An Exploration of Locality, Conservation Laws, and Spin

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Conservation rules are central to our understanding of the physical world, they place restrictions on how particles can move and dictate what can occur during an interaction. However, it is often taken for granted how a conservation law is implemented. For example, "conservation of momentum" is the condition that the sum of incoming and outgoing momenta equals zero. In particular, we place a constraint on the momenta by means of a *linear* function. The assumption of a linear conservation rule is intimately linked to both the geometry of momentum space and locality of the corresponding interaction. In this thesis we investigate the link between locality and conservation rules in a variety of settings.

Part 1 is principally concerned with scalar particles. We begin by constructing the interaction vertex for an arbitrary scattering process in a generic spacetime, showing that curvature is not sufficient to induce a non-local interaction. Along the way we develop a notion of covariant Fourier transform which is used to translate between spacetime and momentum space in the presence of a non-trivial geometry. We also explore the effect on quantum fields of explicitly imposing non-locality via the "Relative Locality" framework. It is found that the fields depend, implicitly, on a fixed point in momentum space with fields based at different points related by a non-local transformation. On the other hand, all non-local behavior in the action can be concentrated in the interaction term.

In the second part of this thesis we generalize the analysis of Part 1 to particles with internal structure, specifically spin. Of particular interest was the possibility that the presence of internal degrees of freedom could provide a sufficient modification of the vertex factor to allow for non-local interactions. Utilizing the coadjoint orbit method we develop a classical model of the relativistic spinning particle called the "Dual Phase Space" model (DPS) which allows for a coherent analysis of the vertex factor. We find that in addition to locality in the standard spacetime variable, interactions are "local" in a second "dual" spacetime variable. Inspired by this overt "duality" we show that DPS can be reformulated as a bilocal model. Specifically, DPS can be realized as the relativistic extension of a mechanical system consisting of two particles coupled by a rigid rod with fixed angular momentum about the center of mass. Interpreted in this way the model is easily quantized and yields the correct values for the spin quantum numbers.

Next we consider a spinorial parameterization of DPS which is entirely first class and reveals several insights into how spin affects the dynamics of a relativistic particle. In particular, we find that the spin motion acts as a Lorentz contraction on the four-velocity and that, in addition to proper time, spinning particles posses a second gauge invariant observable which we call proper angle. The notion of a “half-quantum” state is also introduced as a trajectory which violates the classical equations of motion but which does not produce an exponential suppression in the path integral. In the final chapter of the thesis we explore an extension of the Dual Phase Space Model which includes continuous spin particles. This extended model is then generalized to deSitter spacetime where we present a fully covariant parameterization of the model.

Acknowledgements

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On a more personal note, my parents, sister, and Opa have been the solid foundation that I have leaned on, any accomplishments have been made possible by their love and support. I am also grateful for the friends that I have made in Kitchener-Waterloo, I enjoyed the time spent with each one of you. Specifically, to Daniel Bartholomew-Poyser and Ian Edington, thank you.

Dedication

To Opa, whose determination and selflessness have been my inspiration.

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Chapter 1

Introduction and Overview

1.1 Conservation Laws and Locality

A conservation law expresses the invariance of a particular quantity during the evolution of a system, and determines what are considered valid physical processes. They find applications in two distinct realms: motion of a single particle and interactions between multiple particles. For an individual particle, a quantity Q is conserved if it is independent of the particles' motion, a notion which is formalized by saying that Q commutes with the Hamiltonian. In an interaction, on the other hand, Q is conserved if its value, aggregated across all particles, is the same before and after the interaction takes place. To make this idea formal let us focus on four-momentum, which is arguably the most fundamental conserved quantity. For an n-point vertex with incoming momenta p_I^μ, $\mu = 0, 1, 2, 3$ and $I = 1, 2, \ldots, n$, conservation of momentum can be expressed as

$$P^\mu(p_1, p_2, \ldots, p_n) = 0, \tag{1.1}$$

for some functions P^μ. Of course, in almost every case, we choose P^μ to be the linear function $P^\mu(p_1, p_2, \ldots, p_n) = \sum_I p_I^\mu$, but this raises an important question: Why do we reject other forms of P^μ, what makes linear conservation rules special? The answer is related to an idea at the heart of theoretical physics: locality.

Locality is the physical principle which states that interactions only occur if there is a coincidence of multiple particles at a *single* point in spacetime. This seemingly innocuous statement governs our description of interactions, of observables via the S-matrix, and most importantly implies the physical separation of different scales as embodied in the renormalization group. The assumption that spacetime localization is independent of an observers internal state, i.e. quantum numbers, is known as the hypothesis of absolute locality. "Relative Locality" [1, 2] on the other hand, allows for a violation of this hypothesis by positing that spacetime itself may be observer dependent. In this thesis we will utilize the framework of Relative Locality to explore how locality effects the interaction of fundamental probes.

One way to formalize the notion of Relative Locality is to consider momentum space, and not spacetime, as fundamental. In this framework, momentum space is permitted a non-trivial geometry and spacetime emerges as cotangent planes to points in momentum space so that each observer experiences their own, energy dependent, spacetime. Let us return to our previous example of an n-point vertex. This can be described by the sum of a free action, giving the motion of the particles coming into the vertex, and an interaction term, which defines the vertex itself. Let p_I^μ denote the momenta of the incoming particles and x^I_μ the canonically conjugate spacetime coordinates, then the free action is

$$S_{\text{free}} = \sum_I \int_0^1 d\tau \left(x^I_\mu \dot{p}_I^\mu + N_I C_I(p_I) \right), \tag{1.2}$$

where $C_I(p_I)$ is some mass-shell condition. Note that the appearance of a τ derivative on p_I^μ, as opposed to x^I_μ, reflects the choice to treat momentum space as fundamental. The interaction term is determined by conservation of momentum, however since the p_I^μ take values in a non-linear manifold we can not simply add them together, i.e. we must leave P in eq. (1.1) generic. Thus, assuming that all particles are incoming and reach the interaction vertex at $\tau = 1$, the total action is given by

$$S = \sum_I \int_0^1 d\tau \left(x^I_\mu \dot{p}_I^\mu + N_I C_I(p_I) \right) + z_\mu P^\mu \left(p_1(1), p_2(1), \ldots, p_n(1) \right), \tag{1.3}$$

where z_μ is some Lagrange multiplier which imposes the conservation law. The vertex factor can now be computed by taking the variation of S with respect to $p_I^\mu(1)$, we find

$$x^I_\mu(1) = z_\nu \frac{\partial P^\nu}{\partial p_I^\mu}, \tag{1.4}$$

where $x^I_\mu(1)$ is the spacetime coordinate of the interaction vertex for particle I. If P^μ were linear in p_I then eq. (1.4) would yield $x^I_\mu(1) = z_\mu$ and so each particle assigns the same coordinate to the vertex, i.e. the interaction is local. On the other hand, for a generic P^μ it is not necessarily true that the value of $z_\nu \partial P^\nu / \partial p_I^\mu$ is the same for different I and so each particle potentially assigns a *different* coordinate to the interaction vertex, i.e. the interaction is non-local. This shows that locality requires, at minimum, a linear conservation rule, or stated a different way, a non-linear conservation law can lead to non-local interactions.

In Part 1 of this thesis we explore the link between locality and the linearity of conservation rules for the special case of scalar particles. Chapter 2 is based on the paper [3] where we study the coupling of a particle worldline to a gravitational field and investigate its key properties. More specifically, we utilize the worldline formalism[1] to study the behavior of scalar particles undergoing an arbitrary series of interactions while propagating through an arbitrary spacetime geometry.

The chapter begins by considering the standard example of the relativistic particle in a flat spacetime since this allows us to introduce the relevant techniques and make a smooth transition to a more general geometry. We show that, for an arbitrary scattering process, locality is sufficient to generate the expected results: edge momenta is constant, momentum is conserved at each vertex and edge momenta is identical to vertex momenta.

After exhausting this familiar example we generalize the worldline action to allow for non-trivial spacetime geometries. We begin by demonstrating that edge momenta are no longer constant, instead they are carried along the worldline by parallel transport. Next, we introduce a notion of covariant Fourier transform which is then utilized in deriving the vertex factor. We find no modification from the case of flat spacetime, momentum is conserved at each vertex and edge momenta is identical to vertex momenta.

These results are then used to analyze a loop diagram in the presence of a gravitational field. We show that in a generic curved spacetime the loop momenta is entirely determined by the external momenta, presenting an intriguing approach for regulating the ultraviolet divergences which plague loop integrals in standard QFT. Finally, we argue that the semi–classical effects of quantum gravity can be accounted for by modifying the interaction vertex so as to relax strict locality. We then make a particular choice for the de-localized vertex which preserves Lorentz invariance and demonstrate, rather remarkably, that conservation of momentum at a vertex is preserved.

Chapter 3, which is based on the paper [5], considers the extension of the Relative Locality framework to scalar φ^3-theory. We begin with the generating functional for standard φ^3-theory, Fourier transform this into momentum space and extract the corresponding Feynman rules. We then deform these rules to account for the non-trivial geometry on momentum space. With modified momentum space Feynman rules in hand we write down

[1] See [4] for a review of the formalism along with a list of references

the corresponding generating functional and read off the action for our theory. The action will be written in terms of momenta and should be Fourier transformed into spacetime. As we are working in the context of a curved momentum space it will be necessary to utilize the covariant Fourier transform introduced in Chapter 2. It follows that the transformed fields depend, implicitly, on a fixed point in momentum space with fields based at different points being related by a non-local transformation. This implies that there are a continuum of quantum field theories, one for each point in momentum space, that can be patched together by a field redefinition which we derive explicitly. The transformed action is also non-local, although the kinetic term can be made local by choosing the base point to be the origin of momentum space. In the limit where the geometry of momentum space becomes trivial we recover standard φ^3 scalar field theory.

Relative Locality represents a radical departure from our usual understanding of spacetime and locality as evidenced by the modification of the interaction vertex discussed above. However, Chapter 2 demonstrates that such modifications are not present in interactions between scalar particles even when spacetime is curved. Most particles are not scalars though, they possess spin which is known (see [6]) to modify the vertex factor. Therefore, by considering interactions between spinning particles we may be able to uncover non-local effects. In Part 2 of this thesis we consider the worldline formulation of the relativistic spinning particle. In particular we focus our attention on understanding at a deeper level the interplay between spin and the notion of localization. Beyond the initial question of understanding the geometry of the interaction vertex in the presence of spin we also discover a fascinating array of relationships between spin and an extended notion of geometry.

1.2 Classical Spinning Particle

The notion of "intrinsic angular momentum" was first discussed in the context of classical general relativity by Cartan [7] in 1922. Spin, as it relates to the description of elementary particles, did not make an appearance until 1925 in the work of Goudsmit and Uhlenbeck [8], who proposed that the splitting of spectral lines in the anomalous Zeeman effect could be explained by attributing an internal angular momentum to the electron. This idea was made rigorous a few years later when Dirac [9] published his famous equation, now universally accepted as the correct quantum-mechanical description of spin-$\frac{1}{2}$ particles. Despite the success Dirac's theory has enjoyed, it offers little insight into the physical origin of spin, referred to by Pauli as a "two-valued quantum degree of freedom." Modern treatments hold to this line of thought, either claiming outright that spin has no classical interpretation [10] or avoiding the topic altogether [11]. That is not to say that attempts have not been made to understand spin from a classical perspective; the literature on the subject is vast, predating even Dirac.[2]

Classical models of spin can be roughly divided into two types: phenomenological and group theoretic. Phenomenological models were the first to appear and took as their starting point some intuition regarding the internal structure of a spinning particle. For example, Frenkel [18], Thomas [19, 20], and Kramer [21, 22] proposed that spin was represented by an antisymmetric tensor $S_{\mu\nu}$ whose interaction with the electromagnetic field $F_{\mu\nu}$ was governed by a covariant generalization of $\partial_t \vec{S} \propto \vec{S} \times \vec{B}$, the equation for precession of a magnetic moment $\vec{S}$ in a magnetic field $\vec{B}$. In contrast, Mathisson [23], Papapetrou [24, 25], and Dixion [26, 27, 28] assumed that all information about the spinning particle is contained in its stress energy tensor $T_{\mu\nu}$ with equations of motion following from conservation of energy, $\nabla_\nu T^{\mu\nu} = 0$. Others characterized a spinning particle by a point charge and dipole moment [29, 30, 31], or as a relativistic fluid [32, 33], while still others proposed semiclassical models [34, 35]. The last of these was quantized and shown to reproduce the Dirac propagator in the path integral formalism [36, 37]. This Lagrangian perspective continues to be developed today [38, 39, 40, 41].

Group theoretic models, on the other hand, connect directly with the quantum description of a spinning particle as irreducible representations of the Poincaré group. The first to attempt such a formulation were Hanson and Regge [42] and Balachandran [43, 44], both of whom assumed that the configuration space of a spinning particle was coordinatized by elements of the Poincaré group. This approach was formalized by Kirillov [45], Kostant [46], and Souriau [47, 48], who showed that the coadjoint orbits of a group form a symplectic manifold and therefore have a natural interpretation as the phase space of some classical system. Several authors [49, 50, 51, 52, 53] have utilized the coadjoint orbit method to construct classical descriptions of spin, with quantization achieved by means of the worldline formalism [54, 55].

This approach is dramatically different from the most common worldline treatment of spinning particles [56, 57, 58, 59, 60, 61], where the spin degrees of freedom are represented by Grassmann variables. The group theoretical approach has in our view the merit of conceptual clarity: it allows the spinning degrees of freedom to be parametrized by variables which possess a semiclassical interpretation while also providing a common treatment of all spins at once. Moreover, Wiegmann [52] has shown the equivalence between the Grassman

[2]For readers interested in the subject, see the review articles [12, 13, 14, 15] or the full-length books [16, 17].

variable treatment and the bosonic group theoretical approach.

In Chapter 4, which is based on the paper [62], we utilize group theoretic methods to develop a worldline description of spinning particles. We begin by introducing a novel parameterization for the phase space of a relativistic spinning particle, called the "Dual Phase Space" model (DPS). In this parametrization, the standard phase space of (x, p) is extended by a second set of canonical variables (χ, π) which span a "spin" or "dual" phase space. Here (x, p) label the standard position and momentum of the particle while (χ, π) determine the internal degrees of freedom. We describe in detail the set of constraints on this dual space that realize the relativistic spinning particle and show that interactions are local not only in x, but in the dual position space χ as well. This dual locality property is one of the main results of the Chapter. We also provide a precise formulation of the on-shell action for a spinning particle.

If one ignores the constraints, the phase space of DPS is identical to that of two scalars, suggesting the spinning particle may have a realization as a composite system. In Chapter 5, which is based on the paper [63], we show that this intuition is accurate and that DPS is equivalent to a bilocal model. The notion that elementary particles posses a finite extension has a long history, dating back to Lorentz's theory of the electron. The advent of local quantum field theory seemed to superseded these early notions, modeling elementary particles as field quanta with no internal geometry. However, in the 1950's, persistent divergences in the description of hadrons prompted Yukawa [64, 65] to reconsider these canonical ideas, showing that particles with an intrinsic extension could be modeled by means of a simple bilocal field theory. Unfortunately, these models possessed a number of undesirable features and ultimately fell out of favor when QCD realized an accurate description of hadrons as point like field quanta. Bilocal models would have been relegated to the history books were it not for the advent of another model which also emerged around this time. String theory began as an attempt to understand certain QCD processes and is by far the most studied model in which elementary particles are considered to have a finite extension. There is an intimate link between string theory and bilocal models, with several varieties of the latter being published [66, 67, 68] following the work of Yukawa. In particular, many of the aforementioned models can be viewed as restrictions on the motion of a classical string [69]. More recently bilocal models have emerged in the context of higher spin theory as a method for deriving interaction vertices [70].

Chapter 5 begins by considering a non-relativistic system of two particles, coupled by a rigid rod with a fixed angular momentum about the center of mass. As a constrained system the model is easily quantized and yields the correct values for the spin operators

$\hat{S}^2$ and $\hat{S}_3$. The relativistic extension is then considered and shown to be equivalent to the representation of spin given by the "Dual Phase Space" model (DPS). This allows results from Chapter 4 to re-interpreted in the bilocal picture, in particular we show that "dual locality" is viewed as locality at each constituent particle. The relativistic model can also be quantized and we find that spin sector behaves as in the non-relativistic case yielding the correct values for the spin quantum numbers.

The group theoretic models of classical spinning particles discussed above can be subdivided into those which parameterize the spinning degrees of freedom with vectors [52, 71, 72, 73] and those which utilize spinors [74, 75, 76, 77, 36, 78]. The Dual Phase Space model falls into the former category, and although it provides a ready physical interpretation of spin, it suffers, like its peers, from a proliferation of second class constraints. These are cumbersome and can obscure the true spinning degrees of freedom. On the other hand, models based on a spinorial parameterization do not suffer from this issue, in fact they can eliminate second class constraints entirely. A particularly notable spinor model, and one that will be important for us, is that of Lyakovich et al. [76], further developed in [79] and generalized to any dimension in [80, 81].

In Chapter 6 we present a spinorial version of the DPS model, equivalent to that of Lyakovich, in which spinors are used to resolve all second class constraints. Although the Chapter is quite technical the new parameterization provides additional insight into how spin affects the dynamics of a relativistic particle. We find that, in addition to proper time, a spinning particle possesses a second gauge invariant observable which we call proper angle. This proper angle can then be interpreted as a measure of the oscillation along the particles' classical trajectory, a phenomena known as Zitterbewegung [82]. We also show that the measure of proper time is affected by the spin motion, experiencing a Lorentz like contraction when the particle undergoes a spin transition. The precise delineation between Zitterbewegung and spin transitions is one of the Chapters' major results.

The thesis concludes with Chapter 7 which begins by considering a generalization of the "Dual Phase Space" model that allows for a description of continuous spin particles. A continuous spin particle (CSP) [83] forms one of the four distinct irreducible representations of the Poincaré group. The other three correspond to massive, massless helicity, and tachyonic particles. A modern and thorough treatment of CSP's is given by Schuster and Toro in the series of papers [84, 85, 86] where they show that a consistent gauge theory of these particles can be constructed in flat space. CSP's have also been linked to aspects of both string and higher spin theory. For example, Mourad has shown [87] that the continuous spin representation of the Poincaré group can be obtained from a classical action which has been generalized to a conformal string action. In addition, Bekaert and Mourad show in [88] that Wigner's equations for describing continuous spin particles can be obtained as the limit of the equations for massive higher spin particles. For the purpose of this thesis, the most relevant property of CSP's is their ability, as shown by Schuster and Toro, to mediate long range forces. This opens the possibility that CSP's are a heretofore unexplored dark matter candidate. Unfortunately, facilitating such an exploration would require an understanding of how CSP's couple to gravity and to date this has proven difficult to obtain.

This shortcoming is addressed in the remainder of Chapter 7 where we utilize the dual phase space formalism to develop a consistent theory of CSP's in deSitter. In fact we generalize the entire Dual Phase Space model to a curved background, but it is the inclusion of continuous spin particles that makes this generalization so challenging. A similar programme was proposed in [89] but that model can not accommodate CSP's. The process proceeds in stages. First we consider the irreducible representations of the deSitter symmetry group $SO(4,1)$ and then restrict our attention to those representations which contract to a well known irrep of the Poincaré group. It is assumed that the physical interpretation of the latter can be assigned to the former which allows us to classify irreps of $SO(4,1)$ into particle types. Next, we utilize the dual phase space paradigm to parameterize the generators of the deSitter group in 5-dimensional embedding space. This initial parameterization has no physical significance and so in subsequent sections we develop a four-dimensional fully covariant version of the model.

Part I

Interaction Vertex for a Scalar Particle

Chapter 2

Worldline Formalism in General Relativity

In this Chapter we explore the connection between locality and the linearity of the interaction vertex for scalar particles propagating in a generic spacetime. A similar investigation in the context of relative locality has been conducted by J. Kowalski–Glikman et al. [90].

2.1 Worldline Action in Minkowski Space

Consider a particle of mass m propagating in a spacetime with flat Minkowski metric η_{ab} and having a worldline given by $X^a(\tau)$, for some parameter τ. The motion of such a particle is governed by the action

$$S[e, X] = \frac{1}{2} \int d\tau \left(\frac{1}{e} \dot{X}^a \dot{X}^b \eta_{ab} - em^2 \right), \tag{2.1}$$

where $\dot{X}^a = dX^a/d\tau$ and $-e^2(\tau)$ is the metric along the worldline. Under a change in parametrization $\tau \to s(\tau)$ we have $e(\tau) \to \tilde{e}(s) = (d\tau/ds)e(\tau)$ and so the worldline metric ensures that $S[e, X]$ is invariant under such re-parameterizations.

It will prove convenient to re-write this action in-terms of the momentum conjugate to $X^a(\tau)$, which we easily calculate to be

$$P_a = \frac{\partial \mathcal{L}}{\partial \dot{X}^a} = \frac{1}{e} \dot{X}^b \eta_{ab}.$$

Taking the variation of $S[e, X]$ with respect to e gives the constraint $X^2/e^2 + m^2 = 0$, and upon substituting for P_a we obtain the standard mass-shell condition

$$P^2 + m^2 = 0. \tag{2.2}$$

A brief calculation shows that the Hamiltonian for this system is $\mathcal{H} = e(P^2 + m^2)/2$; an inverse Legendre transform then gives $\mathcal{L} = \dot{X}^a P_a - \mathcal{H}$ as the Lagrangian. Noting that the action eq. (2.1) is just the time integral of the Lagrangian we find

$$S[X, P, e] = \int_0^1 d\tau \left(\dot{X}^a P_a - \frac{e}{2}(P^2 + m^2) \right). \tag{2.3}$$

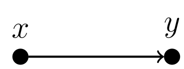
In this formulation the worldline metric behaves like a Lagrange multiplier that imposes the mass shell constraint eq. (2.2). It is conventional to re-label the worldline metric as the lapse function, $e(\tau) = N(\tau)$, so that eq. (2.3) becomes

$$S[X, P, N] = \int_0^1 d\tau \left[\dot{X}^a(\tau) P_a(\tau) - \frac{N(\tau)}{2} \left(P^2(\tau) + m^2 \right) \right]. \tag{2.4}$$

Suppose that the worldline of the particle satisfies $X^a(0) = x^a$ and $X^a(1) = y^a$, i.e. the worldline begins at the point x and terminates at the point y. The amplitude for propagating from x to y is then obtained by taking the path integral of the exponential of the action eq. (2.4), viz

$$G(x, y) = \int \mathcal{D}X \mathcal{D}P \mathcal{D}N \exp \left\{ i \int_0^1 d\tau \left[\dot{X}^a(\tau) P_a(\tau) - \frac{N(\tau)}{2} \left(P^2(\tau) + m^2 \right) \right] \right\}. \tag{2.5}$$

$G(x, y)$ is simply the propagator for the theory and so we will represent it graphically in the usual way

$G(x, y) =$ 

,

where the arrow indicates the direction of momentum flow.

2.2 Propagation Amplitude for an Arbitrary Process

Consider a process in which n_i initial state particles undergo a series of interactions to produce n_f final state particles. No restriction is placed on the number of particles participating in a given interaction, we demand only locality, i.e. interacting particles occupy a single point in spacetime. This evolution can be represented by an oriented graph Γ

in which the edges, labeled e_i, represent the worldlines of the particles and the vertices, labeled v_i, represent interactions. An example with $n_i = 1$ and $n_f = 2$ is given in Figure 2.1.[1]

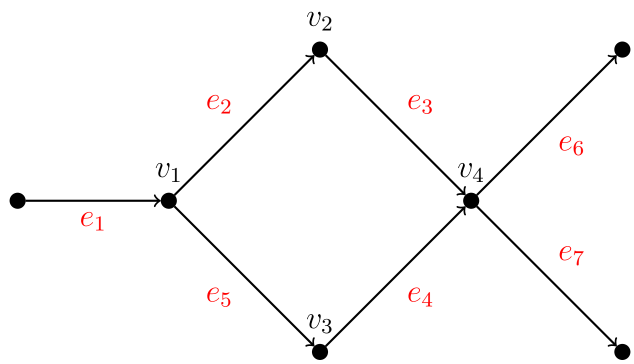


Figure 2.1: A possible graph, Γ, with $n_i = 1$ and $n_f = 2$.

Let $X_e(\tau)$ denote the worldline of a particle propagating along the edge e and $P_e(\tau)$ the momentum it carries. If we reverse the orientation of an edge, $e \to -e$, then $X_e(\tau) = X_{-e}(1-\tau)$ since $X_{-e}(\tau)$ traverses the same path as $X_e(\tau)$ only backwards. Similarly, $P_e(\tau) = -P_{-e}(1-\tau)$, where the overall minus sign takes into account that the direction of momentum flow has be reversed. We will adopt the notation $x_e \equiv X_e(0)$ and $x_{-e} \equiv X_{-e}(0) = X_e(1)$ for the endpoints of the edge e while x_e^{in} and x_{-e}^{out} will denote the coordinates of the initial and final state particles respectively. The amplitude for the graph Γ, denoted $I_\Gamma(x_e^{\text{in}}, x_{-e}^{\text{out}})$, is constructed as follows:

1. Introduce vertex coordinates z_v.

2. Assign a propagator to each edge e and form their product.

3. Integrate over the fiducial coordinates z_v.

Implementing this procedure yields

$$I_\Gamma(x_e^{\text{in}}, x_{-e}^{\text{out}}) = \int \prod_{v\in\Gamma} d^4 z_v \prod_{\text{initial } e} G_e\left(x_e, z_{e,t}\right) \prod_{\text{internal } e} G_e\left(z_{e,s}, z_{e,t}\right) \prod_{\text{final } e} G_e\left(z_{e,s}, x_{-e}\right), \quad (2.6)$$

[1] To emphasize, a vertex is a point having both incoming *and* outgoing momentum. Therefore, where an initial edge originates and where a final edge terminates are not considered vertices.

where $z_{e,s}$ and $z_{e,t}$ are, respectively, the coordinates of the sourcing and terminating vertex of the edge e. The requirement that interactions occur at a single point in spacetime can be made explicit by extracting a delta function for each vertex and re-writing eq. (2.6) as

$$I_\Gamma = \int \prod_{v\in\Gamma} d^4 z_v \prod_{v\in\Gamma} \prod_{\substack{s_e=v\\ t_e=v}} d^4 x_e \delta^{(4)}\left(x_e - z_v\right) \prod_{e\in\Gamma} G_e\left(x_e, x_{-e}\right), \tag{2.7}$$

where the product $\prod_{\substack{s_e=v\\ t_e=v}}$ is taken over all edges sourcing (s_e) from v and terminating (t_e) at v with the latter having their orientation reversed. For example, referring to Figure 2.1 we have

$$\prod_{\substack{s_e=v_4\\ t_e=v_4}} d^4 x_e = d^4 x_{-e_3} d^4 x_{-e_4} d^4 x_{e_6} d^4 x_{e_7}.$$

This type of product will appear repeatedly and it will be convenient to introduce the notation

$$\prod_{v\in\Gamma} \prod_{\substack{s_e=v\\ t_e=v}} \equiv \prod_{v,e}.$$

Returning to our expression for I_Γ in equation eq. (2.7) we take the Fourier transform of the delta functions and expand the G_e using equation eq. (2.5). The result is

$$I_\Gamma = \int \prod_{v\in\Gamma} d^4 z_v \prod_{v,e} d^4 x_e \frac{d^4 p_e}{(2\pi)^4} \prod_{e\in\Gamma} \mathcal{D}a_e \exp\left(-iS_\Gamma\right), \tag{2.8}$$

where $\mathcal{D}a_e = \mathcal{D}X_e \mathcal{D}P_e \mathcal{D}N_e$ and

$$S_\Gamma = -\sum_{e\in\Gamma} \int_0^1 d\tau \left[\dot{X}_e \cdot P_e - N_e\left(P_e^2 + m_e^2\right)\right] + \sum_{v,e} p_e \cdot (x_e - z_v). \tag{2.9}$$

The coordinates, p_e, employed in the Fourier transform are dual to the vertex coordinates z_v, a relationship which suggests the designation "vertex momentum" for the p_e. This should be contrasted with the $P_e(\tau)$ which are dual to the worldline coordinates $X_e(\tau)$ and referred to as edge momenta.

To obtain the equations of motion for this system, and in particular the vertex factor, we simply take the variation of S_Γ:

$$\delta S_\Gamma = \sum_{e\in\Gamma} \int_0^1 d\tau \left[\delta X_e^a \dot{P}_{a,e} - \dot{X}_e^a \delta P_{a,e} - \delta N_e\left(P_e^2 + m_e^2\right) + N_e P_e^a \delta P_{a,e}\right] \\ + \sum_{v,e} \left[(\delta p_{a,e})(x_e^a - z_v^a) + p_{a,e}\delta x_e^a - p_{a,e}\delta z_v^a - P_{a,e}(0)\delta x_e^a\right],$$

where we have assumed that $\delta X_e(0) = 0$ and $\delta X_e(1) = 0$ for incoming and outgoing edges respectively. Setting the variations along the worldline to zero we obtain

$$\dot{P}_{a,e} = 0 \qquad \dot{X}_e^a = N_e \eta^{ab} P_{b,e} \qquad P_e^2 + m_e^2 = 0, \tag{2.10}$$

which hold for all $e \in \Gamma$. The interpretation is standard; momentum is conserved along a linear worldline and the mass-shell condition is satisfied. Turning now to the variations at the vertices we find

$$x_e^a = z_v^a \qquad \forall v \in \Gamma, \tag{2.11}$$

$$p_{a,e} = P_{a,e}(0) \qquad \forall e \in \Gamma, \tag{2.12}$$

$$\sum_{\substack{s_e=v\\t_e=v}} p_{a,e} = 0 \qquad \forall v \in \Gamma, \tag{2.13}$$

and it is assumed that if x_e and z_v appear in the same equation the edge e innervates the vertex v. eq. (2.11) can be easily recognized as the locality condition; all interactions must occur at a single point in spacetime. The subsequent equation relates the vertex momenta to the edge momenta, and noting that the edge momenta is conserved we obtain

$$P_e(0) - P_e(1) = p_e + p_{-e} = 0,$$

where $P_e(1) = -P_{-e}(0)$ was used in the second equality. The locality condition can be combined with this relation and the expression for $\dot{X}_e$ in eq. (2.10) to relate the vertex momenta to a difference in position, viz

$$z_{t_e} - z_{s_e} = \tau_e p_e, \qquad \tau_e \equiv \int_0^1 N_e(\tau) \mathrm{d}\tau. \tag{2.14}$$

The interpretation of the final equation, eq. (2.13), is immediate when combined with equation eq. (2.12), we find

$$\sum_{\substack{s_e=v\\t_e=v}} P_{a,e}(0) = 0, \tag{2.15}$$

which expresses the conservation of edge momentum at each vertex. Having exhausted this simplest example we now consider the case where the geometry of spacetime is non-trivial.

2.3 Worldline Action in Curved Spacetime

Recall the form of the worldline action for a particle propagating in flat spacetime

$$S[X, P, N] = \int d\tau \left[\dot{X}^a(\tau) P_a(\tau) - \frac{N(\tau)}{2} \left(P_a(\tau) P_b(\tau) \eta^{ab} + m^2 \right) \right].$$

We now suppose that $X^a \mapsto X^\mu$ takes values in some generic manifold $\mathcal{M}$ with metric $g_{\mu\nu} = e^a_\mu e^b_\nu \eta^{ab}$. The momentum conjugate to X^μ, say P_μ, takes values in $T_{X(\tau)}\mathcal{M}$, but for convenience we write it in-terms of the flat momentum P_a as $P_\mu = e^a_\mu(X)P_a$. Making the additional replacement $\eta^{ab} \to g^{\mu\nu}$ in the mass shell term we obtain the action

$$S[X,P,N] = \int d\tau \left[\dot{X}^\mu e^a_\mu P_a - \frac{N}{2}\left(P_a P_b \eta^{ab} + m^2\right)\right], \tag{2.16}$$

where we have used $g^{\mu\nu} e^a_\mu e^b_\nu = \eta^{ab}$. To demonstrate that this action is reasonable we will now calculate the equations of motion for X^μ and P_a:

$$\delta S = \int d\tau \Big[-\frac{d}{d\tau}\left(e^a_\mu P_a\right)\delta X^\mu + \dot{X}^\mu e^a_{\mu,\nu} P_a \delta X^\nu + \dot{X}^\mu e^a_\mu \delta P_a - \frac{1}{2}\delta N\left(P_a P_b \eta^{ab} + m^2\right) - N P_b \eta^{ab} \delta P_a \Big].$$

Setting the variations to zero we find

$$-\frac{d}{d\tau}\left(e^a_\mu P_a\right) + e^a_{\nu,\mu} P_a \dot{X}^\nu = 0, \tag{2.17}$$

$$\dot{X}^\mu e^a_\mu - N P_b \eta^{ab} = 0, \tag{2.18}$$

$$P_a P_b \eta^{ab} + m^2 = 0. \tag{2.19}$$

The second equation can be solved for P_a, and after changing variables to proper time $ds = N d\tau$ we obtain

$$P_a = \eta_{ab} e^b_\mu \partial_s X^\mu. \tag{2.20}$$

Substituting this relation into eq. (2.17) gives the evolution equation for X^μ:

$$\partial_s\left(e^a_\mu e^b_\nu \eta_{ab} \partial_s X^\nu\right) - \eta_{ab} e^a_{\nu,\mu} e^b_\alpha \partial_s X^\nu \partial_s X^\alpha = 0. \tag{2.21}$$

The product of tetrads in the second term can be symmetrized over (α,ν) and re-written as, $\partial_\mu(\eta_{ab} e^a_\nu e^b_\alpha)\dot{X}^\nu \dot{X}^\alpha/2$. Making the replacement $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$ in eq. (2.21) then gives

$$\partial_s^2 X^\rho + \frac{1}{2} g^{\rho\mu}\left(g_{\mu\nu,\alpha} + g_{\mu\alpha,\nu} - g_{\nu\alpha,\mu}\right)\partial_s X^\nu \partial_s X^\alpha = 0,$$

which is just the geodesic equation. This is what we expected, free particles in a curved spacetime obey the geodesic equation. It is also enlightening to write the evolution equation eq. (2.17) in terms of P_a as

$$\frac{d}{ds} P_a = e^\mu_a (e^b_{\nu,\mu} - e^b_{\mu,\nu}) P_b \partial_s X^\nu. \tag{2.22}$$

Noting that the above equation is antisymmetric in μ, ν, we obtain that $P^a P_a$ is conserved along the worldline and so the mass-shell constraint is satisfied if it is satisfied initially. Introducing the spin connection, we can write this in the more compact form

$$\frac{d}{ds}P_a - \partial_s X^\mu \omega_\mu{}^b{}_a P_b = 0, \qquad \omega_\mu{}^b{}_a \equiv -(\nabla_\mu e^b_\nu) e^\nu_a. \tag{2.23}$$

Equations of this type can be solved by iterative integration, viz

$$P_a(s) = P_b(0) U(s)^b{}_a, \qquad U(s) = \overrightarrow{\exp} \int_0^s \mathrm{d}\tau \dot{X}^\mu \omega_\mu(X(\tau)), \tag{2.24}$$

where $U(s)$ is the parallel transport operator along the geodesic to which P_a is dual.

Returning to our expression for the worldline action in curved spacetime, eq. (2.16), it follows that the amplitude for a particle to propagate from the point x to the point y is given by

$$\mathcal{G}(x,y) = \int \mathcal{D}X\mathcal{D}P\mathcal{D}N \exp\left(i \int_0^1 d\tau \left[\dot{X}^\mu e^a_\mu(X) P_a - N\left(P_a P_b \eta^{ab} + m^2\right)\right]\right).$$

The development now proceeds as in the previous section. We consider an arbitrary process in which n_i initial state particles undergo a series of interactions to produce n_f final state particles. No restrictions are placed on these interactions other than demanding locality. The process is represented by an oriented graph Γ with a corresponding amplitude given by

$$\mathcal{I}_\Gamma(x^{\text{in}}_e, x^{\text{out}}_{-e}) = \int \prod_{v\in\Gamma} d\mu(z_v) \prod_{\text{initial } e} \mathcal{G}_e\left(x_e, z_{e,t}\right) \prod_{\text{internal } e} \mathcal{G}_e\left(z_{e,s}, z_{e,t}\right) \prod_{\text{final } e} \mathcal{G}_e\left(z_{e,s}, x_{-e}\right),$$

where $d\mu(z_v) = \sqrt{g(z_v)} d^4 z_v$ is the covariant measure on spacetime.

To obtain the form of the interaction vertex we would like to follow the same procedure as in flat spacetime:

1. Make locality explicit by extracting a delta function for each vertex
2. Fourier transform the delta functions
3. Take the variation of the resulting action

Unfortunately the second step in this sequence presents a major impediment. The standard Fourier transform does not respect diffeomorphism invariance and threfore its naive application would break the general covariance of the amplitude $\mathcal{I}_\Gamma$.To proceed we need to define a generalization of the Fourier transform which *does* preserve diffeomorphism covariance; it is to the development of such a "covariant Fourier transform" that we turn in the subsequent section.

2.4 Covariant Fourier Transform

The standard Fourier kernel is given by $\exp(ix \cdot p)$ and manifestly breaks diffeomorphism covariance. In particular, since x^μ transforms as a coordinate and not a contravariant vector, its contraction with the covariant vector p_μ does not transform as a scalar. It is, therefore, the failure of x^μ to behave contravariantly which destroys the covariance of the Fourier kernel. As such, we need a generalization of x^μ which does transform properly, we can then use this generalized "coordinate" to construct a covariant Fourier kernel. The first step is to introduce Synge's world function.

2.4.1 Synge's World-Function

Let $\mathcal{M}$ be endowed with a metric $g_{\mu\nu}$ and a torsionless metric compatible connection $\Gamma^\mu_{\nu\rho}$. Given two points $x, x' \in \mathcal{M}$ connected by a geodesic $\gamma_{xx'}$ Synge's world function [91] is defined as

$$\sigma(x,x') \equiv \frac{1}{2}\left(s'-s\right)^2, \tag{2.25}$$

where $s'-s$ is the arc-length between x and x' as determined by $\gamma_{xx'}$. This definition makes clear that the world function is symmetric upon interchange of its arguments and transforms as a scalar with respect to both x and x'.

Let $\xi^\mu(\lambda)$ be an affine parametrization of $\gamma_{xx'}$ so that the geodesic can be described by the Lagrangian

$$L = \frac{1}{2} g_{\mu\nu} \frac{d\xi^\mu}{d\lambda}\frac{d\xi^\nu}{d\lambda}.$$

Let λ_0 and λ_1 satisfy $\xi(\lambda_0) = x$ and $\xi(\lambda_1) = x'$, then

$$s'-s = \int_s^{s'} ds = \int_s^{s'} \sqrt{g_{\mu\nu} d\xi^\mu d\xi^\nu} = \int_{\lambda_0}^{\lambda_1} \sqrt{2L}\, d\lambda = (\lambda_1-\lambda_0)\sqrt{2L}, \tag{2.26}$$

the final equality follows by noting that L is constant along an affinely parametrized geodesic. Substituting this result into the definition of the world function we obtain

$$\sigma(x,x') = L(\lambda_1-\lambda_0)^2 = (\lambda_1-\lambda_0)\int_{\lambda_0}^{\lambda_1} L\, d\lambda = (\lambda_1-\lambda_0) S(x',\lambda_1; x, \lambda_0), \tag{2.27}$$

where S is Hamilton's principle function. The covariant derivatives of $\sigma(x,x')$ can now be calculated by means of the Hamilton-Jacobi equation; taking the covariant derivative at x we find

$$\sigma_{;\mu} = \sigma_{,\mu} = (\lambda_1-\lambda_0)\frac{\partial S}{\partial x^\mu} = (\lambda_1-\lambda_0)\frac{\partial L}{\partial \dot{x}^\mu} = (\lambda_1-\lambda_0) g_{\mu\nu}\dot{x}^\nu, \tag{2.28}$$

where $\dot{x}^\mu = d\xi^\mu/d\lambda|_{\lambda=\lambda_0}$. A short calculation yields the equation of motion

$$\frac{1}{2}g^{\mu\nu}\sigma_{;\mu}\sigma_{;\nu} = \sigma. \tag{2.29}$$

Swapping the roles of x and x' gives similar expressions for the covariant derivative of the worldfunction at x'

$$\sigma_{;\mu'} = -(\lambda_1 - \lambda_0)g_{\mu'\nu'}\dot{x}^{\nu'} \tag{2.30}$$

$$\frac{1}{2}g^{\mu'\nu'}\sigma_{;\mu'}\sigma_{;\nu'} = \sigma. \tag{2.31}$$

It should be noted that the indices on a tensor indicate the point at which it is evaluated, for example $g_{\mu'\nu'} = g_{\mu'\nu'}(x')$. We can generate implicit expressions for higher derivatives of the world function by differentiating eq. (2.29) and eq. (2.31) repeatedly. In particular, taking one additional derivative we find

$$\sigma^\mu{}_\nu\sigma_\mu = \sigma_\nu \qquad \sigma^{\mu'}{}_{\nu'}\sigma_{\mu'} = \sigma_{\nu'} \tag{2.32}$$

$$\sigma^\mu{}_{\nu'}\sigma_\mu = \sigma_{\nu'} \qquad \sigma^{\mu'}{}_\nu\sigma_{\mu'} = \sigma_\nu, \tag{2.33}$$

where we have omitted the semicolon to simplify notation and will continue to do so. Although these equations were easy to derive they are quite significant. eq. (2.32) demonstrates that the second order derivative of the world function at x or x' behaves like a Kronecker delta when acting on σ_μ or $\sigma_{\mu'}$, respectively. On the other hand eq. (2.33) shows, see Appendix A, that up to a sign the second order mixed derivative of σ behaves like the parallel propagator when acting on σ_μ or $\sigma_{\nu'}$.
One can also examine the behavior of the world-function (and its derivatives) as $x \to x'$ or vice versa. This is known as the "coincidence limit" and is indicated by square brackets, $[\ldots]$; e.g. $[\sigma] = 0$. Besides this rather obvious one, the most common coincidence limits are given by

$$[\sigma_\mu] = [\sigma_{\mu'}] = 0 \tag{2.34}$$

$$[\sigma_{\mu\nu}] = [\sigma_{\mu'\nu'}] = -[\sigma_{\mu\nu'}] = g_{\mu\nu}. \tag{2.35}$$

The coincidence limit will not be of great importance so we refer the reader to [91] for a complete discussion.

The covariant derivatives of $\sigma(x, x')$, being the derivatives of a bi–scalar, behave as contravariant vectors. In particular, $\sigma^\mu(x, x')$ transforms as a scalar at x' and a contravariant vector at x, and vice versa for $\sigma^{\mu'}(x, x')$. Therefore, if $p_{\mu'} \in T^*_{x'}\mathcal{M}$ then $p_{\mu'}\sigma^{\mu'}(x, x')$

transforms as a scalar at both x and x' and so a natural definition of the covariant Fourier kernel is $\exp(ip_{\mu'}\sigma^{\mu'}(x,x'))$.

Before we continue there are some technical issues regarding the domain of the world-function which need to be discussed. Fix the point $x' \in \mathcal{M}$. The definition of $\sigma(x,x')$ assumes the existence of a *unique* geodesic connecting x to x'; a condition which is not, in general, satisfied for two arbitrary points in $\mathcal{M}$. To ensure the world-function remains single valued we need to restrict its domain to a "normal convex neighbourhood" of x', denoted $C_{x'}$. More specifically, $C_{x'}$ is a subset of $\mathcal{M}$ containing x' such that, given another point $x \in C_{x'}$ there exists a unique geodesic, completely contained in $C_{x'}$, connecting x and x'.[2]

2.4.2 Van-Vleck Morette Determinant

Consider the change of variables $x^\mu \to Y'^\mu = \sigma^{\mu'}(x,x')$, where $Y' \in T^*_{x'}\mathcal{M}$ and $g^{-1}Y' \in T_{x'}\mathcal{M}$ is the initial velocity vector of the geodesic going from x to x'. It has Jacobian given by

$$\mathrm{d}^4Y' = \left|\det\left(\sigma^{\mu\nu'}(x,x')\right)\right| \mathrm{d}^4x$$

The Van-Vleck Morette determinant [93],[94],[95] is the bi-scalar obtained from this Jacobian through multiplication by the metric determinant, in particular

$$\mathcal{V}(x,x') \equiv \frac{\left|\det\left(\sigma^{\mu\nu'}(x,x')\right)\right|}{\sqrt{g}_{x'}\sqrt{g}_x}. \tag{2.36}$$

It appears naturally in the symplectic measure when we go from the symplectic coordinates (Y',x') to the end point coordinates (x,x') as

$$\mathrm{d}^4Y' \wedge \mathrm{d}^4x' = \mathcal{V}(x,x')(\sqrt{g}_{x'}\mathrm{d}^4x') \wedge (\sqrt{g}_x\mathrm{d}^4x). \tag{2.37}$$

Note that the change of coordinates $Y' \to x = \exp_{x'}(g^{-1}Y')$ from $T^*_{x'}\mathcal{M}$ to $\mathcal{M}$, is the translated exponential map, and so the inverse Van-Vleck Morette determinant is the Jacobian for this transformation:

$$(\sqrt{g}_x\mathrm{d}^4x) = \mathcal{V}^{-1}(x,x')\left(\frac{\mathrm{d}^4Y'}{\sqrt{g}_{x'}}\right), \tag{2.38}$$

[2]The existence of such a neighborhood for any $x' \in \mathcal{M}$ is guaranteed by Whiteheads theorem [92].

which highlights an important property of the Van-Vleck Morette determinant. If $x \in \mathcal{M}$ is such that $\mathcal{V}^{-1}(x, x') = 0$ then a change in Y' produces no change in x which is equivalent to making a change in the geodesic emanating from x' but no change in the point at which the geodesic terminates; i.e. x is a caustic[3]. The reverse situation, where $\mathcal{V}(x, x') = 0$, is impossible since one cannot change the terminating point of a geodesic (x) without altering the geodesics tangent vector at the sourcing point (Y'). Therefore, while the Van-Vleck Morette determinant is non-zero for all $x \in \mathcal{M}$ it does diverge at caustics. As a final note we observe that $\mathcal{V}(x, x')$ satisfies

$$[\mathcal{V}] = 1. \tag{2.39}$$

2.4.3 Implementing the Fourier Transform

Heuristically, we expect the covariant Fourier transform to take functions on $\mathcal{M}$ and map them to functions on $T^*_{x'}\mathcal{M}$. It is natural then to introduce the notation

$$\mathcal{M}_{x'} \equiv T^*_{x'}\mathcal{M}, \tag{2.40}$$

which express that the cotangent plane at x' acts as a "spacetime" at x' for the Fourier transform. To formalize this initial expectation we fix a point $x' \in \mathcal{M}$ and choose a normal convex neighborhood $C_{x'}$ as the domain of $\sigma(x, x')$. The measures on $\mathcal{M}$ and $\mathcal{M}_{x'}$ are given by

$$d\mu(x) = \sqrt{g}_x \mathrm{d}^4 x, \qquad d\nu_{x'}(p) = g_{x'}^{-1/2} \mathrm{d}^4 p,$$

respectively. Let $\mathcal{L}^2_\mu(C_{x'})$ denote the space of all functions on $\mathcal{M}$ which are square integrable with respect to $d\mu$ and vanish outside of $C_{x'}$. The covariant Fourier transform (see [96, 97] for earlier implementation of this object in a different context) is the map, $\mathcal{F}_{x'}$, given by

$$\begin{aligned} \mathcal{F}_{x'} : \mathcal{L}^2_\mu(C_{x'}) &\to \mathcal{L}^2_{\nu_{x'}}(\mathcal{M}_{x'}) \\ f(x) &\mapsto \hat{f}_{x'}(p), \end{aligned}$$

where

$$\hat{f}_{x'}(p) \equiv \int_{C_{x'}} d\mu(x) \mathcal{V}^{1/2}(x, x') \exp\left(-ip_{\mu'}\sigma^{\mu'}(x, x')\right) f(x). \tag{2.41}$$

[3]Recall that Y' is the tangent vector to the geodesic emanating from x'.

