Reliability Analysis Using the Method of Multiplicative Dimensional Reduction

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

Traditional engineering analyses and designs are based on deterministic input variables, and variability seen in the real world are often ignored to simplify the work. Formal reliability analyses are generally avoided by engineers due to large computational costs associated with the traditional methods, such as simulations. Analysis done by engineers in this age of advanced technology are done using finite element analysis which further increase the computational cost of analyzing a reliability problem . Using reliability methods such as Monte Carlo Simulation (MCS) with a finite element analysis requires thousands of trials to be done. This ultimately is not feasible for a complex problem which takes long computational time. Multiplicative Dimensional Reduction Method (MDRM) is a tool which can be used to calculate the statistical parameters of the response of a function with a large reduction in computational efforts. This method has not been applied to uncertainty analysis, geomechanics and fire resistant design problems to determine if this method is indeed worth using over traditional reliability methods (MCS). The Cubature method is another tool which can be used to calculate the statistical moments of a response. This method will be compared to MCS and MDRM to determine its effectiveness.

The research objectives in this thesis are therefore 1) to determine if the code developed to use MDRM provides accurate results, 2) to compare the results of MDRM and Cubature to MCS to see how accurate the results of MDRM and Cubature are based on equation based problems, 3) to determine the feasibility of using MDRM with uncertainty analysis problems (where epistemic and aleatory variables are defined), 4) to determine the feasibility of solving a MDRM reliability analysis for fire resistant design problems and 5) to determine the feasibility and computational efficiency of using MDRM for geomechanics problems which are both equation based and finite element analysis.

To perform the first objective a problem from Zhang & Pandey (2013) was redone using the code that was developed to make sure the results matched. The second objective was performed by solving steam generator tube failure problem and a time to leak of a pipe problem. The third objective was performed by solving the time to leak of a pipe problem again but this time designating one variable as epistemic and another as aleatory and comparing results between MDRM and MCS. To perform the fourth objective a performance based approach is outlined on how to calculate fire resistant design of a protected and unprotected beam. The results from MDRM and MCS are compared. The fifth and final objective is performed by first showing a step by step method on how to apply MDRM while solving a uni-dimensional consolidation example (settlement of foundation). Lastly two finite element analysis problems are solved to show the application of MDRM with the combination of a finite element analysis. The first problem is of vertical drains and the second problem is of a concrete infinite beam on an elastic foundation. These problems are done using MDRM and MCS and the results and computational effort are compared.

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Dedication

To:

Karma, for giving me the opportunity to live this life, and guiding me to become the person writing this today. To be born to such amazing and wonderful parents who have given me so much unconditional love that I don’t know where I would be without it. To be given the opportunity to live through all the experiences I have had in my life so far, whether good or bad, and learn and grow from them.

Table of Contents

List of Figures

List of Tables

1 Introduction

1.1 Motivation

Uncertainties are unavoidable in a real world problem, therefore deterministic results do not provide much value to designers and engineers, and this makes it necessary to apply reliability analysis for quantifying the structural safety. Changing deterministic problems into reliability problems by making a variable uncertain as well as an integration of reliability analysis with the finite element analysis (FEA) is becoming popular in engineering practice.

Basic issues of reliability analysis are that it takes too many function evaluations to estimate as accurately as possible, the probability distribution of the structural response. For instance, if using Monte Carlo Simulation (MCS), the major advantage is that accurate solutions can be obtained for any problem but the method can become computationally expensive depending on the number of random variables in the problem. Most reliability methods can be applied to simple structural systems which contain a small number of random variables. Even if we are able to calculate the probability statistics of the response (i.e. Mean, standard deviations, etc.) we have little knowledge of the probability distribution of the response.

Thus the main motivation behind this research is to use a method that is computationally efficient, robust, and easy to implement method that can be compared to the accuracy of MCS. The method that will be used is the Multiplicative Dimensional Reduction Method (MDRM) which was developed by Zhang (2013). This method has been implemented before but now the focus is to use it for uncertainty problems with an epistemic variable, fire resistance problems and geomechanics problems to determine the effectiveness of MDRM.

1.2 Objective and Research Significance

The goal of this research investigation is to compare the Multiplicative Dimensional Reduction Method to Monte Carlo simulation and also check out how Cubature Formulae match up with the Multiplicative Dimensional Reduction Method. The specific objectives of this research are:

- To estimate the probability distribution of the structural response using the MDRM along with the maximum entropy principle.
- To apply the MDRM to problems considering that all random variables are simply uncertain and compare with MCS and Cubature formulae.
- To compare the efficiency and accuracy of MDRM with MCS and/or Cubature formulae.
- To apply MDRM to a problem considering that one or more random variables are epistemic and compare with MCS.
- To apply MDRM to fire safety design questions and compare with MCS.
- To apply MDRM to geomechanics problems, specifically finite element analysis and compare with MCS.

1.3 Outline of Thesis

Chapter 2 provides an extensive literature review in reliability analysis, the Multiplicative Dimensional Reduction Method as well as Cubature method. The basic concepts and mathematical equations are provided for both of these methods. The required steps for applying both these methods are also provided.

Chapter 3 presents a couple verification examples. The first example is a code check example to make sure that the developed code works correctly and provides similar results to problems done by others. The second example shows that the MDRM as well as the Cubature method provide similar results to Monte Carlo Simulation results. These are both equation based problems.

Chapter 4 presents the applicability of MDRM for an uncertainty analysis. The first example is done considering all random variables are simply uncertain meaning there is no distinction between an aleatory or epistemic variable. The second example is done considering one variable is an aleatory random variable while another is an epistemic random variable. These are both equation based problems. The results of the first example are used for the sake of accuracy comparison between MDRM, Cubature and MCS. Whereas for the second example Cubature method is not used and the results are used for the sake of accuracy comparison between MDRM and MCS.

Chapter 5 presents the applicability of MDRM for fire resistant design of structures. The problem solved here is of a beam under fire load using a performance based approach. This problem is done twice, once for an unprotected beam and once for a protected beam. This is an equation based problem. The results are then used for the sake of accuracy comparison between MDRM and MCS.

Chapter 6 presents the applicability of MDRM for geomechanics problems. The first two examples are equation based problems of 1D consolidation. The first problem goes over a step by step detailed procedure on how to solve a problem using MDRM. Results of both these problems are used for the sake of accuracy comparison between MDRM and MCS. The second set of two problems are finite element analysis based problems. The first problem is a vertical drain problem solved using ABAQUS and a FORTRAN code developed by Dipanjan Basu. The second problem is of a concrete infinite beam on elastic foundation which was solved using a MATLAB code provided by Hesham Elhuni. This problem was done using two different foundation models (two parameter Pasternak Model and Modified Vlasov Model). These two problems were used for the sake of computational efficiency and accuracy comparison between MDRM and MCS.

2 Literature Review

2.1 Reliability Analysis

Engineering design and analysis problems are often confounded by uncertainties (Cornell & Benjamin, 1970). There are two types of uncertainty, aleatory and epistemic (Tang & Ang, 2006). An aleatory uncertainty is one that is presumed to be the intrinsic randomness of a phenomenon. An epistemic uncertainty is one that is presumed as being caused by lack of knowledge (data) (Ditlevsen & Der Kiureghian, 2009). To incorporate these uncertainties in the analysis, a probabilistic analysis can be used since it allows characterizing the deterministic values as random variables (Madsen & Ditlevsen, 1996).

An engineering design done following a reliability-based methodology requires consideration of uncertainty in the system parameters (Jeffers et al., 2012):

$$X = (X_1, X_2, \dots, X_n) \tag{2.1}$$

Structural resistance, R, and load demand, S, are both random variables which depend on X and characterized by its moments (mean, μ, and standard deviation, σ) and the probability distribution, f.To determine the reliability, define the following performance function (Jeffers et al., 2012):

$$G(X) = R(X) - S(X) \tag{2.2}$$

Which states that as when as the resistance is less than the load on the structure there will be failure ($G(X) < 0$). Therefore, the probability of failure is defined as the probability that $G(X) < 0$:

$$P_f = P(G(X) < 0) \tag{2.3}$$

This can be rewritten as the joint probability density functions over the failure region (Saouma & Puatatsananon, 2006):

$$P_f = \int_{G(X)<0} f_X(X)dX \tag{2.4}$$

Where f_X is the probability density function for the random variables, X_i. This integral is however too complex to solve analytically in most cases, therefore numerical methods such as MCS have to be applied. When the probability of failure is less than the acceptable limit that is when a safe design is achieved.

