{\Psi}_2) + 1152D(\Psi_2)\bar{\Psi}_2\bar{\rho} \\ &-360\delta(\Psi_2)\bar{\Psi}_2\bar{\kappa} + 18\bar{\phi}_0\epsilon\Delta(\phi_0) + 12\bar{\phi}_0\phi_0 D(\gamma) + 216\Psi_2\bar{\Psi}_2\bar{\rho}^2 \\ &+216\bar{\Psi}_2\Psi_2\rho^2 + 144\phi_0\Phi_{11}\bar{\phi}_0 - 144\Psi_2\Phi_{00}\bar{\Psi}_2 + 88\phi_0\beta\bar{\phi}_0\bar{\beta} \\ &+64\phi_0\epsilon\bar{\phi}_0\bar{\gamma} + 40\phi_0\alpha\bar{\phi}_0\bar{\alpha} + 40\phi_0\pi\bar{\phi}_0\bar{\pi} - 48\phi_0\bar{\phi}_0\bar{\epsilon}\bar{\gamma} \\ &+24\phi_0\bar{\phi}_0\bar{\kappa}\bar{\nu} - 48\phi_0\bar{\phi}_0\bar{\pi}\bar{\beta} - 48\phi_0\bar{\phi}_0\bar{\mu}\bar{\epsilon} + 648\bar{\Psi}_2\Psi_2\pi\kappa \\ &-216\bar{\Psi}_2\Psi_2\rho\bar{\epsilon} - 216\bar{\Psi}_2\Psi_2\rho\epsilon + 12\bar{\phi}_0\phi_0\bar{\rho}\mu - 48\bar{\phi}_0\phi_0\mu\epsilon \\ &+24\bar{\phi}_0\phi_0\lambda\sigma - 48\bar{\phi}_0\phi_0\pi\beta + 24\bar{\phi}_0\phi_0\kappa\nu - 48\bar{\phi}_0\phi_0\epsilon\gamma \\ &+12\phi_0\bar{\phi}_0\bar{\mu}\rho + 24\phi_0\bar{\phi}_0\bar{\lambda}\bar{\sigma} - 48\bar{\phi}_0\phi_0\beta\alpha + 12\phi_0\bar{\phi}_0\bar{\tau}\bar{\alpha} \\ &+12\phi_0\bar{\phi}_0\bar{\beta}\tau + 12\phi_0\bar{\phi}_0\bar{\pi}\bar{\tau} + 12\bar{\phi}_0\phi_0\tau\alpha + 12\bar{\phi}_0\phi_0\beta\bar{\tau} \\ &+12\bar{\phi}_0\phi_0\pi\tau + 12\phi_0\bar{\phi}_0\bar{\gamma}\rho + 12\phi_0\bar{\phi}_0\bar{\rho}\bar{\gamma} + 12\phi_0\bar{\phi}_0\bar{\rho}\bar{\mu} \\ &-48\phi_0\bar{\phi}_0\bar{\beta}\bar{\alpha} + 12\bar{\phi}_0\phi_0\gamma\bar{\rho} + 12\bar{\phi}_0\phi_0\rho\gamma + 12\bar{\phi}_0\phi_0\rho\mu = 0 \end{aligned} \tag{A.7.3}$$

[1] the zero function in a neighbourhood of the event under investigation

$$
\begin{aligned}
o_{(ABC}\iota_{D)}\bar{\iota}_{\dot{A}\dot{B}\dot{C}\dot{D}} \quad & -72\Psi_2\bar{\Psi}_2\bar{\delta}(\bar{\sigma}) - 144\Psi_2\bar{\sigma}\bar{\delta}(\bar{\Psi}_2) - 432\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\sigma} - 24\bar{\phi}_0\bar{\delta}(\phi_0)\lambda \\
& -12\bar{\phi}_0\phi_0\bar{\delta}(\lambda) + 32\phi_0\lambda\bar{\delta}(\bar{\phi}_0) - 1152\bar{\Psi}_2\bar{\sigma}\Psi_2\pi + 360\Psi_2\lambda\bar{\Psi}_2\bar{\kappa} \\
& +216\Psi_2\bar{\Psi}_2\bar{\tau}\bar{\sigma} + 216\Psi_2\bar{\Psi}_2\bar{\beta}\bar{\sigma} - 72\Psi_2\bar{\Psi}_2\bar{\sigma}\alpha + 36\bar{\phi}_0\phi_0\pi\lambda \\
& +12\bar{\phi}_0\phi_0\alpha\lambda - 52\bar{\phi}_0\phi_0\lambda\bar{\beta} - 12\bar{\phi}_0\phi_0\nu\bar{\sigma} = 0
\end{aligned}
\tag{A.7.4}
$$

$$
\begin{aligned}
o_{(ABC}\iota_{D)}\bar{o}_{(\dot{A}}\bar{\iota}_{\dot{B}\dot{C}\dot{D})} \quad & -1152\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\tau} - 72\Psi_2\bar{\delta}(\bar{\Psi}_2)\bar{\beta} + 1152\Psi_2\pi\bar{\delta}(\bar{\Psi}_2) - 360\Psi_2\lambda D(\bar{\Psi}_2) \\
& +360\Delta(\Psi_2)\bar{\Psi}_2\bar{\sigma} - 72\bar{\Psi}_2\bar{\delta}(\Psi_2)\bar{\beta} + 72\Psi_2\bar{\delta}(\bar{\Psi}_2)\alpha + 72\Psi_2\bar{\Psi}_2\Delta(\bar{\sigma}) \\
& -32\phi_0\nu\bar{\delta}(\bar{\phi}_0) - 216\bar{\Psi}_2\Psi_2\pi^2 - 72\Psi_2\Delta(\bar{\Psi}_2)\bar{\sigma} + 216\bar{\Psi}_2\Psi_2\bar{\delta}(\pi) \\
& +144\bar{\Psi}_2\pi\bar{\delta}(\Psi_2) + 72\bar{\Psi}_2\bar{\delta}(\Psi_2)\alpha - 216\Psi_2\bar{\Psi}_2\bar{\delta}(\bar{\tau}) - 216\Psi_2\bar{\Psi}_2\bar{\tau}^2 \\
& -32\phi_0\lambda\Delta(\bar{\phi}_0) - 144\Psi_2\bar{\tau}\bar{\delta}(\bar{\Psi}_2) - 72\bar{\Psi}_2\Psi_2 D(\lambda) + 72\bar{\Psi}_2 D(\Psi_2)\lambda \\
& +144\Psi_2\Phi_{20}\bar{\Psi}_2 + 432\bar{\delta}(\Psi_2)\bar{\delta}(\bar{\Psi}_2) + 1368\Psi_2\lambda\bar{\Psi}_2\bar{\rho} - 3456\Psi_2\pi\bar{\Psi}_2\bar{\tau} \\
& -720\Psi_2\nu\bar{\Psi}_2\bar{\kappa} + 1368\Psi_2\mu\bar{\Psi}_2\bar{\sigma} + 76\phi_0\nu\bar{\phi}_0\bar{\beta} + 52\phi_0\lambda\bar{\phi}_0\bar{\mu} \\
& +216\bar{\Psi}_2\Psi_2\pi\alpha - 648\bar{\Psi}_2\Psi_2\lambda\rho - 216\bar{\Psi}_2\Psi_2\pi\bar{\beta} - 648\Psi_2\bar{\Psi}_2\bar{\mu}\bar{\sigma} \\
& +216\Psi_2\bar{\Psi}_2\bar{\tau}\bar{\beta} - 216\Psi_2\bar{\Psi}_2\bar{\tau}\alpha - 216\bar{\Psi}_2\Psi_2\lambda\epsilon + 72\bar{\Psi}_2\Psi_2\bar{\epsilon}\lambda \\
& -216\Psi_2\bar{\Psi}_2\bar{\gamma}\bar{\sigma} + 72\Psi_2\bar{\Psi}_2\gamma\bar{\sigma} + 52\phi_0\lambda\bar{\phi}_0\bar{\gamma} + 24\bar{\phi}_0\lambda\Delta(\phi_0) \\
& +24\bar{\phi}_0\bar{\delta}(\phi_0)\nu + 12\bar{\phi}_0\phi_0\bar{\delta}(\nu) + 12\bar{\phi}_0\phi_0\Delta(\lambda) - 36\bar{\phi}_0\phi_0\pi\nu \\
& -12\bar{\phi}_0\phi_0\nu\alpha - 12\bar{\phi}_0\phi_0\lambda\gamma - 36\bar{\phi}_0\phi_0\mu\lambda + 12\bar{\phi}_0\phi_0\nu\bar{\tau} \\
& +72\bar{\Psi}_2\bar{\delta}(\bar{\delta}(\Psi_2)) + 72\Psi_2\bar{\delta}(\bar{\delta}(\bar{\Psi}_2)) = 0
\end{aligned}
\tag{A.7.5}
$$

$$
\begin{aligned}
o_{(AB}\iota_{CD)}\bar{o}_{(\dot{A}}\bar{\iota}_{\dot{B}\dot{C}\dot{D})} \quad & -12\bar{\phi}_0\phi_0\bar{\delta}(\mu) + 144\Psi_2\bar{\delta}(\bar{\Psi}_2)\epsilon - 24\bar{\phi}_0\mu\bar{\delta}(\phi_0) - 72\bar{\Psi}_2\bar{\delta}(\Psi_2)\epsilon \\
& -12\phi_0\bar{\phi}_0\bar{\delta}(\bar{\gamma}) + 432\Psi_2\delta(\bar{\Psi}_2)\bar{\sigma} - 144\Psi_2\bar{\delta}(\bar{\Psi}_2)\bar{\epsilon} + 26\bar{\delta}(\phi_0)\bar{\phi}_0\bar{\mu} \\
& -432\Psi_2\bar{\kappa}\Delta(\bar{\Psi}_2) - 144\Psi_2\bar{\Psi}_2\Delta(\bar{\kappa}) + 72\bar{\Psi}_2\bar{\delta}(\Psi_2)\bar{\epsilon} - 216\bar{\Psi}_2\Psi_2 D(\pi) \\
& -576\Psi_2 D(\bar{\Psi}_2)\bar{\tau} + 576\Psi_2\bar{\rho}\bar{\delta}(\bar{\Psi}_2) + 38\Delta(\phi_0)\bar{\phi}_0\bar{\beta} - 1152 D(\Psi_2)\bar{\Psi}_2\bar{\tau} \\
& -504\Delta(\Psi_2)\bar{\Psi}_2\bar{\kappa} + 504\delta(\Psi_2)\bar{\Psi}_2\bar{\sigma} + 1152\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\rho} - 30\phi_0\bar{\delta}(\bar{\phi}_0)\bar{\mu} \\
& +26\phi_0\lambda\delta(\bar{\phi}_0) - 12\bar{\phi}_0\phi_0\mu\bar{\tau} + 36\bar{\phi}_0\phi_0\pi\gamma - 24\bar{\phi}_0\phi_0\lambda\tau \\
& +24\bar{\phi}_0\phi_0\nu\epsilon - 12\bar{\phi}_0\phi_0\bar{\rho}\nu - 12\bar{\phi}_0\phi_0\gamma\bar{\tau} - 30\phi_0\bar{\delta}(\bar{\phi}_0)\bar{\gamma} \\
& +144\Psi_2\bar{\Psi}_2\delta(\bar{\sigma}) - 18\phi_0\bar{\beta}\Delta(\bar{\phi}_0) - 360\bar{\Psi}_2\pi D(\Psi_2) - 144\Psi_2\bar{\delta}(D(\bar{\Psi}_2)) \\
& +6\phi_0\Delta(\bar{\delta}(\bar{\phi}_0)) + 6\phi_0\bar{\delta}(\Delta(\bar{\phi}_0)) - 16\bar{\delta}(\phi_0)\Delta(\bar{\phi}_0) + 6\bar{\phi}_0\bar{\delta}(\Delta(\phi_0)) \\
& -360\bar{\delta}(\Psi_2)D(\bar{\Psi}_2) + 360 D(\Psi_2)\bar{\delta}(\bar{\Psi}_2) + 6\bar{\phi}_0\Delta(\bar{\delta}(\phi_0)) + 72\bar{\Psi}_2\bar{\delta}(D(\Psi_2)) \\
& +144\Psi_2 D(\bar{\delta}(\bar{\Psi}_2)) - 72\bar{\Psi}_2 D(\bar{\delta}(\Psi_2)) - 16\Delta(\phi_0)\bar{\delta}(\bar{\phi}_0) + 3888\Psi_2\rho\bar{\Psi}_2\bar{\tau} \\
& -1224\Psi_2\tau\bar{\Psi}_2\bar{\sigma} + 3888\Psi_2\pi\bar{\Psi}_2\bar{\rho} - 1224\Psi_2\mu\bar{\Psi}_2\bar{\kappa} - 216\bar{\Psi}_2\Psi_2\pi\epsilon \\
& +216\bar{\Psi}_2\Psi_2\bar{\epsilon}\pi + 432\Psi_2\bar{\Psi}_2\bar{\kappa}\bar{\gamma} + 144\Psi_2\bar{\Psi}_2\gamma\bar{\kappa} - 432\Psi_2\bar{\Psi}_2\bar{\sigma}\bar{\alpha} \\
& +144\Psi_2\bar{\Psi}_2\beta\bar{\sigma} + 432\Psi_2\bar{\Psi}_2\bar{\pi}\bar{\sigma} - 432\Psi_2\bar{\Psi}_2\bar{\kappa}\bar{\mu} + 432\Psi_2\bar{\Psi}_2\bar{\tau}\bar{\epsilon} \\
& -432\Psi_2\bar{\Psi}_2\bar{\tau}\epsilon - 432\Psi_2\bar{\Psi}_2\bar{\rho}\bar{\beta} - 432\Psi_2\bar{\Psi}_2\bar{\rho}\alpha - 64\phi_0\nu\bar{\phi}_0\bar{\epsilon} \\
& -76\phi_0\mu\bar{\phi}_0\bar{\beta} - 52\phi_0\pi\bar{\phi}_0\bar{\gamma} - 64\phi_0\alpha\bar{\phi}_0\bar{\mu} - 40\phi_0\lambda\bar{\phi}_0\bar{\pi} \\
& -64\phi_0\alpha\bar{\phi}_0\bar{\gamma} - 40\phi_0\lambda\bar{\phi}_0\bar{\alpha} - 88\phi_0\gamma\bar{\phi}_0\bar{\beta} - 52\phi_0\pi\bar{\phi}_0\bar{\mu} \\
& +216\bar{\Psi}_2\Psi_2\alpha\rho + 216\bar{\Psi}_2\Psi_2\bar{\beta}\rho + 24\bar{\phi}_0\phi_0\lambda\beta - 24\bar{\phi}_0\phi_0\rho\nu \\
& +36\bar{\phi}_0\phi_0\mu\alpha + 48\phi_0\bar{\phi}_0\bar{\mu}\bar{\beta} + 48\phi_0\bar{\phi}_0\bar{\beta}\bar{\gamma} - 24\phi_0\bar{\phi}_0\bar{\nu}\bar{\sigma} \\
& -12\phi_0\bar{\phi}_0\bar{\tau}\bar{\mu} - 12\phi_0\bar{\phi}_0\bar{\gamma}\bar{\tau} + 24\bar{\phi}_0\phi_0\alpha\gamma + 48\bar{\phi}_0\phi_0\pi\mu \\
& -30\bar{\phi}_0\delta(\phi_0)\lambda - 30\bar{\phi}_0 D(\phi_0)\nu - 12\bar{\phi}_0\phi_0\Delta(\pi) - 24\bar{\phi}_0\pi\Delta(\phi_0) \\
& -12\bar{\phi}_0\phi_0\delta(\lambda) - 12\phi_0\bar{\phi}_0\bar{\delta}(\bar{\mu}) + 6\phi_0\Delta(\bar{\phi}_0)\bar{\tau} - 12\phi_0\bar{\phi}_0\Delta(\bar{\beta}) \\
& -18\bar{\phi}_0\alpha\Delta(\phi_0) - 18\bar{\phi}_0\bar{\delta}(\phi_0)\gamma - 12\bar{\phi}_0\phi_0\bar{\delta}(\gamma) + 38\phi_0\gamma\bar{\delta}(\bar{\phi}_0) \\
& +32\phi_0\pi\Delta(\bar{\phi}_0) + 26\bar{\delta}(\phi_0)\bar{\phi}_0\bar{\gamma} + 38\phi_0\alpha\Delta(\bar{\phi}_0) + 32\phi_0\mu\bar{\delta}(\bar{\phi}_0) \\
& -432\Psi_2\bar{\Psi}_2 D(\bar{\tau}) + 144\Psi_2 D(\bar{\Psi}_2)\alpha + 144\Psi_2 D(\bar{\Psi}_2)\bar{\beta} + 432\Psi_2\bar{\Psi}_2\bar{\delta}(\bar{\rho}) \\
& +26\phi_0\nu D(\bar{\phi}_0) - 72\phi_0\Phi_{21}\bar{\phi}_0 - 1440\Psi_2\pi D(\bar{\Psi}_2) - 1440\Psi_2\rho\bar{\delta}(\bar{\Psi}_2) \\
& -12\bar{\phi}_0\phi_0 D(\nu) - 12\bar{\phi}_0\phi_0\Delta(\alpha) - 216\bar{\Psi}_2\Psi_2\bar{\delta}(\rho) - 72\bar{\Psi}_2 D(\Psi_2)\alpha \\
& -72\bar{\Psi}_2 D(\Psi_2)\bar{\beta} + 6\bar{\phi}_0\Delta(\phi_0)\bar{\tau} - 360\bar{\Psi}_2\bar{\delta}(\Psi_2)\rho = 0
\end{aligned}
\tag{A.7.6}
$$

$$o_{ABCD}\bar{o}_{(\dot{A}\dot{B}\dot{C}}\bar{\iota}_{\dot{D})} \quad \begin{aligned}&360\bar{\Psi}_2\bar{\nu}\Psi_2\lambda - 1152\Psi_2\nu\bar{\Psi}_2\bar{\mu} - 216\bar{\Psi}_2\Psi_2\mu\nu - 216\bar{\Psi}_2\Psi_2\nu\gamma\\ &-72\bar{\Psi}_2\Psi_2\nu\bar{\gamma} - 144\bar{\Psi}_2\nu\Delta(\Psi_2) - 72\bar{\Psi}_2\Psi_2\Delta(\nu) - 432\Psi_2\nu\Delta(\bar{\Psi}_2) = 0\end{aligned} \tag{A.7.7}$$

$$o_{ABCD}\bar{o}_{\dot{A}\dot{B}\dot{C}\dot{D}} \quad -432\Psi_2\nu\bar{\Psi}_2\bar{\nu} = 0 \tag{A.7.8}$$

$$\iota_{ABCD}\bar{o}_{(\dot{A}}\bar{\iota}_{\dot{B}\dot{C}\dot{D})} \quad \begin{aligned}&360\bar{\Psi}_2\bar{\kappa}\Psi_2\sigma - 1152\Psi_2\kappa\bar{\Psi}_2\bar{\rho} - 216\bar{\Psi}_2\Psi_2\rho\kappa - 12\phi_0\bar{\phi}_0\bar{\beta}\sigma\\ &-12\phi_0\bar{\phi}_0\bar{\alpha}\bar{\rho} - 12\bar{\phi}_0\phi_0\sigma\alpha - 12\bar{\phi}_0\phi_0\beta\bar{\rho} - 12\bar{\phi}_0\phi_0\pi\sigma\\ &-12\phi_0\bar{\phi}_0\bar{\mu}\kappa - 40\phi_0\epsilon\bar{\phi}_0\bar{\pi} - 40\phi_0\epsilon\bar{\phi}_0\bar{\alpha} - 64\phi_0\beta\bar{\phi}_0\bar{\epsilon}\\ &-216\bar{\Psi}_2\Psi_2\epsilon\kappa - 72\bar{\Psi}_2\Psi_2\kappa\bar{\epsilon} - 12\bar{\phi}_0\phi_0\kappa\gamma + 72\bar{\phi}_0\phi_0\epsilon\beta\\ &-12\bar{\phi}_0\phi_0\kappa\mu + 6\phi_0D(\delta(\bar{\phi}_0)) + 6\bar{\phi}_0D(\delta(\phi_0)) - 16D(\phi_0)\delta(\bar{\phi}_0)\\ &-16\delta(\phi_0)D(\bar{\phi}_0) + 6\phi_0\delta(D(\bar{\phi}_0)) + 6\bar{\phi}_0\delta(D(\phi_0)) + 48\phi_0\bar{\phi}_0\bar{\epsilon}\bar{\alpha}\\ &-12\phi_0\bar{\phi}_0\bar{\gamma}\kappa - 24\phi_0\bar{\phi}_0\bar{\lambda}\bar{\kappa} + 48\phi_0\bar{\phi}_0\bar{\pi}\bar{\epsilon} - 12\phi_0\bar{\phi}_0\bar{\rho}\bar{\pi}\\ &-72\phi_0\Phi_{01}\bar{\phi}_0 + 432\Psi_2\kappa D(\bar{\Psi}_2) + 26\phi_0\beta D(\bar{\phi}_0) + 26D(\phi_0)\bar{\phi}_0\bar{\pi}\\ &+26D(\phi_0)\bar{\phi}_0\bar{\alpha} + 26\phi_0\epsilon\delta(\bar{\phi}_0) + 6\phi_0\bar{\delta}(\bar{\phi}_0)\sigma + 72\bar{\Psi}_2\Psi_2D(\kappa)\\ &+144\bar{\Psi}_2\kappa D(\Psi_2) - 30\bar{\phi}_0\epsilon\delta(\phi_0) + 6\bar{\phi}_0\Delta(\phi_0)\kappa - 30\bar{\phi}_0D(\phi_0)\beta\\ &-12\bar{\phi}_0\phi_0D(\beta) - 18\phi_0\bar{\epsilon}\delta(\bar{\phi}_0) - 12\phi_0\bar{\phi}_0D(\bar{\alpha}) - 30\phi_0D(\bar{\phi}_0)\bar{\pi}\\ &+6\phi_0\Delta(\bar{\phi}_0)\kappa - 30\phi_0D(\bar{\phi}_0)\bar{\alpha} + 6\phi_0\delta(\bar{\phi}_0)\bar{\rho} - 12\phi_0\bar{\phi}_0\delta(\bar{\epsilon})\\ &+6\bar{\phi}_0\bar{\delta}(\phi_0)\sigma - 12\bar{\phi}_0\phi_0\delta(\epsilon) + 6\bar{\phi}_0\delta(\phi_0)\bar{\rho} - 12\phi_0\bar{\phi}_0D(\bar{\pi})\\ &+38\delta(\phi_0)\bar{\phi}_0\bar{\epsilon} = 0\end{aligned} \tag{A.7.9}$$

$$o_{ABCD}\bar{o}_{\dot{A}}\bar{\iota}_{\dot{B}\dot{C}\dot{D}} \quad \begin{aligned}&360\bar{\Psi}_2\bar{\sigma}\Psi_2\nu - 1152\Psi_2\lambda\bar{\Psi}_2\bar{\tau} + 216\bar{\Psi}_2\Psi_2\pi\lambda + 216\bar{\Psi}_2\Psi_2\alpha\lambda\\ &-72\bar{\Psi}_2\Psi_2\lambda\bar{\beta} + 24\bar{\phi}_0\phi_0\lambda\nu + 432\Psi_2\lambda\bar{\delta}(\bar{\Psi}_2) + 144\bar{\Psi}_2\lambda\bar{\delta}(\Psi_2)\\ &+72\bar{\Psi}_2\Psi_2\bar{\delta}(\lambda) = 0\end{aligned} \tag{A.7.10}$$

$$o_{ABCD}\bar{\iota}_{\dot{A}\dot{B}\dot{C}\dot{D}} \quad -432\Psi_2\lambda\bar{\Psi}_2\bar{\sigma} - 12\bar{\phi}_0\phi_0\lambda^2 = 0 \tag{A.7.11}$$

$$o_{(A}\iota_{BCD}\bar{o}_{(\dot{A}}\bar{\iota}_{\dot{B}\dot{C}\dot{D})} \quad \begin{aligned}&72\bar{\Psi}_2\Psi_2\Delta(\sigma) - 32\Delta(\phi_0)\bar{\phi}_0\bar{\lambda} - 216\bar{\Psi}_2\Psi_2\tau^2 - 1152\Psi_2\tau\delta(\bar{\Psi}_2)\\ &-216\Psi_2\bar{\Psi}_2\bar{\pi}^2 + 144\Psi_2\Phi_{02}\bar{\Psi}_2 + 432\delta(\Psi_2)\delta(\bar{\Psi}_2) + 72\bar{\Psi}_2\delta(\delta(\Psi_2))\\ &+72\Psi_2\delta(\delta(\bar{\Psi}_2)) + 1368\Psi_2\rho\bar{\Psi}_2\bar{\lambda} - 720\Psi_2\kappa\bar{\Psi}_2\bar{\nu} - 3456\Psi_2\tau\bar{\Psi}_2\bar{\pi}\\ &+1368\Psi_2\sigma\bar{\Psi}_2\bar{\mu} - 36\phi_0\bar{\phi}_0\bar{\pi}\bar{\nu} - 12\phi_0\bar{\phi}_0\bar{\nu}\bar{\alpha} - 12\phi_0\bar{\phi}_0\bar{\lambda}\bar{\gamma}\\ &-360D(\Psi_2)\bar{\Psi}_2\bar{\lambda} + 360\Psi_2\sigma\Delta(\bar{\Psi}_2) + 1152\delta(\Psi_2)\bar{\Psi}_2\bar{\pi} + 24\phi_0\bar{\lambda}\Delta(\bar{\phi}_0)\\ &-72\Psi_2\bar{\Psi}_2D(\bar{\lambda}) + 72\Psi_2D(\bar{\Psi}_2)\bar{\lambda} + 144\Psi_2\bar{\pi}\delta(\bar{\Psi}_2) - 72\bar{\Psi}_2\Delta(\Psi_2)\sigma\\ &+72\bar{\Psi}_2\delta(\Psi_2)\bar{\alpha} - 72\bar{\Psi}_2\delta(\Psi_2)\beta - 216\bar{\Psi}_2\Psi_2\delta(\tau) - 144\bar{\Psi}_2\tau\delta(\Psi_2)\\ &+12\phi_0\bar{\phi}_0\Delta(\bar{\lambda}) + 12\phi_0\bar{\phi}_0\bar{\nu}\tau + 72\Psi_2\bar{\Psi}_2\epsilon\bar{\lambda} - 216\Psi_2\bar{\Psi}_2\bar{\lambda}\bar{\epsilon}\\ &-648\bar{\Psi}_2\Psi_2\mu\sigma + 216\bar{\Psi}_2\Psi_2\tau\beta - 216\bar{\Psi}_2\Psi_2\tau\bar{\alpha} - 216\bar{\Psi}_2\Psi_2\gamma\sigma\\ &+72\bar{\Psi}_2\Psi_2\bar{\gamma}\sigma + 216\Psi_2\bar{\Psi}_2\bar{\pi}\bar{\alpha} - 648\Psi_2\bar{\Psi}_2\bar{\lambda}\bar{\rho} - 216\Psi_2\bar{\Psi}_2\bar{\pi}\beta\\ &+76\phi_0\beta\bar{\phi}_0\bar{\nu} + 52\phi_0\gamma\bar{\phi}_0\bar{\lambda} + 52\phi_0\mu\bar{\phi}_0\bar{\lambda} - 36\phi_0\bar{\phi}_0\bar{\mu}\bar{\lambda}\\ &+72\Psi_2\delta(\bar{\Psi}_2)\bar{\alpha} - 72\Psi_2\delta(\bar{\Psi}_2)\beta - 32\delta(\phi_0)\bar{\phi}_0\bar{\nu} + 216\Psi_2\bar{\Psi}_2\delta(\bar{\pi})\\ &+24\phi_0\delta(\bar{\phi}_0)\bar{\nu} + 12\phi_0\bar{\phi}_0\delta(\bar{\nu}) = 0\end{aligned} \tag{A.7.12}$$