Unless $C_{x'} = \mathcal{M}$, the covariant Fourier transform is not surjective and therefore is not invertible on all of $\mathcal{L}^2_{\nu_{x'}}(\mathcal{M}_{x'})$. This difficulty can be circumvented by restricting to the image of $\mathcal{F}_{x'}$, i.e. $\hat{f}_{x'}(p) \in \mathcal{F}_{x'}(\mathcal{L}_{\nu_{x'}}(C_{x'}))$, which allows us to define the inverse Fourier transform as

$$\mathcal{F}^{-1}_{x'}(\hat{f}_{x'})(x) \equiv \int_{\mathcal{M}_{x'}} d\nu_{x'}(p)\mathcal{V}^{1/2}(x,x') \exp\left(ip_{\mu'}\sigma_{\mu'}(x,x')\right) \hat{f}_{x'}(p), \tag{2.42}$$

for $x \in C_{x'}$ and zero otherwise. Notice that the Fourier transform of a function $\hat{f}_{x'}(p) = \mathcal{F}_{x'}(f(x))(p)$ depends on the choice of base point x'. One does not, therefore, obtain a single Fourier transform but rather a continuum as the base point x' varies throughout $\mathcal{M}$.

As an initial application of this formalism consider the Fourier representation of $\delta(x,y)$, the delta function on $\mathcal{M}$. Assuming $x, y \in C_{x'}$ we posit

$$\delta(x,y) \equiv \int d\nu_{x'}(p)\mathcal{V}^{1/2}(x,x')\mathcal{V}^{1/2}(y,x') \exp\left[ip_{\mu'}\left(\sigma^{\mu'}(x,x') - \sigma^{\mu'}(y,x')\right)\right]. \tag{2.43}$$

This formula is explicitly verified in Appendix B but we note here that the proof depends crucially on the fact, left implicit in the above formula, that the integral is taken over all of $\mathcal{M}_{x'}$. This formulation of the delta function emphasizes the symmetry between x and y, but observing that $\mathcal{V}^{1/2}(x,x')\mathcal{V}^{1/2}(y,x')$ can be factored out of the integral allows us to write

$$\delta(x,y) = \int d\nu_{x'}(p)\mathcal{V}(y,x') \exp\left[ip_{\mu'}\left(\sigma^{\mu'}(x,x') - \sigma^{\mu'}(y,x')\right)\right]. \tag{2.44}$$

In the sequel we will be particularly interested in the special case $y = x'$ for which the delta function becomes

$$\delta(x,x') = \int d\nu_{x'}(p) \exp\left[ip_{\mu'}\sigma^{\mu'}(x,x')\right]. \tag{2.45}$$

A Fourier representation of the delta function on $\mathcal{M}_{x'}$, denoted $\delta_{x'}(p,q)$, can be defined by putting

$$\delta_{x'}(p,q) = \int_{C_{x'}} d\mu(x)\mathcal{V}(x,x') \exp\left[i\left(p_{\mu'} - q_{\mu'}\right)\sigma^{\mu'}(x,x')\right]. \tag{2.46}$$

Note that this is not the the usual delta function unless $C_{x'} = \mathcal{M}$. It is, however, a projector under convolution

$$\delta_{x'}(p,q) = \int_{\mathcal{M}_{x'}} d\nu_{x'}(k)\delta_{x'}(p,k)\delta_{x'}(k,q), \tag{2.47}$$

and as such acts as an identity on the image of the Fourier transform, i.e. on $\mathcal{F}_{x'}(\mathcal{L}_{\nu_{x'}}(C_{x'}))$. A proof that these properties hold is given in Appendix B. Note that a mathematical study of a generalized Fourier transformation in the context of non-commutative SU(2) field theory is presented in [98].

2.5 Interaction Vertex in Curved Spacetime

Having concluded our development of the covariant Fourier transform we are now prepared to continue with the programme suggested at the conclusion of Section 2.3.

2.5.1 Implementing Localization

Recall our set-up: An arbitrary process is represented by an oriented graph Γ with local interactions and relevant amplitude given by

$$\mathcal{I}_\Gamma(x_e^{\text{in}}, x_{-e}^{\text{out}}) = \int \prod_{v\in\Gamma} d\mu(z_v) \prod_{\text{initial } e} \mathcal{G}_e\left(x_e, z_{e,t}\right) \prod_{\text{final } e} \mathcal{G}_e\left(z_{e,s}, x_{-e}\right) \prod_{\text{internal } e} \mathcal{G}_e\left(z_{e,s}, z_{e,t}\right). \tag{2.48}$$

Make locality explicit by extracting a delta function for each edge sourcing or terminating at a vertex

$$\mathcal{I}_\Gamma(x_e^{\text{in}}, x_{-e}^{\text{out}}) = \int \prod_{v\in\Gamma} d\mu(z_v) \prod_{v,e} d\mu(x_e)\delta(x_e, z_v) \prod_{e\in\Gamma} G_e\left(x_e, x_{-e}\right). \tag{2.49}$$

Define $\widetilde{\mathcal{I}}_\Gamma$ to be the quantity obtained from $\mathcal{I}_\Gamma$ by dropping the vertex integrals and fixing the z_v to be *distinct* points in spacetime, then eq. (2.43) gives

$$\begin{aligned}
\widetilde{\mathcal{I}}_\Gamma(x_e^{\text{in}}, x_{-e}^{\text{out}}, z_v) &= \int \prod_{v,e} d\mu(x_e)\delta\left(x_e, z_v\right) \prod_{e\in\Gamma} G_e\left(x_e, x_{-e}\right) \\
&= \int \prod_{v,e} d\mu(x_e) d^4p_e \exp\left(-ip_{a,e}e^a_{\mu_v}(z_v)\sigma^{\mu_v}(x_e, z_v)\right) \prod_{e\in\Gamma} G_e\left(x_e, x_{-e}\right) \\
&= \int \prod_{v,e} d\mu(x_e) d^4p_e \prod_{e\in\Gamma} \mathcal{D}\mu_e \exp(-iS_\Gamma),
\end{aligned}$$

where the action S_Γ is given by

$$S_\Gamma = -\sum_{e\in\Gamma} \int_0^1 d\tau \left[\dot{X}_e^\mu P_{a,e} e^a_\mu - N_e\left(P_{a,e}P_{b,e}\eta^{ab} + m_e^2\right)\right] + \sum_{v,e} p_{a,e}e^a_{\mu_v}\sigma^{\mu_v}(x_e, z_v). \tag{2.50}$$

As in the case of flat spacetime we obtain the vertex factor, along with the kinematical equations of motion, by taking the variation of S_Γ. The equations describing the free evolution of a particle were derived earlier (see eqs. (2.17)–(2.19)) and shown to be consistent with the geodesic equation. As such, we can focus on variations at the vertices which are found to be[4]

$$\sum_{v,e} \big[e^a_{\mu_v}(z_v)\sigma^{\mu_v}(x_e,z_v)\delta p_{a,e} + p_{a,e}e^a_{\mu_v,\nu_v}(z_v)\sigma^{\mu_v}(x_e,z_v)\delta z_v^{\nu_v} + p_{a,e}e^a_{\mu_v}(z_v)\sigma^{\mu_v}{}_{\nu}(x_e,z_v)\delta x_e^{\nu}$$

$$+p_{a,e}e^a_{\mu_v}(z_v)(\partial_{\nu_v}\sigma^{\mu_v}(x_e,z_v))\delta z_v^{\nu_v} - P_{a,e}(0)e^a_{\nu}(x_e)\delta x_e^{\nu}\big] .$$

Setting the variations to zero we obtain the relevant equations of motion

$$e^a_{\mu_v}(z_v)\sigma^{\mu_v}(x_e,z_v) = 0 \qquad \forall v \in \Gamma, \tag{2.51}$$

$$P_{a,e}(0)e^a_{\mu}(x_e) = p_{a,e}\nabla_{x_e^{\mu}}\sigma^a(x_e,z_v) \qquad \forall e \in \Gamma, \tag{2.52}$$

$$\sum_{\substack{s_e=v\\ t_e=v}} p_{a,e}\nabla_{z_v^{\mu}}\sigma^a(x_e,z_v) = 0 \qquad \forall v \in \Gamma, , \tag{2.53}$$

where we have made use of the notation $\sigma^a(x,x') = \sigma^{\mu'}(x,x')e^a_{\mu'}(x')$. Note also that whenever x_e and z_v appear in the same equation the edge e is assumed to innervate the vertex v. From eq. (2.28) we see that the first of these equations requires $x_e = z_v$, which is just the locality condition. Taking the coincidence limit on either side of the remaining equations, making use of eq. (2.35) and multiplying by the inverse tetrad we find

$$P_{a,e}(0) = -p_{a,e} \quad \text{and} \quad \sum_{\substack{s_e=v\\ t_e=v}} p_{a,e} = 0. \tag{2.54}$$

These equations should be supplemented with the equation for conservation of momenta along an edge. As shown in eq. (2.24) the momenta $P_e(1) = -P_{-e}(0)$ at the end of an edge is related to the initial momenta $P_e(0)$ by parallel transport along e, denoted $U_e \equiv \overrightarrow{\exp}(\int_e \mathrm{d}x^{\mu}\omega_{\mu})$. Thus, the equation governing conservation of momentum along an edge is given by

$$p^a_{-e} + (p_e{\cdot}U_e)^a = 0, \tag{2.55}$$

where we denote $(p{\cdot}U)^b = p^a U_a{}^b$. As in the case of flat spacetime we can use the localization condition to relate the vertex momenta to a difference in position, although here the computation is more subtle. Begin with the derivative of the worldfunction evaluated at the endpoints of the edge e, i.e. $\sigma^{\mu_{x_e}}(x_{-e},x_e)$. Equation eq. (2.30) then allows us to write

$$\begin{aligned}\sigma^{\mu_{x_e}}(x_{-e},x_e) &= -(s_1-s_0)\partial_s X^{\mu}_e(0)\\ &= -\partial_s X^{\mu}_e(0)\int_0^1 N_e(\tau)d\tau,\end{aligned}$$

[4]We have assumed that $\delta X_e(0) = 0$ and $\delta X_e(1) = 0$ for incoming and outgoing edges respectively.

where $ds = Nd\tau$ is the proper time along the world line. Now use eq. (2.18) to replace $\partial_s X_e^\mu(0)$ in favour of $P_e(0)$ so that

$$\sigma^{\mu_{x_e}}(x_{-e}, x_e) = -\tau_e P_e^a(0) e_a^\mu(x_e), \tag{2.56}$$

where $\tau_e = \int N_e d\tau$. Finally, the localization equation allows us to identify $x_{-e} = z_{t_e}$ and $x_e = z_{s_e}$ while our relation between edge and vertex momentum yields $P_{b,e}(0) = -p_{a,e}$ and so

$$\sigma^a(z_{t_e}, z_{s_e}) = \tau_e p_e^a. \tag{2.57}$$

These localization equations are compatible with the parallel transport equation of momenta along an edge since

$$\sigma^a(z_{t_e}, z_{s_e}) U_{ea}{}^b = -\sigma^b(z_{s_e}, z_{t_e}); \tag{2.58}$$

see Appendix A for a proof.

2.5.2 Localization on Loops

Let us examine the localization equations eq. (2.55) for a graph that possesses a loop L. Assume that L consists of the edges $L = e_1 e_2 \cdots e_n$, and that $e_i = (i, i+1)$, goes from vertex i to vertex $i+1$. We denote by P_i the external momenta incoming to vertex i and by $p_{e_i} = p_{ii+1}$ the momenta on edge e_i starting at vertex i. This is illustrated in Figure 2.2. The localization equations, eq. (2.55), read

$$p_{i+1i} + p_{ii+1} \cdot U_{ii+1} = 0, \qquad P_i = p_{ii-1} + p_{ii+1}, \tag{2.59}$$

where $i = 1, \ldots, n$ and addition is modulo n. Define $U_{aa+m} \equiv U_{aa+1} U_{a+1a+2} \cdots U_{a+m-1a+m}$ to be the holonomy from a to $a+m$, so that upon summing the above relation we obtain

$$S_n \equiv P_n + P_{n-1} \cdot U_{n-1n} + \cdots P_1 \cdot U_{1n} = p_{n1} \cdot (1 - H_n), \tag{2.60}$$

where $H_n = U_{n1} U_{12} \cdots U_{n-1n}$ is the total holonomy around the loop based at the vertex n. To generalize this relation to an arbitrary base vertex we introduce the momenta transported from the vertex i

$$\hat{P}_i \equiv P_i \cdot U_{in}, \tag{2.61}$$

so that $S_n = \sum_{i=1}^n \hat{P}_i$. We immediately obtain $p_{ii+1} U_{ii+1} = P_i U_{ii+1} + p_{i-1i} U_{i-1i+1}$ which can then be solved iteratively to express p_{ii+1} in terms of the external momenta and p_{n1} as

$$p_{ii+1} \cdot U_{in} = (\hat{P}_i + \cdots \hat{P}_1) + p_{n1} H_n. \tag{2.62}$$

Putting equations eq. (2.60) and eq. (2.62) together we see that the loop momenta p_{ii+1} are entirely determined by the external momenta, which is related to the fact that in presence of gravity the total momenta around a loop is no longer conserved.

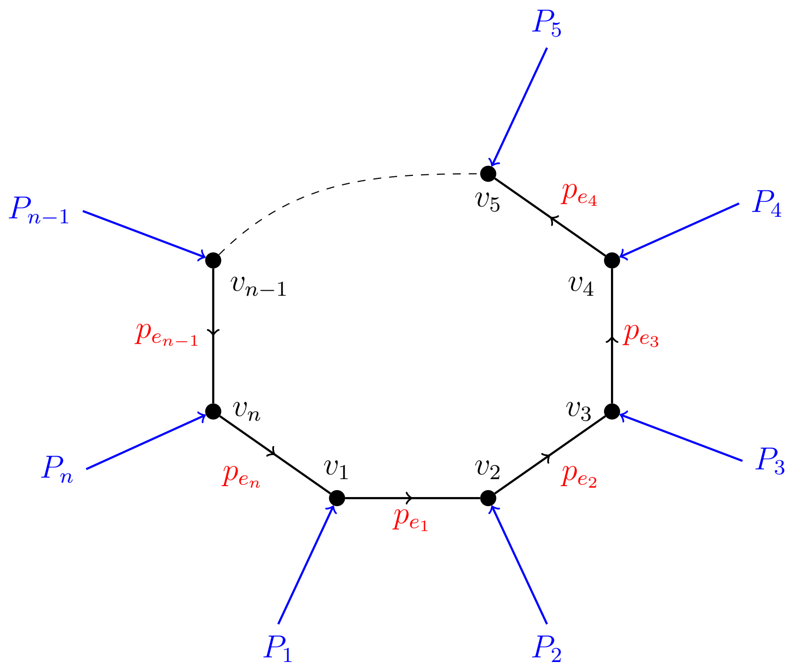


Figure 2.2: *The loop* $L = e_1 e_2 \cdots e_n$

When spacetime is flat $H_n = 1$ and so, the right hand side of eq. (2.60) vanishes, total momentum is conserved and the loop momentum is independent of the external momenta. Consequently, one must integrate over the loop momenta when performing the path integral, leading to the well known problems with ultraviolet divergences. On the other hand, when gravity is present the holonomy will differ from the identity allowing, quite generically, the operator $(1 - H_n)$ to be inverted. In this case we can express all the momenta in terms of the external ones! For a small loop the holonomy approximates to $H^a{}_b = \delta^a_b + R^a{}_{b\mu\nu} \Delta A^{\mu\nu} + \cdots$ where ΔA is the loop area. The invertibility of $(1 - H)$ is therefore related to the invertibility of $R^a{}_b(X)$ for all invertible bivectors X. Thus, in a fully curved background the only way (generically) to have a non invertible $(1 - H)$ is to consider a loop of zero extension, i.e. with $\Delta A = 0$. It is these loops of zero size that give rise to divergences in quantum field theory.

In summary, the effect of a gravitational field on an extended loop is to produce a violation of total momentum conservation. This phenomena is related to the fact that, in the present case, loop momenta can be expressed entirely in-terms of the external momenta. Therefore, if we could argue that quantum gravity requires the expectation value of the holonomy $\langle H \rangle$ to be different from unity for all loops, even those which shrink to an effective size of zero, this would have a dramatic regulating effect on Feynman integrals, at least on their semi-classical evaluation. In particular, we could restrict loop integrals to a finite region of momentum space.

2.5.3 Nonlocal vertex

In the previous section we showed that coupling to a classical gravitational field modifies the loop propagator by introducing a holonomy (around the loop) into the conservation of momentum equation. On the other hand, we saw that momentum conservation at the vertices is unaffected, being identical to the relations derived for flat spacetime.

A pertinent question arises, how do quantum gravity effects alter particle physics amplitudes? It is well known that the inclusion of quantum gravity introduces a new mass scale into the theory, namely the Plank mass. Our question can then be phrased more formally as follows: Suppose we couple gravity to a Feynman integral and compute, by some method, the quantum gravity average, how does this evaluation affect the Feynman integral? It is tempting to assume that the computation, done in any theory of quantum gravity, will correspond to a mass dependent deformation of the standard integral. According to the philosophy presented here, and assuming that new degrees of freedom do not appear, this deformation can in turn be entirely reabsorbed into a deformation of the particle action.

It is natural to assume that this deformation will affect the vertex interaction. Indeed it was the vertex factor $p_a\sigma^a(x_e, z_v)$ which, as we have seen, determined the localization condition $x_e = z_v$. Such exact localization will certainly be relaxed in a theory of quantum gravity. We propose, therefore, to modify the vertex interaction as an effective way to include quantum gravity (de-localizing type) effects.

The simplest such modification is to consider a vertex interaction of the form $p_a\sigma^a(x_e, z_v) - p_a p^a/2M$ where M is the quantum gravity mass scale. In the Euclidean formulation of the theory this amounts to replacing the vertex interaction, $\delta(x, z)$, by a Gaussian weight

$$\delta_M(x, z) = \left(\frac{M}{2\pi}\right)^{\frac{d}{2}} e^{-M\sigma(x,z)}. \tag{2.63}$$

The equations of motion resulting from this de-localized vertex are readily found to be (c.f. eq. (2.51) - eq. (2.53))

$$M\sigma^a(x_e, z_v) = p^a_e \tag{2.64}$$

$$P_{a,e}(0)e^a_\mu(x_e) = p_{a,e}\nabla_{x^\mu_e}\sigma^a(x_e, z_v) \tag{2.65}$$

$$\sum_{\substack{s_e=v\\t_e=v}} [p_{a,e}\nabla_{z^\mu_v}\sigma^a(x_e, z_v)] = 0, \tag{2.66}$$

Observe that the mass scale only enters in the first equation, modifying the locality condition by ensuring that x_e and z_v are no longer identified. Substituting this into eq. (2.66) and using that the Synge function satisfies[5] $\sigma^a\nabla_{z^\mu}\sigma_a(x,z) = e^a_\mu(z)\sigma_a(x,z)$ we obtain

$$\sum_{\substack{s_e=v\\t_e=v}} p^a_e = 0, \tag{2.67}$$

which, as before, is the usual conservation of vertex momenta. Where the modification becomes apparent is in the relationship between the endpoint momenta P and the vertex momenta p; from eq. (2.65) we have

$$P^a_e(0) = M\eta^{ab}e^\mu_b(x_e)\nabla_{x^\mu_e}\sigma_b(x_e, z_v) = M\eta^{ab}\sigma_{\mu_e}(x_e, z_v)e^{\mu_e}_a(x_e). \tag{2.68}$$

Let V_{ev} denote the parallel propagator from z_v to x_e, then $\sigma^{\mu_e} = -[V_{ev}]^{\mu_e}{}_{\nu_v}\sigma^{\nu_v}(x_e, z_v)$ and so

$$P^a_e(0) = -Me^a_{\mu_e}(x_e)[V_{ev}]^{\mu_e}{}_{\nu_v}\sigma^{\nu_v}(x_e, z_v) \tag{2.69}$$

$$= -M[V_{ev}]^a{}_b\sigma^b(x_e, z_v) \tag{2.70}$$

$$= -(p_e\cdot V_{ev})^a, \tag{2.71}$$

where we have made use of the notation $[V_{ev}]^a{}_b = e^a_{\mu_e}(x_e)[V_{ev}]^{\mu_e}{}_{\nu_z}e^{\nu_z}_b(z_v)$ in the second line. Remarkably, this implies that the conservation of momenta along edges is modified in a trivial manner, viz

$$p_{-e} + p_e\cdot(V_{s_e e}U_eV_{-et_e}) = 0. \tag{2.72}$$

The term in brackets is the full propagator from s_e to t_e indicating that the form of this equation is identical to the one consider for a local vertex, see eq. (2.55). It follows that the de-localization of the vertex does not affect the momenta conservation equations, either at the vertex or along the edges. Its only effect is to modify the relationship between

[5] See eq. (2.33)

momenta and coordinates, e.g. in a flat spacetime the modification enters through a shift in proper time

$$z_{t_e} - z_{s_e} = (\tau_e + 1/2M)p_e. \tag{2.73}$$

A more general modification of the vertex that is quadratic, Lorentz invariant and symmetric under exchanges of momenta would include an additional term proportional to $\sum_{e,e'} p^a_{e'} p_{e,a}/2M$ and would give rise to effects similar to those considered above.

Chapter 3

Scalar Field Theory in a Curved Momentum Space

In this Chapter we consider the effect on scalar field theory of explicitly imposing non-locality via the "Relative Locality" framework; specifically we derive the action for a φ^3 theory living in a curved momentum space. Along the way we will utilize some of the tools derived in the preivous Chapter, namely the notion of a covariant Fourier transform.

3.1 Geometry of Momentum Space

In what follows we take momentum space to be a non-linear manifold $\mathcal{P}$ and phase space the cotangent bundle $T^*\mathcal{P}$. Spacetime then emerges as cotangent planes to points in momentum space $T^*_p\mathcal{P}$. We will now embark on a self-contained review of momentum space geometry; the presentation will be as general as possible, although in later sections we will be forced to give up some of this generality for the sake of coherence and ease of calculation.

3.1.1 Combination of Momenta

Conservation of momentum requires that we postulate a rule, $\oplus$, for combining momenta and to keep this rule as general as possible we will allow the physics to tell us what properties are mathematically acceptable. Interaction with a zero momentum object will produce no change in momenta and so 0 should be an identity for $\oplus$, in addition, we need a method for turning an incoming particle into an outgoing one and so our rule should have an inverse. We will not, however, assume this rule is linear and so there is no reason to demand either commutativity or associativity either. Formally, we define our rule as a C^∞ map:

$$\begin{aligned} \oplus : \mathcal{P} \times \mathcal{P} &\to \mathcal{P} \\ (p, q) &\mapsto p \oplus q, \end{aligned} \tag{3.1}$$

having identity 0

$$0 \oplus p = p \oplus 0 = p \qquad \forall p \in \mathcal{P}, \tag{3.2}$$

and inverse $\ominus$

$$(\ominus p) \oplus p = p \oplus (\ominus p) = 0 \qquad \forall p \in \mathcal{P}. \tag{3.3}$$

Note that we assume a unique inverse; if $p, q \in \mathcal{P}$ are such that $q \oplus p = p \oplus q = 0$ then $q = \ominus p$.

Equipped with this combination rule we can enforce the conservation of energy and momentum at each interaction. We will write this as[1]

$$\mathcal{K}_\mu(p^I) = 0, \tag{3.4}$$

where $I = 1, 2, \dots$ runs over the number of particles participating in the interaction. For example, a process with two incoming particles p, q and one outgoing particle k may have

$$\mathcal{K}_\mu = (p \oplus (q \ominus k))_\mu, \tag{3.5}$$

where we have made use of the obvious notation $q \ominus k = q \oplus (\ominus k)$ and have adopted the convention that all momenta are taken to be incoming. Observe that (3.5) is just one of twelve possible choices for $\mathcal{K}$ all of which are distinct if $\oplus$ is neither commutative nor associative. Differences arising from alternate choices of the conservation law are explored in detail in [99].

Suppose we are given a generic conservation law $p \oplus (q \oplus k) = 0$. For this to be meaningful it must be possible to solve for any one of the momenta uniquely in terms of the other two. To address this issue we introduce left (L_p) and right (R_p) translation operators

$$L_p(q) \equiv p \oplus q \qquad \text{and} \qquad R_p(q) \equiv q \oplus p, \tag{3.6}$$

[1] In special relativity $\mathcal{K}_\mu(p^I) = \sum_I p^I_\mu$

which allow the conservation law to be re-written as

$$R_{q\oplus k}(p) = L_p(R_k(q)) = L_p(L_q(k)) = 0. \tag{3.7}$$

The existence of a unique solution for each momenta then reduces to the requirement that the left and right translation operators be invertible. It is therefore assumed that L_p^{-1} and R_p^{-1} exist for all $p \in \mathcal{P}$ and so the solutions of our conservation law are given by

$$p = \ominus(q \oplus k) \qquad q = R_k^{-1}(\ominus p) \qquad k = L_q^{-1}(\ominus p), \tag{3.8}$$

where we have used that $L_p^{-1}(0) = R_p^{-1}(0) = \ominus p$, by the uniqueness of the inverse. Note that we are not assuming the composition law $\oplus$ is left or right invertible; doing so would be equivalent to setting $L_p^{-1} = L_{\ominus p}$ and $R_p^{-1} = R_{\ominus p}$ respectively.

3.1.2 Curvature and Torsion

The algebra induced on momentum space by our composition rule determines a connection on $\mathcal{P}$ via

$$\Gamma_\rho^{\mu\nu}(0) = \frac{\partial}{\partial p_\mu}\frac{\partial}{\partial q_\nu}(p \oplus q)_\rho\Big|_{p,q=0}. \tag{3.9}$$

The torsion is the anti-symmetric part of $\Gamma_\rho^{\mu\nu}$ and measures the extent to which the combination rule fails to commute

$$T_\rho^{\mu\nu}(0) = \Gamma_\rho^{[\mu\nu]}(0) = \frac{\partial}{\partial p_\mu}\frac{\partial}{\partial q_\nu}(p \oplus q - q \oplus p)_\rho\Big|_{p,q=0}. \tag{3.10}$$

Similarly, the curvature of $\mathcal{P}$ is a measure of the lack of associativity of the combination rule

$$R^{\beta\gamma\delta}{}_\mu(0) = -2\frac{\partial}{\partial p_{[\beta}}\frac{\partial}{\partial q_{\gamma]}}\frac{\partial}{\partial k_\delta}\left(p \oplus (q \oplus k) - (p \oplus q) \oplus k\right)_\mu\Big|_{p=q=k=0}. \tag{3.11}$$

Unlike general relativity the connection $\Gamma_\rho^{\mu\nu}$ is not necessarily metric compatible and so $g^{\mu\nu}$ may fail to be covariantly constant. To measure the extent to which the covariant derivative of $g^{\mu\nu}$ deviates from zero we introduce the non-metricity tensor

$$N^{\mu\nu\rho} = \nabla^\mu g^{\nu\rho} = \partial^\mu g^{\nu\rho} - \Gamma_\alpha^{\nu\mu} g^{\alpha\rho} - \Gamma_\alpha^{\rho\mu} g^{\nu\alpha}. \tag{3.12}$$

Let $\left\{{}^{\mu\,\nu}_{\ \rho}\right\}$ denote the standard Levi-Civita connection compatible with the metric $g^{\mu\nu}$. We can then decompose the full connection $\Gamma_\rho^{\mu\nu}$ in-terms of the Levi-Civita connection, the torsion and the non-metricity tensor, viz

$$\Gamma_\rho^{\mu\nu} = \left\{{}^{\mu\,\nu}_{\ \rho}\right\} + \frac{1}{2}T_\rho^{\mu\nu} - \frac{1}{2}g_{\rho\alpha}\left(N^{\mu\nu\alpha} + N^{\nu\mu\alpha} - N^{\alpha\mu\nu} + T^{\alpha\mu\nu} + T^{\alpha\nu\mu}\right), \tag{3.13}$$

where $T^{\mu\nu\rho} = T^{\mu\nu}_{\alpha} g^{\alpha\rho}$. Similarly, we can expand the non-metricity tensor in-terms of the torsion and the symmetric tensor $\mathcal{N}^{\mu\nu}_{\rho} = \Gamma^{(\mu\nu)}_{\rho} - \{ {}^{\mu\,\nu}_{\;\rho} \}$, the result is

$$N^{\mu\nu\rho} = \frac{1}{2}\left(T^{\mu\nu\rho} + T^{\mu\rho\nu}\right) - \mathcal{N}^{\nu\mu}_{\alpha} g^{\alpha\rho} - \mathcal{N}^{\rho\mu}_{\alpha} g^{\alpha\nu}. \tag{3.14}$$

3.1.3 Transport Operators

In order to write the locality equations at each vertex we need to introduce transport operators that arise from the infinitesimal transformation of the addition law. We define the left transport operator as

$$\left(U^{q}_{p\oplus q}\right)^{\mu}_{\nu} = (\mathrm{d}_q L_p)^{\mu}_{\nu} = \frac{\partial (p\oplus q)_{\nu}}{\partial q_{\mu}}, \tag{3.15}$$

and the right transport operator as

$$\left(V^{q}_{q\oplus p}\right)^{\mu}_{\nu} = (\mathrm{d}_q R_p)^{\mu}_{\nu} = \frac{\partial (q\oplus p)_{\nu}}{\partial q_{\mu}}. \tag{3.16}$$

Here the notation $\mathrm{d}_p f \equiv (\partial_{p_\mu} f(p)) dx^{\mu}$ denotes the differential at p of the function f. The most general form of the transport operators, U^{q}_{k} and V^{q}_{k}, from point q to k, can be obtained from the ones defined above by setting $p = R^{-1}_{q}(k)$ and $p = L^{-1}_{q}(k)$ respectively. It will also be useful to give a name to the derivative of the inverse:

$$(I^{p})^{\mu}_{\nu} = (d_p \ominus)^{\mu}_{\nu} = \frac{\partial (\ominus p)_{\nu}}{\partial p_{\mu}}. \tag{3.17}$$

It turns out that these operators are not independent and can be related by

$$V^{p}_{0} = -U^{\ominus p}_{0} I^{p}. \tag{3.18}$$

The proof of this formula is straightforward and requires only the existence of the inverse $\ominus p$:

$$\begin{aligned} 0 &= \frac{\partial}{\partial p}(p \oplus (\ominus p)) \\ &= \frac{\partial}{\partial k}(k \oplus (\ominus p))\Big|_{k=p} + \frac{\partial}{\partial k}(p \oplus k)\Big|_{k=\ominus p} \frac{\partial \ominus p}{\partial p} \\ &= V^{p}_{0} + U^{\ominus p}_{0} I^{p}. \end{aligned}$$

By considering equations of the form $L_p(L^{-1}_p(q)) = q$ and $R_p(R^{-1}_p(q)) = q$ we can also derive formulas for the derivatives of L^{-1}_p and R^{-1}_p:

$$\frac{\partial L^{-1}_p(q)}{\partial q} = \left(U^{L^{-1}_p(q)}_{q}\right)^{-1}, \qquad \frac{\partial L^{-1}_p(q)}{\partial p} = -\left(U^{L^{-1}_p(q)}_{q}\right)^{-1} V^{p}_{q}, \tag{3.19}$$

and

$$\frac{\partial R_p^{-1}(q)}{\partial q} = \left(V_q^{R_p^{-1}(q)}\right)^{-1}, \qquad \frac{\partial R_p^{-1}(q)}{\partial p} = -\left(V_q^{R_p^{-1}(q)}\right)^{-1} U_q^p. \tag{3.20}$$

Without demanding certain properties of the composition rule we can not say anything further. For the sake of completeness we now now present a collection of results that are applicable if the following conditions on $\oplus$ are fulfilled:

- Composition rule is left invertible, i.e. $L_p^{-1} = L_{\ominus p}$:

$$\left(U_{p\oplus q}^q\right)^{-1} = U_q^{p\oplus q} \qquad \text{and} \qquad V_{\ominus p\oplus q}^{\ominus p} I^p = -U_{\ominus p\oplus q}^q V_q^p$$

- Composition rule is right invertible, i.e. $R_p^{-1} = R_{\ominus p}$:

$$\left(V_{q\oplus p}^q\right)^{-1} = V_q^{q\oplus p} \qquad \text{and} \qquad U_{q\ominus p}^{\ominus p} I^p = -V_{q\ominus p}^q U_q^p$$

3.1.4 Metric and Distance Function

It is assumed that the metric on momentum space, $g^{\mu\nu}(p)$, is known. It is then a standard result that the distance between two points $p_0, p_1 \in \mathcal{P}$ along a path $\gamma(\tau)$ is given by:

$$D_\gamma(p_0, p_1) = \int_a^b \sqrt{g^{\mu\nu}\left(\gamma(\tau)\right) \frac{d\gamma_\mu}{d\tau} \frac{d\gamma_\nu}{d\tau}} d\tau, \tag{3.21}$$

where $\gamma(a) = p_0$ and $\gamma(b) = p_1$. Of all the paths connecting p_0 and p_1 *geodesics* will be of principle importance, but here we run into trouble. In relative locality, where the non-metricity tensor does not necessarily vanish, there is more than one viable definition of a geodesic, so it is not immediately clear what one means by a "geodesic." This ambiguity is discussed in Appendix C, where we argue that the most appropriate definition of a geodesic is a path which extremizes $D_\gamma(p_0, p_1)$. We will adopt this convention for the remainder of the Chapter and note that if γ is a geodesic we write $D_\gamma(p_0, p_1) = D(p_0, p_1)$.

The standard definition of a particles mass is by means of the dispersion relation $p^2 = -m^2$. To account for the geometry of momentum space we deform this relation and assume that the mass of a particle with momentum p is related to the geodesic distance from p to the origin, i.e.

$$D^2(p) = -m^2, \tag{3.22}$$

where we have used the simplified notation $D(p, 0) = D(p)$.

3.2 φ^3 Scalar Field

We can now utilize the structures introduced above to examine the effects of a delocalized vertex on φ^3 scalar field theory.

3.2.1 Modified Feynman Rules

The starting point for our analysis will be the well known generating functional for standard φ^3-theory:

$$Z(J) = \int \mathcal{D}\varphi \exp\left(i \int d^4x \left[-\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 + \frac{1}{3!}g\varphi^3 + J\varphi\right]\right). \tag{3.23}$$

This is the position space representation of $Z(J)$ which is ill-suited for our purposes. Relative locality treats momentum space as fundamental and so we should Fourier transform $Z(J)$ so that all integrals are over momenta. Denote by $\mathcal{F}$ the Fourier transform of the argument of the exponential, then[2]

$$\begin{aligned}\mathcal{F} = i \int \frac{d^4p}{(2\pi)^4} \left(-\frac{1}{2}\left(p^2 + m^2\right)\varphi(p)\varphi(-p) + J(p)\varphi(-p)\right) \\ + i\frac{(2\pi)^4 g}{3!} \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \delta(p+k+q)\varphi(p)\varphi(q)\varphi(k)\end{aligned}$$

Following the standard procedure we extract the interaction terms from $Z(J)$ and re-write them as functional derivatives with respect to J acting on the remainder of $Z(J)$. We can then separate out the J dependent terms from the functional by completing the square, in the end we find

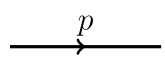
$$\begin{aligned}Z(J) = \exp\left(-\frac{(2\pi)^4 g}{3!} \int d^4p \int d^4q \int d^4k\delta(p+q+k)\frac{\delta}{\delta J(p)}\frac{\delta}{\delta J(q)}\frac{\delta}{\delta J(k)}\right) \\ \times \exp\left(\frac{i}{2}\int \frac{d^4p}{(2\pi)^4} J(p)\left(p^2+m^2\right)^{-1} J(-p)\right) \\ \times \int \mathcal{D}\varphi \exp\left(-\frac{i}{2}\int \frac{d^4p}{(2\pi)^4}\left(p^2+m^2\right)\varphi(p)\varphi(-p)\right).\end{aligned} \tag{3.24}$$

[2] Normally we would denote the Fourier transformed fields as $\hat{\varphi}(p)$, $\hat{J}(p)$ but since we will be regarding the momentum space representation as fundamental we will drop the hat.

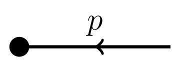
Having successfully removed all J dependence from the functional integral we can evaluate it to obtain some C-number. However, if we insist on the normalization $Z(0) = 1$ we can ignore this number and simply impose the normalization by hand. Hence,

$$
\begin{aligned}
Z(J) \propto \exp &\left(-\frac{(2\pi)^4 g}{3!} \int d^4p \int d^4q \int d^4k \delta(p+q+k) \frac{\delta}{\delta J(p)} \frac{\delta}{\delta J(q)} \frac{\delta}{\delta J(k)} \right) \\
&\times \exp \left(\frac{i}{2} \int \frac{d^4p}{(2\pi)^4} J(p) \left(p^2+m^2\right)^{-1} J(-p) \right).
\end{aligned}
\tag{3.25}
$$

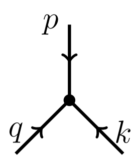
This generating functional can now be expanded as a sum of all possible Feynman diagrams having E external points, P propagators and V vertices where $E = 3V - 2P$. Each diagram is then assigned a value by means of the following Feynman rules:

1. To each propagator, 

$= \dfrac{i}{(2\pi)^4(p^2+m^2)}$;

2. To each external point, 

$= J(p)$;

3. To each vertex, 

$= -g(2\pi)^4 \delta(p+q+k)$;

4. Integrate over all momenta;

5. Divide by the symmetry factor.

We now consider how these rules are modified in presence of a curved momentum space. Let us begin with rule 4), integrate over all momenta. This is equivalent to introducing a measure on momentum space, call it $d\mu(p)$. For the time being we will make no assumptions about the measure other than demanding it reduce to the standard Lebesgue measure in the limit when momentum space becomes a linear manifold[3]. Given $d\mu(p)$ we define $\delta(p, q)$ to be a delta function compatible with this measure, that is:

$$
\int d\mu(p) \delta(p,q) f(p) = f(q)
\tag{3.26}
$$

for any function $f : \mathcal{P} \to \mathcal{P}$. Note that this delta function is assumed to be symmetric upon interchange of its arguments, i.e. $\delta(p, q) = \delta(q, p)$.

[3]An obvious choice would be $d\mu(p) = \sqrt{g(p)} d^4p$.

In deriving the original Feynman rules we tacitly assumed that the change of variables $p \to -p$ has unit Jacobian. In relative locality the equivalent change of variables is $p \to \ominus p$ which has Jacobian $|\det(d_p \ominus)| = |\det(I^p)|$.[4] A priori this quantity could differ from unity which amounts to breaking the symmetry associated with flipping the direction of a propagator. Therefore, diagrams which are related by such a transformation should be regarded as inequivalent, see Figure 3.1.

Figure 3.1: Feynman diagrams related by switching the direction of a propagator are inequivalent.

Diagrams do, however, still posses a symmetry under relabelling of propagators, for example the diagrams shown in Figure 3.2 are equivalent.

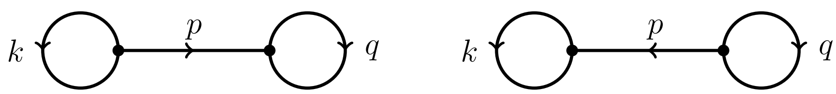


Figure 3.2: Feynman diagrams related by relabelling of propagators are equivalent.

All of this implies that we must propose a different interpretation of the symmetry factor, rule 5). A bit of thought suggests the following modification: Divide by 2^P, where P is the number of propagators appearing in the diagram, then divide by a factor associated with any residual symmetries of the diagram. The diagrams in Figures 3.1, 3.2 have no residual symmetry whereas those in Figure 3.3 have residual symmetry factors of 3! and 2! respectively, given by relabelling the propagators.

[4]Note that the assumption of a unique inverse is critical here; it is equivalent to demanding that $\ominus$ be invertible which in turn is necessary to even define this change of variables.

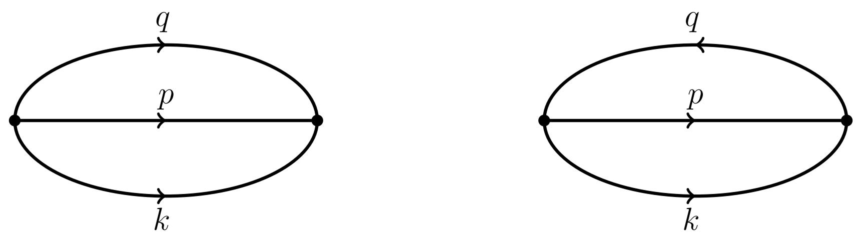


Figure 3.3: Relabelling the propagators gives a residual symmetry factor of 3! for the left diagram and 2! for the right.

We turn next to Rule 1), the factor associated with the propagator.[5] The propagator must have a single simple pole at the particles mass which, given the definition of mass in Relative Locality c.f. eq. (3.22), suggest that we make the following replacement:

$$p^2 + m^2 \to D^2(p) + m^2. \tag{3.27}$$

where $D(p)$, we recall, is the distance of p from the origin as measured by the momentum space metric $g(p)$.

Rule 2) requires no modification and so we come to rule 3), the factor assigned to a vertex. What properties should the modified factor posses? First, it should reduce to the original in the case where momentum space is a linear manifold. Second, it should respect the statistics of our particles. It is well known that in standard QFT scalar particles obey Bose statistics. In our case since we modify the addition rule and relax the notion of locality, we could also relax the bose statistics and investigate non-trivial field statistics. At present we will take the simplest hypothesis and assume that we have Bose statistics in the current framework as well. Therefore, the vertex factor must be symmetric upon interchange of momentum labels. Given that the combination rule is neither associative nor commutative there are several choices we could make, we will consider three of them in detail. Assuming all particles are incoming to the vertex the first of these is:

$$\begin{aligned}\Delta_1 = \frac{1}{6}\big[&\delta(p \oplus (q \oplus k)) + \delta(p \oplus (k \oplus q)) + \delta(q \oplus (p \oplus k)) + \delta(q \oplus (k \oplus p)) \\ &+ \delta(k \oplus (p \oplus q)) + \delta(k \oplus (q \oplus p))\big],\end{aligned} \tag{3.28}$$

where we have used the simplified notation $\delta(p, 0) = \delta(p)$. In this option we always assume that the second and third terms in the sum are grouped together.[6] The second choice

[5] In what follows we will drop all factors of $(2\pi)^4$.

[6] Another, nearly equivalent, choice would be to group the first two terms together.

includes all possible groupings and we write it as:

$$\Delta_2 = \frac{1}{12} \sum_{\mathcal{K}(p,q,k)} \delta\left(\mathcal{K}(p,q,k)\right), \tag{3.29}$$

where $\mathcal{K}(p,q,k)$ represents a possible ordering of momenta. The final option is similar to Δ_1 but we move the grouped factors to the other side of the delta function, this gives

$$\begin{aligned}\Delta_3 = \frac{1}{6}\big[&\delta(p, \ominus(q \oplus k)) + \delta(p, \ominus(k \oplus q)) + \delta(q, \ominus(p \oplus k)) + \delta(q, \ominus(k \oplus p)) \\ &+ \delta(k, \ominus(p \oplus q)) + \delta(k, \ominus(q \oplus p))\big].\end{aligned} \tag{3.30}$$

The difference between Δ_1 and Δ_2 is related to the discrepancy between $\delta(p \oplus q, 0)$ and $\delta(q \oplus p, 0)$ whereas the difference between Δ_1 and Δ_3 is related to the discrepancy between $\delta(p \oplus q, 0)$ and $\delta(p, \ominus q)$. To gain some understanding of these discrepancies let us integrate these delta functions against an arbitrary function $f(p)$, we start with $\delta(p \oplus q)$:

$$\int d\mu(p)\delta(p \oplus q, 0)f(p) = \left|\det\left(V_0^{\ominus q}\right)\right|^{-1} f\left(\ominus q\right)).$$

The calculation for $\delta(q \oplus p)$ is identical and yields:

$$\int d\mu(p)\delta(q \oplus p, 0)f(p) = \left|\det\left(U_0^{\ominus q}\right)\right|^{-1} f\left(\ominus q\right).$$

Obviously these results would be interchanged if we had instead integrated over q. It remains to consider the value obtained from $\delta(p, \ominus q)$:

$$\int d\mu(p)\delta(p, \ominus q)f(p) = f(\ominus q).$$

Note that if we interchanged the roles of p and q in the previous integral we would obtain:

$$\int d\mu(p)\delta(q, \ominus p)f(p) = \left|\det\left(I^q\right)\right| f(\ominus q).$$

We see that the differences between the Δ_i is governed by the extent to which the determinant of the left or right transport operator differs from unity.

It still remains to choose which Δ_i to use as a vertex factor. To motivate this choice let us imagine conserving momentum at a "two point vertex", see figure 3.4.

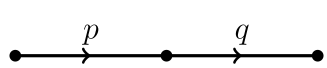


Figure 3.4: *Conserving momentum at a two point vertex.*

Our prescription for conserving momentum should give $p = q$, i.e. $\int d\mu(q)\Delta_i(p,q) = 1$. Both Δ_1 and Δ_2 yield a factor of

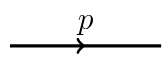
$$\frac{1}{2}\int d\mu(q)\left(\delta(p\ominus q)+\delta(\ominus q\oplus p)\right)=\frac{1}{2}\left|\det\left(I^p\right)\right|^{-1}\left(\left|\det\left(U_0^{\ominus p}\right)\right|^{-1}+\left|\det\left(V_0^{\ominus p}\right)\right|^{-1}\right),$$

whereas Δ_3 gives

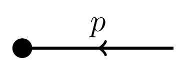
$$\int d\mu(q)\delta(p,q) = 1.$$

This strongly suggests that we adopt Δ_3 as our vertex factor and we will do so for the remainder of the Chapter. To keep notation simple we drop the 3 and denote our vertex factor by $-g\Delta(p,q,k)$.