The likelihood of an event occurring denotes probability (Melchers, 1987), thus, the probability of failure denotes the probability that a structure or other object will stop working as required and fail at a specific time. Whereas reliability can be defined as follows (Lind, Krenk and Madsen, 2006):

$$Reliability = 1 - p_f \tag{2.5}$$

Overall, reliability analysis helps engineers determine whether or not the structure has been designed adequately to last its desired lifetime (Lind, Krenk and Madsen, 2006).

A probabilistic treatment of a problem requires (Jeffers et al., 2012):

1) The identification and characterization of the random variables.
2) Definition of appropriate performance function(s) by which failure can be evaluated.
3) The system reliability which is expressed by the probability of failure, P_f.

2.2 Monte Carlo Simulation

Monte Carlo Simulation (MCS) is a method used to determine the probability of failure of a function by simulating random variables. This method requires the use of a random number generator that can generate many random (pseudo) numbers (Botev, Taimre and Kroese, 2011). The evolution of computers has made this method widely applicable (Sobol, 1994).

There are 3 basic steps to using MCS (Balomenos. 2015):

i. Select the distribution type for each random variable
ii. Generate random numbers based on the selected distribution
iii. Conduct simulations based on the generated random numbers

The more trials/simulations performed the greater the accuracy of the estimation. MCS can be used to calculate the probability of failure with analysis of the function, which is a great advantage of this method along with the fact that it is simple to execute. On the other hand, many simulations are required to achieve an accurate probability of failure, which can be computationally expensive. This is just a brief overview on MCS, more information can be found in the following references, (Tang & Ang, 1984; Melchers, 1987).

2.3 Multiplicative Dimensional Reduction Method

The Multiplicative Dimensional Reduction Method (MDRM) is an alternative to Monte Carlo Simulation (MCS). MDRM provides a considerable advantage in terms of efficiency while still maintaining the accuracy of MCS.

Multiple methods have been derived for dealing with the statistical analysis of multivariate problems in order to avoid the high computational cost of MCS. The first and second –order reliability methods (FORM and SORM) are considered the most popular methods for efficient reliability analysis of structures in the past several years (Hasofer and Lind, 1974), which are based on the first and second-order moments of performance functions. These two methods do however suffer from the problems of inaccuracy of the reliability assessment when the performance functions are strongly nonlinear (Zhang & Li, 2010) and numerical difficulties in searching for design points (Ono & Zhao, 2001).

The method of moments can be used to find an approximate solution to a multivariate problem (Taguchi, 1978) by calculating the first four moments of the response which are mean, variance, skewness and kurtosis. Once the first four moments are calculated the parameters of the distribution can be back calculated. The problem here however is that the calculation of moments involves multi-dimensional integrals which are very complex to solve. Various methods have been developed and researched to look at the efficient evaluation of these integrals. These methods include using point estimate methods (Taguchi, 1978), Rosenblueth, 1981), Taylors series approximation and non-classical orthogonal polynomial approximations (Lennox & Kennedy, 2001). Methods such as high-dimensional model representation (Rabitz, Rosenthal and Li, 2001) and the dimensional reduction method (Rahman & Xu, 2004), (Xu & Rahman, 2004) have also been developed in which the multivariate function is decomposed into orthogonal component functions.

To deal with the issue of sensitivity of tail probabilities, the principle of maximum entropy (MaxEnt) was introduced (Jaynes, 1957); this however required a significant amount of computational effort in the moment calculations when dealing with a large number of constraints. To reduce the computational effort required by using the principle of maximum entropy, fractional moments were introduced. A fractional moment is a moment of order of real numbers (Tagliani and Novi Inverardi, 2003).

MDRM (Zhang, 2013) is a combination of fractional moments and the MaxEnt principle. Using MDRM, a kth statistical moment of the response can be approximated as:

$$E[Y^k] = E\left[\left(h(x)\right)^k\right] \approx E\left[\left(h_0^{(1-n)} \times \prod_{i=1}^{n} h_i\,(x_i)\right)^k\right] \tag{2.6}$$

$$Y = h(x) \tag{2.7}$$

Where *n* is the number of random variables, h_0 is the response when all random variables are held to their means, and $h_i(x_i)$ is a one dimensional ith cut function where all random variables are held to their mean value except the one variable which is being changed. This will be discussed more in the gauss quadrature section. Assuming all input random variables are independent, the above equation can be written as:

$$E[Y^k] \approx h_0^{k(1-n)} \prod_{i=1}^{n} E[\,(h_i(x_i)^k] \tag{2.8}$$

The numerical integration can be optimized using Gauss quadrature formulas. The kth moment can be approximated as a weighted sum (Balomenos, 2015):

$$E[(h_i(x_i)^k] = \sum_{j=1}^{L} w_j\,[h_i(x_j)]^{\,k} \tag{2.9}$$

To perform MDRM, all random variables except one are held at their mean value while each variable is changed one at a time depending on the Gaussian quadrature (5 point, 7 point etc.) using the x_j equations listed in Table 2.1. 5-point gauss quadrature is most commonly used, the weights and points are shown in Table 2.2.

Table 2.1. Gaussian integration formula for the one-dimensional fraction moment calculation.

Distribution	Gaussian Quadrature	x_j	Numerical Integration Formula
Uniform	Gauss-Legendre	$\frac{b-a}{2}z_j + \frac{b+a}{2}$	$\sum_{j=1}^{L} w_j\,[\frac{1}{2}h(x_j)]^{\,k}$
Normal	Probabilists' Gauss-Hermite	$\mu + \sigma z_j$	$\sum_{j=1}^{L} w_j\,[h(x_j)]^{\,k}$
Lognormal	Probabilists' Gauss-Hermite	$\exp(\lambda + \zeta z_j)$*	$\sum_{j=1}^{L} w_j\,[h(x_j)]^{\,k}$

Exponential	Gauss-Laguerre	$\frac{z_j}{\lambda}$	$\sum_{j=1}^{L} w_j\ [h(x_j)]^{\ k}$
Weibull	Gauss-Laguerre	$\beta z_j^{(1/\alpha)}$**	$\sum_{j=1}^{L} w_j\ [h(x_j)]^{\ k}$

* $\zeta = \sqrt{\ln(1+\frac{\sigma^2}{\mu^2})}$ (shape parameter) and $\lambda = \ln(\mu) - \frac{1}{2}\zeta^2$(scale parameter)

** β denotes the scale parameter and α denotes the shape parameter.

Note: z_j denotes the gauss points, w_j denotes the associated Gauss weights and L is the Gauss quadrature order used (i.e. 5-point gauss quadrature, L =5)

Table 2.2. Weights and points of the five order Gaussian Quadrature rules.

Gaussian Quadrature	L	1	2	3	4	5
Gauss-Legendre	w_j	0.23693	0.47863	0.56889	0.47863	0.23693
	z_j	-0.90618	-0.53847	0	0.53847	0.90618
Probabilists' Gauss-Hermite	w_j	0.01126	0.22208	0.53333	0.22208	0.01126
	z_j	-2.85697	-1.35563	0	1.35563	2.85697
Gauss-Laguerre	w_j	0.52176	0.39867	0.07594	0.00361	2.34×10^{-5}
	z_j	0.36356	1.4134	3.5964	7.0858	12.641

More orders of Gauss points and weights can be found in the textbook (Schaferkotter & Kythe, 2004).

