Counting Bases

by

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Abstract

A theorem of Edmonds characterizes when a pair of matroids has a common basis. Enumerating the common bases of a pair of matroid is a much harder problem, and includes the #P-complete problem of counting the number of perfect matchings in a bipartite graph. We focus on the problem of counting the common bases in pairs of regular matroids, and describe a class called *Pfaffian matroid pairs* for which this enumeration problem can be solved. We prove that when a pair of regular matroids is non-Pfaffian, there is a set of common bases which certifies this, and that the number of bases in the certificate is linear in the size of the ground set of the matroids. When both matroids in a pair are series-parallel, we prove that determining if the pair is Pfaffian is equivalent to finding an edge signing in an associated graph, and in the case that the pair is non-Pfaffian, we obtain a characterization of this associated graph. Pfaffian bipartite graphs are a class of graphs for which the number of perfect matchings can be determined; we show that the class of series-parallel Pfaffian matroid pairs is an extension of the class of Pfaffian bipartite graphs.

Edmonds proved that the polytope generated by the common bases of a pair of matroids is equal to the intersection of the polytopes generated by the bases for each matroid in the pair. We consider when a similar property holds for the binary space, and give an excluded minor characterization of when the binary space generated by the common bases of two matroids can not be determined from the binary spaces for the individual matroids. As a result towards a description of the lattice of common bases for a pair of matroids, we show that the lattices for the individual matroids determine when all common bases of a pair of matroids intersect a subset of the ground set with fixed cardinality.

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Contents

Chapter 1

Introduction

Given two matrices A_1 and A_2 whose columns are indexed by a common set S, we are interested in counting the number of common column bases of A_1 and A_2. We may therefore assume that both matrices have rank r and, by possibly removing redundant rows, we may assume that A_1 and A_2 each have r rows. In this thesis we explore the Cauchy-Binet Formula as a tool for counting the common bases of A_1 and A_2. The formula states that:

$$\det A_1 A_2^\top = \sum_{X \subseteq S \colon |X|=r} \det A_1[X] A_2[X].$$

Here $A_i[X]$ denotes the submatrix of A_i indexed by the columns X. Observe that if $\det A_1[X]A_2[X] = 1$ for each common basis X of A_1 and A_2, then $\det A_1 A_2^\top$ counts the number of common bases of A_1 and A_2. This condition is highly restrictive, and we will place the additional restriction that A_1 and A_2 are totally unimodular. That is, we will restrict the counting bases problem to the case that each square nonsingular submatrix of A_1 and A_2 has determinant ± 1. We call the pair (A_1, A_2) of matrices *Pfaffian* if A_1 and A_2 are totally unimodular and either

(i) $\det A_1[X]A_2[X] = 1$ for each common basis X of A_1 and A_2, or

(ii) $\det A_1[X]A_2[X] = -1$ for each common basis X of A_1 and A_2.

We say that the pair (A_1, A_2) of totally unimodular matrices has a *Pfaffian signing* if we can scale some of the columns of A_1 and A_2 by -1 so that (A_1, A_2) becomes Pfaffian. Note that in case (ii), scaling a row of A_2 by -1 results in a pair of matrices satisfying (i), and note also that if (A_1, A_2) has a Pfaffian signing, then it suffices to re-sign the columns of A_2, since re-signing a column of A_1 and the corresponding column of A_2 does not affect whether or not the pair (A_1, A_2) is Pfaffian.

The use of the term "Pfaffian" highlights the connection between Pfaffian matrix pairs and *Pfaffian bipartite graphs*, discussed in Section 3.3. Let $G = (V, E)$ be a bipartite graph with bipartition (U, W) and assume that $|U| = |W|$ and that G has no isolated vertices. Let A_1 and A_2 be the $U \times E$ and $W \times E$ incidence matrices of G. If $X \subseteq E$, then X is a basis of A_i if each vertex in A_i is incident with exactly one edge in X, and thus the common bases of A_1 and A_2 are the perfect matchings of G. Since each column of A_1 and A_2 has exactly one nonzero entry, A_1 and A_2 are totally unimodular and moreover, A_1 and A_2 remain totally unimodular if we replace some of their $+1$ entries with -1. In Section 3.3 we show that G admits a Pfaffian orientation if and only if (A_1, A_2) has a Pfaffian signing. The notion of Pfaffian matroid pairs and their relationship with Pfaffian bipartite graphs is due to Dirk Vertigan, personal communication.

While we are explicitly interested in matrices, many of the concepts are matroidal and matroid theory provides a convenient forum for the discussion. We provide the necessary background in Chapter 2, but will assume that the reader is somewhat familiar with matroids during this introduction. A *matroid pair* is a pair $P = (M_1, M_2)$ of matroids M_1 and M_2, where M_1 and M_2 have equal rank and are defined on the same ground set S. We are interested in the common bases of M_1 and M_2, which we call *bases* of P. Edmonds' Matroid Intersection Theorem [10] states that P has a basis if and only if

$$\mathrm{rank}_{M_1}(S_1) + \mathrm{rank}_{M_2}(S_2) \geq \mathrm{rank}_{M_1}(S)$$

for all partitions $\{S_1, S_2\}$ of S.

A matroid pair $P = (M_1, M_2)$ is called *Pfaffian* if there exists a Pfaffian pair (A_1, A_2) of matrices such that A_1 and A_2 are representations over the reals of M_1 and M_2 respectively; we call such a pair (A_1, A_2) a *Pfaffian representation* of P. Since A_1 and A_2 are totally unimodular, M_1 and M_2 are necessarily regular matroids. A theorem of Camion [3] shows that a regular matroid has an essentially unique totally unimodular representation, which implies that if A_1 and A_2 are totally unimodular representations of regular matroids M_1 and M_2, then (M_1, M_2) is Pfaffian if and only if (A_1, A_2) has a Pfaffian signing.

If G is a bipartite graph, then we will denote the associated matroid pair by $P(G)$; such pairs are called *graphic matroid pairs*, and thus a graphic matroid pair is simply a pair of partition matroids on a common ground set. A *matching covered graph* is a connected graph with the property that each edge of the graph is in a perfect matching and it is natural to restrict the study of Pfaffian bipartite graphs to matching covered graphs. We introduce an analogous notion for matroid pairs: a matroid pair $P = (M_1, M_2)$ is *connected* if $\mathrm{rank}_{M_1}(S_1) + \mathrm{rank}_{M_2}(S_2) > \mathrm{rank}_{M_1}(S)$ for each proper partition $\{S_1, S_2\}$ of S. In Theorem 3.8 we show that if G is a bipartite graph with a perfect matching, then G is matching covered if and only if the graphic matroid pair $P(G)$ is connected.

The study of Pfaffian graphs began in 1963 when, in the context of crystal physics, Kastelyn [20] proved that every planar graph is Pfaffian. More recently, Robertson, Seymour, and Thomas [35] and, independently, McCuaig [28] proved that all Pfaffian bipartite graphs can be constructed from planar graphs and copies of a particular non-planar graph called the Heawood graph, by certain compositions across 4-element sets. In addition to providing a good characterization for Pfaffian bipartite graphs, these results also provide an efficient algorithm for recognizing Pfaffian bipartite graphs. The study of Pfaffian pairs is still in its infancy, and we do not yet know if these remarkable results for bipartite graphs extend to matroids.

Preceding the decomposition results mentioned above, Little [22] proved a beautiful partial characterization of Pfaffian bipartite graphs as those bipartite graphs which do not "contain" an odd subdivision of $K_{3,3}$ (see Section 3.3). In the context of matroid pairs, this characterization of Pfaffian bipartite graphs can be viewed as an excluded minor characterization. Let $P = (M_1, M_2)$ be a matroid pair and let X and Y be disjoint subsets of S. If there exists a basis B of P such that $Y \subseteq B$ and $B \cap X = \emptyset$, then we call the pair $P\backslash X/Y = (M_1\backslash X/Y, M_2\backslash X/Y)$ a *minor* of P. The *dual* of a matroid pair $P = (M_1, M_2)$ is the matroid pair $P^* = (M_1^*, M_2^*)$ and Theorems 3.4 and 3.5 show that the class of Pfaffian matroid pairs is closed under duality and minors. With this description of a minor a matroid pair, Little's Theorem then says that a bipartite graph G is non-Pfaffian if and only if the graphic matroid pair $P(G)$ contains a minor $P(H)$ where H is an odd subdivision of $K_{3,3}$. Thus Little's Theorem provides a succinct method for proving that a matroid pair is non-Pfaffian, but it does not help to prove that a pair is Pfaffian.

The problem of finding an excluded minor characterization for Pfaffian matroid pairs remains open. There are already infinitely many excluded minors coming from the class of graphic matroid pairs, and the existence of many other excluded minors for the class of Pfaffian matroid pairs, such as those discussed in Sections 6.6 and 6.7, suggests that characterizing the excluded minors for Pfaffian matroid pairs may be complicated.

The goals of this thesis are two-fold:

(1) Understand how the different Pfaffian signings of a given Pfaffian matroid pair are related.

(2) Find succinct certificates for proving that a given matroid pair is non-Pfaffian.

Equivalent Pfaffian signings

It is well known that any one Pfaffian orientation of a matching covered bipartite graph G can be obtained from any other Pfaffian orientation by reversing the orientation across some cut in G. This property does not extend to general (non-bipartite) Pfaffian graphs, and despite considerable effort, Little's Theorem and the decomposition of Robertson, Seymour, Thomas, and McCuaig have resisted generalization to general graphs. The existence of "inequivalent" Pfaffian orientations is often cited as the root of the difficulty; similar difficulties arise for Pfaffian matroid pairs.

Let $P = (M_1, M_2)$ be a Pfaffian matroid pair on the ground set S and let (A_1, A_2) be a Pfaffian representation of P. By *resigning* (A_1, A_2) on X we mean scaling by a factor of -1 the columns of A_2 indexed by X. A set $X \subseteq S$ is called a *constant-parity intersecting* *set of* P if either $|X \cap B|$ is even for all bases B of P, or $|X \cap B|$ is odd for all bases B of P. Note that resigning a Pfaffian representation on $X \subseteq S$ gives another Pfaffian representation if and only if X is a constant-parity intersecting set of P.

Suppose that $X_1 \subseteq S$ is a *separator* of M_1; that is, $\text{rank}_{M_1}(X_1) + \text{rank}_{M_1}(S \backslash X_1) = \text{rank}(M_1)$. Then for each basis B of M_1, we have $|B \cap X_1| = \text{rank}_{M_1}(X_1)$, and therefore each separator of M_1 or of M_2 is a constant-parity intersecting set of P. Moreover, the symmetric difference of two constant-parity intersecting sets gives another constant-parity intersecting set. However, it is not always the case that a constant-parity intersecting set for P can be expressed as the symmetric difference of separators of M_1 and M_2, as the following example shows: let A_1 and A_2 be the rank 3 totally unimodular matrices given by

$$A_1 = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array} \text{ and } A_2 = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{pmatrix} \end{array}.$$

If M_1 and M_2 are the matroids represented by A_1 and A_2 respectively, then it is easy to check that $X = \{4, 5, 6\}$ is a constant-parity intersecting set of the matroid pair $P = (M_1, M_2)$. However, each separator of M_1 and M_2 has even size, and therefore X cannot be written as the symmetric difference of separators of M_1 and M_2. This matroid pair is an example of a "twined K_4" pair given in Example 2 of Section 3.5 and discussed further in Section 4.4. The following theorem is one of the main contributions of the thesis.

Theorem 1.1. *Let $P = (M_1, M_2)$ be a connected matroid pair on the ground set S where M_1 and M_2 are binary. If P has no twined K_4 minor, then $X \subseteq S$ is a constant-parity intersecting set of P if and only if there exist separators X_1 of M_1 and X_2 of M_2 such that $X = X_1 \Delta X_2$.*

Theorem 1.1 is proved in Section 4.4 as a consequence of a characterization of the binary space generated by the bases of P.

In Chapter 5 we discuss a closely related property to constant-parity intersecting sets called *constant-size intersecting sets*: a subset X of the ground set for the matroid pair P is a constant-size intersecting set of P if there exists an integer k such that $|X \cap B| = k$ for each basis B of P. In Theorem 5.4 we characterize the constant-size intersecting sets of any connected matroid pair $P = (M_1, M_2)$ in terms of the separators of M_1 and M_2. Theorem 5.4, unlike Theorem 1.1, does not require that M_1 and M_2 are binary and does not require the exclusion of twined K_4's.

Certifying non-Pfaffian pairs

Let $P = (M_1, M_2)$ be a matroid pair on the ground set S. We would like to determine whether or not P is a Pfaffian matroid pair. By Seymour's decomposition of regular matroids [39], we can efficiently check that M_1 and M_2 are regular and can then efficiently find totally unimodular representations A_1 and A_2 of M_1 and M_2 respectively. It remains to determine whether (A_1, A_2) has a Pfaffian signing. This problem can, abstractly, be reformulated using linear algebra.

Let $\mathcal{B}(P)$ denote the set of all bases of P and let K be the $\mathcal{B}(P) \times S$ matrix over $GF(2)$ whose rows are the characteristic vectors for the bases of P. For each $B \in \mathcal{B}(P)$ we define α_B and α'_B in $GF(2)$ such that $\det A_1[B] \det A_2[B] = (-1)^{\alpha_B}$ and $\alpha'_B = 1 + \alpha_B$. The matrix pair (A_1, A_2) then has a Pfaffian signing if an only if there exists a vector x over $GF(2)$ such that either $Kx = \alpha$ or $Kx = \alpha'$. Unfortunately this is not a practical reformulation since K may have exponentially many rows and we do not explicitly know the set of bases of P. However, since K has linear rank, this formulation can be used to succinctly certify that a matroid pair is non-Pfaffian.

A set $\mathcal{C} \subseteq \mathcal{B}(P)$ is called *inconsistent* with respect to α for P if there is no set $X \subseteq S$ such that $|X \cap B| \equiv \alpha_B \mod 2$ for all $B \in \mathcal{C}$. It follows that if P has an inconsistent set with respect to α and P has an inconsistent set with respect to α', then P is non-Pfaffian. The union of two such inconsistent sets is called a *non-Pfaffian bases certificate.* The following theorem is proved in Section 3.4:

Theorem 1.2. *If $P = (M_1, M_2)$ is a matroid pair on the ground set S and M_1 and M_2 are both regular matroids, then either P is Pfaffian or P has a non-Pfaffian bases certificate containing at most $2|S| + 2$ bases.*

For graphic matroid pairs, Little's Theorem [22] (Section 3.3) implies the much stronger result that the number of bases required in a non-Pfaffian bases certificate is six. We conjecture that this general linear bound for certifying that a matroid pair is non-Pfaffian can be improved to a constant bound. The constant in our conjecture must be at least six, and as of yet we do not have an example of a non-Pfaffian matroid pair which shows that the constant must be greater than six.

Conjecture 1.3. *There exists a constant c such that if P is the matroid pair (M_1, M_2) and M_1 and M_2 are regular, then either P is Pfaffian or there is a non-Pfaffian bases certificate for P with at most c bases.*

Aesthetically, Little's Theorem is preferable to Theorem 1.2, and our progress toward a better characterization for non-Pfaffian matroid pairs has been hampered by inequivalent signings. In light of Theorem 1.1, it is natural to restrict our focus to the class of matroid pairs with no twined K_4 minor. We impose the stronger condition that neither M_1 nor M_2 contains an $M(K_4)$ minor. That is, we consider the special case where M_1 and

M_2 are series-parallel matroids. This class still contains the graphic matroid pairs and is closed with respect to duality and minors, and for this class we reformulate the problem of determining whether or not P is Pfaffian to questions concerning the cycle space of a certain directed graph. We then apply an unpublished theorem of Gerards to obtain a nicer characterization involving "odd double-cycles"; see Theorem 6.22. (Gerards theorem is a generalization of a well-known theorem of Seymour and Thomasson [38] on even cycles in directed graphs.) While we are unfortunately unable to prove Conjecture 3.12 for the series-parallel class, we make significant progress; see Chapter 6.

Chapter 2

Preliminaries

Most of the matroids considered in this thesis are *representable* matroids, particularly the matroid pairs in the class we call Pfaffian matroids, and thus most of our results are results about matrices. The language of matroids is therefore not required, however the use of matroid terminology allows the application of matroid defined operations like duality, minors, and pivoting. Here we formally define matroids and the matroid intersection problem. Using the *fundamental graph*, a well known graphical representation for a single matroid, we describe a graphical representation of a pair of matroids. This graphical representation of a matroid pair will be used extensively in Chapters 3, 4, and 6.

We explain the connection between totally unimodular matrices and the matroid class we are restricting the counting bases problem to, namely *regular* matroids, and we state some fundamental theorems of Tutte which highlight the essential role of regular matroids within matroid theory. The final section of this chapter is devoted to the notion of connectivity in matroids and the relationship between matroid connectivity and connectivity in the directed graph we use to represent pairs of matroids; in Theorem 2.8 we show that this directed graph is strongly connected if and only if the matroid pair is connected.

2.1 Matroids

A matroid M consists of a ground set S and a nonempty collection $\mathcal{I}$ of subsets of S satisfying the following two axioms:

(i) If $I_1 \subseteq I_2$ and $I_2 \in \mathcal{I}$ then $I_1 \in \mathcal{I}$.

(ii) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$ then there exists $e \in I_2 \backslash I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

The elements of $\mathcal{I}$ are called the *independent* sets of M and a subset of S which is not independent is *dependent*.

A common example of a matroid comes from linear algebra. Let A be a matrix over a fixed field and let S be the set of column indices of A. If $\mathcal{I}$ is defined to be all subsets of columns of A that form an independent set, then $(S, \mathcal{I})$ is a matroid. A matroid that can be represented by a matrix in this way is called *representable* and the representable matroids are a fundamental class in matroid theory. We will be working mainly with *binary* matroids (matroids representable over $GF(2)$) and with *regular* matroids, which are matroids representable over the reals by a *totally unimodular* matrix. A matrix is totally unimodular if and only if each square nonsingular submatrix has determinant $+1$ or -1; totally unimodular matrices are discussed in Section 2.5.

Another fundamental class of matroids comes from graph theory: if G is a graph and a subset of the edges of G is called independent if and only if it induces an acyclic subgraph in G, then the independent sets of edges satisfy axioms (i) and (ii). A matroid defined on the edges of a graph G in this way is called the *cycle matroid* for G, and is denoted $M(G)$. The *graphic matroids* are those matroids which can be represented by a cycle matroid for a graph, and the graphic matroids are a subclass of the regular matroids (see Oxley [31]). The maximal independent sets of a graphic matroid correspond to spanning forests of the graph, and an inclusion minimal dependent set in a graphic matroid corresponds to a circuit in the graph.

Matroids were first considered by Whitney [46] in 1935 as a generalization of this notion of independence in graphs and linear independence. Terms from linear algebra and graph theory transfer to matroid theory: a *basis* of a matroid is an independent set of maximum size, the *rank* of a subset S' of the ground set of a matroid is the maximum size of an independent set in S', and a *circuit* is an inclusion minimal dependent set. If M is a matroid on the ground set S and $S' \subseteq S$, then we will denote the rank of S' by $\text{rank}_M(S')$. The rank of S is called the rank of the matroid and is denoted $\text{rank}(M)$. An element with rank 0 is a *loop*. Axiom (ii) implies that all bases of a matroid have the same number of elements, and since the independent sets in a matroid are all subsets of a basis of the matroid, a matroid is completely determined by its bases.

The operations of duality and minors from graph theory also apply to matroid theory.

Duality: If G is a planar graph and G^* is a planar dual of G, then there is a natural bijection between the edges of G and the edges of G^*. A set F of edges in G is a spanning forest in G if and only if the complement of the corresponding edges in G^* is a spanning forest in G^*. Similarly, if M is a matroid on the ground set S and $B \subseteq S$, then $S \backslash B$ is a basis of the matroid dual M^* for M if and only if B is a basis for M.

Deletion: If e is an element in the ground set for the matroid M, then the bases for the matroid formed by deleting e are exactly those bases of M that do not contain the element e. This matroid is denoted $M \backslash \{e\}$. If A is a matrix and M is the matroid represented by A, then deleting an element of M corresponds to deleting the corresponding column of A. If $M = M(G)$ then $M\backslash\{e\}$ is equivalent to the cycle matroid for $G\backslash\{e\}$.

Contraction: If e is a loop of the matroid M then $M/\{e\} = M\backslash\{e\}$. Otherwise, the notion of contraction in a matroid is defined via duality and deletion: if e is not a loop of M then $M/\{e\} = (M^*\backslash\{e\})^*$. It follows that B is a basis of $M/\{e\}$ if and only if $B \cup \{e\}$ is a basis of M. When $M = M(G)$ then $M/e = M(G/e)$. Let M be the matroid represented by the matrix A and let e index a column of A. If e has one nonzero entry, then contracting e from M corresponds to deleting the row of A containing the nonzero entry for column e and deleting column e. Row operations on A do not affect column dependencies, and thus if A' is obtained from A by a series of row operations then A and A' represent the same matroid. It follows that if e is not a loop of M then to obtain a representation of $M/\{e\}$ we may first perform row operations such that column e in A has exactly one non-zero entry. These row operations are formalized by the matroid operation called *pivoting* which is discussed in Section 2.4.

2.2 Matroid intersection

Given two matroids M and N on the same ground set S, the matroid intersection problem is to find a common independent set of maximum size. If $\{S_1, S_2\}$ is a partition of S and $X \subseteq S$ is independent in both M and N, then

$$|X| = |X \cap S_1| + |X \cap S_2| \leq \operatorname{rank}_M(S_1) + \operatorname{rank}_N(S_1). \tag{2.1}$$

Thus the maximum size of a common independent set is at most the minimum of $\operatorname{rank}_M(S_1) + \operatorname{rank}_N(S_1)$ over all partitions $\{S_1, S_2\}$ of S. A beautiful theorem of Edmonds [10] proves that there is an independent set $X \subseteq S$ and a partition $\{S_1, S_2\}$ of S for which (2.1) is met with equality.

Theorem 2.1 (Edmonds' Matroid Intersection Theorem). *If M and N are matroids on the ground set S then the maximum size of a common independent set in M and N is equal to the minimum of $\operatorname{rank}_M(S_1) + \operatorname{rank}_N(S_2)$ over all partitions $\{S_1, S_2\}$ of S.*

Corollaries of this theorem include König's theorem for the maximum size of a matching in a bipartite graph and Menger's theorem for the maximum number of vertex disjoint directed paths between two vertex subsets of a directed graph. Assuming independence can be checked in polynomial time with respect to the size of the ground set, results of Edmonds [9], [10] prove that there is a polynomial algorithm for finding a common independent set of maximum size in two matroids. If a weight is assigned to each element in the ground set, then the weighted version of the matroid intersection problem is to find a common independent set of maximum weight. This has a similarly elegant solution and will be discussed in Section 3.7.

If M and N are matroids with the same rank on the same ground set S then we call $P = (M, N)$ a *matroid pair*. The rank of P is the common rank of M and N and the *bases* of P are the common bases of M and N. Edmonds' Matroid Intersection Theorem clearly determines if two matroids have a common basis and the problem of deciding if P has a basis can be efficiently solved.

2.3 Fundamental graphs

We present a well known graphical representation of a matroid, called the fundamental graph, and give a method for combining the fundamental graphs for two matroids on the same ground set into one graphical representation of a matroid pair. The transfer of properties of a matroid pair to properties of the graphical representation for the pair are discussed, and fundamental results required for later chapters are given.

Let M be a matroid on the ground set S and let B be a basis of M. The *partial representation* of M with respect to B is the $\{0,1\}$ matrix R with rows indexed by the elements of B, columns indexed by the elements of $S\backslash B$, and $[R]_{i,j} = 1$ if and only if i belongs to the unique circuit in $B \cup \{j\}$. The bipartite graph with adjacency matrix R is called the *fundamental graph* for M, and we denote this graph by $G(M, B)$. If $v \in B$ then deleting row v from R leaves the partial representation of M/v with respect to $B\backslash v$ and if $v \in S\backslash B$ then deleting column v from R leaves the partial representation for $M\backslash v$ with respect to B. Thus

$$G(M,B)\backslash\{v\} = \begin{cases} G(M/v, B\backslash\{v\}), & \text{if } v \in B; \\ G(M\backslash v, B), & \text{if } v \in S\backslash B. \end{cases} \tag{2.2}$$

If H is an induced subgraph of $G(M, B)$ then it follows from (2.2) that there exists a minor M' of M and a basis B' of M' such that $H = G(M', B')$. If M^* is the dual of M and R is the partial representation of M with respect to the basis B, then R^T is the partial representation of M^* with respect to $S\backslash B$. Therefore the fundamental graph $G(M, B)$ is isomorphic to $G(M^*, S\backslash B)$.

When M is a binary matroid, the matrix $[I|R]$ is a representation of M over $GF(2)$. If S is the ground set for the binary matroid M and $F \subseteq S$, it follows that F is a basis of M if and only if $|F| = |B|$ and $R[B\backslash F; F\backslash B]$ is nonsingular. Here $R[B\backslash F; F\backslash B]$ denotes the submatrix of R indexed by rows $B\backslash F$ and columns $F\backslash B$; the union of $B\backslash F$ and $F\backslash B$ is denoted $B\Delta F$. If A is the adjacency matrix for the bipartite graph G then there is a one-to-one correspondence between perfect matchings in G and non-zero terms in the permutation expansion of the determinant of A and thus the following observation is immediate:

Observation 2.2. *If M is a binary matroid on the ground set S and B is a basis of M, then $F \subseteq S$ is a basis of M if and only if the subgraph of $G(M,B)$ induced by $B\Delta F$ has an odd number of perfect matchings.*

Given a matroid pair $P = (M,N)$, and a basis B of P, the directed bipartite graph $G(P,B)$ is constructed as follows: direct the edges of the fundamental graph $G(M,B)$ from B to $S\backslash B$, direct the edges of the fundamental graph $G(N,B)$ from $S\backslash B$ to B, and identify the vertices of $G(M,B)$ and $G(N,B)$. The resulting graph is called the *fundamental graph* for the matroid pair P. For example, let $P = (M,N)$ be the matroid pair with basis B and let M and N have partial representations R_M and R_N with respect to B, as shown below:

$$R_M = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}, \qquad R_N = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

The fundamental graph $G(P,B)$ is isomorphic to the graph in Figure 2.1, where the black vertices correspond to the elements of B.

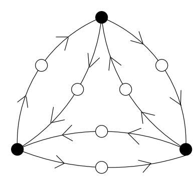

Figure 2.1: $G(P,B)$

Given a matroid pair P on the ground set S, a subset F of S is called *contributing with respect to the basis* B if $B\Delta F$ is a basis of P. A subgraph H of the fundamental graph $G(P,B)$ for P is contributing if $V(FH)$ is contributing with respect to B. Thus there is a bijection between induced contributing subgraphs of $G(P,B)$ and bases of P. When the basis B of P is arbitrary we will denote $G(P,B), G(M,B)$, and $G(N,B)$ by $G(P), G(M)$, and $G(N)$ respectively.

Let P be a matroid pair and let D be a set of vertex disjoint directed circuits in $G(P)$. Then $E(D)$ induces a perfect matching in $G(M)[V(D)]$ and $E(D)$ induces a perfect matching in $G(N)[V(D)]$. Denote these matchings by $T_M(D)$ and $T_N(D)$ respectively. If D' is a set of vertex disjoint directed circuits in $G(P)$ with $V(D) = V(D')$, then $D \neq D'$ if and only if either $T_M(D) \neq T_M(D')$ or $T_N(D) \neq T_N(D')$. Thus D is the unique set of vertex disjoint directed circuits in $G(P)[V(D)]$ if and only if $G(M)[V(D)]$ and $G(N)[V(D)]$ have unique perfect matchings. It follows from Observation 2.2that if D is the unique set of vertex disjoint directed circuits in $G(P)[V(D)]$, then D is contributing. When D is the unique set of vertex disjoint directed circuits in $G(P)[V(D)]$ then we call D a *simple contributing* subgraph of $G(P)$. In Chapter 6 we will consider a class of matroid pairs P for which every contributing circuit in $G(P)$ is simple.

If M^* and N^* are the dual matroids of M and N respectively then the matroid pair (M^*, N^*) is called the *dual* of P and is denoted P^*. Since $G(M^*, S\backslash B)$ is isomorphic to $G(M, B)$ and $G(N^*, S\backslash B)$ is isomorphic to $G(N, B)$, the graph $G(P, B)$ is isomorphic to the graph obtained by reversing the orientation of the edges in $G(P^*, S\backslash B)$.

If P is the matroid pair (M, N) and A_M and A_N are representations of M and N respectively, then (A_M, A_N) is called a representation of P. If A_M does not have full row rank then a row of A_M can be deleted without affecting the column dependencies in M; thus if A_M is a representation of M then we will assume that A_M has full row rank. Similarly, if (A_M, A_N) is a representation of a matroid pair then we assume that both A_M and A_N have full row rank. If B is a basis of P and the matrices R_M and R_N are the partial representations of M and N with respect to B, then (R_M, R_N) is referred to as the partial representation of P with respect to B.

2.4 Pivoting

Given bases B and B' of a matroid M, we would like to know the relationship between $G(M, B)$ and $G(M, B')$. If R is the partial representation of M with respect to B and R' is the partial representations of M with respect to B', then determining the relationship between $G(M, B)$ and $G(M, B')$ is equivalent to determining the relationship between R and R'. The operation that moves from one partial representation of a matroid to another is called *pivoting* and is discussed in this section.

Suppose that M is a binary matroid on the ground set S with basis B. Let $X \subseteq B$ and $Y \subseteq S\backslash B$ be such that $B' = B\Delta(X \cup Y)$ is a basis of M. If R is the partial representation of M with respect to B and $W = R[X; Y]$, then up to permuting rows and columns, we may assume that R has the block decomposition

$$R = \begin{bmatrix} W & C \\ E & D \end{bmatrix}.$$

Pivoting R on $X \cup Y$ is the matrix operation which results in the matrix R', where

$$R' = \begin{bmatrix} W^{-1} & W^{-1}C \\ -EW^{-1} & D - EW^{-1}C \end{bmatrix}.$$

If $A = [I|R]$ then A is a representation of M over $GF(2)$ and pivoting A on $X \cup Y$ results in the matrix A', where

$$\begin{aligned} A' &= \begin{bmatrix} W^{-1} & 0 \\ -EW^{-1} & I \end{bmatrix} A \\ &= \begin{bmatrix} W^{-1} & 0 \\ -EW^{-1} & I \end{bmatrix} \begin{bmatrix} I & 0 & W & C \\ 0 & I & E & D \end{bmatrix} \\ &= \begin{bmatrix} W^{-1} & 0 & I & W^{-1}C \\ -EW^{-1} & I & 0 & D - EW^{-1}C \end{bmatrix}. \end{aligned}$$

Note that the columns of A' indexed by B' are a permutation of the identity matrix, and that a permutation of the columns of A' results in the matrix $[I|R']$. Since A' was obtained by multiplying A by a nonsingular matrix, A' is a representation of M over $GF(2)$ and thus R' is the partial representation of M with respect to B'. We now consider how this pivoting operation affects $G(P, B)$. It suffices to consider the case of pivoting R on $\{u, v\}$ for some $u \in B, v \in S \backslash B$ with $R[u; v] \neq 0$.

Let $u \in B$ index the first row of R and let $v \in S \backslash B$ index the first column of R and assume that

$$R = \begin{bmatrix} 1 & c^T \\ e & D \end{bmatrix}. \tag{2.3}$$

If R' is the matrix obtained by pivoting R on $\{u, v\}$, then over $GF(2)$,

$$R' = \begin{bmatrix} 1 & c^T \\ e & D + ec^T \end{bmatrix}. \tag{2.4}$$

Since $R[u; v] \neq 0$, the set $B' = B \Delta \{u, v\}$ is a basis of M and R' is the partial representation of M with respect to B'.

If x is a vertex in the graph G then let $N(x)$ denote the vertices adjacent to x in G. By considering Equations (2.3) and (2.4) we determine that $G(M, B')$ is obtained from $G(M, B)$ by complementing the edges between $N(u) \backslash \{v\}$ and $N(v) \backslash \{u\}$ and swapping the labels on u and v. Figure 2.2 shows the graphs G and G' where G' is obtained by pivoting on the edge uv in G. Note that the graph obtained by pivoting twice on the edge uv in G is isomorphic to G.

2.5 Totally unimodular graphs

An *edge signing* of a graph is a $\{0, 1\}$ function on the edges of the graph. In this section we consider the relationship between edge signings of the graph $G(P)$ for a matroid pair P and totally unimodular representations of P. These edge signings play a major role in Chapters 3 and 6. We also give some well known and fundamental theorems for regular matroids as a motivation for restricting the counting bases problem to a subclass of regular matroids.

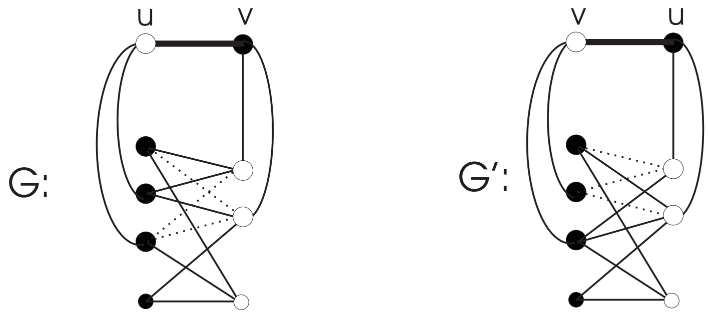

Figure 2.2: Pivoting G on the edge uv

A matrix A over the reals is *totally unimodular* if every square submatrix of A has determinant 0, +1, or −1. A *signing* of a $\{0,1\}$ matrix A is a matrix A' obtained from A by changing some of the nonzero entries to -1; a $\{0,1\}$ matrix is called *regular* if it has a totally unimodular signing. A *scaling* of a is obtained by scaling some of its rows and columns by -1; Camion [3] observed that a totally unimodular signing of a regular matrix is unique up to scaling.

A matroid is regular if it can be represented over $GF(2)$ by a regular matrix. Equivalently, a matroid is regular if it can be represented over the reals by a totally unimodular matrix. The property of being totally unimodular is closed under pivoting (see Cornuéjouls [5]) and since the matrix R is regular if and only if $[I|R]$ is regular, determining if a binary matroid M is regular is equivalent to determining if a partial representation of M is regular. If P is the matroid pair (M,N) then we say that P is regular if M and N are regular. If (A_M, A_N) is a representation of P then (A_M, A_N) is called a totally unimodular representation of P if both A_M and A_N are totally unimodular matrices. Regular matroids are a fundamental class in matroid theory, as the next theorem indicates.

Theorem 2.3 (Tutte [43]). *Let M be a matroid. The following are equivalent:*

1. *M is regular.*

2. *M is representable over every field.*

3. *M is representable over $GF(2)$ and $GF(3)$.*

An edge signing s of the graph G is *even* if $s_e = 0$ for all edges e in G. There is a natural bijection between edge signings s of G and signings A' of the adjacency matrix for G given by $s_e = 1$ if and only if the entry for e in A' is -1. If A' is a totally unimodular signing of the adjacency matrix for G then the associated edge signing of G is called a totally unimodular signing of G. If $P = (M,N)$ is a matroid pair with basis B, then an edge signing of $G(P,B)$ is totally unimodular if the edge signing restricted to $G(M,B)$ is totally unimodular and the edge signing restricted to $G(N,B)$ is totally unimodular. By

Camion's result on the uniqueness of totally unimodular signings, a totally unimodular edge signing of a graph is unique up to resigning across cuts. Thus if the graph G has a totally unimodular edge signing and F is a forest of G, then the totally unimodular edge signing can be fixed on $E(F)$.

Let F be a maximal forest of the graph G and let $\{e_1, e_2, \ldots, e_k\}$ be an ordering of $E(G)\backslash E(F)$ such that for $i = 1, \ldots, k$, there is a circuit C_i through e_i satisfying both of the following:

- $E(C_i) \subseteq E(F) \cup \{e_1, e_2, \ldots, e_i\}$, and
- C_i is an induced circuit in G.

This can be done, for example, by selecting e_1 such that the path in F connecting the end vertices of e_1 is minimal, and for $j \geq 2$ selecting e_j such that the path in $E(F) \cup \{e_1, e_2, \ldots, e_{i-1}\}$ is minimal. If all the edges of $C_i\backslash\{e_i\}$ are signed, then there is a unique sign for e_i that results in a totally unimodular signing of C_i. Thus if G has a totally unimodular signing then we can find such a signing of G by arbitrarily signing the edges of F and then signing each e_i in turn such that the edge signing restricted to C_i is totally unimodular. This algorithm is called *Camion's signing algorithm* and a $\{0, 1\}$ matrix A is regular if and only if Camion's signing algorithm returns a totally unimodular signing of A.

The smallest example of a $\{0, 1\}$ matrix that is not regular comes from the *Fano matroid*, denoted F_7. A partial representation for the Fano matroid is the matrix R, given below:

$$R = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

The bipartite graph G with adjacency matrix R is shown in Figure 2.3. Camion's signing

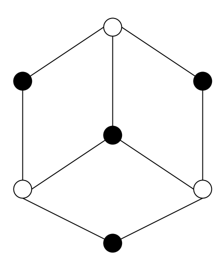

Figure 2.3: $G(F_7, B)$

algorithm applied to G and R results in a signing which is not totally unimodular, and thus R is not a regular matrix and the Fano matroid is not a regular matroid. A theorem of Tutte states that F_7 and the dual of F_7 are the only obstructions for binary matroids to be regular.

Theorem 2.4 (Tutte [43]). *A binary matroid M is regular if and only if M does not contain a minor isomorphic to the Fano or the dual of the Fano.*

There is a short proof of Theorem 2.4 due to Gerards [14].

2.6 Connectivity

We define the well known notion of connectivity for matroids and matroid pairs, and prove that the fundamental graph for a matroid pair (see Section 2.3) is strongly connected if and only if the matroid pair is connected.

A rank r matroid M on the ground set S is *connected* if

$$\text{rank}_M(S_1) + \text{rank}_M(S_2) > r$$

for all partitions $\{S_1, S_2\}$ of S with $S_1 \neq \emptyset$ and $S_1 \neq S$. If $\text{rank}_M(S_1)+\text{rank}_M(S_2) = r$ for some partition $\{S_1, S_2\}$ of S, then S_1 and S_2 are *separators* of M. Clearly both the empty set and S are separators of M, and a separator S_1 of M is called a *nontrivial separator* if $S_1 \neq \emptyset$ and $S_1 \neq S$. If restricting M to a nontrivial separator gives a connected matroid, then the separator is a *component* of M. The components of the graphic matroid $M(G)$ correspond to the blocks (2-connected components) of the graph G, and if M is a representable matroid with matrix representation A then the components of M correspond to the blocks of A. When a matroid is not connected then there is a unique partition of its ground set into components. A common technique in proving results about matroids is to assume the matroids are connected. Here we give an analogous connectivity definition for matroid pairs and we show that for the problem of counting common bases in a pair of matroids, we may assume that the matroid pair is connected.

If P is the matroid pair (M, N), then by Edmonds' Matroid Intersection Theorem (Theorem 2.1), P has a basis if and only if $\text{rank}_M(S_1) + \text{rank}_N(S_2) \geq r$ for all partitions $\{S_1, S_2\}$ of S. We say that P is *connected* if

$$\text{rank}_M(S_1) + \text{rank}_N(S_2) > r$$

for all nontrivial partitions $\{S_1, S_2\}$ of S. If $\{S_1, S_2\}$ is a partition of S such that

$$\text{rank}_M(S_1) + \text{rank}_N(S_2) = r,$$

then S_1 and S_2 are called *separators* of P. In particular, if $S_1 \subseteq S$ is a separator of M and a separator of N, then S_1 is a separator of P, and if S_1 is non-trivial then P is not connected. This notion can be generalized for higher connectivity: the matroid pair $P = (M, N)$ is *k-connected* if

$$\text{rank}_M(S_1) + \text{rank}_N(S_2) > r + k - 1$$

for all partitions $\{S_1, S_2\}$ satisfying $|S_1| \geq k$ and $|S_2| \geq k$.