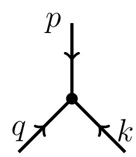
In summary, the modified generating functional is expanded as a sum of all Feynamn diagrams with E external points, P propagators and V vertices, where $E = 3V - 2P$. For each such diagram we include all possible orientations of propagator momenta that are inequivalent under relabelling. A numerical value is then assigned to these diagrams by means of the following Feynman rules:

1. To each propagator, 

$= \dfrac{i}{D^2(p)+m^2}$;

2. To each external point, 

$= J(p)$;

3. To each vertex, 

$= -g\Delta(p,q,k)$

4. Integrate over all momenta using the measure $d\mu(p)$;

5. Divide by 2^P times the residual symmetry factor.

3.2.2 Modified Generating Functional and Action

Having derived a set of Feynman rules we can now write down a generating functional for our theory. It is a straightforward exercise to see that the generating functional for φ^3-theory in relative locality is given by:

$$\begin{aligned} Z_{\mathrm{RL}}(J) \propto \exp\left(-\frac{g}{3!}\int d\mu(p)\int d\mu(q)\int d\mu(k)\Delta(p,q,k)\frac{\delta}{\delta J(p)}\frac{\delta}{\delta J(q)}\frac{\delta}{\delta J(k)}\right) \\ \times \exp\left(\frac{i}{2}\int d\mu(p)J(p)\left(D^2(p)+m^2\right)^{-1}J(\ominus p)\right), \end{aligned} \tag{3.31}$$

where the proportionality constant is fixed by demanding $Z_{\mathrm{RL}}(0) = 1$. The functional derivatives are defined to yield the delta function introduced in the previous section, viz

$$\frac{\delta}{\delta J(p)} J(q) = \delta(p, q). \tag{3.32}$$

To extract an action from this generating functional we need to evaluate the functional derivatives. This can be done by re-introducing scalar fields $\varphi(p)$ as follows:

$$\begin{aligned} Z_{\mathrm{RL}}(J) \propto \exp&\left(-\frac{g}{3!}\int d\mu(p)\int d\mu(q)\int d\mu(k)\Delta(p,q,k)\frac{\delta}{\delta J(p)}\frac{\delta}{\delta J(q)}\frac{\delta}{\delta J(k)}\right)\\ &\times \exp\left(\frac{i}{2}\int d\mu(p)J(p)\left(D^2(p)+m^2\right)^{-1}J(\ominus p)\right)\\ &\times \int \mathcal{D}\varphi \exp\left(-\frac{i}{2}\int d\mu(p)\left(D^2(p)+m^2\right)\varphi(p)\varphi(\ominus p)\right), \end{aligned}$$

where we have used that Z_{RL} is only defined up to a numerical factor. We can now bring the factor containing J into the functional integral and then perform the change of variables $\varphi(p) \to \varphi(p) - J(p)(D^2(p)+m^2)^{-1}$. After some cancellation we find that the argument of the exponential in the path integral is given by

$$\begin{aligned} -\frac{i}{2}\int d\mu(p)\Big[&\varphi(p)\varphi(\ominus p)\left(D^2(p)+m^2\right) - J(p)\varphi(\ominus p) - \varphi(p)J(\ominus p)\frac{D^2(p)+m^2}{D^2(\ominus p)+m^2}\\ &+ J(p)J(\ominus p)\left(\left(D^2(\ominus p)+m^2\right)^{-1} - \left(D^2(p)+m^2\right)^{-1}\right)\Big]. \end{aligned}$$

The non-linear terms in J will cancel if we demand $D^2(p) = D^2(\ominus p)$. This requirement is physically reasonable since $D^2(p)$ yields the squared mass of a particle with momentum p. On the other hand, $\ominus p$ simply represents a reversal in the direction of a particles momentum; it turns an incoming particle into an outgoing one and vice versa. This operation should not alter the mass of the particle and so $D^2(\ominus p) = -m^2 = D^2(p)$. The term quadratic in J now drops out of the integrand and it becomes a simple matter to evaluate the functional derivatives appearing in (3.31). In doing so we will make the assumption $|\det(I^p)| = 1$ as assuming otherwise would make the result untenable. After we evaluate the functional derivatives we can read off the action as the argument of the exponential, we find

$$\begin{aligned} S_{\mathrm{RL}} = &-\frac{1}{2}\int d\mu(p)\left(D^2(p)+m^2\right)\varphi(p)\varphi(\ominus p)\\ &+\frac{g}{3!}\int d\mu(p)\int d\mu(q)\int d\mu(k)\Delta(p,q,k)\varphi(\ominus p)\varphi(\ominus q)\varphi(\ominus k). \end{aligned} \tag{3.33}$$

The fields $\varphi(p)$ commute and so the six terms in $\Delta(p,q,k)$ collapse to $\delta(p, \ominus(q \oplus k)$, which we can then eliminate by integrating over p to obtain

$$\begin{aligned} S_{\text{RL}} = &-\frac{1}{2}\int d\mu(p)\left(D^2(p)+m^2\right)\varphi(p)\varphi(\ominus p) \\ &+\frac{g}{3!}\int d\mu(q)\int d\mu(k)\varphi(q\oplus k)\varphi(\ominus q)\varphi(\ominus k). \end{aligned} \tag{3.34}$$

Finally we require that S_{RL} be real valued and so we impose the reality condition $\varphi(\ominus p) = \varphi^*(p)$; note though that for this prescription to work we also require

$$\ominus(p\oplus q) = (\ominus p)\oplus(\ominus q), \quad \text{or} \quad \ominus(p\oplus q) = (\ominus q)\oplus(\ominus p). \tag{3.35}$$

The first condition demands that $\ominus$ is a morphism while the second that it is an anti-morphism. These are the two conditions that respect the reality condition. Thus, the final form of our action is given by

$$\begin{aligned} S_{\text{RL}} = &-\frac{1}{2}\int d\mu(p)\left(D^2(p)+m^2\right)\varphi(p)\varphi^*(p) \\ &+\frac{g}{3!}\int d\mu(q)\int d\mu(k)\varphi(q\oplus k)\varphi^*(q)\varphi^*(k). \end{aligned} \tag{3.36}$$

One key property of the action is its covariance under momentum space diffeomorphisms. If one assumes that the integration measure is diffeomorphism invariant, i.e. $d\mu(f(p)) = d\mu(p)$ for a diffeomorphism $f : \mathcal{P} \to \mathcal{P}$, that fixes the identity $f(0) = 0$, then the Relative Locality action satisfies

$$S_{\text{RL}}(g,\oplus,\varphi) = S_{\text{RL}}(g_f,\oplus_f,\varphi_f) \tag{3.37}$$

where

$$\varphi_f(p) \equiv \varphi(f(p)), \qquad p\oplus_f q \equiv f^{-1}(f(p)\oplus f(q)), \tag{3.38}$$

while g_f is the pull backed metric.

3.3 Covariant Fourier Transform

The spacetime properties, specifically locality, of S_{RL} can now be obtained by utilizing the covariant Fourier transform presented in Section 2.4. Notice that in the present context it will be p which represents the curved coordinate and so the world-function will depend on points in momentum space. There is also a possible technical difficulty which should be addressed, namely that the connection is not metric compatible and so the definition of geodesic, which is used to define the world-function, is ambiguous. Fortunately, all the properties of the world-function which are relevant for the covariant Fourier transform follow from the fact that the quantity

$$g^{\mu\nu}(\gamma(\tau))\frac{d\gamma_\mu(\tau)}{d\tau}\frac{d\gamma_\nu(\tau)}{d\tau}, \tag{3.39}$$

is constant along a geodesic, where $\gamma(\tau)$ is some path through momentum space. It can be shown that this condition holds for our definition of a geodesic, see Appendix C, and so the results of Section 2.4 can be used without modification. In what follows we will introduce some additional structures which will be useful in re-writing the Fourier transformed action.

3.3.1 Translated World-Function

The covariant Fourier kernel is given by $\exp(ix^{\mu'}\sigma_{\mu'}(p,p'))$, where $p, p' \in \mathcal{P}$ and $x' \in T_{p'}\mathcal{P}$. There is, however, a minor issue with this definition: In the limit where the geometry of momentum space is trivial we have

$$\exp(ix^{\mu'}\sigma_{\mu'}(p,p')) \to \exp(ix^{\mu}(p-p')_{\mu}), \tag{3.40}$$

and the dependence on the fiducial point p' persists. This dependence can be eliminated by introducing a translated version of the world-function and of its derivative at p':

$$\sigma^R(p,p') \equiv \sigma(R_{p'}(p),p'), \qquad \sigma^{R\mu'}(p,p') \equiv \left(\nabla_{p'_\mu}\sigma(p,p')\right)\Big|_{p=R_{p'}(p)}. \tag{3.41}$$

We could have also defined a left translated version of the world-function, $\sigma^L(p,p') \equiv \sigma(L_{p'}(p),p')$, but we chose σ^R for the sake of definiteness[7]. A graphical comparison of $\sigma_{\mu'}(p,p')$ and $\sigma^R_{\mu'}(p,p')$ is given in Figure 3.5. It follows that we can use the kernel $\exp(ix^{\mu'}\sigma^R_{\mu'}(p,p'))$, which is both covariant and limits to $\exp(ix \cdot p)$ in case of flat spacetime, in place of the one originally introduced in Section 2.4.3.

[7]The translated world-function could not be introduced in Section 2.4.1 since there was no rule for combining coordinates on a generic spacetime manifold.

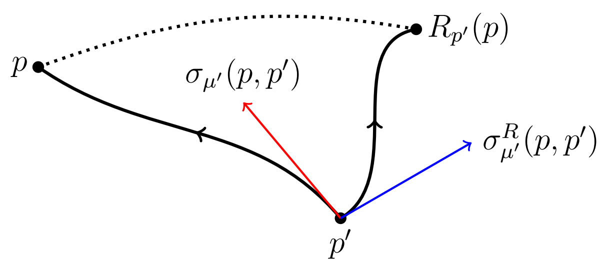


Figure 3.5: *Comparing $\sigma_{\mu'}(p,p')$ and $\sigma^R_{\mu'}(p,p')$. The thick black lines connecting p' to p and $R_{p'}(p)$ represent the unique geodesic interpolating between the two points.*

Some of the technical details regarding the domain of definition of the world-function bear repeating here. Fix the point $p' \in \mathcal{P}$ then the definition of $\sigma(p,p')$ requires that p takes values in a convex normal neighborhood of p', denoted $C_{p'}$ Our primary interest, however, is in the translated world-function $\sigma^R(p,p')$ which will have a domain of definition given by $D_{p'} = R^{-1}_{p'}(C_{p'})$. It is important to note that even if this domain depends on p' it is always a domain centered around the identity, i.e. $0 \in D_{p'}$. See Figure 3.6.

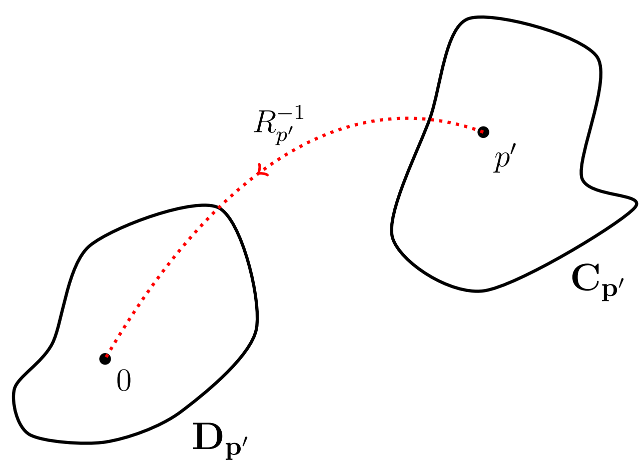


Figure 3.6: *The domain, $C_{p'}$, of $\sigma(p,p')$ is mapped via $R^{-1}_{p'}$ to the domain, $D_{p'}$, of $\sigma^R(p,p')$.*

The Van-Vleck Morette Determinant will now be defined in-terms of the translated world-function

$$\mathcal{V}(p,p') \equiv \frac{\left|\det\left(\sigma^{R\mu\nu'}(p,p')\right)\right|}{\sqrt{g}_{p'}\sqrt{g}_{R_{p'}(p)}}, \tag{3.42}$$

and so

$$\mathcal{V}(0,p') = 1. \tag{3.43}$$

We are now prepared to define the full covariant Fourier transform; to help establish notation it will be beneficial to repeat some of the details presented in Section 2.4. Fix a point $p' \in \mathcal{P}$ and let $\mathcal{M}_{p'} \equiv T^*_{p'}\mathcal{P}$; choose a normal convex neighbourhood $C_{p'}$ giving $D_{p'} \equiv R^{-1}_{p'}(C_{p'})$ as the domain of $\sigma^R(p,p')$. The measure on momentum space, denoted $d\mu(p)$ above, and on the dual spacetime are defined by

$$\begin{aligned} d\mu_{p'}(p) &= \sqrt{g}_{R_{p'}(p)}\mathrm{d}^4 p, \\ d\nu_{p'}(x) &= g_{p'}^{-1/2}\mathrm{d}^4 x, \end{aligned}$$

respectively. Let $\mathcal{L}^2_{\mu_{p'}}(D_{p'})$ denote the space of all functions on $\mathcal{P}$ which are square integrable with respect to $d\mu_{p'}$ and vanish outside of $D_{p'}$. The covariant Fourier transform is then the map, $\mathcal{F}_{p'}$, given by

$$\begin{aligned} \mathcal{F}_{p'} : \mathcal{L}^2_{\mu_{p'}}(D_{p'}) &\to \mathcal{L}^2_{\nu_{p'}}(\mathcal{M}_{p'}) \\ f(p) &\mapsto \hat{f}_{p'}(x), \end{aligned}$$

where

$$\hat{f}_{p'}(x) \equiv \int_{D_{p'}} d\mu_{p'}(p)\mathcal{V}^{1/2}(p,p') \exp\left(-ix^{\mu'}\sigma^R_{\mu'}(p,p')\right) f(p). \tag{3.44}$$

The inverse Fourier transform is now

$$\mathcal{F}^{-1}_{p'}(\hat{f}_{p'})(p) \equiv \int_{\mathcal{M}_{p'}} d\nu_{p'}(x)\mathcal{V}^{1/2}(p,p') \exp\left(ix^{\mu'}\sigma^R_{\mu'}(p,p')\right) \hat{f}_{p'}(x), \tag{3.45}$$

for $p \in D_{p'}$ and zero otherwise. We can also obtain the Fourier representation of the delta function on $\mathcal{P}$

$$\delta(p,q) \equiv \int d\nu_{p'}(x)\mathcal{V}^{1/2}(p,p')\mathcal{V}^{1/2}(q,p') \exp\left[ix^{\mu'}\left(\sigma^R_{\mu'}(p,p') - \sigma^R_{\mu'}(q,p')\right)\right], \tag{3.46}$$

where $p, q \in D_{p'}$. Similarly, the Fourier representation of the delta function on $\mathcal{M}_{p'}$, denoted $\delta_{p'}(x,y)$, is

$$\delta_{p'}(x,y) = \int_{D_{p'}} d\mu(p)\mathcal{V}(p,p') \exp\left[i\sigma^R_{\mu'}(p,p')\left(x^{\mu'} - y^{\mu'}\right)\right]. \tag{3.47}$$

3.3.2 Plane waves

In this section we introduce the notion of plane waves which turn out to be an efficient method for representing the covariant Fourier transform. Formally, we define a plane wave, based at the point $p' \in \mathcal{P}$, to be the function of $p \in D_{p'}$ and $x \in \mathcal{M}_{p'}$ given by

$$e_{p'}(p,x) = \mathcal{V}^{1/2}(R_{p'}(p),p') \exp\left(-ix^{\mu'}\sigma^R_{\mu'}(p,p')\right). \tag{3.48}$$

Recalling the defining differential equation for the world-function, eq. (2.31), a simple calculation shows that $e_{p'}(p,x)$ is an eigenfunction of the Laplacian on $\mathcal{M}_{p'}$,

$$\begin{aligned} g^{\mu'\nu'}(p')\frac{\partial}{\partial x^{\mu'}}\frac{\partial}{\partial x^{\nu'}}e_{p'}(p,x) &= -g^{\mu'\nu'}(p')\sigma^R_{\mu'}(p,p')\sigma_{\nu'}(p,p')e_{p'}(p,x) \\ &= -2\sigma^R(p,p')e_{p'}(p,x) \\ &= -D^2(R_{p'}(p),p')e_{p'}(p,x). \end{aligned}$$

In particular, putting $p' = 0$ we find

$$D^2(p)e_0(p,x) = -\Box_x e_0(p,x); \tag{3.49}$$

a result which will be important in the sequel since it is $D^2(p)$ which appears in the action, S_{RL}. Returning to the definition of $e_{p'}(p,x)$ we see that the covariant Fourier transform, its inverse and the delta functions introduced in the previous section can be re-written as

$$\hat{f}_{p'}(x) = \int_{D_{p'}} d\mu_{p'}(p)e_{p'}(p,x)f(p), \tag{3.50}$$

$$f(p) = \int_{\mathcal{M}_{p'}} d\nu_{p'}(x)e^*_{p'}(p,x)\hat{f}_{p'}(x), \tag{3.51}$$

$$\delta(p,q) = \int_{\mathcal{M}_{p'}} d\nu_{p'}(x)e_{p'}(p,x)e^*_{p'}(q,x), \tag{3.52}$$

$$\delta_{p'}(x,y) = \int_{D_{p'}} d\mu_{p'}(p)e^*_{p'}(p,x)e_{p'}(p,y). \tag{3.53}$$

The advantage of this notation becomes apparent when we attempt to prove the Plancherel formula, which states that

$$\int_{\mathcal{M}_{p'}} d\nu_{p'}(x)\hat{f}_{p'}(x)\hat{f}^*_{p'}(x) = \int_{D_{p'}} d\mu_{p'}(p)f(p)f^*(p), \tag{3.54}$$

provided $\delta_{p'} \circ \hat{f}_{p'} = \hat{f}_{p'}$, which ensures that $\hat{f}_{p'}$ is in the image of the Fourier transform. The proof proceeds as follows, let $\hat{f}_{p'}(x) \in \mathcal{F}_{p'}(\mathcal{L}_{\hat{\mu}_{p'}}(D_{p'}))$ then

$$\int d\nu_{p'}(x)\hat{f}_{p'}(x)\hat{f}^*_{p'}(x) = \int d\nu_{p'}(x)d\mu_{p'}(p)d\mu_{p'}(q)e_{p'}(p,x)e^*_{p'}(q,x)f(p)f^*(q)$$

$$= \int d\mu_{p'}(p)d\mu_{p'}(q)\delta(p,q)f(p)f^*(q)$$
$$= \int d\mu_{p'}(p)f(p)f^*(p),$$

which is the desired result. A similarly straightforward calculation will also verify our claim that (3.45) represents the inverse of $\mathcal{F}_{p'}$.

Observe that the Fourier transform of a function lives in a particular cotangent space designated by p'. To understand the relationship between different choices of p' we define a transport operator $T_{p',q'}(x,y)$ which satisfies

$$\hat{f}_{p'}(x) \equiv \int_{\mathcal{M}_{q'}} d\nu_{q'}(y)T_{p',q'}(x,y)\hat{f}_{q'}(y). \tag{3.55}$$

In other words, $T_{p',q'}$ maps the Fourier transform in one cotangent space to the Fourier transform in another. We can derive an explicit expression for the transport operator by taking the transform of a particular function twice, i.e.

$$\hat{f}_{p'}(x) = \int_{D_{p'}} d\mu_{p'}(p)e_{p'}(p,x)f(p)$$
$$= \int_{D_{p'}\cap D_{q'}} d\mu_{p'}(p)\int_{\mathcal{M}_{q'}} d\nu_{q'}(y)e_{p'}(p,x)e^*_{q'}(p,y)\hat{f}_{q'}(x).$$

In the second line we took the Fourier transform at q' which requires $f(p)$ to vanish outside $D_{q'}$ and so we obtain the stated domain of integration $D_{p'} \cap D_{q'}$. Comparison with the definition of $T_{p',q'}$ in (3.55) yields

$$T_{p',q'}(x,y) = \int_{D_{p'}\cap D_{q'}} d\mu_{p'}(p)e_{p'}(p,x)e^*_{q'}(p,y). \tag{3.56}$$

In the limit where $p' = q'$ this transport operator is simply the delta function $\delta_{p'}(y,x)$, in all other cases $T_{p',q'}$ is a non-local operator.

3.3.3 Star Product

As a final piece of machinery we define a star product on $\mathcal{F}_{p'}(\mathcal{L}^2_{\hat{\mu}_{p'}}(D_{p'}))$ as follows

$$(\hat{f}_{p'} \star_{p'} \hat{g}_{p'})(x) \equiv \int_{\mathcal{M}_{p'}\times\mathcal{M}_{p'}} d\nu_{p'}(y)d\nu_{p'}(z)\omega_{p'}(x,y,z)\hat{f}_{p'}(y)\hat{g}_{p'}(z),$$

where the kernel $\omega_{p'}(x,y,z)$ is given by

$$\omega_{p'}(x,y,z) \equiv \int_{D_{p'}\times D_{p'}} d\mu_{p'}(p)d\mu_{p'}(q)e_{p'}(p\oplus q,x)e^*_{p'}(p,y)e^*_{p'}(q,z). \tag{3.57}$$

Note that the star product is defined only on functions living in the same cotangent spaces $\mathcal{M}_{p'} = T^*_{p'}\mathcal{P}$. Let's take a moment to explore some of the properties this product possesses. First, the product of two plane waves yields the rather pleasing result (see [100, 101] for similar properties in quantum gravity)

$$e_{p'}(p,x) \star_{p'} e_{p'}(q,x) = e_{p'}(p\oplus q,x).$$

Second, explicitly computing the star product of two functions, $(\hat{f}_{p'} \star_{p'} \hat{g}_{p'})(x)$, we find

$$\left(\hat{f}_{p'} \star_{p'} \hat{g}_{p'}\right)(x) = \int d\mu_{p'}(p)d\mu_{p'}(q)e_{p'}(p\oplus q,x)f(p)g(q), \tag{3.58}$$

where $f(p)$ and $g(p)$ have Fourier transforms $\hat{f}_{p'}$ and $\hat{g}_{p'}$ respectively. Furthermore, since $\oplus$ is not commutative we can see that $\star_{p'}$ will also fail to commute. Finally, taking the convolution product of three functions

$$\left(\hat{f}_{p'} \star_{p'} \left(\hat{g}_{p'} \star_{p'} \hat{h}_{p'}\right)\right)(x) = \int d\mu_{p'}(p)d\mu_{p'}(q)d\mu_{p'}(k)e_{p'}(p\oplus(q\oplus k),x)f(p)g(q)h(k), \tag{3.59}$$

which demonstrates that the failure of $\oplus$ to associate propagates a similar failure into $\star_{p'}$.

Let us now investigate the relationship between the star product and the standard point-wise product. Noting that $e_{p'}(0,x) = 1$ we can integrate (3.58) over x to find

$$\int d\nu_{p'}(x)\left(\hat{f}_{p'} \star_{p'} \hat{g}_{p'}\right)(x) = \int d\mu_{p'}(p)\left|\det\left(V_0^{\ominus p}\right)\right|^{-1} f(\ominus p)\, g(p) \tag{3.60}$$

On the other hand, if we compute the integral over the point-wise product $f_{p'}(x)g^*_{p'}(x)$ the Plancherel theorem will give the same result, less the factor of $\det(V)$. By setting $|\det(V_0^p)| = 1$ for all $p \in \mathcal{P}$ it follows that (the integral of) the star product and point-wise product match.[8] In this sense, we can say the star product of two functions is a local object. Performing a similar computation for the star product of three functions we find

$$\int d\nu_{p'}(x)\left(\hat{f}_{p'} \star_{p'} \left(\hat{g}_{p'} \star_{p'} \hat{h}_{p'}\right)\right)(x) = \int d\mu_{p'}(p)d\mu_{p'}(q)f(p\oplus q)g(\ominus p)h(\ominus q), \tag{3.61}$$

where we have also made the change of variables $p,q \to \ominus p, \ominus q$. A bit of thought should convince the reader that (3.61) bears little relation to the integral over the point-wise product of three functions, implying that the star product of three functions is a non-local object. This concludes the additional technical developments and we are now prepared to apply our formalism to the action S_{RL}.

[8] By virtue of (3.18) it follows that $|\det(U_0^p)| = 1$ for all $p \in \mathcal{P}$ as well.

3.3.4 Action in Spacetime

For ease of notation we will not explicitly display the domain of integration in any integrals occurring in this section. Comparing the terms appearing in eq. (3.36) with equations (3.60) and (3.61), and recalling that $\varphi(\ominus p) = \varphi^*(p)$, we can make the following replacements

$$m^2 \int d\mu_{p'}(p)\varphi(p)\varphi^*(p) = m^2 \int d\nu_{p'}(x) \left(\hat{\varphi}_{p'} \star_{p'} \hat{\varphi}_{p'}\right)(x), \tag{3.62}$$

and

$$\int d\mu_{p'}(q)d\mu_{p'}(k)\varphi(q \oplus k)\varphi^*(q)\varphi^*(k) = \int d\nu_{p'}(x) \left(\hat{\varphi}_{p'} \star_{p'} \left(\hat{\varphi}_{p'} \star_{p'} \hat{\varphi}_{p'}\right)\right)(x). \tag{3.63}$$

As discussed in the previous section, the integral appearing in equation (3.62) is local whereas the one appearing in equation (3.63) is not.

The $D^2(p)$ term is more complex and we can not make the simple replacements used above. We proceed by taking the covariant Fourier transform of $\varphi(p)$ and $\varphi^*(p)$

$$\int d\mu_{p'}(p)D^2(p)\varphi(p)\varphi^*(p) = \int d\mu_{p'}(p)d\nu_{p'}(x)d\nu_{p'}(y)D^2(p)e^*_{p'}(p,x)e_{p'}(p,y)\hat{\varphi}_{p'}(x)\hat{\varphi}^*_{p'}(y). \tag{3.64}$$

To proceed we would like to use equation (3.49) and exchange $D^2(p)$ for derivatives of a plane wave, but doing so requires a plane wave based at $p' = 0$. As such we shift $e_{p'}(p, y)$ to $e_0(p, z)$ by introducing the translation operator $T_{p',0}(y, z)$, viz

$$D^2(p)e_{p'}(p,y) = \int d\nu_0(z)D^2(p)T_{p',0}(y,z)e_0(p,z) = -\int d\nu_0(z)T_{p',0}(y,z)\Box_z e_0(p,z)$$

Integrating by parts moves the derivatives onto $T_{p',0}$ which allows us to translate the plane wave back to p' by introducing another translation operator

$$D^2(p)e_{p'}(p,y) = -\int d\nu_0(z)d\nu_{p'}(a)e_{p'}(p,a)T_{0,p'}(z,a)\Box_z T_{p',0}(y,z). \tag{3.65}$$

We can now substitute this back into (3.64) and integrate over p to obtain the delta function $\delta_{p'}(a, x)$, an integration over a then gives

$$\begin{aligned}\int d\mu_{p'}(p)D^2(p)\varphi(p)\varphi^*(p) &= -\int d\nu_{p'}(x)d\nu_{p'}(y)d\nu_0(z)T_{0,p'}(z,x)\Box_z T_{p',0}(y,z)\hat{\varphi}_{p'}(x)\hat{\varphi}^*_{p'}(y) \\ &= -\int d\nu_{p'}(y)d\nu_0(z)\left(\Box_z T_{p',0}(y,z)\right)\hat{\varphi}_0(z)\hat{\varphi}^*_{p'}(y)\end{aligned}$$

$$= -\int d\nu_{p'}(y) d\nu_0(z) T_{p',0}(y,z) \Box_z \hat{\varphi}_0(z) \hat{\varphi}^*_{p'}(y).$$

In the special case $p' = 0$ the translation operator becomes a delta function and integrating over z we obtain the expected (and local) result $-\int d\nu_0(y) \hat{\varphi}^*_0(y) \Box_y \hat{\varphi}_0(y)$. On the other hand, if $p' \neq 0$ the transport operator will be de-localized and the overall result non-local. For ease of notation we will denote $(\Box_y \hat{\varphi})_{p'}(y) = \int d\nu_0(z) T_{p',0}(y,z) \Box_z \hat{\varphi}_0(z)$ and so the $D^2(p)$ term can be written as

$$\int d\mu_{p'}(p) D^2(p) \varphi(p) \varphi^*(p) = -\int d\nu_{p'}(x) \left(\hat{\varphi}_{p'} \star_{p'} (\Box \hat{\varphi})_{p'} \right)(x), \tag{3.66}$$

recalling that the integral over the point-wise product of two functions is identical to the integral over the star product of two functions.

Putting the results of this section together we find that the action for our scalar field theory, in the spacetime $\mathcal{M}_{p'}$, is given by

$$S^{p'}_{RL} = \frac{1}{2} \int d\nu_{p'}(x) \left[\left(\hat{\varphi}_{p'} \star_{p'} (\Box \hat{\varphi})_{p'} \right)(x) - m^2 \left(\hat{\varphi}_{p'} \star_{p'} \hat{\varphi}_{p'} \right)(x) \right] \tag{3.67}$$

$$+ \frac{g}{3!} \int d\nu_{p'}(x) \left(\hat{\varphi}_{p'} \star_{p'} \left(\hat{\varphi}_{p'} \star_{p'} \hat{\varphi}_{p'} \right) \right)(x). \tag{3.68}$$

Observe that the interaction term is non-local for any choice of p' and the m^2 term is local for any choice of p'. The kinetic term on the other hand is local for $p' = 0$ but non-local for any other choice of the base point. This shows that if we denote $\hat{\varphi} \equiv \hat{\varphi}_0$, $d\nu(x) \equiv d\nu_0(x)$ and $\star \equiv \star_0$, the relative locality action becomes, simply

$$S_{RL} = \frac{1}{2} \int d\nu(x) \left[\left(\hat{\varphi} \Box \hat{\varphi} \right)(x) - m^2 \hat{\varphi} \hat{\varphi}(x) \right] + \frac{g}{3!} \int d\nu(x) \left(\hat{\varphi} \star (\hat{\varphi} \star \hat{\varphi}) \right)(x). \tag{3.69}$$

Part II

Interaction Vertex for a Spinning Particle

Chapter 4

Interaction Vertex for Classical Spinning Particle

4.1 Introduction

The framework of Relative Locality represents a radical departure from the usual notions of spacetime and locality as evidenced by the modifications of the vertex factor discussed in the previous Chapter. On the other hand, Chapter 2 shows that these modifications are not apparent in the interactions between scalar particles, even in the presence of a non-trivial background. However, most particles are not scalars, and we know that internal structure, such as spin, modifies the vertex factor [6]. To understand the extent to which this modification affects the locality of interactions we need to develop a worldline formulation of the relativistic spinning particle. This will be done by means of the coadjoint orbit formalsim and the resulting "Dual Phase Space" model will be central to the remainder of this thesis.

4.2 Elementary Classical Systems and Their Quantization

In this section, we discuss the mathematical preliminaries which allow for a classical formulation of the spinning particle. For some readers this might sound paradoxical, since spin is often viewed as a purely quantum object. However, while there are some phenomena, like the relationship between spin and statistics, which are purely quantum, it does not follow that the relativistic spinning particle has no classical description. What it does mean is that this description will only be accurate in the limit of large spins.

It is generally true that one can construct a classical realization of any quantum structure associated with a group G; for spin, the relevant group is the Poincaré group. The procedure for doing so is called the coadjoint orbit method [102] and is outlined below for the case of matrix Lie groups a reasonable simplification, as most groups of interest fall into this category.

Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a matrix Lie group and $\mathfrak{g} \subset \mathrm{Mat}(n, \mathbb{C})$ its Lie algebra. The adjoint action of $g \in G$ on $X \in \mathfrak{g}$ is then matrix conjugation $\mathrm{Ad}(g)X = gXg^{-1}$, and the coadjoint action of G on the dual algebra $\mathfrak{g}^*$ is obtained by taking the dual of Ad. It satisfies

$$\langle \mathrm{Ad}^*(g)\lambda, X\rangle = \langle \lambda, \mathrm{Ad}(g^{-1})X\rangle, \tag{4.1}$$

where $\lambda \in \mathfrak{g}^*$ and $\langle , \rangle$ denotes the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. Each coadjoint orbit $\mathcal{O}_\lambda = \{\mathrm{Ad}^*(g)\lambda \,|\, g \in G\}$ possesses a natural symplectic structure σ_λ and the pair $(\mathcal{O}_\lambda, \sigma_\lambda)$ forms the classical phase space associated with the symmetry group G. To obtain σ_λ explicitly, we let H_λ be the isotropy group for some $\lambda \in \mathfrak{g}^*$, then the bijection $p_\lambda : G/H_\lambda \to \mathcal{O}_\lambda : [g] \to \mathrm{Ad}^*(g)\lambda$ identifies the homogeneous space G/H_λ with the coadjoint orbit through λ. A choice of section $g : G/H_\lambda \to G$ allows us to pull back the Maurer-Cartan form on G to give a symplectic potential on G/H_λ:

$$\theta_\lambda = \langle \lambda, g^{-1}dg\rangle. \tag{4.2}$$

The value of θ_λ depends explicitly on the choice of section. In particular, if $h : G/H_\lambda \to H_\lambda$, the change of section $g \to gh$ yields a corresponding variation $\delta\theta_\lambda = -\langle \lambda, h^{-1}dh\rangle$. Since $Ad^*(H_\lambda)\lambda = \lambda$, this sectional dependence disappears when considering the symplectic form

$$\omega_\lambda = d\theta_\lambda = -\langle \lambda, g^{-1}dg \wedge g^{-1}dg\rangle, \tag{4.3}$$

where the Maurer-Cartan equation $d(g^{-1}dg) = -g^{-1}dg \wedge g^{-1}dg$ has been used. One can now obtain the symplectic form on $\mathcal{O}_\lambda$ by taking the pullback of ω_λ under p_λ^{-1}: $\sigma_\lambda = (p_\lambda^{-1})^*\omega_\lambda$.

We can proceed a bit further. Let $\hat{X}$ denote the extension of the Lie algebra element $X \in \mathfrak{g}$ to a right invariant vector field over G; then

$$\omega_\lambda(\hat{X}, \cdot) = \langle F_\lambda(g), [X, dgg^{-1}]\rangle = d\langle F_\lambda, X\rangle, \tag{4.4}$$

where $F_\lambda(g) := \mathrm{Ad}^*(g)\lambda$ is a generic element of the coadjoint orbit through λ. It follows that the linear function $\mathcal{H}_X(g) := \langle F_\lambda(g), X\rangle$ is a Hamiltonian for the group action and $F_\lambda : G/H_\lambda \to \mathfrak{g}^*$ is its moment map. It follows that the Poisson bracket between two such functions is the commutator $\{\mathcal{H}_X, \mathcal{H}_Y\} = \mathcal{H}_{[X,Y]}$. A classical description of some system is only useful if one can pass to the corresponding quantum version. In the present context, this transition amounts to finding a map between the coadjoint orbits of a group and its irreducible representations. The key idea is that a classical phase space corresponds to a quantum Hilbert space, and a phase space function corresponds to an operator; the symmetry then restricts the mapping almost uniquely. A formal correspondence between a classical system and its quantum counterpart is accomplished via geometric quantization [103], which also forms the basis of the Feynman path integral formulation of quantum mechanics. If the quantum system is finite dimensional, the corresponding phase space has to be compact, since the Hilbert space dimension is related to the phase space volume. Heuristically, the construction proceeds as follows: Let $\mathcal{O}_\lambda$ be a coadjoint orbit of G, and let $X \in \mathfrak{g}$ be a Lie algebra element; the trace of a group element in a unitary irreducible representation $\rho_\lambda : G \to \mathcal{O}_\lambda$ of highest weight λ is then given by

$$\mathrm{Tr}_V\left(\rho_\lambda(e^{iX})\right) = \int \mathcal{D}g\, e^{\frac{i}{\hbar}\int_{S^1}\left[\langle\lambda, g^{-1}\dot{g}\rangle - \langle F_\lambda(g), X\rangle\right]d\tau}, \tag{4.5}$$

where the path integral is taken over all group valued periodic maps $g : S^1 \to G$. This is just a generalization of the usual Feynman path integral quantization where $\mathrm{Tr}e^{i\hat{\mathcal{H}}(\hat{p},\hat{q})}$ is written as

$$\mathrm{Tr}e^{i\hat{\mathcal{H}}(\hat{p},\hat{q})} = \int \mathcal{D}p\mathcal{D}q e^{\frac{i}{\hbar}\int_{S^1}(p\dot{q} - H(p,q))d\tau}, \tag{4.6}$$

and the paths are chosen to be periodic. Here the phase space variables are (p, q), with symplectic potential $p\mathrm{d}q$ and Hamiltonian $\mathcal{H}(p, q)$. In our case, the phase space variables are group elements g, with symplectic potential $\theta_\lambda = \langle\lambda, g^{-1}\mathrm{d}g\rangle$ and Hamiltonian $\mathcal{H}_X(g) = \langle F_\lambda(g), X\rangle$ as discussed above.

This procedure can be reversed, mapping irreducible representations onto coadjoint orbits. To see this, suppose that $\rho : G \to \mathrm{GL}(V)$ is a unitary irreducible representation of G over the vector space V. To each normalized vector $|\Lambda\rangle \in V$, we can associate a linear functional $\lambda \in \mathfrak{g}^*$ by defining

$$\lambda(X) := \hbar\, \langle\Lambda|\, \mathrm{d}\rho(X)\, |\Lambda\rangle\,, \tag{4.7}$$

where $X \in \mathfrak{g}$ and $\mathrm{d}\rho$ is the representation of $\mathfrak{g}$ induced by ρ. H_λ is by definition the subgroup that acts diagonally on Λ, and so, if $h = e^{iH/\hbar} \in H_\lambda$ its action is given by

$$\rho(h)\, |\Lambda\rangle = e^{i\frac{\lambda(H)}{\hbar}}\, |\Lambda\rangle\,. \tag{4.8}$$

It follows that the linear functional associated with $\rho(g)|\Lambda\rangle$ is $\mathrm{Ad}^*(g)\lambda$. If ρ is an irreducible representation, every vector in V can be represented as a linear combination of elements $\rho(g)|\Lambda\rangle$ with $g \in G$; therefore the map

$$V \rightarrow \mathcal{O}_\lambda, \tag{4.9}$$

$$\rho(g)|\Lambda\rangle \mapsto F_\lambda(g) \tag{4.10}$$

identifies rays in V with points in the coadjoint orbits. More explicitly, if we label elements of $\mathcal{O}_\lambda$ by the operators $X_\rho(g) := \rho(g)|\Lambda\rangle\langle\Lambda|\rho^\dagger(g)$, then the symplectic form

$$\omega_\rho := -\hbar \mathrm{Tr}_V(X_\rho \mathrm{d}X_\rho \wedge \mathrm{d}X_\rho) \tag{4.11}$$

simplifies to $\omega_\rho = \hbar\langle\Lambda|\rho(g^{-1})d\rho(g) \wedge \rho(g^{-1})d\rho(g)|\Lambda\rangle$, which is equivalent to the one given in eq. (4.3).

4.3 Coadjoint Orbits of the Poincaré Group

Although we have presented the coadjoint orbit method in general, we are only interested in its application to the Poincaré group $\mathcal{P} = \mathrm{SO}(3,1) \rtimes \mathbb{R}^4$, which is well known to describe the symmetries of a relativistic spinning particle. In this section, we will review the construction of these orbits and show that they are characterized by two quantities which are identified with the particle's mass and spin.

Let $g(\Lambda, x)$ be a generic element of the Poincaré group, where $\Lambda \in \mathrm{SO}(3,1)$ is a Lorentz transformation and $x \in \mathbb{R}^4$ a translation; the group product is given by $(\Lambda_1, x_1)(\Lambda_2, x_2) = (\Lambda_1\Lambda_2, x_1 + \Lambda_1 x_2)$. The generators of translations and Lorentz transformations, which form a basis for the Lie algebra $\mathfrak{p}$, are denoted P_μ and $\mathcal{J}_{\mu\nu} = -\mathcal{J}_{\nu\mu}$, respectively, and satisfy

$$[P_\mu, \mathcal{J}_{\nu\rho}] = \eta_{\mu\nu}P_\rho - \eta_{\mu\rho}P_\nu, \qquad [\mathcal{J}_{\mu\nu}, \mathcal{J}_{\rho\sigma}] = \eta_{\mu\sigma}\mathcal{J}_{\nu\rho} + \eta_{\nu\rho}\mathcal{J}_{\mu\sigma} - \nu_{\mu\rho}\mathcal{J}_{\nu\sigma} - \eta_{\nu\sigma}\mathcal{J}_{\mu\rho}.$$

It is now a straightforward exercise to compute the adjoint action of $g(\Lambda, x)$ on $\mathfrak{p}$, viz.

$$\mathrm{Ad}(g(\Lambda, x))P_\mu = \Lambda^\nu{}_\mu P_\nu, \qquad \mathrm{Ad}(g(\Lambda, x))\mathcal{J}_{\mu\nu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \left(\mathcal{J}_{\rho\sigma} + P_\rho x_\sigma - P_\sigma x_\rho\right). \tag{4.12}$$

Introduce dual generators $\hat{P}^\mu$ and $\hat{\mathcal{J}}^{\mu\nu}$ as a basis for the dual algebra $\mathfrak{p}^*$, and let $\langle,\rangle$ be the natural pairing between $\mathfrak{p}$ and $\mathfrak{p}^*$; then $\langle\hat{P}^\mu, P_\nu\rangle = \delta^\mu_\nu$ and $\langle\hat{\mathcal{J}}^{\mu\nu}, \mathcal{J}_{\rho\sigma}\rangle = 2\delta^\mu_{[\rho}\delta^\nu_{\sigma]}$. The coadjoint action is obtained from eq. (4.12) by recalling its definition in terms of the adjoint action, see eq. (4.1). We find

$$\mathrm{Ad}^*(g(\Lambda, x))\hat{P}^\mu = \Lambda_\nu{}^\mu \left(\hat{P}^\nu - x_\rho \hat{\mathcal{J}}^{\nu\rho}\right), \qquad \mathrm{Ad}^*(g(\Lambda, x))\hat{\mathcal{J}}^{\mu\nu} = \Lambda_\rho{}^\mu \Lambda_\sigma{}^\nu \hat{\mathcal{J}}^{\rho\sigma}, \tag{4.13}$$

where ${\Lambda_\mu}^\nu = {(\Lambda^{-1})^\nu}_\mu$. Elements of the dual algebra $F \in \mathfrak{p}^*$ are parametrized by a vector m_μ and an antisymmetric tensor $M_{\mu\nu}$, $F = (m_\mu, M_{\mu\nu})$. Under the coadjoint action, these components transform as

$$m_\mu \xrightarrow{\mathrm{Ad}^*(g(\Lambda,x))} p_\mu = {\Lambda_\mu}^\nu m_\nu, \tag{4.14}$$

$$M_{\mu\nu} \xrightarrow{\mathrm{Ad}^*(g(\Lambda,x))} J_{\mu\nu} = (x \wedge p)_{\mu\nu} + {\Lambda_\mu}^\rho {\Lambda_\nu}^\sigma M_{\rho\sigma}, \tag{4.15}$$

where $(A \wedge B)_{\mu\nu} = A_\mu B_\nu - A_\nu B_\mu$. The quantities p_μ and $J_{\mu\nu}$ have standard physical interpretations: p_μ represents the total linear momentum of the particle, while $J_{\mu\nu}$ represents the total angular momentum about the origin. Notice that we can split the total angular momentum as $J = L + S$, where $L = x \wedge p$ is the orbital part and $S = (\Lambda M \Lambda^T)$ is the spin angular momenta.

These orbits are characterized by the value of two invariants,[1] one of which is $p^2 = -m^2$, with m representing the mass of the particle. If $m > 0$, the other invariant is $w^2 = m^2 s^2$, where $w_\mu = \frac{1}{2\hbar}\epsilon_{\mu\nu\rho\sigma} p^\nu J^{\rho\sigma}$ is the Pauli-Lubanski vector and s is identified with the particle's spin. The phase space for a relativistic spinning particle of mass m and spin s is then

$$\mathcal{O}_{m,s} = \{(p_\mu, J_{\mu\nu}) \,|\, p^2 = -m^2 \,\text{and}\, w^2 = m^2 s^2\}. \tag{4.16}$$

An arbitrary element $F_{m,s} \in \mathcal{O}_{m,s}$ defines the symplectic form $\sigma_{F_{m,s}}$, and the symplectic manifold $(\mathcal{O}_{m,s}, \sigma_{F_{m,s}})$ constitutes a complete description of the relativistic spinning particle.

If, on the other hand, $m = 0$, then $w^2 = 0$, and since $w \cdot p = 0$, the Pauli-Lubanski vector must be proportional to the momentum $w_\mu = s p_\mu$; the constant of proportionality will be the second orbit invariant. Physically, this represents a massless spinning particle with helicity given by s; the corresponding phase space is denoted $(\mathcal{O}_{0,s}, \sigma_{F_{0,s}})$. There should be no confusion in denoting the spin and helicity by the same variable s as it will be clear from context what is being referred to.

4.4 Models of the Classical Spinning Particle

Given a coadjoint orbit of the Poincaré group, eq. (4.16), a model of the relativistic spinning particle is obtained by making a choice of coordinates on $\mathcal{O}_{m,s}$. There are many viable options, and the resulting theories can seem disparate, but this is only superficial, as one

[1] Quantities which remain unchanged by the coadjoint action of $\mathcal{P}$.

can always find a coordinate transformation between competing models. We demonstrate this explicitly for two popular coordinatizations, those of Balachandran [43] and Wiegmann [52], and in the process examine how the standard quantization condition $2s \in \mathbb{Z}$ arises.

4.4.1 Homogeneous space

With $m > 0$, we can choose $F_{m,s}$ to have components $m_\mu := m\delta^0_\mu$ and $M_{\mu\nu} := 2\hbar s \delta^1_{[\mu}\delta^2_{\nu]}$ which transform under the coadjoint action of $g(\Lambda, x)$ as

$$m\delta^0_\mu \longrightarrow p_\mu = m{\Lambda_\mu}^0, \tag{4.17}$$

$$2s\delta^1_{[\mu}\delta^2_{\nu]} \longrightarrow J_{\mu\nu} = 2mx_{[\mu}{\Lambda_{\nu]}}^0 + 2\hbar s {\Lambda_{[\mu}}^1{\Lambda_{\nu]}}^2. \tag{4.18}$$

The phase space $\mathcal{O}_{m,s}$ is then regarded as a subset of $\mathcal{P}$ coordinatized by $\left\{x^\mu, {\Lambda_\mu}^0, {\Lambda_\mu}^1, {\Lambda_\mu}^2\right\}$. In this parametrization the splitting $J = L + S$ is realized explicitly as

$$L_{\mu\nu} = m\left(x_\mu {\Lambda_\nu}^0 - x_\nu {\Lambda_\mu}^0\right) \qquad \text{and} \qquad S_{\mu\nu} = \hbar s\left({\Lambda_\mu}^1{\Lambda_\nu}^2 - {\Lambda_\nu}^1{\Lambda_\mu}^2\right). \tag{4.19}$$

Comparison with Eqs. (1) and (2) of Ref. [43] shows that this parametrization is identical to that of Balachandran.

To obtain the symplectic potential $\theta_{m,s}$, we first expand the Lie algebra valued one-form $g^{-1}dg$ in the basis $\{P_\mu, \mathcal{J}_{\mu\nu}\}$

$$g^{-1}(\Lambda, x)dg(\Lambda, x) = -{\Lambda_\nu}^\mu dx^\nu P_\mu + \frac{1}{2}\eta_{\rho\sigma}\Lambda^{\rho\mu}d\Lambda^{\sigma\nu}\mathcal{J}_{\mu\nu}.$$

Then, with $F_{m,s}$ as described above, eq. (4.2) gives

$$\theta_{m,s} = -m{\Lambda_\mu}^0 dx^\mu + \frac{\hbar s}{2}\eta^{\mu\nu}\left({\Lambda_\mu}^1 d{\Lambda_\nu}^2 - {\Lambda_\nu}^2 d{\Lambda_\mu}^1\right). \tag{4.20}$$

We can now identify $p_\mu = m{\Lambda_\mu}^0$ with the momentum conjugate to x^μ and write the symplectic form $\omega_{m,s} = d\theta_{m,s}$ as

$$\omega_{m,s} = dx^\mu \wedge dp_\mu + \hbar s \eta^{\mu\nu} d{\Lambda_\mu}^1 \wedge d{\Lambda_\nu}^2. \tag{4.21}$$

Finally, we obtain an action by regarding all coordinates as a function of an auxiliary parameter τ and integrate the symplectic potential, viz.

$$S = \int d\tau \left[p_\mu \dot{x}^\mu - \frac{\hbar s}{2}\eta^{\mu\nu}\left({\Lambda_\mu}^1 {\dot{\Lambda}_\nu}{}^2 - {\Lambda_\nu}^2 {\dot{\Lambda}_\mu}{}^1\right)\right], \tag{4.22}$$

where we have dropped an overall minus sign in the action. Note that we still regard p_μ as being derived from the Lorentz transformation ${\Lambda_\mu}^0$, which implies that this parametrization is explicitly on-shell.

4.4.2 Vector on a sphere

In Ref. [52], Wiegmann parametrizes, in a natural way, the spinning degrees of freedom by a unit vector n_μ orthogonal to the linear momentum p_μ. We now explicitly show that the Wiegmann parametrization is equivalent to Balachandran's. To see how this correspondence comes about, we set $A_\mu = {\Lambda_\mu}^1$ and $B_\mu = {\Lambda_\mu}^2$; then

$$\omega^{\mathrm{S}}_{m,s} = \hbar s \eta^{\mu\nu} dA_\mu \wedge dB_\nu \qquad \text{and} \qquad S_{\mu\nu} = \hbar s (A\wedge B)_{\mu\nu}, \tag{4.23}$$

where $\omega^{\mathrm{S}}_{m,s} = \omega_{m,s} - dx\wedge dp$ is the spin component of the symplectic potential. We introduce the unit momenta $u_\mu = p_\mu/m$ and define $n_\mu = \epsilon_{\mu\nu\rho\sigma}u^\nu A^\rho B^\sigma$; note that n_μ is proportional to the Pauli-Lubanski vector $w_\mu = msn_\mu$. The set $\{u_\mu, n_\mu, A_\mu, B_\mu\}$ forms an orthonormal basis for $\mathbb{R}^4$ adapted to the particle's motion. We can then expand the Minkowski metric as

$$\eta_{\mu\nu} = -u_\mu u_\nu + n_\mu n_\nu + A_\mu A_\nu + B_\mu B_\nu.$$

If we replace the $\eta^{\mu\nu}$ appearing in $\omega^{\mathrm{S}}_{m,s}$ with the expanded version above, we obtain

$$\omega^{\mathrm{S}}_{m,s} = \frac{\hbar s}{2}(A\wedge B)^{\mu\nu}\left(du_\mu \wedge du_\nu - dn_\mu \wedge dn_\nu\right).$$

We can now make use of the relation $(A\wedge B)_{\mu\nu} = -\epsilon_{\mu\nu\rho\sigma}u^\rho n^\sigma$ to eliminate A and B from the expressions for $\omega^{\mathrm{S}}_{m,s}$ and $S_{\mu\nu}$ and obtain a parametrization given entirely in terms of u_μ and n_μ:

$$\omega^{\mathrm{S}}_{m,s} = \frac{\hbar s}{2}\epsilon_{\mu\nu\rho\sigma}u^\mu n^\nu \left(dn^\rho \wedge dn^\sigma - du^\rho \wedge du^\sigma\right), \qquad S^{\mathrm{S}}_{\mu\nu} = -\hbar s \epsilon_{\mu\nu\rho\sigma}u^\rho n^\sigma, \tag{4.24}$$

which corresponds to the Wiegmann expressions [52]. The phase space of this model is coordinatized by $\{x^\mu, p^\mu, n^\mu\}$ subject to the constraints

$$p^2 = -m^2, \qquad n^2 = 1, \qquad p\cdot n = 0, \tag{4.25}$$

which define the on-shell hypersurface. In the rest frame, $u_\mu = \delta^0_\mu$, and the symplectic form $\omega^{\mathrm{S}}_{m,s}$ reduces to

$$\sigma^{\mathrm{S}} = -\frac{\hbar s}{2}\epsilon_{ijk}n^i dn^j \wedge dn^k, \tag{4.26}$$

which is just the area form on a sphere of radius $\hbar s$. It follows that we can regard the two-form eq. (4.24) as a "relativistic generalization" of the symplectic structure on a sphere and n_μ as an S^2 vector boosted in the direction of p_μ.