The probabilists' Gauss-Hermite's points and weights are not readily available online or in textbooks. Physicists' gauss Hermite is what is mainly found but it is just listed as Gauss Hermite. To convert from physicists' to probabilists' the following equations can be used (Sullivan, 2015):

$$w_j^{prob} = \frac{w_j^{phys}}{\sqrt{\pi}} \tag{2.10}$$

$$z_j^{prob} = \sqrt{2} z_j^{phys} \tag{2.11}$$

For example with 4 variables, (a, b, c, d) a total of 21 calculations will be performed (nL+1 $=(4 \times 5) + 1 = 21$). 5 for each of the 4 variables (considering 5-point gauss quadrature) and one when all the variables are held at their respective mean value. The first moment (mean) for each variable is then calculated as follows:

$$\rho_a = \sum_{j=1}^{L=5} w_j \, h(a_j, b_0, c_0, d_0) \tag{2.12}$$

Where $h(a_j, b_0, c_0, d_0)$ is the response value by varying a by each Gaussian point according to the x_j formulas in table 1 and holding b, c, and d to their mean values. The second moment (mean square) is calculated similarly as follows (Pandey, Walbridge and Raimbault, 2015):

$$\theta_b = \sum_{j=1}^{L=5} w_j \, [h(a_0, b_j, c_0, d_0)]^2 \tag{2.13}$$

The mean and mean square of the response of Y can then be approximated as:

$$\mu_Y = E[Y] \approx \; h_0^{(1-n)} \times \prod_{i=1}^{n} \rho_i \tag{2.14}$$

$$\mu_{2Y} = E[Y^2] \approx \; h_0^{(2-2n)} \times \prod_{i=1}^{n} \theta_i \tag{2.15}$$

The variance can then be calculated as:

$$V_Y = \mu_{2Y} - (\mu_Y)^2 \tag{2.16}$$

Where the standard deviation of the response is then the square root of the variance.

Now that the input grid has been created the last step is to use this to find the Lagrange multipliers (λ_i) and fractional exponents (α_i) that define the response. The MaxEnt principle with fractional moment constraints ($[Y^{\alpha}] = M_Y^{\alpha}$) where α is a real number not an integer. This principle states that by maximizing the entropy subjected to the fractional moment constraints, the most unbiased probability distribution can be estimated. The Lagrange multipliers (λ_i) and fractional exponents (α_i) are therefore obtained by applying the following optimization (Balomenos, 2015):

$$\textbf{Randomize: } \{\alpha_i\}_{i=1}^{m} \tag{2.17}$$

$$\textbf{Find: } \{\lambda_i\}_{i=1}^{m} \tag{2.18}$$

by

$$\textbf{Minimizing:} I(\lambda, \alpha) = \ln\left[\int_0^{\infty} \exp\left(-\sum_{i=1}^{m} \lambda_i \, y^{\alpha_i}\right) dy\right] + \sum_{i=1}^{m} \lambda_i \, M_Y^{\alpha_i} \tag{2.19}$$

Where m is the number of fractional moments, $\lambda = [\lambda_0, \lambda_1, \dots, \lambda_m]^T$ are the Lagrange multipliers and $\alpha = [\alpha_0, \alpha_1, \dots, \alpha_m]^T$ are the fractional exponents. This optimization can be implemented in MATLAB using the simplex search method (Wright et. al., 1998). Randomize the α values at least 100 or more times, and find the α and λ values which results in the lowest entropy (function evaluation at the α and λ values which are obtained); the set of values which result in the lowest entropy is the answer to this optimization problem. The best way to randomize α is to set the random values between a bound such as (-1,1), (-5,5) etc. M_Y^{α} can be expanded and replaced in the above equation as follows:

$$M_Y^{\alpha} = E[Y^{\alpha}] \approx h_0^{\alpha(1-n)} \prod_{i=1}^{n} E[\,(h_i(x_i)^{\alpha}] \tag{2.20}$$

$$E[(h_i(x_i)^{\alpha}] = \sum_{j=1}^{L} w_j \, [h_i(x_j)]^{\,\alpha} \tag{2.21}$$

Using 3 fractional moments (m=3) is sufficient since entropy converges very quickly. Now that the α and λ values have been solved for, the estimated probability density function (PDF) of the true PDF can be obtained (Balomenos, 2015):

$$\hat{f}_Y(y) = \exp\left(-\sum_{i=0}^{m} \lambda_i \, y^{\alpha_i}\right) \tag{2.22}$$

For i=0, $\alpha_0 = 0$ and λ_0 is solved for using the following equation:

$$\lambda_0 = \ln\left[\int_0^{\infty} \exp(-\sum_{i=1}^{m} \lambda_i \, y^{\alpha_i}) dy\right] \tag{2.23}$$

For more information on the derivations on the optimization, $\hat{f}_Y(y)$, and λ_0 equations or the MDRM in general please read (Balomenos, 2015; Zhang, 2013). A global sensitivity analysis can also be done using MDRM (Balomenos, 2015; Zhang, 2013).

2.4 Cubature Formulae

There are many cubature formulae with fixed algebraic degree of accuracy developed by mathematicians which help to solve problems efficiently rather than using MCS. Just like MDRM, the cubature formulas can evaluate the statistical moments of the function. However, different cubature formula will give a different degree of accuracy (Xu & Lu, 2017)

Before the cubature formulas can be implemented or discussed some steps need to be taken to transform the random variables as functionals of normal random variables. The Probability Integral Transform is applied to change the variables. Replace the random variable with the corresponding transformation equation from Table 2.3.

Table 2.3. Representation of common univariate distributions as functionals of normal random variables.

Distribution Type	Transformation
Uniform $(\boldsymbol{a}, \boldsymbol{b})$	$a + (b - a)\left(\frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{\xi}{\sqrt{2}}\right)\right)$
Normal $(\boldsymbol{\mu}, \boldsymbol{\sigma})$	$\mu + \sigma\xi$
Lognormal $(\boldsymbol{\mu}, \boldsymbol{\sigma})$	$\exp(\lambda + \zeta\xi)$*
Gamma $(\boldsymbol{a}, \boldsymbol{b})$	$ab\left(\xi\sqrt{\frac{1}{9a}} + 1 - \frac{1}{9a}\right)^3$
Exponential $(\boldsymbol{\lambda})$	$-\frac{1}{\lambda}\log\left(\frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{\xi}{\sqrt{2}}\right)\right)$
Weibull $(\boldsymbol{\alpha}, \boldsymbol{\beta})$	$\beta\left(-\log\left(\frac{1}{2} - \frac{1}{2}\operatorname{erf}\left(\frac{\xi}{\sqrt{2}}\right)\right)\right)^{1/\alpha}$

* $\zeta = \sqrt{\ln(1 + \frac{\sigma^2}{\mu^2})}$ (shape parameter) and $\lambda = \ln(\mu) - \frac{1}{2}\zeta^2$(scale parameter)

Note: ξ is the cubature point corresponding to the cubature formula being used. All equations found from (Isukapalli, 1999) except Weibull equation (Villanueva, Feijóo & Pazos, 2013).

For example, a response function in the form, $h(X) = X_1 + X_2 + X_3$ with X_1 being a normal random variable, X_2 being a lognormal random variable and X_3 being a weibull random variable, is being evaluated. After changing the variables and applying the corresponding equations from Table 2.4, the response function changes to:

$$h(\xi) = [\mu + \sigma\xi] + [\exp(\lambda + \zeta\xi)] + \left[\beta\left(-\log\left(\frac{1}{2} - \frac{1}{2}\operatorname{erf}\left(\frac{\xi}{\sqrt{2}}\right)\right)\right)^{1/\alpha}\right] \tag{2.24}$$

Now that the variables have been changed, the next step is to perform the integral of the form:

$$I(f) = \int_{\mathbb{R}^n} e^{-\xi^T\xi} f(\xi) d\xi \tag{2.25}$$

The integrand *f(x)* is usually not an analytical expression but it is an output from some computational simulation, therefore approximation methods such as the cubature formulae are needed (Lu & Darformal, 2004). This reduces the above integral to the following sum:

$$I(f) \approx \sum_{j=1}^{N} B_j\, f(\xi_{j,1}, \dots, \xi_{j,n}) \tag{2.26}$$

Where B_j and $\xi_j = (\xi_{j,1}, \dots, \xi_{j,n})$ are the weights and quadrature points respectively, and N is the number of quadrature points (Wei, Cui & Chen, 2008).

The first two moments, i.e. Mean and standard deviation, are given as:

$$\mu_Y = \sum_{j=1}^{N} B_j\, h(\xi_j) \tag{2.27}$$

$$\sigma_Y = \sum_{j=1}^{N} B_j \left[h(\xi_j) - \mu_Y\right]^2 \tag{2.28}$$

Where *Y=h(ξ)* is the response function. Derivations for these two equations as well as the equations for the third and fourth moments (i.e. Skewness and Kurtosis) can be found (Xu & Lu, 2017).