The next lemma shows that if we are interested in counting bases of a matroid pair and the pair is not connected, then we can restrict the counting problem to two smaller matroid pairs.

Theorem 2.5. *Let $P = (M, N)$ be a rank r matroid pair on the ground set S, and let $\{S_1, S_2\}$ be a partition of S such that*

$$\operatorname{rank}_M(S_1) + \operatorname{rank}_N(S_2) = r.$$

If $U_1 \subseteq S_1$ and $U_2 \subseteq S_2$, then $U_1 \cup U_2$ is a basis of P if and only if U_1 is a basis of the pair $(M \backslash S_2, N/S_2)$ and U_2 is a basis of the pair $(M/S_1, N \backslash S_1)$.

Proof. Assume that U_1 is a basis of $M \backslash S_2$ and U_2 is a basis of M/S_1, and let $r_1 = \operatorname{rank}_M(S_1)$ and $r_2 = \operatorname{rank}_{M/S_1}(S_2)$. Then $r_1 + r_2 = r$, and $|U_1 \cup U_2| = |U_1| + |U_2| = r$. To show that $U_1 \cup U_2$ is a basis of M it therefore suffices to show that $U_1 \cup U_2$ has rank r in M.

Let $X = S_1 \backslash U_1$ and let $x \in X$. Since

$$\operatorname{rank}_{M/U_1}(x) = \operatorname{rank}_M(U_1 \cup \{x\}) - \operatorname{rank}_M(U_1) = 0,$$

every element of X is a loop of M/U_1 and therefore

$$\operatorname{rank}_{M/U_1}(U_2 \cup X) = \operatorname{rank}_{M/U_1}(U_2). \tag{2.5}$$

Furthermore,

$$\begin{aligned} \operatorname{rank}_{M/U_1}(U_2 \cup X) &= \operatorname{rank}_{(M/U_1)/X}(U_2) + \operatorname{rank}_{M/U_1}(X) \\ &= \operatorname{rank}_{M/S_1}(U_2) + 0 \\ &= r_2. \end{aligned} \tag{2.6}$$

Combining Equations (2.5) and (2.6) gives that $\operatorname{rank}_{M/U1}(U_2) = r_2$. Thus

$$\begin{aligned} \operatorname{rank}_M(U_1 \cup U_2) &= \operatorname{rank}_M(U_1) + \operatorname{rank}_{M/U_1}(U_2) \\ &= r_1 + r_2 \\ &= r, \end{aligned}$$

and therefore $U_1 \cup U_2$ is a basis of M. By symmetry, if U_2 is a basis of $N \backslash S_1$ and U_1 is a basis of N/S_2, then $U_1 \cup U_2$ is a basis of N. Therefore if U_1 is a basis of $(M \backslash S_2, N/S_2)$ and U_2 is a basis of $(M/S_1, N \backslash S_1)$, then $U_1 \cup U_2$ is a basis of P.

Conversely, suppose that $U_1 \cup U_2$ is a basis of P with $U_1 \subseteq S_1$, $U_2 \subseteq S_2$ and let $r_1 = \operatorname{rank}_M(S_1)$ and $r_2 = \operatorname{rank}_N(S_2)$. Since $\operatorname{rank}_M(S_1) + \operatorname{rank}_N(S_2) = e$ by assumption, $r_1 + r_2 = r$. If B is a basis of P, then $|B \cap S_1| \leq r_1$ and $|B \cap S_2| \leq r_2$. Since

$|B \cap S_1| + |B \cap S_2| = r$, it follows that $|B \cap S_1| = r_1$ and $|B \cap S_2| = r_2$ for all bases B of P. In particular, $|(U_1 \cup U_2) \cap S_i| = r_i$ and therefore $|U_i| = r_i$ for $i = 1, 2$.

Since U_1 is independent in $M \backslash S_2$, $\text{rank}_{M\backslash S_2}(U_1) = |U_1| = r_1$ and thus U_1 is a basis of $M\backslash S_2$. Furthermore, $\text{rank}_{M/S_1}(U_2) = \text{rank}_M(U_2 \cup S_1) - \text{rank}_M(S_1) = r - r_1 = r_2$, and thus U_2 is a basis of M/S_1. By symmetry, U_1 is a basis of N/S_2 and U_2 is a basis of $N\backslash S_1$. □

Let $P = (M, N)$ be a rank r matroid on the ground set S and assume that $\{S_1, S_2\}$ is a partition of S with $\text{rank}_M(S_1) + \text{rank}_N(S_2) \le r$. If $P^* = (M^*, N^*)$ is the dual matroid pair of P then P^* has rank $r^* = |S| - r$, and

$$\begin{aligned}\text{rank}_{M^*}(S_2) + \text{rank}_{N^*}(S_1) &= (|S_2| - r + \text{rank}_M(S_1)) + (|S_1| - r + \text{rank}_N(S_2)) \\ &= |S| - 2r + \text{rank}_M(S_1) + \text{rank}_N(S_2) \\ &\le r^*.\end{aligned}$$

Observation 2.6. *The matroid pair P is connected if and only if the dual pair P^* is connected.*

When considering whether a bipartite graph is Pfaffian, an edge which is not in any perfect matching can be deleted. In particular, if an edge is in every perfect matching then we can delete both vertices of the edge and consider the remaining graph. The matroid analog to an edge which is not in any perfect matching is a *loop* of a matroid pair: a loop of the matroid pair P is an element e of the ground set for P such that no basis of P contains e. Similarly, the matroid analog to an edge in every perfect matching of a graph is a *coloop*: an element e of the ground set is a coloop of P if every basis of P contains e. Note that a loop of P is a coloop of P^*.

Let S be the ground set for the rank r matroid pair $P = (M, N)$ and suppose that $e \in S$ is a loop of P. The maximum size of a common independent set of M/e and N/e is then $r - 2$, and by Edmonds' Matroid Intersection Theorem (Theorem 2.1), there exists a partition $\{S_1, S_2\}$ of $S\backslash e$ such that $\text{rank}_{M/e}(S_1) + \text{rank}_{N/e}(S_2) \le r - 2$. Since $\text{rank}_M(S_1 \cup \{e\}) \le \text{rank}_{M/e}(S_1) + 1$ and $\text{rank}_N(S_2) \le \text{rank}_N(S_2 \cup \{e\}) \le \text{rank}_{N/e}(S_2) + 1$, the partition $\{S_1 \cup \{e\}, S_2\}$ of S satisfies

$$\text{rank}_M(S_1 \cup \{e\}) + \text{rank}_N(S_2) \le \text{rank}_{M/e}(S_1) + 1 + \text{rank}_{N/e}(S_2) + 1 \le r,$$

and therefore P is not connected. By duality, if $e \in S$ is a coloop of P then P is not connected, and thus we have the following observation:

Observation 2.7. *If a matroid pair is connected, then it has no loops and no coloops.*

For an undirected graph G and a set $R \subseteq V(G)$, the *edge cut* of G induced by R is denoted $\delta(R)$ and consists of all edges e of G such that exactly one end of e is in R. If G is a directed graph and $R \subseteq V(G)$, then we let

$$\delta(R) = \{vw \in E : v \in R, w \notin R\}.$$

That is, $\delta(R)$ denotes the set of edges directed out of R. The directed graph G is *strongly connected* if $\delta(R) \neq \emptyset$ for all non-empty $R \subset V(G)$, or equivalently, if there exists a directed path from u to v for every $u, v \in V(G)$. Determining if a matroid pair P is connected is equivalent to determining if $G(P)$ is strongly connected, as the next theorem shows.

Theorem 2.8. *The matroid pair P is connected if and only if the fundamental graph $G(P)$ is strongly connected. In particular, if $P = (M, N)$ is a rank r matroid pair and $\{S_1, S_2\}$ is a partition of the ground set S, then $\mathrm{rank}_M(S_1) + \mathrm{rank}_N(S_2) = r$ if and only if $\delta(S_2)$ is empty.*

Proof. Assume that $P = (M, N)$ is a rank r matroid pair on the ground set S, and let (R_M, R_N) be the partial representation of P with respect to the basis B of P. Suppose P is not connected and let $\{S_1, S_2\}$ be a partition of S such that $\mathrm{rank}_M(S_1)+\mathrm{rank}_N(S_2) \leq r$. Since M and N have a common basis, it follows from Edmonds' Matroid Intersection Theorem that $\mathrm{rank}_M(S_1)+\mathrm{rank}_N(S_2) = r$. Let $U_i = B \cap S_i$ for $i = 1, 2$ and let $W_i = S_i \backslash B$. By Theorem 2.5, U_1 is a basis for the matroid pair $(M\backslash S_2, N/S_2)$ and U_2 is a basis for the matroid pair $(M/S_1, N\backslash S_1)$.

Suppose there exists $u \in U_2, v \in W_1$ such that $R_M[u; v] = 1$. Then $B \cup \{v\}\backslash\{u\}$ is a basis for M and therefore $(B \cup \{v\}\backslash\{u\}) \cap S_1 = U_1 \cup \{v\}$ is independent in $M\backslash S_2$, which contradicts that U_1 is a basis of $M\backslash S_2$. Thus every entry of $R_M[U_2; W_1]$ is zero, which implies that there are no edges directed from U_2 to W_1 in $G(P, B)$. By symmetry, every entry of $R_N[U_1; W_2]$ is zero, and therefore there are no edges directed from W_2 to U_1 in $G(P, B)$. Since $\{U_1, U_2\}$ is a partition of B and $\{W_1, W_2\}$ is a partition of $S\backslash B$, it follows that $\delta(W_2 \cup U_2) = \emptyset$ and thus $G(P, B)$ is not strongly connected. (See Figure 2.4.)

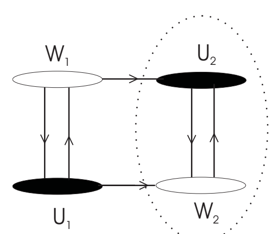

Figure 2.4: A directed cut in $G(P, B)$

Conversely, suppose $G(P, B)$ is not connected and let $\{U_1, U_2\}$ be a partition of B and $\{W_1, W_2\}$ be a partition of $S\backslash B$ such that $\delta(U_2 \cup W_2) = \emptyset$. If $u \in U_2$ and $v \in W_1$ then $R_M[u; v] = 0$ and therefore $B \cup \{v\}\backslash\{u\}$ is not a basis. Thus $\mathrm{rank}_M(U_1 \cup W_1) = \mathrm{rank}_M(U_1) \leq |U_1|$. Symmetrically, $\mathrm{rank}_N(U_2 \cup W_2) = \mathrm{rank}_N(U_2) \leq |U_2|$. Thus

$$\mathrm{rank}_M(U_1 \cup W_1) + \mathrm{rank}_N(W_2) \cup U_2) \leq |U_1| + |U_2| = |B| = r$$

and P is not connected. $\square$

Chapter 3

Pfaffian Matroid Pairs

In this chapter we define *Pfaffian matroid pairs*, which are the primary focus of this thesis. We begin by considering the problem of counting the number of common column bases between two matrices with the same set of columns, and show how the classical Cauchy-Binet formula solves this problem in some cases. The pairs of matrices for which the formula solves the counting problem are called *Pfaffian matrix pairs*. We then formulate the problem of counting bases in terms of matroids, and define Pfaffian matroid pairs in terms of Pfaffian matrix pairs. We give some characterizations of Pfaffian matroid pairs, and we show that the class of Pfaffian matroid pairs contains the class of Pfaffian bipartite graphs.

3.1 Counting common bases

Let A_1 and A_2 be matrices with r rows and k columns. If the columns of A_1 and A_2 are indexed by S and $A_i[X]$ denotes the column submatrix of A_i indexed by the columns $X \subseteq S$, then the classical Cauchy-Binet formula states that

$$\det A_1 A_2^\top = \sum_{X \subseteq S : |X| = r} \det A_1[X] \det A_2[X]. \tag{3.1}$$

Since $\det A_1[X] \det A_2[X] \neq 0$ if and only if X is a common basis of A_1 and A_2, the non-zero terms in the right hand side of Equation (3.1) correspond to the common bases of A_1 and A_2. If A_1 and A_2 are totally unimodular then

$$\det A_1[X] \det A_2[X] \in \{0, \pm 1\}$$

for all $X \subseteq S$ with $|X| = r$ and thus $|\det A_1 A_2^\top|$ gives a lower bound on the number of common bases of A_1 and A_2. If A_1 and A_2 totally unimodular and $|\det A_1 A_2^\top|$ is equal to the number of common bases of A_1 and A_2, then we call the pair (A_1, A_2) a *Pfaffian pair of matrices*. That is, the pair (A_1, A_2) of matrices is Pfaffian if both A_1 and A_2 are totally unimodular and either

(i) $\det A_1[B] = \det A_2[B]$ for all common bases B of A_1 and A_2, or

(ii) $\det A_1[B] = -\det A_2[B]$ for all common bases B of A_1 and A_2.

Let $\tilde{A}_1$ be obtained from A_1 by scaling some of the columns of A_1 by a factor of -1, and let $\tilde{A}_2$ be obtained from A_2 by scaling some of the columns of A_2 by -1. If $(\tilde{A}_1, \tilde{A}_2)$ is a Pfaffian pair of matrices then we say that the matrix pair (A_1, A_2) has a *Pfaffian signing.*

Let P be the matroid pair (M, N). If (A_M, A_N) is a representation of P such that (A_M, A_N) is a Pfaffian matrix pair then (A_M, A_N) is a *Pfaffian representation* of P. By the Cauchy-Binet formula given in Equation (3.1), if (A_M, A_N) is a Pfaffian representation of P, then $|\det A_M A_N^\top|$ is the number of bases of P. We say that a matroid pair is *Pfaffian* whenever it has a Pfaffian representation. If (A_M, A_N) is a Pfaffian representation of P satisfying $\det A_M[B] = -\det A_N[B]$ for all bases B of P, and A'_N is the representation obtained from A_N by scaling a row of A_N, then $\det A'_N[B] = \det A_M[B]$ for all bases B of P. Unless otherwise stated, we therefore assume that if (A_M, A_N) is a Pfaffian representation of P then $\det A_M[B] = \det A_N[B]$ for all bases B of P.

Let (A_M, A_N) be a totally unimodular representation of P. By Camion's theorem [3] for totally unimodular signings, given any other representation (A'_M, A'_N) with the same non-zero entries as (A_M, A_N), the two representations are the same up to resigning across rows and columns. Let (A^i_M, A^i_N) be obtained from (A_M, A_N) by scaling the i^{th} columns of A_M and A_N by -1. If X indexes a square column submatrix in A_M and A_N, then

$$\det A^i_M[X] \det A_N[X] = \det A_M[X] \det A^i_N[X].$$

Thus if (A'_M, A'_N) is a scaling of (A_M, A_N) such that (A'_M, A'_N) is a Pfaffian representation of P, then a Pfaffian representation of P can be obtained from (A_M, A_N) by scaling A_N only.

Suppose A'_N is obtained from A_N by pivoting (Section 2.4). Then there exists a nonsingular matrix D such that $A'_N = DA_N$ and therefore $\det A'_N[B] = \det D \det A_N[B]$ for all bases B. Furthermore, A'_N is a totally unimodular representation of N and thus if (A_M, A_N) is a Pfaffian representation of P then either $\det A_M[B] = \det A'_N[B]$ for all bases B of P or $\det A_M[B] = -\det A'_N[B]$ for all bases B of P. Therefore if A'_N is obtained from A_N by pivoting, then (A_M, A_N) is a Pfaffian representation of P if and only if (A_M, A'_N) is a Pfaffian representation of P.

If B is a basis of P and $\det A_M[B] = \det A_N[B]$, then we say that B is *correctly signed* with respect to (A_M, A_N). A basis which is not correctly signed with respect to (A_M, A_N) is *incorrectly signed.* We make the following observation:

Observation 3.1. *If (A_M, A_N) is a totally unimodular representation of the matroid pair P, then P is Pfaffian if and only if there is a scaling A'_N of A_N such that every basis of P is correctly signed with respect to (A_M, A'_N).*

If (R'_M, R'_N) is a totally unimodular partial representation of P with respect to the basis B, then (R'_M, R'_N) is called a Pfaffian representation of P with respect to B if $([I|R'_M], [I|R'_N])$ is a Pfaffian representation of P. The edge signing of $G(P, B)$ corresponding to (R'_M, R'_N) is a called Pfaffian signing of $G(P, B)$ if (R'_M, R'_N) is a Pfaffian representation of P, and thus P is Pfaffian if and only if $G(P)$ has a Pfaffian signing. Note that (R'_M, R'_N) is a Pfaffian representation of P with respect to B if and only if $\det R'_M[U; W] \det R'_N[U; W] \in \{0, 1\}$ whenever $U \subseteq B$, $W \subseteq S \backslash B$ and $|U| = |W|$. If s is the edge signing of $G(P, B)$ corresponding to (R'_M, R'_M) and $X \subseteq S$ is such that $B \Delta X$ is a basis, then we say that $G(P, B)[X]$ is correctly signed with respect to s if the basis $B \Delta X$ is correctly signed with respect to (R'_M, R'_N). This leads to the next observation.

Observation 3.2. *If P is a matroid pair and s is a totally unimodular edge signing of $G(P)$, then s is a Pfaffian signing of $G(P)$ if and only if all of the contributing subgraphs of $G(P)$ are correctly signed.*

For Observation 3.2 to be useful, a characterization of a Pfaffian signing of the matroid pair P with respect to an edge signing of $G(P)$ is required. This appears to be a difficult problem in general, with the exception of the graphic matroid pairs discussed in Section 3.3.

When s is an edge signing of the graph G and $E' \subseteq E(G)$, let

$$s(E') = \sum_{e \in E'} s_e$$

where the sum is taken over $GF(2)$. In particular, if C is a circuit in G then $s(C)$ is the parity of the number of edges in C that are signed 1. Recall from Section 2.3 that a simple contributing circuit in a fundamental graph is a directed circuit C in $G(P)$ which is the unique circuit cover of $V(C)$ in $G(P)$. When s is a Pfaffian signing of $G(P)$ and C is a simple contributing circuit in $G(P)$ then we observe that $s(C)$ depends only on $|C|$.

Lemma 3.3. *If P is a matroid pair and C is a simple contributing circuit in $G(P)$, then C is correctly signed with respect to the edge signing s if and only if*

$$s(C) \equiv \frac{|C|}{2} + 1 \pmod{2}.$$

Proof. Assume $P = (M, N)$ is a matroid pair on the ground set S with basis B and partial representation (R_M, R_N) with respect to B. Let s be an edge signing of $G(P, B)$ and let (R'_M, R'_N) be the corresponding signing of (R_M, R_N).

Let C be a simple contributing circuit in $G(P, B)$ and define $T_M(C)$ and $T_N(C)$ to be the unique perfect matchings in $G(M, B)[V(C)]$ and $G(N, B)[V(C)]$. Let A_M and A_N be the adjacency matrices for $T_M(C)$ and $T_N(C)$ respectively, and let A'_M and A'_N be the signings of A_M and A_N corresponding to s. Assume $U = V(C) \cap B$ and

$W = V(C)\backslash B$. Since there is a one-to-one correspondence between perfect matchings in $G(M,B)[U \cup W]$ and terms in $\det R'_M[U;W]$, the determinant of $R'_M[U;W]$ is equal to $\det A'_M$. Similarly, $\det R'_N[U;W] = \det A'_N$ and therefore C is correctly signed if and only if $\det A'_M = \det A'_N$.

Assume that $|C| = 2k$ and let $U = \{u_1, u_2, \ldots, u_k\}$ and $W = \{w_1, w_2, \ldots, w_k\}$, where the cyclic order of the vertices on C is $\{u_1, w_1, u_2, w_2, \ldots, u_k, w_k\}$. By simultaneously permuting rows and columns in A_M and A_N we may assume that the i^{th} rows of A_M and A_N are indexed by u_i and that the i^{th} columns of A_M and A_N are indexed by w_i. Since $T_M(C) = \{u_1w_1, u_2w_2, \ldots, u_kw_k\}$, A_M is the k by k identity matrix, and since $T_N = \{w_1u_2, w_2u_3, \ldots, w_ku_1\}$, A_N is a permutation matrix with determinant $(-1)^{k+1}$. Thus $\det A'_M = (-1)^{s(T_M)}$ and $\det A'_N = (-1)^{s(T_N)}(-1)^{(k+1)}$ and therefore $\det A'_M = \det A'_N$ if and only if $s(T_M) = s(T_N) + (k+1)$. Since $s(T_M) + s(T_N) = s(C)$ and $k = \frac{|C|}{2}$, we conclude that C is correctly signed if and only if $s(C) = \frac{|C|}{2} + 1$. □

If P is a matroid pair then Lemma 3.3 implies that it is easy to determine if an edge signing correctly signs the simple contributing circuits of $G(P)$. However, there does not appear to be a similarly easy definition of the sign of a contributing circuit when the circuit in question is not simple.

3.2 Properties of Pfaffian matroid pairs

It is well known that the class of graphic matroids is closed under taking minors, and that the class of regular matroids is closed under duality and under taking minors. In this section we show the class of Pfaffian matroid pairs to be closed under duality and under taking minors.

Theorem 3.4. *If P is a matroid pair then P is Pfaffian if and only if P^* is Pfaffian.*

Proof. Let M and N be matroids on the ground set S and let P be the matroid pair (M,N). Suppose that (A_M, A_N) is a Pfaffian representation of P and let B be a basis of P. We may assume that $A_M = [I|R_M]$ and $A_N = [I|R_N]$, where the rows of A_M and A_N are indexed by B and the columns of R_M and R_N are indexed by $S\backslash B$. The matrices $A^*_M = [R^\top_M|I]$ and $A^*_N = [R^\top_N|I]$ are then totally unimodular representations of M^* and N^* respectively. If B^* is a basis of P^* and $\overline{B^*} = S\backslash B^*$, then $\overline{B^*}$ is a basis of P and

$$\begin{aligned}\det A^*_M[B^*] &= \det R^\top_M[(S\backslash B)\backslash B^*; B \cap B^*]\\ &= \det R_M[B \cap B^*; (S\backslash B)\backslash B^*]\\ &= \det R_M[B\backslash\overline{B^*}; \overline{B^*}\backslash B].\end{aligned}$$

Similarly, $\det A^*_N[B^*] = \det R_N[B\backslash\overline{B^*}; \overline{B^*}\backslash B]$. Since (A_M, A_N) is a Pfaffian representation of P and $\overline{B^*}$ is a basis of P, $\det R_M[B\backslash\overline{B^*}; \overline{B^*}\backslash B] = \det R_N[B\backslash\bar{B}^*; \bar{B}^*\backslash B]$. Thus

(A_M^*, A_N^*) is a Pfaffian representation of P^*, and P is Pfaffian if and only if P^* is Pfaffian. $\square$

Let P be the matroid pair $P = (M, N)$ on the ground set S and let $B \subseteq S$ be a basis of P. If $U \subseteq S \backslash B$ and $V \subseteq B$, then the matroid pair $(M \backslash U/V, N \backslash U/V)$ is called a *minor* of P and is denoted $P \backslash U/V$. Since $((M \backslash U/V)^*, (N \backslash U/V)^*) = (M^*/U \backslash V, M^*/U \backslash V)$, the dual of $P \backslash U/V$ is $P^*/U \backslash V$.

Theorem 3.5. *All minors of a Pfaffian matroid pair are Pfaffian.*

Proof. Let S be the ground set for the matroid pair $P = (M, N)$ and assume $u \in S$ is not a coloop of P. If (A_M, A_N) is a Pfaffian representation of P and A'_M and A'_N are obtained from A_M and A_N respectively by deleting the columns indexed by u, then for all bases B of $(M \backslash u, N \backslash u)$,

$$\det A'_M[B] = \det A_M[B] = \det A_N[B] = \det A'_N[B].$$

Thus (A'_M, A'_N) is a Pfaffian representation for $P \backslash u$. By Theorem 3.4, the pair P/v is Pfaffian whenever v is not a loop of P and thus all minors of P are Pfaffian. $\square$

Let P be a non-Pfaffian matroid pair on the ground set S. If $P \backslash u$ is Pfaffian whenever $u \in S$ is not a coloop of P and P/v is Pfaffian whenever $v \in S$ is not a loop of P, then P is *minimally non-Pfaffian.*

3.3 Pfaffian bipartite graphs

Let G be a bipartite graph and let A be the adjacency matrix for G. Since there is a one-to-one correspondence between perfect matchings in G and nonzero terms in the permutation expansion of $\det A$, the number of perfect matchings in G is at least $|\det A|$. The graph G is a *Pfaffian bipartite graph* if there exists a signing A' of A such that $\det A'$ is exactly the number of perfect matchings in G.

The complete bipartite graph $K_{3,3}$ has exactly six perfect matchings, and the adjacency matrix A for $K_{3,3}$ is the 3 by 3 matrix of all ones. Since it is not possible to replace some of the entries of A with -1 such that the new matrix has determinant ± 6, the graph $K_{3,3}$ is non-Pfaffian. An *odd subdivision* of $K_{3,3}$ is obtained by possibly replacing some of the edges of $K_{3,3}$ with paths of odd length, and every odd subdivision of $K_{3,3}$ is also non-Pfaffian. If the subgraph H of G is an odd subdivision of $K_{3,3}$ and $G \backslash V(H)$ has a perfect matching, then G is said to *contain* $K_{3,3}$. It is straightforward to show that a bipartite graph that contains $K_{3,3}$ is non-Pfaffian (see Robertson, Seymour, and Thomas [35]), and a theorem of Little [22] shows that the converse also holds.

Theorem 3.6 (Little). *A bipartite graph is Pfaffian if and only if it does not contain* $K_{3,3}$.

In this section we present the well known application of matroid intersection as a method of finding a perfect matching in a bipartite graph, and we show that a bipartite graph is Pfaffian if and only if the associated matroid pair is Pfaffian. In Chapter 6 we will show that the graphic matroid pairs for Pfaffian bipartite graphs are a subclass of series-parallel Pfaffian matroid pairs.

If G is a bipartite graph with vertex bipartition $\{U, W\}$ then define the matroids M_U and M_W on the ground set $E(G)$ such that $E' \subseteq E(G)$ is independent in M_U if and only if each vertex in U is incident to at most one edge in E' and $E' \subseteq E(G)$ is independent in M_W if and only if each vertex in W is incident to at most one edge in E'. The matroids M_U and M_W are referred to as the *partition matroids* defined by the bipartition of $V(G)$. The matroid pair $P = (M_U, M_W)$ is denoted $P(G)$ and is called the *graphic matroid pair* for G.

If $E' \subseteq E(G)$ is independent in M_U and in M_W, then E' is a matching of G, and thus the perfect matchings in G correspond to the bases of $P(G)$. Let A_U be the $U \times E(G)$ incidence matrix and let A_W be the $W \times E(G)$ incidence matrix. If A is the adjacency matrix for G, then

$$A = A_U A_W^\top.$$

Furthermore, scaling column e of A_W by -1 corresponds to replacing the entry for e in A with -1, and thus A has a signing A' such that $\det A'$ is the number of perfect matchings in G if and only if A_W has a scaling A'_W such that the determinant of $A_U(A'_W)^\top$ is the number of perfect matchings in G.

Observation 3.7. *The bipartite graph G is Pfaffian if and only if the graphic matroid pair $P(G)$ is Pfaffian, and thus the class of Pfaffian matroid pairs extends the class of Pfaffian bipartite graphs.*

A graph is called *matching covered* if it is connected and every edge in the graph is contained in a perfect matching. It is natural to restrict the study of Pfaffian graphs to matching covered graphs, and the following theorem shows that this is equivalent to restricting graphic matroid pairs to connected matroid pairs.

Theorem 3.8. *Let G be a bipartite graph with a perfect matching. The graphic matroid pair $P(G)$ is connected if and only if G is matching covered.*

Proof. Let $V(G) = \{U, W\}$ be the vertex bipartition of the bipartite graph G, and let M and N be the partition matroids defined on U and W respectively such that $P(G)$ is the graphic matroid pair (M, N). Since G has a perfect matching, $|U| = |W|$ and $P(G)$ has rank equal to $|U|$.

Suppose that $P(G)$ is not connected, and let $\{S_1, S_2\}$ be a non trivial partition of $E(G)$ such that

$$\text{rank}_M(S_1) + \text{rank}_N(S_2) = |U|. \tag{3.2}$$

Since $\mathrm{rank}_M(S_1)$ is the number of vertices of U that are incident to the edges S_1 and $\mathrm{rank}_N(S_2)$ is the number of vertices of W that are incident to the edges S_2, Equation (3.2) implies that there exist sets $U_1 \subseteq U$ and $W_1 \subseteq W$ such that $|U_1| + |W_1| = |U|$ and

$$\delta(U_1) \cup \delta(W_1) = E(G).$$

If $U_2 = U \backslash U_1$, then since $|U| = |W|$, $|U_2| = |W_1|$, and thus every perfect matching of G induces a perfect matching of $G[U_2 \cup W_1]$. It follows that either $\delta(U_2 \cup W_1) = \emptyset$ and G is not connected, or $\delta(U_2 \cup W_1) \neq \emptyset$ and every edge in $\delta(U_2 \cup W_1)$ is not in any perfect matching of G. Hence if $P(G)$ is not connected, then G is not matching covered.

Conversely, suppose that G is not matching covered. If G has an edge e such that e is not in any perfect matching of G, then e is not in any basis of $P(G)$, and thus by definition e is a loop of $P(G)$. By Observation 2.7, $P(G)$ is not connected in this case. Similarly, if G is not connected and $S_1 \subset E(G)$ is such that the subgraph of G induced by S_1 is a component of G, then

$$\mathrm{rank}_M(S_1) + \mathrm{rank}_N(S_2) = |U|$$

and again $P(G)$ is not connected. □

3.4 Certifying non-Pfaffian pairs

Given a Pfaffian representation for a matroid pair, either every basis is correctly signed with respect to the representation, or every basis is incorrectly signed. Thus, if a set of bases of a matroid pair can neither be simultaneously correctly signed, nor simultaneously incorrectly signed, then the matroid pair is non-Pfaffian. We call such a collection of bases a *non-Pfaffian bases certificate.* Since a matroid pair can have an exponential number of bases with respect to the size of the ground set, a non-Pfaffian bases certificate need not be an efficient way to show that the matroid pair is non-Pfaffian. In this section we prove that if a matroid pair is non-Pfaffian, then it has a non-Pfaffian bases certificate where the number of bases in the certificate is linear with respect to the ground set for matroid pair.

Let P be a regular matroid pair on the ground set S and let (A_M, A_N) be a totally unimodular representation of P. Let $X \subseteq S$ and let A'_N be the matrix obtained by scaling by a factor of -1 the columns of A_N indexed by X. If B is a basis of N, then $\det A_N[B] = -\det A'_N[B]$ if and only if $|X \cap B|$ is odd. Thus P is Pfaffian if and only if there exists $X \subseteq S$ satisfying one of the following:

(i) $|X \cap B|$ is even for all correctly signed bases B of P and $|X \cap B'|$ is odd for all incorrectly signed bases B',

(ii) $|X \cap B|$ is odd for all correctly signed bases B of P and $|X \cap B'|$ is even for all incorrectly signed bases B'.

If P has odd rank and $X \subseteq S$ satisfies (ii), then over $GF(2)$,

$$|(S \backslash X) \cap B| = |X \cap B| + |B| = |X \cap B| + 1$$

and thus P is Pfaffian if and only if there exists $X \subseteq S$ satisfying (i). However, a Pfaffian matroid pair P may not have a solution to (i) when P has even rank. For example, let $P = (M, N)$ where M and N have totally unimodular representations A_M and A_N as given below

$$A_M = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}, \quad A_N = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}. \tag{3.3}$$

The correctly signed bases of P with respect to the column indices are

$$\{1,2\},\ \{1,5\},\ \{2,3\},\ \{2,4\},\ \{3,5\},\ \{4,5\}$$

and the only incorrectly signed basis is $\{3,4\}$. Although there is no subset of $\{1,2,3,4,5\}$ which intersects all correctly signed bases with even parity and intersects the incorrectly signed basis with odd parity, P is a Pfaffian matroid pair: scaling A_N across columns 3 and 4 and any row gives a Pfaffian representation of P.

Let (A_M, A_N) be a totally unimodular representation of the matroid pair P and suppose that there exists a set of correctly signed bases $\{B_1, B_2, \ldots, B_{2k+1}\}$ of P and a set of incorrectly signed bases $\{B'_1, B'_2, \ldots, B'_{2m+1}\}$ of P such that

$$B_1 \Delta B_2 \Delta \cdots \Delta B_{2k+1} \Delta B'_1 \Delta B'_2 \Delta B'_{2m+1} = \emptyset. \tag{3.4}$$

For $S_1, S_2 \subseteq S$ and $X \subseteq S$,

$$|X \cap S_1| + |X \cap S_2| = |X \cap (S_1 \Delta S_2)| + 2|X \cap (S_1 \cap S_2)|,$$

and therefore $|X \cap S_1| + |X \cap S_2|$ has the same parity as $|X \cap (S_1 \Delta S_2)|$. Thus for all $X \subseteq S$, Equation (3.4) implies that over $GF(2)$,

$$\begin{aligned} \sum_{i=1}^{2k+1} |X \cap B_i| + \sum_{i=1}^{2m+1} |X \cap B'_i| &= |X \cap (B_1 \Delta B_2 \Delta \cdots \Delta B_{2k+1} \Delta B'_1 \Delta B'_2 \Delta B'_{2m+1})| \\ &= |X \cap \emptyset| \\ &= 0. \end{aligned}$$

If $X \subseteq S$ is such that $|X \cap B|$ is even for all correctly signed bases B and $|X \cap B'|$ is odd for all incorrectly signed bases B', then

$$\sum_{i=1}^{2k+1} |X \cap B_i| + \sum_{i=1}^{2m+1} |X \cap B'_i| = 1. \tag{3.5}$$

Equation (3.5) similarly holds if X intersects all correctly signed bases with odd cardinality and intersects all incorrectly signed bases with even cardinality. Thus the union of an odd number of correctly signed bases with an odd number of incorrectly signed bases which together satisfy (3.4) is a non-Pfaffian bases certificate for P. The following observation will be used to prove that a non-Pfaffian bases certificate of linear size with respect to the ground set exists for any non-Pfaffian matroid pair.

Observation 3.9. *If A is an $m \times n$ matrix over $GF(2)$ and b is an m dimensional vector over $GF(2)$, then exactly one of the following holds:*

1. *$Ax = b$ has a solution.*

2. *There exists $y \in \{0,1\}^m$ such that $y^\top A = 0$ and $y^\top b = 1$.*

Certainly if $y^\top A = 0$ and $y^\top b = 1$ then there is no solution to $Ax = b$, and thus a vector y satisfying $y^\top A = 0$ and $y^\top b = 1$ provides a certificate that $Ax = b$ does not have a solution. The *support* of y, denoted $\mathrm{supp}(y)$ is the set of indices i for which y_i is nonzero. The following lemma shows that if $Ax = b$ has no solution over $GF(2)$, then there exists a certifying vector y such that the size of the support of y is at most one more than the number of columns in A.

Lemma 3.10. *If A is an $m \times n$ matrix over $GF(2)$ and $b \in GF(2)^m$, then either $Ax = b$ has a solution over $GF(2)$ or there exists $y \in \{0,1\}^m$ such that $y^\top A = 0, y^\top b = 1$, and $|\mathrm{supp}(y)| \leq n + 1$.*

Proof. If B is a matrix with k rows and $S \subseteq \{1, 2, \ldots, k\}$, let B_S denote the restriction of B to the rows in S. Suppose $Ax = b$ has no solution over $GF(2)$. By Observation 3.9, there exists $y \in \{0,1\}^m$ such that $y^\top A = 0$ and $y^\top b = 1$. If $S = \mathrm{supp}(y)$ and $\mathbf{1}$ is the vector of all ones, then $\mathbf{1}^\top A_S = 0$ and $\mathbf{1}^\top b_S = 1$. Since A_S has n columns, the rank of A_S is at most n.

Suppose $|\mathrm{supp}(y)| > n + 1$. Then A_S has more than $n + 1$ rows and thus there is a nontrivial subset S_1 of S such that the rows of A_{S_1} are dependent, and therefore $\mathbf{1}^\top A_{S_1} = 0$. If $S_2 = S \backslash S_1$ then

$$\mathbf{1}^\top A_S = \mathbf{1}^\top A_{S_1} + \mathbf{1}^\top A_{S_2} = 0$$

and therefore $\mathbf{1}^\top A_{S_2} = 0$. Similarly,

$$\mathbf{1}^\top b_S = \mathbf{1}^\top b_{S_1} + \mathbf{1}^\top b_{S_2} = 1$$

and thus either $\mathbf{1}^\top b_{S_1} = 1$ or $\mathbf{1}^\top b_{S_2} = 1$. Without loss of generality, assume $\mathbf{1}^\top b_{S_1} = 1$. If $z \in \{0,1\}^m$ is defined by $z_i = 1$ if and only if $i \in S_1$, then $z^\top A = \mathbf{1}^\top A_{S_1} = 0$ and $z^\top b = \mathbf{1}^\top b_{S_1} = 1$. Thus $|\mathrm{supp}(y)| > n + 1$ implies that there exists $z \in \{0,1\}^m$ with $z^\top A = 0$, $z^\top b = 1$, and $|\mathrm{supp}(z)| < |\mathrm{supp}(y)|$. The lemma follows. $\square$

We apply Lemma 3.10 to prove the main theorem of this section: if P is a non-Pfaffian matroid pair then P has a non-Pfaffian bases certificate of linear size with respect to the ground set of P.

Let A be the matrix whose rows are the characteristic vectors for the bases of P. Given a totally unimodular representation (A_M, A_N) of P, let b be the $\{0,1\}$ vector indexed by the bases of P with

$$b_B = \begin{cases} 1, & \text{if } B \text{ is correctly signed with respect to } (A_M, A_N); \\ 0, & \text{else.} \end{cases}$$

If $\bar{b} \equiv b + 1 \pmod 2$, then P is Pfaffian if and only if either $Ax = b$ or $Ax = \bar{b}$ has a solution over $GF(2)$. By Lemma 3.10, if there is no solution to $Ax = b$ then there exists a vector y such that $y^\top A = 0$ and $y^\top b = 1$. This implies that there is a set of bases which can not be simultaneously correctly signed and the number of incorrectly signed bases in the set is odd. Lemma 3.10 further implies that if such a set exists, then there is such a set with at most $|S| + 1$ bases. If the number of correctly signed bases is also odd, then this set certifies that P is non-Pfaffian, and hence there is a non-Pfaffian bases certificate with at most $|S| + 1$ bases.

If $Ax = \bar{b}$ also has no solution then there is a set of at most $|S| + 1$ bases with an odd number of correctly signed bases whose symmetric difference is the empty set. Either this set has an odd number of incorrectly signed bases and therefore certifies that P is non-Pfaffian, or the set for $Ax = b$ can be combined with the set for $Ax = \bar{b}$ to get a non-Pfaffian bases certificate with at most $2|S| + 2$ bases. Thus the following theorem holds:

Theorem 3.11. *If P is a non-Pfaffian matroid pair on the ground set S, then P has a non-Pfaffian bases certificate with at most $2|S| + 2$ bases.*

It follows that the problem of determining if a matroid pair is Pfaffian is in Co-NP: for any matroid pair that is non-Pfaffian, there is a certificate which, in polynomial time with respect to the size of the ground set, verifies that the pair is non-Pfaffian. Theorem 3.11 can be compared with Little's theorem (see Section 3.3) and the problem of determining if a bipartite graph is Pfaffian. By Little's theorem, if a bipartite graph is non-Pfaffian then it has a $K_{3,3}$ minor which certifies that it is non-Pfaffian. In 1999, a decomposition theorem of Robertson, Seymour, and Thomas showed that the problem of determining if a bipartite graph is Pfaffian is also in NP. We do not know if determining if the problem of determining if a matroid pair is Pfaffian is similarly in NP.

