Risk Sharing and Risk Aggregation via Risk Measures

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Actuarial Science

Waterloo, Ontario, Canada, 2017

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Risk measures have been extensively studied in actuarial science in the guise of premium calculation principles for more than 40 years, and recently, they have been the standard tool for financial institutions in both calculating regulatory capital requirement and internal risk management. This thesis focuses on two topics: risk sharing and risk aggregation via risk measures. The problem of risk sharing concerns the redistribution of a total risk among agents using risk measures to quantify risks. Risk aggregation is to study the worst-case value of aggregate risks over all possible dependence structures with given marginal risks.

On the first topic, we address the problem of risk sharing among agents using a two-parameter class of quantile-based risk measures, the so-called Range-Value-at-Risk (RVaR), as their preferences. The family of RVaR includes the Value-at-Risk (VaR) and the Expected Shortfall (ES), the two popular and competing regulatory risk measures, as special cases. We first establish an inequality for RVaR-based risk aggregation, showing that RVaR satisfies a special form of subadditivity. Then, the Pareto-optimal risk sharing problem is solved through explicit construction. We also study risk sharing in a competitive market and obtain an explicit Arrow-Debreu equilibrium. Robustness and comonotonicity of optimal allocations are investigated, and several novel advantages of ES over VaR from the perspective of a regulator are revealed.

Reinsurance, as a special type of risk sharing, has been studied extensively from the perspective of either an insurer or a reinsurer. To take the interests of both parties into consideration, we study Pareto optimality of reinsurance arrangements under general model settings. We give the necessary and sufficient conditions for a reinsurance contract to be Pareto-optimal and characterize all such optimal contracts under more general model assumptions. Sufficient conditions that guarantee the existence of the Pareto-optimal contracts are obtained. When the losses of an insurer and a reinsurer are measured by the ES risk measures, we obtain the explicit forms of the Pareto-optimal reinsurance contracts under the expected value premium principle.

On the second topic, we first study the aggregation of inhomogeneous risks with a special type of model uncertainty, called dependence uncertainty, in individual risk models. We establish general asymptotic equivalence results for the classes of distortion risk measures and convex risk measures under different mild conditions. The results implicitly suggest that it is only reasonable to implement a coherent risk measure for the aggregation of a large number of risks with dependence uncertainty. Then, we bring the well studied dependence uncertainty in individual risk models into collective risk models. We study the worst-case values of the VaR and the ES of the aggregate loss with identically distributed individual losses, under two settings of dependence uncertainty: (i) the counting random variable and the individual losses are independent, and the dependence of the individual losses is unknown; (ii) the dependence of the counting random variable and the individual losses is unknown. Analytical results for the worst-case values of ES are obtained. For the loss from a large portfolio of insurance policies, the asymptotic equivalence of VaR and ES is established, and approximation errors are obtained under the two dependence settings.

Acknowledgements

First of all, I would like to express my deepest gratitude to my supervisors Professor Jun Cai and Professor Ruodu Wang. I am really lucky to be their student, and I benefited a lot from their advice and expertise during my years at the University of Waterloo. Without their guidance and support, I would not have completed this thesis. Professor Cai's critical thinking questions and Professor Wang's sharp insights significantly improved this work. Professor Wang has been very generous with his time, and I have enjoyed the numerous and inspirational meetings we had.

I would also like to thank the rest of my PhD thesis committee: Professor David Laudriault, Professor Xuyang Ma, Professor Etienne Marceau, and Professor Yi Shen, for their valuable time and suggestions.

I was lucky to meet Tiantian at the University of Waterloo, who has been a great friend. Her strong sense of humour made my graduate studies full of laughter and wonderful memories, and I am grateful for her help and encouragement during her stay in Waterloo. I thank Tiantian and Bin for helpful discussions. I would like to thank all of my friends in Waterloo for the enjoyable time we had here.

Finally, but most importantly, I would like to thank my parents and my sisters, in particular my mother, who has always been my cheerleader. Without your unconditional love and support, I would not be where I am today.

Dedicated to my parents

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Chapter 1

Introduction

1.1 Background

What is risk? Risk refers to "hazard, a chance of bad consequences, loss or exposure to mischance" in the Concise Oxford English Dictionary. However, risk means not only possible losses, but also possible gains, which is related to an uncertain future value of a position such as buying a stock. Mathematically, risk is characterized by randomness that can be measured precisely (see e.g. Knight (1921)), and it is modelled by loss random variables. A negative realization of a loss random variable indicates a gain. In this thesis, we consider risks in the context of finance and insurance.

The standard tool to measure risks is risk measures. A risk measure is a mapping from a set of risks to real numbers, and it has to be implemented with certain models, either internal models of a financial institution or external models designed by the regulator. The most popular risk measures in practice are the Value-at-Risk (VaR) and the Expected Shortfall (ES, or Tail-Value-at-Risk). Both are implemented in modern financial and insurance regulation. There have been extensive debates on the comparative advantages of VaR and ES in regulation; the reader is referred to the survey papers Embrechts et al. (2014), Emmer et al. (2015), and Föllmer and Weber (2015). Related debates in regulatory documents by the Basel Committee on Banking Supervision and the International Association of Insurance Supervisors can be found in BCBS (2013) and IAIS (2014). In particular, BCBS (2013) proposed a shift from VaR to ES for determinating capital charges of internal models although it is difficult to back-test ES and ES is not elicitable. Whereas there is a tendency to move from VaR to ES, for a while to come both risk measures will coexist for regulatory purposes. Our results in Chapter 2 add some guidance potentially useful in reaching more widely acceptable solutions.

Denneberg (1990) and Wang (1996) introduced the *distortion risk measures* and later Wang et al. (1997) used an axiomatic approach to characterize the price of an insurance risk as a Choquet integral representation with respect to a distorted probability. For more developments on distortion risk measures, see e.g. Kusuoka (2001), Frittelli and Rosazza Gianin (2002), Song and Yan (2009), Dhaene et al. (2012), Grigorova (2014), Wang et al. (2015) and the references therein. Artzner et al. (1999) presented four axioms for the so called *coherent risk measures* and Kusuoka (2001) characterized *law-invariant* coherent risk measures with the Fatou property. Föllmer and Schied (2002) introduced the concept of *convex risk measures* and proved a corresponding representation theorem, which was further generalized by Frittelli and Rosazza Gianin (2002, 2005) and Kaina and Rüschendorf (2009).

Risk measures have been extensively studied in insurance in the guise of premium calculation principles for more than 40 years (see e.g. Bühlmann (1970), Deprez and Gerber (1985), Wang et al. (1997)). It can be used to determine the insurance premium for transferring part of a risk from an insurer to a reinsurer. Moreover, as the insurer's or the reinsurer's objective functional, risk measures come into play in reinsurance optimization problems. On the one hand, the insurer reduces his or her risk exposure by buying a reinsurance contract. On the other hand, the insurer has to incur additional cost in the form of reinsurance premium payable to the reinsurer. Naturally, the more of a risk is transferred to the reinsurer, the more of the reinsurance premium is. So there is a risk and reward tradeoff faced by the insurer or the reinsurer. Optimal reinsurance designs from either the insurer's perspective or the reinsurer's point of view have been well investigated in the literature. However, as pointed out by Borch (1969), "there are two parties to a reinsurance contract, and that an arrangement which is very attractive to one party, may be quite unacceptable to the other." Hence, an interesting question in optimal reinsurance designs is to consider the interests of both the insurer and the reinsurer; see e.g. Borch (1960). And we study Pareto-optimal reinsurance contracts by minimizing the convex combination of the objective functionals of both parties under a general reinsurance setting. A Pareto-optimal reinsurance policy is one in which neither of the two parties can be better off without making the other worse off, and hence Pareto optimality is a good starting point to study reinsurance problems when an insurer and a reinsurer have conflicting interests.

In the past two decades, risk measures have also been the standard tool for financial institutions in both calculating regulatory capital requirement and internal risk management. The value $\rho(X)$ assigned by the risk measure ρ to a risk X is the amount of cash, or the regulatory capital requirements that a bank or a financial institution has to hold so that taking the risk X is acceptable for the regulator. This risk X could be an individual risk or a sum of individual risks $X_1 + \cdots + X_n$, where $X_1, \dots, X_n$ represent different risks or the claims of a portfolio. Generally, the regulator wants a company to hold sufficiently high level of capital so that the company will meet its obligations, but the company may seek a way such as risk sharing to minimize regulatory capital because holding too much capital will be costly.

The problem of risk sharing concerns the redistribution of a total risk X into n parts $X_1, \dots, X_n$ with $X_1 + \cdots + X_n = X$. For $i = 1, \dots, n$, the redistributed risk X_i is allocated to agent i who is equipped with a *monetary risk measure* ρ_i. The target is to optimize over all possible allocations $(X_1, \dots, X_n)$ such that the following sum

$$\sum_{i=1}^{n} \rho_i(X_i) \quad \text{subject to } X_1 + \cdots + X_n = X, \tag{1.1.1}$$

is minimized. Such an allocation $(X_1, \dots, X_n)$ is called an *optimal allocation* of X, and for monetary risk measures, optimal allocations are Pareto-optimal (see Section 2.4.1).

The risk sharing problem (1.1.1) can be formulated in various contexts; below we list a few interpretations.

(i) **Regulatory capital reduction for a single firm.** In this context, we consider a firm with a one-period total risk X, and a risk measure ρ is used to calculate the regulatory capital needed for holding the risk. Assume that the firm has n separate affiliates, and decides to split the total position X over the n affiliates. Then the total capital this firm is required to hold is $\sum_{i=1}^{n} \rho(X_i)$ subject to $X_1 + \cdots + X_n = X$. The minimization of $\sum_{i=1}^{n} \rho(X_i)$ is a special form of problem (1.1.1). Under this setting as well as the next one, risk measures are regulatory capital principles; this is the the original interpretation of risk measures as introduced in Artzner et al. (1999).

(ii) **Regulatory capital reduction for a group of firms.** In this context, there are n firms in an economy and firm i is required to hold a regulatory capital $\rho(X_i)$ for taking a risk X_i, $i = 1, \dots, n$. The n firms may want to share the total risk X, while minimizing $\sum_{i=1}^{n} \rho(X_i)$ subject to $X_1 + \cdots + X_n = X$, is a special form of problem (1.1.1).

(iii) **Insurance-reinsurance contracts and risk-transfer.** In this context, agent 1 is an insured and agent 2 is an insurer, or agent 1 is an insurance company and agent 2 is a reinsurer (there may also be more than one reinsurer involved, and in that case agents $2, 3, \ldots, n$ represent reinsurers involved). ρ_1 is the disutility functional[1] of the insured and ρ_2 is the pricing function of the insurer. X represents the initial risk the insured faces, and X_2 represents the portion of risk which the insured would like to transfer to the insurer by paying $\rho_2(X_2)$ as the insurance premium. Under this setting, the insured would like to minimize over $\rho_1(X_1 + \rho_2(X_2))$ subject to $X_1 + X_2 = X$, which is the overall disutility of her retained loss. For a monetary risk measure ρ_1, $\rho_1(X_1+\rho_2(X_2)) = \rho_1(X_1)+\rho_2(X_2)$, and hence this set-up corresponds to problem (1.1.1).

(iv) **Risk redistribution among investors.** In this context, investors $1, \ldots, n$ hold respective risks (or assets) $\xi_1, \ldots, \xi_n$. They seek for a redistribution $(X_1, \ldots, X_n)$ of the total risk $X = \xi_1 + \cdots + \xi_n$ so that (1.1.1) is minimized, and each of the investors is better-off compared to their initial position, that is, $\rho(X_i) \leqslant \rho(\xi_i)$, $i = 1, \ldots, n$. Under this setting, $\rho_1, \ldots, \rho_n$ are disutility functionals of the investors.

As the most commonly used families of risk measures, VaR and ES are unified in a more general two-parameter family of risk measures, called the Range-Value-at-Risk (RVaR). The family of RVaR was introduced in Cont et al. (2010) as a *robust* risk measure. More importantly, RVaR can be seen as a bridge connecting VaR and ES. This embedding of VaR and ES into RVaR helps us to understand properties and comparative advantages of the former risk measures, and hence we choose RVaR as the underlying risk measures in the problem of risk sharing discussed in Chapter 2. We will focus on two different and also well connected risk sharing problems: cooperative risk sharing and competitive risk sharing. The former aims to find Pareto-optimal allocations with respect to the sum of individual risk measures or (1.1.1), while the latter aims to find equilibrium allocations with respect to each individual risk measure since generally every agent acts in their own interest.

Besides risk sharing, another topic we are interested in is risk aggregation under dependence uncertainty. Typically, a risk measure has to be implemented with certain models, either internal models of a financial institution or external models designed by the regulator. By specifying a

[1] A disutility functional ρ of an agent describes her preference: for risks X and Y, she prefers X to Y if and only if $\rho(X) \leqslant \rho(Y)$. A risk measure can be interpreted as a disutility functional.

model, uncertainty always arises as an important issue in practice. One particular type of uncertainty that we focus on is the dependence uncertainty in risk aggregation. In the framework of dependence uncertainty, we assume that in a joint model $(X_1, \dots, X_n)$, the marginal distribution of each of $X_1, \dots, X_n$ is known, but the joint distribution is unknown. Denote by $\mathcal{F}$ the set of univariate distribution functions. For $F_1, \dots, F_n \in \mathcal{F}$, let

$$\mathcal{S}_n = \mathcal{S}_n(F_1, \dots, F_n) = \{X_1 + \cdots + X_n : X_i \in L^0,\ X_i \sim F_i,\ i = 1, \dots, n\}.$$

That is, $\mathcal{S}_n$ is the set of aggregate risks with given marginal distributions, but an arbitrary dependence structure.

For a given risk measure ρ and some joint model $(X_1, \dots, X_n)$ with unknown dependence structure, we are interested in the value of the risk aggregation $\rho(X_1 + \cdots + X_n)$. Obviously, $\rho(X_1 + \cdots + X_n)$ lies in a range, and oftentimes the worst-case and the best-case values are of interest. The value $\bar{\rho}(\mathcal{S}_n) := \sup_{S \in \mathcal{S}_n} \rho(S)$ represents the worst-case measurement of the aggregate risk in the presence of dependence uncertainty. If ρ is not coherent, the value of $\bar{\rho}(\mathcal{S}_n)$ is in general difficult to calculate. In Chapter 4, we aim to find an approximation of $\bar{\rho}(\mathcal{S}_n)$ when ρ is a non-coherent distortion or convex risk measure for large n. In other words, we show the asymptotic equivalence of $\bar{\rho}(\mathcal{S}_n)$ and another quantity $\bar{\rho^*}(\mathcal{S}_n)$, where ρ^* is the smallest law-invariant coherent risk measure dominating ρ.

Moreover, we bring the framework of dependence uncertainty into *collective risk models*. Suppose

$$S_N = Y_1 + Y_2 + \cdots + Y_N, \tag{1.1.2}$$

where $Y_1, Y_2, \dots$ are random variables representing claims sizes and N is the number (random or deterministic) of claims that takes values in non-negative integers. Equation (1.1.2) is called a *collective risk model* (an *individual risk model*) when N is random (deterministic). In practice, the claims or losses $Y_1, Y_2, \dots,$ in individual risk models or collective risk models are dependent, and they may also be dependent on the number of claims N. We assume that $Y_1, Y_2, \dots$ are identically distributed, but we do not assume a particular model for the dependence structure among random variables in (1.1.2). Two practical settings of dependence will be considered in Chapter 5:

(i) N is independent of $Y_1, Y_2, \dots$ and the dependence structure of $Y_1, Y_2, \dots$ is unknown.

(ii) The dependence structure of $N, Y_1, Y_2, \ldots$ is unknown.

From the perspective of risk management, we are interested in quantifying S_N by certain risk measures under dependence uncertainty, a crucial concern for risk management in the presence of model uncertainty. In particular, we study the worst-case values of $\mathrm{VaR}_\alpha(S_N)$ and $\mathrm{ES}_\alpha(S_N)$, under the two settings (i) and (ii) above, which may find application in ruin theory.

1.2 Preliminaries

1.2.1 Generalized Inverse and Quantile Functions

Definition 1.1. For a non-decreasing function $F : \mathbb{R} \to \mathbb{R}$ with $F(-\infty) = \lim_{x\downarrow-\infty} F(x)$ and $F(\infty) = \lim_{x\uparrow\infty} F(x)$, the *generalized inverse* $F^{-1} : \mathbb{R} \to \bar{\mathbb{R}} = [-\infty, \infty]$ of F is defined by

$$F^{-1}(y) = \inf\{x \in \mathbb{R} : F(x) \geqslant y\},\ \ y \in \mathbb{R}, \tag{1.2.1}$$

with the convention that $\inf \emptyset = \infty$. If $F : \mathbb{R} \to [0, 1]$ is a distribution function, $F^{-1} : [0, 1] \to \bar{\mathbb{R}}$ is also called the *quantile function* of F.

Proposition 1.1 (Proposition 1 of Embrechts and Hofert (2013))**.** *Let $F : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function with $F(-\infty) = \lim_{x\downarrow-\infty} F(x)$ and $F(\infty) = \lim_{x\uparrow\infty} F(x)$.*

(i) F^{-1} is non-decreasing. If $F^{-1}(y) \in (-\infty, \infty)$, F^{-1} is left continuous at y and admits a limit from the right at y.

(ii) $F^{-1}(F(x)) \leqslant x$. If F is strictly increasing, $F^{-1}(F(x)) = x$.

(iii) Let F be right continuous. Then $F^{-1}(y) < \infty$ implies $F(F^{-1}(y)) \geqslant y$.

(iv) $F(x) \geqslant y$ implies $x \geqslant F^{-1}(y)$. The other implication holds if F is right-continuous. Furthermore, $F(x) < y$ implies $x \leqslant F^{-1}(y)$.

(v) F is continuous if and only if F^{-1} is strictly increasing on $[\inf \mathrm{ran}\ F, \sup \mathrm{ran}\ F]$, where $\mathrm{ran}\ F := \{F(x) : x \in \mathbb{R}\}$ is the range of F.

(vi) F is strictly increasing if and only if F^{-1} is continuous on $\mathrm{ran}\ F$.

For any distribution function F and $\alpha \in (0,1)$, $F^{-1}(\alpha)$ is as defined in (1.2.1). Define U_X as a uniform random variable on $[0,1]$ such that $F^{-1}(U_X) = X$ almost surely, where F is the distribution function of the random variable X. If X is continuously distributed, $U_X = F(X)$ almost surely. For a general random variable X, the existence of U_X is guaranteed; see for instance Proposition 1.3 of Rüschendorf (2013). Moreover, $F^{-1}(U) \overset{d}{=} X$, where U is any U$[0,1]$-distributed random variable.

1.2.2 Common Risk Measures

Assume that all risks are defined on an atomless probability space $(\Omega, \mathcal{A}, \mathbb{P})$ throughout the thesis. A probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is said to be atomless if $A \in \mathcal{A}$ and $\mathbb{P}(A) > 0$ imply that there exists a $B \in \mathcal{A}$ such that $B \subset A$ and $0 < \mathbb{P}(B) < \mathbb{P}(A)$. Let L^p be the set of all random variables in $(\Omega, \mathcal{A}, \mathbb{P})$ with finite p-th moment, $p \in [0, \infty)$, L^∞ be the set of essentially bounded random variables, and L^+ be the set of non-negative random variables. A functional on L^p is said to be L^p-continuous, $p \in [1, \infty]$, if it is continuous with respect to the L^p-norm. We treat almost surely equal random variables as identical.

A *risk measure* is a mapping $\rho : \mathcal{X} \to (-\infty, +\infty]$, where $\mathcal{X}$ is a set of risks and it is a convex cone such that $L^\infty \subset \mathcal{X} \subset L^0$ ($\subset$ is the non-strict set inclusion). $\mathcal{X}$ will be specified for particular risk measures. Below we list some standard properties studied in the literature of risk measures. For any $X, Y \in \mathcal{X}$:

(a) *Monotonicity*: if $X \leqslant Y$ $\mathbb{P}$-a.s, then $\rho(X) \leqslant \rho(Y)$;

(b) *Cash-invariance*: for any $m \in \mathbb{R}$, $\rho(X - m) = \rho(X) - m$;

(c) *Convexity*: for any $\lambda \in [0,1]$, $\rho(\lambda X + (1-\lambda)Y) \leqslant \lambda\rho(X) + (1-\lambda)\rho(Y)$;

(d) *Subadditivity*: $\rho(X + Y) \leqslant \rho(X) + \rho(Y)$;

(e) *Positive homogeneity*: for any $\alpha > 0$, $\rho(\alpha X) = \alpha\rho(X)$;

(f) *Law-invariance*: if X and Y have the same distribution under $\mathbb{P}$, denoted as $X \overset{d}{=} Y$, then $\rho(X) = \rho(Y)$.

We refer to Föllmer and Schied (2016, Chapter 4) and Delbaen (2012) for interpretations of these standard properties of risk measures.

Definition 1.2. A *monetary risk measure* is a risk measure satisfying (a) and (b), a *convex risk measure* is a risk measure satisfying (a)-(c), and a *coherent risk measure* is a risk measure satisfying (a),(b),(d), and (e).

Definition 1.3. The Value-at-Risk (VaR) of a random variable X at confidence level $\alpha \in (0,1)$ is defined as

$$\mathrm{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leqslant x) \geqslant \alpha\}, \quad X \in L^0, \tag{1.2.2}$$

and the Tail-Value-at-Risk (TVaR) or Expected Shortfall (ES) of a random variable X at confidence level $\alpha \in (0,1)$ is defined as

$$\mathrm{TVaR}_\alpha(X) = \mathrm{ES}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_\gamma(X)\mathrm{d}\gamma, \quad X \in L^0, \tag{1.2.3}$$

where $\mathrm{VaR}_\gamma(X)$ is defined in (1.2.2). In general TVaR can be infinite for non-integrable random variables.

Remark 1.1. Note that we use the notation ES in Chapters 2, 4, and 5 and TVaR in Chapter 3 as ES is more often used in finance while TVaR appears more in insurance. Moreover, in Chapters 1, 3, 4, and 5, we adopt Definition 1.3 for VaR and TVaR (or ES) with $\mathrm{VaR}_\alpha(X)$ being the $100\alpha\%$ quantile of the random variable X for $\alpha \in (0,1)$. However, to simplify our results in Chapter 2 (see Remark 2.1), we redefine $\mathrm{VaR}_\alpha(X)$ as the $100(1-\alpha)\%$ quantile of the random variable X and the same notation is applied to ES in Chapter 2.

Both VaR and TVaR are monotone, cash-invariant, positive homogeneous, and law-invariant, but VaR is not necessarily subadditive while TVaR always is. A simple example of VaR being not subaddtive is that for $\alpha \in (0, 0.715)$, $\mathrm{VaR}_\alpha(X_1 + X_2) > \mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2)$, where X_1 and X_2 are independent and identically distributed as exponential distribution with mean 1.

VaR_α has been criticized for taking no consideration of the risk beyond the confidence level α and the lack of subadditivity. TVaR_α is proposed as an alternative risk measure which is a coherent risk measure and takes the values beyond the confidence level α into account. For a continuous random variable X, $\mathrm{TVaR}_\alpha(X)$ is the expected loss given that X exceeds $\mathrm{VaR}_\alpha(X)$. See McNeil et al. (2015) for properties of the two regulatory risk measures and for discussions on the various uses and misuses of VaR as a regulatory risk measure in Quantitative Risk Management.

The following proposition provides some well-known properties of VaR and TVaR which will be used in later chapters.

Proposition 1.2. *For a random variable X and $\alpha \in (0,1)$, we have*

(i) $\mathrm{VaR}_\alpha(X) \leqslant \mathrm{TVaR}_\alpha(X)$ *and* $\mathrm{TVaR}_\alpha(X)$ *is a non-decreasing function of* α*;*

(ii) For any non-decreasing and left continuous function g, $\mathrm{VaR}_\alpha(g(X)) = g(\mathrm{VaR}_\alpha(X))$*;*

(iii) The TVaR *in* (1.2.3) *is equivalent to*

$$\mathrm{TVaR}_\alpha(X) = \mathrm{VaR}_\alpha(X) + \frac{1}{1-\alpha}\mathbb{E}\left[(X - \mathrm{VaR}_\alpha(X))_+\right], \quad X \in L^1. \tag{1.2.4}$$

Moreover, for all $X \in L^1$, *we have*

$$\mathbb{E}[X] = \int_0^1 \mathrm{VaR}_\gamma(X)\mathrm{d}\gamma.$$

Now we introduce the class of *distortion risk measures* (see e.g. Wang et al. (1997)), which includes VaR and TVaR as special cases. Let $\mathcal{H}$ be the set of increasing (in the non-strict sense) functions h supported on $[0,1]$ with $h(0) = h(0^+) = 0$ and $h(1^-) = h(1) = 1$.

Definition 1.4. A distortion risk measure $\rho_h : \mathcal{X} \to (-\infty, \infty]$ with a *distortion function* $h \in \mathcal{H}$ is defined as

$$\rho_h(X) = \int_{\mathbb{R}} x\mathrm{d}h(F(x)), \quad X \in \mathcal{X},\ X \sim F, \tag{1.2.5}$$

provided that (1.2.5) is well-posed for all $X \in \mathcal{X}$. Note that for a given set $\mathcal{X}$, h may need to satisfy some conditions to avoid some ill-posed cases. If $\mathcal{X}$ is either L^∞ or L^+, (1.2.5) is well-posed for all $h \in \mathcal{H}$ and $X \in \mathcal{X}$.

Note that the distortion functions of VaR_α and TVaR_α are $h_1(t) = \mathrm{I}_{\{t \geqslant \alpha\}}$ and $h_2(t) = \mathrm{I}_{\{t \geqslant \alpha\}}\frac{t-\alpha}{1-\alpha}$, respectively. When h is continuous, through a change of variable, ρ_h can be written as

$$\rho_h(X) = \int_0^1 \mathrm{VaR}_t(X)\mathrm{d}h(t), \quad X \in \mathcal{X}. \tag{1.2.6}$$

Any distortion risk measure ρ_h is monotone, cash-invariant, positively homogeneous, and law-invariant. ρ_h is subadditive if and only if h is convex; this dates back to Yaari (1987, Theorem 2). The key feature which characterizes ρ_h is *comonotonic additivity*. Let us first recall the definition of comonotonic random variables.

Definition 1.5. Two random variables X and Y are *comonotonic* if

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geqslant 0 \quad \text{for} \quad (\omega, \omega') \in \Omega \times \Omega \quad (\mathbb{P} \times \mathbb{P})\text{-a.s.}$$

Comonotonicity of X and Y is equivalent to the existence of a random variable $Z \in L^0$ and two non-decreasing functions f and g, such that $X = f(Z)$ and $Y = g(Z)$ almost surely. See Dhaene et al. (2002) for an overview on comonotonicity.

(g) *Comonotonic additivity*: $\rho(X + Y) = \rho(X) + \rho(Y)$ if X and Y are comonotonic.

For a subadditive risk measure ρ interpreted as a tool for capital calculation, comonotonic additivity is particularly important: For comonotonic risks X and Y, the lack of comonotonic additivity (that is, $\rho(X+Y) < \rho(X)+\rho(Y)$) means a diversification benefit (reduction in capital) for non-diversified risks, an undesirable property for risk management.

1.3 Outline of the Thesis

This thesis is organized as follows. In Chapter 2, we first establish a powerful inequality for the RVaR family, which is crucial in proving the main results on quantile-based risk sharing, and it implies that the risk measures RVaR, including VaR and ES as special cases, satisfy a special form of subadditivity. The optimal risk sharing problem with different RVaRs as the risk measures is solved explicitly. A Pareto-optimal allocation is given through an explicit construction. We also study competitive risk sharing in which each agent optimizes their own preferences, regardless of other participants. Under suitable assumptions, we obtain an Arrow-Debreu equilibrium, which is the same as the aforementioned Pareto-optimal allocation. Moreover, the equilibrium pricing rule can be obtained explicitly. Robustness and comonotonicity of optimal allocations are discussed. As a consequence, several novel advantages of ES-based risk management are revealed.

In Chapter 3, we give the necessary and sufficient conditions for a reinsurance contract to be Pareto-optimal and characterize all Pareto-optimal reinsurance contracts under a general reinsurance setting. When the risk measures are chosen as TVaRs, we find explicit Pareto-optimal reinsurance contracts under the expected value premium principle.

In Chapter 4, we study the aggregate risk of inhomogeneous risks with dependence uncertainty, evaluated by a generic risk measure. We first present two examples showing that without some regularity conditions the asymptotic equivalence of two risk measures may fail to hold. Then we study the asymptotic equivalence for distortion or convex risk measures under different conditions. In Chapter 5, we bring the framework of dependence uncertainty in Chapter 4 into collective risk models. We first derive some convex ordering inequalities for collective risk models and thereby obtain analytical formulas for the worst-case values of ES. Asymptotic equivalence of the worst-case values of $\mathrm{VaR}_\alpha(S_N)$ and $\mathrm{ES}_\alpha(S_N)$ under dependence settings (i) and (ii) are given.

Chapter 6 ends the thesis with some concluding remarks.

The thesis resembles the following papers and manuscripts: Embrechts et al. (2017), Cai et al. (2017a,b), and Liu and Wang (2017).

Chapter 2

Quantile-based Risk Sharing

2.1 Introduction

2.1.1 Risk Sharing Problems and Quantile-based Risk Measures

The problem of risk sharing concerns the redistribution of a total risk (random variable) X into n parts $X_1, \dots, X_n$ with $X_1 + \cdots + X_n = X$. Each of the parts is allocated to an agent and for $i = 1, \dots, n$, agent i is equipped with a risk measure ρ_i. In this chapter, $\rho_1, \dots, \rho_n$ are chosen as *monetary risk measures* (see Section 1.2.2). We focus on two different and also well connected risk sharing problems. One concerns cooperative risk sharing, in which we aim to find optimal allocations with respect to the sum of individual risk measures (equivalent to Pareto optimality; see Section 2.4); the other concerns competitive risk sharing, in which we aim to find equilibrium allocations (see Section 2.5).

The risk sharing problem considered in this chapter can be formulated in various contexts. For instance, it may represent regulatory capital reduction within the subsidiaries of a single firm, equilibrium among a group of firms with costs associated with regulatory capital, insurance-reinsurance contracts and risk-transfer, or risk redistribution among investors. Throughout this chapter, we generally refer to a participant in the risk sharing problem as an *agent*, which may represent a subsidiary, a firm, an insured, an insurer, or an investor in different contexts.

The most commonly used families of risk measures in practice are the Value-at-Risk (VaR) and the Expected Shortfall (ES), which are unified in a more general two-parameter family of risk measures, called the Range-Value-at-Risk (RVaR). The family of RVaR was introduced in Cont et al. (2010) as a *robust* risk measure (see Section 2.2). More importantly, RVaR can be seen as a bridge connecting VaR and ES, the two most popular but methodologically very different regulatory risk measures. This embedding of VaR and ES into RVaR helps us to understand many properties and comparative advantages of the former risk measures, and hence motivates our concentration on RVaR as the underlying risk measures in the problem of risk sharing discussed in this chapter.

Since each of VaR, ES and RVaR can be represented as average quantiles of a random variable, we refer to the problems considered in this chapter as *quantile-based* risk sharing. We hope that the methodological results obtained in this chapter will be helpful to risk management and policy makers in designing risk allocations and appropriate regulatory risk measures.

2.1.2 Related Literature

In a seminal paper, Borch (1962) showed that within the context of concave utilities, Pareto-optimal allocations between agents are comonotonic. Since the introduction of coherent and convex risk measures by Artzner et al. (1999), Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2002, 2005), the problem of Pareto-optimal risk sharing has been extensively studied when the underlying risk measures are chosen as convex or coherent. As a relevant mathematical tool, the inf-convolution of convex risk measures was obtained in Barrieu and EI Karoui (2005). For law-invariant *monetary utility functions*, or equivalently, convex risk measures, Jouini et al. (2008) showed the existence of an optimal risk sharing for bounded random variables, which is always comonotonic. This result was generalized to non-monotone risk measures by Acciaio (2007) and Filipović and Svindland (2008), to multivariate risks by Carlier et al. (2012) and to cash-subadditive and quasi-convex risk measures by Mastrogiacomo and Rosazza Gianin (2015). Pareto-optimal risk sharing for Choquet expected utilities is studied by Chateauneuf et al. (2000). See Heath and Ku (2004), Tsanakas (2009) and Dana and Le Van (2010) for more on risk sharing with monetary and convex risk measures. On the design of insurance and reinsurance contracts using risk measures, see Cai et al. (2008), Cui et al. (2013) and Bernard et al. (2015). A summary on problems related to inf-convolution of monetary utility functions can be found in Delbaen (2012). For some recent developments on efficient risk sharing and equilibria of the Arrow and Debreu (1954) type with risk measures and rank-dependent utilities (RDU), see Cherny (2006), Carlier and Dana (2008, 2012), Madan and Schoutens (2012), Xia and Zhou (2016) and Jin et al.

(2016). In particular, Xia and Zhou (2016) studied the existence of Arrow-Debreu equilibria for RDU agents and obtained solutions for the state-price density. As far as we are aware of, there is little existing research on non-convex monetary risk measures in risk sharing, and there are no explicit results on equilibrium allocations under such settings.

The extensive debate on desirable properties of regulatory risk measures, in particular VaR and ES, is summarized in Embrechts et al. (2014) and Emmer et al. (2015); see also BCBS (2016) for a recent discussion concerning market risk under Basel III and Sandström (2010, Chapter 14) for an overview in the context of Solvency II. For a critical voice on risk measures and capital requirements in the case of Solvency II, see Floreani (2013). Whereas there is a tendency to move from VaR to ES, for a while to come both risk measures will coexist for regulatory purposes. Our results add some guidance potentially useful in reaching more widely acceptable solutions. Many quantitative concepts may enter into this discussion; below we highlight some issues relevant for our discussion. An overriding concept no doubt is *model uncertainty* in its various guises. Robustness of risk measures is addressed in Cont et al. (2010), Kou et al. (2013), Krätschmer et al. (2012, 2014) and Embrechts et al. (2015); some recent papers on elicitability[1] and backtesting are Bellini and Bignozzi (2015), Ziegel (2016), Acerbi and Székely (2014), Kou and Peng (2016), and Delbaen et al. (2016); some recent papers addressing model uncertainty in risk aggregation are Embrechts et al. (2013), Bernard and Vanduffel (2015) and Wang et al. (2015), amongst others; the problems of currency exchange and regulatory arbitrage are discussed in Koch-Medina and Munari (2016) and Wang (2016).

An important feature of our contribution is the introduction of a concept of robustness into the problem of risk sharing. It is well-known that various concepts and applications of robustness exist in different fields. In the realm of statistics, Huber and Ronchetti (2009) is an excellent place to start. For a broad discussion of the concept of robustness in economics, see the classic book Hansen and Sargent (2008), and also Gilboa and Schmeidler (1989) and Maccheroni et al. (2006) in the theory of preferences. Within the theory of optimization, a standard reference is Ben-Tal et al. (2009). The concept of robustness in this chapter relates to the practical consideration of model misspecification, and hence it is different from the problem of risk sharing under *robust utility functionals* as in the recent paper Knispel et al. (2016).

[1]The concept of elicitability dates back to Osband (1985). Earlier results on a necessary condition for the elicicitability of risk measures are established in Weber (2006).

2.1.3 Contribution and Structure of the Chapter

First, some basic definitions and preliminaries on the risk measures used in this chapter are given in Section 2.2.

Our theoretical contributions start with establishing a powerful inequality for the RVaR family in Section 2.3. This inequality later serves as a building block for the main results on quantile-based risk sharing; it implies that the risk measures RVaR, including VaR and ES as special cases, satisfy a special form of *subadditivity*.

Section 2.4 contains results on (Pareto-)optimal allocations for agents whose preferences are characterized by the RVaR family. We first solve the optimal risk sharing problem by characterizing the *inf-convolution* of several RVaR measures with different parameters. An optimal allocation is given through an explicit construction.

In Section 2.5, we study competitive risk sharing in which each agent optimizes their own preferences, regardless of other participants. We show that, under suitable assumptions, the optimal allocation obtained in Section 2.4 is an Arrow-Debreu equilibrium. Moreover, the equilibrium pricing rule can be obtained explicitly; it has the form of a mixture of a constant and the reciprocal of the total risk.

We then proceed to discuss some relevant issues on optimal allocation in Section 2.6. In particular, we show that in general, a *robust optimal allocation* exists if and only if none of the underlying risk measures is a VaR, and a *comonotonic optimal allocation* exists only if there is at most one underlying risk measure which is not an ES.

Finally, in Section 2.7 we summarize our main results, and discuss some practical implications of our results for risk management and policy makers. As a consequence, we reveal several novel advantages of ES-based risk management.

2.2 The RVaR Family and Basic Terminology

Let $\mathcal{X}$ be the set of real, integrable random variables (i.e. random variables with finite means) defined on $(\Omega, \mathscr{F}, \mathbb{P})$. We assume that for any $X \in \mathcal{X}$, there exists a $Y \in \mathcal{X}$ independent of X.

To simplify the main results, throughout this chapter, the Value-at-Risk of $X \in \mathcal{X}$ at level

$\alpha \in \mathbb{R}_+ := [0, \infty)$ is defined as the $100(1-\alpha)\%$ (generalized) quantile of X, that is,

$$\mathrm{VaR}_\alpha(X) = \inf\{x \in [-\infty, \infty] : \mathbb{P}(X \leqslant x) \geqslant 1-\alpha\}. \tag{2.2.1}$$

Note that according to (2.2.1), for $\alpha \geqslant 1$, $\mathrm{VaR}_\alpha(X) = -\infty$ for all $X \in \mathcal{X}$. Certainly, only the case $\alpha \in [0, 1)$ is relevant in risk management; we do however allow α to take values greater than 1 in order to unify the main results in this chapter. The risk measures VaR_α, $\alpha \geqslant 0$, are monotone, cash-invariant, positive homogeneous, and law-invariant, but in general not subadditive

The key family of risk measures we study in this chapter is the family of the Range-Value-at-Risk (RVaR), a truncated average quantile of a random variable. For $X \in \mathcal{X}$, the RVaR at level $(\alpha, \beta) \in \mathbb{R}_+^2$ is defined as

$$\mathrm{RVaR}_{\alpha,\beta}(X) = \begin{cases} \frac{1}{\beta}\int_\alpha^{\alpha+\beta} \mathrm{VaR}_\gamma(X)\mathrm{d}\gamma & \text{if } \beta > 0, \\ \mathrm{VaR}_\alpha(X) & \text{if } \beta = 0. \end{cases} \tag{2.2.2}$$

For $X \in \mathcal{X}$ and $\alpha + \beta > 1$, since $\mathrm{VaR}_{\alpha+\beta-\varepsilon}(X) = -\infty$ for all $\varepsilon \in [0, \alpha+\beta-1]$, we have $\mathrm{RVaR}_{\alpha,\beta}(X) = -\infty$.

The family of RVaR is introduced by Cont et al. (2010) as *robust* risk measures, in the sense that for $\alpha > 0$ and $\alpha + \beta < 1$, $\mathrm{RVaR}_{\alpha,\beta}$ is continuous with respect to convergence in distribution (weak convergence). Similar to the case of VaR_α, $\mathrm{RVaR}_{\alpha,\beta}$ is also only relevant in practice for $\alpha + \beta < 1$. An RVaR belongs to the large family of distortion risk measures. Though some of our results hold for the broader class of distortion risk measures, for the reason of practical relevance we restrict our attention to RVaR. This also allows for the explicit derivation of risk sharing formulas.

For all $X \in \mathcal{X}$, $\mathrm{VaR}_\alpha(X)$ is non-increasing and right-continuous in $\alpha \geqslant 0$, and hence we have

$$\mathrm{RVaR}_{\alpha,0}(X) = \mathrm{VaR}_\alpha(X) = \lim_{\beta \to 0^+} \mathrm{RVaR}_{\alpha,\beta}(X), \quad \alpha \geqslant 0.$$

Another special case of RVaR is the Expected Shortfall, defined as

$$\mathrm{ES}_\beta(X) = \mathrm{RVaR}_{0,\beta}(X), \quad \beta \geqslant 0.$$

Different from RVaR and VaR, an ES is subadditive. Therefore, ES_β, $\beta \in [0, 1]$ are law-invariant and coherent risk measures on $\mathcal{X}$. Note that by definition, for all $X \in \mathcal{X}$, $\mathrm{RVaR}_{\alpha,\beta}(X)$ is non-increasing in both $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{R}_+$, and $\mathrm{RVaR}_{\alpha,\beta-\alpha}(X)$ is non-increasing in $\alpha \in [0, \beta]$.

Throughout this chapter, we divide the set of risk measures $\{\mathrm{RVaR}_{\alpha,\beta} : \alpha, \beta \in \mathbb{R}_+\}$ into three subcategories. A risk measure $\mathrm{VaR}_\alpha,\ \alpha > 0$ is called a *true VaR*, a risk measure $\mathrm{RVaR}_{\alpha,\beta},\ \alpha, \beta > 0$ is called a *true RVaR*, and $\mathrm{ES}_\beta,\ \beta \geqslant 0$ is simply called an *ES*.

Remark 2.1. The definition of VaR in this chapter is different from the one in Definition 1.3. Mainly for notational convenience we write $\mathrm{VaR}_\alpha(X)$ for the $100(1-\alpha)\%$ quantile of the random variable X; the same notation is applied to ES_β. Whereas this convention (small $\alpha, \beta > 0$) can be widely found in the academic literature (see for instance Föllmer and Schied (2016) and Delbaen (2012)), we are well aware that in practice the notation $\mathrm{VaR}_\alpha(X)$ typically refers to the $100\alpha\%$ quantile of X (thus α is close to 1) as in Chapters 3, 4, and 5. With this notational convention, our main results like Theorems 2.1 and 2.4 below admit a much more elegant formulation. Moreover, the generic results of this chapter on risk sharing are independent of this notational issue. As a consequence, the applicability for practice remains fully accessible to the (regulatory or industry) end-user.

Recall that for $p \in (0, 1)$ and any distribution function F, $F^{-1}(p)$ is defined in (1.2.1). We say that a random variable with distribution F is *doubly continuous* if both F and F^{-1} are continuous. For any $\beta_1, \dots, \beta_n \in \mathbb{R}$, write $\bigvee_{i=1}^n \beta_i = \max\{\beta_1, \dots, \beta_n\}$ and $\bigwedge_{i=1}^n \beta_i = \min\{\beta_1, \dots, \beta_n\}$.

2.3 Quantile Inequalities

The following theorem establishes the relationship between the individual RVaR and the aggregate RVaR. To unify our results for all possible choices of $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$, from now on the indefinite form $\infty - \infty$ is interpreted as $-\infty$. Note that $\mathrm{RVaR}_{\alpha,\beta}(X) = \infty$ may only happen in the very special case where $X \in \mathcal{X}$ is unbounded above and $\alpha = \beta = 0$.

Theorem 2.1. *For any $X_1, \dots, X_n \in \mathcal{X}$ and any $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \geqslant 0$, we have*

$$\mathrm{RVaR}_{\sum_{i=1}^n \alpha_i, \bigvee_{i=1}^n \beta_i}\left(\sum_{i=1}^n X_i\right) \leqslant \sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i). \tag{2.3.1}$$

Proof. We only show the case of $n = 2$; for $n > 2$, an induction argument is sufficent. For any $X_1, X_2 \in \mathcal{X}$, we consider the following three cases respectively.

(i) $\alpha_1 + \alpha_2 + \beta_1 \vee \beta_2 < 1$.

Let $A_1 = \{U_{X_1} \geqslant 1 - \alpha_1\}$ and $A_2 = \{U_{X_2} \geqslant 1 - \alpha_2\}$. Then $\mathbb{P}(A_1 \cup A_2) \leqslant \mathbb{P}(A_1) + \mathbb{P}(A_2) = \alpha_1 + \alpha_2$. Take

$$Y_1 = \mathrm{I}_{A_1^c} X_1 - m\mathrm{I}_{A_1}, \quad Y_2 = \mathrm{I}_{A_2^c} X_2 - m\mathrm{I}_{A_2}, \tag{2.3.2}$$

where m is a real number satisfying $m > -\min\{\mathrm{VaR}_{\alpha_1+\beta_1}(X_1), \mathrm{VaR}_{\alpha_2+\beta_2}(X_2)\}$. It is straightforward to verify $\mathrm{RVaR}_{\alpha_1,\beta_1}(X_1) = \mathrm{ES}_{\beta_1}(Y_1)$ and $\mathrm{RVaR}_{\alpha_2,\beta_2}(X_2) = \mathrm{ES}_{\beta_2}(Y_2)$. It follows that

$$\mathrm{RVaR}_{\alpha_1,\beta_1}(X_1) + \mathrm{RVaR}_{\alpha_2,\beta_2}(X_2) = \mathrm{ES}_{\beta_1}(Y_1) + \mathrm{ES}_{\beta_2}(Y_2) \geqslant \mathrm{ES}_{\beta_1 \vee \beta_2}(Y_1 + Y_2), \tag{2.3.3}$$

where the last inequality holds since $\mathrm{ES}_\beta(X)$ is subadditive and non-increasing in $\beta \geqslant 0$. Moreover, for $\gamma \in [0, 1]$, we will show

$$\mathrm{VaR}_\gamma(Y_1 + Y_2) \geqslant \mathrm{VaR}_{\gamma+(\alpha_1+\alpha_2)}(X_1 + X_2). \tag{2.3.4}$$

Inequality (2.3.4) holds by the definition of VaR if $\gamma + \alpha_1 + \alpha_2 \geqslant 1$. If $\gamma + \alpha_1 + \alpha_2 < 1$, we have $(Y_1 + Y_2)\mathrm{I}_{A_1^c \cap A_2^c} = (X_1 + X_2)\mathrm{I}_{A_1^c \cap A_2^c}$ and hence for any $x \in \mathbb{R}$,

$$\mathbb{P}(Y_1 + Y_2 > x) \geqslant \mathbb{P}(X_1 + X_2 > x, A_1^c, A_2^c) \geqslant \mathbb{P}(X_1 + X_2 > x) - \mathbb{P}(A_1 \cup A_2).$$

Therefore,

$$\mathrm{VaR}_\gamma(Y_1 + Y_2) \geqslant \mathrm{VaR}_{\gamma+\mathbb{P}(A_1 \cup A_2)}(X_1 + X_2) \geqslant \mathrm{VaR}_{\gamma+(\alpha_1+\alpha_2)}(X_1 + X_2). \tag{2.3.5}$$

Hence (2.3.4) holds. If $\beta_1 \vee \beta_2 > 0$, by (2.3.3) and (2.3.4), we have

$$\begin{aligned}
\mathrm{RVaR}_{\alpha_1,\beta_1}(X_1) + \mathrm{RVaR}_{\alpha_2,\beta_2}(X_2) &\geqslant \mathrm{ES}_{\beta_1 \vee \beta_2}(Y_1 + Y_2) \\
&= \frac{1}{\beta_1 \vee \beta_2} \int_0^{\beta_1 \vee \beta_2} \mathrm{VaR}_\alpha(Y_1 + Y_2) \mathrm{d}\alpha \\
&\geqslant \frac{1}{\beta_1 \vee \beta_2} \int_0^{\beta_1 \vee \beta_2} \mathrm{VaR}_{\alpha+(\alpha_1+\alpha_2)}(X_1 + X_2) \mathrm{d}\alpha \\
&= \mathrm{RVaR}_{\alpha_1+\alpha_2, \beta_1 \vee \beta_2}(X_1 + X_2).
\end{aligned} \tag{2.3.6}$$

If $\beta_1 \vee \beta_2 = 0$, then by using (2.3.6), we have

$$\begin{aligned}
\mathrm{RVaR}_{\alpha_1,0}(X_1) + \mathrm{RVaR}_{\alpha_2,0}(X_2) &= \lim_{\beta \to 0^+} (\mathrm{RVaR}_{\alpha_1,\beta}(X_1) + \mathrm{RVaR}_{\alpha_2,0}(X_2)) \\
&\geqslant \lim_{\beta \to 0^+} \mathrm{RVaR}_{\alpha_1+\alpha_2,\beta}(X_1 + X_2) \\
&= \mathrm{RVaR}_{\alpha_1+\alpha_2,0}(X_1 + X_2).
\end{aligned}$$

In either case,

$$\mathrm{RVaR}_{\alpha_1,\beta_1}(X_1) + \mathrm{RVaR}_{\alpha_2,\beta_2}(X_2) \geqslant \mathrm{RVaR}_{\alpha_1+\alpha_2,\beta_1\vee\beta_2}(X_1 + X_2). \tag{2.3.7}$$

(ii) $\alpha_1 + \alpha_2 < 1$ and $\alpha_1 + \alpha_2 + \beta_1 \vee \beta_2 = 1$.

In this case, (2.3.7) follows from the proof in (i) by using the left-continuity of $\mathrm{RVaR}_{\alpha,\beta}(X)$ in β for $0 < \beta \leqslant 1 - \alpha$.

(iii) $\alpha_1 + \alpha_2 \geqslant 1$ or $\alpha_1 + \alpha_2 + \beta_1 \vee \beta_2 > 1$.

In this case, (2.3.7) holds trivially since $\mathrm{RVaR}_{\alpha_1+\alpha_2,\beta_1\vee\beta_2}(X_1 + X_2) = -\infty$.

In summary, (2.3.1) holds for $n = 2$; the case of $n \geqslant 3$ is obtained by induction. □

By setting $\alpha_1 = \cdots = \alpha_n = 0$ and $\beta_1 = \cdots = \beta_n$, Theorem 2.1 reduces to the classic subadditivity of ES. By setting $\beta_1 = \cdots = \beta_n = 0$, we obtain the following inequality for VaR.

Corollary 2.2. *For any $X_1, \dots, X_n \in \mathcal{X}$ and any $\alpha_1, \dots, \alpha_n \geqslant 0$, we have*

$$\mathrm{VaR}_{\sum_{i=1}^n \alpha_i}\left(\sum_{i=1}^n X_i\right) \leqslant \sum_{i=1}^n \mathrm{VaR}_{\alpha_i}(X_i). \tag{2.3.8}$$

Theorem 2.1 and Corollary 2.2 imply that RVaR and VaR enjoy special forms of *subadditivity* as in (2.3.1) and (2.3.8). For $n = 2$, (2.3.1) reads as

$$\mathrm{RVaR}_{\alpha_1+\alpha_2,\beta_1\vee\beta_2}(X_1 + X_2) \leqslant \mathrm{RVaR}_{\alpha_1,\beta_1}(X_1) + \mathrm{RVaR}_{\alpha_2,\beta_2}(X_2),$$

for all $X_1, X_2 \in \mathcal{X}$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}_+$. This subadditivity involves a combination of the summation of the random variables $X_1, \dots, X_n \in \mathcal{X}$, and the summation of the parameters $(\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n) \in \mathbb{R}_+^2$ with respect to the two-dimensional additive operation $(+, \vee)$. Note that $\vee$-operation is known as the *tropical addition* in the max-plus algebra; see Richter-Gebert et al. (2005) and also Remark 2.4.

Remark 2.2. Recall that $\mathcal{X}$ is the set of integrable random variables in Theorem 2.1 and Corollary 2.2. For non-integrable random variables, the definition of VaR in (2.2.1) is still valid, and it is straightforward to see that (2.3.8) in Corollary 2.2 holds for all random variables $X_1, \dots, X_n$. For the case of RVaR, the definition (2.2.2) may involve ill-posed cases such as $\infty - \infty$. For instance, the integral $\int_0^1 \mathrm{VaR}_\gamma(X)\mathrm{d}\gamma = \mathbb{E}[X]$ is only properly defined on $\mathcal{X}$. Therefore, to make all results consistent throughout this chapter, we focus on integrable random variables.

2.4 Optimal Allocations in Quantile-based Risk Sharing

In this section we study (Pareto-)optimal allocations in a risk sharing problem where the objectives of agents are described by the RVaR family, and the target is to minimize the aggregate risk value defined below. This setting is the most suitable if one assumes that the agents collectively work with each other to reach optimality. This may be interpreted as, for instance, the case where a single firm redistributes an aggregate risk among its divisions, which are assessed regulatory capital separately (e.g. these divisions are geographical and regulated by different national authorities). Competitive optimality, in which each agent optimizes their own objective without cooperation, will be discussed in Section 2.5.

2.4.1 Inf-convolution and Pareto-optimal Allocations

Given $X \in \mathcal{X}$, we define the set of *allocations* of X as

$$\mathbb{A}_n(X) = \left\{ (X_1, \dots, X_n) \in \mathcal{X}^n : \sum_{i=1}^n X_i = X \right\}. \tag{2.4.1}$$

In a risk sharing problem, there are n agents equipped with respective risk measures $\rho_1, \dots, \rho_n$ and they will share a risk X by splitting it into an allocation $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$. Throughout, we refer to $\rho_1, \dots, \rho_n$ in a risk sharing problem as the *underlying risk measures*, X as the *total risk*, and for an allocation $(X_1, \dots, X_n)$, we refer to $\sum_{i=1}^n \rho_i(X_i)$ as the *aggregate risk value*. The problem we consider here is an unconstrained allocation problem, that is, $X_1, \dots, X_n$ in (2.4.1) can be chosen over all integrable random variables.

The *inf-convolution* of n risk measures $\rho_1, \dots, \rho_n$ is a risk measure defined as

$$\mathop{\square}_{i=1}^{n} \rho_i(X) := \inf \left\{ \sum_{i=1}^n \rho_i(X_i) : (X_1, \dots, X_n) \in \mathbb{A}_n(X) \right\}, \quad X \in \mathcal{X}. \tag{2.4.2}$$

That is, the inf-convolution of n risk measures is the infimum over aggregate risk values for all possible allocations.

Definition 2.1. For risk measures $\rho_1, \dots, \rho_n$ and $X \in \mathcal{X}$,

(i) an n-tuple $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ is called an *optimal allocation* of X if $\sum_{i=1}^n \rho_i(X_i) = \square_{i=1}^n \rho_i(X)$;

(ii) an n-tuple $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ is called a *Pareto-optimal allocation* of X if for any $(Y_1, \dots, Y_n) \in \mathbb{A}_n(X)$ satisfying $\rho_i(Y_i) \leqslant \rho_i(X_i)$ for all $i = 1, \dots, n$, we have $\rho_i(Y_i) = \rho_i(X_i)$ for all $i = 1, \dots, n$.

In this chapter, whenever an optimal allocation is mentioned, it is with respect to some underlying risk measures which should be clear from the context. The following statement, unifying optimal allocations and Pareto-optimal ones, can be found in Barrieu and EI Karoui (2005) and Jouini et al. (2008) in the case of convex risk measures. One can easily check that the statement also holds for all monetary risk measures.