$$o_{(AB}\iota_{CD)}\overline{o}_{\dot{A}\dot{B}\dot{C}\dot{D}} \quad \begin{aligned}&-12\phi_0\bar{\phi}_0\bar{\nu}^2+1440\Psi_2\mu\bar{\Psi}_2\bar{\lambda}+1440\Psi_2\tau\bar{\Psi}_2\bar{\nu}+432\Psi_2\bar{\Psi}_2\bar{\pi}\bar{\nu}\\&-432\Psi_2\bar{\Psi}_2\bar{\mu}\bar{\lambda}+432\Psi_2\bar{\Psi}_2\bar{\lambda}\bar{\gamma}-144\Psi_2\bar{\Psi}_2\bar{\lambda}\gamma-432\Psi_2\bar{\Psi}_2\bar{\nu}\bar{\alpha}\\&-144\Psi_2\bar{\Psi}_2\bar{\nu}\beta+144\Psi_2\bar{\Psi}_2\Delta(\bar{\lambda})-144\Psi_2\bar{\Psi}_2\delta(\bar{\nu})+360\Delta(\Psi_2)\bar{\Psi}_2\bar{\lambda}\\&-360\delta(\Psi_2)\bar{\Psi}_2\bar{\nu}=0\end{aligned} \tag{A.7.13}$$

$$o_{(AB}\iota_{CD)}\overline{o}_{(\dot{A}\dot{B}}\overline{\iota}_{\dot{C}\dot{D})} \quad \begin{aligned}&3888\Psi_2\pi\bar{\Psi}_2\bar{\mu}+3888\Psi_2\mu\bar{\Psi}_2\bar{\tau}-1224\Psi_2\nu\bar{\Psi}_2\bar{\rho}-1224\Psi_2\lambda\bar{\Psi}_2\bar{\pi}\\&-64\phi_0\nu\bar{\phi}_0\bar{\mu}-432\bar{\Psi}_2\Psi_2\mu\bar{\beta}-432\bar{\Psi}_2\Psi_2\mu\alpha+144\bar{\Psi}_2\Psi_2\bar{\epsilon}\nu\\&-76\phi_0\nu\bar{\phi}_0\bar{\gamma}-432\bar{\Psi}_2\Psi_2\lambda\beta+216\Psi_2\bar{\Psi}_2\gamma\bar{\tau}-432\bar{\Psi}_2\Psi_2\pi\bar{\gamma}\\&+432\bar{\Psi}_2\Psi_2\pi\gamma+432\bar{\Psi}_2\Psi_2\lambda\tau-216\Psi_2\bar{\Psi}_2\bar{\gamma}\bar{\tau}-52\phi_0\lambda\bar{\phi}_0\bar{\nu}\\&+432\bar{\Psi}_2\Psi_2\nu\epsilon+144\bar{\Psi}_2\Psi_2\bar{\alpha}\lambda-432\bar{\Psi}_2\Psi_2\rho\nu+144\bar{\Psi}_2\Delta(\bar{\delta}(\Psi_2))\\&-144\bar{\Psi}_2\bar{\delta}(\Delta(\Psi_2))-72\Psi_2\Delta(\bar{\delta}(\bar{\Psi}_2))+72\Psi_2\bar{\delta}(\Delta(\bar{\Psi}_2))+360\bar{\delta}(\Psi_2)\Delta(\bar{\Psi}_2)\\&-360\Delta(\Psi_2)\bar{\delta}(\bar{\Psi}_2)+216\Psi_2\bar{\Psi}_2\bar{\mu}\bar{\beta}+216\Psi_2\bar{\Psi}_2\alpha\bar{\mu}+36\bar{\phi}_0\phi_0\mu\nu\\&+12\bar{\phi}_0\phi_0\nu\gamma-72\Psi_2\bar{\delta}(\bar{\Psi}_2)\gamma+72\Psi_2\bar{\delta}(\bar{\Psi}_2)\bar{\gamma}+72\Psi_2\Delta(\bar{\Psi}_2)\bar{\beta}\\&+360\Psi_2\bar{\mu}\bar{\delta}(\bar{\Psi}_2)+216\Psi_2\bar{\Psi}_2\bar{\delta}(\bar{\mu})+144\bar{\Psi}_2\bar{\delta}(\Psi_2)\gamma+432\bar{\Psi}_2\Psi_2\Delta(\pi)\\&-144\bar{\Psi}_2\bar{\delta}(\Psi_2)\bar{\gamma}-144\bar{\Psi}_2\Delta(\Psi_2)\alpha-144\bar{\Psi}_2\Delta(\Psi_2)\bar{\beta}-432\bar{\Psi}_2\lambda\delta(\Psi_2)\\&-576\bar{\Psi}_2\bar{\delta}(\Psi_2)\mu-144\bar{\Psi}_2\Psi_2\delta(\lambda)+1440\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\mu}+576\bar{\Psi}_2\pi\Delta(\Psi_2)\\&+360\Psi_2\Delta(\bar{\Psi}_2)\bar{\tau}+216\Psi_2\bar{\Psi}_2\Delta(\bar{\tau})+72\Psi_2\Delta(\bar{\Psi}_2)\alpha+1440\Delta(\Psi_2)\bar{\Psi}_2\bar{\tau}\\&+1152\Psi_2\pi\Delta(\bar{\Psi}_2)-504\Psi_2\lambda\delta(\bar{\Psi}_2)+504\Psi_2\nu D(\bar{\Psi}_2)-1152\Psi_2\mu\bar{\delta}(\bar{\Psi}_2)\\&+432\bar{\Psi}_2 D(\Psi_2)\nu+144\bar{\Psi}_2\Psi_2 D(\nu)+32\phi_0\nu\Delta(\bar{\phi}_0)-432\bar{\Psi}_2\Psi_2\bar{\delta}(\mu)\\&-12\bar{\phi}_0\phi_0\Delta(\nu)-24\bar{\phi}_0\Delta(\phi_0)\nu=0\end{aligned} \tag{A.7.14}$$

$$o_{(AB}\iota_{CD)}\overline{o}_{\dot{A}\dot{B}\dot{C}\dot{D}} \quad \begin{aligned}&-144\Psi_2\bar{\nu}\Delta(\bar{\Psi}_2)-432\Delta(\Psi_2)\bar{\Psi}_2\bar{\nu}+360\Psi_2\nu\bar{\Psi}_2\bar{\lambda}-1152\bar{\Psi}_2\bar{\nu}\Psi_2\mu\\&-216\Psi_2\bar{\Psi}_2\bar{\mu}\bar{\nu}-216\Psi_2\bar{\Psi}_2\bar{\nu}\bar{\gamma}-72\Psi_2\bar{\Psi}_2\bar{\nu}\gamma-72\Psi_2\bar{\Psi}_2\Delta(\bar{\nu})=0\end{aligned} \tag{A.7.15}$$

$$o_{ABCD}\overline{o}_{(\dot{A}\dot{B}}\overline{\iota}_{\dot{C}\dot{D})} \quad \begin{aligned}&-12\bar{\phi}_0\phi_0\nu^2-360\Psi_2\nu\bar{\delta}(\bar{\Psi}_2)+1440\Psi_2\nu\bar{\Psi}_2\bar{\tau}+1440\Psi_2\lambda\bar{\Psi}_2\bar{\mu}\\&-432\bar{\Psi}_2\Psi_2\mu\lambda+432\bar{\Psi}_2\Psi_2\pi\nu+432\bar{\Psi}_2\Psi_2\lambda\gamma-144\bar{\Psi}_2\Psi_2\lambda\bar{\gamma}\\&-432\bar{\Psi}_2\Psi_2\nu\alpha-144\bar{\Psi}_2\Psi_2\nu\bar{\beta}-144\bar{\Psi}_2\Psi_2\bar{\delta}(\nu)+144\bar{\Psi}_2\Psi_2\Delta(\lambda)\\&+360\Psi_2\lambda\Delta(\bar{\Psi}_2)=0\end{aligned} \tag{A.7.16}$$

$$\iota_{ABCD}\overline{o}_{(\dot{A}\dot{B}\dot{C}}\overline{\iota}_{\dot{D})} \quad \begin{aligned}&32\delta(\phi_0)\bar{\phi}_0\bar{\lambda}-144\bar{\Psi}_2\sigma\delta(\Psi_2)-432\Psi_2\sigma\delta(\bar{\Psi}_2)-72\bar{\Psi}_2\Psi_2\delta(\sigma)\\&-24\phi_0\delta(\bar{\phi}_0)\bar{\lambda}-12\phi_0\bar{\phi}_0\delta(\bar{\lambda})+360\bar{\Psi}_2\bar{\lambda}\Psi_2\kappa-1152\Psi_2\sigma\bar{\Psi}_2\bar{\pi}\\&+216\bar{\Psi}_2\Psi_2\tau\sigma+12\phi_0\bar{\phi}_0\bar{\alpha}\bar{\lambda}+36\phi_0\bar{\phi}_0\bar{\pi}\bar{\lambda}-12\phi_0\bar{\phi}_0\bar{\nu}\sigma\\&+216\bar{\Psi}_2\Psi_2\beta\sigma-72\bar{\Psi}_2\Psi_2\sigma\bar{\alpha}-52\phi_0\beta\bar{\phi}_0\bar{\lambda}=0\end{aligned} \tag{A.7.17}$$

$$
\begin{aligned}
o_{(A}\iota_{BCD)}\overline{o}_{(\dot{A}\dot{B}}\overline{\iota}_{\dot{C}\dot{D})} \quad & 6\bar{\phi}_0\Delta(\delta(\phi_0)) - 16\delta(\phi_0)\Delta(\bar{\phi}_0) - 16\Delta(\phi_0)\delta(\bar{\phi}_0) - 360D(\Psi_2)\delta(\bar{\Psi}_2) \\
& +360\delta(\Psi_2)D(\bar{\Psi}_2) + 72\Psi_2\delta(D(\bar{\Psi}_2)) + 6\bar{\phi}_0\delta(\Delta(\phi_0)) + 144\bar{\Psi}_2D(\delta(\Psi_2)) \\
& -144\bar{\Psi}_2\delta(D(\Psi_2)) + 6\phi_0\delta(\Delta(\bar{\phi}_0)) - 72\Psi_2D(\delta(\bar{\Psi}_2)) + 6\phi_0\Delta(\delta(\bar{\phi}_0)) \\
& -72\phi_0\Phi_{12}\bar{\phi}_0 + 32\delta(\phi_0)\bar{\phi}_0\bar{\mu} - 12\phi_0\bar{\phi}_0\delta(\bar{\mu}) - 24\phi_0\bar{\mu}\delta(\bar{\phi}_0) \\
& -432\bar{\Psi}_2\kappa\Delta(\Psi_2) - 144\bar{\Psi}_2\delta(\Psi_2)\epsilon + 144\bar{\Psi}_2\delta(\Psi_2)\bar{\epsilon} + 432\bar{\Psi}_2\bar{\delta}(\Psi_2)\sigma \\
& +144\bar{\Psi}_2D(\Psi_2)\bar{\alpha} + 144\bar{\Psi}_2D(\Psi_2)\beta - 30\bar{\phi}_0\delta(\phi_0)\mu - 1224\Psi_2\kappa\bar{\Psi}_2\bar{\mu} \\
& -1224\Psi_2\sigma\bar{\Psi}_2\bar{\tau} + 3888\Psi_2\tau\bar{\Psi}_2\bar{\rho} + 3888\Psi_2\rho\bar{\Psi}_2\bar{\pi} - 40\bar{\phi}_0\phi_0\alpha\bar{\lambda} \\
& +48\bar{\phi}_0\phi_0\beta\gamma - 64\bar{\phi}_0\phi_0\bar{\alpha}\gamma - 216\Psi_2\bar{\Psi}_2\bar{\pi}\bar{\epsilon} + 216\Psi_2\bar{\Psi}_2\epsilon\bar{\pi} \\
& -432\bar{\Psi}_2\Psi_2\sigma\alpha + 144\bar{\Psi}_2\Psi_2\bar{\beta}\sigma - 64\bar{\phi}_0\phi_0\bar{\alpha}\mu + 48\bar{\phi}_0\phi_0\mu\beta \\
& -24\bar{\phi}_0\phi_0\nu\sigma - 40\bar{\phi}_0\phi_0\bar{\lambda}\pi + 216\Psi_2\bar{\Psi}_2\bar{\alpha}\bar{\rho} + 216\Psi_2\bar{\Psi}_2\beta\bar{\rho} \\
& -88\phi_0\bar{\phi}_0\beta\bar{\gamma} + 24\phi_0\bar{\phi}_0\bar{\lambda}\bar{\beta} - 24\phi_0\bar{\phi}_0\bar{\rho}\bar{\nu} + 36\phi_0\bar{\phi}_0\bar{\pi}\bar{\gamma} \\
& -64\phi_0\bar{\phi}_0\epsilon\bar{\nu} - 12\phi_0\bar{\phi}_0\bar{\mu}\tau - 52\phi_0\bar{\phi}_0\bar{\pi}\gamma - 12\phi_0\bar{\phi}_0\bar{\gamma}\tau \\
& -24\phi_0\bar{\phi}_0\bar{\lambda}\bar{\tau} + 24\phi_0\bar{\phi}_0\bar{\alpha}\bar{\gamma} + 24\phi_0\bar{\phi}_0\bar{\nu}\bar{\epsilon} + 432\bar{\Psi}_2\Psi_2\kappa\gamma \\
& +144\bar{\Psi}_2\Psi_2\bar{\gamma}\kappa - 12\bar{\phi}_0\phi_0\tau\mu - 12\bar{\phi}_0\phi_0\gamma\tau - 12\phi_0\bar{\phi}_0\bar{\nu}\rho \\
& -52\phi_0\bar{\phi}_0\bar{\pi}\mu + 36\phi_0\bar{\phi}_0\bar{\mu}\bar{\alpha} - 76\phi_0\bar{\phi}_0\bar{\mu}\beta + 432\bar{\Psi}_2\Psi_2\pi\sigma \\
& -432\bar{\Psi}_2\Psi_2\kappa\mu - 432\bar{\Psi}_2\Psi_2\rho\beta - 432\bar{\Psi}_2\Psi_2\rho\bar{\alpha} - 432\bar{\Psi}_2\Psi_2\tau\bar{\epsilon} \\
& +432\bar{\Psi}_2\Psi_2\tau\epsilon + 48\phi_0\bar{\phi}_0\bar{\pi}\bar{\mu} - 216\Psi_2\bar{\Psi}_2\delta(\bar{\rho}) - 18\bar{\phi}_0\beta\Delta(\phi_0) \\
& +38\bar{\phi}_0\Delta(\phi_0)\bar{\alpha} - 30\bar{\phi}_0\delta(\phi_0)\gamma + 26\bar{\phi}_0\bar{\delta}(\phi_0)\bar{\lambda} - 1440D(\Psi_2)\bar{\Psi}_2\bar{\pi} \\
& -1440\delta(\Psi_2)\bar{\Psi}_2\bar{\rho} + 144\bar{\Psi}_2\Psi_2\bar{\delta}(\sigma) + 576\bar{\Psi}_2\rho\delta(\Psi_2) - 432\bar{\Psi}_2\Psi_2D(\tau) \\
& -576\bar{\Psi}_2D(\Psi_2)\tau - 72\Psi_2D(\bar{\Psi}_2)\bar{\alpha} - 72\Psi_2D(\bar{\Psi}_2)\beta - 360\Psi_2\delta(\bar{\Psi}_2)\bar{\rho} \\
& -12\phi_0\bar{\phi}_0\bar{\delta}(\bar{\lambda}) + 32\Delta(\phi_0)\bar{\phi}_0\bar{\pi} + 38\delta(\phi_0)\bar{\phi}_0\bar{\gamma} + 6\bar{\phi}_0\Delta(\phi_0)\tau \\
& -12\bar{\phi}_0\phi_0\Delta(\beta) - 30\phi_0D(\bar{\phi}_0)\bar{\nu} - 18\phi_0\delta(\bar{\phi}_0)\bar{\gamma} + 6\phi_0\Delta(\bar{\phi}_0)\tau \\
& -12\phi_0\bar{\phi}_0\Delta(\bar{\alpha}) + 26\phi_0\delta(\bar{\phi}_0)\gamma - 18\phi_0\bar{\alpha}\Delta(\bar{\phi}_0) - 144\bar{\Psi}_2\Psi_2\Delta(\kappa) \\
& -12\phi_0\bar{\phi}_0D(\bar{\nu}) - 24\phi_0\bar{\pi}\Delta(\bar{\phi}_0) - 12\phi_0\bar{\phi}_0\Delta(\bar{\pi}) - 360\Psi_2\bar{\pi}D(\bar{\Psi}_2) \\
& -216\Psi_2\bar{\Psi}_2D(\bar{\pi}) + 72\Psi_2\delta(\bar{\Psi}_2)\epsilon - 72\Psi_2\delta(\bar{\Psi}_2)\bar{\epsilon} - 504\Psi_2\kappa\Delta(\bar{\Psi}_2) \\
& +504\Psi_2\sigma\bar{\delta}(\bar{\Psi}_2) + 1152\Psi_2\rho\delta(\bar{\Psi}_2) - 1152\Psi_2\tau D(\bar{\Psi}_2) + 26D(\phi_0)\bar{\phi}_0\bar{\nu} \\
& -12\bar{\phi}_0\phi_0\delta(\gamma) - 12\bar{\phi}_0\phi_0\delta(\mu) + 432\bar{\Psi}_2\Psi_2\delta(\rho) - 30\phi_0\bar{\delta}(\bar{\phi}_0)\bar{\lambda} \\
& +38\phi_0\Delta(\bar{\phi}_0)\beta + 26\phi_0\delta(\bar{\phi}_0)\mu - 12\phi_0\bar{\phi}_0\delta(\bar{\gamma}) = 0
\end{aligned}
\tag{A.7.18}
$$

$$
\begin{aligned}
\iota_{ABCD}\overline{o}_{(\dot{A}\dot{B}}\overline{\iota}_{\dot{C}\dot{D})} \quad & 12\phi_0\bar{\phi}_0\delta(\bar{\pi}) + 360\Psi_2\kappa\delta(\bar{\Psi}_2) - 12\phi_0\bar{\phi}_0\bar{\alpha}^2 + 36\phi_0\Phi_{02}\bar{\phi}_0 \\
& -36\bar{\phi}_0\phi_0\beta^2 - 24\phi_0\bar{\phi}_0\bar{\pi}^2 + 144\bar{\Psi}_2\Psi_2\delta(\kappa) + 12\phi_0\bar{\phi}_0D(\bar{\lambda}) \\
& -144\bar{\Psi}_2\Psi_2D(\sigma) + 1440\Psi_2\sigma\bar{\Psi}_2\bar{\rho} + 1440\Psi_2\kappa\bar{\Psi}_2\bar{\pi} - 432\bar{\Psi}_2\Psi_2\rho\sigma \\
& +432\bar{\Psi}_2\Psi_2\tau\kappa + 52\phi_0\beta\bar{\phi}_0\bar{\pi} + 64\phi_0\beta\bar{\phi}_0\bar{\alpha} + 40\phi_0\epsilon\bar{\phi}_0\bar{\lambda} \\
& -432\bar{\Psi}_2\Psi_2\kappa\beta - 144\bar{\Psi}_2\Psi_2\kappa\bar{\alpha} + 432\bar{\Psi}_2\Psi_2\sigma\epsilon - 144\bar{\Psi}_2\Psi_2\sigma\bar{\epsilon} \\
& -24\phi_0\bar{\phi}_0\bar{\lambda}\bar{\epsilon} + 12\phi_0\bar{\phi}_0\bar{\mu}\sigma + 12\phi_0\bar{\phi}_0\bar{\nu}\kappa - 36\phi_0\bar{\phi}_0\bar{\pi}\bar{\alpha} \\
& -252\phi_0^2\bar{\Psi}_2 + 16\delta(\phi_0)\delta(\bar{\phi}_0) - 6\bar{\phi}_0\delta(\delta(\phi_0)) - 6\phi_0\delta(\delta(\bar{\phi}_0)) \\
& +12\phi_0\bar{\phi}_0\bar{\gamma}\sigma + 24\phi_0\bar{\phi}_0\bar{\lambda}\bar{\rho} + 12\bar{\phi}_0\phi_0\gamma\sigma + 12\bar{\phi}_0\phi_0\mu\sigma \\
& +30\phi_0D(\bar{\phi}_0)\bar{\lambda} + 24\phi_0\bar{\pi}\delta(\bar{\phi}_0) - 26\phi_0\beta\delta(\bar{\phi}_0) - 26D(\phi_0)\bar{\phi}_0\bar{\lambda} \\
& -360\Psi_2\sigma D(\bar{\Psi}_2) - 32\delta(\phi_0)\bar{\phi}_0\bar{\pi} - 38\delta(\phi_0)\bar{\phi}_0\bar{\alpha} + 18\phi_0\delta(\bar{\phi}_0)\bar{\alpha} \\
& -6\phi_0\Delta(\bar{\phi}_0)\sigma + 12\phi_0\bar{\phi}_0\delta(\bar{\alpha}) + 30\bar{\phi}_0\delta(\phi_0)\beta - 6\bar{\phi}_0\Delta(\phi_0)\sigma \\
& +12\bar{\phi}_0\phi_0\delta(\beta) = 0
\end{aligned}
\tag{A.7.19}
$$

$$
o_{(AB^{\iota}CD)}\bar{\iota}_{\dot{A}\dot{B}\dot{C}\dot{D}} \quad
\begin{aligned}
&-252\bar{\phi}_0^2\Psi_2 + 16\bar{\delta}(\phi_0)\bar{\delta}(\bar{\phi}_0) - 6\bar{\phi}_0\bar{\delta}(\bar{\delta}(\phi_0)) - 6\phi_0\bar{\delta}(\bar{\delta}(\bar{\phi}_0))\\
&+1440\Psi_2\pi\bar{\Psi}_2\bar{\kappa} + 1440\Psi_2\rho\bar{\Psi}_2\bar{\sigma} - 432\Psi_2\bar{\Psi}_2\bar{\rho}\bar{\sigma} + 432\Psi_2\bar{\Psi}_2\bar{\tau}\bar{\kappa}\\
&+52\phi_0\pi\bar{\phi}_0\bar{\beta} - 432\Psi_2\bar{\Psi}_2\bar{\kappa}\bar{\beta} + 64\phi_0\alpha\bar{\phi}_0\bar{\beta} + 40\phi_0\lambda\bar{\phi}_0\bar{\epsilon}\\
&-144\Psi_2\bar{\Psi}_2\bar{\sigma}\epsilon - 144\Psi_2\bar{\Psi}_2\bar{\kappa}\alpha + 432\Psi_2\bar{\Psi}_2\bar{\sigma}\bar{\epsilon} + 24\bar{\phi}_0\phi_0\lambda\rho\\
&-24\bar{\phi}_0\phi_0\lambda\epsilon + 36\phi_0\Phi_{20}\bar{\phi}_0 - 24\bar{\phi}_0\phi_0\pi^2 - 12\bar{\phi}_0\phi_0\alpha^2\\
&-36\phi_0\bar{\phi}_0\bar{\beta}^2 - 6\bar{\phi}_0\Delta(\phi_0)\bar{\sigma} + 12\bar{\phi}_0\phi_0\bar{\delta}(\alpha) + 24\bar{\phi}_0\pi\bar{\delta}(\phi_0)\\
&+12\bar{\phi}_0\phi_0\bar{\delta}(\pi) + 12\bar{\phi}_0\phi_0D(\lambda) + 30\phi_0\bar{\delta}(\bar{\phi}_0)\bar{\beta} - 6\phi_0\Delta(\bar{\phi}_0)\bar{\sigma}\\
&+12\phi_0\bar{\phi}_0\bar{\delta}(\bar{\beta}) + 30\bar{\phi}_0D(\phi_0)\lambda + 18\bar{\phi}_0\bar{\delta}(\phi_0)\alpha + 360\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\kappa}\\
&-360D(\Psi_2)\bar{\Psi}_2\bar{\sigma} - 38\phi_0\alpha\bar{\delta}(\bar{\phi}_0) - 26\bar{\delta}(\phi_0)\bar{\phi}_0\bar{\beta} + 144\Psi_2\bar{\Psi}_2\bar{\delta}(\bar{\kappa})\\
&-32\phi_0\pi\bar{\delta}(\bar{\phi}_0) - 144\Psi_2\bar{\Psi}_2D(\bar{\sigma}) - 26\phi_0\lambda D(\bar{\phi}_0) + 12\bar{\phi}_0\phi_0\mu\bar{\sigma}\\
&-36\bar{\phi}_0\phi_0\pi\alpha + 12\bar{\phi}_0\phi_0\bar{\kappa}\nu + 12\phi_0\bar{\phi}_0\bar{\gamma}\bar{\sigma} + 12\phi_0\bar{\phi}_0\bar{\mu}\bar{\sigma}\\
&+12\bar{\phi}_0\phi_0\gamma\bar{\sigma} = 0
\end{aligned} \tag{A.7.20}
$$