4.4.3 Quantization condition

As presented above, the quantity s, which represents the particle's spin, is permitted to assume any real value. Recovering the usual restriction $2s \in \mathbb{N}$ one demands that the symplectic form $\omega/\hbar$ be integral; i.e., the integral of $\omega/\hbar$ over a nontrivial two-cycle is an integer multiple of 2π. Consider what this means for the model of Sec. 4.4.2 where there is a single non-trivial two cycle, namely the sphere S^2. In the rest frame, the quantization condition says

$$\frac{1}{\hbar}\int_{S^2}\omega = s\int_{S^2}\frac{1}{2}\epsilon_{ijk}n^i dn^j \wedge dn^k \in 2\pi\mathbb{N}.$$

The quantity under the integral sign is the area form on the two-sphere and evaluates to 4π, which immediately gives the expected result $2s \in \mathbb{N}$.

A more intuitive approach is as follows: Let $\mathcal{C}$ denote the worldline of a spinning particle. Then one can attempt to define an action as the integral over the symplectic potential, i.e. $S = \int_{\mathcal{C}} \theta_{m,s}$. Unfortunately, this is not well defined, since the symplectic form is not exact, and so $\theta_{m,s}$ does not exist globally. Instead, we need to define S as the integral of $\omega_{m,s}$ over some surface of which $\mathcal{C}$ is a boundary:

$$S = \int_{\mathcal{C}} \theta_{m,s} = \int_{\mathcal{S}} \omega_{m,s},$$

where $\partial\mathcal{S} = \mathcal{C}$. The choice of $\mathcal{S}$ is ambiguous, but if we demand that different surfaces change S by a multiple of $2\pi\hbar$, then the path integral will be well defined, since it is $e^{\frac{i}{\hbar}S}$, which is the relevant quantity. For the vector on a sphere, $\mathcal{C} = S^1$, and so $\mathcal{S}$ can be either the upper or lower half sphere. In the rest frame we have

$$\int_{S^2_{\text{upper}}} \omega^{\text{S}}_{m,s} = \int_{S^2} \omega^{\text{S}}_{m,s} + \int_{S^2_{\text{lower}}} \omega^{\text{S}}_{m,s},$$

and so we demand that $\int_{S^2} \omega^{\text{S}}_{m,s} = 2\pi\hbar$, which is the same condition arrived at in the more formal approach.

4.5 Dual Phase Space Model

The previous section presented a sampling of possible parameterizations for the coadjoint orbits of the Poincaré group. There are many other options, all of which are equivalent and can be used interchangeably depending on what aspect of the theory is to be emphasized. Presently, our interest is in analyzing the interaction vertex, and so we introduce a parametrization that is particularly well suited to this task.

4.5.1 Choosing the coordinates

To define this parametrization, we introduce a length scale λ and an energy scale ϵ such that $\lambda\epsilon = \hbar$; otherwise these scales are arbitrary constants. We recall the parametrization presented in Sec. 4.4.1 and define variables $\chi_\mu = \lambda {\Lambda_\mu}^1$ and $\pi_\mu = \epsilon s {\Lambda_\mu}^2$ so that the symplectic form [eq. (4.21)] is written as[2]

$$\omega = dx^\mu \wedge dp_\mu + d\chi^\mu \wedge d\pi_\mu. \tag{4.27}$$

We now forget that p_μ, χ^μ, and π_μ are components of a Lorentz transformation and instead regard them as variables on a classical phase space coordinatized by $\{x^\mu, p_\mu, \chi^\mu, \pi_\mu\}$. It follows from eq. (4.27) that (x^μ, p_μ) and (χ^μ, π_μ) form pairs of canonically conjugate variables with Poisson brackets:

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu, \qquad \{\chi^\mu, \pi_\nu\} = \delta^\mu_\nu, \tag{4.28}$$

wiht all others vanishing. From this perspective χ^μ and π_μ span a "dual" phase space, separate from the standard phase space of x^μ and p_μ, which encodes information about the particle's spin. The internal angular momentum, $S_{\mu\nu}$, further bears out this duality, since in these variables it assumes the form [see eq. (4.19)]

$$S_{\mu\nu} = (\chi \wedge \pi)_{\mu\nu}, \tag{4.29}$$

in direct analogy to orbital angular momentum $L_{\mu\nu} = (x \wedge p)_{\mu\nu}$. It is for this reason that we have called this formulation the *dual phase space model* or DPS and view χ_μ and π_μ as a dual "coordinate" and "momenta," respectively.

It remains to explicitly impose relations among the phase space variables that were implicit in their origin as Lorentz transformations. These constraints will define the dynamics of our theory and are given by

$$\left(p^2 = -m^2,\;\; \pi^2 = \epsilon^2 s^2\right), \qquad \left(p\cdot\pi = 0,\;\; p\cdot\chi = 0\right), \qquad \left(\chi^2 = \lambda^2,\;\; \chi\cdot\pi = 0\right). \tag{4.30}$$

We have grouped the constraints in this manner to emphasize the duality mentioned above. The first pair are mass shell conditions, one in standard phase space $p^2 = -m^2$ and one in dual phase space $\pi^2 = \epsilon^2 s^2$. In this description, the spin is proportional to the length of the dual momenta. In the second set, we see that the two phase spaces are not independent; rather, dual phase space is orthogonal to the canonical momenta. The final

[2] From now on, we will drop subscripts on the symplectic form.

two constraints emphasize the dramatic difference between standard phase space and dual phase space, since in the former x is totally unconstrained, while χ is constrained to live on a two-sphere.

As presently formulated, DPS assumes $m \neq 0$; recall that we made this assumption at the outset of Sec. 4.4.1. This restriction can easily be lifted, as all aspects of the current formulation, both Poisson brackets and constraints, are well defined in the limit $m \to 0$.

An important point to emphasize is that this parametrization is invariant under an $\mathrm{SL}(2,\mathbb{R})$ global symmetry, since any transformation of the form

$$(\chi_\mu, \pi_\mu) \to (A\chi_\mu + B\pi_\mu, C\chi_\mu + D\pi_\mu), \qquad AD - BC = 1 \tag{4.31}$$

does not alter the Poisson brackets [eq. (4.28)] or the angular momenta [eq. (4.29)]. Part of this symmetry can be fixed by imposing the orthogonality condition $\pi \cdot \chi = 0$; the remaining symmetry consists of a rescaling $(\chi_\mu, \pi_\mu) \to (\alpha\chi, \alpha^{-1}\pi)$ as well as a rotation

$$(\chi_\mu, \pi_\mu) \to (\cos\theta\chi_\mu + \tfrac{\lambda}{\epsilon s}\sin\theta\pi_\mu, \cos\theta\pi_\mu - \tfrac{\epsilon s}{\lambda}\sin\theta\chi_\mu). \tag{4.32}$$

These demonstrate, respectively, that the choice of scales λ and ϵ as well as the initial direction of the dual momenta are immaterial; only the product $\lambda\epsilon$ is physically meaningful. We now assume that a choice of scale and axis has been made.

A brief note before we continue: The parametrization presented in this section is identical to the one used by Wigner in his description of continuous spin particles [104] (see also [105] for a classical realization which emphasis the similarity). However, to the authors' knowledge it has never been used in the context of standard spinning particles.

4.5.2 Action and equations of motion

An action for DPS is obtained by making the appropriate change of variables to eq. (4.22), and explicitly implementing the constraints eq. (4.30) by means of Lagrange multipliers, viz.

$$\begin{aligned} S = \int d\tau \Big[& p_\mu \dot{x}^\mu + \pi_\mu \dot{\chi}^\mu - \frac{N}{2}(p^2 + m^2) - \frac{M}{2}\left(\frac{\pi^2}{\epsilon^2} + \frac{s^2\chi^2}{\lambda^2} - 2s^2\right) \\ & - \frac{N_1}{2}\left(\frac{s^2\chi^2}{\lambda^2} - \frac{\pi^2}{\epsilon^2}\right) - N_2\,(\chi\cdot\pi) - N_3\,(p\cdot\pi) - N_4\,(p\cdot\chi)\Big], \end{aligned} \tag{4.33}$$

where we have combined some of the constraints in anticipation of the upcoming constraint analysis. Computing the constraint algebra we find, for $ms \neq 0$, there are two *first-class* *constraints*,

$$\Phi_{\rm m} := \frac{1}{2}\left(p^2+m^2\right), \qquad \Phi_{\rm s} := \frac{1}{2}\left(\frac{\pi^2}{\epsilon^2}+\frac{s^2\chi^2}{\lambda^2}\right)-s^2, \tag{4.34}$$

and four *second-class constraints*,

$$\Phi_1 = \frac{1}{2}\left(\frac{s^2\chi^2}{\lambda^2}-\frac{\pi^2}{\epsilon^2}\right), \qquad \Phi_2 = \chi\cdot\pi, \tag{4.35}$$

$$\Phi_3 = p\cdot\pi, \qquad \Phi_4 = p\cdot\chi. \tag{4.36}$$

The latter satisfy the algebra

$$\{\Phi_1,\Phi_2\}\approx 2s^2, \qquad \{\Phi_3,\Phi_4\}\approx m^2,$$

where $\approx$ denotes equality on the constraint surface and all other commutators vanish.[3] This means that (Φ_1,Φ_2) form a canonical pair whenever $s\neq 0$, as do (Φ_3,Φ_4) when $m\neq 0$. Furthermore, when $m=0$, the constraints Φ_3 and Φ_4 become first class, and so a massless spinning particle is described by *four* first-class constraints and *two* second-class constraints. For completeness, we have included an explicit expression for the Dirac brackets in Appendix E.

The momentum constraint $\Phi_{\rm m}$ generates, as usual, the reparametrization invariance of the worldline $\delta x_\mu = -Np_\mu$. On the other hand, the spin constraint $\Phi_{\rm s}$ generates a $U(1)$ gauge transformation of the χ and π variables. This transformation rotates the dual variables while preserving their normalization constraints Φ_i:

$$\delta\pi_\mu = +\left(\frac{s^2M}{\lambda^2}\right)\chi_\mu, \qquad \delta\chi_\mu = -\left(\frac{M}{\epsilon^2}\right)\pi_\mu. \tag{4.41}$$

[3] The off-shell algebra is a semi-direct product of $\mathrm{SL}(2,\mathbb{R})$ with the two-dimensional Heisenberg algebra H_2. The $\mathrm{SL}(2,\mathbb{R})$ algebra consists of $\hbar(\Phi_{\rm s}+s^2)$, $\hbar\Phi_1$ and Φ_2:

$$\{\Phi_1,\Phi_2\} = 2(\Phi_{\rm s}+s^2), \quad \{\Phi_{\rm s},\Phi_1\} = -2\Phi_2/\hbar^2, \quad \{\Phi_{\rm s},\Phi_2\} = 2\Phi_1. \tag{4.37}$$

These in turn act naturally on Φ_3 and Φ_4:

$$\{\Phi_{\rm s},\Phi_3\} = \frac{\Phi_4}{\epsilon^2}, \quad \{\Phi_1,\Phi_3\} = \frac{\Phi_4}{\epsilon^2}, \quad \{\Phi_2,\Phi_3\} = \Phi_3. \tag{4.38}$$

$$\{\Phi_{\rm s},\Phi_4\} = -\frac{s^2}{\lambda^2}\Phi_3, \quad \{\Phi_1,\Phi_4\} = \frac{\Phi_3}{\lambda^2}, \quad \{\Phi_2,\Phi_4\} = -\Phi_4. \tag{4.39}$$

while together Φ_3 and Φ_4 satisfy

$$\{\Phi_3,\Phi_4\} = (m^2-2\Phi_{\rm m}). \tag{4.40}$$

Massive spinning particle

Let us now assume that $m \neq 0$; then the constraints Φ_i, $i = 1, \dots, 4$ are second class, and so the associated Lagrange multipliers N_1, N_2, N_3, N_4 must vanish. The resulting Hamiltonian is given by

$$H = N\Phi_{\rm m} + M\Phi_{\rm s} = \frac{N}{2}\left(p^2 + m^2\right) + \frac{M}{2}\left(\frac{\pi^2}{\epsilon^2} + \frac{s^2\chi^2}{\lambda^2} - 2s^2\right) \tag{4.42}$$

and defines time evolution in the standard fashion: $\dot{A} = \{H, A\}$. The equations of motion are easily integrated; we find

$$x_\mu(\tau) \;= X_\mu - NP_\mu\tau, \qquad \chi_\mu(\tau) \;= \lambda\left(A_\mu \cos\left(\frac{Ms}{\hbar}\tau\right) + B_\mu \sin\left(\frac{Ms}{\hbar}\tau\right)\right), \tag{4.43}$$

where X_μ, P_μ, A_μ, and B_μ are constant vector solutions of $P^2 = -m^2$, $A^2 = B^2 = 1$, and $A \cdot P = B \cdot P = 0$. The momenta are simply given by

$$p_\mu = -\frac{\dot{x}_\mu}{N} = P_\mu, \qquad \pi_\mu = -\frac{\epsilon^2\dot{\chi}_\mu}{M}. \tag{4.44}$$

This motion is expected, the coordinate x_μ evolves like a free particle while the dual coordinate χ_μ undergoes oscillatory motion of frequency $Ms/\hbar$ in the plane orthogonal to P_μ. Furthermore, the motion is such that both orbital and spin angular momentum are constants of motion, specifically: $L_{\mu\nu} = (X \wedge P)_{\mu\nu}$ and $S_{\mu\nu} = \hbar s(A \wedge B)_{\mu\nu}$.

Massive second order formalism

Further insights into the nature of DPS become apparent when we consider the second-order formalism which is obtained from eq. (4.33) by integrating out the momenta and Lagrange multipliers. Only the main results will be presented here, for a more detailed analysis see Appendix D. We begin by computing the equations of motion for the momenta and dual momenta which can be solved for p_μ and π_μ and then substituted back into the action. We find

$$S \;=\; \int d\tau \left[\frac{\rho}{(N\tilde{N} - N_3^2)} - \frac{\tilde{M}}{2}(\chi^2 - \lambda^2) - \frac{N}{2}m^2 + \frac{\tilde{N}}{2}\epsilon^2 s^2\right], \tag{4.45}$$

where ρ is given by

$$\rho \;:=\; \frac{1}{2}\left[\tilde{N}(\dot{x} - N_4\chi)^2 + N(\dot{\chi} - N_2\chi)^2 - 2N_3(\dot{\chi} - N_2\chi)\cdot(\dot{x} - N_4\chi)\right], \tag{4.46}$$

and we have introduced

$$\tilde{N} = \frac{(M - N_1)}{\epsilon^2}, \qquad \tilde{M} = \frac{s^2(M + N_1)}{\lambda^2}. \tag{4.47}$$

We can now solve for N_2 and N_4, which amounts to making the replacements

$$\dot{x}_\mu - N_4\chi_\mu \longrightarrow D_t x_\mu := \dot{x}_\mu - \frac{(\dot{x}\cdot\chi)}{\chi^2}\chi_\mu, \tag{4.48}$$

$$\dot{\chi}_\mu - N_2\chi_\mu \longrightarrow D_t \chi_\mu := \dot{\chi}_\mu - \frac{(\dot{\chi}\cdot\chi)}{\chi^2}\chi_\mu, \tag{4.49}$$

where D_t is the time derivative projected orthogonal to χ. It remains to integrate out the Lagrange multipliers N, $\tilde{N}$, and N_3; after some algebra we obtain the following form for the action:

$$S = \int \mathrm{d}\tau \left[\alpha\sqrt{\epsilon^2 s^2 (D_t\chi)^2 - m^2(D_t x)^2 - 2s\epsilon m\beta\,|(D_t x)\wedge(D_t\chi)|} - \frac{\tilde{M}}{2}(\chi^2 - \lambda^2)\right], \tag{4.50}$$

where $|(D_t x)\wedge(D_t\chi)| = \sqrt{(D_t x\cdot D_t\chi)^2 - (D_t x)^2(D_t\chi)^2}$ is a coupling between the particle motion and the spin motion, and $\alpha, \beta = \pm 1$ are signs used to define the square roots. Observe that we cannot integrate out the final Lagrange multiplier, since the variation of S with respect to $\tilde{M}$ is just the constraint $\chi^2 = \lambda^2$. It can be checked that the momenta $p_x = \partial S/\partial\dot{x}$ and $\pi_\chi = \partial S/\partial\dot{\chi}$ satisfy the constraints

$$p_x^2 = -m^2, \quad \pi_\chi^2 = \epsilon^2 s^2, \quad \pi_\chi\cdot\chi = 0, \quad p_x\cdot\pi_\chi = 0, \quad p_x\cdot\chi = 0. \tag{4.51}$$

Moreover, when evaluated onshell the action simplifies drastically and becomes

$$S = \alpha\int d\tau\,|m|\dot{x}| - \beta\epsilon s|\dot{\chi}||\,, \tag{4.52}$$

where we have defined $|\dot{x}| = \sqrt{-\dot{x}^2}$ and $|\dot{\chi}| = \sqrt{\dot{\chi}^2}$. As expected, if $s = 0$ eq. (4.52) reduces to the action of a relativistic scalar particle.

Massless spinning particle

As mentioned earlier, a massless particle has four first-class constraints, with Φ_3 and Φ_4 appearing in addition to Φ_s and Φ_m, and so the relevant Hamiltonian is given by

$$H = \frac{N}{2}p^2 + \frac{M}{2}\left(\frac{\pi^2}{\epsilon^2} + \frac{s^2\chi^2}{\lambda^2} - 2s\right) + \frac{N_3}{\epsilon}(p\cdot\pi) + \frac{sN_4}{\lambda}(p\cdot\chi). \tag{4.53}$$

Again, the equations of motion are easily integrated; we find

$$\chi^\mu(\tau) = \lambda\left(A^\mu \cos\left(\frac{Ms}{\hbar}\tau\right) + B^\mu \sin\left(\frac{Ms}{\hbar}\tau\right) - \frac{N_4}{Ms}P^\mu\right), \tag{4.54}$$

$$x^\mu(\tau) = X^\mu + \tau\left(\frac{N_3^2+N_4^2}{M} - N\right)P_\mu + \frac{\epsilon}{M}\left(N_3\chi^\mu(t) + \frac{N_4\hbar}{Ms}\dot{\chi}^\mu(t)\right)\Bigg|_{t=0}^{t=\tau}, \tag{4.55}$$

where $X_\mu, P_\mu, A_\mu,$ and B_μ are constant vector solutions of $P^2 = 0$, $A^2 = B^2 = 1$ and $A \cdot P = B \cdot P = 0$. The momenta are given by

$$p_\mu = P_\mu, \qquad \pi_\mu(\tau) = -\frac{\epsilon}{M}\left(\epsilon\dot{\chi}^\mu(\tau) + N_4 P_\mu\right). \tag{4.56}$$

Apart from a constant offset proportional to P_μ, the evolution of π_μ and χ_μ is identical to the massive particle. This is not the case for x_μ, where, in addition to the expected linear evolution along P_μ, there is oscillatory motion in the hyperplane orthogonal to P_μ of frequency $Ms/\hbar$ and amplitude $\hbar\sqrt{N_3^2+N_4^2}/M$. This latter quantity, we note, is pure gauge, being a function of only the Lagrange multipliers N_3, N_4, and M.

4.6 Coupling to Electromagnetism

At this point DPS describes the free propagation of a relativistic spinning particle. Although our goal is to consider interactions between such particles, it is important to show that DPS can be consistently coupled to electromagnetism. A coupling prescription is said to be consistent if it leaves the constraint structure invariant, lest the introduction of a background field fundamentally alter the system dynamics.

At leading order, we have the minimal coupling prescription [34, 36, 56, 106, 107]

$$p_\mu \to P_\mu = p_\mu + eA_\mu(x), \tag{4.57}$$

which modifies the Poisson bracket of P_μ with itself $\{P_\mu, P_\nu\} = -eF_{\mu\nu}$. Note that the pure spin constraints Φ_s, Φ_1, and Φ_2 are unaffected by this adjustment. We can also include a higher-order term via the spin-orbit coupling $F_{\mu\nu}S^{\mu\nu}$ by making the replacement

$$\Phi_\mathrm{m} = \frac{1}{2}(P^2 + m^2) \to \Phi_{\mathrm{m},g} = \Phi_\mathrm{m} + \frac{eg}{4}F_{\mu\nu}S^{\mu\nu},$$

where g is the gyromagnetic ratio and $S_{\mu\nu} = (\chi \wedge \pi)_{\mu\nu}$ the spin bivector. These modifications alter the algebra of constraints which now reads

$$\{\Phi_3, \Phi_4\} = -\left(P^2 - \frac{e}{2}F^{\mu\nu}S_{\mu\nu}\right) = \widetilde{m}^2 - 2\Phi_{\mathrm{m},g}, \tag{4.58}$$

$$\{\Phi_{\mathrm{m},g}, \Phi_3\} = e(\pi_\mu K^\mu), \qquad \{\Phi_{\mathrm{m},g}, \Phi_4\} = e(\chi_\mu K^\mu), \tag{4.59}$$

where we have introduced an "electromagnetic mass" $\widetilde{m}$ and an "acceleration" vector K_μ:

$$\widetilde{m}^2 := m^2 + \frac{e(g+1)}{2} F^{\mu\nu} S_{\mu\nu}, \qquad K^\mu := F^{\mu\nu} P_\nu - \frac{g}{2}\left(F^{\mu\nu} P_\nu - \frac{1}{2}\partial^\mu F^{\nu\rho} S_{\nu\rho}\right). \tag{4.60}$$

This vector enters the commutator

$$\{\Phi_{\mathrm{m},g}, P_\mu\} = e\left(K_\mu + \frac{g}{2} F_{\mu\nu} P^\nu\right). \tag{4.61}$$

One can now check that, for a massive particle, this prescription does not change the number of degrees of freedom. The theory still possesses two first-class and four second-class constraints. In particular, Φ_{s} remains first class since, the spin sector is unmodified, while the other first-class constraint is given by

$$\Phi_{EM} := \widetilde{m}^2 \Phi_{\mathrm{m},g} - e(\chi_\mu K^\mu)\Phi_3 + e(\pi_\mu K^\mu)\Phi_4. \tag{4.62}$$

The remaining four constraints will be second class, and so the total Hamiltonian is given by

$$H := N\Phi_{EM} + M\Phi_{\mathrm{s}}, \tag{4.63}$$

and it is straightforward to show that H preserves all constraints. In standard phase space, the resulting equations of motion are given by

$$\dot{x}^\mu = -N\left[\widetilde{m}^2 P^\mu + e(SK)^\mu\right], \tag{4.64}$$

$$\dot{P}_\mu = Ne\left[\widetilde{m}^2\left(K_\mu + \frac{g}{2}(FP)_\mu\right) + e(FSK)_\mu\right]. \tag{4.65}$$

where we have denoted $(SK)^\nu = S^{\nu\rho} K_\rho$, $(FSK)_\mu = F_{\mu\nu} S^{\nu\rho} K_\rho$, etc. The equations of motion in dual phase space lead to[4]

$$\dot{S}_{\mu\nu} = Ne\left[P_\mu (SK)_\nu + \frac{g\widetilde{m}^2}{2}(FS)_{\mu\nu} - (a \leftrightarrow b)\right]. \tag{4.67}$$

In the limit of weak ($\widetilde{m}^2 \approx m^2$) and constant electromagnetic field, section 4.6 reduces to the Frenkel-Nyborg equation [18, 12].

[4] They are explicitly given by

$$\dot{\chi}_\mu = -\frac{M}{\epsilon^2}\pi_\mu + eN\left(P_\mu K_\nu + \frac{g\widetilde{m}^2}{2} F_{\mu\nu}\right)\chi^\nu, \quad \dot{\pi}_\mu = \frac{s^2 M}{\lambda^2}\chi_\mu + eN\left(P_\mu K_\nu + \frac{g\widetilde{m}^2}{2} F_{\mu\nu}\right)\pi^\nu. \tag{4.66}$$

For a massless particle, we can see that it is impossible to introduce an electromagnetic field while keeping Φ_3 and Φ_4 first class, since their commutator involves the vector K_μ. This means that the minimal coupling prescription for a massless particle is inconsistent it would change the number of degrees of freedom. This is hardly a surprise, since it is well known that one cannot give a photon or a graviton an electromagnetic charge.

4.7 Interaction Vertex for Classical Spinning Particle

We now come to the central result of this chapter: the interaction vertex for a relativistic spinning particle. In general, interactions between classical point particles are governed by a system of ten equations: conservation of linear momentum (four), and conservation of total angular momentum (six). The latter is represented in the DPS model by $J = x \wedge p + \chi \wedge \pi$ and is a constant of motion. For simplicity, we restrict our attention to a trivalent vertex with one incoming and two outgoing particles, see Fig. 4.1. The particles have phase space coordinates $(x_i, p_i), (\chi_i, \pi_i)$, $i = 1, 2, 3$, and so the conservation equations are given explicitly by

$$p_1 = p_2 + p_3, \tag{4.68}$$

$$(x_1 \wedge p_1 + \chi_1 \wedge \pi_1) = (x_2 \wedge p_2 + \chi_2 \wedge \pi_2) + (x_3 \wedge p_3 + \chi_3 \wedge \pi_3). \tag{4.69}$$

The coordinate x_i denotes the spacetime location assigned to the interaction by particle i, and since one assumes that interactions are local in spacetime, we should have that $x_1 = x_2 = x_3 = x$. Conservation of orbital angular momentum now follows immediately from locality and eq. (4.68); to be explicit,

$$x_1 \wedge p_1 - x_2 \wedge p_2 - x_3 \wedge p_3 = x \wedge (p_1 - p_2 - p_3) = 0. \tag{4.70}$$

Thus, the system of equations we need to solve reduces to

$$x_1 = x_2 = x_3 = x, \qquad p_1 = p_2 + p_3, \tag{4.71}$$

$$\chi_1 \wedge \pi_1 = \chi_2 \wedge \pi_2 + \chi_3 \wedge \pi_3. \tag{4.72}$$

Eq. (4.71) is standard, expressing the locality of interactions, which as mentioned in the Introduction, goes hand in hand with the conservation of linear momentum. The second equation, which expresses the conservation of spin angular momentum, requires some additional work to be properly interpreted.

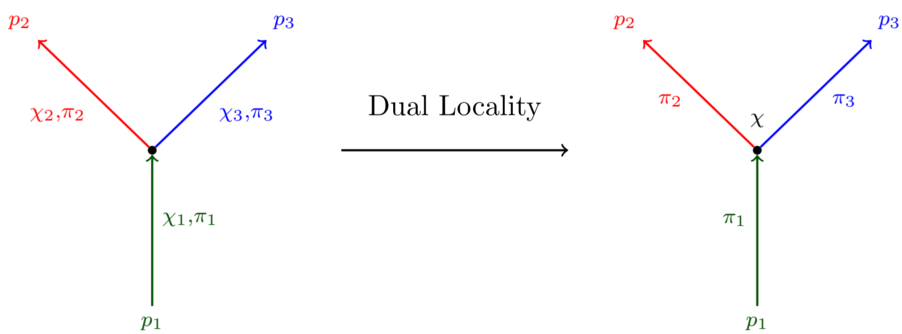


Figure 4.1: Three particle interaction in DPS, with and without the assumption of dual locality

4.7.1 Dual locality

We propose that conservation of spin angular momentum [eq. (4.72)] can be understood as an expression of the "dual locality" of the interaction vertex; i.e., interactions are "local" in dual phase space. Specifically, we assume that there exists a four-vector χ_μ such that $\chi^2 = \lambda^2$ and

$$\chi_1 = \chi_2 = \chi_3 = \chi, \tag{4.73}$$

see Fig. 4.1. It follows from eqs. (4.72) and (4.73) that $\pi_1 = \pi_2 + \pi_3 + \alpha\chi$ for some constant α; contracting both sides with χ, we get $\alpha\lambda^2 = \chi \cdot (\pi_1 - \pi_2 - \pi_3)$; the constraints $\chi_i \cdot \pi_i = \chi \cdot \pi_i = 0$ then imply χ is orthogonal to π_i, and so $\alpha = 0$. Thus, dual locality plus conservation of spin angular momentum intimates the conservation of dual momentum

$$\pi_1 = \pi_2 + \pi_3. \tag{4.74}$$

This, we note, is an exact analogue of the results in standard phase space, further emphasizing the duality of the dual phase space formulation.

To show that dual locality is a viable ansatz, we must demonstrate that it is consistent with the constraints in eq. (4.30), which need to be satisfied for each particle and are enumerated below:

i) $p_1 \cdot \chi = 0$,	iv) $p_1 \cdot \pi_1 = 0$,	vii) $\chi \cdot \pi_2 = 0$,	x) $\pi_2^2 = s_2^2$,
ii) $p_2 \cdot \chi = 0$,	v) $p_2 \cdot \pi_2 = 0$,	viii) $\chi \cdot \pi_3 = 0$,	xi) $\pi_3^2 = s_3^2$,
iii) $\chi^2 = \lambda^2$,	vi) $p_3 \cdot \pi_3 = 0$,	ix) $\pi_1^2 = s_1^2$,	xii) $\pi_1 = \pi_2 + \pi_3$.

Notice that we have included the conservation of dual momentum in this list, constraint xii, since it will be convenient to have all restrictions on dual phase space variables collected in one spot. To proceed, we use the fact that conservation of momenta [eq. (4.71)] implies that $\{p_1, p_2, p_3\}$ span a two-plane, denoted p. We introduce $\{e_0, e_1\}$ as an orthonormal basis for p, where it is assumed that e_0 is timelike. We can then extend this to an orthonormal basis for $\mathbb{R}^4$ by including two additional vectors $\{e_2, e_3\}$. It will also be convenient to define a Hodge dual in p, denoted

$$(\tilde{q})_\mu := \epsilon_{\mu\nu\rho\sigma} e_2^\nu e_3^\rho q^\sigma \tag{4.75}$$

for $q \in \mathsf{p}$.

We now systematically solve the constraints beginning with i-iii which are easily seen to have the solution

$$\chi = \lambda(\cos\phi e_2 + \sin\phi e_3) \tag{4.76}$$

for some arbitrary angle ϕ. Constraints iv – vi imply that the dual momenta π_i lies in the hyperplane orthogonal to p_i, hence we can expand π_i as

$$\pi_i = \alpha_i \tilde{p}_i + A_i e_2 + B_i e_3. \tag{4.77}$$

The Hodge dual of eq. (4.71) implies $\tilde{p}_1 = \tilde{p}_2 + \tilde{p}_3$, and so projecting constraint xii into the plane p and using eq. (4.77) gives

$$(\alpha_1 - \alpha_2)\tilde{p}_2 + (\alpha_1 - \alpha_3)\tilde{p}_3 = 0. \tag{4.78}$$

Thus, if p_2 and p_3 are linearly independent, we get $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$. On the other hand, projecting xii orthogonal to p and using eq. (4.77) again requires $A_1 = A_2 + A_3$ and $B_1 = B_2 + B_3$. Constraints vii and viii are then easily solved by setting $A_2 = -\beta\sin\phi$, $B_2 = \beta\cos\phi$ and $A_3 = -\gamma\sin\phi$, $B_3 = \gamma\cos\phi$ respectively. In summary, we have

$$\pi_1 = \alpha\tilde{p}_1 + (\beta + \gamma)\chi^\perp, \tag{4.79}$$

$$\pi_2 = \alpha\tilde{p}_2 + \beta\chi^\perp, \tag{4.80}$$

$$\pi_3 = \alpha\tilde{p}_3 + \gamma\chi^\perp, \tag{4.81}$$

where $\chi^\perp = -\sin\phi e_2 + \cos\phi e_3$ is orthogonal to χ. It remains to consider constraints ix – xi which are seen to give

$$m_1^2\alpha^2 + (\beta + \gamma)^2 = s_1^2, \tag{4.82}$$

$$m_2^2\alpha^2 + \beta^2 = s_2^2, \tag{4.83}$$

$$m_3^2\alpha^2 + \gamma^2 = s_3^2. \tag{4.84}$$

Before showing that the above equations possess a consistent solution, we need to recall some restrictions on the mass and spin of the constituent particles, namely

$$m_2 + m_3 \leq m_1, \tag{4.85}$$

$$|s_2 - s_3| \leq s_1 \leq s_2 + s_3. \tag{4.86}$$

The first inequality is well known, and easily derived from momentum conservation [eq. (4.71)]. Eq. (4.86), on the other hand, is a quantum-mechanical result derived by considering the eigenvalues of the total angular momentum operator in a composite system. Here we will show that it follows from the assumption of dual locality. We begin by squaring eq. (4.74) to obtain

$$s_1^2 = s_2^2 + s_3^2 + 2\pi_2 \cdot \pi_3.$$

As π_i is spacelike, we can apply the Cauchy-Schwartz inequality with impunity:

$$2|\pi_2 \cdot \pi_3| \leq 2|\pi_2||\pi_3| = 2s_2 s_3.$$

Substituting this result into the previous equation gives $(s_2 - s_3)^2 \leq s_1^2 \leq (s_2 + s_3)^2$, and the desired result follows after taking square roots.

With this in mind we return to Eqs. (4.82)–(4.84). The latter two can be used to solve for β and γ in terms of α and the result substituted into eq. (4.82). After rearranging and taking the square, we get a consistency condition for α:

$$(s_2^2 - m_2^2\alpha^2)(s_3^2 - m_3^2\alpha^2) = (S^2 - M^2\alpha^2)^2, \tag{4.87}$$

where $2M^2 := m_1^2 - m_2^2 - m_3^2$ and $2S^2 := s_1^2 - s_2^2 - s_3^2$. It is not enough to simply solve this equation for α, since it is immediately obvious from Eqs. (4.82)–(4.84) that $\alpha^2 \leq r_i^2$, where $r_i = s_i/m_i$ for $m_i \neq 0$. As such, we introduce variables θ_2 and θ_3 which satisfy

$$\alpha = r_2 \cos\theta_2 = r_3 \cos\theta_3, \tag{4.88}$$

and without loss of generality suppose $r_3 \leq r_2$. Note that we can choose the signs of θ_2 and θ_3 so that $\beta = s_2 \sin\theta_2$ and $\gamma = s_3 \sin\theta_3$. The consistency equation on α now reads $F(\theta_3) = 0$, where

$$F(\theta) := (S^2 - M^2 r_3^2 \cos^2\theta)^2 - s_2^2 s_3^2 \sin^2\theta \left(1 - \tfrac{r_3^2}{r_2^2} + \left(\tfrac{r_3}{r_2}\sin\theta\right)^2\right). \tag{4.89}$$

It suffices, therefore, to show that $F(\theta)$ has a zero in the interval $[-\pi/2, \pi/2]$, and so we note that

$$F(0) = (S^2 - M^2 r_3^2)^2 \geq 0, \qquad F(\pm\pi/2) = -[(s_2+s_3)^2 - s_1^2][s_1^2 - (s_2-s_3)^2] \leq 0,$$

where the second equality follows from eq. (4.86). By the intermediate value theorem, there exists $\bar{\theta} \in [0, \pi/2]$ such that $F(\pm\bar{\theta}) = 0$, and so $\alpha = r_3 \cos\bar{\theta}$ satisfies eq. (4.87). It follows for massive particles that there are two solutions to the dual locality equations for which $\alpha > 0$. These two solutions are related by a change of orientation in the plane orthogonal to p; if (α, β, γ) is a solution, then $(\alpha, -\beta, -\gamma)$ is also a solution. Note that by parity invariance, $(-\alpha, -\beta, -\gamma)$ and $(-\alpha, \beta, \gamma)$ are also solutions.

The case where $m_2 = 0$ can be obtained from the above by allowing $r_2 \to \infty$ in eq. (4.89), and one can again obtain a solution for α by using the intermediate value theorem. In the remaining case[5] $m_2 = m_3 = 0$, Eqs (4.83) and (4.84) are solved immediately as $\beta = \epsilon_2 s_2$ and $\gamma = \epsilon_3 s_3$ where $\epsilon_i = \pm 1$. We then obtain for α

$$\alpha^2 = \frac{1}{m_1^2}\left(s_1^2 - (\epsilon_2 s_2 + \epsilon_3 s_3)^2\right),$$

where eq. (4.86) implies that $\epsilon_2\epsilon_3 = -1$ and we again find four solutions belonging to two sectors related by parity. This completes our analysis of the three-particle interaction, showing that dual locality ensures a consistent vertex for any viable combination of spinning particles.

4.7.2 Universality of dual locality

Having established established dual locality as a sufficient condition to ensure a consistent three-point vertex we now show its necessity. The key point is that when the spin is nonzero, we have an additional gauge symmetry in the system which corresponds to a rotation in the (χ, π) plane; recall eq. (4.41):

$$R_\theta(\chi_\mu, \pi_\mu) = (\cos\theta\chi_\mu + \tfrac{\lambda}{\epsilon s}\sin\theta\pi_\mu, \cos\theta\pi_\mu - \tfrac{\epsilon s}{\lambda}\sin\theta\chi_\mu). \tag{4.90}$$

Such a gauge transformation does not change the value of the spin bivector $R_\theta(\chi) \wedge R_\theta(\pi) = \chi \wedge \pi$. Therefore, if $(\chi_i, \pi_i)_{i=1,2,3}$ is a solution of eq. (4.72), then $(R_{\theta_i}(\chi_i), R_{\theta_i}(\pi_i))_{i=1,2,3}$ is also a solution for arbitrary θ_i. This is simply an expression of the gauge symmetry of the theory. The main claim we now want to prove is that *any* solution of the spin

[5] It is impossible to have three massless interacting particles.

conservation equation [eq. (4.72)] is gauge equivalent to a solution satisfying dual locality. In other words, if $(\chi_i, \pi_i)_{i=1,2,3}$ is a solution of eq. (4.72), then there exists $(\chi', \pi'_i)_{i=1,2,3}$ with $\pi'_1 = \pi'_2 + \pi'_3$, and θ_i such that

$$(\chi_i, \pi_i) = (R_{\theta_i}(\chi'), R_{\theta_i}(\pi'_i)), \text{ for } i = 1, 2, 3. \tag{4.91}$$

Note that in addition to the rotation eq. (4.41), DPS is invariant under the global rescaling $\lambda \to \alpha\lambda$ and $\epsilon \to \alpha^{-1}\lambda$. Therefore, we can assume that all λ_i and ϵ_i have been rescaled to some common values λ and ϵ.

Suppose that we have a solution to Eqs. (4.71) and (4.72), including all accompanying constraints. It is always possible to choose χ_i orthogonal to the plane p. To see why consider χ_2: By construction $\chi_2 \cdot p_2 = 0$, and so we need only ensure that it is orthogonal to p_3, since then conservation of momentum guarantees that it will be orthogonal to p_1 as well. Hence, if $\chi_2 \cdot p_3 \neq 0$, a gauge rotation with $\cot\theta = \lambda\pi_2 \cdot p_3/(s_2\chi_2 \cdot p_3)$, will ensure that the new χ_2 is orthogonal to p_3. A similar argument holds for the other χ_i, and the claim is justified, thereby allowing us to write $\chi_i = \lambda(\cos\phi_i e_2 + \sin\phi_i e_3)$, since $\chi_i^2 = \lambda^2$. Now, we contract eq. (4.72) with (p_1, p_2, p_3) to obtain

$$\chi_2(p_3 \cdot \pi_2) + \chi_3(p_2 \cdot \pi_3) = 0, \tag{4.92}$$

$$\chi_1(p_2 \cdot \pi_1) - \chi_3(p_2 \cdot \pi_3) = 0, \tag{4.93}$$

$$\chi_1(p_3 \cdot \pi_1) - \chi_2(p_3 \cdot \pi_2) = 0. \tag{4.94}$$

There are two cases to consider. Either $(p_i \cdot \pi_j)_{i\neq j}$ are all vanishing or they are all nonvanishing. Indeed, if $p_3 \cdot \pi_2 = 0$, the above equations imply that $p_2 \cdot \pi_3 = p_2 \cdot \pi_1 = p_3 \cdot \pi_1 = 0$, which in turn, via momentum conservation, yields $p_1 \cdot \pi_2 = p_1 \cdot \pi_3 = 0$.

Let us first assume that $p_i \cdot \pi_j = 0$. As argued above, χ_i and π_i are orthogonal to p and therefore can be expanded as

$$\chi_i = \lambda(\cos\phi_i e_2 + \sin\phi_i e_3), \qquad \pi_i = s_i\epsilon(-\sin\phi_i e_2 + \cos\phi_i e_3).$$

A further gauge transformation with $\theta_i = -\phi_i$ can now be performed to give $\chi_i = \lambda e_2$, $\pi_i = s_i\epsilon e_3$, which proves the proposition.

In the generic case we have $(p_i \cdot \pi_j)_{i\neq j} \neq 0$. We contract eq. (4.92) with π_3 to obtain $\pi_3 \cdot \chi_2 = 0$; repeating this for the other π_i we find that $(\chi_i)_{i=1,2,3}$ is orthogonal to $(\pi_j)_{i=1,2,3}$. With this established, we can return to eq. (4.72), contract with χ_1 and then χ_2, and combine the results to eliminate the terms proportional to π_1:

$$0 = \left[(\chi_1 \cdot \chi_2)^2 - \lambda^4\right]\pi_2 + \left[(\chi_1 \cdot \chi_3)(\chi_1 \cdot \chi_2) - \lambda^2(\chi_2 \cdot \chi_3)\right]\pi_3.$$

Note that π_2 and π_3 cannot be parallel, since then $\pi_2 \cdot p_3 \propto \pi_3 \cdot p_3 = 0$, which is contrary to the original assumption $\pi_2 \cdot p_3 \neq 0$. Hence, the previous equation implies that

$$|\chi_1 \cdot \chi_2| = \lambda^2.$$

As χ_i are spacelike vectors which satisfy $\chi_i^2 = \lambda^2$, the Cauchy-Schwartz inequality implies that χ_1 and χ_2 are parallel, hence $\chi_1 = \pm\chi_2$. We can repeat the above procedure, contracting eq. (4.72) with χ_1 and χ_3 to obtain $\chi_1 = \pm\chi_3$, and so

$$\epsilon_1\chi_1 = \epsilon_2\chi_2 = \epsilon_3\chi_3 = \chi,$$

where $\epsilon_i = \pm$. This is not exactly what we want. All we have to do is perform another set of gauge transformations by the angle $(1 - \epsilon_i)\pi/2$ to transform $(\chi_i, \pi_i) \to (\epsilon_i\chi_i, \epsilon_i\pi_i)$. Note that these gauge transformations do not affect any of the orthogonality properties established before and so we obtain the dual locality property

$$\chi_1 = \chi_2 = \chi_3 = \chi, \qquad \pi_1 = \pi_2 + \pi_3. \tag{4.95}$$

This completes the proof, showing that a solution to Eqs. (4.71) and (4.72) implies that dual locality holds, up to a gauge relabeling.

4.7.3 An alternative view of dual locality

The universality of dual locality is an important result further emphasizing the symmetry between standard and dual phase space. As such, it will be beneficial to see how dual locality arises from one of the alternative models presented earlier in this Chapter. In particular, we select the parametrization of Sec. 4.4.2, where spin is represented by a single vector n_μ. Recall that n_μ has the interpretation of an S^2 vector boosted in the direction of the particles' momenta, and the spinning part of angular momentum is given by $S_{\mu\nu} = s{*}(n \wedge u)_{\mu\nu}$. Consider again a three-particle interaction with one particle incoming and the other two outgoing. In what follows we will assume $m \neq 0$.

Interactions, as previously discussed, are governed by the conservation of linear momentum [eq. (4.71)] and the conservation of spin angular momentum. The latter, after taking the Hodge dual and making use of eq. (4.71), can be written as

$$[(r_1 n_1 - r_2 n_2) \wedge p_2] + [(r_1 n_1 - r_3 n_2) \wedge p_3] = 0, \tag{4.96}$$

where $r_i = s_i/m_i$. Let $A^{\perp_{\mathsf{p}}}$ denote the projection of a vector A onto the plane orthogonal to p. Applying this projection to eq. (4.96) yields[6]

$$r_1 n_1^{\perp_{\mathsf{p}}} = r_2 n_2^{\perp_{\mathsf{p}}} = r_3 n_3^{\perp_{\mathsf{p}}}. \tag{4.97}$$

A further condition on the n_i is obtained by contracting eq. (4.96) with $\tilde{p}_2 \wedge \tilde{p}_3$, viz.

$$r_1 n_1 \cdot \tilde{p}_1 = r_2 n_2 \cdot \tilde{p}_2 + r_3 n_3 \cdot \tilde{p}_3. \tag{4.98}$$

The previous two equations provide a natural method for defining variables χ and π_i which satisfy dual locality, in particular

$$\chi = \frac{\lambda}{|n_i^{\perp_{\mathsf{p}}}|} * (e_0 \wedge e_1 \wedge n_i) \qquad \text{and} \qquad \pi_i = \frac{\epsilon r_i}{\lambda} * (\chi \wedge p_i \wedge n_i).$$

It follows from eq. (4.97) that χ is independent of i, while eq. (4.98) can be used to show $\pi_1 = \pi_2 + \pi_3$. The necessary constraints (i–xi) are also satisfied, as one can easily check. Note that the above definitions are ambiguous up to a sign, although the same sign must be chosen for all π_i, and so we see again that there are four possible solutions belonging to two parity-related sectors. In summary, the conservation of angular momentum requires that the vectors $r_i n_i$ be equal when projected into the plane $\mathsf{p}^\perp$. The dual position χ is then the unique (up to a sign) vector of length λ lying in the plane $\mathsf{p}^\perp$ which is orthogonal to $r_i n_i^{\perp_{\mathsf{p}}}$. In turn, the dual momenta π_i is the unique (up to a sign) vector of length ϵs_i orthogonal to p_i, n_i, and χ.

[6]Assuming p_2 and p_3 are linearly independent.

Chapter 5

A Mechanical Model for the Relativistic Spinning Particle

5.1 Introduction

If one ignores the constraints, the phase space of the "Dual Phase Space" (DPS) model is identical to that of two scalar particles which suggests that it can be reformulated as a composite system. In this Chapter we will formalize this observation and show that the relativistic spinning particle can be realized as a simple bilocal model that is equivalent to the original DPS formalism.

