Transitive Factorizations of Permutations and Eulerian Maps in the Plane

by

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A thesis

presented to the University of Waterloo

in fulfilment of the

thesis requirement for the degree of

Master of Mathematics

in

Combinatorics and Optimization

Waterloo, Ontario, Canada, 2005

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners. I understand that my thesis may be made electronically available to the public.

Abstract

The problem of counting ramified covers of a Riemann surface up to homeomorphism was proposed by Hurwitz in the late 1800's. This problem translates combinatorially into factoring a permutation with a specified cycle type, with certain conditions on the cycle types of the factors, such as minimality and transitivity.

Goulden and Jackson have given a proof for the number of minimal, transitive factorizations of a permutation into transpositions. This proof involves a partial differential equation for the generating series, called the Join-Cut equation. Furthermore, this argument is generalized to surfaces of higher genus. Recently, Bousquet-Mélou and Schaeffer have found the number of minimal, transitive factorizations of a permutation into arbitrary unspecified factors. This was proved by a purely combinatorial argument, based on a direct bijection between factorizations and certain objects called m-Eulerian trees.

In this thesis, we will give a new proof of the result by Bousquet-Mélou and Schaeffer, introducing a simple partial differential equation. We apply algebraic methods based on Lagrange's theorem, and combinatorial methods based on a new use of Bousquet-Mélou and Schaeffer's m-Eulerian trees. Some partial results are also given for a refinement of this problem, in which the number of cycles in each factor is specified. This involves Lagrange's theorem in many variables.

Acknowledgements

First and foremost, I would like to thank my advisor, Ian Goulden, for being a great mentor, both in the academic and the personal level. I have no words to express how fortunate I have been to receive his support, encouragement, and wisdom.

I would also like to thank Chris Godsil and David Jackson for being the readers of my thesis. David Jackson for all his advice, inspiration, and the many exciting conversations from which I learned a lot. Many thanks to Chris Godsil, for giving me the opportunity and introducing me to the beauty of research, and for his encouragement and wise words throughout all these years. I am very grateful to Brian Forrest and Frank Zorzitto for their help and support during my undergraduate years, to David McKinnon, Douglas Park, and Ravi Vakil, for their time, and valuable help in geometry, and to Mary Falk, John Borwein, Ada Chan, Adrian Lewis, and Bruce Richmond for being crucial in certain stages of my career, and to John and Amarpreet for the interesting math conversations we had.

I have felt very fortunate to be part of the Department of Combinatorics and Optimization at the University of Waterloo, which has supported me, and to the professors and staff for creating such a great environment. In particular, I would like to thank Paul Schellenberg, Bill Cunningham, and Marg Feeney.

I would like to thank my mom for all these years of love, encouragement, and support, for teaching me so many things in life. My grandma, for always being there for me, and for spoiling me just like every grandma should do. I love you both with all my heart, las quiero mucho. To my friends, especially Alejandro for being the best best friend in the entire world and for his unconditional support, to Martha who has been very special to me all these years, to Federico for all his help and advice, and to Santiago, Camilo, Jose Luis, Changhee, Tony, Matt, Chris, Pat, Carlos, Ryan, Linden, Masoud, Sun, and many more. Finally, I would like to thank Melanie Heu for being the special person she is in my life, for always being there for me, and for everything I have learned from her.

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Chapter 1

Introduction and Background

The subject of this thesis is a mathematical problem that has attracted significant attention in the last decade, with contributions from researchers in algebraic combinatorics, algebraic geometry, and mathematical physics. Its origins are in the work of Hurwitz [7], dating from the 1890's. Here, we emphasize the algebraic combinatorial approach - the key ingredients from this point of view are the ring of formal power series, especially Lagrange's theorem, and a particular class of factorizations of the symmetric group, called transitive factorizations. The background required for these ingredients is described briefly in this first chapter, with a special focus on the translation of Hurwitz's original geometric problem into combinatorial terms. The chapter concludes with an outline of the thesis.

1.1 Background Material

1.1.1 The Symmetric Group

The *symmetric group* S_n is the group of permutations on n symbols, where the product $\sigma\phi$ for $\sigma, \phi \in S_n$ is defined as the permutation obtained by applying ϕ to the symbols $1, \cdots, n$ and then applying σ to the result. This group has order $n!$. A permutation σ in S_n is said to have *degree* $|\sigma| = n$. An element $c \in S_n$ is a cycle of degree $|c| = k$ if it permutes distinct symbols $a_1, \cdots, a_k$

in the following way: for $1 \leq i \leq k-1$, a_i is mapped to a_{i+1}, and a_k is mapped to a_1. If $\sigma \in S_n$, we can write σ uniquely as a product of disjoint cycles $c_1, \cdots, c_m$, and we define the *length* of σ to be m, the number of cycles, and denote it by $l(\sigma)$. Since the cycles are pairwise mutually disjoint, then

$$|c_1| + \cdots + |c_m| = n,$$

so the numbers $|c_1|, \cdots, |c_m|$ form a partition of $\{1, \cdots, n\}$. In fact, the set of partitions of n is a natural index set for conjugacy classes on S_n, and we call a partition the *cycle type* of the permutations in the corresponding class. We write $\alpha \vdash n$ to indicate that α is a partition of $\{1, \ldots, n\}$. Moreover, α is said to have *size* n, and *length* $l(\alpha) = m$ if it has m parts. The size of this conjugacy class is denoted by h^α . If α has d_i cycles of length i for $i = 1, 2, \cdots$, by a simple counting argument, we have

$$h^\alpha = \frac{n!}{\prod_{i\geq 1} i^{d_i} d_i!}. \tag{1.1}$$

1.1.2 Formal Power Series

Let R be a ring with unity, and $\mathbf{x} = (x_1, x_2, \cdots)$ be a set of commutative indeterminates. We define the *ring of formal power series* in $\mathbf{x}$ as

$$R[[\mathbf{x}]] = \left\{ \sum_{\mathbf{i}\geq \mathbf{0}} a_{\mathbf{i}} \mathbf{x}^{\mathbf{i}} \quad : \quad a_{\mathbf{i}} \in R, \mathbf{i} \geq \mathbf{0} \right\},$$

with the usual addition and multiplication. The *coefficient operator* $[\mathbf{x}^{\mathbf{i}}]$ is defined as $[\mathbf{x}^{\mathbf{i}}] \sum_{\mathbf{j}} a_{\mathbf{j}} \mathbf{x}^{\mathbf{j}} = a_{\mathbf{i}}$. We also define the following two subsets of $R[[\mathbf{x}]]$:

$$R[[\mathbf{x}]]_0 = \left\{ f \in R[[\mathbf{x}]] \quad | \quad [\mathbf{x}^{\mathbf{0}}] f = 0 \right\},$$

and

$$R[[\mathbf{x}]]_1 = \left\{ f \in R[[\mathbf{x}]] \quad | \quad [\mathbf{x}^{\mathbf{0}}] f \text{ is a unit in } R \right\}:$$

We also consider the *ring* $R((\mathbf{x}))$ *of formal Laurent series* in $\mathbf{x}$, with the coefficient operator defined analogously. The properties of these rings are given, for example, in [5].

1.1.3 Lagrange Inversion

The following two theorems, as they appear in [5], are used throughout this thesis.

Theorem 1 ***(Lagrange)*** *Let $\phi(\lambda) \in R[[\lambda]]_1$. Then there exists a unique formal power series $w(t) \in R[[t]]_0$ such that $w = t\phi(w)$. Moreover,*

1. if $f(\lambda) \in R((\lambda))$, then for $n \neq 0$,

$$[t^n]f(w) = \frac{1}{n}[\lambda^{n-1}]f'(\lambda)\phi^n(\lambda);$$

2. if $F(\lambda) \in R[[\lambda]]$, then for $n \geq 0$,

$$[t^n]\frac{F(w)}{1 - t\phi'(w)} = [\lambda^n]F(\lambda)\phi^n(\lambda).$$

Theorem 2 ***(Multivariate Lagrange)*** *Let $f(\boldsymbol{\lambda}) \in R((\boldsymbol{\lambda}))$ and $\phi_1(\boldsymbol{\lambda}), \cdots, \phi_m(\boldsymbol{\lambda}) \in R[[\boldsymbol{\lambda}]]_1$ where $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_m)$. Suppose that $w_i = t_i\phi_i(\boldsymbol{w})$ for $i = 1, \cdots, m$, where $\boldsymbol{w} = (w_1, \ldots, w_m)$. Let $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_m)$ and $\boldsymbol{t} = (t_1, \ldots, t_m)$. Then for all $\boldsymbol{k} = (k_1, \ldots, k_m)$,*

$$[\boldsymbol{t}^{\boldsymbol{k}}]f(\boldsymbol{w}(\boldsymbol{t})) = [\boldsymbol{\lambda}^{\boldsymbol{k}}]\left(f(\boldsymbol{\lambda})\Phi^{\boldsymbol{k}}(\boldsymbol{\lambda})\left\|\delta_{i,j} - \frac{\lambda_j}{\phi_i(\boldsymbol{\lambda})}\frac{\partial\phi_i(\boldsymbol{\lambda})}{\partial\lambda_j}\right\|_{m\times m}\right).$$

1.2 The Hurwitz Problem

Let $f : M \to S^2$ be a non-constant meromorphic function on a compact connected Riemann surface M of genus $g \geq 0$. This function is called a *ramified n-sheeted covering* where $n \geq 1$ is the integer that satisfies $|f^{-1}(p)| = n$ for all but a finite number of points $p \in S^2$, called the *critical values*, or *branch points* . We call n the *degree* of f. Two coverings f_1 and f_2 are *equivalent* if there exist homeomorphisms $\pi : M \to M$ and $\rho : S^2 \to S^2$ such that $\rho \circ f_1 = f_2 \circ \pi$.

The Hurwitz problem, first stated in [7], is to find the number of inequivalent, connected, covers. Particular instances of this problem arise by specifying choices for the behaviour at the branch points.

In this thesis, instances of the Hurwitz problem are considered, using an equivalent combinatorial problem, for which some notation is required. For $x \in M$ let $v(x, f)$ be the *multiplicity* with which f takes the value $f(x)$ at the point x. The *branching order* of f at the point x is defined as $b(x, f) = v(x, f) - 1$. Note that if x is not a branch point, then $b(x, f) = 0$, so the branching order is zero at all but a finite number of points. Therefore, the total branching order of f, given by

$$b(f) = \sum_{x \in X} b(f, x),$$

is well defined.

The following Theorem, as it appears in [3], is useful to combinatorialize the problem.

Theorem 3 ***(Riemann-Hurwitz Formula)*** *Let X and Y be compact Riemann surfaces with genera g and g', respectively. Let $f : X \to Y$ be an n-sheeted holomorphic covering mapping between X and Y, with total branching order $b(f)$. Then*

$$g = \frac{b(f)}{2} + n(g' - 1) + 1. \tag{1.2}$$

As a first observation, note that if $y \in Y$ has ramification type σ, then

$$\sum_{x \in f^{-1}(y)} b(f, x) = \sum_{x \in f^{-1}(y)} [v(f, x) - 1] = n - l(\sigma). \tag{1.3}$$

In particular, if y is *unramified* (*i.e.* , not a branch point), then $l(\sigma) = n$, so

$$\sum_{x \in f^{-1}(y)} b(f, x) = 0.$$

Now we use Hurwitz's approach [9] to combinatorialize the Hurwitz problem. Let f be a ramified cover of the sphere with n sheets. For every point $x \in S^2$ there is a permutation $\pi \in S_n$ such that as one walks around x counterclockwise, starting at a sheet j, one ends at sheet $\pi(j)$.

The permutation π is called the *ramification* at the point x, and the cycle type of π is called the *ramification type* at x. Note that for every point that is not a branch point, the ramification is the identity. If the permutation corresponding to this branch point is a transposition, the branch point is called a *simple branch point.* The covers we shall consider have $m+1$ branch points $y_1, \ldots, y_m, y_{m+1}$, of ramification types $\sigma_1, \ldots, \sigma_m, \alpha$, respectively, where the branch point y_{m+1} of ramification type α will be given special consideration – the partition α is defined to be the ramification type of the cover.

Any homeomorphism of the sphere preserves the ramification of every point at its image. Thus, the list $\pi_1, \ldots, \pi_{m+1}$ defines the cover f up to homeomorphism. Now, the following consistency condition must be satisfied by these permutations: if one selects a point y which is not a branch point, and cuts around the great circles that join y to $y_1, \ldots, y_{m+1}$, then after walking around y counterclockwise, one must start and finish on the same sheet. This consistency condition is defined as *monodromy* of the sheets, and its combinatorial interpretation is

$$\iota = \pi_1 \cdots \pi_{m+1}, \tag{1.4}$$

where ι is the identity permutation in S_n. Therefore, by writing $\pi_{m+1} = \pi^{-1}$ and noticing that both π and π^{-1} have cycle type α, the consistency condition becomes

$$\pi = \pi_1 \cdots \pi_m. \tag{1.5}$$

We say that $\pi_1 \cdots \pi_m$ is an ordered factorization of π.

Furthermore, for the cover to be connected one needs to be able to move from sheet j to sheet k, by walking around suitable branch points, for every value of j and k. This means the group generated by $\pi_1, \ldots, \pi_m$ must be a transitive subgroup of S_n. This condition is called *transitivity* , and we say that such an m-tuple $\pi_1, \ldots, \pi_m$, satisfying (1.5), is a *transitive* factorization of π.

The final condition is on the cycle types of the factors. Note that by Theorem 3,

$$
\begin{aligned}
2g &= b(f) + 2n(g'-1) + 2 \\
&= \left(\sum_{x\in X} b(f,x)\right) + 2n(g'-1) + 2 \\
&= \left(\sum_{y\in Y}\sum_{x\in f^{-1}(y)} b(f,x)\right) + 2n(g'-1) + 2 \\
&= \left(\sum_{i=1}^{m} n - l(\sigma_i)\right) - l(\alpha) + n(g'-1) + 2 \quad \text{by (1.3).}
\end{aligned} \tag{1.6}
$$

In the case that Y is S^2, we have $g' = 0$, so (1.6) becomes

$$
2g = \left(\sum_{i=1}^{m} n - l(\sigma_i)\right) - l(\alpha) - n + 2. \tag{1.7}
$$

Thus, the combinatorial problem that is equivalent to Hurwitz's Problem is the following.

Problem 1 ***(Hurwitz)*** *Given partitions $\alpha, \sigma_1, \ldots, \sigma_m$ that satisfy (1.7), and a permutation $\pi \in S_n$ of cycle type α, find the number of transitive factorizations of π with factors $\pi_1, \ldots, \pi_m$, where π_i has cycle type σ_i for every $1 \leq i \leq m$.*

Since the smallest value of g is 0, corresponding to the sphere, then a factorization that satisfies (1.7) for $g = 0$ is called a *minimal* factorization.

1.3 Outline of the Thesis

In this thesis we shall be concerned with two special instances of the Hurwitz problem. The first is the case in which all the branch points are simple branch points, except for one of them. In this case, all the factors π_i are transpositions and π has arbitrary cycle type, say α. Thus let H_α^g be the number of minimal transitive factorizations of a fixed permutation of cycle type α into transpositions, which we refer to as the *Hurwitz number* . We shall consider the minimal case of this problem, namely finding H_α^0, which we write as H_α. In geometric terms, this is the number of ramified covers of the sphere by a sphere. Note that if σ_i is a transposition, then $l(\alpha) = n - 1$, so by setting $g = 0$ in the Riemann-Hurwitz formula for the sphere (1.7), the minimality condition in this instance is

$$m = n + l(\alpha) - 2. \tag{1.8}$$

An explicit formula for H_α has been given by Goulden and Jackson in [5], and we describe their solution in Chapter 2. Central to this presentation is a partial differential equation for the generating series for the H_α called the Join-Cut equation, which has a simple combinatorial interpretation in terms of permutations. We conclude Chapter 2 with another, much simpler partial differential equation for the series, but for which we have no combinatorial interpretation. For both equations, the presentation has a substantial algebraic component, using Lagrange's theorem in an essential and nontrivial way.

The second case on which we shall focus is that in which all the factors π_i are arbitrary, and unspecified permutations. Thus, let $G_\alpha^g(m)$ be the number of transitive factorizations of a permutation of cycle type α into m factors. Again, we shall consider the minimal case of this problem, namely finding $G_\alpha^0(m)$, which we write as $G_\alpha(m)$. Note that if the factors have cycle types $\sigma_1, \ldots, \sigma_m$, then they have $l(\sigma_1), \ldots, l(\sigma_m)$ cycles, respectively, so by setting $g = 0$ in the Riemann-Hurwitz formula for the sphere (1.7), the minimality condition in this instance is

$$0 = \sum_{i=1}^{m} [n - l(\sigma_i)] - l(\alpha) - n + 2,$$

which gives

$$\sum_{i=1}^{m} l(\sigma_i) = n(m-1) - l(\alpha) + 2. \tag{1.9}$$

An explicit formula for G_α has been given by Bousquet-Mélou and Schaeffer in [1]. Their method is to give a direct bijection between the appropriate factorizations and a class of trees called m-Eulerian trees, via intermediate combinatorial objects that are maps in the plane. In this thesis we give a substantially different proof, which parallels the presentation of Chapter 2. In particular, we introduce Lagrange's Theorem in an essential way, and determine a new simple partial differential equation for the generating series. Our combinatorial interpretation for this partial differential equation is based on Bousquet-Mélou and Schaeffer's m-Eulerian trees. We have been unable to find a combinatorial interpretation of this partial differential equation in terms of permutations

In Chapter 4, we extend the method of Chapter 3 in order to keep track of the number of cycles of the factors $\sigma_1, \ldots, \sigma_m$. This involves the multivariable form of Lagrange's Theorem. Although our results are incomplete, we hope that, eventually, they will contribute to a full solution of this refined problem.

Table 1.1: Table of Notation

$R[[\mathbf{x}]]$	The ring of formal power series in $\mathbf{x}$	2
$R((\mathbf{x}))$	The ring of formal Laurent series in $\mathbf{x}$	3
$[\mathbf{x}^\alpha]f$	Coefficient of $\mathbf{x}^\alpha$ in the power series f	2
S_n	The symmetric group on n symbols	1
$\alpha \vdash n$	A partition α of n	2
$\|\alpha\|$	Size of a partition α	1
$l(\alpha)$	Length of a partition α	2
$\|\pi\|$	Degree of a permutation π	1
$l(\pi)$	Length of a permutation π	2
$v(f, x)$	Multiplicity of f at the point x	4
$b(f, x)$	Branching order of f at the point x	4
h^α	Size of the conjugacy class of permutations of type α	2
$\mathbf{p}$	Vector $(p_1, p_2, \ldots)$	9
H_α, H_α^0	Hurwitz number for genus 0	6
H_α^g	Hurwitz number for genus g	6
$H, \widehat{H}, \widetilde{H}$	Generating series for the Hurwitz numbers	10
$G_\alpha(m)$	Number of minimal, transitive factorizations of π of cycle type α	25
$G, \widehat{G}, \widetilde{G}$	Generating series for the numbers G_α	26
$T_\alpha(m)$	Number of m-Eulerian trees of type α	53
T	Generating series for the numbers T_α	53

Chapter 2

Factorizations of a Permutation into Transpositions

2.1 Generating Series for the Hurwitz Numbers

The Hurwitz number H_α is defined in Section 1.2. Our aim is to prove the following theorem.

Theorem 4 *Let α be a partition of n with d_i parts equal to i, $i \geq 1$. Then*

$$H_\alpha = n^{l(\alpha)-3}[n + l(\alpha) - 2]! \prod_{i \geq 1} \left(\frac{i^i}{(i-1)!} \right)^{d_i}.$$

We now present the proof given by Goulden and Jackson in [5]. This proof has two components, one algebraic, and the other combinatorial. The algebraic portion makes use of Lagrange's Theorem, and the combinatorial portion is based on a recursive analysis of multiplying transpositions. First, let $p_1, p_2, \ldots$ be indeterminates. Define the vector $\mathbf{p} = (p_1, p_2, \ldots)$ and for a partition α, let $p_\alpha = p_{\alpha_1} p_{\alpha_2} \cdots$. Recall that $h^\alpha = \frac{n!}{\prod_{i \geq 1} d_i! i^{d_i}}$ is the size of the conjugacy class of permutations of cycle type α.

To begin, define $\widehat{H}_\alpha$ by

$$\widehat{H}_\alpha = n^{l(\alpha)-3}[n+l(\alpha)-2]! \prod_{i\geq 1} \left(\frac{i^i}{(i-1)!}\right)^{d_i},$$

and the generating series H and $\widehat{H}$ by

$$H = \sum_{n\geq 1}\sum_{\alpha\vdash n} H_\alpha \frac{z^n}{n!}\frac{1}{[n+l(\alpha)-2]!}h^\alpha p_\alpha, \quad \widehat{H} = \sum_{n\geq 1}\sum_{\alpha\vdash n} \widehat{H}_\alpha \frac{z^n}{n!}\frac{1}{[n+l(\alpha)-2]!}h^\alpha p_\alpha.$$

Note that, in the generating series H, z is an exponential indeterminate marking the elements in S_n and p_i is an ordinary indeterminate marking for the cycles of length i in π.

2.2 A Partial Differential Equation

Our aim is to prove Theorem 4 by proving that $H = \widehat{H}$. More specifically, we shall prove that H and $\widehat{H}$ satisfy the same partial differential equation with the same initial conditions. We begin with the algebraic portion of the argument, by determining a partial differential equation for $\widehat{H}$. We refer to this equation as the *Join-Cut equation* , for reasons that will be clear later.

Theorem 5 *The generating series $\widehat{H}$ satisfies the equation*

$$\frac{1}{2}\sum_{i,j\geq 1}\left(p_{i+j}\, i\frac{\partial \widehat{H}}{\partial p_i}\, j\frac{\partial \widehat{H}}{\partial p_j} + (i+j)p_ip_j\frac{\partial \widehat{H}}{\partial p_{i+j}}\right) = z\frac{\partial \widehat{H}}{\partial z} + \sum_{k\geq 1} p_k\frac{\partial \widehat{H}}{\partial p_k} - 2\widehat{H}. \tag{2.1}$$

The proof of Theorem 5 is delayed, since it requires some preliminary technical results.