$$
o_{(ABC^{\iota}D)}\bar{o}_{(\dot{A}\dot{B}\dot{C}}\bar{\iota}_{\dot{D})} \quad
\begin{aligned}
&-720\Psi_2\lambda\bar{\Psi}_2\bar{\lambda} + 1368\Psi_2\pi\bar{\Psi}_2\bar{\nu} + 1368\Psi_2\nu\bar{\Psi}_2\bar{\pi} - 3456\Psi_2\mu\bar{\Psi}_2\bar{\mu}\\
&+648\bar{\Psi}_2\Psi_2\nu\tau - 216\bar{\Psi}_2\Psi_2\mu\gamma - 216\bar{\Psi}_2\Psi_2\mu\bar{\gamma} + 216\bar{\Psi}_2\Psi_2\mu^2\\
&+216\Psi_2\bar{\Psi}_2\bar{\mu}^2 - 144\Psi_2\Phi_{22}\bar{\Psi}_2 + 72\bar{\Psi}_2\Psi_2\delta(\nu) - 72\bar{\Psi}_2\delta(\Psi_2)\nu\\
&-144\bar{\Psi}_2\mu\Delta(\Psi_2) + 72\Psi_2\bar{\Psi}_2\bar{\delta}(\bar{\nu}) - 72\Psi_2\bar{\delta}(\bar{\Psi}_2)\bar{\nu} - 144\Psi_2\bar{\mu}\Delta(\bar{\Psi}_2)\\
&-72\Psi_2\Delta(\bar{\Psi}_2)\bar{\gamma} - 72\Psi_2\Delta(\bar{\Psi}_2)\gamma - 216\Psi_2\bar{\Psi}_2\Delta(\bar{\mu}) - 216\bar{\Psi}_2\Psi_2\Delta(\mu)\\
&-72\bar{\Psi}_2\Delta(\Psi_2)\gamma - 72\bar{\Psi}_2\Delta(\Psi_2)\bar{\gamma} - 432\Delta(\Psi_2)\Delta(\bar{\Psi}_2) - 72\bar{\Psi}_2\Delta(\Delta(\Psi_2))\\
&-72\Psi_2\Delta(\Delta(\bar{\Psi}_2)) + 64\phi_0\nu\bar{\phi}_0\bar{\nu} + 216\bar{\Psi}_2\Psi_2\nu\beta + 72\bar{\Psi}_2\Psi_2\bar{\alpha}\nu\\
&+216\Psi_2\bar{\Psi}_2\bar{\nu}\bar{\beta} + 72\Psi_2\bar{\Psi}_2\alpha\bar{\nu} + 648\Psi_2\bar{\Psi}_2\bar{\nu}\bar{\tau} - 216\Psi_2\bar{\Psi}_2\bar{\mu}\bar{\gamma}\\
&-216\Psi_2\bar{\Psi}_2\bar{\mu}\gamma + 360\Psi_2\nu\delta(\bar{\Psi}_2) - 1152\Psi_2\mu\Delta(\bar{\Psi}_2) + 360\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\nu}\\
&-1152\Delta(\Psi_2)\bar{\Psi}_2\bar{\mu} = 0
\end{aligned} \tag{A.7.21}
$$

$$
\iota_{ABCD}\bar{\iota}_{\dot{A}\dot{B}\dot{C}\dot{D}} \quad
\begin{aligned}
&-6\bar{\phi}_0D(D(\phi_0)) - 6\phi_0D(D(\bar{\phi}_0)) + 16D(\phi_0)D(\bar{\phi}_0) + 12\phi_0\bar{\phi}_0\bar{\pi}\bar{\kappa}\\
&+12\phi_0\bar{\phi}_0\bar{\beta}\kappa + 12\phi_0\bar{\phi}_0\bar{\kappa}\bar{\alpha} + 12\bar{\phi}_0\phi_0\beta\bar{\kappa} + 12\bar{\phi}_0\phi_0\pi\kappa\\
&+12\bar{\phi}_0\phi_0\kappa\alpha + 40\bar{\phi}_0\phi_0\epsilon\bar{\epsilon} - 432\Psi_2\kappa\bar{\Psi}_2\bar{\kappa} - 36\bar{\phi}_0\phi_0\epsilon^2\\
&-36\phi_0\bar{\phi}_0\bar{\epsilon}^2 + 36\phi_0\Phi_{00}\bar{\phi}_0 - 26\phi_0D(\bar{\phi}_0)\epsilon - 6\phi_0\delta(\bar{\phi}_0)\bar{\kappa}\\
&-6\phi_0\bar{\delta}(\bar{\phi}_0)\kappa + 30\phi_0D(\bar{\phi}_0)\bar{\epsilon} + 12\bar{\phi}_0\phi_0D(\epsilon) - 26\bar{\phi}_0D(\phi_0)\bar{\epsilon}\\
&-6\bar{\phi}_0\bar{\delta}(\phi_0)\kappa - 6\bar{\phi}_0\delta(\phi_0)\bar{\kappa} + 30\bar{\phi}_0D(\phi_0)\epsilon + 12\phi_0\bar{\phi}_0D(\bar{\epsilon}) = 0
\end{aligned} \tag{A.7.22}
$$

$$
o_{(A^{\iota}BCD)}\bar{\iota}_{\dot{A}\dot{B}\dot{C}\dot{D}} \quad
\begin{aligned}
&6\phi_0\bar{\delta}(D(\bar{\phi}_0)) + 6\bar{\phi}_0\bar{\delta}(D(\phi_0)) + 6\phi_0D(\bar{\delta}(\bar{\phi}_0)) + 6\bar{\phi}_0D(\bar{\delta}(\phi_0))\\
&+360\Psi_2\kappa\bar{\Psi}_2\bar{\sigma} - 1152\bar{\Psi}_2\bar{\kappa}\Psi_2\rho - 216\Psi_2\bar{\Psi}_2\bar{\rho}\bar{\kappa} - 64\phi_0\epsilon\bar{\phi}_0\bar{\beta}\\
&-40\phi_0\pi\bar{\phi}_0\bar{\epsilon} - 40\phi_0\alpha\bar{\phi}_0\bar{\epsilon} - 72\Psi_2\bar{\Psi}_2\bar{\kappa}\epsilon - 72\phi_0\Phi_{10}\bar{\phi}_0\\
&+432D(\Psi_2)\bar{\Psi}_2\bar{\kappa} + 26D(\phi_0)\bar{\phi}_0\bar{\beta} + 38\phi_0\epsilon\bar{\delta}(\bar{\phi}_0) + 26\phi_0\pi D(\bar{\phi}_0)\\
&+26\bar{\delta}(\phi_0)\bar{\phi}_0\bar{\epsilon} + 26\phi_0\alpha D(\bar{\phi}_0) + 72\Psi_2\bar{\Psi}_2D(\bar{\kappa}) + 144\Psi_2\bar{\kappa}D(\bar{\Psi}_2)\\
&-18\bar{\phi}_0\epsilon\bar{\delta}(\phi_0) - 12\bar{\phi}_0\phi_0D(\alpha) - 30\bar{\phi}_0D(\phi_0)\pi + 6\bar{\phi}_0\Delta(\phi_0)\bar{\kappa}\\
&-30\bar{\phi}_0D(\phi_0)\alpha + 6\phi_0\delta(\bar{\phi}_0)\bar{\sigma} - 30\phi_0D(\bar{\phi}_0)\bar{\beta} - 12\phi_0\bar{\phi}_0\bar{\delta}(\bar{\epsilon})\\
&-30\phi_0\bar{\epsilon}\bar{\delta}(\bar{\phi}_0) + 6\phi_0\bar{\delta}(\bar{\phi}_0)\rho - 12\bar{\phi}_0\phi_0D(\pi) + 6\phi_0\Delta(\bar{\phi}_0)\bar{\kappa}\\
&-12\phi_0\bar{\phi}_0D(\bar{\beta}) + 6\bar{\phi}_0\delta(\phi_0)\bar{\sigma} - 216\Psi_2\bar{\Psi}_2\bar{\epsilon}\bar{\kappa} + 48\bar{\phi}_0\phi_0\epsilon\alpha\\
&-12\bar{\phi}_0\phi_0\gamma\bar{\kappa} - 24\bar{\phi}_0\phi_0\lambda\kappa + 48\bar{\phi}_0\phi_0\pi\epsilon + 72\phi_0\bar{\phi}_0\bar{\epsilon}\bar{\beta}\\
&-12\phi_0\bar{\phi}_0\bar{\sigma}\bar{\alpha} - 12\phi_0\bar{\phi}_0\bar{\beta}\rho - 12\phi_0\bar{\phi}_0\bar{\pi}\bar{\sigma} - 12\bar{\phi}_0\phi_0\bar{\kappa}\mu\\
&-12\phi_0\bar{\phi}_0\bar{\kappa}\bar{\gamma} - 12\phi_0\bar{\phi}_0\bar{\kappa}\bar{\mu} - 12\bar{\phi}_0\phi_0\rho\pi - 12\bar{\phi}_0\phi_0\beta\bar{\sigma}\\
&-12\bar{\phi}_0\phi_0\alpha\rho - 16D(\phi_0)\bar{\delta}(\bar{\phi}_0) - 16\bar{\delta}(\phi_0)D(\bar{\phi}_0) + 6\bar{\phi}_0\bar{\delta}(\phi_0)\rho\\
&-12\bar{\phi}_0\phi_0\bar{\delta}(\epsilon) = 0
\end{aligned} \tag{A.7.23}
$$

$$
\begin{aligned}
o_{(AB^{\iota}CD)}\overline{o}_{(\dot{A}\dot{B}^{\overline{\iota}}\dot{C}\dot{D})}\quad
&-4752\Psi_2\mu\bar{\Psi}_2\bar{\rho}-4752\Psi_2\tau\bar{\Psi}_2\bar{\tau}-4752\Psi_2\pi\bar{\Psi}_2\bar{\pi}-4752\Psi_2\rho\bar{\Psi}_2\bar{\mu}\\
&-864\bar{\Psi}_2\Psi_2\kappa\nu+1728\bar{\Psi}_2\Psi_2\rho\mu-432\bar{\Psi}_2\Psi_2\pi\beta+432\bar{\Psi}_2\Psi_2\bar{\alpha}\pi\\
&+432\bar{\Psi}_2\Psi_2\rho\gamma+864\bar{\Psi}_2\Psi_2\lambda\sigma-36\phi_0\bar{\phi}_0\bar{\mu}\bar{\gamma}-24\phi_0\bar{\phi}_0\bar{\nu}\bar{\beta}\\
&+88\phi_0\gamma\bar{\phi}_0\bar{\gamma}+76\phi_0\gamma\bar{\phi}_0\bar{\mu}+76\phi_0\mu\bar{\phi}_0\bar{\gamma}-432\bar{\Psi}_2\Psi_2\tau\alpha\\
&+432\bar{\Psi}_2\Psi_2\bar{\beta}\tau-36\bar{\phi}_0\phi_0\mu\gamma-24\bar{\phi}_0\phi_0\nu\beta-1728\bar{\Psi}_2\Psi_2\pi\tau\\
&-1728\Psi_2\bar{\Psi}_2\bar{\pi}\bar{\tau}-864\Psi_2\bar{\Psi}_2\bar{\kappa}\bar{\nu}+432\Psi_2\bar{\Psi}_2\bar{\rho}\bar{\gamma}+432\Psi_2\bar{\Psi}_2\gamma\bar{\rho}\\
&-432\Psi_2\bar{\Psi}_2\bar{\tau}\bar{\alpha}+432\Psi_2\bar{\Psi}_2\beta\bar{\tau}+864\Psi_2\bar{\Psi}_2\bar{\lambda}\bar{\sigma}+1728\Psi_2\bar{\Psi}_2\bar{\rho}\bar{\mu}\\
&+432\bar{\Psi}_2\Psi_2\mu\epsilon+432\bar{\Psi}_2\Psi_2\bar{\epsilon}\mu+432\Psi_2\bar{\Psi}_2\bar{\mu}\bar{\epsilon}+432\Psi_2\bar{\Psi}_2\epsilon\bar{\mu}\\
&-432\Psi_2\bar{\Psi}_2\bar{\pi}\bar{\beta}+432\Psi_2\bar{\Psi}_2\alpha\bar{\pi}+64\phi_0\nu\bar{\phi}_0\bar{\alpha}-144\bar{\Psi}_2D(\Psi_2)\gamma\\
&-24\phi_0\bar{\phi}_0\bar{\mu}^2-24\bar{\phi}_0\phi_0\mu^2-12\phi_0\bar{\phi}_0\bar{\gamma}^2-12\bar{\phi}_0\phi_0\gamma^2\\
&+36\phi_0\Phi_{22}\bar{\phi}_0-1152\Lambda\Psi_2\bar{\Psi}_2+144\Psi_2\Delta(\bar{\Psi}_2)\bar{\epsilon}-26\bar{\delta}(\phi_0)\bar{\phi}_0\bar{\nu}\\
&+144\Psi_2\bar{\delta}(\bar{\Psi}_2)\bar{\alpha}-144\Psi_2\delta(\bar{\delta}(\bar{\Psi}_2))-396\bar{\delta}(\Psi_2)\delta(\bar{\Psi}_2)+396D(\Psi_2)\Delta(\bar{\Psi}_2)\\
&+396\Delta(\Psi_2)D(\bar{\Psi}_2)-396\delta(\Psi_2)\bar{\delta}(\bar{\Psi}_2)-144\bar{\Psi}_2\delta(\bar{\delta}(\Psi_2))-144\bar{\Psi}_2\bar{\delta}(\delta(\Psi_2))\\
&+144\Psi_2D(\Delta(\bar{\Psi}_2))-144\Psi_2\bar{\delta}(\delta(\bar{\Psi}_2))+144\bar{\Psi}_2D(\Delta(\Psi_2))+144\bar{\Psi}_2\Delta(D(\Psi_2))\\
&+144\Psi_2\Delta(D(\bar{\Psi}_2))-6\phi_0\Delta(\Delta(\bar{\phi}_0))-6\bar{\phi}_0\Delta(\Delta(\phi_0))+16\Delta(\phi_0)\Delta(\bar{\phi}_0)\\
&+64\phi_0\mu\bar{\phi}_0\bar{\mu}+52\phi_0\nu\bar{\phi}_0\bar{\pi}+52\phi_0\pi\bar{\phi}_0\bar{\nu}+40\phi_0\lambda\bar{\phi}_0\bar{\lambda}\\
&+64\phi_0\alpha\bar{\phi}_0\bar{\nu}+24\bar{\phi}_0\phi_0\nu\tau+24\phi_0\bar{\phi}_0\bar{\nu}\bar{\tau}+432\bar{\Psi}_2\Psi_2\bar{\gamma}\rho\\
&+12\phi_0\bar{\phi}_0\bar{\delta}(\bar{\nu})-144\bar{\Psi}_2D(\Psi_2)\bar{\gamma}+144\bar{\Psi}_2\bar{\delta}(\Psi_2)\bar{\alpha}-144\bar{\Psi}_2\bar{\delta}(\Psi_2)\beta\\
&+144\bar{\Psi}_2D(\Psi_2)\mu-144\bar{\Psi}_2\pi\delta(\Psi_2)-432\bar{\Psi}_2\Psi_2\delta(\pi)-432\bar{\Psi}_2\Psi_2\Delta(\rho)\\
&-144\bar{\Psi}_2\rho\Delta(\Psi_2)+144\bar{\Psi}_2\bar{\delta}(\Psi_2)\tau-144\Psi_2D(\bar{\Psi}_2)\gamma-144\Psi_2D(\bar{\Psi}_2)\bar{\gamma}\\
&+144\Psi_2\delta(\bar{\Psi}_2)\bar{\tau}-432\Psi_2\bar{\Psi}_2\Delta(\bar{\rho})-144\Psi_2\bar{\rho}\Delta(\bar{\Psi}_2)+432\Psi_2\bar{\Psi}_2\delta(\bar{\tau})\\
&-144\Psi_2\bar{\delta}(\bar{\Psi}_2)\beta+432\bar{\Psi}_2\Psi_2D(\mu)+144\bar{\Psi}_2\Delta(\Psi_2)\bar{\epsilon}+144\bar{\Psi}_2\Delta(\Psi_2)\epsilon\\
&+144\Psi_2\Delta(\bar{\Psi}_2)\epsilon-144\Psi_2\bar{\pi}\bar{\delta}(\bar{\Psi}_2)+144\Psi_2D(\bar{\Psi}_2)\bar{\mu}+432\Psi_2\bar{\Psi}_2D(\bar{\mu})\\
&+144\Psi_2\delta(\bar{\Psi}_2)\alpha-144\Psi_2\delta(\bar{\Psi}_2)\bar{\beta}-432\Psi_2\bar{\Psi}_2\bar{\delta}(\bar{\pi})-26\phi_0\nu\delta(\bar{\phi}_0)\\
&+12\bar{\phi}_0\phi_0\delta(\nu)+30\bar{\phi}_0\delta(\phi_0)\nu+24\bar{\phi}_0\mu\Delta(\phi_0)+30\phi_0\bar{\delta}(\bar{\phi}_0)\bar{\nu}\\
&+24\phi_0\bar{\mu}\Delta(\bar{\phi}_0)-38\Delta(\phi_0)\bar{\phi}_0\bar{\gamma}-38\phi_0\gamma\Delta(\bar{\phi}_0)-32\Delta(\phi_0)\bar{\phi}_0\bar{\mu}\\
&-32\phi_0\mu\Delta(\bar{\phi}_0)+144\bar{\Psi}_2\delta(\Psi_2)\alpha-144\bar{\Psi}_2\delta(\Psi_2)\bar{\beta}+432\bar{\Psi}_2\Psi_2\bar{\delta}(\tau)\\
&+18\phi_0\Delta(\bar{\phi}_0)\bar{\gamma}+12\phi_0\bar{\phi}_0\Delta(\bar{\gamma})+12\bar{\phi}_0\phi_0\Delta(\mu)+18\bar{\phi}_0\Delta(\phi_0)\gamma\\
&+12\bar{\phi}_0\phi_0\Delta(\gamma)+12\phi_0\bar{\phi}_0\Delta(\bar{\mu})-1224\Delta(\Psi_2)\bar{\Psi}_2\bar{\rho}-1224\Psi_2\pi\delta(\bar{\Psi}_2)\\
&+1224\Psi_2\tau\bar{\delta}(\bar{\Psi}_2)+1224D(\Psi_2)\bar{\Psi}_2\bar{\mu}-1224\Psi_2\rho\Delta(\bar{\Psi}_2)-1224\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\pi}\\
&+1224\Psi_2\mu D(\bar{\Psi}_2)+1224\delta(\Psi_2)\bar{\Psi}_2\bar{\tau}=0
\end{aligned}
\tag{A.7.24}
$$

$$
\begin{aligned}
o_{(AB^{\iota}CD)}\overline{o}_{(\dot{A}\dot{B}\dot{C}^{\overline{\iota}}\dot{D})}\quad
&-1224\Psi_2\rho\bar{\Psi}_2\bar{\nu}-1224\Psi_2\pi\bar{\Psi}_2\bar{\lambda}+3888\Psi_2\mu\bar{\Psi}_2\bar{\pi}+3888\Psi_2\tau\bar{\Psi}_2\bar{\mu}\\
&+12\phi_0\bar{\phi}_0\bar{\nu}\bar{\gamma}-76\phi_0\bar{\phi}_0\bar{\nu}\gamma-52\phi_0\bar{\phi}_0\bar{\lambda}\nu+432\Psi_2\bar{\Psi}_2\bar{\lambda}\bar{\tau}\\
&-432\Psi_2\bar{\Psi}_2\bar{\pi}\gamma+432\Psi_2\bar{\Psi}_2\bar{\pi}\bar{\gamma}-432\Psi_2\bar{\Psi}_2\bar{\rho}\bar{\nu}-432\Psi_2\bar{\Psi}_2\bar{\mu}\bar{\alpha}\\
&-432\Psi_2\bar{\Psi}_2\bar{\mu}\beta+216\bar{\Psi}_2\Psi_2\mu\beta+216\bar{\Psi}_2\Psi_2\bar{\alpha}\mu-216\bar{\Psi}_2\Psi_2\gamma\tau\\
&-144\Psi_2\Delta(\bar{\Psi}_2)\beta-432\Psi_2\bar{\lambda}\bar{\delta}(\bar{\Psi}_2)-576\Psi_2\delta(\bar{\Psi}_2)\bar{\mu}-432\Psi_2\bar{\Psi}_2\delta(\bar{\mu})\\
&+432\Psi_2D(\bar{\Psi}_2)\bar{\nu}+576\Psi_2\bar{\pi}\Delta(\bar{\Psi}_2)+432\Psi_2\bar{\Psi}_2\Delta(\bar{\pi})-12\phi_0\bar{\phi}_0\Delta(\bar{\nu})\\
&-24\phi_0\Delta(\bar{\phi}_0)\bar{\nu}-360\delta(\Psi_2)\Delta(\bar{\Psi}_2)+360\Delta(\Psi_2)\delta(\bar{\Psi}_2)+144\Psi_2\Delta(\delta(\bar{\Psi}_2))\\
&-144\Psi_2\delta(\Delta(\bar{\Psi}_2))+72\bar{\Psi}_2\delta(\Delta(\Psi_2))-72\bar{\Psi}_2\Delta(\delta(\Psi_2))+32\Delta(\phi_0)\bar{\phi}_0\bar{\nu}\\
&-144\Psi_2\delta(\bar{\Psi}_2)\gamma+144\Psi_2\delta(\bar{\Psi}_2)\bar{\gamma}+1440\Psi_2\tau\Delta(\bar{\Psi}_2)+1440\Psi_2\mu\delta(\bar{\Psi}_2)\\
&-144\Psi_2\Delta(\bar{\Psi}_2)\bar{\alpha}-504\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\lambda}+504D(\Psi_2)\bar{\Psi}_2\bar{\nu}+1152\Delta(\Psi_2)\bar{\Psi}_2\bar{\pi}\\
&-1152\delta(\Psi_2)\bar{\Psi}_2\bar{\mu}+360\bar{\Psi}_2\mu\delta(\Psi_2)+216\bar{\Psi}_2\Psi_2\delta(\mu)+72\bar{\Psi}_2\Delta(\Psi_2)\bar{\alpha}\\
&+72\bar{\Psi}_2\Delta(\Psi_2)\beta+360\bar{\Psi}_2\Delta(\Psi_2)\tau+72\bar{\Psi}_2\delta(\Psi_2)\gamma-72\bar{\Psi}_2\delta(\Psi_2)\bar{\gamma}\\
&+216\bar{\Psi}_2\Psi_2\Delta(\tau)+432\Psi_2\bar{\Psi}_2\bar{\nu}\bar{\epsilon}+144\Psi_2\bar{\Psi}_2\epsilon\bar{\nu}-432\Psi_2\bar{\Psi}_2\bar{\lambda}\bar{\beta}\\
&+144\Psi_2\bar{\Psi}_2\alpha\bar{\lambda}-64\phi_0\mu\bar{\phi}_0\bar{\nu}+216\bar{\Psi}_2\Psi_2\bar{\gamma}\tau+36\phi_0\bar{\phi}_0\bar{\mu}\bar{\nu}\\
&+144\Psi_2\bar{\Psi}_2D(\bar{\nu})-144\Psi_2\bar{\Psi}_2\bar{\delta}(\bar{\lambda})=0
\end{aligned}
\tag{A.7.25}
$$

A.8 The $o_{(AB}\iota_{CD)}\overline{o}_{(\dot{A}\dot{B}}\overline{\iota}_{\dot{C}\dot{D})}$ component equation of the 4-index Condition

The following is the $o_{(AB}\iota_{CD)}\overline{o}_{(\dot{A}\dot{B}}\overline{\iota}_{\dot{C}\dot{D})}$ component equation of the 4-index Condition. It is used Chapter 5.