5.2 Non-relativistic Two Particle Model

5.2.1 Hamiltonian Formulation

Let's consider a system comprised of two non-relativistic point particles with masses m_1 and m_2. The corresponding phase space is parametrized by the position and momenta of each particle $(\vec{x}_1, \vec{p}_1)$ and $(\vec{x}_2, \vec{p}_2)$ with standard Poisson bracket structure

$$\left\{x_i^a, p_j^b\right\} = \delta_{ij}\delta^{ab}, \qquad i,j = 1,2 \text{ and } a,b = 1,2,3. \tag{5.1}$$

Let $M = m_1 + m_2$ be the total mass of the system and $\mu = m_1 m_2 / M$ the reduced mass, then we can introduce:

$$\vec{X} = \frac{m_1}{M}\vec{x}_1 + \frac{m_2}{M}\vec{x}_2, \qquad \Delta\vec{x} = \vec{x}_1 - \vec{x}_2, \tag{5.2}$$

where $\vec{X}$ are the coordinates of the center of mass and $\Delta\vec{x}$ is the relative displacement between the particles. Momenta conjugate to these coordinates are given by

$$\vec{P} = \vec{p}_1 + \vec{p}_2, \qquad \Delta\vec{p} = \frac{\mu}{m_1}\vec{p}_1 - \frac{\mu}{m_2}\vec{p}_2, \tag{5.3}$$

respectively. These definitions imply the following non-vanishing Poisson brackets

$$\left\{X^a, P^b\right\} = \delta^{ab}, \qquad \left\{\Delta x^a, \Delta p^b\right\} = \delta^{ab}. \tag{5.4}$$

The coordinates introduced above can also be used to decompose the total angular momentum of the two particle system as the sum of the total and relative angular momenta

$$\vec{J} := \vec{x}_1 \times \vec{p}_1 + \vec{x}_2 + \vec{p}_2 \tag{5.5}$$

$$= \vec{X} \times \vec{P} + \Delta\vec{x} \times \Delta\vec{p}. \tag{5.6}$$

Note that the second equality shows that $\vec{J} = \vec{L} + \vec{S}$, where $\vec{L} = \vec{X} \times \vec{P}$ is the "external" angular momentum associated with motion of the system as a whole while $\vec{S} = \Delta\vec{x} \times \Delta\vec{p}$ is the "internal" angular momentum resulting from the rotation around the center of mass. This internal rotation represents the spin degrees of freedom.

At this point we have a pair of free non-relativistic particles and it remains to impose

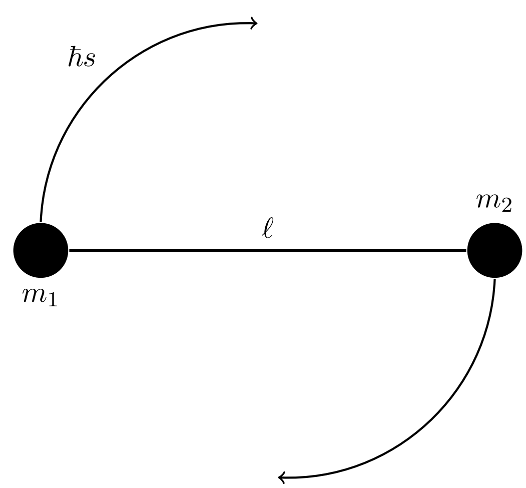


Figure 5.1: Two particles connected by rigid rod of length ℓ and pictured in the center of mass frame where the total angular momentum has magnitude $\hbar s$.

some structure on the system which will make contact with intuitions we have regarding the nature of spinning particles. Classically, a spinning particle is a rigid object with a fixed, non-zero value for its "internal" angular momentum. The former condition can be implemented by demanding that the two particles are coupled by a rigid rod of length ℓ and the latter by setting the magnitude of the angular momentum in the center of mass frame to be $\hbar s$, for some dimensionless constant s. This amounts to imposing the constraints

$$(\Delta\vec{x})^2 = \ell^2, \qquad \text{and} \qquad (\Delta\vec{x} \times \Delta\vec{p})^2 = \hbar^2 s^2, \tag{5.7}$$

see Figure 5.1. These constraints satisfy a closed algebra. A Hamiltonian can now be constructed by adding the constraints in eq. (5.7) to the standard Hamiltonian for a system of two free particles[1]

$$H = \frac{1}{2m_1}\vec{p}_1^{\,2} + \frac{1}{2m_2}\vec{p}_2^{\,2} + \frac{\lambda_1}{2}\left[(\Delta\vec{x})^2 - \ell^2\right] + \frac{\lambda_2}{2}\left[(\Delta\vec{x} \times \Delta\vec{p})^2 - \hbar^2 s^2\right], \tag{5.8}$$

where λ_1 and λ_2 are Lagrange multipliers. To ensure that the constraints are stationary under the evolution defined by H we need to include $\Delta\vec{x} \cdot \Delta\vec{p} = 0$ which allows us to re-write the *full* Hamiltonian as

$$H = \frac{1}{2M}\vec{P}^{\,2} + \frac{1}{2\mu}(\Delta\vec{p})^2 + \frac{\lambda_1}{2}\left[(\Delta\vec{x})^2 - \ell^2\right] + \frac{\lambda_2}{2}\left[(\Delta\vec{p})^2 - \epsilon^2 s^2\right] + \lambda_3 \Delta\vec{x} \cdot \Delta\vec{p}, \tag{5.9}$$

where ϵ has units of energy and satisfies $\epsilon\ell = \hbar$. No further constraints are required but due to the second class nature of the constraints imposed, the condition that all the constraints are preserved under time evolution imposes the following relations between Lagrange multipliers:

$$\lambda_2 = \frac{\ell^2}{\epsilon^2 s^2}\lambda_1 - \frac{1}{\mu} \qquad \text{and} \qquad \lambda_3 = 0. \tag{5.10}$$

The final form of the non-relativistic *restricted* Hamiltonian is therefore, up to a constant term $\epsilon^2 s^2/2\mu$, given by

$$H = \frac{1}{2M}\vec{P}^2 + \lambda\left[\frac{1}{2}\left(\frac{\Delta\vec{p}}{\epsilon}\right)^2 + \frac{s^2}{2}\left(\frac{\Delta\vec{x}}{\ell}\right)^2 - s^2\right], \tag{5.11}$$

where $\lambda = \lambda_1 \ell^2/s^2$. As one can see from H there is a single first class constraint

$$\ell^2(\Delta\vec{p})^2 + \epsilon^2 s^2 (\Delta\vec{x})^2 = 2\hbar^2 s^2, \tag{5.12}$$

and two second class constraints

$$(\Delta\vec{x}) \cdot (\Delta\vec{p}) = 0 \qquad \text{and} \qquad \epsilon^2 s^2 (\Delta\vec{x})^2 - \ell^2(\Delta\vec{p})^2 = 0. \tag{5.13}$$

[1] A similar model appeared in a different context in [108].

The dimension of the reduced phase space is therefore $12 - 1 \times 2 - 2 \times 1 = 8$ for a total of 4 physical degrees of freedom; as expected for a spinning particle (3 for position and 1 for the spin). The motion of the composite system can be deduced by examining the Hamiltonian eq. (5.11). The unconstrained part of H indicates that the center of mass evolves like a free particle, while the single first class constraint is a harmonic oscillator potential acting on the relative separation, and so the latter will execute periodic motion with frequency $\omega \propto s$.

5.2.2 Lagrangian Formulation

It is a straightforward exercise to compute the Lagrangian for this model, beginning with H as given in eq. (5.9) we put $L = \vec{P} \cdot \dot{\vec{X}} + \Delta\vec{p} \cdot \Delta\dot{\vec{x}} - H$. We can now integrate out the momenta, after which the Lagrange multiplier λ_3 enters quadratically and therefore can also be integrated without difficulty. One obtains

$$L = \frac{M}{2}\dot{\vec{X}}^2 + \frac{1}{2}\frac{\mu}{(1+\lambda_2\mu)}(D_t\Delta\vec{x})^2 + \frac{\lambda_2}{2}\epsilon^2 s^2 - \frac{\lambda_1}{2}\left[(\Delta\vec{x})^2 - \ell^2\right], \tag{5.14}$$

where

$$D_t\Delta\vec{x} \coloneqq \Delta\dot{\vec{x}} - \frac{(\Delta\dot{\vec{x}} \cdot \Delta\vec{x})}{(\Delta\vec{x})^2}\Delta\vec{x}, \tag{5.15}$$

is a covariant time derivative which preserves the constraint $(\Delta\vec{x})^2 = \ell^2$. It projects the relative motion $\Delta\dot{\vec{x}}$ orthogonal to $\Delta\vec{x}$. The Lagrange multiplier λ_2 doesn't enter quadratically but we can still solve for it at the classical level. The solution space possesses two branches which are labelled by a sign $\alpha := \mathrm{sign}(1 + \lambda_2\mu)$. Encoding this sign into the spin by $s := \alpha|s|$, we see that the Lagrangian can be expressed purely in terms of the configuration variables and is given by $L = L_s + \frac{\lambda_1}{2}\left[(\Delta\vec{x})^2 - \ell^2\right] - \frac{1}{2}\frac{\epsilon s}{\mu}$ where the spin Lagrangian is simply

$$L_s = \frac{M}{2}\dot{\vec{X}}^2 + \epsilon s|D_t\Delta\vec{x}|. \tag{5.16}$$

We see that the inclusion of spin amounts to a modification of the kinetic energy which is linear in the velocity instead of quadratic. The spin s itself entering as a "stiffness" parameter multiplying the spin kinetic energy $|D_t\Delta\vec{x}|$. The final Lagrange multiplier λ_1 imposes the constraint $(\Delta\vec{x})^2 = \ell^2$ which can be solved by introducing new variables $\vec{y}$ defined implicitly via

$$\Delta\vec{x} = \frac{\ell}{|\vec{y}|}\vec{y}. \tag{5.17}$$

The Lagrangian eq. (5.16) then becomes

$$L = \frac{M}{2}\dot{\vec{X}}^2 + \frac{\hbar s}{|\vec{y}\,|}|\mathcal{D}_t\vec{y}\,| - \frac{1}{2}\frac{\epsilon s}{\mu}, \tag{5.18}$$

where $\mathcal{D}_t\vec{y}$ is the derivative $\dot{\vec{y}}$ projected orthogonally to $\vec{y}$. It satisfies $\mathcal{D}_t(\rho\vec{y}) = \rho\mathcal{D}_t\vec{y}$. Notice that the reduced mass enters only in an overall constant factor.

5.2.3 Quantizing the Non-Relativistic Model

In this section we will quantize the non-relativistic model and show that it reproduces the expected results for a non-relativistic spinning particle. Start with the Lagrangian eq. (5.18) and compute the momenta conjugate to $\vec{X}$ and $\vec{y}$, viz

$$\vec{P}_X = M\dot{\vec{X}}, \qquad \vec{P}_y = \frac{\hbar s}{|\vec{y}\,||\mathcal{D}_t\vec{y}\,|}\mathcal{D}_t\vec{y}. \tag{5.19}$$

It is straightforward to verify that $\vec{P}_y$ satisfies the constraints

$$\vec{P}_y \cdot \vec{y} = 0, \qquad \vec{P}_y^2 - \frac{\hbar^2 s^2}{\vec{y}^2} = 0, \tag{5.20}$$

and so the Hamiltonian is given as

$$H = \frac{\vec{P}_X^2}{2M} + \lambda_1\left(\vec{P}_y \cdot \vec{y}\right) + \frac{\lambda_2}{2}\left(\vec{P}_y^2 - \frac{\hbar^2 s^2}{\vec{y}^2}\right). \tag{5.21}$$

The Poisson brackets are standard

$$\left\{X_i, P_X^j\right\} = \delta_i^j \qquad \left\{y_i, P_y^j\right\} = \delta_i^j \tag{5.22}$$

and can be used to show that the constraints eq. (5.20) are first class.

The absence of second class constraints in conjunction with eq. (5.22) implies that we can quantize by making the standard replacements

$$\hat{X}_i\Psi = X_i\Psi, \qquad \hat{P}_X^i\Psi = -i\hbar\frac{\partial}{\partial X_i}\Psi, \tag{5.23}$$

$$\hat{y}_i\Psi = y_i\Psi, \qquad \hat{P}_y^i\Psi = -i\hbar\frac{\partial}{\partial y_i}\Psi, \tag{5.24}$$

where $\Psi = \Psi(\vec{X}, \vec{y}, t)$. Observe that the unconstrained part of H acts only on the variables $\vec{X}$ while the constraints act only on the $\vec{y}$. This suggests that we separate variables

$\Psi(\vec{X},\vec{y},t) = \Psi_1(\vec{X},t)\Psi_2(\vec{y})$, then the condition $H\Psi = i\hbar\partial_t\Psi$ splits into three differential equations

$$-\frac{\hbar^2}{2M}\nabla_X^2\Psi_1 = i\hbar\frac{\partial\Psi_1}{\partial t}, \tag{5.25}$$

$$\sum_i y^i\frac{\partial\Psi_2}{\partial y^i} = 0, \tag{5.26}$$

$$\nabla_y^2\Psi_2 + \frac{s^2}{\vec{y}^2}\Psi_2 = 0. \tag{5.27}$$

The first equality is just Schrödinger's equation for a free particle indicating that the internal variables continue to evolve as a free particle even in the quantum theory. The remaining equations correspond to the first class constraints imposed on the internal variables and are most easily solved by switching to spherical coordinates. Make the replacements $\vec{y} = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$ and $\Psi_2(\vec{y}) = \psi(r,\theta,\phi)$, then equation eq. (5.26) becomes

$$r\frac{\partial\psi}{\partial r} = 0 \quad\Longrightarrow\quad \psi(r,\theta,\phi) = \psi(\theta,\phi)$$

and so ψ doesn't depend on r. The remaining equation (5.27) now takes the form

$$\Delta\psi := \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = -s^2\psi. \tag{5.28}$$

Here Δ is the Laplacian on the unit sphere S^2 spanned by $\Delta\vec{x}/|\Delta\vec{x}|$. It is well known that the solutions of this equation for functions on the sphere are given by the so called Spherical Harmonics, which represent *integer* spins[2]:

$$\psi(\theta,\phi) = Y_\ell^m(\theta,\phi), \qquad \ell\in\mathbb{N}, \quad m = -\ell,-\ell+1,\dots,\ell-1,\ell, \tag{5.29}$$

where $s^2 = \ell(\ell+1)$. The "internal" angular momentum (spin) operator is $\hat{\vec{S}} = \hat{\vec{y}}\times\hat{\vec{P}}_y$ and one can verify that

$$\hat{S}_3 Y_\ell^m = m\hbar Y_\ell^m \qquad \text{and} \qquad S^2 Y_\ell^m = \hbar^2\ell(\ell+1)Y_\ell^m, \tag{5.30}$$

which is precisely the expected result. Overall the total wave function is given by

$$\Psi(x_1,x_2) = \Psi_1(x_1+x_2)Y\left(\frac{x_1-x_2}{|x_1-x_2|}\right)\delta(|x_1-x_2|-\ell). \tag{5.31}$$

This wave function cannot be split into a product $\phi_1(x_1)\phi_2(x_2)$ showing that the two constituents are fundamentally entangled by the spin constraint. The scalar product between such functions is simply given by $\|\Psi\|^2 = \int_{\mathbb{R}^3}\mathrm{d}^3x|\psi_1|^2(x)\int_{S^2}\mathrm{d}n|Y|^2(n)$.

[2] As discussed in Appendix F, the most general solution of this equation which is regular for $\theta\in[0,\pi]$ and $\phi\in[0,2\pi]$ are the fermionic spherical harmonics Y_ℓ^m for $\ell\in\frac{\mathbb{N}}{2}$, see [109, 110, 111]. In this case however the functionals cannot be understood as depending continuously on the sphere variables Δx.

5.3 Relativistic Two Particle Model

The non-relativistic model presented in the previous section captures our intuition of how a spinning particle should behave, but a truly viable description needs to be relativistic. We begin by replacing the position and momentum variables with their four-vector counterparts $\vec{x}_i \to x_i^\mu$ and $\vec{p}_i \to p_i^\mu$, now assumed to be functions of some auxiliary parameter τ. These have the standard transformation properties under elements of the Poincaré group (Λ, y)

$$x_i \to \Lambda x_i + y \qquad \text{and} \qquad p_i \to \Lambda p_i, \tag{5.32}$$

where Λ is a Lorentz transformation and y a translation. There is also a natural extension of the Poisson bracket structure in equation eq. (5.1) to

$$\{x_i^\mu, p_i^\nu\} = \delta_{ij}\eta^{\mu\nu}, \qquad i, j = 1, 2, \tag{5.33}$$

where $\eta = \mathrm{diag}(-1, 1, 1, 1)$. As in the previous section we can introduce "center of mass"[3] and relative displacement coordinates. In doing so it will be convenient to specialize to the case where the particles are of equal mass $m_1 = m_2 = m$, whence

$$\begin{aligned} X^\mu &= \frac{1}{2}\left(x_1^\mu + x_2^\mu\right), & \Delta x^\mu &= x_1^\mu - x_2^\mu, \\ P^\mu &= p_1^\mu + p_2^\mu, & \Delta p^\mu &= \frac{1}{2}\left(p_1^\mu - p_2^\mu\right). \end{aligned} \tag{5.34}$$

Surprisingly, the case of unequal masses is significantly more complex than in the non-relativistic case and since it is not relevant for the bulk of our current analysis we have relegated its treatment to Appendix G. The variables in eq. (5.34) transform under the Poincaré group as

$$X \to \Lambda X + y, \qquad P,\, \Delta p,\, \Delta x \to \Lambda P,\, \Lambda\Delta p,\, \Lambda\Delta x, \tag{5.35}$$

and one can check that (X^μ, P^μ) and $(\Delta x^\mu, \Delta p^\mu)$ form canonically conjugate pairs. The total angular momentum $\vec{J} = \vec{L} + \vec{S}$ is generalized to an anti-symmetric tensor $J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}$ with

$$L_{\mu\nu} = (X \wedge P)_{\mu\nu} \quad \text{and} \quad S_{\mu\nu} = (\Delta x \wedge \Delta p)_{\mu\nu}, \tag{5.36}$$

where $(A \wedge B)_{\mu\nu} = A_\mu B_\nu - A_\nu B_\mu$. Again $L_{\mu\nu}$ represents the "external" angular momentum of the system as whole while $S_{\mu\nu}$ represents "internal" rotations.

[3] The center of mass is not a relativistically invariant quantity, hence the use of inverted commas.

The relativistic Hamiltonian is a straightforward generalization of the non-relativistic one, see eq. (5.11), in particular the restricted Hamiltonian is

$$H = \frac{N}{2\epsilon}\left[P^2 + 4(m^2 + \epsilon^2 s^2)\right] + \widetilde{N}\left[\frac{1}{2}\left(\frac{\Delta p}{\epsilon}\right)^2 + \frac{s^2}{2}\left(\frac{\Delta x}{\ell}\right)^2 - s^2\right], \tag{5.37}$$

where N and $\widetilde{N}$ are Lagrange multipliers.

To see how eq. (5.37) comes about return to the non-relativistic Hamiltonian eq. (5.8). In the relativistic theory the free part becomes two mass shell constraints, recall that we are assuming particles of equal mass

$$\frac{1}{2m}\vec{p}_i^{\,2} \to (p_i^2 + m)^2, \qquad i = 1, 2. \tag{5.38}$$

Each of these defines an evolution that must preserve the other two constraints eq. (5.7), now written as

$$(\Delta x)^2 = \ell^2 \qquad \text{and} \qquad (\Delta x \wedge \Delta p)^2 = \hbar^2 s^2. \tag{5.39}$$

We can still interpret the first constraint as a rigidity condition, although now it fixes the spacetime interval between the two particles. Similarly, the second constraint can be seen as fixing the square of the "internal" angular momentum tensor, see equation eq. (5.36). To ensure that both constraints are stationary, under the time evolution of each constituent, we need to include $p_1 \cdot \Delta x = 0$ and $p_2 \cdot \Delta x = 0$, which then allows us to write the relativistic Hamiltonian as the following sum of six constraints

$$\begin{aligned} H = \frac{N_1}{2}\left(p_1^2 + m^2\right) + \frac{N_2}{2}(p_2^2 + m^2) + \frac{\lambda_1}{2}\left((\Delta x)^2 - \ell^2\right) \\ + \frac{\lambda_2}{2}\left((\Delta p)^2 - \epsilon^2 s^2\right) + \lambda_3(p_1 \cdot \Delta x) + \lambda_4(p_2 \cdot \Delta x). \end{aligned} \tag{5.40}$$

No further constraints need to be added but demanding that the existing constraints Poisson commute with H imposes the following conditions among the Lagrange multipliers

$$\lambda_3 = \lambda_4 = 0, \qquad N_1 = N_2, \qquad \lambda_2 = \frac{\ell^2}{\epsilon^2 s^2}\lambda_1 - (N_1 + N_2). \tag{5.41}$$

After making these substitutions in eq. (5.40) we obtain the Hamiltonian presented at the outset of this section, see eq. (5.37). As can be easily verified, the relativistic model possesses two first class constraints

$$\Phi_{\mathcal{M}} = P^2 + 4(m^2 + \epsilon^2 s^2), \qquad \Phi_S = \ell^2(\Delta p)^2 + \epsilon^2 s^2 (\Delta x)^2 - 2\hbar^2 s^2, \tag{5.42}$$

and four second class constraints

$$P \cdot \Delta x = 0, \quad \Delta p \cdot \Delta x = 0, \quad P \cdot \Delta p = 0, \quad \ell^2 (\Delta p)^2 - \epsilon^2 s^2 (\Delta x)^2 = 0. \tag{5.43}$$

Thus, the reduced phase space has dimension $16 - 2 \times 2 - 4 \times 1 = 8$ yielding 4 physical degrees of freedom, as in the non-relativistic model. Note that the primary constraints eq. (5.39) are identical to those considered in the previous section if one transforms to the rest frame of the "center of mass" $P = (m, \vec{0})$ and implements $P \cdot \Delta x = P \cdot \Delta p = 0$.

The equations of motion are obtained from Hamilton's equation $\dot{A} = \{H, A\}$, we find

$$\begin{aligned} \frac{dX^\mu}{d\tau} &= -NP^\mu, & \frac{d\Delta x^\mu}{d\tau} &= -\widetilde{N}\ell^2 \Delta p^\mu, \\ \frac{dP^\mu}{d\tau} &= 0, & \frac{d\Delta p^\mu}{d\tau} &= \widetilde{N}\epsilon^2 s^2 \Delta x^\mu, \end{aligned} \tag{5.44}$$

which are easily integrated to give

$$\begin{aligned} X^\mu(\tau) &= X_0^\mu - N\tau P_0^\mu, & \Delta x^\mu(\tau) &= \ell \left[A^\mu \cos(\Omega\tau) + B^\mu \sin(\Omega\tau) \right], \\ P^\mu(\tau) &= P_0^\mu, & \Delta p^\mu(\tau) &= \epsilon s \left[A^\mu \sin(\Omega\tau) - B^\mu \cos(\Omega\tau) \right], \end{aligned} \tag{5.45}$$

where $\Omega = \widetilde{N}\hbar s$. The constant vectors A^μ, B^μ and P_0^μ satisfy $A^2 = B^2 = 1$, $P_0^2 = 4(m^2 + \epsilon^2 s^2)$ and $A \cdot P_0 = B \cdot P_0 = A \cdot B = 0$. As we can see, the "center of mass" propagates as a free particle while the relative displacement executes circular motion with frequency Ω. This result conforms with our intuition about the system since in the original set-up both particles were free but constrained to rotate with constant "internal" angular momentum. The angle between p_1 and p_2, denoted θ, can be computed from

$$p_1 \cdot p_2 = -|p_1||p_2| \cosh\theta \qquad \Longrightarrow \qquad \cosh\theta = 1 + \frac{2\epsilon^2 s^2}{m^2}. \tag{5.46}$$

The evolution is pictured in Figure 5.2. In Figure 5.3 we plot the position and momentum of each particle at $\tau = 0$ projected into the planes defined by $\{A, B\}$, $\{A, P_0\}$ and $\{B, P_0\}$. Both figures assume $X_0 = 0$. This completes our construction of a bi-local model, its relation to the relativistic spinning particle will be explored in the subsequent section.

5.4 Re-interpreting the Model

As the analysis in the previous section made apparent, the most natural variables for describing this two particle system are not the individual coordinates (x_1, p_1) and (x_2, p_2) but rather the "center of mass" (X, P) and the relative displacement $(\Delta x, \Delta p)$. This suggests that we could re-interpret the model as a *single* particle whose trajectory is determined by (X, P) but which possesses internal degrees of freedom described by $(\Delta x, \Delta p)$. This re-interpretation is more than just a curiosity, it is an exact realization of the relativistic spinning particle.

The "Dual Phase Space" Model (DPS) developed in Chapter 4, provides a classical realization of the relativistic spinning particle by means of the coajoint orbit method [102]. In particular, the naive phase space is parameterized by two pairs of canonically conjugate four-vectors, $(\mathbf{x}^\mu, \mathbf{p}^\mu)$ which describe the position and linear momentum of the particle and $(\boldsymbol{\chi}^\mu, \boldsymbol{\pi}^\mu)$ which encode the internal degrees of freedom associated with the spin. Note that

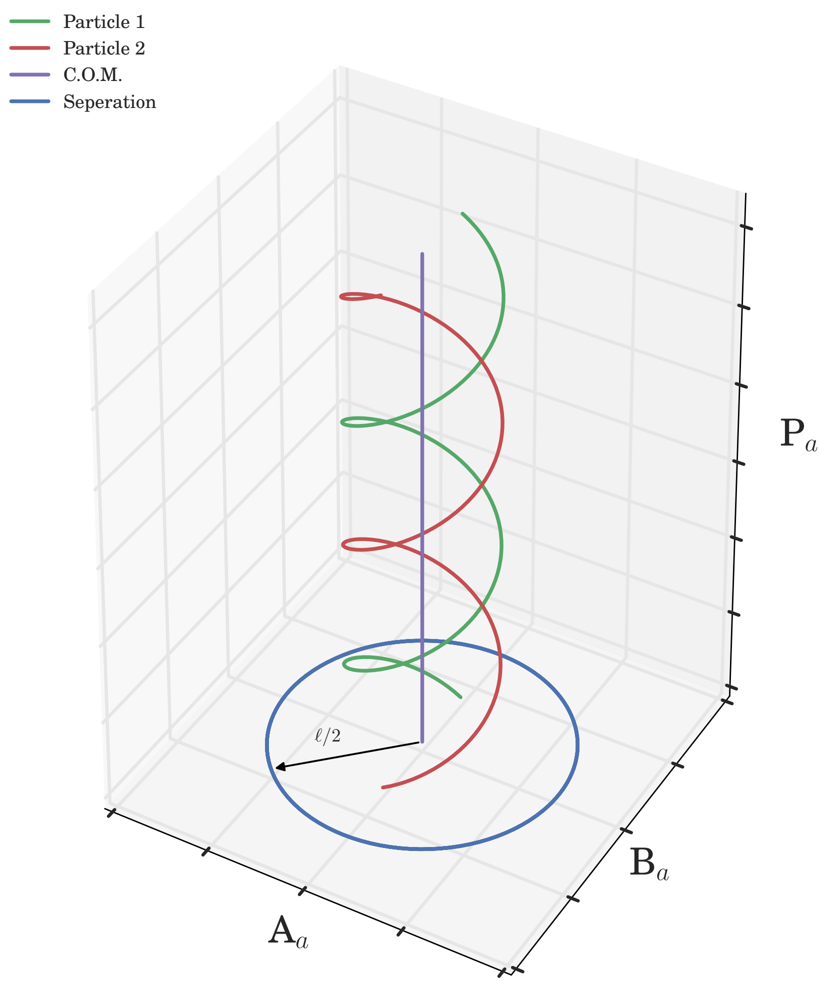


Figure 5.2: Particle trajectories plotted over two periods in the hyper-plane defined by the triplet of orthogonal vectors (A, B, P_0).

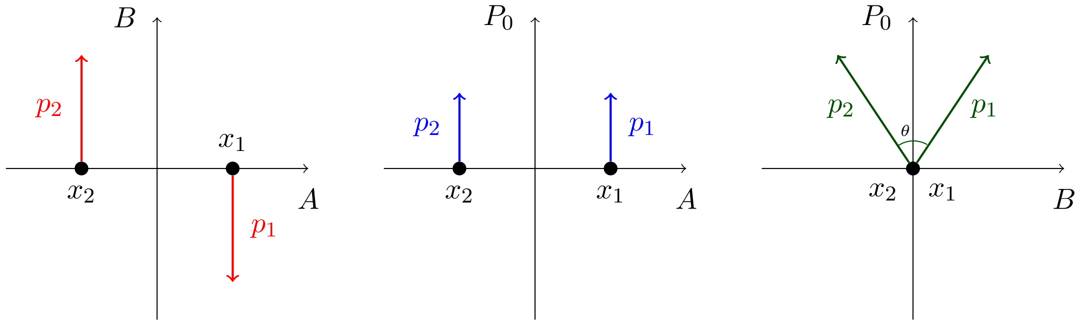


Figure 5.3: Projections, at $\tau = 0$, of (x_1, p_1) and (x_2, p_2) into the indicated planes. The angle θ is given in eq. (5.46).

we use bold faced characters to denote quantities originating in the DPS model. The Poisson brackets are trivial $\{\mathbf{p}^\mu, \mathbf{x}^\nu\} = \eta^{\mu\nu}$ and $\{\boldsymbol{\pi}^\mu, \boldsymbol{\chi}^\nu\} = \eta^{\mu\nu}$ while transformations under elements of the Poincaré group (Λ, y) are given by

$$\mathbf{x} \to \Lambda\mathbf{x} + y \qquad \text{and} \qquad \mathbf{p},\, \boldsymbol{\pi},\, \boldsymbol{\chi} \to \Lambda\mathbf{p},\, \Lambda\boldsymbol{\pi},\, \Lambda\boldsymbol{\chi}. \tag{5.47}$$

The dynamics of DPS are defined by two first class and four second class constraints, given respectively by

$$\mathbf{p}^2 = -\mathbf{M}^2, \quad \boldsymbol{\lambda}^2\boldsymbol{\pi}^2 + \boldsymbol{\epsilon}^2\mathbf{s}^2\boldsymbol{\chi}^2 = 2\hbar^2\mathbf{s}^2, \tag{5.48}$$

$$\mathbf{p}\cdot\boldsymbol{\pi} = 0, \quad \mathbf{p}\cdot\boldsymbol{\chi} = 0, \quad \boldsymbol{\pi}\cdot\boldsymbol{\chi} = 0, \quad \boldsymbol{\lambda}^2\boldsymbol{\pi}^2 - \boldsymbol{\epsilon}^2\mathbf{s}^2\boldsymbol{\chi}^2 = 0, \tag{5.49}$$

where $\mathbf{m}$ and $\mathbf{s}$ are the mass and spin of the particle while $\boldsymbol{\epsilon}$ and $\boldsymbol{\lambda}$ are arbitrary energy and length scales which satisfy $\boldsymbol{\epsilon}\boldsymbol{\lambda} = \hbar$. Comparing DPS to the relativistic two particle model presented in Section 5.3 shows an exact match under the following identifications

$$\begin{aligned} &\mathbf{p} = P, &&\mathbf{x} = X, &&\boldsymbol{\epsilon} = \epsilon, &&\mathbf{s} = s, \\ &\boldsymbol{\pi} = \Delta p, &&\boldsymbol{\chi} = \Delta x, &&\boldsymbol{\lambda} = \ell, &&\mathbf{M}^2 = 4(m^2 + s^2\epsilon^2). \end{aligned} \tag{5.50}$$

It is particularly interesting to note that the mass of the spinning particle $\mathbf{m}$ is larger than the sum of the constituent masses. A mass defect is the hallmark of a confined system, but that is not what we have here. Instead there is a mass surplus, indicating the presence of entanglement[4] with the entangled state having a higher energy than the

[4]In the standard quantum mechanical treatment of two entangled electrons there is a constraint on the total angular momentum, namely $J = 0$. Similarly, the entanglement in the bilocal picture is a result of the spin constraint Φ_S, c.f. eq. (5.42).

sum of its constituents. The extra energy is exactly the energy present in the spin motion; it is given by $\hbar s/\ell$ and can be lowered by having the pairs separate. Consequently, this constituent picture suggests that massive particles of higher integer spin are unstable and it is energetically favored to lower the spin towards a spinless particle. A conclusion not contradicted by nature.

We also see that the limit $m \to 0$ of massless constituents can be taken without incident, in which case the entire mass of the spinning particle arises as "entanglement energy" from the spin constraint. In this limit the particle radius can be expressed as

$$r = \frac{\ell}{2} = \frac{\hbar s}{\mathbf{M}}, \tag{5.51}$$

which scales inversely with the mass in the same manner as the Bohr radius of an atom. The limit of massless constituent particles also provides a possible resolution to a long standing problem regarding the center of mass of a spinning particle. The center of mass of an extended rotating object is not relativistically invariant and any classical model of spin which views a spinning particle as possessing some non-zero extension encounters this problem, see [40, 41] for a detailed analysis. In the case of massless constituent particles this is a moot point since a system of massless particles does not have a center of mass and one is forced to consider the geometric centroid instead, which is precisely what X^μ is in the relativistic case.

If we assume physical constituents with positive mass square, the bilocal model can only describe particles whose mass is greater than its spin, since we have the relationship

$$\mathbf{M}^2 = \frac{4\hbar^2 s^2}{\ell^2} + 4m^2. \tag{5.52}$$

If the mass of the constituents are fixed this gives rise to a trajectory which is similar in spirit but different in details from a Regge trajectory where the mass square is linearly related to the spin $\mathbf{M}^2 \geq \alpha' J + \beta$. To go beyond the bound $\mathbf{M} \geq \frac{2\hbar s}{\ell}$ and describe massless particles $\mathbf{M} = 0$ requires that the constituents be tachyons with $m^2 = -\hbar^2 s^2/\ell^2$.

5.5 Interactions

Given the mapping eq. (5.50) between DPS and the two particle model, results from [62] can be imported directly and re-interpreted in the two particle picture. For example, interaction with a background electromagnetic field is achieved via the minimal coupling prescription

$$p_1 \to p_1 + \frac{q}{2}A(x_1 + x_2) \qquad p_2 \to p_2 + \frac{q}{2}A(x_1 + x_2),$$

where q is the total charge of the spinning particle. It follows that each constituent particle carries half the total charge while the electromagnetic field couples to the center of mass coordinate X^μ. This formulation also suggests that one could investigate a generalization of the coupling of electromagnetism to spinning particles where the location of the field interaction for the constituents 1 and 2 are not the same.

Interactions between spinning particles were a focal point of Chapter 4 with the Chapter culminating in the formulation of a necessary and sufficient condition for a consistent three-point vertex. In detail, suppose a vertex has one incoming and two out going particles with coordinates $(\mathbf{x}_i, \mathbf{p}_i)$, $(\boldsymbol{\pi}_i, \boldsymbol{\chi}_i)$, where $i = 1, 2, 3$ and it is assumed that particle #1 is incoming. The vertex is governed by conservation of linear and angular momentum along with the requirement that interactions are local in space-time, i.e. $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}_3$. It turns out that consistency is possible if and only if there exists a choice of $\boldsymbol{\chi}$ variables such that the interaction is also local in the dual space. That is one has to impose $\boldsymbol{\chi}_1 = \boldsymbol{\chi}_2 = \boldsymbol{\chi}_3$, a condition we referred to as "dual locality". The conservation equations then become

$$\mathbf{p}_1 = \mathbf{p}_2 + \mathbf{p}_3 \qquad \text{and} \qquad \boldsymbol{\pi}_1 = \boldsymbol{\pi}_2 + \boldsymbol{\pi}_3, \tag{5.53}$$

which can be solved by elementary methods. In the two particle picture these notions have concrete interpretations: Locality plus "dual locality" becomes the condition that interactions are local for each constituent particle, while equation eq. (5.53) implies conservation of momentum at each particle. This is pictured in Figures 5.4–5.6, where we have used the notation $p_i^{(j)}$ to indicate the i-th constituent of particle j, $i = 1, 2$, $j = 1, 2, 3$. In Figure 5.6 each spinning particle is represented by a string of length ℓ and it is seen that the interaction splits the incoming strip into two halves. The resulting worldsheet is not a smooth manifold but a branched 2 dimensional surface. This form of the interaction vertex is very different from the string inspired interaction which has been explored in the literature on massless particles [70].

5.6 Quantization and Other Bilocal Models

Before examining the quantization of the relativistic two particle model it is interesting to note the relationship between DPS and other bilocal models appearing in the literature. A

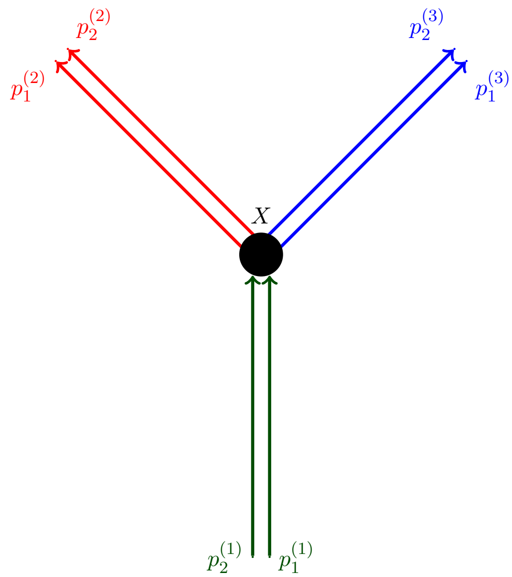


Figure 5.4: Three-point interaction vertex.

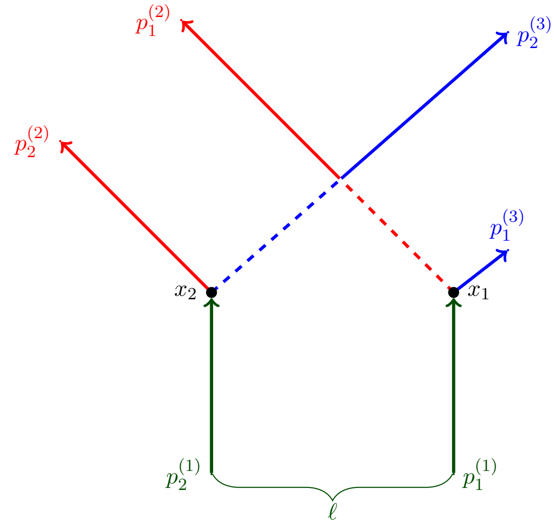


Figure 5.5: Detailed view of three-point interaction vertex

popular model introduced by Takabayasi [66] and known as the "Simple Relativistic Oscillator Model" (SROM) is obtained by combining $\Phi_{\mathcal{M}}$ and Φ_S and dropping all remaining constraints that don't involve P_μ. In particular,

$$\Phi = \Phi_{\mathcal{M}} + \frac{4}{\ell^2}\Phi_S, \qquad \Phi_1 = P \cdot \Delta p, \qquad \Phi_2 = P \cdot \Delta x. \tag{5.54}$$

For a model to be interpreted as "bilocal" the two constituent particles need a well defined mass which means that the values of p_i^2 must be specified by the constraints. As $p_1, p_2 = P/2 \pm \Delta p$ we need to specify at least, $P^2 + 4(\Delta p)^2$ and $P \cdot \Delta p$. The SROM is therefore a minimally constrained bilocal model that has non-trivial kinematics in the relative separation.

A similar model has been proposed by Casalbuoni and Longhi [68]. It imposes the primary constraints $P^2 + (\Delta p)^2 + (\Delta x/\alpha')^2 = 0$, where α' is the inverse string tension, supplemented by $\Phi_1 = \Phi_2 = 0$ and $(\Delta p \cdot \Delta x) = 0$. This model is obtained from a truncation of string theory, by restricting the string motion to excite only one oscillator. It corresponds to a limit of our model in which $m = 0$, $s = 0$ and the separation $\ell = 0$ also vanish. More precisely the relationship between the string tension and spinning particle tension is given in the limit $s \to 0$ by $\ell^2 \sim \hbar\alpha' s^2$. Our description does not really survive this limit since we need a non-zero separation length, so this string model is really a different model. In this limit the vertex of interaction is derived from the string vertex and

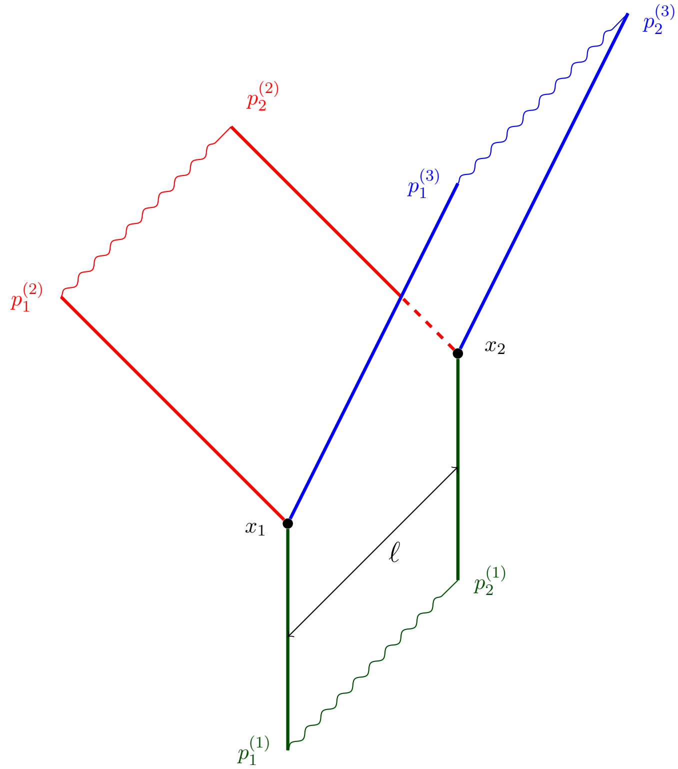


Figure 5.6: Expanded view of three-point interaction vertex.

has a geometry very different from the vertex we described (c.f. Figure 1 in [70]).

Another class of models arise by setting the total mass $\mathbf{M}$ to zero or equivalently fixing $m^2 = -4\epsilon^2 s^2$, in which case we have tachyonic constituents. We can obtain several versions of massless higher spin particles, see the discussion by Bengtsson in [70]. The massless case is special, since $\mathbf{M} = 0$ implies that the constraints

$$\Phi_1 = P \cdot \Delta p, \qquad \Phi_2 = P \cdot \Delta x \tag{5.55}$$

are first class.

By considering only the constraints $\Phi_{\mathcal{M}}$, Φ_1 and Φ_2 we obtain a theory which describes a reducible tower of higher spin massless gauge fields. Including $\Delta p \cdot \Delta x = 0$ and $\ell^2(\Delta p)^2 = \epsilon^2 s^2 (\Delta x)^2$ makes this tower irreducible and adding Φ_S as well gives a single higher spin massless gauge field. In all these models the issue of the interaction vertex is still open.

5.6.1 Quantizing the Relativistic Model

To quantize the relativistic two particle model we will first obtain a Lagrangian description as we did in the non-relativistic case. This analysis has already been done for DPS, see eq. (4.50) in Chapter 4, and since the two models are equivalent we can simply import the result. We find

$$L_s = \epsilon \sqrt{\frac{s^2}{y^2}(\mathcal{D}_\tau y)^2 - \mathbf{M}^2 (\mathcal{D}_\tau X)^2 - \frac{2ms}{|y|}\sqrt{(\mathcal{D}_t X \cdot \mathcal{D}_t y)^2 - (\mathcal{D}_t X)^2 (\mathcal{D}_t y)^2}}, \tag{5.56}$$

where $\epsilon = \pm$ and the sign of s is not fixed. These signs come from defining the square roots and

$$\Delta x^\mu = \ell y^\mu / |y|, \quad \mathcal{D}_\tau A^\mu = \dot{A}^\mu - \frac{\dot{A} \cdot y}{y^2} y^\mu, \quad \mathbf{M}^2 = 4(m^2 + \epsilon^2 s^2).$$

The momenta conjugate to X^μ and y^μ, denoted P_X^μ and P_y^μ respectively, can be obtained in the standard fashion by varying the action with respect to $\dot{X}$ and $\dot{y}$ respectively. There is no need to know their exact form, it is sufficient to note that they satisfy the following constraints

$$P_X^2 = -\mathbf{M}^2, \quad P_y^2 = \frac{s^2}{|y|^2}, \quad P_y \cdot y = 0 \tag{5.57}$$

$$P_X \cdot y = 0, \quad P_X \cdot P_y = 0. \tag{5.58}$$

The first three constraints are first class[5] and are strikingly similar to those appearing in the non-relativistic model, see eq. (5.20). The final two constraints are second class which will complicate the quantization procedure since we must first implement Dirac brackets before promoting to commutators. Forgoing some details, we find that the commutator algebra which takes into account the second class constraints is given by

$$\left[\hat{X}^\mu, \hat{X}^\nu\right] = \frac{i}{\mathbf{M}^2}\hat{S}^{\mu\nu}, \quad \left[\hat{X}^\mu, \hat{P}^\nu_X\right] = i\eta^{\mu\nu}, \quad \left[\hat{X}^\mu, \hat{y}^\nu\right] = \frac{i}{\mathbf{M}^2}\hat{y}^\mu \hat{P}^\nu_X, \tag{5.59}$$

$$\left[\hat{X}^\mu, \hat{P}^\nu_y\right] = \frac{i}{\mathbf{M}^2}\hat{P}^\mu_y \hat{P}^\nu_X, \quad \left[\hat{y}^\mu, \hat{P}^\nu_y\right] = i\left(\eta^{\mu\nu} + \frac{1}{\mathbf{M}^2}\hat{P}^\mu_X \hat{P}^\nu_X\right), \tag{5.60}$$

where $S_{\mu\nu} = (y \wedge p)_{\mu\nu}$ is the spin tensor and $\mathbf{M}^2 := -P_X^2$. It can be checked directly that commutators of the second class constraints either vanish directly or are proportional to the mass-shell constraints $(\hat{P}_X^2 + \mathbf{M}^2) = 0$.

Let $\mathcal{H} = L^2(\mathbb{R}^4 \times \mathbb{R}^4)$ be the Hilbert space of square integrable functions $\Psi(X, y)$. An action of the operators on $\mathcal{H}$ which respects the preceding commutation relations can be defined as follows

$$\hat{X}^\mu \Psi = \left(X^\mu + \frac{i}{\mathbf{M}^2} S^{\mu\nu}\frac{\partial}{\partial X^\nu}\right)\Psi, \qquad \hat{P}^\mu_X \Psi = -\, i\frac{\partial}{\partial X_\mu}\Psi, \tag{5.61}$$

$$\hat{y}^\mu \Psi = \mathcal{P}^{\mu\nu} y_\nu \Psi, \qquad \hat{P}^\mu_y \Psi = -\, i\mathcal{P}^{\mu\nu}\frac{\partial}{\partial y^\nu}\Psi, \tag{5.62}$$

where

$$S^{\mu\nu} = -i\left(y^\mu \frac{\partial}{\partial y_\nu} - y^\nu \frac{\partial}{\partial y_\mu}\right), \qquad \mathcal{P}^{\mu\nu} = \eta^{\mu\nu} - M^{-2}\frac{\partial^2}{\partial X_\mu \partial X_\nu}. \tag{5.63}$$

It is easily verified that the operator identities $\hat{P}_X \cdot \hat{y} = \hat{P}_X \cdot \hat{P}_y = 0$ are satisfied and so we turn our attention to the first class constraints, eq. (5.57). The action of these constraints on the Hilbert space $\mathcal{H}$ yields the following differential equations

$$\Box_X \Psi = \mathbf{M}^2 \Psi, \tag{5.64}$$

$$y^\mu \frac{\partial}{\partial y_\nu}\mathcal{P}_{\mu\nu}\Psi = 0, \tag{5.65}$$

$$y^\mu y^\nu \frac{\partial^2}{\partial y_\alpha \partial y_\beta}\mathcal{P}_{\mu\nu}\mathcal{P}_{\alpha\beta}\Psi = -s^2\Psi. \tag{5.66}$$

Assuming separation of variables $\Psi(X, y) = \Psi_X(X)\Psi_y(y)$, eq. (5.64) is just the Klein-Gordon equation for $\Psi_X(X)$ which is easily solved in momentum space and $\Psi_X(X) =$

[5] We have the standard Poisson brackets $\{X^\mu, P^\nu_X\} = \eta^{\mu\nu}$ and $\{y^\mu, P^\nu_y\} = \eta^{\mu\nu}$.