In equation 2.25 and 2.26, $f(\xi)$ represents the arbitrary integrand, for example for σ_Y, $f(\xi) = \left[h(\xi_j) - \mu_Y\right]^2$, or $f(\xi) = h(\xi_j)$ for μ_Y.

In this section the 5 most efficient known cubature formulae of degree 5 will be used. Formulas of degree 5 are well developed and particularly useful for Gaussian weighted integration (Wei, Cui & Chen, 2008).

Cubature Formula 1

This is a formula that is valid for $2 \le n \le 7$, where n is the number of random variables, given by Stroud (1966) for numerical integration over infinite regions:

$$
\begin{aligned}
I(f) = & a\left[f\left(\sqrt{2}\eta, \sqrt{2}\eta, \dots, \sqrt{2}\eta\right) + f\left(-\sqrt{2}\eta, -\sqrt{2}\eta, \dots, -\sqrt{2}\eta\right)\right] \\
& + b\left[\sum_{permutation} f\left(\sqrt{2}\lambda, \sqrt{2}\xi, \dots, \sqrt{2}\xi\right) + f\left(-\sqrt{2}\lambda, -\sqrt{2}\xi, \dots, -\sqrt{2}\xi\right)\right] \\
& + c\left[\sum_{permutation} f\left(\sqrt{2}\mu, \sqrt{2}\mu, \sqrt{2}\gamma, \dots, \sqrt{2}\gamma\right) + f\left(-\sqrt{2}\mu, -\sqrt{2}\mu, -\sqrt{2}\gamma, \dots, -\sqrt{2}\gamma\right)\right]
\end{aligned} \tag{2.29}
$$

Where the values of the 8 parameters, $(a, b, c, \eta, \lambda, \xi, \mu, \gamma)$, are given by (Stroud, 1966) and are shown in Table 2.5, and where the summations are taken over all distinct permutations of the input variables. This formula requires a total of $n^2 + n + 2$ points. It contains fewer points than any other 5th degree formula when $n \ge 4$ (Wei, Cui & Chen, 2008).

Cubature Formula 2

This is a formula that is valid for $n > 3$, where n is the number of random variables, which is derived by Mysovskikh (1980) for the surface of the sphere. This formula is expressed as:

$$
\begin{aligned}
I(f) = & \frac{2}{n+2} f(0) + \frac{n^2(7-n)}{2(n+1)^2(n+2)^2} \sum_{j=1}^{n+1} \left[f\left(\sqrt{n+2} \times a^{(j)}\right) + f\left(-\sqrt{n+2} \times a^{(j)}\right)\right] \\
& + \frac{2(n-1)}{(n+1)^2(n+2)^2} \sum_{j=1}^{n(n+1)/2} \left[f\left(\sqrt{n+2} \times b^{(j)}\right) + f\left(-\sqrt{n+2} \times b^{(j)}\right)\right]
\end{aligned} \tag{2.30}
$$

Where $a^{(j)}$ and $b^{(j)}$ are:

$$a^{(j)} = \left(a_1^{(j)}, a_2^{(j)}, \dots, a_n^{(j)}\right), j = 1,2,\dots,n+1 \tag{2.31}$$

$$b^{(j)} = \sqrt{\frac{n}{2(n-1)}}\left(a^{(k)} + a^{(l)}\right), k < l, l = 1,2,\dots,n+1 \tag{2.32}$$

$$a_i^{(j)} = \begin{cases} -\left(\frac{n+1}{n(n-i+2)(n-i+1)}\right)^{1/2}, for\ i < j \\ \left(\frac{(n+1)(n-j+1)}{n(n-j+2)}\right)^{1/2}, \quad for\ i = j \\ 0, \quad for\ i > j \end{cases} \tag{2.33}$$

This formula requires a total of $n^2 + 3n + 3$ points. When $n < 7$ the weights are all positive but for $n > 7$ the weights end up being negative (Xu, Chen & Li, 2012).

Cubature Formula 3

A formula proposed in Stroud & Secrest (1963) which requires the use of $2n^2 + 1$ points is:

$$I(f) = \frac{2}{n+2}f(0) + \frac{4-n}{2(n+2)^2}\sum_{full\ sym} f(\pm\sqrt{n+2}, 0, \dots, 0) + \frac{1}{(n+2)^2}\sum_{full\ sym} f(\pm\sqrt{\frac{n}{2}+1}, \pm\sqrt{\frac{n}{2}+1}, 0, \dots, 0) \tag{2.34}$$

Where the summation is done over all the different reflections and permutations and this formula holds for arbitrary dimensions(n).

Cubature Formula 4

A formula constructed by McNamee & Stenger (1967) and Phillips (1980) which requires the use of $2n^2 + 1$ points is:

$$I(f) = \frac{n^2 - 7n + 18}{18} f(0) + \frac{4-n}{18} \sum_{full\ sym} f(\pm\sqrt{3}, 0, \dots, 0)$$
$$+ \frac{1}{36} \sum_{full\ sym} f(\pm\sqrt{3}, \pm\sqrt{3}, 0, \dots, 0) \quad (2.35)$$

Where the summation is done over all the different reflections and permutations and this formula holds for arbitrary dimensions (n).

Cubature Formula 5

Victoir (2004) constructed some cubature formulae called the quasi-symmetric point method (Q-SPM). The Q-SPM is efficient because it only uses parts of points in the same symmetric point set and the points in the same point set possess the same weight value (Xu, Chen & Li, 2012). The second class of Q-SPM integral points are given by:

$$x_0 = (hr, 0, \dots, 0), x_1 = (hs, hs, \dots, hs) \quad (2.36)$$

Where:

$$r = \sqrt{\frac{n+2}{2}}, s = \sqrt{\frac{n+2}{n-2}} \quad (2.37)$$

Where h is the permutation of ±1, x_0 and x_1 represent the two symmetric point sets where N_1 and N_2 points are involved respectively. The total number of points required is $(N = N_1 + N_2,$ where $N_1 = 2n$ and $N_2 << 2^n)$ is listed in Table 2.6, when the number of random variables varies from 3 to 20. The weights for the points are:

$$\alpha_0 = \frac{4}{(n+2)^2}, \alpha_1 = \frac{(n-2)^2}{(n+2)^2(N-2n)} \quad (2.38)$$

Where N can be found in Table 3. There the integration equation is (Xu & Lu, 2017):

$$I(f) = \alpha_0 \sum f(x_0) + \alpha_1 \sum f(x_1) \tag{2.39}$$

All 5 of the above formulae are efficient, where depending on the number of random variables, tens of points or a few hundred are needed. As stated before different formulae will give different results, one formula might give a more accurate result for 3 random variables while another might give a more accurate result for 4 random variables.

Xu & Lu (2017) work on multiple examples to choose the "best" cubature formula out of these 5. The criterion used to determine the "best" cubature formula is one that is able to efficiently achieve the statistical moments with the highest accuracy. Out of the 5 cubature formulae, formula 1 is chosen as the "best" the most amount of times.

Table 2.4. Values of parameters depending on number of random variables from 2 to 7.

n	Parameter	Value
2	η	0.446103183094540
	λ	1.36602540378444
	ξ	−0.366025403784439
	μ	1.98167882945871
	γ	---
	a	0.328774019778636
	b	0.0833333333333333
	c	0.00455931355 69736
3	η	0.476731294622796
	λ	0.935429018879534
	ξ	−0.731237647787132
	μ	0.433155309477649
	γ	2.66922328697744
	a	0.24200 00000 00000
	b	0.081000 00000 00000
	c	0.0050000 00000 00000
4	η	0.523945658287507
	λ	1.19433782552719
	ξ	-0.398112608509063
	μ	-0.318569372920112

	γ	1.85675837424096
	a	0.155502116982037
	b	0.0777510584910183
	c	0.00558227484231506
5	η	2.14972564378798
	λ	4.64252986016289
	ξ	-0.623201054093728
	μ	-0.447108700673434
	γ	0.812171426076331
	a	0.000487749259189752
	b	0.000487749259189752
	c	0.0497073504444862
6	η	1.0000000000000000
	λ	1.41421356237309
	ξ	0
	μ	-1.0000000000000000
	γ	1.0000000000000000
	a	0.0078125000000000
	b	0.0625000000000000
	c	0.0078125000000000
7	η	0
	λ	0.959724318748357
	ξ	-0.772326488820521
	μ	-1.41214270131942
	γ	0.319908106249452
	a	0.111111111111111
	b	0.013888888888889
	c	0.013888888888889

Table 2.5. Number of points required for Q-SPM, based on the number of random variables.

n	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
Points	14	24	42	44	78	144	146	276	278	280	282	284	286	288	546	548	550	552

3 MDRM Analysis: Verification Examples

3.1 Introduction

3.1.1 Objective

The objective of this chapter is to determine if the code that has been developed works properly and gives the values required. Also this chapter will go over a simple problem to demonstrate how to solve a problem using MDRM and Cubature. The results will then be compared with MCS to see the accuracy and efficiency of MDRM and Cubature.