$$
\begin{aligned}
&96\phi_1\bar\phi_1\bar\rho\bar\mu - 24\phi_1\bar\phi_2\beta\bar\sigma - 72\phi_1\bar\phi_2\bar\rho\bar\beta + 24\phi_0\bar\phi_0\bar\nu\bar\tau - 36\phi_2\bar\phi_2\bar\rho\bar\epsilon + 48\phi_2\bar\phi_1\bar\rho\bar\pi \\
&-36\phi_2\bar\phi_0\bar\pi\bar\alpha + 96\bar\phi_1\phi_1\rho\mu - 24\bar\phi_1\phi_0\bar\alpha\lambda - 48\bar\phi_1\phi_0\lambda\beta - 72\bar\phi_1\phi_0\mu\alpha + 24\bar\phi_1\phi_2\rho\bar\alpha \\
&-72\bar\phi_1\phi_2\rho\beta + 24\phi_1\bar\phi_1\bar\tau\bar\alpha - 24\phi_1\bar\phi_1\beta\bar\tau + 24\phi_1\bar\phi_0\beta\bar\gamma - 96\bar\phi_1\phi_2\rho\tau + 48\bar\phi_1\phi_0\lambda\tau \\
&-48\bar\phi_1\phi_0\alpha\gamma - 48\bar\phi_1\phi_0\nu\epsilon - 24\bar\phi_1\phi_0\bar\gamma\alpha + 432\bar\Psi_2\Psi_2\bar\gamma\rho - 12\phi_0\bar\phi_2\bar\beta^2 - 48\phi_1\bar\phi_0\bar\nu\bar\epsilon \\
&-24\phi_1\bar\phi_0\bar\alpha\gamma + 24\phi_1\bar\phi_1\gamma\bar\rho + 24\phi_1\bar\phi_1\bar\rho\bar\gamma + 144\phi_1\bar\phi_1\bar\kappa\bar\nu + 24\phi_1\bar\phi_1\bar\mu\bar\epsilon + 24\phi_1\bar\phi_1\epsilon\bar\mu \\
&+48\phi_1\bar\phi_2\bar\pi\bar\sigma - 48\phi_1\bar\phi_2\bar\kappa\bar\gamma - 24\phi_1\bar\phi_2\epsilon\bar\beta - 36\bar\phi_0\phi_2\tau\beta + 24\bar\phi_0\phi_2\mu\sigma - 24\bar\phi_0\phi_2\gamma\sigma \\
&-36\phi_0\bar\phi_0\bar\mu\bar\gamma - 24\phi_0\bar\phi_0\bar\nu\bar\beta + 48\phi_0\bar\phi_1\bar\tau\bar\mu + 72\phi_0\bar\phi_1\bar\nu\bar\sigma - 36\phi_0\bar\phi_2\bar\tau\bar\beta + 24\phi_0\bar\phi_2\bar\mu\bar\sigma \\
&-24\phi_0\bar\phi_2\bar\gamma\bar\sigma + 12\phi_2\bar\phi_1\bar\pi\bar\epsilon + 72\phi_2\bar\phi_1\bar\lambda\bar\kappa - 24\phi_2\bar\phi_2\bar\kappa\bar\alpha + 24\phi_2\bar\phi_2\bar\pi\bar\kappa + 12\bar\phi_2\phi_1\pi\epsilon \\
&+72\bar\phi_2\phi_1\lambda\kappa - 24\bar\phi_2\phi_2\kappa\alpha + 24\bar\phi_2\phi_2\pi\kappa + 52\phi_1\rho\bar\phi_2\bar\beta - 1008\phi_1^2\bar\Psi_2 - 1008\bar\phi_1^2\Psi_2 \\
&+24\bar\phi_1\phi_1\pi\beta - 24\bar\phi_1\phi_1\bar\alpha\pi - 48\phi_1\bar\phi_0\bar\lambda\bar\beta + 48\phi_1\bar\phi_0\bar\rho\bar\nu - 72\phi_1\bar\phi_0\bar\mu\bar\alpha + 48\bar\phi_2\phi_1\rho\pi \\
&-24\bar\phi_2\phi_0\lambda\epsilon - 36\bar\phi_2\phi_0\pi\alpha + 12\phi_2\bar\phi_1\bar\alpha\bar\rho + 24\phi_2\bar\phi_0\bar\lambda\bar\rho - 24\phi_2\bar\phi_0\bar\lambda\bar\epsilon - 96\phi_1\bar\phi_2\bar\rho\bar\tau \\
&+48\phi_1\bar\phi_0\bar\lambda\bar\tau - 48\phi_1\bar\phi_0\bar\alpha\bar\gamma + 144\bar\phi_1\phi_1\lambda\sigma + 12\bar\phi_1\bar\delta(\phi_0)\bar\gamma - 144\bar\Psi_2 D(\Psi_2)\gamma + 12\phi_2\bar\phi_0\delta(\bar\pi) \\
&+12\phi_2\bar\phi_1 D(\bar\pi) - 24\phi_0\bar\phi_0\bar\mu^2 - 24\bar\phi_0\phi_0\mu^2 - 12\phi_2\bar\phi_2\bar\epsilon^2 - 24\phi_2\bar\phi_0\bar\pi^2 - 12\bar\phi_2\phi_2\epsilon^2 \\
&-24\bar\phi_2\phi_0\pi^2 - 24\phi_2\bar\phi_2\bar\rho^2 - 12\phi_2\bar\phi_0\bar\alpha^2 - 12\bar\phi_0\phi_2\beta^2 - 4752\Psi_2\mu\bar\Psi_2\bar\rho - 4752\Psi_2\tau\bar\Psi_2\bar\tau \\
&-4752\Psi_2\pi\bar\Psi_2\bar\pi - 4752\Psi_2\rho\bar\Psi_2\bar\mu - 864\bar\Psi_2\Psi_2\kappa\nu - 36\bar\phi_2\phi_2\rho\epsilon + 1728\bar\Psi_2\Psi_2\rho\mu - 432\bar\Psi_2\Psi_2\pi\beta \\
&+432\bar\Psi_2\Psi_2\bar\alpha\pi + 432\bar\Psi_2\Psi_2\rho\gamma + 864\bar\Psi_2\Psi_2\lambda\sigma + 24\bar\phi_1\phi_2\epsilon\bar\alpha - 48\bar\phi_1\phi_2\epsilon\beta - 48\bar\phi_1\phi_2\sigma\alpha \\
&+48\bar\phi_1\phi_2\kappa\mu - 96\bar\phi_1\phi_0\pi\mu - 24\phi_1\bar\phi_2 D(\bar\beta) - 24\bar\phi_1\phi_2\delta(\epsilon) + 12\bar\phi_1 D(\phi_2)\bar\alpha - 36\bar\phi_1 D(\phi_2)\beta \\
&-36\bar\phi_1\epsilon\delta(\phi_2) + 72\bar\phi_1 D(\phi_1)\mu + 72\bar\phi_1\delta(\phi_1)\pi + 24\bar\phi_1\phi_1\delta(\pi) - 60\bar\phi_1\sigma\bar\delta(\phi_2) + 24\phi_1\bar\phi_0\delta(\bar\mu) \\
&+24\phi_1\bar\delta(\bar\phi_1)\bar\alpha - 24\phi_1\bar\delta(\bar\phi_1)\beta + 48\phi_1\bar\mu\delta(\bar\phi_0) - 48\phi_1\bar\delta(\bar\phi_2)\bar\rho + 60\phi_1\bar\delta(\bar\phi_0)\bar\lambda - 24\phi_1\bar\phi_2\delta(\bar\sigma) \\
&+12\phi_0\bar\delta(\bar\phi_1)\bar\gamma - 20\bar\delta(\phi_0)\bar\phi_1\bar\mu + 40\phi_1\rho\Delta(\bar\phi_1) - 76\phi_0\alpha\Delta(\bar\phi_1) + 32\phi_2\rho D(\bar\phi_2) - 26\phi_0\lambda D(\bar\phi_2) \\
&-26D(\phi_2)\bar\phi_0\bar\lambda + 32D(\phi_2)\bar\phi_2\bar\rho + 76\phi_2\epsilon\delta(\bar\phi_1) - 40\phi_1\pi\delta(\bar\phi_1) + 38\bar\delta(\phi_0)\bar\phi_2\bar\beta + 20\phi_1\rho\bar\delta(\bar\phi_2) \\
&-38\phi_0\alpha\bar\delta(\bar\phi_2) - 38\Delta(\phi_0)\bar\phi_0\bar\gamma + 20\Delta(\phi_0)\bar\phi_1\bar\tau - 38\phi_0\gamma\Delta(\bar\phi_0) - 144\Psi_2\delta(\bar\delta(\bar\Psi_2)) - 396\bar\delta(\Psi_2)\delta(\bar\Psi_2) \\
&+396D(\Psi_2)\Delta(\bar\Psi_2) + 396\Delta(\Psi_2)D(\bar\Psi_2) - 396\delta(\Psi_2)\bar\delta(\bar\Psi_2) - 12\phi_1\delta(\Delta(\bar\phi_0)) - 144\bar\Psi_2\delta(\bar\delta(\Psi_2)) - 6\phi_2\delta(\delta(\bar\phi_0)) \\
&-12\phi_1\Delta(\delta(\bar\phi_0)) - 12\phi_1 D(\bar\delta(\bar\phi_2)) - 12\bar\phi_1\delta(D(\phi_2)) - 24\phi_1\delta(\bar\delta(\bar\phi_1)) - 12\phi_0\Delta(\bar\delta(\bar\phi_1)) - 12\phi_2\delta(D(\bar\phi_1)) \\
&-24\bar\phi_1\delta(\bar\delta(\phi_1)) - 12\phi_0\bar\delta(\Delta(\bar\phi_1)) - 6\phi_2 D(D(\bar\phi_2)) - 12\bar\phi_2 D(\bar\delta(\phi_1)) + 16D(\phi_2)D(\bar\phi_2) + 32D(\phi_1)\bar\delta(\bar\phi_2) \\
&+32\delta(\phi_2)D(\bar\phi_1) + 32\Delta(\phi_0)\bar\delta(\bar\phi_1) + 32\delta(\phi_1)\Delta(\bar\phi_0) - 144\bar\Psi_2\bar\delta(\delta(\Psi_2)) - 12\phi_1\bar\delta(D(\bar\phi_2)) - 12\bar\phi_0\delta(\Delta(\phi_1)) \\
&+32\Delta(\phi_1)\delta(\bar\phi_0) + 64\delta(\phi_1)\bar\delta(\bar\phi_1) + 64\Delta(\phi_1)D(\bar\phi_1) + 16\delta(\phi_2)\delta(\bar\phi_0) + 64D(\phi_1)\Delta(\bar\phi_1) + 64\bar\delta(\phi_1)\delta(\bar\phi_1) \\
&+32\bar\delta(\phi_0)\Delta(\bar\phi_1) + 32\bar\delta(\phi_1)D(\bar\phi_2) + 32D(\phi_2)\delta(\bar\phi_1) + 16\bar\delta(\phi_0)\bar\delta(\bar\phi_2) + 144\Psi_2 D(\Delta(\bar\Psi_2)) - 144\Psi_2\bar\delta(\delta(\bar\Psi_2)) \\
&+144\bar\Psi_2 D(\Delta(\Psi_2)) + 144\bar\Psi_2\Delta(D(\Psi_2)) + 144\Psi_2\Delta(D(\bar\Psi_2)) - 6\bar\phi_0\delta(\delta(\phi_2)) - 24\phi_1 D(\Delta(\bar\phi_1)) - 12\bar\phi_1\bar\delta(\Delta(\phi_0)) \\
&-24\bar\phi_1\Delta(D(\phi_1)) - 24\phi_1\bar\delta(\delta(\bar\phi_1)) - 6\bar\phi_2\bar\delta(\bar\delta(\phi_0)) - 24\bar\phi_1 D(\Delta(\phi_1)) - 24\bar\phi_1\bar\delta(\delta(\phi_1)) - 24\phi_1\Delta(D(\bar\phi_1)) \\
&-6\phi_0\Delta(\Delta(\bar\phi_0)) - 12\bar\phi_0\Delta(\delta(\phi_1)) - 6\bar\phi_0\Delta(\Delta(\phi_0)) - 6\phi_0\bar\delta(\bar\delta(\bar\phi_2)) - 12\phi_2 D(\delta(\bar\phi_1)) - 12\bar\phi_1 D(\delta(\phi_2)) \\
&-12\bar\phi_2\bar\delta(D(\phi_1)) - 12\bar\phi_1\Delta(\bar\delta(\phi_0)) + 52\phi_0\alpha\bar\phi_1\bar\mu - 88\phi_0\alpha\bar\phi_2\bar\beta - 52\phi_1\tau\bar\phi_0\bar\gamma + 16\phi_1\tau\bar\phi_1\bar\tau \\
&+88\phi_0\gamma\bar\phi_0\bar\gamma - 52\phi_0\gamma\bar\phi_1\bar\tau - 52\phi_2\epsilon\bar\phi_1\bar\pi + 88\phi_2\epsilon\bar\phi_2\bar\epsilon + 16\phi_1\pi\bar\phi_1\bar\pi - 52\phi_1\pi\bar\phi_2\bar\epsilon \\
&-28\phi_2\kappa\bar\phi_1\bar\mu + 64\phi_2\kappa\bar\phi_2\bar\beta + 40\phi_0\pi\bar\phi_1\bar\mu - 76\phi_0\pi\bar\phi_2\bar\beta + 40\phi_1\mu\bar\phi_0\bar\pi - 28\phi_1\mu\bar\phi_2\bar\kappa \\
&-76\phi_2\beta\bar\phi_0\bar\pi + 64\phi_2\beta\bar\phi_2\bar\kappa - 40\phi_1\tau\bar\phi_0\bar\mu + 28\phi_1\tau\bar\phi_2\bar\sigma + 76\phi_0\gamma\bar\phi_0\bar\mu - 64\phi_0\gamma\bar\phi_2\bar\sigma \\
&-64\phi_2\sigma\bar\phi_0\bar\gamma + 28\phi_2\sigma\bar\phi_1\bar\tau + 76\phi_0\mu\bar\phi_0\bar\gamma - 40\phi_0\mu\bar\phi_1\bar\tau - 432\bar\Psi_2\Psi_2\tau\alpha + 432\bar\Psi_2\Psi_2\bar\beta\tau \\
&+24\phi_1\bar\phi_2\alpha\bar\epsilon - 48\phi_1\bar\phi_2\bar\epsilon\bar\beta - 48\phi_1\bar\phi_2\bar\sigma\bar\alpha + 48\phi_1\bar\phi_2\bar\kappa\bar\mu - 96\phi_1\bar\phi_0\bar\pi\bar\mu + 24\phi_1\bar\phi_1\bar\pi\bar\beta \\
&+144\phi_1\bar\phi_1\bar\lambda\bar\sigma - 24\phi_1\bar\phi_1\alpha\bar\pi - 36\bar\phi_0\phi_0\mu\gamma - 24\bar\phi_0\phi_0\nu\beta + 48\bar\phi_0\phi_1\tau\mu + 72\bar\phi_0\phi_1\nu\sigma \\
&-1728\bar\Psi_2\Psi_2\pi\tau - 1728\Psi_2\bar\Psi_2\bar\pi\bar\tau - 864\Psi_2\bar\Psi_2\bar\kappa\bar\nu + 432\Psi_2\bar\Psi_2\bar\rho\bar\gamma + 432\Psi_2\bar\Psi_2\gamma\bar\rho - 432\Psi_2\bar\Psi_2\bar\tau\bar\alpha \\
&+432\Psi_2\bar\Psi_2\beta\bar\tau + 864\Psi_2\bar\Psi_2\bar\lambda\bar\sigma + 1728\Psi_2\bar\Psi_2\bar\rho\bar\mu + 432\bar\Psi_2\Psi_2\mu\epsilon + 432\bar\Psi_2\Psi_2\bar\epsilon\mu + 432\Psi_2\bar\Psi_2\bar\mu\bar\epsilon \\
&+432\Psi_2\bar\Psi_2\epsilon\bar\mu - 432\Psi_2\bar\Psi_2\bar\pi\bar\beta + 432\Psi_2\bar\Psi_2\alpha\bar\pi - 76\phi_2\tau\bar\phi_0\bar\alpha + 40\phi_2\tau\bar\phi_1\bar\rho + 64\phi_0\nu\bar\phi_0\bar\alpha \\
&-28\phi_0\nu\bar\phi_1\bar\rho - 52\phi_2\sigma\bar\phi_0\bar\mu + 40\phi_2\sigma\bar\phi_2\bar\sigma + 64\phi_0\mu\bar\phi_0\bar\mu - 52\phi_0\mu\bar\phi_2\bar\sigma - 64\phi_2\tau\bar\phi_0\bar\pi \\
&+52\phi_2\tau\bar\phi_2\bar\kappa + 52\phi_0\nu\bar\phi_0\bar\pi - 40\phi_0\nu\bar\phi_2\bar\kappa + 52\phi_1\mu\bar\phi_0\bar\alpha - 16\phi_1\mu\bar\phi_1\bar\rho - 88\phi_2\beta\bar\phi_0\bar\alpha \\
&+52\phi_2\beta\bar\phi_1\bar\rho - 40\phi_2\kappa\bar\phi_0\bar\nu + 52\phi_2\kappa\bar\phi_2\bar\tau + 52\phi_0\pi\bar\phi_0\bar\nu - 64\phi_0\pi\bar\phi_2\bar\tau - 52\phi_2\rho\bar\phi_0\bar\lambda
\end{aligned}
$$

$$
\begin{aligned}
&+64\phi_2\rho\bar\phi_2\bar\rho + 40\phi_0\lambda\bar\phi_0\bar\lambda - 52\phi_0\lambda\bar\phi_2\bar\rho - 28\phi_1\rho\bar\phi_0\bar\nu + 40\phi_1\rho\bar\phi_2\bar\tau + 64\phi_0\alpha\bar\phi_0\bar\nu \\
&-76\phi_0\alpha\bar\phi_2\bar\tau - 40\phi_2\rho\bar\phi_1\bar\pi + 76\phi_2\rho\bar\phi_2\bar\epsilon + 28\phi_0\lambda\bar\phi_1\bar\pi - 64\phi_0\lambda\bar\phi_2\bar\epsilon - 64\phi_2\epsilon\bar\phi_0\bar\lambda \\
&+76\phi_2\epsilon\bar\phi_2\bar\rho + 28\phi_1\pi\bar\phi_0\bar\lambda - 40\phi_1\pi\bar\phi_2\bar\rho - 16\phi_1\rho\bar\phi_1\bar\mu + 24\phi_1\bar\phi_0\epsilon\bar\nu - 24\phi_1\bar\phi_2\bar\tau\epsilon \\
&+24\bar\phi_1\phi_1\tau\alpha - 24\bar\phi_1\phi_1\bar\beta\tau + 24\bar\phi_1\phi_0\bar\beta\gamma - 24\bar\phi_1\phi_0\pi\bar\gamma + 24\bar\phi_1\phi_2\bar\gamma\kappa + 24\bar\phi_1\phi_0\bar\epsilon\nu \\
&+96\bar\phi_1\phi_1\pi\tau - 24\bar\phi_1\phi_2\tau\bar\epsilon - 72\bar\phi_1\phi_2\tau\epsilon + 48\bar\phi_1\phi_0\rho\nu + 24\bar\phi_1\phi_0\mu\bar\beta - 24\bar\phi_1\phi_2\bar\beta\sigma \\
&+96\phi_1\bar\phi_1\bar\pi\bar\tau - 24\phi_1\bar\phi_0\bar\pi\gamma - 72\phi_1\bar\phi_0\bar\pi\bar\gamma - 72\phi_1\bar\phi_2\bar\tau\bar\epsilon + 24\phi_1\bar\phi_2\gamma\bar\kappa + 12\bar\phi_0\phi_1\mu\beta \\
&+24\bar\phi_0\phi_0\nu\tau + 12\bar\phi_0\phi_1\gamma\tau - 24\phi_1\bar\phi_0\alpha\bar\lambda + 24\phi_1\bar\phi_2\bar\rho\alpha + 12\bar\phi_2\phi_1\alpha\rho - 72\bar\phi_1\phi_0\pi\gamma \\
&+48\bar\phi_1\phi_2\pi\sigma - 48\bar\phi_1\phi_2\kappa\gamma - 24\bar\phi_1\phi_2\bar\epsilon\beta + 24\bar\phi_2\phi_0\lambda\rho + 12\phi_0\bar\phi_1\bar\gamma\bar\tau + 24\bar\phi_1\phi_1\rho\gamma \\
&+24\bar\phi_1\phi_1\bar\gamma\rho + 144\bar\phi_1\phi_1\kappa\nu + 12\phi_0\bar\phi_1\bar\mu\bar\beta + 24\bar\phi_1\phi_1\mu\epsilon + 24\bar\phi_1\phi_1\bar\epsilon\mu + 24\phi_1\bar\phi_0\bar\mu\beta \\
&-24\bar\phi_2\phi_2\rho^2 - 12\phi_0\bar\phi_0\bar\gamma^2 - 24\bar\phi_0\phi_2\tau^2 - 12\bar\phi_0\phi_0\gamma^2 - 12\bar\phi_2\phi_0\alpha^2 - 24\phi_0\bar\phi_2\bar\tau^2 \\
&-504\bar\phi_0\bar\phi_2\Psi_2 - 504\phi_0\phi_2\bar\Psi_2 + 144\phi_2\Phi_{01}\bar\phi_1 + 36\phi_2\Phi_{02}\bar\phi_0 + 144\phi_1\Phi_{10}\bar\phi_2 + 36\phi_0\Phi_{20}\bar\phi_2 \\
&+36\phi_2\Phi_{00}\bar\phi_2 + 36\phi_0\Phi_{22}\bar\phi_0 + 576\phi_1\Phi_{11}\bar\phi_1 + 144\phi_0\Phi_{21}\bar\phi_1 + 144\phi_1\Phi_{12}\bar\phi_0 - 1152\Lambda\Psi_2\bar\Psi_2 \\
&-12\bar\phi_0\phi_2\delta(\beta) + 144\Psi_2\Delta(\bar\Psi_2)\bar\epsilon - 52\phi_0\lambda\delta(\bar\phi_1) - 26\bar\delta(\phi_0)\bar\phi_0\bar\nu + 64\bar\delta(\phi_1)\bar\phi_2\bar\rho + 144\Psi_2\bar\delta(\bar\Psi_2)\bar\alpha \\
&-40\bar\delta(\phi_1)\bar\phi_1\bar\pi + 76\bar\delta(\phi_1)\bar\phi_2\bar\epsilon + 52\phi_2\kappa\Delta(\bar\phi_1) - 36\phi_1\bar\epsilon\bar\delta(\bar\phi_2) + 64\phi_2\rho\delta(\bar\phi_1) - 12\phi_2\bar\phi_1\delta(\bar\rho) \\
&+72\phi_1 D(\bar\phi_1)\bar\mu + 72\phi_1\bar\delta(\bar\phi_1)\bar\pi + 24\phi_1\bar\phi_1\bar\delta(\bar\pi) - 60\phi_1\bar\sigma\delta(\bar\phi_2) + 32\bar\delta(\phi_0)\bar\phi_2\bar\tau + 12\phi_0\bar\phi_0\bar\delta(\bar\nu) \\
&-48\phi_1 D(\bar\phi_2)\bar\tau + 24\phi_1\bar\phi_0\Delta(\bar\pi) + 48\phi_1\bar\pi\Delta(\bar\phi_0) - 144\bar\Psi_2 D(\Psi_2)\bar\gamma - 64\phi_0\pi\Delta(\bar\phi_1) - 52\bar\delta(\phi_1)\bar\phi_0\bar\lambda \\
&+144\bar\Psi_2\bar\delta(\Psi_2)\bar\alpha - 144\bar\Psi_2\bar\delta(\Psi_2)\beta + 144\bar\Psi_2 D(\Psi_2)\mu - 144\bar\Psi_2\pi\delta(\Psi_2) - 432\bar\Psi_2\Psi_2\delta(\pi) - 432\bar\Psi_2\Psi_2\Delta(\rho) \\
&-144\bar\Psi_2\rho\Delta(\Psi_2) + 144\bar\Psi_2\bar\delta(\Psi_2)\tau - 144\Psi_2 D(\bar\Psi_2)\gamma - 144\Psi_2 D(\bar\Psi_2)\bar\gamma + 144\Psi_2\delta(\bar\Psi_2)\bar\tau - 432\Psi_2\bar\Psi_2\Delta(\bar\rho) \\
&-144\Psi_2\bar\rho\Delta(\bar\Psi_2) + 432\Psi_2\bar\Psi_2\delta(\bar\tau) - 144\Psi_2\bar\delta(\bar\Psi_2)\beta + 432\bar\Psi_2\Psi_2 D(\mu) + 144\bar\Psi_2\Delta(\Psi_2)\bar\epsilon + 144\bar\Psi_2\Delta(\Psi_2)\epsilon \\
&+144\Psi_2\Delta(\bar\Psi_2)\epsilon - 144\Psi_2\pi\bar\delta(\bar\Psi_2) + 144\Psi_2 D(\bar\Psi_2)\bar\mu + 432\Psi_2\bar\Psi_2 D(\bar\mu) + 144\Psi_2\delta(\bar\Psi_2)\alpha - 144\Psi_2\delta(\bar\Psi_2)\bar\beta \\
&-432\Psi_2\bar\Psi_2\bar\delta(\bar\pi) - 76\Delta(\phi_1)\bar\phi_0\bar\alpha + 40\Delta(\phi_1)\bar\phi_1\bar\rho + 32\phi_2\tau\delta(\bar\phi_0) - 26\phi_0\nu\delta(\bar\phi_0) - 64\delta(\phi_1)\bar\phi_0\bar\mu \\
&+52\delta(\phi_1)\bar\phi_2\bar\sigma + 52\phi_2\sigma\bar\delta(\bar\phi_1) - 64\phi_0\mu\bar\delta(\bar\phi_1) - 64\Delta(\phi_1)\bar\phi_0\bar\pi + 52\Delta(\phi_1)\bar\phi_2\bar\kappa + 64\phi_2\tau D(\bar\phi_1) \\
&-52\phi_0\nu D(\bar\phi_1) - 38\delta(\phi_2)\bar\phi_0\bar\alpha + 20\delta(\phi_2)\bar\phi_1\bar\rho - 20\phi_1\mu\delta(\bar\phi_0) + 38\phi_2\beta\delta(\bar\phi_0) - 52D(\phi_1)\bar\phi_0\bar\nu \\
&+64D(\phi_1)\bar\phi_2\bar\tau + 12\bar\phi_0\phi_0\delta(\nu) - 30\bar\phi_0\sigma\Delta(\phi_2) + 30\bar\phi_0\delta(\phi_0)\nu + 24\bar\phi_0\mu\Delta(\phi_0) - 24\bar\phi_0\delta(\phi_2)\tau \\
&-12\bar\phi_0\phi_2\delta(\tau) + 36\bar\phi_0\delta(\phi_1)\mu - 12\bar\phi_0\Delta(\phi_1)\beta - 30\phi_0\bar\sigma\Delta(\bar\phi_2) + 30\phi_0\bar\delta(\bar\phi_0)\bar\nu + 24\phi_0\bar\mu\Delta(\bar\phi_0) \\
&-24\phi_0\bar\delta(\bar\phi_2)\bar\tau - 12\phi_0\bar\phi_2\bar\delta(\bar\tau) + 36\phi_0\bar\delta(\bar\phi_1)\bar\mu - 12\phi_0\Delta(\bar\phi_1)\bar\beta - 30\phi_2\bar\kappa\delta(\bar\phi_2) - 18\phi_2 D(\bar\phi_2)\bar\epsilon \\
&-12\phi_2\bar\phi_2 D(\bar\epsilon) + 36\phi_2 D(\bar\phi_1)\bar\pi + 12\bar\phi_2\phi_1 D(\pi) - 30\bar\phi_2\kappa\bar\delta(\phi_2) - 18\bar\phi_2 D(\phi_2)\epsilon - 12\bar\phi_2\phi_2 D(\epsilon) \\
&+36\bar\phi_2 D(\phi_1)\pi + 30\bar\phi_2 D(\phi_0)\lambda - 12\bar\phi_2\phi_2 D(\rho) - 12\bar\phi_2\bar\delta(\phi_1)\epsilon + 24\bar\phi_2\pi\bar\delta(\phi_0) - 24\bar\phi_2 D(\phi_2)\rho \\
&+12\bar\phi_2\phi_0 D(\lambda) + 18\phi_2\delta(\bar\phi_0)\bar\alpha + 30\phi_2 D(\bar\phi_0)\bar\lambda - 36\phi_2\bar\rho\delta(\bar\phi_1) + 12\phi_2\bar\phi_0\delta(\bar\alpha) + 60\phi_1 D(\bar\phi_0)\bar\nu \\
&+36\phi_1\delta(\bar\phi_0)\bar\gamma - 72\phi_1\bar\rho\Delta(\bar\phi_1) - 72\phi_1\delta(\bar\phi_1)\bar\tau + 24\phi_1\bar\phi_0\Delta(\bar\alpha) - 24\phi_1\bar\phi_1\Delta(\bar\rho) + 12\phi_1\delta(\bar\phi_0)\gamma \\
&+36\phi_1\bar\alpha\Delta(\bar\phi_0) - 12\phi_1\bar\delta(\bar\phi_2)\epsilon - 60\phi_1\bar\kappa\Delta(\bar\phi_2) + 24\phi_1\bar\phi_1 D(\bar\mu) + 16\Delta(\phi_0)\Delta(\bar\phi_0) - 6\bar\phi_2 D(D(\phi_2)) \\
&-20D(\phi_2)\bar\phi_1\bar\pi + 38D(\phi_2)\bar\phi_2\bar\epsilon + 38\phi_2\epsilon D(\bar\phi_2) - 20\phi_1\pi D(\bar\phi_2) - 40D(\phi_1)\bar\phi_1\bar\mu + 76D(\phi_1)\bar\phi_2\bar\beta \\
&+26\phi_2\kappa\bar\delta(\bar\phi_2) - 32\phi_0\pi\bar\delta(\bar\phi_2) - 32\delta(\phi_2)\bar\phi_0\bar\pi + 26\delta(\phi_2)\bar\phi_2\bar\kappa - 40\phi_1\mu D(\bar\phi_1) + 76\phi_2\beta D(\bar\phi_1) \\
&-32\Delta(\phi_0)\bar\phi_0\bar\mu + 26\Delta(\phi_0)\bar\phi_2\bar\sigma + 40\phi_1\tau\bar\delta(\bar\phi_1) - 76\phi_0\gamma\bar\delta(\bar\phi_1) - 76\delta(\phi_1)\bar\phi_0\bar\gamma + 40\delta(\phi_1)\bar\phi_1\bar\tau \\
&+26\phi_2\sigma\Delta(\bar\phi_0) - 32\phi_0\mu\Delta(\bar\phi_0) + 144\bar\Psi_2\delta(\Psi_2)\alpha - 144\bar\Psi_2\delta(\Psi_2)\bar\beta + 432\bar\Psi_2\Psi_2\bar\delta(\tau) - 24\phi_1\bar\phi_2\bar\delta(\bar\epsilon) \\
&+12\phi_1 D(\bar\phi_2)\alpha - 36\phi_1 D(\bar\phi_2)\bar\beta - 24\phi_1\bar\phi_2\Delta(\bar\kappa) - 18\bar\phi_0\delta(\phi_2)\beta + 12\bar\phi_0\phi_1\delta(\mu) - 24\phi_1\Delta(\bar\phi_1)\bar\epsilon \\
&-24\phi_1\Delta(\bar\phi_1)\epsilon + 24\phi_1\bar\phi_0 D(\bar\nu) - 24\phi_1\bar\phi_2 D(\bar\tau) - 24\bar\phi_1\phi_1\bar\delta(\tau) - 12\bar\phi_1\Delta(\phi_0)\bar\beta + 24\bar\phi_1\phi_0\bar\delta(\gamma) \\
&+24\bar\phi_1 D(\phi_1)\gamma + 24\bar\phi_1 D(\phi_1)\bar\gamma - 12\bar\phi_2\phi_1\bar\delta(\rho) + 36\bar\phi_1\alpha\Delta(\phi_0) - 18\phi_0\bar\delta(\bar\phi_2)\bar\beta + 12\phi_0\bar\phi_1\bar\delta(\bar\mu) \\
&-12\phi_0\bar\phi_2\bar\delta(\bar\beta) - 12\phi_2\bar\phi_2 D(\bar\rho) - 12\phi_2\delta(\bar\phi_1)\bar\epsilon + 12\phi_2\bar\phi_0 D(\bar\lambda) - 12\bar\phi_1\delta(\phi_2)\bar\epsilon - 60\bar\phi_1\kappa\Delta(\phi_2) \\
&+24\bar\phi_1\phi_1 D(\mu) - 24\bar\phi_1\phi_2 D(\beta) - 12\bar\phi_2\phi_2\bar\delta(\kappa) + 12\bar\phi_2 D(\phi_1)\alpha + 12\bar\phi_2\phi_0\bar\delta(\pi) - 36\bar\phi_2\rho\bar\delta(\phi_1) \\
&+18\phi_0\Delta(\bar\phi_0)\bar\gamma + 12\phi_0\bar\phi_0\Delta(\bar\gamma) - 12\phi_0\bar\phi_1\Delta(\bar\tau) + 12\bar\phi_0\delta(\phi_1)\gamma - 12\bar\phi_0\phi_2\Delta(\sigma) - 36\bar\phi_0\Delta(\phi_1)\tau \\
&+12\bar\phi_0\phi_0\Delta(\mu) + 18\bar\phi_0\Delta(\phi_0)\gamma + 12\bar\phi_0\phi_0\Delta(\gamma) - 12\bar\phi_0\phi_1\Delta(\tau) + 24\phi_1\delta(\bar\phi_1)\alpha - 24\phi_1\delta(\bar\phi_1)\bar\beta \\
&+24\phi_1\bar\phi_0\bar\delta(\bar\lambda) - 24\phi_1\bar\phi_2\delta(\bar\rho) + 18\bar\phi_2\bar\delta(\phi_0)\alpha + 12\bar\phi_2\phi_0\bar\delta(\alpha) - 24\bar\phi_1\Delta(\phi_1)\bar\epsilon - 24\bar\phi_1\Delta(\phi_1)\epsilon \\
&+24\bar\phi_1\phi_0 D(\nu) + 48\bar\phi_1\pi\Delta(\phi_0) - 48\bar\phi_1 D(\phi_2)\tau - 24\bar\phi_1\phi_2 D(\tau) + 24\bar\phi_1\phi_0\bar\delta(\mu) + 24\bar\phi_1\delta(\phi_1)\alpha \\
&-24\bar\phi_1\delta(\phi_1)\bar\beta - 24\bar\phi_1\phi_2\bar\delta(\sigma) + 24\phi_1 D(\bar\phi_1)\gamma + 24\phi_1 D(\bar\phi_1)\bar\gamma - 12\phi_0\bar\phi_2\Delta(\bar\sigma) - 36\phi_0\Delta(\bar\phi_1)\bar\tau \\
&+12\phi_0\bar\phi_0\Delta(\bar\mu) + 24\phi_2\bar\pi\delta(\bar\phi_0) - 12\phi_2\bar\phi_2\delta(\bar\kappa) + 12\phi_2 D(\bar\phi_1)\bar\alpha - 24\phi_2 D(\bar\phi_2)\bar\rho + 24\bar\phi_1\bar\delta(\phi_1)\bar\alpha \\
&-24\bar\phi_1\bar\delta(\phi_1)\beta + 48\bar\phi_1\mu\bar\delta(\phi_0) - 48\bar\phi_1\delta(\phi_2)\rho + 24\bar\phi_1\phi_0\delta(\lambda) + 60\bar\phi_1\delta(\phi_0)\lambda - 24\bar\phi_1\phi_2\delta(\rho) \\
&-24\phi_1\bar\phi_1\delta(\bar\tau) - 12\phi_1\Delta(\bar\phi_0)\beta + 24\phi_1\bar\phi_0\delta(\bar\gamma) + 60\bar\phi_1 D(\phi_0)\nu + 36\bar\phi_1\bar\delta(\phi_0)\gamma - 72\bar\phi_1\rho\Delta(\phi_1) \\
&-72\bar\phi_1\bar\delta(\phi_1)\tau + 24\bar\phi_1\phi_0\Delta(\alpha) - 24\bar\phi_1\phi_1\Delta(\rho) - 1224\Delta(\Psi_2)\bar\Psi_2\bar\rho - 1224\Psi_2\pi\delta(\bar\Psi_2) + 1224\Psi_2\tau\bar\delta(\bar\Psi_2) \\
&+1224D(\Psi_2)\bar\Psi_2\bar\mu - 1224\Psi_2\rho\Delta(\bar\Psi_2) - 1224\bar\delta(\Psi_2)\bar\Psi_2\bar\pi + 1224\Psi_2\mu D(\bar\Psi_2) + 1224\delta(\Psi_2)\bar\Psi_2\bar\tau + 20\phi_1\tau\Delta(\bar\phi_0) \\
&+24\bar\phi_1\phi_0\Delta(\pi) - 24\bar\phi_1\phi_2\Delta(\kappa) = 0
\end{aligned}
\tag{A.8.1}
$$