Proposition 2.3. *For any monetary risk measures $\rho_1, \dots, \rho_n$, an allocation is Pareto-optimal if and only if it is optimal.*

Proof. It is trivial to check that an optimal allocation is always Pareto-optimal. To show the other direction, suppose that $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ is not optimal. Then there exists an allocation $(Y_1, \dots, Y_n) \in \mathbb{A}_n(X)$ such that $\sum_{i=1}^n \rho_i(Y_i) < \sum_{i=1}^n \rho_i(X_i)$. Take $c_i = \rho_i(X_i) - \rho_i(Y_i)$, $i = 1, \dots, n$ and $c = \sum_{i=1}^n c_i > 0$. Then we have

$$(Y_1 + c_1 - c/n, \dots, Y_n + c_n - c/n) \in \mathbb{A}_n(X),$$

and

$$\rho_i(Y_i + c_i - c/n) < \rho_i(Y_i + c_i) = \rho_i(X_i).$$

Therefore, $(X_1, \dots, X_n)$ is not Pareto-optimal. □

In the sequel, we do not distinguish between optimal allocations and Pareto-optimal ones. In order to find an optimal allocation, we simply need to minimize the aggregate risk value over all allocations. In some situations, the n agents in a sharing problem have initial risks $\xi_1, \dots, \xi_n$, respectively, and the total risk is $X = \xi_1 + \dots + \xi_n$. With a given total risk X, the initial risks $\xi_1, \dots, \xi_n$ do not affect Pareto-optimality and we do not take them into account in this section. They do play a role in the formulation of a competitive equilibrium; see Section 2.5.

2.4.2 Optimal Allocations

In this section, we find an optimal allocation of $X \in \mathcal{X}$ such that (2.4.2) is attained when $\rho_1, \dots, \rho_n$ are from the family of RVaR. The main result is the following theorem.

Theorem 2.4. *For $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \geqslant 0$, we have*

$$\mathop{\square}_{i=1}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(X) = \mathrm{RVaR}_{\sum_{i=1}^n \alpha_i, \bigvee_{i=1}^n \beta_i}(X), \quad X \in \mathcal{X}. \tag{2.4.3}$$

Moreover, if $p := \sum_{i=1}^n \alpha_i + \bigvee_{i=1}^n \beta_i < 1$, then, assuming $\beta_n = \bigvee_{i=1}^n \beta_i$, an optimal allocation $(X_1, \dots, X_n)$ of $X \in \mathcal{X}$ is given by

$$X_i = (X - m)\,\mathrm{I}_{\{1-\sum_{k=1}^i \alpha_k < U_X \leqslant 1-\sum_{k=1}^{i-1} \alpha_k\}}, \quad i = 1, \dots, n-1, \tag{2.4.4}$$

$$X_n = (X - m)\,\mathrm{I}_{\{U_X \leqslant 1-\sum_{k=1}^{n-1} \alpha_k\}} + m, \tag{2.4.5}$$

where $m \in (-\infty, \mathrm{VaR}_p(X)]$ is a constant and U_X is defined as in Section 1.2.1.

Proof. Write $\rho_i = \mathrm{RVaR}_{\alpha_i,\beta_i}$, $i = 1, \dots, n$. Since the order of $(\alpha_i,\ \beta_i)$, $i = 1, \dots, n$ is irrelevant in (2.4.3), we may assume without loss of generality $\beta_n = \bigvee_{i=1}^n \beta_i$. To show (2.4.3), it suffices to show

$$\mathop{\square}_{i=1}^{n} \rho_i(X) \leqslant \mathrm{RVaR}_{\sum_{i=1}^n \alpha_i, \beta_n}(X); \tag{2.4.6}$$

indeed, Theorem 2.1 guarantees the other direction of the inequality. In all the following cases, take $(X_1, \dots, X_n)$ in (2.4.4)-(2.4.5) with some $m \in \mathbb{R}$. It is easy to see $X_1 + \dots + X_n = X$, and for $i = 1, \dots, n-1$, we have $\rho_i(X_i) \leqslant 0$ since $\mathbb{P}(X_i > 0) \leqslant \alpha_i$. We discuss the following four cases.

(i) $p < 1$.

Take $m \leqslant \mathrm{VaR}_p(X)$. It is easy to verify $\rho_n(X_n) = \mathrm{RVaR}_{\sum_{i=1}^n \alpha_i, \beta_n}(X)$. Thus,

$$\mathop{\square}_{i=1}^{n} \rho_i(X) \leqslant \sum_{i=1}^{n} \rho_i(X_i) \leqslant \mathrm{RVaR}_{\sum_{i=1}^n \alpha_i, \beta_n}(X).$$

Therefore (2.4.6) holds, and $(X_1, \dots, X_n)$ is an optimal allocation.

(ii) $p > 1$.

Take $m < 0$. If $\alpha_n + \beta_n > 1$ then (2.4.6) holds trivially since $\sum_{i=1}^n \rho_i(X_i) = -\infty$. If

$\alpha_n + \beta_n \leqslant 1$, using the subadditivity of ES, we have

$$\begin{aligned}
\rho_n(X_n) &= \mathrm{RVaR}_{\alpha_n,\beta_n}\left(X\mathrm{I}_{\{U_X \leqslant 1-\sum_{k=1}^{n-1}\alpha_k\}} + m\mathrm{I}_{\{U_X > 1-\sum_{k=1}^{n-1}\alpha_k\}}\right) \\
&\leqslant \mathrm{ES}_{\alpha_n+\beta_n}\left(X\mathrm{I}_{\{U_X \leqslant 1-\sum_{k=1}^{n-1}\alpha_k\}} + m\mathrm{I}_{\{U_X > 1-\sum_{k=1}^{n-1}\alpha_k\}}\right) \\
&\leqslant \mathrm{ES}_{\alpha_n+\beta_n}\left(X\mathrm{I}_{\{U_X \leqslant 1-\sum_{k=1}^{n-1}\alpha_k\}}\right) + \mathrm{ES}_{\alpha_n+\beta_n}\left(m\mathrm{I}_{\{U_X > 1-\sum_{k=1}^{n-1}\alpha_k\}}\right) \\
&= \begin{cases} \mathrm{ES}_{\alpha_n+\beta_n}\left(X\mathrm{I}_{\{U_X \leqslant 1-\sum_{k=1}^{n-1}\alpha_k\}}\right) + m\frac{p-1}{\alpha_n+\beta_n} & \text{if } \sum_{i=1}^n \alpha_i < 1 \\ m & \text{if } \sum_{i=1}^n \alpha_i \geqslant 1 \end{cases} \\
&\to -\infty \qquad \text{as } m \to -\infty.
\end{aligned}$$

This shows $\square_{i=1}^n \rho_i(X_i) = -\infty$ and (2.4.6) holds.

(iii) $p = 1$, $\beta_n = 0$.

Since $\mathbb{P}(X_n > m) \leqslant \alpha_n$, one has $\mathrm{VaR}_{\alpha_n}(X_n) \leqslant m \to -\infty$ as $m \to -\infty$. This shows $\square_{i=1}^n \rho_i(X_i) = -\infty$ and (2.4.6) holds.

(iv) $p = 1$, $\beta_n > 0$.

If $\alpha_n + \beta_n = 1$ then $\rho_n(X_n) = \rho_n(X) = \mathrm{RVaR}_{\alpha_n,\beta_n}(X)$, and (2.4.6) holds.

If $\alpha_n + \beta_n < 1$, take $m = \mathrm{VaR}_q(X)$ for $q \in (\alpha_n + \beta_n, 1) \cap (1 - \beta_n, 1)$. We have

$$\begin{aligned}
\rho_n(X_n) &= \mathrm{RVaR}_{\alpha_n,\beta_n}\left(X\mathrm{I}_{\{U_X \leqslant \alpha_n+\beta_n\}} + \mathrm{VaR}_q(X)\mathrm{I}_{\{U_X > \alpha_n+\beta_n\}}\right) \\
&= \frac{1}{\beta_n}\left(\int_{1-\beta_n}^{q} \mathrm{VaR}_\gamma(X)\mathrm{d}\gamma + (1-q)\mathrm{VaR}_q(X)\right) \\
&\to \frac{1}{\beta_n}\int_{1-\beta_n}^{1} \mathrm{VaR}_\gamma(X)\mathrm{d}\gamma \qquad \text{as } q \to 1.
\end{aligned}$$

This shows $\square_{i=1}^n \rho_i(X_i) \leqslant \frac{1}{\beta_n}\int_{1-\beta_n}^1 \mathrm{VaR}_\gamma(X)\mathrm{d}\gamma = \mathrm{RVaR}_{1-\beta_n,\beta_n}(X)$ and (2.4.6) holds.

Combining the cases (i)-(iv), the proof is complete. $\square$

If $X \geqslant 0$, then by setting $m = 0$ in (2.4.4)-(2.4.5), the optimal allocation is

$$X_i = X\mathrm{I}_{\{1-\sum_{k=1}^{i}\alpha_k < U_X \leqslant 1-\sum_{k=1}^{i-1}\alpha_k\}}, \quad i = 1, \ldots, n-1, \tag{2.4.7}$$

$$X_n = X\mathrm{I}_{\{U_X \leqslant 1-\sum_{k=1}^{n-1}\alpha_k\}}. \tag{2.4.8}$$

The interpretation of the above allocation is clear: for each $i = 1, \dots, n-1$, agent i takes a risk X_i with probability of loss $\mathbb{P}(X_i > 0) = \alpha_i$. This implies $\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) = 0$. The last agent (agent n) takes the rest of the risk, and $\mathrm{RVaR}_{\alpha_n,\beta_n}(X_n) = \mathrm{RVaR}_{\sum_{i=1}^n \alpha_i,\beta_n}(X)$ which is positive if $X > 0$. For each agent i, the parameter β_i can be seen as the sensitivity with respect to a loss exceeding the α_i-probability level. In view of the above discussion, we will refer to β_i as the *tolerance parameter* of agent i, and agent n as the *remaining-risk bearer*, who has the largest tolerance parameter among all agents.

Remark 2.3. Some observations on the optimal allocation in Theorem 2.4:

(i) Assuming $p < 1$ in Theorem 2.4, each $X_1, \dots, X_n$ is a function of U_X in the optimal allocation (2.4.4)-(2.4.5). If X is continuously distributed, then $X_1, \dots, X_n$ are also functions of X, since U_X can be taken as $F(X)$ where F is the distribution of X. In this case, the optimal allocation in (2.4.4)-(2.4.5) can be written as

$$X_i = (X - m)\,\mathrm{I}_{\{F^{-1}(1-\sum_{k=1}^{i}\alpha_k) < X \leqslant F^{-1}(1-\sum_{k=1}^{i-1}\alpha_k)\}}, \quad i = 1, \dots, n-1, \text{ and} \tag{2.4.9}$$

$$X_n = (X - m)\,\mathrm{I}_{\{X \leqslant F^{-1}(1-\sum_{k=1}^{n-1}\alpha_k)\}} + m, \tag{2.4.10}$$

where $m \in (-\infty, \mathrm{VaR}_p(X)]$.

(ii) If $\alpha_i = \beta_i = 0$ for some $i = 1, \dots, n$, assuming $n \geqslant 2$, one can always choose $X_i = 0$ in an optimal risk sharing $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$. This is because for any $\alpha, \beta \in \mathbb{R}_+$ and $X_1, X_2 \in \mathcal{X}$,

$$\mathrm{RVaR}_{\alpha,\beta}(X_1 + X_2) + \mathrm{VaR}_0(0) \leqslant \mathrm{RVaR}_{\alpha,\beta}(X_1 + \mathrm{VaR}_0(X_2)) = \mathrm{RVaR}_{\alpha,\beta}(X_1) + \mathrm{VaR}_0(X_2).$$

That is, it is not beneficial to allocate any risk to agent i, since she is extremely averse to taking any risk. This is already reflected in the construction in (2.4.4).

(iii) If $\sum_{i=1}^n \alpha_i + \bigvee_{i=1}^n \beta_i > 1$, as $\mathrm{RVaR}_{\sum_{i=1}^n \alpha_i, \bigvee_{i=1}^n \beta_i}(X) = -\infty$, no optimal allocation exists. There exists an allocation $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ such that $\sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\ \beta_i}(X_i) < -m$ for any $m \in \mathbb{R}$. If $\sum_{i=1}^n \alpha_i + \bigvee_{i=1}^n \beta_i = 1$, from the proof of Theorem 2.4 parts (iii) and (iv), it follows that, depending on the choice of (α_i, β_i), $i = 1, \dots, n$, an optimal allocation may or may not exist.

The following corollary for VaR follows directly from Theorem 2.4.

Corollary 2.5. *For $\alpha_1, \ldots, \alpha_n \geqslant 0$, we have*

$$\mathop{\square}_{i=1}^{n} \mathrm{VaR}_{\alpha_i}(X) = \mathrm{VaR}_{\sum_{i=1}^n \alpha_i}(X), \quad X \in \mathcal{X}.$$

Moreover, if $p := \sum_{i=1}^n \alpha_i < 1$, an optimal allocation of $X \in \mathcal{X}$ is given by (2.4.4)-(2.4.5) *where* $m \in (-\infty, \mathrm{VaR}_p(X)]$.

Similarly to Corollary 2.2, Corollary 2.5 also holds for non-integrable random variables; see Remark 2.2.

Remark 2.4. From Theorem 2.4 and Corollary 2.5, the subset $\mathcal{G}$ of risk measures on $\mathcal{X}$,

$$\mathcal{G} = \left\{ \mathrm{RVaR}_{\alpha,\beta} : (\alpha, \beta) \in \mathbb{R}_+^2 \right\},$$

forms a commutative monoid (semi-group) equipped with the addition $\square$. Moreover, this monoid is isomorphic to the monoid $\mathbb{R}_+^2$ equipped with the addition $(+, \vee)$. The identity element in the monoid $(\mathcal{G}, \square)$ is $\mathrm{RVaR}_{0,0} = \mathrm{ES}_0 = \mathrm{VaR}_0$, and the identity element in the monoid $(\mathbb{R}_+^2, (+, \vee))$ is simply $(0, 0)$. The submonoid $\mathcal{G}_V = \{\mathrm{VaR}_\alpha : \alpha \in \mathbb{R}_+\}$ of $(\mathcal{G}, \square)$ is isomorphic to the monoid $(\mathbb{R}_+, +)$, and the submonoid $\mathcal{G}_E = \{\mathrm{ES}_\beta : \beta \in \mathbb{R}_+\}$ of $(\mathcal{G}, \square)$ is isomorphic to the monoid $(\mathbb{R}_+, \vee)$.

2.5 Competitive Equilibria

In Section 2.4, (Pareto-)optimal allocations are obtained for the quantile-based risk sharing problem, which is more suitable for the study of cooperative games. If the agents represent a group of individual firms, there might not be a central coordination for these self-interested firms to reach Pareto-optimality. In this section, we investigate settings of non-cooperative equilibria. We shall see that the optimal allocation obtained in Section 2.4 is indeed part of an Arrow-Debrew equilibrium under a condition on the distribution function of X..

2.5.1 An Arrow-Debreu Equilibrium

We consider a classic Arrow-Debreu economic equilibrium model (Arrow and Debreu (1954)) for agents whose objectives are characterized by the RVaR family. All discussions are based on the underlying risk measures $\mathrm{RVaR}_{\alpha_1,\beta_1}, \ldots, \mathrm{RVaR}_{\alpha_n,\beta_n}$, $\alpha_i, \beta_i \in [0, 1)$ satisfying

$$\sum_{i=1}^{n} \alpha_i + \bigvee_{i=1}^{n} \beta_i < 1, \qquad \beta_n = \bigvee_{i=1}^{n} \beta_i. \tag{2.5.1}$$

Note that we are assuming without loss of generality that the n-th agent has the largest tolerance parameter among all agents.

For $i = 1, \ldots, n$, assume that agent i has an initial risk $\xi_i \in \mathcal{X}$. Let $X = \sum_{i=1}^n \xi_i$ be the total risk, and assume $X \geqslant 0$. Let Ψ be the set of bounded non-negative random variables ψ. The random variable $\psi \in \Psi$ presents the pricing rule for the microeconomic market among the agents, that is, by taking a risk Y in this market, one receives a monetary payment of $\mathbb{E}[\psi Y]$.

For each $i = 1, \ldots, n$, agent i may trade the initial risk ξ_i for a new position $X_i \in \mathcal{X}$, and receive the monetary amount $\mathbb{E}[\psi(X_i - \xi_i)]$. For a given $\psi \in \Psi$, the agent's objective is

$$\text{to minimize} \quad \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i - \mathbb{E}[\psi(X_i - \xi_i)]) \tag{2.5.2}$$

over $X_i \in \mathcal{X}$ satisfying $0 \leqslant X_i \leqslant X$; that is, one is not allowed to take more than the total risk, or take less than zero[2]. Obviously, ξ_i is irrelevant in optimizing (2.5.2). By cash-invariance of RVaR, (2.5.2) is equivalent to

$$\begin{aligned} \text{to minimize} \quad & \mathcal{V}_i(X_i) = \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) - \mathbb{E}[\psi X_i] \\ \text{over} \quad & X_i \in \mathcal{X}, \quad 0 \leqslant X_i \leqslant X, \end{aligned} \qquad i = 1, \ldots, n. \tag{2.5.3}$$

To reach an equilibrium, the market clearing equation

$$\sum_{i=1}^n X_i^* = X = \sum_{i=1}^n \xi_i$$

needs to be satisfied, where X_i^* solves (2.5.3), $i = 1, \ldots, n$.

The constraint $0 \leqslant X_i \leqslant X$ is essential to the optimization (2.5.3). Note that the functional $X_i \mapsto \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) - \mathbb{E}[\psi X_i]$ is positively homogeneous. If we allow X_i to be taken over the full set $\mathcal{X}$, then the infimum value of (2.5.3) will always be either 0 or $-\infty$ (one cannot expect a non-trivial equilibrium to exist). In view of this, we consider non-negative random variables and write $\mathcal{X}_+ = \{X \in \mathcal{X} : X \geqslant 0\}$. Below we adopt Definition 3.60 of Föllmer and Schied (2016) for an Arrow-Debreu equilibrium[3].

[2] In the formulation of competitive equilibria, typically there is a budget constraint of the form $\mathbb{E}[\psi Y_i] \geqslant \mathbb{E}[\psi \xi_i]$ in the minimization of a target $\mathcal{V}_i(Y_i)$. Note that in (2.5.2), $Y_i = X_i - \mathbb{E}[\psi(X_i - \xi_i)]$ and it always satisfies $\mathbb{E}[\psi Y_i] = \mathbb{E}[\psi \xi_i]$. Hence we omit the budget constraint in our formulation. Optimization over Y with the constraint $0 \leqslant Y \leqslant X$ is a generalization of the problem considered in Schied (2004), where $0 \leqslant Y \leqslant K$ for a constant $K \in \mathbb{R}$ is studied. See p.180 of Föllmer and Schied (2016) for equilibria under this constraint.

[3] There are several different models for the Arrow-Debreu equilibria in finance (see e.g. Starr (2011) and Xia and Zhou (2016)), often involving an extra individual consumption optimization. In the definition we adopt (following Föllmer and Schied (2016)), individual consumptions are omitted for simplicity.

Definition 2.2 (Arrow-Debreu equilibrium)**.** Let $X \in \mathcal{X}_+$. A pair $(\psi, (X_1^*, \dots, X_n^*)) \in \Psi \times \mathbb{A}_n(X)$ is *an Arrow-Debreu equilibrium* for (2.5.3) if

$$X_i^* \in \arg\min \{\mathcal{V}_i(X_i) : X_i \in \mathcal{X},\ 0 \leqslant X_i \leqslant X\},\ i = 1, \dots, n. \tag{2.5.4}$$

The pricing rule ψ in an Arrow-Debreu equilibrium is called an *equilibrium pricing rule*, and the allocation $(X_1^*, \dots, X_n^*)$ in an Arrow-Debreu equilibrium is called an *equilibrium allocation*.

For an introduction of Arrow-Debreu equilibria in finance, see Föllmer and Schied (2016, Section 3.6). Certainly, the equilibrium pricing rule ψ, assuming it exists, is arbitrary on the set $\{X = 0\}$. Explicit solutions of Arrow-Debreu equilibria for non-convex objectives (or non-concave objectives in the framework of utility maximization), including the RVaR family, are very limited in the literature. We are not aware of any explicit solutions. For some recent development on Arrow-Debreu equilibria for rank-dependent utilities, see Xia and Zhou (2016) and Jin et al. (2016).

We first establish a connection between an Arrow-Debreu equilibrium and an optimal allocation.

Proposition 2.6. *Let $X \in \mathcal{X}_+$ and assume* (2.5.1) *holds. Suppose that $(\psi, (X_1^*, \dots, X_n^*)) \in \Psi \times \mathbb{A}_n(X)$ is an Arrow-Debreu equilibrium for* (2.5.3)*. Then $(X_1^*, \dots, X_n^*)$ is necessarily an optimal allocation for $\mathrm{RVaR}_{\alpha_1,\beta_1}, \dots, \mathrm{RVaR}_{\alpha_n,\beta_n}$.*

Proof. By the construction in (2.4.7)-(2.4.8), there exists $(Y_1, \dots, Y_n) \in \mathbb{A}_n(X)$ such that

$$\sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(Y_i) = \mathrm{RVaR}_{\alpha,\beta}(X)$$

and $0 \leqslant Y_i \leqslant X$, $i = 1, \dots, n$, where $\alpha = \sum_{i=1}^n \alpha_i$ and $\beta = \bigvee_{i=1}^n \beta_i$. Since $(\psi, (X_1^*, \dots, X_n^*))$ is an Arrow-Debreu equilibrium, we have for $i = 1, \dots, n$,

$$\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i^*) - \mathbb{E}[\psi X_i^*] \leqslant \mathrm{RVaR}_{\alpha_i,\beta_i}(Y_i) - \mathbb{E}[\psi Y_i].$$

It follows from $\sum_{i=1}^n X_i^* = X = \sum_{i=1}^n Y_i$ that

$$\begin{aligned}\sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i^*) - \mathbb{E}[\psi X] &= \sum_{i=1}^n (\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i^*) - \mathbb{E}[\psi X_i^*]) \\ &\leqslant \sum_{i=1}^n (\mathrm{RVaR}_{\alpha_i,\beta_i}(Y_i) - \mathbb{E}[\psi Y_i]) = \mathrm{RVaR}_{\alpha,\beta}(X) - \mathbb{E}[\psi X].\end{aligned}$$

Therefore $\sum_{i=1}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i^*) \leqslant \mathrm{RVaR}_{\alpha,\beta}(X)$. By Theorem 2.4, $(X_1^*,\dots,X_n^*)$ is an optimal allocation. $\square$

Proposition 2.6 is a special version of the *First Welfare Economics Theorem*[4] for the optimization (2.5.3), stating that an equilibrium allocation is Pareto-optimal under suitable assumptions (see e.g. Arrow (1951) and Arrow and Debreu (1954)).

Next we shall see that, with an extra condition on the value of $\mathbb{P}(X > 0)$, the optimal allocation in Theorem 2.4 is indeed an equilibrium allocation, and the corresponding equilibrium pricing rule is explicit. Recall that for $X \geqslant 0$ and assuming (2.5.1), an optimal allocation in Theorem 2.4 is given by

$$X_i^* = X\mathrm{I}_{\{1-\sum_{k=1}^{i}\alpha_k < U_X \leqslant 1-\sum_{k=1}^{i-1}\alpha_k\}}, \quad i = 1,\dots,n-1, \tag{2.5.5}$$

$$X_n^* = X\mathrm{I}_{\{U_X \leqslant 1-\sum_{k=1}^{n-1}\alpha_k\}}. \tag{2.5.6}$$

The following theorem establishes an explicit Arrow-Debreu equilibrium for (2.5.3).

Theorem 2.7. *Write $\alpha = \sum_{i=1}^{n}\alpha_i$, $\underline{\alpha} = \bigwedge_{i=1}^{n}\alpha_i$ and $\beta = \bigvee_{i=1}^{n}\beta_i = \beta_n$. Assume $\alpha+\beta<1$, and $X \in \mathcal{X}_+$ satisfies $\mathbb{P}(X>0) \leqslant \max\{\underline{\alpha}+\beta, \alpha\}$. Let $(X_1^*,\dots,X_n^*)$ be given by* (2.5.5)-(2.5.6), *and*

$$\psi = \min\left\{\frac{x}{X\beta}, \frac{1}{\beta}\right\}\mathrm{I}_{\{X\beta>0\}} \qquad \textit{where } x = \mathrm{VaR}_\alpha(X). \tag{2.5.7}$$

Then $(\psi,(X_1^*,\dots,X_n^*))$ is an Arrow-Debreu equilibrium for (2.5.3).

Proof. Recall that $\mathrm{RVaR}_{\alpha_n,\beta_n}(X_n^*) = \mathrm{RVaR}_{\alpha,\beta}(X)$ and $\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i^*) = 0$ for $i = 1,\dots,n-1$. We consider two cases separately.

(i) Suppose $\mathbb{P}(X>0) \leqslant \alpha$. This implies $\mathrm{RVaR}_{\alpha_n,\beta_n}(X_n^*) = \mathrm{RVaR}_{\alpha,\beta}(X) = 0$, $x = 0$ and $\psi = 0$. On the other hand, for any $0 \leqslant X_i \leqslant X$, we have $\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) - \mathbb{E}[\psi X_i] = \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) \geqslant 0$. Thus X_i^* satisfies (2.5.4), and hence $(\psi,(X_1^*,\dots,X_n^*))$ is an Arrow-Debreu equilibrium.

(ii) Suppose $\alpha < \mathbb{P}(X>0) \leqslant \underline{\alpha}+\beta$. This implies $x,\beta>0$. For $i = 1,\dots,n$, take any $X_i \in \mathcal{X}$ such that $0 \leqslant X_i \leqslant X$. Note that by definition, $\psi X \leqslant x/\beta$. We have

$$\mathbb{E}[\psi \mathrm{I}_{\{U_{X_i}\geqslant 1-\alpha_i\}} X_i] \leqslant \mathbb{E}[\psi \mathrm{I}_{\{U_{X_i}\geqslant 1-\alpha_i\}} X] \leqslant \mathbb{E}\left[\frac{x}{\beta}\mathrm{I}_{\{U_{X_i}\geqslant 1-\alpha_i\}}\right] = \frac{x\alpha_i}{\beta}. \tag{2.5.8}$$

[4] The First Welfare Economics Theorem has many versions, and it usually requires completeness of the market, which is not the case for our formulation.

On the other hand, using $\psi \leqslant 1/\beta$ and $\mathbb{P}(X_i > 0) \leqslant \mathbb{P}(X > 0) \leqslant \alpha_i + \beta$,

$$\begin{aligned}\mathbb{E}[\psi \mathrm{I}_{\{U_{X_i} < 1-\alpha_i\}} X_i] &\leqslant \frac{1}{\beta}\mathbb{E}[\mathrm{I}_{\{U_{X_i} < 1-\alpha_i\}} X_i] \\ &= \frac{1}{\beta}\int_{\alpha_i}^{1} \mathrm{VaR}_\gamma(X_i)\mathrm{d}\gamma \\ &= \frac{1}{\beta}\int_{\alpha_i}^{\alpha_i+\beta} \mathrm{VaR}_\gamma(X_i)\mathrm{d}\gamma = \mathrm{RVaR}_{\alpha_i,\beta}(X_i) \leqslant \mathrm{RVaR}_{\alpha_i\beta_i}(X_i). \qquad (2.5.9)\end{aligned}$$

Combining (2.5.8) and (2.5.9), we have

$$\mathbb{E}[\psi X_i] \leqslant \frac{x\alpha_i}{\beta} + \mathrm{RVaR}_{\alpha_i\beta_i}(X_i).$$

Equivalently,

$$\mathrm{RVaR}_{\alpha_i\beta_i}(X_i) - \mathbb{E}[\psi X_i] \geqslant -\frac{x\alpha_i}{\beta}.$$

Next we verify that $\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i^*) - \mathbb{E}[\psi X_i^*]$ is equal to $-x\alpha_i/\beta$. Write

$$A_i = \left\{1 - \sum_{k=1}^{i}\alpha_k < U_X \leqslant 1 - \sum_{k=1}^{i-1}\alpha_k\right\} \subset \{U_X \geqslant 1-\alpha\}.$$

Note that $\psi = \frac{x}{X\beta}\mathrm{I}_{\{U_X \geqslant 1-\alpha\}} + \frac{1}{\beta}\mathrm{I}_{\{U_X < 1-\alpha\}}$. We have $X_i^* = X\mathrm{I}_{A_i}$ for $i = 1,\dots,n-1$, and $X_n^* = X\mathrm{I}_{A_n} + X\mathrm{I}_{\{U_X < 1-\alpha\}}$. For $i = 1,\dots,n-1$,

$$\begin{aligned}\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i^*) - \mathbb{E}[\psi X_i^*] = -\mathbb{E}[\psi X_i^*] &= -\mathbb{E}\left[\frac{x}{X\beta}\mathrm{I}_{\{U_X \geqslant 1-\alpha\}} X \mathrm{I}_{A_i}\right] \\ &= -\mathbb{E}\left[\frac{x}{\beta}\mathrm{I}_{A_i}\right] = -\frac{x\alpha_i}{\beta}.\end{aligned}$$

For the last agent, we have

$$\begin{aligned}\mathbb{E}[\psi X_n^*] &= \mathbb{E}\left[\frac{x}{X\beta}\mathrm{I}_{\{U_X \geqslant 1-\alpha\}} X \mathrm{I}_{A_n}\right] + \mathbb{E}\left[\frac{1}{\beta}\mathrm{I}_{\{U_X < 1-\alpha\}} X\right] \\ &= \frac{x\alpha_n}{\beta} + \frac{1}{\beta}\int_{\alpha}^{1}\mathrm{VaR}_\gamma(X)\mathrm{d}\gamma \\ &= \frac{x\alpha_n}{\beta} + \frac{1}{\beta}\int_{\alpha}^{\alpha+\beta}\mathrm{VaR}_\gamma(X)\mathrm{d}\gamma = \frac{x\alpha_n}{\beta} + \mathrm{RVaR}_{\alpha,\beta}(X).\end{aligned}$$

Rearranging the above equation, and using $\mathrm{RVaR}_{\alpha_n,\beta_n}(X_n^*) = \mathrm{RVaR}_{\alpha,\beta}(X)$, we obtain

$$\mathrm{RVaR}_{\alpha_n,\beta_n}(X_n^*) - \mathbb{E}[\psi X_n^*] = \mathrm{RVaR}_{\alpha,\beta}(X) - \mathbb{E}[\psi X_n^*] = -\frac{x\alpha_n}{\beta}.$$

In summary, for $i = 1, \ldots, n$,

$$\mathrm{RVaR}_{\alpha_i \beta_i}(X_i) - \mathbb{E}[\psi X_i] \geqslant -\frac{x\alpha_i}{\beta} = \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i^*) - \mathbb{E}[\psi X_i^*].$$

Therefore, $(\psi, (X_1^*, \ldots, X_n^*))$ is an Arrow-Debreu equilibrium. $\square$

From Theorem 2.7, there are two cases for the equilibrium pricing rule ψ on $\{X > 0\}$:

(i) if $\mathbb{P}(X > 0) \leqslant \alpha$, then $\psi = 0$;

(ii) if $\alpha < \mathbb{P}(X > 0) \leqslant \underline{\alpha} + \beta$, then

$$\psi = \min\left\{\frac{x}{X\beta}, \frac{1}{\beta}\right\} = \frac{x}{X\beta}\mathrm{I}_{\{U_X \geqslant 1-\alpha\}} + \frac{1}{\beta}\mathrm{I}_{\{U_X < 1-\alpha\}} \quad \text{where } x = \mathrm{VaR}_\alpha(X). \tag{2.5.10}$$

The above case (i) is perhaps less interesting. In this case, each agent takes a "free-lunch" risk $X\mathrm{I}_{A_i}$ which does not contribute to their measure of risk. It is then not surprising to see that the equilibrium price of any risk is zero.

On the other hand, the above case (ii) is somewhat remarkable. The equilibrium pricing rule ψ in (2.5.10) consists of two parts. If $X \geqslant x$, the pricing rule is given by $\psi = \frac{x}{X\beta}$, a constant times the reciprocal of X. This form of equilibrium pricing rule is found in the Arrow-Debreu equilibrium for log utility maximizers (see e.g. Example 3.63 of Föllmer and Schied (2016)). If $0 < X < x$, ψ is equal to the constant $1/\beta$. If $X = 0$, as mentioned before, ψ is arbitrary and its value does not affect the optimization problem. For simplicity one can take $\psi = 1/\beta$ to unify with the previous case, so that ψ is a non-increasing function of X. The distribution of the equilibrium pricing rule ψ is a mixture of a scaled reciprocal of X given $X \geqslant x$ and a constant $1/\beta$ given $X < x$. We are not aware of any existing literature containing this particular form of equilibrium pricing rules.

Remark 2.5. The condition $\mathbb{P}(X > 0) \leqslant \max\{\underline{\alpha} + \beta, \alpha\}$ is crucial for the above Arrow-Debreu equilibrium. One can verify that if $\mathbb{P}(X > 0) > \max\{\underline{\alpha} + \beta, \alpha\}$, then $(\psi, (X_1^*, \ldots, X_n^*))$ in (2.5.5)-(2.5.7) is no longer an Arrow-Debreu equilibrium. It is not clear yet whether an Arrow-Debreu equilibrium exists in this case. We conjecture that the solution depends on other distributional properties of X.

2.5.2 An Equilibrium Model for Expected Profit minus Cost of Capital

Let $\alpha_i, \beta_i, \xi_i, \psi$, $i = 1, \ldots, n$ be as in Section 2.5.1. Previously, we considered an Arrow-Debreu equilibrium in which each agent's objective is to minimize their risk measure $\mathrm{RVaR}_{\alpha_i,\beta_i}$. This can be interpreted as a setting of minimizing each firm's regulatory capital, where the regulatory capital is calculated by $\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i - \mathbb{E}[\psi(X_i - \xi_i)])$ for firm i to bear the risk position X_i. Admittedly, it is simplistic to suggest that regulatory capital is the only concern of a firm in managing its risk. A more comprehensive model for the objective of firm i, for $i = 1, \ldots, n$, may be chosen as

$$\text{to maximize} \quad \mathcal{U}_i(X_i) = \mathbb{E}\left[u_i\Big(w_i + \mathbb{E}[\psi(X_i - \xi_i)] - X_i - c_i\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i)\Big)\right], \tag{2.5.11}$$

where $w_i \in \mathbb{R}$ is the initial wealth of firm i, $u_i : \mathbb{R} \to \mathbb{R}$ is the utility function of the firm, and $c_i > 0$ is a constant which represents the cost of raising one unit of capital for this firm[5]. In other words, the objective in (2.5.11) is to maximize the expected utility of the initial wealth w_i plus the cash received $\mathbb{E}[\psi(X_i - \xi_i)]$, minus the cost of capital $c_i\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i)$ and the potential loss X_i.

Solving an Arrow-Debreu equilibrium for general preferences modeled by (2.5.11) is quite challenging and is beyond the scope of this chapter. Below we shall illustrate that, in the special case when $u_1, \ldots, u_n$ are linear, the equilibrium allocation in Section 2.5.1 is again an equilibrium allocation under a slightly stronger condition.

Assume that $u_1, \ldots, u_n$ are linear utility functions. Under this setting, w_i and ξ_i in (2.5.11) can be omitted, and the optimization problem (2.5.11) is equivalent to

$$\begin{array}{ll} \text{to minimize} & \mathcal{V}_i(X_i) = \mathbb{E}[X_i] + c_i\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) - \mathbb{E}[\psi X_i] \\ & \text{over} \quad X_i \in \mathcal{X}, \quad 0 \leqslant X_i \leqslant X, \end{array} \qquad i = 1, \ldots, n. \tag{2.5.12}$$

It is not surprising that the cost-of-capital coefficients $c_1, \ldots, c_n$ play a non-negligible role in an equilibrium for (2.5.12). In the following, let $d_i = \beta_i/c_i$ represent the *tolerance-to-cost ratio* of agent i, $i = 1, \ldots, n$. Without loss of generality, we assume $d_n = \bigvee_{i=1}^n d_i$. That is, an agent with the largest tolerance-to-cost ratio is rearranged to be the n-th agent.

[5]Here, for simplicity, we assume that the cost of the regulatory capital is $c_i\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i)$ regardless of the initial wealth and the cash received.

Theorem 2.8. *Write* $\alpha = \sum_{i=1}^n \alpha_i$, $\eta = \bigwedge_{i=1}^n (\alpha_i + \beta_i)$ *and* $d = \bigvee_{i=1}^n d_i = d_n$. *Assume* $\alpha + \bigvee_{i=1}^n \beta_i < 1$, *and* $X \in \mathcal{X}_+$ *satisfies* $\mathbb{P}(X > 0) \leqslant \max\{\eta, \alpha\}$. *Let* $(X_1^*, \dots, X_n^*)$ *be given by* (2.5.5)-(2.5.6), *and*

$$\psi = 1 + \min\left\{\frac{x}{Xd}, \frac{1}{d}\right\} \mathrm{I}_{\{Xd>0\}} \qquad \text{where } x = \mathrm{VaR}_\alpha(X). \tag{2.5.13}$$

Then $(\psi, (X_1^*, \dots, X_n^*))$ *is an Arrow-Debreu equilibrium for* (2.5.12).

The proof of Theorem 2.8 is similar to that of Theorem 2.7 and is given in Section 2.8.2. Theorem 2.8 suggests that for the objectives in (2.5.12), there exists an Arrow-Debreu equilibrium in which the allocation is again (2.5.5)-(2.5.6), albeit the remaining-risk bearer (see Remark 2.3) in this problem is the agent with the largest tolerance-to-cost ratio, instead of the one with the largest tolerance parameter as in Theorem 2.7.

Remark 2.6. Noting $\eta = \bigwedge_{i=1}^n (\alpha_i + \beta_i) \leqslant \bigwedge_{i=1}^n \alpha_i + \bigvee_{i=1}^n \beta_i$, the constraint $\mathbb{P}(X > 0) \leqslant \max\{\eta, \alpha\}$ is slightly stronger than the one in Theorem 2.7, where $\mathbb{P}(X > 0) \leqslant \max\{\bigwedge_{i=1}^n \alpha_i + \bigvee_{i=1}^n \beta_i, \alpha\}$ is required. This technical condition was caused by the introduction of the possibly different coefficients $c_1, \dots, c_n$, and does not seem to be dispensable.

2.5.3 A Numerical Example

To illustrate our results, in this section we present a simple example, in which three agents share a homogenous credit risk portfolio, to illustrate our result. Take $\alpha_i = 0.01$, $\beta_i = 0.25$, $i = 1, 2, 3$ and $c_1 = 0.1$, $c_2 = 0.08$, $c_3 = 0.05$. This setting corresponds to that the three agents are subject to the same regulatory risk measure $\mathrm{RVaR}_{0.01,0.25}$ with different cost-of-capital coefficients. The use of $\mathrm{RVaR}_{0.01,0.25}$ may be seen as a robust approximation of $\mathrm{ES}_{0.26}$ as suggested by Cont et al. (2010). Agent 3 has the smallest cost-of-capital coefficient, therefore the largest tolerance-to-cost ratio.

Let $\xi_i = \sum_{j=1}^{100} L_{i,j}$ where $L_{i,j}$, $i = 1, 2, 3$, $j = 1, \dots, 100$ are iid Bernoulli random variables with parameter $p = 0.001$. Each $L_{i,j}$ represents the loss from a single credit event and for simplicity we assume that they are all iid. The total risk $X = \xi_1 + \xi_2 + \xi_3$ has a $\mathrm{Bin}(300, 0.001)$ distribution, and we can calculate $\mathbb{P}(X > 0) = 0.259293$.

We consider the competitive equilibrium in Section 2.5.2. The following quantities are straightforward from Theorem 2.8: $d = 5$, $x = \mathrm{VaR}_{0.03}(X) = 2$ and $\psi = 1 + \frac{2}{5X}\mathrm{I}_{\{X\geqslant 2\}} + \frac{1}{5}\mathrm{I}_{\{X\leqslant 1\}}$. A

		$i=1$	$i=2$	$i=3$
initial expected loss	$\mathbb{E}[\xi_i]$	0.1	0.1	0.1
initial risk measure	$\mathrm{RVaR}_{\alpha_i,\beta_i}(\xi_i)$	0.3412	0.3412	0.3412
initial price of risk	$\mathbb{E}[\psi\xi_i]$	0.1197	0.1197	0.1197
equilibrium expected loss	$\mathbb{E}[X_i^*]$	0.0239	0.0200	0.2561
equilibrium risk measure	$\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i^*)$	0	0	0.9444
equilibrium price of risk	$\mathbb{E}[\psi X_i^*]$	0.0279	0.0240	0.3073
initial objective value	$\mathbb{E}[\xi_i]+c_i\mathrm{RVaR}_{\alpha_i,\beta_i}(\xi_i)$	0.1341	0.1273	0.1171
equilibrium objective value	$\mathbb{E}[X_i^*]+c_i\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i^*)-\mathbb{E}[\psi(X_i^*-\xi_i)]$	0.1157	0.1157	0.1157
initial total risk measure	$\sum_{i=1}^3 \mathrm{RVaR}_{\alpha_i,\beta_i}(\xi_i)$	1.0236		
equilibrium total risk measure	$\sum_{i=1}^3 \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i^*)$	0.9444		

Table 2.1: Numerical comparison between the initial allocation and the equilibrium allocation.

comparison between the initial allocation and the equilibrium allocation are reported in Table 2.1[6]. From Table 2.1, each of the agents has an improved objective value, with agent 3 being the remaining-risk bearer, whose improvement is the smallest among the three agents. It is not a coincidence that the equilibrium objective values of the three agents in Table 2.1 are identical. This is because $\mathbb{E}[X_i^*]+c_i\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i^*)-\mathbb{E}[\psi(X_i^*-\xi_i)]=\mathbb{E}[\psi\xi_i]-\frac{x\alpha_i}{d}$ (see (2.8.9)) which is the same for $i=1,2,3$.

2.6 Model Misspecification, Robustness and Comonotonicity in Risk Sharing

As shown in Sections 2.4 and 2.5, the optimal allocations in (2.4.4)-(2.4.5) are prominent to various settings of risk sharing and equilibria when using the RVaR family of risk measures. In this section we discuss a few issues related to the above optimal allocations. If an allocation $(X_1,\dots,X_n)$ is determined by X, it can be written as $(X_1,\dots,X_n)=(f_1(X),\dots,f_n(X))\in\mathbb{A}_n(X)$ for some functions $f_1,\dots,f_n$. We denote by $\mathbb{F}_n$ the set of *sharing principles* $(f_1,\dots,f_n)$ where each

[6] $\mathrm{RVaR}_{\alpha_i,\beta_i}(\xi_i)$, $\mathbb{E}[X_i^*]$, $i=1,2,3$ and $\mathrm{RVaR}_{\alpha_3,\beta_3}(X_3^*)$ are calculated by the average of 100 repetitions of simulations of size 100,000. Other quantities are calculated analytically.

$f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, n$, has at most finitely many points of discontinuity, $f_1(x) + \cdots + f_n(x) = x$ for all $x \in \mathbb{R}$, and $f_i(X) \in \mathcal{X}$ for $X \in \mathcal{X}$, $i = 1, \ldots, n$. As discussed in Remark 2.3, the cases in which $\sum_{i=1}^n \alpha_i + \bigvee_{i=1}^n \beta_i < 1$ and $\alpha_i + \beta_i > 0$ for each $i = 1, \ldots, n$ are most relevant for the existence of an optimal allocation, and we shall make this assumption in the following discussions.

2.6.1 Robust Allocations

In this section we discuss risk sharing in the presence of model uncertainty by studying the resulting aggregate risk value when the distribution of the total risk $X \in \mathcal{X}$ is misspecified. We will see that this in general implies serious problems for VaR but not for RVaR or ES. This relates to the issue of the robustness of VaR and RVaR; for a relevant discussion on robustness properties for risk measures, see Cont et al. (2010), Kou et al. (2013), Krätschmer et al. (2014) and Embrechts et al. (2015); see also Remark 2.8 below. In contrast to the above literature, we are interested in the robustness of the optimal allocation instead of the robustness of the risk measures themselves.

Definition 2.3. For given risk measures $\rho_1, \ldots, \rho_n$ on $\mathcal{X}$, $X \in \mathcal{X}$ and a pseudo-metric[7] π defined on $\mathcal{X}$, an allocation $(f_1(X), \ldots, f_n(X)) \in \mathbb{A}_n(X)$ with $(f_1, \ldots, f_n) \in \mathbb{F}_n$ is *π-robust* if the functional $Z \to \sum_{i=1}^n \rho_i(f_i(Z))$ is continuous at $Z = X$ with respect to π.

Commonly used pseudo-metrics π in risk management include the L^q metric for $q \geqslant 1$, the L^∞ metric (assuming X is bounded), or the Lévy metric π_W[8], which metrizes weak convergence (convergence in distribution). As we take the common domain $\mathcal{X}$ as the set of integrable random variables, we shall analyze the cases $\pi = L^1$, L^∞ and π_W in the following.

In Definition 2.3, X represents an agreed-upon underlying risk. The n agents design a sharing principle $(f_1, \ldots, f_n)$ based on the knowledge of a model X. The true risk Z is unknown to the agents, and can be slightly different from the model X. If an optimal allocation is robust in the sense of Definition 2.3, then under a small model misspecification, the true aggregate risk value $\sum_{i=1}^n \rho_i(f_i(Z))$ would not be too far away from the optimized value for X. On the other hand, for a non-robust optimal allocation, a small model misspecification would destroy the optimality of the allocation.

[7]A pseudo-metric is similar to a metric except that the distance between two distinct points can be zero. For instance, a metric on the set of distributions, such as the Lévy metric, induces a pseudo-metric on $\mathcal{X}$.

[8]More rigorously, π_W is the pseudo-metric on $\mathcal{X}$ induced by the the Lévy metric on the set of distributions.

Proposition 2.9. *Let $X \in \mathcal{X}$ be a continuously distributed random variable. Suppose that $Z_j \to X$ weakly as $j \to \infty$, then for $\alpha_i, \beta_i \in [0, 1)$, $\alpha_i + \beta_i < 1$, $i = 1, \dots, n$, and $(f_1, \dots, f_n) \in \mathbb{F}_n$, we have*

$$\liminf_{j \to \infty} \sum_{i=1}^{n} \mathrm{RVaR}_{\alpha_i, \beta_i}(f_i(Z_j)) \geqslant \sum_{i=1}^{n} \mathrm{RVaR}_{\alpha_i, \beta_i}(f_i(X)).$$

The proof of Proposition 2.9 is given in Section 2.8.3. Proposition 2.9 suggests that if the actual risk Z is misspecified as X, then the aggregate risk value for an allocation of Z is asymptotically larger than that for an allocation of X. Proposition 2.9 remains valid if weak convergence is strengthened to L^1-convergence or L^∞-convergence.

The next proposition discusses the connection between the robustness property of the inf-convolution risk measure and that of the optimal allocation.

Proposition 2.10. *For given risk measures $\rho_1, \dots, \rho_n$ on $\mathcal{X}$, $X \in \mathcal{X}$ and a pseudo-metric π defined on $\mathcal{X}$, if there exists a π-robust optimal allocation of X, then $\square_{i=1}^{n} \rho_i$ is π-upper-semicontinuous at X.*

The proof of Proposition 2.10 is given in Section 2.8.4. In Section 2.6.2 below we shall see that π-continuity (stronger than π-upper-semicontinuity) of $\square_{i=1}^{n} \rho_i$ is not sufficient for the existence of a π-robust optimal allocation. More discussions on the relationship in Proposition 2.10 for the RVaR family and convex risk measures are presented in Remark 2.8.

Remark 2.7. Recently, Krätschmer et al. (2012, 2014) and Zähle (2016) developed robustness properties for statistical functionals (including law-invariant risk measures) on Orlicz hearts with respect to ψ-weak topologies. These concepts are well suitable for studying convex risk measures; see Cheridito and Li (2009) for more on risk measures on Orlicz hearts. For $\mathrm{RVaR}_{\alpha,\beta}$ with $\alpha > 0$, the tail distribution of a risk beyond its $(1 - \alpha)$-quantile level does not play a role, and hence the notions of Orlicz hearts and ψ-weak convergence are hardly relevant. In the case of $\mathrm{ES}_\beta = \mathrm{RVaR}_{0,\beta}$, the corresponding Orlicz heart is L^1 and the corresponding gauge function ψ is linear; see Krätschmer et al. (2014).

2.6.2 Robust Allocations for Quantile-based Risk Measures

In the following we characterize robust optimal allocations in the RVaR family. For technical reasons, we assume that the total risk X under study is doubly continuous; this includes practically all models used in risk management and robust statistics. Note that this does not imply that the random variables in an optimal allocation are continuously distributed.

Theorem 2.11. *For risk measures* $\mathrm{RVaR}_{\alpha_1,\beta_1},\dots,\mathrm{RVaR}_{\alpha_n,\beta_n}$*,* $\alpha_i,\beta_i\in[0,1)$*,* $\alpha_i+\beta_i>0$*,* $i=1,\dots,n$*,* $\sum_{i=1}^n\alpha_i+\bigvee_{i=1}^n\beta_i<1$ *and a doubly continuous random variable* $X\in\mathcal{X}$*, the following hold.*

(i) There exists an L^1*-robust optimal allocation of* X *if and only if* $\beta_1,\dots,\beta_n>0$*.*

(ii) If X *is bounded, then there exists an* L^∞*-robust optimal allocation of* X *if and only if* $\beta_1,\dots,\beta_n>0$*.*

(iii) There exists a π_W*-robust optimal allocation of* X *if and only if* $\beta_1,\dots,\beta_n>0$ *and* $\alpha_i>0$ *for some* $i=1,\dots,n$*.*

A proof of Theorem 2.11 is given in Section 2.8.5. From Theorem 2.11, if all of the underlying risk measures are true RVaR or ES, then an L^1-robust optimal allocation can be obtained. More interestingly, as soon as one of the underlying risk measures is a true VaR, not only the allocation in (2.4.9)-(2.4.10) is non-robust, but any optimal allocation is non-robust with respect to any commonly used metrics.

A true RVaR is known to have a strong form of robustness (π_W-continuity), and hence it is not surprising that the strongest robustness in the optimal allocation is found for true RVaR. On the contrary, if one of $\beta_1,\dots,\beta_n$ is zero, even if $\square_{i=1}^n\mathrm{RVaR}_{\alpha_i,\beta_i}$ is π_W-continuous, and each of $\mathrm{RVaR}_{\alpha_i,\beta_i}$ is π_W-continuous at X (a VaR is π_W-continuous at any doubly continuous random variable), an L^∞-optimal allocation does not exist, not to say L^1- or π_W-robust ones. Thus, individual robustness of the underlying risk measures does not imply the existence of robust optimal allocations.

Remark 2.8. In the literature of risk measures, there is a well-known conflict between convexity and robustness. This is due to the fact that any convex risk measure is not π_W-upper-semicontinuous on the set of bounded random variables (see Bäuerle and Müller (2006) and Cont et al. (2010)). If the underlying risk measures $\rho_1,\dots,\rho_n$ are convex risk measures, then $\square_{i=1}^n\rho_i$ is also a convex risk measure (Barrieu and EI Karoui (2005)). In this case, there does not exist a π_W-robust optimal allocation by Proposition 2.10. On the other hand, from Theorem 2.11 (iii), for a π_W-robust optimal allocation to exist, some of the underlying risk measures can be convex (ES), at least one of them is a true RVaR, which is not convex. To summarize, the conflict between convexity and robustness still exists, and this only applies to weak convergence, not to L^∞ and L^1 metrics; to allow for a robust optimal allocation, some (but not all) of the underlying risk measures may be convex.

2.6.3 Comonotonicity in Optimal Allocations

Another important concept in the literature of risk sharing is comonotonicity, which relates to a type of moral hazard among collaborative agents sharing a risk. As we have seen from (2.4.4)-(2.4.5) in Theorem 2.4, the optimal allocation we construct may not be comonotonic. If the allocations are constrained to be comonotonic, general results on risk sharing for a general class of risk measures including RVaR are already known in the literature; see Jouini et al. (2008) and Cui et al. (2013). In this section we discuss whether an optimal allocation in a quantile-based risk sharing problem can be chosen as comonotonic.

In the following theorem, we show that, in a quantile-based risk sharing problem, a comonotonic optimal allocation exists if and only if all underlying risk measures are ES except for the one with the largest tolerance parameter.

Theorem 2.12. *For risk measures* $\mathrm{RVaR}_{\alpha_1,\beta_1},\dots,\mathrm{RVaR}_{\alpha_n,\beta_n}$*,* $\alpha_i,\beta_i\in[0,1)$*,* $\alpha_i+\beta_i\leqslant 1$*,* $i=1,\dots,n$*, and any continuously distributed random variable* $X\in\mathcal{X}$*, there exists a comonotonic optimal allocation of* X *if and only if there exists* $i=1,\dots,n$*, such that for all* $j=1,\dots,n$*,* $j\neq i$*,* $\alpha_j=0$ *and* $\beta_i\geqslant\beta_j$*.*

To prove Theorem 2.12, we need some results on risk sharing problems with allocations confined to comonotonic ones; see Section 2.8.1. The proof of the theorem is given in Section 2.8.6.

Remark 2.9. Comonotonicity is closely related to convex-order consistency and convexity (see Rüschendorf (2013) and Föllmer and Schied (2016)). Within the RVaR family, the latter two properties are only satisfied by ES. In view of this, it is not surprising that the existence of comonotonic optimal allocations relies on the presence of ES as the underlying risk measures.

2.7 Summary and Discussions

2.7.1 Summary of Main Results

For underlying risk measures $\mathrm{RVaR}_{\alpha_1,\beta_1}, \dots, \mathrm{RVaR}_{\alpha_n,\beta_n}$, $\alpha_i, \beta_i \geqslant 0$, $i = 1, \dots, n$, we solve the optimal risk sharing problem of a total risk $X \in \mathcal{X}$ and construct corresponding Arrow-Debreu Equilibria. The mathematical results are summarized below.

We first establish an inequality in Theorem 2.1,

$$\mathrm{RVaR}_{\sum_{i=1}^n \alpha_i, \bigvee_{i=1}^n \beta_i}\left(\sum_{i=1}^n X_i\right) \leqslant \sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i),$$

which applies to all random variables $X_1, \dots, X_n \in \mathcal{X}$ and $\alpha_i, \beta_i \in \mathbb{R}_+$, $i = 1, \dots, n$.