Theorem 6 *Let s be defined by the equation*

$$s = z\exp f(s), \quad \textit{where} \quad f(s) = \sum_{i\geq 1}\frac{i^i}{i!}p_i s^i. \tag{2.2}$$

Then the following hold:

$$\left(z\frac{\partial}{\partial z}\right)^2 \widehat{H} = \sum_{i\geq 1}\frac{i^i}{i!}p_i s^i, \tag{2.3}$$

$$z\frac{\partial \widehat{H}}{\partial z} = \sum_{i\geq 1}\frac{i^{i-1}}{i!}p_i s^i - \frac{1}{2}\left(\sum_{i\geq 1}\frac{i^i}{i!}p_i s^i\right)^2, \tag{2.4}$$

and for $k \geq 1$*,*

$$z\frac{\partial^2 \widehat{H}}{\partial p_k \partial z} = \frac{k^{k-1}}{k!}s^k, \tag{2.5}$$

$$\frac{\partial \widehat{H}}{\partial p_k} = \frac{k^{k-2}}{k!}s^k - \frac{k^{k-1}}{k!}\sum_{i\geq 1}\frac{i^{i+1}}{i!}p_i\frac{s^{k+i}}{k+i}. \tag{2.6}$$

PROOF. Note that we can reexpress the generating series in the following way.

$$\begin{aligned}
\widehat{H} &= \sum_{n\geq 1}\sum_{\alpha\vdash n}\widehat{H}_\alpha\frac{z^n}{n!}\frac{1}{\mu(\alpha)!}h^\alpha p_\alpha \\
&= \sum_{n\geq 1}\sum_{d_1+2d_2+\cdots=n} n^{l(\alpha)-3}[n+l(\alpha)-2]!\prod_{i\geq 1}\left[\frac{i^i}{(i-1)!}\right]^{d_i}\frac{z^n}{n!}\frac{1}{[n+l(\alpha)-2]!}\frac{n!}{\prod_{i\geq 1}d_i!i^{d_i}}\prod_{i\geq 1}p_i^{d_i} \\
&= \sum_{n\geq 1}\frac{z^n}{n^3}\sum_{d_1+2d_2+\cdots=n}\prod_{i\geq 1}\left[\frac{i^i np_i}{(i-1)!i}\right]^{d_i}\frac{1}{d_i!} \\
&= \sum_{n\geq 1}\frac{z^n}{n^3}[\lambda^n]\prod_{i\geq 1}\left[\sum_{k\geq 0}\frac{1}{k!}\left(\frac{i^i np_i\lambda^i}{i!}\right)^k\right] \\
&= \sum_{n\geq 1}\frac{z^n}{n^3}[\lambda^n]\prod_{i\geq 1}\exp\left(\frac{i^i np_i\lambda^i}{i!}\right) \\
&= \sum_{n\geq 1}\frac{z^n}{n^3}[\lambda^n]\left(\exp\sum_{i\geq 1}\frac{i^i p_i\lambda^i}{i!}\right)^n.
\end{aligned}$$

By (2.2) we have

$$\begin{aligned}
\widehat{H} &= \sum_{n\geq 1}\frac{z^n}{n^3}[\lambda^n]\{\exp f(\lambda)\}^n \\
&= \sum_{n\geq 1}\frac{z^n}{n^3}\frac{1}{n}[\lambda^{n-1}]n\exp[f(\lambda)]f^{'}(\lambda)\{\exp f(\lambda)\}^{n-1} \\
&= \sum_{n\geq 1}\frac{z^n}{n^2}\frac{1}{n}[\lambda^{n-1}]f^{'}(\lambda)\{\exp f(\lambda)\}^n,
\end{aligned}$$

so by Theorem 1,

$$\widehat{H} = \sum_{n \geq 1} \frac{z^n}{n^2}[z^n]f(s), \tag{2.7}$$

from which (2.3) follows immediately.

To prove (2.4), apply $z\frac{\partial}{\partial z}$ to (2.2), giving

$$z\frac{\partial s}{\partial z} = z \exp f(s) + z \exp[f(s)]f'(s)z\frac{\partial s}{\partial z},$$

and using (2.2) again, obtain

$$z\frac{\partial s}{\partial z} = s + sf'(s)z\frac{\partial s}{\partial z},$$

or in a simpler form,

$$\frac{s}{z} = [1 - sf'(s)]\frac{\partial s}{\partial z}. \tag{2.8}$$

Now, apply $\int \frac{1}{z}dz$ to (2.3) and change variables to s, to get,

$$\begin{aligned}
\left(z\frac{\partial}{\partial z}\right)\widehat{H} &= \int_0^z \frac{1}{z}\left(z\frac{\partial}{\partial z}\right)^2 \widehat{H}dz \\
&= \int_0^z \frac{1}{z}f(s)dz \\
&= \int_0^z \frac{s}{z}\frac{f(s)}{s}dz \\
&= \int_0^z [1 - sf'(s)]\frac{\partial s}{\partial z}\frac{f(s)}{s}dz \quad \text{using (2.8)} \\
&= \int_0^s \frac{f(s)}{s}ds - \int_0^s f(s)f'(s)ds \\
&= \int_0^s \sum_{i\geq 1}\frac{i^i p_i s^{i-1}}{i!}ds - \frac{1}{2}f(s)^2 \\
&= \sum_{i\geq 1}\frac{i^{i-1}p_i s^i}{i!} - \frac{1}{2}\left(\sum_{i\geq 1}\frac{i^i p_i s^i}{i!}\right).
\end{aligned}$$

This proves (2.4).

To prove (2.5), apply $\frac{\partial}{\partial p_k}$ to (2.2), to obtain

$$\begin{aligned}\frac{\partial s}{\partial p_k} &= z\exp[f(s)]\frac{\partial}{\partial p_k}f(s)\\ &= s\left(\frac{k^k}{k!}s^k+\sum_{i\geq 1}\frac{i^i}{i!}p_i\frac{\partial s}{\partial p_k}\right),\end{aligned}$$

and solve for $\frac{\partial s}{\partial p_k}$ to get

$$\begin{aligned}\frac{\partial s}{\partial p_k} &= \frac{k^k}{k!}s^{k+1}\left(1-\sum_{i\geq 1}\frac{i^{i+1}}{i!}p_i s^i\right)^{-1}\\ &= \frac{k^k}{k!}s^{k+1}[1-f'(s)].\end{aligned}$$

Comparing this expression and (2.8), we obtain

$$\frac{\partial s}{\partial p_k}=\frac{k^k}{k!}s^k z\frac{\partial s}{\partial z}. \tag{2.9}$$

Now, apply $z\frac{\partial}{\partial z}$ to (2.3), and replace using (2.2), to obtain

$$z\frac{\partial \widehat{H}}{\partial z}=\sum_{n\geq 1}\frac{1}{n}z^n[z^n]\log\left(\frac{s}{z}\right), \tag{2.10}$$

and then apply $\frac{\partial}{\partial p_k}$, to get, for $n \geq 1, k \geq 1$,

$$\begin{aligned}[z^n]z\frac{\partial^2\widehat{H}}{\partial p_k\partial z} &= \frac{1}{n}[z^n]\frac{1}{s}\frac{\partial s}{\partial p_k}\\ &= \frac{1}{n}[z^n]\frac{1}{s}\frac{k^k}{k!}s^k z\frac{\partial s}{\partial z} \quad \text{by (2.9)}\\ &= \frac{1}{n}[z^{n-1}]\frac{k^k}{k!}s^{k-1}\frac{\partial s}{\partial z}\\ &= [z^n]\int_0^z\frac{k^k}{k!}s^{k-1}\frac{\partial s}{\partial z}dz\\ &= [z^n]\frac{k^{k-1}}{k!}s^k.\end{aligned}$$

This proves (2.5).

To prove (2.6), apply $\int \frac{1}{z} dz$ to (2.5) with respect to z to obtain

$$\begin{aligned}
\frac{\partial}{\partial p_k}\widehat{H} &= \int_0^z \frac{1}{z}\frac{k^{k-1}}{k!}s^k dz \\
&= \frac{k^{k-1}}{k!}\int_0^z \frac{s}{z}s^{k-1}dz \\
&= \frac{k^{k-1}}{k!}\int_0^z \left[1-sf'(s)\right]\frac{\partial s}{\partial z}s^{k-1}dz \quad \text{by (2.8)} \\
&= \frac{k^{k-1}}{k!}\int_0^s s^{k-1}ds - \frac{k^{k-1}}{k!}\int_0^s s^k f'(s)ds \\
&= \frac{k^{k-1}}{k!}\frac{s^k}{k} - \frac{k^{k-1}}{k!}\int_0^s s^k \sum_{i\geq 1}\frac{i^{i+1}}{i!}p_i s^{i-1}ds \\
&= \frac{k^{k-2}}{k!} - \frac{k^{k-1}}{k!}\sum_{i\geq 1}\frac{i}{i+k}\frac{i^i}{i!}p_i s^{i+k},
\end{aligned}$$

thus proving (2.6). □

In order to prove Theorem 5, we also need the following theorem.

Theorem 7 *1. For $m \geq 1$, let*

$$S_m = \sum_{\substack{i,j\geq 1\\ i+j=m}} \frac{i^i}{i!}\frac{j^{j-1}}{j!}.$$

Then

$$S_m = \frac{m^m}{m!} - \frac{m^{m-1}}{m!}.$$

2. For $k, m \geq 1$, let

$$T_{k,m} = \frac{k^{k+1}}{k!}\sum_{\substack{i\geq 1, j\geq 0\\ i+j=m}} \frac{i^i}{i!}\frac{j^j}{j!}\frac{1}{k+j}.$$

Then

$$T_{k,m} + T_{m,k} = \frac{(k+m)^{k+m}}{(k+m)!}.$$

PROOF. 1. Define w to be the unique formal solution of $w = ze^w$. Therefore by Theorem 1, $w = \sum_{i\geq 1} \frac{i^{i-1}}{i!} z^i$. Applying $z\frac{\partial}{\partial z}$, we get

$$\begin{aligned} z\frac{\partial w}{\partial z} &= ze^w + z^2 e^w \frac{\partial w}{\partial z} \\ &= w + z^2 e^w \frac{\partial w}{\partial z}. \end{aligned}$$

Solving for $z\frac{\partial w}{\partial z}$, we get

$$z\frac{\partial w}{\partial z} = \frac{w}{1-w} = \sum_{i\geq 1} \frac{i^i}{i!} z^i. \tag{2.11}$$

Therefore,

$$\begin{aligned} S_m &= [z^m]\left(\sum_{i\geq 1} \frac{i^{i-1}}{i!} z^i\right)\left(\sum_{i\geq 1} \frac{i^i}{i!} z^i\right) \\ &= [z^m]\frac{w}{1-w} w = [z^m]\frac{w}{1-w} - w \\ &= \frac{m^m}{m!} - \frac{m^{m-1}}{m!}, \end{aligned}$$

as desired.

2. Let $u = w(x)$ and $v = w(y)$, so

$$u = xe^u, \quad v = ye^v. \tag{2.12}$$

Using the same reasoning as for (2.11) above, we obtain

$$\frac{v}{1-v} = \sum_{j\geq 1} \frac{j^i}{j!} y^j. \tag{2.13}$$

Adding 1 to both sides and multiplying by y^{k-1} we obtain

$$\frac{y^{k-1}}{1-v} = \sum_{j\geq 0} \frac{j^j}{j!} y^{j+k-1},$$

and using the fact that $\int_0^y y^{j+k-1}dy = \frac{y^{j+k}}{j+k}$, we get

$$\frac{1}{y^k}\int_0^y \frac{y^{k-1}}{1-v}dy = \sum_{j\geq 0}\frac{j^i}{j!}\frac{y^j}{j+k}. \tag{2.14}$$

By a similar reasoning as in (2.11), we get

$$\frac{v}{1-v}\frac{\partial y}{\partial v} = y,$$

so multiplying both sides of the equation by y^k and rearranging, we obtain

$$\begin{aligned} \frac{y^{k-1}}{1-v}dy &= y^k\frac{dv}{v} \\ &= v^{k-1}e^{-kv}dv \quad \text{by (2.12).} \end{aligned}$$

Now, integrate to obtain

$$\begin{aligned} \int_0^y \frac{y^{k-1}}{1-v}dy &= \int_0^v v^{k-1}e^{-kv}dv \\ &= k^{-k}\int_0^{kv} t^{k-1}e^{-t}dt \\ &= \frac{k!}{k^{k+1}}\left(1 - e^{-kv}\sum_{i=1}^{k}\frac{k^{k-i}}{(k-i)!}v^{k-i}\right) \\ &= \frac{k!}{k^{k+1}}\left(1 - y^k\sum_{i=1}^{k}\frac{k^{k-i}}{(k-i)!}\frac{1}{v^i}\right). \end{aligned} \tag{2.15}$$

Now, define $T(x,y)$ as

$$
\begin{aligned}
T(x,y) &= \sum_{k,m\geq 1} T_{k,m} x^k y^m \\
&= \sum_{k\geq 1} \frac{k^{k+1}}{k!} x^k \left(\sum_{i\geq 1} \frac{i^i}{i!} y^i \right) \left(\sum_{j\geq 0} \frac{j^i}{j!} \frac{y^j}{j+k} \right) \\
&= \sum_{k\geq 1} \frac{k^{k+1}}{k!} x^k \frac{v}{1-v} \frac{1}{y^k} \int_0^y \frac{y^{k-1}}{1-v} dy \quad \text{from (2.14)} \\
&= \sum_{k\geq 1} \frac{k^{k+1}}{k!} x^k \frac{v}{1-v} \frac{1}{y^k} \frac{k!}{k^{k+1}} \left(1 - y^k \sum_{i=1}^{k} \frac{k^{k-i}}{(k-i)!} \frac{1}{v^i} \right) \quad \text{from (2.15)} \\
&= \frac{v}{1-v} \sum_{k\geq 1} \left(\left(\frac{x}{y} \right)^k - x^k \sum_{i=1}^{k} \frac{k^{k-i}}{(k-i)!} \frac{1}{v^i} \right) \\
&= \frac{v}{1-v} \left(\frac{x}{y-x} - \sum_{k\geq 1} x^k [\lambda^k] \left(e^{k\lambda} \frac{\lambda}{v-\lambda} \right) \right)
\end{aligned}
$$

Then, by Theorem 1, we have

$$
[\lambda^k] \frac{\lambda}{v-\lambda} e^{k\lambda} = [x^k] \frac{\frac{u}{v-u}}{1-xe^u} = [x^k] \frac{\frac{u}{v-u}}{1-u},
$$

so

$$
T(x,y) = \frac{v}{1-v} \left(\frac{x}{y-x} - \frac{u}{v-u} \frac{1}{1-u} \right).
$$

Therefore

$$
\begin{aligned}
T(x,y) + T(y,x) &= \frac{1}{y-x} \left(\frac{xv}{1-v} - \frac{yu}{1-u} \right) \\
&= \sum_{l\geq 1} \frac{l^l}{l!} \frac{xy^l - yx^l}{y-x} \quad \text{by (2.13)} \\
&= \sum_{l\geq 1} \frac{l^l}{l!} \sum \overset{r,s\geq 1}{r+s=lx^r y^s},
\end{aligned}
$$

so

$$
T_{k,m} + T_{m,k} = [x^k y^m](T(x,y) + T(y,x)) = \frac{(k+m)^{k+m}}{(k+m)!}
$$

as desired. □

Now we are able to prove Theorem 5. Note that subtracting the right hand side from the left hand side of (2.1), and applying $z\frac{\partial}{\partial z}$ gives

$$\frac{1}{2}\sum_{i,j\geq 1} p_{i+j}\left(i\frac{z\partial^2\widehat{H}}{\partial p_i\partial z}j\frac{\partial\widehat{H}}{\partial p_j}+i\frac{\partial\widehat{H}}{\partial p_i}j\frac{z\partial^2\widehat{H}}{\partial p_j\partial z}\right)+\frac{1}{2}\sum_{i,j\geq 1} z\frac{\partial^2\widehat{H}}{\partial p_{i+j}\partial z}$$

$$-\left(z\frac{\partial}{\partial z}\right)^2\widehat{H}-\sum_{i\geq 1} p_i\frac{z\partial^2\widehat{H}}{\partial p_i\partial z}+2z\frac{\partial\widehat{H}}{\partial z}.$$

After using the symmetry between i and j to cancel the first factor of $\frac{1}{2}$ and applying (2.3), (2.4), (2.5), and (2.6), we get

$$\sum_{i,j\geq 1} p_{i+j}\left(i\frac{i^{i-2}}{i!}s^i-\frac{i^{i-1}}{i!}\sum_{k\geq 1}\frac{k^{k+1}}{k!}p_k\frac{s^{i+k}}{i+k}\right)j\frac{j^{j-1}}{j!}s^j$$

$$+\frac{1}{2}\sum_{i,j\geq 1}(i+j)p_ip_j\frac{(i+j)^{i+j-1}}{(i+j)!}s^{i+j}$$

$$-\sum_{i\geq 1}\frac{i^i}{i!}p_is^i-\sum_{i\geq 1}p_i\frac{i^{i-1}}{i!}s^i+s\left[\sum_{i\geq 1}\frac{i^{i-1}}{i!}p_is^i-\frac{1}{2}\left(\sum_{i\geq 1}\frac{i^i}{i!}p_is^i\right)^2\right].$$

Extracting coefficients and doing some algebraic manipulation, the above expression becomes

$$\sum_{m\geq 1}\left(\sum_{\substack{i,j\geq 1\\ i+j=m}}\frac{i^i}{i!}\frac{j^{j-1}}{j!}-\frac{m^m}{m!}+\frac{m^{m-1}}{m!}\right)p_ms^m+\sum_{m\geq 1}\left(\frac{1}{2}\frac{(2m)^{2m}}{(2m)!}-\frac{m^{m+1}}{m!}\sum_{\substack{i\geq 1,j\geq 0\\ i+j=m}}\frac{i^i}{i!}\frac{j^j}{j!}\frac{1}{m+j}\right)p_m^2s^{2m}$$

$$+\sum_{\substack{k,m\geq 1\\ k\neq m}}\left(\frac{(k+m)^{k+m}}{(k+m)!}-\frac{k^{k+1}}{k!}\sum_{\substack{i\geq 1,j\geq 0\\ i+j=m}}\frac{i^i}{i!}\frac{j^j}{j!}\frac{1}{k+j}-\frac{m^{m+1}}{m!}\sum_{\substack{i\geq 1,j\geq 0\\ i+j=k}}\frac{i^i}{i!}\frac{j^j}{j!}\frac{1}{m+j}\right)p_kp_ms^{k+m}.$$

Finally, recalling the definitions of S_m and $T_{k,m}$ we get

$$\sum_{m\geq 1}\left(S_m - \frac{m^m}{m!} + \frac{m^{m-1}}{m!}\right)p_m s^m + \left(\frac{1}{2}\frac{(2m)^{2m}}{(2m)!} - T_{m,m}\right)p_m^2 s^{2m} + \sum_{\substack{k,m\geq\\ k\neq m}}\left(\frac{(k+m)^{k+m}}{(k+m)!} - T_{k,m} - T_{m,k}\right)p_k p_m s^{k+m},$$

which by Theorem 7 is equal to 0.

Note that we have applied $z\frac{\partial}{\partial z}$ in the above analysis, so to establish that (2.1) holds, we must check that both sides agree at $z = 0$. Indeed, because $\widehat{H} = 0$ at $z = 0$, both sides of (2.1) are equal to 0 at $z = 0$, which proves (2.1), and completes the proof of Theorem 5.

2.3 The Join-Cut Equation

Now we give the combinatorial portion of our argument, by proving that H also satisfies the Join-Cut equation.

Theorem 8 *The generating series H satisfies the Join-Cut equation (2.1).*

PROOF. First, we need to analyse the action of multiplying by a transposition. Let π be a permutation in S_n. Let σ be the transposition (a, b). We shall consider two cases, namely when a and b belong to the same cycle, or to different cycles in the disjoint cycle representation of π. Note that if a and b belong to the same cycle in π, then multiplying by σ cuts this cycle into two cycles, so $l(\sigma\pi) = l(\pi) + 1$. In this case, σ is called a *cut* . If a and b belong to different cycles, then multiplying by σ joins these two cycles into one, so $l(\sigma\pi) = l(\pi) - 1$. In this case, σ is called a *join* .

For a transitive factorization $(\sigma_m, \sigma_{m-1}, \ldots, \sigma_1)$ of π into transpositions, define the graph $G(\sigma_m, \sigma_{m-1}, \ldots, \sigma_1)$ in the following way: The vertex set is $\{1, 2, \ldots, n\}$ and if transposition σ_k exchanges vertices r and s, there is an edge labelled k between the vertices labelled r and s, for $k = 1, \ldots, m$.

Now construct a spanning tree T of G by adding edges in increasing order starting from σ_1 as long as the edge added doesn't create a cycle. Therefore each edge joins two different connected components, so its permutation is a join. Since the factorization is transitive, then G is connected, which implies that T has $n-1$ edges, all joins.

Note that in this setting, z is an exponential indeterminate marking the vertices in G, and p_i is an ordinary indeterminate marking the cycles of length i in the permutation π. We consider the action of removing the transposition σ_m from the product $\pi = \sigma_m \cdots \sigma_1$.

Since we have established in (1.8) that the number of transpositions needed is $m = n+l(\alpha)-2$, then this action can be written as

$$z\frac{\partial G}{\partial z} + \sum_{k\geq 1} p_k \frac{\partial G}{\partial p_k} - 2G. \tag{2.16}$$

Now we consider two cases, namely when the transposition is a join or a cut.

If the transposition is a join, then the graph $G(\sigma_{m-1}, \ldots, \sigma_1)$ has two components, so the edge corresponding to σ_m is in the spanning tree T. The two disjoint graphs obtained by deleting the edge corresponding to σ_m are minimal, transitive factorizations of a permutation on the set of vertices of each subgraph, and so the contribution to the generating series is

$$\frac{1}{2}\sum_{i,j\geq 1} p_{i+j}\left(i\frac{\partial G}{\partial p_i}\right)\left(j\frac{\partial G}{\partial p_j}\right), \tag{2.17}$$

where i and j are the lengths of the cycles that we are joining, and there are ij ways of choosing the pair of vertices that were joined by the deleted edge in each cycle.

If the transposition is a cut, then σ_m corresponds to an edge in the graph which is not in the spanning tree T, and since it is cutting a cycle say of length $i+j$ into two cycles of lengths i and j, then the contribution to the generating series is

$$\frac{1}{2}\sum_{i,j\geq 1} p_i p_j (i+j)\frac{\partial G}{\partial p_{i+j}}. \tag{2.18}$$

The result follows from (2.16), (2.17), and (2.18). □

Thus, from Theorems 5 and 8, we can conclude that H and $\widehat{H}$ satisfy the same partial differential equation. Now we must prove that they satisfy the same initial conditions. Note that the initial condition is when $n = 1$ and $\alpha = (1)$. Combinatorially, this corresponds to the number of empty factorizations of the identity permutation in S_1, which is clearly 1, so

$$[zp_1]H = 1.$$

Algebraically, from the definition of $\widehat{H}_\alpha$, we see that

$$[zp_1]\widehat{H} = 1.$$

Thus $H = \widehat{H}$, which completes the proof of Theorem 4.

2.4 Generalizations to Higher Genus and the ELSV Equation

In order to generalize to surfaces of higher genus, as has been done by Goulden and Jackson in [6], a third operator must be considered. Recall that the join operator represents the action of multiplying two cycles that haven't already been joined, by a transposition. The new join operator represents the action of multiplying two cycles that have already been joined, by a transposition, and thus it can be written as

$$p_{i+j}\frac{\partial^2}{\partial p_i \partial p_j}.$$

As a consequence of the Riemann-Hurwitz formula (1.6), there are g of these new joins, where g is the genus of the surface. Therefore, we shall add a variable x marking this operator, and keeping track of the genus of the surface. Also, note that in the context of section 2.3, since we are joining two vertices in different cycles of lengths i, and j, there are ij ways of choosing these pair of vertices. Therefore, defining H^g by

$$H^g = \sum_{n\geq 1}\sum_{\alpha\vdash n} H^g_\alpha \frac{z^n}{n!}\frac{1}{[n+l(\alpha)+2g-2]!}h^\alpha p_\alpha,$$

the differential equation that must be considered is

$$z\frac{\partial H^g}{\partial z}+\sum_{k\geq 1}p_k\frac{\partial H^g}{\partial p_k}+(2g-2)H^g = \frac{1}{2}\sum_{i,j\geq 1}\left(ijp_{i+j}x\frac{\partial^2 H^g}{\partial_i\partial p_j}+ijp_{i+j}\frac{\partial H^g}{\partial p_i}\frac{\partial H^g}{\partial p_j}+(i+j)p_ip_j\frac{\partial H^g}{\partial_{i+j}}\right). \tag{2.19}$$

This generating series has been considered in [6], where the following result has been obtained.