In Section 6.6 we give a class of matroid pairs for which the linear bound in the size of the non-Pfaffian certificate can be improved to a bound of constant size. We conjecture that the linear bound in Theorem 3.11 can be improved to a constant bound for all matroid pairs.

Conjecture 3.12. *There exists a constant c such that every non-Pfaffian regular matroid pair has a non-Pfaffian bases certificate containing at most c bases.*

Conjecture 3.12 is true for graphic matroid pairs: if G is a bipartite graph, then Little's Theorem [22] implies that the graphic matroid $P(G)$ has a non-Pfaffian bases certificate containing exactly six bases.

3.5 Examples

We give three examples of Pfaffian matroid pairs and three examples of non-Pfaffian pairs, and note the significance of each. Each of the non-Pfaffian matroid pairs is minimally non-Pfaffian.

Example 1: *trivial Pfaffian pair.*

A simple example of a Pfaffian matroid pair is the matroid pair $P = (M, M)$ where M is any regular matroid. If A_M is any totally unimodular representation of M then (A_M, A_M) is clearly a Pfaffian representation of P. It follows that the number of bases of a regular matroid pair can be easily determined.

Example 2: *twined* K_4 *pair.*

Let D_3 be the graph composed of three circuits, each of length 2, and let K_4 be the complete graph on four vertices. Consider the matroid pair $P = (M(D_3), M(K_4))$, where the edges in D_3 and K_4 are labeled as shown in Figure 3.1(a) and 3.1(b) respectively.

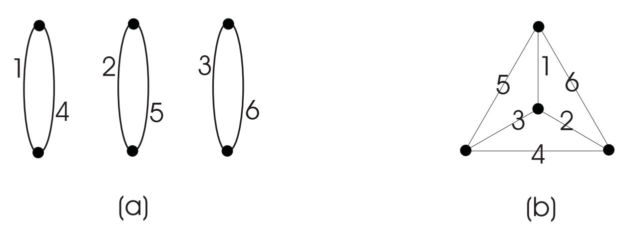

Figure 3.1: edge labels for D_3 and K_4

We refer to P as a *twined* K_4 *pair*. The matrices A_M and A_N below are totally unimodular representations of $M(D_3)$ and $M(K_4)$ respectively:

$$A_M = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \qquad A_N = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix}.$$

For the basis B of P corresponding to the first three columns of A_M and A_N, the graph $G(P, B)$ is shown in Figure 3.2. Since $G(P, B)$ is strongly connected, Theorem 2.8 implies that P is a connected matroid pair. Note that P has four bases, namely B, $B_1 = \{1, 5, 6\}$,

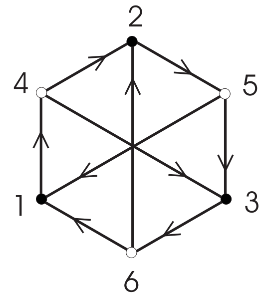

Figure 3.2: $G(P, B)$

$B_2 = \{2, 4, 6\}$ and $B_3 = \{3, 4, 5\}$. The determinant of $A_M A_N^T$ is 4 and therefore (A_M, A_N) is a Pfaffian representation of P. If

$$A'_N = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix},$$

then A'_N is obtained from A_N by scaling columns 4, 5, and 6 by -1, and since the determinant of $A_M A'^T_N$ is 4, the representation (A_M, A'_N) of P is also a Pfaffian representation. The twined K_4 pair has a pivotal role in our analysis of unique signings (Section 3.6) and binary spaces (Chapter 4).

Example 3: *skew-symmetric matroid pair.*

A matrix A is *skew-symmetric* if

$$A^\top = -A.$$

Note that if A is an $n \times n$ skew-symmetric matrix, then

$$\det A = \det A^T = \det(-A) = (-1)^n \det A,$$

and thus skew-symmetric matrices have even rank.

Let M be the regular matroid with representation $[I|I]$ and let N be a regular matroid with totally unimodular representation $[I|A]$ for some skew-symmetric matrix A. We call the matroid pair $P = (M, N)$ a *skew-symmetric matroid pair.* The bases of P correspond to those square submatrices of I and A that are nonsingular in both. Since the only nonsingular submatrices of I are principal submatrices, the bases of P correspond to principal nonsingular submatrices of A. It can be shown that skew-symmetric matrices have non-negative determinant, and since a principal submatrix of a skew-symmetric matrix is skew-symmetric, $\det A[X; X] \geq 0$ for all row and column subsets X. Furthermore, $\det I[X; X] = 1$ for all row and column subsets X, and thus the representation

$([I|I],[I|A])$ is a Pfaffian representation of P. From the observation above that all non-singular skew-symmetric matrices have even rank, it follows that all bases of P intersect the columns of A an even number of times, and therefore resigning across these columns gives another Pfaffian representation of P. We note that this example is a generalization of Example 2.

Example 4: *smallest non-Pfaffian pair.*

Let P be the regular matroid pair on the ground set $\{1,2,3,4,5\}$ with totally unimodular representation (A_M, A_N) as shown below:

$$A_M = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}, \qquad A_N = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

Note that the submatrix indexed by columns 3 and 4 has determinant $+1$ in A_M and determinant -1 in A_N, and thus the basis $\{3,4\}$ of P is incorrectly signed. The matroid pair P has six bases, three of which are incorrectly signed and three which are correctly signed.

Correctly signed Bases:	$\{1,2\},\{1,5\},\{2,5\}$
Incorrectly signed Bases:	$\{3,4\},\{3,5\},\{4,5\}$

Since the symmetric difference of the three correctly signed bases is the empty set and the symmetric difference of the three incorrectly signed bases is also the empty set, the set of all bases of P forms a non-Pfaffian bases certificate for P. With respect to rank and size of the ground set, this example and its dual are the smallest non-Pfaffian matroid pairs.

Example 5: $\mathbf{K_{3,3}}$.

If M and N are the two partition matroids defined by the vertex bipartition of $K_{3,3}$ (Section 3.3) then A_M and A_N shown in (3.6) are totally unimodular representations of M and N respectively. Since $K_{3,3}$ is a non-Pfaffian graph, the matroid pair $P=(M,N)$ is non-Pfaffian by Observation 3.7. The six bases of P correspond to the six perfect matchings of $K_{3,3}$, and since each edge in $K_{3,3}$ is in exactly two perfect matchings, the symmetric difference of these six bases is the empty set. Under the representation (A_M, A_N), three of the bases are correctly signed and three are incorrectly signed.

$$A_M = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \qquad A_N = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \tag{3.6}$$

Correctly signed Bases:	$\{1,2,3\},\{4,6,9\},\{5,7,8\}$
Incorrectly signed Bases:	$\{1,6,7\},\{2,8,9\},\{3,4,5\}$

The graph of $G(P)$ in Figure 3.3 is an example of a double cycle. Double cycles and

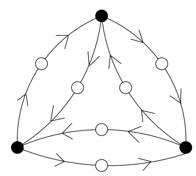

Figure 3.3: $G(P)$ corresponding to $K_{3,3}$

their relationship to the problem of counting bases are analyzed in Section 6.7.

Example 6: *neutral circuit.*

Let the matroid pair P have basis $B = \{1,2,3,4,5,6\}$ and partial representation (R_M, R_N) with respect to B, as shown in (3.7). Note that $G(P)$ shown in Figure (3.4) has a Hamiltonian circuit C whose vertices can be covered with two disjoint induced directed circuits and whose vertices can also be covered with three disjoint induced directed circuits. A *neutral* circuit in a directed graph is a directed circuit whose vertices can be covered with an even number of disjoint contributing circuits and can also be covered with an odd number of disjoint contributing circuits. Thus the Hamiltonian circuit C in $G(P)$ is neutral.

$$R_M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad R_N = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{3.7}$$

Correctly signed Bases:	$\{1,2,3,4,5,6\},\{1,2,3,10,11,12\},\{4,5,6,7,8,9\}$
Incorrectly signed Bases:	$\{1,2,5,6,9,10\},\{1,3,4,6,8,11\},\{2,3,4,5,7,12\}$

In Theorem 6.20 we show that if M and N are series-parallel matroids and $G(M,N)$ has a neutral circuit, then the matroid pair (M,N) is non-Pfaffian and has an non-Pfaffian bases certificate containing at most 32 bases. In this example, P has a non-Pfaffian bases certificate with six bases.

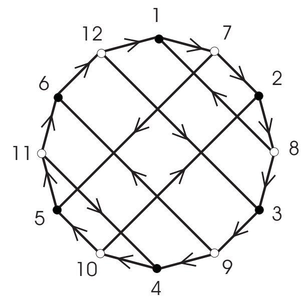

Figure 3.4: A neutral circuit

3.6 Unique signing

In Section 3.3 we defined a Pfaffian bipartite graph to be a bipartite graph G for which there exists a signing A' of the adjacency matrix A for G such that the determinant of A' is equal to the number of perfect matchings of G. This signing of A corresponds to a $\{0, 1\}$ signing of the edges of G, where an edge is signed 1 if and only if the corresponding entry of A is signed -1. Given a perfect matching T of G, and an edge signing of s of G, call a T-alternating circuit C of G *odd* with respect to s if the sum of s on the edges of C is odd. Pfaffian bipartite graphs can equivalently be defined as the bipartite graphs with an edge signing such that with respect to a fixed perfect matching T of G, all T alternating circuits in the graph are odd. Similar to Camion's [3] observation that a totally unimodular signing of a $\{0, 1\}$ matrix is unique up to resigning across rows and columns, it follows from this definition of a Pfaffian bipartite graph that resigning a Pfaffian signing across cuts in G creates another Pfaffian signing. The Pfaffian signing can therefore be fixed on a spanning tree of G, and this partial signing uniquely determines the sign on all of the remaining edges which are in an T alternating circuit. Edges not in such a circuit can be removed from the graph without affecting whether G is Pfaffian, and so we may assume that all edges in G are in a perfect matching.

Up to resigning across cuts, a Pfaffian bipartite graph therefore has a unique Pfaffian signing. Similarly, up to resigning across rows and columns, a regular matrix has a unique totally unimodular signing. This uniqueness in both cases is critical for constructing an algorithm to find either a Pfaffian signing of a bipartite graph or a totally unimodular signing of a regular matrix. For a Pfaffian matroid pair there are two types of trivial resignings of a Pfaffian representation of the pair, which we describe next. In this section we consider whether a Pfaffian representation is unique up to such trivial resignings.

Let $P = (M, N)$ be a rank r Pfaffian matroid pair on the ground set S and let (R_M, R_N) be a Pfaffian representation of P with respect to the basis B. The first trivial resigning of (R_M, R_N) occurs when P is not connected.

Suppose that $\{S_1, S_2\}$ is a non-trivial partition of S such that

$$\mathrm{rank}_M(S_1) + \mathrm{rank}_N(S_2) = r.$$

By Theorem 2.5, if $B' \subseteq S$ is a basis of P then $B' \cap S_1$ is a basis of $M \backslash S_2$ and thus $|B' \cap S_1| = \mathrm{rank}_M(S_1)$ for all bases B' of P. It follows that if R'_N is the representation obtained from R_N by scaling the rows and columns corresponding to S_1, then (R_M, R'_N) is a Pfaffian representation of P. The edge resigning of $G(P)$ corresponding to the signing R'_N resigns the edges from $G(N)$ in the cut $\delta(S_1)$. Theorem 2.5 proved that these edges correspond to a directed cut in $G(P)$, and therefore there are no directed circuits in $G(P)$ that intersect $\delta(S_1)$. Thus such a resigning does not affect the sign on any directed circuit in $G(P)$.

The second trivial resigning occurs when at least one of the matroids in the pair is not connected. Let $X \subset S$ index a component of M, and let R'_N be obtained from R_N by resigning across the rows and columns corresponding to X. Since $|X \cap B'| = \mathrm{rank}_M(X)$ for all bases B' of P,

$$\det R_N[U; W] = (-1)^{\mathrm{rank}_M(X)} \det R_N[U; W]$$

whenever $(B \backslash U) \cup W$ is a basis of P, and therefore (R_M, R'_N) is a Pfaffian representation of P. Graphically, this resigning corresponds to resigning the edges of $G(N)$ across the cut $\delta(X)$ in $G(P)$. Since X is a component of $G(M)$, all the edges of $\delta(X) \cup \delta(S \backslash X)$ are edges of $G(N)$ and thus all the edges with exactly one end in X are resigned. It again follows that such a resigning does not affect the sign on any directed circuit in $G(P)$.

Unfortunately, there are Pfaffian matroid pairs for which a Pfaffian signing of the fundamental graph can be resigned across edges that do not correspond to such cuts; in particular, a skew-symmetric matroid pair as defined in Examples 3 of Section 3.5 can have resignings that do not correspond to cuts in its fundamental graph. We conjecture that if a Pfaffian matroid pair is sufficiently connected and does not contain a skew-symmetric minor, then up to resigning across cuts of the fundamental graph for the pair, the Pfaffian signing is unique. Recall from Section 2.6 that the matroid pair (M, N) is *k-connected* if

$$\mathrm{rank}_M(S_1) + \mathrm{rank}_N(S_2) > r + k - 1$$

whenever the partition $\{S_1, S_2\}$ satisfies $|S_1| \geq k$ and $|S_2| \geq k$.

Conjecture 3.13. *If the Pfaffian matroid pair $P = (M, N)$ is 3-connected and does not contain a skew-symmetric pair as a minor, then up to resigning across separators of M and N, the Pfaffian representation of P is unique.*

3.7 Exact basis problem

Given a weight function on the ground set for a matroid pair, the exact basis problem is to determine if the matroid pair has a basis of a particular weight. Although there are optimality conditions and an efficient algorithm by Edmonds [10] for finding a basis of maximum or minimum weight, there are no efficient algorithms to solve the exact basis problem. Here we present an algebraic formulation of the exact basis problem which leads to an efficient solution when the matroid pair is Pfaffian. An application of the exact basis problem is the corresponding exact matching problem in an edge weighted graph. Thus, given a weighting on the edges of a Pfaffian bipartite graph, the formulation presented here can be used to determine if the graph has a perfect matching of a specific weight.

The main tool of the formulation is a linear algebra identity. Let A_1 and A_2 be square matrices with rows and columns indexed by the integers 1 through m. If $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_k\}$ are subsets of $\{1, 2, \ldots, m\}$ then define $\text{sign}(X, Y)$ by

$$\text{sign}(X, Y) = (-1)^{\sum_{i=1}^{k} x_i + y_i}.$$

A standard identity from linear algebra states that the determinant of $A_1 + A_2$ can be obtained by the following summation, where $\bar{X}$ denotes the complement of X:

$$\det(A_1 + A_2) = \sum_{X \subseteq Z} \sum_{Y \subseteq Z : |X| = |Y|} \text{sign}(X, Y) \det A_1[X; Y] \det A_2[\bar{X}; \bar{Y}]. \tag{3.8}$$

Suppose that P is a rank r Pfaffian matroid pair on the ground set S with weights $w(e)$ defined for all $e \in S$. Let (A_M, A_N) be a Pfaffian representation of P and assume that the rows of A_M and A_N are indexed by the basis B. Let A be the square matrix formed from A_M and A_N as shown in (3.9), such that the rows and columns of A are indexed by $B \cup S$ to match the row and column indices of A_M and A_N.

$$A = \left[\begin{array}{c|c} 0 & A_M \\ \hline -A_N^{\top} & 0 \end{array}\right] \tag{3.9}$$

Let z be an indeterminant and let Z be the diagonal matrix with the same row and column indices as A, as shown in (3.10).

$$Z = \left[\begin{array}{c|cccc} 0 & & 0 & & \\ \hline & z^{w(e_1)} & & & \\ 0 & & z^{w(e_2)} & & \\ & & & \ddots & \\ & & & & z^{w(n)} \end{array}\right] \tag{3.10}$$

Note that the only nonsingular submatrices of Z are principal submatrices $Z[X;X]$ with $X \subseteq S$. Furthermore, for all $X \subseteq S$,

$$\det Z[X;X] = \sum_{e \in X} z^{w(e)} = z^{w(X)}.$$

Since $\text{sign}(X,X) = 1$ for all $X \subseteq S$, identity (3.8) implies that

$$\begin{aligned}\det(A+Z) &= \sum_{X \subseteq S} \det A[B \cup X; B \cup X] \det Z[S \backslash X; S \backslash X] \\ &= \sum_{X \subseteq Z} \det A_M[X] \det A_N[X] z^{w(S)-w(X)}\end{aligned}$$

The assumption that (A_M, A_N) is a Pfaffian representation of P therefore implies that the coefficient of $z^{w(S)-k}$ in $\det(A+Z)$ is exactly the number of bases of P with weight k. It follows that the exact basis problem can be solved for a pair of Pfaffian matroids. In particular, when M is a regular matroid and w is a weight function on the ground set of M, then for all weights k this formulation determines if M has a basis of weight k. We note that the weights given by w must be unitary for this formulation to lead to an efficient algorithm, otherwise $\det(A+Z)$ can have exponentially many terms. For non unitary weights, the problem remains hard.

Chapter 4

Binary Spaces

A major obstacle to characterizing Pfaffian matroid pairs is the existence of "inequivalent" Pfaffian signings: given two Pfaffian representations of the same matroid pair, we do not have a simple description of how these two signings differ. (See Section 3.6 and Examples 2 and 3 in Section 3.5.)

Given a matroid P on the ground set S, a set $X \subseteq S$ is called a *constant-parity* intersecting set if either

(i) $|B \cap X|$ is even for all bases B of P, or

(ii) $|B \cap X|$ is odd for all bases B of P.

It follows that resigning a Pfaffian representation of P on $X \subseteq S$ gives another Pfaffian representation if and only if X is a constant-parity intersecting set. Thus the problem of describing the relationship between multiple Pfaffian representations of the same matroid pair is equivalent to the problem of describing the constant-parity intersecting sets of the matroid pair.

In this chapter we approach the problem of describing the constant-parity sets for a matroid pair P through a characterization of the *binary space* generated by the bases of P; if A is a set of $\{0,1\}$ vectors then the binary space generated by A, denoted $\mathrm{Bin}(A)$, is all linear combinations over $GF(2)$ of the vectors in A.

Let M be a matroid on the n element ground set S and let Q^M and R^M be the subsets of $GF(2)^n$ given by

$$Q^M = \{x \in GF(2)^n : x(S_i) = 0 \text{ for all } 1 \leq i \leq c\} \text{ and}$$
$$R^M = \{x \in GF(2)^n : x(S_i) = \mathrm{rank}(S_i) \text{ for all } 1 \leq i \leq c\}.$$

If P is the connected matroid pair (M, N), then we show in Theorem 4.8 that

$$\mathrm{Bin}(\mathcal{B}(P)) \subseteq (Q^M \cap Q^N) \cup (R^M \cap R^N). \tag{4.1}$$

Our main result is Theorem 4.14, where we show that by excluding the twined K_4 pair (Example 2 of Section 3.5) as a matroid minor, the inclusion in Equation (4.1) above is tight. This proof of this main theorem considers a binary space defined on the vertices of a directed graph. In Section 4.2 we determine the dimension of this binary space using a variant of ear decompositions, and in Section 4.3 this dimension is used to characterize $\mathrm{Bin}(\mathcal{B}(P))$ for a matroid pair P in terms of the fundamental graph for P.

4.1 The binary space of a matroid

If M is a matroid on the ground set S and S' is a component of M, then for all bases B of M, $|B \cap S'| = \mathrm{rank}_M(S')$, and thus the parity of $|B \cap S'|$ is equal to the parity of the rank of S'. If $x \in \mathrm{Bin}(\mathcal{B}(M))$ is the sum of k basis vectors, then it follows that $|x(S')| = k \cdot \mathrm{rank}_M(S')$ over $GF(2)$. This simple necessary condition completely characterizes the vectors in $\mathrm{Bin}(\mathcal{B}(M))$, as the next theorem shows.

Theorem 4.1 (Rieder). *Let M be a matroid with c components on the ground set S and for $1 \leq i \leq c$, let $S_i \subseteq S$ be the ground set for the i^{th} component of M. If*

$$Q^M = \{x \in GF(2)^S : x(S_i) = 0 \textit{ for all } 1 \leq i \leq c\} \textit{ and}$$

$$R^M = \{x \in GF(2)^S : x(S_i) = \mathrm{rank}(S_i) \textit{ for all } 1 \leq i \leq c\},$$

then $\mathrm{Bin}(\mathcal{B}(M)) = Q^M \cup R^M$.

Proof. If $y \in \mathrm{Bin}(\mathcal{B}(M))$ then there exist $b_1, b_2, \ldots, b_k \in \mathcal{B}(M)$ such that $y = \sum_{j=1}^{k} b_j$, and therefore for $1 \leq i \leq c$,

$$y(S_i) = \sum_{j=1}^{k} b_j(S_i) = k\mathrm{rank}(S_i) = \begin{cases} 0, & \text{if } k \text{ is even;} \\ \mathrm{rank}(S_i), & \text{if } k \text{ is odd.} \end{cases}$$

Thus if $y \in \mathrm{Bin}(\mathcal{B}(M))$, then $y \in Q^M \cup R^M$.

Let $1 \leq i \leq c$ be such that $|S_i| \geq 2$ and let $u, v \in S_i$. Since M restricted to S_i is connected, there exists a cycle C in M with $C \subseteq S_i$ and $u, v \in C$. The independent set $C \backslash u$ can be extended to a basis B_v of M where $u \notin B_v$. The dependent set $B_v \cup u$ then contains a unique cycle, namely C, and therefore $B_u = (B_v \cup u) \backslash v$ is also a basis of M. Let e_j denote the j^{th} standard basis vector. The sum over $GF(2)$ of the characteristic vector for B_v with the characteristic vector for B_u is then $e_v + e_u$, and since i, u and v were arbitrary, it follows that $e_u + e_v \in \mathrm{Bin}(\mathcal{B}(M))$ whenever u and v are elements in the same component of M. Thus if x satisfies $x(S_i) = 0$ for all $1 \leq i \leq c$, then x can be expressed as a linear combination of basis vectors, and therefore

$$Q^M \subseteq \mathrm{Bin}(\mathcal{B}(M)). \tag{4.2}$$

Let $x \in R^M$ and $y \in \mathcal{B}(M)$, and let $z = x + y$. Then $z(S_i) = 0$ for $1 \leq i \leq c$, and thus by (4.2), $z \in \text{Bin}(\mathcal{B}(M))$. It follows that $z + y = x \in \text{Bin}(\mathcal{B}(M))$, and therefore $R^M \subseteq \text{Bin}(\mathcal{B}(M))$. The theorem follows. □

A description of the lattice generated by $\mathcal{B}(M)$ contains a description of the binary space generated by $\mathcal{B}(M)$ and thus Theorem 4.1 is a corollary of Rieder's [33] characterization of $\text{Lat}(\mathcal{B}(M))$ (Theorem 5.2). If $\dim(\text{Bin}(\mathcal{B}(M)))$ denotes the dimension of the binary space of $\text{Bin}(\mathcal{B}(M))$, then Theorem 4.1 implies that $\dim(\text{Bin}(\mathcal{B}(M)))$ is determined by the number of components of M and the parity of the rank of these components.

Corollary 4.2. *If M is a matroid on n elements and M has c connected components, then*

$$\dim(\text{Bin}(\mathcal{B}(M))) = \begin{cases} n - c, & \textit{if all components of } M \textit{ have even rank;} \\ n - c + 1, & \textit{if a component of } M \textit{ has odd rank.} \end{cases}$$

Proof. Let S be the ground set for M and for $1 \leq i \leq c$ let S_i be the subset of S corresponding to the i^{th} component of M. Let A be the matrix over $GF(2)$ such that the i^{th} row of A is the characteristic vector for S_i. Since A has n columns and c independent rows, the nullspace of A has dimension $n-c$. Since $Q^M = \{x : Ax = 0\}$, the dimension of Q^M is $n - c$. If all components of M have even rank then by Theorem 4.1, $\text{Bin}(\mathcal{B}(M)) = Q^M$ and thus $\text{Bin}(\mathcal{B}(M))$ has dimension $n - c$.

Suppose instead that a component of M has odd rank. Let $r \in \{0,1\}^c$ be such that r_i is equivalent to $\text{rank}_M(S_i)$ over $GF(2)$, and let $y \in \mathcal{B}(M)$. Then $Ay = r \neq 0$, and therefore y is not in the nullspace of A. It follows that the dimension of $\text{Bin}(\mathcal{B}(M))$ is at least $n - c + 1$. If $\{z_1, z_2, \ldots, z_{n-c}\}$ is a basis for the nullspace of A and $x \in \mathcal{B}(M)$, then $A(x + y) = r + r = 0$ and thus there exist $\alpha_i \in \{0,1\}$ such that $x + y = \sum_{i=1}^{n-c} \alpha_i z_i$. Therefore x is in the span of $\{y, z_1, \ldots, z_{n-c}\}$ whenever $x \in \mathcal{B}(M)$, and it follows that $\text{Bin}(\mathcal{B}(M))$ has dimension $n - c + 1$ whenever a component of M has odd rank. □

For a matroid pair $P = (M, N)$, we would like a description of $\text{Bin}(\mathcal{B}(P))$ in terms of the intersection of $\text{Bin}(\mathcal{B}(M))$ and $\text{Bin}(\mathcal{B}(N))$. Our approach requires some theorems about cycle spaces, which we present next.

4.2 Cycle spaces in directed graphs

Given a connected matroid pair $P = (M, N)$ with sets Q^M, Q^N, R^M, and R^N defined as in Theorem 4.1, the binary space generated by $\mathcal{B}(P)$ is characterized by the directed circuits in the fundamental graph for P, as the following theorem shows:

Theorem *If $P = (M, N)$ is a connected matroid pair, then*

$$\mathrm{Bin}(\mathcal{B}(P)) = (Q^M \cap Q^N) \cup (R^M \cap R^N)$$

if and only if $V(C) \in \mathrm{Bin}(\mathcal{B}(P))$ for every directed cycle C in $G(P)$.

We prove this theorem in Section 4.3, and our proof uses the dimension of the vector space generated by the vertices of the directed circuits in the fundamental graph for the matroid pair. In this section we determine the dimension of that vector space for any strongly connected bipartite graph.

Let $G = (V, E)$ be a directed graph. A *circulation* in G is a function on the edges of G such that the total flow out of any vertex is equal to the total flow into the vertex. That is, a circulation in G over the field $\mathbb{F}$ is a vector $f \in \mathbb{F}^E$ such that

$$f(\delta(v)) = f(\delta(V \backslash v)) \qquad \forall v \in V.$$

The set of circulations of a graph G is called the *cycle space* of G and is denoted $\mathcal{C}(G)$. The cycle space is a well studied vector space and its dimension is well known. (See Bondy and Murty [2].)

Theorem 4.3. *If $G = (V, E)$ is a strongly connected directed graph and $\mathbb{F}$ is a field, then the dimension of $\mathcal{C}(G)$ over $\mathbb{F}$ is $|E| - |V| + 1$.*

Note that if C is a circuit in G then there is an associated circulation f for C where

$$f(e) = \begin{cases} 1, & \text{if } e \text{ is a forward edge in } C \text{ with respect to an orientation of } C; \\ -1, & \text{if } e \text{ is a backward edge in } C \text{ with respect to the same orientation;} \\ 0, & \text{otherwise.} \end{cases}$$

Every circulation in $\mathcal{C}(G)$ can be expressed as a weighted sum of circulations for circuits (see Bondy and Murty [2]).

If G is a directed graph then for each $f \in \mathcal{C}(G)$ we define a function f_V on the vertices of G such that $f_V(v)$ is the total amount of flow out of, and therefore the total amount of flow into, vertex v:

$$f_V(v) = \sum_{vu \in \delta(v)} f(vu) \qquad \forall\, v \in V(G).$$

The *vertex cycle space* of G, denoted $\mathcal{C}_V(G)$, is the set of f_V over all $f \in \mathcal{C}(G)$. In our characterization of the binary space for a connected matroid pair P we require the dimension of the vertex cycle space over $GF(2)$ for $G(P)$. In this section we therefore determine the dimension of $\mathcal{C}_V(G)$ over $GF(2)$ when G is a strongly connected and bipartite directed graph. When G has vertex bipartition $\{X, Y\}$ we partition the edges of G into those directed out of X and those directed into X, and we consider the two subgraphs induced by these edges. We first determine the dimension of the subspace of $C_V(G)$ containing those vectors which are zero both on every component of $G[\delta(X)]$ and on every component of $G[\delta(Y)]$.

Lemma 4.4. *Let G be a strongly connected bipartite directed graph with vertex bipartition $\{X, Y\}$ and let c_X and c_Y be the number of components of $G[\delta(X)]$ and $G[\delta(Y)]$ respectively. Let $\mathcal{W}$ be the set of $W \subseteq X \cup Y$ such that $G[W]$ is a component of either $G[\delta(X)]$ or $G[\delta(Y)]$. If $\mathbb{F}$ is a field and*

$$R = \{x \in \mathbb{F}^V : x(W) = 0 \quad \textit{for all} \quad W \in \mathcal{W}\},$$

then R has dimension $|X \cup Y| - c_X - c_Y + 1$.

Proof. Let A be the matrix over $\mathbb{F}$ whose rows are the characteristic vectors for the subsets in $\mathcal{W}$. Then $R = \{x : Ax = 0\}$, and the dimension of R is equal to the dimension of the nullspace of A. Since G is strongly connected, every vertex of G is incident to an arc in $\delta(X)$. The components of $G[\delta(X)]$ therefore partition the vertices of G, and summing the rows of A over all components of $G[\delta(X)]$ gives the all ones vector. Similarly, the sum of the rows of A indexed by the components of $G[\delta(Y)]$ is the all ones vector. Thus the rows of A are dependent and the rank of A is at most $|\mathcal{W}|-1$. If a strict subset of the rows of A are dependent, then there is a set of the components of $G[\delta(X)]$ whose union covers the same proper subset of V as a set of components of $G[\delta(Y)]$. This contradicts that G is connected, and therefore the rank of A is equal to $|\mathcal{W}| - 1 = c_X + c_Y - 1$. Since A has $|V|$ columns, the nullspace of A has dimension $|V| - c_X - c_Y + 1$, and the lemma follows. ☐

Let G be a strongly connected directed graph and let H be a subgraph of G. An *ear* of H is a directed path Q in G with both ends of Q in $V(H)$ and no interior vertices of Q in $V(H)$. The ends of an ear need not be distinct. An *ear decomposition* of G is a sequence of strongly connected subgraphs $G_0, G_1, \ldots, G_k$ of G such that G_0 is a directed circuit, $G_k = G$, and for $1 \le i \le k$, there exists an ear Q_{i-1} of G_{i-1} such that $G_i = G_{i-1} \cup Q_{i-1}$. Hence

$$|E(G_i)| - |V(G_i)| = |E(G_{i-1})| - |V(G_{i-1})| + 1$$

for $1 \le i \le k$, and therefore

$$k = |E(G)| - |V(G)|. \tag{4.3}$$

It follows that all ear decompositions of G contain the same number of ears. Ear decompositions were first used by Hetyei [17] and are a common tool in graph theory.

For directed bipartite graphs we describe a variant of an ear decomposition using *acyclic ears*: if H is a subgraph of the strongly connected bipartite directed graph G, and G has vertex bipartition $\{X, Y\}$, then we call an ear Q of H acyclic if the subgraph of G induced by $\delta(X)\cap E(H\cup Q)$ is acyclic and the subgraph of G induced by $\delta(Y)\cap E(H\cup Q)$ is also acyclic. If H is the subgraph induced by the bold edges in Figure 4.1(a), then the directed path shown with dashed lines in Figure 4.1(b) is an acyclic ear of H.

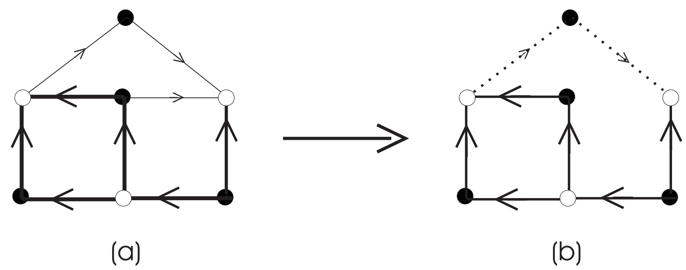

Figure 4.1: An acyclic ear of a subgraph

Let G be a strongly connected directed bipartite graph and let $d \geq 0$ be such that there exist subgraphs $H_0, H_1, \ldots, H_d$ of G where H_0 is a directed circuit in G and for $1 \leq i \leq d-1$, H_{i+1} is formed by the addition of an acyclic ear to H_i. Note that if a subgraph H has an acyclic ear then H restricted to $G[\delta(X)]$ is a forest and H restricted to $G[\delta(Y)]$ is a forest. However, when H restricted to $G[\delta(X)]$ is a forest and H restricted to $G[\delta(Y)]$ is a forest, it need not be that H has an acyclic ear in G. If H_d does not have an acyclic ear, then we call the sequence $H_0, \ldots, H_d$ an *acyclic ear decomposition* of G.

Ear decompositions and acyclic ear decompositions share some similar properties. For example, just as all ear decompositions of a strongly connected graph have the same number of ears, all acyclic ear decompositions of a strongly connected directed bipartite graph have the same number of acyclic ears, as shown in the next lemma.

Lemma 4.5. *Let G be a strongly connected bipartite directed graph with vertex bipartition $\{X, Y\}$ and let c_X and c_Y be the number of components of $G[\delta(X)]$ and $G[\delta(Y)]$ respectively. If $H_0, \ldots, H_d$ is an acyclic ear decomposition of G, then*

$$d = |X \cup Y| - c_X - c_Y.$$

Proof. If H_d is not a spanning subgraph of G then there exists an edge uv of G such that $u \in V(H_d)$ and $v \notin V(H_d)$. For any shortest path Q from v to $V(H_d)$, the ear $Q_d = \{uv\} \cup Q$ is an acyclic ear for H_d. This contradicts the definition of an acyclic ear decomposition and thus H_d is a spanning subgraph of G.

Suppose the subgraph of H_d induced by $E(H_d) \cap \delta(X)$ is not a spanning subgraph of $G[\delta(X)]$, and let $v \in V(G)$ be such that v is not incident to an edge in $E(H_d) \cap \delta(X)$. Since G is strongly connected, v is incident to an edge $e \in \delta(X)$. The subgraph of G

induced by $\delta(X) \cap E(H_d \cup e)$ is therefore acyclic and by definition, the ear $Q_d = e$ is an acyclic ear of H_d. Thus H_d must be a spanning forest of $G[\delta(X)]$, and it follows that

$$|E(H_d) \cap \delta(X)| = |V(H_d)| - c_X = |X \cup Y| - c_X. \tag{4.4}$$

Similarly, the subgraph of H_d induced by $\delta(Y)$ is a spanning forest of $G[\delta(Y)]$, and therefore

$$|E(H_d) \cap \delta(Y)| = |X \cup Y| - c_Y. \tag{4.5}$$

It follows from Equations (4.4) and (4.5) that

$$|E(H_d)| = 2|X \cup Y| - c_X - c_Y, \tag{4.6}$$

and since $H_0, H_1, \ldots, H_d$ is an ear decomposition of H_d,

$$d = |E(H_d)| - |X \cup Y|. \tag{4.7}$$

Equations (4.6) and (4.7) imply that $d = |X \cup Y| - c_X - c_Y$. $\square$

Given an ear decomposition $G_0, G_1, \ldots, G_k$ of G and paths Q_{i-1} with $G_i = G_{i-1} \cup Q_{i-1}$ for $1 \leq i < k$, let $C_0 = G_0$ and for $1 \leq i \leq k$ let C_i be a directed circuit in G_i with $Q_{i-1} \subseteq C_i$. For all $0 \leq i \leq k$, there exists $e \in E(C_i)$ such that $e \notin E(C_j)$ for all $0 \leq j < i$, and therefore the characteristic vectors for $E(C_0), \ldots, E(C_k)$ are independent. Since $k = |E(G)| - |V(G)|$, Theorem 4.3 implies that $\{C_0, \ldots, C_k\}$ is a basis for $\mathcal{C}(G)$. Such a basis is called a cycle basis of $\mathcal{C}(G)$ with respect to the ear decomposition $G_0, G_1, \ldots, G_k$. An analogous set of independent vectors exists for the vertex cycle space $\mathcal{C}_V(G)$ using acyclic ear decompositions, as we show next. Note that the cycle basis for $\mathcal{C}(G)$ is defined with respect to the *edges* of G where as the vertex cycle space is defined with respect to the *vertices* of G.

Lemma 4.6. *Let G be a strongly connected bipartite directed graph with acyclic ear decomposition $H_0, \ldots H_d$ and let $C_0, \ldots, C_d$ be a cycle basis of $\mathcal{C}(H_d)$ with respect to $H_0, \ldots, H_d$. The characteristic vectors for $V(C_0), \ldots, V(C_d)$ are independent over $GF(2)$.*

Proof. For $0 \leq i \leq d$, let c_i be the characteristic vector for $V(C_i)$ and let $f_{C_i} \in \mathcal{C}$ be the associated circulation for C_i. Assume that $\alpha_i \in \{0, 1\}$ are such that $\sum_{i=1}^{d} \alpha_i c_i = 0$ over $GF(2)$, and let $f \in \mathcal{C}(G)$ be given by

$$f = \sum_{i=0}^{d} \alpha_i f_{C_i}.$$

Suppose $\alpha_i \neq 0$ for some $0 \leq i \leq d$. Since $f_{C_1}, \ldots, f_{C_d}$ are independent over $GF(2)$, $f \neq 0$ and there exists an edge h of G such that $f(h) \neq 0$. Without loss of generality, assume that $h \in \delta(X)$ and consider the subgraph H of H_d where $e \in E(H)$ if and only if $f(e) \neq 0$. Thus $h \in E(H)$, and by the assumption that $\sum_{i=1}^{d} \alpha_i c_i = 0$, the sum of the weights of all edges out of each vertex is zero, as is the sum of the weights of all edges into each vertex. So $H[\delta(X)]$ is a nonempty subgraph with each vertex of degree at least two, and therefore $H[\delta(X)]$ contains a circuit. But H is a subgraph of H_d and by construction $H_d[\delta(X)]$ is acyclic. Thus $\alpha_i = 0$ for all i, and $\{c_1, \ldots, c_d\}$ are independent over $GF(2)$. $\square$

We can now prove the main result of this section.

Theorem 4.7. *If G is a strongly connected bipartite directed graph with vertex bipartition $\{X, Y\}$, and c_X and c_Y are the number of connected components in $G[\delta(X)]$ and $G[\delta(Y)]$ respectively, then the dimension of $\mathcal{C}_V(G)$ over $GF(2)$ is*

$$|X \cup Y| - c_X - c_Y + 1.$$

Proof. Let $C \subseteq E(G)$ be a directed circuit in G and let x be the characteristic vector over $GF(2)$ for $V(C)$. Since $C \cap \delta(X)$ is a matching in $G[\delta(X)]$, $V(C)$ intersects each component in $G[\delta(X)]$ with even parity, and therefore $x(W) = 0$ whenever $W \subseteq V(G)$ is a component of $G[\delta(X)]$. Similarly, $C \cap \delta(Y)$ is a matching in $G[\delta(Y)]$ and if $W \subseteq V(G)$ is a component of $G[\delta(Y)]$ then $x(W) = 0$.