3.1.2 Organization

The organization of this chapter is as follows. Section 3.2, firstly presents a MDRM code check to provide evidence that the MATLAB code that was developed was implemented correctly. A problem is done from Zhang & Pandey (2013) to compare results. Section 3.4 presents a steam generator failure problem. This is an equation based problem that is solved using MDRM, MCS and Cubature and the results from each methods is compared. The conclusions are then summarized in Section 3.4.

3.2 MDRM Code Verification

To ensure the code was implemented correctly a problem from Zhang & Pandey (2013) was done to see if the entropy values and the probability density function matched up. The entropy values matching up shows that the α and λ don't have to match up with what Zhang & Pandey (2013) determined them to be in order to obtain the same results. Making sure the graphs match ensures that even though the α and λ don't match up with what Zhang & Pandey (2013) determined them to be the probability density function produced is still the same.

The problem that was done from Zhang & Pandey (2013) involved figuring out the ultimate bending moment of resistance of a reinforced beam using the following equation:

$$M_U(X) = X_1X_2X_3 - X_4\frac{X_1^2X_2^2}{X_5X_6} \tag{3.1}$$

The distributions for each of the 6 random variables are given in Table 3.1 below:

Table 3.1. Random variables in the reinforced concrete beam example.

Variable	Description	Distribution	Mean	St.Deviation
$\boldsymbol{X_1}$	Area of reinforcement	Lognormal	1260	252
$\boldsymbol{X_2}$	Yield stress of reinforcement	Lognormal	300	60
$\boldsymbol{X_3}$	Effective depth of reinforcement	Lognormal	770	154
$\boldsymbol{X_4}$	Stress-Strain factor of concrete	Lognormal	0.35	0.035
$\boldsymbol{X_5}$	Compressive strength of concrete	Weibull	25	5
$\boldsymbol{X_6}$	Width of beam	Normal	200	40

Using this information, the entropy was found to be 5.9143 whereas the entropy given in Zhang & Pandey (2013) is 5.9147 which results in a relative error of 0.0068% showing that the entropy found in this paper is approximately equal to the entropy found in Zhang & Pandey (2013). Next the probability distribution found in this paper is shown below in Figure 3.1.

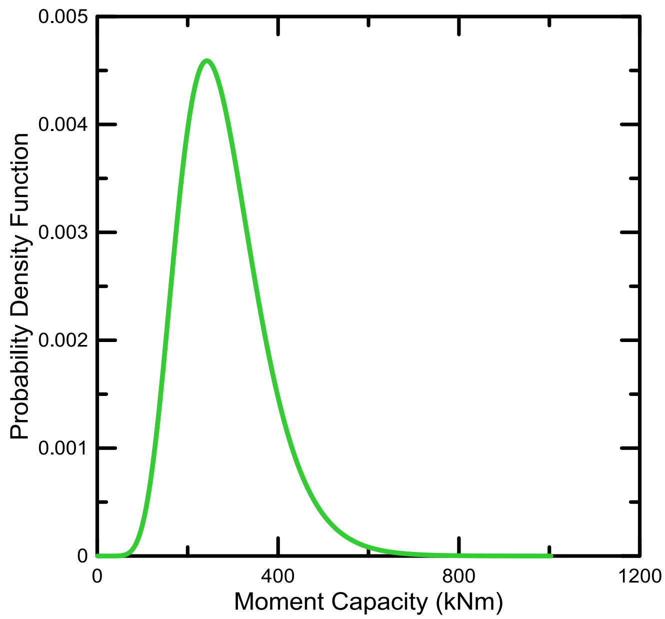

Figure 3.1. Probability density function (PDF) of the ultimate bending moment (M_U)

This probability density function is exactly the same as Fig 2 in Zhang & Pandey (2013). They both have a maximum value of about 0.0045, start having non zero values at about 60 kNm and go back to zero at about 660 kNm.

In summary the entropy values and the PDF's match exactly with what is determined in Zhang & Pandey (2013) thus providing evidence that the code developed in this paper is correct and can be used to solve other problems.

3.3 Steam Generator Tube Failure Problem

Steam generator tubes in pressurized water reactors serve as a pressure boundary and a containment boundary. Rupture of a steam generator tube can have significant safety and environmental consequences. Fretting between steam generator tubes and secondary support structures or between neighbouring steam generator tubes is a widespread degradation mechanism, which can ultimately cause the retirement of a nuclear power plant (Duan, Wang and Kozluk, 2015).

Consider the following empirical equation for the predicted capacity expressed as the failure pressure, P_{FP}, of CANDU steam generator tubes with a fretting flaw (Kozluk, Mills, & Pagan, 2006):

$$P_{FP} = \left[-0.3668 + 1.334\sqrt{1-\frac{a}{h}} + 2.277\left(\frac{a}{h}\right)\left(\frac{h}{2L}\right) + \varepsilon\right] \times \frac{2h}{D_0 - h}\sigma_f \tag{3.2}$$

Where a is the flaw depth, h is the wall thickness, $2L$ is the flaw length, ε is the model uncertainty, D_0 is the outside diameter and σ_f is the flow strength (Duan, Wang and Kozluk, 2015). The flaw depth a at the end of the evaluation period is defined as follows:

$$\frac{a}{h} = \frac{a_0}{h} + \frac{a_g}{h} + \frac{a_e}{h} \tag{3.3}$$

Where a_0 is the beginning of the evaluation period flaw depth, a_g is the flaw growth during the evaluation period, and a_e is the Non Destructive Examination (NDE) measurement error of the flaw depth a_0 (Duan, Wang and Kozluk, 2015). This problem will be solved considering that all random variables are simply uncertain with no specific designation as either epistemic or aleatory, this is a first order random variable probabilistic model definition. Table 3.2 defines each variable in the failure pressure equation.

Table 3.2. Variables.

Variable	Type of Distribution	Parameters of Distribution
$\boldsymbol{\varepsilon}$	Normal	(0, 0.0185) – (mean, st.dev)
$\boldsymbol{\sigma_f}$	Normal	(452, 11) – (mean, st.dev)
$\boldsymbol{a_e/h}$	Normal	(0, 2.551067) – (mean, st.dev)
$\boldsymbol{a_0/h}$	Lognormal	(28,0.2) – (mean, st.dev)
$\boldsymbol{a_g/h}$	Weibull	(1.43595, 0.319793 months) – (shape, scale)
$\boldsymbol{h}$	Constant	1.13 mm

$\mathbf{2L}$	Constant	40 mm
$\mathbf{D_0}$	Constant	15.94 mm

This problem was solved using Monte Carlo Simulation (MCS), the Multiplicative Dimensional Reduction Method (MDRM) and the 5 Cubature formulas.

Using MCS, 10^6 simulations were done for each random variable and each corresponding failure pressure was calculated. Figure 3.2 shows the cumulative distribution of failure pressure, P_{FP}. The mean and standard deviation were calculated as 54.1319 and 2.2237 respectively.

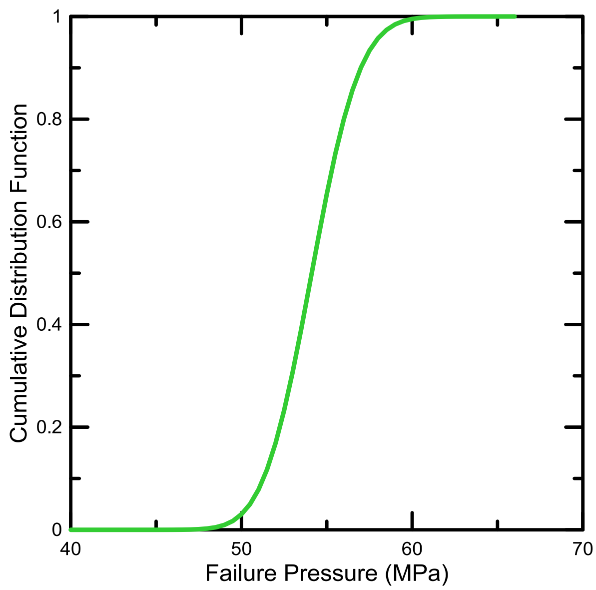

Figure 3.2. Cumulative distribution function of failure pressure.