Theorem 9 *For $\alpha\vdash n$, with $\alpha=(\alpha_1,\dots,\alpha_m)$,*

$$H^1_\alpha = \frac{h^\alpha}{24n!}(n+m)!\left(\prod_{i=1}^m\frac{\alpha_i^{\alpha_i}}{(\alpha_i-1)!}\right)\left(n^m-n^{m-1}-\sum_{i=2}^m(i-2)!e_in^{m-i}\right)$$

where e_i is the i-th elementary symmetric function in $\alpha_1,\dots,\alpha_m$ and $e_1=\alpha_1+\cdots+\alpha_m=n$.

The following is a remarkable connection between the number of ramified covers of the sphere and some properties of the moduli space of curves.

Let $\mathcal{M}_{g,n}$ be the *moduli space of n-pointed smooth curves of genus g* . Let $\overline{\mathcal{M}}_{g,n}$ be its *Deligne-Mumford compactification* . This compactification is defined by adding the points corresponding to nodal curves, namely those curves in which the points are either smooth, or contained in a neighbourhood analytically isomorphic to the neighbourhood around the origin of the curve $xy=0$ in $\mathbb{C}^2$. This space is smooth, compact, irreducible, and it has dimension $3g-3+n$. The *stability condition*

$$2g-2+n$$

assures that the automorphism group is finite.

Let c_i be the i-th *Chern class* . There are n natural line bundles in $\overline{\mathcal{M}}_{g,n}$, namely $\mathbb{L}_1,\dots,\mathbb{L}_n$. These are the one-dimensional cotangent spaces to the curve at each one of the n marked points. Define the ψ-classes by $\psi_i=c_i(\mathbb{L}_i)$. There is also a natural vector bundle on $\overline{\mathcal{M}}_{g,n}$ called the *Hodge bundle* . It corresponds to the g-dimensional space of differentials of a curve. As the definition is extended over the boundary, one obtains a rank g vector bundle $\mathbb{E}$ over $\overline{\mathcal{M}}_{g,n}$. Define the λ-classes by $\lambda_k = c_k(\mathbb{E})$. The intersections of ψ classes and λ classes are called *Hodge integrals* . The following remarkable formula, found by Ekedahl, Lando, Shapiro, and Vainshtein in [2], shows the relation between Hurwitz numbers and Hodge integrals.

Theorem 10 ***(ELSV Formula)*** *Let H^g_α be the number of ramified coverings of a surface of genus g with n sheets and ramification α, for $\alpha \vdash n$, where $\alpha = (\alpha_1, \ldots, \alpha_m)$. Let $r = n + m + 2(g-1)$. Then*

$$H^g_\alpha = \frac{r!}{|Aut(\alpha)|} \prod_{i=1}^{m} \frac{\alpha_i^{\alpha_i}}{\alpha_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{1 - \lambda_1 + \cdots + (-1)^g \lambda_g}{\prod_i (1 - \alpha_i \psi_i)},$$

where the denominator should be interpreted as a formal power series in the ϕ_i.

2.5 A Simple Quadratic Partial Differential Equation

The Join-Cut equation is not the only quadratic partial differential equation that holds for $\widehat{H}$. The following result gives a simple quadratic partial differential equation for $\widehat{H}$, that follows immediately from Theorem 6.

Theorem 11

$$z\frac{\partial \widehat{H}}{\partial z} = \sum_{k \geq 1} z p_k \frac{\partial^2 \widehat{H}}{\partial p_k \partial z} - \frac{1}{2}\left[\left(z\frac{\partial}{\partial z}\right)^2 \widehat{H}\right]^2.$$

PROOF. Substitute (2.3) and (2.5), after adding over all k, into (2.4). □

Theorem 11 is due to Goulden and Jackson (private communication). Now substitute $p_i \mapsto u p_i$, for $i \geq 1$, into $\widehat{H}$ to obtain $\widetilde{H}$, and obtain a second partial differential equation in two variables.

Theorem 12

$$z\frac{\partial}{\partial z}\widetilde{H} = uz\frac{\partial^2}{\partial u \partial z}\widetilde{H} - \frac{1}{2}\left[\left(z\frac{\partial}{\partial z}\right)^2 \widetilde{H}\right]^2.$$

PROOF. Note that the map between $\widehat{H}$ and $\widetilde{H}$ also affects the operators in the following way:

$$p_k \frac{\partial \widehat{H}}{\partial p_k} \mapsto q_k \frac{\partial \widetilde{H}}{\partial q_k}, \tag{2.20}$$

where $q_k = up_k$. But

$$q_k \frac{\partial}{\partial q_k} = up_k \frac{\partial}{u \partial p_k} = p_k \frac{\partial}{\partial p_k}.$$

The result follows by substituting (2.20) into Theorem 11. □

Note that in this context, u keeps track of the cycles in π. The apparent simplicity of Theorem 12, especially as it involves only the two variables z and u, suggests that there may be a simple combinatorial interpretation. This would provide an alternative to the Join-Cut analysis in the combinatorial portion of our argument. We have looked for such a combinatorial interpretation, but have been unable to find one.

Chapter 3

Factorizations of a Permutation into Factors of Arbitrary Type

3.1 Generating Series

The number $G_\alpha(m)$ is defined in Section 1.2. Our aim is to prove the following theorem.

Theorem 13 *Let α be a partition of n with d_i cycles of length i for every $i \geq 0$. The number $G_\alpha(m)$ of minimal, transitive factorizations of a permutation π of cycle type α into m factors $\pi_1, \ldots, \pi_m$ is*

$$G_\alpha(m) = m\frac{\{(m-1)n-1\}!}{\{(m-1)n-l(\alpha)+2\}!}\prod_{i\geq 1}\left\{i\binom{mi-1}{i}\right\}^{d_i}.$$

Theorem 13 has been proved by Bousquet-Mélou and Schaeffer in [1], by giving a direct bijection between these factorizations, and some particular objects called m-Eulerian trees, followed by a counting argument for these trees. Bousquet-Mélou and Schaeffer's proof has essentially no algebraic component, and in fact does not consider any generating series for the number $G_\alpha(m)$. In this thesis, we give a different proof, which differs in the details, but is parallel to the proof of Theorem 4 in Chapter 2, for Hurwitz numbers. Our proof has an algebraic portion, applying Lagrange's theorem to a generating series, and a combinatorial portion, which is a recursive analysis of Bousquet-Mélou and Schaeffer's m-Eulerian trees. This recursive analysis is also not part of Bousquet-Mélou and Schaeffer's presentation. To begin, define $\widehat{G}_\alpha(m)$ by

$$\widehat{G}_\alpha(m) = m\frac{\{(m-1)n-1\}!}{\{(m-1)n-l(\alpha)+2\}!}\prod_{i\geq 1}\left\{i\binom{mi-1}{i}\right\}^{d_i},$$

and the generating series G by

$$G = \sum_{n\geq 1}\sum_{\alpha\vdash n} G_\alpha(m)\frac{z^n}{n!}u^{l(\alpha)}h^\alpha p_\alpha v^{c(\alpha)}, \quad \widetilde{G} = \sum_{n\geq 1}\sum_{\alpha\vdash n}\widehat{G}_\alpha(m)\frac{z^n}{n!}u^{l(\alpha)}h^\alpha p_\alpha v^{c(\alpha)}.$$

Note that, in the generating series G, z is an exponential indeterminate marking the elements in S_n, u is an ordinary indeterminate marking the cycles in the permutation, p_i is an ordinary indeterminate marking the cycles of length i in π, and v is an ordinary indeterminate marking the total number of cycles of the factors $\pi_1,\ldots,\pi_m$, say $c(\alpha)$. Note that by the minimality condition, $c(\alpha) = (m-1)n + l(\alpha) - 2$.

3.2 A Partial Differential Equation

Our aim is to prove Theorem 13 by proving that $G = \widetilde{G}$. More specifically, we shall prove that G and $\widetilde{G}$ satisfy the same partial differential equation with the same initial conditions. We begin with the algebraic portion of the argument, by determining a partial differential equation for $\widetilde{G}$. This is a new result. Note the similarity in form to the partial differential equation in Theorem 12 for $\widetilde{H}$.

Theorem 14 *The generating series $\widehat{G}$ satisfies the partial differential equation*

$$(m-1)\left(z\frac{\partial}{\partial z}v\frac{\partial}{\partial v}\widetilde{G}\right)^2 = 2m\left(z\frac{\partial}{\partial z}\right)\left(u\frac{\partial}{\partial u}-1\right)\widetilde{G}. \tag{3.1}$$

PROOF. In order to simplify the calculations, we first let $u = v = 1$ in $\widetilde{G}$ to obtain $\widehat{G}$. Note that we can reexpress the generating series in the following way:

$$
\begin{aligned}
[z^n]\widehat{G} &= \sum_{k\geq 0} \sum_{\substack{\alpha\vdash n \\ l(\alpha)=k}} \widehat{G}_\alpha(m)\frac{1}{n!}h^\alpha p_\alpha \\
&= \sum_{k\geq 0} \sum_{\substack{d_1+2d_2+\cdots=n \\ d_1+d_2+\cdots=k}} m\frac{\{(m-1)n-1\}!}{\{(m-1)n-k+2\}!}\prod_{i\geq 1}\left\{i\binom{mi-1}{i}\right\}^{d_i}\frac{1}{n!}h^\alpha \prod_{i\geq 1} p_i^{d_i} \\
&= \sum_{k\geq 0} \sum_{\substack{d_1+2d_2+\cdots=n \\ d_1+d_2+\cdots=k}} m\frac{\{(m-1)n-1\}!}{\{(m-1)n-k+2\}!}\prod_{i\geq 1}\frac{1}{d_i!}\left\{\binom{mi-1}{i}p_i\right\}^{d_i} \\
&= \sum_{k\geq 0} m\frac{\{(m-1)n-1\}!}{\{(m-1)n-k+2\}!}[\lambda^n x^k]\prod_{i\geq 1}\sum_{j\geq 0}\frac{1}{j!}\left\{\binom{mi-1}{i}x\lambda^i p_i\right\}^j \\
&= \sum_{k\geq 0} m\frac{\{(m-1)n-1\}!}{\{(m-1)n-k+2\}!}[\lambda^n x^k]\exp\left\{\sum_{i\geq 1}\binom{mi-1}{i}x\lambda^i p_i\right\} \\
&= \sum_{k\geq 0} m\frac{\{(m-1)n-1\}!}{\{(m-1)n-k+2\}!}[\lambda^n]\frac{\{\sum_{i\geq 1}\binom{mi-1}{i}\lambda^i p_i\}^k}{k!} \\
&= \frac{m}{(m-1)n\{(m-1)n+1\}\{(m-1)n+2\}}[\lambda^n]\sum_{k\geq 0}\binom{(m-1)n+2}{k}\left\{\sum_{i\geq 1}\binom{mi-1}{i}\lambda^i p_i\right\}^k \\
&= \frac{m}{(m-1)n\{(m-1)n+1\}\{(m-1)n+2\}}[\lambda^n]\left\{1+\sum_{i\geq 1}\binom{mi-1}{i}\lambda^i p_i\right\}^{(m-1)n+2}.
\end{aligned}
$$

Let $A(\lambda) = \sum_{i\geq 1}\binom{mi-1}{i}p_i\lambda^i$, so we have

$$
\begin{aligned}
[z^n]\widehat{G} &= \frac{m}{(m-1)n\{(m-1)n+1\}\{(m-1)n+2\}}[\lambda^n]\{1+A(\lambda)\}^{(m-1)n+2} \qquad (3.2)\\
&= \frac{m}{(m-1)n\{(m-1)n+1\}}[\lambda^n]\frac{1}{(m-1)n+2}\{1+A(\lambda)\}^{(m-1)n+2} \\
&= \frac{m}{(m-1)n\{(m-1)n+1\}}\frac{1}{n}[\lambda^{n-1}]\{1+A(\lambda)\}^{(m-1)n+1}\frac{\partial A}{\partial\lambda} \\
&= \frac{m}{(m-1)n\{(m-1)n+1\}}\frac{1}{n}[\lambda^{n-1}]\{(1+A(\lambda))\frac{\partial A}{\partial\lambda}\}\{(1+A(\lambda))^{m-1}\}^n.
\end{aligned}
$$

Now use Theorem 1 with

$$F(\lambda) = \left(A(\lambda) + \frac{1}{2}A(\lambda)^2\right),$$

which implies

$$F^{'}(\lambda) = \{1 + A(\lambda)\}\frac{\partial A}{\partial \lambda},$$

and

$$w = z\phi(w), \quad \text{where} \quad \phi(\lambda) = \{1 + A(\lambda)\}^{m-1}, \tag{3.3}$$

in order to obtain

$$\begin{aligned}
[z^n]\widehat{G} &= \frac{m}{(m-1)n\{(m-1)n+1\}}[z^n]F(w) \\
&= \frac{m}{(m-1)n\{(m-1)n+1\}}[z^n]\left(A(w) + \frac{1}{2}A(w)^2\right) \\
n\{(m-1)n+1\}[z^n]\widehat{G} &= \frac{m}{m-1}[t^n]\left(A(w) + \frac{1}{2}A(w)^2\right) \\
[z^n]\left(z\frac{\partial}{\partial z}\right)\left\{(m-1)(z\frac{\partial}{\partial z}) + 1\right\}\widehat{G} &= \frac{m}{m-1}[t^n]\left(A(w) + \frac{1}{2}A(w)^2\right).
\end{aligned}$$

Therefore, we conclude that

$$\left(z\frac{\partial}{\partial z}\right)\left\{(m-1)z\frac{\partial}{\partial z} + 1\right\}G = \frac{m}{m-1}A(w) + \frac{m}{2(m-1)}A(w)^2. \tag{3.4}$$

Now we apply $p_k\frac{\partial}{\partial p_k}$ to (3.2), noticing that $p_k\frac{\partial A}{\partial p_k} = \binom{mk-1}{k}\lambda^k p_k$, to obtain

$$\begin{aligned}
[z^n]p_k\frac{\partial}{\partial p_k}\widehat{G} &= \frac{m}{(m-1)n\{(m-1)n+1\}\{(m-1)n+2\}}[\lambda^n]\{(m-1)n+2\}\{1+A(\lambda)\}^{(m-1)n+1}p_k\frac{\partial A}{\partial p_k} \\
&= \frac{m}{(m-1)n\{(m-1)n+1\}}\binom{mk-1}{k}p_k[\lambda^{n-k}]\{1+A(\lambda)\}^{(m-1)n+1} \\
&= \frac{m}{(m-1)n\{(m-1)n+1\}}\binom{mk-1}{k}p_k\frac{1}{n-k}[\lambda^{n-k-1}]\{(m-1)n+1\}\{1+A(\lambda)\}^{(m-1)n}\frac{\partial A}{\partial \lambda} \\
&= \frac{m}{(m-1)(n-k)}\binom{mk-1}{k}p_k\frac{1}{n}[\lambda^{n-1}]\lambda^k\frac{\partial A}{\partial \lambda}\{\{1+A(\lambda)\}^{m-1}\}^n.
\end{aligned} \tag{3.5}$$

Now, define

$$F(\lambda) = \sum_{i\geq 1} \frac{i}{i+k}\binom{mi-1}{i}\lambda^{i+k}p_i,$$

so

$$\begin{aligned} F'(\lambda) &= \sum_{i\geq 1} i\binom{mi-1}{i}\lambda^{i+k-1}p_i \\ &= \lambda^k \frac{\partial A}{\partial \lambda}. \end{aligned} \tag{3.6}$$

Thus, using Theorem 1 with (3.3) and (3.6), and substituting the expression obtained into (3.5), we obtain

$$\begin{aligned} [z^n]p_k\frac{\partial \widehat{G}}{\partial p_k} &= \frac{m}{(m-1)(n-k)}\binom{mk-1}{k}p_k[t^n]\sum_{i\geq 1}\frac{i}{i+k}\binom{mi-1}{i}w^{i+k}p_i \\ n[z^n]p_k\frac{\partial \widehat{G}}{\partial p_k} - k[z^n]p_k\frac{\partial \widehat{G}}{\partial p_k} &= \frac{m}{m-1}[t^n]\sum_{i\geq 1}\frac{i}{i+k}\binom{mi-1}{i}w^ip_i\binom{mk-1}{k}w^kp_k \\ [z^n]\{t\frac{\partial}{\partial z}p_k\frac{\partial}{\partial p_k} - kp_k\frac{\partial}{\partial p_k}\}\widehat{G} &= \frac{m}{m-1}[z^n]\sum_{i\geq 1}\frac{i}{i+k}\binom{mi-1}{i}w^ip_i\binom{mk-1}{k}w^kp_k. \end{aligned}$$

Summing over all k, and using the symmetry between i and k, we get

$$\begin{aligned} \sum_{k\geq 1}\left\{z\frac{\partial}{\partial z}p_k\frac{\partial}{\partial p_k} - kp_k\frac{\partial}{\partial p_k}\right\}\widehat{G} &= \sum_{k\geq 1}\frac{m}{m-1}\sum_{i\geq 1}\frac{i}{i+k}\binom{mi-1}{i}w^ip_i\binom{mk-1}{k}w^kp_k \\ &= \frac{1}{2}\frac{m}{m-1}\sum_{\substack{i\geq 1\\ k\geq 1}}\binom{mi-1}{i}w^ip_i\binom{mk-1}{k}w^kp_k \\ &= \frac{m}{2(m-1)}\left(\sum_{i\geq 1}\binom{mi-1}{i}w^ip_i\right)^2 \\ &= \frac{m}{2(m-1)}A(w)^2. \end{aligned}$$

Noticing the fact that $z\frac{\partial \widehat{G}}{\partial z} = \sum_{k\geq 1} p_k \frac{\partial \widehat{G}}{\partial p_k}$, we have

$$\left(z\frac{\partial}{\partial z}\right)\left(\sum_{k\geq 1} p_k \frac{\partial}{\partial p_k} - 1\right)\widehat{G} = \frac{m}{2(m-1)}A(w)^2. \tag{3.7}$$

Now, subtracting (3.7) from (3.4), we get

$$\left(z\frac{\partial}{\partial z}\right)\left\{(m-1)z\frac{\partial}{\partial z} - \sum_{k\geq 1} p_k \frac{\partial}{\partial p_k} + 2\right\}\widehat{G} = \frac{m}{m-1}A(w). \tag{3.8}$$

Squaring (3.8), scaling, and comparing with (3.7), we obtain

$$(m-1)\left\{\left(z\frac{\partial}{\partial z}\right)\left((m-1)z\frac{\partial}{\partial z} - \sum_{k\geq 1} p_k \frac{\partial}{\partial p_k} + 2\right)\widehat{G}\right\}^2 = 2m\left(z\frac{\partial}{\partial z}\right)\left(\sum_{k\geq 1} p_k \frac{\partial}{\partial p_k} - 1\right)\widehat{G}. \tag{3.9}$$

Now substitute $p_i \mapsto p_i u v^{-1}$ and $z \mapsto z v^{m-1}$ into $\widehat{G}$ and multiply by v^2 to obtain $\widetilde{G}$. Note that for $\alpha \vdash n$, the monomial $z^n \mathbf{p}^\alpha$ is replaced by $z^n \mathbf{p}^\alpha u^{l(\alpha)} v^{n(m-1)-l(\alpha)+2} = z^n \mathbf{p}^\alpha u^{l(\alpha)} v^{c(\alpha)}$, so we indeed obtain $\widetilde{G}$ after this substitution. Furthermore,

$$\begin{aligned}
[z^n \mathbf{p}^\alpha u^{l(\alpha)} v^{c(\alpha)}] u\frac{\partial \widetilde{G}}{\partial u} &= l(\alpha)[z^n \mathbf{p}^\alpha u^{l(\alpha)} v^{c(\alpha)}]\widetilde{G} \\
&= [z^n \mathbf{p}^\alpha u^{l(\alpha)} v^{c(\alpha)}]\sum_{k\geq 1} p_k \frac{\partial \widetilde{G}}{\partial p_k} \\
&= [z^n \mathbf{p}^\alpha]\sum_{k\geq 1} p_k \frac{\partial \widehat{G}}{\partial p_k},
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
[z^n \mathbf{p}^\alpha u^{l(\alpha)} v^{c(\alpha)}] v\frac{\partial \widetilde{G}}{\partial v} &= c(\alpha)[z^n \mathbf{p}^\alpha u^{l(\alpha)} v^{c(\alpha)}]\widetilde{G} \\
&= [z^n \mathbf{p}^\alpha u^{l(\alpha)} v^{c(\alpha)}]\left((m-1)z\frac{\partial}{\partial z} - \sum_{k\geq 1} p_k \frac{\partial}{\partial p_k} + 2\right)\widetilde{G} \\
&= [z^n \mathbf{p}^\alpha]\left((m-1)z\frac{\partial}{\partial z} - \sum_{k\geq 1} p_k \frac{\partial}{\partial p_k} + 2\right)\widehat{G}.
\end{aligned} \tag{3.11}$$

Substituting (3.10) and (3.11) into (3.9), we obtain (3.1), which completes the proof. □

3.3 Constellations and Eulerian Trees

Our goal in this section is to begin the combinatorial portion of our argument. We use some particular combinatorial objects, called *m-Eulerian trees* in [1]. Later, in Section 3.4, we will be able to show, by a recursive analysis, that the generating series G satisfies the partial differential equation (3.1) given in Theorem 14. The results in this section are all given, explicitly or implicitly, in [1].

For $m \geq 2$, an *m-constellation* is a 2-face-coloured planar map with black and white faces, such that every white face has degree mi for some integer i that can vary from face to face, and every black face has degree m. Furthermore, each face is adjacent only to faces of the opposite colour. If one of the edges is distinguished, the constellation is rooted. In this constellation, label the vertices from 1 to m and the black faces from 1 to n, where n is the number of black faces, in the following way. The ends of the root edge are coloured 1 and 2 in clockwise order with respect to the white face they belong. In every white face, the labels increase clockwise and in every black face they increase counterclockwise, both modulo m. Every black face receives a label from 1 to n, making sure that the black face containing the root edge is labeled 1. This can be seen in Figure 3.1 for $m = 3$.

The first result we shall prove towards our goal is the following:

Theorem 15 *There is a one to one bijection between rooted constellations with n black faces of degree m, and m+1-tuples of permutations $(\pi_1, \ldots, \pi_m, \pi)$ where $\pi_1 \cdots \pi_m$ is a minimal transitive factorization of π. Furthermore, each white face of the constellation of degree mi corresponds to a cycle in π of length i.*

PROOF. Let C be such a constellation. Construct π_i in the following way. For any vertex of label i, the labels of the black faces around it form a cycle, once they are taken in clockwise order. Furthermore, they are disjoint for each vertex of label i, since each black face only has one vertex of label i for every i. Therefore, putting all these cycles together gives a permutation of

$\{1, \ldots, n\}$, which we call π_i. Now, note that as we multiply the π_i, we are taking each black face, say of label j, and rotating it around the white face that contains its edge with ends 1 and m. This is done m times, so for each face of degree mi we obtain a cycle of length i in π. This process can be seen in Figure 3.1, where we show the constellation corresponding to the factorization $\pi = \pi_1\pi_2\pi_3$, where

$$\begin{aligned}
\pi &= (13)(2)(4), \\
\pi_1 &= (12)(3)(4), \\
\pi_2 &= (134)(2), \\
\pi_3 &= (243)(1).
\end{aligned}$$

Now, since the constellation is connected, then for every pair of black faces of labels i and j, we can find a sequence of vertices such that when we rotate the faces around those vertices, we get from face i to face j. Therefore, for any i and j, we can find a sequence of permutations (namely those at the vertices we have just mentioned) such that the product of them sends i to j, which means the group generated by the α_i is transitive.