Assuming $\sum_{i=1}^n \alpha_i + \bigvee_{i=1}^n \beta_i < 1$, a Pareto-optimal allocation $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ can be constructed explicitly as in Theorem 2.4, with the aggregate risk value

$$\sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) = \mathrm{RVaR}_{\sum_{i=1}^n \alpha_i, \bigvee_{i=1}^n \beta_i}(X).$$

This optimal allocation turns out to be an Arrow-Debreu equilibrium allocation in the settings of Theorems 2.7 and 2.8, and the equilibrium pricing rule is obtained explicitly.

Some properties of the above optimal allocation are further characterized. In particular, in Theorems 2.11 and 2.12 we show that, to allow for an L^1-robust optimal allocation of X, the underlying risk measures should all be ES or true RVaR, and to allow for a comonotonic optimal allocation of X, all but one of the underlying risk measures should be ES.

2.7.2 Implications for the Choice of a Suitable Regulatory Risk Measure

As mentioned in the introduction, there has recently been an extensive debate on the desirability of regulatory risk measures, and in particular, VaR or ES, in banking and insurance. It is a fact that currently VaR and ES coexist as regulatory risk measures throughout the broader financial industry. For example, within banking, where VaR used to rule as "the benchmark" (see Jorion (2006)), ES as an alternative is strongly gaining ground. This is for instance the case for internal models within the new regulatory guidelines for the trading book; see BCBS (2014).

The "coexistence" becomes clear from the fact that Credit Risk is still falling under the VaR-regime. For Operational Risk we are at the moment in a transitionary phase where VaR-based internal models within the Advanced Measurement Approach (AMA) may be scaled down fully; see BCBS (2016). This less quantitative modeling approach towards Operational Risk is already standard in insurance regulation like the Swiss Solvency Test (SST) and Solvency II. Within the latter regulatory landscapes, we also witness a coexistence of VaR (Solvency II) and ES (SST) making the results of this chapter more relevant.

Below we discuss some relevant implications of our results to the above regulatory debates on risk measures. In particular, we discover some new advantages of ES, supporting the transition initiated by the Basel Committee on Banking Supervision. We like to stress however that, through various explicit formulas, our results are relevant for the ongoing discussion on the use of risk measures within Quantitative Risk Management more generally.

(i) Capturing tail risk "Tail risk" is currently of crucial concern for banking regulation. Below we quote the Basel Committee on Banking Supervision, Page 1 of BCBS (2016), Executive Summary:

> "*... A shift from Value-at-Risk (VaR) to an Expected Shortfall (ES) measure of risk under stress. Use of ES will help to ensure a more prudent capture of "tail risk" and capital adequacy during periods of significant financial market stress.*"

From our results in Section 2.4, for any risk $X \geqslant 0$ with $\mathbb{P}(X > 0) < n\alpha$, one has $\square_{i=1}^{n} \mathrm{VaR}_\alpha(X) = \mathrm{VaR}_{n\alpha}(X) = 0$. Therefore, in the optimization of risks under true VaR (or true RVaR), there is a part of the loss which is undertaken by the firms, but its riskiness is completely ignored; this should be also clear from the optimal allocation presented in Theorem 2.4.

Although the fact that VaR cannot capture tail risk is often argued, our results explain this fact mathematically for the first time in the framework of risk sharing and optimization. Within the RVaR family, to completely avoid such a phenomenon, one requires $\alpha_i = 0$, $i = 1, \ldots, n$, which means that the regulator needs to impose ES as the regulatory risk measure.

(ii) Model misspecification Due to model uncertainty, a non-robust allocation may lead to a significantly higher aggregate risk value for the agents, that is, far away from the optimal one. Any model for the total risk X suffers from model uncertainty, be it at the level of statistical (parameter) uncertainty or at the level of the analytic structure of the model (e.g. which economic factors to include). The 2007 - 2009 financial crisis (unfortunately) gave ample proof of this, especially in the context of the rating of mortgage based derivatives; see, for instance, Donnelly and Embrechts (2010).

From our results in Section 2.6, as soon as one of underlying risk measures is a true VaR, an optimal allocation cannot be robust. Therefore, a true RVaR or an ES is a better choice than a VaR in the presence of model uncertainty. Our conclusion is consistent with the observations in Cont et al. (2010) that RVaR has advantages in robustness properties over VaR and ES, albeit our results come from a different mathematical setting. Remarkably, ES is more robust than VaR in our settings of risk sharing.

(iii) Understanding the least possible total capital Let ρ be a regulatory risk measure in use for a given jurisdiction. Note that, via sharing, be it cooperative (e.g. fragmentation of a single firm; see Section 2.4) or competitive (see Section 2.5), the total risk in the economy remains the same while the total regulatory capital is reduced.

The mathematical results obtained in the chapter give a guideline for calculating the least possible aggregate capital $\sum_{i=1}^n \rho(X_i)$ in the economy, when the regulatory risk measure is chosen within the RVaR family. In practice, a regulator may not know how risks are (will be) distributed among firms before she designs a regulatory risk measure; there are many possibilities. Our results can be seen as a worst-case scenario (least amount) total regulatory capital in the economy. Our results also suggest that, within a VaR-based regulatory system, constraints on the within-firm fragmentation have to be imposed; otherwise the total regulatory capital may be artificially reduced.

2.8 Technical Details

2.8.1 Comonotonic Risk Sharing for Distortion Risk Measures

For $\alpha, \beta \in [0,1)$ and $\alpha+\beta \leqslant 1$, $\mathrm{RVaR}_{\alpha,\beta}$ belongs to the class of *distortion risk measures* of the form

$$\rho_{\bar{h}}(X) = \int_0^1 \mathrm{VaR}_\alpha(X)\mathrm{d}\bar{h}(\alpha), \quad X \in \mathcal{X}, \tag{2.8.1}$$

for some non-decreasing and left-continuous function $\bar{h} : [0,1] \to [0,1]$ satisfying $\bar{h}(0) = 0$ and $\bar{h}(1) = 1$, such that the above integral is properly defined, where VaR is defined in (2.2.1). Note that equations (2.8.1) and (1.2.6) are identical with $\bar{h}(t) = 1 - h(1-t)$, and $\bar{h}$ is also a distortion function, called the *dual distortion function* of h. In this section, whenever we mention a distortion function, it is $\bar{h}$ in (2.8.1). A distortion risk measure $\rho_{\bar{h}}$ is coherent if $\bar{h}$ is concave. For $\alpha, \beta \in [0,1)$ and $\alpha + \beta \leqslant 1$, the distortion function of $\mathrm{RVaR}_{\alpha,\beta}(X)$ is given by

$$\bar{h}^{(\alpha,\beta)}(t) := \begin{cases} \min\{\mathrm{I}_{\{t>\alpha\}}\frac{t-\alpha}{\beta}, 1\} & \text{if } \beta > 0, \\ \mathrm{I}_{\{t>\alpha\}} & \text{if } \beta = 0, \end{cases} \quad t \in [0,1]. \tag{2.8.2}$$

The set of comonotonic allocations is defined as

$$\mathbb{A}_n^+(X) = \{(X_1, \dots, X_n) \in \mathbb{A}_n(X) : \; X_i \uparrow X, \; i = 1, \dots, n\},$$

where $X_i \uparrow X$ means that X_i and X are comonotonic.

The *constrained inf-convolution* of risk measures $\rho_1, \dots, \rho_n$ is defined as

$$\mathop{\boxplus}_{i=1}^{n} \rho_i(X) := \inf\left\{\sum_{i=1}^{n} \rho_i(X_i) : (X_1, \dots, X_n) \in \mathbb{A}_n^+(X)\right\}.$$

Definition 2.4. Let $\rho_1, \dots, \rho_n$ be risk measures and $X \in \mathcal{X}$. An n-tuple $(X_1, X_2, \dots, X_n) \in \mathbb{A}_n^+(X)$ is called an *optimal constrained allocation* of X if $\sum_{i=1}^{n} \rho_i(X_i) = \boxplus_{i=1}^{n} \rho_i(X)$.

It is obvious that $\square_{i=1}^{n} \rho_i(X) \leqslant \boxplus_{i=1}^{n} \rho_i(X)$. Hence, if an optimal allocation of X is comonotonic, then it is also an optimal constrained allocation, and $\square_{i=1}^{n} \rho_i(X) = \boxplus_{i=1}^{n} \rho_i(X)$. In Jouini et al. (2008) it is shown that for law-invariant convex risk measures on L^∞, optimal constrained allocations are also optimal allocations. This statement remains true if the underlying risk measures preserve convex order; this is based on the *comonotone improvement* in Landsberger and Meilijson (1994) and Ludkovski and Rüschendorf (2008).

A solution to the optimal constrained allocation can be found in Jouini et al. (2008) for convex risk measures and in Cui et al. (2013) for general distortion risk measures in the context of the design of optimal reinsurance contracts. We give a self-contained proof here which we believe is simpler than the existing ones in the literature.

Proposition 2.13. *For n distortion functions $\bar{h}_1, \dots, \bar{h}_n$ such that $\rho_{\bar{h}_i}$ is finite on $\mathcal{X}$ for $i = 1, \dots, n$, we have*

$$\mathop{\boxplus}_{i=1}^{n} \rho_{\bar{h}_i}(X) = \int_0^1 \mathrm{VaR}_\alpha(X) \mathrm{d}\bar{h}(\alpha), \quad X \in \mathcal{X}, \tag{2.8.3}$$

where $\bar{h}(t) = \min\{\bar{h}_1(t), \ldots, \bar{h}_n(t)\}$*. Moreover, an optimal constrained allocation* $(X_1, \ldots, X_n)$ *of* $X \in \mathcal{X}$ *is given by* $X_i = f_i(X)$*,* $i = 1, \ldots, n$*, where*

$$f_i(x) = \int_0^x g_i(t)\mathrm{d}t, \quad x \in \mathbb{R},$$

and

$$g_i(t) = \begin{cases} 0 & \text{if } \bar{h}_i(1-F(t)) > \bar{h}(1-F(t)), \\ 1/k(t) & \text{otherwise,} \end{cases}$$

for $t \in \mathbb{R}$ *and* $k(t) = \#\{j = 1, \ldots, n : \bar{h}_j(1-F(t)) = \bar{h}(1-F(t))\}$*.*

Proof. We first show

$$\mathop{\boxplus}_{i=1}^{n} \rho_{\bar{h}_i}(X) \geqslant \int_0^1 \mathrm{VaR}_\alpha(X)\mathrm{d}\bar{h}(\alpha). \tag{2.8.4}$$

For two left-continuous distortion functions f and g, we have $\rho_f(X) \leqslant \rho_g(X)$ if $f \leqslant g$ (see Lemma A.1 of Wang et al. (2015)). Therefore, for any $(X_1, X_2, \ldots, X_n) \in \mathbb{A}_n^+(X)$, by the comonotonic additivity of VaR, we have

$$\int_0^1 \mathrm{VaR}_\alpha(X)\mathrm{d}\bar{h}(\alpha) = \int_0^1 \left(\mathrm{VaR}_\alpha(X_1) + \cdots + \mathrm{VaR}_\alpha(X_n)\right)\mathrm{d}\bar{h}(\alpha) \leqslant \sum_{i=1}^n \rho_{\bar{h}_i}(X_i).$$

Thus, (2.8.4) holds. Conversely, let F be the distribution of X. Since $f_1(t), \ldots, f_n(t)$ are Lipschitz continuous and non-decreasing, we have

$$\begin{aligned}
\sum_{i=1}^n \rho_{\bar{h}_i}(f_i(X)) &= \sum_{i=1}^n \int_0^1 \mathrm{VaR}_t(f_i(X))\mathrm{d}\bar{h}_i(t) \\
&= \sum_{i=1}^n \int_0^1 f_i(\mathrm{VaR}_t(X))\mathrm{d}\bar{h}_i(t) \\
&= \sum_{i=1}^n \int_0^1 \int_0^{\mathrm{VaR}_t(X)} g_i(s)\mathrm{d}s\mathrm{d}\bar{h}_i(t) \\
&= \sum_{i=1}^n \left(\int_0^\infty \bar{h}_i(1-F(s))g_i(s)\mathrm{d}s - \int_{-\infty}^0 (1-\bar{h}_i(1-F(s)))g_i(s)\mathrm{d}s\right) \\
&= \int_0^\infty \bar{h}(1-F(s))\mathrm{d}s - \int_{-\infty}^0 (1-\bar{h}(1-F(s)))\mathrm{d}s = \rho_{\bar{h}}(X),
\end{aligned}$$

where the fourth equality follows from Fubini's Theorem and the last equality

$$\rho_{\bar{h}}(X) = \int_0^\infty \bar{h}(1 - F(x))\mathrm{d}x - \int_{-\infty}^0 (1 - \bar{h}(1 - F(x)))\mathrm{d}x \tag{2.8.5}$$

is given in, for instance, Theorem 6 of Dhaene et al. (2012). Thus,

$$\mathop{\boxplus}_{i=1}^{n} \rho_{\bar{h}_i}(X) \leqslant \int_0^1 \mathrm{VaR}_\alpha(X)\mathrm{d}\bar{h}(\alpha).$$

The desired result follows. □

Since RVaRs belong to the family of distortion risk measures, their optimal constrained allocations can be constructed analogously, as summarized in the following corollary.

Corollary 2.14. *For $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in [0, 1)$ such that $\alpha_i + \beta_i \leqslant 1$, $i = 1, \dots, n$, we have*

$$\mathop{\boxplus}_{i=1}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(X) = \int_0^1 \mathrm{VaR}_\alpha(X)\mathrm{d}\bar{h}(\alpha), \quad X \in \mathcal{X}, \tag{2.8.6}$$

where $\bar{h}(t) = \min\{\bar{h}^{(\alpha_1,\beta_1)}(t), \dots, \bar{h}^{(\alpha_n,\beta_n)}(t)\}$, $t \in [0, 1]$.

2.8.2 Proof of Theorem 2.8

Proof. Similarly to the proof of Theorem 2.7, we consider two cases separately.

(i) Suppose $\mathbb{P}(X > 0) \leqslant \alpha$. This implies $\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i^*) = 0$ for $i = 1, \dots, n$, and $\psi = 1$. On the other hand, for any $0 \leqslant X_i \leqslant X$, we have $\mathbb{E}[X_i] + c_i\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) - \mathbb{E}[\psi X_i] = c_i\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) \geqslant 0$. Thus X_i^* satisfies (2.5.4), and hence $(\psi, (X_1^*, \dots, X_n^*))$ is an Arrow-Debreu equilibrium.

(ii) Suppose $\alpha < \mathbb{P}(X > 0) \leqslant \eta$. This implies one of $\beta_1, \dots, \beta_n$ is positive, and therefore $d > 0$. For $i = 1, \dots, n$, take any $X_i \in \mathcal{X}$ such that $0 \leqslant X_i \leqslant X$. Note that by definition, $(\psi - 1)X \leqslant x/d$. We have

$$\mathbb{E}[(\psi - 1)\mathrm{I}_{\{U_{X_i} \geqslant 1-\alpha_i\}}X_i] \leqslant \mathbb{E}[(\psi - 1)\mathrm{I}_{\{U_{X_i} \geqslant 1-\alpha_i\}}X] \leqslant \mathbb{E}\left[\frac{x}{d}\mathrm{I}_{\{U_{X_i} \geqslant 1-\alpha_i\}}\right] = \frac{x\alpha_i}{d}. \tag{2.8.7}$$

On the other hand, using $(\psi-1)\leqslant 1/d$ and $\mathbb{P}(X_i>0)\leqslant\mathbb{P}(X>0)\leqslant\alpha_i+\beta_i$,

$$\begin{aligned}\mathbb{E}[(\psi-1)\mathrm{I}_{\{U_{X_i}<1-\alpha_i\}}X_i]\leqslant\frac{1}{d}\mathbb{E}[\mathrm{I}_{\{U_{X_i}<1-\alpha_i\}}X_i]&\leqslant\frac{c_i}{\beta_i}\int_{\alpha_i}^{1}\mathrm{VaR}_\gamma(X_i)\mathrm{d}\gamma\\&=\frac{c_i}{\beta_i}\int_{\alpha_i}^{\alpha_i+\beta_i}\mathrm{VaR}_\gamma(X_i)\mathrm{d}\gamma\\&=c_i\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i).\end{aligned}\tag{2.8.8}$$

Combining (2.8.7) and (2.8.8), we have

$$\mathbb{E}[(\psi-1)X_i]\leqslant\frac{x\alpha_i}{d}+c_i\mathrm{RVaR}_{\alpha_i\beta_i}(X_i).$$

Equivalently,

$$\mathbb{E}[X_i]+c_i\mathrm{RVaR}_{\alpha_i\beta_i}(X_i)-\mathbb{E}[\psi X_i]\geqslant-\frac{x\alpha_i}{d}.$$

Next we verify that $\mathcal{V}_i(X_i^*)$ is equal to $-x\alpha_i/d$. Write $A_i=\{1-\sum_{k=1}^{i}\alpha_k<U_X\leqslant 1-\sum_{k=1}^{i-1}\alpha_k\}\subset\{U_X\geqslant 1-\alpha\}$. We have $X_i^*=X\mathrm{I}_{A_i}$ for $i=1,\dots,n-1$, and $X_n^*=X\mathrm{I}_{A_n}+X\mathrm{I}_{\{U_X<1-\alpha\}}$. For $i=1,\dots,n-1$,

$$\begin{aligned}\mathbb{E}[X_i]+c_i\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i^*)-\mathbb{E}[\psi X_i^*]=\mathbb{E}[(1-\psi)X_i^*]&=-\mathbb{E}\left[\frac{x}{Xd}\mathrm{I}_{\{U_X\geqslant 1-\alpha\}}X\mathrm{I}_{A_i}\right]\\&=-\mathbb{E}\left[\frac{x}{d}\mathrm{I}_{A_i}\right]=-\frac{x\alpha_i}{d}.\end{aligned}$$

For the last agent, we have

$$\begin{aligned}\mathbb{E}[(\psi-1)X_n^*]&=\mathbb{E}\left[\frac{x}{Xd}\mathrm{I}_{\{U_X\geqslant 1-\alpha\}}X\mathrm{I}_{A_n}\right]+\mathbb{E}\left[\frac{1}{d}\mathrm{I}_{\{U_X<1-\alpha\}}X\right]\\&=\frac{x\alpha_n}{d}+\frac{1}{d}\int_\alpha^1\mathrm{VaR}_\gamma(X)\mathrm{d}\gamma\\&=\frac{x\alpha_n}{d}+\frac{c_n}{\beta_n}\int_\alpha^{\alpha+\beta}\mathrm{VaR}_\gamma(X)\mathrm{d}\gamma\\&=\frac{x\alpha_n}{d}+c_n\mathrm{RVaR}_{\alpha,\beta}(X).\end{aligned}$$

Therefore,

$$\mathbb{E}[X_n^*]+c_n\mathrm{RVaR}_{\alpha_n,\beta_n}(X_n^*)-\mathbb{E}[\psi X_n^*]=c_n\mathrm{RVaR}_{\alpha,\beta}(X)-\mathbb{E}[(\psi-1)X_n^*]=-\frac{x\alpha_n}{d}.$$

In summary, for $i=1,\dots,n$,

$$\mathcal{V}_i(X_i)\geqslant-\frac{x\alpha_i}{d}=\mathcal{V}_i(X_i^*).\tag{2.8.9}$$

By definition, $(\psi,(X_1^*,\dots,X_n^*))$ is an Arrow-Debreu equilibrium for (2.5.12). $\square$

2.8.3 Proof of Proposition 2.9

Proof. For fixed $i = 1, \dots, n$, we will show that for any $\alpha, \beta \in [0, 1)$, $\alpha + \beta < 1$, the inequality

$$\liminf_{j\to\infty} \mathrm{RVaR}_{\alpha,\beta}(f_i(Z_j)) \geqslant \mathrm{RVaR}_{\alpha,\beta}(f_i(X)). \tag{2.8.10}$$

holds. Then the proposition follows from taking $(\alpha, \beta) = (\alpha_i, \beta_i)$ in (2.8.10) and summing up over $i = 1, \dots, n$.

Since X is continuously distributed, by the Continuous Mapping Theorem, we have $f_i(Z_j) \to f_i(X)$ weakly. Then, $\mathrm{VaR}_\gamma(f_i(Z_j)) \to \mathrm{VaR}_\gamma(f_i(X))$ for almost every $\gamma \in (0, 1)$. By noting that $\mathrm{VaR}_{\alpha+\beta}(X) > -\infty$, we have that $\mathrm{VaR}_{\alpha+\beta}(Z_j)$ is bounded below for $j \in \mathbb{N}$, and hence Fatou's Lemma gives us

$$\liminf_{j\to\infty} \mathrm{RVaR}_{\alpha,\beta}(f_i(Z_j)) \geqslant \frac{1}{\beta}\int_\alpha^{\alpha+\beta} \liminf_{j\to\infty} \mathrm{VaR}_\gamma(f_i(Z_j))\mathrm{d}\gamma = \mathrm{RVaR}_{\alpha,\beta}(f_i(X)), \quad \beta > 0. \tag{2.8.11}$$

For any $\gamma > 0$, since $\mathrm{VaR}_\gamma(X)$ is non-increasing in $\gamma \in [0, 1)$, using (2.8.11), we have

$$\liminf_{j\to\infty} \mathrm{VaR}_\alpha(f_i(Z_j)) \geqslant \liminf_{j\to\infty} \mathrm{RVaR}_{\alpha,\gamma}(f_i(Z_j)) \geqslant \mathrm{RVaR}_{\alpha,\gamma}(f_i(X)).$$

By letting $\gamma \downarrow 0$, we obtain

$$\liminf_{j\to\infty} \mathrm{VaR}_\alpha(f_i(Z_j)) \geqslant \mathrm{VaR}_\alpha(f_i(X)). \tag{2.8.12}$$

Therefore, (2.8.10) follows from (2.8.11)-(2.8.12). □

2.8.4 Proof of Proposition 2.10

Proof. Let $(f_1(X), \dots, f_n(X)) \in \mathbb{A}_n(X)$ be a π-robust optimal allocation of X. For any $Z_j \to X$ in π as $n \to \infty$, we have

$$\square_{i=1}^n \rho_i(Z_j) \leqslant \sum_{i=1}^n \rho_i(f_i(Z_j)) \to \sum_{i=1}^n \rho_i(f_i(X)) = \square_{i=1}^n \rho_i(X).$$

Therefore, $\square_{i=1}^n \rho_i$ is π-upper-semicontinuous at X. □

2.8.5 Proof of Theorem 2.11

Proof. Since the risk sharing problem is invariant under a constant shift in X, without loss of generality we may assume $\mathrm{VaR}_p(X) = 0$, where $p = \sum_{i=1}^n \alpha_i + \bigvee_{i=1}^n \beta_i < 1$. Similar to the proof of Theorem 2.4, we may also assume $\beta_n = \bigvee_{i=1}^n \beta_i$. Let F be the distribution of X.

Part 1. We first show that, in all cases (i)-(iii), the optimal allocation in (2.4.9)-(2.4.10) is robust. The optimal allocation in (2.4.9)-(2.4.10) can be written as $(f_1(X), \dots, f_n(X))$, where

$$f_i(x) = x\mathrm{I}_{\{F^{-1}(1-\sum_{k=1}^i \alpha_k) < x \leqslant F^{-1}(1-\sum_{k=1}^{i-1} \alpha_k)\}}, \quad i = 1, \dots, n-1, \ x \in \mathbb{R}, \text{ and} \tag{2.8.13}$$

$$f_n(x) = x\mathrm{I}_{\{x \leqslant F^{-1}(1-\sum_{k=1}^{n-1} \alpha_k)\}}, \quad x \in \mathbb{R}. \tag{2.8.14}$$

To show the cases (i) and (ii), suppose that $\beta_1, \dots, \beta_n > 0$. Let $Z_j \in \mathcal{X}, \ j \in \mathbb{N}$, be a sequence of random variables such that $Z_j \to X$ in L^1, $j \to \infty$. Note that this implies that $\{Z_j : j \in \mathbb{N}\}$ is uniformly integrable. By the Continuous Mapping Theorem, we have $f_i(Z_j) \to f_i(X)$ in probability. For each $i = 1, \dots, n$, since $f_i(x) \leqslant x\mathrm{I}_{\{x \geqslant 0\}}$ and $\{Z_j : j \in \mathbb{N}\}$ is uniformly integrable, $\{f_i(Z_j) : j \in \mathbb{N}\}$ is also uniformly integrable. Hence, we have $f_i(Z_j) \to f_i(X)$ in L^1. Note that $\mathrm{RVaR}_{\alpha,\beta}, \ \alpha, \beta > 0$, is continuous with respect to weak convergence (see Cont et al. (2010)) and $\mathrm{ES}_\beta, \ \beta > 0$ is continuous with respect to L^1-convergence (see Emmer et al. (2015)). Therefore, as $j \to \infty$, for $i = 1, \dots, n$,

$$\mathrm{RVaR}_{\alpha_i,\beta_i}(f_i(Z_j)) \to \mathrm{RVaR}_{\alpha_i,\beta_i}(f_i(X)). \tag{2.8.15}$$

Thus, $(f_1(X), \dots, f_n(X))$ is an L^1-robust optimal allocation of X. Note that if X is bounded, then L^∞-robustness is weaker than L^1 robustness, and hence $(f_1(X), \dots, f_n(X))$ is an L^∞-robust optimal allocation of X.

To show the case (iii), suppose that $\beta_1, \dots, \beta_n > 0$ and $\alpha_1 > 0$ without loss of generality (in fact, if $\alpha_1 = 0$, then $f_1(X) = 0$ and we can proceed to consider the next agent). Let $Z_j \in \mathcal{X}, \ j \in \mathbb{N}$, be a sequence of random variables such that $Z_j \to X$ in π_W, $j \to \infty$. By the Continuous Mapping Theorem, we have $f_i(Z_j) \to f_i(X)$ weakly. Since $\mathrm{RVaR}_{\alpha,\beta}, \ \alpha, \beta > 0$, is continuous with respect to weak convergence, we have

$$\mathrm{RVaR}_{\alpha_1,\beta_1}(f_1(Z_j)) \to \mathrm{RVaR}_{\alpha_1,\beta_1}(f_1(X)). \tag{2.8.16}$$

Note that all $f_i(X)$, $i = 2, \dots, n$ are bounded above by $\mathrm{VaR}_{\alpha_1}(X)$. By a simple argument of the Dominated Convergence Theorem, we have, for $i = 2, \dots, n$, regardless of whether $\alpha_i = 0$,

$$\mathrm{RVaR}_{\alpha_i,\beta_i}(f_i(Z_j)) \to \mathrm{RVaR}_{\alpha_i,\beta_i}(f_i(X)). \tag{2.8.17}$$

Thus, $(f_1(X), \dots, f_n(X))$ is an π_W-robust optimal allocation of X.

Part 2. Next we show the other direction of the statements in (i)-(iii).

(1) (i) and (ii), $n = 2$: Suppose that $\beta_k = 0$ and $\alpha_k > 0$ for some $k = 1, \dots, n$. We first look at the case $n = 2$, and we may assume that the first agent uses a true VaR. That is, $\alpha_1 > 0$ and $\beta_1 = 0$. Recall that we have assumed $\mathrm{VaR}_{\alpha_1+\alpha_2+\beta_2}(X) = 0$.

Suppose that (X_1, X_2) is an optimal allocation of X where $X_1 = f_1(X)$ and $X_2 = f_2(X)$ for some $(f_1, f_2) \in \mathbb{F}_2$. Since $(X_1 + c, X_2 - c)$ is also optimal for any $c \in \mathbb{R}$ and the robustness property of $(X_1 + c, X_2 - c)$ is the same as (X_1, X_2), we may assume without loss of generality $\mathrm{VaR}_{\alpha_1}(X_1) = 0$. As (X_1, X_2) is optimal, we have, from Theorem 2.4,

$$\mathrm{VaR}_{\alpha_1}(X_1) + \mathrm{RVaR}_{\alpha_2,\beta_2}(X_2) = \mathrm{RVaR}_{\alpha_1+\alpha_2,\beta_2}(X). \tag{2.8.18}$$

Writing (2.8.18) in an integral form, we have

$$\beta_2 \mathrm{VaR}_{\alpha_1}(X_1) + \int_0^{\beta_2} \mathrm{VaR}_{\alpha_2+\beta}(X_2) \mathrm{d}\beta = \int_0^{\beta_2} \mathrm{VaR}_{\alpha_1+\alpha_2+\beta}(X) \mathrm{d}\beta. \tag{2.8.19}$$

Note that from Corollary 2.2, we have, for any $\beta \geqslant 0$,

$$\mathrm{VaR}_{\alpha_1}(X_1) + \mathrm{VaR}_{\alpha_2+\beta}(X_2) \geqslant \mathrm{VaR}_{\alpha_1+\alpha_2+\beta}(X). \tag{2.8.20}$$

Therefore, the inequalities in (2.8.20) are equalities for almost every $\beta \geqslant 0$. By noting that both sides of (2.8.20) are right-continuous, the inequalities in (2.8.20) are indeed equalities for all $\beta \geqslant 0$. In particular, we have

$$\mathrm{VaR}_{\alpha_1}(X_1) + \mathrm{VaR}_{\alpha_2+\beta_2}(X_2) = \mathrm{VaR}_{\alpha_1+\alpha_2+\beta_2}(X). \tag{2.8.21}$$

Thus,

$$\mathrm{VaR}_{\alpha_1}(X_1) = \mathrm{VaR}_{\alpha_2+\beta_2}(X_2) = \mathrm{VaR}_{\alpha_1+\alpha_2+\beta_2}(X) = 0, \tag{2.8.22}$$

which implies $\mathbb{P}(X_2 > 0) \leqslant \alpha_2 + \beta_2$.

Let $A_1 = \{U_{X_1} > 1 - \alpha_1\}$, $A_2 = \{U_{X_2} > 1 - \alpha_2\}$ and $A = \{U_{X_2} > 1 - \alpha_2 - \beta_2\}$. Note that by (2.8.22), $\{X_1 > 0\} \subseteq A_1$ and $\{X_2 > 0\} \subseteq A$. However, since $\mathbb{P}(X > 0) = \alpha_1 + \alpha_2 + \beta_2$, and

$$\{X > 0\} \subseteq (\{X_1 > 0\} \cup \{X_2 > 0\}),$$

we have

$$\begin{aligned}\alpha_1+\alpha_2+\beta_2 \leqslant \mathbb{P}(\{X_1>0\}\cup\{X_2>0\}) &\leqslant \mathbb{P}(X_1>0)+\mathbb{P}(X_2>0)\\ &\leqslant \mathbb{P}(A_1)+\mathbb{P}(A)=\alpha_1+\alpha_2+\beta_2. \qquad (2.8.23)\end{aligned}$$

Therefore, all the inequalities in (2.8.23) are equalities, and in particular, $\mathbb{P}(\{X_1>0\}\cup\{X_2>0\})=\mathbb{P}(X_1>0)+\mathbb{P}(X_2>0)$ implies

$$\mathbb{P}(X_1>0, X_2>0)=0. \qquad (2.8.24)$$

From (2.3.6) in the proof of Theorem 2.1, we can see that (2.8.18) implies that the inequalities in (2.3.5) are equalities for almost every $\gamma\in[0,\beta_2]$, where Y_1, Y_2 are defined in (2.3.2) and m is some constant. In particular, by taking $\gamma\downarrow 0$ in

$$\mathrm{VaR}_\gamma(Y_1+Y_2)=\mathrm{VaR}_{\gamma+\alpha_1+\alpha_2}(X) \quad \text{for almost every } \gamma\in[0,\beta_2],$$

and since both sides are right-continuous in γ, we have

$$\mathrm{VaR}_0(Y_1+Y_2)=\mathrm{VaR}_{\alpha_1+\alpha_2}(X).$$

That is, $X\leqslant \mathrm{VaR}_{\alpha_1+\alpha_2}(X)$ almost surely on $A_1^c\cap A_2^c$, and equivalently,

$$\{X>\mathrm{VaR}_{\alpha_1+\alpha_2}(X)\}\subset(A_1\cup A_2) \quad \text{a.s.}$$

It follows that

$$\alpha_1+\alpha_2=\mathbb{P}(X>\mathrm{VaR}_{\alpha_1+\alpha_2}(X))\leqslant\mathbb{P}(A_1\cup A_2)\leqslant\mathbb{P}(A_1)+\mathbb{P}(A_2)=\alpha_1+\alpha_2,$$

and therefore all the inequalities above are equalities. In particular, we have $\mathbb{P}(X>\mathrm{VaR}_{\alpha_1+\alpha_2}(X))=\mathbb{P}(A_1\cup A_2)$ and hence

$$\{X>\mathrm{VaR}_{\alpha_1+\alpha_2}(X)\}=(A_1\cup A_2) \quad \text{a.s.}$$

From $\mathbb{P}(X_1>0, X_2>0)=0$ in (2.8.24), $X_2\leqslant 0$ almost surely on A_1. Finally, since $X_1=X-X_2$, we have $X_1>\mathrm{VaR}_{\alpha_1+\alpha_2}(X)$ almost surely on A_1, and this further implies

$$\{X_1>\mathrm{VaR}_{\alpha_1+\alpha_2}(X)\}=A_1 \quad \text{a.s.} \qquad (2.8.25)$$

We consider the cases $\beta_2>0$ and $\beta_2=0$ separately:

(a) If $\beta_2 > 0$, then since $\mathrm{VaR}_\gamma(X)$ is strictly decreasing in $\gamma \in [0,1]$ (implied by the continuity of F; see Proposition 1.1), we have $\mathrm{VaR}_{\alpha_1+\alpha_2}(X) > \mathrm{VaR}_{\alpha_1+\alpha_2+\beta_2}(X) = 0$.

(b) If $\beta_2 = 0$, since f_1 and f_2 have at most finitely many discontinuity points, there is a constant $c \in (0, \mathrm{VaR}_0(X))$ such that f_1 and f_2 are continuous on the interval $(0,c)$. Since $\mathrm{VaR}_\gamma(X) = F^{-1}(1-\gamma)$ is continuous and strictly decreasing in γ, we have that for any subinterval $(a,b) \subset (0,c)$, one has $\mathbb{P}(X \in (a,b)) > 0$. From (2.8.24), we have $\mathbb{P}(f_1(X) > 0, f_2(X) > 0) = 0$, and hence for almost every $x \in (0, \mathrm{VaR}_0(X))$, $f_1(x) > 0$ implies $f_2(x) \leqslant 0$. Moreover, since $f_1(x) + f_2(x) = x$, $x \in (0,c)$, we know that $f_1(x)$ and $f_2(x)$ cannot be in the interval $(0,x)$. By the continuity of f_1 and f_2, we know that either $f_1(x) \leqslant 0$ for all $x \in (0,c)$ or $f_2(x) \leqslant 0$ for all $x \in (0,c)$. Without loss of generality, assume $f_1(x) \leqslant 0$ for all $x \in (0,c)$. Then, together with $\mathbb{P}(f_1(X) > 0, f_2(X) > 0) = 0$, we have $\{X_1 > c\} = A_1$ almost surely.

In both (a) and (b), there is a constant $c_0 > 0$ such that $\{X_1 > c_0\} = A_1$ almost surely. Define

$$B = \{x \in \mathbb{R} : f_1(x) > c_0\},$$

and thus $\{X \in B\} = \{X_1 > c_0\}$. From (2.8.25), $\mathbb{P}(X \in B) = \mathbb{P}(X_1 > c_0) = \mathbb{P}(A_1) = \alpha_1$. For $\varepsilon > 0$, let Y_ε be a Uniform$[-\varepsilon, \varepsilon]$ random variable independent of X and

$$Z_\varepsilon = X + Y_\varepsilon \mathrm{I}_{\{X \notin B\}}.$$

We can easily see that $Z_\varepsilon \to X$ in L^1 (in L^∞ if X is bounded) as $\varepsilon \downarrow 0$, and $\mathbb{P}(Z_\varepsilon \in B) > \alpha_1$ which means $\mathrm{VaR}_{\alpha_1}(f_1(Z_\varepsilon)) \geqslant c_0$. On the other hand, from (2.8.10), we have

$$\liminf_{\varepsilon \downarrow 0} \mathrm{RVaR}_{\alpha_2,\beta_2}(f_2(Z_\varepsilon)) \geqslant \mathrm{RVaR}_{\alpha_2,\beta_2}(f_2(X)),$$

and hence

$$\begin{aligned}&\liminf_{\varepsilon \downarrow 0} \left(\mathrm{VaR}_{\alpha_1}(f_1(Z_\varepsilon)) + \mathrm{RVaR}_{\alpha_2,\beta_2}(f_2(Z_\varepsilon))\right) - \left(\mathrm{VaR}_{\alpha_1}(f_1(X)) + \mathrm{RVaR}_{\alpha_2,\beta_2}(f_2(X))\right)\\ &\geqslant c_0 > 0.\end{aligned}$$

Thus, $(f_1(X), f_2(X))$ is not L^1-robust (and not L^∞-robust if X is bounded).

(2) (i) and (ii), $n > 2$: We may assume $\alpha_1 > 0$, $\beta_1 = 0$, that is, the first agent uses a true VaR. Suppose that $(f_1(X), \dots, f(X_n))$ is an optimal allocation of X where $(f_1, f_2, \dots, f_n) \in \mathbb{F}_n$.

Write $\alpha = \sum_{i=2}^{n} \alpha_i$, $\beta = \bigvee_{i=2}^{n} \beta_i$ and $g(x) = f_2(x) + \cdots + f_n(x)$, $x \in \mathbb{R}$; it is easy to see that $(f_1, g) \in \mathbb{F}_2$. From Theorems 2.1 and 2.4,

$$\begin{aligned}\mathrm{RVaR}_{\sum_{i=1}^{n}\alpha_i, \bigvee_{i=1}^{n}\beta_i}(X) = \sum_{i=1}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(f_i(X)) &\geqslant \mathrm{VaR}_{\alpha_1}(f_1(X)) + \mathrm{RVaR}_{\alpha,\beta}(g(X)) \\ &\geqslant \mathrm{RVaR}_{\sum_{i=1}^{n}\alpha_i, \bigvee_{i=1}^{n}\beta_i}(X).\end{aligned}$$

Hence, the above inequalities are all equalities, and in particular,

$$\mathrm{VaR}_{\alpha_1}(f_1(X)) + \mathrm{RVaR}_{\alpha,\beta}(g(X)) = \mathrm{RVaR}_{\alpha+\alpha_1,\beta}(f_1(X) + g(X)).$$

Thus, $(f_1(X), g(X))$ is an optimal allocation of X for the underlying risk measures VaR_{α_1} and $\mathrm{RVaR}_{\alpha,\beta}$. From part (ii), we know that there exists Z_ε, such that $Z_\varepsilon \to X$ in L^1 as $\varepsilon \downarrow 0$ and

$$\liminf_{\varepsilon\downarrow 0} \left(\mathrm{VaR}_{\alpha_1}(f_1(Z_\varepsilon)) + \mathrm{RVaR}_{\alpha,\beta}(g(Z_\varepsilon))\right) - \left(\mathrm{VaR}_{\alpha_1}(f_1(X)) + \mathrm{RVaR}_{\alpha,\beta}(g(X))\right) > 0.$$

Using Theorem 2.1 again, we have, for $\varepsilon > 0$,

$$\mathrm{VaR}_{\alpha_1}(f_1(Z_\varepsilon)) + \sum_{i=2}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(f_i(Z_\varepsilon)) \geqslant \mathrm{VaR}_{\alpha_1}(f_1(Z_\varepsilon)) + \mathrm{RVaR}_{\alpha,\beta}(g(Z_\varepsilon)).$$

Therefore,

$$\liminf_{\varepsilon\downarrow 0} \left(\mathrm{VaR}_{\alpha_1}(f_1(Z_\varepsilon)) + \sum_{i=2}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(f_i(Z_\varepsilon))\right) - \mathrm{RVaR}_{\sum_{i=1}^{n}\alpha_i, \bigvee_{i=1}^{n}\beta_i}(X) > 0.$$

Thus, $(f_1(X), \dots, f_n(X))$ is not robust (and not L^∞-robust if X is bounded).

(3) (iii): Suppose that there exists a π_W-robust optimal allocation. Since π_W-robustness is stronger than L^1-robustness, we know that $\beta_1, \dots, \beta_n > 0$. If $\alpha_1 = \dots = \alpha_n = 0$, then $\square_{i=1}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i} = \mathrm{ES}_{\beta_n}(X)$. As ES_{β_n} is not upper-semicontinuous at any X with respect to weak convergence (see Cont et al. (2010)), by Proposition 2.10 there cannot exist any π_W-robust optimal allocation. Hence, in order to allow for a π_W-robust optimal allocation, all of $\beta_1, \dots, \beta_n$ have to be positive, and at least one of $\alpha_1, \dots, \alpha_n$ has to be positive. $\square$

2.8.6 Proof of Theorem 2.12

Proof. For the "if" part, take $X_i = X$ and $X_j = 0$ for $j \neq i$. We can see that

$$\sum_{j=1}^{n} \mathrm{RVaR}_{\alpha_j,\beta_j}(X_j) = \mathrm{RVaR}_{\alpha_i,\beta_i}(X) = \mathop{\square}_{i=1}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(X)$$

and thus the "if" part holds.

In the following we show the "only-if" part. Suppose that there exists a comonotonic optimal allocation. This implies

$$\mathop{\square}_{i=1}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(X) = \mathop{\boxplus}_{i=1}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(X).$$

By Theorem 2.4 and Corollary 2.14, we have

$$\mathop{\square}_{i=1}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(X) = \mathrm{RVaR}_{\sum_{i=1}^{n}\alpha_i, \bigvee_{i=1}^{n}\beta_i}(X),$$

and

$$\mathop{\boxplus}_{i=1}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(X) = \int_0^1 \mathrm{VaR}_\alpha(X)\mathrm{d}h(\alpha),$$

where h is given in Corollary 2.14.

Let $\alpha = \sum_{i=1}^{n} \alpha_i$, $\beta = \max\{\beta_i : i = 1, \dots, n\}$, and $g(t) = h^{(\alpha,\beta)}(t)$, $t \in [0, 1]$. It is easy to see $h(t) \geqslant g(t)$. By (2.8.5), we have

$$0 = \int_0^1 \mathrm{VaR}_\gamma(X)\mathrm{d}h(\gamma) - \int_0^1 \mathrm{VaR}_\gamma(X)\mathrm{d}g(\gamma) = \int_{-\infty}^{+\infty} \left(h(1 - F(x)) - g(1 - F(x))\right) \mathrm{d}x,$$

where F is the distribution of X. Since $h(t) \geqslant g(t)$, we have $h(1 - F(x)) = g(1 - F(x))$ for almost every $x \in \mathbb{R}$, and as X is continuously distributed, this leads to $h(t) = g(t)$ for almost every $t \in [0, 1]$. Thus, $h^{(\alpha,\beta)}(t) = \min\{h^{(\alpha_1,\beta_1)}(t), \dots, h^{(\alpha_n,\beta_n)}(t)\}$. Simple algebra shows that there exists $i \in \{1, \dots, n\}$ such that for all $j \neq i$, $\alpha_j = 0$ and $\beta_i \geqslant \beta_j$. $\square$

Chapter 3

Pareto-optimal Reinsurance Arrangements

3.1 Introduction

Reinsurance, as a type of risk sharing, has been extensively studied in actuarial science. Generally, there are two parties in a reinsurance contract, an insurer and a reinsurer. Suppose that the insurer faces a nonnegative ground-up loss $X \in \mathcal{X}$, where $\mathcal{X}$ is a set of random variables containing all random variables involved in the reinsurance contract. The reinsurer agrees to cover part of the loss X, say $I(X)$, and the insurer will pay a reinsurance premium $\pi\left(I\left(X\right)\right)$ to the reinsurer. The function $I \in \mathcal{I}_0 : \mathbb{R}_+ \to \mathbb{R}_+$ is called the ceded loss function, where $\mathbb{R}_+ = [0, \infty)$ and $\mathcal{I}_0$ is a non-empty set of all feasible reinsurance contracts. With the reinsurance contract (function) I and the premium principle $\pi : \mathcal{X} \to \mathbb{R}$, the loss random variables

$$C_I = C_I(X) = X - I(X) + \pi\left(I\left(X\right)\right) \qquad \text{and} \qquad R_I = R_I(X) = I(X) - \pi\left(I\left(X\right)\right) \tag{3.1.1}$$

represent the risk exposures of the insurer and the reinsurer under the reinsurance contract, respectively. Note that $C_I + R_I = X$ and the set $\{(C_I, R_I) : I \in \mathcal{I}_0\}$ is a constraint subset of the set of allocations in (2.4.1) when $n = 2$. Furthermore, let $\rho_1 : \mathcal{X} \to \mathbb{R}$ and $\rho_2 : \mathcal{X} \to \mathbb{R}$ be the objective functionals of the insurer and the reinsurer, respectively. The functionals describe the preferences of the insurer and the reinsurer. Precisely, the insurer prefers X over Y if and only if $\rho_1(X) \leqslant \rho_1(Y)$, and the reinsurer prefers X over Y if and only if $\rho_2(X) \leqslant \rho_2(Y)$. We call the

5-tuple $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$ a *reinsurance setting*. In this setting, the general objective functionals ρ_1 and ρ_2 can be risk measures, variances, and disutility functionals. Moreover, up to a sign change, the objective functionals ρ_1 and ρ_2 can also be mean-variance functionals, expected utilities, rank-dependent expected utilities, and so on.

An optimal reinsurance design under the setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$ can be formulated as an optimization problem that tries to find an optimal contract $I^* \in \mathcal{I}_0$ such that an objective function is minimized at I^*. Optimal reinsurance designs from either the insurer's perspective (e.g. $\min_{I\in\mathcal{I}_0} \rho_1(C_I)$) or the reinsurer's point of view (e.g. $\min_{I\in\mathcal{I}_0} \rho_2(R_I)$) have been well investigated in the literature. However, as pointed out by Borch (1969), "there are two parties to a reinsurance contract, and that an arrangement which is very attractive to one party, may be quite unacceptable to the other." Hence, an interesting question in optimal reinsurance designs is to consider both the insurer's preference and the reinsurer's preference. To address this issue, Borch (1960) derived the optimal retentions of the quota-share and stop-loss reinsurances by maximizing the product of the expected utility functions of the two parties' terminal wealth; Hürlimann (2011) obtained the optimal retentions of the combined quota-share and stop-loss reinsurances by minimizing the sum of the variances of the losses of the insurer and the reinsurer; Cai et al. (2013) proposed the joint survival and profitable probabilities of an insurer and a reinsurer as optimization criteria to determine optimal reinsurances; Cai et al. (2016) developed optimal reinsurances that minimize the convex combination of the VaRs of the losses of an insurer and a reinsurer under certain constraints; and Lo (2017b) discussed the generalized problems of Cai et al. (2016) by using the Neyman-Pearson approach.

Obviously, an insurer and a reinsurer have conflicting interests in a reinsurance contract. A celebrated economic concept used in optimal decision problems with conflicting interests is Pareto optimality, which has been well studied under various settings in insurance and risk management. For instance, Gerber (1978) discussed Pareto-optimal risk exchanges and Golubin (2006) studied Pareto-optimal insurance policies when both the insurer and the reinsurer are risk averse. In addition, Pareto-optimality in risk sharing with different risk measures can be found in Jouini et al. (2008), Filipović and Svindland (2008), Embrechts et al. (2017), and references therein. Most of the existing results in optimal risk sharing/exchange can not be used to determine optimal reinsurance contracts since the model settings for reinsurance designs are usually different from the ones for risk sharing problems. In particular, a reinsurance setting often has practical constraints such as the constraint that the shared risks should be non-negative and comonotonic or the condition that the risk measure of the insurer's loss is not larger than a given value, or the requirement that the expected net profit of an reinsurer is not less than a given amount or the restriction that the reinsurance premium is not bigger than an insurer's budget.

In this chapter, we will use the concept of Pareto-optimality to study Pareto-optimal reinsurance contracts under a general reinsurance setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$. Generally speaking, a Pareto-optimal reinsurance policy is one in which neither of the two parties can be better off without making the other worse off and it can be defined mathematically as follows. The following definition of Pareto optimality is a special form of Definition 2.1 under the setting of reinsurance.

Definition 3.1. Let $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$ be a reinsurance setting. A reinsurance contract $I^* \in \mathcal{I}_0$ is called *Pareto-optimal* under $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$, if there is no $I \in \mathcal{I}_0$ such that $\rho_1(C_I) \leqslant \rho_1(C_{I^*})$ and $\rho_2(R_I) \leqslant \rho_2(R_{I^*})$, and at least one of the two inequalities is strict, where C_I and R_I are defined in (3.1.1).

First, similar to Pareto-optimal problems in other fields such as risk exchanges (e.g. Gerber (1978)) and risk allocations (e.g. Barrieu and Scandolo (2008)), it is easy to see that a Pareto-optimal reinsurance contract exists if there is a contract that minimizes the convex combination of the objective functionals of the insurer and the reinsurer. Indeed, the following proposition gives a sufficient condition for a reinsurance contract to be Pareto-optimal in a general reinsurance setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$.

Proposition 3.1. *In a reinsurance setting* $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$*, if*

$$I^* \in \underset{I \in \mathcal{I}_0}{\arg\min}\{\lambda\rho_1(C_I) + (1-\lambda)\rho_2(R_I)\}, \tag{3.1.2}$$

for some $\lambda \in (0,1)$*, then* I^* *is a Pareto-optimal reinsurance contract under the setting* $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$.

Proof. If I^* is not Pareto-optimal, then there exists an $\hat{I} \in \mathcal{I}_0$ such that $\rho_1(C_{\hat{I}}) \leqslant \rho_1(C_{I^*})$ and $\rho_2(R_{\hat{I}}) \leqslant \rho_2(R_{I^*})$, and at least one of the two inequalities is strict. Then $\lambda\rho_1(C_{\hat{I}}) + (1-\lambda)\rho_2(R_{\hat{I}}) < \lambda\rho_1(C_{I^*}) + (1-\lambda)\rho_2(R_{I^*})$. Thus, $I^* \notin \arg\min_{I \in \mathcal{I}_0}\{\lambda\rho_1(C_I) + (1-\lambda)\rho_2(R_I)\}$, a contradiction. $\square$

Proposition 3.1 holds without any assumptions on $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$. Nevertheless, the minimization problem in (3.1.2) for $\lambda \in (0,1)$ may have no solutions. Furthermore, the conditions in Proposition 3.1 are not necessary for a Pareto-optimal reinsurance contract. Indeed, there are other Pareto-optimal reinsurance contracts that are not the solutions to the minimization problem $\min_{I\in\mathcal{I}_0}\{\lambda\rho_1(C_I)+(1-\lambda)\rho_2(R_I)\}$ for $\lambda\in(0,1)$. In fact, as showed in Theorem 3.2, under certain assumptions on $(X,\rho_1,\rho_2,\pi,\mathcal{I}_0)$, Pareto-optimal reinsurance contracts also exist in the solutions to the minimization problems $\min_{I\in\mathcal{I}_0}\{\rho_1(C_I)\}$ and $\min_{I\in\mathcal{I}_0}\{\rho_2(R_I)\}$, and all Pareto-optimal reinsurance contracts are included in the solutions to the minimization problem

$$\min_{I\in\mathcal{I}_0}\{\lambda\rho_1(C_I)+(1-\lambda)\rho_2(R_I)\},\quad \lambda\in[0,1]. \tag{3.1.3}$$

Therefore, the key to find Pareto-optimal reinsurance contracts is to solve the problem (3.1.3). Theorem 3.5 establishes the sufficient conditions that guarantee the existence of the solutions to the problem (3.1.3) or for the existence of Pareto-optimal reinsurance contracts under the setting $(X,\rho_1,\rho_2,\pi,\mathcal{I}_0)$.

The problem (3.1.3) itself is also of interest. Mathematically, when $\lambda=1$ and $\lambda=0$, the problem (3.1.3) is reduced to the problems of finding the optimal reinsurance contracts that minimize an insure's objective functional and a reinsurance's objective functional, respectively. In addition, from an economical point of view, if the reinsurer is designing a contract based on the solutions to the problem (3.1.3), such a contract will be more attractive to the insurer than ones designed based on the solutions to the problem $\min_{I\in\mathcal{I}_0}\rho_2(R_I)$. On the other hand, if the insurer is asking the reinsurer to sell a contract based on the solutions to the problem (3.1.3), the reinsurer is more willing to sell such a contact than ones designed based on the solutions to the problem $\min_{I\in\mathcal{I}_0}\rho_1(C_I)$.

Although Theorem 3.5 gives the conditions such that the solutions to the problem (3.1.3) exist, it is not a trivial work to find the solutions to the problem (3.1.3) even for simple choices of ρ_1, ρ_2, and π. In the literature, many researchers studied the problem (3.1.3) in the case of $\lambda=0$ or $\lambda=1$ with special choices of ρ_1, ρ_2, π, and $\mathcal{I}_0$. See e.g. Chi and Tan (2011), Bernard and Tian (2009), Cui et al. (2013), Cheung et al. (2014), Cheung and Lo (2015), and Lo (2017a) for minimization of Value-at-Risk (VaR) / Tail-Value-at-Risk (TVaR), tail risk measures, general distortion risk measures, general law-invariant convex risk measures, and insurer's risk–adjusted liability, respectively, Kaluszka and Okolewski (2008) and Cai and Wei (2012) for maximization of the expected utility, and Bernard et al. (2015) for maximization of rank-dependent expected utility. For the problem (3.1.3) with $\lambda\in[0,1]$, Cai et al. (2016) solved the problem with certain constraints when the functionals ρ_1 and ρ_2 are VaRs; Jiang et al. (2017) discussed the problem without constraints when the functionals ρ_1 and ρ_2 are VaRs, and Lo (2017b) investigated the problem using the Neyman-Pearson approach. In this chapter, we will also solve the problem when the functionals ρ_1 and ρ_2 are TVaRs. Although the approach proposed in Lo (2017b) can solve the problem (3.1.3) for several special cases, the approach does not work for the problem (3.1.3) when the functionals ρ_1 and ρ_2 are TVaRs as pointed out in Lo (2017b). In addition, note that there are many Pareto-optimal reinsurance contacts under the setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$. In this chapter, we will also use numerical examples to discuss how to choose the weight λ in (3.1.2) so that the feasible deals of Pareto-optimal contracts can be made from the practice purpose.

The rest of the chapter is organized as follows. In Sections 3.2, we give the necessary and sufficient conditions for a reinsurance contract to be Pareto-optimal and characterize all Pareto-optimal reinsurance contracts under a more general setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$. We also obtain the sufficient conditions that guarantee the existence of the solutions to the minimization problem (3.1.3). In Sections 3.3, we solve the problem (3.1.3) explicitly when the functionals ρ_1 and ρ_2 are TVaRs and π is the expected value premium principle. In Section 3.4, we use two numerical examples to illustrate the solutions derived in Section 3.3 and discuss how to choose the weight λ in (3.1.2) to obtain the feasible Pareto-optimal reinsurance contracts from the practice purpose. Some conclusions are drawn in Section 3.5. Some technical proofs are in Section 3.6.