The MDRM was done using the fifth order (L=5) Gauss quadrature and considering five input random variables (n=5). An input grid is generated to evaluate the response which is seen in table 3.3. The Gauss Hermite and Gauss Laguerre formulas are adopted since 4 random variables follow Normal/Lognormal distribution and the other follows Weibull distribution. In total there are $(5 \times 5) + 1 = 26$ response evaluations. For each evaluation point, the other random variable is fixed to its mean value.

Table 3.3. Input Grid for the response evaluation.

Random Variable	Trial	z_j	a_e/h	a_0/h	a_g/h	σ_f	ε	a/h	P_{FP}
	1	-2.857	-7.2884	28	0.29036	452	0	0.21002	57.41342
	2	-1.3556	-3.45823	28	0.29036	452	0	0.248321	55.57616
a_e/h	3	0	0	28	0.29036	452	0	0.282904	53.87291
	4	1.3556	3.458227	28	0.29036	452	0	0.317486	52.12433
	5	2.857	7.288399	28	0.29036	452	0	0.355788	50.13052
	6	-2.857	0	27.4337	0.29036	452	0	0.277241	54.15485
	7	-1.3556	0	27.72948	0.29036	452	0	0.280198	54.00774
a_0/h	8	0	0	27.99929	0.29036	452	0	0.282896	53.87327
	9	1.3556	0	28.27171	0.29036	452	0	0.285621	53.73721
	10	2.857	0	28.57653	0.29036	452	0	0.288669	53.58464
	11	0.26356	0	28	0.126348	452	0	0.281263	53.95469
	12	1.4134	0	28	0.406925	452	0	0.284069	53.81473
a_g/h	13	3.5964	0	28	0.779788	452	0	0.287798	53.62827
	14	7.0858	0	28	1.250493	452	0	0.292505	53.39213
	15	12.641	0	28	1.871333	452	0	0.298713	53.07935
	16	-2.857	0	28	0.29036	420.573	0	0.282904	50.1272
	17	-1.3556	0	28	0.29036	437.0884	0	0.282904	52.09563
σ_f	18	0	0	28	0.29036	452	0	0.282904	53.87291
	19	1.3556	0	28	0.29036	466.9116	0	0.282904	55.65019
	20	2.857	0	28	0.29036	483.427	0	0.282904	57.61863
	21	-2.857	0	28	0.29036	452	-0.05285	0.282904	50.22727
	22	-1.3556	0	28	0.29036	452	-0.02508	0.282904	52.14312
ε	23	0	0	28	0.29036	452	0	0.282904	53.87291
	24	1.3556	0	28	0.29036	452	0.025079	0.282904	55.60271
	25	2.857	0	28	0.29036	452	0.052855	0.282904	57.51855

Fixed mean values	26	N/A	0	28	0.29036	452	0	0.282904	53.87291

Note: z_j denotes the Gauss Laguerre and Gauss Hermite points.

As many significant figures as possible are shown in all the tables in case these problems in case the problems are to be repeated. The next step is to calculate the mean (ρ_i) and the mean square (θ_i) of an i^{th} cut function is approximated as a weighted sum (Table 3.4). Then the MDRM approximation is used to calculate the statistical moment of the response function (Table 3.5).

Table 3.4. Output Grid for each cut function evaluation.

Random Variable	**Trial**	w_j	P_{FP}	$w_j \times P_{FP}$	ρ_i	$w_j \times {P_{FP}}^2$	θ_i
	1	1.13E-02	57.41342	0.65		37.11	
	2	0.22208	55.57616	12.34		685.94	
a_e/h	3	0.53333	53.87291	28.73	53.86	1547.88	2902.59
	4	0.22208	52.12433	11.58		603.38	
	5	1.13E-02	50.13052	0.56		28.29	
	6	1.13E-02	54.15485	0.61		33.01	
	7	0.22208	54.00774	11.99		647.77	
a_0/h	8	0.53333	53.87327	28.73	53.87	1547.90	2902.30
	9	0.22208	53.73721	11.93		641.30	
	10	1.13E-02	53.58464	0.60		32.32	
	11	0.52176	53.95469	28.15		1518.90	
	12	0.39867	53.81473	21.45		1154.56	
a_g/h	13	7.59E-02	53.62827	4.07	53.87	218.41	2902.23
	14	3.61E-03	53.39213	0.19		10.30	
	15	2.34E-05	53.07935	0.00		0.07	
	16	1.13E-02	50.1272	0.56		28.29	
	17	0.22208	52.09563	11.57		602.72	
σ_f	18	0.53333	53.87291	28.73	53.87	1547.88	2904.02
	19	0.22208	55.65019	12.36		687.77	
	20	1.13E-02	57.61863	0.65		37.37	
	21	1.13E-02	50.22727	0.57		28.40	
	22	0.22208	52.14312	11.58		603.81	
ε	23	0.53333	53.87291	28.73	53.87	1547.88	2903.93
	24	0.22208	55.60271	12.35		686.60	
	25	1.13E-02	57.51855	0.65		37.24	
Fixed mean value	26	N/A	53.87291				

Note: w_j denotes the Gauss Laguerre and Gauss Hermite weights.

Table 3.5. Statistical Moments of the response.

P_{FP}	MDRM (26 Trials)	MCS (10^6 Simulations)	Relative Error (%)
First Moment	53.8609	54.1319	0.5
Second Moment	2905.918	2935.205	0.998
Standard Deviation	2.2188	2.2237	0.22
COV	0.04119	0.04108	0.268

Note: Relative Error (%) $= \frac{|MCS-MDRM|}{MCS} \times 100$

Table 3.5 shows the agreeability of these two methods. The relative errors are all within 1% thus proving that the MDRM is a very good alternative to the high computational cost of MCS.

The output responses obtained using MDRM are combined with the MaxEnt principle with fractional moment constraints, in order to estimate the response probability distribution. Table 3.6 provides the Lagrange multipliers (λ_i) and the fractional exponents (α_i) which are used to estimate the probability distribution of the response. The number of fractional moments used are m=2, m=3, and m=4.

Table 3.6. MaxEnt parameters for failure pressure.

Fractional Moments	Entropy	i	0	1	2	3	4
		λ_i	704.381	-175.621	0.04756		
m=2	2.2061	α_i		0.4025	2.0544		
		$M_X^{\alpha_i}$		2.765E-09	78167.41		
		λ_i	704.379	50.8772	-138.566	49.1863	
m=3	2.2060	α_i		-2.7479	0.8101	1.0137	
		$M_X^{\alpha_i}$		1.398E+08	1.029E-07	7.878E+06	
		λ_i	704.378	22.6168	47.3913	0.03417	-114.8718
m=4	2.2060	α_i		1.0743	-4.4391	1.5810	0.7581
		$M_X^{\alpha_i}$		3.556E-08	0.8104	2.918E-07	0.02064

The estimated probability distribution of the failure pressure is compared to the MCS (Figure 3.3). Then the probability of failure is estimated by plotting the probability of exceedance (POE). From these two figures it is seen that MDRM provides highly accurate approximation for almost the entire range of the output response distribution (Figure 3.4).

Cubature was done using the 5 formulas stated earlier. With Formula 1, 32 points were used to determine the mean and standard deviation, Formula 2 required the use of 43 points. Formula 3 and 4 both used 51 points, and Formula 5 used 42 points.

Data points were simulated using equation (3.2) and a probability paper plot was done. From the probability paper plot, it was determined that a Normal distribution most accurately depicted the probability density function for equation (3.2). Once the means and standard deviations were found using the cubature method, the shape and scale factors were calculated and a MCS of 5000 simulations was done to determine the PDF and POE of each Formula (1, 2, 3, 4, and 5). Table 3.7 compares the means and standard deviations of the Cubature Formulas, MCS and MDRM.

Table 3.7. Means and Standard Deviations of the response.