Furthermore, since the number of vertices of the constellation is $v = \sum_{i=1}^{m} l(\alpha_i)$, the number of edges is $e = mn$ and the number of faces is $f = n + l(\alpha)$, then by the Euler characteristic formula, *i.e.* $v - e + f = 2$ we derive the equation

$$\sum_{i=1}^{m} l(\alpha_i) - mn + n + l(\alpha) = 2$$

so, by (1.9), the factorization is minimal.

To prove the bijection in the other direction, pick a permutation $\alpha \in S_n$ and a minimal transitive factorization $\alpha = \alpha_1 \cdots \alpha_m$ of it. For each $i = 1, \ldots, n$ create a black m-gon with its vertices labelled 1 to m. For each i from 1 to m, identify the vertices of label i following the cycles in α_i. This yields the desired constellation, and as we have seen above, the transitivity condition ensures the map is connected, and the minimality condition ensures that the Euler characteristic

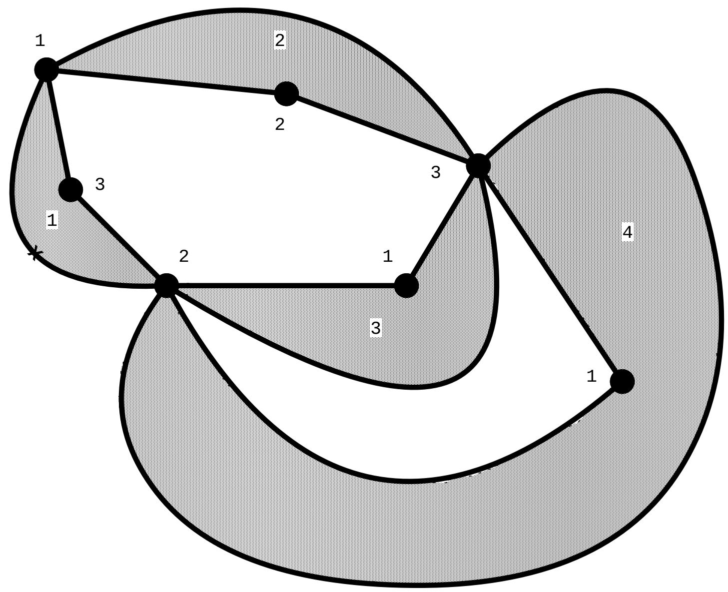

Figure 3.1: A Constellation

is two, so the map is planar.

It is routine to prove that these two mappings compose to the identity. □

Now, in order to count constellations, one must consider their planar dual. A planar dual is given in Figure 3.2. In order to study this dual, the following combinatorial object will be easier to handle.

An *m-Eulerian planted tree* is a coloured tree where the colours of the vertices are alternately black and white, and has the following two properties:

- Every inner black vertex has total degree m and inner degree 1 or 2.
- Every inner white vertex has total degree mi for some $i \geq 1$, and exactly $i - 1$ neighbours of inner degree 1.

An example can be seen in Figure 3.3, where *inner vertices* (vertices incident to more than one edge) are drawn as squares, and leaves are drawn as circles.

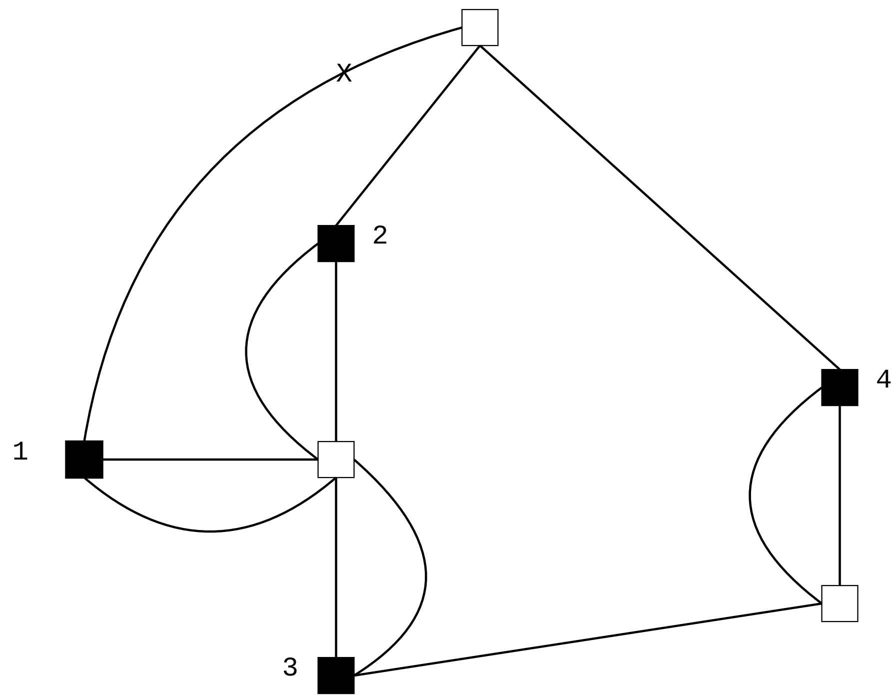

Figure 3.2: The Dual Graph of the Constellation in Figure 3.1

Now let d_i be the number of inner white vertices of degree i for each i. If $n = \sum_{i\geq 1} id_i$ and $l = \sum_{i\geq 1} d_i$, then an m-Eulerian tree has the following:

- l inner white vertices,
- $n-1$ inner black vertices,
- $(m-1)n - l - m + 2$ white leaves,
- $(m-1)n - l + 2$ black leaves.

The similarity of these variables n, m, l, and d_i with those at the beginning of Chapter 3 is a consequence of the combinatorial construction. In fact, $(m-1)n - l + 2$ is recognizable in the definition of $\widehat{G}_\alpha(m)$.

These trees are important, because they are in bijection with constellations. To see one side of the bijection, we pair up the black and white leaves of the tree by walking around the tree counter-clockwise starting by the root. For every white edge, we open a bracket, and for every black edge,

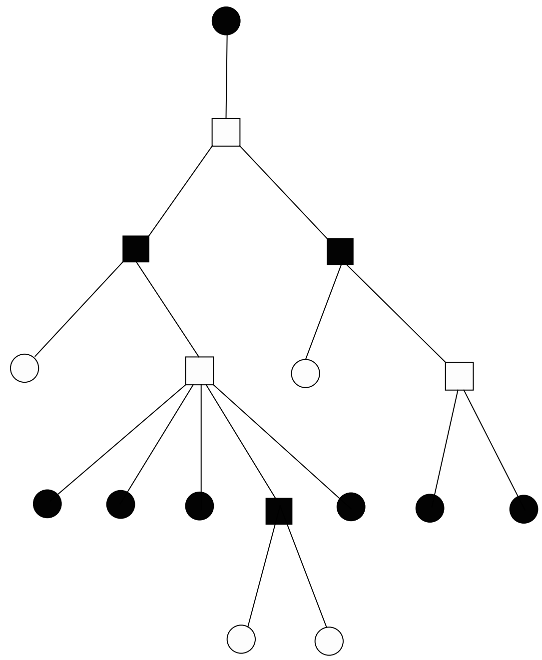

Figure 3.3: Eulerian Tree

we close a bracket, thus the tree in Figure 3.3 gives the following word:)()))(()()). Consider this word cyclically, and for any two brackets that correspond to each other in the expression, pair up the corresponding leaves with a dotted line. As there are m more black leaves than white leaves, the unpaired m black leaves are paired up with m new white leaves that we add to the graph, which are all emanating from a new black inner vertex. This process can be seen in Figure 3.4.

As a last step, we remove the leaves and join the edges incident to them, to obtain a graph (not necessarily simple), that consists of the dual of the constellation. For example, the graph obtained by removing the leaves of the tree in Figure 3.4 is the graph in Figure 3.2. Thus, by taking the dual we obtain the constellation in Figure 3.1, and the requirements on the degrees of the inner vertices in the Eulerian tree ensures that the map obtained is a constellation. Denote this map from m-Eulerian trees to constellations by Φ.

Now we describe the inverse of this bijection, *i.e.* , given a constellation C, we obtain its corresponding m-Eulerian tree T. All we need is to find the right edges in the constellation that

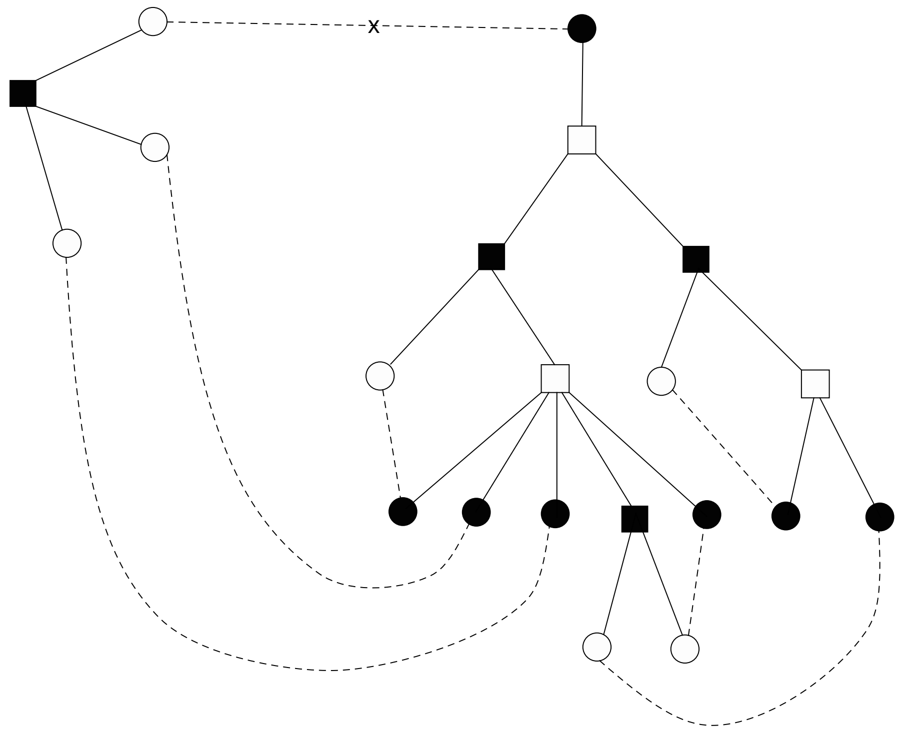

Figure 3.4: Closing the Edges of the Eulerian Tree

we need to turn into dotted edges, to obtain the tree. These edges form a special graph, formed of an m-gon (formed by the duals of the edges adjacent to the star), and a tree emanating from each of the vertices of the m-gon, since the remaining graph is connected. So we begin by considering the m-gon that is dual to the special vertex corresponding to the star that we add in T to get the dual. From now on it shall be called the root m-gon. Now we draw all the dual edges to the dotted edges that pair the black and white leaves, thus obtaining m trees $T_i, i = 1, \ldots, m$, as it can be seen in Figure 3.5. This will be called the *covering forest* of the dual graph, and it is the key for the bijection, since it has very special properties that shall help us identify it directly from the constellation. Notice that since we are taking the duals of edges in the dual graph of the constellation, then the covering forest F covers the vertices in C, which is the reason for its name.

We begin by defining the *rank* of a vertex in the forest, as the distance from it to the root

m-gon, which we label $r(v)$. We also define the *right-left prefix order* in the tree as a recursive way to traverse the vertices of the tree, by starting from the root, and then traversing all the trees adjacent to it using the right-left prefix order, starting from the rightmost neighbour and moving to the left. The following theorem is the key to the bijection.

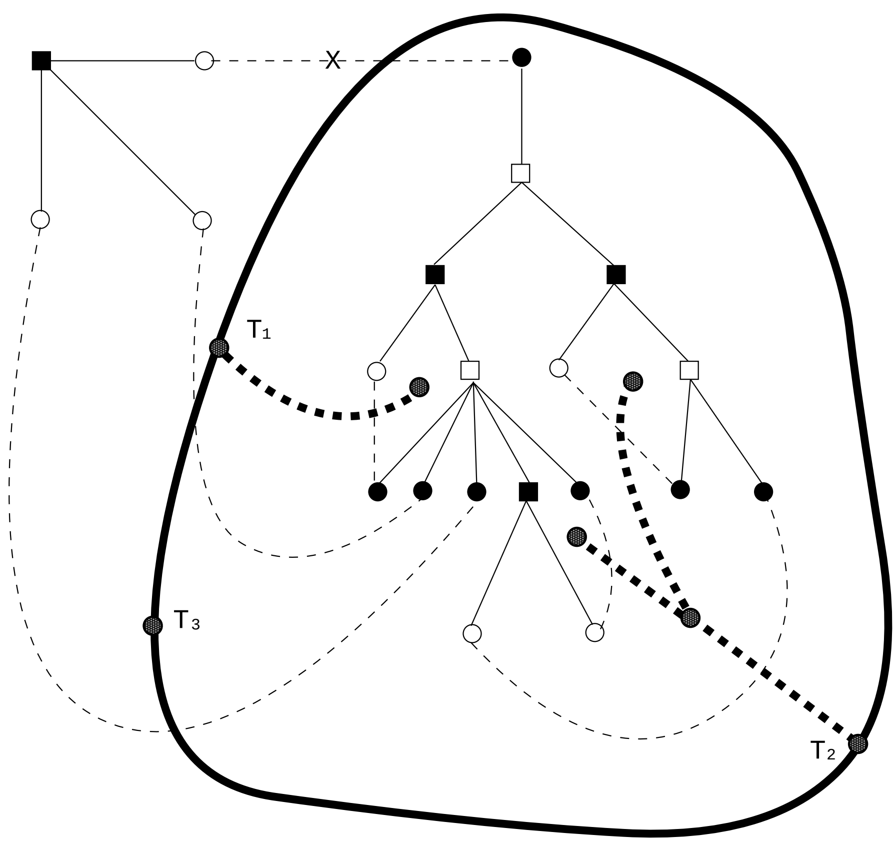

Figure 3.5: The Covering Forest

Theorem 16 *Let C be a constellation and $\bar{C}$ the map obtained by orienting the edges in C clockwise around the white faces (which is also counterclockwise around the black edges). Then there is a unique covering forest with the following properties:*

1. The orientation of the edges of F induced by the trees T_i taken from the roots to the leaves is the same orientation as in $\bar{C}$.

2. The rank increases by one along each edge of F, so the depth of a vertex of T_i is given by its rank.

3. For a vertex v not in the root m-gon, all the vertices with rank $r(v)-1$ and label $l(v)-1$ (labels taken modulo m), occur in the same tree T_i. If we visit them in right-left prefix order, the first one that is adjacent to u is the father of u in T_i.

4. Let v be the father of u in T_i and e be the edge of T_i that links v to its father. If we visit the edges of C adjacent to v in clockwise order, starting from e, the first one that ends at u belongs to T_i.

This unique covering forest is called the rank forest of C. Note that properties 1 and 2 imply that if a vertex u belongs to a tree T_i, then $i = l(u) - r(u)$ mod m.

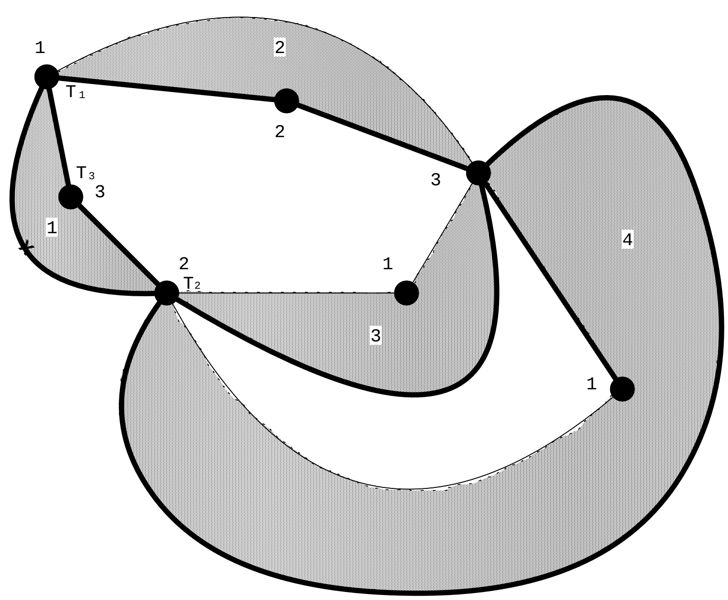

Figure 3.6: The Covering Forest of a constellation

Proof. We construct F inductively, in the following way. In step 0, we consider the vertices of the m-gon as the trees. If in step k we have not covered all the vertices, then pick a vertex v

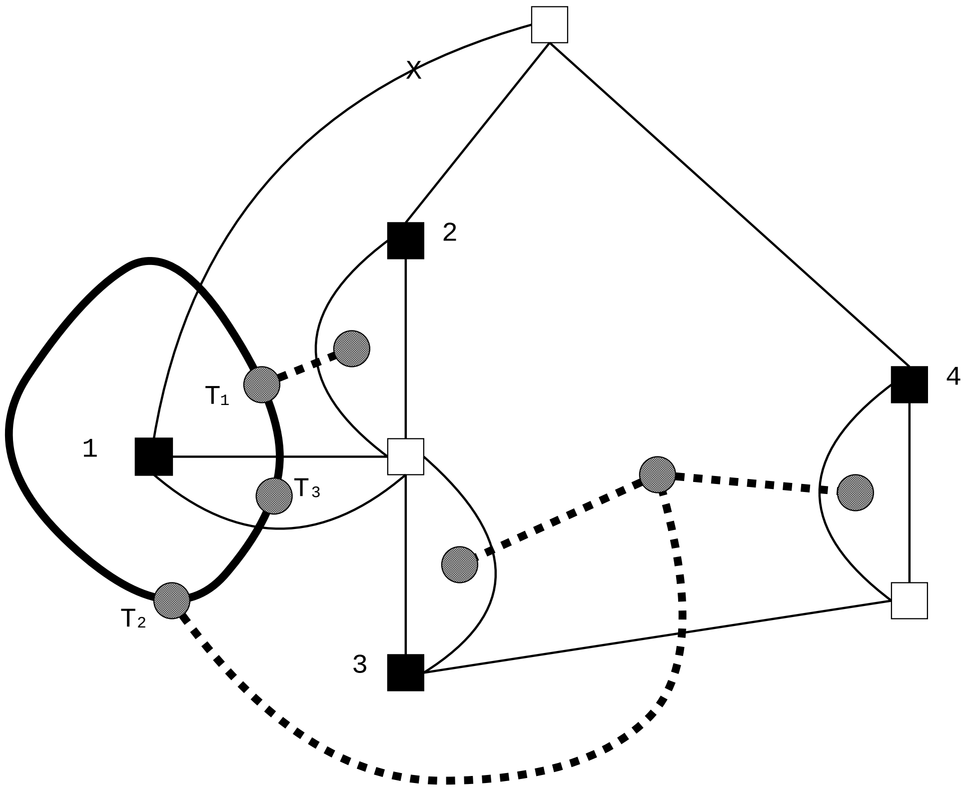

Figure 3.7: The Covering Forest of the Dual Graph of the Constellation in Figure 3.6

that has not been covered. By induction, all the vertices of rank $r(k) - 1$ and label $l(u) - 1$ all belong to the same tree, so we put u in this tree. The father of u and the edge connecting it to its father are picked as in properties 3 and 4, completing the proof. □

The rank forest of the constellation in Figure 3.1 is given in Figure 3.6.

Now, to construct the Eulerian tree T, all we need is to remove the edges that are duals to those of the forest. Denote this mapping from constellations to trees by Ψ. The following result is our main goal.

Theorem 17 *Let C be a constellation. Then $T = \Psi(C)$ is an m-Eulerian tree, and $\Phi(T) = C$.*

For this proof, we need some preliminary propositions.

Lemma 1 *Let T be a balanced m-Eulerian tree, and $C = \Phi(T)$ its corresponding constellation. Let E be the dual map of C. Let S the set of edges of E that were not edges of T, and S' be* *its dual set of edges. Then $S^{'}$ is formed by the root m-gon and the rank forest of C. Namely, $\Psi \circ \Phi = id$.*

Proof. Clearly, the edges dual to those that connect the star to the tree T form an m-gon in S. Furthermore, there is no other cycle in S, because every cycle in S would separate the dual into two connected components, and the union of T and the star has only two connected components. Therefore, the set S is formed by an m-gon and trees emanating from each of its vertices. Let this graph be G. Clearly, it is a covering forest, since every vertex belongs to it, so all we need to prove is that it is indeed the rank forest. We shall prove that it satisfies each of the four properties in the statement of Theorem 16. Let the trees attached to this m-cycle be $T_i, i = 1, \ldots, m$.

To prove Property 1, take any edge $e^{'}$ in one of the trees, say T_a. Note that the dual of $e^{'}$ is a dotted edge e. Now, recall that the way we join the black and white leaves in the tree T is by starting at a white leaf, then walking clockwise around the tree and pairing it with a black leaf. Therefore, if we stand at a white leaf and walk around it counterclockwise, we find the infinite face, the edge e and the finite face, as seen in Figure 3.8. Therefore, if we walk over the edge $e^{'}$ from the m-gon towards the tree, we cross it in the counterclockwise direction with respect to the white leaf. This means that, in the dual of the constellation, after removing the leaves and joining the inner vertices, we are walking on the edge $e^{'}$ counterclockwise with respect to the black vertex adjacent to that white leaf. This means that, in the constellation, the edge e is directed in the counterclockwise direction with respect to the black face to which it belongs. This is the way we have defined the orientation in $\bar{C}$, which means the orientation of the edges in T_a respects that of $\bar{C}$, thus proving that Property 1 is satisfied.

Now, notice that this edge $e^{'}$ cuts the tree T into two trees T_1 and T_2 planted in a black leaf and a white leaf, respectively, as in Figure 3.9. Since the property of being m-Eulerian is local, namely it is satisfied on every vertex, then T_1 is m-Eulerian if and only if this property is satisfied at the white vertex adjacent to its new black root. The property that we want to satisfy is that of the mi neighbours of this vertex, exactly $i-1$ are black inner vertices of inner degree 1. Therefore, if the inner black vertex in e has inner degree 2, then T_1 is m-Eulerian and it has

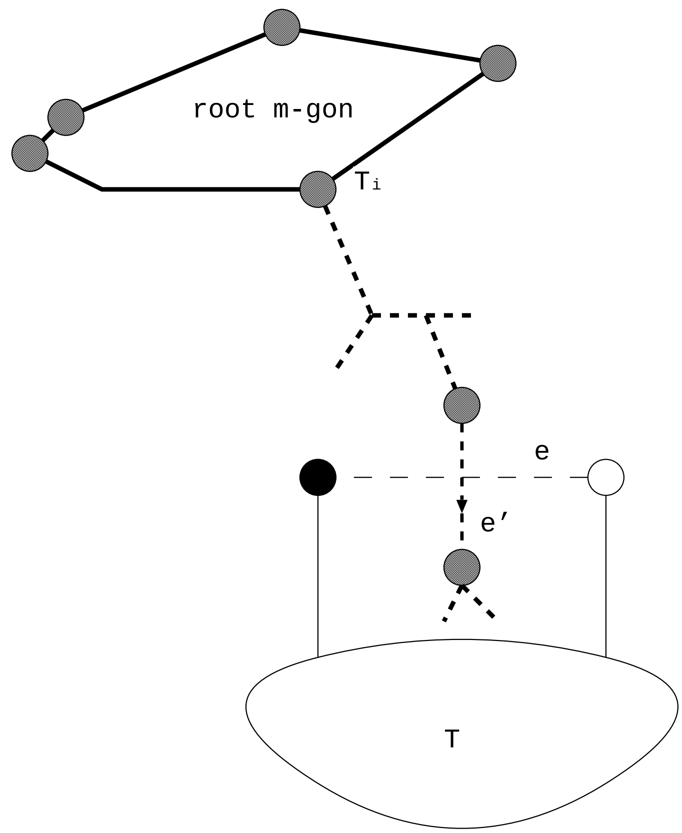

Figure 3.8: The Covering Forest of a constellation

m more black leaves than white leaves. If it has inner degree 1, then the white inner vertex now has $i-2$ neighbours of inner degree 1 and degree mi, which means the tree T_1 has $2m$ more black leaves than white leaves. In summary, T_1 has αm more black leaves than white leaves, where α is 1 or 2.