$\int \mathrm{d}k e^{ik\cdot X}\tilde{\Psi}_X(k)\delta(k^2+m^2)$ is the general solution. It follows that

$$\mathcal{P}^{\mu\nu}\Psi = \left(\eta^{\mu\nu} + \frac{1}{\mathbf{M}^2}k^\mu k^\nu\right)\Psi \equiv \mathcal{P}_k^{\mu\nu}\Psi, \tag{5.67}$$

where $\mathcal{P}_k^{\mu\nu}$ is the projection operator onto the hyper-plane orthogonal to k_μ. Let us introduce the coordinate $y_k^\mu = \mathcal{P}_k^{\mu\nu}y_\nu$, then we can assume a further separation of variables for $\Psi_y(y)$, namely

$$\Psi_y(y) = \Psi_0(y\cdot k)\Psi_{y_k}(y_k). \tag{5.68}$$

We can now express eqs. (5.65)–(5.66) as follows

$$y_k^\mu \frac{\partial}{\partial y_k^\mu}\Psi_{y_k} = 0, \tag{5.69}$$

$$\Box_{y_k}\Psi_{y_k} + \frac{s^2}{y_k^2}\Psi_{y_k} = 0. \tag{5.70}$$

For k^μ timelike the vector y_k^μ takes values in a three dimensional spacelike hyperplane orthogonal to k^μ. As such eqs. (5.69)–(5.70) have the same solution as their non-relativistic counterparts eqs. (5.26)–(5.27), i.e. $\Phi_{y_k}(y_k) = Y_\ell^m$ where Y_ℓ^m is a spherical harmonic. As the Hamiltonian is a sum of the first class constraints this completes the quantization of the relativistic two-particle model. The solutions are characterized by three quantum numbers M, ℓ and m where $M \in \mathbb{R}$, $\ell \in \mathbb{N}$ and $m = -\ell, -\ell+1, \ldots, \ell-1, \ell$; wavefunctions are written as

$$\Psi_{M,\ell,m} = \Psi_0\Psi_k^M Y_\ell^m, \tag{5.71}$$

where Ψ_0 is undetermined.

Chapter 6

First Order Parameterization of Spinning Particle

6.1 Introduction

In Chapter 4 we introduced the "Dual Phase Space" (DPS) model as a natural setting for understanding the relativistic spinning particle. However, one of the challenges with DPS was the presence of second class constraints, recall that for a massive particle four of the six defining constraints were of this type. The present Chapter seeks to simplify the constraint structure of DPS by utilizing spinors to parameterize the spinning degrees of freedom and thereby resolve all second class constraints. The resulting model provides valuable insights into the effect of spin on motion through spacetime. To keep the Chapter self contained we will review some of the results from previous Chapters.

6.2 Overview

As this Chapter is rather technical we include at the outset an overview of the relevant results. The "Dual Phase Space" model parameterizes the phase space of the relativistic particle of mass m and spin s in terms of two pairs of canonically conjugate four vectors (x^μ, p^μ) and (χ^μ, π^μ). The pair (x^μ, p^μ) represent the standard position and momentum of the particle while (χ^μ, π^μ) are "dual" variables which encode the spinning degrees of freedom. The dynamics of the particle are then determined by a set of six real constraints, and if the particle is massive four of these are second class. These constraints can be presented most straightforwardly by introducing a complex "spin" vector

$$\ell^\mu = \frac{\pi^\mu}{\epsilon} + is\frac{\chi^\mu}{\lambda}, \tag{6.1}$$

where λ and ϵ are fundamental length and energy scales respectively and satisfy $\lambda\epsilon = \hbar$.

The first class constraints which define mass and spin are simply restrictions on the length of the momenta and spin vector

$$p^2 + m^2 = 0, \qquad \ell\ell^* = 2s^2. \tag{6.2}$$

These are then supplemented by additional constraints which form a second class system. The first result of this Chapter is to construct a purely first class model by using the spinor formalism to solve the second class constraints. The purpose of such a re-parameterization is two fold, first it provides greater control over the action while allowing for a better understanding of the effect of spin on particle dynamics. Secondly, it makes a connection with the standard Dirac formalism and therefore should permit a description of fermions[1].

We find that the general solution to the second class constraints is obtained by setting

$$\ell = \frac{|\xi\rangle\langle\xi|p}{m}, \tag{6.3}$$

where $\xi_\alpha = |\xi\rangle$ is a spinor, $\langle\xi| = \epsilon^{\alpha\beta}\xi_\beta$ is the transposed spinor, and p is the momenta represented as a 2×2 hermitian operator. The resulting first order action has two undetermined Lagrange multipliers corresponding to the mass shell and spin constraint. We can interpret the Lagrange multipliers as generators of two gauge invariant quantities, proper time $\tau(t)$ (dual to the mass shell) and proper angle $\phi(t)$ (dual to the spin constraint) and we show that the action is of the form

$$S = m\tau(t) + 2\hbar s\phi(t). \tag{6.4}$$

The spin velocity $|\dot{\xi}\rangle$ can be expressed as a function of two complex coefficients $(\mathfrak{a}, \mathfrak{b})$ that characterize the spin motion and which are defined by the expansion

$$|\dot{\xi}\rangle = \mathfrak{a}|\xi\rangle + \frac{\mathfrak{b}m}{2\hbar s}\,\dot{x}|\xi], \tag{6.5}$$

where $|\xi] = \langle\xi|^\dagger$. The proper time and the proper angle are then explicitly given by

$$\dot{\tau} = |\dot{x}|\sqrt{1-|\mathfrak{b}|^2}, \qquad \dot{\phi} = \mathrm{I}m(\mathfrak{a}). \tag{6.6}$$

[1] It was shown in Chapter 5 that, upon quantization, DPS only yields integer spins.

where $|\dot{x}| \equiv \sqrt{-\dot{x}^2}$. The fact that the action is independent of $\mathrm{Re}(\mathfrak{a})$ means that it is invariant under spin rescaling $|\xi\rangle \to \alpha|\xi\rangle$, with $\alpha \in \mathbb{R}^+$ which is essentially the expression of Lorentz invariance from the spin point of view. Comparing this to the standard action for the spinless relativistic particle one notices that the four-velocity is modified by a factor of the form $\sqrt{1-|\mathfrak{b}|^2}$. This shows that spin motion can be viewed as inducing a Lorentz contraction of the four-velocity! In addition, one notes that there is a maximal speed of spin propagation encoded into the causality condition $|\mathfrak{b}| \leq 1$. As shown in Section 6.4.2, violating this bound would yield a spacelike velocity $\dot{x}^2 > 0$. Note that the parameter $\mathfrak{b}$ measures the propensity of spin to flip along the motion of the particle.

Further analysis reveals an even more stringent restriction on the classical spin motion: If a relativistic spinning particle has an initial configuration given by (x, ξ) and x' is in the future light cone of x , then there is a classical path connecting (x, ξ) and (x', ξ') only if $\xi' \propto \xi$, that is only if the spin state does not evolve under classical motion. It follows that[2] there are trajectories which have $\dot{\xi} \neq 0$ but which still satisfy the causality constraint $|\mathfrak{b}| \leq 1$. These "half-quantum" states are interesting because although they are not classical they are not exponentially suppressed in the path integral either. This possibility explains why spin and its motion can only be fully understood as a quantum object since the boundary between quantum and classical is not as sharply defined as it is for spacetime motion.

We complete the first order formulation by computing explicitly the commutators among the position and spin variable and we witness that the presence of spin renders the position variable non-commutative (as already noticed in [71, 6, 112]). The calculation is involved but is simplified by considering the symmetries of the symplectic potential/form. In particular we find that the position coordinates acts a type of boost generator on the spin variables:

$$\{\xi_\alpha, x_{\beta\dot{\beta}}\} = \frac{p_{\alpha\dot{\beta}}\xi_\beta}{m^2}. \tag{6.7}$$

From our analysis we can clearly see two new phenomena associated with spin, the existence of a spin causality constraint and the possibility of"half-quantum" states, both of which are not discussed in the literature.

[2]See Section 6.4.1 for a complete discussion

6.3 First Class Formalism

In the DPS model the phase space of the relativistic spinning particle is parameterized by two pairs of canonically conjugate four vectors (x_μ, p_ν) and (χ_μ, π_ν) with Poisson brackets

$$\{x_\mu, p_\nu\} = \hbar\eta_{\mu\nu}, \qquad \{\chi_\mu, \pi_\nu\} = \hbar\eta_{\mu\nu}. \tag{6.8}$$

The fundamental length and energy scales λ and ϵ allow for the unification of the "dual" position χ_μ and "dual" momenta π_μ into a single complex vector

$$\ell_\mu \equiv \frac{\pi_\mu}{\epsilon} + is\frac{\chi_\mu}{\lambda}, \qquad \{\ell_\mu, \ell^*_\nu\} = 2s\eta_{\mu\nu}. \tag{6.9}$$

The dynamics of the spinning particle are then characterized by two sets of constraints with the first being obtained from a simple restriction on the lengths of p and ℓ:

$$p^2 + m^2 = 0, \quad \ell\ell^* = 2s^2. \tag{6.10}$$

The remaining constraints are then given by two pairs of orthogonality conditions

$$p \cdot \ell = 0, \qquad \ell^2 = 0, \tag{6.11}$$

and their conjugates.

It is easy to verify that the constraints in eq. (6.10) are first class. However, although the constraints in eq. (6.11) commute with each other they do not commute with their conjugates unless m or s vanish. The first major result of this Chapter is to solve this second class system by means of the spinor formalism leaving us with a purely first class representation of the relativistic spinning particle.

Let us begin with the null condition $\ell^2 = 0$, which is solved by noting that any complex null vector can be represented by a product of spinors ξ_α, $\bar\zeta_{\dot\alpha}$, viz

$$\ell_{\alpha\dot\alpha} = \xi_\alpha\bar\zeta_{\dot\alpha}, \qquad \ell = |\xi\rangle[\zeta|. \tag{6.12}$$

Here and in what follows we will utilize the spinor formalism quite extensively (see [74, 113, 114, 115]). For the readers' convenience this formalism is described at length in Chapter H, and we give a short overview here as well.

Denote by χ^α, $\alpha = 0, 1$, a two-dimensional complex spinor and $\bar\chi^{\dot\alpha} = (\chi^\alpha)^\dagger$ its complex conjugate. Indices are raised and lowered with the epsilon tensor $\epsilon^{\alpha\beta}$ which is the skew symmetric tensor normalized by $\epsilon^{01} = 1$, i.e.

$$\chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta, \quad \chi_\alpha = \epsilon_{\alpha\beta}\chi^\beta, \quad \bar\chi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\chi_{\dot\beta}, \quad \bar\chi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\chi^{\dot\beta}.$$

These quantities are represented as bras and kets (see also [116]) via

$$|\chi\rangle = \chi_\alpha, \qquad \langle\chi| = \chi^\alpha, \qquad |\chi] = \bar{\chi}^{\dot{\alpha}}, \qquad [\chi| = \bar{\chi}_{\dot{\alpha}}, \tag{6.13}$$

with the notation being specifically designed to distinguish a spinor from its conjugate. Note also that we have adopted a convention in which the epsilon tensor satisfies $\epsilon_{\alpha\gamma}\epsilon^{\gamma\beta} = \delta_\alpha^\beta$. The $\mathrm{SL}(2,\mathbb{C})$ invariant contractions between spinors are denoted by a rocket:

$$\langle\zeta|\xi\rangle := \zeta^\alpha\xi_\alpha, \qquad [\zeta|\xi] := \bar{\zeta}_{\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}, \qquad [\zeta|\xi] = -\langle\zeta|\xi\rangle^*. \tag{6.14}$$

Let $(\sigma^a)_{\alpha\dot{\alpha}} = (\mathbb{1}_{\alpha\dot{\alpha}}, \vec{\sigma}_{\alpha\dot{\alpha}})$ be the standard four vector of sigma matrices, and $(\bar{\sigma}^a)^{\dot{\alpha}\alpha} \equiv (\sigma^a)_{\beta\dot{\beta}}\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}$ the same vector but with indices raised. Given a real vector p^a we can construct the two by two hermitian operators $p_{\alpha\dot{\alpha}} := p_a(\sigma^a)_{\alpha\dot{\alpha}}$ and $\bar{p} = p_a\bar{\sigma}^a$ as well as the hermitian pairing $\langle\xi|p|\zeta] = [\zeta|\bar{p}|\xi\rangle$. This completes the brief introduction to the spinor formalism.

Given the parameterization in eq. (6.12) the remaining second class constraint is equivalent to $[\zeta|\bar{p}|\xi\rangle = 0$ which has the general solution $[\zeta|\bar{p} \propto \langle\xi|$. The normalization can be chosen arbitrarily and to keep the spinor dimensionless we put $m[\zeta| = \langle\xi|p$, provided that $m \neq 0$. Thus, the general solution of eq. (6.11) is

$$\ell_{\alpha\dot{\alpha}} = \frac{\xi_\alpha\xi^\beta p_{\beta\dot{\alpha}}}{m} \qquad \text{or} \qquad \ell = \frac{|\xi\rangle\langle\xi|p}{m}. \tag{6.15}$$

The first class constraints, eq. (6.10), are expressed in-terms of these new variables as

$$\Phi_m = \frac{1}{2}\mathrm{Tr}(p\bar{p}) - m^2, \qquad \Phi_s = \frac{1}{2}\langle\xi|p|\xi] - ms. \tag{6.16}$$

To obtain an action we add these constraints to the symplectic potential $\Theta = p_\mu dx^\mu + \pi_\mu d\chi^\mu$. As a one-form on phase space the symplectic potential can be evaluated on the second class constraints eq. (6.11) and the spin part $\pi_\mu d\chi^\mu$ expressed entirely in terms of the spinor variable ξ^α, viz

$$\Theta = -\frac{1}{2}\mathrm{Tr}(\bar{p}dx) + \frac{i\hbar}{2m}\frac{\langle\xi|p|\xi]}{2ms}\,\bar{p}^{\alpha\dot{\alpha}}\left(\xi_\alpha \mathrm{d}\bar{\xi}_{\dot{\alpha}} - \bar{\xi}_{\dot{\alpha}}\mathrm{d}\xi_\alpha\right). \tag{6.17}$$

It is interesting to see that the symplectic structure already depends on a unit of action through $\hbar$ even if it is a classical entity. This expresses mathematically the notion that spin blurs the sharp distinction between classical and quantum that we are familiar with. Without loss of generality and up to a redefinition of Lagrange multipliers we can implement the first class constraints in Θ. It will therefore be convenient to work with the simpler version

$$\Theta_{m,s} = -\frac{1}{2}\mathrm{Tr}(\bar{p}dx) + \frac{i\hbar}{2m}\,\bar{p}^{\alpha\dot{\alpha}}\left(\xi_\alpha \mathrm{d}\bar{\xi}_{\dot{\alpha}} - \bar{\xi}_{\dot{\alpha}}\mathrm{d}\xi_\alpha\right), \tag{6.18}$$

which summarizes the symplectic structure of the relativistic spinning particle.

6.4 Classical Action for the Relativistic Spinning Particle

6.4.1 First Order Action and Half-Quantum States

The action $S_{m,s} = \int_0^\tau dt L_{m,s}(t)$ for the relativistic spinning particle has Lagrangian $L_{m,s} = \Theta_{m,s} + \frac{N}{m}\Phi_m + \frac{M}{m}\Phi_s$. Introducing the quantity $\theta(\xi) := i\hbar\left(|\dot{\xi}\rangle[\xi| - |\xi\rangle[\dot{\xi}|\right)$ we find that the Lagrangian is expressed explicitly as

$$L_{m,s} = -\frac{1}{2m}\mathrm{Tr}\left(\bar{p}\left(m\dot{x} + \theta(\xi) - M|\xi\rangle[\xi|\right) - \frac{N}{2}p\bar{p}\right) - \frac{Nm}{2} - Ms. \tag{6.19}$$

The first class constraints Φ_m and Φ_s generate time translations (parameterized by α) and local spin rotations (parameterized by β) respectively. Both of these gauge transformations leave the Lagrangian invariant and act on the phase space variables as

$$\delta_{(\alpha,\beta)}N := \dot{\alpha}, \qquad \delta_{(\alpha,\beta)}M := \dot{\beta}, \qquad \delta_{(\alpha,\beta)}x := \frac{\alpha p}{m}, \qquad \delta_{(\alpha,\beta)}|\xi\rangle := -\frac{i\beta}{2\hbar}|\xi\rangle, \tag{6.20}$$

while $\delta_{(\alpha,\beta)}p = 0$ and $\delta_{(\alpha,\beta)}\theta(\xi) = \dot{\beta}|\xi\rangle[\xi|$. These transformations can also be used to fix the Lagrange multipliers N and M to constant values, giving rise to two gauge invariant observables, the *proper time* τ and the *proper angle* ϕ:

$$\tau(t) = \int_0^t \mathrm{d}t' N(t'), \qquad \phi(t) = \frac{1}{2\hbar}\int_0^t \mathrm{d}t' M(t'). \tag{6.21}$$

The appearance of a new type of observable in addition to proper time is one of the most relevant facts about spin from the perspective of this Chapter.

To obtain the first order action we need to solve the equation of motion for p which is given by $Np = m\dot{x} + \theta(e^{i\phi}\xi)$. Inserting this into the Lagrangian we find

$$L_{m,s} = -\frac{1}{4Nm}\mathrm{Tr}\left[\left(m\dot{x} + \theta(e^{i\phi}\xi)\right)\left(m\dot{\bar{x}} + \bar{\theta}(e^{i\phi}\xi)\right)\right] - \frac{Nm}{2} - 2\hbar s\dot{\phi}, \tag{6.22}$$

where we have used that $2\hbar\dot{\phi} = M$. We can further expand $L_{m,s}$ by means of the identities

$$\mathrm{Tr}\left[\dot{x}\bar{\theta}(\xi)\right] = -2\hbar\mathrm{Im}\left(\langle\dot{\xi}|\dot{x}|\xi]\right) \qquad \text{and} \qquad \mathrm{Tr}\left[\theta(\xi)\bar{\theta}(\xi)\right] = -2\hbar^2|\langle\xi|\dot{\xi}\rangle|^2, \tag{6.23}$$

while also making use of the re-parametrization invariant spinor velocity $|\partial_\tau\xi\rangle = |\dot{\xi}\rangle/N$. We find

$$L_{m,s} = \underbrace{\frac{1}{2\tilde{N}}\dot{x}^2 - \frac{\tilde{N}}{2}\left(m^2 - |\hbar\langle\xi|\partial_\tau\xi\rangle|^2\right)}_{\text{Modified Mass-shell}} + \overbrace{\hbar\,\mathrm{Im}\left(\langle\partial_\tau\xi|\dot{x}|\xi]\right)}^{\text{Spin Potential}} + \underbrace{\hbar\dot{\phi}\left(\langle\xi|\dot{x}|\xi] - 2s\right)}_{\text{Spin Constraint}}, \tag{6.24}$$

where we have defined $\tilde{N} \equiv N/m$. Written in this form, the Lagrangian is valid in the massless limit as well.

As seen above there are three terms which make up the Lagrangian: A modified mass-shell with effective mass M given by

$$M^2 = m^2 - |\hbar\langle\xi|\partial_\tau\xi\rangle|^2, \tag{6.25}$$

a potential that couples the linear velocity to the spin, and finally the spin constraint

$$\langle\xi|\dot{x}|\xi] = 2\hbar s. \tag{6.26}$$

The minus sign appearing in the modified mass-shell, eq. (6.25), imposes a *causality constraint*: At a classical level the linear velocity must be timelike or null, i.e $M^2 \geq 0$, and so the spin motion must satisfy

$$\hbar|\langle\xi|\partial_\tau\xi\rangle| \leq m. \tag{6.27}$$

Therefore, while the component of $\dot{x}$ along $|\xi]$ is fixed by eq. (6.26), the causality constraint restricts the spin velocity $|\langle\xi|\dot{\xi}\rangle|$ to be bounded from above. Of course, this is a classical restriction and can be violated at the quantum level. These are the virtual processes whose amplitudes will be suppressed in the path integral.

We can extend this analysis to the semi-classical level and see more clearly the delineation between which processes will experience an exponential suppression and those which will not. Specifically, let us examine the trajectories defined by the classical equations of motion. For x we find that the evolution is characterized by

$$m\dot{x} + \theta(\xi) = NP + \dot{\phi}|\xi\rangle[\xi|, \tag{6.28}$$

where P^a is a constant of motion. It follows that the particle will undergo oscillatory motion, known as Zitterbewegung [82], due to the rotation of the spin, on top of the standard linear evolution. On the other hand, the equation of motion for ξ reduces to $\xi(\tau) = e^{i\phi(\tau)}\xi(0)$ which yields an even more stringent restriction on the spin motion than eq. (6.27), see Section I.1 for more details. In particular, it implies that the spin state can only change by an internal phase during classical evolution.

Observe that it is possible to violate the restriction on the spin evolution while still satisfying both the causality constraint eq. (6.27) and the classical equation of motion eq. (6.28) for x . Such "half-quantum" states represent trajectories which are not fully classical yet will not be exponentially suppressed in the path integral. Normally motion is classified as either classical, in which case the classical equations of motion are satisfied, or quantum, in which case the classical action is imaginary. Little is known about "half-quantum" states and they deserve further exploration; it is possible that they represent entanglement.

6.4.2 Second-Order Action

The second order action can be obtained from eq. (6.24) by integrating out $\tilde{N}$ and $\dot{\phi}$. For $\tilde{N}$ we proceed in the usual fashion by solving its equation of motion,

$$\tilde{N}^2 = -\frac{\dot{x}^2}{(m^2 - |\hbar\langle\xi|\partial_\tau\xi\rangle|^2)} \tag{6.29}$$

and substituting the result back into $L_{m,s}$. The integration over ϕ, on the other hand, imposes the spin constraint eq. (6.26). In order to solve it we introduce a spinor ρ_α, free of constraints, and which is related to ξ via[3]

$$|\xi] = |\rho]\sqrt{\frac{2Ns}{\langle\rho|\dot{x}|\rho]}}. \tag{6.30}$$

Combining these transformations gives the second order action

$$S = m\int_0^1 \mathrm{d}\tau\sqrt{-\dot{x}^2\left(1 - \left|\frac{2\hbar s}{m}\frac{\langle\rho|\dot{\rho}\rangle}{\langle\rho|\dot{x}|\rho]}\right|^2\right)} + 2\hbar s\int_0^1 \mathrm{d}\tau\left(\frac{\mathrm{Im}\langle\dot{\rho}|\dot{x}|\rho]}{\langle\rho|\dot{x}|\rho]}\right). \tag{6.31}$$

As in the first order case, this action is invariant under time translations and spin rotations, now expressed as

$$\delta_{(\alpha,\beta)}x = \alpha\dot{x}, \qquad \delta_{(\alpha,\beta)}|\rho\rangle = \alpha|\dot{\rho}\rangle + i\beta|\rho\rangle, \tag{6.32}$$

where $\alpha, \beta \in \mathbb{R}$. The proper time and proper angle can be identified with the first and second term of eq. (6.31) respectively, viz.

$$\tau(t) \equiv \int_0^t \mathrm{d}t'\sqrt{-\dot{x}^2\left(1 - \frac{|2\hbar s\langle\rho|\dot{\rho}\rangle|^2}{m^2\langle\rho|\dot{x}|\rho]^2}\right)}, \tag{6.33}$$

$$\phi(t) \equiv \mathrm{Im}\int_0^t \mathrm{d}t'\left(\frac{\langle\dot{\rho}|\dot{x}|\rho]}{\langle\rho|\dot{x}|\rho]}\right). \tag{6.34}$$

[3]We assume that $\langle\rho|\dot{x}|\rho] > 0$.

6.4.3 Decomposing Spin Velocity

The spinors $|\rho\rangle$ and $\dot{x}|\rho]$ form a basis for spinor space provided that $\langle\rho|\dot{x}|\rho] \neq 0$. Therefore, we can expand the spin velocity in this basis by introducing two complex functions $(\mathfrak{a}(\tau), \mathfrak{b}(\tau))$, with $|\mathfrak{b}| < 1$:

$$|\dot{\rho}\rangle = \mathfrak{a}|\rho\rangle + \frac{\mathfrak{b}m}{2\hbar s}\dot{x}|\rho]. \tag{6.35}$$

It is straightforward to solve for $\mathfrak{a}$ and $\mathfrak{b}$

$$\mathfrak{a} = \frac{\langle\dot{\rho}|\dot{x}|\rho]}{\langle\rho|\dot{x}|\rho]}, \qquad \mathfrak{b} = \frac{2\hbar s}{m}\frac{\langle\rho|\dot{\rho}\rangle}{\langle\rho|\dot{x}|\rho]}, \tag{6.36}$$

from which it follows that

$$S = m\int d\tau|\dot{x}|\sqrt{1-|\mathfrak{b}|^2} + 2\hbar s\int d\tau \mathrm{Im}(\mathfrak{a}). \tag{6.37}$$

We see that knowledge of the spin velocity at all times uniquely determines the proper time and proper angle. In particular, if the spin velocity has a component along $\dot{x}|\rho]$ the proper time runs at a slower pace and so $\sqrt{1-|\mathfrak{b}|^2}$ can be viewed as a time contraction factor due to the spin motion. In the appendix we extend this analysis a bit further and derive the equations of motion associated with the second order action eq. (6.31).

The action in eq. (6.31) is a special case of the one derived by Lyankhovich et. al. in [79]. The difference between the two comes from the inclusion of a term in Lyankovich's model which allows for the description of continuous spin particles (CSP's). As DPS is equivalent to the restricted version of the latter model (as established in this Chaoter) it is reasonable to assume that there is a generalization of the Dual Phase Space model which will also permit the inclusion of CSP's. We explore this possibility more fully in Chapter 7.

6.5 Poisson Brackets

Computing the Poisson algebra is rather tedious but can be simplified somewhat by first considering the symmetries of the symplectic potential/form. From eq. (6.17) we have that the symplectic potential is expressed in-terms of the original spinor ξ_α as

$$\Theta = -\frac{1}{2}\mathrm{Tr}(p\mathrm{d}\bar{x}) + \frac{i\hbar}{2m}\frac{\langle\xi|p|\xi]}{2ms}\left(\langle\xi|p|\mathrm{d}\xi] - \langle\mathrm{d}\xi|p|\xi]\right). \tag{6.38}$$

The symmetry group of Θ is the Poincaré group, which factors as the semi-direct product of the translation group and the group of left and right rotations. Let the infinitesimal generators of right and left rotations be denoted by ${\rho_\alpha}^\beta$ and ${\bar\rho^{\dot\alpha}}_{\dot\beta}$, respectively. Under these rotations the phase space variables transform in the following manner:

$$\delta^R_\rho x = \rho x, \qquad \delta^R_\rho p = \rho p, \qquad \delta^R_\rho |\xi\rangle = \rho|\xi\rangle, \qquad \delta^R_\alpha [\xi| = 0, \tag{6.39}$$

$$\delta^L_{\bar\rho} x = x\bar\rho, \qquad \delta^L_{\bar\rho} p = p\bar\rho, \qquad \delta^L_{\bar\rho} |\xi\rangle = 0, \qquad \delta^L_{\bar\rho} [\xi| = [\xi|\bar\rho. \tag{6.40}$$

We do not require ρ or $\bar\rho$ to be traceless and so rotations have a non-trivial action on the epsilon tensor, in particular

$$\delta^R_\rho \epsilon_{\alpha\beta} = {\rho_\alpha}^\gamma \epsilon_{\gamma\beta} + {\rho_\beta}^\gamma \epsilon_{\alpha\gamma}, \qquad \delta^R_\rho \epsilon^{\alpha\beta} = -\epsilon^{\gamma\beta}{\rho_\gamma}^\alpha - \epsilon^{\alpha\gamma}{\rho_\gamma}^\beta, \tag{6.41}$$

where the second equality follows by demanding invariance of ${\delta^\alpha}_\beta$. Identical results hold for left rotations of $\bar\epsilon$. Thus, the action of left and right rotations on quantities with raised indices can be obtained from eqs. (6.39)–(6.40) by adding a minus sign and moving the rotation matrix to the other side, e.g. $\delta^R_\rho\langle\xi| = -\langle\xi|\rho$, which implies that the rocket $\langle\xi|\xi\rangle$ is $\mathrm{SL}(2,\mathbb{C})$ invariant. We can also denote the infinitesimal generator of translations as $a_{\alpha\dot\alpha}$, which acts only on the positional coordinate x as $\delta_a x = a$. The Hamiltonian vector fields associated with these transformations are given by

$$R_\rho \equiv -(\bar x\rho)^{\dot\alpha\alpha}\frac{\partial}{\partial \bar x^{\dot\alpha\alpha}} + (\rho p)_{\alpha\dot\alpha}\frac{\partial}{\partial p_{\alpha\dot\alpha}} - (\langle\xi|\rho)^\alpha\frac{\partial}{\partial\xi^\alpha}, \tag{6.42}$$

$$V_{\bar\rho} \equiv -(\bar\rho\bar x)^{\dot\alpha\alpha}\frac{\partial}{\partial \bar x^{\dot\alpha\alpha}} + (p\bar\rho)_{\alpha\dot\alpha}\frac{\partial}{\partial p_{\alpha\dot\alpha}} - (\bar\rho|\xi])^{\dot\alpha}\frac{\partial}{\partial\bar\xi^{\dot\alpha}}, \tag{6.43}$$

$$T_a \equiv \bar a^{\dot\alpha\alpha}\frac{\partial}{\partial \bar x^{\dot\alpha\alpha}}, \tag{6.44}$$

respectively. We can now compute the corresponding Hamiltonian by considering the action of the symplectic form $\Omega = d\Theta$, viz.

$$\Omega(R_\rho,\cdot) = \mathrm{d}\mathrm{Tr}(\rho J), \qquad \Omega(L_{\bar\rho},\cdot) = d\mathrm{Tr}(\bar J\bar\rho), \qquad \Omega(T_a,\cdot) = d\mathrm{Tr}(\bar a p/2), \tag{6.45}$$

where

$$J = -\left[\frac{p\bar x}{2} + i\hbar\frac{p|\xi]\langle\xi|}{2m}\frac{\langle\xi|p|\xi]}{2ms}\right], \qquad \bar J = -\left[\frac{\bar x p}{2} - i\hbar\frac{\langle\xi|p|\xi]}{2ms}\frac{|\xi]\langle\xi|p}{2m}\right]. \tag{6.46}$$

It should be noted that the left and right rotations include left and right dilations. These are obtained by taking ρ and $\bar\rho$ proportional to the identity; the generators are

$$D = -\frac{1}{2}\mathrm{Tr}(p\bar x), \qquad R = \hbar\frac{\langle\xi|p|\xi]^2}{m^2 s}. \tag{6.47}$$

On the other hand, rotations associated with traceless ρ and $\bar{\rho}$ correspond to left and right Lorentz transformations.

As noted in eq. (6.45) the Hamiltonians for right rotations, left rotations, and translations are given by J, $\bar{J}$ and $p/2$, respectively. As such, we can write down the following brackets

$$\{A_\alpha\,, J_\beta{}^\gamma\} = \delta^\gamma{}_\alpha A_\beta, \qquad \left\{\bar{B}_{\dot\alpha}\,, \bar{J}^{\dot\beta}{}_{\dot\gamma}\right\} = \delta^{\dot\beta}{}_{\dot\alpha}\bar{B}_{\dot\gamma}, \tag{6.48}$$

$$\{A^\alpha\,, J_\beta{}^\gamma\} = -\delta^\alpha{}_\beta A^\gamma, \qquad \left\{\bar{B}^{\dot\alpha}\,, \bar{J}^{\dot\beta}{}_{\dot\gamma}\right\} = -\delta^{\dot\alpha}{}_{\dot\gamma}\bar{B}^{\dot\beta}, \tag{6.49}$$

$$\{x_{\alpha\dot\alpha}, p_{\beta\dot\beta}\} = 2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}, \qquad \{\bar{x}^{\dot\alpha\alpha}, p_{\beta\dot\beta}\} = 2\delta^\alpha_\beta\delta^{\dot\alpha}_{\dot\beta}, \tag{6.50}$$

where A (respectively $\bar{B}$) is any quantity with a single undotted (dotted) index and an unspecified number of dotted (undotted) indices. Note that J commutes with any quantity possessing only undotted indices and vice versa for $\bar{J}$, furthermore since $p_{\alpha\dot\alpha}$, ξ^α, and $\bar{\xi}^{\dot\alpha}$ are invariant under translations they must commute with $p_{\alpha\dot\alpha}$. Commutators between the J and $\bar{J}$ follow from the Jacobi identity

$$\{J_\alpha{}^\beta, J_\gamma{}^\rho\} = \delta^\rho_\alpha J_\gamma{}^\beta - \delta^\beta_\gamma J_\alpha{}^\rho, \quad \left\{\bar{J}^{\dot\alpha}{}_{\dot\beta}, \bar{J}^{\dot\gamma}{}_{\dot\rho}\right\} = \delta^{\dot\gamma}_{\dot\beta}\bar{J}^{\dot\alpha}{}_{\dot\rho} - \delta^{\dot\alpha}_{\dot\rho}\bar{J}^{\dot\gamma}{}_{\dot\beta}, \quad \left\{J_\alpha{}^\beta, \bar{J}^{\dot\alpha}{}_{\dot\beta}\right\} = 0. \tag{6.51}$$

Before we continue, it will be convenient to introduce the null "position" vector

$$\bar{\Delta} = \frac{\hbar}{2ms}\frac{\langle\xi|p|\xi]}{m}|\xi]\langle\xi|, \tag{6.52}$$

which allow us, c.f. eq. (6.46), to parameterize J and $\bar{J}$ as

$$J = -\frac{1}{2}p(\bar{x} + i\bar{\Delta}), \qquad \bar{J} = -\frac{1}{2}\left(\bar{x} - i\bar{\Delta}\right)p. \tag{6.53}$$

These expressions can now be inverted to obtain $\bar{x}$ and $\bar{\Delta}$ in-terms of variables whose Poisson brackets we already know

$$\bar{x} = -\frac{1}{m^2}\left(\bar{p}J + \bar{J}\bar{p}\right), \qquad \bar{\Delta} = \frac{i}{m^2}\left(\bar{p}J - \bar{J}\bar{p}\right). \tag{6.54}$$

Using these results we can compute the remaining Poisson brackets, as detailed in Appendix I.3. One finds that x acts as a generator of translations in momentum space while also rotating the spin variable along an axis determined by p, viz

$$\{\bar{x}^{\dot\alpha\alpha}, p_{\beta\dot\beta}\} = 2\delta^\alpha_\beta\delta^{\dot\alpha}_{\dot\beta}, \qquad \left\{\xi^\alpha, \bar{x}^{\dot\beta\beta}\right\} = \frac{1}{m^2}\bar{p}^{\dot\beta\alpha}\xi^\beta. \tag{6.55}$$

Furthermore, the position variable itself is observed to be non-commutative, with the deviation from commutativity being proportional to the spin content. This fundamental modification to the notion of localization is one of the main features of spin, and has been exploited in previous works [6]. Explicitly, the x commutation relations read

$$\left\{\bar{x}^{\dot{\alpha}\alpha}, \bar{x}^{\dot{\beta}\beta}\right\} = \frac{i\hbar}{2ms}\frac{\langle\xi|p|\xi]}{m^3}\left(\bar{p}^{\dot{\alpha}\beta}\bar{\xi}^{\dot{\beta}}\xi^{\alpha} - \bar{p}^{\dot{\beta}\alpha}\bar{\xi}^{\dot{\alpha}}\xi^{\beta}\right). \tag{6.56}$$

Last but not least, we witness that the spinor variables behave as creation and annihilation operators: The holomorphic spinors commute with each other, $\left\{\xi^{\alpha}, \xi^{\beta}\right\} = 0$, whereas a spinor and its conjugate do not

$$\left\{\xi^{\alpha}, \bar{\xi}^{\dot{\beta}}\right\} = -\frac{is}{\hbar\langle\xi|p|\xi]^2}\left(2\langle\xi|p|\xi]\bar{p}^{\dot{\beta}\alpha} - m^2\xi^{\alpha}\bar{\xi}^{\dot{\beta}}\right). \tag{6.57}$$

This concludes our analysis.

Chapter 7

Continuous Spin Partilces in deSitter

7.1 Introduction

In quantum field theory, elementary particles form irreducible representations of the Poincaré group [83] and in four dimensions these irreps fall into four distinct types. Two of these correspond to the standard massive particle and massless particle of definite helicity while the other two give rise to tachyons $m^2 < 0$ and the so called continuous spin particle (CSP). It is well known that the appearance of a tachyon indicates an instablity in the underlying theory, but CSP's present a unique unexplored opportunity. In particular, Schuster and Toro have shown [84, 85, 86] that CSP's mediate long range forces and therefore provide for a possible dark matter candidate. However, to fully address this possibility requires an understanding of how CSP's couple to gravity, a task which has proven difficult. In this Chapter we will address this shortcoming by utilizing the Dual Phase Space formalism to construct a consistent theory of CSP's in deSitter.

7.2 Continuous Spin Particles in the DPS Model

In Section 4.3 we showed that the coadjoint orbits of the Poincaré group are characterized by two invariants, the squared momentum p^2 which determines the mass, and the square of the Pauli-Lubanski vector w^2 which determines the spin, c.f. the discussion preceding eq. (4.16). In our treatment of massless particles, $p^2 = 0$, we assumed that $w^2 = 0$ implying $w \propto p$ with the constant of proportionality interpreted as the helicity. This is not the most general scenario though, in fact as shown by Wigner [83], it is possible to have $p^2 = 0$ and

$w^2 = \rho^2$ resulting in the so called continuous spin particle (CSP). At a group theoretic level, recall that the little group for massless representations of the Poincaré group is the Euclidean group $E(2)$. This group is infinite dimensional, unless one restricts to the subspace where the translation generators are trivial. The latter case corresponds to the usual massless helicity particle, whereas the more general case yields the continuous spin particle. In this section we will show how to generalize the DPS model to incorporate CSP's.

7.2.1 Generalizing the Model

In the Dual Phase Space model, the angular momentum operator is parameterized as $J_{\mu\nu} = (x \wedge p)_{\mu\nu} + (\chi \wedge \pi)_{\mu\nu}$, and so the square of the Pauli-Lubanski vector $w_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} p^\nu J^{\rho\sigma}$ can be written as

$$w^2 = -p^2(\chi^2\pi^2 - (\chi \cdot \pi)^2) + (p \cdot \chi)^2\pi^2 + (p \cdot \pi)^2\chi^2 - 2(p \cdot \chi)(p \cdot \pi)(\chi \cdot \pi). \tag{7.1}$$

In the massless case the first term drops out, and so achieving $w^2 \neq 0$ requires at least one of $p \cdot \chi$ or $p \cdot \pi$ be non-zero. For definiteness let's put $p \cdot \chi = \lambda\rho$ and leave the remaining DPS constraints unaltered, c.f eq. (4.16). The value of w^2 is now

$$w^2 = s^2\rho^2, \tag{7.2}$$

where we have used that $\lambda\epsilon = 1$. It turns out that this straightforward modification is sufficient to allow for a description of CSP's, see [79, 105]. In summary, the generalized DPS model is represented by the set of six constraints

$$\begin{gathered} \Phi_m = \frac{1}{2}\left(p^2 + m^2\right), \quad \Phi_\pi = \frac{1}{2}\left(\pi^2 - s^2\epsilon^2\right), \quad \Phi_\chi = \frac{1}{2}\left(\chi^2 - \lambda^2\right), \\ \Phi_{\chi\pi} = \chi \cdot \pi, \quad \Phi_{p\pi} = p \cdot \pi, \quad \Phi_{p\chi} = p \cdot \chi - \lambda\rho, \end{gathered} \tag{7.3}$$

and it is assumed that $\rho = 0$ if $m \neq 0$.

7.2.2 Equations of Motion

It is interesting to note that for a CSP, as for a massive particle, the six constraints eq. (7.3) split into two first class and four second class, with the former given by

$$\Phi_{m=0}, \qquad \Phi_s = \lambda^2\rho\Phi_\pi + \epsilon s^2\Phi_{p\chi}. \tag{7.4}$$

It follows that a CSP has four physical degrees of freedom despite being a massless particle; recall that a massless helicity particle has only three physical degrees of freedom. It is also interesting to note that the spin constraint Φ_s is quite different from the one appearing in the description of massive and massless helicity particles, c.f. eq. (4.34). This discrepancy is born out in the equations of motion, which follow from the Hamiltonian $H = N\Phi_p + M\Phi_s$, viz.

$$p_\mu(\tau) = P_\mu \tag{7.5}$$

$$\pi_\mu(\tau) = A_\mu - M\epsilon s^2 \tau P_\mu \tag{7.6}$$

$$\chi_\mu(\tau) = B_\mu + M\lambda^2 \rho\tau A_\mu - \frac{M^2}{2}\lambda\rho s^2 \tau^2 P_\mu \tag{7.7}$$

$$x_\mu(\tau) = X_\mu + \left(NP_\mu + M\epsilon s^2 B_\mu\right)\tau + \frac{M^2}{2}\lambda s^2 \rho\tau^2 A_\mu - \frac{M^3}{6} s^4 \rho\tau^3 P_\mu, \tag{7.8}$$

where P_μ, A_μ, B_μ, and X_μ are constant vectors which satisfy $A^2 = \lambda^2$, $B^2 = \epsilon^2 s^2$, $P\cdot A = \lambda\rho$ and $P\cdot A = A\cdot B = 0$. Strikingly, these equations exhibit *no oscillatory motion*, something which is a feature of the massive and massless helicity particles. At this time we have no intuition for why the behavior of CSP's is so dramatically different from the other particles we have considered.

Continuous spin particles can be minimally coupled to an electromagnetic field via

$$\pi_\mu \to \Pi_\mu \equiv \pi_\mu + eA_\mu(\chi). \tag{7.9}$$

A higher order term can also be included by making the replacement

$$\Pi^2 - \epsilon^2 \to \Pi^2 + gF_{\mu\nu}L^{\mu\nu} - \epsilon^2, \tag{7.10}$$

where g is a "gyromagnetic ratio" and $L_{\mu\nu} = (x \wedge p)_{\mu\nu}$. Observe that this coupling occurs entirely in the dual space, as opposed to the case of a massive particle, see Section 4.6, which has exactly the opposite behavior. The interaction vertex between CSP's is an ongoing topic of research and we have yet to find a consistent prescription as was done for the massive and massless particles in the original DPS model.

7.3 Representations of the deSitter Group

The study of particles in a curved background presents an interesting challenge. Translations are no longer a symmetry of the underlying spacetime and therefore there is no generator which can naturally be identified with the momentum. Consequently, the definition of "mass" is ambiguous and so the classification of particles into massive and massless is open for debate. In this Chapter we are principally interested in deSitter which has symmetry group SO$(4,1)$ and many of its irreducible representations contract to representations of the Poincaré group. We will therefore adopt the convention that an irrep of SO$(4,1)$ will have the same physical interpretation as the representation of the Poincaré group to which it contracts. We begin by examining the irreps of the deSitter group.

Let $\eta_{AB} = \text{diag}(-,+,+,+,+)$ then SO$(4,1)$ is the group consisting of 5×5 matrices $X^A{}_B$ satisfying $X^A{}_C X^B{}_D \eta_{AB} = \eta_{CD}$. The generators of SO$(4,1)$ are denoted J_{AB}, where $A, B = 0, 1, \ldots, 4$; they satisfy the standard algebra

$$[J_{AB}, J_{CD}] = (\eta_{AC} J_{BD} - \eta_{AD} J_{BC} + \eta_{BD} J_{AC} - \eta_{AC} J_{BD}), \tag{7.11}$$

and provide a parameterization of the SO$(4,1)$ Casimirs via

$$C_2 = \frac{1}{2} J_{AB} J^{AB}, \qquad C_4 = W_A W^A, \tag{7.12}$$

where $W_A = \frac{1}{8}\epsilon_{ABCDE} J^{BC} J^{DE}$. The radius of curvature of the manifold underlying SO$(4,1)$ will be denoted $R > 0$.

Let $\Pi_{q,s}$ denote the unitary irreducible representations of G. The quantities q and s characterize the representation [117]; they can be thought of as representing "mass" and "spin" and parametrize the eigenvalues of the Casimirs:

$$C_2 = -\left[s(s+1) + (q+1)(q-2)\right], \tag{7.13}$$

$$C_4 = -s(s+1)q(q-1). \tag{7.14}$$

In what follows, we will ignore representations of SO$(4,1)$ which do not contract to a representation of the Poincaré group since such a representation would correspond to a model without a ready physical interpretation.

There are two classes of representation to consider, the Principle Series and the Discrete Series [118, 117, 119]. In the former case we put $q = \frac{1}{2} + i\nu$ where $\nu \in \mathbb{R}$, so that the relevant members of the Principle Series satisfy

$$s = 0, 1, 2, \ldots\ , \qquad \nu \geq 0, \tag{7.15}$$

$$s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\ , \qquad \nu > 0. \tag{7.16}$$

Define $m \equiv \nu/R$, then a group contraction ($R \to \infty$) yields the massive representations of the Poincaré group, a fact evidenced by the Casimirs

$$\mathcal{C}_2 = m^2 + \frac{1}{R^2}\left(\frac{9}{4} - s(s+1)\right) \xrightarrow[R\to\infty]{} m^2 \tag{7.17}$$

$$\mathcal{C}_4 = s(s+1)\left(m^2 + \frac{1}{4R^2}\right) \xrightarrow[R\to\infty]{} m^2 s(s+1), \tag{7.18}$$

where $\mathcal{C}_i = C_i/R^2$, $i = 2, 4$. On the other hand, if we put $\nu = s = n$ where $2n \in \mathbb{N}$ satisfies $n^2 = R\rho$ for some $\rho > 0$, a group contraction yields the continuous spin representation of the Poincaré group [120]. We can see this at the level of the Casimirs as well

$$\mathcal{C}_2 = \frac{1}{R^2}\left(\frac{9}{4} - \sqrt{R\rho}\right) \xrightarrow[R\to\infty]{} 0, \tag{7.19}$$

$$\mathcal{C}_4 = \rho^2 + \frac{1}{4R^2}(5R\rho + 1) \xrightarrow[R\to\infty]{} \rho^2, \tag{7.20}$$

where ρ is the CSP scale, see eq. (7.2) and the preceeding discussion.

The Discrete Series, is defined by

$$s = 1, 2, 3, \dots\,, \qquad q = s, s-1, \dots, 0 \tag{7.21}$$

$$s = \frac{1}{2}, \frac{3}{2}, \dots\,, \qquad q = s, s-1, \dots, \frac{1}{2}, \tag{7.22}$$

and the relevant members have $q = s$. These contract to massless representations of the Poincare group with helicity s; the Casimirs behave as expected

$$\mathcal{C}_2 = -\frac{2}{R^2}(s+1)(s-1) \xrightarrow[R\to\infty]{} 0, \tag{7.23}$$

$$\mathcal{C}_4 = -\frac{1}{R^2}s^2(s+1)(s-1) \xrightarrow[R\to\infty]{} 0. \tag{7.24}$$

7.4 Dual Phase Space Parametrization

To construct a classical model of elementary particles in deSitter we need to examine the coadjoint orbits of SO(4, 1). Forgoing the details, these are parameterized by the generators J_{AB} and are uniquely determined by the values of the quadratic and quartic Casimirs eq. (7.12). The Poisson brackets between the J_{AB} are given by the right hand side of eq. (7.11). In five dimensions any anti-symmetric tensor can be written as the sum of two simple bi-vectors, and so J_{AB} admits a dual phase space parameterization

$$J_{AB} = (S \wedge T)_{AB} + (U \wedge V)_{AB}. \tag{7.25}$$

We emphasize that no physical interpretation should be attached to the variables S, T, U, V, they are simply dynamical quantities which parameterize the classical phase space. It will be assumed that S, T and U, V form canonically conjugate pairs so the only non-vanishing Poisson brackets are $\{S_A, T_B\} = \{U_A, V_B\} = \eta_{AB}$. Furthermore, since J_{AB} represents an angular momentum S, U will have units of length while T, V will have units of linear momentum.