P_{FP}	**MDRM (26 Trials)**	**MCS (10^6 Trials)**	**Formula 1**	**Formula 2**	**Formula 3**	**Formula 4**	**Formula 5**
Mean	53.8609	54.1319	53.8602	32.88	53.8604	53.8604	53.245
St. Dev	2.2188	2.2237	2.2341	16.9785	2.2340	2.2340	2.1676
M_RE(%)	0.5	N/A	0.502	39.26	0.502	0.502	1.64
S_RE(%)	0.22	N/A	0.468	663.5	0.463	0.463	2.53

Note: M_RE is the mean relative error and S_RE is the standard deviation relative error, compared to MCS, where Relative Error (%) $= \frac{|MCS-x|}{MCS} \times 100$

From Table 3.7 it is seen that Formula 1, 3 and 4 give the closest values to MCS whereas Formula 5 is not as close. Formula 2 is not close at all, there is definitely some error that results in the values being so far off. A possible error is that the formula was not applied correctly or was interpreted incorrectly. Formula 1 was selected as the "best" cubature formula in terms of being able to efficiently be able to achieve the statistical moments with the highest accuracy for multiple examples (Xu & Lu, 2017). This is clearly seen here between the 5 formulas and also on the PDF and POE plots below, Figure 3.3 and 3.4.

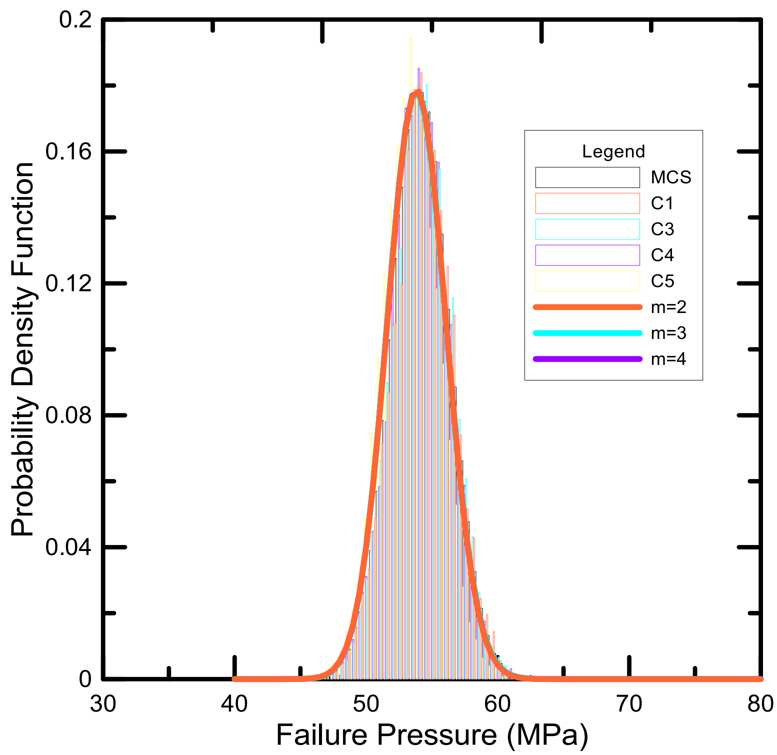

Figure 3.3. Probability density function of the response.

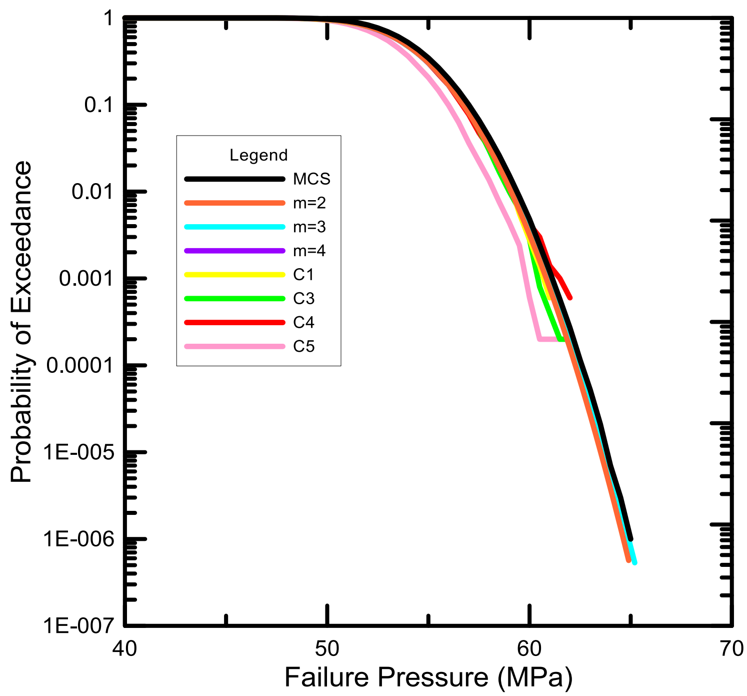

Figure 3.4. Probability of Exceedance (POE) of the response.

In this example, there were 5 random variables which causes Formula 1 to use 32 trials whereas MDRM uses only 26 while providing better efficiency which can be seen in Table 3.7 through the relative errors and Figures 3.3 and 3.4.

3.4 Conclusions

In conclusion, the code was developed correctly as the results of entropy and probability density function from this paper matched up with Zhang & Pandey (2013). Also, the results from MDRM and Cubature formulae show very good agreement with the MCS results. In this one example MDRM uses the least amount of evaluation points to solve the problem and provides the closest results to MCS. Of all the cubature formulae, Formula 1 provided the most accurate results with the least number of points needed.

4 MDRM for Uncertainty Analysis

4.1 Introduction

Uncertainty analysis investigates the uncertainty of variables that are used in the analysis or decision-making problems in which observations and models represent the knowledge base. In other words, uncertainty analysis aims to determine the uncertainty of the variable and the type of random variable (epistemic or aleatory).

The objective of this chapter is to examine the applicability, accuracy of the MDRM and comparing it to MCS and Cubature Formulae to determine what method is the "best." This problem will first be solved considering that all random variables are simply uncertain with no specific designation as either epistemic or aleatory and the second time it will be solved considering one random variable is epistemic while another is aleatory.

4.1.1 Organization

The organization of this chapter is as follows. Section 4.2 presents a simply uncertain problem of a time to leak for a pipe. This is an equation based problem where no distinction is made for the random variables as to whether they are aleatory or epistemic. This problem is done using MDRM, MCS and Cubature methods and the results are compared. Section 4.3 solves the same problem as Section 4.2 but this time one random variable is designated as aleatory and another is designated as epistemic. This time the problem is only done using MDRM and MCS and the results are compared. Finally, the conclusions are summarized in Section 4.4.

4.2 Simple Uncertain Problems

4.2.1 Time to Leak for a Pipe Problem

Consider the following simple model for the time to leak for a pipe, for example from stress corrosion cracking, as:

$$T_L = T_I + \frac{W}{R} \tag{4.1}$$

where T_L is the time to leak (months), T_I is the time to crack initiation (months), W is the wall thickness (mm), and R is the crack growth rate (mm/month). $\frac{W}{R}$ represents the time it takes for an initiated crack to grow through the pipe wall, which results in a leak. Assuming all variables are deterministic and known precisely, this problem can be solved directly without any uncertainty. However, in reality this is not the case, many of the parameters are unknown and hence described as uncertain or random variables. (Jyrkama & Pandey, 2016)

There are two types of random variables, epistemic and aleatory. An aleatory uncertainty is one that is presumed to be the intrinsic randomness of a phenomenon. An epistemic uncertainty is one that is presumed as being caused by lack of knowledge (data) (Ditlevsen & Kiureghian, 2009).

Figure 4.1 shows the difference between a first order random variable probabilistic model definition and a second order random variable probabilistic model definition.