Now let u be a vertex. We shall prove that Properties 2, 3, and 4 are satisfied by induction on the rank of u. Let $r(u) = k$. Note that if $k = 0$, then u belongs to the root m-gon and the properties are satisfied trivially. If $k \geq 1$, then assume the properties are satisfied for every vertex of rank less than or equal to $k-1$. Since u has rank k, then its distance from the m-gon is k, and as the edges are directed as in the constellation, then its label must be at least k, say $k+d$ with $d \geq 0$. Now we shall prove that $a \equiv l(u) - r(u) (\bmod\ m)$. Let b be the congruence modulo m of $l(u) - r(u)$ and

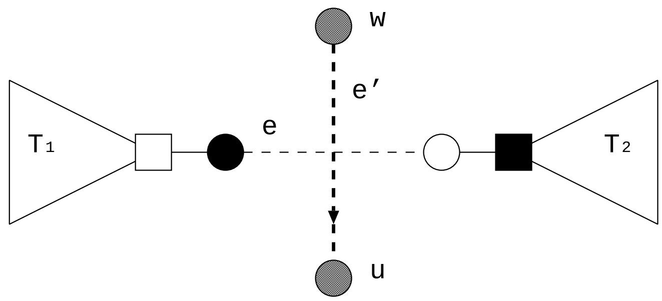

Figure 3.9: The edge splits the tree in two

assume, for a contradiction, that b is not congruent to a modulo m. Since u has rank k, then there is a vertex v connected to u with rank $k-1$ and label $l(u)-1$, and by the induction hypothesis, it must belong to the tree of label $l(u)-1-(k-1)=l(u)-1-(r(u)-1)=l(u)-r(u)=b$. Furthermore, $r(v)=k-1$ so v has height $k-1$ in the tree T_b. Therefore, since u and v belong to different trees, then the edge $e^{'}$ doesn't belong to $S^{'}$.

Now, as it can be seen in Figure 3.10, every edge in the path from w to the root of the tree T_b is the dual of some edge that connects a black leaf in T_1 with a white leaf in T_2, so since the rank of w is k, then there must be k of these pairs of leaves. Similarily (but not in the figure for the sake of simplicity), since the rank of u is $k+d$ then there must be $k+d$ white leaves if T_1 paired up with black leaves in T_1. Now, consider the m unpaired black leaves in T, and let r of them be in T_1, so $r \leq m$. Then note that the difference between the number of black and white leaves of T_1 is $(k-1)+1+r-(k+d)=\alpha m$ which means $r=d+\alpha m$ and as α can only be 1 or 2, we have $\alpha=1$ and $r=m$. Therefore T_2 has no unpaired black leaves. But since $a \neq b$, then there is at least one edge in the path from the root of T_a to that of T_b in the direction of T_2, as in the figure, and this edge is the dual of some edge that connects T_2 to the root vertex, as in the figure. This means T_2 has at least one unpaired black leaf, which gives a contradiction. Therefore $a=b$, which means w and u belong to the same tree, say T_a.

Now we shall prove that in this tree T_a, u has depth $r(u)=k$, thus proving it satisfies Property 2. Since the rank is k, then there is a vertex w of rank $k-1$ and label $l(u)-1$ that is connected to v by an edge $e^{'}$. By the induction hypothesis, v has depth $k-1$ in the tree T_a. If $e^{'}$ is an edge

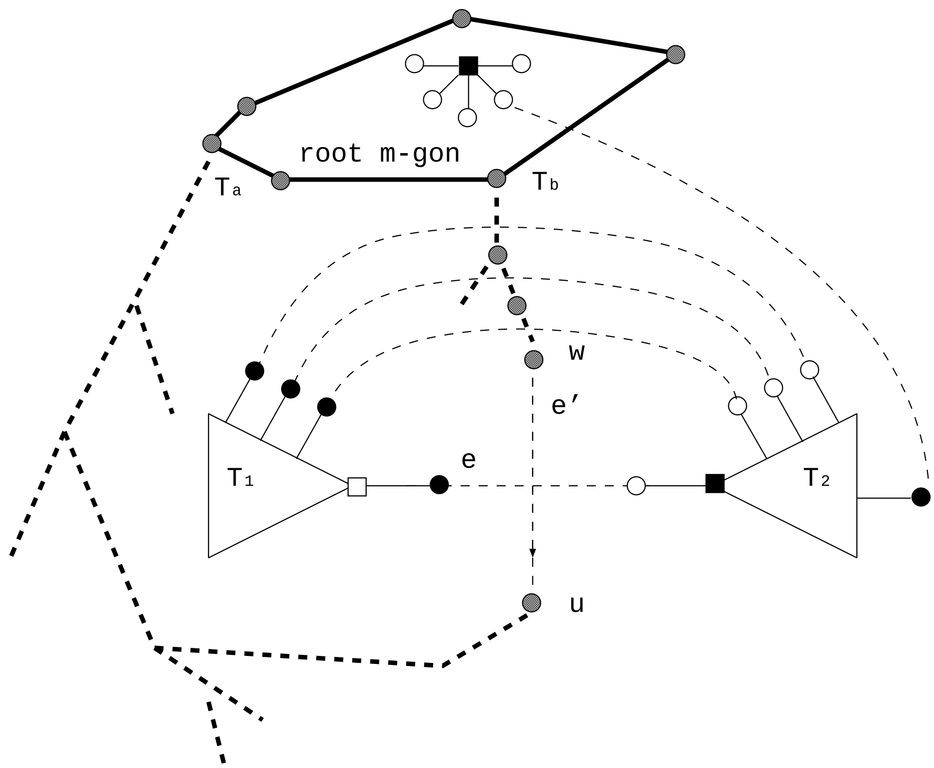

Figure 3.10: If a is not b

in T_a, then u has depth k in T_a thus proving the result. If it doesn't, then let x be the common ancestor of u and w of greatest depth, say δ. As seen in Figure 3.11 and for the same reason as before, there are δ black leaves in T_1 which are paired up with white leaves of T_1, by those dotted edges dual to the δ edges that connect x to the root of T_a. Therefore the difference between the number of black and white leaves of T_a is $(k-1-\delta)+r+1-(k+d-\delta) = \alpha m$. Thus, $r = d+\alpha m$, and since $r \leq m$, then $d = 0$, $\alpha = 1$, and $r = m$. Recall that the depth of u in T_a is $k+d$, so we can conclude that the depth of u is k, therefore proving it satisfies Property 2. Since $r = m$, we can conclude that all the single black leaves of T lie in T_i which means that T lies in the infinite face of the graph formed by the edge e and the tree T_a, and T_2 lies in the inside. Therefore, if the father v of w in T_a is not w, then it can not come after (since it can not cross T_1 so it must come before w in the right-left prefix order of T_a as in Figure . If $v = w$, then the edge that connects

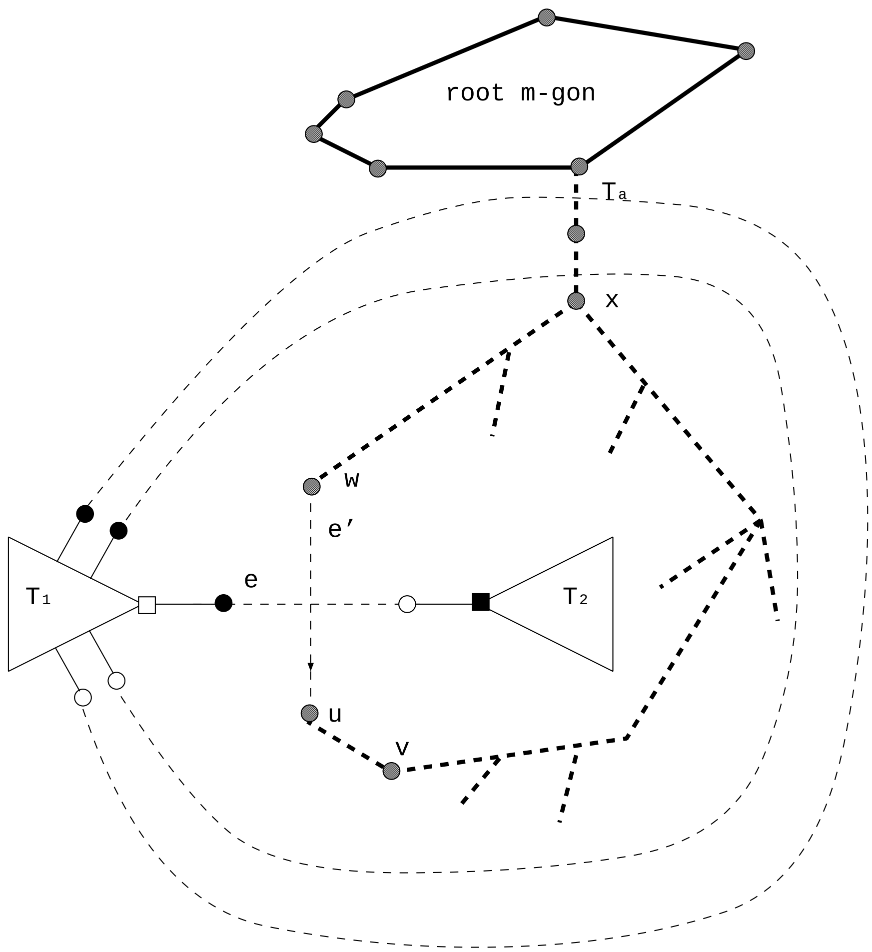

Figure 3.11: the father of u is the right-most vertex v

them must come before e' as in Figure 3.12, satisfying Property 4. □

The following lemmas prove that $\Psi(C)$ is equal to T.

Lemma 2 *Let C be a constellation. Then $\Psi(C)$ is a planted tree.*

PROOF. Let U be the graph formed by deleting the edges in S from the dual of C. Note that the dual of U is the graph formed by the edges in S', and since this last graph is connected, then U has no cycles (since a cycle in U would disconnect its dual). Therefore U is a forest. Note that the connected components of U are the star dual to the root m-gon, and the components of $\Psi(C)$, so to prove that $\Psi(C)$ is a tree, all we have to prove is that U only has two connected

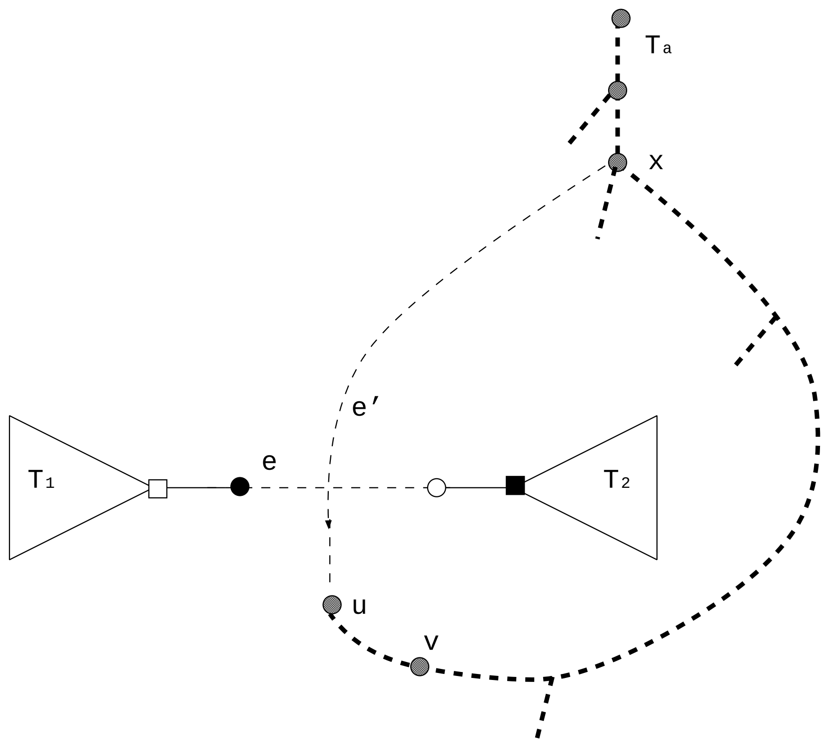

Figure 3.12: The edge connecting u and its father is at the left of $e^{'}$

components.

Note that the number of connected components in a forest is the difference between the number of vertices, and edges. Therefore, the number of connected components of U is $v(U) - e(U)$. But note that since the covering forest F has m trees, then it has $v(C) - m$ edges, so the size of $S^{'}$ is $v(C)$. Therefore $v(U) = v(E) + 2v(C)$ and $e(U) = e(E) + v(C)$ so $v(U) - e(U) = v(E) + v(C) - e(E) = v(E) + f(E) - e(E) = 2$ by Euler's characteristic formula. Therefore U has two components, and the result follows. □

Lemma 3 *Let C be a constellation, and let v be an inner black vertex of $\Psi(C)$. Then v has degree 1 or 2.*

PROOF. We shall prove the equivalent, namely that in the constellation C, every black m-gon (different than the root m-gon) has at most two edges that do not belong to the covering forest

F.

Let B be an m-gon in C. Note that if u and v are adjacent vertices in B, then their ranks can differ by at most 1, since if say $r(u) \geq r(v) + 1$, we can move v to the same tree with u and decrease its rank. We consider two cases.

Case 1. There is an edge e in B between vertices u and v such that $r(u) = r(v) + 1$. We call e a "special edge". We shall prove that there can only be at most one special edge in B.

Since u and v are taken in clockwise order, then $l(u) = l(v) + 1$, so $r(u) - l(u) = r(v) - l(v)$. Thus we conclude that u and v belong to the same tree, say T_a. So we can have two cases, if v is the father of u or if not.

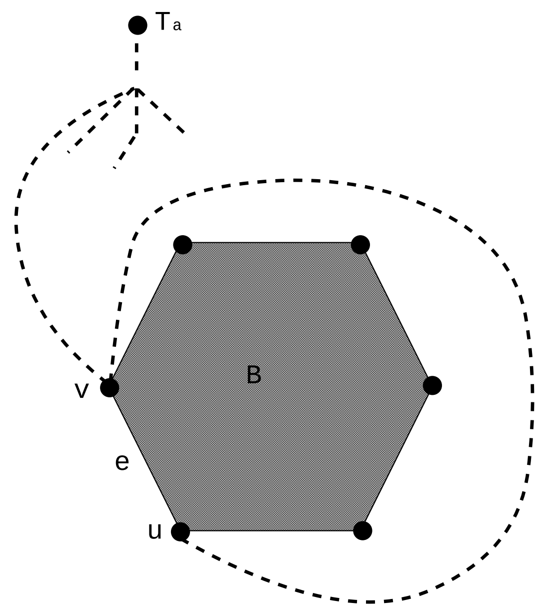

Figure 3.13: The special edge connecting u and its father v is at the left of e

If v is the father of u, then by construction, we know that the edge that connects v and u is the first one in the right-left prefix order, and as in Figure 3.13 we can see that there can be no other special edge, and all the vertices in B belong to T_a (because if a vertex doesn't belong, then the path that connects it to the root of its tree must cross the edge that connects u and v).

If v is not the father of u, then say w is the father. Then w comes before v in the right-left

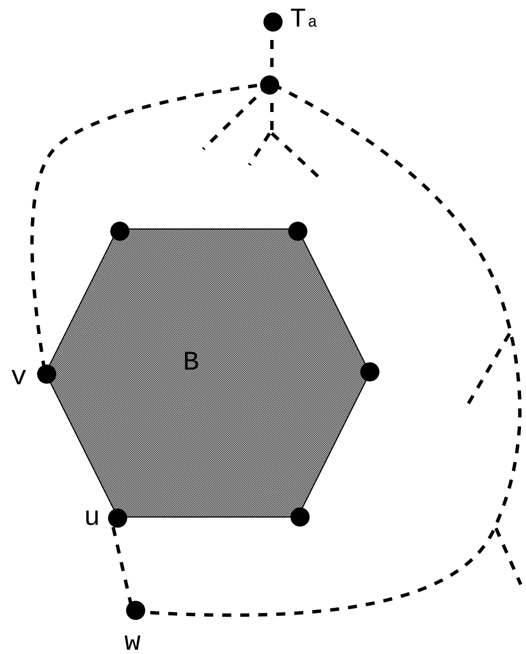

Figure 3.14: The father of u is to the right of v

prefix order of T_a, and by Figure 3.14 and a similar reasoning as before, we can see that there can be no other special edge and all the vertices in B belong to T_a.

Thus, we can conclude that there cannot be two special edges in B, and all vertices in B belong to the same tree T_a. Since they all have different labels around the face, and the difference between the label and the rank at every vertex is constant, then they must also have different ranks. So let v_1 be the vertex of smallest rank in B, and denote $v_2, \ldots, v_m$ the remaining vertices, in clockwise order. Let e_i be the edge connecting v_i and v_{i+1}. Since $r(v_k) \leq r(v_{k+1}) + 1$ for all k, because they are connected, and all the ranks must be different, then $r(v_k) = r(v_{k+1}) + 1$, so $r_{m-i} = r_1 + i + 1$ for all $0 \leq i \leq m-1$. Now, we know there is at most one special edge, say e_j. Since the ranks are all consecutive from v_1 to v_j, then all the edges $e_2, \ldots, e_{j-1}$ must belong to T_a. For the same reason, $e_{j+1}, \ldots, e_m$ must also belong to T_a. This only leaves e_j and e_1 as the only possible edges that do not belong to the forest, as Figure 3.15 shows, completing the proof for Case 1.

Case 2. There is no edge e in B between two vertices u and v taken in clockwise order, such

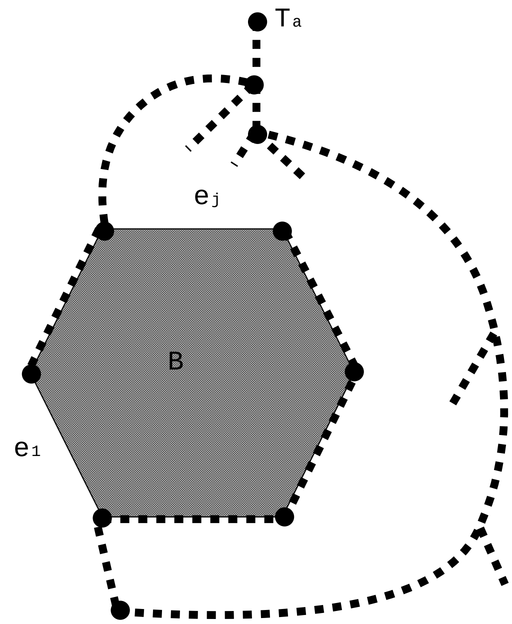

Figure 3.15: At most two edges not in F

that $r(u) = r(v) + 1$, namely, every edge e with ends u and v satisfies $r(u) < r(v) + 1$. We use the same notation for the edges and vertices as in Case 1. In order to prove this by contradiction, we assume there are three edges in B which don't belong to the forest F, say e_1, e_i, and e_j with $1 < i < j \leq m$.

Since e_1, e_i, and e_j do not belong to the forest F, then $r(v_k) < r(v_{k+1}) + 1$ for $k = 1, i, j$. For all the other vertices, $r(v_k) \leq r(v_{k+1}) + 1$. So we have

$$r(v_1) < r(v_2) + 1 \leq r(v_3) + 1 \leq \cdots \leq r(v_i) + i + 1 < r(v_{i+1}) + 1 \leq \cdots$$

$$\leq r(v_j) + j + 1 < r(v_{j+1}) + 1 \leq \cdots \leq r(v_1) + m$$

and since the difference between the label and the rank is the label of the tree, which increases by at most one as we go around the face, then $l(v_k) = l(v_1 - k + 1)$, so

$$l(v_1) - r(v_1) > l(v_1) - 1 - r(v_2) \geq \cdots \geq l(v_1) - i + 1 > l(v_1) - i - r(v_{i+1}) \geq \cdots$$

$$\geq l(v_1) - j + 1 > l(v_1) - j - r(v_{j+1}) \geq \cdots \geq l(v_1) - m + r(v_1)$$

Now, let v_1, v_i, and v_j belong to the trees T_a, T_b, and T_c, respectively, and pick v_1 in such a way that $a \leq b$ and $a \leq c$. Since for a vertex v belonging to T_a we have $l(v) - r(v) = a$ modulo m, then there exist integers k_1, k_2, and k_3 such that $l(v_1) - r(v_1) = a + k_1 m$, $l(v_1) - i + 1 - r(v_i) = b + k_2 m$, and $l(v_1) - j + 1 - r(v_j) = c + k_3 m$. Therefore, by the inequality above, we have $a + k_1 m > b + k_2 m > c + k_3 m$ and since a, b, and c are in between 0 and $m - 1$, then we have $k_1 = k_2 = k_3$ and $a > b > c$.

But note that if we walk around the vertices in a white face f in the clockwise direction, we meet vertices of the tree T_1, then T_2, until we meet vertices of the tree T_m. This is because as we walk on f, the rank either stays the same, or increases by 1. As we go from say a vertex u of a tree T_a to its immediate neighbour v, the label increases by 1. So if the rank increases by 1, then $l(u) - r(u) = l(v) - r(v)$ so they belong to the same tree. If the rank doesn't increase, then $l(u) - r(u) = l(v) - r(v) + 1$, so v belongs to T_{a+1}.

Therefore, we cannot have three trees around the cycle going clockwise such that their labels are decreasing, which contradicts the choice of a, b, and c above. This proves the theorem. □

Lemma 4 *Let T be an m-Eulerian tree, and let u be a white inner vertex of $\Psi(T)$ of degree mi. Then exactly $i - 1$ of its neighbours are black inner vertices of inner degree 1.*

Proof. We shall prove the equivalent, namely that if W is a white face of degree mi in a constellation C, then W contains exactly $i - 1$ edges e that satisfy the following property. e is adjacent to a black face B_e, and in this black face, every edge belongs to the forest F except e.

Let z be the number of edges in the face W with this desired property. We shall prove that $z = i - 1$, first by proving that $z \leq i$, then by obtaining a contradiction when $z = i$, and then by proving that $z \geq i$.

First we shall prove that $z \leq i$. Label the vertices of W as $v_1, v_2, \ldots, v_{mi}$ in clockwise order, and denote the edge e_k as before. We have seen that $r(v_{k+1}) \geq r(v_k) + 1$. Now, note that if an edge e_k connecting v_k and v_{k+1} is the only edge not belonging to the forest F in its corresponding black face B_{e_k}, then since the rank of the vertices increases by one on every tree edge, and B_{e_k} has

m edges, then $r(v_k) = r(v_{k+1}) + m - 1$. Therefore, for every k, we can write $r(v_k) \geq r(v_{k+1}) + a_k$ where $a_k = m - 1$ if e_k is the only edge on B_{e_k} not belonging to F, and $a_k = -1$ otherwise.