3.2 Pareto Optimality in Reinsurance Policy Design

3.2.1 Model Assumptions

In a reinsurance setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$ with $X \in \mathcal{X}$, to avoid moral hazard, a reinsurance contract $I \in \mathcal{I}_0$ should satisfy that $I(0) = 0$ and $0 \leqslant I(x) - I(y) \leqslant x - y$ for all $0 \leqslant y \leqslant x$. We denote by $\mathcal{I}$ the set of all contracts that satisfy this property, namely,

$$\mathcal{I} := \{I : \mathbb{R}_+ \to \mathbb{R}_+ \mid I(0) = 0 \text{ and } 0 \leqslant I(x) - I(y) \leqslant x - y, \text{ for all } 0 \leqslant y \leqslant x\}.$$

Hence, we have $\mathcal{I}_0 \subset \mathcal{I}$. Note that $\mathcal{I}_0$ does not have to be equal to $\mathcal{I}$. Such a set $\mathcal{I}_0$ can be a finite set of contracts or an infinite set of contracts such as the set of stop-loss contracts, multi-layer contracts, quota-share contracts, or all the contracts in $\mathcal{I}$ with some budget/solvency constraints. Moreover, for any $I \in \mathcal{I}$ and any nonnegative random variable X, the random variables X, $X - I(X)$, and $I(X)$ are comonotonic. Recall that random variables $X_1, \ldots, X_n$ with $n \geqslant 2$ are comonotonic if there exist non-decreasing functions $f_1, \ldots, f_n$ and a random variable $Z \in L^0$ such that $X_i = f_i(Z)$ almost surely for $i = 1, \ldots, n$.

Throughout we let $\mathcal{X}$ be a convex cone of random variables containing L^∞ satisfying $I(X) \in \mathcal{X}_+$ for all $X \in \mathcal{X}_+$ and $I \in \mathcal{I}$, where $\mathcal{X}_+ = \{X \in \mathcal{X} : X \geqslant 0\}$. Note that $\mathcal{X}_+$ is still a convex cone. The set $\mathcal{X}$ is the set of all random losses that are of our interest. In the context of reinsurance, $\mathcal{X}$ may be chosen as L^1, L^∞ or L^0 depending on the specific problems.

For any random loss $X \in \mathcal{X}_+$, a reinsurance contract $I \in \mathcal{I}$, and a premium principle $\pi : \mathcal{X} \to \mathbb{R}$, the two loss random variables C_I and R_I defined in (3.1.1) are in $\mathcal{X}$, but they may not be in $\mathcal{X}_+$. In particular, if $\pi(I(X)) > 0$, then $C_I \in \mathcal{X}_+$ but R_I may not be in $\mathcal{X}_+$.

For a given $X \in \mathcal{X}$, all random variables involved in a reinsurance contract under the setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$, such as C_I and R_I, are in the set

$$\mathcal{C}(X) = \{Y \in \mathcal{X} : Y,\ X - Y \text{ and } X \text{ are comonotonic}\}. \tag{3.2.1}$$

Also, we have $I(X) \in \mathcal{C}(X)$ for $I \in \mathcal{I}$.

In the following we aim to establish necessary and sufficient conditions for the existence of the Perato-optimal reinsurance policies in a general reinsurance setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$.

3.2.2 Necessary and Sufficient Conditions for Pareto-optimal Contracts

To obtain the necessary condition for a reinsurance contract to be Pareto-optimal in a general reinsurance setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$, we have to make some assumptions on $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$. To do so, we introduce the following definition and notation. A functional ρ is said to be *semilinear* on a set $\mathcal{Y}$ if $\rho(\lambda X + Y) = \lambda\rho(X) + \rho(Y)$ for all $\lambda > 0$, $X, Y \in \mathcal{Y}$. For a reinsurance setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$, denote

$$K(\lambda) = \underset{I \in \mathcal{I}_0}{\arg\min}\{\lambda\rho_1(C_I) + (1-\lambda)\rho_2(R_I)\}, \quad \lambda \in [0,1], \tag{3.2.2}$$

$$K^*(0) = \underset{I \in K(0)}{\arg\min}\{\rho_1(C_I)\}, \quad K^*(1) = \underset{I \in K(1)}{\arg\min}\{\rho_2(R_I)\}, \quad \text{and} \quad K^*(\lambda) = K(\lambda),\ \lambda \in (0,1), \tag{3.2.3}$$

where C_I and R_I are defined in (3.1.1). Note that $K(0)$ (resp. $K(1)$) is the set of contracts minimizing the objective functional of the reinsurer (resp. insurer) while $K^*(0)$ (resp. $K^*(1)$) is the set of the contracts that are in $K(0)$ (resp. $K(1)$) and minimize the objective functional of the insurer (resp. reinsurer). For $\lambda \in (0,1)$, $K(\lambda)$ is the set of contracts minimizing the convex combination of the objective functionals of the insurer and the reinsurer.

As shown in the following theorem, the sets $K^*(\lambda)$, $\lambda \in [0,1]$, characterize all Pareto-optimal contracts in the reinsurance setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$. The proof of the following theorem follows the ideas similar to those used in Gerber (1978) for Pareto-optimal risk exchanges.

Theorem 3.2. *Let $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$ be a reinsurance setting. If π is semilinear on $\mathcal{C}(X)$, ρ_1, ρ_2 are convex on $\mathcal{C}(X)$, and $\mathcal{I}_0$ is a convex set, then $I^* \in \mathcal{I}_0$ is a Pareto-optimal contract under the setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$ if and only if there exists $\lambda \in [0,1]$ such that $I^* \in K^*(\lambda)$, where $K^*(\lambda)$ is defined in* (3.2.3).

Proof. "$\Longrightarrow$" Define the set $S = \{(\rho_1(C_I),\ \rho_2(R_I)) : I \in \mathcal{I}_0\} \subset \mathbb{R}^2$. For any set $T \subset \mathbb{R}^2$, we say that $(x^*, y^*) \in T$ is a Pareto-optimal point of T if there is no $(x,y) \in T$ such that $(x,y) \neq (x^*, y^*)$ and $(x,y) \leqslant (x^*, y^*)$; here and below the inequality between vectors are component-wise inequalities. Let $\bar{S}$ be the convex hull of S. The agenda for the proof is the following. (a) First, we verify that for any $(\bar{x}, \bar{y}) \in \bar{S}$, there exists a point $(x,y) \in S$ such that $(x,y) \leqslant (\bar{x}, \bar{y})$. (b) Second, use (a) to show that for any Pareto-optimal point (x^*, y^*) of S, there exists $\lambda \in [0,1]$ such that $(x^*, y^*) \in \arg\min_{(x,y)\in S}\{\lambda x + (1-\lambda)y\}$. (c) Third, use (a) and (b) to prove the necessary conditions for a contract to be Pareto-optimal.

For any $I_1, I_2 \in \mathcal{I}_0$ and $\theta \in [0,1]$, let $I = \theta I_1 + (1-\theta) I_2 \in \mathcal{I}_0$. The convexity of ρ_1 and the semilinearity of π on $\mathcal{C}(X)$ imply

$$\begin{aligned}
\theta\rho_1(C_{I_1}) + (1-\theta)\rho_1(C_{I_2}) &\geqslant \rho_1(\theta C_{I_1} + (1-\theta) C_{I_2}) \\
&= \rho_1(X - (\theta I_1(X) + (1-\theta) I_2(X)) + \theta\pi(I_1(X)) + (1-\theta)\pi(I_2(X))) \\
&= \rho_1(X - I(X) + \pi(I(X))). \qquad (3.2.4)
\end{aligned}$$

Similarly,

$$\theta\rho_2(R_{I_1}) + (1-\theta)\rho_2(R_{I_2}) \geqslant \rho_2(I(X) - \pi(I(X))). \qquad (3.2.5)$$

Therefore, for any $(\bar{x}, \bar{y}) \in \bar{S}$, there exists $(x,y) = (\rho_1(C_I), \rho_2(R_I)) \in S$ such that $(x,y) \leqslant (\bar{x}, \bar{y})$.

Next we take a Pareto-optimal point (x^*, y^*) of S. If there exists $(\bar{x}, \bar{y}) \in \bar{S}$ such that $(\bar{x}, \bar{y}) \leqslant (x^*, y^*)$ then from the second statement above, we have, there exists $(x,y) \in S$ with $(x,y) \leqslant (\bar{x}, \bar{y})$. From the Pareto-optimality of (x^*, y^*) in S we know $(x,y) = (\bar{x}, \bar{y}) = (x^*, y^*)$. This shows that (x^*, y^*) is a Pareto-optimal point of $\bar{S}$.

Define $T = \{(x, y) \in \mathbb{R}^2 : (x, y) \leqslant (x^*, y^*)\}$. Note that both T and $\bar{S}$ are convex sets, and by the Pareto-optimality of (x^*, y^*) in $\bar{S}$, the interiors of $\bar{S}$ and T are disjoint. By the Hyperplane Separation Theorem (e.g. Theorem 11.3 of Rockafellar (1970)), there exists a vector $(a, b) \in \mathbb{R}^2$, $(a, b) \neq (0, 0)$ such that $\sup_{(x,y)\in T}\{ax + by\} \leqslant \inf_{(x,y)\in\bar{S}}\{ax + by\}$. Note that $\sup_{(x,y)\in T}\{ax+by\} < \infty$ and for any $(x, y) \in T$, we have $(x-1, y) \in T$. It follows that

$$\sup_{(x,y)\in T}\{ax + by\} \geqslant \sup_{(x,y)\in T}\{a(x-1) + by\} = \sup_{(x,y)\in T}\{ax + by\} - a,$$

which implies $a \geqslant 0$. Similarly we have $b \geqslant 0$. Therefore $\sup_{(x,y)\in T}\{ax + by\} = ax^* + by^* \leqslant \inf_{(x,y)\in\bar{S}}\{ax + by\}$. This shows that (x^*, y^*) minimizes $ax + by$ over $(x, y) \in \bar{S}$.

Now, suppose that I^* is Pareto-optimal for the reinsurance design problem. Then $(\rho_1(C_{I^*}), \rho_2(R_{I^*}))$ is a Pareto-optimal point of S. The aforementioned arguments suggest that there exist $a, b \geqslant 0$, $a + b > 0$ such that

$$a\rho_1(C_{I^*}) + b\rho_2(R_{I^*}) = \min_{(x,y)\in S}\{ax + by\} = \min_{I\in\mathcal{I}_0}\{a\rho_1(C_I) + b\rho_2(R_I)\}.$$

By setting $\lambda = a/(a + b)$ in the above equation, we conclude that $I^* \in K(\lambda)$ for some $\lambda \in [0, 1]$. If $\lambda \in (0, 1)$, then $K^*(\lambda) = K(\lambda)$ and $I^* \in K^*(\lambda)$. Below suppose $\lambda = 0$ and take any $I \in K(0)$. By the definition of $K(0)$, $\rho_2(R_I) = \rho_2(R_{I^*})$. From the Pareto optimality of I^*, we have $\rho_1(C_{I^*}) \leqslant \rho_1(C_I)$. Therefore, $I^* \in \arg\min_{I\in K(0)}\{\rho_1(C_I)\} = K^*(0)$. The case $\lambda = 1$ is analogous. To summarize, $I^* \in K^*(\lambda)$ for some $\lambda \in [0, 1]$.

" $\Longleftarrow$ " Suppose $I^* \in K(\lambda)$ for some $\lambda \in [0, 1]$. For $\lambda \in (0, 1)$, one can obtain from Proposition 3.1 that I^* is Pareto-optimal. If $\lambda = 0$, take $I \in \mathcal{I}_0$ such that $\rho_1(C_I) \leqslant \rho_1(C_{I^*})$ and $\rho_2(R_I) \leqslant \rho_2(R_{I^*})$. By the definition of $K(0)$ and noting that $I^* \in K(0)$, we have $\rho_2(R_I) = \rho_2(R_{I^*})$, thus $I \in K(0)$. Further, by the definition of $K^*(0)$, we have $\rho_1(C_{I^*}) \leqslant \rho_1(C_I)$. Therefore, $\rho_1(C_I) = \rho_1(C_{I^*})$ and $\rho_2(R_I) = \rho_2(R_{I^*})$. This shows that I^* is Pareto-optimal. □

We point out that the assumptions in Theorem 3.2 are easily satisfied by many functionals of ρ_1 and ρ_2, premium principles of π, and feasible sets of $\mathcal{I}_0$, including many practical choices considered in the literature (see discussions below). In addition, in Theorem 3.2, the functionals ρ_1, ρ_2 and π are assumed to satisfy the corresponding properties on the subset $\mathcal{C}(X) \subset \mathcal{X}$. In fact, in many applications, the specified functionals ρ_1, ρ_2 and π can satisfy the corresponding properties globally or on $\mathcal{X}$. We first give the definitions of comonotonic-semilinearity and comonotonic-convexity for a functional as follows.

Definition 3.2. A functional $\rho : \mathcal{X} \to \mathbb{R}$ is said to be *comonotonic-semilinear* if $\rho(\lambda X + Y) = \lambda\rho(X) + \rho(Y)$ for any comonotonic random variables $X, Y \in \mathcal{X}$ and $\lambda > 0$ and to be *comonotonic-convex* if $\rho(\lambda X + (1-\lambda)Y) \leqslant \lambda\rho(X) + (1-\lambda)\rho(Y)$ for any comonotonic random variables $X, Y \in \mathcal{X}$ and $\lambda \in [0, 1]$.

The property of comonotonic-convexity has been studied and characterized in Song and Yan (2009). Now, we can reformulate Theorem 3.2 based on the global properties of the functionals below.

Corollary 3.3. *Let $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$ be a reinsurance setting. If π is comonotonic-semilinear, ρ_1, ρ_2 are comonotonic-convex, and $\mathcal{I}_0$ is a convex set, then $I^* \in \mathcal{I}_0$ is Pareto-optimal under the setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$ if and only if there exists $\lambda \in [0, 1]$ such that $I^* \in K^*(\lambda)$, where K^* is defined in* (3.2.3).

Proof. Note that $C(X) \subset \mathcal{X}$ and every element in $\mathcal{C}(X)$ is comonotonic with X. Hence, the assumptions of Corollary 3.3 imply that the assumptions of Theorem 3.2 hold. □

Below we make a few observations on the conditions assumed in Theorem 3.2 and Corollary 3.3.

(i) Comonotonic-convexity is a weaker property than comonotonic-semilinearity or convexity. If functional ρ is comonotonic-semilinear or convex, then it is comonotonic-convex. This property of comonotonic-convexity can be satisfied by many functionals studied in the literature such as distortion risk measures, convex risk measures, convex functionals including concave expected utilities (up to a sign change), and so on.

(ii) The comonotonic-semilinearity of π is essential to Theorem 3.2 and Corollary 3.3, and it cannot be weakened to comonotonic-convexity. The reason is that $\rho(C_I)$ has a positive term $\pi(I(X))$ while $\rho(R_I)$ has a negative term $-\pi(I(X))$. To obtain both inequalities (3.2.4) and (3.2.5), one needs to assume that π has a linear structure in these values. The property of comonotonic-semilinearity can be satisfied by the expected value premiums, Wang's premiums, and others.

(iii) In Theorem 3.2, we assume that the set of contracts $\mathcal{I}_0 \subset \mathcal{I}$ is convex. The convex assumption on $\mathcal{I}_0$ allows us to consider the minimization problem (3.1.3) with constraints if the constraints form a convex subset of I. Interesting examples of such constraints include $I_0 = \{I: \ \rho_1(C_I) \leqslant r\}$, where ρ_1 is a convex risk measure and $r \in \mathbb{R}$ is an acceptable risk level under the risk measure ρ_1 (see Cai et al. (2016) and Lo (2017b)), or $I_0 = \{I: \ \pi(C_I) \leqslant p\}$, where π is a convex premium principle and $p \in \mathbb{R}$ is an acceptable budget for the insurer, or $I_0 = \{I: \ \mathbb{E}[\pi(C_I) - I(X)] \geqslant w\}$, where π is a convex premium principle and $w \in \mathbb{R}$ is an acceptable amount for the reinsurer's expected net profit. Also note that $\mathcal{I}$ itself is a convex set.

3.2.3 Existence of Pareto-optimal Reinsurance Contracts

By Theorem 3.2, we know that the sets of contracts $K^*(\lambda)$, $\lambda \in [0,1]$, characterize all Pareto-optimal contracts in a reinsurance setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$. However, we do not know whether $K^*(\lambda)$ is non-empty. Hence, it would be important to give some conditions under which $K^*(\lambda)$ is not empty, or in other words, to give the conditions under which the minimization problem (3.1.3) has solutions (minimizers). To obtain such conditions, we recall the following definition. We say that a set of functions is *pw-closed* if it is closed with respect to point-wise convergence. Note that $\mathcal{I}$ is pw-closed.

Lemma 3.4. *Let $\mathcal{I}_0 \subset \mathcal{I}$ be pw-closed. For any sequence $\{I_n, \, n \in \mathbb{N}\} \subset \mathcal{I}_0$, there exists a subsequence $\{I_{n_k}, \, k \in \mathbb{N}\}$ pointwise converging to an $I^* \in \mathcal{I}_0$.*

Proof. Define $\mathcal{G} = \left\{ g : [0, \infty) \to [0, 1) \mid g(\cdot) = 1 - \frac{1}{I(\cdot)+1}, \ I \in \mathcal{I}_0 \right\}$. Since any $I \in \mathcal{I}$ is continuous and increasing, so is any $g \in \mathcal{G}$. For any sequence $\{I_n, \, n \in \mathbb{N}\} \subset \mathcal{I}_0$, $\{g_n := 1 - \frac{1}{I_n + 1}, \, n \in \mathbb{N}\} \subset \mathcal{G}$ is uniformly bounded and by Helly's theorem (see e.g. Klenke (2013)), there exists a function g^* and a subsequence $\{g_{n_k}, \, k \in \mathbb{N}\}$ such that $\{g_{n_k}, \, k \in \mathbb{N}\}$ pointwise converges to g^*. For any $x \in [0, \infty)$, $I_{n_k}(x) \leqslant x$ and $0 \leqslant g_{n_k}(x) \leqslant 1 - \frac{1}{x+1} < 1$, therefore $\{I_{n_k}(x) = \frac{1}{1 - g_{n_k}(x)} - 1, \, k \in \mathbb{N}\}$ converges to $I^*(x) := \frac{1}{1 - g^*(x)} - 1$. Since $\mathcal{I}_0$ is closed with respect to pointwise convergence, we have $I^* \in \mathcal{I}_0$. Therefore, there exists a subsequence $\{I_{n_k}, \, k \in \mathbb{N}\} \subset \mathcal{I}_0$ pointwise converging to I^*. □

Furthermore, we say that a functional ρ is *as-continuous* on a set $\mathcal{Y} \subset L^0$ if ρ is continuous with respect to almost sure convergence for sequences in $\mathcal{Y}$. We say that a reinsurance setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$ is *proper* if $\inf_{I \in \mathcal{I}_0} \rho_1(C_I) > -\infty$ and $\inf_{I \in \mathcal{I}_0} \rho_2(R_I) > -\infty$. One can easily verify that if ρ_1, ρ_2, π are non-decreasing functionals on $\mathcal{C}(X)$, then $\rho_1(C_I) \geqslant \rho_1(\pi(0))$ and $\rho_2(R_I) \geqslant$

$\rho_2(-\pi(X))$, thus both $\rho_1(C_I)$ and $\rho_2(R_I)$ are bounded from below, and hence the corresponding reinsurance setting is proper.

Theorem 3.5. *Let $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$ be a proper reinsurance setting. If π, ρ_1, ρ_2 are as-continuous on $\mathcal{C}(X)$ and $\mathcal{I}_0$ is pw-closed, then $K^*(\lambda)$ is non-empty for each $\lambda \in [0, 1]$.*

Proof. Define the set $S = \{(\rho_1(C_I),\ \rho_2(R_I)) : I \in \mathcal{I}_0\} \subset \mathbb{R}^2$. Since the reinsurance design problem is proper, there exists $M \in \mathbb{R}$ such that $(x, y) \geqslant (M, M)$ for all $(x, y) \in S$, and also note that S is not empty. Next, for $K^*(\lambda)$ to be non-empty, it suffices to verify that S is a closed set. For a sequence of $\{I_n \in \mathcal{I}_0,\ n \in \mathbb{N}\}$ such that $\rho_1(C_{I_n}) \to a$ and $\rho_2(R_{I_n}) \to b$, where $a, b \in \mathbb{R}$, it suffices to show that there exists $I^* \in \mathcal{I}_0$ such that $\rho_1(C_{I^*}) = a$ and $\rho_2(R_{I^*}) = b$. By Lemma 3.4, there exists a subsequence $\{I_{n_k},\ k \in \mathbb{N}\}$ of $\{I_n,\ n \in \mathbb{N}\}$ converging pointwise to, say $I^* \in \mathcal{I}_0$. Therefore, $\{I_{n_k}(X)\}$ converges to $I^*(X)$ almost surely (indeed, for all $\omega \in \Omega$). The limits $\rho_1(C_{I^*}) = a$ and $\rho_2(R_{I^*}) = b$ follow from the assumed continuity of ρ_1, ρ_2 and π. □

Similar to Corollary 3.3, we can replace the condition of the as-continuity on $\mathcal{C}(X)$ in Theorem 3.5 by a global condition of L^p-continuity on L^p if X is in L^p as stated in the following corollary.

Corollary 3.6. *Let $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$ be a proper reinsurance setting, in which $X \in \mathcal{X} = L^p$ for some $p \in [1, \infty]$. If π, ρ_1, ρ_2 are L^p-continuous on L^p and $\mathcal{I}_0$ is pw-closed, then $K^*(\lambda)$ is non-empty for each $\lambda \in [0, 1]$.*

Proof. We need to verify that $\{I_{n_k}(X)\}$ in the proof of Theorem 3.5 converges to $I^*(X)$ in L^p. This is implied by the the dominated convergence theorem, noting that $I_{n_k}(X)$ is dominated by $X \in L^p$. □

Below we make a few observations on the conditions assumed in Theorem 3.5 and Corollary 3.6.

(i) For $X \in L^p$, $p \in [1, \infty]$, the as-continuity on $\mathcal{C}(X)$ in Theorem 3.5 is weaker than L^p-continuity in Corollary 3.6.

(ii) If ρ_1 and ρ_2 are monetary risk measures (monotone and cash-invariant), then they are L^∞-continuous. Thus, for monetary risk measures, the continuity assumption can be removed if X is bounded.

(iii) If ρ_1 and ρ_2 are finite-valued convex risk measures on L^p, $p \in [1,\infty]$, then they are L^p-continuous, see e.g. Kaina and Rüschendorf (2009). Hence, all finite-valued convex risk measures satisfy the conditions for ρ_1 and ρ_2 in Theorems 3.2 and 3.5.

3.2.4 Special Cases: VaR and TVaR

Throughout this chapter, VaR and TVaR are defined as in Definition 1.3. In this section we have a closer look at the two popular risk measures, VaR and TVaR (see Section 1.2.2 for their properties), and put them into the framework of Theorems 3.2 and 3.5. For $\alpha \in (0,1)$, both VaR_α and TVaR_α, considered as functionals mapping a set $\mathcal{X} = L^0$ or $\mathcal{X} = L^1$ to $\mathbb{R}$, are comonotonic-semilinear, and they are monetary risk measures. In addition, TVaR is also convex and subadditive.

Now we put VaR and TVaR into the context of Theorems 3.2 and 3.5. Since TVaR_α, $\alpha \in (0,1)$ is L^1-continuous and comonotonic-semilinear, for any X in L^1, TVaR_α satisfies the conditions for ρ_1 and ρ_2 in Theorems 3.2 and 3.5. For the case of VaR, for any X in L^0, noting that $\mathrm{VaR}_\alpha(I(X)) = I(\mathrm{VaR}_\alpha(X))$ for any continuous and increasing function I, VaR_α is continuous with respect to the almost sure convergence $I_{n_k}(X)$ to $I^*(X)$. Thus, for any X in L^0, VaR_α satisfies the conditions for ρ_1 and ρ_2 in Theorems 3.2 and 3.5. We summarize our findings above on VaR and TVaR in the proposition below. Write $\mathcal{R}_1 = \{\mathrm{VaR}_\alpha : \alpha \in (0,1)\}$ and $\mathcal{R}_2 = \{\mathrm{TVaR}_\alpha : \alpha \in (0,1)\}$.

Proposition 3.7. *Suppose that $\rho_1, \rho_2 \in \mathcal{R}_1 \cup \mathcal{R}_2$, $X \in L^0$ ($X \in L^1$ if at least one of ρ_1, ρ_2 is in $\mathcal{R}_2$), π is an additive and as-continuous functional on $\mathcal{C}(X)$ and $\mathcal{I}_0$ is convex and pw-closed. Then, the following assertions hold.*

(i) $I^* \in \mathcal{I}_0$ is Pareto-optimal under the setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$ if and only if $I^* \in K^*(\lambda)$ for some $\lambda \in [0,1]$.

(ii) For each $\lambda \in [0,1]$, $K^*(\lambda)$ is non-empty.

3.3 Pareto-optimal Reinsurance Contracts under TVaRs

In this section, we solve the minimization problem (3.1.3) when the functionals ρ_1 and ρ_2 are TVaRs and find the explicit forms of optimal reinsurance contracts. More precisely, in this section, in the reinsurance setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$, we choose the feasible set to be $\mathcal{I}_0 = \mathcal{I}$. Furthermore, assume that for any reinsurance contract $I \in \mathcal{I}$, the reinsurance premium $\pi(I(X))$ is determined by the expected value principle, namely $\pi(I(X)) = (1+\theta)\mathbb{E}[I(X)]$, where θ is a positive risk loading factor. Suppose that the insurer and the reinsurer use TVaR_α and TVaR_β, respectively, to measure their own risk, where $\alpha, \beta \in (0,1)$. Thus, the problem (3.1.3) reduces to the following minimization problem

$$\min_{I \in \mathcal{I}} \{\lambda\, \mathrm{TVaR}_\alpha(X - I(X) + \pi(I(X))) + (1-\lambda)\, \mathrm{TVaR}_\beta(I(X) - \pi(I(X)))\}, \tag{3.3.1}$$

where $\lambda \in [0,1]$.

We use the following notation henceforth

$$\theta^* = \frac{1}{1+\theta}, \tag{3.3.2}$$

$$m = m(\lambda) = \frac{\lambda}{1-\alpha} + (1-2\lambda)(1+\theta), \tag{3.3.3}$$

$$p = p(\lambda) = 1 - (1-\lambda)/m, \tag{3.3.4}$$

$$q = q(\lambda) = 1 - \frac{\lambda}{\frac{1-\lambda}{1-\beta} - (1-2\lambda)(1+\theta)}. \tag{3.3.5}$$

The following Theorems 3.8 and 3.9 give explicit solutions to the problem (3.3.1). Theorem 3.8 deals with the case $\alpha \leqslant \beta$ and Theorem 3.9 handles the case $\alpha \geqslant \beta$.

Theorem 3.8. *For $0 < \alpha \leqslant \beta < 1$, $\lambda \in [0,1]$, and a non-negative integrable ground-up loss random variable X, optimal reinsurance contracts $I^* = I^*_\lambda$ to problem (3.3.1) are given as follows:*

(i) If $0 \leqslant \lambda < \frac{1}{2}$ and $\frac{1-\lambda}{1-\beta} > m$, then

$$I^*(x) = \begin{cases} x \wedge \mathrm{VaR}_{p(\lambda)}(X), & \text{if } (1-\alpha)(1+\theta) \geqslant 1; \\ x \wedge \mathrm{VaR}_{1-\theta^*}(X), & \text{if } (1-\alpha)(1+\theta) < 1. \end{cases}$$

(ii) If $0 \leqslant \lambda < \frac{1}{2}$ and $\frac{1-\lambda}{1-\beta} = m$, then $I^*(x) = \left(x \wedge \mathrm{VaR}_{p(\lambda)}(X)\right) \mathbb{I}_{\{x \leqslant \mathrm{VaR}_\beta(X)\}} + I(x)\, \mathbb{I}_{\{x > \mathrm{VaR}_\beta(X)\}}$, where I can be any function such that $I^* \in \mathcal{I}$.

(iii) If $0 \leqslant \lambda < \frac{1}{2}$ and $\frac{1-\lambda}{1-\beta} < m$, then $I^*(x) = x$.

(iv) If $\lambda = \frac{1}{2}$, then

$$I^*(x) = \begin{cases} I(x) \wedge I(\mathrm{VaR}_\alpha(X)), & \text{if } \alpha < \beta; \\ I(x), & \text{if } \alpha = \beta, \end{cases}$$

where I can be any function such that $I^* \in \mathcal{I}$.

(v) If $\frac{1}{2} < \lambda < 1$ and $\frac{1-\lambda}{1-\beta} > m > 0$, then

$$I^*(x) = \begin{cases} 0, & \text{if } (1-\alpha)(1+\theta) \geqslant 1; \\ (x - \mathrm{VaR}_{1-\theta^*}(X))_+ \wedge \left(\mathrm{VaR}_{p(\lambda)}(X) - \mathrm{VaR}_{1-\theta^*}(X)\right), & \text{if } (1-\alpha)(1+\theta) < 1. \end{cases}$$

(vi) If $\frac{1}{2} < \lambda < 1$ and $\frac{1-\lambda}{1-\beta} = m > 0$, then

$$I^*(x) = \left[(x - \mathrm{VaR}_{1-\theta^*}(X))_+ \wedge (\mathrm{VaR}_\beta(X) - \mathrm{VaR}_{1-\theta^*}(X))\right] \mathbb{I}_{\{x \leqslant \mathrm{VaR}_\beta(X)\}} + \; I(x)\, \mathbb{I}_{\{x > \mathrm{VaR}_\beta(X)\}},$$

where I can be any function such that $I^* \in \mathcal{I}$.

(vii) If $\frac{1}{2} < \lambda \leqslant 1$ and $\frac{1-\lambda}{1-\beta} < m$, then $I^*(x) = (x - \mathrm{VaR}_{1-\theta^*}(X))_+$.

(viii) If $\frac{1}{2} < \lambda < 1$ and $m = 0$, then $I^*(x) = 0$.

(ix) If $\lambda = 1$ and $m = 0$, then $I^*(x) = I(x)\, \mathbb{I}_{\{x > \mathrm{VaR}_\alpha(X)\}}$, where I can be any function such that $I^* \in \mathcal{I}$.

(x) If $\frac{1}{2} < \lambda \leqslant 1$ and $m < 0$, then $I^*(x) = 0$.

Proof. The proof of each case is similar. We only give the proof of case (i) here, from which the reader can grasp the main idea of the proof. The proof for the rest cases is in Section 3.6.1.

For any $I \in \mathcal{I}$, define $V(I) = \lambda \mathrm{TVaR}_\alpha (X - I(X) + \pi(I(X))) + (1-\lambda)\mathrm{TVaR}_\beta(I(X) - \pi(I(X)))$. Since $X - I(X)$ and $I(X)$ are comonotonic, by comonotonic additivity and cash invariance of TVaR, we have

$$V(I) = \lambda \mathrm{TVaR}_\alpha(X) - \lambda \mathrm{TVaR}_\alpha(I(X)) + (1-\lambda)\mathrm{TVaR}_\beta(I(X)) + (2\lambda - 1)(1+\theta)\mathbb{E}[I(X)].$$

With the expression (1.2.4) for TVaR, we have

$$\begin{aligned} V(I) = \lambda \mathrm{TVaR}_\alpha(X) - \lambda\Big\{ \mathrm{VaR}_\alpha(I(X)) + \frac{1}{1-\alpha}\mathbb{E}\left[(I(X) - \mathrm{VaR}_\alpha(I(X)))_+\right] \Big\} \\ + (1-\lambda)\Big\{ \mathrm{VaR}_\beta(I(X)) + \frac{1}{1-\beta}\mathbb{E}\left[(I(X) - \mathrm{VaR}_\beta(I(X)))_+\right] \Big\} + (2\lambda-1)(1+\theta)\mathbb{E}[I(X)]. \end{aligned} \tag{3.3.6}$$

Note that for any integrable random variable Y, $\mathbb{E}[Y] = \int_0^1 \mathrm{VaR}_r(Y)\mathrm{d}r$. Thus, $V(I)$ can be rewritten as

$$
\begin{aligned}
V(I) &= \lambda \mathrm{TVaR}_\alpha(X) - \lambda I(\mathrm{VaR}_\alpha(X)) - \frac{\lambda}{1-\alpha}\int_0^1 \left[I(\mathrm{VaR}_r(X)) - I(\mathrm{VaR}_\alpha(X))\right]_+ \mathrm{d}r \\
&\quad + (1-\lambda)I(\mathrm{VaR}_\beta(X)) + \frac{1-\lambda}{1-\beta}\int_0^1 \left[I(\mathrm{VaR}_r(X)) - I(\mathrm{VaR}_\beta(X))\right]_+ \mathrm{d}r \\
&\quad + (2\lambda-1)(1+\theta)\int_0^1 I(\mathrm{VaR}_r(X))\mathrm{d}r \\
&= \lambda \mathrm{TVaR}_\alpha(X) - (1-2\lambda)(1+\theta)\int_0^\alpha I(\mathrm{VaR}_r(X))\mathrm{d}r - m\int_\alpha^\beta I(\mathrm{VaR}_r(X))\mathrm{d}r \\
&\quad + \left(\frac{1-\lambda}{1-\beta} - m\right)\int_\beta^1 I(\mathrm{VaR}_r(X))\mathrm{d}r. \qquad (3.3.7)
\end{aligned}
$$

Let $\xi_a = I(\mathrm{VaR}_\alpha(X))$ and $\xi_b = I(\mathrm{VaR}_\beta(X))$. Clearly $\xi_a \leqslant \xi_b$ and $\mathrm{VaR}_\alpha(X) - \xi_a \leqslant \mathrm{VaR}_\beta(X) - \xi_b$ as $I(x)$ and $x - I(x)$ are nondecreasing for all $x \geqslant 0$ and $\alpha \leqslant \beta$. Note that $0 \leqslant \xi_a \leqslant \mathrm{VaR}_\alpha(X)$ and $0 \leqslant \xi_b \leqslant \mathrm{VaR}_\beta(X)$ since $0 \leqslant I(x) \leqslant x$ for all $x \geqslant 0$. Recall the definition of m in (3.3.3). Equality (3.3.6) reduces to

$$
\begin{aligned}
V(I) &= \lambda \mathrm{TVaR}_\alpha(X) + (1-2\lambda)I(\mathrm{VaR}_\alpha(X)) + (1-\lambda)\left[I(\mathrm{VaR}_\beta(X)) - I(\mathrm{VaR}_\alpha(X))\right] \\
&\quad - \frac{\lambda}{1-\alpha}\int_{I(\mathrm{VaR}_\alpha(X))}^\infty \mathbb{P}(I(X) > z)\mathrm{d}z + \frac{1-\lambda}{1-\beta}\int_{I\left(\mathrm{VaR}_\beta(X)\right)}^\infty \mathbb{P}(I(X) > z)\mathrm{d}z \\
&\quad + (2\lambda-1)(1+\theta)\int_0^\infty \mathbb{P}(I(X) > z)\mathrm{d}z \\
&= \lambda \mathrm{TVaR}_\alpha(X) + (1-2\lambda)\xi_a + (1-\lambda)(\xi_b - \xi_a) - (1-2\lambda)(1+\theta)\int_0^{\xi_a} \mathbb{P}(I(X) > z)\mathrm{d}z \\
&\quad - m\int_{\xi_a}^{\xi_b} \mathbb{P}(I(X) > z)\mathrm{d}z + \left(\frac{1-\lambda}{1-\beta} - m\right)\int_{\xi_b}^\infty \mathbb{P}(I(X) > z)\mathrm{d}z. \qquad (3.3.8)
\end{aligned}
$$

(i) If $0 \leqslant \lambda < \frac{1}{2}$ and $\frac{1-\lambda}{1-\beta} > m$, then $m > 0$. For the above $I \in \mathcal{I}$, define

$$
\hat{I}(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant \xi_a; \\ \xi_a & \text{if } \xi_a \leqslant x \leqslant \mathrm{VaR}_\alpha(X); \\ x - \mathrm{VaR}_\alpha(X) + \xi_a & \text{if } \mathrm{VaR}_\alpha(X) \leqslant x \leqslant \xi_b - \xi_a + \mathrm{VaR}_\alpha(X); \\ \xi_b & \text{if } x \geqslant \xi_b - \xi_a + \mathrm{VaR}_\alpha(X). \end{cases} \qquad (3.3.9)
$$

The relationship between $I(x)$ and $\hat{I}(x)$ is illustrated by Figure 3.1. One can show that

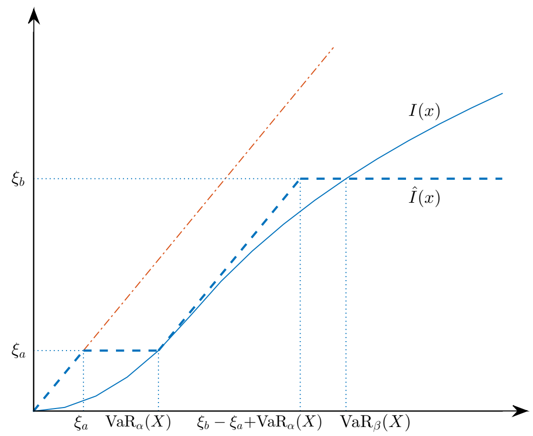

Figure 3.1: Relationship between $I(x)$ and $\hat{I}(x)$ in case (i)

$\hat{I}(x) \in \mathcal{I}$ and $V(I) \geqslant V(\hat{I})$ for any $I \in \mathcal{I}$. Indeed, from Figure 3.1, we conclude that for $0 \leqslant x \leqslant \mathrm{VaR}_\beta(X)$, $I(x) \leqslant \hat{I}(x)$, and for $x \geqslant \mathrm{VaR}_\beta(X)$, $I(x) \geqslant \hat{I}(x)$. Moreover, since $-(1-2\lambda)(1+\theta)<0$, $-m<0$, and $\frac{1-\lambda}{1-\beta}-m>0$, we have

$$
\begin{aligned}
-(1-2\lambda)(1+\theta)\int_0^\alpha \hat{I}(\mathrm{VaR}_r(X))\mathrm{d}r &\leqslant -(1-2\lambda)(1+\theta)\int_0^\alpha I(\mathrm{VaR}_r(X))\mathrm{d}r,\\
-m\int_\alpha^\beta \hat{I}(\mathrm{VaR}_r(X))\mathrm{d}r &\leqslant -m\int_\alpha^\beta I(\mathrm{VaR}_r(X))\mathrm{d}r,\\
\left(\frac{1-\lambda}{1-\beta}-m\right)\int_\beta^1 \hat{I}(\mathrm{VaR}_r(X))\mathrm{d}r &\leqslant \left(\frac{1-\lambda}{1-\beta}-m\right)\int_\beta^1 I(\mathrm{VaR}_r(X))\mathrm{d}r.
\end{aligned}
$$

Hence, it follows immediately from (3.3.7) that $V(I) \geqslant V(\hat{I})$, where the inequality is strict if I and $\hat{I}$ are not identical almost everywhere, which means that the optimal reinsurance contract can only take the form of (3.3.9) in case (i). The equivalence of (3.3.7) and (3.3.8) implies that $\min_{I\in\mathcal{I}} V(I) = \min_{\{\xi_a,\xi_b\}} V(\hat{I})$, where $V(\hat{I})$ is the expression in (3.3.8).

Next, it remains to find the values of ξ_a and ξ_b such that $V(\hat{I})$ is minimized. Let $s=\xi_a$ and $t=\xi_b-\xi_a$. Clearly $0\leqslant s\leqslant \mathrm{VaR}_\alpha(X)$ and $0\leqslant t\leqslant \mathrm{VaR}_\beta(X)-\mathrm{VaR}_\alpha(X)$. Since

$$
\mathbb{P}\left(\hat{I}(X)>x\right)=\begin{cases}
\mathbb{P}\left(X>x\right) & \text{if } 0\leqslant x<\xi_a;\\
\mathbb{P}\left(X>x+\mathrm{VaR}_\alpha(X)-\xi_a\right) & \text{if } \xi_a\leqslant x<\xi_b;\\
0 & \text{if } x\geqslant \xi_b,
\end{cases}
$$

equation (3.3.8) reduces to

$$
\begin{aligned}
V(\hat{I}) = \lambda \mathrm{TVaR}_\alpha(X) + (1-2\lambda)s - (1-2\lambda)(1+\theta)\int_0^s \mathbb{P}(X>z)\mathrm{d}z \\
+ (1-\lambda)t - m\int_{\mathrm{VaR}_\alpha(X)}^{t+\mathrm{VaR}_\alpha(X)} \mathbb{P}(X>z)\mathrm{d}z. \qquad (3.3.10)
\end{aligned}
$$

Denote by

$$
\begin{aligned}
& f(s) = \lambda \mathrm{TVaR}_\alpha(X) + (1-2\lambda)s - (1-2\lambda)(1+\theta)\int_0^s \mathbb{P}(X>z)\mathrm{d}z, \\
& g(t) = (1-\lambda)t - m\int_{\mathrm{VaR}_\alpha(X)}^{t+\mathrm{VaR}_\alpha(X)} \mathbb{P}(X>z)\mathrm{d}z, \\
& \mathcal{A} = \{s\in\mathbb{R} \mid 0 \leqslant s \leqslant \mathrm{VaR}_\alpha(X)\}, \\
& \mathcal{B} = \{t\in\mathbb{R} \mid 0 \leqslant t \leqslant \mathrm{VaR}_\beta(X) - \mathrm{VaR}_\alpha(X)\}, \\
& \mathcal{C} = \left\{(s,t)\in\mathbb{R}^2 \mid s\in\mathcal{A}, t\in\mathcal{B}\right\}.
\end{aligned}
$$

Then $V(\hat{I}) = f(s) + g(t)$. Lebesgue differentiation theorem implies that f and g are continuous in s and t, respectively. Suppose that there exist $s^* \in \mathcal{A}$ and $t^* \in \mathcal{B}$ such that $\min_{s\in\mathcal{A}} f(s) = f(s^*)$ and $\min_{t\in\mathcal{B}} g(t) = g(t^*)$. Then $\min_{(s,t)\in\mathcal{C}} V(\hat{I}) = f(s^*) + g(t^*)$ because $\min_{(s,t)\in\mathcal{C}} V(\hat{I}) = \min_{(s,t)\in\mathcal{C}}\{f(s)+g(t)\} \geqslant \min_{s\in\mathcal{A}} f(s) + \min_{t\in\mathcal{B}} g(t) = f(s^*) + g(t^*) \geqslant \min_{(s,t)\in\mathcal{C}}\{f(s)+g(t)\} = \min_{(s,t)\in\mathcal{C}} V(\hat{I})$. Therefore, it remains to find s^* and t^*, or the corresponding ξ_a^* and ξ_b^*, where $\xi_a^* = s^*$ and $\xi_b^* = t^* + s^*$.

If $(1-\alpha)(1+\theta) \geqslant 1$, then $p \geqslant \alpha$ since $m = \frac{\lambda}{1-\alpha} + (1-2\lambda)(1+\theta) \geqslant \frac{\lambda}{1-\alpha} + (1-2\lambda)\frac{1}{1-\alpha} = \frac{1-\lambda}{1-\alpha}$. For $0 \leqslant s_1 < s_2 < \mathrm{VaR}_\alpha(X)$, as $\lambda < 1/2$ and $\mathbb{P}(X \leqslant s_2) < \alpha$, we have

$$
\begin{aligned}
f(s_1) - f(s_2) &= (1-2\lambda)\Big((1+\theta)\int_{s_1}^{s_2} \mathbb{P}(X>z)\mathrm{d}z - (s_2 - s_1)\Big) \\
&\geqslant (1-2\lambda)(1+\theta)(s_2-s_1)\left[\mathbb{P}\left(X > s_2\right) - \theta^*\right] > 0,
\end{aligned}
$$

which, together with the continuity of f, implies that $f(s)$ is strictly decreasing for $s \in \mathcal{A}$ and $\xi_a^* = s^* = \mathrm{VaR}_\alpha(X)$. On the other hand, $\frac{1-\lambda}{1-\beta} > m$ implies $\beta > 1 - \frac{1-\lambda}{m}$, that is, $\beta > p$. Thus, $\mathrm{VaR}_p(X) - \mathrm{VaR}_\alpha(X) \in \mathcal{B}$. Note that if $\mathrm{VaR}_\alpha(X) = \mathrm{VaR}_\beta(X)$, then $\mathrm{VaR}_p(X) = \mathrm{VaR}_\alpha(X)$ since $\alpha \leqslant p < \beta$. For $0 \leqslant t_1 < t_2 < \mathrm{VaR}_p(X) - \mathrm{VaR}_\alpha(X)$, as $m > 0$

and $\mathbb{P}(X \leqslant t_2 + \mathrm{VaR}_\alpha(X)) < p = 1 - \frac{1-\lambda}{m}$, we have

$$
\begin{aligned}
g(t_1) - g(t_2) &= m \int_{t_1+\mathrm{VaR}_\alpha(X)}^{t_2+\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z - (1-\lambda)(t_2 - t_1) \\
&\geqslant \Big(\mathbb{P}(X > t_2 + \mathrm{VaR}_\alpha(X)) - (1-\lambda)/m\Big)(t_2 - t_1)m \\
&> \Big(1 - p - \frac{1-\lambda}{m}\Big)(t_2 - t_1)m = 0.
\end{aligned}
$$

Therefore, $g(t)$ is strictly decreasing for $0 \leqslant t \leqslant \mathrm{VaR}_p(X) - \mathrm{VaR}_\alpha(X)$. Similarly, for $\mathrm{VaR}_p(X) - \mathrm{VaR}_\alpha(X) < t_1 < t_2 \leqslant \mathrm{VaR}_\beta(X) - \mathrm{VaR}_\alpha(X)$, as $m > 0$, we have

$$
\begin{aligned}
g(t_1) - g(t_2) &= m \int_{t_1+\mathrm{VaR}_\alpha(X)}^{t_2+\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z - (1-\lambda)(t_2 - t_1) \\
&\leqslant \Big(\mathbb{P}(X > t_1 + \mathrm{VaR}_\alpha(X)) - (1-\lambda)/m\Big)(t_2 - t_1)m \\
&\leqslant \Big(\mathbb{P}(X > \mathrm{VaR}_p(X)) - (1-\lambda)/m\Big)(t_2 - t_1)m \\
&= \Big(1 - (1-\lambda)/m - \mathbb{P}(X \leqslant \mathrm{VaR}_p(X))\Big)(t_2 - t_1)m \leqslant 0.
\end{aligned}
$$

Thus, $g(t)$ is increasing for $\mathrm{VaR}_p(X) - \mathrm{VaR}_\alpha(X) \leqslant t \leqslant \mathrm{VaR}_\beta(X) - \mathrm{VaR}_\alpha(X)$. Thus $t^* = \mathrm{VaR}_p(X) - \mathrm{VaR}_\alpha(X)$ minimizes g, $\xi_a^* = \mathrm{VaR}_\alpha(X)$, $\xi_b^* = \mathrm{VaR}_p(X)$, and the optimal reinsurance contract is $I^*(x) = x \wedge \mathrm{VaR}_p(X)$.

If $(1-\alpha)(1+\theta) < 1$, then $p < \alpha$. Similarly, we can show that $\xi_a^* = s^* = \mathrm{VaR}_{1-\theta^*}(X)$ and $t^* = 0$. Therefore, $\xi_a^* = \xi_b^* = \mathrm{VaR}_{1-\theta^*}(X)$ and the optimal reinsurance contract is $I^*(x) = x \wedge \mathrm{VaR}_{1-\theta^*}(X)$. □

Remark 3.1. When $\lambda = 1$, cases (vii), (ix) and (x) of Theorem 3.8 recover Theorem 3.3 of Cheung et al. (2014). We point out that Theorem 3.3 of Cheung et al. (2014) holds under the assumption that the ground-up loss random variable X has a continuous and strictly increasing distribution function. However, Theorem 3.8 does not require the assumption.

Theorem 3.9. *For $0 < \beta \leqslant \alpha < 1$, $\lambda \in [0,1]$ and a non-negative integrable ground-up loss random variable X, optimal reinsurance contracts $I^* = I^*_\lambda$ to problem (3.3.1) are given as follows:*

(i) If $\lambda = 0$ and $(1-\beta)(1+\theta) > 1$, then $I^*(x) = x$.

(ii) If $\lambda = 0$ and $(1-\beta)(1+\theta) = 1$, then $I^*(x) = x\,\mathbb{I}_{\{x \leqslant \mathrm{VaR}_\beta(X)\}} + I(x)\,\mathbb{I}_{\{x > \mathrm{VaR}_\beta(X)\}}$, where I can be any function such that $I^* \in \mathcal{I}$.

(iii) *If* $0 \leqslant \lambda < \frac{1}{2}$ *and* $\frac{1-\lambda}{1-\beta} > m$*, then* $I^*(x) = x \wedge \mathrm{VaR}_{1-\theta^*}(X)$*.*

(iv) *If* $0 < \lambda < \frac{1}{2}$ *and* $\frac{1-\lambda}{1-\beta} = m$*, then* $I^*(x) = (x \wedge \mathrm{VaR}_{1-\theta^*}(X))\, \mathbb{I}_{\{x \leqslant \mathrm{VaR}_\alpha(X)\}} + I(x)\, \mathbb{I}_{\{x > \mathrm{VaR}_\alpha(X)\}}$*, where* I *can be any function such that* $I^* \in \mathcal{I}$*.*

(v) *If* $0 < \lambda < \frac{1}{2}$ *and* $(1-2\lambda)(1+\theta) < \frac{1-\lambda}{1-\beta} < m$*, then*

$$I^*(x) = \begin{cases} x, & \text{if } (1-\beta)(1+\theta) \geqslant 1; \\ x \wedge \mathrm{VaR}_{1-\theta^*}(X) + \big(x - \mathrm{VaR}_{q(\lambda)}(X)\big)_+, & \text{if } (1-\beta)(1+\theta) < 1. \end{cases}$$

(vi) *If* $0 < \lambda < \frac{1}{2}$ *and* $(1-2\lambda)(1+\theta) = \frac{1-\lambda}{1-\beta}$*, then*

$$I^*(x) = x\, \mathbb{I}_{\left\{x \leqslant \mathrm{VaR}_\beta(X) \text{ or } x \geqslant \mathrm{VaR}_\alpha(X)\right\}} + I(x)\, \mathbb{I}_{\left\{\mathrm{VaR}_\beta(X) \leqslant x \leqslant \mathrm{VaR}_\alpha(X)\right\}},$$

where I *can be any function such that* $I^* \in \mathcal{I}$*.*

(vii) *If* $0 < \lambda < \frac{1}{2}$ *and* $\frac{1-\lambda}{1-\beta} < (1-2\lambda)(1+\theta)$*, then* $I^*(x) = x$*.*

(viii) *If* $\lambda = \frac{1}{2}$*, then*

$$I^*(x) = \begin{cases} I(x)\, \mathbb{I}_{\{x \leqslant \mathrm{VaR}_\beta(X)\}} + (x - \mathrm{VaR}_\beta(X) + I(\mathrm{VaR}_\beta(X)))\, \mathbb{I}_{\{x > \mathrm{VaR}_\beta(X)\}}, & \text{if } \alpha > \beta; \\ I(x), & \text{if } \alpha = \beta, \end{cases}$$

where I *can be any function such that* $I^* \in \mathcal{I}$*.*

(ix) *If* $\frac{1}{2} < \lambda \leqslant 1$ *and* $\frac{1-\lambda}{1-\beta} > m,$*, then* $I^*(x) = 0$*.*

(x) *If* $\frac{1}{2} < \lambda \leqslant 1$ *and* $\frac{1-\lambda}{1-\beta} = m$*, then* $I^*(x) = I(x)\, \mathbb{I}_{\{x > \mathrm{VaR}_\alpha(X)\}}$*, where* I *can be any function such that* $I^* \in \mathcal{I}$*.*

(xi) *If* $\frac{1}{2} < \lambda \leqslant 1$ *and* $\frac{1-\lambda}{1-\beta} < m$*, then*

$$I^*(x) = \begin{cases} \big(x - \mathrm{VaR}_{q(\lambda)}(X)\big)_+, & \text{if } (1-\beta)(1+\theta) \geqslant 1; \\ (x - \mathrm{VaR}_{1-\theta^*}(X))_+, & \text{if } (1-\beta)(1+\theta) < 1. \end{cases}$$

The proof of Theorem 3.9 is in Section 3.6.2.

3.4 Best Pareto-optimal Reinsurance Contracts under TVaRs

Under a reinsurance setting $(X, \rho_1, \rho_2, \pi, \mathcal{I})$ with $\rho_1 = \mathrm{TVaR}_\alpha$, $\rho_2 = \mathrm{TVaR}_\beta$, and $\pi(I(X)) = (1+\theta)\,\mathbb{E}(I(X))$ for $I \in \mathcal{I}$, by Proposition 3.1 or Theorem 3.2, we know that for any $\lambda \in (0,1)$, the reinsurance contract $I^* = I_\lambda^*$ given in Theorems 3.8 or 3.9 is a Pareto-optimal contract for the case of $\alpha \leqslant \beta$ or the case of $\alpha > \beta$. However, an interesting and practical question is that what the Pareto-optimal contracts I_λ^* for $\lambda \in (0,1)$ are the best ones in the sense that the Pareto-optimal contracts I_λ^* could be accepted by both the insurer and the reinsurer. To address this issue, let us recall that one of the main reasons for an insurer to buy a reinsurance contact is to reduce its risk measure (required reserves/capitals), while the goal of a reinsurer as the seller of a contract is to make profits. Before reinsurance, the risk of the insurer is X and its risk measure is $\mathrm{TVaR}_\alpha(X)$. Under a Pareto-optimal contract I_λ^*, the risk of the insurer is $C_{I_\lambda^*} = C_{I_\lambda^*}(X) = X - I_\lambda^*(X) + \pi(I_\lambda^*(X))$, and the insurer expects its risk measure to be reduced at least $100(1-\gamma)\%$ under the Pareto-optimal reinsurance I_λ^*, namely

$$\mathrm{TVaR}_\alpha(C_{I_\lambda^*}(X)) \leqslant \gamma\,\mathrm{TVaR}_\alpha(X) \tag{3.4.1}$$

for $0 < \gamma < 1$. On the other hand, under the Pareto-optimal reinsurance I_λ^*, the reinsurer has an expected gross income $\mathbb{E}(\pi(I_\lambda^*(X)))$ and an expected net profit $\mathbb{E}(\pi(I_\lambda^*(X)) - I_\lambda^*(X))$, and the reinsurer would like the expected net profit at least to be $100\sigma\%$ of the expected gross income, namely

$$\mathbb{E}(\pi(I_\lambda^*(X)) - I_\lambda^*(X)) \geqslant \sigma\,\mathbb{E}(\pi(I_\lambda^*(X))) \tag{3.4.2}$$

for $0 < \sigma < 1$. In addition, the reinsurer also has a concern about the TVaR of its risk. Assume that the reinsurer wishes that under a Pareto-optimal contract I^*, the maximum TVaR of its risk is not bigger than $100\kappa\%$ of the TVaR of X if the reinsurer acts as the insurer and has the ground-up loss X, namely

$$\mathrm{TVaR}_\beta(R_{I_\lambda^*}(X)) \leqslant \kappa\,\mathrm{TVaR}_\beta(X) \tag{3.4.3}$$

for $0 < \kappa < 1$. Therefore, the best Pareto-optimal reinsurance contacts I_λ^* for $\lambda \in (0,1)$ should be those such that all of the three conditions (3.4.1), (3.4.2), and (3.4.3) hold.