The quadratic and quartic Casimirs can be expressed in-terms of these variables by substituting eq. (7.25) into eq. (7.12). The value of C_2 and C_4 can be determined by introducing ten constraints corresponding to each possible inner product between the vectors S, T, U, V. There is no need to consider the most general version of such constraints, instead we find that the following are sufficient for our purposes:

$$\begin{gathered} S^2 = \ell^2, \quad T^2 = -\mu^2\nu^2, \quad U^2 = \lambda^2(1+\rho R), \quad V^2 = \epsilon^2 s^2, \quad S\cdot T = \sqrt{\rho R}, \\ U\cdot V = 0, \quad T\cdot U = 0, \quad T\cdot V = 0, \quad S\cdot U = -\ell\lambda\sqrt{\rho R}, \quad S\cdot V = 0. \end{gathered} \tag{7.26}$$

The parameters introduced above require some explanation: ℓ, λ have units of length, μ, ϵ, ρ have units of mass, ν, s are dimensionless, and $\ell\mu = \lambda\epsilon = 1$. The values of the quadratic and quartic Casimirs are now given by

$$C_2 = -\nu^2 + s^2, \tag{7.27}$$

$$C_4 = \nu^2 s^2 + R\rho\left(s^2 + R\rho\right). \tag{7.28}$$

It remains to determine how the particle classification (massive, massless, or continuous spin) is related to the choice of model parameters. To proceed we will use insights from the previous section while also demanding two key properties:

1. The degrees of freedom for each model should match its flat space counterpart, i.e. massive and continuous spin particles will have four and massless helicity three.

2. In the limit of flat spacetime the Casimirs should reduce to their Poincaré values.

Massive Spinning Particle

This is the easiest model to identify and we simply choose

$$\nu = Rm, \qquad \rho = 0. \tag{7.29}$$

The Casimirs are seen to be

$$\mathcal{C}_2 = -m^2 + \frac{s^2}{R^2}, \qquad \mathcal{C}_4 = m^2 s^2, \tag{7.30}$$

which have the appropriate limits. From the ten constraints eq. (7.26) one can show that two are first class and eight second class, so the reduced phase space has dimension $20 - 2\times 2 - 8\times 1 = 8$, giving four physical degrees of freedom.

Massless Spinning Particle - Helicity

In the previous section we showed that a massless particle was one which had equal "mass" and "spin" quantum numbers. In the present context this amounts to setting $\nu^2 = s^2$. c.f eq. (7.27). To ensure that the number of degrees of freedom are correct we require $\rho = 0$, and so the model is given by

$$\nu^2 = s^2, \qquad \rho = 0, \tag{7.31}$$

with Casimirs

$$\mathcal{C}_2 = 0, \qquad \mathcal{C}_4 = \frac{s^4}{R^2}. \tag{7.32}$$

It is straightforward to verify that the model has four first class and six second class constraints, giving the expected three physical degrees of freedom.

Massless Spinning Particle - CSP

A CSP is a massless particle and therefore the choice of ν should be identical to the massless helicity case. Where a CSP differs is that, in the limit $R \to \infty$, the quartic Casimir will not vanish and so we must have $\rho \neq 0$. It follows that the model is defined by

$$\nu^2 = s^2, \qquad \rho \neq 0, \tag{7.33}$$

with the Casimirs behaving as required

$$\mathcal{C}_2 = 0, \qquad \mathcal{C}_4 = \rho^2 + \frac{R\rho + s^2}{R^2} s^2. \tag{7.34}$$

One can verify that there are two first class and eight second class constraints, giving four physical degrees of freedom.

Observe that a massless helicity particle can be obtained from both the massive and continuous spin models by setting $m = s/R$ and $\rho = 0$ respectively. This behavior is consistent with both intuition and the original DPS model (in the $R \to \infty$ limit).

7.5 Four Dimensional Model

The models constructed in the previous section offer little insight into the behavior of spinning particles in a curved background; recall that the variables S, T, U, V lacked any physical interpretation. To address this issue let us split the generators J_{AB} as follows:

$$J_{AB} \longrightarrow P_\mu \equiv \frac{1}{R} J_{4\mu},\ J_{\mu\nu}, \tag{7.35}$$

where $\mu, \nu = 0, 1, 2, 3$. These satisfy the algebra

$$\{P_\mu, P_\nu\} = \frac{1}{R^2} J_{\mu\nu}, \quad \{P_\mu, J_{\nu\rho}\} = \eta_{\mu\nu} P_\rho - \eta_{\mu\rho} P_\nu, \tag{7.36}$$

$$\{J_{\mu\nu}, J_{\rho\sigma}\} = \eta_{\mu\rho} J_{\nu\sigma} + \eta_{\nu\sigma} J_{\mu\rho} - \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\nu\rho} J_{\mu\sigma}, \tag{7.37}$$

which can be seen to reduce to the Poincaré algebra in the the limit $R \to \infty$. Therefore, we can view P_μ and $J_{\mu\nu}$ as curved generalizations of the Minkowski momentum and angular momentum, respectively. Taking inspiration from the original DPS model, see Section 4.5.1, we parameterize $J_{\mu\nu}$ as

$$J_{\mu\nu} = (X \wedge P)_{\mu\nu} + (\widetilde{\chi} \wedge \widetilde{\pi})_{\mu\nu}. \tag{7.38}$$

Here X^μ is interpreted as a position coordinate, while $\widetilde{\chi}^\mu$ and $\widetilde{\pi}^\mu$ can be viewed as curved versions of the dual position and dual momentum. In what follows we will refer to $X, P, \widetilde{\chi}, \widetilde{\pi}$ as "curved DPS" coordinates. It should be noted though that this choice of coordinates is not unique, any modification having the same $R \to \infty$ will have a similar interpretation.

These new coordinates can be related to the variables of the previous section S, T, U, V by observing that the latter were defined to satisfy $J_{AB} = (S \wedge T)_{AB} + (U \wedge V)_{AB}$. Comparison with eq. (7.35) and eq. (7.38) immediately yields

$$P_\mu = \frac{1}{R} \left(S_4 T_\mu - S_\mu T_4 + U_4 V_\mu - U_\mu V_4 \right), \tag{7.39}$$

$$(S \wedge T)_{\mu\nu} + (U \wedge V)_{\mu\nu} = (X \wedge P)_{\mu\nu} + (\widetilde{\chi} \wedge \widetilde{\pi})_{\mu\nu}. \tag{7.40}$$

Without loss of generality[1] we assume that $S_4 \neq 0$ and solve eq. (7.39) for T_μ

$$T_\mu = \frac{1}{S_4} \left(R P_\mu + T_4 S_\mu - U_4 V_\mu + V_4 U_\mu \right). \tag{7.41}$$

This can now be substituted into eq. (7.40) which strongly suggests that we make the following identifications

$$X_\mu = \frac{R}{S_4} S_\mu, \qquad \widetilde{\chi}_\mu = U_\mu - \frac{U_4}{S_4} S_\mu, \qquad \widetilde{\pi}_\mu = V_\mu - \frac{V_4}{S_4} S_\mu. \tag{7.42}$$

[1] As we are not interested in cases where $P_\mu = 0$ at least one of S_4, T_4, U_4, V_4 must be non-zero and if it isn't S_4 we can just re-label so it is.

The Poisson brackets between the curved DPS variables can now be computed from the S, T, U, V algebra, we find[2]

$$\{X_\mu, P_\nu\} = \eta_{\mu\nu} + \frac{1}{R^2} X_\mu X_\nu, \quad \{\widetilde{\chi}_\mu, \widetilde{\pi}_\nu\} = \eta_{\mu\nu} + \frac{1}{R^2} X_\mu X_\nu \tag{7.43}$$

$$\{P_\mu, P_\nu\} = \frac{1}{R^2} \left[(X \wedge P)_{\mu\nu} + (\widetilde{\chi} \wedge \widetilde{\pi})_{\mu\nu} \right], \tag{7.44}$$

$$\{P_\mu, \widetilde{\chi}_\nu\} = -\frac{1}{R^2} \widetilde{\chi}_\mu X_\nu, \quad \{P_\mu, \widetilde{\pi}_\nu\} = -\frac{1}{R^2} \widetilde{\pi}_\mu X_\nu. \tag{7.45}$$

It is now possible to re-write the constraints of the previous section eq. (7.26) in-terms of the curved DPS coordinates and the fourth component of the original variables. Explicitly writing out these constraints isn't particularly illuminating but we note that the presence of S_4, T_4, U_4, and V_4 is undesirable and a complete physical model requires that these dependencies be removed.

Eliminating the four degrees of freedom associated with S_4, T_4, U_4 and V_4 can be accomplished by solving four of the constraints in eq. (7.26). To maintain consistency of the model the constraints we choose must form a closed second class subset of the original ten. Furthermore, the remaining six constraints should, in the limit $R \to \infty$, reduce to the ones used in the original DPS model, see eq. (4.30). This last condition requires that any X^μ dependence in the remaining constraints be suppressed by some inverse power of R. Therefore, since $S^\mu \sim X^\mu$ we choose to solve the four S-constraints, $S^2, S \cdot T, S \cdot U$, and $S \cdot V$, which are easily seen to form a closed, second class set.

7.5.1 Solving the Constraints

Begin by re-writing the S-constraints explicitly as restrictions on S_4, T_4, U_4, V_4, viz.

$$\begin{gathered}\Phi_S = S_4 - \frac{\ell}{\sigma^{1/2}}, \quad \Phi_T = T_4 + \frac{1}{S_4 \sigma} \left(P \cdot X + \frac{1}{R} (V_4 X \cdot \widetilde{\chi} - U_4 X \cdot \widetilde{\pi}) - \sqrt{\rho R} \right), \\ \Phi_U = U_4 + \frac{1}{S_4 \sigma} \left(\frac{S_4}{R} X \cdot \widetilde{\chi} + \ell \lambda \sqrt{\rho R} \right), \quad \Phi_V = V_4 + \frac{1}{R \sigma} X \cdot \widetilde{\pi},\end{gathered} \tag{7.46}$$

where we have introduced $\sigma \equiv 1 + X^2/R^2$. Surprisingly, the corresponding Dirac brackets between X^μ, $\widetilde{\chi}^\mu$, $\widetilde{\pi}^\mu$, and P^μ are identical to their Poisson brackets and so we can strongly implement the constraints eq. (7.46) without modifying the algebra eqs. (7.43)–(7.45).

[2] A similar calculation will yield the Poisson brackets between the curved DPS variables and the fourth components of S, T, U, V.

The remaining constraints can now be written entirely in-terms of the physical variables $X^\mu, P^\mu, \widetilde{\chi}^\mu, \widetilde{\pi}^\mu$; we begin with those which do not involve P^μ:

$$\Phi_{\widetilde{\chi}} = \widetilde{\chi}^2 - \lambda^2 - \frac{(X \cdot \widetilde{\chi})^2}{R^2\sigma}, \tag{7.47}$$

$$\Phi_{\widetilde{\pi}} = \widetilde{\pi}^2 - \epsilon^2 s^2 - \frac{(X \cdot \widetilde{\pi})^2}{R^2\sigma}, \tag{7.48}$$

$$\Phi_{\widetilde{\chi}\widetilde{\pi}} = \widetilde{\chi} \cdot \widetilde{\pi} - \frac{X \cdot \widetilde{\pi} X \cdot \widetilde{\chi}}{R^2\sigma}. \tag{7.49}$$

As required, to leading order in R^{-1} these are identical to their flat space counterparts, the same will be true for those involving P_μ but the higher order modifications are significantly more complex. We have

$$\Phi_{P\widetilde{\chi}} = P \cdot \widetilde{\chi} + \frac{\lambda\sqrt{\rho R}}{R\sigma^{1/2}}\left(\Phi_{\widetilde{\chi}\widetilde{\pi}} - \sqrt{\rho R}\right) - \frac{1}{R^2\sigma}\left(P \cdot X X \cdot \widetilde{\chi} + X \cdot \widetilde{\pi}\Phi_{\widetilde{\chi}} - X \cdot \widetilde{\chi}\Phi_{\widetilde{\chi}\widetilde{\pi}} + \lambda^2 X \cdot \widetilde{\pi}\right),$$

$$\Phi_{P\widetilde{\pi}} = P \cdot \widetilde{\pi} - \frac{\lambda\sqrt{\rho R}}{R\sigma^{1/2}}\left(\Phi_{\widetilde{\pi}} + \epsilon^2 s^2\right) - \frac{1}{R^2\sigma}\left(P \cdot X X \cdot \widetilde{\pi} + X \cdot \widetilde{\pi}\Phi_{\widetilde{\chi}\widetilde{\pi}} + X \cdot \widetilde{\chi}\Phi_{\widetilde{\pi}} - \epsilon^2 s^2 X \cdot \widetilde{\chi}\right),$$

and the "mass-shell" constraint

$$\Phi_P = P^2 + \frac{2\lambda\sqrt{\rho R}}{R\sigma^{1/2}}\Phi_{P\widetilde{\pi}} - \frac{1}{R^2\sigma}\left(-\nu^2 - \rho R + (P \cdot X)^2 + 2P \cdot \widetilde{\chi} X \cdot \widetilde{\pi} - 2P \cdot \widetilde{\pi} X \cdot \widetilde{\chi}\right)$$
$$- \frac{\lambda^2\rho}{R\sigma}\left(\Phi_{\widetilde{\pi}} + \epsilon^2 s^2\right) + \frac{1}{R^4\sigma^2}\left((X \cdot \widetilde{\chi})^2\widetilde{\pi}^2 + (X \cdot \widetilde{\pi})^2\widetilde{\chi}^2 - 2X \cdot \widetilde{\chi} X \cdot \widetilde{\pi}\widetilde{\chi} \cdot \widetilde{\pi}\right). \tag{7.50}$$

This completes the construction of the physical model. Unfortunately, the constraints, especially those involving P_μ, can't be easily interpreted as some straightforward modification of their flat space counterparts. The main reason for this is that we have not treated the variables covariantly, instead raising and lowering indices with the Minkowski metric $\eta_{\mu\nu}$.

Unlike the previous two models the first class constraints are not identical in form to their flat counterparts

$$\Phi_p + \frac{\epsilon}{R^2}\left(E^2\Phi_\chi + \lambda^2 s^2 \Phi_\pi\right), \tag{7.51}$$

$$\lambda^2\rho\Phi_\pi - 2\epsilon E\Phi_{p\chi} + \frac{1}{\sqrt{R\rho}}\left(\frac{E^2}{\sqrt{R}}\Phi_\chi + 2\rho\Phi_{\chi\pi} + 2\epsilon s^2\lambda\Phi_{P\pi}\right). \tag{7.52}$$

7.6 Covariant Model

The first step in implementing a fully covariant model is to express the deSitter metric in-terms of X^μ. It is easiest to begin with the S_A variables, these satisfy $S^2 = \ell^2$ and so we can write the line element as

$$ds^2 = \frac{R^2}{\ell^2}\left(dS^\mu dS_\mu + (dS_4)^2\right). \tag{7.53}$$

Making the substitution $S^\mu = S_4 X^\mu / R$ and $S_4 = \ell/\sqrt{\sigma}$ we find that the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$, are given by

$$g_{\mu\nu} = \frac{1}{\sigma}\left(\eta_{\mu\nu} - \frac{1}{R^2\sigma}X_\mu X_\nu\right), \qquad g^{\mu\nu} = \sigma\left(\eta^{\mu\nu} + \frac{1}{R^2}X^\mu X^\nu\right). \tag{7.54}$$

Let us now introduce covariant versions of the curved DPS coordinates, whose indices will be raised and lowered by the above metric. In keeping with standard conventions we define the position x^μ and dual position χ^μ as contravariant vectors while the momenta p_μ and dual momenta π_μ will be covariant. These "covariant DPS" coordinates are defined in-terms of the curved DPS coordinates via

$$x^\mu \equiv X^\mu, \qquad \chi^\mu \equiv \sigma^{1/2}\widetilde{\chi}^\mu, \qquad \pi_\mu = \sigma^{1/2} g_{\mu\nu}\widetilde{\pi}^\nu, \tag{7.55}$$

$$p_\mu \equiv \sigma g_{\mu\nu}\left(P^\nu - \frac{1}{R^2\sigma}(\widetilde{\chi}\wedge\widetilde{\pi})^{\nu\rho}X_\rho\right). \tag{7.56}$$

Observe that these have the same $R \to \infty$ limit as the curved DPS coordinates and therefore are still interpretable as generalizations of the standard DPS coordinates to a curved spacetime. The metric and σ can now be written as functions of x^μ:

$$\sigma = 1 + \frac{1}{R^2}\eta_{\mu\nu}x^\mu x^\nu, \qquad g_{\mu\nu} = \frac{1}{\sigma}\left(\eta_{\mu\nu} - \frac{1}{R^2\sigma}\eta_{\mu\alpha}\eta_{\nu\beta}x^\alpha x^\beta\right), \tag{7.57}$$

$$g^{\mu\nu} = \sigma\left(\eta^{\mu\nu} + \frac{1}{R^2}x^\mu x^\nu\right). \tag{7.58}$$

The Christoffel symbols and Riemann curvature tensor are

$$\Gamma^\rho_{\mu\nu} = -\frac{1}{R^2\sigma}x^\alpha\left(\eta_{\alpha\mu}\delta^\rho_\nu + \eta_{\alpha\nu}\delta^\rho_\mu\right), \tag{7.59}$$

$$R^\sigma{}_{\rho\mu\nu} = \frac{1}{R^2}\left(\delta^\sigma_\mu g_{\rho\nu} - \delta^\sigma_\nu g_{\rho\mu}\right), \tag{7.60}$$

and provide for a succinct expression of the Poisson brackets between the covariant DPS coordinates, viz.

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu, \qquad \{\chi^\mu, \pi_\nu\} = \delta^\mu_\nu, \tag{7.61}$$

$$\{p_\mu, \chi^\nu\} = \Gamma^\nu_{\mu\rho}\chi^\rho, \qquad \{p_\mu, \pi_\nu\} = -\Gamma^\rho_{\mu\nu}\pi_\rho, \tag{7.62}$$

$$\{p_\mu, p_\nu\} = R^\sigma{}_{\rho\mu\nu}\chi^\rho\pi_\sigma. \tag{7.63}$$

The real virtue of the covariant DPS coordinates comes when examining the constraints of the previous section, now written as

$$\Phi_\chi = g_{\mu\nu}\chi^\mu\chi^\nu - \lambda^2, \quad \Phi_\pi = g^{\mu\nu}\pi_\mu\pi_\nu - \epsilon^2 s^2, \quad \Phi_{\chi\pi} = \chi^\mu\pi_\mu, \tag{7.64}$$

$$\Phi_{p\chi} = \chi^\mu p_\mu - \lambda\rho, \quad \Phi_{p\pi} = g^{\mu\nu}p_\mu\pi_\nu + \sqrt{\frac{\rho}{R}}\epsilon, \tag{7.65}$$

$$\Phi_p = g^{\mu\nu}p_\mu p_\nu + \frac{1}{R^2}(\nu^2 + \rho R). \tag{7.66}$$

To leading order in R^{-1} these are obtained from their Minkowski counterparts by making the replacement $\eta \to g$, which is precisely the straightforward generalization we sought. Consequently, we can assign to them the same physical interpretation as in the original DPS model, see the discussion following eq. (4.30). The corresponding constraint algebra is tedious to calculate, but in the end we find

$$\{\Phi_\chi, \Phi_\pi\} \simeq 0, \qquad \{\Phi_\chi, \Phi_{\chi\pi}\} \simeq 2\lambda^2, \tag{7.67}$$

$$\{\Phi_\pi, \Phi_{\chi\pi}\} \simeq -2\epsilon^2 s^2, \qquad \{\Phi_{p\chi}, \Phi_\chi\} \simeq 0, \tag{7.68}$$

$$\{\Phi_{p\pi}, \Phi_\chi\} \simeq -2\rho\lambda, \qquad \{\Phi_p, \Phi_\chi\} \simeq 0, \tag{7.69}$$

$$\{\Phi_{p\chi}, \Phi_\pi\} \simeq -2\sqrt{\frac{\rho}{R}}\epsilon, \qquad \{\Phi_{p\pi}, \Phi_\pi\} \simeq 0, \tag{7.70}$$

$$\{\Phi_p, \Phi_\pi\} \simeq 0, \qquad \{\Phi_{p\chi}, \Phi_{\chi\pi}\} \simeq \rho\lambda, \tag{7.71}$$

$$\{\Phi_{p\pi}, \Phi_{\chi\pi}\} \simeq \sqrt{\frac{\rho}{R}}\epsilon, \qquad \{\Phi_p, \Phi_{\chi\pi}\} \simeq 0, \tag{7.72}$$

$$\{\Phi_p, \Phi_{p\chi}\} \simeq -\frac{2\lambda}{R^3}\sqrt{\rho R}, \qquad \{\Phi_p, \Phi_{p\pi}\} \simeq -\frac{2\rho\epsilon s^2}{R^2}, \tag{7.73}$$

$$\{\Phi_{p\chi}, \Phi_{p\pi}\} \simeq \frac{1}{R^2}(\nu^2 + \rho R - s^2). \tag{7.74}$$

We are now prepared to write down the complete covariant model for type of particle.

Massive Particle

The massive particle is defined by the following parameter values

$$\nu = Rm, \qquad \rho = 0. \tag{7.75}$$

The first class constraints are identical in form to the massive DPS model in flat space, c.f. eq. (4.34)

$$\Phi_p, \qquad \epsilon^2 s^2 \Phi_\chi + \lambda^2 \Phi_\pi. \tag{7.76}$$

The flat space limit is as expected

$$p^2 = -m^2, \quad \chi^2 = \lambda^2 s^2, \quad \pi^2 = \epsilon^2 s^2, \quad \chi \cdot \pi = 0, \quad p \cdot \chi = 0, \quad p \cdot \pi = 0. \tag{7.77}$$

Massless Helicity Particle

For the massless helicity particle we have

$$\nu^2 = s^2, \qquad \rho = 0, \tag{7.78}$$

with first class constraints

$$\Phi_p, \quad \Phi_{p\cdot\chi}, \quad \Phi_{p\cdot\pi}, \quad \epsilon^2 s^2 \Phi_\chi + \lambda^2 \Phi_\pi. \tag{7.79}$$

Again these are identical in form to the flat DPS model, with an exact match appearing in the flat space limit

$$p^2 = 0, \quad \chi^2 = \lambda^2, \quad \pi^2 = \epsilon^2 s^2, \quad \chi \cdot \pi = 0, \quad P \cdot \chi = 0, \quad P \cdot \pi = 0. \tag{7.80}$$

Continuous Spin Particle

A CSP has the same parameter values as a massless particle except $\rho \neq 0$, in particular

$$\nu^2 = s^2, \qquad \rho \neq 0. \tag{7.81}$$

Unlike the previous two models the first class constraints are not identical in form to their flat counterparts

$$\Phi_p + \frac{1}{R^2}\left(\epsilon^2 s^2 \Phi_\chi + \lambda^2 \Phi_\pi\right), \tag{7.82}$$

$$\lambda^2 \rho \left(1 - \frac{1}{R\rho}\right) \Phi_\pi + \frac{\epsilon^2}{R}(1 - s^2)\Phi_\chi + 2\sqrt{\frac{\rho}{R}}\Phi_{\chi\pi} + 2\epsilon s^2 \Phi_{p\chi} - \frac{2\lambda}{\sqrt{\rho R}}\Phi_{p\pi}. \tag{7.83}$$

Despite this discrepancy the constraints do retain the correct flat space limit

$$p^2 = 0, \quad \chi^2 = \lambda^2 s^2, \quad \pi^2 = E^2, \quad \chi \cdot \pi = 0, \quad p \cdot \pi = 0, \quad p \cdot \chi = \lambda\rho. \tag{7.84}$$

At this time we have no intuition for why the CSP model differs so greatly from the original; further investigation is required.

Chapter 8

Conclusion

In this thesis we have explored the relationship between conservation laws and locality. Chapters 2–3 focused on scalar particles in both a curved spacetime and a curved momentum space. In the former case we found that curvature was not sufficient to introduce non-local behavior while in the latter all non-locality in the quantum field theory could be absorbed into the interaction term. The bulk of the thesis, though, was contained in Part 2 where we examined the effect of spin on the interaction vertex. We were motivated by the possibility that spin could sufficiently modify the vertex factor to introduce non-local behavior. The speculation was that if locality is violated at some energy scale spin could retain the remnants of this violation. Although we did not see any non-locality in the interaction vertex, we did show in Chapter 5 that spin can be realized as a bilocal model. This is a rather different kind of non-locality than we expected but it did show that the purely quantum picture of spin as given by the Dirac equation is incomplete. This was further emphasized by the results obtained in Chapter 6, where we demonstrated that the motion of a spinning particle is described by two gauge invariant quantities, the usual proper time and a proper angle. The latter was then interpreted as the amount of Zitterbewegung along the particles' trajectory and contrasted to the notion of a spin transition which was shown to induce a Lorentz contraction of the proper time. That Chapter also provided a better understanding of the delineation between a classical understanding of spin, as developed in the present work, and the usual quantum interpretation, demonstrating that a rich understanding of spin emerges when we explore its classical realization.

The work presented here is by no means exhaustive and leaves open entire vistas of future directions. Although we did quantize the DPS model in Chapter 5 we should explore the quantum field theory obtained by taking the path integral of the DPS action. This would give us a better understanding of how the interaction vertex obtained in Chapter 4 is related to the usual vertices appearing in standard quantum field theories. Some of the surprising effects spin has on the motion of particles, as discussed in Chapter 6, may be observable although the predicted effects would need to be formalized. The spinoral parameterization introduced in that Chapter could also provide a framework for incorporating fermions into DPS.

One of the least developed ideas presented in this thesis was the incorporation of continuous spin particles into the Dual Phase Space framework. As shown in Chapter 7, CSP's behave much differently than either the massive or massless helicity particle, and we have no intuition for why that should be. Schuster and Toro [84, 85, 86] have developed a field theoretic description of CSP's and it would be interesting to see how our formulation is related to theirs. Additional study of the curved spacetime formalism developed in Chapter 7 is also warranted. In particular, we would like to explore the equations of motion for each type of spinning particle with special attention payed to how the behavior of the dual coordinate χ^μ differs from flat space. Finally, it is reasonable to assume that a bilocal version of the curved DPS model can be developed as per Chapter 5, which may provide a path to quantization.

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Appendices

Appendix A

The Worldfunction and the Parallel Propagator

In what follows we provided a detailed exploration of the relationship between the worldfunction and the parallel propagator. Consider two points in spacetime $x, x' \in \mathcal{M}$ joined by a geodesic $\gamma_{xx'}$. The parallel propagator, denoted $U^{\mu}{}_{\mu'}(x, x')$, is the operator which takes a vector field at x' and parallel transports along $\gamma_{xx'}$ to a vector field at x. By definition, a geodesic is a curve which parallel transports its own tangent vector, i.e. $T^{\mu}\nabla_{\mu}T_{\nu} = 0$, where $T^{\mu} = d\gamma^{\mu}_{xx'}/d\tau$ for some affine parameter τ. In terms of the parallel propagator this becomes

$$T^{\mu}(x) = U^{\mu}{}_{\mu'}(x, x')T^{\mu'}(x), \qquad T^{\mu'}(x') = U^{\mu'}{}_{\mu}(x, x')T^{\mu}(x). \tag{A.1}$$

The tangent vectors $T^{\mu}(x)$ and $T^{\mu'}(x')$ are related to the derivative of the worldfunction at x and x' via eq. (2.28) and eq. (2.30), respectively. In particular,

$$\sigma^{\mu}(x, x') = T^{\mu} = U^{\mu}{}_{\mu'}T^{\mu'}(x') = -U^{\mu}{}_{\mu'}\sigma^{\mu'}(x, x'), \tag{A.2}$$

with a similar computation holding for $\sigma^{\mu'}$. Substituting into eq. (A.1) we find

$$\sigma^{\mu}(x, x') + U^{\mu}{}_{\mu'}(x, x')\sigma^{\mu'}(x, x') = 0, \tag{A.3}$$

$$\sigma^{\mu'}(x, x') + U^{\mu'}{}_{\mu}(x, x')\sigma^{\mu}(x, x') = 0, \tag{A.4}$$

which justifies the statement made in the text that, when acting on σ_{μ} or $\sigma_{\mu'}$, the second order mixed derivative of the worldfunction behaves like the parallel propagator (up to a sign). These equations can be written in the tetrad basis, e^{a}, by making a couple of observations. Focusing on eq. (A.3), we have

$$\sigma^{\mu}(x,x') = \nabla_{x_\mu}\sigma(x,x') = \nabla_{x_\mu}\sigma(x,x') \quad \Longrightarrow \quad e^a_\mu(x)\sigma^{\mu}(x,x') = \sigma^a(x,x'), \tag{A.5}$$

$$U^{\mu}{}_{\mu'}\sigma^{\mu'} = U^{\mu}{}_{\mu'}e^{\mu'}_a(x')e^a_{\nu'}(x')\sigma^{\nu'} = U^{\mu}{}_{\mu'}e^{\mu'}_a(x')\sigma^a, \tag{A.6}$$

with similar results holding for (A.4). Thus, we obtain the form of these equations used in the text

$$\sigma^a(x,x') + U^a{}_b(x,x')\sigma^b(x,x') = 0, \tag{A.7}$$

$$\sigma^a(x,x') + U^a{}_b(x,x')\sigma^b(x',x) = 0. \tag{A.8}$$

Appendix B

Covariant Fourier Representation of Delta Function

In this appendix we explicitly verify some technical details regarding the covariant Fourier transform of the delta function. Let us begin with eq. (2.43) which gives the Fourier representation of $\delta(x, y)$; the delta function on $\mathcal{M}$. Assuming $x, y \in \mathcal{C}_{x'}$ and $f(x) \in \mathcal{L}^2_{\nu_{x'}}(C_{x'})$ we put

$$\begin{aligned}\tilde{f}(y) &\equiv \int_{C_{x'}} d\mu(x)\delta(x,y)f(x) \\ &= \int_{C_{x'}} d\mu(x) \int d\nu_{x'}(p)\mathcal{V}^{1/2}(x,x')\mathcal{V}^{1/2}(y,x') \\ &\qquad \times \exp\left[ip_{\mu'}\left(\sigma^{\mu'}(x,x') - \sigma^{\mu'}(y,x')\right)\right] f(x),\end{aligned} \tag{B.1}$$

where the second equality follows by using eq. (2.43). The integral over p covers the entire cotangent space $\mathcal{M}_{x'}$ and therefore turns the exponential into $\delta(\sigma_{\mu'}(x,x') - \sigma_{\mu'}(y,x'))$ which can be decomposed in the standard fashion. To do this we note that the uniqueness of the geodesic connecting x' to x and x' to y implies that $\sigma_{\mu'}(x,x') = \sigma_{\mu'}(y,x')$ if an only if $x = y$, and so

$$\delta\left(\sigma_{\mu'}(x,x') - \sigma_{\mu'}(y,x')\right) = \frac{\sqrt{g}_x}{|\det(\sigma^{\mu}{}_{\nu'})|}\delta(x,y) = \sqrt{g}_{x'}\mathcal{V}^{-1}(x,x')\delta(x,y), \tag{B.2}$$

where the definition of the Van–Vleck Morette determinant along with the indentity $|g_{x'}\det(\sigma^{\mu}{}_{\nu'})| = |\det(\sigma^{\mu\nu'})|$ were used in the last equality. Substituting eq. (B.2) into our expression for $\tilde{f}(y)$ and noting that the presence of $\delta(x,y)$ allows us to replace all occurrences of y with x we find

$$\tilde{f}(y) = \int_{C_{x'}} d\mu(x)\delta(x,y)f(x) = f(y), \tag{B.3}$$

where we used $x, y \in C_{x'}$ in the second equality. This demonstrates the validity of eq. (2.43) as a representation of the delta function.

The Fourier representation of $\delta_{x'}(p,q)$, the "delta function" on $\mathcal{M}_{x'}$, is given in eq. (2.46). Unless $\mathcal{C}_{x'} = \mathcal{M}$ this representaion will not correspond to the standard delta function; however there are at least two important properties it should satisfy:

1. $\delta_{x'}(p,q)$ is a projector.
2. The image of $\delta_{x'}(p,q)$ is identical to the image of $\mathcal{F}_{x'}$.

To demonstrate the first item, make the change of variables $x^{\mu} \to Y^{\mu'} = \sigma^{\mu'}(x,x')$ in eq. (2.46) to find

$$\delta_{x'}(p,q) = \int_{\Sigma_{x'}} \frac{d^4Y'}{\sqrt{g}_{x'}} \exp\left[iY^{\mu'}(p_{\mu'} - q_{\mu'})\right], \tag{B.4}$$

where $C_{x'} \to \Sigma_{x'}$ under the coordinate change. The convolution product of $\delta_{x'}$ with itself can now be expressed as:

$$\begin{aligned}\int_{\mathcal{M}_{x'}} \mathrm{d}\nu_{x'}(q)\delta_{x'}(p,q)\delta_{x'}(q,k) &= \int_{C_{x'}\times C_{x'}} \frac{\mathrm{d}^4Y'\mathrm{d}^4Z'}{|g_{x'}|}\left(\int_{\mathcal{M}_{x'}} \mathrm{d}\nu_{x'}(q)e^{iq_{\mu'}(Y^{\mu'}-Z^{\mu'})}\right) \\ &\quad\times e^{ip_{\mu'}Z^{\mu'}}e^{-ik_{\mu'}Y^{\mu'}} \\ &= \int_{C_{x'}\times C_{x'}} \frac{\mathrm{d}^4Y'\mathrm{d}^4Z'}{\sqrt{g}_{x'}}\delta(Y',Z')e^{ip_{\mu'}Z^{\mu'}}e^{-ik_{\mu'}Y^{\mu'}} \\ &= \delta_{p'}(x,z),\end{aligned}$$

which confirms that $\delta_{x'}(p,q)$ is a projector, i.e. identity onto its image.

For the second item, suppose $\hat{f}_{x'}(p) \in \mathcal{F}_{x'}(\mathcal{L}^2_{\nu_{x'}}(C_{x'}))$ so there exists a function $f(x) \in \mathcal{L}^2_{\mu}(C_{x'})$ such that

$$\hat{f}_{x'}(p) = \int_{C_{x'}} d\mu(x)\mathcal{V}^{1/2}(x,x')\exp\left(-ip_{\mu'}\sigma^{\mu'}(x,x')\right)f(x). \tag{B.5}$$

Evaluating the convolution of $\delta_{x'}$ with $\hat{f}_{x'}$ we find

$$(\delta_{x'} \circ \hat{f}_{x'})(q) = \int_{\mathcal{M}_{x'}} d\nu_{x'}(p)\int_{C_{x'}} d\mu(x)\mathcal{V}(x,x')\exp\left[i\sigma^{\mu'}(x,x')(p_{\mu'} - q_{\mu'})\right]$$

$$
\begin{aligned}
&\times \int_{C_{x'}} d\mu(y)\mathcal{V}^{1/2}(y,x')\exp\left(-ip_{\mu'}\sigma^{\mu'}(y,x')\right) f(y).\\
&= \int_{C_{x'}} d\mu(x)d\mu(y)\mathcal{V}^{1/2}(x,x')\exp\left(-iq_{\mu'}\sigma^{\mu'}(x,x')\right)\\
&\quad \times \int_{\mathcal{M}_{p'}} d\nu_{x'}(p)\mathcal{V}^{1/2}(x,x')\mathcal{V}^{1/2}(y,x')\exp\left[ip_{\mu'}\left(\sigma^{\mu'}(x,x')-\sigma^{\mu'}(y,x')\right)\right] f(y)\\
&= \int_{C_{x'}} d\mu(x)d\mu(y)\mathcal{V}^{1/2}(x,x')\exp\left(-iq_{\mu'}\sigma^{\mu'}(x,x')\right)\delta(x,y)f(y)\\
&= \int_{C_{x'}} d\mu(x)\mathcal{V}^{1/2}(x,x')\exp\left(-iq_{\mu'}\sigma^{\mu'}(x,x')\right) f(x)\\
&= \hat{f}_{x'}(q),
\end{aligned}
$$

where we have used the Fourier representation of $\delta(x,y)$ in going from the third line to the fourth. This shows that the image of $\delta_{x'}$ under convolution is identical with the image of $\mathcal{F}_{x'}$, as required.

Appendix C

Geodesics in Relative Locality

In this appendix we provide additional details on the definition of a geodesic in Relative Locality. A geodesic can be defined as a path, $p(\tau)$, which parallel transports its own tangent vector. This requires $\dot{p}_\alpha \nabla^\alpha \dot{p}_\mu = 0$ and so the geodesic equation is given by:

$$\frac{d^2 p_\mu}{d\tau^2} + \Gamma_\mu^{\alpha\beta} \frac{dp_\alpha}{d\tau} \frac{dp_\beta}{d\tau} = 0. \tag{C.1}$$

Alternatively, we can define a geodesic as a path which extremizes the distance between two points on the manifold. In general relativity, where the connection is metric compatible, these definitions are equivalent. This is not the case in relative locality where the connection is derived, not from a metric, but from the combination rule $\oplus$. In choosing between these definitions we note that the distance function $D_\gamma(p_0, p_1)$ is tied to the notion of mass and features prominently in the structure of relative locality. As such, it is natural to have a definition of geodesic which extremizes D_γ, and so we make this choice. We will now present a detailed derivation of the geodesic equation and explore some of its properties.

Following the argument given in [91], suppose we have two points $P, Q \in \mathcal{P}$ and an infinity of curves, $p_\mu(u, v)$ interpolating between P and Q. The parameter v indicates which curve is being considered while u parametrizes the selected curve. We assume that u varies between u_0 and u_1 so that P, Q have coordinates $p_\mu(u_0, v)$ and $p_\mu(u_1, v)$ respectively. A geodesic is then a curve which gives a stationary value to the following integral for variations which leave the endpoints fixed[1]:

$$I(v) = \frac{1}{2} \int_{u_0}^{u_1} g^{\mu\nu} \frac{dp_\mu}{du} \frac{dp_\nu}{du} du. \tag{C.2}$$

[1]Such a curve will also give a stationary value to D_γ so we are justified in considering the simpler function $I(v)$.

Introduce the tangent vectors $U_\mu = \partial p_\mu/\partial u$ and $V_\mu = \partial p_\mu/\partial u$, where V_μ vanishes at $u = u_0, u_1$. We then define the covariant derivative along the path p_μ by

$$\frac{DA_\mu}{du} = \frac{dA_\mu}{du} + \Gamma^{\alpha\beta}_{\mu} A_\alpha U_\beta \qquad \text{and} \qquad \frac{DA_\mu}{dv} = \frac{dA_\mu}{dv} + \Gamma^{\alpha\beta}_{\mu} A_\alpha V_\beta, \tag{C.3}$$

where these definitions are extended to arbitrary tensors in the standard way. A brief calculation shows that $DU_\mu/dv = DV_\mu/du$, which we will make use of shortly. Demanding that $I(v)$ be stationary under variations which leave the end–points fixed is equivalent to the condition: $dI(v)/dv = 0$ for V_μ arbitrary, except at the end–points. Thus we proceed by differentiating $I(v)$, making use of the fact that d/dv and D/dv are interchangeable when applied to a scalar:

$$\frac{dI(v)}{dv} = \frac{1}{2}(u_1 - u_0)\int_{u_0}^{u_1}\left(\nabla^\rho g^{\mu\nu} V_\rho U_\mu U_\nu + 2g^{\mu\nu} U_\nu \frac{DU_\mu}{dv}\right) du \tag{C.4}$$

$$= \frac{1}{2}(u_1 - u_0)\int_{u_0}^{u_1}\left(\left[N^{\rho\mu\nu} - 2N^{\mu\rho\nu}\right] V_\rho U_\mu U_\nu - 2g^{\mu\nu} V_\mu \frac{DU_\nu}{du}\right) du. \tag{C.5}$$

Setting this to zero and expanding DU_ν/du using (C.3) we find the geodesic equation:

$$\frac{dU_\alpha}{du} = \frac{1}{2} g_{\rho\alpha}\left[N^{\rho\mu\nu} - 2N^{\mu\rho\nu}\right] U_\mu U_\nu - \Gamma^{\mu\nu}_{\alpha} U_\mu U_\nu. \tag{C.6}$$

This result can be simplified using equation (3.14) which gives

$$\left[N^{\rho\mu\nu} - 2N^{\mu\rho\nu}\right] U_\mu U_\nu = 2\left[T^{\rho\mu\nu} + \mathcal{N}^{\mu\nu}_{\alpha} g^{\alpha\rho}\right] U_\mu U_\nu.$$

Substituting this back into (C.6), noting that $\Gamma^{\mu\nu}_{\rho} U_\mu U_\nu = \Gamma^{(\mu\nu)}_{\rho} U_\mu U_\nu$ and using $\mathcal{N}^{\mu\nu}_{\alpha} = \Gamma^{(\mu\nu)}_{\alpha} - \left\{ {}^{\mu\,\nu}_{\ \alpha} \right\}$ we find

$$\frac{dU_\alpha}{du} = \Big(g_{\rho\alpha} T^{\rho\mu\nu} - \left\{ {}^{\mu\,\nu}_{\ \alpha} \right\}\Big) U_\mu U_\nu, \tag{C.7}$$

which is the final form of the geodesic equation.

A particularly useful feature of geodesics in the case of a metric compatible connection is that the quantity $L = g^{\mu\nu} U_\mu U_\nu$ is constant along a geodesic. It turns out that this holds for our definition as well:

$$\begin{aligned}
\frac{d}{du}\left(g^{\mu\nu} U_\mu U_\nu\right) &= \partial^\rho g^{\mu\nu} U_\rho U_\mu U_\nu + 2g^{\mu\nu} U_\nu \frac{dU_\mu}{du} \\
&= \left(\partial^\rho g^{\beta\nu} + 2T^{\beta\rho\nu} - 2g^{\mu\beta}\left\{ {}^{\rho\,\nu}_{\ \mu} \right\}\right) U_\beta U_\nu U_\rho \\
&= \left(\partial^\rho g^{\beta\nu} - 2g^{\mu\beta}\left\{ {}^{\rho\,\nu}_{\ \mu} \right\}\right) U_\beta U_\nu U_\rho \\
&= 0.
\end{aligned}$$

This is extremely fortunate because it allows us to relate the distance function $D^2_{p(\tau)}(P,Q)$, c.f. eq. (3.21), directly to the integral $I(v)$, in particular

$$I = \frac{1}{2} D^2_{p(\tau)}(P,Q). \tag{C.8}$$

Appendix D

Second Order Formulation of DPS

In this appendix we show how the second-order formulation of the DPS action can be obtained from eq. (4.33) by integrating out the momenta and all Lagrange multipliers. Begin by rewriting the action as

$$S = \int d\tau \left[p_\mu(\dot{x}^\mu - N_4\chi^\mu) + \pi_\mu(\dot{\chi}^\mu - N_2\chi^\mu) - \frac{N}{2}(p^2 + m^2) - \frac{\tilde{N}}{2}(\pi^2 - \epsilon^2 s^2) - N_3(p \cdot \pi) \right.$$
$$\left. - \frac{\tilde{M}}{2}(\chi^2 - \lambda^2) \right], \tag{D.1}$$

where we have introduced

$$\tilde{N} = \frac{(M - N_1)}{\epsilon^2}, \qquad \tilde{M} = \frac{s^2(M + N_1)}{\lambda^2}. \tag{D.2}$$

The equations of motion for the momenta read

$$N p_\mu + N_3 \pi_\mu = (\dot{x}_\mu - N_4 \chi_\mu), \tag{D.3}$$
$$N_3 p_\mu + \tilde{N} \pi_\mu = (\dot{\chi}_\mu - N_2 \chi_\mu), \tag{D.4}$$

and upon inverting these we obtain

$$(N\tilde{N} - N_3^2) p_\mu = \tilde{N}(\dot{x}_\mu - N_4 \chi_\mu) - N_3(\dot{\chi}_\mu - N_2 \chi_\mu), \tag{D.5}$$
$$(N\tilde{N} - N_3^2) \pi_\mu = -N_3(\dot{x}_\mu - N_4 \chi_\mu) + N(\dot{\chi}_\mu - N_2 \chi_\mu). \tag{D.6}$$

Substituting this result into Eq. (D.1), we find

$$S = \int d\tau \left[\frac{\rho}{(N\tilde{N} - N_3^2)} - \frac{\tilde{M}}{2}(\chi^2 - \lambda^2) - \frac{N}{2} m^2 + \frac{\tilde{N}}{2} \epsilon^2 s^2 \right], \tag{D.7}$$

where ρ is given by

$$\rho \ := \ \frac{1}{2}\left[\tilde{N}(\dot{x}-N_4\chi)^2+N(\dot{\chi}-N_2\chi)^2-2N_3(\dot{\chi}-N_2\chi)\cdot(\dot{x}-N_4\chi)\right]. \tag{D.8}$$

We can now start integrating out the constraints, beginning by varying Chapter D with respect to N_2 and N_4, then

$$N_2\chi^2=\dot{\chi}\cdot\chi, \qquad N_4\chi^2=\dot{x}\cdot\chi. \tag{D.9}$$

This suggests the notation

$$D_t x_\mu := \dot{x}_\mu - \frac{(\dot{x}\cdot\chi)}{\chi^2}\chi_\mu, \qquad D_t\chi_\mu := \dot{\chi}_\mu - \frac{(\dot{\chi}\cdot\chi)}{\chi^2}\chi_\mu, \tag{D.10}$$

where D_t is the time derivative projected orthogonal to χ. We can now compute the variation with respect to the Lagrange multipliers $N, \tilde{N}$, and N_3; after some algebra we find

$$(D_t\chi)^2 \ = \ \tilde{N}^2\epsilon^2 s^2 - N_3^2 m^2, \tag{D.11}$$

$$(D_t x)^2 \ = \ N_3^2\epsilon^2 s^2 - N^2 m^2, \tag{D.12}$$

$$(D_t\chi)\cdot(D_t x) \ = \ N_3\tilde{N}\epsilon^2 s^2 - N_3 N m^2. \tag{D.13}$$

To solve for these equations, it will be convenient to define

$$D := (N\tilde{N}-N_3^2)s\epsilon m, \qquad T := (\tilde{N}\epsilon^2 s^2 - N m^2), \tag{D.14}$$

which allow us to rewrite Eqs. (D.11)–(D.13) as

$$(D_t\chi)^2=\tilde{N}T+\tfrac{mD}{s\epsilon}, \qquad (D_t\chi)\cdot(D_t x)=N_3 T, \qquad (D_t x)^2 \ = \ NT-\tfrac{s\epsilon D}{m}. \tag{D.15}$$

These relations are straightforward to invert, and we find

$$D \ = \ \beta\sqrt{[(D_t\chi)\cdot(D_t x)]^2-(D_t x)^2(D_t\chi)^2}=\beta\left|(D_t x)\wedge(D_t\chi)\right|, \tag{D.16}$$

$$T \ = \ \alpha\sqrt{\epsilon^2 s^2(D_t\chi)^2-m^2(D_t x)^2-2\beta s\epsilon m\left|(D_t x)\wedge(D_t\chi)\right|}, \tag{D.17}$$

where $\alpha=\pm1$ and $\beta=\pm1$ are signs needed to define the square root. For definiteness, we choose both signs to be positive from now on. Thus, after integration of N_2, N_4 and $N, \tilde{N}$ and N_3 the action becomes

$$S=\int \mathrm{d}\tau\left[\alpha\sqrt{\epsilon^2 s^2(D_t\chi)^2-m^2(D_t x)^2-2s\epsilon m\beta\left|(D_t x)\wedge(D_t\chi)\right|}-\frac{\tilde{M}}{2}(\chi^2-\lambda^2)\right]. \tag{D.18}$$

Observe that we cannot integrate out the final Lagrange multiplier, since the variation of S with respect to $\tilde{M}$ is just the constraint $\chi^2 = \lambda^2$. We can, however, obtain expressions for some of the other Lagrange multipliers, viz.

$$N = \frac{m(D_t x)^2 + s\epsilon\beta|(D_t x) \wedge (D_t \chi)|}{mT}, \tag{D.19}$$

$$\tilde{N} = \frac{s\epsilon(D_t \chi)^2 - m\beta|(D_t x) \wedge (D_t \chi)|}{s\epsilon T}, \tag{D.20}$$

$$N_3 = \frac{[(D_t x) \cdot (D_t \chi)]}{T}. \tag{D.21}$$

The conjugate momenta are now obtained via the standard prescription $p_x = \partial S/\partial \dot{x}$ and $\pi_\chi = \partial S/\partial \dot{\chi}$. We find

$$p_{x,\mu} = -\frac{m}{T}\left(mD_t x_\mu + \frac{\beta s\epsilon}{|D_t x \wedge D_t \chi|}\left[(D_t x \cdot D_t \chi) D_t \chi_\mu - (D_t \chi)^2 D_t x_\mu\right]\right), \tag{D.22}$$

$$\pi_{\chi,\mu} = \frac{s\epsilon}{T}\left(s\epsilon D_t \chi_\mu - \frac{m\beta}{|D_t x \wedge D_t \chi|}\left[(D_t x \cdot D_t \chi) D_t x_\mu - (D_t x)^2 D_t \chi_\mu\right]\right). \tag{D.23}$$

It can be checked that these momenta satisfy the constraints

$$p_x^2 = -m^2, \quad \pi_\chi^2 = \epsilon^2 s^2, \quad \pi_\chi \cdot \chi = 0, \quad p_x \cdot \pi_\chi = 0, \quad p_x \cdot \chi = 0. \tag{D.24}$$

The variation of the action with respect to x_μ and χ_μ determines the Lagrange equations of motion, in particular

$$\dot{p}_{x,\mu} = 0, \qquad \dot{\pi}_{\chi,\mu} = -(\chi \cdot \dot{x}) p_{x,\mu} - (\chi \cdot \dot{\chi}) \pi_{\chi,\mu} - \tilde{M} \chi_\mu. \tag{D.25}$$

Provided we implement $\chi^2 = \lambda^2$, these equations preserve $p_x^2 = -m^2$ and $\pi_\chi^2 = \epsilon^2 s^2$; demanding that $\pi_\chi \cdot \chi = 0$ also be preserved in time determines the Lagrange multiplier $\tilde{M}$:

$$\tilde{M} = \frac{\epsilon^2 s^2}{\lambda^2} \tilde{N}. \tag{D.26}$$

On the other hand, for the remaining two constraints we have

$$\frac{d}{dt}(p_x \cdot \chi) = -\frac{m^2}{T}(D_t x) \cdot (D_t \chi), \qquad \frac{d}{dt}(p_x \cdot \pi_\chi) = m^2 (\chi \cdot \dot{x}). \tag{D.27}$$

Therefore, ensuring that these quantities are stationary in time requires that we impose constraints on the initial conditions, specifically $(D_t \chi) \cdot (D_t x) = \dot{x} \cdot \chi = 0$. These are equivalent, when $\chi^2 = \lambda^2$, to $\dot{x} \cdot \chi = \dot{x} \cdot \dot{\chi} = 0$ which implies that the dual motion is always orthogonal to the particle velocity. Once these extra constraints are imposed, the action simplifies to the one quoted in the main text [Eq. (4.52)],

$$S = \alpha \int d\tau \, |m|\dot{x}| - \beta\epsilon s|\dot{\chi}||\,, \tag{D.28}$$

where we have defined $|\dot{x}| = \sqrt{-\dot{x}^2}$ and $|\dot{\chi}| = \sqrt{\dot{\chi}^2}$.