A.9 Component Equations of the 5-index Condition for Case 4

This appendix contains the component equations of the 5-index Condition used in Chapter 4 with the following substitutions made:

1. **Case 4 assumptions**

$$\phi_1 = 0, \qquad \bar{\phi}_1 = 0, \phi_2 = 0, \qquad \bar{\phi}_2 = 0.$$

2. Suitable conformal and dyad transformations have been chosen such that

$$\phi_0 = \Psi_2.$$

3. The following equalities follow from the Case 4 assumption the Newman-Penrose field equations, Bianchi identities, and the 0-index to the 4-index necessary conditions. (See the proof of Proposition 4.4.1.)

$$\begin{aligned}
&\bar{\kappa} = -\kappa, \\
&\lambda = \nu = \mu = \gamma = \Phi_{12} = \Phi_{22} = 0, \\
&\Delta(\Psi_2) = \Delta(\alpha) = \Delta(\pi) = \Delta(\Lambda) = \Delta(\Phi_{02}) = \Delta(\Phi_{11}) = 0.
\end{aligned}$$

The above substitutions are made before displaying the component equations in order to save printing space, since the full versions of these equations are several times as long as what appears below. The following are the actual form of the component equations from the 5-index condition that are used explicitly in Chapter 4.

$$
{}^{o}(ABCD^{\iota}E)\,{}^{\bar{o}}(\dot{A}\dot{B}^{\bar{\iota}}\dot{C}\dot{D}\dot{E})\qquad
\begin{aligned}
&\tfrac{36}{25}D(\bar{\Psi}_2)\Psi_2\Delta(\beta)-\tfrac{36}{5}\bar{\delta}(\bar{\Psi}_2)\bar{\pi}\delta(\Psi_2)-\tfrac{36}{25}\delta(\bar{\Psi}_2)\Psi_2\Delta(\epsilon)\\
&+\tfrac{18}{25}\bar{\delta}(\Psi_2)\delta(\bar{\Psi}_2)\beta-\tfrac{54}{25}\bar{\delta}(\Psi_2)\bar{\alpha}\delta(\bar{\Psi}_2)+\tfrac{18}{25}\delta(\bar{\Psi}_2)\bar{\delta}(\Psi_2)\tau\\
&+\tfrac{108}{5}\Psi_2^2\bar{\Psi}_2\bar{\pi}-\tfrac{36}{25}\delta(\bar{\Psi}_2)\Psi_2\delta(\alpha)-\tfrac{108}{25}\bar{\Psi}_2\Delta(\bar{\rho})\delta(\Psi_2)\\
&+\tfrac{72}{25}\bar{\Psi}_2\Phi_{02}\Psi_2\pi+\tfrac{36}{25}\bar{\Psi}_2\bar{\pi}\bar{\delta}(\delta(\Psi_2))+\tfrac{216}{25}\bar{\Psi}_2\bar{\pi}\Psi_2\tau^2\\
&-\tfrac{18}{25}D(\bar{\Psi}_2)\Delta(\delta(\Psi_2))+\tfrac{72}{25}\bar{\Psi}_2\Phi_{02}\Psi_2\alpha-\tfrac{216}{25}\bar{\Psi}_2\bar{\pi}\Phi_{11}\Psi_2\\
&-\tfrac{36}{5}\bar{\delta}(\bar{\Psi}_2)\Psi_2\beta^2+\tfrac{72}{25}\Delta(\bar{\delta}(\bar{\Psi}_2))\Psi_2\epsilon-\tfrac{72}{25}\Psi_2\Phi_{02}\bar{\Psi}_2\bar{\alpha}\\
&+\tfrac{756}{5}\bar{\Psi}_2\Psi_2^2\tau+\tfrac{216}{25}\bar{\Psi}_2\bar{\tau}\Psi_2\beta^2+\tfrac{288}{25}\Psi_2\tau\bar{\Psi}_2\bar{\alpha}^2\\
&-\tfrac{144}{25}\Psi_2\pi\bar{\Psi}_2\bar{\pi}^2+\tfrac{432}{25}\Psi_2\tau\bar{\Psi}_2\bar{\alpha}\bar{\pi}-\tfrac{36}{25}\bar{\delta}(\delta(\bar{\Psi}_2))\delta(\Psi_2)\\
&+\tfrac{54}{25}\bar{\delta}(\bar{\Psi}_2)\Phi_{02}\Psi_2-\tfrac{72}{25}\bar{\delta}(\Psi_2)\bar{\pi}\delta(\bar{\Psi}_2)+\tfrac{126}{25}\delta(\Psi_2)\bar{\delta}(\bar{\Psi}_2)\beta\\
&+\tfrac{108}{25}\delta(\bar{\Psi}_2)\Psi_2\alpha\bar{\alpha}-\tfrac{72}{25}\delta(\Psi_2)\bar{\Psi}_2\bar{\beta}\bar{\pi}-\tfrac{72}{25}\Psi_2\tau\bar{\Psi}_2\delta(\bar{\pi})\\
&-\tfrac{216}{25}\Psi_2\tau\bar{\alpha}\delta(\bar{\Psi}_2)+\tfrac{72}{25}\delta(\Psi_2)\bar{\Psi}_2\bar{\alpha}\beta+\tfrac{36}{25}\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\pi}\beta\\
&+\tfrac{144}{25}\Psi_2\pi\bar{\pi}\delta(\bar{\Psi}_2)-\tfrac{108}{25}\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\alpha}\bar{\pi}+\tfrac{108}{25}\delta(\bar{\Psi}_2)\Psi_2\pi\bar{\alpha}\\
&-\tfrac{144}{25}\Psi_2\tau\bar{\pi}\delta(\bar{\Psi}_2)-\tfrac{108}{25}\delta(\bar{\Psi}_2)\Psi_2\pi\beta-\tfrac{72}{25}\Psi_2\tau\bar{\Psi}_2\delta(\bar{\alpha})\\
&+\tfrac{144}{25}\delta(\bar{\Psi}_2)\Psi_2\alpha\bar{\pi}+\tfrac{324}{25}\delta(\bar{\Psi}_2)\Psi_2\beta\bar{\tau}+\tfrac{108}{25}\delta(\Psi_2)\bar{\Psi}_2\bar{\alpha}\bar{\tau}\\
&-\tfrac{108}{25}\bar{\Psi}_2\bar{\tau}\beta\delta(\Psi_2)+\tfrac{72}{25}\Psi_2\tau\delta(\bar{\Psi}_2)\beta+\tfrac{108}{25}\bar{\delta}(\bar{\Psi}_2)\Psi_2\beta\bar{\alpha}\\
&+\tfrac{216}{25}\bar{\Psi}_2\delta(\bar{\pi})\Psi_2\alpha+\tfrac{216}{25}\Psi_2\pi\bar{\Psi}_2\delta(\bar{\pi})+\tfrac{144}{25}\bar{\Psi}_2\bar{\pi}\delta(\Psi_2)\tau\\
&+\tfrac{72}{5}\bar{\delta}(\bar{\Psi}_2)\bar{\pi}\Psi_2\beta-\tfrac{72}{25}\bar{\delta}(\bar{\Psi}_2)\tau\Psi_2\beta+\tfrac{216}{25}\bar{\Psi}_2\Delta(\bar{\rho})\Psi_2\beta\\
&-\tfrac{108}{25}\bar{\Psi}_2\tau\bar{\tau}\delta(\Psi_2)-\tfrac{36}{25}\Psi_2\tau\delta(\bar{\Psi}_2)\alpha-\tfrac{36}{25}\Psi_2\tau\delta(\bar{\Psi}_2)\pi\\
&-\tfrac{108}{25}\delta(\bar{\Psi}_2)\Psi_2\beta\alpha-\tfrac{72}{25}\bar{\Psi}_2\bar{\pi}\Psi_2\bar{\delta}(\beta)+\tfrac{72}{25}\delta(\bar{\Psi}_2)\Psi_2\beta\bar{\beta}\\
&-\tfrac{108}{25}\bar{\Psi}_2\delta(\bar{\pi})\bar{\delta}(\Psi_2)-\tfrac{72}{25}\bar{\Psi}_2\bar{\pi}\Psi_2\Delta(\sigma)+\tfrac{216}{25}\bar{\Psi}_2\delta(\bar{\tau})\Psi_2\beta\\
&+\tfrac{216}{25}\bar{\Psi}_2\bar{\delta}(\bar{\pi})\Psi_2\beta+\tfrac{144}{25}\delta(\Psi_2)\tau\bar{\Psi}_2\bar{\alpha}-\tfrac{72}{25}\Psi_2\Delta(\sigma)\bar{\Psi}_2\bar{\alpha}\\
&-\tfrac{216}{25}\bar{\Psi}_2\bar{\pi}\Psi_2\alpha\beta+\tfrac{144}{25}\bar{\Psi}_2\bar{\pi}\Psi_2\beta\bar{\beta}+\tfrac{216}{25}\bar{\Psi}_2\bar{\pi}\Psi_2\alpha\bar{\alpha}\\
&+\tfrac{216}{25}\bar{\Psi}_2\tau\bar{\tau}\Psi_2\beta+\tfrac{216}{25}\bar{\Psi}_2\bar{\pi}\Psi_2\delta(\tau)+\tfrac{144}{25}\Psi_2\tau\bar{\Psi}_2\bar{\pi}^2\\
&-\tfrac{144}{25}\Psi_2\tau\bar{\Psi}_2\bar{\pi}\beta-\tfrac{216}{25}\Psi_2\pi\bar{\Psi}_2\bar{\pi}\beta+\tfrac{18}{25}\delta(\bar{\Psi}_2)\Delta(D(\Psi_2))\\
&-\tfrac{72}{25}\bar{\Psi}_2\bar{\pi}\Phi_{02}\Psi_2+\tfrac{72}{25}\delta(\Psi_2)\bar{\Psi}_2\bar{\pi}\beta+\tfrac{216}{25}\Psi_2\pi\bar{\Psi}_2\bar{\alpha}\bar{\pi}\\
&-\tfrac{216}{25}\bar{\Psi}_2\bar{\tau}\Psi_2\beta\bar{\alpha}-\tfrac{72}{25}\delta(\Psi_2)\tau\delta(\bar{\Psi}_2)+\tfrac{216}{25}\Psi_2\tau^2\bar{\Psi}_2\bar{\alpha}\\
&-\tfrac{108}{25}\Psi_2\tau\Phi_{02}\bar{\Psi}_2+\tfrac{216}{25}\Psi_2\delta(\tau)\bar{\Psi}_2\bar{\alpha}-\tfrac{144}{25}\bar{\Psi}_2\bar{\pi}^2\Psi_2\alpha\\
&-\tfrac{108}{25}\Phi_{11}\Psi_2\delta(\bar{\Psi}_2)-\tfrac{36}{25}\delta(\delta(\bar{\Psi}_2))\bar{\delta}(\Psi_2)+\tfrac{36}{25}\bar{\delta}(\bar{\Psi}_2)\Psi_2\delta(\beta)\\
&-\tfrac{144}{25}\Psi_2\tau\bar{\Psi}_2\bar{\alpha}\beta-\tfrac{72}{25}\bar{\Psi}_2\bar{\pi}\delta(\delta(\Psi_2))-\tfrac{36}{25}\bar{\Psi}_2\Phi_{02}\bar{\delta}(\Psi_2)\\
&+\tfrac{72}{25}\bar{\delta}(\delta(\bar{\Psi}_2))\Psi_2\beta-\tfrac{108}{25}\bar{\Psi}_2\delta(\bar{\tau})\delta(\Psi_2)-\tfrac{54}{25}\delta(\Psi_2)\bar{\delta}(\bar{\Psi}_2)\bar{\alpha}\\
&-\tfrac{72}{25}\delta(\Psi_2)\bar{\Psi}_2\bar{\alpha}^2-\tfrac{18}{25}\bar{\delta}(\bar{\Psi}_2)\delta(\delta(\Psi_2))+\tfrac{18}{25}\delta(\bar{\Psi}_2)\delta(\bar{\delta}(\Psi_2))\\
&-\tfrac{54}{5}\Psi_2^2\delta(\bar{\Psi}_2)+\tfrac{36}{25}\delta(\delta(\Psi_2))\delta(\bar{\Psi}_2)+\tfrac{36}{25}\delta(\bar{\delta}(\bar{\Psi}_2))\delta(\Psi_2)\\
&-\tfrac{36}{25}\Delta(\delta(\bar{\Psi}_2))D(\Psi_2)+\tfrac{36}{25}\Delta(D(\bar{\Psi}_2))\delta(\Psi_2)+\tfrac{36}{25}\Psi_2\Phi_{02}\delta(\bar{\Psi}_2)\\
&-\tfrac{36}{25}\delta(\Psi_2)\delta(\bar{\Psi}_2)\beta+\tfrac{36}{25}\Psi_2\tau\delta(\delta(\bar{\Psi}_2))+\tfrac{72}{25}\delta(\delta(\bar{\Psi}_2))\Psi_2\alpha\\
&-\tfrac{36}{25}\delta(\Psi_2)\delta(\bar{\Psi}_2)\bar{\beta}-\tfrac{36}{25}\delta(\bar{\Psi}_2)\Psi_2\delta(\pi)-\tfrac{108}{25}\Psi_2\delta(\tau)\delta(\bar{\Psi}_2)\\
&+\tfrac{36}{25}\delta(\Psi_2)\bar{\alpha}\delta(\bar{\Psi}_2)+\tfrac{36}{25}\Psi_2\Delta(\sigma)\delta(\bar{\Psi}_2)-\tfrac{72}{25}\delta(\delta(\Psi_2))\bar{\Psi}_2\bar{\alpha}\\
&+\tfrac{36}{25}\bar{\delta}(\bar{\Psi}_2)\tau\delta(\Psi_2)+\tfrac{72}{25}\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\pi}^2-\tfrac{72}{25}\delta(\Psi_2)\bar{\Psi}_2\bar{\alpha}\bar{\pi}\\
&-\tfrac{72}{25}\Delta(D(\bar{\Psi}_2))\Psi_2\beta-\tfrac{162}{25}\delta(\Psi_2)\delta(\bar{\Psi}_2)\bar{\tau}-\tfrac{1188}{25}\bar{\Psi}_2\Psi_2\delta(\Psi_2)\\
&-\tfrac{108}{25}\bar{\Psi}_2\bar{\delta}(\bar{\pi})\delta(\Psi_2)-\tfrac{72}{25}\delta(\bar{\delta}(\bar{\Psi}_2))\Psi_2\beta+\tfrac{72}{25}\bar{\Psi}_2\pi\delta(\delta(\bar{\Psi}_2))\\
&-\tfrac{108}{25}\Psi_2\tau^2\delta(\bar{\Psi}_2)=0
\end{aligned}
\qquad (\mathrm{A.9.1})
$$

$O_{(ABC^{L}DE)}\bar{O}_{(\dot{A}\dot{B}\dot{C}^{\bar{L}}\dot{D}\dot{E})}$
(to be continued)