In the rest of this section, we will use two examples to illustrate how to determine the best Pareto-optimal reinsurance contacts $I_{\lambda^*}^*$ among the available Pareto-optimal reinsurance contracts I_λ^* with $\lambda \in (0,1)$, such that all of the three conditions (3.4.1), (3.4.2), and (3.4.3) hold.

Note that if $\pi(I(X)) = (1+\theta)\,\mathbb{E}(I(X))$ for $I \in \mathcal{I}$, then $\mathbb{E}(\pi(I_\lambda^*(X)) - I_\lambda^*(X)) = \frac{\theta}{1+\theta}\,\mathbb{E}(\pi(I_\lambda^*(X))$ for any $\lambda \in (0, 1)$ and (3.4.2) holds if and only if $\sigma \leqslant \frac{\theta}{1+\theta}$.

In the following two examples, we let the safety loading factor be $\theta = 0.2$. Thus, $\theta^* = 1/(1+\theta) = 0.8333$. Furthermore, let $\sigma \leqslant \frac{\theta}{1+\theta} = 0.16667$. Thus, (3.4.2) holds for any I_λ^*. Moreover, we let $\kappa = 0.8$ and discuss the impacts of the parameter γ and the distribution of the ground-up loss random variable X on the best Pareto-optimal contracts by setting $\gamma = 0.5, 0.7, 0.8$ and assuming that X has an exponential distribution and a Pareto distribution, respectively.

Example 3.1. Suppose that the ground-up loss X follows an exponential distribution with distribution function $F(x) = 1 - e^{-0.001x}$ for $x \geqslant 0$. Then $\mathbb{E}(X) = 1000$, $\mathrm{VaR}_\alpha(X) = -1000\ln(1-\alpha)$, and $\mathrm{TVaR}_\alpha(X) = 1000\,[1 - \ln(1-\alpha)]$. Thus, $\mathrm{VaR}_{1-\theta^*}(X) = 182.32$.

If $\alpha < \beta$ with $\alpha = 0.95$ and $\beta = 0.99$, then $(1-\alpha)(1+\theta) < 1$. When $\lambda = 0.5$, which is the case (iv) of Theorem 3.8, by taking $I(x) = x \wedge \mathrm{VaR}_{1-\theta^*}(X) = x \wedge 182.32$ in (iv) of Theorem 3.8, we have that the Pareto-optimal reinsurance can be $I^*(x) = x \wedge 182.32$. When $\lambda = 0.84$, which is the case (vi) of Theorem 3.8, by taking $I(x) = \mathrm{VaR}_{p(0.84)}(X) - 182.32$ in (vi) of Theorem 3.8, we see that the Pareto-optimal reinsurance can be $I^*(x) = (x - 182.32)_+ \wedge (\mathrm{VaR}_{p(0.84)}(X) - 182.32)$. When $\lambda \in (0,\ 0.5)$, $\lambda \in (05,\ 0.84)$, and $\lambda \in (0.84,\ 1)$, the Pareto-optimal reinsurance contracts are of cases (i), (v), and (vii) of Theorem 3.8, respectively. Therefore, the Pareto-optimal reinsurance contracts are

$$I_\lambda^*(x) = \begin{cases} x \wedge 182.32 & \text{if } \lambda \in (0,\ 0.5]; \\ (x - 182.32)_+ \wedge (\mathrm{VaR}_{p(\lambda)}(X) - 182.32) & \text{if } \lambda \in (0.5,\ 0.84]; \\ (x - 182.32)_+ & \text{if } \lambda \in (0.84,\ 1). \end{cases} \tag{3.4.4}$$

It is easy to verify that when $\lambda \in (0.5,\ 0.84]$, $\mathrm{TVaR}_\alpha(C_{I_\lambda^*}(X))$ is decreasing in λ, while $\mathrm{TVaR}_\beta(R_{I_\lambda^*}(X))$ and $\mathbb{E}\big(\pi(I_\lambda^*(X))\big)$ are increasing in λ. In addition, they are all constants for $\lambda \in (0, 0.5]$ and $\lambda \in (0.84, 1)$, respectively. The values of $\mathrm{TVaR}_\alpha(X)$, $\mathrm{TVaR}_\alpha(C_{I_\lambda^*}(X))$, $\mathrm{TVaR}_\beta(X)$, $\mathrm{TVaR}_\beta(R_{I_\lambda^*}(X))$, and $\mathbb{E}[\pi(I_\lambda^*(X))]$ are the key for one to find $\lambda^* \in (0,1)$ such that the Pareto-optimal contracts $I_{\lambda^*}^*$ satisfy (3.4.1)-(3.4.3). These key values are presented in Table 3.1.

If $\alpha > \beta$ with $\alpha = 0.99$ and $\beta = 0.95$, then $(1-\beta)(1+\theta) < 1$. When $\lambda = 0.1599$, which is the case (iv) of Theorem 3.9, the Pareto-optimal reinsurance can be $I^*(x) = x \wedge 182.32$ by taking $I(x) = \mathrm{VaR}_{1-\theta^*}(X) = 182.32$ in (iv) of Theorem 3.9. When $\lambda = 0.5$, which is the case (viii) of Theorem 3.9, then the Pareto-optimal reinsurance can be $I^*(x) = x$ by taking $I(x) = x$ in (viii) of Theorem 3.9. When $\lambda \in (0, 0.1599)$, $\lambda \in (0.1599, 0.5)$, and $\lambda \in (0.5, 1)$ and the Pareto-optimal contracts are of cases (iii), (v), and(xi) of Theorem 3.9, respectively. Thus, the Pareto-optimal reinsurance contracts are

$$I_\lambda^*(x) = \begin{cases} x \wedge 182.32 & \text{if } \lambda \in (0,\ 0.1599]; \\ x \wedge 182.32 + \big(x - \mathrm{VaR}_{q(\lambda)}(X)\big)_+ & \text{if } \lambda \in (0.1599,\ 0.5]; \\ (x - 182.32)_+ & \text{if } \lambda \in (0.5,\ 1). \end{cases} \tag{3.4.5}$$

When $\lambda \in (0.1599,\ 0.5]$, $\mathrm{TVaR}_\alpha(C_{I_\lambda^*}(X))$ is decreasing in λ, while $\mathrm{TVaR}_\beta(R_{I_\lambda^*}(X))$ and $\mathbb{E}\big(\pi\big(I_\lambda^*(X)\big)\big)$ are increasing in λ. The key values in this case are given in Table 3.2.

If $\alpha = \beta = 0.95$, then $(1-\alpha)(1+\theta) < 1$. By Theorem 3.8 or 3.9, the Pareto-optimal reinsurance contracts are

$$I_\lambda^*(x) = \begin{cases} x \wedge 182.32 & \text{if } \lambda \in (0,\ 0.5); \\ ax,\ a \in [0,1] & \text{if } \lambda = 0.5; \\ (x - 182.32)_+ & \text{if } \lambda \in (0.5,\ 1). \end{cases} \tag{3.4.6}$$

We point out that for the case $\alpha = \beta$ and $\lambda = 1/2$, the Pareto-optimal contract can be any contract in $\mathcal{I}$. To simplify the discussion of how to determine the best Pareto-optimal contracts, we consider all of quota-share reinsurances and find the best quota-share reinsurances as the best Pareto-optimal contracts for the case $\alpha = \beta$ and $\lambda = 1/2$. The corresponding key values are given in Table 3.3.

Based on those values given in Tables 3.1-3.3 and the forms of Pareto-optimal reinsurance contracts given in (3.4.4)-(3.4.6), we can easily find $\lambda^* \in (0,1)$ such that the corresponding Pareto-optimal reinsurance contracts $I_{\lambda^*}^*$ satisfy (3.4.1)-(3.4.3). Such values of λ^* are summarized in Table 3.4.

From Table 3.4 and (3.4.4), we see that if $\alpha < \beta$ with $\alpha = 0.95$ and $\beta = 0.99$, then the limited stop-loss reinsurances $I_{\lambda^*}^*(x) = (x - 182.32)_+ \wedge (\mathrm{VaR}_{p(\lambda^*)}(X) - 182.32)$ and the stop-loss reinsurance $I^*(x) = (x - 182.32)_+$ are the best Pareto-optimal reinsurance contracts for all the three cases of γ, where the values of λ^* for each of the three cases of γ are given in Table 3.4.

If $\alpha > \beta$ with $\alpha = 0.99$ and $\beta = 0.95$, then, from Table 3.4 and (3.4.5), we find that the stop-loss contract $I^*(x) = (x - 182.32)_+$ is the best Pareto-optimal reinsurance for all the three cases of γ. Besides, the contracts $I_{\lambda^*}^*(x) = x \wedge 182.32 + \big(x - \mathrm{VaR}_{q(\lambda^*)}(X)\big)_+$ are also the best Pareto-optimal reinsurance contracts for the cases of $\gamma = 0.7, 0.8$, where the values of λ^* for each of the two cases of γ are given in Table 3.4.

Table 3.1: Key values with an exponential ground-up loss and $\alpha = 0.95 < \beta = 0.99$

	$\mathrm{TVaR}_\alpha(X)$	$\mathrm{TVaR}_\alpha\big((C_{I^*_\lambda}(X)\big)$	$\mathrm{TVaR}_\beta(X)$	$\mathrm{TVaR}_\beta\big((R_{I^*_\lambda}(X)\big)$	$\mathbb{E}\big(\pi(I^*_\lambda(X))\big)$
$\lambda \in (0,\ 0.5]$	3995.73	4013.41	5605.17	-17.68	200
$\lambda \in (0.5,\ 0.84]$	3995.73	(2122.32, 1370.51]	5605.17	(1873.41, 3433.86]	(940, 987.99]
$\lambda \in (0.84,\ 1)$	3995.73	1182.32	5605.17	4422.85	1000

Table 3.2: Key values with an exponential ground-up loss and $\alpha = 0.99 > \beta = 0.95$

	$\mathrm{TVaR}_\alpha(X)$	$\mathrm{TVaR}_\alpha(C_{I^*_\lambda}(X))$	$\mathrm{TVaR}_\beta(X)$	$\mathrm{TVaR}_\beta\big((R_{I^*_\lambda}(X)\big)$	$\mathbb{E}\big(\pi(I^*_\lambda(X))\big)$
$\lambda \in (0,\ 0.1599]$	5605.17	5622.85	3995.73	-17.68	200
$\lambda \in (0.1599,\ 0.5]$	5605.17	(4634.55, 3073.41]	3995.73	(170.33, 922.32]	(212.04, 260]
$\lambda \in (0.5,\ 1)$	5605.17	1182.32	3995.73	2813.41	1000

Table 3.3: Key values with an exponential ground-up loss and $\alpha = \beta = 0.95$

	$\mathrm{TVaR}_\alpha(X)$	$\mathrm{TVaR}_\alpha(C_{I^*_\lambda}(X))$	$\mathrm{TVaR}_\beta(X)$	$\mathrm{TVaR}_\beta\big((R_{I^*_\lambda}(X)\big)$	$\mathbb{E}\big(\pi(I^*_\lambda(X))\big)$
$\lambda \in (0,\ 0.5)$	3995.73	4013.41	3995.73	-17.68	200
$\lambda = 0.5,\ I^*_\lambda(X) = a\,x,\ a \in [0,1]$	3995.73	[3995.73, 1200]	3995.73	[0, 2795.73]	[0, 1200]
$\lambda \in (0.5,\ 1)$	3995.73	1182.32	3995.73	2813.41	1000

Table 3.4: Best Pareto-optimal reinsurance contracts $I^*_{\lambda^*}$ with an exponential ground-up loss

$\kappa = 0.8$	$\alpha = 0.95,\ \beta = 0.99$	$\alpha = 0.99,\ \beta = 0.95$	$\alpha = \beta = 0.95$
$\gamma = 0.5$	$\lambda^* \in [0.5376,\ 1)$	$\lambda^* \in [0.5,\ 1)$	$\lambda^* \in (0.5,\ 1)$ or $\lambda^* = 0.5$ with $a^* \in [0.7146, 1]$
$\gamma = 0.7$	$\lambda^* \in (0.5,\ 1)$	$\lambda^* \in [0.2845,\ 1)$	$\lambda^* \in (0.5,\ 1)$ or $\lambda^* = 0.5$ with $a^* \in [0.4288, 1]$
$\gamma = 0.8$	$\lambda^* \in (0.5,\ 1)$	$\lambda^* \in [0.1817,\ 1)$	$\lambda^* \in (0.5,\ 1)$ or $\lambda^* = 0.5$ with $a^* \in [0.2858, 1]$

If $\alpha = \beta = 0.95$, then, from Table 3.4 and (3.4.6), we see that the stop-loss reinsurance $I^*(x) = (x - 182.32)_+$ and the quota-share reinsurances $I^*(x) = a^*\,x$ are the best Pareto-optimal reinsurance contracts for all the three cases of γ, where the values of a^* for each case are given in Table 3.4. $\square$

Example 3.2. Suppose that the ground loss X follows a Pareto distribution with distribution function $1 - \left(\frac{2000}{x+2000}\right)^3$ for any $x \geqslant 0$. Thus $\mathbb{E}(X) = 1000$, which is the same mean as the exponential distribution assumed in Example 3.1. In addition, $\mathrm{VaR}_\alpha(X) = 2000\big((1-\alpha)^{-1/3} - 1\big)$ and $\mathrm{TVaR}_\alpha(X) = 3000(1-\alpha)^{-1/3} - 2000$. Hence, $\mathrm{VaR}_{1-\theta^*}(X) = 125.32$. By using the arguments similar to those for Example 3.1, we obtain the (best) Pareto-optimal reinsurances for the Pareto distribution as follows.

If $\alpha < \beta$ with $\alpha = 0.95$ and $\beta = 0.99$, then $(1-\alpha)(1+\theta) < 1$. By Theorem 3.8, the Pareto-optimal reinsurance contracts are

$$I^*_\lambda(x) = \begin{cases} x \wedge 125.32 & \text{if } \lambda \in (0,\, 0.5]; \\ (x-125.32)_+ \wedge (\mathrm{VaR}_{p(\lambda)}(X) - 125.32) & \text{if } \lambda \in (0.5,\, 0.84]; \\ (x-125.32)_+ & \text{if } \lambda \in (0.84,\, 1). \end{cases} \tag{3.4.7}$$

When $\lambda \in (0.5,\, 0.84]$, $\mathrm{TVaR}_\alpha(C_{I^*_\lambda}(X))$ is decreasing in λ, while $\mathrm{TVaR}_\beta(R_{I^*_\lambda}(X))$ and $\mathbb{E}\big(\pi(I^*_\lambda(X))\big)$ are increasing in λ. The key values for this case are given in Table 3.5.

If $\alpha > \beta$ with $\alpha = 0.99$ and $\beta = 0.95$, then $(1-\beta)(1+\theta) < 1$. By Theorem 3.9, the Pareto-optimal reinsurance contracts are

$$I^*_\lambda(x) = \begin{cases} x \wedge 125.32 & \text{if } \lambda \in (0,\, 0.1599]; \\ x \wedge 125.32 + \big(x - \mathrm{VaR}_{q(\lambda)}(X)\big)_+ & \text{if } \lambda \in (0.1599,\, 0.5]; \\ (x-125.32)_+ & \text{if } \lambda \in (0.5,\, 1). \end{cases} \tag{3.4.8}$$

When $\lambda \in (0.1599,\, 0.5]$, $\mathrm{TVaR}_\alpha(C_{I^*_\lambda}(X))$ is decreasing in λ, while $\mathrm{TVaR}_\beta(R_{I^*_\lambda}(X))$ and $\mathbb{E}\big(\pi\big(I^*_\lambda(X)\big)\big)$ are increasing in λ. The key values are given in Table 3.6.

If $\alpha = \beta = 0.95$, then $(1-\alpha)(1+\theta) < 1$. By Theorem 3.8 or 3.9 , the Pareto-optimal reinsurance contracts are

$$I^*_\lambda(x) = \begin{cases} x \wedge 125.32 & \text{if } \lambda \in (0,\, 0.5); \\ ax,\ a \in [0,1] & \text{if } \lambda = 0.5; \\ (x-125.32)_+ & \text{if } \lambda \in (0.5,\, 1). \end{cases} \tag{3.4.9}$$

The corresponding key values are given in Table 3.7.

Based on those values given in Table 3.5-3.7, and the forms of Pareto-optimal reinsurance contracts given in (3.4.7)-(3.4.9), we can easily find $\lambda^* \in (0,1)$ such that the corresponding Pareto-optimal reinsurance contracts $I^*_{\lambda^*}$ satisfy (3.4.1)-(3.4.3). Such values of λ^* are summarized in Table 3.8.

If $\alpha < \beta$ with $\alpha = 0.95$ and $\beta = 0.99$, from Table 3.8 and (3.4.7), we see that the limited stop-loss reinsurances $I^*_{\lambda^*}(x) = (x-125.32)_+ \wedge (\mathrm{VaR}_{p(\lambda^*)}(X) - 125.32)$ are the best Pareto-optimal reinsurances for all the three cases of γ, where the values of λ^* for each case are given in Table 3.8.

Table 3.5: Key values with a Pareto ground-up loss when $\alpha = 0.95 < \beta = 0.99$

	$\mathrm{TVaR}_\alpha(X)$	$\mathrm{TVaR}_\alpha(C_{I_\lambda^*}(X))$	$\mathrm{TVaR}_\beta(X)$	$\mathrm{TVaR}_\beta\big((R_{I_\lambda^*}(X)\big)$	$\mathbb{E}[\pi(I_\lambda^*(X))]$
$\lambda \in (0, 0.5]$	6143.25	6155.27	11924.77	-12.02	137.34
$\lambda \in (0.5, 0.84]$	6143.25	(3739.53, 2061.18]	11924.77	(2403.72, 6147.84]	(899.79, 1006.92]
$\lambda \in (0.84, 1)$	6143.25	1187.98	11924.77	10736.79	1062.66

Table 3.6: Key values with a Pareto ground-up loss when $\alpha = 0.99 > \beta = 0.95$

	$\mathrm{TVaR}_\alpha(X)$	$\mathrm{TVaR}_\alpha(C_{I_\lambda^*}(X))$	$\mathrm{TVaR}_\beta(X)$	$\mathrm{TVaR}_\beta\big((R_{I_\lambda^*}(X)\big)$	$\mathbb{E}[\pi(I_\lambda^*(X))]$
$\lambda \in (0,\ 0.1599]$	11924.77	11936.79	6143.25	-12.02	137.34
$\lambda \in (0.1599,\ 0.5]$	11924.77	(7349.98, 3603.72]	6143.25	(860.77, 2539.53]	(193.05, 300.21]
$\lambda \in (0.5,\ 1)$	11924.77	1187.98	6143.25	4955.27	1062.66

Table 3.7: Key values a Pareto ground-up loss when $\alpha = \beta = 0.95$

	$\mathrm{TVaR}_\alpha(X)$	$\mathrm{TVaR}_\alpha(C_{I_\lambda^*}(X))$	$\mathrm{TVaR}_\beta(X)$	$\mathrm{TVaR}_\beta\big((R_{I_\lambda^*}(X)\big)$	$\mathbb{E}[\pi(I^*(X))]$
$\lambda \in (0,\ 0.5)$	6143.25	6155.27	6143.25	-12.02	137.34
$\lambda = 0.5,\ I_\lambda^*(X) = a \cdot x,\ a \in [0,1]$	6143.25	[6143.25, 1200]	6143.25	[0, 4943.25]	[0, 1200]
$\lambda \in (0.5,\ 1)$	6143.25	1187.98	6143.25	4955.27	1062.66

Table 3.8: Best Pareto-optimal reinsurance contracts $I_{\lambda^*}^*$ with a Pareto ground-up loss

$\kappa = 0.8$	$\alpha = 0.95,\ \beta = 0.99$	$\alpha = 0.99,\ \beta = 0.95$	$\alpha = \beta = 0.95$
$\gamma = 0.5$	$\lambda^* \in [0.6173,\ 0.84]$	$\lambda^* \in [0.2392,\ 0.5]$	$\lambda^* = 0.5$ with $a^* \in [0.6214,\ 0.9942]$
$\gamma = 0.7$	$\lambda^* \in (0.5,\ 0.84]$	$\lambda^* \in (0.1599,\ 0.5]$	$\lambda^* = 0.5$ with $a^* \in [0.3728,\ 0.9942]$
$\gamma = 0.8$	$\lambda^* \in (0.5,\ 0.84]$	$\lambda^* \in (0.1599,\ 0.5]$	$\lambda^* = 0.5$ with $a^* \in [0.2486,\ 0.9942]$

If $\alpha > \beta$ with $\alpha = 0.99$ and $\beta = 0.95$, then, from Table 3.8 and (3.4.8), we find that the contracts $I_{\lambda^*}^*(x) = x \wedge 125.32 + \big(x - \mathrm{VaR}_{q(\lambda^*)}(X)\big)_+$ are the best Pareto-optimal reinsurance contracts for all the three cases of γ, where the values of λ^* for each case are given in Table 3.8.

If $\alpha = \beta = 0.95$, then, from Table 3.8 and (3.4.9), we see that the quota-share reinsurances $I^*(x) = a^*\, x$ are the best Pareto-optimal reinsurance contracts for all the three cases of γ, where the values of a^* for each case are given in Table 3.8. □

Both Tables 3.4 and 3.8 show that the higher of the insurer's requirement (such as a smaller value of γ), the less of the choices of the best Pareto-optimal reinsurances or the best values of λ^*. Moreover, the distribution of the ground-up loss random variable and the confidence levels of TVaRs also have significant influences in the best Pareto-optimal contracts. If $\alpha \leqslant \beta$, which means that the TVaR standard of the reinsurer is not lower than the insurer, then the riskier of the ground-up loss (such as the Pareto loss), the less of the choices of the best Pareto-optimal contracts or the best values of λ^*. However, if $\alpha > \beta$ or the insurer has a higher standard on TVaR than the reinsurer, then a more riskier ground-up loss (the Pareto loss) will result in a more conservative best Pareto-optimal contract (such as the reinsurance with a limit) for the reinsurer, while for a less risker ground-up loss (the exponential loss), an unlimited contract such as the stop-loss reinsurance can be the best Pareto-optimal contract for the reinsurer. All these observations or findings reflect the conflicting interests between the insurer and the reinsurer.

In addition, we also point out that if an insurer or an insurance has a 'greedy' requirement in a reinsurance contact, such as a very small value of γ or a very large value of σ and κ in Examples 3.1 and 3.2, then the best Pareto-optimal reinsurance contracts may not exist. Indeed, the insurer and the reinsurer are not able to make a deal of reinsurance if any of the two parties has a 'greedy' requirement in a reinsurance contact.

3.5 Conclusions

In this chapter, we give a comprehensive study of Pareto-optimal reinsurance arrangements and show that under general model settings and assumptions, a Pareto-optimal reinsurance contract is an optimizer of the convex combination of both parties' preferences, and such optimizers always exist. This result helps to justify many existing research techniques on the joint optimization problems for an insurer and a reinsurer. Moreover, we show how to solve an optimal reinsurance problem by minimizing the convex combination of TVaRs of the insurer's and the reinsurer's losses and to find the best Pareto-optimal reinsurance contracts in the sense that both the insurer's aim and the reinsurer's goal can be satisfied.

3.6 Technical Details

The proofs of Theorems 3.8 and 3.9 are given in this section.

3.6.1 Proof of Theorem 3.8

Proof. (i) This case is proved in Section 3.3.

(ii) If $0 \leqslant \lambda < \frac{1}{2}$ and $\frac{1-\lambda}{1-\beta} = m$, then $m > 0$. For any $I \in \mathcal{I}$, define

$$\hat{I}(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant \xi_a; \\ \xi_a & \text{if } \xi_a \leqslant x \leqslant \mathrm{VaR}_\alpha(X); \\ x - \mathrm{VaR}_\alpha(X) + \xi_a & \text{if } \mathrm{VaR}_\alpha(X) \leqslant x \leqslant \xi_b - \xi_a + \mathrm{VaR}_\alpha(X); \\ \xi_b & \text{if } \xi_b - \xi_a + \mathrm{VaR}_\alpha(X) \leqslant x \leqslant \mathrm{VaR}_\beta(X); \\ \tilde{I}(x) & \text{if } x > \mathrm{VaR}_\beta(X), \end{cases}$$

where $\tilde{I}$ can be any function such that $\hat{I} \in \mathcal{I}$.

The conditions $0 \leqslant \lambda < \frac{1}{2}$ and $\frac{1-\lambda}{1-\beta} = m$ imply $(1-\alpha)(1+\theta) \geqslant 1$. Then $\xi_a^* = \mathrm{VaR}_\alpha(X)$ and $\xi_b^* = \mathrm{VaR}_p(X)$. The optimal reinsurance contract is $I^*(x) = (x \wedge \mathrm{VaR}_p(X))\mathbb{I}_{\{x \leqslant \mathrm{VaR}_\beta(X)\}} + \tilde{I}(x)\mathbb{I}_{\{x > \mathrm{VaR}_\beta(X)\}}$.

(iii) If $0 \leqslant \lambda < \frac{1}{2}$ and $\frac{1-\lambda}{1-\beta} < m$, then $m > 0$. For any $I \in \mathcal{I}$, define

$$\hat{I}(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant \xi_a; \\ \xi_a & \text{if } \xi_a \leqslant x \leqslant \mathrm{VaR}_\alpha(X); \\ x - \mathrm{VaR}_\alpha(X) + \xi_a & \text{if } \mathrm{VaR}_\alpha(X) \leqslant x \leqslant \mathrm{VaR}_\alpha(X) + \xi_b - \xi_a; \\ \xi_b & \text{if } \mathrm{VaR}_\alpha(X) + \xi_b - \xi_a \leqslant x \leqslant \mathrm{VaR}_\beta(X); \\ x - \mathrm{VaR}_\beta(X) + \xi_b & \text{if } x \geqslant \mathrm{VaR}_\beta(X). \end{cases}$$

One can show that $\hat{I}(x) \in \mathcal{I}$, $V(I) \geqslant V(\hat{I})$ from (3.3.7), and

$$\mathbb{P}\left(\hat{I}(X) > x\right) = \begin{cases} \mathbb{P}\left(X > x\right) & \text{if } 0 \leqslant x < \xi_a; \\ \mathbb{P}\left(X > x + \mathrm{VaR}_\alpha(X) - \xi_a\right) & \text{if } \xi_a \leqslant x < \xi_b; \\ \mathbb{P}\left(X > x + \mathrm{VaR}_\beta(X) - \xi_b\right) & \text{if } x \geqslant \xi_b. \end{cases}$$

Let $s = \xi_a$ and $t = \xi_b - \xi_a$. It follows from (3.3.8)

$$\begin{aligned} V(\hat{I}) = {} & \lambda \mathrm{TVaR}_\alpha(X) + (1-2\lambda)s + (1-\lambda)t - (1-2\lambda)(1+\theta)\int_0^s \mathbb{P}(X > z)\mathrm{d}z \\ & - m\int_{\mathrm{VaR}_\alpha(X)}^{t+\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z + \left(\frac{1-\lambda}{1-\beta} - m\right)\int_{\mathrm{VaR}_\beta(X)}^\infty \mathbb{P}(X > z)\mathrm{d}z. \end{aligned}$$

Let

$$g(t) = (1-\lambda)t - m\int_{\mathrm{VaR}_\alpha(X)}^{t+\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z.$$

Then for $0 < t_1 < t_2 < \mathrm{VaR}_\beta(X) - \mathrm{VaR}_\alpha(X)$, as $\mathbb{P}(X \leqslant t_2 + \mathrm{VaR}_\alpha(X)) < \beta$, we have

$$\begin{aligned} g(t_1) - g(t_2) &= (1-\lambda)(t_1 - t_2) + m\int_{t_1+\mathrm{VaR}_\alpha(X)}^{t_2+\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z \\ &\geqslant (t_2 - t_1)\left(m\mathbb{P}(X > t_2 + \mathrm{VaR}_\alpha(X)) - (1-\lambda)\right) > 0. \end{aligned}$$

The conditions $0 \leqslant \lambda < \frac{1}{2}$ and $\frac{1-\lambda}{1-\beta} < m$ implies $(1-\alpha)(1+\theta) > 1$, $s^* = \mathrm{VaR}_\alpha(X)$ and $t^* = \mathrm{VaR}_\beta(X) - \mathrm{VaR}_\alpha(X)$ minimize $V(\hat{I})$. Therefore, $\xi_a^* = \mathrm{VaR}_\alpha(X)$ and $\xi_b^* = \mathrm{VaR}_\beta(X)$ minimize $V(\hat{I})$ and the optimal reinsurance contract is $I^*(x) = x$.

(iv) If $\lambda = \frac{1}{2}$, then $m = \frac{1}{2(1-\alpha)}$ and $p = \alpha$. Furthermore, if $\alpha = \beta$, then $V(I) = \frac{1}{2}\mathrm{TVaR}_\alpha(X)$ for any $I \in \mathcal{I}$; if $\alpha < \beta$, then for any $I \in \mathcal{I}$,

$$V(I) = \frac{1}{2}\mathrm{TVaR}_\alpha(X) - \frac{1}{2(1-\alpha)}\int_\alpha^\beta I(\mathrm{VaR}_r(X))\mathrm{d}r + \frac{1}{2}\left(\frac{1}{1-\beta} - \frac{1}{1-\alpha}\right)\int_\beta^1 I(\mathrm{VaR}_r(X))\mathrm{d}r, \tag{3.6.1}$$

or equivalently

$$\begin{aligned} V(I) = &\frac{1}{2}\mathrm{TVaR}_\alpha(X) + \frac{1}{2}(\xi_b - \xi_a) - \frac{1}{2(1-\alpha)}\int_{\xi_a}^{\xi_b} \mathbb{P}(I(X) > z)\mathrm{d}z \\ &+ \frac{1}{2}\left(\frac{1}{1-\beta} - \frac{1}{1-\alpha}\right)\int_{\xi_b}^\infty \mathbb{P}(I(X) > z)\mathrm{d}z. \end{aligned} \tag{3.6.2}$$

where $\xi_a = I(\mathrm{VaR}_\alpha(X))$ and $\xi_b = I(\mathrm{VaR}_\beta(X))$. Define

$$\hat{I}(x) = \begin{cases} \tilde{I}(x) & \text{if } 0 \leqslant x \leqslant \mathrm{VaR}_\alpha(X); \\ x - \mathrm{VaR}_\alpha(X) + \xi_a & \text{if } \mathrm{VaR}_\alpha(X) \leqslant x \leqslant \xi_b - \xi_a + \mathrm{VaR}_\alpha(X); \\ \xi_b & \text{if } x \geqslant \xi_b - \xi_a + \mathrm{VaR}_\alpha(X), \end{cases}$$

where $\tilde{I}$ can be any function such that $\hat{I} \in \mathcal{I}$. According to (3.6.1), it is easy to show that $V(I) \geqslant V(\hat{I})$. By (3.6.2), we have

$$V(\hat{I}) = \frac{1}{2}\mathrm{TVaR}_\alpha(X) + \frac{1}{2}(\xi_b - \xi_a) - \frac{1}{2(1-\alpha)}\int_{\mathrm{VaR}_\alpha(X)}^{\mathrm{VaR}_\alpha(X)+\xi_b-\xi_a} \mathbb{P}(X > z)\mathrm{d}z.$$

Let $t = \xi_b - \xi_a$ and $g(t) = t - \frac{1}{(1-\alpha)}\int_{\mathrm{VaR}_\alpha(X)}^{t+\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z$. For $t_1, t_2 \in [0, \mathrm{VaR}_\beta(X) -$

$\mathrm{VaR}_\alpha(X)]$ and $t_1 \geqslant t_2$,

$$\begin{aligned} g(t_2) - g(t_1) &= t_2 - t_1 + \frac{1}{1-\alpha} \int_{t_2+\mathrm{VaR}_\alpha(X)}^{t_1+\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z \\ &\leqslant (t_1 - t_2)\left[\frac{1}{1-\alpha}\mathbb{P}(X > t_2 + \mathrm{VaR}_\alpha(X)) - 1\right] \leqslant 0. \end{aligned}$$

Therefore, g is increasing in $t \in [0, \mathrm{VaR}_\beta(X) - \mathrm{VaR}_\alpha(X)]$ and $t^* = 0$ minimizes g. The optimal solution is $I^*(x) = \tilde{I}(x) \wedge \tilde{I}(\mathrm{VaR}_\alpha(X))$, where $\tilde{I}$ can be any function such that $I^* \in \mathcal{I}$.

(v) If $\frac{1}{2} < \lambda < 1$ and $\frac{1-\lambda}{1-\beta} > m > 0$, for any $I \in \mathcal{I}$, define

$$\hat{I}(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant \mathrm{VaR}_\alpha(X) - \xi_a; \\ x - \mathrm{VaR}_\alpha(X) + \xi_a & \text{if } \mathrm{VaR}_\alpha(X) - \xi_a \leqslant x \leqslant \xi_b - \xi_a + \mathrm{VaR}_\alpha(X); \\ \xi_b & \text{if } x \geqslant \xi_b - \xi_a + \mathrm{VaR}_\alpha(X). \end{cases} \tag{3.6.3}$$

We can show that $\hat{I}(x) \in \mathcal{I}$, $V(I) \geqslant V(\hat{I})$ from (3.3.7), and

$$\mathbb{P}\left(\hat{I}(X) > x\right) = \begin{cases} \mathbb{P}\left(X > x + \mathrm{VaR}_\alpha(X) - \xi_a\right) & \text{if } 0 \leqslant x < \xi_b; \\ 0 & \text{if } x \geqslant \xi_b. \end{cases}$$

Let $s = \xi_a$ and $t = \xi_b - \xi_a$. It follows from (3.3.8)

$$\begin{aligned} V(\hat{I}) = {}& \lambda \mathrm{TVaR}_\alpha(X) + (1-2\lambda)s + (1-\lambda)t - (1-2\lambda)(1+\theta)\int_{\mathrm{VaR}_\alpha(X)-s}^{\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z \\ & - m\int_{\mathrm{VaR}_\alpha(X)}^{t+\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z. \end{aligned}$$

If $(1-\alpha)(1+\theta) \geqslant 1$, then $1 - \theta^* \geqslant \alpha$ and $s^* = 0$. Moreover,

$$(1-\alpha)(1+\theta) \geqslant 1 \Longleftrightarrow p \leqslant \alpha. \tag{3.6.4}$$

Indeed, recall that $p = 1 - \frac{1-\lambda}{m}$ and $m = \frac{\lambda}{1-\alpha} + (1-2\lambda)(1+\theta) > 0$. It is equivalent to show that $(1-\alpha)m \leqslant 1-\lambda$, i.e., $\lambda + (1-2\lambda)(1+\theta)(1-\alpha) \leqslant 1-\lambda$, which is $(1-2\lambda)(1+\theta)(1-\alpha) \leqslant 1-2\lambda$, or $(1+\theta)(1-\alpha) \geqslant 1$ since $1-2\lambda < 0$. Thus, $p \leqslant \alpha$. One can show that $t^* = 0$. As a result, $\xi_a^* = 0$ and $\xi_b^* = 0$. Thus, $I^*(x) = 0$ is an optimal contract.

If $(1-\alpha)(1+\theta) < 1$, then $1-\theta^* < \alpha$ and from (3.6.4), we know $p > \alpha > 1-\theta^*$ and $\frac{1-\lambda}{1-\beta} > m$ implies $p < \beta$. Therefore, $s^* = \mathrm{VaR}_\alpha(X) - \mathrm{VaR}_{1-\theta^*}(X)$ and $t^* = \mathrm{VaR}_p(X) -$

$\mathrm{VaR}_{\alpha}(X)$ minimize $V(\hat{I})$, which implies $\xi_a^* = \mathrm{VaR}_{\alpha}(X) - \mathrm{VaR}_{1-\theta^*}(X)$ and $\xi_b^* = \mathrm{VaR}_p(X) - \mathrm{VaR}_{1-\theta^*}(X)$. Hence, $I^*(x) = (x - \mathrm{VaR}_{1-\theta^*}(X))_+ \wedge (\mathrm{VaR}_p(X) - \mathrm{VaR}_{1-\theta^*}(X))$ is an optimal reinsurance contract.

(vi) If $\frac{1}{2} < \lambda < 1$ and $\frac{1-\lambda}{1-\beta} = m > 0$ and note that $\xi_b - \xi_a + \mathrm{VaR}_{\alpha}(X) \leqslant \mathrm{VaR}_{\beta}(X)$, for any $I \in \mathcal{I}$, define $\hat{I}(x)$ the same as in (3.6.3) for $x \leqslant \mathrm{VaR}_{\beta}(X)$ and define $\hat{I}(x) = \tilde{I}(x)$ for $x \geqslant \mathrm{VaR}_{\beta}(X)$, where $\tilde{I}$ can be any function such that $\hat{I} \in \mathcal{I}$. If $\alpha = \beta$, then $\frac{1-\lambda}{1-\beta} = m$ implies $(1-\alpha)(1+\theta) = 1$ and $p = \alpha$, the optimal reinsurance contract is $I^*(x) = \tilde{I}(x)\mathbb{I}_{\{x > \mathrm{VaR}_{\beta}(X)\}}$. If $\alpha < \beta$, then $\frac{1-\lambda}{1-\beta} = m$ implies $(1-\alpha)(1+\theta) < 1$, the optimal contract is

$$I^*(x) = \left[(x - \mathrm{VaR}_{1-\theta^*}(X))_+ \wedge (\mathrm{VaR}_p(X) - \mathrm{VaR}_{1-\theta^*}(X))\right] \mathbb{I}_{\{x \leqslant \mathrm{VaR}_{\beta}(X)\}} + \tilde{I}(x)\mathbb{I}_{\{x > \mathrm{VaR}_{\beta}(X)\}}. \tag{3.6.5}$$

Hence, in either case, the optimal contract is given in (3.6.5). Note $p = \beta$ since $\frac{1-\lambda}{1-\beta} = m$.

(vii) If $\frac{1}{2} < \lambda \leqslant 1$ and $\frac{1-\lambda}{1-\beta} < m$, for any $I \in \mathcal{I}$, define

$$\hat{I}(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant \mathrm{VaR}_{\alpha}(X) - \xi_a; \\ x - \mathrm{VaR}_{\alpha}(X) + \xi_a & \text{if } \mathrm{VaR}_{\alpha}(X) - \xi_a \leqslant x \leqslant \xi_b - \xi_a + \mathrm{VaR}_{\alpha}(X); \\ \xi_b & \text{if } \xi_b - \xi_a + \mathrm{VaR}_{\alpha}(X) \leqslant x \leqslant \mathrm{VaR}_{\beta}(X); \\ x - \mathrm{VaR}_{\beta}(X) + \xi_b & \text{if } x \geqslant \mathrm{VaR}_{\beta}(X). \end{cases}$$

One can show that $\hat{I}(x) \in \mathcal{I}$, $V(I) \geqslant V(\hat{I})$ from (3.3.7), and

$$\mathbb{P}\left(\hat{I}(X) > x\right) = \begin{cases} \mathbb{P}\left(X > x + \mathrm{VaR}_{\alpha}(X) - \xi_a\right) & \text{if } 0 \leqslant x < \xi_b; \\ \mathbb{P}\left(X > x + \mathrm{VaR}_{\beta}(X) - \xi_b\right) & \text{if } x \geqslant \xi_b. \end{cases}$$

Let $s = \xi_a$ and $t = \xi_b - \xi_a$. It follows from (3.3.8)

$$\begin{aligned} V(\hat{I}) = {} & \lambda \mathrm{TVaR}_{\alpha}(X) + (1-2\lambda)s + (1-\lambda)t - (1-2\lambda)(1+\theta)\int_{\mathrm{VaR}_{\alpha}(X)-s}^{\mathrm{VaR}_{\alpha}(X)} \mathbb{P}(X > z)\mathrm{d}z \\ & - m\int_{\mathrm{VaR}_{\alpha}(X)}^{t+\mathrm{VaR}_{\alpha}(X)} \mathbb{P}(X > z)\mathrm{d}z + \left(\frac{1-\lambda}{1-\beta} - m\right)\int_{\mathrm{VaR}_{\beta}(X)}^{\infty} \mathbb{P}(X > z)\mathrm{d}z. \end{aligned}$$

As $1 - \beta \leqslant \mathbb{P}\left(X > t + \mathrm{VaR}_{\alpha}(X)\right) \leqslant 1 - \alpha$ and $m > 0$, one can show that $t^* = \mathrm{VaR}_{\beta}(X) - \mathrm{VaR}_{\alpha}(X)$. The conditions $\frac{1}{2} < \lambda \leqslant 1$ and $\frac{1-\lambda}{1-\beta} < m$ implies $(1-\alpha)(1+\theta) < 1$, thus $1 - \theta^* < \alpha \leqslant \beta$. $s^* = \mathrm{VaR}_{\alpha}(X) - \mathrm{VaR}_{1-\theta^*}(X)$ minimizes f. Hence, $\xi_a^* = \mathrm{VaR}_{\alpha}(X) - \mathrm{VaR}_{1-\theta^*}(X)$ and $\xi_b^* = \mathrm{VaR}_{\beta}(X) - \mathrm{VaR}_{1-\theta^*}(X)$ minimize $V(\hat{I})$. Thus, $I^*(x) = (x - \mathrm{VaR}_{1-\theta^*}(X))_+$ is an optimal reinsurance contract.

(viii) If $\frac{1}{2} < \lambda < 1$ and $m = 0$, then $\frac{1-\lambda}{1-\beta} - m > 0$. For any $I \in \mathcal{I}$, define

$$\hat{I}(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant \mathrm{VaR}_\alpha(X) - \xi_a; \\ x - \mathrm{VaR}_\alpha(X) + \xi_a & \text{if } \mathrm{VaR}_\alpha(X) - \xi_a \leqslant x \leqslant \mathrm{VaR}_\alpha(X); \\ \tilde{I}(x) & \text{if } \mathrm{VaR}_\alpha(X) \leqslant x \leqslant \mathrm{VaR}_\beta(X); \\ \xi_b & \text{if } x \geqslant \mathrm{VaR}_\beta(X), \end{cases}$$

where $\tilde{I}$ can be any function such that $\hat{I} \in \mathcal{I}$. It follows from (3.3.8)

$$\begin{aligned} V(\hat{I}) = {} & \lambda \mathrm{TVaR}_\alpha(X) + (1-2\lambda)\xi_a + (1-\lambda)(\xi_b - \xi_a) \\ & - (1-2\lambda)(1+\theta) \int_{\mathrm{VaR}_\alpha(X)-\xi_a}^{\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z. \end{aligned}$$

Let $s = \xi_a$ and $t = \xi_b - \xi_a$. Then $t^* = 0$ as $1 - \lambda > 0$. Let

$$f(s) = \lambda \mathrm{TVaR}_\alpha(X) + (1-2\lambda)s - (1-2\lambda)(1+\theta) \int_{\mathrm{VaR}_\alpha(X)-s}^{\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z.$$

For $0 < s_1 < s_2 \leqslant \mathrm{VaR}_\alpha(X)$,

$$\begin{aligned} f(s_2) - f(s_1) &= (2\lambda - 1)\left[(1+\theta)\int_{\mathrm{VaR}_\alpha(X)-s_2}^{\mathrm{VaR}_\alpha(X)-s_1} \mathbb{P}(X > z)\mathrm{d}z - (s_2 - s_1)\right] \\ &\geqslant (2\lambda - 1)(s_2 - s_1)\left[(1+\theta)\mathbb{P}(X > \mathrm{VaR}_\alpha(X) - s_1) - 1\right] \\ &> (2\lambda - 1)(s_2 - s_1)\left[(1+\theta)(1-\alpha) - 1\right] > 0, \end{aligned}$$

where the last inequality holds since $m = 0$ implies $(1-\alpha)(1+\theta) = \frac{\lambda}{2\lambda - 1} > 1$. Therefore, $s^* = 0$ is the unique minimizer of f. Thus, $\xi_a^* = \xi_b^* = 0$ and the optimal contract is $I^* = 0$.

(ix) If $\lambda = 1$ and $m = 0$, then $(1-\alpha)(1+\theta) = 1$. For any $I \in \mathcal{I}$, define

$$\hat{I}(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant \mathrm{VaR}_\alpha(X) - \xi_a; \\ x - \mathrm{VaR}_\alpha(X) + \xi_a & \text{if } \mathrm{VaR}_\alpha(X) - \xi_a \leqslant x \leqslant \mathrm{VaR}_\alpha(X); \\ \tilde{I}(x) & \text{if } x \geqslant \mathrm{VaR}_\alpha(X), \end{cases}$$

where $\tilde{I}$ can be any function such that $\hat{I} \in \mathcal{I}$. It follows from (3.3.8)

$$V(\hat{I}) = \mathrm{TVaR}_\alpha(X) - \xi_a + (1+\theta) \int_{\mathrm{VaR}_\alpha(X)-\xi_a}^{\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z.$$

It is easy to show that $\xi_a^* = 0$ and the optimal reinsurance contract is $I^* = \tilde{I}(x)\mathbb{I}_{\{x \geqslant \mathrm{VaR}_\alpha(X)\}}$.

(x) If $\frac{1}{2} < \lambda \leqslant 1$ and $m < 0$, then $\frac{1-\lambda}{1-\beta} > m$. For any $I \in \mathcal{I}$, define

$$\hat{I}(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant \mathrm{VaR}_\alpha(X) - \xi_a; \\ x - \mathrm{VaR}_\alpha(X) + \xi_a & \text{if } \mathrm{VaR}_\alpha(X) - \xi_a \leqslant x \leqslant \mathrm{VaR}_\alpha(X); \\ \xi_a & \text{if } \mathrm{VaR}_\alpha(X) \leqslant x \leqslant \mathrm{VaR}_\beta(X) - (\xi_b - \xi_a); \\ x - \mathrm{VaR}_\beta(X) + \xi_b & \text{if } \mathrm{VaR}_\beta(X) - (\xi_b - \xi_a) \leqslant x \leqslant \mathrm{VaR}_\beta(X); \\ \xi_b & \text{if } x \geqslant \mathrm{VaR}_\beta(X). \end{cases} \tag{3.6.6}$$

One can show that $\hat{I}(x) \in \mathcal{I}$, $V(I) \geqslant V(\hat{I})$ from (3.3.7),, and

$$\mathbb{P}\left(\hat{I}(X) > x\right) = \begin{cases} \mathbb{P}\left(X > x + \mathrm{VaR}_\alpha(X) - \xi_a\right) & \text{if } 0 \leqslant x < \xi_a; \\ \mathbb{P}\left(X > x + \mathrm{VaR}_\beta(X) - \xi_b\right) & \text{if } \xi_a < x < \xi_b; \\ 0 & \text{if } x \geqslant \xi_b. \end{cases}$$

Let $s = \xi_a$ and $t = \xi_b - \xi_a$. It follows from (3.3.8)

$$\begin{aligned} V(\hat{I}) = {} & \lambda \mathrm{TVaR}_\alpha(X) + (1-2\lambda)s + (1-\lambda)t - (1-2\lambda)(1+\theta)\int_{\mathrm{VaR}_\alpha(X)-s}^{\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z \\ & - m\int_{\mathrm{VaR}_\beta(X)-t}^{\mathrm{VaR}_\beta(X)} \mathbb{P}(X > z)\mathrm{d}z. \end{aligned}$$

Note that $m < 0$. The conditions $\frac{1}{2} < \lambda \leqslant 1$ and $m < 0$ imply $(1-\alpha)(1+\theta) > 1$. Then $s^* = 0$ and $t^* = 0$. Hence, $\xi_a^* = \xi_b^* = 0$ and $I^*(x) = 0$ is an optimal reinsurance contract. $\square$

3.6.2 Proof of Theorem 3.9

Proof. The proof is similar to that of Theorem 3.8. For any $I \in \mathcal{I}$, define

$$V(I) = \lambda \mathrm{TVaR}_\alpha\left(X - I(X) + \pi\left(I\left(X\right)\right)\right) + (1-\lambda)\mathrm{TVaR}_\beta(I(X) - \pi\left(I\left(X\right)\right))$$

and denote by $\xi_a = I\left(\mathrm{VaR}_\alpha(X)\right)$ and $\xi_b = I(\mathrm{VaR}_\beta(X))$. Clearly $\xi_a - \xi_b \leqslant \mathrm{VaR}_\alpha(X) - \mathrm{VaR}_\beta(X)$ and $\xi_b \leqslant \mathrm{VaR}_\beta(X)$. Note that $V(I)$ can be written as

$$\begin{aligned} V(I) = {} & \lambda \mathrm{TVaR}_\alpha(X) - \lambda(\xi_a - \xi_b) + (1-2\lambda)\xi_b - (1-2\lambda)(1+\theta)\int_0^{\xi_b} \mathbb{P}(I(X) > z)\mathrm{d}z \\ & + \left[\frac{1-\lambda}{1-\beta} - (1-2\lambda)(1+\theta)\right]\int_{\xi_b}^{\xi_a} \mathbb{P}(I(X) > z)\mathrm{d}z + \left(\frac{1-\lambda}{1-\beta} - m\right)\int_{\xi_a}^{\infty} \mathbb{P}(I(X) > z)\mathrm{d}z \end{aligned} \tag{3.6.7}$$

or equivalently

$$V(I) = \lambda \mathrm{TVaR}_\alpha(X) - (1-2\lambda)(1+\theta)\int_0^\beta I(\mathrm{VaR}_r(X))\mathrm{d}r$$
$$+\left(\frac{1-\lambda}{1-\beta} - (1-2\lambda)(1+\theta)\right)\int_\beta^\alpha I(\mathrm{VaR}_r(X))\mathrm{d}r + \left(\frac{1-\lambda}{1-\beta} - m\right)\int_\alpha^1 I(\mathrm{VaR}_r(X))\mathrm{d}r. \tag{3.6.8}$$

Let $s = \xi_b$ and $t = \xi_a - \xi_b$. Then $0 \leqslant s \leqslant \mathrm{VaR}_\beta(X)$, $0 \leqslant t \leqslant \mathrm{VaR}_\alpha(X) - \mathrm{VaR}_\beta(X)$.

If $\lambda = 0$, then $m = 1+\theta > 0$. Equations (3.6.7) and (3.6.8) reduce to

$$\begin{aligned} V(I) &= \xi_b - (1+\theta)\int_0^{\xi_b} \mathbb{P}(I(X) > z)\mathrm{d}z + \left[\frac{1}{1-\beta} - (1+\theta)\right]\int_{\xi_b}^\infty \mathbb{P}(I(X) > z)\mathrm{d}z \\ &= -(1+\theta)\int_0^\beta I(\mathrm{VaR}_r(X))\mathrm{d}r + \left(\frac{1}{1-\beta} - (1+\theta)\right)\int_\beta^1 I(\mathrm{VaR}_r(X))\mathrm{d}r. \end{aligned}$$

(i) If $\lambda = 0$ and $(1-\beta)(1+\theta) > 1$, for any $I \in \mathcal{I}$, define

$$\hat{I}(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant \xi_b; \\ \xi_b & \text{if } \xi_b \leqslant x \leqslant \mathrm{VaR}_\beta(X); \\ x - \mathrm{VaR}_\beta(X) + \xi_b & \text{if } x \geqslant \mathrm{VaR}_\beta(X). \end{cases}$$

We have $\hat{I} \in \mathcal{I}$, $V(I) \geqslant V(\hat{I})$ from (3.6.8), and

$$\mathbb{P}\left(\hat{I}(X) > x\right) = \begin{cases} \mathbb{P}(X > x) & \text{if } 0 \leqslant x < \xi_b; \\ \mathbb{P}(X > x + \mathrm{VaR}_\beta(X) - \xi_b) & \text{if } x \geqslant \xi_b. \end{cases}$$

It follows from (3.6.7)

$$V(\hat{I}) = \xi_b - (1+\theta)\int_0^{\xi_b} \mathbb{P}(X > z)\mathrm{d}z + \left[\frac{1}{1-\beta} - (1+\theta)\right]\int_{\mathrm{VaR}_\beta(X)}^\infty \mathbb{P}(X > z)\mathrm{d}z.$$

It is easy to show that $V(\hat{I})$ is strictly decreasing in ξ_b and $\xi_b^* = \mathrm{VaR}_\beta(X)$ minimizes $V(\hat{I})$. The optimal reinsurance contract is $I^*(x) = x$.

(ii) If $\lambda = 0$ and $(1-\beta)(1+\theta) = 1$, for any $I \in \mathcal{I}$, define

$$\hat{I}(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant \xi_b; \\ \xi_b & \text{if } \xi_b \leqslant x \leqslant \mathrm{VaR}_\beta(X); \\ \tilde{I}(x) & \text{if } x \geqslant \mathrm{VaR}_\beta(X), \end{cases}$$

where $\tilde{I}$ can be any function such that $\hat{I} \in \mathcal{I}$. Clearly $V(I) \geqslant V(\hat{I})$ from (3.6.8), and $\mathbb{P}\left(\hat{I}(X) > x\right) = \mathbb{P}(X > x)$ for $0 \leqslant x < \xi_b$. It follows from the above case that $\xi_b^* = \mathrm{VaR}_\beta(X)$ minimizes $V(\hat{I})$. The optimal reinsurance contract is $I^*(x) = x\mathbb{I}_{\{x \leqslant \mathrm{VaR}_\beta(X)\}} + \tilde{I}(x)\mathbb{I}_{\{x > \mathrm{VaR}_\beta(X)\}}$.