Since each function in $\mathcal{C}_V(G)$ can be expressed as the sum of directed circuits, it follows that $y(W) = 0$ for all $y \in \mathcal{C}_V(G)$, and by Lemma 4.4, the dimension of $\mathcal{C}_V(G)$ over $GF(2)$ is at most $|V| + 1 - c_X - c_Y$. By Lemma 4.6, there exist $|V| + 1 - c_X - c_Y$ directed circuits that are independent with respect to vertices, and the theorem follows. $\square$

4.3 The binary space for a pair of matroids

Let $P = (M, N)$ be a connected matroid pair. Since $\mathcal{B}(P) = \mathcal{B}(M) \cap \mathcal{B}(N)$,

$$\mathrm{Bin}(\mathcal{B}(P)) \subseteq \mathrm{Bin}(\mathcal{B}(M)) \cap \mathrm{Bin}(\mathcal{B}(N)). \tag{4.8}$$

We consider when this containment is strict and show that this is related to the vertex cycle space (Section 4.2) for the fundamental graph of P.

Let M and N be matroids on the ground set S and assume that M has c_M connected components and that N has c_N connected components. For $1 \leq i \leq c_M$ and $1 \leq j \leq c_N$, let $S_i^M \subseteq S$ and $S_j^N \subseteq S$ be the ground sets corresponding to the i^{th} and j^{th} components of M and N respectively. Let Q^M and R^M be subspaces defined over $GF(2)$ with

$$Q^M = \{x \in \{0,1\}^S : x(S_i^M) = 0 \text{ for all } 1 \leq i \leq c_M\} \text{ and}$$
$$R^M = \{x \in \{0,1\}^S : x(S_i^M) = \mathrm{rank}_M(S_i^M) \text{ for all } 1 \leq i \leq c_M\}.$$

If we define Q^N and R^N similarly, then by Theorem 4.1 and Equation (4.8),

$$\mathrm{Bin}(\mathcal{B}(M,N)) \subseteq (Q^M \cap Q^N) \cup (Q^M \cap R^N) \cup (R^M \cap Q^N) \cup (R^M \cap R^N). \tag{4.9}$$

This containment can be tightened can be P is connected, as the next theorem shows.

Theorem 4.8. *If P is the connected matroid pair (M,N), then*

$$\mathrm{Bin}(\mathcal{B}(P)) \subseteq (Q^M \cap Q^N) \cup (R^M \cap R^N). \tag{4.10}$$

Proof. If every component of M is even, then $Q^M = R^M$ and therefore Equations (4.9) and (4.10) are equivalent. Similarly, if $Q^N = R^N$ then (4.9) and (4.10) are equivalent. Thus we may assume that there exists a component $X_M \subseteq S$ of M such that $\mathrm{rank}_M(X_M)$ is odd and a component $X_N \subseteq S$ of N such that $\mathrm{rank}_M(X_N)$ is odd. If $y \in \mathrm{Bin}(\mathcal{B}(P))$ then there exist $b_1, \dots b_d \in \mathcal{B}(P)$ such that $y = \sum_{i=1}^{d} b_i$ and therefore

$$y(X_M) = \sum_{i=1}^{d} b_i(X_M) = d = \sum_{i=1}^{d} b_i(X_N) = y(X_N). \tag{4.11}$$

If $x \in R^M$ then $x(X_M) = 1$ and if $x \in Q^N$ then $x(X_N) = 0$, and therefore Equation 4.11 implies that $(R^M \cap Q^N) \cap \mathrm{Bin}(\mathcal{B}(P)) = \emptyset$. Similarly, $(Q^M \cap R^N) \cap \mathrm{Bin}(\mathcal{B}(P)) = \emptyset$, and (4.9) simplifies to

$$\mathrm{Bin}(\mathcal{B}(P)) \subseteq (Q^M \cap Q^N) \cup (R^M \cap R^N).$$

□

Theorem 4.8 suggests characterizing the matroid pairs $P = (M,N)$ for which $\mathrm{Bin}(\mathcal{B}(P))$ is equal to $(Q^M \cap Q^N) \cup (R^M \cap R^N)$, rather than characterizing the matroid pairs for which $\mathrm{Bin}(\mathcal{B}(P)) = \mathrm{Bin}(\mathcal{B}(M)) \cap \mathrm{Bin}(\mathcal{B}(N))$.

Corollary 4.9. *If $P = (M,N)$ is a connected matroid pair with even rank and M has a component of odd rank, then*

$$\mathrm{Bin}(\mathcal{B}(P)) \neq \mathrm{Bin}(\mathcal{B}(M)) \cap \mathrm{Bin}(\mathcal{B}(N)).$$

Proof. If M has a component of odd rank then the proof of Theorem 4.8 shows that $R^M \cap Q^N$ and $\mathrm{Bin}(\mathcal{B}(P))$ have no common elements. Since $R^M \cap Q^N$ is contained in $\mathrm{Bin}(\mathcal{B}(M)) \cap \mathrm{Bin}(\mathcal{B}(N))$, the corollary follows if $R^M \cap Q^N$ is nonempty.

If $R^M \cap Q^N$ is empty, then there does not exists a vector x over $GF(2)$ which satisfies both $x(S_i^M) = \mathrm{rank}_M(S_i^M)$ for all $1 \leq i \leq c_M$ and $x(S_i^N) = 0$ for all $1 \leq i \leq c_N$. By Observation 3.9, there exists $I_M \subseteq \{1, \dots, c_M\}$ and $I_N \subseteq \{1, \dots, c_N\}$ such that

$$\bigcup_{i \in I_M} S_i^M = \bigcup_{i \in I_N} S_i^N \tag{4.12}$$

and

$$\sum_{i \in I_M} \mathrm{rank}_M(S_i^M) = 1 \pmod 2. \tag{4.13}$$

Since P is connected, (4.12) is satisfied only if

$$\bigcup_{i \in I_M} S_i = S.$$

This implies that $\sum_{i \in I_M} \mathrm{rank}_M(S_i^M) = x(S)$ which contradicts (4.13) since P has even rank by assumption. Thus $R^M \cap Q^N$ is nonempty and the corollary follows. □

Before characterizing those matroid pairs $P = (M, N)$ for which $\mathrm{Bin}(\mathcal{B}(P))$ is equal to $(Q^M \cap Q^N) \cup (R^M \cap R^N)$, we first explain why such pairs, if Pfaffian, have essentially unique Pfaffian signings. Recall from the introduction to this chapter that a constant-parity intersecting set of the matroid pair P is a subset S of the ground set for P such that either $|B \cap X|$ is even for all bases B of P, or $|B \cap X|$ is odd for all bases B of P.

Theorem 4.10. *If $P = (M, N)$ is a strongly connected matroid pair on the ground set S and*

$$\mathrm{Bin}(\mathcal{B}(P)) = (Q^M \cap Q^N) \cup (R^M \cap R^N),$$

then the set $X \subseteq S$ is a constant-parity intersecting set for P if and only if there exists a separator X_M of M and a separator X_N of N such that $X = X_M \Delta X_N$.

Proof. Let $P = (M, N)$ be a strongly connected matroid pair on the ground set S. If X_M and X_N are separators of M and N respectively and B is a basis of P, then over $GF(2)$,

$$|B \cap (X_M \Delta X_N)| = |B \cap X_M| + |B \cap X_N| = \mathrm{rank}_M(X_M) + \mathrm{rank}_N(X_N).$$

Since $\mathrm{rank}_M(X_M)+\mathrm{rank}_N(X_N)$ has fixed parity, $X_M \Delta X_N$ is a constant-parity intersecting set for P.

Suppose instead that $X \subseteq S$ is a constant-parity intersecting set for P and let A be the matrix over $GF(2)$ whose rows are the characteristic vectors for the components of M and N. Note that the vector space $Q^M \cap Q^N$ is then equal to the nullspace of A.

First consider the case that $|B \cap X|$ is even for all bases B of P. Then $x(X) = 0$ for all x in the nullspace of A, and thus X is the symmetric difference of components of M and N. It follows that $X = X_M \Delta X_N$ for some separator X_M of M and separator X_N of N.

Next consider the case that $|B \cap X|$ is odd for all bases B of P. If some component of M or N has odd rank, then $Q^M \cap Q^N$ is the set of all binary combinations of an even number of bases from $\mathcal{B}(P)$, and $R^M \cap R^N$ is the set of all binary combinations of an odd number of bases from $\mathcal{B}(P)$. Since $x(X)$ is odd if $x \in R^M \cap R^N$, if $x \in Q^M \cap Q^N$

then $x(X)$ is even and thus $x(X) = 0$ for all x in the nullspace of A. It again follows that $X = X_M \Delta X_N$ for some separator X_M of M and separator X_N of N.

It remains to consider the case that $|B \cap X|$ is odd for all bases B of P and every component of M and N has even rank. In this case, $R^M \cap R^N = Q^M \cap Q^N$ and the binary space generated by $\mathcal{B}(P)$ is equal to the nullspace of A. Since some vectors in the nullspace of A correspond to the sum of an even number of bases and some correspond to an odd number of bases, it is no longer the case that $x(X) = 0$ for all x in the nullspace of A. We show in fact that such a case can not occur.

Since every component of M has even rank, P has even rank and thus $X \neq S$. Let $u \in S$ be such that $u \notin X$ and let M' be obtained from M by replacing the element u by the two elements u_1 and u_2 such that

(i) u_1 and u_2 are in series in M',

(ii) u_2 is a loop of M'/u_1, and

(iii) $M/u_2 = M$.

Let $S' = S\Delta\{u, u_1, u_2\}$ and $r' = r + 1$, where r is the rank of M. Then the rank of M' is r', and if $T \subseteq S'$ then

$$\operatorname{rank}_{M'}(T) = \begin{cases} \operatorname{rank}_M(T), & \text{if } |T \cap \{u_1, u_2\}| = 0; \\ \operatorname{rank}_M(T \backslash \{u_1, u_2\}) + 1, & \text{if } |T \cap \{u_1, u_2\}| = 1; \\ \operatorname{rank}_M(T\Delta\{u, u_1, u_2\}) + 1, & \text{if } |T \cap \{u_1, u_2\}| = 2. \end{cases} \tag{4.14}$$

If we define N' similarly and let P' be the matroid pair (M', N'), then since P is a connected matroid pair it follows from Equation (4.14) that P' is a connected matroid pair. We show that $|B' \cap X|$ is odd for all bases B' of P', and that this leads to a contradiction.

If B' is a basis of P' then since u_1 and u_2 are in series, $|B' \cap \{u_1, u_2\}| \geq 1$. Suppose first that $u_2 \in B'$ and $u_1 \in B'$. Then $B = B'\Delta\{u_1, u_2, u\}$ is a basis of P and since $|B \cap X|$ is odd and $\{u_1, u_2, u\} \cap X$ is empty, $|B' \cap X|$ is odd. Next, suppose that $u_2 \in B'$ and $u_1 \notin B'$. Then $B = B'\Delta\{u_2\}$ is a basis of P which implies that $|B \cap X|$ is odd and again $|B' \cap X|$ is odd. Finally, suppose that $u_2 \notin B'$. Then $u_1 \in B'$ and $B = B'\Delta\{u, u_1\}$ is a basis of P. Since $|B \cap X|$ is odd and $\{u, u_1\} \cap X$ is empty, it follows that $|B' \cap X|$ is odd.

Thus $|B' \cap X|$ is odd for all bases B' of P, and since P' is connected it follows that $X = X_{M'} \Delta X_{N'}$ for some separator $X_{M'}$ of M' and $X_{N'}$ of N'. Exactly one of $X_{M'}$ and $X_{N'}$ must have odd rank for $|B' \cap X|$ to be odd, and since only the component containing u_1 and u_2 has odd rank in either M' or N', this implies that $\{u_1 \cup u_2\}$ is in exactly one of $X_{M'}$ and $X_{N'}$. This in turn implies that $\{u_1, u_2\} \subset X$ and since $\{u_1, u_2\} \cap S = \emptyset$, this contradicts the assumption that $X \subseteq S$. $\square$

We now give a characterization in terms of fundamental graphs for when the binary space generated by ($\mathcal{B}(M,N)$) is equal to $(Q^M \cap Q^N) \cup (R^M \cap R^N)$. We require one lemma concerning the dimension of $(Q^M \cap Q^N) \cup (R^M \cap R^N)$, the proof of which follows from the results of Section 4.2.

Lemma 4.11. *Let $P = (M,N)$ be a connected matroid pair on the ground set S, and assume M and N have c_M and c_N components respectively. If all components of M and N have even rank then $(Q^M \cap Q^N) \cup (R^M \cap R^N)$ has dimension $|S| - (c_M + c_N) + 1$, and if a component of M or N has odd rank then $(Q^M \cap Q^N) \cup (R^M \cap R^N)$ has dimension $|S| - (c_M + c_N) + 2$.*

Proof. Let A be the matrix over $GF(2)$ whose rows are the characteristic vectors for the components of M and N. The sum of the rows of A is zero, and since P is connected, no proper subset of the rows of A sums to zero. Thus A has rank $c_M + c_N - 1$ and therefore the nullspace of A has dimension $|S| - c_M - c_N + 1$. Since $Q^M \cap Q^N$ is equal to the nullspace of A, the dimension of $Q^M \cap Q^N$ is $|S| - c_M - c_N + 1$. If every component of M and N has even rank, then $R^M = Q^M$ and $R^N = Q^N$ and the lemma follows.

Suppose a component of M or N has odd rank, and let y be the characteristic vector for a common basis of M and N. Then $Ay \neq 0$ and thus the dimension of $(Q^M \cap Q^N) \cup (R^M \cap R^N)$ is at least $|S| - c_M - c_N + 2$. Let r be such that $Ay = r$ and let $x \in R^M \cap R^N$. Then $Ax = r = Ay$, and thus $A(x+y) = 0$. Hence $x + y$ is in the nullspace of A for all $x \in R^M \cap R^N$, and it follows that $(Q^M \cap Q^N) \cup (R^M \cap R^N)$ has dimension $|S| - c_M - c_N + 2$ whenever M or N has a component of odd rank. $\square$

We can now prove the main theorem of this section.

Theorem 4.12. *If $P = (M,N)$ is a connected matroid pair, then*

$$\mathrm{Bin}(\mathcal{B}(P)) = (Q^M \cap Q^N) \cup (R^M \cap R^N) \tag{4.15}$$

if and only if $V(C) \in \mathrm{Bin}(\mathcal{B}(P))$ for every directed cycle C in $G(P)$.

Proof. Let C be a directed circuit in $G(P)$. Since $E(C)$ is a matching in $G(M)$, $V(C)$ intersects each component in $G(M)$ with even parity and thus $V(C) \in Q^M$. Similarly, $V(C) \in Q^N$ and therefore $V(C) \in Q^M \cap Q^N$. Thus if Equation (4.15) holds, then $V(C) \in \mathrm{Bin}(\mathcal{B}(P))$ whenever C is a directed circuit in $G(P)$.

Conversely, suppose $V(C) \in \mathrm{Bin}(\mathcal{B}(P))$ for all directed cycles C in $G(P)$, and let $G = G(P)$. Then $G[\delta(B)] = G(M)$ and $G[\delta(S \backslash B)] = G(N)$. By Theorem 2.8, G is strongly connected. Furthermore, $G[\delta(B)]$ has c_M components and $G[\delta(S \backslash B)]$ has c_N components, and by Theorem 4.7, the dimension of $\mathcal{C}_V(G)$ is $|S| - c_M - c_N + 1$. Since $\mathcal{C}_V(G) \subseteq \mathrm{Bin}(\mathcal{B}(P))$ by assumption, the dimension of $\mathrm{Bin}(\mathcal{B}(P))$ is at least $|S| - c_M - c_N + 1$. If B is a common basis for M and N and M or N has a component of odd rank, then $B \notin \mathcal{C}_V(G)$, and thus $\mathrm{Bin}(\mathcal{B}(P))$ has dimension at least $|S| - c_M - c_N + 2$ whenever M or N has a component of odd rank. By Lemma 4.11, the dimension of $\mathrm{Bin}(\mathcal{B}(P))$ is therefore at least the dimension of $(Q^M \cap Q^N) \cup (R^M \cap R^N)$. Since $\mathrm{Bin}(\mathcal{B}(P)) \subseteq (Q^M \cap Q^N) \cup (R^M \cap R^N)$, and both $\mathrm{Bin}(\mathcal{B}(P))$ and $(Q^M \cap Q^N) \cup (R^M \cap R^N)$ are binary spaces, equality holds. $\square$

In Chapter 5 we discuss the set of all *integer* linear combinations of the characteristic vectors for the bases in $\mathcal{B}(P)$, which we call the *lattice* generated by $\mathcal{B}(P)$. We denote this set by $\mathrm{Lat}(\mathcal{B}(P))$. Most of the steps in the proof of Theorem 4.12 apply to the lattice as well, the key difference being that the binary space is a vector space. Hence if $\mathrm{Bin}(A)$ and $\mathrm{Bin}(A')$ have the same dimension and $\mathrm{Bin}(A) \subseteq \mathrm{Bin}(A')$, then $\mathrm{Bin}(A) = \mathrm{Bin}(A')$. The corresponding statement is not true for the integer lattice. For example, if

$$A = \{(2,0),\ (0,2)\} \text{ and } B = \{(1,0),\ (0,1)\},$$

then $\mathrm{Lat}(A) \subseteq \mathrm{Lat}(B)$ and both $\mathrm{Lat}(A)$ and $\mathrm{Lat}(B)$ have dimension 2, but clearly $\mathrm{Lat}(A) \neq \mathrm{Lat}(B)$. Thus our results for the binary space do not immediately apply to the lattice.

4.4 An excluded minor characterization

A natural problem suggested by Theorem 4.12 is to characterize the connected matroid pairs P for which there exists a directed circuit C in $G(P)$ with $V(C) \notin \mathrm{Bin}(\mathcal{B}(P))$. We give such a characterization in this section, and show that there is essentially only one matroid pair $P = (M, N)$ for which $\mathrm{Bin}(\mathcal{B}(P))$ is not completely described by $\mathrm{Bin}(\mathcal{B}(M))$ and $\mathrm{Bin}(\mathcal{B}(N))$.

Let M be the cycle matroid for three circuits of length 2 and let N be the cycle matroid for K_4. Let A_M and A_N be the representations for M and N given by

$$A_M = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \qquad A_N = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix}.$$

If the elements of the ground set S for M and N are labeled with respect to the columns of A_M and A_N, then the bases of the matroid pair $P = (M, N)$ are

$$\begin{aligned} B_1 &= \{1,2,3\}, \quad B_2 = \{1,5,6\}, \\ B_3 &= \{2,4,6\}, \quad B_4 = \{3,4,5\}. \end{aligned}$$

Since $|B \cap \{4,5,6\}|$ is even for all bases B of P and $|S \cap \{4,5,6\}|$ is odd, $S \notin \mathrm{Bin}(\mathcal{B}(M,N))$. We note that P is the twined K_4 pair from Example 2 in Section 3.5 and that the same pair is used in Chapter 5 to show that the lattice for $\mathcal{B}(M,N)$ need not be equal to $\mathrm{Lat}(\mathcal{B}(M)) \cap \mathrm{Lat}(\mathcal{B}(N))$. If we consider the fundamental graph $G(P, B_1)$ in Figure 4.2, we see that $G(P, B_1)$ has two Hamiltonian directed circuits. Thus there exists a circuit

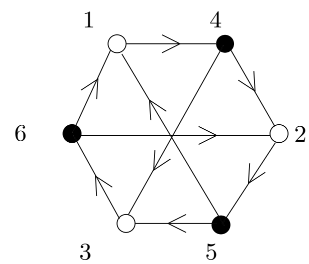

Figure 4.2: $G(P, B_1)$

C in $G(P, B_1)$ with $V(C) = S$ and hence there exists a circuit C in $G(P, B_1)$ with $V(C) \notin \mathrm{Bin}(\mathcal{B}(P))$.

Let C_1 and C_2 be the two Hamiltonian circuits in $G(P, B_1)$ and let G be a graph obtained by subdividing some edges in $E(C_1) \cap E(C_2)$ into directed paths of odd length. If the matroid pair P has a basis B such that $G(P, B) = G$, then we call P a *twined K_4 pair.* In this section we show that by excluding the twined K_4 pair as a minor, the binary space for the matroid pair (M, N) is completely characterized by $\mathrm{Bin}(\mathcal{B}(M))$ and $\mathrm{Bin}(\mathcal{B}(N))$.

Given two directed circuits C and C' in a directed graph G, the components of $C \cap C'$ are directed paths which we call the *intersection paths* between C and C'. If the cyclic order of these paths on C is the reverse of the cyclic order of these paths on C', then we say that C and C' are *twined* circuits. In particular, the two Hamiltonian circuits of $G(P, B_1)$ in Figure 4.2 are twined. We require one observation about twined circuits.

Lemma 4.13. *If C_1 and C_2 are directed circuits with at least two intersection paths, then there exists a directed circuit $C_3 \subset C_1 \cup C_2$ such that C_1 and C_3 are twined.*

Proof. Let $\{P_1, P_2, \ldots, P_k\}$ be the intersection paths of C_1 and C_2. We may assume the paths are indexed such that their cyclic order on C_1 is $P_k, P_{k-1}, \ldots, P_1$. Let the cyclic order of the paths on C_2 be $P_{\alpha(1)}, P_{\alpha(2)}, \ldots, P_{\alpha(k)}$ with $\alpha(1) = 1$. If $\alpha(i) = i$ for $2 \leq i \leq k$, then C_1 and C_2 are twined. Suppose instead that $\alpha(i) \neq i$ for some $2 \leq i \leq k$. Let m be minimum such that

$$\alpha(1) < \alpha(2) < \cdots < \alpha(m-1) > \alpha(m),$$

and let $1 \leq n < m - 1$ be maximum such that $\alpha(n) < \alpha(m)$. Define $P_{m,n}$ to be the directed path in C_1 from the head of $P_{\alpha(m)}$ to the tail of $P_{\alpha(n)}$, and let $P_{n,m}$ be the directed path in C_2 from the tail of $P_{\alpha(n)}$ to the head of $P_{\alpha(m)}$. Then $C_3 = P_{m,n}P_{n,m}$ is a directed circuit in $C_1 \cup C_2$, and C_1 and C_3 are twined. $\square$

Using Theorem 4.12, we can now show that when we exclude the twined K_4 pair as a minor of the matroid pair P, we can completely characterize the binary space generated by the bases of P.

Theorem 4.14. *If M and N are binary matroids and the connected matroid pair $P = (M, N)$ does not have a twined K_4 pair as a minor, then*

$$\mathrm{Bin}(\mathcal{B}(P)) = (Q^M \cap Q^N) \cup (R^M \cap R^N).$$

Proof. Assume $\mathrm{Bin}(\mathcal{B}(P)) \neq (Q^M \cap Q^N) \cup (R^M \cap R^N)$, and let B be a basis of P. By Theorem 4.12, there is a directed circuit C in $G(P, B)$ with $V(C) \notin \mathrm{Bin}(\mathcal{B}(P))$. Over all bases B and all circuits C in $G(P, B)$ with $C \notin \mathrm{Bin}(\mathcal{B}(P))$, choose B and C to minimize $|C|$. Let G be the subgraph of $G(P, B)$ induced by $V(C)$ and let G_M and G_N be the subgraphs of G induced by $E(G(M, B))$ and $E(G(N, B))$ respectively. If e is a chord of C then we let C_e denote the unique directed circuit in $C \cup \{e\}$ with $e \in C_e$.

Claim 1: If C' is a directed circuit in G and C and C' are twined, then $V(C') = V(C)$ and there are an odd number of edges in $C' \backslash C$.
Proof. Let C' be a directed circuit in G such that C and C' are twined, and let $E(C') \backslash E(C) = \{1, \ldots, k\}$ with the edge labels respecting the cyclic order of the edges on C'. Then $|C_i| < |C|$ for $1 \leq i \leq k$, and thus $V(C_i) \in \mathrm{Bin}(\mathcal{B}(P))$ for $1 \leq i \leq k$. Since C and C' are twined,

$$V(C) = V(C') \Delta V(C_1) \Delta V(C_2) \Delta \cdots \Delta V(C_k),$$

and therefore $V(C') \notin \mathrm{Bin}(\mathcal{B}(M, N))$. Since $V(C') \subseteq V(C)$, the minimality of $|C|$ implies that $V(C') = V(C)$. If k is even, as in Figure 4.3, then $V(C') = V(C)$ implies

$$V(C) = V(C_2) \Delta \cdots \Delta V(C_k).$$

This contradicts that $V(C) \notin \mathrm{Bin}(\mathcal{B}(M, N))$, and thus k must be odd. The claim follows.

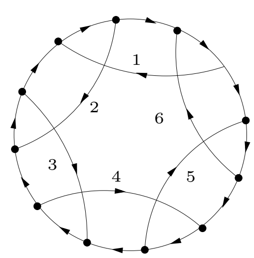

Figure 4.3: $V(C) = V(C_2) \Delta V(C_4) \Delta V(C_6)$

Let T_M be the perfect matching in G_M with $T_M = E(G_M) \cap C$, and similarly let T_N be the perfect matching in G_N with $T_N = E(G_N) \cap C$. Since $V(C) \notin \mathrm{Bin}(\mathcal{B}(P))$ and $B \in \mathcal{B}(P)$, $V(C) \Delta B$ is not a common basis of M and N. Without loss of generality, we may assume that $V(C)\Delta B$ is not a basis of N. By Observation 2.2, G_N has a perfect matching T_N^* with $T_N^* \neq T_N$. Since T_M and T_N^* are both perfect matchings of G, $T_M \cup T_N^*$ covers $V(G)$ with disjoint directed circuits. Let $T_M \cup T_N^* = C_1 \cup \ldots \cup C_m$, where C_i is a directed circuit for $1 \leq i \leq m$. If $m \geq 1$, then $|C_i| < |C|$ for all $1 \leq i \leq m$, and by the minimality of C, $V(C_i) \in \mathrm{Bin}(\mathcal{B}(M, N))$. This is a contradiction, since $V(C) = V(C_1)\Delta \cdots \Delta V(C_m)$. Thus $T_M \cup T_N^*$ is a directed circuit C^* with $V(C^*) = V(C)$. Furthermore, C^* and C have at least two intersection paths, and therefore by Lemma 4.13, G has a directed circuit that is twined with C. Over all circuits in G that are twined with C, choose the circuit C' to minimize $|C' \backslash C|$.

Let $k = |C' \backslash C|$ and let $P_1, P_2, \ldots, P_k$ be the intersection paths between C and C', where we may assume that the indices of the intersection paths respect their cyclic ordering on C'. Let x_i and y_i be the head and tail respectively of P_i. Then

$$C \backslash C' = \{y_i x_{i-1} : 1 \leq i \leq k\} \quad \text{and} \quad C' \backslash C = \{y_i x_{i+1} : 1 \leq i \leq k\},$$

where the subscripts are taken modulo k.

Label the edges of $C' \backslash C$ with 1 through k such that for all i the edge $y_i x_{i+1}$ is labeled i. Without loss of generality we may assume that $x_1 \in B$. If S is the ground set for P then $V(G(P))$ has bipartition $\{B, S \backslash B\}$, and since $y_2 x_1$ is an edge of $G(P)$, the assumption that $x_1 \in B$ implies that $y_2 \notin B$. Similarly, $y_2 \notin B$ implies that $x_3 \in B$, since $y_2 x_3 \in E(C')$. Since k is odd, we conclude that $x_i \in B$ for all $1 \leq i \leq k$ and $y_i \notin B$ for all $1 \leq i \leq k$. See Figure 4.4, where C is the outer circuit. Note that all edges in

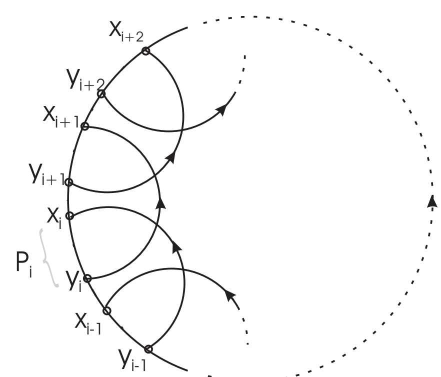

Figure 4.4: $C \cup C'$

$C\Delta C'$ are directed from $S \backslash B$ to B, and thus $E(C\Delta C') \subseteq E(G_N)$.

Each directed path P_i has odd length, and to prove that the graph G corresponds to a twined K_4 pair, it remains to show that $G = C \cup C'$ and $k = 3$. We first consider edges in $G \backslash (C \cup C')$.

Claim 2: If e is an edge in G and $e \notin C \cup C'$, then e is a backwards arc of an intersection path of C and C'.

Proof. Let $e = xy \in E(G)\backslash(C \cup C')$. Since $V(C) = V(C')$, there exists $1 \leq i, j \leq k$ such that $x \in V(P_i)$ and $y \in V(P_j)$. Suppose $i \neq j$ and let Q' be the directed path in C' from y to x. (See Figure 4.5.) Without loss of generality we may assume that $j < i$ and thus if C^* is the directed circuit in G formed by the union of Q' with e, then C^* and C are twined with $i - j + 1$ intersection paths. Furthermore, $V(C^*) \neq V(C)$: if $j \not\equiv i + 1 \pmod{k}$ then $v \notin V(C^*)$ for all $v \in V(P_{i+1})$, and if $j \equiv i + 1 \pmod{k}$ then either the head of P_i or the tail of P_j is not in $V(C^*)$. This contradicts Claim 1, and therefore we may assume that x and y are in the same intersection path P_i of C and C'.

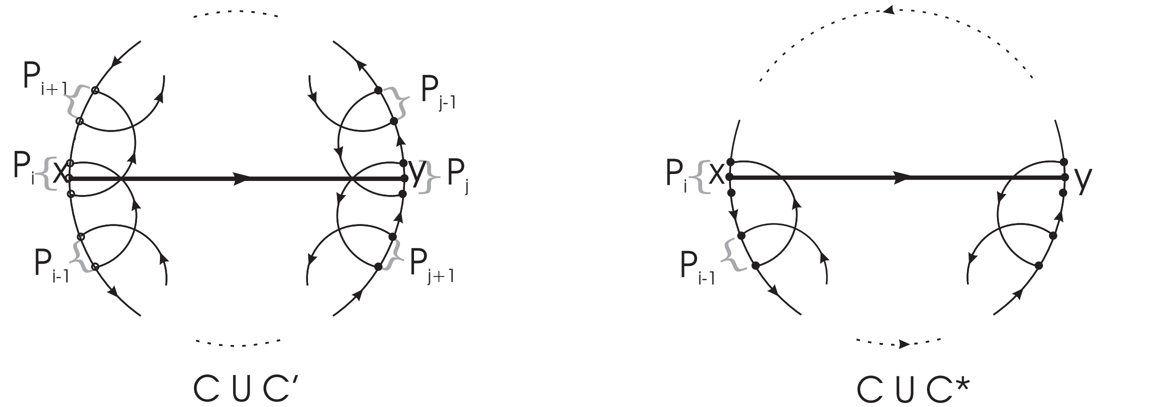

Figure 4.5: A forbidden chord of G

Suppose that e is a shortcut arc of P_i, as in Figure 4.6. Let Q' be the directed path

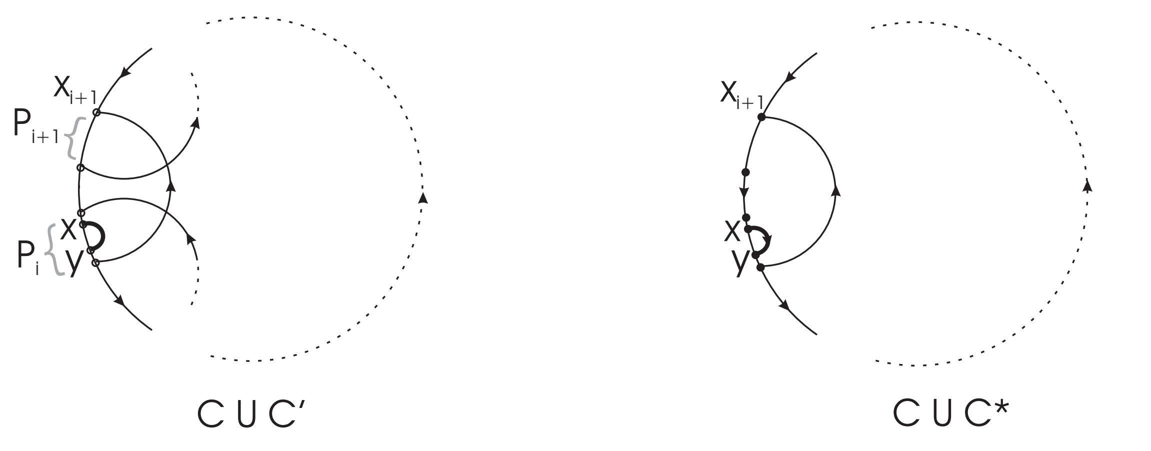

Figure 4.6: A second type of forbidden chord of G

from y to x_{i+1} in C' and let Q be the directed path in C from x_{i+1} to x. If C^* is the directed circuit in G formed by the union of Q' with Q and e, then C^* and C are twined with two intersection paths. Furthermore, $v \notin V(C^*)$ for all $v \in V(C)\backslash(V(P_i) \cup V(P_{i+1}))$

and thus $V(C^*) \neq V(C)$. This contradicts Claim 1 and therefore e is a backwards arc of an intersection path of C and C'.

If $G \neq C \cup C'$, then there exists $xy \in E(G)\backslash(C \cup C')$ such that C_{xy} is induced. Suppose first that $x = y_i$ and $y = x_i$ for some $1 \leq i \leq k$. Without loss of generality we may assume $i = 1$, as in Figure 4.7, and since k is odd,

$$V(C) = V(C_{y_1x_1})\Delta V(C_2)\Delta V(C_4)\Delta \cdots \Delta V(C_{k-1}). \tag{4.16}$$

Since $V(C_{y_1x_1}) \in \mathrm{Bin}(\mathcal{B}(P))$ and $V(C_i) \in \mathrm{Bin}(\mathcal{B}(P))$ for all $1 \leq i \leq k$, Equation 4.16

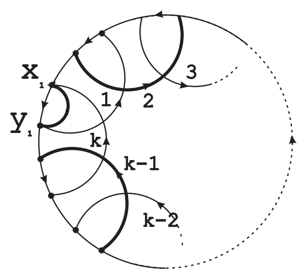

Figure 4.7: $xy = y_1x_1$

contradicts the assumption that $V(C) \notin \mathrm{Bin}(\mathcal{B}(P))$. Therefore $xy \neq y_ix_i$ for any $1 \leq i \leq k$.

Since C_{xy} is an induced circuit of G, C_{xy} is a contributing subgraph of $G(P)$ and $B' = B\Delta C_{xy}$ is a basis of P. Let G' be the subgraph of $G(P, B')$ induced by $V(C)$, and let $x', y' \in V(G)$ be such that $x'y, xy' \in E(C)$. (See Figure 4.8.) Let Q be the directed

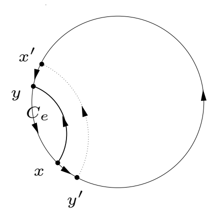

Figure 4.8: The pivot circuit C_{xy}

path in C from y' to x'. The next claim implies that if $e_1 \in E(C_{xy})$ and $e_2 \in E(Q)$, then e_1 and e_2 are not in a decomposing circuit together, and thus $E(Q) \subseteq E(G')$.

Claim 3: Let $xy \in E(G)\backslash(C \cup C')$ with $xy \neq y_ix_i$ for any $1 \leq i \leq k$. If e_1 is an edge in C_{xy}, and e_2 is an internal edge in the directed path of C from x to y, then e_1 and e_2 are not in the same decomposing circuit.

Proof. Suppose by contradiction that e_1 and e_2 are edges in a decomposing circuit K of G and let Q be the directed path in C from x to y. Since e_1 is an edge in C_{xy} and

e_2 is an internal edge in Q, e_1 and e_2 are not incident. Let $V(K) = \{u_1, v_1, u_2, v_2\}$ and assume that the edges and vertices of K are labeled as in Figure 4.9. Since $xy \notin C \cup C'$,

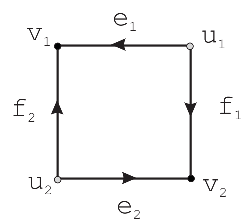

Figure 4.9: The decomposing circuit C^* of Claim 3

Claim 2 implies that there exists an intersection path P_i of C and C' such that xy is a backwards arc of P_i. Since $xy \neq y_i x_i$ and $e_1 \in E(C_{xy})$, either $f_1 \notin C \cup C'$ or $f_2 \notin C \cup C'$. If $f_1 \notin C \cup C'$ then by Claim 2, $v_2 \in V(P_i)$ and v_2 precedes u_1 along P_i. Since $e_2 = u_2 v_2$ is an edge of Q, the edge $f_2 = u_2 v_1$ contradicts Claim 2. Similarly, if $f_2 \notin C \cup C'$ then by Claim 2, $u_2 \in V(P_i)$ and v_1 precedes u_2 along P_i. Since $e_2 = u_2 v_2$ is an edge of Q, the edge $f_1 = u_1 v_2$ contradicts Claim 2. Therefore e_1 and e_2 are not in a decomposing circuit of G and the claim follows.

Claim 2 implies that $x'y' \notin E(G)$, and therefore $x'y' \in E(G')$. It follows that $C^* = Q, x'y'$ is a directed circuit in G', and

$$V(C) = V(C^*) \Delta V(C_{xy}). \tag{4.17}$$

Since $V(C^*) \in \mathrm{Bin}(\mathcal{B}(P))$ and $V(C_{xy}) \in \mathrm{Bin}(\mathcal{B}(P))$, Equation 4.17 contradicts the assumption that $V(C) \notin \mathrm{Bin}(\mathcal{B}(P))$, and therefore $G = C \cup C'$.

To prove that G corresponds to a twined K_4 pair, it remains to show that $|C \backslash C'| = k = 3$. Since $G = C \cup C'$, the directed circuit C_1 is induced in G and is therefore contributing in $G(P)$, and thus $B' = B \Delta V(C_1)$ is a basis of P. Let G' be the subgraph of $G(P, B')$ induced by $V(C)$, and let Q be the directed path in C from x_k to y_3. If $k \geq 5$ then there are no decomposing circuits in G, and thus Q is a directed path in G'. Since $y_1 x_2$, $y_1 x_k$ and $y_3 x_2$ are all edges in G_N, pivoting on $y_1 x_2$ creates the edge $x_3 y_k$ in G'. Thus the union of Q and $x_3 y_k$ is a directed circuit C^* in G', and

$$V(C) = V(C_1) \Delta V(C^*). \tag{4.18}$$

Since $V(C^*) \in \mathrm{Bin}(\mathcal{B}(P))$ and $V(C_1) \in \mathrm{Bin}(\mathcal{B}(P)$, Equation 4.18 contradicts the assumption that $V(C) \notin \mathrm{Bin}(\mathcal{B}(M, N)$. Therefore $k = 3$, and the minor of P corresponding to $G = C \cup C'$ is a twined K_4 pair. $\square$

We note that the converse of Theorem 4.14 is not true. For example, let P be the matroid pair on the ground set $S = \{1, 2, \ldots, 8\}$ with basis $B = \{1, 2, 3, 4\}$ for which

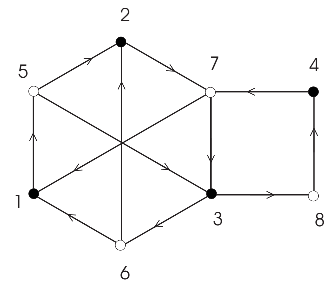

Figure 4.10: $G(P, B)$ has a twined K_4 pair as a minor

$G(P, B)$ is shown in Figure 4.10. Note that $G(P, B)$ has six directed circuits. The vertex set for each of the six circuits is contained in $\mathrm{Bin}(\mathcal{B}(P))$, and thus by Theorem 4.12, $\mathrm{Bin}(\mathcal{B}(P)) = (Q^M \cap Q^N) \cup (R^M \cap R^N)$. However, the matroid pair $P/\{4\}\backslash\{8\}$ is a twined K_4 pair.

A corollary of Theorem 4.14 is that Pfaffian matroid pairs with no twined K_4 minor have essentially unique Pfaffian signings.