$$
\begin{aligned}
&-\tfrac{36}{25}\Psi_2\tau\bar{\pi}\bar{\delta}(\bar{\Psi}_2)-\tfrac{72}{25}\bar{\Psi}_2\bar{\tau}\beta\bar{\delta}(\Psi_2)-\tfrac{18}{5}\bar{\Psi}_2\bar{\tau}\alpha\delta(\Psi_2)\\
&+\tfrac{54}{25}\bar{\delta}(\bar{\Psi}_2)\Psi_2\beta\bar{\beta}-\tfrac{126}{25}\bar{\delta}(\bar{\Psi}_2)\Psi_2\alpha\beta+\tfrac{162}{25}\bar{\delta}(\bar{\Psi}_2)\Psi_2\alpha\bar{\pi}\\
&-\tfrac{36}{25}\bar{\Psi}_2\bar{\tau}\pi\delta(\Psi_2)-\tfrac{36}{25}\bar{\Psi}_2\bar{\tau}\Psi_2\delta(\pi)+\tfrac{36}{25}\Psi_2\pi\bar{\alpha}\bar{\delta}(\bar{\Psi}_2)\\
&+\tfrac{162}{25}\Psi_2\pi\bar{\pi}\bar{\delta}(\bar{\Psi}_2)+\tfrac{36}{25}\Psi_2\sigma\bar{\Psi}_2\Delta(\bar{\beta})+\tfrac{162}{25}\delta(\Psi_2)\bar{\Psi}_2\bar{\pi}\pi\\
&+\tfrac{18}{25}\Psi_2\tau\Delta(D(\bar{\Psi}_2))+\tfrac{18}{25}\bar{\Psi}_2\bar{\tau}\Delta(D(\Psi_2))-\tfrac{36}{25}\bar{\Psi}_2\bar{\tau}^2\delta(\Psi_2)\\
&+\tfrac{18}{25}\Psi_2\tau\bar{\delta}(\delta(\bar{\Psi}_2))-\tfrac{18}{25}\delta(\Psi_2)\delta(\bar{\Psi}_2)\alpha-\tfrac{18}{25}\bar{\delta}(\Psi_2)\bar{\alpha}\bar{\delta}(\bar{\Psi}_2)\\
&-\tfrac{18}{25}\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\alpha}^2-\tfrac{18}{25}\bar{\delta}(\Psi_2)\bar{\Psi}_2\delta(\bar{\alpha})+\tfrac{18}{25}\bar{\Psi}_2\bar{\tau}\delta(\bar{\delta}(\Psi_2))\\
&+\tfrac{18}{25}\delta(\Psi_2)\bar{\Psi}_2\delta(\bar{\beta})-\tfrac{18}{25}\delta(\bar{\Psi}_2)\Psi_2\bar{\delta}(\alpha)-\tfrac{18}{25}\delta(\bar{\Psi}_2)\Psi_2\alpha^2\\
&-\tfrac{18}{25}\Psi_2\sigma\Delta(\bar{\delta}(\bar{\Psi}_2))+\tfrac{18}{25}\bar{\Psi}_2\bar{\rho}\Delta(\bar{\delta}(\Psi_2))+\tfrac{27}{25}\bar{\delta}(\bar{\Psi}_2)\bar{\pi}\delta(\Psi_2)\\
&-\tfrac{18}{25}\bar{\Psi}_2\bar{\sigma}\Delta(\delta(\Psi_2))+\tfrac{18}{25}\bar{\Psi}_2\Delta(\bar{\sigma})\delta(\Psi_2)-\tfrac{36}{25}\Psi_2\pi\delta(\bar{\delta}(\bar{\Psi}_2))\\
&-\tfrac{54}{25}\bar{\Psi}_2\bar{\delta}(\bar{\pi})\bar{\delta}(\Psi_2)-\tfrac{81}{25}\delta(\bar{\Psi}_2)\pi\delta(\Psi_2)-\tfrac{27}{25}\bar{\delta}(\Psi_2)\delta(\bar{\Psi}_2)\beta\\
&-\tfrac{9}{25}\bar{\delta}(\Psi_2)\bar{\alpha}\delta(\bar{\Psi}_2)+\tfrac{54}{25}\Phi_{11}\Psi_2\bar{\delta}(\bar{\Psi}_2)+\tfrac{36}{25}\bar{\Psi}_2\bar{\pi}\delta(\bar{\delta}(\Psi_2))\\
&-\tfrac{36}{25}\Psi_2\pi\Delta(D(\bar{\Psi}_2))-\tfrac{18}{5}\delta(\bar{\Psi}_2)\bar{\delta}(\Psi_2)\tau+\tfrac{18}{25}\bar{\delta}(\bar{\Psi}_2)\Psi_2\bar{\delta}(\beta)\\
&-\tfrac{36}{25}\Delta(\delta(\Psi_2))\bar{\Psi}_2\bar{\epsilon}-\tfrac{36}{25}\bar{\delta}(\delta(\Psi_2))\bar{\Psi}_2\bar{\alpha}-\tfrac{36}{25}\Delta(\bar{\delta}(\bar{\Psi}_2))\Psi_2\epsilon\\
&-\tfrac{36}{25}\delta(\bar{\delta}(\bar{\Psi}_2))\Psi_2\beta-\tfrac{108}{25}\Psi_2\pi^2\bar{\Psi}_2\bar{\pi}-\tfrac{54}{25}\Psi_2\Delta(\rho)\delta(\bar{\Psi}_2)\\
&-\tfrac{36}{25}\bar{\Psi}_2\bar{\pi}\bar{\delta}(\delta(\Psi_2))-\tfrac{18}{5}\Psi_2\tau\bar{\alpha}\bar{\delta}(\bar{\Psi}_2)-\tfrac{36}{25}\Psi_2\tau\bar{\Psi}_2\Delta(\bar{\epsilon})\\
&-\tfrac{36}{25}\delta(\Psi_2)\bar{\Psi}_2\bar{\pi}\bar{\tau}-\tfrac{36}{25}\Psi_2\tau\delta(\bar{\Psi}_2)\bar{\tau}+\tfrac{72}{25}\delta(\Psi_2)\bar{\Psi}_2\bar{\beta}\tau\\
&-\tfrac{36}{25}\bar{\Psi}_2\bar{\tau}\bar{\delta}(\Psi_2)\tau-\tfrac{36}{25}\bar{\Psi}_2\bar{\tau}\Psi_2\Delta(\epsilon)-\tfrac{36}{25}\Psi_2\pi\bar{\delta}(\bar{\Psi}_2)\tau\\
&+\tfrac{72}{25}\bar{\delta}(\bar{\Psi}_2)\Psi_2\beta\bar{\tau}-\tfrac{72}{25}\Psi_2\tau\delta(\bar{\Psi}_2)\bar{\beta}-\tfrac{36}{25}\Psi_2\tau\bar{\Psi}_2\bar{\delta}(\bar{\alpha})\\
&+\tfrac{36}{25}\delta(\Psi_2)\bar{\Psi}_2\bar{\alpha}\alpha+\tfrac{36}{25}\bar{\Psi}_2\bar{\sigma}\Psi_2\Delta(\beta)+\tfrac{36}{25}\delta(\Psi_2)\bar{\Psi}_2\bar{\pi}\alpha\\
&-\tfrac{36}{25}\Psi_2\tau\bar{\Psi}_2\bar{\delta}(\bar{\pi})+\tfrac{189}{25}\bar{\Psi}_2^2\delta(\Psi_2)+\tfrac{18}{25}D(\bar{\Psi}_2)\Delta(\delta(\Psi_2))\\
&+\tfrac{1782}{25}\bar{\Psi}_2^2\Psi_2\bar{\pi}+\tfrac{18}{25}\Psi_2\Delta(\sigma)\bar{\delta}(\bar{\Psi}_2)+\tfrac{108}{25}\bar{\Psi}_2\bar{\delta}(\bar{\tau})\Psi_2\beta\\
&+\tfrac{108}{25}\Psi_2\delta(\tau)\bar{\Psi}_2\bar{\beta}+\tfrac{36}{5}\tau\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\alpha}+\tfrac{108}{25}\Psi_2\Delta(\rho)\bar{\Psi}_2\bar{\alpha}\\
&+\tfrac{72}{25}\Psi_2\tau^2\bar{\Psi}_2\bar{\beta}+\tfrac{72}{25}\bar{\Psi}_2\bar{\tau}^2\Psi_2\beta-\tfrac{108}{25}\Psi_2\pi\bar{\Psi}_2\bar{\alpha}^2\\
&-\tfrac{36}{25}\bar{\Psi}_2\Phi_{20}\Psi_2\beta-\tfrac{36}{25}\Psi_2\Phi_{02}\bar{\Psi}_2\bar{\beta}-\tfrac{108}{25}\bar{\Psi}_2\bar{\tau}\Phi_{11}\Psi_2\\
&-\tfrac{108}{25}\Psi_2\tau\Phi_{11}\bar{\Psi}_2-\tfrac{108}{25}\Psi_2\pi\bar{\Psi}_2\bar{\pi}^2+\tfrac{18}{25}\Psi_2\rho\Delta(\delta(\bar{\Psi}_2))\\
&+\tfrac{18}{25}\bar{\delta}(\bar{\Psi}_2)\Phi_{02}\Psi_2-\tfrac{18}{25}\bar{\delta}(\Psi_2)\bar{\pi}\delta(\bar{\Psi}_2)-\tfrac{27}{25}\delta(\Psi_2)\bar{\delta}(\bar{\Psi}_2)\beta\\
&-\tfrac{18}{5}\delta(\Psi_2)\bar{\Psi}_2\bar{\beta}\bar{\pi}+\tfrac{54}{25}\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\pi}\beta+\tfrac{54}{25}\Psi_2\pi\bar{\pi}\delta(\bar{\Psi}_2)\\
&-\tfrac{18}{25}\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\alpha}\bar{\pi}+\tfrac{54}{25}\delta(\bar{\Psi}_2)\Psi_2\pi\bar{\alpha}-\tfrac{54}{25}\delta(\bar{\Psi}_2)\Psi_2\pi\beta\\
&-\tfrac{36}{25}\delta(\Psi_2)\bar{\Psi}_2\bar{\alpha}\bar{\tau}+\tfrac{54}{25}\bar{\delta}(\bar{\Psi}_2)\tau\Psi_2\beta+\tfrac{36}{25}\Psi_2\tau\delta(\bar{\Psi}_2)\alpha\\
&-\tfrac{18}{25}\bar{\Psi}_2\delta(\bar{\pi})\bar{\delta}(\Psi_2)+\tfrac{108}{25}\Psi_2\pi\bar{\Psi}_2\bar{\pi}\beta+\tfrac{18}{25}\delta(\bar{\Psi}_2)\Delta(D(\Psi_2))\\
&-\tfrac{216}{25}\Psi_2\pi\bar{\Psi}_2\bar{\alpha}\bar{\pi}-\tfrac{18}{5}\Psi_2\pi\bar{\delta}(\bar{\Psi}_2)\beta+\tfrac{54}{25}\bar{\Psi}_2\bar{\tau}\delta(\Psi_2)\bar{\beta}\\
&-\tfrac{54}{25}\Psi_2\delta(\tau)\bar{\delta}(\bar{\Psi}_2)+\tfrac{54}{25}\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\pi}\pi-\tfrac{18}{25}\delta(\bar{\Psi}_2)\Psi_2\alpha\pi\\
&+\tfrac{54}{25}\delta(\bar{\Psi}_2)\Psi_2\pi\bar{\beta}+\tfrac{54}{25}\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\pi}\alpha-\tfrac{36}{25}\Psi_2\tau\bar{\delta}(\bar{\Psi}_2)\alpha\\
&-\tfrac{54}{25}\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\pi}\bar{\beta}+\tfrac{36}{25}\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\alpha}\bar{\tau}+\tfrac{9}{25}\delta(\delta(\bar{\Psi}_2))\bar{\delta}(\Psi_2)\\
&+\tfrac{108}{25}\Psi_2\pi\bar{\Psi}_2\bar{\beta}\bar{\pi}-\tfrac{216}{25}\Psi_2\pi\bar{\Psi}_2\bar{\pi}\alpha+\tfrac{18}{25}\bar{\Psi}_2\Phi_{20}\delta(\Psi_2)\\
&+\tfrac{36}{25}\bar{\delta}(\delta(\bar{\Psi}_2))\Psi_2\alpha-\tfrac{27}{25}\delta(\bar{\Psi}_2)\Phi_{20}\Psi_2+\tfrac{1782}{25}\bar{\Psi}_2\Psi_2^2\pi\\
&-\tfrac{27}{25}\bar{\Psi}_2\Phi_{02}\bar{\delta}(\Psi_2)-\tfrac{27}{25}\bar{\delta}(\Psi_2)\bar{\beta}\delta(\bar{\Psi}_2)-\tfrac{36}{25}\bar{\delta}(\bar{\Psi}_2)\delta(\Psi_2)\bar{\tau}\\
&+\tfrac{18}{25}\bar{\delta}(\Psi_2)\bar{\delta}(\bar{\Psi}_2)\tau+\tfrac{27}{25}\bar{\delta}(\bar{\Psi}_2)\pi\delta(\Psi_2)+\tfrac{378}{25}\bar{\Psi}_2^2\Psi_2\tau
\end{aligned}
$$

to be continued . . .

$o_{(ABC^{\iota}DE)}\,\bar{o}_{(\dot{A}\dot{B}\dot{C}^{\bar{\iota}}\dot{D}\dot{E})}$ (continued)

$$
\begin{aligned}
&\dots \text{ continued}\\
&+\tfrac{36}{25}\bar\delta(\Psi_2)\bar\delta(\bar\Psi_2)\beta-\tfrac{27}{25}\delta(\Psi_2)\bar\beta\bar\delta(\bar\Psi_2)+\tfrac{9}{5}\delta(\Psi_2)\bar\delta(\bar\Psi_2)\alpha\\
&-\tfrac{18}{25}\delta(\bar\Psi_2)\Psi_2\bar\delta(\pi)-\tfrac{18}{25}\bar\delta(\Psi_2)\delta(\bar\Psi_2)\pi-\tfrac{9}{25}\bar\delta(\Psi_2)\delta(\bar\Psi_2)\alpha\\
&+\tfrac{9}{5}\delta(\Psi_2)\bar\delta(\bar\Psi_2)\bar\alpha+\tfrac{18}{25}\delta(\bar\delta(\bar\Psi_2))\bar\delta(\Psi_2)+\tfrac{18}{25}\bar\delta(\bar\Psi_2)\delta(\delta(\Psi_2))\\
&-\tfrac{18}{25}\delta(\bar\Psi_2)\delta(\bar\delta(\Psi_2))-\tfrac{9}{25}\delta(\bar\delta(\bar\Psi_2))\delta(\Psi_2)+\tfrac{18}{25}\Delta(D(\bar\Psi_2))\bar\delta(\Psi_2)\\
&-\tfrac{9}{25}\bar\delta(\bar\Psi_2)\bar\delta(\delta(\Psi_2))+\tfrac{9}{25}\delta(\bar\Psi_2)\bar\delta(\bar\delta(\Psi_2))+\tfrac{18}{25}\Delta(\bar\delta(\bar\Psi_2))D(\Psi_2)\\
&+\tfrac{18}{25}\bar\delta(\bar\delta(\bar\Psi_2))\delta(\Psi_2)-\tfrac{18}{25}\bar\delta(\delta(\bar\Psi_2))\bar\delta(\Psi_2)+\tfrac{648}{25}\bar\Psi_2\Psi_2\bar\delta(\Psi_2)\\
&-\tfrac{36}{25}\Delta(D(\Psi_2))\bar\Psi_2\bar\alpha+\tfrac{36}{25}\delta(\Psi_2)\delta(\bar\Psi_2)\bar\beta-\tfrac{54}{25}\delta(\bar\Psi_2)\Psi_2\delta(\pi)\\
&-\tfrac{54}{25}\bar\Psi_2\Delta(\bar\tau)D(\Psi_2)-\tfrac{36}{25}\delta(\delta(\Psi_2))\bar\Psi_2\bar\beta+\tfrac{36}{25}\delta(\bar\delta(\Psi_2))\bar\Psi_2\bar\alpha\\
&-\tfrac{36}{25}\bar\delta(\bar\Psi_2)\tau\delta(\Psi_2)+\tfrac{648}{25}\bar\Psi_2\Psi_2\delta(\bar\Psi_2)+\tfrac{378}{25}\Psi_2^2\bar\Psi_2\bar\tau\\
&+\tfrac{36}{25}\Psi_2\pi\bar\delta(\delta(\bar\Psi_2))-\tfrac{81}{25}\bar\delta(\Psi_2)\bar\delta(\bar\Psi_2)\bar\pi-\tfrac{72}{25}\Psi_2\tau\bar\Psi_2\bar\pi\alpha\\
&-\tfrac{54}{25}\bar\Psi_2\bar\delta(\bar\tau)\delta(\Psi_2)+\tfrac{108}{25}\bar\Psi_2\delta(\bar\tau)\Psi_2\alpha-\tfrac{36}{25}\Psi_2\Delta(\sigma)\bar\Psi_2\bar\beta\\
&+\tfrac{36}{5}\bar\Psi_2\bar\pi\tau\bar\delta(\Psi_2)+\tfrac{18}{25}\bar\delta(\delta(\Psi_2))\delta(\bar\Psi_2)+\tfrac{108}{25}\Psi_2\bar\delta(\tau)\bar\Psi_2\bar\alpha\\
&-\tfrac{54}{25}\bar\Psi_2\Delta(\bar\rho)\bar\delta(\Psi_2)+\tfrac{18}{25}\delta(\Psi_2)\delta(\bar\Psi_2)\bar\tau-\tfrac{54}{25}\Psi_2\bar\delta(\tau)\delta(\bar\Psi_2)\\
&+\tfrac{54}{25}\Phi_{11}\bar\Psi_2\delta(\Psi_2)-\tfrac{54}{25}\Psi_2\Delta(\tau)D(\bar\Psi_2)+\tfrac{108}{25}\bar\Psi_2\bar\pi\Psi_2\Delta(\rho)\\
&+\tfrac{108}{25}\Psi_2\pi\bar\Psi_2\Delta(\bar\rho)-\tfrac{54}{25}\bar\Psi_2\delta(\bar\tau)\bar\delta(\Psi_2)-\tfrac{36}{25}\bar\Psi_2\bar\pi\Delta(D(\Psi_2))\\
&-\tfrac{36}{25}\delta(\bar\delta(\bar\Psi_2))\Psi_2\alpha-\tfrac{18}{5}\bar\tau\delta(\bar\Psi_2)\bar\delta(\Psi_2)-\tfrac{36}{25}\Delta(D(\bar\Psi_2))\Psi_2\alpha\\
&+\tfrac{252}{25}\Psi_2\tau\bar\Psi_2\bar\alpha\bar\beta-\tfrac{72}{25}\Psi_2\tau\bar\Psi_2\bar\alpha\alpha+\tfrac{108}{25}\Psi_2\pi\bar\Psi_2\bar\alpha\beta\\
&-\tfrac{108}{25}\Psi_2\tau\bar\Psi_2\bar\beta\beta-\tfrac{72}{25}\bar\Psi_2\bar\tau\Psi_2\alpha\bar\alpha+\tfrac{108}{25}\bar\Psi_2\bar\pi\Psi_2\alpha\bar\beta\\
&+\tfrac{54}{25}\bar\delta(\Psi_2)\bar\Psi_2\bar\alpha\beta+\tfrac{36}{25}\bar\delta(\bar\Psi_2)\Psi_2\alpha\bar\alpha+\tfrac{54}{25}\delta(\Psi_2)\bar\Psi_2\bar\beta\beta\\
&-\tfrac{36}{25}\bar\Psi_2\bar\tau\Psi_2\delta(\alpha)+\tfrac{162}{25}\delta(\Psi_2)\bar\Psi_2\bar\alpha\pi+\tfrac{54}{25}\delta(\bar\Psi_2)\Psi_2\alpha\bar\beta\\
&-\tfrac{126}{25}\delta(\Psi_2)\bar\Psi_2\bar\alpha\bar\beta+\tfrac{189}{25}\Psi_2^2\bar\delta(\bar\Psi_2)+\tfrac{108}{25}\Psi_2\pi\bar\Psi_2\bar\delta(\bar\pi)\\
&-\tfrac{36}{25}\Psi_2\tau^2\bar\delta(\bar\Psi_2)+\tfrac{144}{25}\Psi_2\tau\bar\Psi_2\bar\pi\bar\beta-\tfrac{108}{25}\bar\Psi_2\bar\tau\Psi_2\beta\bar\beta\\
&+\tfrac{252}{25}\bar\Psi_2\bar\tau\Psi_2\beta\alpha+\tfrac{144}{25}\bar\Psi_2\bar\tau\Psi_2\beta\pi-\tfrac{72}{25}\bar\Psi_2\bar\tau\Psi_2\pi\bar\alpha\\
&+\tfrac{72}{25}\Psi_2\tau\bar\Psi_2\bar\pi\bar\tau+\tfrac{72}{25}\Psi_2\tau\bar\Psi_2\bar\alpha\bar\tau+\tfrac{72}{25}\bar\Psi_2\bar\tau\Psi_2\pi\tau\\
&+\tfrac{72}{25}\bar\Psi_2\bar\tau\Psi_2\alpha\tau-\tfrac{108}{25}\bar\Psi_2\bar\pi\Psi_2\alpha^2-\tfrac{36}{25}\bar\Psi_2\Delta(\bar\sigma)\Psi_2\beta\\
&+\tfrac{108}{25}\Psi_2\Delta(\tau)\bar\Psi_2\bar\epsilon+\tfrac{108}{25}\bar\Psi_2\bar\pi\Psi_2\delta(\pi)+\tfrac{108}{25}\Psi_2\pi\bar\Psi_2\delta(\bar\tau)\\
&+\tfrac{36}{5}\Psi_2\pi\bar\tau\delta(\bar\Psi_2)+\tfrac{108}{25}\Psi_2\delta(\pi)\bar\Psi_2\bar\alpha+\tfrac{108}{25}\bar\Psi_2\Delta(\bar\tau)\Psi_2\epsilon\\
&+\tfrac{36}{5}\bar\tau\delta(\bar\Psi_2)\Psi_2\alpha+\tfrac{108}{25}\bar\Psi_2\bar\delta(\bar\pi)\Psi_2\alpha+\tfrac{108}{25}\bar\Psi_2\Delta(\bar\rho)\Psi_2\alpha\\
&+\tfrac{108}{25}\bar\Psi_2\bar\pi\Psi_2\bar\delta(\tau)=0
\end{aligned}
\tag{A.9.2}
$$

$$
\begin{aligned}
o_{(AB^{\iota}CDE)}\bar{o}_{\dot{A}\dot{B}\dot{C}\dot{D}\dot{E}} \quad & \tfrac{36}{5}D(\Psi_2)\bar{\Psi}_2\bar{\delta}(\bar{\beta}) + \tfrac{36}{5}\bar{\Psi}_2\bar{\delta}(\bar{\sigma})\bar{\delta}(\Psi_2) + \tfrac{72}{5}\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\beta}\epsilon \\
& + \tfrac{36}{5}\bar{\Psi}_2\bar{\sigma}\alpha\bar{\delta}(\Psi_2) - \tfrac{108}{5}\bar{\Psi}_2\bar{\sigma}\bar{\delta}(\Psi_2)\bar{\beta} + \tfrac{108}{5}D(\Psi_2)\bar{\Psi}_2\bar{\beta}\alpha \\
& + \tfrac{108}{5}\Psi_2\rho\bar{\beta}\bar{\delta}(\bar{\Psi}_2) + \tfrac{324}{5}\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\beta}\rho + \tfrac{108}{5}\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\epsilon}\alpha \\
& - \tfrac{36}{5}\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\epsilon}\bar{\beta} - \tfrac{72}{5}\Psi_2\pi\bar{\Psi}_2D(\bar{\beta}) + \tfrac{144}{5}\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\epsilon}\pi \\
& + \tfrac{72}{5}\Psi_2\Phi_{20}\bar{\Psi}_2\bar{\epsilon} - \tfrac{108}{5}\bar{\Psi}_2\bar{\tau}\bar{\sigma}\bar{\delta}(\Psi_2) - \tfrac{144}{5}\bar{\sigma}\bar{\delta}(\bar{\Psi}_2)\Psi_2\alpha \\
& - \tfrac{72}{5}\Psi_2\pi\bar{\Psi}_2\bar{\delta}(\bar{\sigma}) + \tfrac{216}{5}\Psi_2\bar{\delta}(\rho)\bar{\Psi}_2\bar{\beta} - \tfrac{108}{5}\Psi_2\tau\bar{\sigma}\bar{\delta}(\bar{\Psi}_2) \\
& + \tfrac{216}{5}\Psi_2\bar{\delta}(\pi)\bar{\Psi}_2\bar{\epsilon} + \tfrac{216}{5}\Psi_2D(\pi)\bar{\Psi}_2\bar{\beta} + 72D(\Psi_2)\pi\bar{\Psi}_2\bar{\beta} \\
& - \tfrac{72}{5}\bar{\Psi}_2\bar{\delta}(\bar{\sigma})\Psi_2\alpha - \tfrac{144}{5}\Psi_2\pi\bar{\sigma}\bar{\delta}(\bar{\Psi}_2) - \tfrac{72}{5}\bar{\Psi}_2\bar{\sigma}\Psi_2\alpha^2 \\
& - \tfrac{216}{5}\Psi_2\rho\bar{\Psi}_2\bar{\beta}^2 - \tfrac{108}{5}\bar{\Psi}_2\Phi_{10}\Psi_2\pi - \tfrac{36}{5}\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\alpha}\bar{\sigma} \\
& - \tfrac{36}{5}\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\pi}\bar{\sigma} - \tfrac{72}{5}\bar{\Psi}_2\bar{\sigma}\delta(\Psi_2)\bar{\beta} - \tfrac{72}{5}\Psi_2\pi\bar{\delta}(\bar{\Psi}_2)\epsilon \\
& - \tfrac{108}{5}\Psi_2\pi D(\bar{\Psi}_2)\alpha + \tfrac{36}{5}\Psi_2\pi\bar{\beta}D(\bar{\Psi}_2) + \tfrac{108}{5}\Psi_2\rho\bar{\delta}(\bar{\Psi}_2)\alpha \\
& - \tfrac{144}{5}\Psi_2\pi^2\bar{\Psi}_2\bar{\epsilon} + \tfrac{72}{5}\bar{\delta}(\bar{\delta}(\Psi_2))\bar{\Psi}_2\bar{\epsilon} - \tfrac{108}{5}\Psi_2\bar{\delta}(\pi)D(\bar{\Psi}_2) \\
& + \tfrac{72}{5}D(\bar{\delta}(\Psi_2))\bar{\Psi}_2\bar{\beta} - \tfrac{72}{5}\bar{\Psi}_2\bar{\sigma}\Psi_2\pi\alpha - \tfrac{36}{5}D(\bar{\delta}(\Psi_2))\bar{\delta}(\bar{\Psi}_2) \\
& - \tfrac{36}{5}\Psi_2\Phi_{20}D(\bar{\Psi}_2) - \tfrac{54}{5}\bar{\Psi}_2\Phi_{10}\bar{\delta}(\Psi_2) + \tfrac{216}{5}\Psi_2\pi\bar{\Psi}_2\bar{\tau}\bar{\sigma} \\
& + \tfrac{18}{5}\bar{\sigma}\delta(\bar{\Psi}_2)\bar{\delta}(\Psi_2) + \tfrac{216}{5}\Psi_2\pi\bar{\Psi}_2\bar{\epsilon}\alpha + \tfrac{216}{5}\Psi_2\tau\bar{\sigma}\bar{\Psi}_2\bar{\beta} \\
& + \tfrac{36}{5}\Psi_2\pi D(\bar{\delta}(\bar{\Psi}_2)) + \tfrac{216}{5}\bar{\Psi}_2\bar{\tau}\bar{\sigma}\Psi_2\alpha - \tfrac{36}{5}\bar{\delta}(\Psi_2)\bar{\Psi}_2\bar{\delta}(\bar{\epsilon}) \\
& + \tfrac{216}{5}\bar{\Psi}_2\bar{\sigma}\Psi_2\pi\bar{\beta} - \tfrac{36}{5}\bar{\delta}(\bar{\delta}(\Psi_2))D(\bar{\Psi}_2) - \tfrac{216}{5}\Psi_2\rho\bar{\Psi}_2\bar{\beta}\alpha \\
& + \tfrac{36}{5}\bar{\delta}(D(\Psi_2))\bar{\delta}(\bar{\Psi}_2) + \tfrac{216}{5}\bar{\Psi}_2\bar{\sigma}\Psi_2\alpha\bar{\beta} + \tfrac{54}{5}\bar{\Psi}_2\Phi_{20}D(\Psi_2) \\
& - \tfrac{108}{5}\Psi_2D(\pi)\bar{\delta}(\bar{\Psi}_2) - \tfrac{36}{5}D(\Psi_2)\bar{\Psi}_2\bar{\beta}^2 + \tfrac{144}{5}\Psi_2\pi\bar{\Psi}_2\bar{\beta}\epsilon \\
& - \tfrac{18}{5}\bar{\delta}(\bar{\delta}(\bar{\Psi}_2))D(\Psi_2) - \tfrac{72}{5}\bar{\delta}(\Psi_2)D(\bar{\Psi}_2)\pi - \tfrac{216}{5}\Psi_2\pi\bar{\Psi}_2\bar{\epsilon}\bar{\beta} \\
& + \tfrac{18}{5}\bar{\delta}(D(\bar{\Psi}_2))\bar{\delta}(\Psi_2) - \tfrac{162}{5}\bar{\delta}(\Psi_2)\bar{\delta}(\bar{\Psi}_2)\rho - \tfrac{18}{5}\bar{\delta}(\Psi_2)\bar{\beta}D(\bar{\Psi}_2) \\
& - \tfrac{1188}{5}\bar{\Psi}_2^2\bar{\sigma}\Psi_2 - \tfrac{108}{5}\Psi_2\bar{\delta}(\rho)\bar{\delta}(\bar{\Psi}_2) - \tfrac{36}{5}\bar{\delta}(\Psi_2)\bar{\delta}(\bar{\Psi}_2)\epsilon \\
& + \tfrac{72}{5}\Psi_2\pi^2D(\bar{\Psi}_2) + \tfrac{72}{5}\bar{\sigma}\bar{\delta}(\bar{\Psi}_2)\bar{\delta}(\Psi_2) - 36D(\Psi_2)\pi\bar{\delta}(\bar{\Psi}_2) \\
& - \tfrac{54}{5}\bar{\delta}(\Psi_2)D(\bar{\Psi}_2)\alpha + \tfrac{54}{5}D(\Psi_2)\bar{\beta}\bar{\delta}(\bar{\Psi}_2) - \tfrac{54}{5}D(\Psi_2)\bar{\delta}(\bar{\Psi}_2)\alpha \\
& - \tfrac{72}{5}\bar{\delta}(D(\Psi_2))\bar{\Psi}_2\bar{\beta} + \tfrac{36}{5}\bar{\delta}(\bar{\Psi}_2)\delta(\Psi_2)\bar{\sigma} = 0
\end{aligned} \tag{A.9.3}
$$