(iii) $0 \leqslant \lambda < \frac{1}{2}$ and $\frac{1-\lambda}{1-\beta} > m$. If $\lambda = 0$, then $\frac{1-\lambda}{1-\beta} > m$ implies $(1-\beta)(1+\theta) < 1$, for any $I \in \mathcal{I}$, define

$$\hat{I}(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant \xi_b; \\ \xi_b & \text{if } x \geqslant \xi_b. \end{cases}$$

We have $\hat{I} \in \mathcal{I}$, $V(I) \geqslant V(\hat{I})$ from (3.6.8), and

$$\mathbb{P}\left(\hat{I}(X) > x\right) = \begin{cases} \mathbb{P}(X > x) & \text{if } 0 \leqslant x < \xi_b; \\ 0 & \text{if } x \geqslant \xi_b. \end{cases}$$

It follows that $\xi_b^* = \mathrm{VaR}_{1-\theta^*}(X)$ and the optimal contract is $I^*(x) = x \wedge \mathrm{VaR}_{1-\theta^*}(X)$.

If $0 < \lambda < \frac{1}{2}$ and $\frac{1-\lambda}{1-\beta} > m$, then $m > 0$ and $\frac{1-\lambda}{1-\beta} - (1-2\lambda)(1+\theta) > 0$. For any $I \in \mathcal{I}$, define

$$\hat{I}(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant \xi_b; \\ \xi_b & \text{if } \xi_b \leqslant x \leqslant \mathrm{VaR}_\alpha(X) - (\xi_a - \xi_b); \\ x - \mathrm{VaR}_\alpha(X) + \xi_a & \text{if } \mathrm{VaR}_\alpha(X) - (\xi_a - \xi_b) \leqslant x \leqslant \mathrm{VaR}_\alpha(X); \\ \xi_a & \text{if } x \geqslant \mathrm{VaR}_\alpha(X). \end{cases}$$

We have $\hat{I} \in \mathcal{I}$, $V(I) \geqslant V(\hat{I})$ from (3.6.8), and

$$\mathbb{P}\left(\hat{I}(X) > x\right) = \begin{cases} \mathbb{P}(X > x) & \text{if } 0 \leqslant x < \xi_b; \\ \mathbb{P}(X > x + \mathrm{VaR}_\alpha(X) - \xi_a) & \text{if } \xi_b \leqslant x < \xi_a; \\ 0 & \text{if } x \geqslant \xi_a. \end{cases}$$

It follows from (3.6.7)

$$\begin{aligned} V(\hat{I}) &= \lambda \mathrm{TVaR}_\alpha(X) - \lambda t + (1-2\lambda)s - (1-2\lambda)(1+\theta)\int_0^s \mathbb{P}(X > z)\mathrm{d}z \\ &\quad + \left[\frac{1-\lambda}{1-\beta} - (1-2\lambda)(1+\theta)\right]\int_{\mathrm{VaR}_\alpha(X)-t}^{\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z. \end{aligned} \tag{3.6.9}$$

The conditions $0 < \lambda < \frac{1}{2}$ and $\frac{1-\lambda}{1-\beta} > m$ imply $(1-\beta)(1+\theta) < 1$. Then $s^* = \mathrm{VaR}_{1-\theta^*}(X)$ and $t^* = 0$ minimize $V(\hat{I})$. The optimal reinsurance contract is $I^*(x) = x \wedge \mathrm{VaR}_{1-\theta^*}(X)$.

(iv) If $0 < \lambda < \frac{1}{2}$ and $\frac{1-\lambda}{1-\beta} = m$, then $\frac{1-\lambda}{1-\beta} - (1-2\lambda)(1+\theta) > 0$. For any $I \in \mathcal{I}$, define

$$\hat{I}(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant \xi_b; \\ \xi_b & \text{if } \xi_b \leqslant x \leqslant \mathrm{VaR}_\alpha(X) - (\xi_a - \xi_b); \\ x - \mathrm{VaR}_\alpha(X) + \xi_a & \text{if } \mathrm{VaR}_\alpha(X) - (\xi_a - \xi_b) \leqslant x \leqslant \mathrm{VaR}_\alpha(X); \\ \tilde{I}(x) & \text{if } x \geqslant \mathrm{VaR}_\alpha(X), \end{cases}$$

where $\tilde{I}$ can be any function such that $\hat{I} \in \mathcal{I}$. Clearly $V(I) \geqslant V(\hat{I})$ from (3.6.8), and $V(\hat{I})$ is the same as (3.6.9). The conditions $0 < \lambda < \frac{1}{2}$ and $\frac{1-\lambda}{1-\beta} = m$ imply $(1-\beta)(1+\theta) \leqslant 1$. It follows that the optimal reinsurance contract is $I^*(x) = x \wedge \mathrm{VaR}_{1-\theta^*}(X)\mathbb{I}_{\{x \leqslant \mathrm{VaR}_\alpha(X)\}} + \tilde{I}(x)\mathbb{I}_{\{x > \mathrm{VaR}_\alpha(X)\}}$.

(v) If $0 < \lambda < \frac{1}{2}$ and $(1-2\lambda)(1+\theta) < \frac{1-\lambda}{1-\beta} < m$, for any $I \in \mathcal{I}$, define

$$\hat{I}(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant \xi_b; \\ \xi_b & \text{if } \xi_b \leqslant x \leqslant \mathrm{VaR}_\alpha(X) - (\xi_a - \xi_b); \\ x - \mathrm{VaR}_\alpha(X) + \xi_a & \text{if } x \geqslant \mathrm{VaR}_\alpha(X) - (\xi_a - \xi_b). \end{cases}$$

One can show that $\hat{I}(x) \in \mathcal{I}$, $V(I) \geqslant V(\hat{I})$ from (3.6.8), and

$$\mathbb{P}\left(\hat{I}(X) > x\right) = \begin{cases} \mathbb{P}\left(X > x\right) & \text{if } 0 \leqslant x < \xi_b; \\ \mathbb{P}\left(X > x + \mathrm{VaR}_\alpha(X) - \xi_a\right) & \text{if } x \geqslant \xi_b. \end{cases}$$

It follows from (3.6.7)

$$\begin{aligned} V(\hat{I}) &= \lambda \mathrm{TVaR}_\alpha(X) - \lambda t + (1-2\lambda)s - (1-2\lambda)(1+\theta)\int_0^s \mathbb{P}(X > z)\mathrm{d}z \\ &\quad + \left[\frac{1-\lambda}{1-\beta} - (1-2\lambda)(1+\theta)\right]\int_{\mathrm{VaR}_\alpha(X)-t}^{\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z + \left(\frac{1-\lambda}{1-\beta} - m\right)\int_{\mathrm{VaR}_\alpha(X)}^{\infty} \mathbb{P}(X > z)\mathrm{d}z. \end{aligned}$$

If $(1-\beta)(1+\theta) \geqslant 1$, $s^* = \mathrm{VaR}_\beta(X)$, $t^* = \mathrm{VaR}_\alpha(X) - \mathrm{VaR}_\beta(X)$, and the optimal reinsurance contract is $I^*(x) = x$.

If $(1-\beta)(1+\theta) < 1$, $s^* = \mathrm{VaR}_{1-\theta^*}(X)$ minimizes f. Recall q defined in (3.3.5). We claim that $\beta < q < \alpha$. Indeed, $0 < \frac{1-\lambda}{1-\beta} - (1-2\lambda)(1+\theta) < \frac{\lambda}{1-\alpha}$ implies $1 - \alpha < \frac{\lambda}{\frac{1-\lambda}{1-\beta} - (1-2\lambda)(1+\theta)}$, that is, $q < \alpha$. Besides, $\beta < q$ is equivalent to $\lambda < 1 - \lambda - (1-2\lambda)(1+\theta)(1-\beta)$, which is $0 < (1-2\lambda)\left[1 - (1+\theta)(1-\beta)\right]$, that is, $(1-\beta)(1+\theta) < 1$ as $1 - 2\lambda > 0$. Thus, $\beta < q < \alpha$. Hence, $t^* = \mathrm{VaR}_\alpha(X) - \mathrm{VaR}_q(X)$ and $I^*(x) = x \wedge \mathrm{VaR}_{1-\theta^*}(X) + \left(x - \mathrm{VaR}_q(X)\right)_+$ is an optimal reinsurance contract.

(vi) If $0 < \lambda < \frac{1}{2}$ and $(1-2\lambda)(1+\theta) = \frac{1-\lambda}{1-\beta}$, then $\frac{1-\lambda}{1-\beta} < m$. For any $I \in \mathcal{I}$, define

$$\hat{I}(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant \xi_b; \\ \xi_b & \text{if } \xi_b \leqslant x \leqslant \mathrm{VaR}_\beta(X); \\ \tilde{I}(x) & \text{if } \mathrm{VaR}_\beta(X) \leqslant x \leqslant \mathrm{VaR}_\alpha(X); \\ x - \mathrm{VaR}_\alpha(X) + \xi_a & \text{if } x \geqslant \mathrm{VaR}_\alpha(X), \end{cases}$$

where $\tilde{I}$ can be any function such that $\hat{I} \in \mathcal{I}$. Then $V(I) \geqslant V(\hat{I})$ from (3.6.8), and

$$\mathbb{P}\left(\hat{I}(X) > x\right) = \begin{cases} \mathbb{P}\left(X > x\right) & \text{if } 0 \leqslant x < \xi_b; \\ \mathbb{P}\left(X > x + \mathrm{VaR}_\alpha(X) - \xi_a\right) & \text{if } x \geqslant \xi_a. \end{cases}$$

It follows from (3.6.7)

$$\begin{aligned} V(\hat{I}) &= \lambda \mathrm{TVaR}_\alpha(X) - \lambda t + (1-2\lambda)s - (1-2\lambda)(1+\theta)\int_0^s \mathbb{P}(X > z)\mathrm{d}z \\ &\quad + \left(\frac{1-\lambda}{1-\beta} - m\right)\int_{\mathrm{VaR}_\alpha(X)}^\infty \mathbb{P}(X > z)\mathrm{d}z. \end{aligned}$$

Clearly $t^* = \mathrm{VaR}_\alpha(X) - \mathrm{VaR}_\beta(X)$. Since $(1-2\lambda)(1+\theta) = \frac{1-\lambda}{1-\beta}$ implies $(1-\beta)(1+\theta) = \frac{1-\lambda}{1-2\lambda} = 1 + \frac{\lambda}{1-2\lambda} > 1$, similar analysis to the case (iii) yields $s^* = \mathrm{VaR}_\beta(X)$ and the optimal reinsurance contract is $I^*(x) = x\,\mathbb{I}_{\{x \leqslant \mathrm{VaR}_\beta(X) \text{ or } x \geqslant \mathrm{VaR}_\alpha(X)\}} + \tilde{I}(x)\,\mathbb{I}_{\{\mathrm{VaR}_\beta(X) \leqslant x \leqslant \mathrm{VaR}_\alpha(X)\}}$.

(vii) If $0 < \lambda < \frac{1}{2}$ and $\frac{1-\lambda}{1-\beta} < (1-2\lambda)(1+\theta)$, then $\frac{1-\lambda}{1-\beta} - m < 0$. For any $I \in \mathcal{I}$, define

$$\hat{I}(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant \xi_b; \\ \xi_b & \text{if } \xi_b \leqslant x \leqslant \mathrm{VaR}_\beta(X); \\ x - \mathrm{VaR}_\beta(X) + \xi_b & \text{if } \mathrm{VaR}_\beta(X) \leqslant x \leqslant \mathrm{VaR}_\beta(X) + \xi_a - \xi_b; \\ \xi_a & \text{if } \mathrm{VaR}_\beta(X) + \xi_a - \xi_b \leqslant x \leqslant \mathrm{VaR}_\alpha(X); \\ x - \mathrm{VaR}_\alpha(X) + \xi_a & \text{if } x \geqslant \mathrm{VaR}_\alpha(X). \end{cases}$$

One can show that $\hat{I}(x) \in \mathcal{I}$, $V(I) \geqslant V(\hat{I})$ from (3.6.8),

$$\mathbb{P}\left(\hat{I}(X) > x\right) = \begin{cases} \mathbb{P}\left(X > x\right) & \text{if } 0 \leqslant x < \xi_b; \\ \mathbb{P}\left(X > x + \mathrm{VaR}_\beta(X) - \xi_b\right) & \text{if } \xi_b \leqslant x < \xi_a; \\ \mathbb{P}\left(X > x + \mathrm{VaR}_\alpha(X) - \xi_a\right) & \text{if } x \geqslant \xi_a. \end{cases}$$

It follows from (3.6.7)

$$
\begin{aligned}
V(\hat{I}) = \lambda \mathrm{TVaR}_\alpha(X) - \lambda t + (1-2\lambda)s - (1-2\lambda)(1+\theta)\int_0^s \mathbb{P}(X > z)\mathrm{d}z \\
+ \left[\frac{1-\lambda}{1-\beta} - (1-2\lambda)(1+\theta)\right]\int_{\mathrm{VaR}_\beta(X)}^{\mathrm{VaR}_\beta(X)+t} \mathbb{P}(X > z)\mathrm{d}z \\
+ \left(\frac{1-\lambda}{1-\beta} - m\right)\int_{\mathrm{VaR}_\alpha(X)}^{\infty} \mathbb{P}(X > z)\mathrm{d}z.
\end{aligned}
$$

The conditions $0 < \lambda < \frac{1}{2}$ and $\frac{1-\lambda}{1-\beta} < (1-2\lambda)(1+\theta)$ imply $1-\beta(1+\theta) > \frac{1-\lambda}{1-2\lambda} > 1$. Hence, $s^* = \mathrm{VaR}_\beta(X)$ and $t^* = \mathrm{VaR}_\alpha(X) - \mathrm{VaR}_\beta(X)$ minimize $V(\hat{I})$. The optimal reinsurance contract is $I^*(x) = x$.

(viii) If $\lambda = \frac{1}{2}$, then $m = \frac{1}{2(1-\alpha)}$ and $q = \beta$. Furthermore, if $\alpha = \beta$, then $V(I) = \frac{1}{2}\mathrm{TVaR}_\alpha(X)$ for any $I \in \mathcal{I}$; if $\alpha > \beta$, then for any $I \in \mathcal{I}$,

$$
V(I) = \frac{1}{2}\mathrm{TVaR}_\alpha(X) + \frac{1}{2(1-\beta)}\int_\beta^\alpha I(\mathrm{VaR}_r(X))\mathrm{d}r + \frac{1}{2}\left(\frac{1}{1-\beta} - \frac{1}{1-\alpha}\right)\int_\alpha^1 I(\mathrm{VaR}_r(X))\mathrm{d}r \tag{3.6.10}
$$

or equivalently

$$
\begin{aligned}
V(I) = \frac{1}{2}\mathrm{TVaR}_\alpha(X) - \frac{1}{2}(\xi_a - \xi_b) + \frac{1}{2(1-\beta)}\int_{\xi_b}^{\xi_a} \mathbb{P}(I(X) > z)\mathrm{d}z \\
+ \frac{1}{2}\left(\frac{1}{1-\beta} - \frac{1}{1-\alpha}\right)\int_{\xi_a}^{\infty} \mathbb{P}(I(X) > z)\mathrm{d}z.
\end{aligned} \tag{3.6.11}
$$

where $\xi_a = I(\mathrm{VaR}_\alpha(X))$ and $\xi_b = I(\mathrm{VaR}_\beta(X))$. Define

$$
\hat{I}(x) = \begin{cases} \tilde{I}(x) & \text{if } 0 \leqslant x \leqslant \mathrm{VaR}_\beta(X); \\ \xi_b & \text{if } \mathrm{VaR}_\beta(X) \leqslant x \leqslant \mathrm{VaR}_\alpha(X) - (\xi_a - \xi_b); \\ x - \mathrm{VaR}_\alpha(X) + \xi_a & \text{if } x \geqslant \mathrm{VaR}_\alpha(X) - (\xi_a - \xi_b), \end{cases}
$$

where $\tilde{I}$ can be any function such that $\hat{I} \in \mathcal{I}$. According to (3.6.10), it is easy to show that $V(I) \geqslant V(\hat{I})$. By (3.6.11), we have

$$
\begin{aligned}
V(\hat{I}) = \frac{1}{2}\mathrm{TVaR}_\alpha(X) - \frac{1}{2}(\xi_a - \xi_b) + \frac{1}{2(1-\beta)}\int_{\mathrm{VaR}_\alpha(X)-(\xi_a-\xi_b)}^{\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z \\
+ \frac{1}{2}\left(\frac{1}{1-\beta} - \frac{1}{1-\alpha}\right)\int_{\mathrm{VaR}_\alpha(X)}^{\infty} \mathbb{P}(X > z)\mathrm{d}z.
\end{aligned}
$$

Let $t = \xi_a - \xi_b$ and $g(t) = -t + \frac{1}{(1-\beta)} \int_{\mathrm{VaR}_\alpha(X)-t}^{\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z$. For $t_1, t_2 \in [0, \mathrm{VaR}_\alpha(X) - \mathrm{VaR}_\beta(X)]$ and $t_1 \geqslant t_2$,

$$\begin{aligned} g(t_1) - g(t_2) &= t_2 - t_1 + \frac{1}{1-\beta} \int_{\mathrm{VaR}_\alpha(X)-t_1}^{\mathrm{VaR}_\alpha(X)-t_2} \mathbb{P}(X > z)\mathrm{d}z \\ &\leqslant (t_1 - t_2) \left[\frac{1}{1-\beta} \mathbb{P}(X > \mathrm{VaR}_\alpha(X) - t_1) - 1 \right] \leqslant 0. \end{aligned}$$

Therefore, g is decreasing in $t \in [0, \mathrm{VaR}_\alpha(X) - \mathrm{VaR}_\beta(X)]$ and $t^* = \mathrm{VaR}_\alpha(X) - \mathrm{VaR}_\beta(X)$ minimizes g. The optimal solution is

$$I^*(x) = \tilde{I}(x)\, \mathbb{I}_{\{x \leqslant \mathrm{VaR}_\beta(X)\}} + \Big(x - \mathrm{VaR}_\beta(X) + \tilde{I}(\mathrm{VaR}_\beta(X)) \Big)\, \mathbb{I}_{\{x > \mathrm{VaR}_\beta(X)\}},$$

where $\tilde{I}$ can be any function such that $I^* \in \mathcal{I}$.

(ix) If $\frac{1}{2} < \lambda \leqslant 1$ and $\frac{1-\lambda}{1-\beta} > m$, then $(1-2\lambda)(1+\theta) < 0$ and $\frac{1-\lambda}{1-\beta} - (1-2\lambda)(1+\theta) > 0$. For any $I \in \mathcal{I}$, define

$$\hat{I}(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant \mathrm{VaR}_\beta(X) - \xi_b; \\ x - \mathrm{VaR}_\beta(X) + \xi_b & \text{if } \mathrm{VaR}_\beta(X) - \xi_b \leqslant x \leqslant \mathrm{VaR}_\beta(X); \\ \xi_b & \text{if } \mathrm{VaR}_\beta(X) \leqslant x \leqslant \mathrm{VaR}_\alpha(X) - (\xi_a - \xi_b); \\ x - \mathrm{VaR}_\alpha(X) + \xi_a & \text{if } \mathrm{VaR}_\alpha(X) - (\xi_a - \xi_b) \leqslant x \leqslant \mathrm{VaR}_\alpha(X); \\ \xi_a & \text{if } x \geqslant \mathrm{VaR}_\alpha(X). \end{cases}$$

One can show that $\hat{I}(x) \in \mathcal{I}$, $V(I) \geqslant V(\hat{I})$ from (3.6.8), and

$$\mathbb{P}\left(\hat{I}(X) > x\right) = \begin{cases} \mathbb{P}\left(X > x + \mathrm{VaR}_\beta(X) - \xi_b\right) & \text{if } 0 \leqslant x < \xi_b; \\ \mathbb{P}\left(X > x + \mathrm{VaR}_\alpha(X) - \xi_a\right) & \text{if } \xi_b \leqslant x < \xi_a; \\ 0 & \text{if } x \geqslant \xi_a. \end{cases}$$

It follows from (3.6.7)

$$\begin{aligned} V(\hat{I}) = \lambda \mathrm{TVaR}_\alpha(X) - \lambda t + (1-2\lambda)s - (1-2\lambda)(1+\theta) \int_{\mathrm{VaR}_\beta(X)-s}^{\mathrm{VaR}_\beta(X)} \mathbb{P}(X > z)\mathrm{d}z \\ + \left[\frac{1-\lambda}{1-\beta} - (1-2\lambda)(1+\theta) \right] \int_{\mathrm{VaR}_\alpha(X)-t}^{\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z. \end{aligned}$$

One can show that $t^* = 0$ minimizes $V(\hat{I})$. The conditions $\frac{1}{2} < \lambda \leqslant 1$ and $\frac{1-\lambda}{1-\beta} > m$ imply $(1-\beta)(1+\theta) \geqslant 1$. Then $s^* = 0$ and the optimal reinsurance contract is $I^*(x) = 0$.

(x) If $\frac{1}{2} < \lambda \leqslant 1$ and $\frac{1-\lambda}{1-\beta} = m$, then $\frac{1-\lambda}{1-\beta} - (1-2\lambda)(1+\theta) > 0$. For any $I \in \mathcal{I}$, define

$$\hat{I}(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant \mathrm{VaR}_\beta(X) - \xi_b; \\ x - \mathrm{VaR}_\beta(X) + \xi_b & \text{if } \mathrm{VaR}_\beta(X) - \xi_b \leqslant x \leqslant \mathrm{VaR}_\beta(X); \\ \xi_b & \text{if } \mathrm{VaR}_\beta(X) \leqslant x \leqslant \mathrm{VaR}_\alpha(X) - (\xi_a - \xi_b); \\ x - \mathrm{VaR}_\alpha(X) + \xi_a & \text{if } \mathrm{VaR}_\alpha(X) - (\xi_a - \xi_b) \leqslant x \leqslant \mathrm{VaR}_\alpha(X); \\ \tilde{I}(x) & \text{if } x \geqslant \mathrm{VaR}_\alpha(X), \end{cases}$$

where $\tilde{I}$ can be any function such that $\hat{I} \in \mathcal{I}$. Then $V(I) \geqslant V(\hat{I})$ from (3.6.8), and

$$\mathbb{P}\left(\hat{I}(X) > x\right) = \begin{cases} \mathbb{P}\left(X > x + \mathrm{VaR}_\beta(X) - \xi_b\right) & \text{if } 0 \leqslant x < \xi_b; \\ \mathbb{P}\left(X > x + \mathrm{VaR}_\alpha(X) - \xi_a\right) & \text{if } \xi_b \leqslant x < \xi_a. \end{cases}$$

Then by (3.6.7), we have

$$\begin{aligned} V(\hat{I}) = {} & \lambda \mathrm{TVaR}_\alpha(X) - \lambda t + (1-2\lambda)s - (1-2\lambda)(1+\theta)\int_{\mathrm{VaR}_\beta(X)-s}^{\mathrm{VaR}_\beta(X)} \mathbb{P}(X > z)\mathrm{d}z \\ & + \left[\frac{1-\lambda}{1-\beta} - (1-2\lambda)(1+\theta)\right]\int_{\mathrm{VaR}_\alpha(X)-t}^{\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z. \end{aligned}$$

The conditions $\frac{1}{2} < \lambda \leqslant 1$ and $\frac{1-\lambda}{1-\beta} = m$ imply $(1-\beta)(1+\theta) \geqslant 1$. It follows from case (ix) that the optimal reinsurance contract is $I^*(x) = \tilde{I}(x)\mathbb{I}_{\{x \geqslant \mathrm{VaR}_\alpha(X)\}}$.

(xi) If $\frac{1}{2} < \lambda \leqslant 1$and $\frac{1-\lambda}{1-\beta} < m$, then $(1-2\lambda)(1+\theta) < 0$ and $\frac{1-\lambda}{1-\beta} - (1-2\lambda)(1+\theta) > 0$. For any $I \in \mathcal{I}$, define

$$\hat{I}(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant \mathrm{VaR}_\beta(X) - \xi_b; \\ x - \mathrm{VaR}_\beta(X) + \xi_b & \text{if } \mathrm{VaR}_\beta(X) - \xi_b \leqslant x \leqslant \mathrm{VaR}_\beta(X); \\ \xi_b & \text{if } \mathrm{VaR}_\beta(X) \leqslant x \leqslant \mathrm{VaR}_\alpha(X) - (\xi_a - \xi_b); \\ x - \mathrm{VaR}_\alpha(X) + \xi_a & \text{if } x \geqslant \mathrm{VaR}_\alpha(X) - (\xi_a - \xi_b). \end{cases}$$

One can show that $\hat{I}(x) \in \mathcal{I}$, $V(I) \geqslant V(\hat{I})$ from (3.6.8), and

$$\mathbb{P}\left(\hat{I}(X) > x\right) = \begin{cases} \mathbb{P}\left(X > x + \mathrm{VaR}_\beta(X) - \xi_b\right) & \text{if } 0 \leqslant x < \xi_b; \\ \mathbb{P}\left(X > x + \mathrm{VaR}_\alpha(X) - \xi_a\right) & \text{if } x \geqslant \xi_b. \end{cases}$$

Then by (3.6.7), we have

$$
\begin{aligned}
V(\hat{I}) = {} & \lambda \mathrm{TVaR}_\alpha(X) - \lambda t + (1-2\lambda)s - (1-2\lambda)(1+\theta)\int_{\mathrm{VaR}_\beta(X)-s}^{\mathrm{VaR}_\beta(X)} \mathbb{P}(X > z)\mathrm{d}z \\
& + \left[\frac{1-\lambda}{1-\beta} - (1-2\lambda)(1+\theta)\right]\int_{\mathrm{VaR}_\alpha(X)-t}^{\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z \\
& + \left(\frac{1-\lambda}{1-\beta} - m\right)\int_{\mathrm{VaR}_\alpha(X)}^{\infty} \mathbb{P}(X > z)\mathrm{d}z.
\end{aligned}
$$

If $(1-\beta)(1+\theta) \geqslant 1$, then $s^* = 0$. Note that $\frac{1-\lambda}{1-\beta} < m$ implies $q < \alpha$. In addition, $\lambda > 1/2$ and $(1-\beta)(1+\theta) \geqslant 1$ imply $\beta \leqslant q$. Thus, $t^* = \mathrm{VaR}_\alpha(X) - \mathrm{VaR}_q(X)$ minimizes $V(\hat{I})$ and $I^*(x) = (x - \mathrm{VaR}_q(X))_+$ is an optimal contract.

If $(1-\beta)(1+\theta) < 1$, then $s^* = \mathrm{VaR}_\beta(X) - \mathrm{VaR}_{1-\theta^*}(X)$. Note that $\mathbb{P}(X > \mathrm{VaR}_\alpha(X) - t) \leqslant 1-\beta$ implies $t^* = \mathrm{VaR}_\alpha(X) - \mathrm{VaR}_\beta(X)$. Thus, the optimal reinsurance contract is $I^*(x) = (x - \mathrm{VaR}_{1-\theta^*}(X))_+$. $\square$

Chapter 4

Asymptotic Equivalence of Risk Measures under Dependence Uncertainty

4.1 Introduction

An event is uncertain or ambiguous if its probability is unknown. In this chapter, one particular type of uncertainty that we focus on is the dependence uncertainty in risk aggregation. In the framework of dependence uncertainty, we assume that in a joint model $(X_1, \ldots, X_n)$, the marginal distribution of each of $X_1, \ldots, X_n$ is known, but the joint distribution is unknown. This is due to statistical and modeling challenges in obtaining precise information on the dependence structure of a joint model; see Embrechts et al. (2014) for more illustrations. Denote by $\mathcal{F}$ the set of univariate distribution functions. For $F_1, \ldots, F_n \in \mathcal{F}$, let

$$\mathcal{S}_n = \mathcal{S}_n(F_1, \ldots, F_n) = \{X_1 + \cdots + X_n : X_i \in L^0,\ X_i \sim F_i,\ i = 1, \ldots, n\}.$$

That is, $\mathcal{S}_n$ is the set of aggregate risks with given marginal distributions, but an arbitrary dependence structure. Some properties of the set $\mathcal{S}_n$ are given in Bernard et al. (2014).

For a given risk measure $\rho : \mathcal{X} \to (-\infty, +\infty]$, where the set $\mathcal{X}$ is a convex cone of risks, we are interested in the value of the risk aggregation $\rho(X_1 + \cdots + X_n)$ for some joint model

$(X_1, \ldots, X_n)$ with unknown dependence structure. When we implement the risk measure ρ to the aggregate risk $X = X_1 + \cdots + X_n$, dependence uncertainty always arises as an important issue in practice. Obviously, $\rho(X_1 + \cdots + X_n)$ lies in a range, and often the worst-case value and the best-case value are of particular interest. The value $\bar{\rho}(\mathcal{S}_n) := \sup_{S \in \mathcal{S}_n} \rho(S)$ represents the worst-case measurement of the aggregate risk in the presence of dependence uncertainty. If ρ is not convex, the value of $\bar{\rho}(\mathcal{S}_n)$ is in general difficult to calculate. For the case of VaR, some analytical results are given in Wang et al. (2013) and Jakobsons et al. (2016). It is common to calculate $\overline{\mathrm{VaR}}_p(\mathcal{S}_n)$ by numerical calculation and a popular algorithm is the Rearrangement Algorithm in Embrechts et al. (2013). If partial dependence information is available, one can study the values of risk measures in constrained subsets of $\mathcal{S}_n$; see Bernard et al. (2017a,b,c), Bernard and Vanduffel (2015), Bignozzi et al. (2015) and Puccetti et al. (2017) for research along this direction. In this chapter we focus on the full set $\mathcal{S}_n$, that is, no dependence information.

We are particularly interested in the case where n goes to infinity, that is, a very large number of risks. On the one hand, this setting provides mathematical tractability for the behaviour of risk aggregation; on the other hand, dependence uncertainty among a very large number of risks is a practical setting due to the statistical challenges arising in high-dimensional models.

Under this setting, an elegant result is that the VaR and the ES at the same confidence level are asymptotically equivalent. That is, for a given sequence of distributions $F_1, F_2 \ldots$ and $p \in (0, 1)$,

$$\lim_{n\to\infty} \frac{\sup_{S\in\mathcal{S}_n} \mathrm{VaR}_p(S)}{\sup_{S\in\mathcal{S}_n} \mathrm{ES}_p(S)} = 1 \tag{4.1.1}$$

holds under some conditions.

The equivalence (4.1.1) is known to hold under particular conditions. (4.1.1) is first shown under a homogeneous setting (that is, $F_1 = F_2 = \cdots$) in Puccetti and Rüschendorf (2014) under an assumption of *complete mixability* (Wang and Wang (2011)). It is then generalized by for instance Puccetti et al. (2013) and Wang and Wang (2015), among others, under different conditions. The inhomogeneous case is finally obtained in Embrechts et al. (2015) under general moment conditions on the marginal distributions $F_1, F_2, \ldots$.

An immediate question is whether the asymptotic equivalence in (4.1.1) is not only true for the pair (VaR_p, ES_p), but it also holds for larger classes of risk measures. We say that a risk measure ρ^* dominates ρ if they are defined on the same set $\mathcal{X}$ and $\rho \leqslant \rho^*$ on $\mathcal{X}$. It is well known that ES_p is the smallest law-invariant coherent risk measure dominating VaR_p; see Kusuoka (2001).

For a law-invariant risk measure ρ, denote by ρ^* the smallest law-invariant coherent risk measure dominating ρ, if such a risk measure exists. It is natural to ask whether the following equivalence

$$\lim_{n\to\infty} \frac{\sup_{S\in\mathcal{S}_n} \rho(S)}{\sup_{S\in\mathcal{S}_n} \rho^*(S)} = 1, \tag{4.1.2}$$

holds and under what conditions. A result of type (4.1.2) is called an *asymptotic equivalence* for risk measures ρ and ρ^*.

We focus on two popular classes of risk measures in this chapter. The class of *distortion risk measures*, including VaR and ES, is extensively studied as tools for capital calculation (see e.g. Acerbi (2002) and Cont et al. (2010)), insurance premium calculation (see e.g. Wang et al. (1997)), and decision making (see e.g. Yaari (1987)). The class of *convex risk measures*, introduced by Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2002) as an extension of coherent risk measures, is able to reflect non-linearity in the increase of the size of risks, such as risky positions in a financial market with limited liquidity. See Section 1.2.2 for definitions.

The main results in Wang et al. (2015) imply that (4.1.2) holds in the homogeneous model ($F_1 = F_2 = \cdots$) if ρ is a distortion risk measure or a convex risk measure. The assumption of homogeneity is nice for mathematical analysis; however, it is not a realistic assumption for practical applications. In this chapter, our aim is to show (4.1.2) in inhomogeneous models for general risk measures. This requires regularity conditions on the marginal distributions, which we will specify later.

The asymptotic equivalence in (4.1.2) has two practical merits. First, it suggests that using a non-coherent risk measure would lead to roughly the same worst-case value as its coherent partner if the dependence structure is unknown for a joint model of high dimension; therefore a regulator may want to directly implement a coherent risk measure instead. This point is relevant for the search of risk measures in the recent regulatory documents BCBS (2013) and IAIS (2014). Second, the value $\bar{\rho}^*(\mathcal{S}_n)$ can be analytically calculated without specifying a dependence structure, as the worst-case value for ρ^* is often simply the sum of the values of $\rho^*(X_1), \ldots, \rho^*(X_n)$ with corresponding marginal distributions $X_i \sim F_i$, $i = 1, \ldots, n$. As a consequence, (4.1.2) can be used to approximate $\bar{\rho}(\mathcal{S}_n)$ if needed. These merits provide a powerful tool for evaluating model uncertainty for risk aggregation with non-coherent risk measures.

Mathematically, the main result in this chapter generalizes not only the results in Embrechts et al. (2015) for VaR and ES, but also those in Wang et al. (2015) for general risk measures in the homogeneous setting. More importantly, our methods unify the two streams of research in this field. A significant mathematical challenge arises as the method in Wang et al. (2015) relies on the study of the quantity

$$\Gamma_\rho(X) = \lim_{n\to\infty} \frac{1}{n} \sup\{\rho(S) : S \in \mathcal{S}_n(F,\dots,F)\}, \quad X \sim F,$$

which cannot be naturally generalized to an inhomogeneous setting. In this chapter, we use an alternative method by constructing a specific $S_n \in \mathcal{S}_n$ such that $\rho(S_n)$ and $\rho^*(S_n)$ are close. It should be noted that the case of distortion risk measures is technically much more involved than the case of convex risk measures, because we know that the worst-case dependence structure for convex risk measures is comonotonicity, but not for non-coherent distortion risk measures in general. The main theorem and its proof reveal the worst-case dependence structure for general distortion risk measures (Choquet integrals). This dependence structure is valuable to many other fields where probability distortion is involved, for instance in decision theory (see for instance Yaari (1987) and Quiggin (1993)), behavioral finance (see for instance He and Zhou (2016)), reinsurance (see for instance Bernard et al. (2015)) and insurance pricing (see for instance Wang et al. (1997)).

The structure of this chapter is as follows. In Section 4.2, we present two examples showing that without some regularity conditions, the asymptotic equivalence may fail to hold. In Section 4.3, we study the asymptotic equivalence for distortion risk measures under some regularity conditions. In Section 4.4, we study the asymptotic equivalence for convex risk measures under general conditions. Conclusions are stated in Section 4.5. A proof of the main theorem of Section 4.3 is in Section 4.6.

4.2 Vanishing Risks and Exploding Risks

Before we move on to the main result of this chapter, we present two counter-examples of asymptotic equivalence to help the reader understand the nature of the problem. Let Γ be the set of all pairs (ρ_1, ρ_2) where ρ_1 is a non-coherent monetary risk measure on $\mathcal{X}$ and ρ_2 is a coherent risk measure on $\mathcal{X}$ dominating ρ_1. For $(\rho, \rho^*) \in \Gamma$, in order to have the general asymptotic equivalence

$$\lim_{n\to\infty} \frac{\sup_{S\in\mathcal{S}_n} \rho(S)}{\sup_{S\in\mathcal{S}_n} \rho^*(S)} = 1, \tag{4.2.1}$$

some regularity conditions have to be imposed to avoid the following cases of vanishing and exploding risks. Note that both cases are typically irrelevant in practice. In particular, we can let $(\rho, \rho^*) = (\mathrm{VaR}, \mathrm{ES})$ and the random variable $X \sim U[0,1]$ in the following examples.

Example 4.1 (Vanishing risks)**.** For a pair $(\rho, \rho^*) \in \Gamma$, take $X \in \mathcal{X}$ such that $0 < \rho(X) < \rho^*(X)$; such X always exists since ρ is not coherent and hence $\rho \neq \rho^*$ for some subset of $\mathcal{X}$. Write $a = \rho(X)$ and $b = \rho^*(X)$. Let F_1 be the distribution of X. For $i = 2, 3, \ldots$, let F_i be a distribution supported in $[0, k_i]$, where $\{k_i,\ i = 2, 3, \ldots\}$ is a sequence of positive numbers such that $\sum_{i=2}^{\infty} k_i < (b-a)/2$. From the monotonicity and cash-invariance of ρ and ρ^*, we have

$$\sup_{S \in \mathcal{S}_n} \rho(S) \leqslant \rho(X_1) + \sum_{i=2}^{n} k_i \leqslant a + \frac{1}{2}(b-a) = \frac{1}{2}(a+b)$$

and

$$\sup_{S \in \mathcal{S}_n} \rho^*(S) \geqslant \rho^*(X_1) = b.$$

Then for $n \in \mathbb{N}$,

$$\frac{\sup_{S \in \mathcal{S}_n} \rho(S)}{\sup_{S \in \mathcal{S}_n} \rho^*(S)} \leqslant \frac{a+b}{2b} < 1.$$

That is, (4.2.1) does not hold. This example suggests that for (4.2.1) to hold, a regularity condition has to be imposed to avoid vanishing risks, that is, the scale of individual risks shrinks too fast as $n \to \infty$.

Example 4.2 (Exploding risks)**.** For illustration we take $(\rho, \rho^*) \in \Gamma$ where ρ is positive homogeneous. This example includes, for instance, a distortion risk measure and its dominating coherent distortion risk measure; see Section 4.3 below. Take a random variable $X \in \mathcal{X}$ supported on a compact interval $[0,1]$ such that $\rho(X) < \rho^*(X)$; such X always exists as both ρ and ρ^* are positive homogeneous and $\rho \neq \rho^*$ for some subset of $\mathcal{X}$. Write $a = \rho(X)$ and $b = \rho^*(X)$. Now, let $\{k_i,\ i \in \mathbb{N}\}$ be a sequence of positive numbers such that $k_1 = 1$ and $2\sum_{i=1}^{n} k_i < (b-a)k_{n+1}$ for all $n \in \mathbb{N}$. Let F_i be the distribution of $k_i X$ for $i \in \mathbb{N}$.

From the monotonicity and the cash-invariance of ρ and ρ^*, we have

$$\sup_{S \in \mathcal{S}_n} \rho(S) \leqslant k_n \rho(X) + \sum_{i=1}^{n-1} k_i = k_n a + \sum_{i=1}^{n-1} k_i < k_n a + \frac{1}{2} k_n (b-a) = \frac{1}{2} k_n (a+b)$$

and

$$\sup_{S \in \mathcal{S}_n} \rho^*(S) \geqslant \rho^*(X_n) = k_n \rho^*(X) = k_n b.$$

Therefore,

$$\frac{\sup_{S\in\mathcal{S}_n}\rho(S)}{\sup_{S\in\mathcal{S}_n}\rho^*(S)} \leqslant \frac{k_n(a+b)}{2k_nb} = \frac{a+b}{2b} < 1.$$

That is, (4.2.1) does not hold. This example suggests that for (4.2.1) to hold, a regularity condition has to be imposed to avoid exploding risks. Here, the scale of individual risks grows too fast as $n \to \infty$.

4.3 Asymptotic Equivalence for Distortion Risk Measures

Throughout this section, we take $\mathcal{X} = L^+$. As monetary risk measures are cash-invariant, this assumption is technically equivalent to assuming that each risk is uniformly bounded from below (bounded gain). Gains are typically not relevant when regulatory risk measures such as VaR and ES are applied, and hence this is a common assumption in risk management. Throughout this chapter, VaR and ES are defined as in Definition 1.3 and a distortion risk measure is defined in Definition 1.4.

4.3.1 Some Lemmas

Before stating the main result of this section, we first provide some necessary lemmas on distortion risk measures and on the set $\mathcal{S}_n$. A key object for our analysis is the largest convex distortion function dominated by h, defined as

$$h^*(t) = \sup\{g(t)\in[0,1]:\ g \text{ is increasing and convex on } [0,1] \text{ and } g \leqslant h\}, \quad t\in[0,1]. \quad (4.3.1)$$

We will use the notation h^* throughout Section 4.3.

The first lemma formulates an order in two distortion risk measures from the order in their respective distortion functions.

Lemma 4.1. *For two distortion functions $h_1, h_2 \in \mathcal{H}$, if $h_1(t) \leqslant h_2(t)$ for all $t\in[0,1]$, then*

$$\rho_{h_1}(X) \geqslant \rho_{h_2}(X), \quad X\in\mathcal{X}.$$

Proof. Let F be the distribution of $X\in\mathcal{X}$. For $x\in\mathbb{R}$ and $i=1,2$, let $g_i(x) = h_i(F(x+)) = \lim_{y\to x^+} h_i(F(y))$, that is, g_i is the right-continuous correction of $h_i\circ F$. Since $h_1 \leqslant h_2$ on $[0,1]$, we have $g_1 \leqslant g_2$ on $\mathbb{R}$. Let Y_i be a random variable with distribution function g_i, $i = 1, 2$. Then we have $\mathbb{E}[Y_1] \geqslant \mathbb{E}[Y_2]$ from $g_1 \leqslant g_2$. Finally, we obtain

$$\rho_{h_1}(X) = \int_{\mathbb{R}} x\mathrm{d}(h_1 \circ F)(x) = \int_{\mathbb{R}} x\mathrm{d}(h_1 \circ F)(x+) = \int_{\mathbb{R}} x\mathrm{d}g_1(x) = \mathbb{E}[Y_1] \geqslant \mathbb{E}[Y_2] = \rho_{h_2}(X),$$

as desired, where the second equality is due to the facts that the integrand $x \to x \in \mathbb{R}$ is continuous, $X \in L^+$, and $h_1 \circ F$ is increasing. □

The next lemma gives ρ_{h^*} as the smallest coherent distortion risk measure dominating ρ_h. It was given in Wang et al. (2015) based on Lemma 4.1 for right coutinuous $h \in \mathcal{H}$; since Lemma 4.1 is true for all $h \in \mathcal{H}$, the next lemma also holds for all $h \in \mathcal{H}$. It is also shown in Wang et al. (2015) that ρ_{h^*} is the smallest law-invariant coherent risk measure dominating ρ_h.

Lemma 4.2 (Lemma 3.1 of Wang et al. (2015))**.** *For any $h \in \mathcal{H}$, h^* as in* (4.3.1) *is a continuous distortion function. Moreover, the smallest coherent distortion risk measure dominating ρ_h exists and has distortion function h^*, that is,*

$$\rho_{h^*}(X) = \int_0^1 \mathrm{VaR}_t(X)\mathrm{d}h^*(t), \quad X \in \mathcal{X}. \tag{4.3.2}$$

The following lemma provides a building block for the dependence structure that we need for the asymptotic equivalence.

Lemma 4.3 (Corollary A.3 of Embrechts et al. (2015))**.** *Suppose that $\{F_i,\ i \in \mathbb{N}\}$ is a sequence of distributions with bounded support, then there exist random variables $X_i \sim F_i$, $i \in \mathbb{N}$ such that for each $n \in \mathbb{N}$,*

$$|S_n - \mathbb{E}[S_n]| \leqslant L_n, \tag{4.3.3}$$

where $S_n = X_1 + \cdots + X_n$ and L_n is the largest length of the support of F_i, $i = 1, \ldots, n$.

Finally, the following lemma from convex analysis provides an important geometric property of the pair (h, h^*).

Lemma 4.4 (Lemma 5.1 of Brighi and Chipot (1994))**.** *Suppose $h \in \mathcal{H}$ is continuous and h^* is defined in* (4.3.1)*. The set $\{t \in [0, 1] : h(t) \neq h^*(t)\}$ is the union of some disjoint open intervals, and h^* is linear on each of the intervals.*

4.3.2 Asymptotic Equivalence for Distortion Risk Measures

For a given $h \in \mathcal{H}$ and h^* defined in (4.3.1), we list two conditions for a sequence of distribution functions $\{F_i,\ i \in \mathbb{N}\}$ that we work with. In the following, $X_i \sim F_i$, $i \in \mathbb{N}$.

Condition A1. $\liminf\limits_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \rho_{h^*}(X_i) > 0$.

Condition A2. $\lim_{q\to 1} \sup_{i\in\mathbb{N}} \int_q^1 F_i^{-1}(t)\mathrm{d}h^*(t) = 0$.

Condition A1 requires that ρ_{h^*} of the marginal risks does not vanish, and therefore eliminates the case of vanishing risks as in Example 4.1. Condition A2 requires that the marginal risks be uniformly integrable with respect to h^*, and therefore eliminates the case of exploding risks as in Example 4.2. A1 automatically holds for marginal risks uniformly bounded below away from zero and A2 automatically holds for marginal risks uniformly bounded above. The following theorem contains the main result of this chapter.

Theorem 4.5. *For $h \in \mathcal{H}$ and a sequence of distribution functions $\{F_i,\ i \in \mathbb{N}\}$ supported in $\mathbb{R}_+$ and satisfying Conditions A1-A2, we have*

$$\lim_{n\to\infty} \frac{\sup\{\rho_h(S) : S \in \mathcal{S}_n\}}{\sup\{\rho_{h^*}(S) : S \in \mathcal{S}_n\}} = 1, \tag{4.3.4}$$

where h^* is defined in (4.3.1).

Proof. The proof of this theorem is technical and depends on the geometrical relationship between h and h^*. Here we give the proof for the following nice case, from which the reader should be able to grasp the main ideas. A full proof can be found in Section 4.6.

Case 1. Assume that h is continuous and there exists $p \in (0,1)$ such that $h(t) = h^*(t)$ for all $t \in [p,1]$.

Proof of the Theorem for Case 1. Since h is continuous, we directly work with (1.2.6). From Lemma 4.4, there exist disjoint open intervals (a_k, b_k), $k \in K \subset \mathbb{N}$ on which $h \neq h^*$, and furthermore, p can be taken as $p = \sup_{k\in K} b_k < 1$. Note that $h(t) = h^*(t)$ for $t \in [p,1]$ and h^* is linear on each of $[a_k, b_k]$, $k \in K$. Define $\mathrm{I}_k = (a_k, b_k)$, $k \in K$. For some $U \sim \mathrm{U}[0,1]$, let

$$S_n^c = F_1^{-1}(U) + \cdots + F_n^{-1}(U), \tag{4.3.5}$$

and

$$R_n = \begin{cases} F_1^{-1}(U) + \cdots + F_n^{-1}(U), & \text{if } U \notin \cup_{k\in K} \mathrm{I}_k, \\ \mathbb{E}\left[F_1^{-1}(U) + \cdots + F_n^{-1}(U) \mid U \in \mathrm{I}_k\right], & \text{if } U \in \mathrm{I}_k,\ k \in K. \end{cases} \tag{4.3.6}$$

Clearly, $F_i^{-1}(U) \sim F_i$, $i = 1, \ldots, n$, and hence $S_n^c \in \mathcal{S}_n$. As

$$\mathbb{E}\left[F_i^{-1}(U) \mid U \in \mathrm{I}_k\right] = \frac{\int_{(a_k,b_k)} F_i^{-1}(t)\mathrm{d}t}{b_k - a_k} \quad \text{and} \quad F_{S_n^c}^{-1}(t) = \sum_{i=1}^n F_i^{-1}(t) \quad \text{for } t \in (0,1),$$

we have

$$\begin{aligned} &\int_{(a_k,b_k)} F_{S_n^c}^{-1}(t)\mathrm{d}h^*(t) - \int_{(a_k,b_k)} F_{R_n}^{-1}(t)\mathrm{d}h^*(t) \\ &= \frac{h^*(b_k) - h^*(a_k)}{b_k - a_k} \sum_{i=1}^n \int_{(a_k,b_k)} F_i^{-1}(t)\mathrm{d}t - \sum_{i=1}^n \frac{\int_{(a_k,b_k)} F_i^{-1}(t)\mathrm{d}t}{b_k - a_k} \int_{(a_k,b_k)} \mathrm{d}h^*(t) = 0. \end{aligned}$$

It follows that

$$\begin{aligned} \rho_{h^*}(S_n^c) - \rho_{h^*}(R_n) &= \int_0^p F_{S_n^c}^{-1}(t)\mathrm{d}h^*(t) - \int_0^p F_{R_n}^{-1}(t)\mathrm{d}h^*(t) \\ &= \sum_{k\in K} \left[\int_{a_k}^{b_k} F_{S_n^c}^{-1}(t)\mathrm{d}h^*(t) - \int_{a_k}^{b_k} F_{R_n}^{-1}(t)\mathrm{d}h^*(t)\right] = 0, \end{aligned} \tag{4.3.7}$$

that is, $\rho_{h^*}(S_n^c) = \rho_{h^*}(R_n)$. As $F_i^{-1}(U)$ is bounded for $U \in \mathrm{I}_k$, $k \in K$, by Lemma 4.3, for each k, we can find random variables $Y_{1k}, \ldots, Y_{nk}$, independent of U, such that Y_{ik} is identically distributed as $F_i^{-1}(U)|U \in \mathrm{I}_k$ and independent of U, $i = 1, \ldots, n$, and

$$\left|Y_{1k} + \cdots + Y_{nk} - \mathbb{E}\left[F_1^{-1}(U) + \cdots + F_n^{-1}(U) \mid U \in \mathrm{I}_k\right]\right| \leqslant \max_{i=1,\ldots,n}\{F_i^{-1}(b_k) - F_i^{-1}(a_k)\}. \tag{4.3.8}$$

Let $X_i^* = F_i^{-1}(U)\mathrm{I}_{\{U \notin \cup_{k\in K}\mathrm{I}_k\}} + \sum_{k\in K} Y_{ik}\mathrm{I}_{\{U\in \mathrm{I}_k\}}$, $i = 1, \ldots, n$. It is easy to check that $X_i^* \sim F_i$, $i = 1, \ldots, n$. Denote by

$$S_n^* = X_1^* + \cdots + X_n^*. \tag{4.3.9}$$

Clearly, $S_n^* \in \mathcal{S}_n$ and

$$|R_n - S_n^*| \leqslant \max_{i=1,\ldots,n}\{F_i^{-1}(b_k) - F_i^{-1}(a_k)\} \leqslant \max_{i=1,\ldots,n}\left\{F_i^{-1}(p)\right\}. \tag{4.3.10}$$

As $h^* \leqslant h$ and ρ_{h^*} is coherent and hence subadditive, by Lemma 4.1, we have $\rho_h(S_n^*) \leqslant \rho_{h^*}(S_n^*) \leqslant \rho_{h^*}(S_n^c)$. Integration by parts yields

$$\begin{aligned}\int_0^p F_{R_n}^{-1}(t)\mathrm{d}h^*(t) - \int_0^p F_{R_n}^{-1}(t)\mathrm{d}h(t) &= \int_0^p \left(h(t) - h^*(t)\right)\mathrm{d}F_{R_n}^{-1}(t) \\ &= \sum_{k\in K}\int_{(a_k,b_k)} \left(h(t) - h^*(t)\right)\mathrm{d}F_{R_n}^{-1}(t) = 0 \qquad (4.3.11)\end{aligned}$$

where the last equality follows since $F_{R_n}^{-1}(t)$ is a constant for t in each (a_k, b_k). Since $h(t) = h^*(t)$ on $[p, 1]$, we have

$$\begin{aligned}\rho_{h^*}(S_n^c) - \rho_h(S_n^*) &= \int_0^p F_{S_n^c}^{-1}(t)\mathrm{d}h^*(t) - \int_0^p F_{S_n^*}^{-1}(t)\mathrm{d}h(t) \\ &= \left(\int_0^p F_{S_n^c}^{-1}(t)\mathrm{d}h^*(t) - \int_0^p F_{R_n}^{-1}(t)\mathrm{d}h^*(t)\right) \\ &\quad + \left(\int_0^p F_{R_n}^{-1}(t)\mathrm{d}h^*(t) - \int_0^p F_{R_n}^{-1}(t)\mathrm{d}h(t)\right) \\ &\quad + \left(\int_0^p F_{R_n}^{-1}(t)\mathrm{d}h(t) - \int_0^p F_{S_n^*}^{-1}(t)\mathrm{d}h(t)\right) \\ &\leqslant \max_{i=1,\dots,n}\left\{F_i^{-1}(p)\right\},\end{aligned}$$

where the last inequality follows from (4.3.7), (4.3.10)), and (4.3.11). Condition A2 implies that for any $\varepsilon > 0$, there exists $q > p$ such that

$$\sup_{i\in\mathbb{N}}\int_q^1 F_i^{-1}(t)\mathrm{d}h^*(t) < \varepsilon.$$

Hence, by noting that $h^*(q) < 1$,

$$\max_{i=1,\dots,n}\left\{F_i^{-1}(p)\right\} \leqslant \max_{i=1,\dots,n}\left\{F_i^{-1}(q)\right\} < \frac{\varepsilon}{1-h^*(q)}.$$

By Condition A1, $\lim_{n\to\infty}\sum_{i=1}^n \rho_{h^*}(X_i) = \infty$. Therefore, as $n \to \infty$,

$$\left|\frac{\sup\left\{\rho_h(S) : S \in \mathcal{S}_n\right\}}{\sup\left\{\rho_{h^*}(S) : S \in \mathcal{S}_n\right\}} - 1\right| \leqslant \frac{\max_{i=1,\dots,n}\{F_i^{-1}(p)\}}{\sum_{i=1}^n \rho_{h^*}(X_i)} \to 0.$$

The desired result follows. □

From the above proof, we can see that for this nice case, Condition A1 can be weakened to $\lim_{n\to\infty}\sum_{i=1}^n \rho_{h^*}(X_i)=\infty$ and Condition A2 can be weakened to $\max_{i=1,\ldots,n}\left\{F_i^{-1}(p)\right\}<\infty$. Conditions A1 and A2 are necessary for the proofs of other cases discussed in Section 4.6. For Case 1, indeed we can give a more intuitive condition which is also easy to verify.