Corollary 4.15. *If $P = (M, N)$ is a connected and Pfaffian matroid pair on the ground set S and P does not have a twined K_4 pair as a minor, then $X \subseteq S$ is a constant-parity intersecting set of P if and only if $X = X_M \cap X_N$ for some separator X_M of M and X_N of N.*

Proof. This follows from Theorems 4.10 and 4.14. $\square$

Chapter 5

Lattices

A description of the convex hull generated by the characteristic vectors for a combinatorial structure can lead to efficient algorithms for problems defined on the underlying combinatorial structure. For example, the convex hull generated by the characteristic vectors for independent sets in a matroid pair has a beautiful description due to Edmonds [10], and this description leads to an efficient method of finding an independent set of maximum weight. In this chapter we focus on a discrete analogue of convex hulls called *lattices*. Lattices associated with combinatorial structures were first studied by Lovász [26], [25], in the context of perfect matchings.

Given a set of vectors A, the lattice of A contains all integer linear combinations of vectors in A. Formally, if $A = \{a_1, \ldots, a_m\}$ with $a_i \in \mathbb{R}^n$ for $1 \leq i \leq m$, then the lattice generated by A, denoted $\mathrm{Lat}(A)$, is defined as

$$\mathrm{Lat}(A) = \left\{ \sum_{i=1}^{m} \lambda_i a_i : \lambda_i \in \mathbb{Z} \right\}.$$

When A is a matrix, $\mathrm{Lat}(A)$ denotes the lattice generated by the columns of A. A basis of a lattice is a minimum set of vectors that generates the lattice, and if the vectors in the set A are rational, then a basis of $\mathrm{Lat}(A)$ can be found in polynomial time (Lovász [26]).

Lovász [25] showed that when G is bipartite and $\mathcal{M}(G)$ is the set of characteristic vectors for the perfect matchings of G then the lattice generated by $\mathcal{M}(G)$ is the set of integer points in the space generated by all linear combinations of vectors in $\mathcal{M}(G)$. A description of the lattice generated by the characteristic vectors of a combinatorial structure is an extremely powerful tool for studying combinatorial problems; here we consider the lattice generated by the characteristic vectors for the bases of a matroid pair.

If P is a matroid pair then we denote the set of characteristic vectors for the bases of P by $\mathcal{B}(P)$. Our interest in $\mathrm{Lat}(\mathcal{B}(P))$ is motivated by Pfaffian signings: the *constant-parity intersecting sets* of P are those subsets X of the ground set of P for which either

$|B \cap X|$ is even for all bases B of P, or $|B \cap X|$ is odd for all bases B of P. Two Pfaffian representations of P are equivalent up to resigning across a constant-parity intersecting set, and a characterization of $\mathrm{Lat}(\mathcal{B}(P))$ implies a characterization of the constant-parity intersecting sets. Characterizing these sets is therefore a vital step towards constructing an algorithm to find a Pfaffian representation of P. The *constant-size intersecting sets* are closely related to the constant-parity intersecting sets: a subset X of the ground set of P is a constant-parity intersecting set of P if and only if there exists an integer k such that $|X \cap B| = k$ for every basis B of P. In this chapter we characterize the constant-size intersecting sets for any connected matroid pair (M, N) in terms of the separators for M and N.

5.1 Matroid polytopes

When M is a matroid on the ground set S, the convex hull of $\mathcal{B}(M)$, denoted $\mathrm{conv}(\mathcal{B}(M))$, has a simple description. If $x \in \mathrm{conv}(\mathcal{B}(M))$ and $T \subseteq S$, then let $x(T)$ denote the sum of x_s over all $s \in T$. Note that for all $x \in \mathrm{conv}(\mathcal{B}(M))$, x is nonnegative, $x(S) = r$, and $x(T) \leq \mathrm{rank}_M(T)$ for all $T \subseteq S$. The Matroid Polytope Theorem of Edmonds [10] states that these necessary conditions are sufficient, and therefore

$$\mathrm{conv}(\mathcal{B}(M)) = \{x \in \mathbb{R}^S : x(T) \leq \mathrm{rank}_M(T)\ \forall\, T \subseteq S,\ x(S) = r,\ x \geq 0\}.$$

If P is the matroid pair (M, N), then $\mathcal{B}(P) = \mathcal{B}(M) \cap \mathcal{B}(N)$ and therefore

$$\mathrm{conv}(\mathcal{B}(P)) \subseteq \mathrm{conv}(\mathcal{B}(M)) \cap \mathrm{conv}(\mathcal{B}(N)). \tag{5.1}$$

The Matroid Intersection Polytope Theorem of Edmonds [10] states that the containment in 5.1 is met with equality.

Theorem 5.1 (Matroid Intersection Polytope Theorem). *If M and N are rank r matroids on the same ground set S and P is the matroid pair (M, N), then*

$$\begin{aligned}\mathrm{conv}(\mathcal{B}(P)) = \{x \in \mathbb{R}^S :\quad & x(T) \leq \mathrm{rank}_M(T) \quad \forall\, T \subseteq S,\\ & x(T) \leq \mathrm{rank}_N(T) \quad \forall\, T \subseteq S,\\ & x(S) = r,\\ & x \geq 0\}\end{aligned}$$

and the linear system which defines $\mathrm{conv}(\mathcal{B}(P))$ *is totally dual integral.*

A rational linear system $Ax \leq b$ is *totally dual integral* if the linear program

$$\min\{y^\top b : y^\top A = w,\ y \geq 0\}$$

has an integral optimal solution y whenever w is integral and an optimal solution exists. Suppose that $Ax \leq b$ is a totally dual integral system. If the vector b is integral and the polytope given by $\{x : Ax \leq b\}$ is a rational polytope, then $\max\{w^\top x : Ax \leq b\}$ has an integral optimal solution for all integral vectors w. Totally dual integral systems are therefore extremely valuable in linear programming.

When P is the matroid pair (M, N),

$$\mathrm{Lat}(\mathcal{B}(P)) \subseteq \mathrm{Lat}(\mathcal{B}(M)) \cap \mathrm{Lat}(\mathcal{B}(N)),$$

and therefore it is natural to consider if the Matroid Intersection Polytope Theorem has an analogous theorem in lattices. However, there are matroid pairs P for which $\mathrm{Lat}(\mathcal{B}(P))$ is strictly contained in $\mathrm{Lat}(\mathcal{B}(M)) \cap \mathrm{Lat}(\mathcal{B}(N))$. For example, let M and N be the matroids with representations A_M and A_N over the reals, as given below:

$$A_M = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \qquad A_N = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix}.$$

Since $(1,1,1,0,0,0) \in \mathcal{B}(M)$ and $(0,0,0,1,1,1) \in \mathcal{B}(M)$, the vector of all ones is contained in $\mathrm{Lat}(\mathcal{B}(M))$. Similarly, $(1,1,0,1,0,0)$ and $(0,0,1,0,1,1)$ are both elements in $\mathcal{B}(N)$ and thus the vector of all ones is contained in $\mathrm{Lat}(\mathcal{B}(N))$. It follows that $(1,1,1,1,1,1) \in \mathrm{Lat}(\mathcal{B}(M)) \cap \mathrm{Lat}(\mathcal{B}(N))$. However, if P is the matroid pair (M, N), then

$$\mathcal{B}(P) = \{(1,1,1,0,0,0),\ (1,0,0,0,1,1),\ (0,1,0,1,0,1),\ (0,0,1,1,1,0)\}$$

and if the i^{th} columns of A_M and A_N correspond to element i in the ground set for M and N, then we see that $x(\{4,5,6\})$ is even for each $x \in \mathcal{B}(P)$. Thus $x(\{4,5,6\})$ is even for all $x \in \mathrm{Lat}(\mathcal{B}(P))$ and this implies that $(1,1,1,1,1,1) \notin \mathrm{Lat}(\mathcal{B}(P))$. It follows that $\mathrm{Lat}(\mathcal{B}(P)) \neq \mathrm{Lat}(\mathcal{B}(M)) \cap \mathrm{Lat}(\mathcal{B}(N))$ in this case. We note that P is a twined-K_4 pair, as given in Example 2 of Section 3.5.

As a partial result towards a description of $\mathrm{Lat}(\mathcal{B}(P))$, we show that if S' is a constant-size intersecting set of P, then S' can be characterized by the separators of P. We note that lattice generated by $\mathcal{B}(M)$ has been well characterized by Rieder [33] and depends only on the components of M and the rank of these components.

Theorem 5.2 (Rieder). *Let M be a rank r matroid on the ground set S and let $\{S_1, S_2, \ldots, S_k\}$ be the partition of S corresponding to the components of M. Then $\mathrm{Lat}(\mathcal{B}(M))$ is the set of $x \in \mathbb{Z}^S$ satisfying all of the following:*

(i) $x(S) \equiv 0 \pmod{r}$,

(ii) $x(S_i) \equiv 0 \pmod{\mathrm{rank}_M(S_i)}\ \forall\, 1 \leq i \leq k$,

(iii) $\mathrm{rank}_M(S_j)x(S_i) = \mathrm{rank}_M(S_i)x(S_j)\ \forall\, 1 \leq i, j \leq k$.

5.2 Separators and constant-size intersecting sets

Let M and N be rank r matroids on the ground set S and let P be the matroid pair (M, N). Recall from Section 2.6 that a *separator* of M is a subset T of S such that

$$\mathrm{rank}_M(T) + \mathrm{rank}_M(S \backslash T) = r$$

and a *non-trivial* separator is a non-empty separator that is not equal to the ground set. A *component* of M is a nontrivial inclusion minimal separator. Similarly, a separator of P is a subset T of S such that

$$\mathrm{rank}_M(T) + \mathrm{rank}_N(S \backslash T) = r.$$

We wish to characterize the constant-size intersecting sets of P, that is, we wish to characterize those subsets T of S for which $x(T)$ is constant for all $x \in \mathcal{B}(P)$. When P is the pair (M, M) the characterization of the constant-size intersecting sets is simple: there exists a basis of M that intersects $T \subseteq S$ in $\mathrm{rank}_M(T)$ elements and thus if $|T \cap B|$ is constant for all bases B of M then $|T \cap B| = \mathrm{rank}_M(T)$ for all B. Furthermore, when $|T \cap B|$ is constant for all bases B then $|(S \backslash T) \cap B|$ is constant and therefore all bases intersect $S \backslash T$ with cardinality equal to the rank of $S \backslash T$. Thus

$$|T \cap B| + |(S \backslash T) \cap B| = \mathrm{rank}_M(T) + \mathrm{rank}_M(S \backslash T).$$

Since $|T \cap B| + |(S \backslash T) \cap B| = |B| = r$, it follows that the constant-size intersecting sets of the matroid pair (M, M) are exactly the separators of M. When $M \neq N$ and $T \subseteq S$ has the property that $|T \cap B| = k$ for all bases B of P, then certainly $k \leq \mathrm{rank}_M(T)$ and $k \leq \mathrm{rank}_N(T)$, but it need not be that $k = \mathrm{rank}_M(T)$ or $k = \mathrm{rank}_N(T)$. Using the Matroid Intersection Polytope Theorem (Theorem 5.1) and linear programming, we characterize the constant-size intersecting sets for all connected matroid pairs. Unlike the matroid pairs discussed in most of this thesis, in this section we neither assume the matroids to be regular or representable.

The total dual integrality of the linear system for the convex hull of $\mathcal{B}(P)$ given by Edmonds' Matroid Intersection Polytope Theorem implies that the linear program

$$\begin{array}{c} \text{Maximize} \quad c^T x \\ \text{subject to} \\ x \in \mathrm{conv}(\mathcal{B}(P)) \end{array} \tag{5.2}$$

has an optimal solution that is the characteristic vector for a basis of P. From the characterization of $\mathrm{conv}(\mathcal{B}(P))$ given by the Matroid Intersection Polytope Theorem, the dual of the maximization problem (5.2) is

$$\begin{gathered}\text{Minimize} \quad \sum_{T\subseteq S}(y_T^M \operatorname{rank}_M(T) + y_T^N \operatorname{rank}_N(T)) + zr \\ \text{subject to} \\ \sum_{T\subseteq S: e\in T}(y_T^M + y_T^N) + z \geq c_e \quad \forall e \in S, \\ y^M, y^N \geq 0.\end{gathered} \tag{5.3}$$

If the basis B of P gives an optimal solution to (5.2) and y^M, y^N, z is an optimal solution to (5.3), then the complementary slackness conditions between (5.2) and (5.3) imply that

$$\begin{gathered}\sum_{T\subseteq S: e\in T}(y_T^M + y_T^N) + z = c_e \; \forall\, e \in B, \\ |T\cap B| = \operatorname{rank}_M(T) \text{ if } y_T^M > 0, \\ |T\cap B| = \operatorname{rank}_N(T) \text{ if } y_T^N > 0.\end{gathered}$$

If $T \subseteq S$ is a separator of M then $|B\cap T| = \operatorname{rank}_M(T)$ for all bases B of M and therefore $|B\cap T| = \operatorname{rank}_M(T)$ for all bases B of P. The next lemma shows that the converse holds when P is connected.

Lemma 5.3. *If $P = (M, N)$ is a connected pair of matroids on the ground set S and $T \subseteq S$ is such that $|T\cap B| = \operatorname{rank}_M(T)$ for all bases B of P, then T is a separator of M.*

Proof. Let $\mathcal{S}_M$ contain the subsets T of S which satisfy $|T\cap B| = \operatorname{rank}_M(T)$ for all bases B of P and let $\mathcal{S}_N$ contain all $U \subseteq S$ such that $|U\cap B| = \operatorname{rank}_N(U)$ for all bases B of P.

Claim 1: $\mathcal{S}_M$ and $\mathcal{S}_N$ are closed under union and intersection.
Proof. If $T, T' \in \mathcal{S}_M$ and B is a basis of P, then

$$\begin{aligned}\operatorname{rank}_M(T\cup T') + \operatorname{rank}_M(T\cap T') &\geq |(T\cup T')\cap B| + |(T\cap T')\cap B| \\ &= |T\cap B| + |T'\cap B| \\ &= \operatorname{rank}_M(T) + \operatorname{rank}_M(T').\end{aligned}$$

The submodularity of the rank function implies that

$$\operatorname{rank}_M(T) + \operatorname{rank}_M(T') \geq \operatorname{rank}_M(T\cup T') + \operatorname{rank}_M(T\cap T')$$

and therefore $|(T\cup T')\cap B| = \operatorname{rank}_M(T\cup T')$ and $|(T\cap T')\cap B| = \operatorname{rank}_M(T\cap T')$. Since B was arbitrary, $T\cup T' \in \mathcal{S}_M$ and $T\cap T' \in \mathcal{S}_M$ and thus $\mathcal{S}_M$ is closed under intersection and union. The symmetric argument for N shows that $\mathcal{S}_N$ is similarly closed under intersection and union.

Claim 2: If $T \in \mathcal{S}_M$ is nontrivial, then $S\backslash T \notin \mathcal{S}_N$.
Proof. Assume that P has rank r and let $T \subseteq S$ be such that $T \in \mathcal{S}_M$ and $S\backslash T \in \mathcal{S}_N$. For all bases B of P

$$r = |T \cap B| + |(S\backslash T) \cap B| = \text{rank}_M(T) + \text{rank}_N(S\backslash T).$$

The assumption that P is connected therefore implies that either $T = S$ or $T = \emptyset$, and $S\backslash T \notin \mathcal{S}_N$ whenever $T \in \mathcal{S}_M$ is nontrivial.

Claim 3: If $T \in \mathcal{S}_M$ and $S\backslash T \in \mathcal{S}_M$ then T is a separator of M.
Proof. Assume that T and $S\backslash T$ are both elements in $\mathcal{S}_M$. If B be is a basis of P, then

$$r = |T \cap B| + |(S\backslash T) \cap B| = \text{rank}_M(T) + \text{rank}_M(S\backslash T)$$

and thus T is a separator of M whenever $T \in \mathcal{S}_M$ and $S\backslash T \in \mathcal{S}_M$.

Let $R \in \mathcal{S}_M$ and consider the linear program

$$\begin{aligned} \text{Maximize} \quad & x(S\backslash R) \\ \text{subject to} & \\ x \in \text{conv}(\mathcal{B}(P)) & \end{aligned} \tag{5.4}$$

and its dual

$$\begin{gathered} \text{Minimize} \quad \sum_{T \subseteq S} (y_T^M \text{rank}_M(T) + y_T^N \text{rank}_N(T)) + zr \\ \text{subject to} \\ \sum_{T \subseteq S : e \in T} (y_T^M + y_T^N) \geq \begin{cases} 1 - z, & \text{if } e \notin R; \\ -z, & \text{if } e \in R; \end{cases} \\ y^M, y^N \geq 0. \end{gathered} \tag{5.5}$$

We show that $S\backslash R \in \mathcal{S}_M$ and thus by Claim 3, R is a separator of M.

If $x \in \mathcal{B}(P)$, then

$$x(S\backslash R) = x(S) - x(R) = r - \text{rank}_M(R)$$

and therefore $x(S\backslash R) = r - \text{rank}_M(R)$ for all $x \in \text{conv}(\mathcal{B}(P))$. Every feasible solution for (5.4) is therefore optimal and in particular every basis of P is optimal for (5.4). For all $e \in S$ the connectivity of P therefore implies an optimal solution $x \in \mathcal{B}(P)$ with $x_e = 1$.

Let y^M, y^N, z be an optimal solution to (5.5). By the total dual integrality of the linear system for conv($\mathcal{B}(P)$) we may assume that y^M, y^N and z are integral and thus we can let $\mathcal{R}$ be the multiset defined on the subsets of S such that $T \subseteq S$ has multiplicity $y^M_T + y^N_T$ in $\mathcal{R}$. The complementary slackness conditions between (5.4) and (5.5) imply the following:

$$\sum_{T \subseteq S: e \in T} (y^M_T + y^N_T) = \begin{cases} 1 - z, & \text{if } e \in S \backslash R; \\ -z, & \text{else;} \end{cases} \tag{5.6}$$

$$\text{If } y^N_T > 0, \text{ for some } T \subseteq S, \text{ then } |B \cap T| = \text{rank}_M(T) \text{ for all } B \in \mathcal{B}(P), \tag{5.7}$$

$$\text{If } y^N_T > 0, \text{ for some } T \subseteq S, \text{ then } |B \cap T| = \text{rank}_N(T) \text{ for all } B \in \mathcal{B}(P). \tag{5.8}$$

Conditions (5.7) and (5.8) imply that the sets in $\mathcal{R}$ are from $\mathcal{S}_M \cup \mathcal{S}_N$. Condition (5.6) implies that each element of $S \backslash R$ is covered by exactly $-z+1$ sets in $\mathcal{R}$ and each element in R is covered exactly $-z$ times by $\mathcal{R}$. (Note that the nonnegativity requirement of y^M and y^N implies that $-z \geq 0$.) Over all multisets on $\mathcal{S}_M \cup \mathcal{S}_N$ that cover the elements of $S \backslash R$ exactly $k+1$ times and cover the elements of R exactly k times for some integer k, choose the multiset $\mathcal{R}'$ to minimize k. Let $\mathcal{R}_M = \mathcal{R}' \cap \mathcal{S}_M$ and let $\mathcal{R}_N = \mathcal{R}' \backslash \mathcal{R}_M$.

Claim 4: We may assume that $\mathcal{R}_M$ is such that either $T \subseteq T'$ or $T' \subseteq T$ for all $T, T' \in \mathcal{R}_M$.

Proof. If $T, T' \in \mathcal{R}_M$ are such that $T \backslash T'$ and $T' \backslash T$ are both nonempty, then let

$$\mathcal{R}'_M = \mathcal{R}_M \cup \{T \cup T', T \cap T'\} \backslash \{T, T'\}.$$

Each $e \in S$ is covered by the same number of sets of $\mathcal{R}'_M$ as by $\mathcal{R}_M$ and since $\mathcal{S}_M$ is closed under union and intersection, the elements of $\mathcal{R}'_M$ are all contained in $\mathcal{S}_M$.

We can similarly assume that $T \subseteq T'$ or $T' \subseteq T$ for all $T, T' \in \mathcal{R}_N$. The minimality of k implies that $S \notin \mathcal{R}_M$ and $S \notin \mathcal{R}_N$ and thus neither $\mathcal{R}_M$ nor $\mathcal{R}_N$ is a cover of S.

Claim 5: $\mathcal{R}_N$ is empty.

Proof. If $\mathcal{R}_N$ is not empty then let $\mathcal{R}'_M = \mathcal{R}_M \cup R$ and consider the multiset $\mathcal{R}'_M \cup \mathcal{R}_N$ which covers each element of S exactly $k+1$ times. If there exists $T, T' \in \mathcal{R}'_M$ with $T \backslash T' \neq \emptyset$ and $T' \backslash T \neq \emptyset$ then $\mathcal{R}' \backslash \{T, T'\} \cup \{T \cup T', T \cap T'\}$ is also a $k+1$ cover of S with all elements from $\mathcal{R}_M \cup \mathcal{R}_N$. Thus we may again assume that $T \subseteq T'$ or $T' \subseteq T$ for all $T, T' \in \mathcal{R}'_M$, although we can no longer use minimality to conclude that S is not contained in $\mathcal{R}'_M$.

Let T be a minimal element of $\mathcal{R}'_M$ and let U be a maximal element of $\mathcal{R}_N$. If $|\mathcal{R}'_M| = m$ and $e \in S \backslash (T \cup U)$, then e is covered by at most $m-1$ elements of $\mathcal{R}'_M$ and

e is not covered by $\mathcal{R}_N$. This contradicts the assumption that $\mathcal{R}'$ is a $k+1$ cover, since each element of T is covered m times by $\mathcal{R}'_M$. It follows that $T \cup U = S$ and since the minimality assumption implies that $U \neq S$, there exists $e \in T \backslash U$. Any element in $T \backslash U$ is covered exactly m times in $\mathcal{R}'$ and thus $\mathcal{R}'$ is an m cover of all elements in S. If $e \in T \cap U$ then e is covered m times by $\mathcal{R}'_M$ and at least once by $\mathcal{R}_N$ and thus $T \cap U$ is empty and $\{T, U\}$ is a partition of S. Since T and U are nontrivial, this partition contradicts the assumption that P is connected, and thus $\mathcal{R}_N$ is empty.

Since $\mathcal{R}_N$ is empty, $\mathcal{R}_M$ covers all elements in $S \backslash R$ exactly $k+1$ times and all elements in R exactly k times. Since $\mathcal{R}_M$ is not a cover of S, $k = 0$ and therefore $m = 1$ and $\mathcal{R}_M = S \backslash R$. It follows that $S \backslash R \in \mathcal{S}_M$. Since $R \in \mathcal{S}_M$ by assumption, Claim 3 implies that R is a separator of M. □

By symmetry, Lemma 5.3 implies that if $P = (M, N)$ is a connected matroid pair on the ground set S and $U \subseteq S$ is such that $|U \cap B| = \text{rank}_N(U)$ for all bases B of P, then U is a separator of N.

Suppose that T and U are separators of M and N respectively with $T \cup U$ spanning S. If $R = T \cap U$ and M and N have rank r, then for all bases B of P,

$$\begin{aligned} |B \cap R| &= |B \cap (T \cap U)| \\ &= |B \cap T| + |B \cap U| - |B \cap (T \cup U)| \\ &= \text{rank}_M(T) + \text{rank}_N(U) - r. \end{aligned}$$

Thus every basis of P intersects R with fixed cardinality. We claim that whenever P is connected and $R \subseteq S$ is a constant-size intersecting set for P, then there exists a partition $\{R_1, R_2, \ldots, R_k\}$ of R and separators T_i of M and U_i of N such that $R_i = T_i \cap U_i$ and $T_i \cup U_i = S$ for $1 \leq i \leq k$. A constant-size intersecting set R is *minimal* if there does not exist a non-trivial subset R' of R such that R' is also a constant-size intersecting set and it therefore suffices to consider minimal constant-size intersecting sets.

Theorem 5.4. *Let $P = (M, N)$ be a connected matroid pair on the ground set S and suppose that $R \subseteq S$ is a constant-size intersecting set of P. If R is minimal, then there exist separators T and U of M and N respectively such that $T \cap U = R$ and $T \cup U = S$.*

Proof. Consider the linear program

$$\begin{array}{c} \text{Maximize} \quad x(R) \\ \text{subject to} \\ x \in \text{conv}(\mathcal{B}(P)). \end{array} \tag{5.9}$$

If $|B \cap R|$ is constant for all bases B of P then $x(R)$ is constant for all $x \in \text{conv}(\mathcal{B}(P))$, and therefore every feasible solution to (5.9) is optimal. In particular, every basis of P

is an optimal solution. If P has rank r, then the dual of (5.9) is

$$\begin{gathered} \text{Minimize} \quad \sum_{T\subseteq S}(y_T^M \operatorname{rank}_M(T) + y_T^N \operatorname{rank}_N(T)) + zr \\ \text{subject to} \\ \sum_{T\subseteq S: e\in T}(y_T^M + y_T^N) + z \geq \begin{cases} 1, & \text{if } e \in R, \\ 0, & \text{else,} \end{cases} \\ y^M, y^N \geq 0. \end{gathered} \tag{5.10}$$

Let y^M, y^N, z be an optimal solution to (5.10). By the total dual integrality of the linear system for $\operatorname{conv}(\mathcal{B}(P))$ we may assume that y^M, y^N and z are integral. If $e \in S$ then the connectivity of P implies a basis B of P with $e \in B$ and thus for all $e \in S$ there exists an optimal solution x to (5.9) with $x_e > 0$. The complementary slackness conditions between (5.9) and (5.10) therefore imply that

$$\sum_{T\subseteq S: e\in T}(y_T^M + y_T^N) = \begin{cases} 1 - z, & \text{if } e \in R; \\ -z, & \text{else,} \end{cases} \tag{5.11}$$

$$y_T^M > 0 \quad \text{only if} \quad |B \cap T| = \operatorname{rank}_M(T) \quad \text{for all bases } B \text{ of } P, \tag{5.12}$$

$$y_U^N > 0 \quad \text{only if} \quad |B \cap U| = \operatorname{rank}_N(U) \quad \text{for all bases } B \text{ of } P. \tag{5.13}$$

Since P is connected, Lemma 5.3 simplifies conditions (5.12) and (5.13) to

$$\begin{aligned} y_T^M > 0 &\quad \text{only if } T \text{ is a separator of } M \\ y_U^N > 0 &\quad \text{only if } U \text{ is a separator of } N \end{aligned} \tag{5.14}$$

Let $\mathcal{R}_M$ be the multiset on the separators of M such that the separator $T \subseteq S$ of M has multiplicity y_T^M in $\mathcal{R}_M$. Similarly, let $\mathcal{R}_N$ be the multiset on the separators of N such that the separator $U \subseteq S$ of N has multiplicity y_U^N in $\mathcal{R}_N$. Condition (5.6) implies that the multiset $\mathcal{R}_M \cup \mathcal{R}_N$ covers all $e \in R$ exactly $-z+1$ times and e is covered exactly $-z$ times by $\mathcal{R}_M \cup \mathcal{R}_N$ when $e \in S \backslash R$. Thus there exists an integer k and a multiset $\mathcal{R}$ such that the elements of $\mathcal{R}$ are separators of either M or N and $e \in S$ is covered exactly $k+1$ times by $\mathcal{R}$ when $e \in R$ and e is covered exactly k times by $\mathcal{R}$ when $e \in S \backslash R$. Over all such multisets, choose $\mathcal{R}$ to minimize k and let $\mathcal{R}_M = \mathcal{R} \cap \mathcal{S}_M$ and let $\mathcal{R}_N = \mathcal{R} \backslash \mathcal{R}_M$.

If there exist $T, T' \in \mathcal{R}_M$ with both $T \backslash T'$ and $T' \backslash T$ nonempty then let the multiset $\mathcal{R}'_M$ be defined by

$$\mathcal{R}'_M = \mathcal{R}_M \cup \{T \cup T', T \cap T'\} \backslash \{T, T'\}.$$

Each element of S is covered by the same number of sets in $\mathcal{R}'_M$ as it is covered by in $\mathcal{R}_M$. Furthermore, if T and T' are separators of M, then $T \cup T'$ and $T \cap T'$ are separators of

M and thus the elements of $\mathcal{R}'_M$ are all separators of M. We may therefore assume that either $T \subseteq T'$ or $T' \subseteq T$ for all $T, T' \in \mathcal{R}_M$ and if $|\mathcal{R}_M| = m$ then $\mathcal{R}_M = \{T_1, T_2, \ldots, T_m\}$ with $T_i \subseteq T_{i+1}$ for $1 \leq i \leq m-1$. If $|\mathcal{R}_N| = n$ then we can similarly assume that $\mathcal{R}_N = \{U_1, U_2, \ldots, U_n\}$ with $U_i \subseteq U_{i+1}$ for $1 \leq i \leq n-1$. The minimality of k implies that $T_m \neq S$ and $U_n \neq S$.

If $e \in T_1$ then e is covered m times by $\mathcal{R}_M$ and is therefore covered at least m times by $\mathcal{R}$, and hence $m \leq k+1$. Suppose $m = k+1$. Since all elements of T_1 are therefore covered $k+1$ times in $\mathcal{R}_M$ and only elements of R are covered $k+1$ times by $\mathcal{R}$, every element in T_1 must be contained in R. Since T_1 is a separator of M, all bases of P intersect T_1 with fixed cardinality, and since R is minimal with this property, $T_1 \subseteq R$ implies $T_1 = R$ and thus R is a separator of M when $m = k+1$. The elements in $S \backslash U_n$ are covered only by $\mathcal{R}_M$ and are therefore covered at most m times by $\mathcal{R}$ implying that $m \geq k$, and thus either $m = k$ or R is a separator of M. If R is a separator of M then the lemma follows by taking the trivial separator $U = S$ of N. The symmetric argument with $\mathcal{R}_N$ shows that either $n = k$ or R is a separator of N.

Hence we may assume that $n = m = k$. Since all elements of S are covered at least k times by $\mathcal{R}$ and all elements in $S \backslash (T_1 \cup U_k)$ are covered at most $k-1$ times by $\mathcal{R}$, the set $T_1 \cup U_k$ spans S. Since $T_1 \neq S$ and $U_k \neq S$, both T_1 and U_k are nontrivial. If $T_1 \cap U_k$ is empty then $\{T_1, U_k\}$ is a partition of S and for all bases B of P,

$$r = |B \cap T_1| + |B \cap U_k| = \mathrm{rank}_M(T_1) + \mathrm{rank}_N(U_k). \tag{5.15}$$

Since T_1 and U_k are nontrivial, (5.15) contradicts the assumption that P is connected and thus $T_1 \cap U_k$ is nonempty. All elements in $T_1 \cap U_k$ are covered by at least $k+1$ elements of $\mathcal{R}$ and are thus covered by exactly $k+1$ elements of $\mathcal{R}$ and therefore $T_1 \cap U_k \subseteq R$. Since T_1 and U_k are separators of M and N respectively and $T_1 \cup U_k = S$, all bases of P intersect $T_1 \cap U_k$ with fixed cardinality and the minimality of R with respect to this property implies that $R = T_1 \cap U_k$. Thus either R is a separator of M or N or there exist separators T and U of M and N respectively such that $T \cup U$ spans S and $R = T \cap U$. $\square$

Chapter 6

Series-Parallel Matroids

Many of the difficulties associated with finding a Pfaffian representation of a regular matroid pair are simplified in the case of graphic matroids pairs $P = P(G)$ where G is a bipartite graph and the matroids in the pair $P(G)$ are the partition matroids for a bipartite graph, as defined in Section 3.3. For example, finding a totally unimodular edge signing of the fundamental graph for P is easy in this case since the even edge signing (all edges are signed 0, Section 2.5) of $G(P)$ is totally unimodular when P is a graphic matroid pair. Recall from Section 2.3 that if S is the ground set for P and B is a basis for P, then the *contributing* sets of $G(P, B)$ correspond to the subsets $H \subseteq S$ for which $B \Delta H$ is a basis of P. When $P = (M, N)$ is a graphic matroid pair, then the fundamental graphs $G(M)$ and $G(N)$ are both acyclic, and thus by the observation that $H \subseteq S$ is contributing if and only if $G(M)[H \Delta B]$ and $G(N)[H \Delta B]$ each have an odd number of perfect matchings (Observation 2.2), identifying the contributing sets for $G(P, B)$ is simple: $X \Delta B$ is a basis of P if and only if there is a unique set D of disjoint directed circuits in $G(P, B)$ such that $V(D) = X$.

In this section we consider a minor and duality closed class of regular matroid pairs for which both of these properties hold; namely the even edge signing is a totally unimodular signing and the contributing sets are easily identified in the fundamental graph. The matroids in this class are called *series-parallel* matroid pairs. By exploiting these properties of series-parallel matroid pairs we get some characterizations of $G(P)$ when P is non-Pfaffian, and we show that when P is Pfaffian then a Pfaffian signing of $G(P)$ has a simple description in terms of signings of directed circuits. We first require some definitions and characterizations of series-parallel matroids.

Let G be a graph. Recall that for $W \subseteq V(G)$, the edges in $E(G)$ with exactly one end in W are denoted $\delta(W)$, and $\delta(W)$ is called an *edge cut* of G. The edge cut $\delta(W)$ is *minimal* if there is no nonempty edge cut $\delta(W')$ with $\delta(W') \subset \delta(W)$. Two edges e and f in G are said to be in *series* if $\{e, f\}$ is a minimal edge cut of G, and if e and f are in series in G and the graph H is such that $H = G/e$, then G is a *series extension* of H. Two edges in

G are *parallel* if they are incident to exactly the same two vertices and *parallel extensions* of graphs have an analogous definition to series extensions: if e and f are parallel edges in G and $G \backslash e = H$, then G is a *parallel extension* of H. A *series-parallel graph* is a graph which can be obtained from a single edge by a sequence of parallel extensions and series extensions. Electrical networks constructed by adding resistors in series and in parallel correspond to series-parallel graphs, and thus series-parallel graphs are studied for their applications in electrical engineering. The recursive structure of series-parallel graphs is useful in implementing algorithms, and is another motivation for the study of series-parallel graphs.

If H is a 2-connected graph then all series and parallel extensions of H are 2-connected. Since the graph consisting of a single edge is 2-connected, all series-parallel graphs are 2-connected. (Note that the path of length 2 is not a series extension of the single edge, since the two edge cut in the path is not a minimal cut.) Let K_4 be the complete graph on four vertices. Since K_4 does not contain edges in series and K_4 does not have any parallel edges, K_4 is not series-parallel. Dirac [7], Adam [1] and Duffin [8] independently showed that a graph is series-parallel if and only if it is 2-connected and does not have a K_4 minor.

A matroid M is series-parallel if $M = M(G)$ and each connected component of the graph G is series-parallel. Since minors of $M(G)$ correspond to minors of G, it follows from the definition that series-parallel matroids do not have a $M(K_4)$ minor. A stronger characterization holds.

Theorem 6.1. *A matroid is series-parallel if and only if it does not have a minor isomorphic to $M(K_4)$ or $U_{2,4}$.*

The matroid $U_{2,4}$ is the rank two matroid on four elements such that any two elements of the ground set form a basis. Since series-parallel matroids are graphic and graphic matroids are binary, one direction of Theorem 6.1 follows from Tutte's [43] characterization of binary matroids together with the excluded minor characterization for series-parallel graphs. The converse direction uses a connectivity result and Tutte's Wheels and Whirls theorem [42]. For a proof of Theorem 6.1, including a proof of the excluded minor characterization of series-parallel graphs, see Oxley [31], page 363.

The matroid pair $P = (M, N)$ is series-parallel if M and N are series-parallel matroids. When P is series-parallel and minimally non-Pfaffian we show that either $G(P)$ has a Hamiltonian directed circuit (Section 6.6) or $G(P)$ has a "weak odd double cycle" (defined in Section 6.7) as a spanning subgraph (Section 6.7). In the case that $G(P)$ has a Hamiltonian directed circuit there is a non-Pfaffian certificate of constant size that proves that P is Pfaffian and when $G(P)$ has a weak odd double cycle then the number of circuits composing the double cycle is small.

Let G be a bipartite graph with vertex bipartition $\{U, W\}$ and let M_U and N_W be the partition matroids for U and W respectively, such that $(M_U, M_W) = P(G)$ (see Section 3.3). If T is a perfect matching of G and R_U is the partial representation of M_U

with respect to T, then each column in R_U is parallel to a column of the identity matrix and therefore each element in $S \backslash T$ is parallel with an element of T. It follows that R_U is the partial representation for the cycle matroid of a graph formed by adding parallel edges to a spanning tree. Since such a graph clearly has no K_4 minor, R_U and R_W are partial representations of series-parallel matroids, and thus the class of series-parallel matroid pairs contains the class of bipartite graphs. In particular, Pfaffian series-parallel matroid pairs are an extension of the class of Pfaffian bipartite graphs.

6.1 Fundamental graphs of series-parallel matroids

Using Theorem 6.1 we deduce some properties of the fundamental graph $G(P)$ when P is a series-parallel matroid pair. These properties allow a simple characterization of Pfaffian signings of P with respect to totally unimodular signings of $G(P)$.

A *domino graph* is a bipartite graph formed by adding one chord to a cycle of length 6, as in Figure 6.1. Graphs that do not have a domino graph as an induced minor are called *domino free.* If G is a directed bipartite graph with bipartition $\{U, W\}$ then G is domino free if the underlying undirected graph for $G[\delta(U)]$ is domino free and the underlying undirected graph for $G[\delta(W)]$ is domino free.

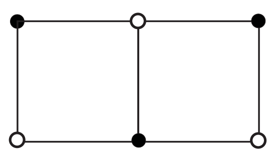

Figure 6.1: domino graph

If B is a basis for $M(K_4)$ and B corresponds to a path of length 3 in K_4, then up to permuting rows and columns, the partial representation R_{K_4} of $M(K_4)$ with respect to B is given by

$$R_{K_4} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Thus $G(M(K_4), B)$ is isomorphic to a domino graph. It follows that if the matroid M does not have an $M(K_4)$ minor then $G(M)$ is domino free. By Theorem 6.1, $G(M)$ is domino free whenever M is series-parallel and thus if P is a series-parallel matroid pair, then $G(P)$ is domino free.

Let W_k denote the wheel with k spokes (Figure 6.2). If B is the basis of $M(W_k)$ corresponding to the spanning tree of W_k containing all the spoke edges, then up to permuting rows and columns, the partial representation R_{W_k} of $M(W_k)$ with respect to

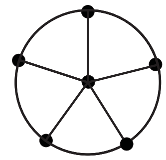

Figure 6.2: The wheel with 5 spokes, W_5

B is given by

$$R_{W_k} = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ & & & & \vdots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}.$$

Thus $G(M(W_k), B)$ is isomorphic to a circuit of length $2k$. It follows that if the matroid M has an induced circuit of length $2k$ then M has a minor isomorphic to the cycle matroid for W_k. Since W_3 is isomorphic to K_4 and W_3 is a minor of W_k whenever $3 \leq k$, if M does not have a $M(K_4)$ minor then $G(M)$ does not have an induced circuit of length greater than 4. Thus we have the following observation:

Observation 6.2. *If M is a series-parallel matroid then all induced circuits in $G(M)$ have length 4 and $G(M)$ is domino free.*

We apply this observation to Camion's signing algorithm. Let M be a series-parallel matroid and let F be a maximal forest of $G(M)$. Let $E(G(M))\backslash E(F) = \{e_1, \ldots, e_k\}$ where for $i = 1, \ldots, k$ there exists an induced circuit C_i of $G(M)$ with $e_i \in E(C_i)$ and $E(C_i) \subseteq E(F) \cup \{e_1, \ldots, e_i\}$. Since M is graphic it is regular and therefore $G(M)$ has a totally unimodular signing. All induced cycles in $G(M)$ have length 4, and thus Camion's algorithm implies that an edge signing of $G(M)$ is a totally unimodular signing if and only if the signing restricted to every circuit of length 4 in $G(M)$ is totally unimodular. Since $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ is the adjacency matrix for a circuit of length 4, a totally unimodular edge signing s of a circuit C of length 4 satisfies $s(C) = 0$ and thus any edge signing of $G(M)$ for which every circuit of length 4 is even will be a totally unimodular signing of $G(M)$. Recalling that the even edge signing of a graph assigns 0 to each edge, we have the following observation:

Observation 6.3. *If M is a series-parallel matroid and R is the partial representation of M with respect to the basis B, then $[I|R]$ defined over the reals is totally unimodular and the even edge signing of $G(M)$ is totally unimodular.*

The edges e and f are in parallel in the graph G if and only if every spanning tree of G contains at most one edge from $\{e, f\}$, and thus e and f are in parallel in the graphic matroid $M(G)$ if and only if $|B \cap \{e, f\}| \leq 1$ for all bases B of $M(G)$. The edges e and f are in series in G if and only if every spanning tree of G contains at least one edge from $\{e, f\}$, and thus e and f are in series in $M(G)$ if and only if $|B \cap \{e, f\}| \geq 1$ for all bases B of $M(G)$. More generally, given a matroid M and two elements x and y in the ground set for M, the elements x and y are said to be parallel if $|B \cap \{x, y\}| \leq 1$ for all bases B and x and y are in series if $|B \cap \{x, y\}| \geq 1$ for all bases B. Thus x and y are parallel in M if and only if x and y are in series in the dual M^*. If x and y are parallel in M and the basis B of M contains x, then $B\Delta\{x, y\}$ is a basis of M. By duality, if x and y are in series in M and B is a basis of M with $x \in B$, $y \notin B$, then $B\Delta\{x, y\}$ is a basis of M.