$$
{}^{O}(AB^{t}CDE)\,{}^{\bar O}(\dot{A}\dot{B}\dot{C}\dot{D}^{\bar t}\dot{E}) \qquad
\begin{aligned}
&\tfrac{36}{25}\Psi_2\pi\delta(\bar\delta(\bar\Psi_2)) + \tfrac{36}{25}\bar\delta(\Psi_2)\bar\delta(\bar\Psi_2)\alpha + \tfrac{36}{25}\delta(\Psi_2)\bar\Psi_2\bar\delta(\bar\beta)\\
&-\tfrac{36}{25}\bar\Psi_2\bar\delta(\bar\pi)\bar\delta(\Psi_2) + \tfrac{72}{25}\bar\delta(\bar\delta(\Psi_2))\bar\Psi_2\bar\alpha - \tfrac{72}{25}\bar\delta(\delta(\Psi_2))\bar\Psi_2\bar\beta\\
&+\tfrac{72}{5}\delta(\Psi_2)\pi\bar\Psi_2\bar\beta + \tfrac{216}{25}\Psi_2\bar\delta(\tau)\bar\Psi_2\bar\beta + \tfrac{216}{25}\Psi_2\bar\delta(\pi)\bar\Psi_2\bar\alpha\\
&+\tfrac{216}{25}\bar\Psi_2\bar\delta(\bar\tau)\Psi_2\alpha - \tfrac{72}{25}\bar\Psi_2\Delta(\bar\sigma)\Psi_2\alpha + \tfrac{144}{25}\bar\delta(\bar\Psi_2)\bar\tau\Psi_2\alpha\\
&-\tfrac{72}{25}\Psi_2\pi\bar\Psi_2\Delta(\bar\sigma) - \tfrac{108}{25}\Psi_2\tau\bar\tau\bar\delta(\bar\Psi_2) + \tfrac{216}{25}\Psi_2\Delta(\rho)\bar\Psi_2\bar\beta\\
&+\tfrac{216}{25}\bar\Psi_2\bar\pi\Psi_2\bar\delta(\pi) - \tfrac{108}{25}\bar\Psi_2\bar\tau\Phi_{20}\Psi_2 - \tfrac{216}{25}\Psi_2\pi\Phi_{11}\bar\Psi_2\\
&+\tfrac{72}{25}\Psi_2\Phi_{20}\bar\Psi_2\bar\alpha - \tfrac{72}{25}\bar\Psi_2\Phi_{20}\Psi_2\alpha - \tfrac{72}{25}\bar\Psi_2\Phi_{20}\Psi_2\pi\\
&+\tfrac{216}{25}\bar\Psi_2\bar\tau^2\Psi_2\alpha + \tfrac{216}{25}\Psi_2\pi\bar\Psi_2\bar\tau^2 - \tfrac{144}{25}\Psi_2\pi^2\bar\Psi_2\bar\alpha\\
&+\tfrac{216}{25}\Psi_2\tau\bar\Psi_2\bar\beta^2 + \tfrac{288}{25}\bar\Psi_2\bar\tau\Psi_2\alpha^2 + \tfrac{144}{25}\bar\Psi_2\bar\tau\Psi_2\pi^2\\
&+\tfrac{72}{25}\bar\Psi_2\bar\pi\Phi_{20}\Psi_2 - \tfrac{144}{25}\Psi_2\pi^2\bar\Psi_2\bar\pi - \tfrac{108}{25}\Psi_2\tau\bar\beta\bar\delta(\bar\Psi_2)\\
&+\tfrac{108}{25}\delta(\Psi_2)\bar\Psi_2\bar\beta\alpha + \tfrac{72}{25}\bar\delta(\Psi_2)\bar\Psi_2\bar\beta\beta - \tfrac{72}{25}\Psi_2\pi\bar\Psi_2\delta(\bar\beta)\\
&+\tfrac{144}{25}\bar\delta(\Psi_2)\bar\Psi_2\bar\alpha\pi + \tfrac{324}{25}\bar\delta(\Psi_2)\bar\Psi_2\bar\beta\tau - \tfrac{108}{25}\bar\delta(\Psi_2)\bar\Psi_2\bar\alpha\bar\beta\\
&+\tfrac{108}{25}\bar\delta(\Psi_2)\bar\Psi_2\bar\alpha\alpha - \tfrac{72}{25}\bar\Psi_2\bar\tau\Psi_2\bar\delta(\pi) - \tfrac{72}{25}\Psi_2\pi\bar\delta(\bar\Psi_2)\alpha\\
&+\tfrac{72}{25}\Psi_2\pi\bar\beta\bar\delta(\bar\Psi_2) - \tfrac{144}{25}\bar\Psi_2\bar\tau\pi\bar\delta(\Psi_2) - \tfrac{36}{25}\bar\Psi_2\bar\tau\bar\delta(\Psi_2)\bar\pi\\
&-\tfrac{216}{25}\bar\Psi_2\bar\tau\alpha\bar\delta(\Psi_2) + \tfrac{72}{25}\bar\delta(\bar\Psi_2)\Psi_2\alpha\bar\beta + \tfrac{72}{25}\bar\Psi_2\bar\tau\bar\delta(\Psi_2)\bar\beta\\
&-\tfrac{72}{25}\bar\Psi_2\bar\tau\Psi_2\bar\delta(\alpha) + \tfrac{144}{25}\Psi_2\pi\bar\delta(\bar\Psi_2)\bar\tau - \tfrac{72}{25}\Psi_2\pi\bar\delta(\bar\Psi_2)\beta\\
&-\tfrac{72}{25}\bar\Psi_2\bar\tau\delta(\Psi_2)\bar\beta + \tfrac{144}{25}\bar\delta(\Psi_2)\bar\Psi_2\bar\pi\pi - \tfrac{108}{25}\delta(\bar\Psi_2)\Psi_2\alpha\pi\\
&+\tfrac{36}{25}\delta(\bar\Psi_2)\Psi_2\pi\bar\beta + \tfrac{108}{25}\bar\delta(\Psi_2)\bar\Psi_2\bar\pi\alpha + \tfrac{108}{25}\Psi_2\tau\bar\delta(\bar\Psi_2)\alpha\\
&-\tfrac{108}{25}\bar\delta(\Psi_2)\bar\Psi_2\bar\pi\bar\beta - \tfrac{36}{25}\bar\delta(\Psi_2)\bar\Psi_2\bar\alpha\bar\tau + \tfrac{216}{25}\Psi_2\pi\bar\Psi_2\bar\delta(\bar\tau)\\
&+\tfrac{216}{25}\Psi_2\delta(\pi)\bar\Psi_2\bar\beta + \tfrac{216}{25}\Psi_2\pi\bar\Psi_2\bar\alpha\alpha + \tfrac{108}{5}\bar\Psi_2^2\Psi_2\pi\\
&+\tfrac{216}{25}\Psi_2\tau\bar\tau\bar\Psi_2\bar\beta + \tfrac{144}{25}\Psi_2\pi\bar\Psi_2\bar\beta\beta + \tfrac{756}{5}\Psi_2\bar\Psi_2^2\bar\tau\\
&-\tfrac{216}{25}\Psi_2\pi\bar\Psi_2\bar\alpha\bar\beta - \tfrac{108}{25}\Psi_2\Delta(\rho)\bar\delta(\bar\Psi_2) - \tfrac{216}{25}\Psi_2\tau\bar\Psi_2\bar\beta\alpha\\
&+\tfrac{72}{25}\bar\Psi_2\bar\pi\bar\delta(\bar\delta(\Psi_2)) - \tfrac{144}{25}\bar\Psi_2\bar\tau\Psi_2\alpha\bar\beta - \tfrac{72}{25}\Psi_2\pi\bar\delta(\bar\delta(\bar\Psi_2))\\
&+\tfrac{432}{25}\bar\Psi_2\bar\tau\Psi_2\alpha\pi + \tfrac{72}{25}\Delta(\bar\delta(\Psi_2))\bar\Psi_2\bar\epsilon - \tfrac{216}{25}\Psi_2\pi\bar\Psi_2\bar\beta\bar\pi\\
&-\tfrac{72}{25}\bar\delta(\bar\Psi_2)\bar\tau\bar\delta(\Psi_2) - \tfrac{1188}{25}\Psi_2\bar\Psi_2\bar\delta(\bar\Psi_2) + \tfrac{216}{25}\Psi_2\pi\bar\Psi_2\bar\pi\alpha\\
&-\tfrac{108}{25}\bar\Psi_2\bar\tau^2\bar\delta(\Psi_2) - \tfrac{72}{25}\Delta(D(\Psi_2))\bar\Psi_2\bar\beta - \tfrac{144}{25}\bar\Psi_2\bar\tau\Psi_2\pi\bar\beta\\
&+\tfrac{54}{25}\bar\Psi_2\Phi_{20}\delta(\Psi_2) + \tfrac{36}{25}D(\Psi_2)\bar\Psi_2\Delta(\bar\beta) - \tfrac{108}{25}\bar\Psi_2\bar\delta(\bar\tau)\bar\delta(\Psi_2)\\
&+\tfrac{36}{25}\bar\Psi_2\Delta(\bar\sigma)\bar\delta(\Psi_2) - \tfrac{36}{25}\delta(\bar\Psi_2)\Phi_{20}\Psi_2 - \tfrac{108}{25}\Phi_{11}\bar\Psi_2\bar\delta(\Psi_2)\\
&+\tfrac{36}{25}\bar\Psi_2\Phi_{20}\bar\delta(\Psi_2) - \tfrac{36}{25}\bar\delta(\Psi_2)\bar\Psi_2\Delta(\bar\epsilon) + \tfrac{18}{25}\bar\delta(\Psi_2)\bar\beta\delta(\bar\Psi_2)\\
&-\tfrac{36}{25}\bar\delta(\Psi_2)\bar\Psi_2\bar\delta(\bar\alpha) + \tfrac{36}{25}\bar\delta(\bar\Psi_2)\delta(\Psi_2)\bar\tau - \tfrac{162}{25}\bar\delta(\Psi_2)\bar\delta(\bar\Psi_2)\tau\\
&-\tfrac{36}{5}\bar\delta(\bar\Psi_2)\pi\delta(\Psi_2) - \tfrac{36}{25}\bar\delta(\Psi_2)\bar\delta(\bar\Psi_2)\beta + \tfrac{126}{25}\delta(\Psi_2)\bar\beta\bar\delta(\bar\Psi_2)\\
&-\tfrac{54}{25}\delta(\Psi_2)\bar\delta(\bar\Psi_2)\alpha - \tfrac{108}{25}\delta(\bar\Psi_2)\Psi_2\bar\delta(\pi) - \tfrac{72}{25}\bar\delta(\Psi_2)\delta(\bar\Psi_2)\pi\\
&-\tfrac{54}{25}\bar\delta(\Psi_2)\delta(\bar\Psi_2)\alpha - \tfrac{36}{5}\delta(\Psi_2)\bar\Psi_2\bar\beta^2 + \tfrac{72}{25}\delta(\bar\Psi_2)\Psi_2\pi^2\\
&-\tfrac{36}{25}\bar\delta(\Psi_2)\bar\beta\bar\delta(\bar\Psi_2) - \tfrac{72}{25}\bar\delta(\bar\Psi_2)\Psi_2\alpha^2 + \tfrac{72}{25}\delta(\bar\delta(\Psi_2))\bar\Psi_2\bar\beta\\
&+\tfrac{36}{25}\bar\Psi_2\bar\tau\bar\delta(\bar\delta(\Psi_2)) + \tfrac{18}{25}\Delta(D(\bar\Psi_2))\bar\delta(\Psi_2) - \tfrac{54}{5}\bar\Psi_2^2\bar\delta(\Psi_2)\\
&+\tfrac{36}{25}\bar\delta(\bar\Psi_2)\bar\delta(\delta(\Psi_2)) - \tfrac{36}{25}\delta(\bar\Psi_2)\bar\delta(\bar\delta(\Psi_2)) + \tfrac{36}{25}\Delta(D(\Psi_2))\bar\delta(\bar\Psi_2)\\
&-\tfrac{36}{25}\delta(\bar\delta(\Psi_2))\bar\delta(\bar\Psi_2) - \tfrac{36}{25}\Delta(\bar\delta(\Psi_2))D(\bar\Psi_2) + \tfrac{36}{25}\bar\delta(\bar\delta(\bar\Psi_2))\bar\delta(\Psi_2)\\
&-\tfrac{18}{25}\Delta(\bar\delta(\bar\Psi_2))D(\Psi_2) - \tfrac{18}{25}\bar\delta(\bar\delta(\bar\Psi_2))\delta(\Psi_2) + \tfrac{18}{25}\bar\delta(\delta(\bar\Psi_2))\bar\delta(\Psi_2)\\
&-\tfrac{72}{25}\bar\delta(\bar\delta(\bar\Psi_2))\Psi_2\alpha - \tfrac{108}{25}\bar\Psi_2\delta(\pi)\bar\delta(\bar\Psi_2) + \tfrac{18}{25}\bar\tau\delta(\bar\Psi_2)\bar\delta(\Psi_2)\\
&-\tfrac{108}{25}\Psi_2\bar\delta(\tau)\bar\delta(\bar\Psi_2) = 0
\end{aligned}
\tag{A.9.4}
$$

Appendix B

Component Equations for Type D Symmetric Spacetimes

A symmetric spacetime satisfies (5.2.2), (5.2.3), and (5.2.4). This appendix contains their component equations with respect to a dyad canonical to the Weyl spinor.

The component equations of (5.2.4) are

$$\iota_A \bar{\iota}_{\dot{A}} \qquad D(\Lambda) = 0 \tag{B.0.1}$$

$$o_A \bar{o}_{\dot{A}} \qquad \Delta(\Lambda) = 0 \tag{B.0.2}$$

$$\iota_A \bar{o}_{\dot{A}} \qquad \delta(\Lambda) = 0 \tag{B.0.3}$$

$$o_A \bar{\iota}_{\dot{A}} \qquad \bar{\delta}(\Lambda) = 0 \tag{B.0.4}$$

The component equations of (5.2.3), $\Psi_{ABCD;E\dot{E}} = 0$, are as follows:

$$\iota_{(C}\iota_B o_A \iota_{D)}\bar{o}_{\dot{E}} o_E \qquad -12\tau\Psi_2 = 0 \tag{B.0.5}$$

$$\bar{\iota}_{\dot{E}}\iota_{(C}\iota_B o_A \iota_{D)} o_E \qquad 12\rho\Psi_2 = 0 \tag{B.0.6}$$

$$\iota_E\iota_{(C}\iota_B o_A \iota_{D)}\bar{o}_{\dot{E}} \qquad 12\sigma\Psi_2 = 0 \tag{B.0.7}$$

$$\iota_E\bar{\iota}_{\dot{E}}\iota_{(C}\iota_B o_A \iota_{D)} \qquad -12\kappa\Psi_2 = 0 \tag{B.0.8}$$

$$o_{(A} o_B \iota_D o_{C)}\bar{o}_{\dot{E}} o_E \qquad 12\nu\Psi_2 = 0 \tag{B.0.9}$$

$$\bar{\iota}_{\dot{E}} o_{(A} o_B \iota_D o_{C)} o_E \qquad -12\lambda\Psi_2 = 0 \tag{B.0.10}$$

$$\iota_E o_{(A} o_B \iota_D o_{C)}\bar{o}_{\dot{E}} \qquad -12\mu\Psi_2 = 0 \tag{B.0.11}$$

$$\iota_E\bar{\iota}_{\dot{E}} o_{(A} o_B \iota_D o_{C)} \qquad 12\pi\Psi_2 = 0 \tag{B.0.12}$$

$$\iota_E\bar{\iota}_{\dot{E}} o_{(A} o_B \iota_{C)}\iota_D \qquad 6D(\Psi_2) = 0 \tag{B.0.13}$$

$$o_{(A} o_B \iota_C \iota_{D)}\bar{o}_{\dot{E}} o_E \qquad 6\Delta(\Psi_2) = 0 \tag{B.0.14}$$

$$\iota_E o_{(A} o_B \iota_C \iota_{D)}\bar{o}_{\dot{E}} \qquad -6\delta(\Psi_2) = 0 \tag{B.0.15}$$

$$\bar{\iota}_{\dot{E}} o_{(A} o_B \iota_C \iota_{D)} o_E \qquad -6\bar{\delta}(\Psi_2) = 0 \tag{B.0.16}$$

The component equations of (5.2.2), $\Phi_{AB\dot{A}\dot{B};C\dot{C}} = 0$, are as follows:

$$o_{(A}\iota_{B)}\bar{o}_{(\dot{A}}\bar{\iota}_{\dot{B})} o_C \bar{o}_{\dot{C}} \qquad \Delta(\Phi_{11}) = \nu\Phi_{01} + \bar{\nu}\Phi_{10} - \Phi_{21}\tau - \Phi_{12}\bar{\tau} \tag{B.0.17}$$

$$o_{(A}\iota_{B)}\bar{o}_{(\dot{A}}\bar{\iota}_{\dot{B})} \iota_C \bar{o}_{\dot{C}} \qquad \delta(\Phi_{11}) = \Phi_{01}\mu - \Phi_{12}\bar{\rho} - \sigma\Phi_{21} + \bar{\lambda}\Phi_{10} \tag{B.0.18}$$

$$o_{(A}\iota_{B)}\bar{o}_{(\dot{A}}\bar{\iota}_{\dot{B})} o_C \bar{\iota}_{\dot{C}} \qquad \bar{\delta}(\Phi_{11}) = \Phi_{10}\bar{\mu} - \Phi_{21}\rho - \bar{\sigma}\Phi_{12} + \lambda\Phi_{01} \tag{B.0.19}$$

$$o_{(A}\iota_{B)}\bar{o}_{(\dot{A}}\bar{\iota}_{\dot{B})} \iota_C \bar{\iota}_{\dot{C}} \qquad D(\Phi_{11}) = -\kappa\Phi_{21} - \bar{\kappa}\Phi_{12} + \Phi_{01}\pi + \Phi_{10}\bar{\pi} \tag{B.0.20}$$

$$o_{(A}\iota_{B)}\bar{\iota}_{\dot{A}}\bar{\iota}_{\dot{B}} o_C \bar{o}_{\dot{C}} \qquad \Delta(\Phi_{10}) = -\Phi_{20}\tau + \nu\Phi_{00} - 2\Phi_{11}\bar{\tau} + 2\Phi_{10}\bar{\gamma} \tag{B.0.21}$$

$$o_{(A}\iota_{B)}\bar{\iota}_{\dot{A}}\bar{\iota}_{\dot{B}} \iota_C \bar{o}_{\dot{C}} \qquad \delta(\Phi_{10}) = -2\Phi_{11}\bar{\rho} + \Phi_{00}\mu + 2\Phi_{10}\bar{\alpha} - \sigma\Phi_{20} \tag{B.0.22}$$

$$o_B o_A \bar{o}_{(\dot{A}}\bar{\iota}_{\dot{B})} o_C \bar{\iota}_{\dot{C}} \qquad \bar{\delta}(\Phi_{21}) = -2\Phi_{21}\alpha - \bar{\sigma}\Phi_{22} + 2\lambda\Phi_{11} + \Phi_{20}\bar{\mu} \tag{B.0.23}$$

$$o_{(A}\iota_{B)}\bar{\iota}_{\dot{A}}\bar{\iota}_{\dot{B}} o_C \bar{\iota}_{\dot{C}} \qquad \bar{\delta}(\Phi_{10}) = -\Phi_{20}\rho - 2\Phi_{11}\bar{\sigma} + \Phi_{00}\lambda + 2\Phi_{10}\bar{\beta} \tag{B.0.24}$$

$$o_{(A}\iota_{B)}\bar{\iota}_{\dot{A}}\bar{\iota}_{\dot{B}} \iota_C \bar{\iota}_{\dot{C}} \qquad D(\Phi_{10}) = -2\Phi_{11}\bar{\kappa} + \Phi_{00}\pi + 2\Phi_{10}\bar{\epsilon} - \Phi_{20}\kappa \tag{B.0.25}$$

$$o_{(A}\iota_{B)}\bar{o}_{\dot{A}}\bar{o}_{\dot{B}} o_C \bar{o}_{\dot{C}} \qquad \Delta(\Phi_{12}) = 2\Phi_{11}\bar{\nu} - \Phi_{22}\tau + \Phi_{02}\nu - 2\Phi_{12}\bar{\gamma} \tag{B.0.26}$$

$$o_B o_A \bar{o}_{\dot{A}} \bar{o}_{\dot{B}} \iota_C \bar{\iota}_{\dot{C}} \qquad D(\Phi_{22}) = -2\Phi_{22}\bar{\epsilon} - 2\Phi_{22}\epsilon + 2\Phi_{21}\bar{\pi} + 2\Phi_{12}\pi \tag{B.0.27}$$

$$o_{(A}\iota_{B)} \bar{o}_{\dot{A}} \bar{o}_{\dot{B}} \iota_C \bar{o}_{\dot{C}} \qquad \delta(\Phi_{12}) = -2\Phi_{12}\bar{\alpha} - \Phi_{22}\sigma + 2\Phi_{11}\bar{\lambda} + \Phi_{02}\mu \tag{B.0.28}$$

$$\iota_A \iota_B \bar{o}_{(\dot{A}} \bar{\iota}_{\dot{B})} o_C \bar{\iota}_{\dot{C}} \qquad \bar{\delta}(\Phi_{01}) = -2\Phi_{11}\rho + \Phi_{00}\bar{\mu} + 2\Phi_{01}\alpha - \bar{\sigma}\Phi_{02} \tag{B.0.29}$$

$$o_{(A}\iota_{B)} \bar{o}_{\dot{A}} \bar{o}_{\dot{B}} \iota_C \bar{\iota}_{\dot{C}} \qquad D(\Phi_{12}) = -\kappa\Phi_{22} + \Phi_{02}\pi - 2\Phi_{12}\bar{\epsilon} + 2\Phi_{11}\bar{\pi} \tag{B.0.30}$$

$$o_{(A}\iota_{B)} \bar{o}_{\dot{A}} \bar{o}_{\dot{B}} o_C \bar{\iota}_{\dot{C}} \qquad \bar{\delta}(\Phi_{12}) = -\Phi_{22}\rho - 2\Phi_{12}\bar{\beta} + \lambda\Phi_{02} + 2\Phi_{11}\bar{\mu} \tag{B.0.31}$$

$$\iota_A \iota_B \bar{\iota}_{\dot{A}} \bar{\iota}_{\dot{B}} \iota_C \bar{\iota}_{\dot{C}} \qquad D(\Phi_{00}) = 2\Phi_{00}\bar{\epsilon} + 2\Phi_{00}\epsilon - 2\Phi_{01}\bar{\kappa} - 2\Phi_{10}\kappa \tag{B.0.32}$$

$$\iota_A \iota_B \bar{\iota}_{\dot{A}} \bar{\iota}_{\dot{B}} o_C \bar{\iota}_{\dot{C}} \qquad \bar{\delta}(\Phi_{00}) = 2\Phi_{00}\alpha - 2\Phi_{01}\bar{\sigma} - 2\Phi_{10}\rho + 2\Phi_{00}\bar{\beta} \tag{B.0.33}$$

$$\iota_A \iota_B \bar{o}_{\dot{A}} \bar{o}_{\dot{B}} \iota_C \bar{o}_{\dot{C}} \qquad \delta(\Phi_{02}) = -2\Phi_{02}\bar{\alpha} + 2\Phi_{01}\bar{\lambda} - 2\Phi_{12}\sigma + 2\Phi_{02}\beta \tag{B.0.34}$$

$$\iota_A \iota_B \bar{\iota}_{\dot{A}} \bar{\iota}_{\dot{B}} o_C \bar{o}_{\dot{C}} \qquad \Delta(\Phi_{00}) = -2\Phi_{10}\tau - 2\Phi_{01}\bar{\tau} + 2\Phi_{00}\bar{\gamma} + 2\Phi_{00}\gamma \tag{B.0.35}$$

$$\iota_A \iota_B \bar{\iota}_{\dot{A}} \bar{\iota}_{\dot{B}} \iota_C \bar{o}_{\dot{C}} \qquad \delta(\Phi_{00}) = 2\Phi_{00}\bar{\alpha} - 2\sigma\Phi_{10} - 2\Phi_{01}\bar{\rho} + 2\Phi_{00}\beta \tag{B.0.36}$$

$$o_A o_B \bar{\iota}_{\dot{A}} \bar{\iota}_{\dot{B}} \iota_C \bar{\iota}_{\dot{C}} \qquad D(\Phi_{20}) = 2\Phi_{20}\bar{\epsilon} - 2\Phi_{20}\epsilon - 2\Phi_{21}\bar{\kappa} + 2\Phi_{10}\pi \tag{B.0.37}$$

$$o_A o_B \bar{\iota}_{\dot{A}} \bar{\iota}_{\dot{B}} o_C \bar{o}_{\dot{C}} \qquad \Delta(\Phi_{20}) = 2\nu\Phi_{10} - 2\Phi_{20}\gamma + 2\Phi_{20}\bar{\gamma} - 2\Phi_{21}\bar{\tau} \tag{B.0.38}$$

$$\iota_A \iota_B \bar{o}_{\dot{A}} \bar{o}_{\dot{B}} o_C \bar{o}_{\dot{C}} \qquad \Delta(\Phi_{02}) = 2\Phi_{01}\bar{\nu} - 2\Phi_{02}\bar{\gamma} + 2\Phi_{02}\gamma - 2\Phi_{12}\tau \tag{B.0.39}$$

$$o_A o_B \bar{\iota}_{\dot{A}} \bar{\iota}_{\dot{B}} \iota_C \bar{o}_{\dot{C}} \qquad \delta(\Phi_{20}) = 2\Phi_{20}\bar{\alpha} - 2\Phi_{20}\beta - 2\Phi_{21}\bar{\rho} + 2\Phi_{10}\mu \tag{B.0.40}$$

$$o_B o_A \bar{o}_{\dot{A}} \bar{o}_{\dot{B}} o_C \bar{\iota}_{\dot{C}} \qquad \bar{\delta}(\Phi_{22}) = -2\Phi_{22}\alpha + 2\Phi_{21}\bar{\mu} + 2\lambda\Phi_{12} - 2\Phi_{22}\bar{\beta} \tag{B.0.41}$$

$$o_B o_A \bar{o}_{\dot{A}} \bar{o}_{\dot{B}} \iota_C \bar{o}_{\dot{C}} \qquad \delta(\Phi_{22}) = -2\Phi_{22}\bar{\alpha} + 2\Phi_{12}\mu + 2\Phi_{21}\bar{\lambda} - 2\Phi_{22}\beta \tag{B.0.42}$$

$$o_B o_A \bar{o}_{\dot{A}} \bar{o}_{\dot{B}} o_C \bar{o}_{\dot{C}} \qquad \Delta(\Phi_{22}) = 2\Phi_{12}\nu + 2\Phi_{21}\bar{\nu} - 2\Phi_{22}\gamma - 2\Phi_{22}\bar{\gamma} \tag{B.0.43}$$

$$o_A o_B \bar{\iota}_{\dot{A}} \bar{\iota}_{\dot{B}} o_C \bar{\iota}_{\dot{C}} \qquad \bar{\delta}(\Phi_{20}) = -2\Phi_{20}\alpha + 2\Phi_{10}\lambda - 2\Phi_{21}\bar{\sigma} + 2\Phi_{20}\bar{\beta} \tag{B.0.44}$$

$$\iota_A \iota_B \bar{o}_{(\dot{A}} \bar{\iota}_{\dot{B})} \iota_C \bar{o}_{\dot{C}} \qquad \delta(\Phi_{01}) = -\Phi_{02}\bar{\rho} - 2\sigma\Phi_{11} + \bar{\lambda}\Phi_{00} + 2\Phi_{01}\beta \tag{B.0.45}$$

$$\iota_A \iota_B \bar{o}_{(\dot{A}} \bar{\iota}_{\dot{B})} o_C \bar{o}_{\dot{C}} \qquad \Delta(\Phi_{01}) = -\Phi_{02}\bar{\tau} + \bar{\nu}\Phi_{00} - 2\Phi_{11}\tau + 2\Phi_{01}\gamma \tag{B.0.46}$$

$$o_B o_A \bar{o}_{(\dot{A}} \bar{\iota}_{\dot{B})} \iota_C \bar{\iota}_{\dot{C}} \qquad D(\Phi_{21}) = -\bar{\kappa}\Phi_{22} + \Phi_{20}\bar{\pi} - 2\Phi_{21}\epsilon + 2\Phi_{11}\pi \tag{B.0.47}$$

$$o_B o_A \bar{o}_{(\dot{A}} \bar{\iota}_{\dot{B})} o_C \bar{o}_{\dot{C}} \qquad \Delta(\Phi_{21}) = 2\nu\Phi_{11} - \Phi_{22}\bar{\tau} + \bar{\nu}\Phi_{20} - 2\Phi_{21}\gamma \tag{B.0.48}$$

$$o_B o_A \bar{o}_{(\dot{A}} \bar{\iota}_{\dot{B})} \iota_C \bar{o}_{\dot{C}} \qquad \delta(\Phi_{21}) = -\Phi_{22}\bar{\rho} - 2\Phi_{21}\beta + \bar{\lambda}\Phi_{20} + 2\Phi_{11}\mu \tag{B.0.49}$$

$$\iota_A \iota_B \bar{o}_{(\dot{A}} \bar{\iota}_{\dot{B})} \iota_C \bar{\iota}_{\dot{C}} \qquad D(\Phi_{01}) = -2\kappa\Phi_{11} + \Phi_{00}\bar{\pi} + 2\Phi_{01}\epsilon - \bar{\kappa}\Phi_{02} \tag{B.0.50}$$

$$\iota_A \iota_B \bar{o}_{\dot{A}} \bar{o}_{\dot{B}} \iota_C \bar{\iota}_{\dot{C}} \qquad D(\Phi_{02}) = 2\Phi_{02}\epsilon - 2\Phi_{02}\bar{\epsilon} - 2\kappa\Phi_{12} + 2\Phi_{01}\bar{\pi} \tag{B.0.51}$$

$$\iota_A \iota_B \bar{o}_{\dot{A}} \bar{o}_{\dot{B}} o_C \bar{\iota}_{\dot{C}} \qquad \bar{\delta}(\Phi_{02}) = 2\Phi_{02}\alpha - 2\Phi_{02}\bar{\beta} - 2\Phi_{12}\rho + 2\Phi_{01}\bar{\mu} \tag{B.0.52}$$

Appendix C

MAPLE (NPspinor) Code

This appendix contains the NPspinor code used to generate the component equations of the spinor equations (2.3.10), ... , (2.3.13).