Condition A3. For a pre-assigned $p\in(0,1)$,

$$\lim_{n\to\infty}\frac{\max_{i=1,\ldots,n}\left\{\mathrm{VaR}_p(X_i)\right\}}{\sum_{i=1}^n \mathrm{VaR}_p(X_i)}=0. \tag{4.3.12}$$

Condition A3 simply says that there is no single risk which dominates the sum of all other risks in terms of VaR_p, a reasonable assumption for a joint model of high dimension. A3 is not strictly comparable to A1 and A2, but it has an important merit: it does not depend on h or h^* except for a point $p\in(0,1)$ given beforehand, which may be based on h and h^*. For a practical choice of $\{F_i,\ i\in\mathbb{N}\}$, it is often that (4.3.12) holds for all $p\in(0,1)$.

Theorem 4.6. *Suppose that $h\in\mathcal{H}$ is continuous and there exists $p\in(0,1)$ such that $h(t)=h^*(t)$ for all $t\in[p,1]$. For a sequence of distribution functions $\{F_i,\ i\in\mathbb{N}\}$ supported in $\mathbb{R}_+$ satisfying Condition A3, we have*

$$\lim_{n\to\infty}\frac{\sup\left\{\rho_h(S):S\in\mathcal{S}_n\right\}}{\sup\left\{\rho_{h^*}(S):S\in\mathcal{S}_n\right\}}=1, \tag{4.3.13}$$

where h^* is defined in (4.3.1).

Proof. Following the same proof in Case 1 of the above theorem, we obtain

$$0\leqslant\rho_{h^*}(S_n^c)-\rho_h(S_n^*)\leqslant\max_{i=1,\ldots,n}\left\{\mathrm{VaR}_p(X_i)\right\}.$$

As

$$\rho_{h^*}(X_i)=\int_0^1\mathrm{VaR}_t(X_i)\mathrm{d}h^*(t)\geqslant\int_p^1\mathrm{VaR}_t(X_i)\mathrm{d}h^*(t)\geqslant\mathrm{VaR}_p(X_i)(1-h^*(p)),$$

we have

$$\begin{aligned}\left|\frac{\sup\left\{\rho_h(S):S\in\mathcal{S}_n\right\}}{\sup\left\{\rho_{h^*}(S):S\in\mathcal{S}_n\right\}}-1\right|&\leqslant\frac{\max_{i=1,\ldots,n}\{\mathrm{VaR}_p(X_i)\}}{\sum_{i=1}^n\rho_{h^*}(X_i)}\\&\leqslant\frac{\max_{i=1,\ldots,n}\{\mathrm{VaR}_p(X_i)\}}{(1-h^*(p))\sum_{i=1}^n\mathrm{VaR}_p(X_i)}\to 0\text{ as }n\to\infty\end{aligned}$$

by (4.3.12). $\square$

Remark 4.1. The worst-case dependence structure for general distortion risk measures is revealed via the construction of S_n^*. For $n \to \infty$, to obtain a sum of S_n^*, one needs comonotonicity on the set $\left(\bigcup_{k\in K} \mathrm{I}_k\right)^c$ and an *extreme negative dependence* conditional on each of the intervals I_k, $k \in K$. For a fixed n, the worst-case dependence structure for a general distortion risk measure is still not clear, because an extreme negative dependence may not be properly defined for fixed n unless the marginal distributions satisfies a notion of *joint mixability*; see Puccetti and Wang (2015) for related discussions on the above two notions of negative dependence.

4.3.3 Remarks on the Conditions

In addition to Examples 4.1 and 4.2, we give a more subtle example to show that the uniform integrability condition A2 is essential. We compare our conditions with the ones in Embrechts et al. (2015) for VaR and ES. Theorem 3.3 of Embrechts et al. (2015) shows that

$$\lim_{n\to\infty} \frac{\sup\left\{\mathrm{VaR}_p(S) : S \in \mathcal{S}_n\right\}}{\sup\left\{\mathrm{ES}_p(S) : S \in \mathcal{S}_n\right\}} = 1, \tag{4.3.14}$$

if for $X_i \sim F_i$, $i \in \mathbb{N}$, the following two conditions are satisfied:

(a*) $\sup_{i\in\mathbb{N}} \mathbb{E}[|X_i|^k] < \infty$ for some $k > 1$,

(b*) $\liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \mathrm{ES}_p(X_i) > 0$.

A natural question is whether k in (a*) can be taken as 1, that is,

(a') $\sup_{i\in\mathbb{N}} \mathbb{E}[|X_i|] < \infty$.

In comparison with the conditions in Embrechts et al. (2015), another question is whether A2 can be weakened to

(A2') $\sup_{i\in\mathbb{N}} \rho_{h^*}(X_i) < \infty$.

For the pair $(\rho_h, \rho_{h^*}) = (\mathrm{VaR}_p, \mathrm{ES}_p)$, (b*) is equivalent to our condition A1, and (a') is equivalent to A2' if we only consider $\mathcal{X} = L^+$.

The answer to both questions turns out to be negative. In the following example, Conditions A1and A2' are satisfied; in other words, conditions (a') and (b*) are satisfied. We will see that (4.3.14) fails to hold for all $p \in (0, 1)$.

Example 4.3. Suppose that the probability space is the Lebesgue unit interval $([0,1], \mathcal{B}([0,1]), \mathbb{P})$, where $\mathbb{P}$ is the Lebesgue measure. Let

$$F_i(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 - \frac{1}{i^2} & \text{if } 0 \leqslant x < i^2, \\ 1 & \text{if } i^2 \leqslant x. \end{cases}$$

Clearly the support of F_i is nonnegative, $i \in \mathbb{N}$. One can calculate

$$\mathrm{VaR}_\alpha(X_i) = i^2 \mathrm{I}_{\{\alpha \in (1-1/i^2, 1)\}}, \quad i \in \mathbb{N}.$$

For $X_i \sim F_i, i \in \mathbb{N}$, $\sup_{i \in \mathbb{N}} \mathbb{E}[X_i] = 1 < \infty$. One can also check that for $i \geqslant 1/\sqrt{1-p}$, $\mathrm{ES}_p(X_i) = \frac{1}{1-p}$. As a consequence,

$$\lim_{n\to\infty} \sup \{\mathrm{ES}_p(S/n) : \ S \in \mathcal{S}_n\} = \lim_{n\to\infty} \frac{\sum_{i=1}^n \mathrm{ES}_p(X_i)}{n} = \frac{1}{1-p}.$$

Thus, (a'), (b*), A1 and A2' are all satisfied.

Next we will show that

$$\lim_{n\to\infty} \sup \{\mathrm{VaR}_p(S/n) : S \in \mathcal{S}_n\} = 0.$$

Note that $\sum_{i=1}^{\infty} \frac{1}{i^2} < \infty$. For any $\varepsilon > 0$, which we choose as $\varepsilon = 1 - p$, there exists an N such that for $n \geqslant N$, we have

$$\sum_{i=n}^{\infty} \frac{1}{i^2} < \varepsilon. \tag{4.3.15}$$

Take a fixed number $k > N$ such that $\sum_{i=1}^{N} i^2 < k^2$, we have for any $n > N$,

$$\begin{aligned} \mathbb{P}(S_n > k^2) &= \mathbb{P}(X_1 + \cdots + X_N + X_{N+1} + \cdots + X_n > k^2) \\ &\leqslant \mathbb{P}(\text{at least one } X_i > 0, \ i = N+1, \ldots, n) \\ &\leqslant \sum_{i=N+1}^{n} \mathbb{P}(X_i > 0) = \sum_{i=N+1}^{n} \frac{1}{i^2} < \varepsilon. \end{aligned}$$

Thus, $\mathrm{VaR}_{1-\varepsilon}(S_n) \leqslant k^2$. Therefore,

$$0 \leqslant \lim_{n\to\infty} \sup \{\mathrm{VaR}_p(S/n) : S \in \mathcal{S}_n\} = \lim_{n\to\infty} \frac{\sup \{\mathrm{VaR}_p(S) : S \in \mathcal{S}_n\}}{n} \leqslant \lim_{n\to\infty} \frac{k^2}{n} = 0.$$

In summary,

$$\lim_{n\to\infty} \frac{\sup \{\mathrm{VaR}_p(S) : S \in \mathcal{S}_n\}}{\sup \{\mathrm{ES}_p(S) : S \in \mathcal{S}_n\}} = 0.$$

4.4 Asymptotic Equivalence for Convex Risk Measures

In this section we study asymptotic equivalence for convex risk measures. Compared to the previous section, the result in this section is less technically involved as the worst-case dependence structure for convex risk measures is explicitly known as comonotonicity. We assume $\mathcal{X} = L^1$, as the canonical space for law-invariant convex risk measures is L^1; see Filipović and Svindland (2012).

4.4.1 Some Lemmas

First, we recall the Kusuoka representation of law-invariant convex risk measures as established in Frittelli and Rosazza Gianin (2005) for $\mathcal{X} = L^\infty$. The extension of the representation to L^p, $p \in [1, \infty)$ is established in Svindland (2008). The Fatou property (FP) has to be assumed throughout Section 4.4 for the representation to hold. A risk measure ρ is said to satisfy the (L^1-)*Fatou property* if for $X, X_1, X_2, \ldots \in L^1$, $X_n \xrightarrow{L^1} X$ as $n \to \infty$ implies $\liminf_{n\to\infty} \rho(X_n) \geqslant \rho(X)$.

Lemma 4.7 (Lemma 2.14 of Svindland (2008))**.** *A law-invariant convex risk measure ρ mapping L^1 to $\mathbb{R}$ with the Fatou property has a representation*

$$\rho(X) = \sup_{\mu \in \mathcal{P}} \left\{ \int_0^1 \mathrm{ES}_p(X) \mathrm{d}\mu(p) - v(\mu) \right\}, \qquad X \in L^1, \tag{4.4.1}$$

where $\mathcal{P}$ is the set of all probability measures on $[0,1]$ and v is a function from $\mathcal{P}$ to $\mathbb{R} \cup \{+\infty\}$, called a penalty function of ρ.

From now on, we denote by ρ^v a convex risk measure with penalty function v which maps L^1 to $\mathbb{R}$. For a law-invariant convex risk measure, without loss of generality we can assume $\rho^v(0) = 0$, or equivalently, in (4.4.1), $\inf\{v(\mu) : \mu \in \mathcal{P}\} = 0$. If one is interested in a law-invariant convex risk measure ρ with $\rho(0) = c \neq 0$, one can define $\tilde{\rho}(\cdot) = \rho(\cdot) - c$ so that $\tilde{\rho}$ is a law-invariant convex risk measure and $\tilde{\rho}(0) = 0$. A result on $\tilde{\rho}$ would simply lead to a result on ρ.

Similarly to the case of distortion risk measures, a convex risk measure is dominated by a coherent risk measure. The following simple lemma is a combination of Theorem 4.1 and Corollary 4.2 of Wang et al. (2015).

Lemma 4.8 (Wang et al. (2015)). *The smallest law-invariant coherent risk measure dominating ρ^v exists, and it is given by*

$$\rho^{v*}(X) = \sup_{\mu \in \mathcal{P}_v} \left\{ \int_0^1 \mathrm{ES}_p(X) \mathrm{d}\mu(p) \right\}, \qquad X \in L^1, \tag{4.4.2}$$

where $\mathcal{P}_v = \{\mu \in \mathcal{P} : v(\mu) < +\infty\}$.

Remark 4.2. A popular subclass of law-invariant convex risk measures is the class of *convex shortfall risk measures* in Föllmer and Schied (2016). It is shown that for all convex shortfall risk measures ρ^v, the smallest dominating coherent risk measure ρ^{v*} is always a coherent *expectile*; see Proposition 4.3 of Wang et al. (2015).

Unlike the case of general distortion risk measures, the dependence structure of $(X_1, \ldots, X_n)$ which gives the maximum value of $\rho^v(X_1 + \cdots + X_n)$ for given marginal distributions is always comonotonicity. Hence, an explicit expression of $\sup\{\rho^v(S_n) : S_n \in \mathcal{S}_n\}$ can be obtained. This is technically convenient to study asymptotic equivalence for convex risk measures.

Lemma 4.9. *For a sequence of distribution functions $\{F_i,\ i \in \mathbb{N}\}$,*

$$\sup\{\rho^v(S) : S \in \mathcal{S}_n\} = \sup_{\mu \in \mathcal{P}} \left\{ \sum_{i=1}^n \int_0^1 \mathrm{ES}_p(X_i) \mathrm{d}\mu(p) - v(\mu) \right\}, \tag{4.4.3}$$

where $X_i \sim F_i$, $i = 1, \ldots, n$.

Proof. Let $Y_1, \ldots, Y_n \in L^1$ be comonotonic random variables such that $Y_i \sim F_i$, $i = 1, \ldots, n$. We have $\rho^v(X_1 + \cdots + X_n) \leqslant \rho^v(Y_1 + \cdots + Y_n)$; see Lemma 5.2 of Bäuerle and Müller (2006). It follows from Lemma 4.7 that

$$\begin{aligned} \sup\{\rho^v(S) : S \in \mathcal{S}_n\} &= \rho^v(Y_1 + \cdots + Y_n) \\ &= \sup_{\mu \in \mathcal{P}} \left\{ \int_0^1 \mathrm{ES}_p \left(\sum_{i=1}^n Y_i \right) \mathrm{d}\mu(p) - v(\mu) \right\} \\ &= \sup_{\mu \in \mathcal{P}} \left\{ \sum_{i=1}^n \int_0^1 \mathrm{ES}_p\left(Y_i\right) \mathrm{d}\mu(p) - v(\mu) \right\}. \end{aligned}$$

We obtain (4.4.3) since $\mathrm{ES}_p(X_i) = \mathrm{ES}_p(Y_i)$, $p \in (0,1)$, $i = 1, \ldots, n$. $\square$

Lemma 4.10. *For given $\varepsilon > 0$, $n \in \mathbb{N}$, and a sequence of distribution functions $\{F_i,\ i \in \mathbb{N}\}$ such that* $\sup\{\rho^{v*}(S) : S \in \mathcal{S}_n\} < \infty$*, there exists $\mu_n \in \mathcal{P}_v$ such that*

$$\sup\left\{\rho^{v*}(S) : S \in \mathcal{S}_n\right\} - \sum_{i=1}^{n} \int_0^1 \mathrm{ES}_p(X_i)\mathrm{d}\mu_n(p) < \varepsilon, \tag{4.4.4}$$

where $X_i \sim F_i$, $i \in \mathbb{N}$.

Proof. By applying Lemma 4.9 to the coherent risk measure ρ^{v*}, we obtain

$$\sup\left\{\rho^{v*}(S) : S \in \mathcal{S}_n\right\} = \sup_{\mu \in \mathcal{P}_v} \left\{ \sum_{i=1}^{n} \int_0^1 \mathrm{ES}_p(X_i)\mathrm{d}\mu(p) \right\}.$$

By definition, there exists $\mu_n \in \mathcal{P}_v$ such that (4.4.4) holds. $\square$

4.4.2 Asymptotic Equivalence for Convex Risk Measures

Similarly to Section 4.3, we need to assume some conditions on a sequence of distribution functions $\{F_i,\ i \in \mathbb{N}\}$ for the asymptotic equivalence to hold. In the following, $X_i \sim F_i$, $i \in \mathbb{N}$.

Condition B1. $\sum_{i=1}^{n} \mathbb{E}[X_i] \to \infty$ as $n \to \infty$.

Condition B2. $\rho^{v*}(\sum_{i=1}^{n} F_i^{-1}(U)) < \infty$ for some $U \sim \mathrm{U}[0,1]$ and all $n \in \mathbb{N}$.

Condition B3. There exist $\varepsilon > 0$ and a sequence $\mu_n \in \mathcal{P}_v,\ n \in \mathbb{N}$ satisfying (4.4.4), such that

$$\lim_{n\to\infty} \frac{v(\mu_n)}{\sum_{i=1}^{n} \int_0^1 \mathrm{ES}_p(X_i)\mathrm{d}\mu_n(p)} \to 0.$$

Condition B1 is assumed to avoid the vanishing risks in Example 4.1. Condition B2 is trivial as we need the denominator in the asymptotic equivalence (4.1.2) to be finite for any given n. Condition B3 is a technical condition to guarantee convergence in our proof. Note that if $v(\mu)$ is bounded for $\mu \in \mathcal{P}_v$, then B3 is automatically satisfied when B1 holds.

Theorem 4.11. *Given a sequence of distribution functions $\{F_i,\ i \in \mathbb{N}\}$ satisfying Conditions B1-B3, we have*

$$\lim_{n\to\infty} \frac{\sup\left\{\rho^{v}(S) : S \in \mathcal{S}_n\right\}}{\sup\left\{\rho^{v*}(S) : S \in \mathcal{S}_n\right\}} = 1. \tag{4.4.5}$$

Proof. First note that for any $S_n \in \mathcal{S}_n$, due to Lemma 4.7 and B2, we have

$$\infty > \rho^{v*}(S_n) \geqslant \rho^{v}(S_n) \geqslant \sum_{i=1}^{n} \mathbb{E}[X_i],$$

and hence both $\sup\{\rho^v(S) : S \in \mathcal{S}_n\}$ and $\sup\{\rho^{v*}(S) : S \in \mathcal{S}_n\}$ are positive for large n, and

$$\lim_{n\to\infty} \frac{\sup\{\rho^v(S) : S \in \mathcal{S}_n\}}{\sup\{\rho^{v*}(S) : S \in \mathcal{S}_n\}} \leqslant 1. \tag{4.4.6}$$

Write $\lambda_n = \sum_{i=1}^n \int_0^1 \mathrm{ES}_p(X_i)\mathrm{d}\mu_n(p) \geqslant \sum_{i=1}^n \mathbb{E}[X_i]$. We have $\lambda_n \to \infty$ as $n \to \infty$ from Condition B1. From Lemmas 4.9 and 4.10, we have

$$\lim_{n\to\infty} \frac{\sup\{\rho^v(S) : S \in \mathcal{S}_n\}}{\sup\{\rho^{v*}(S) : S \in \mathcal{S}_n\}} \geqslant \lim_{n\to\infty} \frac{\lambda_n - v(\mu_n)}{\lambda_n + \varepsilon} = \lim_{n\to\infty} \frac{\lambda_n}{\lambda_n + \varepsilon} = 1. \tag{4.4.7}$$

Combining (4.4.6) and (4.4.7) we obtain (4.4.5). □

4.5 Conclusions

In this chapter, we show that the asymptotic equivalence of VaR and ES in Embrechts et al. (2015) and preceding papers can be generalized to general risk measures for inhomogeneous models under some regularity conditions. The risk measures that we study include the class of distortion risk measures and the class of convex risk measures. The main result in this chapter is that under dependence uncertainty in the aggregation of a large number of risks, the worst-case value of a non-coherent risk measure is asymptotically equivalent to that of a corresponding coherent risk measure. This result helps to analyze risk aggregation under dependence uncertainty for financial regulation and internal risk management.

4.6 Full Proof of Theorem 4.5

Proof. We show the theorem in two steps. First we assume that h is continuous, and then we approximate the general case by the result for continuous h.

For some intervals $\{\mathrm{I}_k,\ k \in K\}$ which will be specified later, let S_n^c, R_n, and S_n^* be as defined in (4.3.5), (4.3.6) and (4.3.9).

The proof in the case of continuous $h \in \mathcal{H}$.

Depending on the set $\{t \in [0,1] : h(t) \neq h^*(t)\}$, we have the following three cases:

Case 1: For some $p \in (0,1)$, $h(t) = h^*(t)$ for all $t \in [p,1]$. This case is dealt with in Section 4.3.

Case 2: $h \neq h^*$ in the intervals (a_k, b_k), $k \in K \subset \mathbb{N}$, where $\sup_{k\in K} b_k = 1$. Moerover, for all $p \in (0,1)$, there exist $t_0, t_1 \in (p,1)$ such that $h^*(t_0) = h(t_0)$ and $h^*(t_1) \neq h(t_1)$.

Condition A2 and the above property of h and h^* impliy that for any $\varepsilon > 0$, there exists q such that

$$\sup_{i\in\mathbb{N}} \int_q^1 F_i^{-1}(t)\mathrm{d}h^*(t) < \varepsilon \quad \text{and} \quad h(q) = h^*(q). \tag{4.6.1}$$

Let I_k in (4.3.6) be $(a_k, b_k) \cap [0,q]$, $k \in K$.Then $\rho_{h^*}(S_n^c) = \rho_{h^*}(R_n)$ and

$$|S_n^* - R_n| \leqslant \max_{i=1,\dots,n} \{F_i^{-1}(q)\},$$

which implies

$$\left| \int_0^q F_{S_n^*}^{-1}(t)\mathrm{d}h(t) - \int_0^q F_{R_n}^{-1}(t)\mathrm{d}h(t) \right| \leqslant \max_{i=1,\dots,n} \{F_i^{-1}(q)\}.$$

$$\begin{aligned}
&\left| \int_0^q F_{R_n}^{-1}(t)\mathrm{d}h(t) - \int_0^q F_{R_n}^{-1}(t)\mathrm{d}h^*(t) \right| \\
&= \left| F_{R_n}^{-1}(t)[h(t) - h^*(t)] \,|_0^q - \int_0^q [h(t) - h^*(t)]\mathrm{d}F_{R_n}^{-1}(t) \right| \\
&= \left| F_{R_n}^{-1}(q)[h(q) - h^*(q)] - \sum_{k\in K} \int_{\mathrm{I}_k} [h(t) - h^*(t)]\mathrm{d}F_{R_n}^{-1}(t) \right| = 0.
\end{aligned}$$

By (4.3.7),

$$\left| \int_0^q F_{R_n}^{-1}(t)\mathrm{d}h^*(t) - \int_0^q F_{S_n^c}^{-1}(t)\mathrm{d}h^*(t) \right| = \left| \sum_{k\in K} \left[\int_{\mathrm{I}_k} F_{R_n}^{-1}(t)\mathrm{d}h^*(t) - \int_{\mathrm{I}_k} F_{S_n^c}^{-1}(t)\mathrm{d}h^*(t) \right] \right| = 0.$$

Thus,

$$
\begin{aligned}
&\left|\int_0^q F_{S_n^*}^{-1}(t)\mathrm{d}h(t) - \int_0^q F_{S_n^c}^{-1}(t)\mathrm{d}h^*(t)\right| \\
&\leqslant \left|\int_0^q F_{S_n^*}^{-1}(t)\mathrm{d}h(t) - \int_0^q F_{R_n}^{-1}(t)\mathrm{d}h(t)\right| + \left|\int_0^q F_{R_n}^{-1}(t)\mathrm{d}h(t) - \int_0^q F_{R_n}^{-1}(t)\mathrm{d}h^*(t)\right| \\
&\quad + \left|\int_0^q F_{R_n}^{-1}(t)\mathrm{d}h^*(t) - \int_0^q F_{S_n^c}^{-1}(t)\mathrm{d}h^*(t)\right| \\
&\leqslant \max_{i=1,\dots,n}\{F_i^{-1}(q)\}.
\end{aligned}
\tag{4.6.2}
$$

On the other hand,

$$
\left|\int_q^1 F_{S_n^*}^{-1}(t)\mathrm{d}h(t) - \int_q^1 F_{S_n^c}^{-1}(t)\mathrm{d}h^*(t)\right| \leqslant \int_q^1 F_{S_n^c}^{-1}(t)\mathrm{d}h^*(t) = \sum_{i=1}^n \int_q^1 F_i^{-1}(t)\mathrm{d}h^*(t). \tag{4.6.3}
$$

By Condition A1, $s := \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \rho_{h^*}(X_i) > 0$. Then for the above $\varepsilon > 0$, there exists $N > 0$ such that for $n > N$,

$$
\frac{\sum_{i=1}^n \rho_{h^*}(X_i)}{n} > s - \varepsilon. \tag{4.6.4}
$$

Hence, for any $\varepsilon > 0$ and $n > \max\{N, 1/(1-h^*(q))$, from (4.6.1)–(4.6.4), we have

$$
\begin{aligned}
\left|\frac{\sup\{\rho_h(S) : S \in \mathcal{S}_n\}}{\sup\{\rho_{h^*}(S) : S \in \mathcal{S}_n\}} - 1\right| &\leqslant \frac{|\rho_h(S_n^*) - \rho_{h^*}(S_n^c)|}{\sum_{i=1}^n \rho_{h^*}(X_i)} \\
&\leqslant \frac{\max_{i=1,\dots,n}\{F_i^{-1}(q)\}}{\sum_{i=1}^n \rho_{h^*}(X_i)} + \frac{\sum_{i=1}^n \int_q^1 F_i^{-1}(t)\mathrm{d}h^*(t)}{\sum_{i=1}^n \rho_{h^*}(X_i)} \\
&\leqslant \frac{\varepsilon}{n(1-h^*(q))(s-\varepsilon)} + \frac{\varepsilon}{(s-\varepsilon)} \\
&\leqslant \frac{2\varepsilon}{(s-\varepsilon)}.
\end{aligned}
$$

As ε is arbitrary, (4.3.4) follows.

Case 3: $h \neq h^*$ in the intervals (a_k, b_k), $k \in K \subset \mathbb{N}$, where $\sup_{k\in K} b_k = 1$. Moreover, there exists a $p \in (0,1)$ such that $h(t) \neq h^*(t)$ for all $t \in [p, 1)$ and h^* is linear on $[p, 1]$ with slope $c > 0$.

Recall that $h(1^-) = h(1) = 1$ and $h^*(1^-) = h^*(1) = 1$. For any $\varepsilon > 0$, take $q \in [p, 1]$ such that

$$|h(q) - 1| < \frac{\varepsilon}{2}, \qquad |h^*(q) - 1| < \frac{\varepsilon}{2}, \tag{4.6.5}$$

$$\sup_{i\in\mathbb{N}} \int_q^1 F_i^{-1}(t)\mathrm{d}h^*(t) < c\varepsilon. \tag{4.6.6}$$

Equation (4.6.6) implies that

$$(1-q)\sup_{i\in\mathbb{N}} F_i^{-1}(q) < \varepsilon.$$

Let I_k in (4.3.6) be $(a_k, b_k) \cap [0, q]$. Then $\rho_{h^*}(S_n^c) = \rho_{h^*}(R_n)$. Similarly to Case 2, we have

$$\begin{aligned}
\left|\int_0^q F_{S_n^*}^{-1}(t)\mathrm{d}h(t) - \int_0^q F_{R_n}^{-1}(t)\mathrm{d}h(t)\right| &\leqslant \max_{i=1,\dots,n}\{F_i^{-1}(q)\},\\
\left|\int_0^q F_{R_n}^{-1}(t)\mathrm{d}h^*(t) - \int_0^q F_{S_n^c}^{-1}(t)\mathrm{d}h^*(t)\right| &= 0,\\
\left|\int_q^1 F_{S_n^*}^{-1}(t)\mathrm{d}h(t) - \int_q^1 F_{S_n^c}^{-1}(t)\mathrm{d}h^*(t)\right| &\leqslant \sum_{i=1}^n \int_q^1 F_i^{-1}(t)\mathrm{d}h^*(t).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\left|\int_0^q F_{R_n}^{-1}(t)\mathrm{d}h(t) - \int_0^q F_{R_n}^{-1}(t)\mathrm{d}h^*(t)\right| &= F_{R_n}^{-1}(q)\,|h(q) - h^*(q)|\\
&\leqslant F_{R_n}^{-1}(q)\varepsilon = \varepsilon \sum_{i=1}^n \frac{\int_p^1 F_i^{-1}(t)\mathrm{d}t}{1-p},
\end{aligned}$$

where the last inequality follows by (4.6.5). Thus, for any $\varepsilon > 0$, $n > \max\{N, 1/(1-q)\}$,

$$\begin{aligned}
&\left|\frac{\sup\{\rho_h(S) : S \in \mathcal{S}_n\}}{\sup\{\rho_{h^*}(S) : S \in \mathcal{S}_n\}} - 1\right|\\
&\leqslant \frac{\max_{i=1,\dots,n}\{F_i^{-1}(q)\}}{\sum_{i=1}^n \rho_{h^*}(X_i)} + \frac{\frac{\sum_{i=1}^n \int_p^1 F_i^{-1}(t)\mathrm{d}t}{1-p}\varepsilon}{\sum_{i=1}^n \rho_{h^*}(X_i)} + \frac{\sum_{i=1}^n \int_q^1 F_i^{-1}(t)\mathrm{d}h^*(t)}{\sum_{i=1}^n \rho_{h^*}(X_i)}\\
&\leqslant \frac{\varepsilon}{n(s-\varepsilon)(1-q)} + \frac{\frac{\varepsilon}{1-p}\sum_{i=1}^n \int_p^q F_i^{-1}(t)\mathrm{d}t + (\frac{\varepsilon}{1-p} + c)\sum_{i=1}^n \int_q^1 F_i^{-1}(t)\mathrm{d}t}{n(s-\varepsilon)}\\
&\leqslant \frac{\varepsilon}{n(s-\varepsilon)(1-q)} + \frac{\frac{\varepsilon n(q-p)}{1-p}\sup_{i\in\mathbb{N}} F_i^{-1}(q) + (\frac{\varepsilon}{1-p} + c)n\varepsilon}{n(s-\varepsilon)}\\
&\leqslant \frac{\varepsilon}{s-\varepsilon} + \frac{\varepsilon^2\frac{q-p}{(1-p)(1-q)} + (\frac{\varepsilon}{1-p} + c)\varepsilon}{s-\varepsilon}.
\end{aligned}$$

As $\varepsilon > 0$ is arbitrary, the result follows.

The proof in the case of general $h \in \mathcal{H}$.

Denote $\bar{\rho}_h(n) = \sup\{\rho_h(S) : S \in \mathcal{S}_n\}$ for any $h \in \mathcal{H}$. S_n^c is defined as in (4.3.5). Clearly $\rho_{h^*}(S_n^c) = \bar{\rho}_{h^*}(n)$. For any $h \in \mathcal{H}$, let $h_\delta \in \mathcal{H}$ be continuous such that $h_\delta \geqslant h$ on $[0,1]$ and $h_\delta \to h$ weakly as $\delta \to 0^+$. The existence of such h_δ is an exercise for mathematical analysis.

By Lemma A.5 of Wang et al. (2015), for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sup_{t \in [0,1]} |h_\delta^*(t) - h^*(t)| \leqslant \varepsilon. \tag{4.6.7}$$

Condition A2 implies that for any $\varepsilon > 0$, there exists $q \in (0,1)$ such that

$$\sup_{i \in \mathbb{N}} \int_q^1 F_i^{-1}(t) \mathrm{d}h^*(t) < \varepsilon.$$

Note that $\sup_{i \in \mathbb{N}} F_i^{-1}(q) < \frac{\varepsilon}{1-h^*(q)} < \infty$. Take $M = \sup_{i \in \mathbb{N}} F_i^{-1}(q)$. Then

$$\rho_{h^*}(S_n^c I_{\{S_n^c > Mn\}}) = \int_{\{S_n^c > Mn\}} F_{S_n^c}^{-1}(t) \mathrm{d}h^*(t) \leqslant \int_q^1 F_{S_n^c}^{-1}(t) \mathrm{d}h^*(t) = \sum_{i=1}^n \int_q^1 F_i^{-1}(t) \mathrm{d}h^*(t) \leqslant n\varepsilon.$$

Condition A1 implies that for $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ and $s > 0$ such that for $n \geqslant N_1$, $\rho_{h^*}(S_n^c) > ns$. By comonotonic additivity and monotonicity of distortion risk measures,

$$\rho_{h^*}(S_n^c) = \rho_{h^*}(S_n^c \wedge (Mn)) + \rho_{h^*}((S_n^c - Mn) I_{\{S_n^c > Mn\}}) \leqslant \rho_{h^*}(S_n^c \wedge (Mn)) + n\varepsilon.$$

Thus,

$$\frac{\rho_{h^*}(S_n^c \wedge (Mn))}{\rho_{h^*}(S_n^c)} \geqslant 1 - \frac{\varepsilon}{s}, \quad \text{for all } n \geqslant N_1. \tag{4.6.8}$$

Let $Y = S_n^c \wedge (Mn)$.

$$\begin{aligned}
\rho_{h^*}(S_n^c \wedge (Mn)) - \rho_{h_\delta^*}(S_n^c \wedge (Mn)) &= \int_0^1 F_Y^{-1}(t) \mathrm{d}h^*(t) - \int_0^1 F_Y^{-1}(t) \mathrm{d}h_\delta^*(t) \\
&= \int_0^1 F_Y^{-1}(t) \mathrm{d}(-(1-h^*(t))) - \int_0^1 F_Y^{-1}(t) \mathrm{d}(-(1-h_\delta^*(t))) \\
&= \int_0^1 [h_\delta^*(t) - h^*(t)] \mathrm{d}F_Y^{-1}(t) \leqslant \varepsilon M n,
\end{aligned}$$

where the last inequality follows from (4.6.7). Thus,

$$\frac{\rho_{h^*}(S_n^c \wedge (Mn)) - \rho_{h^*_\delta}(S_n^c \wedge (Mn))}{\rho_{h^*}(S_n^c \wedge (Mn))} \leqslant \frac{\varepsilon Mn}{(1-\varepsilon/s)ns} = \frac{\varepsilon M}{s-\varepsilon},$$

which implies

$$\frac{\rho_{h^*_\delta}(S_n^c \wedge (Mn))}{\rho_{h^*}(S_n^c \wedge (Mn))} \geqslant 1 - \frac{\varepsilon M}{s-\varepsilon} \quad \text{for all } n \geqslant N_1.$$

As $\rho_{h^*_\delta}(S_n^c \wedge (Mn)) \leqslant \rho_{h^*_\delta}(S_n^c)$ and by the above inequality, we have

$$\frac{\rho_{h^*_\delta}(S_n^c)}{\rho_{h^*}(S_n^c \wedge (Mn))} \geqslant 1 - \frac{\varepsilon M}{s-\varepsilon} \quad \text{for all } n \geqslant N_1. \tag{4.6.9}$$

From the first half of the proof, for any $\varepsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that for $n \geqslant N_2$,

$$\frac{\bar{\rho}_{h_\delta}(n)}{\bar{\rho}_{h^*_\delta}(n)} \geqslant 1 - \varepsilon. \tag{4.6.10}$$

Thus for any $\varepsilon > 0$, there exist $\delta > 0$ and $N = N_1 \vee N_2$ such that for $n \geqslant N$,

$$\begin{aligned}
\frac{\bar{\rho}_{h_\delta}(n)}{\rho_{h^*}(S_n^c)} &= \frac{\bar{\rho}_{h_\delta}(n)}{\bar{\rho}_{h^*_\delta}(n)} \times \frac{\bar{\rho}_{h^*_\delta}(n)}{\rho_{h^*}(S_n^c \wedge (Mn))} \times \frac{\rho_{h^*}(S_n^c \wedge (Mn))}{\rho_{h^*}(S_n^c)} \\
&= \frac{\bar{\rho}_{h_\delta}(n)}{\bar{\rho}_{h^*_\delta}(n)} \times \frac{\rho_{h^*_\delta}(S_n^c)}{\rho_{h^*}(S_n^c \wedge (Mn))} \times \frac{\rho_{h^*}(S_n^c \wedge (Mn))}{\rho_{h^*}(S_n^c)} \\
&\geqslant (1-\varepsilon)\left(1 - \frac{\varepsilon M}{s-\varepsilon}\right)\left(1 - \frac{\varepsilon}{s}\right) \\
&\geqslant 1 - \left(1 + \frac{M}{s-\varepsilon} + \frac{1}{s}\right)\varepsilon,
\end{aligned}$$

where the inequality follows from (4.6.8)-(4.6.10). Note that $\bar{\rho}_h(n) \geqslant \bar{\rho}_{h_\delta}(n)$. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geqslant N$,

$$\frac{\bar{\rho}_h(n)}{\bar{\rho}_{h^*}(n)} \geqslant 1 - \left(1 + \frac{M}{s-\varepsilon} + \frac{1}{s}\right)\varepsilon,$$

that is,

$$\lim_{n\to\infty} \frac{\sup\{\rho_h(S) : S \in \mathcal{S}_n\}}{\sup\{\rho_{h^*}(S) : S \in \mathcal{S}_n\}} = 1. \qquad \square$$

Chapter 5

Collective Risk Models with Dependence Uncertainty

5.1 Introduction

In the previous chapter, we showed the asymptotic equivalence results of the classes of distortion risk measures and convex risk measures under dependence uncertainty for *individual risk models*. In this chapter, we address a similar problem of measuring large insurance portfolios using the risk measure VaR under model uncertainty at the level of the dependence among individual claims and the number of claims.

The aggregate loss of an insurance company (the total amount paid on all claims occurring over a fixed period) is often modelled by a sum of random variables,

$$S_N = Y_1 + \cdots + Y_N, \tag{5.1.1}$$

where $Y_1, Y_2, \ldots$ are non-negative random variables representing the individual claims and N (random or deterministic), the number of claims, takes values in non-negative integers.

When N is a non-random positive integer, (5.1.1) is called an *individual risk model*, in which $Y_1, Y_2, \ldots$ represent losses from each individual policy and N is the number of policies. When N itself is random, (5.1.1) is called a *collective risk model*. For portfolio analysis, individual risk models are a priori the most natural, whereas for ruin theoretic problems, collective risk models are more natural. In the classic treatment of collective risk models, $Y_1, Y_2, \ldots$ are iid random variables representing individual claim sizes, and the counting random variable N is assumed to be independent of $(Y_1, Y_2, \ldots)$. This classic assumption on the independence of $N, Y_1, Y_2, \ldots$ provides great mathematical convenience and elegance, as well as nice interpretations.

In some situations, the claims or losses $Y_1, Y_2, \ldots,$ in individual risk models or collective risk models are dependent, and they may also be dependent on the number of claims N. Think about, for instance, the losses from wind and flood damage in a certain region; see Kousky and Cooke (2009) for related real-life examples. In the context of collective risk models or the closely related setting of compound Poisson processes, certain types of dependence among $N, Y_1, Y_2, \ldots$ are studied. For instance, see Cheung et al. (2010), Albrecher et al. (2014) and Landriault et al. (2014) for recent development on dependent Sparre Anderson risk models.

Due to the high dimensionality of the joint model and sometimes limited data, it is often difficult to accurately model or justify a dependence structure. When N in (5.1.1) is a non-random number n, see Chapter 4 for risk aggregation under dependence uncertainty in individual risk models. In this chapter, we bring the framework of dependence uncertainty into collective risk models. We assume that $Y_1, Y_2, \ldots$ are identically distributed as in classic collective risk models, but we do not assume a particular model for the dependence structure among random variables in (5.1.1). Two different practical settings will be considered:

(i) N is independent of $Y_1, Y_2, \ldots$ and the dependence structure of $Y_1, Y_2, \ldots$ is unknown.

(ii) The dependence structure of $N, Y_1, Y_2, \ldots$ is unknown.

From the perspective of risk management, we are particularly interested in quantifying S_N by certain *risk measures* under dependence uncertainty, a crucial concern for risk management in the presence of model uncertainty. Risk measures for individual and collective risk models are well studied; see for instance Cai and Tan (2007) for optimal stop-loss reinsurance for these models under VaR and ES, and Hürlimann (2003) for ES bound for compound Poisson risks. It is well-known that an analytical calculation of the distribution of S_N, as well as $\mathrm{VaR}_\alpha(S_N)$ and $\mathrm{ES}_\alpha(S_N)$, is generally unavailable (see Klugman et al. (2012)). Approximation, simulation or numerical calculation is often needed.

In this chapter, we study the worst-case values of $\mathrm{VaR}_\alpha(S_N)$ and $\mathrm{ES}_\alpha(S_N)$, under the two settings (i) and (ii) above. The recent literature on dependence uncertainty has focused on the individual risk model, in which $N = n$ in (5.1.1) is non-random; see Section 4.1 for details. Meanwhile, as a well-known result, the worst-case value of $\mathrm{ES}_\alpha(S_N)$ for a non-random $N = n$ is simply equal to the sum of the individual ES_α values, and this worst-case value is attained by comonotonic $Y_1, \ldots, Y_n$.

Assuming N is bounded by a fixed number $n \in \mathbb{N}$, a collective risk model can be reduced to an individual risk model as

$$S_N = \sum_{i=1}^{N} Y_i = \sum_{i=1}^{n} Y_i \mathrm{I}_{\{N \geqslant i\}}.$$

Due to the dependence induced by N among $Y_i \mathrm{I}_{\{N \geqslant i\}}$, $i = 1, \ldots, n$, using a collective model as in setting (i) can be seen as one of the ways to introduce partial dependence information into risk aggregation for individual risk models; see Section 5.7 for details and a comparison.

The main contributions of this chapter are summarized as follows. Based on the classic theory of stochastic orders, we first derive some convex ordering inequalities for collective risk models and thereby obtain analytical formulas for the worst-case values of ES. Using the results on ES for collective risk models, we are able to study the worst-case values of $\mathrm{VaR}_\alpha(S_N)$ and $\mathrm{ES}_\alpha(S_N)$ as $\mathbb{E}[N]$ increases to infinity, that is, a very large insurance portfolio. For simplicity the reader may think of the case where N is Poisson-distributed with parameter $\mathbb{E}[N]$, the most classic choice for the counting random variable N. In both settings (i) and (ii), under some moment and convergence conditions, we show that the worst-case values of $\mathrm{VaR}_\alpha(S_N)$ and $\mathrm{ES}_\alpha(S_N)$ enjoy very nice asymptotic properties. In particular, one can approximate them using the asymptotic equivalent $\mathbb{E}[N]\mathrm{ES}_\alpha(Y_1)$ and the convergence rates are obtained in both settings. The results can be used to approximate VaR and ES of a large insurance portfolio since it is straightforward to calculate $\mathbb{E}[N]\mathrm{ES}_\alpha(Y_1)$. Mathematically, our results generalize the asymptotic equivalence results for homogeneous individual risk models in Wang and Wang (2015).

The rest of this chapter is organized as follows. In Section 5.2, we present basic notation and definitions, stochastic orders, and some preliminary results on VaR-ES risk aggregation with dependence uncertainty. In Section 5.3, we study collective risk models with dependence uncertainty and obtain formulas for the worst-case ES. In Section 5.4, we establish asymptotic equivalence results under setting (i) and give the convergence rate under this setting. In Section 5.5, asymptotic equivalence results under setting (ii) are given, albeit stronger regularity conditions are needed compared to the case of setting (i). A brief conclusion is drawn in Section 5.6. Additional discussions are given in Section 5.7.

5.2 Preliminaries

5.2.1 Notation

Assume that the atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is rich enough such that for any random variable X that appears in this chapter, there exists a random variable independent of X. For a distribution F, let $\mathcal{X}_F$ be the set of random variables with distribution F, and for $N \in L^0$, let $\mathcal{X}_F^N$ be the set of random variables in $\mathcal{X}_F$ independent of N. Let $\mathcal{X}_0$ be the set of counting random variables (i.e. taking value in $\{0, 1, \dots, \}$). Throughout this chapter, VaR and ES are defined as in Definition 1.3.

For a sequence $\mathbf{Y} = (Y_i,\ i \in \mathbb{N})$, we write (with a slight abuse of notation) $\mathbf{Y} \subset \mathcal{X}_F$ if $Y_i \in \mathcal{X}_F,\ i \in \mathbb{N}$, and similarly for $\mathbf{Y} \subset \mathcal{X}_F^N$. Denote by $\mathcal{Y}_F^N$ the set of random sequences with marginal distribution F and independent of N, that is,

$$\mathcal{Y}_F^N = \{(Y_1, Y_2, \dots) \subset \mathcal{X}_F : (Y_1, Y_2, \dots) \text{ is independent of } N\}.$$

Note that for a sequence $\mathbf{Y} = (Y_i,\ i \in \mathbb{N})$, there is a subtle difference between $\mathbf{Y} \subset \mathcal{X}_F^N$ and $\mathbf{Y} \in \mathcal{Y}_F^N$: the latter requires independence between the sequence $\mathbf{Y}$ and N, whereas the former only requires pair-wise independence between N and Y_i for $i \in \mathbb{N}$.

Throughout this chapter, for $N \in \mathcal{X}_0$ and $\mathbf{Y} = (Y_i,\ i \in \mathbb{N}) \subset L^0$, write

$$S_N = \sum_{i=1}^{N} Y_i,$$

where by convention $\sum_{i=1}^{0} Y_i = 0$. In the following, whenever S_N or S_n appears, it implicitly depends on $\mathbf{Y} = (Y_i,\ i \in \mathbb{N})$ which should be clear from the context.

In collective risk models, $Y_i,\ i \in \mathbb{N}$ are always assumed to be identically distributed, since Y_i represents the claim size of the i-th claim from a pool of policies, not the loss from a specific policy. We also assume $Y_i,\ i \in \mathbb{N}$ to be integrable; otherwise $\mathrm{ES}_\alpha(Y_1)$ is infinite for $\alpha \in (0, 1)$. In the case when the claim size Y_1 is not integrable, ES is not a proper risk measure to use in insurance practice.

5.2.2 Stochastic Orders

Definition 5.1. For $X, Y \in L^1$, X is said to be smaller than Y in *convex order* (resp. *increasing convex order*), denoted by $X \leqslant_{\mathrm{cx}} Y$ (resp. $X \leqslant_{\mathrm{icx}} Y$), if $\mathbb{E}[f(X)] \leqslant \mathbb{E}[f(Y)]$ for all convex functions (resp. increasing convex functions) $f : \mathbb{R} \to \mathbb{R}$, provided that the above expectations exist (can be infinity).

For a general introduction to convex order and increasing convex order, see Müller and Stoyan (2002) and Shaked and Shanthikumar (2007). Convex order is closely associated with the concept of *comonotonicity* (see Section 1.2.2 for definition).

Given random variables $X_1, X_2, \ldots, X_n$, the following lemma presents an upper bound for sums $S_n = X_1 + X_2 + \cdots + X_n$ in the sense of convex order; see Theorem 7 of Dhaene et al. (2002) and Theorem 3.5 of Rüschendorf (2013). In particular, Rüschendorf (2013, Chapter 3) contains two different proofs and a brief history of this celebrated result.

Lemma 5.1. *For any random vector $(X_1, \ldots, X_n) \in (L^1)^n$ we have*

$$X_1 + \cdots + X_n \leqslant_{\mathrm{cx}} X_1^c + \cdots + X_n^c,$$

where $X_i^c \stackrel{d}{=} X_i$, $i = 1, \ldots, n$, and $X_1^c, \ldots, X_n^c \in L^1$ are comonotonic.

Another property about increasing convex order and comonotonicity is given in the following lemma, which is Corollary 3.28 (c) of Rüschendorf (2013).

Lemma 5.2. *For $X, Y, X^c, Y^c \in L^1$ such that X^c, Y^c are comonotonic, $X \stackrel{d}{=} X^c$, $Y \stackrel{d}{=} Y^c$ and $X^cY^c \in L^1$, we have*

$$XY \leqslant_{\mathrm{icx}} X^cY^c.$$

The stochastic inequality in the above lemma holds for every monotonic supermodular function of X and Y; see Theorem 2 of Tchen (1980) and Theorem 2.1 of Puccetti and Wang (2015).

In this chapter, we will frequently use some well-known properties of ES; see Section 1.2.2. The following lemma is well known in the literature of convex order (see Theorem 4.A.3 of Shaked and Shanthikumar (2007)).

Lemma 5.3. *For $X, Y \in L^1$, $X \leqslant_{\mathrm{icx}} Y$ if and only if $\mathrm{ES}_\alpha(X) \leqslant \mathrm{ES}_\alpha(Y)$ for all $\alpha \in (0, 1)$.*

As a consequence of Lemma 5.3, for $\alpha \in (0, 1)$, ES_α preserves increasing convex order (and hence convex order). Another property that will be used later is the L^1-continuity of ES below; for a proof of this property, see, for instance, Svindland (2008).

Lemma 5.4. *For $\alpha \in (0, 1)$, $\mathrm{ES}_\alpha : L^1 \to \mathbb{R}$ is continuous with respect to the L^1-norm.*

Recalling the definition of the L^1-continuity, the above lemma means that for a sequence of random variables $X_1, X_2, \ldots$ and $X \in L^1$, as $n \to \infty$, $\mathbb{E}[|X_n - X|] \to 0$ implies that $\mathrm{ES}_\alpha(X_n) \to \mathrm{ES}_\alpha(X)$.

5.2.3 VaR-ES Asymptotic Equivalence in Risk Aggregation

We give some preliminary results on the VaR-ES asymptotic equivalence in risk aggregation, which will be useful in the proofs of the main results in this chapter.

Lemma 5.5 (Corollary 3.7 of Wang and Wang (2015))**.** *For any distribution F and $Y \in \mathcal{X}_F$,*

$$\lim_{n\to\infty} \frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_\alpha\left(S_n\right)}{n} = \mathrm{ES}_\alpha\left(Y\right), \qquad \alpha \in (0,1). \tag{5.2.1}$$

The result in (5.2.1) can be rewritten as

$$\lim_{n\to\infty} \frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_\alpha\left(S_n\right)}{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{ES}_\alpha\left(S_n\right)} = 1, \qquad \alpha \in (0,1), \tag{5.2.2}$$

provided that $0 < \mathrm{ES}_\alpha(Y) < \infty$, $Y \in \mathcal{X}_F$. The convergence rate of (5.2.2) is given in the following lemma.

Lemma 5.6 (Corollary 3.8 of Wang and Wang (2015))**.** *Suppose that the distribution F has finite p-th moment, $p \geqslant 1$, and* ES *at level $\alpha \in (0,1)$ is non-zero. Then as $n \to \infty$,*

$$\frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_\alpha\left(S_n\right)}{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{ES}_\alpha\left(S_n\right)} = 1 - o\left(n^{1/p-1}\right).$$

5.3 Collective Risk Models with Dependence Uncertainty

5.3.1 Setup and a Motivating Example

In this section, we study the worst-case values of VaR and ES for collective risk models. As mentioned in the introduction, we consider two different settings of dependence uncertainty:

(i) the number of claims N is independent of the claim sizes $Y_1, Y_2, \ldots$ and the dependence structure of $Y_1, Y_2, \ldots$ is unknown;

(ii) the dependence structure of $N, Y_1, Y_2, \ldots$ is unknown.

We refer to the setting (i) as the *classic* collective risk model with dependence uncertainty and to the setting (ii) as the *generalized* collective risk model with dependence uncertainty. Using the notation introduced in Section 5.2, for some distribution F on $\mathbb{R}_+$ (i.e. non-negative claim sizes), setting (i) reads as $\mathbf{Y} \in \mathcal{Y}_F^N$ and setting (ii) reads as $\mathbf{Y} \subset \mathcal{X}_F$. The quantities of interest in setting (i) are

$$\sup_{\mathbf{Y}\in\mathcal{Y}_F^N} \mathrm{VaR}_\alpha(S_N) \quad \text{and} \quad \sup_{\mathbf{Y}\in\mathcal{Y}_F^N} \mathrm{ES}_\alpha(S_N), \tag{5.3.1}$$

and the quantities of interest in setting (ii) are

$$\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_\alpha(S_N) \quad \text{and} \quad \sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{ES}_\alpha(S_N). \tag{5.3.2}$$

It turns out that under both settings (i) and (ii), the worst-case value of ES is straightforward to calculate, whereas an analytical formula for the worst-case value of VaR is not available. This is similar to the well-studied case of individual risk models; see Embrechts et al. (2014) for a review on worst-case VaR aggregation when N is non-random.

Before we carry out a theoretical treatment, we illustrate with a simple example in the theory of loss models by comparing an individual risk model and a corresponding collective risk model formulation. Assume both models admit dependence uncertainty, and we evaluate worst-case ES for both models as in (5.3.2). We shall see that the ES bound is largely reduced by knowing the distribution of the claim frequency, as opposed to an *uncertain distribution of the claim frequency* implied by the individual risk model with dependence uncertainty.

Example 5.1. Let $n = 40000$. Consider an individual risk model

$$S = \sum_{i=1}^{n} X_i,$$

where for $i = 1, \ldots, n$, X_i follows a distribution F such that $\mathbb{P}(X_i > x) = \frac{1}{1000} e^{-x}$, $x \geqslant 0$. If we assume that $X_1, \ldots, X_n$ are independent, then the collective reformulation of S is given by

$$S_N = \sum_{i=1}^{N} Y_i,$$

where N follows the Poisson distribution with parameter $\lambda = 40$ (denoted by Pois(40)), Y_i follows an Exponential distribution with mean 1 (denoted by Expo(1)), $i \in \mathbb{N}$, and $N, Y_1, Y_2, \dots$ are independent. Below we assume that only N and $(Y_i,\ i \in \mathbb{N})$ are independent, but the dependence among $X_1, \dots, X_n$ and the dependence among $Y_1, Y_2, \dots,$ are uncertain. Take $\alpha = 0.95$. To evaluate the corresponding worst-case ES_α values, we have[1]

$$\sup_{\mathbf{Y} \in \mathcal{Y}_F^N} \mathrm{ES}_\alpha \left(\sum_{i=1}^N Y_i \right) = 164.09,$$
$$\sup_{X_i \in \mathcal{X}_F, i \leqslant n} \mathrm{ES}_\alpha \left(\sum_{i=1}^n X_i \right) = 800.$$

As we can see from the numerical results, the knowledge of $N \sim$ Pois(40) greatly reduces the worst-case ES value, as compared to the individual risk model. In the sequel, we shall investigate the VaR and ES bounds for collective risk models under dependence uncertainty.

5.3.2 VaR and ES Bounds for Collective Risk Models

In this section we establish some explicit formulas for VaR and ES bounds in (5.3.1) and (5.3.2). We first provide a simple result on convex order for collective risk models with unknown dependence.

Lemma 5.7. *Suppose that $(Y_i, N) \in L^1 \times \mathcal{X}_0$, $i \in \mathbb{N}$, have identical joint distributions and $NY_1 \in L^1$. We have*

$$\sum_{i=1}^N Y_i \leqslant_{\mathrm{cx}} NY_1. \tag{5.3.3}$$

Proof. First, one can easily verify $\mathbb{E}[\sum_{i=1}^N Y_i] = \mathbb{E}[NY_1]$ and hence both sides of (5.3.3) are in L^1. Let $D = \{n \in \{0, 1, \dots\} : \mathbb{P}(N = n) > 0\}$ be the range of N. Denote by F_n the conditional distribution of Y_1 given $N = n$ for $n \in D$. Let f be a convex function such that both $\mathbb{E}[f(\sum_{i=1}^N Y_i)]$ and $\mathbb{E}[f(NY_1)]$ are properly defined. For $n \in D$, there exist some U[0,1]-distributed random variables $U_1^n, \dots, U_n^n$ such that

$$\mathbb{E}[f(Y_1 + \dots + Y_n) | N = n] = \mathbb{E}[f(F_n^{-1}(U_1^n) + \dots + F_n^{-1}(U_n^n))].$$

[1] the first value is calculated via Theorem 5.9 (see below) and the average of 100 repetitions of simulation with a sample of size 100,000, and the second value is calculated analytically.