Therefore, we have

$$r(v_1) \geq r(v_2) + a_1 \geq r(v_3) + a_1 + a_2 \geq \cdots \geq r_{mi} + a_1 + \cdots + a_{mi-1} \geq r(v_1) + a_1 + \cdots + a_{mi}.$$

This means $a_1 + \cdots + a_{mi} \leq 0$. But since z of the a_i are $m - 1$ and the rest are -1, then we have $z(m-1) + (-1)(mi - z) \leq 0$, which implies $z \leq i$.

Now we shall rule out the case $z = i$. Note that if $z = i$, then $a_1 + \cdots + a_{mi} = i(m-1) + (-1)(mi - i) = 0$ so there must be an equality in the equations above, which means for every k we have $r(v_k) = r(v_{k+1}) + a_k$. This means for all k that are not the only non-forest edge on its corresponding black face, we have $r(v_{k+1}) = r(v_k) + 1$. As $l(v_{k+1}) = l(v) + 1$, then the difference between the label and the rank remains constant, so all the vertices in W must belong to the same tree, say T_a.

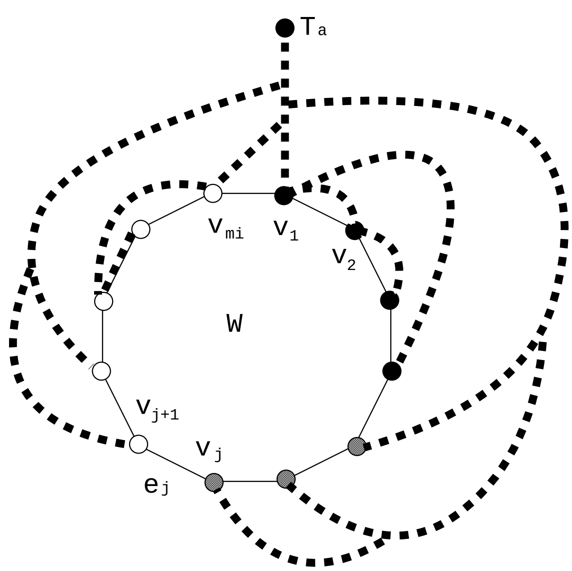

Figure 3.16: The right and left vertices of a white face

Now let v_1 be a vertex of the white face W of smallest rank, ie $r(v_1) \leq r(v_k)$ for $k = 1, \ldots, mi$.

We split the vertices of W into three types:

- Type I: v_1, and those that come before v_1 in the right-left prefix order of T_a.
- Type II: Those that belong to a subtree of T_a rooted at v_1, and whose rooted edge is in between the first edge of the path that connects v_1 and the root of T_a, and the edge that connects v_1 and v_2, taken in clockwise order.
- Type III: All the other vertices of W.

In Figure 3.16 we can see that the vertices of Type I are coloured black, those of Type II are coloured grey, and those of Type III are coloured white.

Now we define the vertices of Type I and II "right vertices", and those of Type III, left vertices (this labels should be clear from the drawing). Note that by the definition, every right vertex comes before every left vertex in the right-left prefix order of T_a. Now consider an edge e_j which is the only edge of its corresponding black face which is not in F. As there is a path between v_j and v_{j+1} all in T_a, then v_{j+1} is also a right vertex, and $j \neq mi$. Now let v_j be the right vertex of largest label, so v_{j+1} is a left vertex. So e_j cannot be the only edge of its corresponding black face which is not in F, which means $r(v_{j+1}) = r(v_j) + 1$.

Now consider the father of v_j in T_a, say v. So $r(v) = r(v_j) - 1 = r(v_{j+1})$. Consider two cases, if $v = v_j$ and if $v \neq v_j$. If $v = v_j$ then v comes before v_j in the right-left prefix order of T_a, so v_{j+1} also comes before v_j in this prefix order, but this is a contradiction since v_{j+1} is a left vertex and it must come after any right vertex. If $v \neq v_j$, then v must be v_1, because v_1 is the only right vertex that can have left sons (since the son of every other right vertex also comes before v_1 in the right-left prefix order, so it is a right vertex). But this implies that v_2 is a left vertex, which is a contradiction, since v_2 is clearly of Type II, by the definition.

Therefore, z is at most $i-1$. Now we use a counting argument to prove that $z = i-1$. As before, let d_i be the number of white vertices of T of degree mi. Let b_1 be the number of inner black vertices of T of inner degree 1, and b_2 the number of those of inner degree 2. Since every white vertex is adjacent to at most $i-1$ inner black vertices of degree 1, then $b_1 \leq \sum_i (i-1)d_i$. The equality holds if and only if every white vertex is adjacent to exactly $i-1$ inner black vertices of inner degree 1, so we shall prove that equality holds in the sum in order to prove the result.

Let l_b be the number of black leaves, and l_w the number of white leaves in T. Since there are m unpaired black leaves, then $l_b = l_w + m$. We shall count the number of edges on T. Note that since every edge has exactly one white end, then there are as many edges as white vertices, i.e., $e(T) = l_w + m\sum_i id_i$. Since every edge has exactly one black end, then $e(T) = l_b + m(b_1 + b_2)$. Therefore, $b_1 + b_2 = \sum_i id_i - 1$. Since every inner white vertex is the child of an inner black vertex, except for the white vertex adjacent to the root of T (which is a black leaf), then $\sum_i d_i = b_2 + 1$, so $b_1 = -1 + \sum_i id_i + 1 - \sum_i d_i = \sum_i (i-1)d_i$, which completes the proof. □

Now we know that $T = \Psi(C)$ is an m-Eulerian tree. The only remaining steps are to prove that the tree is balanced, and that $\Phi \circ \Psi = id$, which is the next Lemma.

Lemma 5 *Let C be a constellation, then $T = \Psi(C)$ is a balanced m-Eulerian tree, and $\Phi(T) = C$.*

Proof. Recall that in the bijection that sends C to $T = \Psi(C)$, we have taken the dual of C and then deleted the edges in the set S. So we need to prove that the edges that we have not deleted form the tree T. This shall prove that $\Phi(T) = C$ and that T is balanced, since the root edge of T would be deleted, so the root of T would be an unpaired leaf.

So basically, we need to prove that the edges that haven't been deleted form a set of trees that satisfy the properties stated in the definition of the rank forest. But all these properties are clear except for the one that states that T lies to the left of the edge e as we visit it from the white end to the black end. But note that the dual of e, say e', belongs to a tree T_a of the rank forest F. But T_a and T don't intersect, so since the orientation of the edges in the trees T_a coincides with the orientation of the edges in C, then the tree T must be to the left of e. This completes the proof. □

3.4 A Bijection for Eulerian Trees

The results given in Section 3.3 give us enough tools to complete the combinatorial portion of our argument. In this section we give a bijective argument proving that the generating series G satisfies the partial differential equation (3.1) of Theorem 14. The proof is based on a similar equation satisfied by the generating series for the number of m-Eulerian trees. For $\alpha \vdash n$, let $T_\alpha(m)$ be the number of m-Eulerian trees with:

- $n-1$ black inner vertices,
- $l(\alpha)$ white inner vertices, d_i of them of degree mi,
- $c(\alpha) = (m-1)n - l(\alpha) + 2$ black leaves,
- $(m-1)n - l(\alpha) - m + 2$ white leaves.

Recall that each inner white vertex of degree mi has $i-1$ inner black neighbours of inner degree 1, and each inner black vertex has inner degree 1 or 2, and total degree m.

Note that the number of inner black vertices of degree 1 with exactly 1 inner white neighbour is equal to

$$\sum_{i\geq 1} d_i(i-1) = \sum_{i\geq 1} id_i - \sum_{i\geq 1} d_i = n - l(\alpha).$$

Also note that there are exactly $(n-1) - (n - l(\alpha)) = l(\alpha) - 1$ inner black vertices with more than 1 (*i.e.* exactly 2) inner white neighbours. We call these black vertices *crowded* vertices. Now define the generating series T by

$$T = \sum_{n\geq 1}\sum_{\alpha\vdash n} T_\alpha(m)\frac{z^n}{n!}\frac{u^{l(\alpha)}}{l(\alpha)!}h^\alpha p_\alpha v^{c(\alpha)},$$

where z is an exponential indeterminate marking the black inner vertices, u is an exponential indeterminate marking the white inner vertices, p_i an ordinary indeterminate marking the white inner vertices of degree mi, and v an ordinary indeterminate marking the black leaves. We begin by proving that T satisfies a partial differential equation very similar to (3.1).

Theorem 18 *The generating series T for the number of m-Eulerian trees satisfies the partial differential equation*

$$\left(u\frac{\partial}{\partial u} - 1\right) T = 2(m-1)\left(v\frac{\partial}{\partial v}T\right)^2.$$

Proof. Consider two m-Eulerian trees. In each one, mark a leaf, thus obtaining $\left(v\frac{\partial}{\partial v}T\right)^2$. Now identify these two leaves, and note that the tree obtained has the properties of an m-Eulerian tree on every vertex except for the new one. Furthermore, the new vertex is an inner black vertex with inner degree two, i.e. a crowded vertex. And all we need to do to satisfy the conditions of an m-Eulerian tree is to attach $m-2$ leaves to it. This can be done in $m-1$ ways, since all we need is to decide on which side of the tree we want the leaf to go. This process determines a new Eulerian tree, with a selected crowded vertex, from which we obtain $\left(u\frac{\partial}{\partial u}-1\right)T$. Note that this bijection constructs every Eulerian tree in two possible ways, by permuting the trees in its preimage, which accounts for the factor of 2. This process, which is clearly bijective, is illustrated in Figure 3.17. □

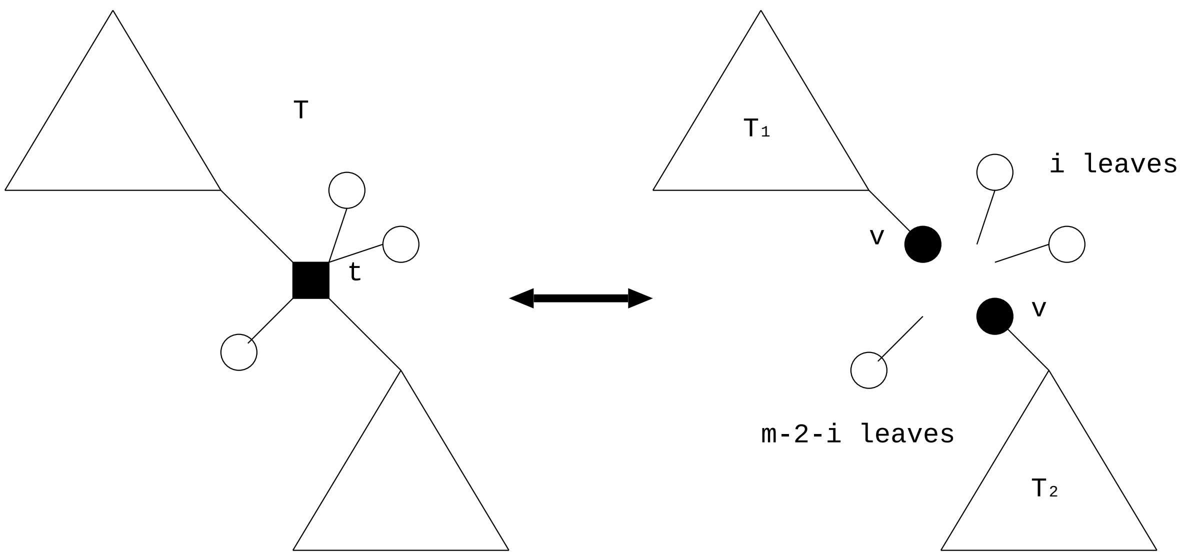

Figure 3.17: Cutting the Tree

We now provide a link between the tree generating series T and the transitive factorization generating series G.

Theorem 19

$$mT = z\frac{\partial G}{\partial z}.$$

Proof. Note that, by the bijection in [1], every factorization gives us n m-Eulerian trees, since we are at liberty to choose which element of the factorization is the star, with the remaining ones forming the tree. This tree gives us m factorizations, since we need to pick one edge between the star and the tree to root it (which is equivalent to picking a starting point for the factors that are already cyclically ordered). The result follows. □

Now, we can prove that G satisfies the same partial differential equation as $\widetilde{G}$.

Theorem 20 *The generating series G satisfies the partial differential equation (3.1).*

PROOF. By Theorems 19 and 18 we can conclude that G satisfies

$$(m-1)\left(z\frac{\partial}{\partial z}v\frac{\partial}{\partial v}G\right)^2 = 2m\left(z\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial u}-1\right)G.$$

□

To complete the proof of Theorem 13, we must prove that G and $\widetilde{G}$ satisfy the same initial conditions. Note that the initial condition occurs when $n = 1$ and $\alpha = (1)$. Combinatorially, this corresponds to factorizations of the identity permutation in S_1 into m factors $\sigma_1, \ldots, \sigma_m$, satisfying the minimality condition (1.9). This last condition becomes $\sum_{i=1}^{m} l(\sigma_i) = m$, and thus there is a single factorization in this case, with $\sigma_i = (1)$ for $i = 1, \ldots, m$, so we have

$$[zp_1uv^m]G = 1.$$

Algebraically, from the definition of $\widetilde{G}$, we can see that

$$[zp_1uv^m]\widetilde{G} = 1.$$

Thus $G = \widetilde{G}$, which completes the proof of Theorem 13.

3.5 Direct Proof by Lagrange's Theorem

In this section we give a different, more direct proof of Theorem 13. This proof has an algebraic portion using Lagrange's Theorem, and a combinatorial portion using m-Eulerian trees, but in this case, the combinatorial argument is direct, rather than recursive.

Recall that in the proof of Theorem 14 we have defined $A(w)$ by

$$A(w) = \sum_{i \geq 1} \binom{mi-1}{i} p_i w^i \tag{3.12}$$

where

$$w = z[1 + A(w)]^{m-1} \tag{3.13}$$

Consider an m-Eulerian planted tree, with a selected black leaf. Now, replace this leaf by a black inner vertex, with $m-1$ white leaves emanating from it. Select one of these leaves, and plant the new tree on that. We call this a *pseudo-Eulerian planted tree* .

Consider a vector with m entries, the first one of them, a black inner vertex. The remaining $m-1$ entries are $m-1$ cells, and in each cell we can have either a black leaf, or a pseudo-Eulerian planted tree. We call this a *w-vector* .

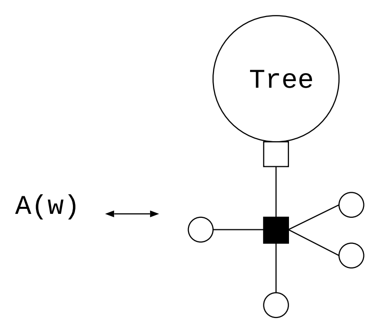

Figure 3.18: A pseudo-Eulerian Tree

Theorem 21 *The series $A(w)$ is the generating series for pseudo-Eulerian planted trees, where z is an exponential variable marking of the black inner vertices, u an ordinary variable marking of the white inner vertices, and p_i an ordinary variable marking of the white inner vertices of degree mi for all i, as illustrated in Figure 3.18. The series w is the generating series for these vectors, where the variables z, u, and p_i act as before, as illustrated in Figure 3.19.*

Proof. Note that Equation (3.13) is clear, since z marks the first entry of the vector, namely the black inner vertex, and for the remaining $m-1$ entries, we have a choice of a black leaf which

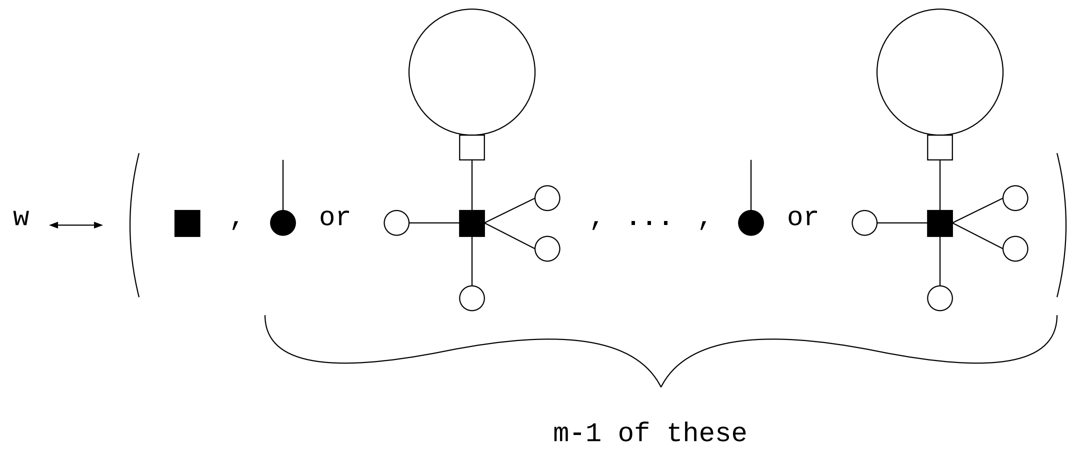

Figure 3.19: A w-vector

is marked by a 1 or a pseudo-Eulerian planted tree, which is marked by an $A(w)$.

Equation (3.12) can be explained in the following way. Note that a pseudo-Eulerian planted tree is planted on a white leaf, which is adjacent to a black inner vertex of inner degree 1, marked by z. This inner vertex is adjacent to exactly one white vertex. If the degree of this vertex is mi, then this vertex is marked by a up_i, for all $i \geq 1$. Now, for the remaining $mi-1$ edges that emanate from this white vertex, $i-1$ of them must be adjacent to a black vertex of inner degree 1, and they can be chosen in $\binom{mi-1}{i-1}$ ways. Each one of them is marked by z. So we have an ordered set of i copies of the variable z, starting from the black vertex adjacent to the root, and going clockwise around the white vertex specified before. Now, on the remaining $mi-i$ edges adjacent to this white vertex, we must attach either a black leaf, or a pseudo-Eulerian tree. Again, taking them clockwise, we have an ordered list of $(m-1)i$ objects, each one being a choice of a black leaf of a pseudo-Eulerian tree. Note that all these, together with the m black vertices of inner degree one above, can be arranged into an ordered list of i w-vectors, in the canonical way. This process can be seen more clearly in Figure 3.20. □

Now by a slight change in the procedure used to prove Theorem 14, we find the number of pseudo-Eulerian trees. Note that as before, by Lagrange's Theorem, and (3.12),

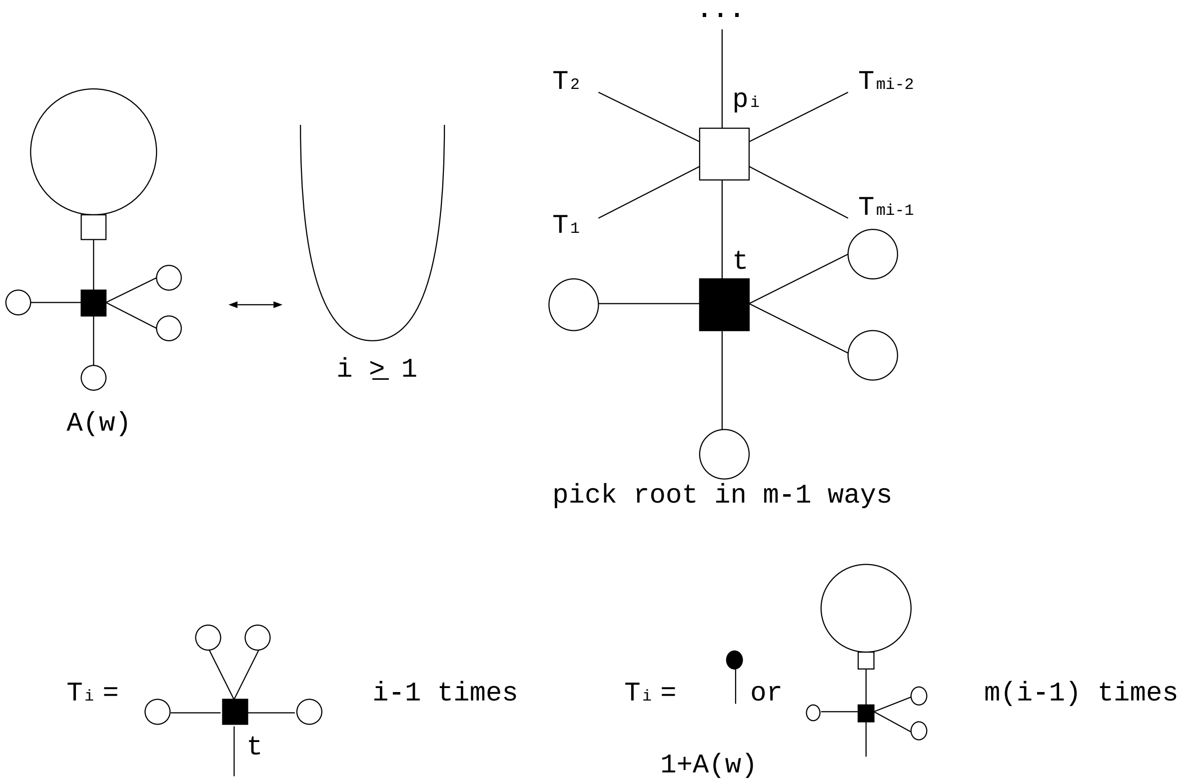

Figure 3.20: Decomposition of pseudo-Eulerian trees

$$\begin{aligned}
[z^n]A(w) &= \frac{1}{n}[\lambda^{n-1}]\frac{\partial A}{\partial \lambda}\{1+A(\lambda)\}^{(m-1)n} \\
&= [\lambda^n]\frac{1}{(m-1)n+1}\{1+A(\lambda)\}^{(m-1)n+1} \\
&= \frac{1}{(m-1)n+1}[\lambda^n]\sum_{k\geq 1}\binom{(m-1)n+1}{k}A(\lambda)^k \\
&= \sum_{k\geq 1}\frac{\{(m-1)n\}!}{\{(m-1)n-k+1\}!}[\lambda^n]\frac{1}{k!}\sum_{i\geq 1}\binom{mi-1}{i}p_i\lambda^i \\
&= \sum_{k\geq 1}\frac{\{(m-1)n\}!}{\{(m-1)n-k+1\}!}[\lambda^n x^k]\exp\left\{\sum_{i\geq 1}\binom{mi-1}{i}p_i\lambda^i\right\} \\
&= \sum_{k\geq 1}\frac{\{(m-1)n\}!}{\{(m-1)n-k+1\}!}[\lambda^n x^k]\prod_{i\geq 1}\exp\left\{\binom{mi-1}{i}p_i\lambda^i\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k\geq 1} \frac{\{(m-1)n\}!}{\{(m-1)n-k+1\}!}[\lambda^n x^k] \prod_{i\geq 1}\sum_{j\geq 0}\frac{1}{j!}\left\{\binom{mi-1}{i}p_i\lambda^i\right\}^j \\
&= \sum_{k\geq 1} \frac{\{(m-1)n\}!}{\{(m-1)n-k+1\}!} \sum_{\substack{d_1+d_2+\cdots=k\\ d_1+2d_2+\cdots=n}} \frac{1}{d_i!}\binom{mi-1}{i}_i^d p_i^{d_i} \\
&= \sum_{k\geq 1}\sum_{l(\alpha)=k} \frac{\{(m-1)n\}!}{\{(m-1)n-k+1\}!}\prod_{i\geq 1}\frac{1}{d_i!}\binom{mi-1}{i}^{d_i} p_i^{d_i}.
\end{aligned}
$$

Thus, the number of pseudo-Eulerian planted trees is

$$
[z^n u^{l(\alpha)} p^\alpha]A(w) = \frac{\{(m-1)n\}!}{\{(m-1)n-k+1\}!}\prod_{i\geq 1}\frac{1}{d_i!}\binom{mi-1}{i}^{d_i}
$$

Note that the number of planted pseudo-Eulerian trees is equal to the number of m-Eulerian trees multiplied by a factor of $(m-1)n-k+2$, in order to pick which black leaf we turn into a black inner vertex, so

$$
T_\alpha(m) = \frac{[(m-1)n]!}{[m-1n-k+2]!}\prod_{i\geq 1}\frac{1}{d_i!}\binom{mi-1}{i}^{d_i}.
$$

Recall that m-Eulerian trees correspond bijectively to m-Constellations. As in each constellation we can label the non-root m-gons in $(n-1)!$ ways, and each of these corresponds to $h_\alpha = \frac{n!}{\prod_{i\geq 1} i^{d_i} d_i!}$ permutations with d_i cycles of length i, then Theorem 13 follows directly. □

Chapter 4

A Multivariate Extension that Preserves Cycle Numbers

4.1 A Colour Extension to Preserve Cycles

In this chapter we study a refinement of the bijection in Section 3.5, combinatorially using colours that mark the cycles of each factor, as suggested at the end of [1]. This allows us to find the number of minimal, transitive, factorizations of a permutation, in which the number of cycles of each factor is specified. This is equivalent to finding the number of ramified covers of the sphere with a specified number of cycles at each ramification point. In the algebraic treatment, we must consider Lagrange's theorem in many variables.