Let M be a series-parallel matroid on the ground set S and let R be the partial representation of M with respect to the basis B. Since M is series-parallel, there exist $e, f \in S$ such that either e and f are parallel or e and f are in series.

Suppose that e and f are parallel, and therefore the columns for e and f in the representation $[I|R]$ are equal. Since two parallel elements can not be in the same basis, we may assume that $f \notin B$. If $e \in B$ then e is a column in the identity matrix and therefore the column for f in R has exactly one non-zero entry. Since R is the adjacency matrix for $G(M, B)$, it follows that the degree of f in $G(M, B)$ is one. If e and f are parallel in M and neither e nor f is contained in B, then e and f index equal columns in R and thus e and f have the same neighbours in $G(M, B)$. A pair of vertices with the same set of neighbours in a graph are *twin vertices*.

Suppose instead that $e, f \in S$ are in series in M. By duality, e and f are parallel in M^* and thus one of the following holds:

(i) e and f are twin vertices in $G(M^*, S \backslash B)$, or

(ii) e or f has degree one in $G(M^*, S \backslash B)$.

Since $G(M^*, S \backslash B)$ is isomorphic to $G(M, B)$, one of the following hold:

(i) e and f are twin vertices in $G(M, B)$, or

(ii) e or f has degree one in $G(M, B)$.

We summarize these observations:

Observation 6.4. *If M is a series-parallel matroid, then either $G(M)$ has a pair of twin vertices or $G(M)$ has a vertex of degree one. If B is a basis of M and u and v are twin vertices in $G(M, B)$ then u and v are parallel if and only if $u, v \notin B$, and u and v are in series if and only if $u, v \in B$. If u is a degree one vertex in $G(M, B)$ and $uv \in E(G(M, B))$ then u and v are parallel if $u \notin B$, and u and v are in series if $u \in B$.*

Observation 6.4 is used in the proof of the next lemma, which in turn will be used to identify contributing sets in $G(P)$ when P is a series-parallel matroid pair.

Lemma 6.5. *If M is a series-parallel matroid on the ground set S then $X \subseteq S$ is contributing if and only if $G(M)[X]$ has a unique perfect matching.*

Proof. Let B be a basis of M. If the subgraph of $G(M, B)$ induced by X has a unique perfect matching then $G(M, B)[X]$ has an odd number of perfect matchings. Since M is binary, $X \Delta B$ is a basis of M. Conversely, suppose $X \Delta B$ is a basis of M and let G be the subgraph of $G(M, B)$ induced by X. Since $X \Delta B$ is a basis of M, G has at least one perfect matching and thus $|X|$ is even. We will use induction on $|X|$ to show that G has a unique perfect matching.

If $|X| = 2$ then since G has a perfect matching and G is simple, G has a unique perfect matching. Suppose $|X| = 2k$ with $k > 1$ and assume that the theorem holds for smaller k. Let $\bar{B} = S \backslash B$ and $M' = M/(B \backslash X) \backslash (\bar{B} \backslash X)$, and let $B' = B \cap X$ and $X' = X \backslash B$. Since $B' \cup (B \backslash X) = B$ and $X' \cup (B \backslash X) = B \Delta X$, both B' and X' are bases of M'. Furthermore, $G = G(M', B')$. Since M' is a minor of a series-parallel matroid, M' is series-parallel and by Observation 6.4 either G has a vertex of degree one or G has a pair of twin vertices.

Suppose that G has twin vertices u and v. Since $V(G)$ has bipartition $\{B', X'\}$, either $u, v \in B'$ or $u, v \in X'$. If $u, v \in B'$ then u and v are in series in M' and since $|X' \cap \{u, v\}| = 0$, this contradicts that X' is a basis of M'. If $u, v \in X'$ then u and v are parallel in M' and since $|X' \cap \{u, v\}| = 2$, this again contradicts that X' is a basis of M'. Therefore there are no twin vertices in G and G must have a vertex of degree one.

Let u be a vertex of G with degree one and let $v \in V(G)$ be such that $uv \in E(G)$. Every perfect matching of G must use uv and thus G has a unique perfect matching if and only if $G \backslash \{u, v\}$ has a unique perfect matching. Since $G \backslash \{u, v\}$ is the subgraph of $G(M, B)$ induced by $X \backslash \{u, v\}$, by the induction hypothesis it is sufficient to prove that $(X \backslash \{u, v\}) \Delta B$ is a basis of M.

Observation 6.4 implies that either u and v are parallel in M' or u and v are in series in M'. Since X' is a basis of M' with $|X' \cap \{u, v\}| = 1$, it follows that $X' \Delta \{u, v\}$ is a basis of M', and therefore $(X' \Delta \{u, v\}) \cup (B \backslash X)$ is a basis of M. Since $(X' \Delta \{u, v\}) \cup (B \backslash X) = X \backslash \{u, v\} \Delta B$, the graph $G \backslash \{u, v\}$ has a unique perfect matching by the induction hypothesis and therefore G has a unique perfect matching. $\square$

Let $P = (M, N)$ be a series-parallel matroid pair. If D is a set of vertex disjoint directed circuits in $G(P)$ then D induces a perfect matching $T_M(D)$ in $G(M)[V(D)]$ and D induces a perfect matching $T_N(D)$ in $G(N)[V(D)]$. Furthermore, if D' is a set of vertex disjoint circuits in $G(P)$ with $V(D) = V(D')$, then $D \neq D'$ if and only if either $T_M(D) \neq T_M(D')$ or $T_N(D) \neq T_N(D')$. It follows that if H is a subgraph of $G(P)$ then there is a unique set of disjoint directed circuits D in $G(P)$ with $V(D) = V(H)$ if and only if $G(M)[V(H)]$ and $G(N)[V(H)]$ have unique perfect matchings. By Lemma 6.5, H is contributing if and only if there is a unique set D of vertex disjoint directed circuits in $G(P)$ with $V(D) = V(H)$. In particular, we have the following observation.

Observation 6.6. *If P is a series-parallel matroid pair then all contributing circuits in $G(P)$ are simple contributing circuits.*

6.2 Contributing circuits and decomposing circuits

Using the structure of $G(P)$ when P is a series-parallel matroid pair, we show that a totally unimodular signing of $G(P)$ is a Pfaffian signing if and only all simple contributing circuits are correctly signed. Since the even edge signing is a totally unimodular signing of $G(P)$ whenever P is a series-parallel matroid pair, this signing result for contributing circuits gives an easy way to recognize if a signing of $G(P)$ is non-Pfaffian.

A *decomposing circuit* K in a directed graph is a circuit of length 4 such that the orientation of the edges in K alternates, as in Figure 6.3. Let C be a directed circuit

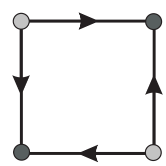

Figure 6.3: A decomposing circuit

in G and let K be a decomposing circuit in G with $|E(K) \cap E(C)| = 2$. The two arcs in $E(K) \cap E(C)$ are called the *C-edge* arcs of K and the two arcs in $E(K)\backslash E(C)$ are called the *C-chord* arcs of K. If e is a directed chord in the directed circuit C then we let C_e denote the directed circuit in $C \cup \{e\}$ through the arc e. If e and k are the C-chord arcs of K then e and k are chords of C and the circuits C_e and C_k are vertex disjoint with $V(C) = V(C_e) \cup V(C_k)$. We say that K *decomposes* C into C_e and C_k. Decomposing circuits are used extensively in our analysis of series-parallel matroids, and the next lemma is applied frequently.

Lemma 6.7. *If P is a series-parallel matroid pair and D is a set of vertex disjoint directed circuits in $G(P)$, then D is contributing if and only if $G(P)$ does not have a decomposing circuit K with $|E(K) \cap E(D)| = 2$.*

Proof. Let P be a series-parallel matroid pair and let D be a set of vertex disjoint directed circuits in $G(P)$. The matching $T_M(D)$ is the unique matching of $G(M)[V(D)]$ if and only if $G(M)$ does not contain a $T_M(D)$ alternating circuit, and if K is a decomposing circuit in $G(P)$ with $|E(K) \cap E(D)| = 2$, then K is either a $T_M(D)$ alternating circuit in $G(M)$ or a $T_N(D)$ alternating circuit in $G(N)$. Thus if D is contributing then there is no decomposing circuit K with $|E(K) \cap E(D)| = 2$. Conversely, suppose D is not contributing in $G(P)$. Without loss of generality we may assume that $G(M)$ has a

$T_M(D)$ alternating circuit C. If $|C| \neq 4$ then since M is series-parallel, C has a chord e, and $C \cup \{e\}$ has a $T_M(D)$ alternating circuit C' through e with $|C'| < |C|$. Therefore there is a $T_M(D)$ alternating circuit K in $G(M)$ with length 4 and thus K is a decomposing circuit in $G(P)$ with $|E(K) \cap E(D)| = 2$. □

Let K and K' be decomposing circuits in the directed bipartite graph G. If K and K' share exactly one edge, then $K \cup K'$ is a domino, and thus if G is domino free then $K \cup K'$ is not an induced subgraph of G. There are two edges e and e' in the subgraph induced by $K \cup K'$ such that $K \cup K' \cup \{e\}$ is not a domino and $K \cup K' \cup \{e'\}$ is not a domino, and these edges are called the *joining edges* between K and K' shown (see Figure 6.4). Note that one joining edge is directed from K to K' and the other joining edge is directed from K' to K.

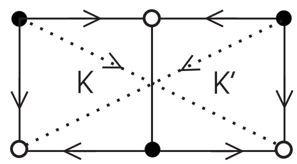

Figure 6.4: Two possible joining edges in a domino

Given a matroid pair P and a signing of $G(P)$, we would like a simple way to recognize if the signing is Pfaffian. The main theorem of this section states that if P is series-parallel and a signing of $G(P)$ is totally unimodular, then to determine if the signing is a Pfaffian signing it is sufficient to check that the contributing circuits in $G(P)$ are correctly signed. Recall that all contributing circuits in $G(P)$ are simple when P is series-parallel (Lemma 6.5), and that a simple contributing circuit C in $G(P)$ is correctly signed by the edge signing s if $s(C)$ has parity $\frac{|C|}{2} + 1$ (Lemma 3.3).

Theorem 6.8. *If P is a series-parallel matroid pair and s is a totally unimodular signing of $G(P)$, then s is a Pfaffian signing of P if and only if all contributing circuits in $G(P)$ are correctly signed.*

Proof. Let $P = (M, N)$ be a series-parallel matroid pair on the ground set S and let B be a basis of P. Let (R_M, R_N) be the partial representation of P with respect to B. If s is a Pfaffian signing of $G(P, B)$ then all contributing circuits are correctly signed, so it suffices to show that if H is a contributing subgraph of $G(P, B)$ and all contributing circuits of $G(P, B)$ are correctly signed, then H is correctly signed.

Let s be a totally unimodular signing of $G(P, B)$ and let (R'_M, R'_N) be the corresponding signing of (R_M, R_N). Let $U \subseteq B$ and $W \subseteq S \backslash B$ be such that $G(P, B)[U \cup W]$ is contributing and let $D = \{C_1, C_2, \ldots, C_n\}$ be the unique set of disjoint directed circuits in $G(P, B)$ with $V(D) = U \cup W$. Let A_M and A_N be the adjacency matrices for $T_M(D)$ and $T_N(D)$ respectively. Since $T_M(D)$ is the unique perfect matching in $G(M, B)[U \cup W]$

and there is a one-to-one correspondence between the perfect matchings and terms in the determinant of $R_M[U;W]$, the determinant of $R_M[U;W]$ is equal to the determinant of A_M. Thus if A'_M is the signing of A_M corresponding to s then $\det R'_M[U;W] = \det A'_M$. Similarly, if A'_N is the signing of A_N corresponding to s then $\det R'_N[U;W] = \det A'_N$.

Let $U_i = U \cap V(C_i)$ and $W_i = W \cap V(C_i)$. By simultaneously permuting rows and columns we may assume that A'_M and A'_N are block diagonal with i^{th} block indexed by rows U_i and columns W_i. Thus

$$\det R'_M[U;W] = \det A'_M = \sum_{i=1}^{k} \det A'_M[U_i;W_i] = \sum_{i=1}^{k} \det R'_M[U_i;W_i]$$

and similarly,

$$\det R'_N[U;W] = \sum_{i=1}^{k} \det R'_N[U_i;W_i].$$

Since C_i is a contributing circuit with $V(C_i) = U_i \cup W_i$, if the contributing circuits in $G(P,B)$ are correctly signed then $\det R'_M[U_i;W_i] = \det R'_N[U_i;W_i]$ for $1 \le i \le k$. Thus $\det R'_M[U;W] = \det R'_N[U;W]$ if all contributing circuits are correctly signed and therefore all contributing subgraphs are correctly signed whenever all contributing circuits are correctly signed. $\square$

To determine if P is a Pfaffian pair, it therefore suffices to determine if $G(P)$ has a totally unimodular signing s such that $s(C) = \frac{|C|}{2} + 1$ for all contributing circuits C in $G(P)$. Recall that when P is a Pfaffian matroid pair, a Pfaffian signing of $G(P)$ can be obtained by resigning any totally unimodular signing of $G(P)$ across a cut in $G(N)$. Since the even edge signing of $G(P)$ is a totally unimodular signing, if $G(P)$ has a Pfaffian signing then it can be obtained by resigning the even edge signing across a cut in $G(N)$.

6.3 Vertex signings of the fundamental graph

We use *vertex signings* to define a fundamental class of non-Pfaffian series-parallel matroid pairs, and in Section 6.6 we show that the matroids in this class have a non-Pfaffian certificate with at most 32 bases. A vertex signing of a graph is a $\{0,1\}$ function on the vertices of the graph. If $\alpha_C \in \{0,1\}$ is defined for all directed circuits in the directed graph G, then a vertex signing t is a *proper vertex signing* with respect to α if $t(V(C)) = \alpha_C$ over $GF(2)$ for each directed circuits C in G. Similarly, a *proper edge signing* of G with respect to α is an edge signing s such that $s(C) = \alpha_C$ for all directed circuits C. When P is series-parallel, we show that the problem of finding a proper edge signing of $G(P)$ is equivalent to finding a proper vertex signing of $G(P)$. This equivalence leads naturally to a class of non-Pfaffian series-parallel matroid pairs, and simplifies the analysis of these non-Pfaffian matroid pairs in Section 6.6.

Let $P = (M, N)$ be a series-parallel matroid pair on the ground set S, and let t be a vertex signing of $G(P)$. Let $X \subseteq S$ be such that $t_v = 1$ if and only if $v \in X$, and let s be the totally unimodular edge signing of $G(P)$ obtained from the even edge signing by resigning across the cut $\delta(X)$ in $G(N)$. Assume that C is a directed circuit in $G(P)$. If T is the matching in $G(N)$ induced by C and uv is an edge in T, then $s_{uv} = t_u + t_v$. Since $V(T) = V(C)$ and $s_e = 0$ unless e is an edge in $G(N)$,

$$s(C) = s(T) = t(V(T)) = t(V(C)).$$

Thus for all vertex signings t of $G(P)$ there is a totally unimodular edge signing s of $G(P)$ with $s(C) = t(C)$ for all directed circuits C in $G(P)$. If P is a Pfaffian series-parallel matroid then $G(P)$ has a Pfaffian signing obtained by resigning the even signing across the edges in a cut $\delta(X)$ of $G(N)$. Thus Theorem 6.8 implies the following observation:

Observation 6.9. *The series-parallel matroid pair P is Pfaffian if and only if $G(P)$ has a vertex signing t satisfying*

$$t(V(C)) = \frac{|C|}{2} + 1$$

for each contributing circuit C of $G(P)$.

Let t be a vertex signing of the directed graph G satisfying $t(V(C)) = \frac{|C|}{2} + 1$ for all contributing circuits C in G. If $\{C_1, C_2, \dots, C_k\}$ is a decomposition of the directed circuit C into contributing circuits, then over $GF(2)$,

$$\begin{aligned} t(V(C)) &= \sum_{i=1}^{k} t(V(C_i)) \\ &= \sum_{i=1}^{k} \frac{|C_i|}{2} + 1 \\ &= \frac{|C|}{2} + k \end{aligned} \tag{6.1}$$

It follows that if C has a decomposition into k contributing circuits and C also has a decomposition into k' contributing circuits, then k and k' must have the same parity. If C is a directed circuit in a graph and all decompositions of C have the same parity k, then we define the *decomposing parity* of C to be k modulo 2. The decomposing parity of C is denoted $\mathrm{dp}(C)$ and in particular, $\mathrm{dp}(C) = 1$ when C is contributing. If P is a series-parallel matroid pair and the directed circuit C in $G(P)$ has a decomposition into an even number of contributing circuits and a decomposition into an odd number of circuits, then it follows from Equation (6.1) and Observation 6.9 that P is non-Pfaffian. A circuit with a decomposition into both an even number of contributing circuits and an odd number of contributing circuits is called a *neutral* circuit.

If C is a directed circuit in the graph G and C is not neutral, then we say that an edge signing s of G *respects the decomposing parity of* C if $s(C) = \frac{|C|}{2} + \mathrm{dp}(C)$. Similarly, a vertex signing t of G respects the decomposing parity of C if $t(V(C)) = \frac{|C|}{2} + \mathrm{dp}(C)$. If a signing respects the decomposing parity of all directed circuits in G, then the signing is said to respect the decomposing parity of G. We therefore have an alternative form of Theorem 6.8:

Theorem 6.10. *If P is a series-parallel matroid pair and $G(P)$ does not have any neutral circuits, then P is Pfaffian if and only if $G(P)$ has a totally unimodular edge signing that respects the decomposing parity of all directed circuits in $G(P)$.*

In the next section we prove that the totally unimodular requirement can be omitted from Theorem 6.10.

6.4 Proper edge signings of the fundamental graph

The obstructions to a graph having a proper edge signings have been well characterized by Gerards [15], as we will see in Section 6.7. When P is a series-parallel matroid we would like to use this characterization together with Theorem 6.10 to get some structural results about $G(P)$ whenever $G(P)$ does not have an edge signing that respects the decomposing parity of $G(P)$. However, if $G(P)$ has an edge signing that respects the decomposing parity of $G(P)$, such an edge signing need not be a Pfaffian signing: if $s(C) = \frac{|C|}{2} + \mathrm{dp}(C)$ for all directed circuits C but s is not a totally unimodular signing of $G(P, B)$, then from the results so far we can not conclude that P is Pfaffian. In this section we show that if P is connected and $G(P)$ does not have any neutral circuits, then determining if P is Pfaffian is equivalent to determining if $G(P)$ has an edge signing which respects the decomposing parity of the directed circuits in $G(P)$. The main theorem of this section is the following:

Theorem 6.11. *If P is a connected series-parallel matroid and $G(P)$ has no neutral circuits, then P is Pfaffian if and only if there is an edge signing of $G(P)$ that respects the decomposing parity of $G(P)$.*

Let s be an edge signing of G and let $E' \subseteq E(G)$ be a cut of G. The edge signing s' defined by

$$s'_e = \begin{cases} s_e + 1, & \text{if } e \in E'; \\ s_e, & \text{otherwise} \end{cases}$$

is said to be obtained by resigning s across E'. Since all circuits in G intersect E' an even number of times, $s'(C) = s(C)$ for all directed circuits C in G. Thus if α is a circuit signing for G, then s' is a proper edge signing of G with respect to α if and only if s is a proper edge signing with respect to α. If the circuit signing α of G satisfies

$\alpha_C = \alpha_{C_1} + \alpha_{C_2}$ whenever a directed circuit C has a decomposition into C_1 and C_2, then we say α is a good circuit signing. Clearly α must be a good circuit signing if G has a proper vertex signing with respect to α. The proof of Theorem 6.11 requires the following two lemmas about decomposing circuits in a proper edge signing of a directed graph.

Lemma 6.12. *Let K be a decomposing circuit in the directed graph G and assume that $ab, cd \in E(K)$ are such that G has a directed path from b to a that is disjoint from a directed path from d to c. If α is a good circuit signing of G and s is a proper edge signing of G with respect to α, then $s(K) = 0$.*

Proof. Assume that α is a good circuit signing of G and let s be a proper edge signing of G with respect to α. By resigning across cuts in G we may assume that $s(ab) = s(ad) = s(cb) = 0$. Let P_{ba} be a directed path in G from b to a and let P_{dc} be a directed path in G from d to c and assume P_{ba} and P_{dc} are vertex disjoint, as shown in Figure 6.5. If C

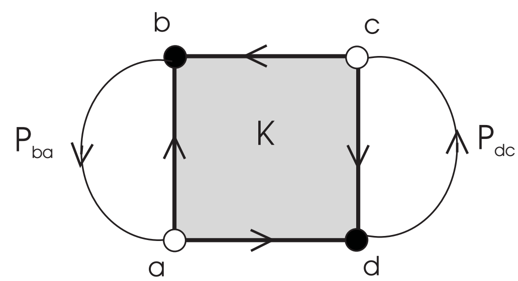

Figure 6.5: The decomposing circuit K for Lemma 6.12

is the directed circuit in $P_{ba} \cup P_{dc} \cup \{ad, cb\}$, then

$$s(C) = s(P_{ba}) + s(P_{dc}) + s(ad) + s(cb) = s(P_{ba}) + s(P_{dc}). \tag{6.2}$$

Furthermore, K decomposes C into C_{ab} and C_{cd}, and since α is a good circuit signing of G and s is a proper edge signing with respect to α,

$$\begin{aligned} s(C) &= s(C_{ab}) + s(C_{cd}) \\ &= (s(ab) + s(P_{ba})) + (s(cd) + s(P_{dc})) \\ &= s(P_{ab}) + s(P_{dc}) + s(cd). \end{aligned} \tag{6.3}$$

Equations (6.2) and (6.3) imply that $s(cd) = 0$ and thus $s(K) = 0$. □

Lemma 6.13. *Let G be a bipartite, domino free, strongly connected bipartite graph, and let α be a good circuit signing of G. If s is a proper edge signing of G with respect to α and $\alpha_C = \frac{|C|}{2} + 1$ whenever C is a contributing circuit, then $s(K) = 0$ for all decomposing circuits K in G.*

Proof. Let s be a proper edge signing of G with respect to α. We will say that $E' \subseteq E(G)$ is *even* if $s(E') = 0$ and *odd* if $s(E') = 1$. Assume that G has an odd decomposing circuit K with $V(K) = \{a, b, c, d\}$ and $ab, cd \in E(K)$. Since G is strongly connected, there is a directed path P_{ba} from b to a and a directed path P_{dc} from d to c. in G. Over all odd decomposing circuits in G, choose K, P_{ba}, and P_{dc} to minimize $|P_{ba}|+|P_{dc}|$. By resigning s across cuts in G, we may assume that $s(ab) = s(ad) = s(cb) = 0$. Since K is odd, $s(cd) = 1$.

Lemma 6.12 implies that P_{ba} and P_{dc} are not disjoint. Let $x, y \in V(P_{ba}) \cap V(P_{dc})$ be such that the distance from b to x along P_{ba} is minimum and the distance from y to a along P_{ba} is minimum. Suppose $x \neq y$ and y precedes x along P_{dc}. The choice of x and y implies that the subpaths of P_{ba} from b to x and from y to a are edge disjoint from the subpaths of P_{dc} from d to y and from x to c. (See Figure 6.6, where the bold lines indicate edges, the thin lines indicate paths, and only the relevant subpaths of P_{ba} and P_{dc} are shown.) Thus if P_{da} is the directed path formed by the union of the subpath of

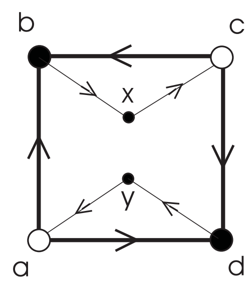

Figure 6.6: y precedes x along P_{dc}

P_{dc} from d to y with the subpath of P_{ba} from y to a, and P_{bc} is the directed path formed from the union of the subpath of P_{ba} from b to x with the subpath of P_{dc} from x to c, then P_{da} and P_{bc} are vertex disjoint. This contradicts Lemma 6.12, and therefore either $y = x$ or x precedes y along P_{dc}. By the minimality of $|P_{ba}| + |P_{dc}|$, we may assume that the subpath of P_{ba} from x to y is equal to the subpath of P_{dc} from x to y, as shown in Figure 6.7. The minimality of $|P_{ba}| + |P_{dc}|$ implies that cd is not an edge in P_{ba}, but x need not be distinct from b or d, and y need not be distinct from a or c.

For each edge e in K, let $C(e)$ be the unique directed circuit in $K \cup P_{ba} \cup P_{dc}$ with $e \in C(e)$. Each edge in $P_{ba} \Delta P_{dc}$ is in exactly two such circuits, each edge in $P_{ba} \cap P_{dc}$ is in all four such circuits, and each edge in K is in exactly one circuit. It follows that over $GF(2)$,

$$\sum_{e \in K} s(C(e)) = s(K) = 1, \tag{6.4}$$

and

$$\sum_{e \in K} |C(e)| = 2(|P_{ba}| + |P_{dc}|) + |K|. \tag{6.5}$$

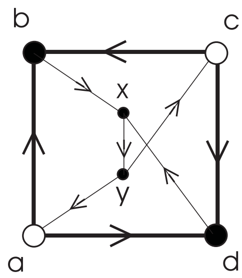

Figure 6.7: x precedes y along P_{dc}

Since a and b are in opposite sides of the vertex bipartition of G, both $|P_{ba}|$ and $|P_{dc}|$ are odd. Equation (6.5) therefore implies that

$$\sum_{e \in K} |C(e)| \equiv 0 \pmod 4. \tag{6.6}$$

If $C(e)$ is contributing for all $e \in K$ then

$$\sum_{e \in K} s(C(e)) = \sum_{e \in K} \left(\frac{|C(e)|}{2} + 1\right)$$

and thus by (6.6), $\sum_{e \in K} s(C(e))$ is even. This contradicts (6.4), and thus without loss of generality we may assume that $C(ab)$ has a decomposing circuit K_1. Let $V(K_1) = \{a_1, b_1, c_1, d_1\}$ and label the vertices of K_1 such that a_1b_1 and c_1d_1 are the $C(ab)$-edge arcs of K_1 and b_1 precedes d_1 along P_{ba}. If $a_1b_1 \neq ab$, then a_1d_1 is a shortcut arc of P_{ba}, which contradicts the minimality of $|P_{ba}|$. Thus $a_1b_1 = ab$. Note that since cd is not an edge in P_{ba}, either $d_1 \neq d$ or $c_1 \neq c$.

If $d_1 = d$ then $s(ad_1) = s(ad) = 0$, and if $d_1 \neq d$ then $d_1 \notin V(K)$ and resigning s across $\delta(d_1)$ does not affect the sign of any edges in K. Thus we may assume $s(ad_1) = 0$. Similarly, either $c_1 = c$ and $s(c_1b) = 0$, or we may resign s across $\delta(c_1)$ without affecting s on either the edges of K or on $s(ad_1)$ and thus we may assume $s(c_1b) = 0$. Let P_{bc_1} be the subpath of P_{ba} from b to c_1 and let P_{d_1a} be the subpath of P_{ba} from d_1 to a. Then P_{bc_1} is a directed b_1, c_1 path and P_{d_1a} is a directed d_1, a_1 path, and since $P_{b_1c_1}$ and $P_{d_1a_1}$ are disjoint, Lemma 6.12 implies that $s(K_1) = 0$. Thus $s(c_1d_1) = 0$, as shown in Figure 6.8(a).

If $d_1 = d$ then there is an odd decomposing cycle K_2 with $cd, c_1b \in E(K_2)$. Since P_{dc} is disjoint from P_{bc_1}, the cycle K_2 contradicts Lemma 6.12. Similarly, if $c_1 = c$ then there is an odd decomposing cycle K_2 with $cd, ad_1 \in E(K_2)$. The path P_{dc} is disjoint from P_{d_1a}, and thus K_2 again contradicts Lemma 6.12. Hence $d_1 \neq d$ and $c_1 \neq c$, and therefore K and K_1 share exactly one edge, ab. Since G is domino free, either $cd_1 \in E(G)$, or $c_1d \in E(G)$. By the symmetry from reversing the orientation of G we may assume that $cd_1 \in E(G)$. By considering where d_1 lies on P_{ba} relative to y, we show that $s(cd_1) = 0$.

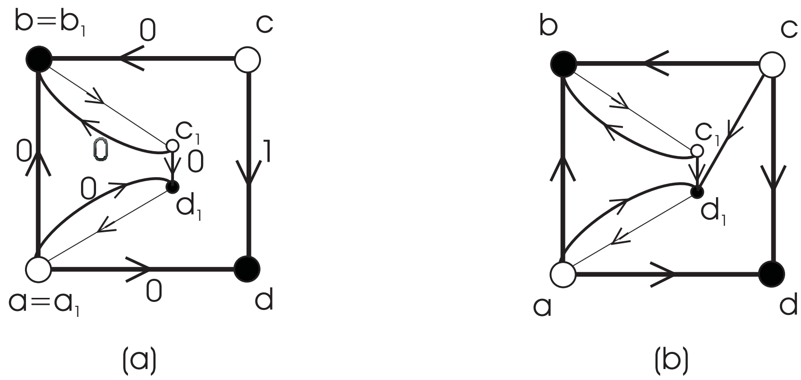

Figure 6.8: The decomposing circuit K_1 of $C(ab)$

If $d_1 = y$ or d_1 precedes y along P_{ba}, then let K_2 be the decomposing circuit with edges cd_1 and c_1b, as shown in bold in Figure 6.9(a). Let P_{d_1c} be the subpath of P_{dc} from d_1 to c. Since P_{bc_1} and P_{d_1c} are disjoint, Lemma 6.12 implies that K_2 is even and therefore $s(cd_1) = 0$. If y precedes d_1 along P_{ba}, then let K_2 be the decomposing circuit with edges cd_1 and ab, as shown in bold in Figure 6.9(b). Let P_{bc} be the directed path formed from

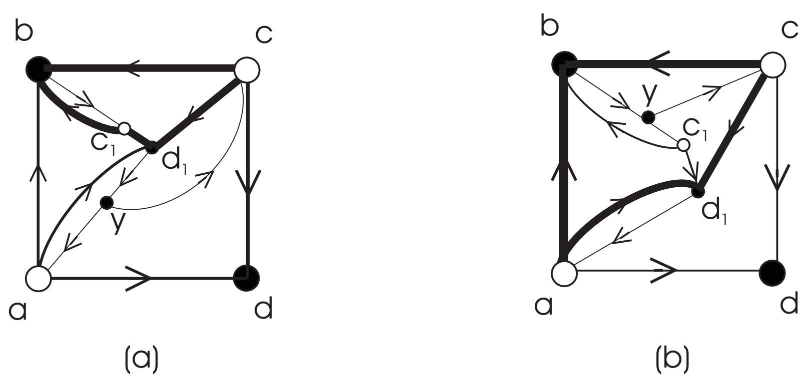

Figure 6.9: y relative to d_1 on P_{ba}

the union of the subpath of P_{ba} from b to y with the subpath of P_{dc} from y to c. Since P_{bc} and P_{d_1a} are disjoint, Lemma 6.12 implies that K_2 is even and therefore $s(cd_1)$ is again 0. Thus if $cd_1 \in E(G)$, then $s(cd_1) = 0$. However there is then an odd decomposing circuit in G with edges ad_1 and cd, and P_{d_1a} is a strict subpath of P_{ba}, contradicting the minimality of $|P_{ba}| + |P_{dc}|$ in our choice of K. Thus $cd_1 \notin E(G)$ and since this edge is necessary for G to be domino free, it follows that $s(K) = 0$ for all decomposing circuits in G. $\square$

We can now prove the main theorem of this section.

Proof of Theorem 6.11. Let P be a connected series-parallel matroid pair and suppose that $G(P)$ does not have any neutral circuits. If the edge signing s is a Pfaffian signing of $G(P)$, then by Theorem 6.10, s respects the decomposing parity of all directed circuits in $G(P)$. Conversely, let α be the circuit signing of $G(P)$ defined by $\alpha_C = \frac{|C|}{2} + \mathrm{dp}(C)$ for all directed circuits C in $G(P)$, and suppose s is a proper signing of $G(P)$ with respect to α. If C decomposes into C_1 and C_2, then $|C| = |C_1| + |C_2|$ and since $G(P)$ does not have any neutral circuits, $\mathrm{dp}(C) = \mathrm{dp}(C_1) + \mathrm{dp}(C_2)$. It follows that $\alpha_C = \alpha_{C_1} + \alpha_{C_2}$ and therefore α is a good circuit signing of $G(P)$. By Lemma 6.13, $s(K) = 0$ for all decomposing circuits K. Thus s is a totally unimodular edge signing of $G(P)$ which respects the decomposing parity of $G(P)$, and Theorem 6.8 implies that s is a Pfaffian signing.

□

If $G(P)$ does not have any neutral circuits, then by Theorem 6.11 determining if P is Pfaffian is equivalent to determining if $G(P)$ has an edge signing s satisfying $s(C) = \frac{|C|}{2} + \mathrm{dp}(C)$ for all directed circuits C in $G(P)$. We use this result in Section 6.7 to obtain a partial description of $G(P)$ when P is non-Pfaffian and $G(P)$ has no neutral circuits. In Section 6.6 we show that if $G(P)$ has a neutral circuit, then there exists a set of at most 28 bases which certifies that P is non-Pfaffian.

6.5 Accordions

In this section we describe a decomposition of a directed graph into contributing circuits. The decomposition, which we call an *accordion*, is used in Sections 6.6 and 6.7 as a method for combining disjoint contributing circuits into a larger contributing subgraph.

Let G be a directed graph and let R and Q be vertex disjoint directed paths in G. A *bridge circuit* between R and Q is a decomposing circuit in G that contains one edge of R and one edge of Q. Suppose K_1 and K_2 are bridge circuits between R and Q and $E(K_1) \cap E(R)$ precedes $E(K_2) \cap E(R)$ along R. If $E(K_2) \cap E(Q)$ precedes $E(K_1) \cap E(Q)$ along Q then K_1 and K_2 are *parallel* (Figure 6.10(a)), and K_1 and K_2 are *strongly crossing* if $E(K_1) \cap E(Q)$ precedes $E(K_2) \cap E(Q)$ along Q (Figure 6.10(b)). When K_1 and K_2 are

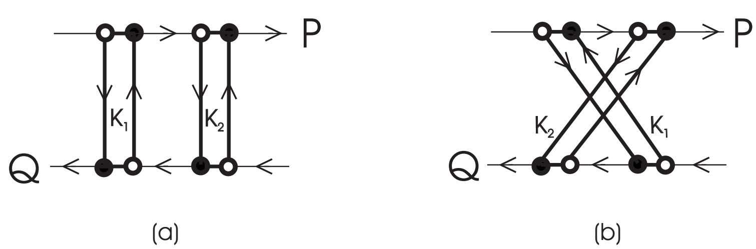

Figure 6.10: Parallel and strongly crossing bridge circuits

parallel then the directed circuit in the subgraph $R \cup Q \cup K_1 \cup K_2$ is called an *accordion circuit* with ends K_1 and K_2 (Figure 6.11(a)).

Let e be an arc with both endpoints in a directed path R with e not in R. If the tail of e precedes its head, then e is a *shortcut* arc in R, and e is a *backwards* arc otherwise. When K_1 and K_2 share an edge of R and there is a joining edge g between K_1 and K_2 that is a backwards arc of Q, then the directed circuit in $Q \cup g$ is also called an accordion circuit with ends K_1 and K_2 (Figure 6.11(b)).

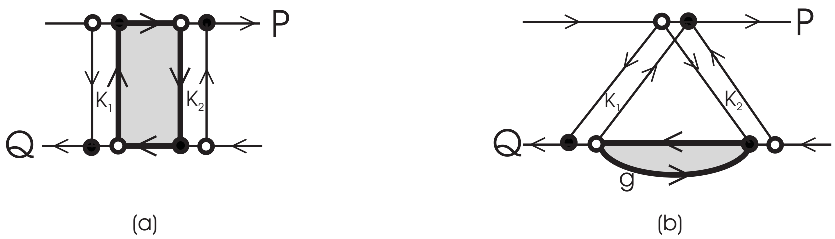

Figure 6.11: Accordion circuits

Let $K_1, K_2, \ldots, K_m$ be distinct bridge circuits between R and Q and for $1 \leq i \leq m$ let $e_i = E(K_i) \cap E(R)$ and $f_i = E(K_i) \cap E(Q)$. Suppose that either e_i precedes e_{i+1} along R or $e_i = e_{i+1}$, and that either f_{i+1} precedes f_i along Q or $f_i = f_{i+1}$. If there is an accordion circuit C_i with ends K_i and K_{i+1} for all $1 \leq i \leq m-1$, then the union of R, Q, $K_1, K_2, \ldots, K_m$ and $C_1, C_2, \ldots, C_{m-1}$ is called an *accordion* between R and Q. The circuits $K_1, K_2, \ldots, K_m$ are the bridge circuits of the accordion and the bridge circuits K_1 and K_m are the ends of the accordion. Figure 6.12 is an example of an accordion with 8 bridge circuits and seven accordion circuits. If there is at most one bridge circuit between R and Q then the paths R and Q together with possibly one bridge circuit are called a *trivial accordion.* If $\mathcal{H}$ is an accordion and all the accordion circuits in $\mathcal{H}$ are

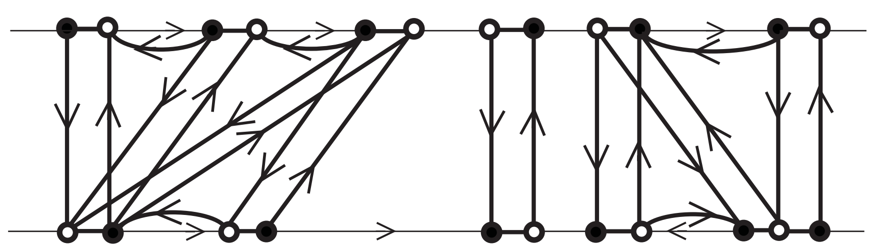

Figure 6.12: An accordion

contributing, then $\mathcal{H}$ is a *contributing accordion.*

The two lemmas in this section describe when accordion circuits in a contributing accordion $\mathcal{H}$ can be combined in a trivial way into contributing subgraphs of $\mathcal{H}$. These lemmas will be used in Section 6.6 to find a small set of bases that certify that a matroid pair is non-Pfaffian. If $\mathcal{H}$ is a contributing accordion between the directed paths R and

Q then there are three types of contributing circuits in $\mathcal{H}$, as seen in Figure 6.12: those that intersect only $V(R)$, those that intersect only $V(Q)$, and those that intersect both $V(R)$ and $V(Q)$. We first consider the contributing circuits that intersect only $V(R)$. A shortcut arc e in R is an *internal* shortcut arc if both ends of e are internal vertices in R.