Appendix E

Dirac Brackets for DPS

We include here an explicit formulation of the Dirac brackets for DPS. Assuming $m \neq 0$ a direct computation gives

$$\begin{aligned} \{f,g\}_{\mathrm{DB}} = \{f,g\} &+ \frac{1}{2s^2}\left(\{f,\Phi_1\}\{\Phi_2,g\} - \{f,\Phi_2\}\{\Phi_1,g\}\right) \\ &+ \frac{1}{m^2}\left(\{f,\Phi_3\}\{\Phi_4,g\} - \{f,\Phi_4\}\{\Phi_3,g\}\right). \end{aligned} \tag{E.1}$$

The commutation relations between the phase space variables are now given by

$$\{x^\mu, p^\nu\}_{\mathrm{DB}} = \eta^{\mu\nu}, \qquad \{x^\mu, x^\nu\}_{\mathrm{DB}} = \frac{1}{m^2}(\chi \wedge \pi)^{\mu\nu}, \tag{E.2}$$

$$\{x^\mu, \chi^\nu\}_{\mathrm{DB}} = \frac{1}{m^2}\chi^\mu p^\nu, \qquad \{\chi^\mu, \chi^\nu\}_{\mathrm{DB}} = -\frac{1}{2\epsilon^2 s^2}\left(\chi \wedge \pi\right)^{\mu\nu}, \tag{E.3}$$

$$\{x^\mu, \pi^\nu\}_{\mathrm{DB}} = \frac{1}{m^2}\pi^\mu p^\nu, \qquad \{\pi^\mu, \pi^\nu\}_{\mathrm{DB}} = -\frac{s^2}{2\lambda^2}(\chi \wedge \pi)^{\mu\nu}, \tag{E.4}$$

$$\{\chi^\mu, \pi^\nu\}_{\mathrm{DB}} = \eta^{\mu\nu} - \frac{s}{2\lambda^2}\chi^\mu\chi^\nu - \frac{1}{2\epsilon^2 s^2}\pi^\mu\pi^\nu + \frac{1}{m^2}p^\mu p^\nu. \tag{E.5}$$

To obtain the brackets for a massless particle let $m \to \infty$ in the above relations.

Appendix F

Fermionic Spherical Harmonics

In this appendix we include a brief discussion on "fermionic spherical harmonics" $Y_\ell^m(\theta,\phi)$ which allow for half-integer values of m, ℓ, see [109, 110]. We begin with the standard differential equation

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y(\theta,\phi)=-\lambda Y(\theta,\phi), \tag{F.1}$$

which is separable and we make the assumption that $\lambda \geq 0$. Putting $Y(\theta,\phi)=\Theta(\theta)\Phi(\phi)$ we find

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+(\lambda\sin^2\theta-\kappa)\Theta=0 \tag{F.2}$$

$$\frac{d^2\Phi}{d\phi^2}=-\kappa\Phi \tag{F.3}$$

where κ is the separation constant. The second equation is straightforward to solve

$$\Phi_m(\phi)=\alpha_1 e^{im\phi}+\alpha_2 e^{-im\phi}. \tag{F.4}$$

where $m^2=\kappa$ and α_1, α_2 are integration constants. It is standard to argue that m should be an integer since ϕ has period 2π and $\Phi(\phi)$ must be single valued, however this reasoning is spurious. It is only the probability density $|\Phi(\phi)|$ which needs to be single valued since it is this quantity which has a physical interpretation. Under this less restrictive assumption we only require that $\Phi_m(\phi)$ is periodic and therefore that $2m \in \mathbb{N}$.

Put $\lambda=\ell(\ell+1)$ in equation (F.2) and make the change of variables $x=\cos\theta$ to obtain

$$(1-x^2)\ddot{\Theta}-2x\dot{\Theta}+\left(\ell(\ell+1)-\frac{m^2}{1-x^2}\right)\Theta=0, \tag{F.5}$$

where a dot indicates a derivative with respect to x. Notice that since λ is assumed to be non-negative ℓ is real valued. This is the associated Legendre equation and it's solution is well known, namely $\Theta(x) = \beta_1 P_\ell^m(x) + \beta_2 Q_\ell^m(x)$ for some constants β_1, β_2. To have a normalizable wavefunction it is sufficient to require that $\Theta(x)$ be regular on the interval $[-1, 1]$; to this end let us examine the behavior of $P_\ell^m(x)$ and $Q_\ell^m(x)$ as $x \to 1^-$. As eq. (F.2) is invariant under $m \to -m$ we can restrict to $m \geq 0$ without loss of generality, we find

$$P_\ell^m(x) \sim (1-x)^{-m/2}, \qquad m \neq 1, 2, \dots \tag{F.6}$$

$$P_\ell^m(x) \sim (1-x)^{m/2}, \qquad m = 1, 2, \dots, \quad \ell - m \neq -1, -2, \dots \tag{F.7}$$

$$Q_\ell^0(x) \sim \log(1-x), \qquad \ell \neq -1, -2, \dots \tag{F.8}$$

$$Q_\ell^m(x) \sim (1-x)^{-m/2}, \qquad m \neq \frac{1}{2}, \frac{3}{2}, \dots \tag{F.9}$$

$$Q_\ell^m(x) \sim (1-x)^{m/2}, \qquad m = \frac{1}{2}, \frac{3}{2}, \dots, \quad \ell - m \neq -1, -2, \dots. \tag{F.10}$$

It follows that a regular solution is only possible if m is either an integer or half-integer, in the former case we have $\Theta(x) = \beta_1 P_\ell^m(x)$ and in the latter $\Theta(x) = \beta_2 Q_\ell^m(x)$. The values of ℓ are as yet unrestricted, but we still need to consider regularity of the wavefunction as $x \to -1^+$, which can be determined from the following relations

$$P_\ell^m(-x) = \cos((\ell - m)\pi) P_\ell^m(x) - \frac{2}{\pi} \sin((\ell - m)\pi) Q_\ell^m(x). \tag{F.11}$$

$$Q_\ell^m(-x) = -\cos((\ell - m)\pi) Q_\ell^m(x) - \frac{2}{\pi} \sin((\ell - m)\pi) P_\ell^m(x).. \tag{F.12}$$

When m is an integer/half-integer eqs. (F.6)–(F.10) imply that only $P_\ell^m(x)$ respectively $Q_\ell^m(x)$ are finite in the limit $x \to 1^+$. Therefore, if the wavefunction is to be regular as $x \to -1^+$ we require that terms containing the other Legendre function vanish from eq. (F.11)/eq. (F.12). In each case this implies that $\ell - m$ is an integer and so if m is an integer/half-integer ℓ is as well. Furthermore, in each case we have that $\ell - m \geq 0$ and since this should be symmetric with respect to $m \to -m$ we also have $\ell + m \geq 0$, combining these conditions gives $-\ell \leq m \leq \ell$. Noting that for m a half-integer $Q_\ell^m(x) \propto P_\ell^{-m}(x)$ we can write the most general solution to eq. (F.2) as

$$\Theta_\ell^m(x) = \beta P_\ell^{\epsilon_\ell |m|}(x), \qquad \ell = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots, \quad m = -\ell, -\ell + 1, \dots, \ell - 1, \ell \tag{F.13}$$

$$\tag{F.14}$$

where $\epsilon_\ell = (-1)^{2\ell}$. This result can now be combined with $\Phi_m(\phi)$ to obtain the full solution to eq. (F.1) namely $Y_\ell^m(\theta, \phi) = \Theta_\ell^m(\theta)\Phi_m(\phi)$. When m is an integer these are the standard spherical harmonics, however if m is a half-integer we obtain "fermonic" spherical harmonics which change sign under $\phi \to \phi + 2\pi$. As mentioned earlier, a multivalued wavefunction is acceptable provided that the probability density is single valued and it is easy to verify that this property holds for "fermonic" spherical harmonics.

Appendix G

Unequal Massess

This appendix examines the effect of allowing for unequal masses in the relativistic bilocal model of Section 5.3. In the non-relativistic model the form of the final Hamiltonian was independent of any mass difference between the constituent particles. This is decidedly not the case when considering the relativistic setting, as will be explored in the current appendix. We begin by defining the masses $M = m_1 + m_2$ and $\mu = m_1 m_2/(m_1 + m_2)$ and the four-vector coordinates

$$\begin{aligned} X^\mu &= \frac{m_1}{M} x_1^\mu + \frac{m_2}{M} x_2^\mu, & \Delta x^\mu &= x_1^\mu - x_2^\mu, \\ P^\mu &= p_1^\mu + p_2^\mu, & \Delta p^\mu &= \frac{\mu}{m_1} p_1^\mu - \frac{\mu}{m_2} p_2^\mu, \end{aligned} \tag{G.1}$$

which have Poisson brackets $\{X^\mu, P^\nu\} = \{\Delta x^\mu, \Delta p^\nu\} = \eta^{\mu\nu}$ and total angular momenta $J = X \wedge P + \Delta x \wedge \Delta p$. Generalizing the analysis of Section 5.3, there are two mass-shell constraints

$$p_i^2 + m_i^2 = 0, \qquad i = 1, 2 \tag{G.2}$$

both of which must leave $(\Delta x)^2 = \ell^2$ and $(\Delta x \wedge \Delta p) = \hbar^2 s^2$ stationary. Again we find that that the constraints $p_1 \cdot \Delta x = p_2 \cdot \Delta x = 0$ must be included, and noting that $p_1 = \frac{m_1}{M} P + \Delta p$ and $p_2 = \frac{m_2}{M} P - \Delta p$ the full Hamiltonian can be written as

$$\begin{aligned} H \;&= \tfrac{N}{2}\left(P^2 + M^2 + \tfrac{M}{\mu}(\Delta p)^2\right) + \tilde{N}\left((P \cdot \Delta p) - \tfrac{\Delta m}{2\mu}(\Delta p)^2\right) + \tfrac{\lambda_1}{2}\left((\Delta x)^2 - \ell^2\right) \\ &\quad + \tfrac{\lambda_2}{2}\left((\Delta p)^2 - \epsilon^2 s^2\right) + (\lambda_3 m_1 + \lambda_4 m_2)(P \cdot \Delta x) + (\lambda_3 - \lambda_4)(\Delta p \cdot \Delta x), \end{aligned} \tag{G.3}$$

where we have introduced the mass difference $\Delta m = m_1 - m_2$.

We see that the four constraints

$$(P \cdot \Delta x) = 0, \quad (\Delta p \cdot \Delta x) = 0, \quad (\Delta x)^2 = \ell^2, \quad (\Delta p)^2 = \epsilon^2 s^2 \tag{G.4}$$

are identical to the equal mass case, whereas the mass shell and final orthogonality constraint are modified. Specifically, define

$$\mathcal{M}^2 := M^2 + \frac{M}{\mu}\epsilon^2 s^2, \qquad \rho := \frac{\Delta m}{2\mu}\epsilon^2 s^2, \tag{G.5}$$

then the modified constraints are

$$P^2 + \mathcal{M}^2 = 0, \qquad (P \cdot \Delta p) = \rho. \tag{G.6}$$

No further constraints need to be added but demanding that the existing constraints Poisson commute with H imposes the following conditions among the Lagrange multipliers

$$\lambda_3 = \lambda_4 = 0, \tag{G.7}$$

$$\left(N\frac{M}{\mu} - \tilde{N}\frac{\Delta m}{2M} + \lambda_2\right) = \frac{\lambda_1 \ell^2}{\epsilon^2 s^2} = \tilde{N}\frac{\mathcal{M}^2}{\rho}. \tag{G.8}$$

It follows that the reduced Hamiltonian involves two unconstrained Lagrange multipliers which correspond to the first class constraints

$$\Phi_P = P^2 + \mathcal{M}^2, \tag{G.9}$$

$$\Phi_S = \frac{(\Delta p)^2}{\epsilon^2}s^2 + \frac{(\Delta x)^2}{\ell^2} - 2\hbar^2 s^2 + \frac{\rho}{\mathcal{M}^2}\left[(P \cdot \Delta p) - \rho\right]. \tag{G.10}$$

There are an additional four second class constraints: a modified one $P \cdot \Delta p = \rho$ and three unmodified

$$P \cdot \Delta x = 0, \quad \Delta p \cdot \Delta x = 0, \quad \epsilon^2 s^2 (\Delta x)^2 - \ell^2 (\Delta p)^2 = 0. \tag{G.11}$$

The key difference from the equal mass case is the fact that $P \cdot \Delta p \neq 0$ which gives rise to the addtional complexity in the spin cosntraint Φ_S.

From these expressions it is clear that the case of continuous spin particles[1] [83, 105, 84] can then be obtained in the limit where $\mathcal{M} \to 0$ while keeping ρ fixed. Indeed, in this limit we recover the constraints

$$P^2 = 0, \quad P \cdot \Delta x = 0, \quad P \cdot \Delta p = \rho \tag{G.12}$$

[1] The idea of continuous spin particles in the DPS framework will be discussed more fully in Chapter 7.

together with $\epsilon^2 s^2(\Delta x)^2 + \ell^2(\Delta p)^2 = 2\hbar^2 s^2$ and $\Delta p \cdot \Delta x = 0$, $\epsilon^2 s^2(\Delta x)^2 = \ell^2(\Delta p)^2$. These are the constraints for a continuous spin particle.

At the outset of this appendix we put X_μ as the "center of mass" but this choice was arbitrary. Another option is to look for a definition of X' which leads to a vanishing mixing parameter ρ. Note that in order to keep the canonical algebra, changing X also means that we are changing Δp. Lets consider

$$X' = X - \frac{\Delta m}{2\mu}\frac{\epsilon^2 s^2}{\mathcal{M}^2}P, \qquad \Delta p' = \Delta p + \frac{\Delta m}{2\mu}\frac{\epsilon^2 s^2}{\mathcal{M}^2}P, \tag{G.13}$$

which preserve the canonical algebra by construction and satisfy $P \cdot \Delta p' = 0$. This change of coordinates can be seen as a redefinition of the effective spin, which is now given by $\epsilon^2 s'^2 = (\Delta p')^2$, while also rendering the position coordinate X' momentum dependent. For example, imagine coupling the massive spinning particle to an external electromagnetic field: With a vanishing mixing parameter it is natural to consider the coupling $A(X')$, however when expressed in the CSP frame where the mixing doesn't vanish this reads $A(X + \alpha P)$ and the location of the coupling is now momentum dependent.

Appendix H

Spinor Formalism

In this appendix we present a brief review of the spinor helicity formalism, see [113, 114, 115]. Let χ^α be a complex spinor and $\bar{\chi}^{\dot{\alpha}} = (\chi^\alpha)^\dagger$ it's complex conjugate. Indices are raised and lowered with the epsilon tensor $\epsilon_{\alpha\beta}$, which is totally skew symmetric and normalized by $\epsilon_{01} = 1$, i.e.

$$\chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta, \quad \chi_\alpha = \epsilon_{\alpha\beta}\chi^\beta, \quad \bar{\chi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\chi}_{\dot{\beta}}, \quad \bar{\chi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\chi}^{\dot{\beta}},$$

and these quantities are represented as

$$|\chi\rangle = \chi_\alpha, \qquad \langle\chi| = \chi^\alpha, \qquad |\chi] = \bar{\chi}^{\dot{\alpha}}, \qquad [\chi| = \bar{\chi}_{\dot{\alpha}}. \tag{H.1}$$

so we see that if $|\xi\rangle$ is our spinor, the hermitian conjugate spinor is denoted by $[\xi|$ as usual while $|\xi]$ denotes the same spinor but with indices raised. Note that we adopt a convention in which the epsilon tensor satisfies $\epsilon_{\alpha\gamma}\epsilon^{\gamma\beta} = \delta_\alpha^\beta$. Contractions between spinors are simply

$$\langle\zeta|\xi\rangle \equiv \zeta^\alpha\xi_\alpha, \qquad [\zeta|\xi] \equiv \bar{\zeta}_{\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}, \qquad [\zeta|\xi] = -\langle\zeta|\xi\rangle^*. \tag{H.2}$$

Let $(\sigma^a)_{\alpha\dot{\alpha}} = (\mathbb{1}_{\alpha\dot{\alpha}}, \vec{\sigma}_{\alpha\dot{\alpha}})$ be the standard four vector of sigma matrices, and $(\bar{\sigma}^a)^{\dot{\alpha}\alpha} \equiv (\sigma^a)_{\beta\dot{\beta}}\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}$ the same vector but with indices raised, then the following relations hold

$$\mathrm{Tr}(\sigma^a\bar{\sigma}^b) = -2\eta^{ab}, \qquad \eta_{ab}(\sigma^a)_{\alpha\dot{\alpha}}(\sigma^b)_{\beta\dot{\beta}} = -2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}. \tag{H.3}$$

Generically, a matrix with an overbar is assumed to have upper indices $\bar{M}^{\dot{\alpha}\alpha}$, whereas an unadorned matrix will have lower indices $M_{\alpha\dot{\alpha}}$. In matrix notation we have that $\bar{M} = \epsilon M^t\epsilon^{-1}$ and $\det(M) = -\frac{1}{2}\mathrm{Tr}(M\bar{M})$. Multiplication between a matrix and a spinor is denoted by juxtaposition

$$M_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = M|\chi], \quad \chi^\alpha M_{\alpha\dot{\alpha}} = \langle\chi|M, \quad \bar{M}^{\dot{\alpha}\alpha}\chi_\alpha = \bar{M}|\chi\rangle, \quad \bar{\chi}_{\dot{\alpha}}\bar{M}^{\dot{\alpha}\alpha} = [\chi|\bar{M}. \tag{H.4}$$

Any vector p^μ can be represented as a matrix by contracting it with the vector of sigma matrices

$$p_\mu = -\frac{1}{2}(\bar{\sigma}_\mu)^{\dot{\alpha}\alpha} p_{\alpha\dot{\alpha}} \quad \Longleftrightarrow \quad p_{\alpha\dot{\alpha}} = p_\mu(\sigma^\mu)_{\alpha\dot{\alpha}}. \tag{H.5}$$

It follows from eq. (H.3) that $p_{\alpha\dot{\alpha}}\bar{p}^{\dot{\alpha}\beta} = -p^2\delta_\alpha^\beta$ and $\bar{p}^{\dot{\alpha}\alpha}p_{\alpha\dot{\beta}} = -p^2\delta_{\dot{\beta}}^{\dot{\alpha}}$, so that the inner product of two vectors p^μ and q^μ is given by

$$p_\mu q^\mu = -\frac{1}{2}\mathrm{Tr}(p\bar{q}).$$

Let $\Lambda^\mu{}_\nu b^\nu$ be a Lorentz transformation, then the action of Λ on a spinor is represented by matrices $(L_\alpha{}^\beta, \bar{L}^{\dot{\alpha}}{}_{\dot{\beta}})$, that is

$$|\xi\rangle \to L|\xi\rangle, \qquad \langle\xi| \to \langle\xi|L^{-1}, \tag{H.6}$$

$$|\xi] \to (L^{-1})^\dagger|\xi], \qquad [\xi| \to [\xi|L^\dagger. \tag{H.7}$$

The relationship between Λ and $(L, \bar{L})$ is obtained through

$$\bar{L}^{-1}\bar{\sigma}^\mu L = \Lambda^\mu{}_\nu\bar{\sigma}^\nu, \qquad L^{-1}\sigma^\mu\bar{L} = \Lambda^\mu{}_\nu\sigma^\nu, \tag{H.8}$$

with the $(L, \bar{L})$ satisfying

$$\bar{L} = (L^{-1})^\dagger, \qquad \epsilon^{\alpha\alpha'}L_{\alpha'}{}^{\beta'}\epsilon_{\beta'\beta} = ([L^{-1}])_\beta{}^\alpha, \qquad \epsilon_{\dot{\alpha}\dot{\alpha}'}(\bar{L})^{\dot{\alpha}'}{}_{\dot{\beta}'}\epsilon^{\dot{\beta}'\dot{\beta}} = (\bar{L}^{-1})^{\dot{\beta}}{}_{\dot{\alpha}} = (\bar{L}^\dagger)^{\dot{\beta}}{}_{\dot{\alpha}}. \tag{H.9}$$

Observe that the contractions we have introduced above are indeed Lorentz invariant.
Let us now introduce a structure that involves the contraction of two conjugate spinors along a vector

$$p^{\dot{\alpha}\alpha}\bar{\zeta}_{\dot{\alpha}}\xi_\alpha = [\zeta|\bar{p}|\xi\rangle = \langle\xi|p|\zeta]. \tag{H.10}$$

Although this contraction is only invariant under Lorentz transformations that fix p, it does have the advantage of defining a hermitian form

$$[\zeta|p|\xi\rangle^* = [\xi|p^\dagger|\zeta\rangle = [\xi|p|\zeta\rangle. \tag{H.11}$$

Furthermore, if p is a timelike vector $p^2 + m^2 = 0$, this contraction defines a norm $[\xi|\bar{p}|\xi\rangle$ and in the center of mass frame this norm square is simply given by $\pm m(|\xi_0|^2 + |\xi_1|^2)$. The sign of the this scalar product is the sign of the energy $\pm = \mathrm{sign}(p_0)$.
The next thing to consider is the spinorial expression of a bivector. We begin by defining the rotation matrices

$$(\sigma^{\mu\nu})_\alpha{}^\beta \equiv \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)_\alpha{}^\beta, \qquad (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \equiv \frac{i}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)^{\dot{\alpha}}{}_{\dot{\beta}}, \tag{H.12}$$

which can be used to expand the anti-symmetric combination of Pauli matrices

$$\sigma^{[\mu}_{\alpha\dot{\alpha}}\sigma^{\nu]}_{\beta\dot{\beta}} = i\epsilon_{\dot{\alpha}\dot{\beta}}(\sigma^{\mu\nu})_{\alpha\beta} - i\epsilon_{\alpha\beta}(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}, \tag{H.13}$$

$$\bar{\sigma}^{\dot{\alpha}\alpha}_{[\mu}\bar{\sigma}^{\dot{\beta}\beta}_{\nu]} = -i\epsilon_{\dot{\alpha}\dot{\beta}}(\sigma^{\mu\nu})_{\alpha\beta} + i\epsilon_{\alpha\beta}(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}. \tag{H.14}$$

The rotation matrices possess self-duality properties

$$(*\sigma)^{\mu\nu} = i\sigma^{\mu\nu}, \qquad (*\bar{\sigma})^{\mu\nu} = -i\sigma^{\mu\nu}, \tag{H.15}$$

where $(*M)^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}M_{\rho\sigma}$ and we have assumed $\epsilon^{0123} = 1$. A bi-vector $S_{\mu\nu}$ can be decomposed into self-dual $S_\alpha{}^\beta = S_{\mu\nu}(\sigma^{\mu\nu})_\alpha{}^\beta$ and anti self-dual $\bar{S}^{\dot{\alpha}}{}_{\dot{\beta}} = S_{\mu\nu}(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}$ parts, specifically

$$S_{\mu\nu}\sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}} = (iS_{(\alpha\beta)}\epsilon_{\dot{\alpha}\dot{\beta}} - i\bar{S}_{(\dot{\alpha}\dot{\beta})}\epsilon_{\alpha\beta}), \tag{H.16}$$

$$(*S)_{\mu\nu}\sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}} = -(S_{(\alpha\beta)}\epsilon_{\dot{\alpha}\dot{\beta}} + \bar{S}_{(\dot{\alpha}\dot{\beta})}\epsilon_{\alpha\beta}). \tag{H.17}$$

With the spinor indices raised the decomposition is the negative of the one presented above. If the bivector is simple, i.e. $S_{\mu\nu} = (\chi \wedge \pi)_{\mu\nu}$, then we have

$$S_\alpha{}^\beta = \frac{i}{2}(\chi\bar{\pi} - \pi\bar{\chi})_\alpha{}^\beta, \quad \bar{S}^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{i}{2}(\bar{\chi}\pi - \bar{\pi}\chi)^{\dot{\alpha}}{}_{\dot{\beta}} \tag{H.18}$$

or

$$S_{\alpha\beta} = -i(\pi\bar{\chi})_{(\alpha\beta)}, \quad \bar{S}_{\dot{\alpha}\dot{\beta}} = i(\bar{\chi}\pi)_{(\dot{\alpha}\dot{\beta})}. \tag{H.19}$$

In other words we can express the matrix product of two vectors as

$$(\chi\bar{\pi})_\alpha{}^\beta = -(\chi_\mu\pi^\mu)\delta^\beta_\alpha - i(\chi\wedge\pi)_\alpha{}^\beta, \qquad (\bar{\chi}\pi)^{\dot{\alpha}}{}_{\dot{\beta}} = -(\chi_\mu\pi^\mu)\delta^{\dot{\alpha}}_{\dot{\beta}} - i(\chi\wedge\pi)^{\dot{\alpha}}{}_{\dot{\beta}}. \tag{H.20}$$

The matrix corresponding to a vector p_μ can be expressed explicitly as

$$p_{\alpha\dot{\alpha}} = \begin{pmatrix} (p_0+p_3) & (p_1-ip_2) \\ (p_1+ip_2) & (p_0-p_3) \end{pmatrix}, \qquad \bar{p}^{\dot{\alpha}\alpha} = \begin{pmatrix} (p_0-p_3) & -(p_1-ip_2) \\ -(p_1+ip_2) & (p_0+p_3) \end{pmatrix}. \tag{H.21}$$

We see that the bar operator corresponds to parity reversal, that is, if we denote the parity transformed vector $\tilde{p}_\mu \equiv (p_0, -p_i)$ then $\bar{p} = \tilde{p}$ as matrices. We also find that

$$(\chi\bar{\pi})_\alpha{}^\beta = (\chi_\mu\pi^\mu)1 + i\begin{pmatrix} J_3+iK_3 & (J_1+iK_1)-i(J_2+iK_2) \\ (J_1+iK_1)+i(J_2+iK_2) & -(J_3+iK_3) \end{pmatrix} \tag{H.22}$$

where we have defined

$$K_i = (\chi\wedge\pi)_{i0}, \qquad J_i = \epsilon_{ijk}(\chi\wedge\pi)^{jk}, \tag{H.23}$$

as "boost" and "rotation" generators respectively.

Appendix I

Details on the Spinoral Parameterization of DPS

This appendix provides extra details for some of the results presented in Chapter 6.

I.1 Classical Spin Motion

We show explictly that the spin state does not evolve during classical motion. The equations of motion associated with the Lagrangian eq. (6.22) are given by

$$\frac{d}{dt} P = 0, \tag{I.1}$$

$$\frac{d}{dt} \left(\langle \xi | P \right) = - \Big(\langle \dot{\xi} | + 2i\dot{\phi} \langle \xi | \Big) P. \tag{I.2}$$

where we have defined

$$P := \frac{1}{N} \left(\dot{x} + \theta(e^{i\phi} \xi) \right). \tag{I.3}$$

The first of these implies that $P_{\alpha\dot{\alpha}}$ is constant, and inserting this result into the second equation gives

$$\Big(\langle \dot{\xi} | + i\dot{\phi} \langle \xi | \Big) C = 0 \qquad \Longrightarrow \qquad \frac{d}{dt} \left(e^{i\phi} \langle \xi | \right) = 0. \tag{I.4}$$

It follows that $e^{i\phi} \xi$ is a constant of motion and so $\theta(e^{i\phi} \xi)$ vanishes on-shell.

I.2 Second Order Equations of Motion

In what follows we derive the equations of motion resulting from the second order actioneq. (6.31). Begin by defining the momentum p_μ conjugate to x_μ in the usual manner $p_\mu = \delta S/\delta \dot{x}^\mu$, then the equation of motion for x is determined by conservation of momenta $\dot{p} = 0$. As an important aside, the relationship between $p_{\alpha\dot{\alpha}}$ and $\delta S/\delta \bar{x}^{\dot{\alpha}\alpha}$ isn't quite as expected, specifically

$$p_{\alpha\dot{\alpha}} = (\sigma^\mu)_{\alpha\dot{\alpha}} p_\mu = (\sigma^\mu_{\alpha\dot{\alpha}}) \frac{\delta L}{\delta \dot{x}^\mu} = (\sigma^\mu_{\alpha\dot{\alpha}}) \frac{\delta L}{\delta \dot{\bar{x}}^{\dot{\beta}\beta}} \frac{\partial \dot{\bar{x}}^{\dot{\beta}\beta}}{\partial \dot{x}^\mu} = -2 \frac{\delta L}{\delta \dot{\bar{x}}^{\dot{\alpha}\alpha}}. \tag{I.5}$$

Recalling the definitions of $\mathfrak{a}$ and $\mathfrak{b}$ from eq. (6.36) we find that the momenta $p_{\alpha\dot{\alpha}}$ is given by

$$\begin{aligned} -\frac{1}{2}p = m\frac{\dot{x}}{2|\dot{x}|}\sqrt{1-|\mathfrak{b}|^2} - \frac{i\hbar s}{\langle\rho|\dot{x}|\rho]}\Big(|\dot{\rho}\rangle[\rho| - |\rho\rangle[\dot{\rho}|\Big) \\ - \frac{2\hbar s}{\langle\rho|\dot{x}|\rho]}\mathrm{Im}(\mathfrak{a})|\rho\rangle[\rho| + \frac{m|\mathfrak{b}|^2}{\sqrt{1-|\mathfrak{b}|^2}}\frac{|\rho\rangle|\dot{x}|[\rho|}{\langle\rho|\dot{x}|\rho]}. \end{aligned} \tag{I.6}$$

It follows from this lengthy expression that

$$\dot{x}|\rho] = -\frac{|\dot{x}|}{\sqrt{1-|\mathfrak{b}|^2}}\Big(\hat{p}|\rho] + i\mathfrak{b}^*|\rho\rangle\Big),$$

where $\hat{p}$ is the unit momenta $\hat{p} = p/m$. The spin equations of motion can now be written in matrix form as

$$\partial_\tau \begin{pmatrix} |\rho\rangle \\ \hat{p}|\rho] \end{pmatrix} = \begin{pmatrix} \mathfrak{a} - \frac{im}{2\hbar s}\frac{|\mathfrak{b}|^2}{\sqrt{1-|\mathfrak{b}|^2}}|\dot{x}| & -\frac{m\mathfrak{b}}{2\hbar s}\frac{|\dot{x}|}{\sqrt{1-|\mathfrak{b}|^2}} \\ \frac{m\mathfrak{b}^*}{2\hbar s}\frac{|\dot{x}|}{\sqrt{1-|\mathfrak{b}|^2}} & \mathfrak{a}^* + \frac{im}{2\hbar s}\frac{|\mathfrak{b}|^2}{\sqrt{1-|\mathfrak{b}|^2}}|\dot{x}| \end{pmatrix} \begin{pmatrix} |\rho\rangle \\ \hat{p}|\rho] \end{pmatrix}.$$

To simplify the presentation we introduce the notation

$$\begin{gathered} \boldsymbol{\rho} = \begin{pmatrix} |\rho\rangle \\ \hat{p}|\rho] \end{pmatrix}, \quad a_0 = \mathrm{Re}(\mathfrak{a}), \quad a_1 = -\frac{m}{2\hbar s}\frac{|\dot{x}|}{\sqrt{1-|\mathfrak{b}|^2}}\mathrm{Im}(\mathfrak{b}), \\ a_2 = -\frac{m}{2\hbar s}\frac{|\dot{x}|}{\sqrt{1-|\mathfrak{b}|^2}}\mathrm{Re}(\mathfrak{b}), \quad a_3 = \mathrm{Im}(\mathfrak{a}) - \frac{m}{2\hbar s}\frac{|\mathfrak{b}|^2}{\sqrt{1-|\mathfrak{b}|^2}}|\dot{x}|, \end{gathered}$$

and so the spin equations of motion become

$$\partial_\tau \boldsymbol{\rho} = \begin{pmatrix} a_0 + ia_3 & ia_1 + a_2 \\ ia_1 - a_2 & a_0 - ia_3 \end{pmatrix} \boldsymbol{\rho}, \tag{I.7}$$

$$= \left(a_0 \mathbb{1} + i\vec{a}\cdot\vec{\sigma}\right)\boldsymbol{\rho}, \tag{I.8}$$

where $\vec{a} = (a_1, a_2, a_3)$. To solve this equation we introduce a vector $\boldsymbol{\xi}$ which satisfies

$$\boldsymbol{\rho} = e^{\int_0^\tau a_0(t)dt}\boldsymbol{\xi}. \tag{I.9}$$

Eq. (I.8) then implies $\partial_\tau \boldsymbol{\xi} = i\vec{a}\cdot\vec{\sigma}\boldsymbol{\xi}$, which has the formal solution

$$\boldsymbol{\xi} = \mathrm{T}\exp\left(i\vec{\sigma}\cdot\int_0^\tau \vec{a}(t)dt\right)\boldsymbol{\xi}(0). \tag{I.10}$$

It is only in the special case where at most one component of $\vec{a}$ is non-zero that we could obtain an explicit expression for $\boldsymbol{\xi}$.

I.3 Poisson Brackets

Here we include the explicit derivation of the Poisson brackets between $x^{\alpha\dot{\alpha}}$, ξ^α, and $\bar{\xi}^{\dot{\alpha}}$. Equations (6.48)–(6.51) together with the expression for x and Δ in terms of J given by (6.54) imply that

$$\left\{(\bar{x}+i\bar{\Delta})^{\dot{\alpha}\alpha}, (\bar{x}+i\bar{\Delta})^{\dot{\beta}\beta}\right\} = 0, \qquad \left\{(\bar{x}+i\bar{\Delta})^{\dot{\alpha}\alpha}, (\bar{x}-i\bar{\Delta})^{\dot{\beta}\beta}\right\} = \frac{4i}{m^2}\bar{p}^{\dot{\alpha}\beta}\bar{\Delta}^{\dot{\beta}\alpha}, \tag{I.11}$$

$$\left\{(\bar{x}-i\bar{\Delta})^{\dot{\alpha}\alpha}, (\bar{x}-i\bar{\Delta})^{\dot{\beta}\beta}\right\} = 0, \qquad \left\{(\bar{x}-i\bar{\Delta})^{\dot{\alpha}\alpha}, (\bar{x}+i\bar{\Delta})^{\dot{\beta}\beta}\right\} = -\frac{4i}{m^2}\bar{p}^{\dot{\beta}\alpha}\bar{\Delta}^{\dot{\alpha}\beta}, \tag{I.12}$$

which combine to give

$$\left\{\bar{x}^{\dot{\alpha}\alpha}, \bar{x}^{\dot{\beta}\beta}\right\} = \left\{\bar{\Delta}^{\dot{\alpha}\alpha}, \bar{\Delta}^{\dot{\beta}\beta}\right\} = \frac{i}{m^2}\left(\bar{p}^{\dot{\alpha}\beta}\bar{\Delta}^{\dot{\beta}\alpha} - \bar{p}^{\dot{\beta}\alpha}\bar{\Delta}^{\dot{\alpha}\beta}\right), \tag{I.13}$$

$$\left\{\bar{x}^{\dot{\alpha}\alpha}, \bar{\Delta}^{\dot{\beta}\beta}\right\} = -\frac{1}{m^2}\left(\bar{p}^{\dot{\alpha}\beta}\bar{\Delta}^{\dot{\beta}\alpha} + \bar{p}^{\dot{\beta}\alpha}\bar{\Delta}^{\dot{\alpha}\beta}\right). \tag{I.14}$$

Noting that the brackets in eq. (I.13) are anti-symmetric under interchange of $(\alpha, \dot{\alpha})$ with $(\beta, \dot{\beta})$ allows us to re-write them in a more revealing form

$$\left\{\bar{x}^{\dot{\alpha}\alpha}, \bar{x}^{\dot{\beta}\beta}\right\} = \left\{\bar{\Delta}^{\dot{\alpha}\alpha}, \bar{\Delta}^{\dot{\beta}\beta}\right\} = \frac{i}{m^2}\left[\epsilon^{\dot{\alpha}\dot{\beta}}(p\bar{\Delta})^{(\alpha\beta)} - \epsilon^{\alpha\beta}(\bar{\Delta}p)^{(\dot{\alpha}\dot{\beta})}\right]. \tag{I.15}$$

A further application of eqs. (6.48)–(6.49) to ξ and $\bar{\xi}$ in conjunction with the decomposition (6.54) yields

$$\left\{\xi^\alpha, \bar{x}^{\dot{\beta}\beta}\right\} = \frac{1}{m^2}\bar{p}^{\dot{\beta}\alpha}\xi^\beta, \qquad \left\{\bar{\xi}^{\dot{\alpha}}, \bar{x}^{\dot{\beta}\beta}\right\} = \frac{1}{m^2}\bar{p}^{\dot{\alpha}\beta}\bar{\xi}^{\dot{\beta}}, \tag{I.16}$$

$$\left\{\xi^{\alpha},\bar{\Delta}^{\dot{\beta}\beta}\right\}=-\frac{i}{m^2}\bar{p}^{\dot{\beta}\alpha}\xi^{\beta},\qquad\left\{\bar{\xi}^{\dot{\alpha}},\bar{\Delta}^{\dot{\beta}\beta}\right\}=\frac{i}{m^2}\bar{p}^{\dot{\alpha}\beta}\bar{\xi}^{\dot{\beta}}.\tag{I.17}$$

It remains to compute the brackets between ξ and $\bar{\xi}$. We begin by substituting the definition of Δ, see eq. (6.52), into eq. (I.17), whence

$$\frac{\hbar}{2m^2 s}\langle\xi|p|\xi]\left(\xi^{\beta}\left\{\xi^{\alpha},\bar{\xi}^{\dot{\beta}}\right\}+\bar{\xi}^{\dot{\beta}}\left\{\xi^{\alpha},\xi^{\beta}\right\}\right)=-\frac{i}{m^2}\bar{p}^{\dot{\beta}\alpha}\xi^{\beta}+\frac{i}{2\langle\xi|p|\xi]}\xi^{\alpha}\xi^{\beta}\bar{\xi}^{\dot{\beta}}.\tag{I.18}$$

Contract either side with ξ_{β} to obtain $\left\{\xi^{\alpha},\xi^{\beta}\right\}\xi_{\beta}=0$ which, by virtue of the anti–symmetry of the bracket, implies

$$\left\{\xi^{\alpha},\xi^{\beta}\right\}=0.\tag{I.19}$$

Upon substituting the above result into eq. (I.18) and contracting with $(p|\xi])_{\beta}$ we obtain

$$\left\{\xi^{\alpha},\bar{\xi}^{\dot{\beta}}\right\}=-\frac{is}{\hbar\langle\xi|p|\xi]^2}\left(2\langle\xi|p|\xi]\bar{p}^{\dot{\beta}\alpha}-m^2\xi^{\alpha}\bar{\xi}^{\dot{\beta}}\right).\tag{I.20}$$

Similar results hold for $\bar{\xi}$, in particular

$$\left\{\bar{\xi}^{\dot{\alpha}},\bar{\xi}^{\dot{\beta}}\right\}=0,\qquad\left\{\bar{\xi}^{\dot{\alpha}},\xi^{\beta}\right\}=\frac{is}{\hbar\langle\xi|p|\xi]^2}\left(2\langle\xi|p|\xi]\bar{p}^{\dot{\alpha}\beta}-m^2\xi^{\beta}\bar{\xi}^{\dot{\alpha}}\right).\tag{I.21}$$

Appendix J

Constraint Algebra for the Covariant DPS Model

In what follows we provide additional details on the computation of the constraint algebra for the covariant DPS model discussed in Chapter 7. The algebra between χ and π constraints is straighforward to compute

$$\{\Phi_\chi, \Phi_\pi\} = 4\Phi_{\chi\pi} \simeq 0 \tag{J.1}$$

$$\{\Phi_\chi, \Phi_{\chi\pi}\} = 2\left(\Phi_\chi + \lambda^2 s^2\right) \simeq 2\lambda^2 s^2 \tag{J.2}$$

$$\{\Phi_\pi, \Phi_{\chi\pi}\} = -2(\Phi_\pi + E^2) \simeq -2E^2 \tag{J.3}$$

For constraints involving p the computation is much more challenging, lets proceed systematically by first computing the Poisson brackets of these constraints with the coordinate vectors χ^μ, π_μ and p_μ. In doing so it will be useful to know the Poisson bracket between p_μ and the metric

$$\{p_\mu, g_{\nu\rho}\} = \frac{2\epsilon}{R^2\sigma} g_{\nu\rho}\eta_{\mu\alpha}x^\alpha - g_{\mu\gamma}\Gamma^\gamma_{\nu\rho}, \tag{J.4}$$

$$\{p_\mu, g^{\nu\rho}\} = -\frac{2\epsilon}{R^2\sigma} g^{\nu\rho}\eta_{\mu\alpha}x^\alpha + g_{\mu\gamma}\Gamma^\gamma_{\alpha\beta}g^{\alpha\nu}g^{\beta\rho}. \tag{J.5}$$

Keeping in mind that indices are raised and lowered by the metric and its inverse, we find

$$\{\chi^\mu, \Phi_{p\chi}\} = -\Gamma^\mu_{\nu\rho}\chi^\nu\chi^\rho,$$

$$\{\pi_\mu, \Phi_{p\chi}\} = -p_\mu + \Gamma^\rho_{\mu\nu}\pi_\rho\chi^\nu,$$

$$\{p_\mu, \Phi_{p\chi}\} = \Gamma^\nu_{\mu\rho}\chi^\rho p_\nu + R^\sigma{}_{\rho\mu\nu}\pi_\sigma\chi^\rho\chi^\nu,$$

and

$$\begin{aligned}
\{\chi^\mu, \Phi_{p\pi}\} &= p^\mu - \pi^\nu \Gamma^\mu_{\nu\rho}\chi^\rho, \\
\{\pi_\mu, \Phi_{p\pi}\} &= \pi^\nu \pi_\alpha \Gamma^\rho_{\mu\nu}, \\
\{p_\mu, \Phi_{p\pi}\} &= -\frac{2\epsilon}{R^2\sigma} p_\nu \pi^\nu \eta_{\mu\rho} x^\rho + g_{\mu\nu}\Gamma^\nu_{\rho\gamma} p^\rho \pi^\gamma \\
&\quad + R^\rho{}_{\gamma\mu\nu}\pi_\rho \chi^\gamma \pi^\nu - \Gamma^\rho_{\mu\nu} p^\nu \pi_\rho,
\end{aligned}$$

and finally

$$\begin{aligned}
\{\chi^\mu, \Phi_p\} &= -2\Gamma^\mu_{\nu\rho}\chi^\rho p^\nu, \\
\{\pi_\mu, \Phi_p\} &= 2\Gamma^\rho_{\mu\nu} p^\nu p_\rho, \\
\{p_\mu, \Phi_p\} &= -\frac{2\epsilon}{R^2\sigma} p^\nu p_\nu \eta_{\mu\rho} x^\rho + g_{\mu\nu}\Gamma^\nu_{\rho\gamma} p^\rho p^\gamma \\
&\quad + 2R^\sigma{}_{\rho\mu\nu}\pi_\sigma \chi^\rho p^\nu.
\end{aligned}$$

It is now a tedious but straightforward task to compute the constraint algebra. We find

$$\{\Phi_\chi, \Phi_{p\chi}\} = 0, \tag{J.6}$$

$$\{\Phi_\pi, \Phi_{p\chi}\} = -2\Phi_{p\pi} - \frac{2\epsilon f}{R} E \simeq -\frac{2\epsilon f}{R} E, \tag{J.7}$$

$$\{\Phi_{\chi\pi}, \Phi_{p\chi}\} = -\Phi_{p\chi} + \frac{\epsilon f^2}{R}\lambda \simeq \frac{\epsilon f^2}{R}\lambda, \tag{J.8}$$

and

$$\{\Phi_\chi, \Phi_{p\pi}\} = 2\Phi_{p\chi} - \frac{2\epsilon f^2}{R}\lambda \simeq -\frac{2\epsilon f^2}{R}\lambda, \tag{J.9}$$

$$\{\Phi_\pi, \Phi_{p\pi}\} = 0, \tag{J.10}$$

$$\{\Phi_{\chi\pi}, \Phi_{p\pi}\} = \Phi_{p\pi} + \frac{\epsilon f}{R} E \simeq \frac{\epsilon f}{R} E, \tag{J.11}$$

and

$$\{\Phi_\chi, \Phi_p\} = 0, \tag{J.12}$$

$$\{\Phi_\pi, \Phi_p\} = 0, \tag{J.13}$$

$$\{\Phi_{\chi\pi}, \Phi_p\} = 0, \tag{J.14}$$

and

$$\{\Phi_p, \Phi_{p\chi}\} = \frac{2\epsilon}{R^2}\left(\left(\Phi_{p\pi} + \frac{\epsilon f}{R} E\right)\left(\Phi_\chi + \lambda^2 s^2\right) - \Phi_{\chi\pi}\left(\Phi_{p\chi} - \frac{\epsilon f^2}{R}\lambda\right)\right) \tag{J.15}$$

$$\simeq \frac{2fs^2}{R^3}\lambda, \tag{J.16}$$

$$\{\Phi_p, \Phi_{p\pi}\} = \frac{2\epsilon}{R^2}\left(\Phi_{\chi\pi}\left(\Phi_{p\pi} + \frac{\epsilon f}{R}E\right) - \left(\Phi_{p\chi} - \frac{\epsilon f^2}{R}\lambda\right)\left(\Phi_\pi + E^2\right)\right) \tag{J.17}$$

$$\simeq \frac{2f^2}{R^3}E \tag{J.18}$$

and finally

$$\{\Phi_{p\chi}, \Phi_{p\pi}\} = \Phi_p + \frac{\epsilon}{R^2}\left(\iota\epsilon\nu^2 - (1+\epsilon)f^2\right) + \frac{\epsilon}{R^2}\left(\Phi_{\chi\pi} - \left(\Phi_\pi + E^2\right)\left(\Phi_\chi + \lambda^2 s^2\right)\right) \tag{J.19}$$

$$\simeq \frac{\epsilon}{R^2}\left(\iota\epsilon\nu^2 - s^2 - (1+\epsilon)f^2\right). \tag{J.20}$$