The particular reason for studying this refinement is that Bousquet-Mélou and Schaeffer have shown how to specialize their result (Theorem 13 of Chapter 3), using inclusion-exclusion to remove the identity permutation as a factor, to obtain Goulden and Jackson's result (Theorem 4 of Chapter 2). The refinement considered in this chapter might lead to a stronger result that specializes to both Theorems 4 and 13 symmetrically. We also note that the apparent similarity of the partial differential equations in Theorem 11 and Theorem 14 of Chapter 3 also provides some evidence that looking for a joint generalization is reasonable. Our results are incomplete, but we hope that, eventually, they will contribute to a full solution of this refined problem.

The problem can be stated as follows: Let α be a partition of n, and let $c_1, \ldots, c_m$ be such that $1 \leq c_i \leq n$ for all $i = 1, \ldots m$. Find the number of minimal transitive factorizations $\pi = \pi_1 \cdots \pi_m$, where π is of cycle type α, and $l(\pi_i) = c_i$ for all $i = 1, \ldots m$.

We generalize the recurrence (3.12), given in Chapter 3 for factoring the identity permutation, by the following theorem:

Theorem 22 *Let T be the tree that corresponds to a factorization $\iota = \alpha_1 \cdots \alpha_m$, where ι denotes the identity permutation in S_n. Then*

1. The black vertices in T are leaves, or inner vertices of inner degree at least 2.

2. There is a one to one correspondence between the cycles of the factors and the black leaves of T.

PROOF. 1. We have to prove that there is no black inner vertex of inner degree 1. As we have seen before, every white inner vertex corresponds to a cycle in α, and if the cycle has length i, then the vertex has degree mi and $i - 1$ black neighbours of inner degree 1. Since every cycle in the identity permutation has length 1, then every white inner vertex is has 0 black neighbours of inner degree 1, completing the proof.

2. By the bijection given in [1], to turn T into a constellation that gives the factorization of α, we join the black and white vertices to create faces, as explained in section 3.3. As we join the vertices in this fashion, we note that if an edge e joins vertices w and b which are white and black respectively, as we walk from w to b, the tree T lies to the left of the edge e, in the sense that the map formed by T and e has two faces, and if we turn around w in counterclockwise order, we meet successively the infinite face, the edge e, and the finite face, as explained in the proof of Lemma 1 in Chapter 3. Therefore, if we draw an arrow out of every black vertex that points to this finite face, we obtain a unique arrow for every face, thus pairing up the black leaves and the faces of the map. Now, as we take the dual of the constellations, these faces become the vertices, which as we have seen in the bijection, correspond to the factors in the factorization. Therefore, we have paired up the black leaves in the tree and the factors in the factorization. □

Now as a further extension of the above argument, note that in the constellation, we have numbered the vertices from 1 to m specifying the factor they belong to. If we add these labels to the corresponding black vertices in the tree T, in the form of m different colours, we obtain an m-Eulerian tree that keeps track of the number of cycles in every factor of the factorization. Call this object a coloured m-Eulerian tree.

We also add labels to the inner black vertices, and the rules for adding the labels (colours) to the black vertices can be seen in Figure 4.1.

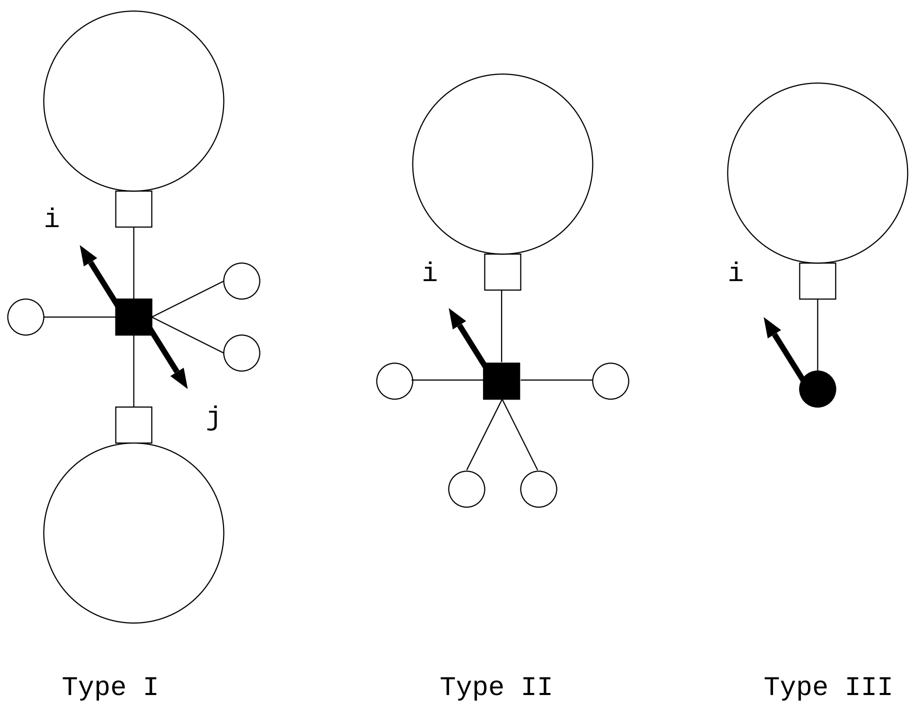

Figure 4.1: Rules for labelling the vertices

We say that these black vertices are of types I, II, and III, as in the figure. We can see that this labelling makes sense by studying a particular case first.

4.1.1 Factoring the Identity Permutation

First we shall study this labelling when π is the identity permutation in S_n. Note that by Theorem 22, all the black vertices of T must be of types I and III. The following result is crucial for our future counting arguments.

Theorem 23 *Let C be the constellation corresponding to a factorization of the identity permutation, D its dual graph, and T its coloured m-Eulerian tree. Every cycle in D has as many labels as the length of the corresponding cycle in the factorization.*

Proof. Consider the graph obtained after closing the edges in T with dotted lines. We consider any black inner vertex, and prove that this vertex adds exactly one label to every face it belongs, and this label contains the colour of the face.

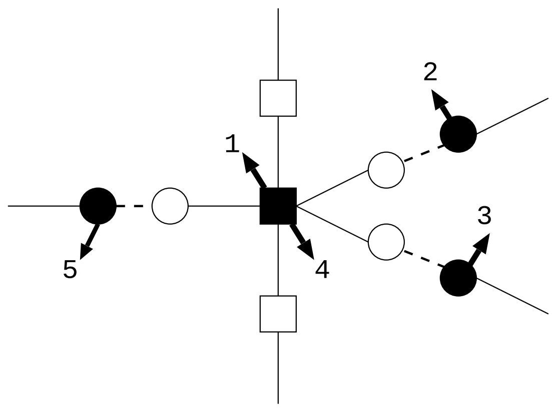

Figure 4.2: Every cycle gets labelled

First note that since we are factoring the identity permutation, then every black inner vertex is of type I, except for the star (*i.e.* , the vertex labelled 1) which of type II.

Those of type I are adjacent to two white inner vertices, and $m - 2$ white leaves. Note that the black inner vertex contains two labels, namely one for each face which is at the right of the edge that goes from the black vertex, to the inner white vertex adjacent to it, in that direction. For the rest of the face, this vertex has no labels, but since all the other neighbours of it are white leaves, then each one of them is connected to a black leaf. This leaf has a marker that points to the right, thus it marks that face. Therefore all the faces around a black vertex are marked for this vertex, and this means each cycle is marked once for each element it contains. An example of this with $m = 5$ can be seen in Figure 4.2.

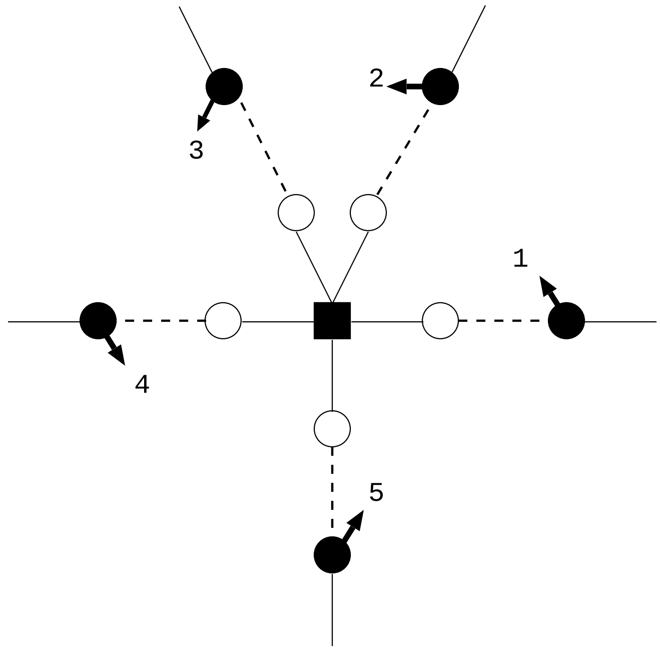

Figure 4.3: Cycles around the special vertex

The only type II vertex, namely the star, is adjacent to m white leaves, each of them connected to a black leaf, which marks the corresponding face, as shown in Figure 4.3. □

Corollary 24 *Every cycle in the dual graph D contains one label corresponding to a black leaf, and the rest corresponding to the black inner vertices.*

Proof. From Theorem 23 we see that each cycle gets marked once for each black inner vertex it contains, and this label can come from either the vertex, or one of the leaves adjacent to it. By Theorem 22, the cycles are in one-to-one correspondence with the leaves, namely there is only one leaf per cycle, so all the remaining labels must come from the inner black vertices. □

4.2 A Multivariate Lagrange Argument

In this section we extend the decomposition of Chapter 3 to coloured m-Eulerian trees, and then use a multivariate Lagrange approach. We start by a straightforward extension of the pseudo-

Eulerian tree obtained by adding a label to the two faces and defining the variables $y_{i,j}$ and $u_{i,j}$ as shown in Figure 4.4.

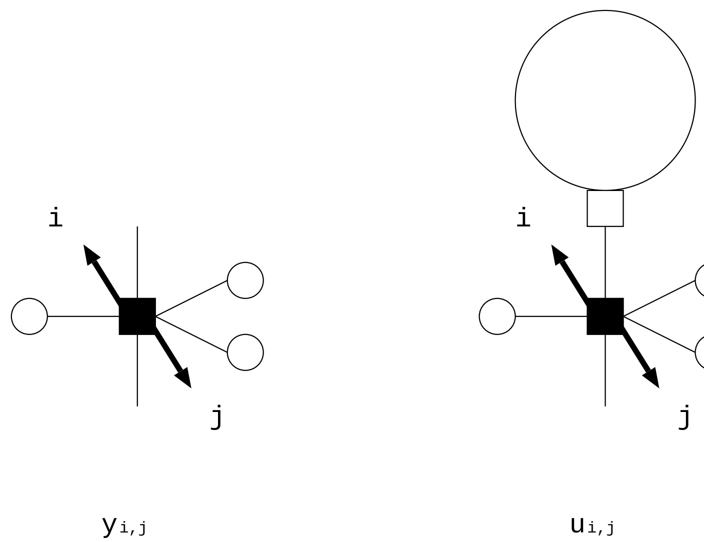

Figure 4.4: Extension of the pseudo-Eulerian Tree

The following extension of the bijection in Chapter 3 gives us a multivariable recursion that we shall solve using the multivariate extension of Lagrange's theorem.

Here we decompose a coloured m-Eulerian tree into a coloured black vertex of type I at the root, and the $m-1$ trees adjacent to the white inner vertex adjacent to this vertex. By Theorem 23, each of these trees is adjacent to a face of colour k for all $k \neq i$, and is denoted T_k. Each T_k could be either a black leaf, which we mark by x_k, or a pseudo-Eulerian tree whose lower label is k, which we mark by a $y_{l,k}$ where l is the colour of the upper label, and it must be different than k, as seen in Figure 4.6. This gives the following equation, which has similar form to that given at the end of [1], but for our purposes it proved easier to handle.

$$u_{i,j} = y_{i,j} \prod_{k \neq i} \left(x_k + \sum_{l \neq k} u_{k,l} \right) \tag{4.1}$$

As in Chapter 3, if we remove the black vertex attached to the root and add a black leaf, which becomes a new root, we get a rooted m-Eulerian tree, which is labelled v. In order to remove the

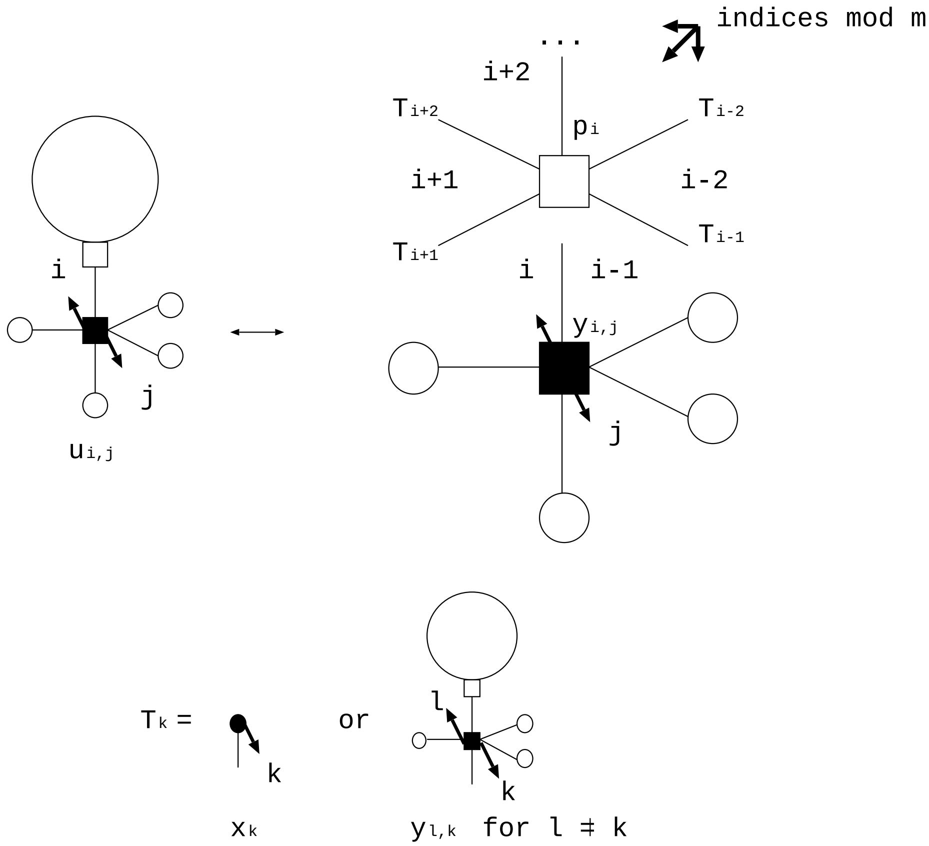

Figure 4.5: Multivariate extension of the bijection

labels on the faces, we must add over all i, and all $j \neq i$, as in Figure 4.6, to get the equation

$$v = \sum_i \sum_{j \neq i} u_{i,j}.$$

Now, define the vector $\mathbf{u} = (u_{i,j}$ for $i \neq j$, $i, j = 1, \ldots, m)$, and let

$$g_{i,j} = \sum_i \sum_{j \neq i} u_{i,j}, \tag{4.2}$$

and

$$f_{i,j}(\mathbf{u}) = \prod_{k \neq i} \left(x_k + \sum_{l \neq k} u_{l,k} \right). \tag{4.3}$$

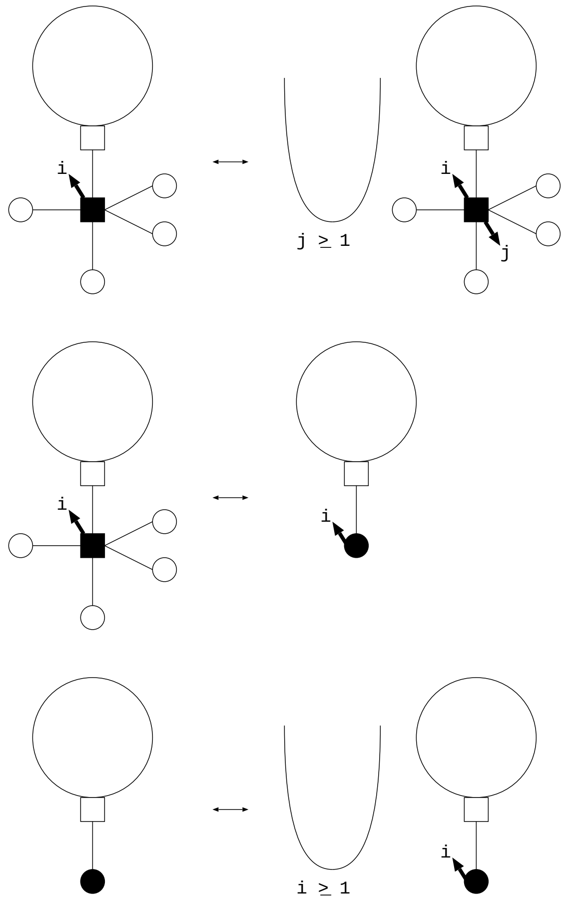

Figure 4.6: Multivariate extension of the bijection

Note that by Corollary 24, a cycle of length r in the i-th factor is marked with r variables, one of them of the form x_i and $r-1$ of them of the form $y_{i,j}$ or $y_{j,i}$ for some $j \neq i$. Therefore, if we let $x_i = 1$, we can still keep track of the number of cycles in the i-th factor by making use of the vector $\mathbf{y} = (y_{i,j}, 1 \leq i \leq m, 1 \leq j \leq m)$. Thus, (4.3) is equivalent to

$$f_{i,j}(\mathbf{u}) = \prod_{k \neq i} \left(1 + \sum_{l \neq k} u_{l,k} \right). \tag{4.4}$$

Therefore we get the system of equations formed by (4.2) and (4.4). By Theorem 2, the solution to this system is

$$[\mathbf{y}^{\alpha}]g(\mathbf{u}) = [\boldsymbol{\lambda}^{\alpha}]g(\boldsymbol{\lambda})f^{\alpha}(\boldsymbol{\lambda})\left|\delta_{(i,j),(l,k)} - \frac{\lambda_{l,k}}{f_{i,j}(\boldsymbol{\lambda})}\frac{\partial f_{i,j}(\boldsymbol{\lambda})}{\partial \lambda_{l,k}}\right|_{(i,j),(l,k)}. \tag{4.5}$$

In order to simplify this, let

$$s_k = 1 + \sum_{l \neq k} \lambda_{l,k}, \quad \text{so} \quad f_{i,j}(\boldsymbol{\lambda}) = \prod_{k \neq i} s_k,$$

and note that

$$\begin{aligned} f^{\alpha}(\boldsymbol{\lambda}) &= \prod_{i \neq j}[f_{i,j}(\boldsymbol{\lambda})]^{\alpha_{i,j}} \\ &= \prod_{i \neq j}\prod_{k \neq i} s_k^{\alpha_{i,j}} \\ &= \prod_{k} s_k^{\sum_{i \neq k}\sum_{j \neq i}\alpha_{i,j}}. \end{aligned}$$

Thus, (4.5) is equivalent to

$$[\mathbf{y}^{\alpha}]g(\mathbf{u}) = \prod_{k} s_k^{\sum_{i \neq k}\sum_{j \neq i}\alpha_{i,j}} \left|(I_{m(m-1)} - L)\right|,$$

where L is the $m(m-1)$ by $m(m-1)$ matrix whose $(i,j),(l,k)$ entry is $\frac{\lambda_{l,k}}{f_{i,j}(\boldsymbol{\lambda})}\frac{\partial f_{i,j}(\boldsymbol{\lambda})}{\partial \lambda_{l,k}}$, for $i \neq j$ and $l \neq k$. Now we calculate the determinant of $I_{m(m-1)} - L$, which for notational convenience, we denote as $|I - L|$. By the definition of s_k,

$$\frac{\partial s_r}{\partial \lambda_{l,k}} = \delta_{r,k}.$$

Therefore, we can calculate the following derivatives:

$$\frac{\partial f_{i,j}(\boldsymbol{\lambda})}{\partial \lambda_{l,k}} = \prod_{\substack{r \neq i \\ r \neq k}} s_r, \quad \text{if} \quad i \neq k,$$

and

$$\frac{\partial f_{i,j}(\boldsymbol{\lambda})}{\partial \lambda_{l,k}} = 0, \quad \text{if} \quad i = k.$$

Thus, the $(i,j),(l,k)$ entry of L is

$$\frac{\lambda_{l,k}}{f_{i,j}(\boldsymbol{\lambda})}\frac{\partial f_{i,j}(\boldsymbol{\lambda})}{\partial \lambda_{l,k}} = \frac{\lambda_{l,k}}{s_k}, \quad \text{if} \quad i \neq k.$$

and

$$\frac{\lambda_{l,k}}{f_{i,j}(\boldsymbol{\lambda})}\frac{\partial f_{i,j}(\boldsymbol{\lambda})}{\partial \lambda_{l,k}} = 0, \quad \text{if} \quad i = k.$$

The determinant of $I - L$ seems to have a path and cycle structure, which prompts the question of wether there is a combinatorial interpretation behind it:

- A *path* p of length $l(p) = n$ is a monomial of the form $\frac{\lambda_{a_1,a_2}}{s_{a_2}}\frac{\lambda_{a_2,a_3}}{s_{a_3}}\cdots\frac{\lambda_{a_{n-2},a_{n-1}}}{s_{a_{n-1}}}\frac{\lambda_{a_{n-1},a_n}}{s_{a_n}}$, and its components are the indices $a_1,\ldots,a_n$.