C.1 The 2-index Condition (2.3.13)

```
# Type D assumptions
#
W0:=0:  W0c:=0:  W1:=0:  W1c:=0:
W3:=0:  W3c:=0:  W4:=0:  W4c:=0:

T1 := del( psi[A,B,K,L], X,Ac ) * eps[K,X]:
T1 := contract( dyad(T1) ):
T1 := del( T1, Y,Bc ) * eps[L,Y]:
T1 := contract( dyad( T1 ) ):
T1 := collect( T1, basis(T1), distributed ):

T2 := psi[A,B,X,Y]*phi[K,L,Ac,Bc]*eps[K,X]*eps[L,Y]:
T2 := contract( dyad(T2) ):
T2 := collect( T2, basis(T2), distributed ):

T3 := 5*F[A,B]*Fc[Ac,Bc]:
T3 := dyad( T3 ):

cIII := T1 + T2 + T3:  cIII := symm(cIII,[A,B]):
```

```
cIII := symm(cIII,[Ac,Bc]):
cIII := findsymm( cIII, [A,B], nice ):

save cIII, `tD-cIII-symm.m`:
```

C.2 The 3-index Condition (2.3.10)

```
# Type D assumptions
#
W0:=0:  W0c:=0:  W1:=0:  W1c:=0:
W3:=0:  W3c:=0:  W4:=0:  W4c:=0:

T1 := 3*del( psi[A,B,C,K], X,Ac )*eps[K,X]:
T1 := contract( dyad( T1 ) ):
T1 := dyad( T1 * Fc[Bc,Cc] ):
T1 := symm( T1, [Ac,Bc,Cc] ):

T2 := del( Fc[Ac,Bc], Cc,K ):
T2 := psi[A,B,C,X] * T2 * eps[K,X]:
T2 := contract( dyad( T2 ) ):
T2 := symm( T2, [Ac,Bc,Cc] ):

cIV := T1 + conj(T1) - T2 - conj(T2):
cIV := findsymm( cIV, [A,B,C], nice ):

save cIV, `tD-cIV-symm.m`:
```

C.3 The 4-index Condition (2.3.11)

C.3.1 Individual Symmetrized Terms

```
# Type D assumptions
#
W0:=0:  W0c:=0:  W1:=0:  W1c:=0:
W3:=0:  W3c:=0:  W4:=0:  W4c:=0:

T1 := del( psi[A,B,C,D], K,Kc ) * del( psic[Ac,Bc,Cc,Dc], X,Xc )
* eps[K,X]: T1 := contract( dyad(T1) ):
T1 := contract( T1 * eps[Kc,Xc] ):
T1 := 3 * T1:
save T1, `cV-t1.m`:

T2 := del(psi[X,A,B,C],D,Ac):
T2 := T2 * del(psic[Bc,Cc,Dc,Lc],K,Yc):
T2 := dyad( T2 ):
T2 := contract( eps[K,X] * eps[Lc,Yc] * T2 ):
T2 := collect( T2, basis(T2), distributed ):
T2 := 4 * (T2 + conj(T2)):
T2 := symm( T2, [A,B,C,D] ):
T2 := symm( T2, [Ac,Bc,Cc,Dc] ):
save T2, `cV-t2.m`:

T3 := eps[K,X]*del(psi[A,B,C,K],X,Ac):
T3 := T3 * eps[Kc,Yc]*del(psic[Bc,Cc,Dc,Kc],Yc,D):
T3 := contract( dyad( T3 ) ):
T3 := collect( T3, basis(T3), distributed ):
T3 := -40 * T3:
T3 := symm( T3, [A,B,C,D] ):
T3 := symm( T3, [Ac,Bc,Cc,Dc] ):
save T3, `cV-t3.m`:

T4 := eps[K,Y]*psi[Y,A,B,C]*del(del(psic[Ac,Bc,Cc,Kc],K,Yc),D,Dc):
T4 := contract( dyad( T4 ) ):
T4 := contract( eps[Kc,Yc] * T4 ):
T4 := collect( T4, basis(T4), distributed ):
```

```
T4 := -4 * ( T4 + conj(T4) ):
T4 := symm( T4, [A,B,C,D] ):
T4 := symm( T4, [Ac,Bc,Cc,Dc] ):
save T4, `cV-t4.m`:

T5 := eps[K,X]*psi[X,A,B,C]*del(del(psic[Ac,Bc,Cc,Kc],D,Xc),K,Dc):
T5 := contract( dyad(T5) ):
T5 := collect( T5, basis(T5), distributed ):
T5 := contract( eps[Kc,Xc] * T5 ):
T5 := collect( T5, basis(T5), distributed ):
T5 := 12 * ( T5 + conj(T5) ):
T5 := symm( T5, [A,B,C,D] ):
T5 := symm( T5, [Ac,Bc,Cc,Dc] ):
save T5, `cV-t5.m`:

T6 := psi[X,A,B,C]*phi[D,K,Kc,Ac]*psic[Bc,Cc,Dc,Xc]:  T6 := contract(
dyad( eps[K,X]*T6 ) ):
T6 := contract( eps[Kc,Xc] * T6 ):
T6 := collect( T6, basis(T6), distributed ):
T6 := - 16 * T6:
T6 := symm( T6, [A,B,C,D] ):
T6 := symm( T6, [Ac,Bc,Cc,Dc] ):
save T6, `cV-t6.m`:

T7 := -32 * dyad( L * psi[A,B,C,D]*psic[Ac,Bc,Cc,Dc] ):
save T7, `cV-t7.m`:

T8 := dyad( del(del(F[A,B],C,Cc),D,Dc)*Fc[Ac,Bc] ):
T8 := collect( T8, basis(T8), distributed ):
T8 := -6 *( T8 + conj(T8) ):
T8 := symm( T8, [A,B,C,D] ):
T8 := symm( T8, [Ac,Bc,Cc,Dc] ):
save T8, `cV-t8.m`:

T9 := dyad( del(F[A,B],C,Cc)*del(Fc[Ac,Bc],Dc,D) ):
T9 := collect( T9, basis(T9), distributed ):
T9 := 16 * T9:
T9 := symm( T9, [A,B,C,D] ):
```

```
T9 := symm( T9, [Ac,Bc,Cc,Dc] ):
save T9, `cV-t9.m`:

T10 := dyad( F[A,B]*F[C,D]*psic[Ac,Bc,Cc,Dc] ):
T10 := collect( T10, basis(T10), distributed ):
T10 := -42 * ( T10 + conj(T10) ):
T10 := symm( T10, [A,B,C,D] ):
T10 := symm( T10, [Ac,Bc,Cc,Dc] ):
save T10, `cV-t10.m`:

T11 := dyad( F[A,B]*phi[C,D,Cc,Dc]*Fc[Ac,Bc] ):
T11 := collect( T11, basis(T11), distributed ):
T11 := 36 * T11:
T11 := symm( T11, [A,B,C,D] ):
T11 := symm( T11, [Ac,Bc,Cc,Dc] ):
save T11, `cV-t11.m`:
```

C.3.2 cV.sum

```
read `cV-t1.m`:
read `cV-t2.m`:
read `cV-t3.m`:
read `cV-t4.m`:
read `cV-t5.m`:
read `cV-t6.m`:
read `cV-t7.m`:
read `cV-t8.m`:
read `cV-t9.m`:
read `cV-t10.m`:
read `cV-t11.m`:

T1 := collect( T1, basis(T1), distributed ):
T2 := collect( T2, basis(T2), distributed ):
T3 := collect( T3, basis(T3), distributed ):
T4 := collect( T4, basis(T4), distributed ):
T5 := collect( T5, basis(T5), distributed ):
T6 := collect( T6, basis(T6), distributed ):
```

```
T7 := collect( T7, basis(T7), distributed ):
T8 := collect( T8, basis(T8), distributed ):
T9 := collect( T9, basis(T9), distributed ):
T10 := collect( T10, basis(T10), distributed ):
T11 := collect( T11, basis(T11), distributed ):

cVnotsymm := T1 + T2 + T3 + T4 + T5 + T6 + T7 + T8 + T9 + T10 + T11:
cV := findsymm( cVnotsymm, [A,B,C,D], nice ):
save cV, `tD-cV-symm.m`:
```

C.4 The 5-index Condition (2.3.12)

C.4.1 cVI-symm-spinor-basis.txt

```
dyadspinor:=[o,i];
dyadspinorc:=[oc,ic];
indexlist:=[A,B,C,D,E];
indexlistc:=[Ac,Bc,Cc,Dc,Ec];

templist:=[]:

for ii from 5 to 0 by -1 do
templist:= [op(templist),[seq(1,mm=1..ii),seq(2,nn=1..5-ii)]];
od:

cVIbasis := []:

for ii from 1 to nops(templist) do
mul(dyadspinor[templist[ii][jj]][indexlist[jj]],jj=1..5);
symm(%,indexlist);
findsymm(%,indexlist,nice);
cVIbasis := [op(cVIbasis),%]:
print(cVIbasis[ii]):
od:

cVIbasisc := []:
for ii from 1 to nops(templist) do
mul(dyadspinorc[templist[ii][jj]][indexlistc[jj]],jj=1..5);
symm(%,indexlistc);
findsymm(%,indexlist,nice);
cVIbasisc := [op(cVIbasisc),%]:
print(cVIbasisc[ii]);
od:

save cVIbasis, cVIbasisc, `cVI-symm-spinor-basis.m`:
save cVIbasis, cVIbasisc, `cVI-symm-spinor-basis.txt`:
```

C.4.2 gen-tD-cVI-symmetrize.txt

```
# re-write in terms of coeffs.  "AA", and symmetrize over un-dotted
indices
#
tempsymm := rewrite(temp,As,'AA'):
tempsymm := symm(tempsymm,[A,B,C,D,E]):

# now re-write the coeffs.  again (as "BB"), and symmetrize over
dotted indices
#
tempsymm := rewrite(tempsymm,Bs,'BB'):
tempsymm := symm(tempsymm,[Ac,Bc,Cc,Dc,Ec]):

# replace the spinor mess with un-evaluated symmetrizations
#
tempsymm := findsymm(tempsymm,[A,B,C,D,E],nice):

# finally, replace all coefficient values and save
#
As := AA: Bs := BB: tempsymm := eval(tempsymm):
```

C.4.3 gen-tD-cVI-footer.txt

```
read `gen-tD-cVI-symmetrize.txt`:

tDcVIt.termnum := eval(tempsymm):

save tDcVIt.termnum, cat(`tD-cVI-t`,termnum,`-symm.m`):
save tDcVIt.termnum, cat(`tD-cVI-t`,termnum,`-symm.txt`):
```

C.4.4 Individual Symmetrized Terms

```
# Type D assumptions
#
W0:=0:  W0c:=0:  W1:=0:  W1c:=0:
```

```
W3:=0:  W3c:=0:  W4:=0:  W4c:=0:

termnum := `1`:
temp :=del(psi[Z,A,B,C],K,Ac)*eps[K,Z]:
temp :=collect(expand(contract(dyad(temp))),basis(temp),distributed):
temp :=dyad(temp*del(del(Fc[Bc,Cc],D,Dc),E,Ec)):
temp :=expand(temp):
temp :=collect(temp,basis(temp),distributed):
tempc:=conj(temp):
temp :=-6*(temp+tempc):
temp := eval(temp):
read `gen-tD-cVI-footer.txt`:

termnum := `2`:
temp := eps[K,Z]*del(psi[A,B,C,Z],D,Dc)*del(del(Fc[Ac,Bc],K,Cc),E,Ec):
temp := contract(dyad(temp)):
temp := collect(temp,basis(temp),distributed):
temp := 6 * ( temp + conj(temp) ):
temp := collect(temp,basis(temp),distributed):
temp := eval(temp):
read `gen-tD-cVI-footer.txt`:

termnum := `3`:
temp := contract(dyad(eps[K,Z]*del(del(psi[Z,A,B,C],K,Ac),D,Dc))):
temp := dyad( temp * del(Fc[Bc,Cc],E,Ec) ):
temp := collect(temp,basis(temp),distributed):
temp := 24 * ( temp + conj(temp) ):
temp := collect(temp,basis(temp),distributed):
temp := eval(temp):
read `gen-tD-cVI-footer.txt`:

termnum := `4`:
temp := contract(dyad( eps[K,Z]*psi[Z,A,B,C]*phi[K,D,Ac,Dc] )):
temp := dyad( temp * del(Fc[Bc,Cc],E,Ec) ):
temp := collect(temp,basis(temp),distributed):
temp := 24 * ( temp + conj(temp) ):
temp := collect(temp,basis(temp),distributed):
temp := eval(temp):
```

```
read `gen-tD-cVI-footer.txt`:

termnum := `5`:
temp := contract(dyad(eps[K,Z]*del(psi[Z,A,B,C],E,Ec)*phi[K,D,Ac,Dc])):
temp := dyad( temp * Fc[Bc,Cc] ):
temp := collect(temp,basis(temp),distributed):
temp := -18 * ( temp + conj(temp) ):
temp := collect(temp,basis(temp),distributed):
temp := eval(temp):
read `gen-tD-cVI-footer.txt`:

termnum := `6`:
temp := contract(dyad( eps[K,Z]*del(psi[Z,A,B,C],K,Ac) )):
temp := dyad( temp * phi[D,E,Dc,Ec] * Fc[Bc,Cc] ):
temp := collect(temp,basis(temp),distributed):
temp := 18 * ( temp + conj(temp) ):
temp := collect(temp,basis(temp),distributed):
temp := eval(temp):
read `gen-tD-cVI-footer.txt`:
read `gen-tD-cVI-header.txt`:

termnum:=`7`:
temp := contract(dyad(eps[K,Z]*psi[Z,A,B,C]*del(psic[Bc,Cc,Dc,Ec],K,Ac))):
temp := dyad( temp * F[D,E] ):
temp := collect(temp,basis(temp),distributed):
temp := -36 * ( temp + conj(temp) ):
temp := collect(temp,basis(temp),distributed):
temp := eval(temp):
read `gen-tD-cVI-footer.txt`:

termnum := `8`:
temp := contract(dyad( eps[K,Z]*del(psi[Z,A,B,C],K,Ac) )):
temp := dyad( temp*psic[Bc,Cc,Dc,Ec]*F[D,E] ):
temp := collect(temp,basis(temp),distributed):
temp := -138 * ( temp + conj(temp) ):
temp := collect(temp,basis(temp),distributed):
temp := eval(temp):
read `gen-tD-cVI-footer.txt`:
```

```
termnum := `9`:
temp := contract(dyad( eps[K,Z]*del(psi[Z,A,B,C],D,Dc)*F[K,E] )):
temp := dyad( temp * psic[Bc,Cc,Ec,Ac] ):
temp := collect(temp,basis(temp),distributed):
temp := 6 * ( temp + conj(temp) ):
temp := collect(temp,basis(temp),distributed):
temp := eval(temp):
read `gen-tD-cVI-footer.txt`:

termnum := `10`:
temp := contract(dyad( eps[Kc,Zc]*psic[Zc,Ac,Bc,Cc]*Fc[Kc,Dc] )):
temp := dyad( temp * del(psi[A,B,C,D],E,Ec) ):
temp := collect(temp,basis(temp),distributed):
temp := 6 * ( temp + conj(temp) ):
temp := collect(temp,basis(temp),distributed):
temp := eval(temp):
read `gen-tD-cVI-footer.txt`:
```

Bibliography

[1] Abraham, R., Marsden, J. E., Ratiu, T., *Manifolds, Tensor Analysis, and Applications*, Springer-Verlag, 1988.

[2] Adams, William W. & Loustaunau, Philippe, *Introduction to Gröbner Bases*, American Mathematical Society, 1994.

[3] Anderson, W. G., *Contributions to the Study of Huygens' Principle for Non-self-adjoint Scalar Wave Equations on Curved Space-times*, Master's Thesis (University of Waterloo), 1991.

[4] Anderson W. G. & McLenaghan, R. G., *On the Validity of Huygens' Principle for Second Order Partial Differential Equations with Four Independent Variables. II. - a Sixth Necessary Condition*, Ann. Inst. Henri Poincaré, **60**, n° 4 (1994), 373-432.

[5] Anderson W. G., McLenaghan, R. G., & Sasse, F. D., *Huygens' Principle for the Non-self-adjoint Scalar Wave Equation on Petrov Type III Space-times*, Ann. Inst. Henri Poincaré, **70**, n° 3 (1999), 259-276.

[6] Asgeirsson, L., *Some Hints on Huygens' Principle and Hadamard's Conjecture*, Comm. Pure & Applied Math., **9**, 307-326.

[7] Cahen, M. & Defrise, L., *Lorentzian* 4*-dimensional manifolds with "local isotropy"*, Comm. Math. Phys., **11**, 56.

[8] Cahen, M. & McLenaghan, R. G., *Métriques des espaces lorentziens symétriques aux dérivées partielles non linéaires*, C. R. Acad. Sci. Paris, t. **266** (1968), p. 1125.

[9] Carminati, J. & McLenaghan, R. G., *An Explicit Determination of the Petrov Type N Space-times on which the Conformally Invariant Scalar*

Wave Equation Satisfies Huygens' Principle, Ann. Inst. Henri Poincaré, **44**, n° 2 (1986), 115-153.

[10] Carminati, J. & McLenaghan, R. G., *An Explicit Determination of the Space-times on which the Conformally Invariant Scalar Wave Equation Satisfies Huygens' Principle. Part II: Petrov Type D Space-times*, Ann. Inst. Henri Poincaré, **47** (1987), 337-354.

[11] Czapor, S. R., *Gröbner Basis Methods for Solving Algebraic Equations*, Doctoral Thesis (University of Waterloo), 1989.

[12] Czapor, S. R., *Solving Algebraic Equation: Combining Buchberger's Algorithm with Multivariate Factorization*, J. Symb. Comp. **7** (1989), 49-55.

[13] Czapor, S. R., McLenaghan, R. G., & Carminati, J., *The Automatic Conversion of Spinor Equations to Dyad Form in MAPLE*, General Relativity and Gravitation, **24**, n^{circ} 9 (1991), 911-928.

[14] Czapor, S. R., McLenaghan, R. G., & Sasse, F. D., *Complete Solution of Hadamard's Problem for the Scalar Wave Equation on Petrov Type III Spacetimes*, Ann. Inst. Henri Poincaré, **71**, n° 6 (1999), 595-620.

[15] De Felice , F. & Clarke, C. J. S, *Relativity on Curved Manifolds*, Cambridge University Press, 1990.

[16] Ehlers, J. & Kundt, W., *Exact Solutions of the Gravitational Field Equations*, Article in *Gravitation - An Introduction to Current Research*, ed. L. Witten, John Wiley and Sons, Toronto, 1962.

[17] Friedlander, F. G., *The Wave Equation on a Curved Space-time*, Cambridge University Press, 1975.

[18] Fröberg, R., *An Introduction to Gröbner Bases*, John Wiley & Sons, 1997.

[19] Frolov, V. P., *The Newman-Penrose Method in Theory of General Relativity*, Problems in the General Theory of Relativity and Theory of Group Representations, edited by Basov, N.G., V.96, Lebedev Physics Institute Series, Consultants Bureau, New York, 1979.

[20] Geddes, K. O., Czapor, S. R., & Labahn, G, *Algorithms for Computer Algebra*, Kluwer Academic Publishers, 1992.

[21] Günther, P., *Zur Gültigkeit des Huygensschen Prinzips bei partiellen Differentialgleichungen vom normal hyperbolischen Typus*, Ber. Verh. Sachs. Akad. Wiss. Leipzig, **100** (1952), 1-43.

[22] Günther, P., *Ein Beispiel einer nichttrivialen Huygensschen Differentialgleichung mit vier unabhängigen Variablen*, Arch. Rational Mech. Anal. **18** (1965), 103-106.

[23] Günther, P., *Huygens' Principle and Hyperbolic Equations*, Academic Press, 1988.

[24] Hadamard, Jacques, *Lectures on Cauchy's Problem in Linear Partial Differential Equations*, Dover Publications, 1952.

[25] Hadamard, Jacques, *The Problem of Diffusion of Waves*, Ann. of Math., **43** (1942), 510-522.

[26] Herglotz, G., *Über die Bestimmung eines Linienelementes in normal Koordinaten aus dem Riemannschen Krümmungstensor*, Math. Ann. **93** (1925), 46-53.

[27] Huygens, Christiaan, *Treatise on Light*, Translated by Thompson, Silvanus P., Dover Publications Inc., 1962.

[28] Kramer, D., Stephani, H., MacCallum, M., & Herlt, E., *Exact Solutions of Einstein's Field Equations*, Cambridge University Press, 1980.

[29] Lidl, Rudolf & Pilz, Günther, *Applied Abstract Algebra*, 2nd ed., Springer-Verlag, 1998

[30] Mathisson, M., *Le problème de M. Hadamard relatif à la diffusion des ondes*, Acta Math., **71** (1939), 249-282.

[31] McLenaghan, R. G., Doctoral Thesis (University of Cambridge), 1967.

[32] McLenaghan, R. G., *An Explicit Determination of the Empty Spacetimes on which the Wave Equation Satisfies Huygens' Principle*, Proc. Camb. Phil. Soc., **65** (1969), 139-155.

[33] McLenaghan, R. G., *On the Validity of Huygens' Principle for Second Order Partial Differential Equations with Four Independent Variables. Part I: Derivation of Necessary Conditions*, Ann. Inst. Henri Poincaré, **20**, n° 2 (1974), 153-188.

[34] McLenaghan, R. G. & Walton, T. F., *An Explicit Determination of the Non-self-adjoint Wave Equations on Curved Space-time that Satisfy Huygens' Principle. Part I: Petrov Type N Background Space-times*, Ann. Inst. Henri Poincaré, **48**, n° 3 (1988), 267-280.

[35] McLenaghan, R. G. & Williams, G. C., *An Explicit Determination of the Petrov D Space-times on which Weyl's Neutrino Equation and Maxwell's Equations Satisfy Huygens' Principle*, Ann. Inst. Henri Poincaré, **53**, n° 2 (1990), 217-223.

[36] Penrose, E. & Penrose, R., *An Approach to Gravitational Radiation by a Method of Spin Coefficients*, J. Math. Physics, **3**, n° 3 (1962), 566-578.

[37] Nicolaescu, Liviu I, *Lectures on the Geometry of Manifolds*, World Scientific, 1999.

[38] O'Neill, Barrett, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, 1983.

[39] Penrose, R., *A Spinor Approach to General Relativity*, Ann. Physics, **10** (1960), 171-201.

[40] Penrose, R. & Rindler, W., *Spinors and Space-time* v.1 & 2, Cambridge University Press, 1986.

[41] Rubenstein, I. & Ruberstein, L., *Partial Differential Equations in Classical Mathematical Physics*, Cambridge University Press, 1993.

[42] Sasse, F. D. *Huygens' Principle for Relativistic Wave Equations on Petrov Type III Space-times*, Doctoral Thesis (University of Waterloo), 1997.

[43] Singer, I. M., & Thorpe, J. A., *Lecture Notes on Elementary Topology and Geometry*, Springer-Verlag, 1967.

[44] Stellmacher, K. L., *Eine Klasse von Huygensschen Differentialgleichungen und ihre Integration*, Math. Ann. **130** (1955), 219-233.

[45] Walton, T. F. *The Validity of Huygens' Principle for the Non-self-adjoint Scalar Wave Equation on Curved Space-times*, Master's Thesis (University of Waterloo), 1988.

[46] Wünsch, V., *Huygens' Principle on Petrov Type D Space-times*, Ann. Physik, **46** (1989), 593-597.