Lemma 6.14. *If $\mathcal{H}$ is a contributing accordion between the directed paths R and Q in the directed graph G and R does not have any internal shortcut arcs, then the set of accordion circuits C in $\mathcal{H}$ with $V(C) \subseteq V(R)$ can be partitioned into two contributing subgraphs.*

Proof. Let $\mathcal{C}$ be the set of accordion circuits C in $\mathcal{H}$ with $V(C) \subseteq V(R)$ and let $\{\mathcal{C}_A, \mathcal{C}_B\}$ be a partition of $\mathcal{C}$ such that if $C_A, C_B \in \mathcal{C}$ and an edge in R is incident to a vertex in C_A and a vertex in C_B then $C_A \in \mathcal{C}_A$ if and only if $C_B \in \mathcal{C}_B$.

Let C_1 and C_2 be circuits in C_A and suppose $C_1 \cup C_2$ is not a contributing cycle of $G(P, B)$. Then there exists a decomposing circuit K in G with $E(K)$ equal to $\{u_1v_1, u_1v_2, u_2v_1, u_2v_2\}$ where $u_1v_1 \in E(C_1)$ and $u_2v_2 \in E(C_2)$, and we may assume that u_1 and v_1 precede u_2 and v_2 along R. Since C_1 and C_2 are circuits in $\mathcal{C}$, the vertices u_1, v_1, u_2, and v_2 are all in R. Furthermore, since $C_1, C_2 \in C_A$, neither u_1v_2 nor u_2v_1 is an edge of R. Since u_1v_2 and u_2v_1 have opposite orientation with respect to R, one of these chords is an internal shortcut arc of R, contradicting the assumption that no such chord exists. Thus $C_1 \cup C_2$ is contributing for any $C_1, C_2 \in \mathcal{C}_A$. The union of all circuits in $\mathcal{C}_A$ is therefore contributing, as is the union of all circuits in $\mathcal{C}_B$, and thus $\mathcal{C}$ is the disjoint union of two contributing subgraphs. $\square$

A bridge circuit between the directed paths R and Q that strongly crosses two parallel bridge circuits is a *twisted* bridge circuit. A twisted bridge circuit that strongly crosses the parallel bridge circuits K_1 and K_2 is *internally twisted* if K_1 and K_2 are edge disjoint from the ends of $\mathcal{H}$. The last lemma of this section shows how to combine contributing circuits that intersect both $V(R)$ and $V(Q)$ into a contributing set.

Lemma 6.15. *Let $\mathcal{H}$ be a contributing accordion between R and Q in the directed graph G where G is domino free and bipartite. If R and Q do not have any shortcut arcs and there are no internally twisted bridge circuits between R and Q, then the set of accordion circuits in $\mathcal{H}$ which intersect both $V(R)$ and $V(Q)$ can be partitioned into two contributing subgraphs.*

Proof. Let $\mathcal{C}$ be the set of accordion circuits C in $\mathcal{H}$ that intersect both $V(R)$ and $V(Q)$ and let $\{\mathcal{C}_A, \mathcal{C}_B\}$ be a partition of $\mathcal{C}$ such that if $C_A, C_B \in \mathcal{C}$ and an edge in R or Q intersects both $V(C_A)$ and $V(C_B)$ then $C_A \in \mathcal{C}_A$ if and only if $C_B \in \mathcal{C}_B$. We claim that $\mathcal{C}_A$ and $\mathcal{C}_B$ are contributing.

Let $C_1 \in \mathcal{C}_A$ have ends L_1 and R_1 and let $C_2 \in \mathcal{C}_A$ have ends L_2 and R_2. Then L_1, R_1, L_2, R_2 are parallel and distinct and we may assume that $E(L_i) \cap E(R)$ precedes

$E(R_i) \cap E(R)$ along R for $i = 1, 2$ and that $E(R_1) \cap E(R)$ precedes $E(L_2) \cap E(R)$ along R. Let l_1 be the edge in L_1 directed from Q to R and for $i = 1, 2$, let r_i be the edge

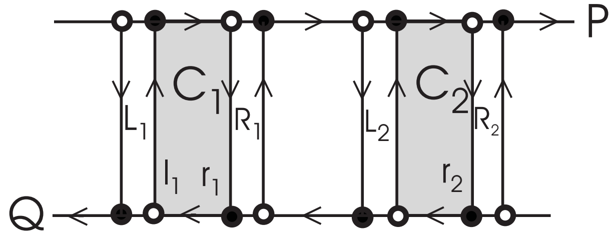

Figure 6.13: Contributing accordion circuits C_1 and C_2 from $\mathcal{C}_A$

in R_i directed from R to Q, as shown in Figure 6.13. If $C_1 \cup C_2$ is not contributing in G then there exists a decomposing circuit K in G with $V(K) = \{u_1, v_1, u_2, v_2\}$ and $u_i v_i = E(K) \cap E(C_i)$ for $i = 1, 2$.

First consider the case that $u_1 v_1$ is an edge in R. Since R does not have any shortcut arcs and $u_1 v_2 \in E(G)$, v_2 must be in $V(Q)$ and thus either $u_2 v_2 = r_2$ or $u_2 v_2 \in E(Q)$. If $u_2 v_2 \in E(Q)$ then K crosses both R_1 and L_2 and is therefore a twisted bridge circuit, contradicting our assumption that no such circuits exists in $\mathcal{H}$. If $u_1 v_1 \in E(R)$ and $u_2 v_2 = r_2$ then K and R_2 share exactly one edge, namely r_2. Both possible joining edges between K and R_2 lead to a contradiction: if a joining edge between K and R_2 is incident to u_1 then R has a shortcut arc, and if a joining edge g between K and R_2 is incident to v_1 then the bridge circuit through g in $K \cup R_2 \cup \{g\}$ crosses both R_1 and L_2. It follows that $u_1 v_1$ is not an edge of R.

Suppose instead that $u_1 v_1 = r_1$. If $v_2 \in V(R)$ then $u_1 v_2$ is a shortcut arc of R and if $u_2 \in V(Q)$ then $u_2 v_1$ is shortcut arc of Q. Thus $u_1 v_1 = r_1$ implies $u_2 v_2 = r_2$ and K and R_2 share the edge r_2. The joining edge between K and R_2 is either a shortcut arc of R or a shortcut arc of Q, and thus $u_1 v_1 \neq r_1$. We may therefore assume that either $u_1 v_1 \in E(Q)$ or $u_1 v_1 = l_1$. By symmetry, we may assume that either $u_2 v_2 \in E(R)$ or $u_2 v_2 = r_2$.

If $u_1 v_1 \in E(Q)$ and $u_2 v_2 \in E(R)$ then K strongly crosses R_2 and L_2, and thus either $u_1 v_1 = l_1$ or $u_2 v_2 = r_2$. If $u_1 v_1 = l_1$ and $u_2 v_2 \in E(R)$ then K and L_1 share the edge l_1. The joining edge between K and L_1 that is directed from L_1 to K is a shortcut arc of R. If the joining edge g between K and L_1 is directed from K to L_1 then the bridge circuit containing g in $L_1 \cup K \cup \{g\}$ strongly crosses R_1 and L_2 and thus contradicts the assumption that there are no twisted bridges. Thus if $u_1 v_1 = l_1$ then $u_2 v_2 \notin E(R)$ and by symmetry $u_1 v_1 \notin E(Q)$ when $u_2 v_2 = r_2$. More generally, there is no decomposing circuit through l_1 and an edge e of R where e is preceded along R by the edge of L_2 in R. The only case that remains is $u_1 v_1 = l_1$ and $u_2 v_2 = r_2$.

If $u_1 v_1 = l_1$ and $u_2 v_2 = r_2$ then K and R_2 share the edge r_2. If the joining edge g between K and R_2 is directed from K to R_2, then there is a decomposing circuit in $K \cup R_2 \cup \{g\}$ that contains l_1 and an edge e of R such that L_2 precedes e along R. Thus the joining edge g between K and R_2 must be directed from R_2 to K. By symmetry the joining edge g' between K and L_1 is directed from L_1 to K. This is a contradiction since there is then a bridge circuit through g and g' that strongly crosses R_1 and L_2.

Thus $C_1 \cup C_2$ is contributing for all circuits $C_1, C_2 \in \mathcal{C}_A$ and $\mathcal{C}_A$ is therefore a contributing set, as is the set of circuits in $\mathcal{C}_B$. Since $\{\mathcal{C}_A, \mathcal{C}_B\}$ is a partition of $\mathcal{C}$, the lemma follows. □

6.6 Neutral circuits and a certificate of constant size

Neutral circuits are directed circuits which have a decomposition into an even number of contributing circuits and a decomposition into an odd number of circuits. Section 6.3 showed that if P is a series-parallel matroid pair and $G(P)$ has a neutral circuit, then P is non-Pfaffian. In this section we prove that if a series-parallel matroid pair has a neutral circuit, then it has an non-Pfaffian bases certificate of at most 32 bases which certifies that the matroid pair is non-Pfaffian. This supports Conjecture 3.12 that there exists a constant c such that every non-Pfaffian matroid pair has a non-Pfaffian bases certificate of size at most c.

Let P be a minimally non-Pfaffian series-parallel matroid pair on the ground set S and let C be a neutral circuit in $G(P)$. The minor of P corresponding to the minor $G(P)[V(C)]$ of $G(P)$ is non-Pfaffian, and the assumption that P is minimally non-Pfaffian therefore implies that $V(C) = S$. Hence all neutral circuits of $G(P)$ are Hamiltonian. If C' is a non-Hamiltonian directed circuit in $G(P)$ then it follows by minimality that the decomposing parity of C' is well defined. For a non-Hamiltonian circuit C' in $G(P)$, define $\text{sign}(C')$ by

$$\text{sign}(C') \equiv \frac{|C|}{2} + \text{dp}(C') \pmod 2.$$

It follows from Theorem 6.10 that if s is a Pfaffian signing of the minor of P corresponding to $G(P)[V(C')]$ then $s(C') = \text{sign}(C')$. If the directed circuit C is neutral in G and the decomposing circuit K decomposes C into C_f and C_k, then let $\text{sign}(C, K) = \text{sign}(C_f) + \text{sign}(C_k)$. Note that C_f and C_k are not Hamiltonian and thus $\text{sign}(C, K)$ is well defined. For all neutral circuits C in $G(P)$ there exist decomposing circuits K and K' of C such that $\text{sign}(C, K) \neq \text{sign}(C, K')$. In particular, if $\{k, h\}$ are the C-chord edges of K and $\{k', h'\}$ are the C-chord edges of K', then

$$\text{sign}(C_k) + \text{sign}(C_h) + \text{sign}(C_{k'}) + \text{sign}(C_{h'}) = 1.$$

Furthermore, $\{C_k, C_h, C_{k'}, C_{h'}\}$ is an even cover of S, and thus a decomposition of $\{C_k, C_h, C_{k'}, C_{h'}\}$ into contributing circuits leads to a non-Pfaffian basis certificate for P. Our approach to finding a certificate of bounded size for P is therefore to show that

C_k, C_h, $C_{k'}$, and $C_{h'}$ each partition into a bounded number of contributing sets. We first require some definitions.

Let f and h be vertex disjoint chords of the directed circuit C. If exactly one end of f is contained in C_h, then f and h *cross.* If K is a decomposing circuit of C and h is a chord of C, then K and h cross if h crosses a C-chord edge of K. If h crosses both C-chord edges in K then h strongly crosses K. If K and K' are distinct decomposing circuits of C and a C-chord edge of K crosses K', then we say that K and K' cross. Two decomposing circuits of C that share an edge are weakly crossing, and crossing decomposing circuits which are not weakly crossing are *strongly crossing.* A pair of decomposing circuits of C are *parallel* if they do not cross. (See Figure 6.14.)

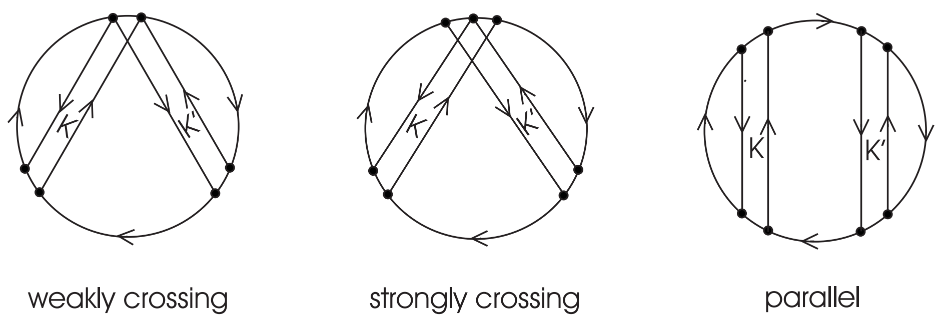

Figure 6.14: Pairs of decomposing circuits

If D is a set of directed circuits in the directed graph G and each vertex in G is in an even number of circuits of D, then D is an *even cover* of G. An even cover D of G is *spanning* if $V(D) = V(G)$ and D and is non-spanning otherwise.

Let D be an even cover of $G(P)$. If the sum of $\text{sign}(C')$ over all directed circuits C' in D is odd, then $G(P)[V(D)]$ does not have a signing which respects the decomposing parity of $G(P)[V(D)]$ and thus the minor of P corresponding to $G(P)[V(D)]$ is non-Pfaffian. The assumption that P is minimally non-Pfaffian therefore implies that D is spanning. Our proof that a neutral circuit has a non-Pfaffian bases certificate containing a bounded number of bases which certifies that the matroid pair is non-Pfaffian uses this observation together with three lemmas relating $\text{sign}(C, K)$ and $\text{sign}(C, K')$ for decomposing circuits K and K' in the neutral circuit C in $G(P)$. The first of the three lemmas considers $\text{sign}(C, K)$ and $\text{sign}(C, K')$ when K and K' are parallel.

Lemma 6.16. *Let P be a minimally non-Pfaffian matroid pair and let C be a neutral circuit in $G(P)$. If K and K' are parallel decomposing circuits of C, then* $\text{sign}(C, K) = \text{sign}(C, K')$.

Proof. Let the C-chord edges of K and K' be $\{f, h\}$ and $\{f', h'\}$ respectively, with C_f and $C_{f'}$ vertex disjoint as shown in Figure 6.6(a). If $C' = (C_h)'_h$ as shown in Figure 6.6(b), then K' decomposes C_h into C' and $C_{f'}$, and K decomposes $C_{h'}$ into C' and C_f. Since

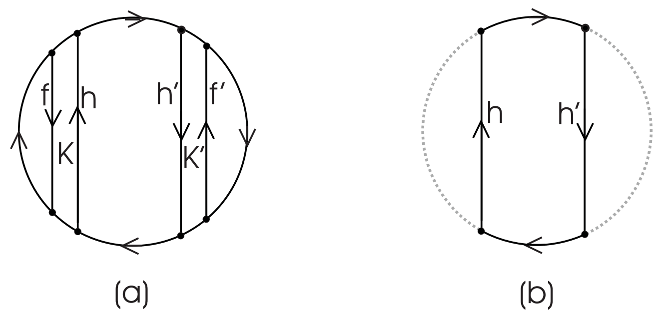

Figure 6.15: Parallel decomposing circuits of C

P is minimally non-Pfaffian, $\mathrm{sign}(C_{h'}) = \mathrm{sign}(C') + \mathrm{sign}(C_f)$ and $\mathrm{sign}(C_h) = \mathrm{sign}(C') + \mathrm{sign}(C_{f'})$, and thus

$$\begin{aligned}\mathrm{sign}(C,K) &= \mathrm{sign}(C_f) + \mathrm{sign}(C_h)\\ &= \mathrm{sign}(C_f) + (\mathrm{sign}(C_{f'}) + \mathrm{sign}(C'))\\ &= \mathrm{sign}(C_f) + \mathrm{sign}(C_{f'}) + (\mathrm{sign}(C_{h'}) + \mathrm{sign}(C_f))\\ &= \mathrm{sign}(C_{f'}) + \mathrm{sign}(C_{h'})\\ &= \mathrm{sign}(C,K').\end{aligned}$$

□

The second of the three lemmas considers $\mathrm{sign}(C,K)$ and $\mathrm{sign}(C,K')$ when the decomposing circuits K and K' of C are weakly crossing.

Lemma 6.17. *Let P be a minimally non-Pfaffian matroid pair and let C be a neutral circuit in $G(P)$. If K and K' are weakly crossing decomposing circuits of C with $e = E(K) \cap E(K')$, then $\mathrm{sign}(C,K) = \mathrm{sign}(C,K')$ if either of the following hold:*

(i) A joining edge g between K and K' is a backwards arc of $C \backslash \{e\}$.

(ii) The C-chord edge f of K is such that C_f is disjoint from K' and the path $C_f \backslash \{f\}$ has a shortcut arc.

Proof. Let $E(K) = \{e, k, f, h\}$ and let $E(K') = \{e, k', f', h'\}$ where the edges of K and K' are labeled as in Figure 6.16(a).

(i) Assume a joining edge g between K and K' is a backwards arc of $C \backslash \{e\}$, as in Figure 6.16(b). Then C_h decomposes into $C_{f'}$ and C_g, and $C_{h'}$ decomposes into C_f and C_g. Thus

$$\begin{aligned}\mathrm{sign}(C_h) &= \mathrm{sign}(C_{f'}) + \mathrm{sign}(C_g), \quad \text{and}\\ \mathrm{sign}(C_{h'}) &= \mathrm{sign}(C_f) + \mathrm{sign}(C_g),\end{aligned}$$

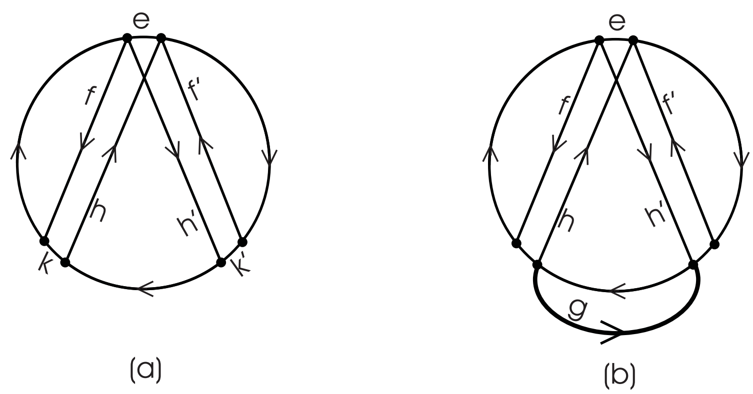

Figure 6.16: Weakly crossing decomposing circuits K and K' of C

and therefore

$$\begin{aligned}\operatorname{sign}(C,K) &= \operatorname{sign}(C_f)+\operatorname{sign}(C_h)\\ &= \operatorname{sign}(C_{f'})+\operatorname{sign}(C_{h'})\\ &= \operatorname{sign}(C,K').\end{aligned}$$

(ii) Let w be a shortcut arc of $C_f\backslash\{f\}$. If D_1 and D_2 are given by

$$D_1=\{(C_f)_w, C_f, (C_{h'})_w, C_{h'}\} \text{ and } D_2=\{(C_f)_w, (C_{h'})_w, C_h, C_{h'}\},$$

then D_1 and D_2 are both non-spanning even cycle covers of $G(P)$, as shown in Figure 6.6(a) and (b) respectively. Since P is minimally non-Pfaffian, $\operatorname{sign}(D_1) =$

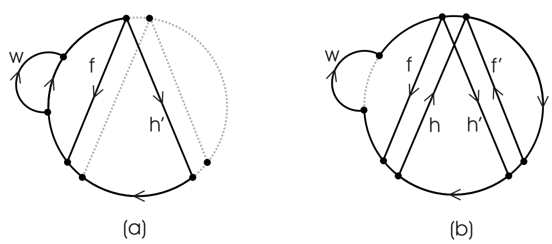

Figure 6.17: Non-spanning even cycle covers of $G(P)$

$\operatorname{sign}(D_2) = 0$ and thus

$$\begin{aligned}\operatorname{sign}(C,K)+\operatorname{sign}(C,K') &= (\operatorname{sign}(C_f)+\operatorname{sign}(C_h))+(\operatorname{sign}(C_{f'})+\operatorname{sign}(C_{h'}))\\ &= \operatorname{sign}(D_1)+\operatorname{sign}(D_2)\\ &= 0.\end{aligned}$$

By reversing the orientation of the edges of G, it follows that if $C_{f'}\backslash\{f'\}$ has a shortcut arc then $\operatorname{sign}(C,K) = \operatorname{sign}(C,K')$.

□

The third and final lemma relating $\mathrm{sign}(C, K)$ and $\mathrm{sign}(C, K')$ considers the case that K and K' are strongly crossing.

Lemma 6.18. *Let P be a minimally non-Pfaffian matroid pair and let C be a neutral circuit in $G(P)$. If K and K' are strongly crossing decomposing circuits of C, then* $\mathrm{sign}(C, K) = \mathrm{sign}(C, K')$ *if either of the following hold:*

(i) A directed path in C that is edge disjoint from K and K' has a shortcut arc.

(ii) A chord of C strongly crosses K and K'.

Proof. Assume $\{f, h\}$ are the C-chord edges of K and $\{f', h'\}$ are the C-chord edges of K', where the edges of K and K' are labeled as shown in Figure 6.18.

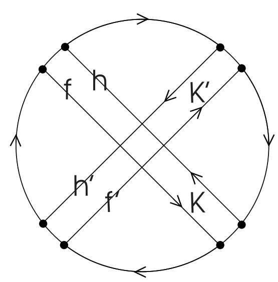

Figure 6.18: Strongly crossing decomposing circuits K and K'

(i) Let Q be a directed path in C that is edge disjoint from $E(K)$ and $E(K')$ and assume that Q has a shortcut arc w. Without loss of generality we may assume that Q is directed from the head of h' to the tail of f. Then $D_1 = \{(C_f)_w, C_h, (C_{h'})_w, C_{f'}\}$ and $D_2 = \{(C_f)_w, C_f, (C_{h'})_w, C_{h'}\}$ (Figures 6.19(a) and (b) respectively) are non-spanning even cycle covers of $G(P)$. Since $G(P)$ is minimally non-Pfaffian, $\mathrm{sign}(D_1) =$

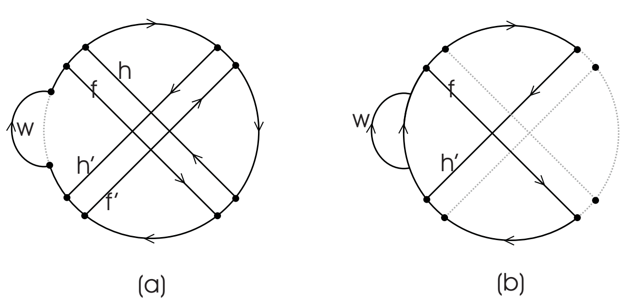

Figure 6.19: Cycle even covers through the shortcut arc w

$\text{sign}(D_2) = 0$ and since

$$\begin{aligned}\text{sign}(D_1) + \text{sign}(D_2) &= \text{sign}(C_h) + \text{sign}(C_f) + \text{sign}(C_{h'}) + \text{sign}(C_{f'}) \\ &= \text{sign}(C, K) + \text{sign}(C, K'),\end{aligned}$$

it follows that $\text{sign}(C, K) + \text{sign}(C, K') = 0$.

(ii) If w is a chord of C that strongly crosses K and K' then without loss of generality we may assume that the tail of w is contained in $V(C_h) \cap V(C_{f'})$ and that the head of w is contained in $V(C_f) \cap V(C_{h'})$, as in Figure 6.20(a). Let C' be the circuit in $C \cup \{f, f', w\}$ with $f, f', w \in C'$ as shown in Figure 6.20(b). Then

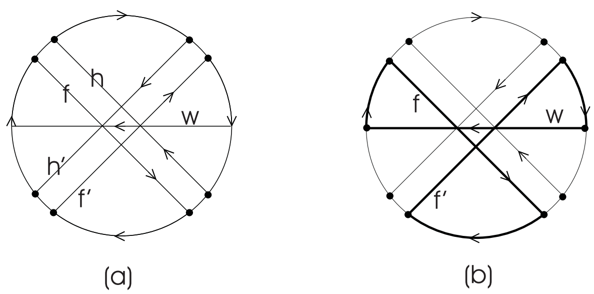

Figure 6.20: A chord of C that strongly crosses K and K'

$D_1 = \{C', C_w, C_f, C_{h'}\}$ and $D_2 = \{C', C_w, C_{f'}, C_h\}$ are both non-spanning cycle even covers of $G(P)$ and thus $\text{sign}(D_1) = 0 = \text{sign}(D_2)$. Then

$$\begin{aligned}\text{sign}(D_1) + \text{sign}(D_2) &= \text{sign}(C_f) + \text{sign}(C_{h'}) + \text{sign}(C_{f'}) + \text{sign}(C_h) \\ &= \text{sign}(C, K) + \text{sign}(C, K')\end{aligned}$$

and thus $\text{sign}(C, K) = \text{sign}(C, K')$.

$\square$

Using Lemmas 6.16, 6.17, and 6.18, we can now show that P has an non-Pfaffian bases certificate containing at most 32 bases. Since C is a neutral circuit in $G(P)$, there exist decomposing circuits F and H of C such that $\text{sign}(C, F) \neq \text{sign}(C, H)$ and by Lemma 6.16 the circuits F and H cross in C. There are two cases to consider, depending on whether F and H strongly cross or weakly cross. We first consider the case that F and H strongly cross, and construct a contributing accordion in C.

Lemma 6.19. *Let C be a neutral directed circuit in $G(P)$ and let F and H be strongly crossing decomposing circuits of C with $\text{sign}(C, F) \neq \text{sign}(C, H)$. If f is a C-chord edge of F, then $V(C_f)$ can be decomposed into at most 8 contributing sets.*

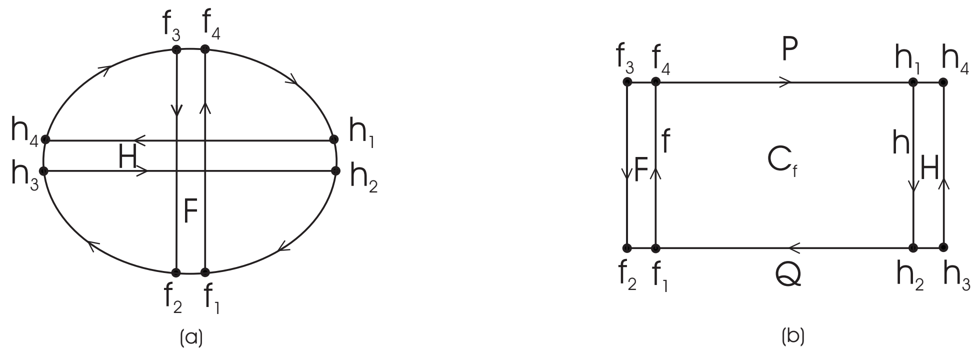

Figure 6.21: strongly crossing decomposing circuits F and H

Proof. Let $V(F) = \{f_1, f_2, f_3, f_4\}$ and $V(H) = \{h_1, h_2, h_3, h_4\}$ where we may assume that the cyclic order of $V(F)$ and $V(H)$ along C is as shown in Figure 6.21(a). Let $f = f_1f_4$ and $h = h_1h_2$. We show that C_f has a decomposition into at most 8 contributing subgraphs.

Let R be the path in C from f_3 to h_1 together with the edge h_1h_4 and let Q be the edge h_3h_2 together with the path in C from h_2 to f_2. Then R and Q are vertex disjoint directed paths and F and H are parallel bridge circuits between R and Q. The circuit C_f is a circuit in $R \cup Q \cup F \cup H$ with ends F and H (see Figure 6.21(b)) and by Lemma 6.18(i), R and Q do not have any internal shortcut arcs. We use induction on $|C_f|$ to prove that there is a contributing accordion between R and Q with ends F and H. For the base case, if C_f has length two then it is contributing and thus there is a trivial accordion between R and Q with ends F and H.

If C_f is contributing then it has a trivial decomposition into one contributing cycle, and thus we may assume that C_f has a decomposing circuit K. Suppose $E(K)$ intersects $E(R)$ and $E(K)$ contains f but not h. Let $k = E(K) \cap E(R)$ with $k = k_1k_2$, and over all such decomposing circuits in C_f, choose K to minimize the distance along R from f_4 to k_1. If $C_{k_1f_4}$ is not contributing then let K' be a decomposing circuit of $C_{k_1f_4}$. If $k_1f_4 \notin E(K')$ then K' is a decomposing circuit of C that is parallel to both F and H, which contradicts Lemma 6.16. If $k_1f_4 \in E(K')$ then let $k' = E(K') \cap E(C)$ with $k' = k'_1k'_2$. Since K' and K share exactly one edge, namely k_1f_4, either k'_1k_2 or $f_1k'_2$ is a joining edge between K and K'. Both joining edges lead to contradictions: k'_1k_2 is a shortcut arc of R and if $f_1k'_2$ is a chord in C then C_f has a decomposing circuit through f and k' that contradicts the choice of K. Thus $C_{k_1f_4}$ is contributing.

The decomposing circuits F and K share the edge f and thus there is a joining edge between K and F. If f_3k_2 is the joining edge then C has a decomposing circuit K' with $f_3f_4, k \in E(K')$. The circuits K' and F weakly cross in C with $E(K') \cap E(F) = f_3f_4$, and the joining edge f_1k_2 between K' and F is a backwards arc of $C\backslash\{f_3f_4\}$. Thus Lemma 6.17(i) implies that $\text{sign}(C, F) = \text{sign}(C, K')$ which then implies that $\text{sign}(C, K') \neq \text{sign}(C, H)$. This is a contradiction of Lemma 6.16, since K' and H

are parallel in C. Thus k_1f_2 is the unique joining edge between F and K, as shown in Figure 6.22. Let F' be the decomposing circuit of C with C-edge arcs $\{f_1f_2, k\}$.

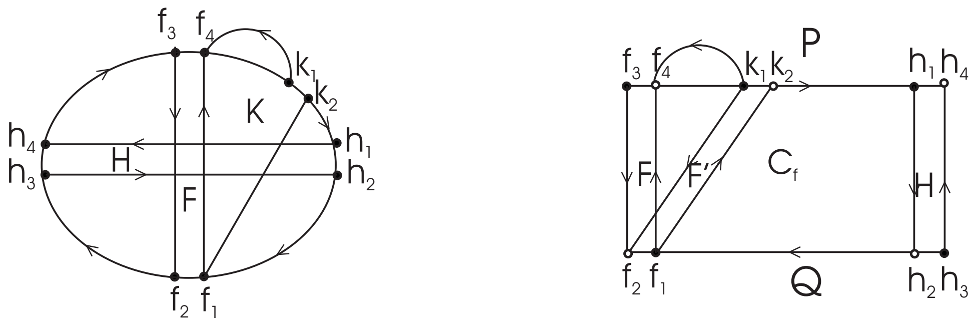

Figure 6.22: Building an accordion

Then F' and F weakly cross in C and F' and H strongly cross. The joining edge k_1f_4 between F and F' implies that $\text{sign}(C, F) = \text{sign}(C, F')$ by Lemma 6.17(i), and thus $\text{sign}(C, F') \neq \text{sign}(C, H)$. By the induction hypothesis, there is a contributing accordion between R and Q with ends F' and H. Adding $E(F)$ and k_1f_4 to this accordion gives a contributing accordion between R and Q with ends F and H. Thus there is an accordion between R and Q with ends F and H whenever a decomposing circuit K of C_f contains an edge of R and contains f but not h. By reversing the orientation of the edges of G, we conclude that there is a contributing accordion between R and Q with ends F and H whenever a decomposing circuit K of C_f contains an edge of Q and the edge f but not h. Let C' be the directed circuit with $E(C') = E(C)\Delta E(F)\Delta E(H)$. Then $C_f = C'_h$ and thus if C_f has a decomposing circuit K with $h \in E(K)$ and $f \notin E(K)$ then C'_h has a decomposing circuit K with $h \in E(K)$ and $f \notin E(K)$. Hence there is a contributing accordion between R and Q whenever a decomposing circuit K of C_f contains the edge f but not the edge h.

So we may assume that any decomposing circuit for C_f contains the edge f if and only if it also contains h. We show that if such a circuit exists then $C_{h_1f_4}$ and $C_{f_1h_2}$ are contributing.

Let K be a decomposing circuit of C_f with $f, h \in E(K)$ and suppose $C_{h_1f_4}$ is not contributing. If K' is a decomposing circuit of $C_{h_1f_4}$ and K' is edge disjoint from K then K' is a decomposing circuit of C that is parallel to both F and H, contradicting Lemma 6.16. Thus we may assume that K' is a decomposing circuit of $C_{h_1f_4}$ with $h_1f_4 \in E(K')$. Let $k' = E(K') \cap E(R)$ with $k' = k'_1k'_2$, as shown in Figure 6.23. If $f_1k'_2$ is a joining edge between K' and K then C_f has a decomposing circuit that intersects C_f in edges f and k', contradicting our assumption that any decomposing circuit of C_f through f also contains h. If k'_1h_2 is a joining edge between K and K' then C_f has a decomposing circuit that intersects C_f in edges h and k', again contradicting our assumption that any decomposing circuit of C_f through h also contains f. It follows that $C_{h_1f_4}$ is contributing, and by symmetry we conclude that $C_{f_1h_2}$ is contributing.

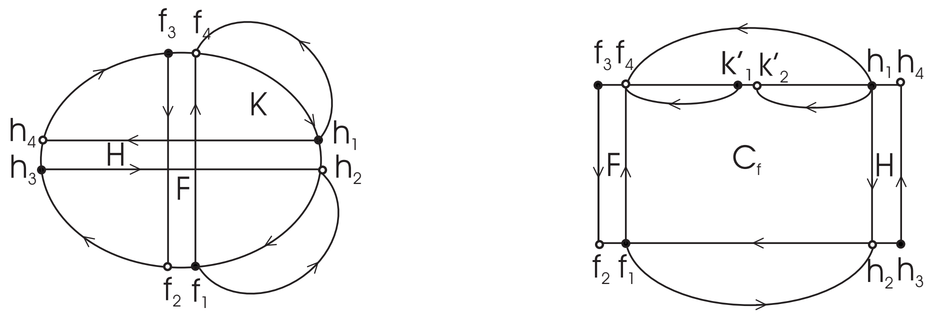

Figure 6.23: A decomposing circuit through f and h

Consider F and K, which share the edge f. By reversing the orientation of G if necessary, we may assume that f_3h_2 is a joining edge between F and K. This creates a decomposing circuit K' of C with $h, f_3f_4 \in E(K')$. Since K' and F are weakly crossing decomposing circuits of C, Lemma 6.17(i) and the joining edge f_1h_2 imply that $\text{sign}(C, K') = \text{sign}(C, F)$ and therefore $\text{sign}(C, K') \neq \text{sign}(C, H)$. Applying Lemma 6.17(i) again implies that h_3f_4 is the unique joining edge between K' and H.

Finally, suppose every decomposing circuit of C_f contains neither f nor h, and let K be a decomposing circuit of C_f. Since Lemma 6.16 implies that K can not be parallel to both F and H, K is a bridge circuit between R and Q. Let $k_1k_2 = E(K) \cap E(R), l_1l_2 = E(K) \cap E(Q)$ and over all such bridge circuits choose K to minimize $|(C_f)_{k_1l_2}|$. Then $(C_f)_{k_1l_2}$ is contributing and K and F are parallel decomposing circuits in C. Lemma 6.16 implies that $\text{sign}(C, K) = \text{sign}(C, F)$. This implies that K and H give opposite signs to C, and by the induction hypothesis there is a contributing accordion between R and Q with ends K and H. Adding the edges of F to this accordion creates a contributing accordion between R and Q with ends F and H. Hence by induction, there is a contributing accordion $\mathcal{H}$ between R and Q with ends F and H.

Let $\mathcal{C}$ be the contributing circuits in $\mathcal{H}$. Let $\mathcal{C}_R$ be the circuits in $\mathcal{C}$ that only use vertices in R and let $\mathcal{C}_Q$ be the circuits in $\mathcal{C}$ that only use vertices in Q. If $\mathcal{C}_{f,h}$ contains the circuits in $\mathcal{C}$ that use f or h and $\mathcal{C}' = \mathcal{C} \backslash \{\mathcal{C}_R, \mathcal{C}_Q, \mathcal{C}_{f,h}\}$ then $\{\mathcal{C}_R, \mathcal{C}_Q, \mathcal{C}_{f,h}, \mathcal{C}'\}$ is a partition of $\mathcal{C}$. Since $\mathcal{C}_{f,h}$ contains at most two circuits, to prove the lemma it remains to show that the circuits in $\{\mathcal{C}_R, \mathcal{C}_q, \mathcal{C}'\}$ can be covered with at most six contributing subgraphs. Since R and Q have no internal shortcut arcs, Lemma 6.14 implies that $\mathcal{C}_R$ and $\mathcal{C}_Q$ can each be partitioned into at most two contributing cycles.

If $R' = R \backslash \{f_3f_4, h_1h_4\}$ and $Q' = Q \backslash \{h_3h_2, f_1f_2\}$ and $\mathcal{H}'$ is the accordion between R' and Q' given by the bridge circuits in $\mathcal{H}$ that are bridge circuits between R' and Q', then the circuits in $\mathcal{C}'$ are all accordion circuits in H'. Since R and Q do not have internal shortcut arcs, R' and Q' do not have shortcut arcs. Suppose there exists an internally twisted bridge circuit K' in H where K' strongly crosses the parallel bridge circuits K_1 and K_2 in H'. Then K' is parallel to F and thus $\text{sign}(C, K') \neq \text{sign}(C, H)$. Furthermore, since K' is internally twisted, both K_1 and K_2 are parallel to F and cross H. If $E(K_1)$

precedes $E(K_2)$ along R then the edge of K_1 directed from the tail of $E(K_1) \cap E(Q)$ to the head of $E(K_1) \cap E(R)$ strongly crosses K and H, as shown in Figure 6.24. This

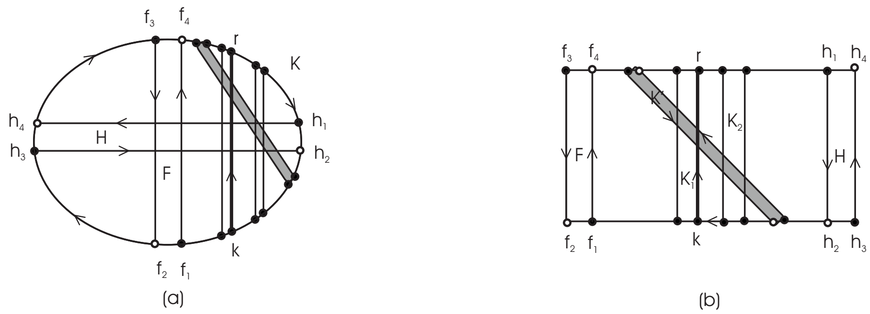

Figure 6.24: An internally twisted bridge circuit

contradicts Lemma 6.18(ii), and thus $\mathcal{H}'$ has no twisted bridge circuits. Lemma 6.15 therefore implies that the circuits in $\mathcal{C}'$ can be partitioned into at most two contributing sets, and thus $\mathcal{C}$ can be partitioned into at most eight contributing sets. It follows that if F and H are strongly crossing decomposing circuits of C with $\mathrm{sign}(C, F) \neq \mathrm{sign}(C, H)$ and f is a C-chord edge of F, then C_f can be partitioned into at most 8 contributing sets. $\square$

We can now prove our main result for neutral circuits.