It follows from Lemma 5.1 that

$$\mathbb{E}[f(Y_1 + \cdots + Y_n)|N = n] \leqslant \mathbb{E}[f(nF_n^{-1}(U_1^n))] = \mathbb{E}[f(nY_1)|N = n].$$

Summing up over $n \in D$ yields

$$\mathbb{E}[f(Y_1 + \cdots + Y_N)] \leqslant \mathbb{E}[f(NY_1)],$$

and hence by definition, (5.3.3) holds. $\square$

As a special case of Lemma 5.7, if N is in L^1 and independent of the identically distributed random variables $Y_1, Y_2, \ldots \in L^1$, then (5.3.3) holds. This particular result will be used later.

To deal with setting (ii) in which the dependence structure between N and $Y_1, Y_2, \ldots$ is unspecified, we give a result in the following lemma on increasing convex order instead of convex order. Note that for $X, Y \in L^1$, $X \leqslant_{\mathrm{cx}} Y$ implies that $\mathbb{E}[X] = \mathbb{E}[Y]$. Since $\mathbb{E}[S_N]$ depends on the dependence structure between N and $Y_1, Y_2, \ldots$, convex order between collective risk models under different dependence structures cannot be expected.

Lemma 5.8. *Suppose that the distribution F on $\mathbb{R}_+$ has finite second moment, and $N \in \mathcal{X}_0 \cap L^2$. For $Y_1, Y_2, \ldots \in \mathcal{X}_F$, we have*

$$\sum_{i=1}^{N} Y_i \leqslant_{\mathrm{icx}} NY,$$

where $Y \in \mathcal{X}_F$ and N, Y are comonotonic.

Proof. Note that $NY \in L^1$ by Hölder's inequality. Define $X_n = \sum_{i=1}^{n} Y_i \mathrm{I}_{\{N \geqslant i\}}$, $n \in \mathbb{N}$ and $X_\infty = \sum_{i=1}^{\infty} Y_i \mathrm{I}_{\{N \geqslant i\}}$. Note that $\mathbb{P}(X_\infty > X_n) \to 0$ as $n \to \infty$, and hence $\mathbb{P}(X_\infty < \infty) = 1$. Thus X_∞ is a finite random variable. Then we have $X_n \to X_\infty$ almost surely and hence $X_n \to X_\infty$ in distribution. Since $F \to F^{-1}(\gamma)$ is weakly continuous at each F_0 for which $s \to F_0^{-1}(s)$ is continuous at $s = \gamma$ (see e.g. Cont et al. (2010)), we have

$$\mathrm{VaR}_\gamma(X_n) \to \mathrm{VaR}_\gamma(X_\infty) \quad \text{almost everywhere in } \gamma \in [0, 1]. \tag{5.3.4}$$

For any $\mathbf{Y} \subset \mathcal{X}_F$ and any $\alpha \in (0,1)$, we have

$$
\begin{aligned}
\mathrm{ES}_\alpha\left(\sum_{i=1}^N Y_i\right) = \mathrm{ES}_\alpha\left(\sum_{i=1}^\infty Y_i \mathrm{I}_{\{N \geqslant i\}}\right) &= \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_\gamma(X_\infty)\mathrm{d}\gamma \\
\text{(by (5.3.4))} \quad &= \frac{1}{1-\alpha}\int_\alpha^1 \lim_{n\to\infty} \mathrm{VaR}_\gamma(X_n)\mathrm{d}\gamma \\
\text{(Fatou's Lemma)} \quad &\leqslant \liminf_{n\to\infty} \mathrm{ES}_\alpha\left(X_n\right) \\
\text{(subadditivity of ES)} \quad &\leqslant \liminf_{n\to\infty} \sum_{i=1}^n \mathrm{ES}_\alpha\left(Y_i \mathrm{I}_{\{N \geqslant i\}}\right) \\
\text{(by Lemmas 5.2 and 5.3)} \quad &\leqslant \liminf_{n\to\infty} \sum_{i=1}^n \mathrm{ES}_\alpha\left(Y \mathrm{I}_{\{N \geqslant i\}}\right) \\
\text{(comonotonic additivity of ES)} \quad &= \liminf_{n\to\infty} \mathrm{ES}_\alpha\left(\sum_{i=1}^n Y \mathrm{I}_{\{N \geqslant i\}}\right) \\
\text{(L^1-continuity of ES)} \quad &= \mathrm{ES}_\alpha\left(NY\right).
\end{aligned}
$$

Since $\mathrm{ES}_\alpha\left(\sum_{i=1}^N Y_i\right) \leqslant \mathrm{ES}_\alpha\left(NY\right)$ for all $\alpha \in (0,1)$, by Lemma 5.3, we have $\sum_{i=1}^N Y_i \leqslant_{\mathrm{icx}} NY$. $\square$

Remark 5.1. Using the same proof, the stochastic inequality in Lemma 5.8 can be generalized to the random sum of non-identically distributed random variables as follows. Suppose that $N \in \mathcal{X}_0$, $Y_i \geqslant 0$, $i \in \mathbb{N}$, and $\sum_{i=1}^N Y_i^c \in L^1$, where $Y_i^c \overset{d}{=} Y_i$, $i \in \mathbb{N}$, and $Y_1^c, Y_2^c, \ldots$ and N are comonotonic. Then we have

$$\sum_{i=1}^N Y_i \leqslant_{\mathrm{icx}} \sum_{i=1}^N Y_i^c.$$

With the help of Lemmas 5.7 and 5.8, we arrive at the worst-case values of ES for collective risk models under dependence uncertainty.

Theorem 5.9. *Suppose that F is a distribution on $\mathbb{R}_+$, $N \in \mathcal{X}_0$ and $Y, Y^* \in \mathcal{X}_F$ such that N, Y are independent and N, Y^* are comonotonic.*

(i) If $Y, N \in L^1$, then

$$\sup_{\mathbf{Y} \in \mathcal{Y}_F^N} \mathrm{ES}_\alpha(S_N) = \mathrm{ES}_\alpha(NY), \quad \alpha \in (0,1). \tag{5.3.5}$$

(ii) If $Y, N \in L^2$, then

$$\sup_{\mathbf{Y} \subset \mathcal{X}_F} \mathrm{ES}_\alpha(S_N) = \mathrm{ES}_\alpha(NY^*), \quad \alpha \in (0,1). \tag{5.3.6}$$

Proof. Note that $NY \in L^1$ since N, Y are independent. Since $\mathbf{Y} \subset \mathcal{X}_F^N$ for any $\mathbf{Y} \in \mathcal{Y}_F^N$, we have

$$\sup_{\mathbf{Y} \in \mathcal{Y}_F^N} \mathrm{ES}_\alpha(S_N) \leqslant \sup_{\mathbf{Y} \subset \mathcal{X}_F^N} \mathrm{ES}_\alpha(S_N).$$

By Lemma 5.3, ES preserves increasing convex order. Further, by Lemmas 5.7 and 5.8, we have

$$\sup_{\mathbf{Y} \in \mathcal{Y}_F^N} \mathrm{ES}_\alpha(S_N) \leqslant \sup_{\mathbf{Y} \subset \mathcal{X}_F^N} \mathrm{ES}_\alpha(S_N) \leqslant \mathrm{ES}_\alpha(NY) \quad \text{and} \quad \sup_{\mathbf{Y} \subset \mathcal{X}_F} \mathrm{ES}_\alpha(S_N) \leqslant \mathrm{ES}_\alpha(NY^*).$$

It suffices to take $Y_1, Y_2, \ldots$ to be identical to $Y \in \mathcal{X}_F^N$ to show that $\sup_{\mathbf{Y} \in \mathcal{Y}_F^N} \mathrm{ES}_\alpha(S_N) \geqslant \mathrm{ES}_\alpha(NY)$ in (i) and to take $Y_1, Y_2, \ldots$ to be identical to $Y^* \in \mathcal{X}_F$ to show $\sup_{\mathbf{Y} \subset \mathcal{X}_F} \mathrm{ES}_\alpha(S_N) \geqslant \mathrm{ES}_\alpha(NY^*)$ in (ii). $\square$

The results in Theorem 5.9 are consistent with simple intuition. Assume that the riskiness of an insurance portfolio is measured by an ES. If the number of claims and the claim sizes are independent, then, in the worst-case dependence scenario, all claims are comonotonic. If the number of claims and the claim sizes are also dependent, then in the worst-case dependence scenario, all claims are comonotonic and they are further comonotonic with the number of claims. This could for instance be close to reality in the case of insurance losses from flood damage in an area, where the claim sizes and the number of claims are largely determined by the magnitude of the flood, and hence they are all positively correlated. Thus, the portfolio of insurance policies with heavy positive dependence has the most dangerous dependence structure, if an ES is the risk measure in use. Note that such an intuition is not valid for the risk measure VaR.

The values of $\mathrm{ES}_\alpha(NY)$ and $\mathrm{ES}_\alpha(NY^*)$ in (5.3.5) and (5.3.6) are straightforward to calculate. For (5.3.5), one needs to calculate the distribution of NY, which is the product of two independent random variables. This involves a one-step convolution after a logarithm transformation. For (5.3.6), note that $NY^* \stackrel{d}{=} G^{-1}(U)F^{-1}(U)$, where U is U$[0,1]$-distributed and G is the distribution of N. In that case, its ES is simply

$$\mathrm{ES}_\alpha(NY^*) = \frac{1}{1-\alpha} \int_\alpha^1 G^{-1}(u)F^{-1}(u)\mathrm{d}u,$$

which is as simple as calculating the ES of any known distribution.

The following corollary gives an ES ordering for an individual risk model with dependence uncertainty, a collective risk model under setting (i), and a collective risk model under setting (ii).

Corollary 5.10. *Suppose that F is a distribution on $\mathbb{R}_+$ with finite first moment, $N \in X_0$ and $\mathbb{E}[N] \in \mathbb{N}$. We have the following orders*

$$\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{ES}_\alpha\left(S_{\mathbb{E}[N]}\right) \leqslant \sup_{\mathbf{Y}\in\mathcal{Y}_F^N} \mathrm{ES}_\alpha(S_N) \leqslant \sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{ES}_\alpha(S_N), \quad \alpha \in (0,1). \tag{5.3.7}$$

Proof. Since $\mathbf{Y} \in \mathcal{Y}_F^N$ implies $\mathbf{Y} \subset \mathcal{X}_F$, the second inequality follows immediately. To show the first inequality, take $Y \in \mathcal{X}_F^N$. Note that from the properties of ES,

$$\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{ES}_\alpha\left(S_{\mathbb{E}[N]}\right) = \mathbb{E}[N]\mathrm{ES}_\alpha(Y) = \mathrm{ES}_\alpha(\mathbb{E}[N]Y),$$

and from Theorem 5.9,

$$\sup_{\mathbf{Y}\in\mathcal{Y}_F^N} \mathrm{ES}_\alpha\left(S_N\right) = \mathrm{ES}_\alpha(NY).$$

By Theorem 3.A.33 of Shaked and Shanthikumar (2007), $\mathbb{E}[N]Y \leqslant_{\mathrm{cx}} NY$. The rest of the proof follows since ES preserves convex order as in Lemma 5.3. $\square$

In the case of $N, Y \in L^2$, the order in (5.3.7) can be formulated as follows. For $N \in \mathcal{X}_0$, $Y \overset{d}{=} Y^*$ such that N, Y are independent and N, Y^* are comonotonic, we have

$$\mathbb{E}[N]\mathrm{ES}_\alpha(Y) \leqslant \mathrm{ES}_\alpha(NY) \leqslant \mathrm{ES}_\alpha(NY^*), \quad \alpha \in (0,1). \tag{5.3.8}$$

As for the problem of the worst-case value of VaR for collective risk models, there is no simple analytical formula, as expected from classic results on dependence uncertainty. Note that VaR_α is dominated by ES_α, for $\alpha \in (0,1)$; thus $\mathrm{VaR}_\alpha(S_N) \leqslant \mathrm{ES}_\alpha(S_N)$ for all model settings. From Theorem 5.9, we have

$$\sup_{\mathbf{Y}\in\mathcal{Y}_F^N} \mathrm{VaR}_\alpha(S_N) \leqslant \mathrm{ES}_\alpha(NY) \quad \text{and} \quad \sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_\alpha(S_N) \leqslant \mathrm{ES}_\alpha(NY^*), \tag{5.3.9}$$

where F, N, Y and Y^* are as in Theorem 5.9. In the next two sections, we will see that

$$\sup_{\mathbf{Y}\in\mathcal{Y}_F^N} \mathrm{VaR}_\alpha(S_N) \approx \mathrm{ES}_\alpha(NY) \quad \text{and} \quad \sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_\alpha(S_N) \approx \mathrm{ES}_\alpha(NY^*),$$

if N is large (in some sense). That is, the inequalities in (5.3.9) are almost sharp and can be used to approximate VaR.

Remark 5.2. In Lemma 5.8 and Theorem 5.9 (ii), we require $Y, N \in L^2$ so that $NY^* \in L^1$; recall that L^1 is the domain of ES_α. One may also use the slightly more general assumption that $N \in L^p$ and $Y \in L^q$ for some $p, q > 1$ such that $1/p + 1/q = 1$.

5.4 Asymptotic Results for Classic Collective Risk Models

5.4.1 Setup and Objectives

The rest of this chapter is dedicated to the study of an analog of the asymptotic equivalence in (5.2.2) for collective risk models. Recall that throughout we write

$$S_{N(v)} = \sum_{i=1}^{N(v)} Y_i, \quad \mathbf{Y} = (Y_1, Y_2, \ldots).$$

For some distribution F on $\mathbb{R}_+$, and a counting random variable $N(v)$ with parameter v, the analog of (5.2.2) in setting (i) is

$$\lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{ES}_\alpha\left(S_{N(v)}\right)} = 1, \tag{5.4.1}$$

and the analog of (5.2.2) in setting (ii) is

$$\lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{ES}_\alpha\left(S_{N(v)}\right)} = 1. \tag{5.4.2}$$

Here, $v \to \infty$ indicates that the expected number of claims goes to infinity. The parameter v is interpreted as the *volume* of the insurance portfolio, and it can be chosen as, for instance, $\mathbb{E}[N(v)]$.

One of the key assumptions we propose is $N(v)/v \to 1$ in L^1. This assumption naturally holds if $N(v)$ is a Poisson random variable with parameter $v > 0$, or $N(v)$ is the partial sum of a short-range dependent stationary sequence (so that a law of large numbers holds). Indeed, the problem we study in this chapter first appeared as a question of measuring large insurance portfolios under dependence uncertainty, where $N(v)$ is a Poisson random variable with a large parameter. Moreover, an insurance company can analyze effects from potential extension of business by measuring the insurance portfolio as v increases.

Results under setting (i) are presented in this section and results under setting (ii) are given in Section 5.5 below. Since $v\mathrm{ES}_\alpha(Y)$ is straightforward to calculate and thus serves as a basis for approximation of the two worst-case values of interest, we present our results in terms of the two ratios

$$\frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha(Y)} \quad \text{and} \quad \frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{ES}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha(Y)}.$$

We also establish convergence rates in both cases.

5.4.2 VaR-ES Asymptotic Equivalence

Theorem 5.11. *Suppose that the distribution F on $\mathbb{R}_+$ has finite first moment, $Y \in \mathcal{X}_F$, and $\{N(v),\ v \geqslant 0\} \subset \mathcal{X}_0$ such that $N(v)/v \to 1$ in L^1 as $v \to \infty$. Then for $\alpha \in (0,1)$,*

$$\lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{v} = \lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{ES}_\alpha\left(S_{N(v)}\right)}{v} = \mathrm{ES}_\alpha\left(Y\right). \tag{5.4.3}$$

Proof. By the independence of $N(v)$ and Y, and $\frac{N(v)}{v} \xrightarrow{L^1} 1$, we have

$$\mathbb{E}\left|\frac{N(v)Y}{v} - Y\right| \leqslant \mathbb{E}\left|\frac{N(v)}{v} - 1\right| \cdot \mathbb{E}\left[Y\right] \to 0, \qquad \text{as } v \to \infty.$$

Hence, $\frac{N(v)Y}{v} \xrightarrow{L^1} Y$. Continuity of ES with respect to the L^1-norm implies

$$\lim_{v\to\infty} \mathrm{ES}_\alpha\left(\frac{N(v)Y}{v}\right) = \mathrm{ES}_\alpha\left(Y\right). \tag{5.4.4}$$

From Theorem 5.9 (i), we have

$$\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{ES}_\alpha(S_{N(v)}) = \mathrm{ES}_\alpha\left(N(v)Y\right). \tag{5.4.5}$$

By (5.4.4) and the positive homogeneity of ES, we have

$$\lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{ES}_\alpha(S_{N(v)})}{v} = \mathrm{ES}_\alpha\left(Y\right).$$

Thus, we obtain the second equality in (5.4.3).

Since L^1-convergence implies convergence in probability, $\frac{N(v)}{v} \xrightarrow{L^1} 1$ yields that for any $\varepsilon > 0$ and $\delta > 0$, there exists an $M_1 > 0$ such that for all $v \geqslant M_1$,

$$\mathbb{P}\left(\left|\frac{N\left(v\right)}{v} - 1\right| > \delta\right) < \varepsilon.$$

Write $S_{N(v)} = Y_1 + \cdots + Y_{N(v)}$. Define

$$S^*_{N(v)} = \begin{cases} S_{N(v)} & \text{if } N(v)/v \geqslant 1-\delta, \\ 0 & \text{if } N(v)/v < 1-\delta. \end{cases}$$

Since $S_{N(v)} \geqslant S^*_{N(v)}$, we have

$$\begin{aligned}
\mathrm{VaR}_{\alpha+\varepsilon}\left(S_{N(v)}\right) \geqslant \mathrm{VaR}_{\alpha+\varepsilon}\left(S^*_{N(v)}\right) &= \inf\left\{t \in \mathbb{R} : \mathbb{P}\left(S^*_{N(v)} \leqslant t\right) \geqslant \alpha+\varepsilon\right\} \\
&= \inf\left\{t \in \mathbb{R} : \mathbb{P}\left(S_{N(v)} \leqslant t, N(v)/v \geqslant 1-\delta\right) + \mathbb{P}\left(0 \leqslant t, N(v)/v < 1-\delta\right) \geqslant \alpha+\varepsilon\right\} \\
&\geqslant \inf\left\{t \in \mathbb{R} : \mathbb{P}\left(S_{N(v)} \leqslant t, N(v)/v \geqslant 1-\delta\right) \geqslant \alpha\right\} \\
&\geqslant \inf\left\{t \in \mathbb{R} : \mathbb{P}\left(S_{\lfloor(1-\delta)v\rfloor} \leqslant t\right) \geqslant \alpha\right\} = \mathrm{VaR}_{\alpha}\left(S_{\lfloor(1-\delta)v\rfloor}\right). \qquad (5.4.6)
\end{aligned}$$

By Lemma 5.5, for any $\varepsilon_2 > 0$, there exists an $M_2 > 1/\varepsilon$ such that for all $v > M_2$,

$$\frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_{\alpha-\varepsilon}\left(S_{\lfloor(1-\delta)v\rfloor}\right)}{\lfloor(1-\delta)v\rfloor} > \mathrm{ES}_{\alpha-\varepsilon}(Y) - \varepsilon_2.$$

Thus, for the above $\varepsilon > 0$ and $v > \max\{M_1, M_2\}$,

$$\begin{aligned}
\frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{VaR}_{\alpha}\left(S_{N(v)}\right)}{v} &\geqslant \frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{VaR}_{\alpha-\varepsilon}\left(S_{\lfloor(1-\delta)v\rfloor}\right)}{v} \\
&= \frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_{\alpha-\varepsilon}\left(S_{\lfloor(1-\delta)v\rfloor}\right)}{\lfloor(1-\delta)v\rfloor} \cdot \frac{\lfloor(1-\delta)v\rfloor}{v} \\
&> \left[\mathrm{ES}_{\alpha-\varepsilon}(Y) - \varepsilon_2\right] \cdot \frac{(1-\delta)v - 1}{v} \\
&> \left[\mathrm{ES}_{\alpha-\varepsilon}(Y) - \varepsilon_2\right](1-\delta-\varepsilon),
\end{aligned}$$

which implies

$$\liminf_{v\to\infty} \frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{VaR}_{\alpha}\left(S_{N(v)}\right)}{v} \geqslant \mathrm{ES}_{\alpha}(Y).$$

On the other hand,

$$\limsup_{v\to\infty} \frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{VaR}_{\alpha}\left(S_{N(v)}\right)}{v} \leqslant \lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{ES}_{\alpha}(S_{N(v)})}{v} = \mathrm{ES}_{\alpha}(Y).$$

Therefore,

$$\lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{VaR}_{\alpha}\left(S_{N(v)}\right)}{v} = \mathrm{ES}_{\alpha}(Y).$$

Thus, we obtain the first equality in (5.4.3). □

Theorem 5.11, together with Lemma 5.5, suggests that for $\alpha \in (0,1)$ and $Y \in \mathcal{X}_F$, the following five quantities are all asymptotically equivalent as $v \to \infty$:

(i) $\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)$; (iii) $\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_\alpha\left(S_{\lfloor v\rfloor}\right)$; (v) $v\mathrm{ES}_\alpha(Y)$.

(ii) $\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{ES}_\alpha\left(S_{N(v)}\right)$; (iv) $\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{ES}_\alpha\left(S_{\lfloor v\rfloor}\right)$;

Hence, one may use (v) above (straightforward to calculate) to approximate the other four quantities. The approximation error, that is, the convergence rate in Theorem 5.11, is studied in the following section.

Remark 5.3. Since $\mathrm{VaR}_\alpha \leqslant \mathrm{ES}_\alpha$, the quantity in (i) is smaller than or equal to the quantity in (ii), and similarly for (iii) and (iv). Another observation is that $\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{ES}_\alpha\left(S_{\lfloor v\rfloor}\right) = \lfloor v\rfloor \mathrm{ES}_\alpha(Y) \leqslant v\mathrm{ES}_\alpha(Y)$. From Corollary 5.10, the quantity in (iv) is smaller than or equal to the quantity in (ii), provided that $\mathbb{E}[N(v)] = \lfloor v\rfloor$. However, there is no general order between (i) and (v) (or (iv)); when we approximate $\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)$ with $v\mathrm{ES}_\alpha(Y)$, it is not clear which one is larger. See Theorem 5.12 below for more detailed analysis on their relationship.

5.4.3 Rate of Convergence

Theorem 5.12. *Suppose that the distribution F on $\mathbb{R}_+$ has finite p-th moment, $p \geqslant 1$, $Y \in \mathcal{X}_F$, $\mathbb{E}[Y] > 0$, and $\limsup_{v\to\infty} v^q \mathbb{E}\left|\frac{N(v)}{v} - 1\right| \leqslant c$ for some $q > 0,\ c > 0$. Then for $\alpha \in (0,1)$,*

$$-2C^{1/2}v^{-q/2} + o\left(v^{1/p-1}\right) + o\left(v^{-q/2}\right) \leqslant \frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha\left(Y\right)} - 1 \leqslant Cv^{-q} + o\left(v^{-q}\right), \tag{5.4.7}$$

and

$$\left|\frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{ES}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha\left(Y\right)} - 1\right| \leqslant Cv^{-q} + o\left(v^{-q}\right), \tag{5.4.8}$$

where $C = \frac{c}{1-\alpha}$.

Proof. Let $\delta = \sqrt{C}v^{-q/2}$, $\eta = \left(v^q\mathbb{E}\left|\frac{N(v)}{v} - 1\right| - c\right)_+$, and $\varepsilon = \frac{c+\eta}{\delta}v^{-q}$. Clearly $\varepsilon = \sqrt{c(1-\alpha)}v^{-q/2} + o(v^{-q/2})$. Note that

$$v^{-q}(c+\eta) \geqslant \mathbb{E}\left|\frac{N(v)}{v} - 1\right| \geqslant \int_{|N(v)/v-1|>\delta}\left|\frac{N(v)}{v} - 1\right| \mathrm{d}\mathbb{P} > \delta\,\mathbb{P}\left(\left|\frac{N\left(v\right)}{v} - 1\right| > \delta\right).$$

Hence,

$$\mathbb{P}\left(\left|\frac{N(v)}{v}-1\right|>\delta\right)<\frac{c+\eta}{\delta}v^{-q}=\varepsilon.$$

This implies $\mathrm{VaR}_\alpha\left(S_{N(v)}\right)\geqslant\mathrm{VaR}_{\alpha-\varepsilon}\left(S_{\lfloor(1-\delta)v\rfloor}\right)$ as shown in (5.4.6). By Lemma 5.6, we have

$$\begin{aligned}\frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}}\mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha(Y)}&\geqslant\frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F}\mathrm{VaR}_{\alpha-\varepsilon}\left(S_{\lfloor(1-\delta)v\rfloor}\right)}{\sup_{\mathbf{Y}\subset\mathcal{X}_F}\mathrm{ES}_{\alpha-\varepsilon}\left(S_{\lfloor(1-\delta)v\rfloor}\right)}\cdot\frac{\lfloor(1-\delta)v\rfloor\mathrm{ES}_{\alpha-\varepsilon}(Y)}{v\mathrm{ES}_\alpha(Y)}\\&\geqslant\left[1-o\left(v^{1/p-1}\right)\right]\cdot\left(1-\delta-v^{-1}\right)\cdot\frac{\mathrm{ES}_{\alpha-\varepsilon}(Y)}{\mathrm{ES}_\alpha(Y)}.\end{aligned}\tag{5.4.9}$$

Note that

$$\left|1-\frac{\mathrm{ES}_{\alpha-\varepsilon}(Y)}{\mathrm{ES}_\alpha(Y)}\right|=\frac{\left(\frac{1}{1-\alpha}-\frac{1}{1-\alpha+\varepsilon}\right)\int_\alpha^1\mathrm{VaR}_\gamma(Y)\,\mathrm{d}\gamma-\frac{1}{1-\alpha+\varepsilon}\int_{\alpha-\varepsilon}^\alpha\mathrm{VaR}_\gamma(Y)\,\mathrm{d}\gamma}{\mathrm{ES}_\alpha(Y)}\leqslant\frac{\varepsilon}{1-\alpha}.$$

Therefore,

$$\frac{\mathrm{ES}_{\alpha-\varepsilon}(Y)}{\mathrm{ES}_\alpha(Y)}\geqslant1-\frac{\varepsilon}{1-\alpha}.$$

Plugging the above inequality into (5.4.9), one has

$$\begin{aligned}\frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}}\mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha(Y)}&\geqslant\left[1-o\left(v^{1/p-1}\right)\right]\cdot\left(1-\delta-v^{-1}\right)\cdot\left(1-\frac{\varepsilon}{1-\alpha}\right)\\&=1-2\sqrt{C}v^{-q/2}-o\left(v^{1/p-1}\right)-o\left(v^{-q/2}\right).\end{aligned}$$

Thus, we obtain the first inequality in (5.4.7).

In the next step we show (5.4.8). From Theorem 5.9 (i),

$$\frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}}\mathrm{ES}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha(Y)}=\frac{\mathrm{ES}_\alpha(N(v)Y)}{v\mathrm{ES}_\alpha(Y)}.$$

By the subadditivity of ES, we have

$$\mathrm{ES}_\alpha(Y)=\mathrm{ES}_\alpha\left(\frac{N(v)}{v}Y+Y-\frac{N(v)}{v}Y\right)\leqslant\mathrm{ES}_\alpha\left(\frac{N(v)}{v}Y\right)+\mathrm{ES}_\alpha\left(Y-\frac{N(v)}{v}Y\right).$$

Similarly, $\mathrm{ES}_\alpha\left(\frac{N(v)}{v}Y\right)\leqslant\mathrm{ES}_\alpha(Y)+\mathrm{ES}_\alpha\left(\frac{N(v)}{v}Y-Y\right)$. It follows that

$$\mathrm{ES}_\alpha(Y)-\mathrm{ES}_\alpha\left(\frac{N(v)}{v}Y\right)\leqslant\mathrm{ES}_\alpha\left(Y-\frac{N(v)}{v}Y\right)\leqslant\mathrm{ES}_\alpha\left(\left|Y-\frac{N(v)}{v}Y\right|\right),$$

and

$$\mathrm{ES}_\alpha\left(\frac{N(v)}{v}Y\right)-\mathrm{ES}_\alpha\left(Y\right)\leqslant\mathrm{ES}_\alpha\left(\frac{N(v)}{v}Y-Y\right)\leqslant\mathrm{ES}_\alpha\left(\left|Y-\frac{N(v)}{v}Y\right|\right).$$

Therefore,

$$\begin{aligned}\left|\mathrm{ES}_\alpha\left(\frac{N(v)}{v}Y\right)-\mathrm{ES}_\alpha\left(Y\right)\right|&\leqslant\mathrm{ES}_\alpha\left(\left|Y-\frac{N(v)}{v}Y\right|\right)\\&\leqslant\frac{1}{1-\alpha}\mathbb{E}\left|\frac{N(v)}{v}-1\right|\cdot\mathbb{E}\left[Y\right],\end{aligned}\tag{5.4.10}$$

which implies

$$\left|\frac{\mathrm{ES}_\alpha\left(\frac{N(v)}{v}Y\right)}{\mathrm{ES}_\alpha(Y)}-1\right|\leqslant Cv^{-q}+o(v^{-q}).$$

Thus we obtain (5.4.8) as

$$\left|\frac{\sup_{\mathbf{Y}\in\mathcal{X}_F^{N(v)}}\mathrm{ES}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha\left(Y\right)}-1\right|\leqslant Cv^{-q}+o(v^{-q}),$$

and the second inequality in (5.4.7) is automatically implied since VaR_α is dominated by ES_α. □

In Example 5.2 of the next section, we will see that $q=1/2$ for Poisson(v)-distributed $N(v)$. In this case, assuming $p\geqslant 4/3$ (typically true), the convergence rate in the left-hand side of (5.4.7) is led by $O(v^{-1/4})$ and the one in (5.4.8) is led by $O(v^{-1/2})$. Admittedly, the convergence rate $O(v^{-1/4})$ is not very fast in general, and its applicability for approximation depends on the models and the magnitude of v. However, for risk management purpose, one should be on the conservative side; as such, the faster rate $O(v^{-q})$ in the right-hand side of (5.4.7) and in (5.4.8) is more important in practice. In Example 5.5 below, we will see that the term $O(v^{-q})$ for the upper bounds in Theorem 5.12 is sharp.

5.4.4 Some Examples

Example 5.2 (Poisson number of claims)**.** As the primary example, suppose that $N(v)$ follows a Poisson distribution with parameter v. We check the conditions and parameters in Theorems 5.11 and 5.12. Clearly, $N(v)/v\to 1$ in L^1 as $v\to\infty$ by the L^1-Law of Large Numbers. Indeed, note that $\mathbb{E}|N(v)-v| = 2e^{-v}\frac{v^{\lfloor v\rfloor+1}}{\lfloor v\rfloor!}$, and further by Stirling's formula and some elementary analysis, one has

$$2e^{-v}\frac{v^{\lfloor v\rfloor+1}}{\lfloor v\rfloor!}v^{-1/2} \to \sqrt{\frac{2}{\pi}},$$

which means

$$\lim_{v\to\infty} v^{1/2}\mathbb{E}\left|\frac{N(v)}{v}-1\right| = \sqrt{\frac{2}{\pi}}.$$

Therefore in Theorem 5.12, $c=\sqrt{\frac{2}{\pi}}$ and $q=1/2$.

Example 5.3 (Non-random number of claims)**.** Suppose that $N(v)$ equals $\lfloor v\rfloor$. Then $q=\infty$ in the conditions of Theorem 5.12, and the lower bound on VaR convergence rate given in (5.4.7) is equivalent to Lemma 5.6.

Example 5.4 (Non-random claim sizes)**.** Suppose that Y is not random and $\limsup_{v\to\infty} v^q\mathbb{E}\left|\frac{N(v)}{v}-1\right| \leqslant c$ for some $q>0$, $c>0$. In this case, we have a convergence rate that is slightly stronger than the one given in (5.4.7),

$$\begin{aligned} 1-2C^{1/2}v^{-q/2}-o\left(v^{-q/2}\right) &\leqslant \frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}}\mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha(Y)} \\ &\leqslant \frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}}\mathrm{ES}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha(Y)} \leqslant 1+Cv^{-q}+o(v^{-q}), \end{aligned} \tag{5.4.11}$$

where $C=\frac{c}{1-\alpha}$. Compared with Theorem 5.12, the term $o(v^{1/p-1})$ in the lower bound for VaR convergence disappears. This is quite natural since the term $o(v^{1/p-1})$ is due to the randomness of Y as suggested by Lemma 5.6. To see the first inequality in (5.4.11), let $\varepsilon=\alpha$ and $\delta=\sqrt{C}v^{-q/2}$. For v large enough,

$$\mathbb{P}\left(\left|\frac{N(v)}{v}-1\right|>\delta\right)<\varepsilon \quad \text{and} \quad \mathbb{P}\left(\frac{N(v)}{v}<1-\delta\right)<\alpha,$$

which imply

$$\mathrm{VaR}_\alpha\left(\frac{N(v)}{v}\right)\geqslant 1-\delta.$$

Therefore,

$$\frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}}\mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha(Y)} = \frac{\mathrm{VaR}_\alpha(N(v))}{v} \geqslant 1-\delta \geqslant 1-2C^{1/2}v^{-q/2}-o\left(v^{-q/2}\right).$$

The rest of (5.4.11) comes from Theorem 5.12.

Example 5.5 (Sharpness of the rate in the right-hand side of (5.4.7) and in (5.4.8))**.** For some $q > 0$, take $N(v) = \lfloor v + v^{1-q} \rfloor$ and let F be a degenerate distribution of a constant, say 1. In this case, $S_{N(v)} = N(v)$ is not random, and obviously

$$\frac{\mathrm{VaR}_\alpha(S_{N(v)})}{v} = \frac{\mathrm{ES}_\alpha(S_{N(v)})}{v} = O(v^{-q}).$$

This shows that the leading term v^{-q} in the right-hand side of (5.4.7) and in (5.4.8) is sharp up to a constant scale, even in the case when $Y_1, Y_2, \ldots$ and $N(v)$ are deterministic.

5.5 Asymptotic Results for Generalized Collective Risk Models

In this section, we study the more complicated setting (ii) in which N and $Y_1, Y_2, \ldots$ are not necessarily independent, and their joint distribution is also uncertain. We have similar results as in Theorem 5.11 and Theorem 5.12 under stronger regularity conditions.

5.5.1 VaR-ES Asymptotic Equivalence

Theorem 5.13. *Suppose that the distribution F on $\mathbb{R}_+$ has finite second moment, $Y \in \mathcal{X}_F$, and $\{N(v),\ v \geqslant 0\} \subset \mathcal{X}_0$ such that $N(v)/v \to 1$ in L^2 as $v \to \infty$. Then for $\alpha \in (0,1)$,*

$$\lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{v} = \lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{ES}_\alpha\left(S_{N(v)}\right)}{v} = \mathrm{ES}_\alpha\left(Y\right). \tag{5.5.1}$$

Proof. From Theorem 5.9, for fixed $v > 0$, we have

$$\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{ES}_\alpha\left(S_{N(v)}\right) = \mathrm{ES}_\alpha\left(N(v)Y^*\right), \tag{5.5.2}$$

where $Y^* \in \mathcal{X}_F$ is comonotonic with $N(v)$. Hölder's inequality implies

$$\mathbb{E}\left|\frac{N(v)Y^*}{v} - Y^*\right| \leqslant \sqrt{\mathbb{E}\left|\frac{N(v)}{v} - 1\right|^2 \cdot \mathbb{E}\left[(Y^*)^2\right]} \to 0, \qquad \text{as } v \to \infty.$$

Hence, $\frac{N(v)Y^*}{v} \xrightarrow{L^1} Y^*$. As a consequence, continuity of ES with respect to the L^1-norm implies

$$\lim_{v\to\infty} \mathrm{ES}_\alpha\left(N(v)Y^*/v\right) = \mathrm{ES}_\alpha\left(Y^*\right) = \mathrm{ES}_\alpha(Y).$$

Therefore,

$$\lim_{v\to\infty}\frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F}\mathrm{ES}_\alpha(S_{N(v)})}{v}=\lim_{v\to\infty}\frac{\mathrm{ES}_\alpha\left(N(v)Y^*\right)}{v}=\mathrm{ES}_\alpha\left(Y\right).$$

Thus we obtain the second equality in (5.5.1).

For the first equality in (5.5.1), $\frac{N(v)}{v}\xrightarrow{L^2}1$ implies that for any $\varepsilon>0$ and $\delta>0$, for v large enough, one has

$$\mathbb{P}\left(\left|\frac{N\left(v\right)}{v}-1\right|>\delta\right)<\varepsilon.$$

Similarly to the proof of Theorem 5.12, we have

$$\frac{\mathrm{ES}_{\alpha-\varepsilon}\left(Y\right)}{\mathrm{ES}_\alpha\left(Y\right)}\geqslant 1-\frac{\varepsilon}{1-\alpha},$$

and

$$\frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F}\mathrm{ES}_\alpha(S_{N(v)})}{v\mathrm{ES}_\alpha\left(Y\right)}\geqslant\frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F}\mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha\left(Y\right)}\geqslant\left[1-o\left(v^{1/p-1}\right)\right]\cdot\left(1-\delta-v^{-1}\right)\cdot\frac{\mathrm{ES}_{\alpha-\varepsilon}\left(Y\right)}{\mathrm{ES}_\alpha\left(Y\right)}.$$

Thus,

$$\lim_{v\to\infty}\frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F}\mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{v}=\mathrm{ES}_\alpha\left(Y\right),$$

and we obtain the first equality in (5.5.1). □

5.5.2 Rate of Convergence

In this section we provide the convergence rate in generalized collective risk models. Similarly to Theorem 5.13, stronger regularity conditions are required as compared to results in Section 5.4.

Theorem 5.14. *Suppose that the distribution F on $\mathbb{R}_+$ has finite p-th moment, $p\geqslant 2$, $Y\in\mathcal{X}_F$, $\mathbb{E}[Y]>0$, and $\limsup_{v\to\infty}v^r\mathbb{E}\left|\frac{N(v)}{v}-1\right|^2\leqslant c$ for some $r>0$ and $c>0$. Then we have*

$$-2\left(\frac{c}{1-\alpha}\right)^{1/3}v^{-r/3}+o\left(v^{1/p-1}\right)+o\left(v^{-r/3}\right)\leqslant\frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F}\mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha\left(Y\right)}-1 \tag{5.5.3}$$

$$\leqslant\left|\frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F}\mathrm{ES}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha\left(Y\right)}-1\right|\leqslant\frac{\sqrt{\mathbb{E}[Y^2]}}{\mathrm{ES}_\alpha(Y)}\frac{\sqrt{c}}{1-\alpha}v^{-r/2}+o\left(v^{-r/2}\right). \tag{5.5.4}$$

Proof. Let $\delta = \left(\frac{c}{1-\alpha}v^{-r}\right)^{1/3}$, $\eta = \left(v^r\mathbb{E}\left|\frac{N(v)}{v}-1\right|^2 - c\right)_+$, and $\varepsilon = \frac{c+\eta}{\delta^2}v^{-r}$. Clearly $\varepsilon = \left(c(1-\alpha)^2 v^{-r}\right)^{1/3} + o(v^{-r/3})$. Similar to the proof of Theorem 5.12, we have

$$\mathbb{P}\left(\left|\frac{N\left(v\right)}{v}-1\right| > \delta\right) < \varepsilon \quad \text{and} \quad \frac{\mathrm{ES}_{\alpha-\varepsilon}\left(Y\right)}{\mathrm{ES}_{\alpha}\left(Y\right)} \geqslant 1 - \frac{\varepsilon}{1-\alpha}.$$

Moreover,

$$\begin{aligned}
\frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F}\mathrm{VaR}_{\alpha}\left(S_{N(v)}\right)}{v\mathrm{ES}_{\alpha}\left(Y\right)} &\geqslant \frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F}\mathrm{VaR}_{\alpha-\varepsilon}\left(S_{\lfloor(1-\delta)v\rfloor}\right)}{\sup_{\mathbf{Y}\subset\mathcal{X}_F}\mathrm{ES}_{\alpha-\varepsilon}\left(S_{\lfloor(1-\delta)v\rfloor}\right)} \cdot \frac{\lfloor(1-\delta)v\rfloor\mathrm{ES}_{\alpha-\varepsilon}\left(Y\right)}{v\mathrm{ES}_{\alpha}\left(Y\right)} \\
&\geqslant \left[1 - o\left(v^{1/p-1}\right)\right]\cdot\left(1-\delta-v^{-1}\right)\left(1-\frac{\varepsilon}{1-\alpha}\right) \\
&\geqslant 1 - 2\left(\frac{c}{1-\alpha}\right)^{1/3} v^{-r/3} - o\left(v^{1/p-1}\right) - o\left(v^{-r/3}\right).
\end{aligned}$$

Thus we obtain (5.5.3). The first inequality in (5.5.4) comes from the fact that ES_α dominates VaR_α.

By (5.4.10) and Hölder's inequality, we have

$$\left|\mathrm{ES}_{\alpha}\left(\frac{N(v)}{v}Y\right) - \mathrm{ES}_{\alpha}\left(Y\right)\right| \leqslant \mathrm{ES}_{\alpha}\left(\left|Y - \frac{N(v)}{v}Y\right|\right) \leqslant \frac{1}{1-\alpha}\sqrt{\mathbb{E}\left|\frac{N(v)}{v}-1\right|^2 \cdot \mathbb{E}\left[Y^2\right]}.$$

As a consequence,

$$\left|\frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F}\mathrm{ES}_{\alpha}\left(S_{N(v)}\right)}{v\mathrm{ES}_{\alpha}\left(Y\right)} - 1\right| \leqslant \frac{\sqrt{\mathbb{E}[Y^2]}}{\mathrm{ES}_{\alpha}(Y)}\frac{\sqrt{c}}{1-\alpha}v^{-r/2} + o\left(v^{-r/2}\right).$$

Thus we obtain the second inequality in (5.5.4). ☐

Example 5.6 (Poisson number of claims, revisited)**.** Suppose that $N(v)$ follows a Poisson distribution with parameter v. We can check the parameters in Theorem 5.14. Since $\mathbb{E}[|N(v)/v-1|^2] = \mathrm{Var}(N(v))/v^2 = 1/v$, we have $r=1$ and $c=1$. Therefore, the leading term in the left-hand side of (5.5.3) is $O(v^{-1/3})$, which converges to zero faster than $O(v^{-1/4})$ as in Example 5.2 under setting (i). This is intuitive as $\sup_{\mathbf{Y}\subset\mathcal{X}_F}\mathrm{VaR}_{\alpha}\left(S_{N(v)}\right) \geqslant \sup_{\mathbf{Y}\subset\mathcal{X}_F^{N(v)}}\mathrm{VaR}_{\alpha}\left(S_{N(v)}\right)$. The right-hand side of (5.5.4) remains the same order $O(v^{-1/2})$.

5.5.3 A Remark on the Dependence of Collective Risk Models

In Sections 5.4 and 5.5, we studied the asymptotic equivalence of VaR and ES in two settings. A natural question that follows would be whether an asymptotic equivalence holds also for specified dependence structures between $N(v)$ and $Y_1, Y_2, \ldots$ other than independence. That is, whether the following limit

$$\lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\subset\hat{\mathcal{X}}_F^{N(v)}} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{\sup_{\mathbf{Y}\subset\hat{\mathcal{X}}_F^{N(v)}} \mathrm{ES}_\alpha\left(S_{N(v)}\right)} = 1 \tag{5.5.5}$$

holds, where $\hat{\mathcal{X}}_F^{N(v)} \subset \mathcal{X}_F$ is the set of random variables with distribution F and a pre-specified dependence structure (copula) with $N(v)$. Note that from Lemma 5.7, the worst-case ES can be calculated as $\mathrm{ES}_\alpha(N(v)Y)$, where $Y \in \hat{\mathcal{X}}_F^{N(v)}$.

In general, the knowledge on the dependence structure of $(N(v), Y_i)$, $i = 1, 2, \ldots$, would put some restrictions on the dependence structure of $(Y_1, Y_2, \ldots)$; the latter was assumed to be arbitrary in our settings (i) and (ii), as well as in the classic setup of dependence uncertainty. With the "effect of dependence uncertainty" demolished, (5.5.5) may no longer hold true. This is evidenced by the following (rather extreme) example where $N(v), Y_i$ are comonotonic for $i = 1, 2, \ldots$ (note that this does not necessarily imply that $Y_1, Y_2, \ldots$ are comonotonic since $N(v)$ is discrete). For other pre-specified dependence structures between $(N(v), Y_i)$, $i = 1, 2, \ldots$, the question of (5.5.5) requires a case-by-case study.

Assume that the distribution F has finite second moment, $\{N(v),\ v \geqslant 0\} \subset \mathcal{X}_0$ such that $N(v)/v \to 1$ in L^2 as $v \to \infty$, and $Y \in \mathcal{X}_F^{c,v}$. Denote by $\mathcal{X}_F^{c,v} \subset \mathcal{X}_F$ the set of random variables with distribution F and comonotonic with $N(v)$. In this case one still has the ES convergence as in Theorem 5.11,

$$\lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F^{c,v}} \mathrm{ES}_\alpha\left(S_{N(v)}\right)}{v} = \mathrm{ES}_\alpha\left(Y\right), \quad \alpha \in (0,1), \tag{5.5.6}$$

whereas the VaR convergence

$$\lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F^{c,v}} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{v} = \mathrm{ES}_\alpha\left(Y\right), \quad \alpha \in (0,1), \tag{5.5.7}$$

may fail to hold.

To see (5.5.6), by Hölder's inequality, we have

$$\mathbb{E}\left|\frac{N(v)Y}{v}-Y\right| \leqslant \sqrt{\mathbb{E}\left|\frac{N(v)}{v}-1\right|^2 \cdot \mathbb{E}\left[Y^2\right]} \to 0, \qquad \text{as } v \to \infty.$$

Hence, $\frac{N(v)Y}{v} \xrightarrow{L^1} Y$. From Lemma 5.7, we have $\sup_{\mathbf{Y}\subset\mathcal{X}_F^{c,v}} \mathrm{ES}_\alpha\left(S_{N(v)}\right) = \mathrm{ES}_\alpha\left(N(v)Y\right)$, and (5.5.6) follows from the continuity of ES with respect to the L^1-norm.

To see that (5.5.7) may not hold true, we simply give a counter-example. Take any $\alpha \in (0,1)$. Let F be a Bernoulli distribution with parameter $(1-\alpha)/2$, and assume that for each $v > 0$, there exists a positive integer f_v such that $\mathbb{P}(N(v) > f_v) = (1-\alpha)/2$. For fixed v and any $Y_1, Y_2, \dots \in X_F^{c,v}$, we have $\{Y_i = 1\} = \{N(v) > f_v\}$ almost surely for each $i = 1, 2, \dots$, and hence $Y_1, Y_2, \dots$ are almost surely equal. As a consequence, there is indeed no dependence uncertainty: $S_{N(v)} = N(v)Y_1$ almost surely. Since $\mathbb{P}(N(v)Y_1 > 0) \leqslant \mathbb{P}(Y_1 > 0) = (1-\alpha)/2$, we have

$$\sup_{\mathbf{Y}\subset\mathcal{X}_F^{c,v}} \mathrm{VaR}_\alpha(S_{N(v)}) = \mathrm{VaR}_\alpha(N(v)Y_1) = 0.$$

Therefore,

$$\lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F^{c,v}} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{v} = 0.$$

Thus (5.5.7) does not hold noting that $\mathrm{ES}_\alpha(Y) > 0$.

5.6 Conclusion

In this chapter, we study the worst-case values of VaR and ES of the aggregate loss in collective risk models under two settings of dependence uncertainty. Analytical formulas for the worst-case values of ES are obtained. For both settings, an asymptotic equivalence of the VaR and ES for a random sum of risks is established under some general moment and regularity conditions. The conditions in our main results are easily satisfied by common models, including the classic compound Poisson collective risk models. Our main results suggest that under dependence uncertainty, we can use $v\mathrm{ES}_\alpha(Y)$ to approximate the worst-case risk aggregation when the risk measure is VaR_α or ES_α and v is large enough; the approximation error is also obtained in terms of some moment and convergence rate of the claim sizes and the claim frequency.

5.7 Additional Discussions

In this section, we discuss the difference between a collective risk model and a corresponding individual risk model. Let N be the counting random variable which is bounded by some $n \in \mathbb{N}$, i.e. $N \leqslant n$, and $Y_i \sim F$, $i \in \mathbb{N}$ (in fact, only $Y_1, \dots, Y_n$ are used). A random sum S_N may be written in two ways: a collective risk model

$$S_N = \sum_{i=1}^{N} Y_i \tag{5.7.1}$$

and an individual risk model

$$S_N = \sum_{i=1}^{n} Y_i \mathrm{I}_{\{N \geqslant i\}} = \sum_{i=1}^{n} Z_i, \tag{5.7.2}$$

where $Z_i = Y_i \mathrm{I}_{\{N \geqslant i\}}$, $i = 1, \dots, n$. Note that this setup is different from the collective reformulation in Example 5.1, where one starts with a homogeneous individual risk model with small probability of loss from each individual risk, and arrives at a Poisson collective risk model.

In the recent literature of dependence uncertainty for an individual risk model, $Z_1, \dots, Z_n$ in (5.7.2) are assumed to have an arbitrary dependence. In our collective risk model, although S_N may be written as in (5.7.2), the dependence among $Z_1, \dots, Z_n$ is not arbitrary anymore, as it is driven by a common random variable N. There are further essential differences, if we look at the two formulations more closely under the two settings of dependence uncertainty studied in this chapter.

(i) N **and the sequence** $Y_1, Y_2, \dots$ **are independent.** In this case, the distribution of Z_i can be determined by that of Y_i and N. Denote this distribution by F_i. We can consider the worst-case risk measure (take an ES for instance) in our model

$$\sup_{\mathbf{Y} \in \mathcal{Y}_F^N} \mathrm{ES}_\alpha \left(\sum_{i=1}^{N} Y_i \right) \tag{5.7.3}$$

and in the classic model

$$\sup_{Z_i \in \mathcal{X}_{F_i}, i \leqslant n} \mathrm{ES}_\alpha \left(\sum_{i=1}^{n} Z_i \right). \tag{5.7.4}$$

Clearly, through (5.7.1) and (5.7.2), the collective risk model formulation $\mathbf{Y} \in \mathcal{Y}_F^N$ in (5.7.3) is a submodel of the individual risk model formulation $Z_i \in \mathcal{X}_{F_i},\ i \leqslant n$ in (5.7.4), and hence the worst-case value in (5.7.3) should be smaller than or equal to the one in (5.7.4). We shall illustrate this difference with a numerical example where one has

$$\sup_{\mathbf{Y}\in\mathcal{Y}_F^N} \mathrm{ES}_\alpha\left(\sum_{i=1}^{N} Y_i\right) < \sup_{Z_i\in\mathcal{X}_{F_i}, i\leqslant n} \mathrm{ES}_\alpha\left(\sum_{i=1}^{n} Z_i\right).$$

See Example 5.7 below.

(ii) **The dependence between N and the sequence $Y_1, Y_2, \ldots$ is also unknown.** In this case, the distribution of Z_i, and the conditional distribution of Z_i given N are both unknown. Hence, no existing result in the literature of dependence uncertainty that we are aware of can be applied to this setting.

Example 5.7. Let $n = 10$. Suppose that for $i = 1, \ldots, n$, Y_i follows an exponential distribution with parameter 1 (denoted by Expo(1)), and N follows the binomial distribution with parameters n and $1/3$ (denoted by $\mathrm{Bin}(n, 1/3)$), independent of $\{Y_i,\ i \in \mathbb{N}\}$. For $i = 1, \ldots, n$, denote the distribution of $Y_i \mathrm{I}_{\{N\geqslant i\}}$ by F_i. Take $\alpha = 0.95$. By Theorem 5.9, we can calculate

$$\sup_{\mathbf{Y}\in\mathcal{Y}_F^N} \mathrm{ES}_\alpha\left(\sum_{i=1}^{N} Y_i\right) = \mathrm{ES}_\alpha(NY_1) = 15.813,$$

$$\sup_{Z_i\in\mathcal{X}_{F_i}, i\leqslant n} \mathrm{ES}_\alpha\left(\sum_{i=1}^{n} Z_i\right) = \sum_{i=1}^{n} \mathrm{ES}_\alpha(Z_i) = 19.026,$$

where the first value is the average of 100 repetitions of simulation with a sample of size 100,000, and the second value is calculated analytically.

The above illustration shows that, under the above setting (i), the collective risk model imposes a special type of dependence through the counting random variable N, and has a smaller worst-case ES value of the aggregate risk as compared to the corresponding individual risk model with dependence uncertainty. Thus, using a collective risk model is one of the many ways of introducing partial dependence information into risk aggregation.

Chapter 6

Concluding Remarks and Future Research

We mainly study risk sharing and risk aggregation via risk measures in the thesis. In Chapter 2, for underlying risk measures RVaR, we solve the optimal risk sharing problem of a total risk and construct a Pareto-optimal allocation in Theorem 2.4 and a corresponding Arrow-Debreu Equilibria in the settings of Theorems 2.7 and 2.8. The condition on the distribution of the total risk in Theorems 2.7 and 2.8 and the definition of RVaR guarantee the existence of an Arrow-Debreu equilibrium. One possible direction for future research is to remove the condition and prove the existence of an Arrow-Debreu equilibrium under a more general setting. Another possible direction is to consider the risk sharing problem given that the agents have different beliefs on the future states of risks.

In Chapter 3, we study Pareto-optimal reinsurance arrangements and show that under general model settings and assumptions, a Pareto-optimal reinsurance contract is an optimizer of the convex combination of both parties' preferences, and such optimizers always exist. In particular, we solve the optimal reinsurance problem explicitly when the preferences are TVaR. A more practical setting is to include constraints such as budget limit on the reinsurance premium in an optimization problem.

Regarding risk aggregation, in Chapter 4, we show the asymptotic equivalence results of the class of distortion risk measures and the class of convex risk measures for inhomogeneous individual risk models under some regularity conditions. The main result is that under dependence uncertainty in the aggregation of a large number of risks, the worst-case value of a non-coherent risk measure is asymptotically equivalent to that of a corresponding coherent risk measure. This result helps to analyze risk aggregation under dependence uncertainty for financial regulation and internal risk management. In Chapter 5, we obtain similar asymptotic equivalence of VaR and ES for homogeneous collective risk models under some moment and regularity conditions. The conditions are easily satisfied by common models such as the compound Poisson models. Besides choosing distortion or convex risk measures in risk aggregation, one possible choice for future research is a rank-dependent utility, which is proposed in Quiggin (1982) to model decision under uncertainty and popular in behavioral finance.

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