- A *cycle* c of length $l(c) = n$ is a monomial of the form $\frac{\lambda_{a_1,a_2}}{s_{a_2}}\frac{\lambda_{a_2,a_3}}{s_{a_3}}\cdots\frac{\lambda_{a_{n-1},a_n}}{s_{a_n}}\frac{\lambda_{a_n,a_1}}{s_{a_1}}$, and its components are the indices $a_1,\ldots,a_n$.

- A *circuit* T the product of a collection of cycles with disjoint sets of components. The length $l(T)$ is defined as the sum of the lengths of its corresponding cycles, and the size $s(T)$ as the number of cycles.

- A *tour* F is the product of one path and a collection of cycles, all of the former with disjoint sets of components. The length $l(F)$ is defined as the sum of the lengths of its corresponding path and cycles, and the size $s(T)$ as the number of paths and cycles.

Theorem 25

$$\det(I-L) = 1 + \sum_{\textit{circuits } T} (-1)^{l(T)-s(T)-1}[l(T)-1]T + \sum_{\textit{tours } F} (-1)^{l(F)-s(F)} F$$

PROOF. First define the following $m(m-1) \times m(m-1)$ matrices, whose rows and columns are indexed by pairs (i,j), $1 \leq i,j \leq m$ and $i \neq j$:

$$\Lambda_{(i,j),(l,k)} = \frac{\lambda_{i,j}}{s_j}\delta_{i,l}\delta_{j,k},$$

and

$$A_{(i,j),(l,k)} = 1 - \delta_{i,k}.$$

Note that Λ is a diagonal matrix, A is a $0,1$-matrix, and $\Lambda A = L$.

Now, let α be a subset of the rows of A. Denote as A_α the submatrix indexed by this set of rows and the corresponding set of columns. Let $\boldsymbol{\lambda}^\alpha$ be the monomial corresponding to the set of indices α on the variables $\lambda_{i,j}$. We use the determinant expansion

$$|I - \Lambda A| = \sum_\alpha \boldsymbol{\lambda}^\alpha |A_\alpha|(-1)^{|\alpha|}, \tag{4.6}$$

where the sum is taken over all the subsets α of rows of A.

First, we shall prove that the only subsets α for which $|A_\alpha| \neq 0$ are those for which $\boldsymbol{\lambda}^\alpha$ is a circuit or a tour. The first step towards this is to prove that if there are two elements in α that share the same first index, for example (i,r) and (i,s), then $|A_\alpha| = 0$. Since for every other index (l,k), the entry $A_{(i,j),(l,k)} = 1 - \delta_{i,k}$ depends only on i and k, then $A_{(i,r),(l,k)} = A_{(i,s),(l,k)}$ so there are two equal rows in A_α, which means $|A_\alpha| = 0$. Similarly, if there are two indices that share the same second element, for example (r,j) and (s,j), then the matrix A_α has two equal columns, so its determinant is zero.

Therefore, we will only consider those α for which every element appears at most once in each entry on the 2-tuples in α. If we consider the graph formed by the vertex set $\{1,\ldots,m\}$ and edges formed by the pairs in α, then this condition is equivalent to the condition of every vertex in the graph having degree at most two. Therefore this graph is a union of paths and cycles, from which we conclude that $\boldsymbol{\lambda}^\alpha$ is a product of paths and cycles. Now we prove that if there are two or more paths in $\boldsymbol{\lambda}^\alpha$, then this determinant of A_α is zero, by studying the form of this matrix.

First notice that since the $(i,j),(l,k)$ entry of A is only zero when $i = k$, then in submatrix corresponding to the paths and cycles, the only zero entries would be in the positions $(i, i+1)$, $(i-1, i)$, Therefore, if $\boldsymbol{\lambda}^\alpha$ is a cycle of length n, then $A_\alpha = c_n$ where

$$c_n = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 1 \end{pmatrix},$$

and if $\boldsymbol{\lambda}^\alpha$ is a path of length n then $A_\alpha = p_n$, where

$$p_n = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 1 \end{pmatrix}.$$

Now, if $\boldsymbol{\lambda}^\alpha$ is a product of paths and cycles, then the submatrices of A_α corresponding to the paths and cycles are the same, and for every entry $(i, j), (l, k)$ for which (i, j) and (l, k) belong to different components (paths or cycles), then $i \neq k$, so the corresponding entry would be $1 - \delta_{i,k} = 1$. Therefore, if $\boldsymbol{\lambda}^\alpha$ consists of r paths of lengths $a_1, \ldots, a_r$ and s cycles of lengths $b_1, \ldots, b_s$, and denoting $\mathbf{1}$ as the rows and columns of the corresponding size where all the entries are 1, then we have

$$A_\alpha = \begin{pmatrix} p_{a_1} & & & & & \\ & \ddots & & & \mathbf{1} & \\ & & p_{a_r} & & & \\ & & & c_{b_1} & & \\ & \mathbf{1} & & & \ddots & \\ & & & & & c_{b_s} \end{pmatrix}.$$

Note that if there are two or more paths, then their corresponding submatrices, say p_i and p_j, both contain one row of ones. Therefore the matrix A_α contains two rows of ones, which makes the determinant zero. Therefore, for the determinant of A_α not to be zero, there can be at most one path, thus implying that $\boldsymbol{\lambda}^\alpha$ is a circuit or a tour.

Now we find the determinant of A_α when $\boldsymbol{\lambda}^\alpha$ is a circuit or a tour. For a circuit of size k, say $\boldsymbol{\lambda}^\alpha = T$ formed by cycles of lengths $r_1, \ldots, r_k$, we have that the length of the circuit is $l(T) = r_1 + \cdots r_k = n$, and

$$A_\alpha = \begin{pmatrix} c_{r_1} & \cdots & \mathbf{1} \\ \vdots & \ddots & \vdots \\ \mathbf{1} & \cdots & c_{r_k} \end{pmatrix}.$$

Now we make some column swaps to turn this matrix into $J_n - I_n$ where J_n is the matrix of all 1's, and I_n is the identity matrix, both of size n. Note that for every submatrix c_{r_i} of size r_i, we need to take the last row and move it to the first one, which requires $r_i - 1$ swaps. Therefore, with $r_1 - 1 + \cdots + r_k - 1 = n - k$ swaps, we have obtained $J_n - I_n$. It is known that $|J_n - I_n| = (-1)^{n-1}(n-1)$, and that each swap multiplies the determinant by -1, so we have

$$\begin{aligned} |A_\alpha| &= (-1)^{n-k}(-1)^{n-1}(n-1) \\ &= (-1)^{k-1}(n-1). \end{aligned}$$

Therefore, the monomial corresponding to α in $|I - \Lambda A|$ is

$$\begin{aligned} (-1)^{l(T)}\boldsymbol{\lambda}^\alpha|A_\alpha| &= (-1)^n T|A_\alpha| \\ &= (-1)^{n-k-1}(n-1)T \\ &= (-1)^{l(T)-s(T)-1}[l(T)-1]T. \end{aligned} \tag{4.7}$$

Now we find the determinant of A_α when $\boldsymbol{\lambda}^\alpha = F$ is a tour of length n and size $k+1$. Similarily, say F is formed by a path of length p and k cycles of lengths $r_1, \ldots, r_k$. Then the length of the tour is $l(F) = r + r_1 + \cdots + r_k = n$, and

$$A_\alpha = \begin{pmatrix} p_r & \mathbf{1} & \cdots & \mathbf{1} \\ \mathbf{1} & c_{r_1} & \cdots & \mathbf{1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{1} & \mathbf{1} & \cdots & c_{r_k} \end{pmatrix}.$$

As before, doing $r - 1 + r_1 - 1 + \cdots + r_k - 1 = n - k - 1$ swaps we obtain the matrix

$$H = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}.$$

Note that if we denote by $\text{cof}(i)$ the cofactor of the i-th entry in the first row, then

$$|H| = \text{cof}(1) - \text{cof}(2) + \cdots + (-1)^n \text{cof}(n).$$

But for the same reason,

$$|J_n - I_n| = 0 - \text{cof}(2) + \cdots + (-1)^n \text{cof}(n)$$

and $\text{cof}(1) = |J_{n-1} - I_{n-1}|$, so

$$\begin{aligned} |H| &= |J_n - I_n| + |J_{n-1} - I_{n-1}| \\ &= (-1)^{n-1}(n-1) + (-1)^{n-2}(n-2) \\ &= (-1)^{n-1}. \end{aligned}$$

As each swap multiplies the determinant by -1, then

$$|A_F| = (-1)^{n-k-1}(-1)^{n-1} = (-1)^k.$$

Therefore, the monomial corresponding to α in $|I - \Lambda A|$ is

$$\begin{aligned}(-1)^{l(F)}\boldsymbol{\lambda}^{\alpha}|A_\alpha| &= (-1)^n F|A_\alpha| \\ &= (-1)^{n-k}F \\ &= (-1)^{l(F)-s(F)}F. \qquad (4.8)\end{aligned}$$

The result then follows from (4.6), (4.7), and (4.8). □

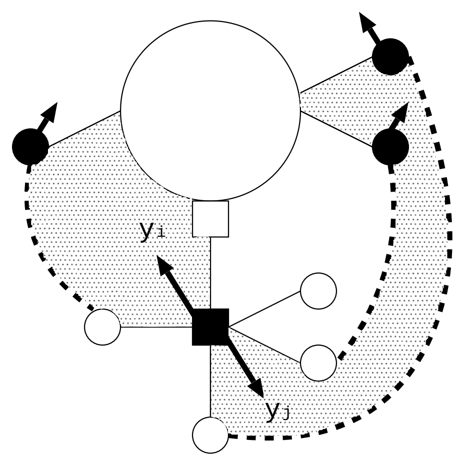

Figure 4.7: Marking each cycle in the factors

Now we shall do the following change of variables Φ. Instead of marking each cycle in the i-th factor with markers of the form $y_{i,j}$ and $y_{j,i}$, for $j \neq i$, we split the marker $y_{i,j}$ into y_i and y_j, each marking the corresponding cycle surrounding the black inner vertex, as seen in Figure 4.7. Now, a cycle of length r in the i-th factor is marked by y_i^{r-1}. Therefore, if the cycles of the i-th factor σ_i have lengths $c_1, \ldots, c_{l(\sigma_i)}$, then the exponent of the variable y_i is $\sum_{i=1}^{l(\sigma_i)}(c_i - 1) = n - l(\sigma_i)$, which still keeps track of the number of cycles of σ_i.

We have been unable to directly apply the change of variables to the solution of the Lagrange equation, but we have some partial results on the determinant of the matrix.

First we find the determinant of $I - M = \Phi(I - L)$ where

$$M_{l,k} = \frac{\lambda_l \lambda_k}{t_k}, \quad \text{if} \quad l \neq k,$$

or

$$M_{l,k} = 0, \quad \text{if} \quad l = k,$$

where t_k is defined as $t_k = \Phi(s_k) = 1 + \sum_{l \neq k} \lambda_l \lambda_k$.

Note that $|I - M| = \Phi(|I - L|)$. The result is summarized in the following theorem.

Theorem 26

$$|I - M| = 1 + \sum_{n \geq 2} (-1)^n (n-1) \frac{\lambda_{a_1}^2 \cdots \lambda_{a_n}^2}{s_{a_1} \cdots s_{a_n}} + \sum_{n \geq 2} (-1)^{n-1} \frac{\lambda_{a_1} \lambda_{a_2}^2 \lambda_{a_3}^2 \cdots \lambda_{a_{n-1}}^2 \lambda_{a_n}}{s_{a_2} s_{a_3} \cdots s_{a_n}}$$

where the sums are taken over all sets $(a_1, \ldots, a_n)$ of positive integers.

Proof. Note that by Theorem 25, every monomial in $|I - L|$ is a circuit or a tour, so the only possible terms in $|I - M|$ are those derived from a circuit or a tour, for example $\frac{\lambda_1^2 \cdots \lambda_n^2}{s_1 \cdots s_n}$ or $\frac{\lambda_1 \lambda_2^2 \lambda_3^2 \cdots \lambda_{n-1}^2 \lambda_n}{s_2 s_3 \cdots s_n}$. By symmetry, all we have to do is determine the coefficient of each one of these two monomials.

If the coefficient comes from a circuit, note that $\frac{\lambda_1^2 \cdots \lambda_n^2}{s_1 \cdots s_n}$ can come from any permutation σ of $(1, 2, \ldots, n)$ with no fixed points, which forms the monomial

$$\frac{\lambda_{1,\sigma(1)}}{s_{\sigma(1)}} \cdots \frac{\lambda_{n,\sigma(n)}}{s_{\sigma(n)}}.$$

If σ has k cycles, then by Theorem 25, the coefficient of $\frac{\lambda_1^2 \cdots \lambda_n^2}{s_1 \cdots s_n}$ is $(-1)^{n-k-1}$. So if Z_k is the subset of S_n of permutations σ that have k cycles and no fixed points, then the desired coefficient is

$$\sum_{k\geq 1}\sum_{\sigma\in Z_k}(n-1)(-1)^{n-k-1} = (-1)^{n-1}(n-1)\sum_{k\geq 1}\sum_{\sigma\in Z_k}(-1)^k \tag{4.9}$$

Note that if we let x be an exponential indeterminate marking the elements and u be an ordinary indeterminate marking the cycles in the permutations, we know that the exponential generating series for the permutations is $\exp\left(u\log\left(\frac{1}{1-x}\right)\right)$. Since we don't want fixed points, we must remove ux from $u\log\left(\frac{1}{1-x}\right)$, so we have

$$\begin{aligned}
\sum_{k\geq 1}\sum_{\sigma\in Z_k}(-1)^k &= \sum_{k\geq 1}(-1)^k\left[\frac{x^n}{n!}u^k\right]\exp\left(u\log\left(\frac{1}{1-x}\right)-ux\right)\\
&= \sum_{k\geq 1}(-1)^k\left[\frac{x^n}{n!}u^k\right](1-x)^{-u}\exp(-ux)\\
&= \left[\frac{x^n}{n!}\right](1-x)^{-u}\exp(-ux)\mid_{u=-1}\\
&= \left[\frac{x^n}{n!}\right](1-x)\exp(x)\\
&= \left[\frac{x^n}{n!}\right]\exp(x)-x\exp(x)\\
&= n!\left(\frac{1}{n!}-\frac{1}{(n-1)!}\right)\\
&= 1-n.
\end{aligned} \tag{4.10}$$

Therefore, by (4.9), the coefficient of the monomial $\frac{\lambda_1^2\cdots\lambda_n^2}{s_1\cdots s_n}$ is $\frac{\lambda_1\lambda_2^2\lambda_3^2\cdots\lambda_{n-1}^2\lambda_n}{s_2 s_3\cdots s_n}$ is

$$(-1)^{n-1}(n-1)(1-n) = (-1)^n(n-1)^2. \tag{4.11}$$

Now, if the coefficient comes from a tour, then it must be of the form $\frac{\lambda_a\lambda_1^2\cdots\lambda_n^2\lambda_b}{s_1\cdots s_n s_b}$. This can come from any tour which is formed by a path and cycles. The path must have as components, a subset $\{d_1,\ldots,d_i\}$ of $\{1,\ldots,n\}$ of size say i, of the form $\frac{\lambda_{d_a,d_1}\lambda_{d_1,d_2}\cdots\lambda_{d_k,d_b}}{s_{d_1},\ldots,s_{d_b}}$. The remaining cycles must have combined length j, where $i+j=n$, and their product must be of the form $\frac{\lambda_{e_1,e_2}\cdots\lambda_{e_j,e_1}}{s_{e_1}\cdots s_{e_j}}$, where $\{e_1,\ldots,e_j\}$ is some permutation of the complement of the set $\{d_1,\ldots,d_i\}$, then the tour has size $k+1$ and length n, so its coefficient is $(-1)^{n-k-1}$ by Theorem 25. Since there are $\binom{n}{i}$ ways to choose the subset of size i, then by the reasoning above, the coefficient of this tour is

$$\begin{aligned}
\sum_{i+j=n}(-1)^{n-j-1}\binom{n}{i}i!\sum_{\sigma\in Z_k}1 &= (-1)^{n-1}\sum_{i+j=n}\binom{n}{i}i!\sum_{\sigma\in Z_k}(-1)^j \\
&= (-1)^{n-1}\left[\frac{x^n}{n!}\right]\left(\sum_{i\geq 0}\frac{i!}{i!}x^i\right)\left(\sum_{j\geq 0}\frac{1-j}{j!}x^j\right) \quad \text{by (4.10)} \\
&= (-1)^{n-1}\left[\frac{x^n}{n!}\right]\frac{1}{1-x}\{\exp(x)-x\exp(x)\} \\
&= (-1)^{n-1}\left[\frac{x^n}{n!}\right]\exp(x) \\
&= (-1)^{n-1}. \qquad (4.12)
\end{aligned}$$

By (4.10) and (4.12), the result follows. □

Now, let e_i be the i-th elementary symmetric function on the variables $x_1, x_2, \ldots$. The following theorem shows a possible route one might consider when trying to evaluate this determinant.

Theorem 27 *If we let $s_i = 1$ for every i, then the determinant factors as*

$$\det(I-M) = (1-e_2-2e_3-3e_4-\cdots)(1-e_2+2e_3-3e_4+\cdots)$$

Proof. For a partition $(a_1, a_2, \ldots)$ of n, denote the monomial symmetric function as the sum of distinct $x_{\sigma(1)}^{a_1}x_{\sigma(2)}^{a_2}\cdots$ over permutations σ of $\{1, 2, \ldots\}$. Note that any monomial in the product $(1-e_2-2e_3-3e_4-\cdots)(1-e_2+2e_3-3e_4+\cdots)$ must be of the form $m_{2,\ldots,2,1,\ldots,1}$. If there are r 2's and s 1's in this partition, then we denote the symmetric function by $m_{2^r1^s}$. We shall consider 5 cases for this, namely:

Case 1. $r \neq 0$ and $r - s = 0$.

As the functions are symmetric, all we need is to find the coefficient of $\lambda_1^2\cdots\lambda_r^2$. Note that this only appears as a product of $-(r-1)\lambda_1\cdots\lambda_r$ in the left factor $(1-e_2-2e_3-3e_4-\cdots)$, and $(-1)^r(r-1)\lambda_1\cdots\lambda_r$ in the right factor $(1-e_2+2e_3-3e_4+\cdots)$. Therefore, its coefficient is $(-1)^r(r-1)^2$.

Case 2. $r \neq 0$ and $r - s = 1$.

By a similar reasoning, there are only two ways this coefficient may appear as a product of coefficients in the left and right factors, namely

$$[-(r-1)\lambda_1\cdots\lambda_r][(-1)^r r\lambda_1\cdots\lambda_{r+1}]+[-r\lambda_1\cdots\lambda_{r+1}][(-1)^{r-1}(r-1)\lambda_1\cdots\lambda_r]$$

Which is clearly 0.

Case 3. $r\neq 0$ and $r-s=2$.

By the same type of reasoning as above, considering the factors from the left and the right, there are four summands that add to the coefficient of the monomial $\lambda_1^2\cdots\lambda_r^2\lambda_{r+1}\lambda_{r+2}$, namely

$$[-(r-1)\lambda_1\cdots\lambda_r][(-1)^{r+1}(r+1)\lambda_1\cdots\lambda_{r+2}],$$

$$[-r\lambda_1\cdots\lambda_r\lambda_{r+1}][(-1)^r r\lambda_1\cdots\lambda_r\lambda_{r+2}],$$

$$[-r\lambda_1\cdots\lambda_r\lambda_{r+2}][(-1)^r r\lambda_1\cdots\lambda_r\lambda_{r+1}],$$

$$[-(r+1)\lambda_1\cdots\lambda_{r+2}][(-1)^{r-1}(r-1)\lambda_1\cdots\lambda_r].$$

These summands add to

$$(-1)^{r+2}(r+1)(r-1)+2(-1)^{r+1}r^2+(-1)^r(r-1)(r-1)=2(-1)^{r-1}.$$

Case 4. $r\neq 0$ and $r-s\geq 2$.

Note that for every subset $\{a_1,\ldots a_i\}$ of $\{r+1,\ldots,r+s\}$, with complement $\{a_{i+1},\ldots a_s\}$ we can get the monomial from a term in the left factor and a term in the right factor as

$$[-(r+i-1)\lambda_1\cdots\lambda_r\lambda_{a_1}\cdots\lambda_{a_i}][(-1)^{r+s-i-1}(r+s-i-1)\lambda_1\cdots\lambda_r\lambda_{a_{i+1}}\cdots\lambda_{a_s}].$$

The coefficient in the product is the sum of these coefficients, which is

$$\begin{aligned}\sum_{i=0}^{k}(-1)^{r+s-1}\binom{s}{i}(r+i-1)(r+s-i-1) &= (-1)^{r+s}(r^2+rs-2r-s+1)\sum_{i=0}^{s}(-1)^i\binom{s}{i} \\ &+(-1)^{r+s}s\sum_{i=0}^{s}(-1)^i\binom{s}{i}i \\ &+(-1)^{r+s-1}\sum_{i=0}^{s}(-1)^i\binom{s}{i}i^2.\end{aligned}$$

The last three summands are zero, because for every $j \leq s$ we have

$$\sum_{i=0}^{s}(-1)^i\binom{s}{i}i^j = \left(x\frac{\partial}{\partial x}\right)^r (x+1)^k \mid_{x=-1}= 0,$$

which means the coefficient is zero.

Case 5. $r = 0$

Note that the only way we can obtain the coefficient $\lambda_1 \cdots \lambda_s$ is by partitioning the set $\{1, \dots s\}$ into two disjoint sets $\{a_1, \dots a_i\}$ and $\{a_{i+1}, \dots a_s\}$, and taking the monomial $\lambda_{a_1} \cdots \lambda_{a_i}$ from the left factor, and the monomial $\lambda_{a_{i+1}}, \dots \lambda_{a_s}$ from the right factor. When $i = 0$ the monomial is

$$(-1)(s-1)\lambda_{a_1}\cdots\lambda_{a_s}.$$

When $1 \leq i \leq s-1$ it is

$$\binom{s}{i}(i-1)\lambda_{a_1}\cdots\lambda_{a_i})(k-i-1)(-1)^{s-i-1}\lambda_{a_{i+1}}\cdots\lambda_{a_s}.$$

Finally, when $i = s$ it is

$$(-1)(k-1)\lambda_1\cdots\lambda_s.$$

These coefficients add to

$$\sum_{i=0}^{s}\binom{s}{i}(i-1)(s-i-1)(-1)^{s-i-1}.$$

The last term can also be written as a linear combination of $\sum_{i=0}^{s}(-1)^i\binom{s}{i}i^j$ for $j = 0, 1$, and 2, so it is equal to 0 by the final observation in Case 4.

Therefore, by putting together Cases 1, 2, 3, 4, and 5, we obtain

$$1 + \sum_{\substack{n \geq 2 \\ \{a_1, \ldots a_n\} \subset \mathbb{Z}}} (-1)^n (n-1) \lambda_{a_1}^2 \cdots \lambda_{a_n}^2 + \sum_{\substack{n \geq 2 \\ \{a_1, \ldots a_n\} \subset \mathbb{Z}}} (-1)^{n-1} \lambda_{a_1} \lambda_{a_2}^2 \lambda_{a_3}^2 \cdots \lambda_{a_{n-1}}^2 \lambda_{a_n}$$
$$= (-1)^{n-1}(n-1) m_{2^n} + (-1)^{n-1} m_{2^{n-2}1^2},$$

and the result follows by Theorem 26. □

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