Theorem 6.20. *If P is a minimally non-Pfaffian series-parallel matroid pair and $G(P)$ has a neutral circuit, then P has a non-Pfaffian bases certificate containing at most* 32 *bases.*

Proof. Let P be a minimally non-Pfaffian matroid pair and let C be a neutral circuit in $G(P)$. Suppose there exist strongly crossing decomposing circuits F and H of C such that $\mathrm{sign}(C, F) \neq \mathrm{sign}(C, H)$. If $\{f, f'\}$ and $\{h, h'\}$ are the C-chord edges of F and H respectively, then by Lemma 6.19, $C_f, C_{f'}, C_h$, and $C_{h'}$ each partition into at most eight contributing sets. Since $\{C_f, C_{f'}, C_h, C_{h'}\}$ is a certificate that $G(P)$ is not signable, there is a set of at most 32 contributing subgraphs that proves that $G(P)$ is not signable. By the bijection between contributing subgraphs and bases of P, there is a non-Pfaffian bases certificate containing at most 32 bases of P.

We may assume that whenever C' is a directed circuit in $G(P)$ and F and H are decomposing circuits of C' with $\mathrm{sign}(C', F) \neq \mathrm{sign}(C', H)$, then F and H are weakly crossing in C'. Let F and H be such a pair of decomposing circuits for C let $\{e, f\}$ and $\{e, h\}$ be the C-chord edges of F and H respectively, where the orientation of G is such that h precedes f on the directed path $C \backslash \{e\}$. For $i = 1, 2$, assume $f_i, e_i, h_i \in V(C)$ are such that $e = e_1e_2$, $f = f_1f_2$, and $h = h_1h_2$. By Lemma 6.17(i), h_1f_2 is the unique joining edge between F and H (see Figure 6.25(a)). Let Q be the path in C from h to f

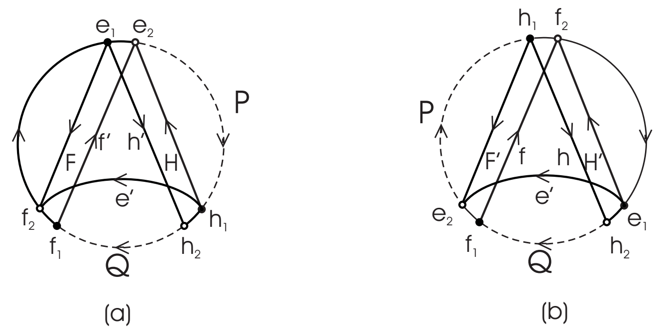

Figure 6.25: Weakly crossing decomposing circuits of C and C'

and over all neutral circuits C and all pairs of weakly crossing decomposing circuits F, H with $\mathrm{sign}(C,F) \neq \mathrm{sign}(C,H)$, choose C, F, and H to minimize $|Q|$.

Let $f' = f_1e_2$ and $h' = e_1h_2$ and let C' be the directed circuit in G with

$$E(C') = E(C)\Delta\{e, f, h\}\Delta\{e', f', h'\}.$$

Then C' has weakly crossing decomposing circuits F' and H' where e' and f' are the C'-edge arcs of F' and E' and h' are the C'-edge arcs of H'. Furthermore, $C_{f'} = C'_h$ and $C_{h'} = C'_f$ (Figure 6.25(b)), and the assumption that $\mathrm{sign}(C,F) \neq \mathrm{sign}(C,H)$ implies that $\mathrm{sign}(C',F') \neq \mathrm{sign}(C',H')$. Let R be the directed path in C from e_2 to h_1, such that $C_{f'} = R, h, Q, f'$.

Claim 1: There is no decomposing circuit of $C_{f'}$ that uses both edge h and edge f'.
Proof of Claim 1: Since $\mathrm{sign}(C,F) \neq \mathrm{sign}(C,H)$, Lemma 6.17(i) implies that f_1h_2 is not an edge in $G(P)$. In particular there is no decomposing circuit of $C_{f'}$ that uses f_1h_2 and therefore there is no decomposing circuit of $C_{f'}$ that uses both edge h and edge f'.

Claim 2: There is no decomposing circuit in $C_{f'}$ that uses edge h.
Proof of Claim 2: Let K be a decomposing circuit in $C_{f'}$ with $C_{f'}$-edge arcs h and k. By Claim 1, either k is an edge of R or k is an edge of Q.

Let $k = k_1k_2$ and suppose first that k is an edge of R. Then K and F are parallel and therefore $\mathrm{sign}(C,K) = \mathrm{sign}(C,F)$ and thus $\mathrm{sign}(C,K) \neq \mathrm{sign}(C,H)$. Since K and H are weakly crossing, Lemma 6.17(i) implies that e_1k_2 is the unique joining edge between K and H and $k_1e_2 \notin E(G(P))$, as shown in Figure 6.26(a). The edge e_1k_2 creates a decomposing circuit K' in C' with $E(K') \cap E(C') = \{h', k\}$ (see Figure 6.26(b)). Since K' and F' are strongly crossing in C', $\mathrm{sign}(C',K') = \mathrm{sign}(C',F')$ by assumption and thus $\mathrm{sign}(C',K') \neq \mathrm{sign}(C',H')$. The decomposing circuits K' and H' are weakly crossing in C' and the joining edge h_1k_2 between K' and H' therefore contradicts Lemma 6.17(i). Hence if h and k are the $C_{f'}$-edge arcs of K then k is not an edge of R.

Suppose instead that k is an edge of Q. Then K and F are again parallel and thus $\mathrm{sign}(C,K) = \mathrm{sign}(C,F)$ and $\mathrm{sign}(C,K) \neq \mathrm{sign}(C,H)$. Since K and H are weakly

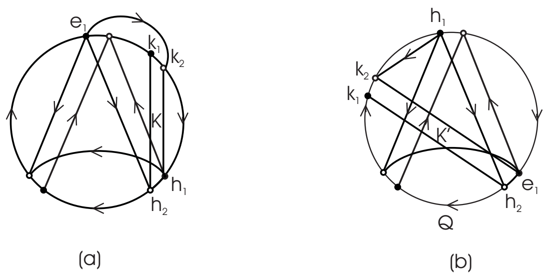

Figure 6.26: Weakly crossing decomposing circuits of C and C'

crossing, Lemma 6.17(i) implies that the edge k_1e_2 is the unique joining edge of K and H and $e_1k_2 \notin E(G)$ (Figure 6.27(a)).

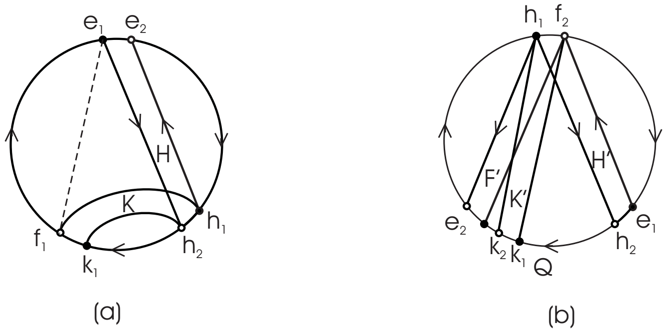

Figure 6.27: K contains h and an edge in Q

The decomposing circuits H' and K share the edge h, and since e_1k_2 is not an edge in $G(P)$, k_1f_2 is the unique joining edge between H' and K. Then C' has a decomposing circuit K' with C'-edge arcs h_1f_2 and k_1k_2, and K' weakly crosses both F' and H'. (See Figure 6.27(b).) This contradicts the minimality of $|Q|$ since the both the directed path in C' from h_2 to k_1 and the directed path in C' from k_2 to f_1 are strict subpaths of Q and either $\text{sign}(C', K') \neq \text{sign}(C', F')$ or $\text{sign}(C', K') \neq \text{sign}(C', H')$.

It follows that there is no decomposing circuit of $C_{f'}$ that contains the edge h and the proof of Claim 2 is complete.

Claim 3: There is no decomposing circuit in $C_{f'}$ that contains the edge h.
Proof of Claim 3: Reversing the orientation of the edges in $G(P)$ and considering C', Claim 2 implies that a decomposing circuit of C'_h does not contain f'. Since $C'_h = C_{f'}$, there is no decomposing circuit of $C_{f'}$ that contains f'.

Claim 4: The directed circuits $C_{f'}$ and $C_{h'}$ are contributing in $G(P)$.
Proof of Claim 4: Suppose $C_{f'}$ is not contributing and let K be a decomposing circuit of $C_{f'}$. By Claims 1 and 2, K is edge disjoint from f' and h. If both $C_{f'}$-edge arcs

of K are in R then K is parallel to both F and H, which contradicts Lemma 6.16. Thus at most one $C_{f'}$-edge arc of K is an edge of R. Similarly, at most one $C_{f'}$-edge arc of K is an edge of Q. Thus K is a bridge circuit between R and Q, and therefore K crosses H and is parallel to F. By Lemma 6.16, $\mathrm{sign}(C, K) = \mathrm{sign}(C, F)$ and thus $\mathrm{sign}(C, K) \neq \mathrm{sign}(C, H)$. This contradicts the assumption C does not contain strongly crossing decomposing circuits which give different signs to C and the circuit $C_{f'}$ must be contributing.

By reversing the orientation of $G(P)$ it follows that $C_{h'}$ is contributing. Since

$$\{C_{e_1 f_2}, C_{f'}, C_{h_1 e_2}, C'_h\}$$

is a certificate that $G(P)$ is not signable, it remains to show that $C_{e_1 f_2}$ and $C_{h_1 e_2}$ can be partitioned into a small number of contributing cycles. By symmetry it suffices to consider $C_{h_1 e_2}$.

Let w be a backwards arc of R and let K be a decomposing circuit of C_w. If $w \notin E(K)$ then K is also a decomposing circuit of $C_{f'}$. This contradicts Claim 3, and thus $w \in E(K)$. Let w' and w'' be the C_w-chord arcs of K. Then w' and w'' are both backwards arcs of R and $V(C_w)$ is the disjoint union of $V(C_{w'})$ and $V(C_{w''})$. Since both $C_{w'}$ and $C_{w''}$ are shorter than C_w, repeating this process leads to a decomposition of C_w into contributing circuits of the form $C_{w'}$ for some backwards arc w' or R . Since h_1e_2 is a backwards arc of R, we can therefore decompose $C_{h_1 e_2}$ into contributing circuits of the form C_w where w is a backwards arc of R. Let $\mathcal{C}$ be the set of contributing circuits in such a decomposition of $C_{h_1 e_2}$ and let $\{\mathcal{C}_A, \mathcal{C}_B\}$ be a partition of $\mathcal{C}$ such that if $D_A, D_B \in \mathcal{C}$ and an edge of R is incident to both a vertex in D_A and a vertex in D_B, then $D_A \in \mathcal{C}_A$ if and only if $D_B \in \mathcal{C}_B$.

Let $D_1, D_2 \in \mathcal{C}_A$. If $D_1 \cup D_2$ is not contributing, then there exists a decomposing circuit K in $G(P)$ with $V(K) = \{u_1, u_2, v_1, v_2\}$ where $u_1v_1 \in E(D_1) \cap E(K)$ and $u_2v_2 \in E(D_2) \cap E(K)$. By the construction of $\mathcal{C}$ we may assume that both u_1 and v_1 precede u_2 and v_2 along R, and by the construction of $\mathcal{C}_A$ neither u_1v_2 nor u_2v_1 is an edge of R. This is a contradiction, since u_1v_2 is a shortcut arc of R. Therefore the union of all contributing circuits in $\mathcal{C}_A$ is a contributing subgraph, as is the union of all contributing circuits in $\mathcal{C}_B$, and hence $V(C_{h_1 e_2})$ is the disjoint union of at most two contributing subgraphs. Reversing the orientation of the edges of $G(P)$ implies that $V(C_{e_1 f_2})$ is also the disjoint union of at most two contributing cycles, and we therefore have a set of at most six contributing cycles which certify that $V(C)$ is not signable. □

The constant of 32 in Theorem 6.20 is not claimed to be optimal. We do not have an example of a neutral circuit that requires more than six bases to certify that it is non-Pfaffian, and in fact we do not have an example of a minimal non-Pfaffian matroid pair that requires more than six bases to certify that it is non-Pfaffian.

6.7 Odd double cycles

There remains one case left to consider for series-parallel matroid pairs, namely that P is a non-Pfaffian series-parallel matroid pair and there are no neutral circuits in $G(P)$. Theorem 6.10 implies that $G(P)$ does not have an edge signing that respects the decomposing parity of all directed circuits in $G(P)$, and we can therefore use a characterization of Gerards [15] to get a structural result for $G(P)$. Unfortunately the structure is not induced in $G(P)$ and so this result on $G(P)$ does not translate back to a good structural characterization of P. Some definitions are required first.

Given two directed circuits C and C', the connected components in $C \cap C'$ are directed paths which we call the intersection paths of C and C'. If there are at least two intersection paths of C and C' and the cyclic order of the intersection paths on C is the inverse of the cyclic order of the paths on C', then we say C and C' are *twined* (Figure 6.28).

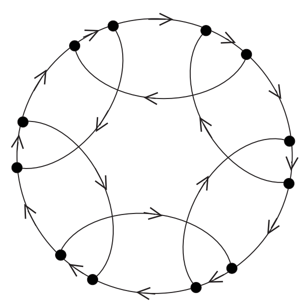

Figure 6.28: Twined circuits

Let C^F and C^B be twined directed circuits in the directed graph G. If C^F and C^B have k intersection paths and $k \geq 2$, then $C^F \cup C^B$ contains k directed circuits $\{C^1, C^2, \ldots, C^k\}$ each distinct from C^F and C^B. The union of these k circuits together with C^F and C^B is called a *weak k-double cycle*. When each intersection path of C^F and C^B is a single vertex, then $C^F \cup C^B$ is a k-double cycle. Let D be a weak k-double cycle and let $\alpha_C \in \{0, 1\}$ be defined for each of the directed circuits in D. If $\alpha_C = 1$ for an odd number of directed circuits in D, then D is an *odd double cycle* with respect to α. Note that D contains $k+2$ directed circuits and each edge in D is in either two or four of these $k+2$ circuits. If D is an odd double cycle in the directed graph G and s is an edge signing of G, then let $s(D)$ be the sum of $s(C)$ over all circuits C in D. Since each $e \in E(D)$ is in an even number of circuits in D, $s(D) = 0$ for all edge signings s of D. It follows that if G has an odd double cycle as a subgraph with respect to some circuit signing α, then G does not have a proper edge signing with respect to α. The structural theorem of Gerards [15] that is used in this section is that the converse also holds.

Theorem 6.21 (Gerards). *If G is a directed graph and $\alpha_C \in \{0, 1\}$ is defined for all directed circuits C in G, then either G has a proper edge signing with respect to α or G*

has a weak double cycle that is odd with respect to α as a subgraph.

If P is a series-parallel non-Pfaffian matroid pair and $G(P)$ does not have a neutral circuit, then Theorem 6.21 together with Theorem 6.11 implies that $G(P)$ has a weak odd double cycle with respect to the circuit signing $\alpha_C = \frac{|C|}{2} + \mathrm{dp}(C)$. The results in this section can be combined with the results Section 6.6 to get the following result for series-parallel matroid pairs:

Theorem 6.22. *If P is a minimally non-Pfaffian series-parallel matroid pair, then either $G(P)$ has a Hamiltonian neutral cycle or $G(P)$ has a spanning weak odd k-double cycle with $k \leq 4$.*

If $G(P)$ has a weak odd k-double cycle D as an induced subgraph and each of the $k+2$ circuits in D is contributing then each circuit corresponds to a basis of P and thus there is a set of at most $k+3$ bases which certifies that P is non-Pfaffian. Unfortunately, the odd double cycle in $G(P)$ that is the obstruction for P to be Pfaffian need not be an induced subgraph of $G(P)$ and therefore the circuits in D need not be contributing. We have been able to show that in most cases, the weak odd double cycle in $G(P)$ has a decomposition into at most 40 contributing circuits, and thus there is a set of at most 40 bases which certifies that P is non-Pfaffian. Based on this partial result and the results of Section 6.6, we make the following weakening of Conjecture 3.12:

Conjecture 6.23. *There exists a constant c such that if P is a series-parallel matroid pair, then either P is Pfaffian or P has a non-Pfaffian bases certificate with at most c bases.*

If this conjecture is true, then it contains some of the power of an excluded minor characterization: if there is a finite set of excluded minors for Pfaffian matroids, then either a matroid pair is Pfaffian or there is a certificate of constant size, namely one of the excluded minors, which proves that the pair is non-Pfaffian. Similarly, if Conjecture 6.23 holds, then either a series-parallel matroid pair is Pfaffian or there is a certificate of constant size, namely a non-Pfaffian bases certificate, which proves that the pair is non-Pfaffian.

Some notation and terminology for weak double cycles is required. Let C^F and C^B be twined directed circuits and let D be the weak double cycle formed by $C^F \cup C^B$. If C^F and C^B have k intersection paths, then let $S = \{C^1, C^2, \ldots, C^k\}$ be the set of distinct directed circuits in D such that each C^i is distinct from C^F and C^B and

$$C^F \cup C^B = C^1 \cup C^2 \cup \cdots \cup C^k.$$

When $k > 2$ then the circuits in S are called the *short circuits* of D and the circuits C^F and C^B are the *long circuits* of D. If D is a weak k-double cycle then the *cycle size* of D is k and the *cycle length* of D is the sum of the lengths of the $k+2$ directed circuits in D.

When D is a weak k-double cycle with short circuits $\{C^1, C^2, \ldots, C^k\}$, we assume that for all i, C^i intersects C^{i+1} and C^{i-1}, where the index addition is modulo k. The intersection path of $C^i \cap C^{i-1}$ is denoted P_i and if C^F and C^B are the long circuits in D then we let F_i be the subpath of $C^F \cap C^i$ and B_i be the subpath of $C^B \cap C^i$ such that F_i, P_i, B_i, and P_{i+1} are all edge disjoint and $C^i = P_i \cup F_i \cup P_{i+1} \cup B_i$. We assume that C^F and C^B are labeled such that the C^F traverses the paths P_i in forward order, $P_1, P_2, \ldots, P_k$, and C^B therefore traverses the paths in reverse order, $P_k, P_{k-1}, \ldots, P_1$ (see Figure 6.29).

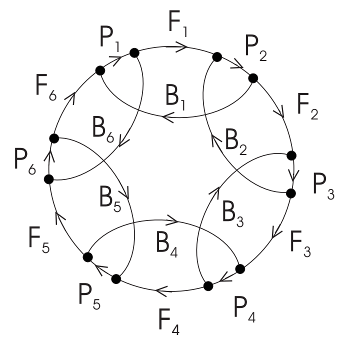

Figure 6.29: The subpath labels for a weak 6-double cycle

Let G be a directed graph and suppose that the weak double cycle D is a subgraph of G. If u and v are vertices in D and $uv \in E(G) \backslash E(D)$, then uv is a *chord* of D. We classify three types of chords of D: forbidden short circuit chords, forbidden long circuit chords, and forbidden between-circuit chords. These chords are collectively referred to as *forbidden chords* of D.

Forbidden short circuit chords: Let uv be a chord in a short circuit C^i of D and suppose that uv is also a chord in a long circuit C^F or C^B of D. If u precedes v along the intersection path between C^i and C^F or C^B, then uv is a forbidden short circuit chord of D. Two examples of forbidden short circuit chords are shown in Figure 6.30(a).

Forbidden long circuit chords: Let uv be a chord in a long circuit of D, where we may assume that uv is a chord of C^F. Let i and j be such that either $u \in P_i$ or u is an internal vertex in F_i, and either $v \in P_{j+1}$ or v is an internal vertex in F_j. If $j \notin \{i, i-1, i-2\}$, then uv is a forbidden long circuit chord of D. A forbidden long circuit chord is shown in Figure 6.30(b).

Forbidden between-circuit chords: Let uv be a chord in D such that after possibly relabeling the circuits in D, u is an internal vertex in F_i, v is an internal vertex in B_j, and $j \neq i$. Further, suppose that if $j = i - 1$ then v is not the second vertex in B_{i-1} (Figure 6.31(a)) and if $j = i + 1$ then u is not the second to last vertex in F_i

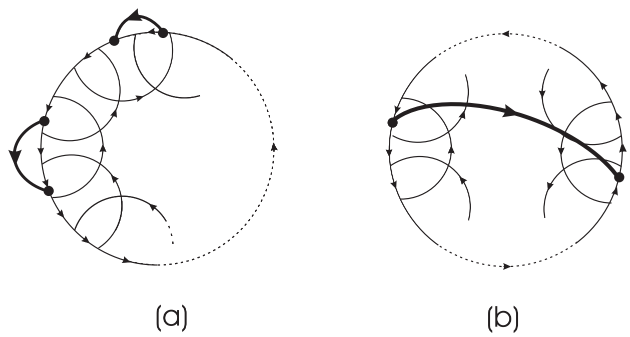

Figure 6.30: Forbidden short (a) and long (b) circuit chords

(Figure 6.31(b)). The chord uv is then a forbidden between-circuit chord of D.

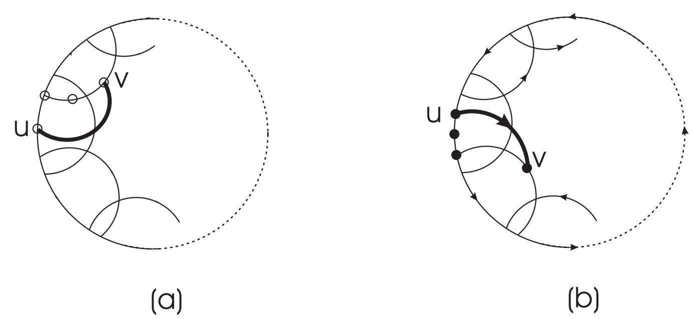

Figure 6.31: Forbidden between-circuit chords

Let G be a directed graph with circuit signing α and suppose that G has an odd weak double cycle with respect to α. Let n be the minimum number of vertices in an odd weak double cycle in G and let k be the minimum cycle size of an odd weak double cycle in G with n vertices. A *minimal weak odd double cycle* in G is a weak k-double cycle D with n vertices and minimum cycle length. We consider where chords of D may lie.

Lemma 6.24. *Let G be a directed graph and let D be a minimal weak odd k-double cycle in G with respect to a circuit signing of G. If $k \geq 5$, then D does not have any forbidden chords.*

Proof. Assume that D has short circuits $\{C^1, C^2 \ldots, C^k\}$ and long circuits $\{C^F, C^B\}$ and let uv be a forbidden short circuit chord of D. By relabeling the circuits in D we may assume that uv is a chord in C^1 and a chord in C^F. The circuits C^1 and C^F_{uv} are twined with two intersection paths and form the weak 2-double cycle $D_1 = \{C^1, C^1_{uv}, C^F, C^F_{uv}\}$. The circuits C^B and C^F_{uv} are also twined with k intersection paths and form the weak k-double cycle D_2 with short circuits $\{C^1_{uv}, C^2, \ldots, C^k\}$. Since D is the symmetric difference of the circuits in D_1 with the circuits in D_2, either D_1 or D_2 is an odd weak double-cycle.

If D_1 is odd then G has an odd weak 2-double-cycle, contradicting the minimality of D. If D_2 is odd then since $|C^1_{uv}| < |C^1|$ and $|C^F_{uv}| < |C^F|$, either D_2 has fewer vertices than D or the cycle length of D_2 is less than the cycle length of D. This again contradicts the minimality of D, and thus D does not contain any forbidden short circuit chords.

Suppose that uv is a forbidden long circuit chord of D. By relabeling the circuits in D, we may assume that either $u \in P_1$ or u is an internal vertex of F_1 and that either $v \in P_{j+1}$ or v is an internal vertex in F_j, with $j \neq k-1, k, 1$. Let P_1' be the directed path in C^1 from the tail of P_1 to u, and let P_{j+1}' be the directed path in C^j from v to the head of P_{j+1}. Let P^* be the directed path in C^B from the head of P_{j+1} to the tail of P_1 and let C^* be the directed circuit in $P^* \cup P_1' \cup \{uv\} \cup P_{j+1}'$.

The circuits C^F_{uv} and C^B are twined with intersection paths $P_{j+1}', P_{j+2}, \ldots, P_k, P_1'$ and form a weak $(k-j+1)$-double cycle D_1. The circuits C^* and C^F are also twined, with intersection paths $P_1', P_2, \ldots, P_j, P_{j+1}'$ and thus form a weak $(j+1)$-double cycle D_2. The short circuits in D_1 are $\{C^{j+1}, C^{j+2}, \ldots, C^k, C^*\}$ and the short circuits in D_2 are $\{C^1, C^2, \ldots, C^j, C^F_{uv}\}$, and therefore D is the symmetric difference of the circuits in D_1 with the circuits in D_2. This implies that either D_1 or D_2 is odd, and since $j \neq k-1, k, 1$ by assumption, either D_1 or D_2 contradicts the minimality of D. Hence D does not contain any forbidden long circuit chords.

Suppose uv is a forbidden between-circuit chord of D. Up to relabeling we may assume that u is an internal vertex of F_1 and v is an internal vertex in B_j for some $j \neq 1$. Let P_1' be the directed path in C^1 from the tail of P_1 to u, and let P_j' be the directed path in C^j from the tail v to the head of P_j. Let P_B' be the directed path in C^B from the head of P_j to the tail of P_1 and similarly let P_F' be the directed path in C^F from the head of P_j to the tail of P_1. Let C^{B*} be the directed circuit in $P_B' \cup P_1' \cup P_j' \cup \{uv\}$ and let C^{F*} be the directed circuit in $P_F' \cup P_1' \cup P_j' \cup \{uv\}$.

The circuits C^{B*} and C^F are twined with intersection paths $P_1', P_2, \ldots, P_j$ and thus form a weak j-double cycle D_1. Similarly, the circuits C^{F*} and C^B are twined with intersection paths $P_j', P_{j+1}, \ldots, P_k, P_1$ and thus form a weak $(k-j+2)$-double cycle D_2. The short circuits in D_1 are $\{C^1, C^2, \ldots, C^{j+1}, C^{F*}\}$ and the short circuits in D_2 are $\{C^j, C^{j+1}, \ldots, C^k, C^{B*}\}$. The symmetric difference of the circuits in D_1 with the circuits in D_2 is D, and therefore either D_1 or D_2 is odd.

If D_1 is odd then since D_1 has cycle size j, the minimality of D implies that $j = k$. The assumption that uv is a forbidden between-circuit chord then implies that v is not the second vertex in B_j and the internal vertices of B^j between the tail of B_j and v are therefore not contained in D_1. This implies that D_1 has fewer vertices than D, contradicting the minimality of D.

If D_2 is odd then since D_2 has cycle size $k-j+2$ and $j \neq 1$, the minimality of the cycle size of D implies that $j = 2$. The assumption that uv is a forbidden between-circuit chord then implies that u is not the second to last vertex of F_1, and thus the internal vertices on F_1 between v and the head of F_1 are not contained in D_2. This implies that D_2 has fewer vertices than D and contradicts the minimality of D. Thus D does not have any forbidden between-circuit chords. □

Suppose D is a minimal weak odd k-double cycle in the directed graph G with $k \geq 5$. We would like a decomposition of each of the circuits in D into contributing circuits. If each circuit is induced then this is easy, and thus we may assume that a circuit in D has a decomposing circuit. Let K be a decomposing circuit in the short circuit C^i in D. If the C^i-edge arcs of K are contained in P_i or P_{i+1} then one of the C^i-chord edges of K is a forbidden short circuit chord. If both C^i-edge arcs of K are contained in F_i or both are contained in B_i then one of the C^i-chord edges of K is a forbidden short circuit chord. Thus all decomposing circuits in C^i are bridge circuits between F_i and B_i. Using this observation we can construct a contributing accordion between F_i and B_i.

Let K be a decomposing circuit of C^i with $V(K) = \{u_1, v_1, u_2, v_2\}$ where u_1v_1 is an edge in F_i and u_2v_2 is an edge in B_i, and over all such decomposing circuits in C^i, choose K to minimize the sum of the distance from the tail of F_i to u_1 with the distance from v_2 to the head of B_i. A decomposing circuit in $C^i_{u_1v_2}$ which is edge disjoint from K contradicts the minimality in the choice of K and thus if $C^i_{u_1v_2}$ is not contributing it has a decomposing circuit K' that contains the arc u_1v_2. Each of the possible joining edges between K and K' either contradicts the minimality of the choice of K or is a forbidden short circuit chord. Thus the choice of K implies that $C^i_{u_1v_2}$ is contributing.

Consider $C^i_{u_2v_1}$, and suppose it has a decomposing circuit K' such that K' contains the edge u_2v_1 and contains an edge uv in F^i. Choose K' to minimize the distance along F^i from v_1 to u. The edge u_1v is a forbidden short circuit chord, and thus uv_2 is the joining edge between K and K'. There is therefore a bridge circuit K'' between F_i and B_i such that K and K'' are the ends of the accordion circuit $C^i_{uv_1}$. The minimality of the length of the path between v_1 and u implies that $C^i_{uv_1}$ is contributing. If $C^i_{u_2v_1}$ has a decomposing circuit which contains u_2v_1 and an edge in B^i then we can similarly choose the decomposing circuit such C^i has a bridge circuit K'' and a contributing accordion circuit between K and K''.

If $C^i_{u_2v_1}$ is not contributing and does not have a decomposing circuit that contains the edge u_2v_1 then there is a decomposing circuit K'' of $C^i_{u_2v_1}$ that is a bridge circuit between F_i and B_i. Let u_3v_3 and u_4v_4 be the $C^i_{u_2v_1}$ edges of K'' and assume with $u_3v_3 \in F_i$ and $u_4v_4 \in B_i$. If K'' is chosen to minimize the sum of the distance along F_i from v_1 to u_3 plus the distance along B_i from v_4 to u_2 then the accordion circuit between K and K'' is contributing.

Hence there is either a contributing accordion or a trivial accordion between F_i and B_i with ends K and K''. Over all contributing accordions between F_i and B_i, choose the accordion H with ends K and K' such that the number of bridges in H is maximum. If k is a C^i-chord edge of K and k' is a C^i-chord edge of K' such that C^i_k and $C^i_{k'}$ are disjoint, then the maximality of the number of bridges in H implies that C^i_k and $C^i_{k'}$ are contributing. The contributing accordion circuits of H together with the circuits C^i_k and $C^i_{k'}$ are a decomposition of C^i which we call an *accordion decomposition* of C^i.

Observation 6.25. *If D is a minimal weak odd k-double cycle in the directed graph G and $k \geq 5$, then every short circuit in D has an accordion decomposition.*

We can now prove the main structural result of this section.

Proof of Theorem 6.22. Let P be a minimally non-Pfaffian series-parallel matroid pair. If $G(P)$ has a neutral circuit C, then the matroid pair corresponding to the subgraph of $G(P)$ induced by $V(C)$ is non-Pfaffian, and thus by the minimality of P the neutral circuit is Hamiltonian. If $G(P)$ does not have a neutral circuit, then Theorems 6.21 and 6.11 imply that $G(P)$ has a weak odd double-cycle D. Let D be a weak odd double cycle in $G(P)$ with cycle size k and suppose that $k \geq 5$. We show that D is not minimal.

Claim 1: The long circuits in D are contributing.
Proof. If C^F is not contributing, then let K be a decomposing circuit of C^F with C^F-edge arcs u_1v_1 and u_2v_2. By relabeling the circuits in D, we may assume that $u_1v_1 \in P_1 \cup F_1$. Suppose u_1v_2 is a chord of D. Since D has no forbidden long circuit chords and no forbidden short circuit chords, v_2 is an internal vertex in the directed path in C^F from the head of P_{k-1} to the tail of P_1 and since $k \geq 5$, u_2v_1 is not an edge in D. Since u_2v_1 is directed from C^k or C^{k-1} to C^1, it is a forbidden long circuit chord in D. If u_1v_2 is an edge of D, then it is the path B_k and thus u_2 is again a vertex in C^{k-1} and u_2v_1 is a forbidden long circuit chord. This contradicts the minimality of D and thus C^F is contributing. By symmetry C^B is also a contributing circuit and thus the long circuits in D are contributing.

For $1 \leq i \leq k$, let H_i be an accordion between F_i and B_i with a maximum number of bridges, and let $\mathcal{C}_i$ be the set of contributing circuits in the accordion decomposition of C^i from H_i. Note that the construction of the accordion is such that exactly one circuit in $\mathcal{C}_i$ intersects P_i and exactly one circuit in $\mathcal{C}_i$ intersects P_{i+1}. Let $A_1 \in \mathcal{C}_1$ and $A_2 \in \mathcal{C}_2$ be such that both A_1 and A_2 intersect P_2.

Claim 2: If $A \in \mathcal{C}_2$ is distinct from A_2 then $A \cup A_1$ is contributing.
Proof. If $A \cup A_1$ is not contributing then there exists a decomposing circuit K in G with edges uv and u_1v_1 such that $uv \in A$ and $u_1v_1 \in A_1$. Suppose $uv \in C^F$. If $u_1v_1 \in C^1$ then since C^F is contributing, $u_1v_1 \in B_1$. The chord u_1v is a forbidden between-circuit chord unless v is the second vertex on F_1. If v is the second vertex on F_1 then since $uv \in D$, u must be the first vertex on F_1, but the first vertex on F_1 is also the tail of P_2, contradicting the assumption that A is vertex disjoint from P_2. Hence if $uv \in C^F$ then $u_1v_1 \notin C^1$. If $u_1v_1 \notin C^1$ then let K_1 be the bridge circuit between F_1 and B_1 such that u_1v_1 is a C^1-chord edge of K_1. Since $u_1v_1 \in A_1$ and A_1 is a contributing circuit

containing P_2, if $u_1v_1 \in K_1$ then $u_1 \in B_1$ and $v_1 \in F_1$. If $uv \in F_2$ then the chord u_1v is a forbidden chord unless v is the second vertex of F_1, but this again leads to a contradiction of the assumption that A and P_2 are vertex disjoint. If $uv \in P_3$ then the two possible joining edges between K and K_1 both contradict the assumption that the long circuits are contributing. Thus $uv \notin C^F$ and by symmetry $uv \notin C^B$. There remain two cases two check: either uv is an edge in a bridge circuit of C^2, or C^2_{uv} is a contributing accordion circuit in $\mathcal{C}_2$.

Suppose uv is an edge in a bridge circuit K_2 of C^2 and assume $u \in B_2$ and $v \in F_2$. Then u_1v is a forbidden chord unless u_1 is an interior vertex in B_1, and uv_1 is a forbidden chord unless v_1 is an interior vertex in F_1. Thus $u_1 \in B_1$ and $v_1 \in F_1$ and there is a bridge circuit K_1 in C^1 such that u_1v_1 is a C^1-chord edge of K_1. Let u_2v_2 be the other C^1-chord edge in K_1. Since K_1 and K share exactly one edge, either u_2v or uv_2 is a joining edge between K_1 and K. The former is a forbidden C^B chord and the latter a forbidden C^F chord, and thus if uv is an edge in the bridge circuit K_2 then $u \in F_2$ and $v \in B_2$. Since A is disjoint from P_2, u is an interior edge of F_2 and v is an interior edge of B_2.

Suppose uv and u_3v_3 are the C^2-chord edges in K_2. Since K and K_2 share exactly one edge, either u_1v_3 or u_3v_1 is a joining edge between K and K_2. The chord u_1v_3 is a forbidden short circuit chord if $u_1 \in P_2$ and is a forbidden long circuit chord if $u_1 \in P_1 \cup F_1$. If $u_1 \in B_1$ then since u is an interior vertex of F_1 the vertex v_3 is not the second vertex on F_2 and thus u_1v_3 is a forbidden between-circuit chord. Hence u_1v_3 is not a joining edge of K and K_2. By symmetry the chord u_3v_1 is forbidden and thus uv is not an edge in a decomposing circuit of C^2.

Suppose uv is such that C^2_{uv} is a contributing circuit in $\mathcal{C}_2$ and assume that uv is a backwards arc of F^2. By the construction of the accordion decomposition of C^2 there exists an edge $u_4v_4 \in B_2$ and edges u_3v, uv_3 in F_2 such that C^2 has bridge circuits K_2 and K_3 with $u_4v_4, u_3v \in E(K_2)$ and $u_4v_4, uv_3 \in E(K_3)$. If u_1v is not a forbidden chord in D then $u_1 \in B_2$ and u_3 is the tail of P_2.

Suppose $u_1v_1 \in C^1$ and first consider the case that $u_3 = u_1$. There is then a decomposing circuit K' through u_3, v_1, u, and v_4 such that K' and K_3 share the edge uv_4, and thus either u_3v_3 or u_4v_1 is a joining edge between K' and K_3. The former is a forbidden short circuit chord and the latter is a forbidden long circuit chord, and thus u_3 and u_1 are distinct, and u_1 is an internal vertex in B_1. Let K' be the decomposing circuit with edges uv and u_4v_3. Then K' and K share the edge uv and the possible joining edges between K' and K are u_4v_1 and u_1v_3. Both joining edges lead to contradictions: the chord u_4v_1 is a forbidden long circuit chord and the chord u_1v_3 is either a forbidden long circuit chord (when v_3 is the tail of F_2) or a forbidden between-circuit chord (when v_3 is internal in F_2). Thus u_1v_1 is not an edge in C^1 and it follows that u_1v_1 is a C^1-chord edge of the bridge circuit K_1 between B_1 and F_1, where the C^1-chord edges of K_1 are u_1v_1 and u_2v_2.

The decomposing circuits K_1 and K share the edge u_1v_1. and since u_2v is a forbidden long circuit chord, uv_2 is the unique joining edge between K_1 and K. This is a contradiction, since this creates a decomposing circuit through uv and $u_1v_4 \in C^1$ which we saw leads to a contradiction.

We conclude that uv is not a backwards arc of F^2, and by reversing the orientation of G we conclude that uv is not a backwards arc of B^2. Hence there is no decomposing circuit K between an edge of A and an edge of A_1. It follows that $A_1 \cup A$ is a contributing circuit whenever $A \in \mathcal{C}_2$ is distinct from A_2.

Claim 3: If $A \in \mathcal{C}_1$ is distinct from A_1, then $A \cup A_2$ is contributing.
Proof. This follows from Claim 2 by symmetry.

Claim 4: If $A \in \mathcal{C}_j$ and $j \neq 1, 2$ then either $A \cup A_1$ is contributing or $A \cup A_2$ is contributing.
Proof. Let $A \in \mathcal{C}_j$ for some $j \neq 1, 2$. If $j \neq k$ and $u_1, v_1 \in V(A_1)$ and $u_j, v_j \in V(C^j)$ then either u_1v_j is a forbidden chord or u_jv_1 is forbidden chord. Thus there is no decomposing circuit K with edges u_1v_1 and u_jv_j and therefore there is no decomposing circuit with an edge in A and an edge in A_1. It follows that $A \cup A_1$ is contributing. If $j = k$ and $u_2, v_2 \in A_2$ and $u_j, v_j \in V(C^j)$ then either u_2v_j is a forbidden chord or u_jv_2 is forbidden chord. Thus there is no decomposing circuit with an edge in A_2 and edge in A and $A \cup A_2$ is contributing.

Let $D' = \{C^F, C^B, \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ be a decomposition of D into contributing circuits. If $C \in D'$ and P_2 is not a subpath in C, then by Claims 2 and 3 either $C \cup A_1$ is contributing or $C \cup A_2$ is contributing, and thus there is a decomposition D^* of D into contributing sets $\{C'_1, C'_2, \ldots, C'_m\}$ such that $V(P_2) \subseteq V(C'_i)$ for all i and

$$V(C'_1)\Delta V(C'_2)\Delta \cdots \Delta V(C'_M) = V(D) = \emptyset.$$

This contradicts the assumption that P is minimally non-Pfaffian. Hence if P is minimally non-Pfaffian and $G(P)$ has no neutral cycles, then for some $k \leq 4$, $G(P)$ has a spanning subgraph D that is an odd weak k-double cycle.

□

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