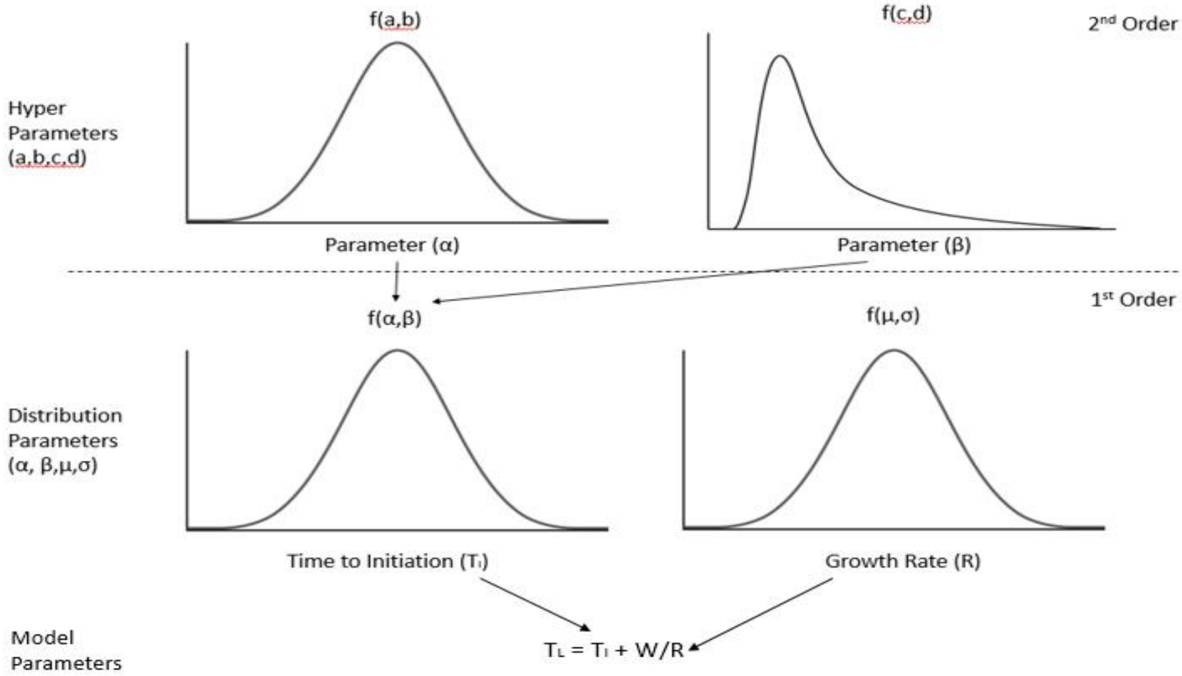

Figure 4.1. First vs second order random variable probabilistic model definition.

This problem will first be solved considering that all random variables are simply uncertain with no specific designation as either epistemic or aleatory, this is a first order random variable probabilistic model definition. Table 4.1 defines each variable in equation (4.1).

Table 4.1. Variables.

Variable	Type of Distribution	Parameters of Distribution
$\boldsymbol{T_I}$	Weibull	(3,480 months) – (shape, scale)
$\boldsymbol{W}$	Constant	40 mm
$\boldsymbol{R}$	Normal	(5 mm/month, 1 mm/month) – (mean, st.dev)

This problem was solved using Monte Carlo Simulation (MCS), the Multiplicative Dimensional Reduction Method (MDRM) and the 5 Cubature formulas.

Using MCS, 1,000,000 simulations were done for each random variable and each corresponding time to leak was calculated. Figure 4.2 shows the cumulative distribution of time to leak, T_L. The mean and standard deviation were calculated as 437.0078 and 155.6737 respectively.

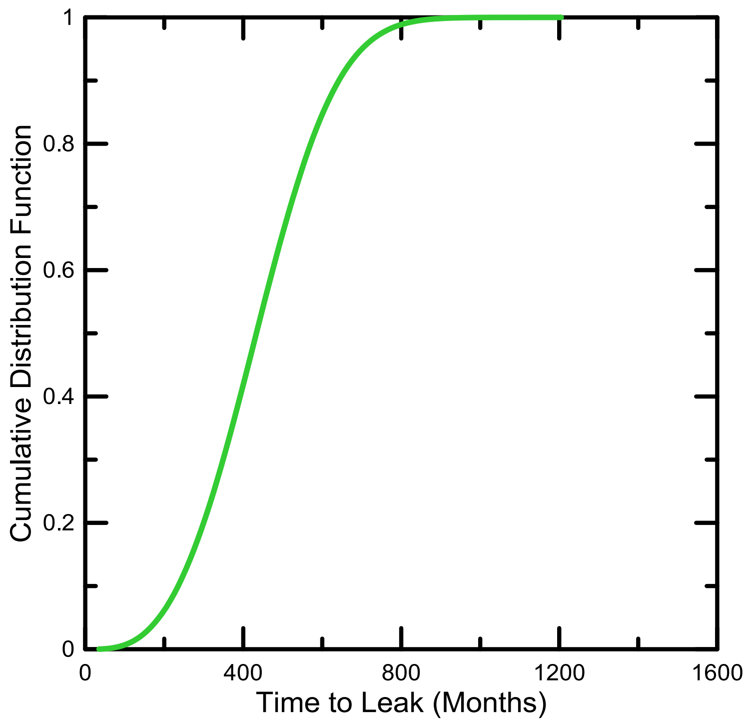

Figure 4.2. Cumulative distribution function of time to leak.

The MDRM was done using the fifteenth order (L=15) Gauss quadrature and considering two input random variables (n=2). An input grid is generated to evaluate the response which can be seen in Table 4.2. The Gauss Hermite and Gauss Laguerre formulas are adopted since one random variable follows Normal distribution and the other follows Weibull distribution. In total there are $(2 \times 15) + 1 = 31$ response evaluations. For each evaluation point, the other random variable is fixed to its mean value.

Table 4.2. Input Grid for the response evaluation.

Random Variable	Trial	z_j	T_i (months)	R (mm/month)	W (mm)	T_l (months)
	1	0.093308	217.7111	5	40	225.7111
	2	0.492692	379.111	5	40	387.111
	3	1.215595	512.2763	5	40	520.2763
	4	2.26995	630.8314	5	40	638.8314
	5	3.667623	740.2347	5	40	748.2347
	6	5.425337	843.4321	5	40	851.4321
	7	7.565916	942.3128	5	40	950.3128
T_i	8	10.12023	1038.257	5	40	1008.93
	9	13.13028	1132.398	5	40	921.9758
	10	16.65441	1225.794	5	40	1275.559
	11	20.77648	1319.568	5	40	1342.948
	12	25.62389	1415.108	5	40	1430.687
	13	31.40752	1514.441	5	40	1526.229
	14	38.53068	1621.226	5	40	1630.748
	15	48.02609	1744.752	5	40	1752.752
	16	-6.36395	428.6302	-1.36	40	399.3035
	17	-5.19009	428.6302	-0.19	40	218.2075
	18	-4.19621	428.6302	0.80	40	478.3943
	19	-3.28908	428.6302	1.71	40	452.0094
	20	-2.43244	428.6302	2.57	40	444.2091
	21	-1.60671	428.6302	3.39	40	440.4181
	22	-0.79913	428.6302	4.20	40	438.152
R	23	-2.32E-16	428.6302	5.00	40	436.6302
	24	0.799129	428.6302	5.80	40	435.5278
	25	1.60671	428.6302	6.61	40	434.6846
	26	2.432437	428.6302	7.43	40	434.012
	27	3.289082	428.6302	8.29	40	433.4558
	28	4.196208	428.6302	9.20	40	432.9798
	29	5.190094	428.6302	10.19	40	432.5555
	30	6.363948	428.6302	11.36	40	432.1501
Fixed Mean Values	31	N/A	428.6302	5.00	40	436.6302

Note: z_j denotes the Gauss Laguerre and Gauss Hermite points.

The next step is to calculate the mean (ρ_i) and the mean square (θ_i) of an i^{th} cut function is approximated as a weighted sum (Table 4.3). Then the MDRM approximation is used to calculate the statistical moment of the response function (Table 4.4).

Table 4.3. Output Grid for each cut function evaluation.

Random Variable	**Trial**	w_j	T_L	w_j x T_L	ρ_i	w_j x T_L^2	θ_i
	1	0.218235	225.711	49.25804		11118.09	
	2	0.34221	387.111	132.4733		51281.87	
	3	2.63E-01	520.276	136.847		71198.26	
	4	0.126426	638.831	80.76478		51595.08	
	5	4.02E-02	748.234	30.08417		22510.02	
	6	0.008564	851.432	7.291561		6208.269	
	7	1.21E-03	950.312	1.152194		1094.944	
T_I	8	1.12E-04	1008.93	0.112672	437.989	113.6778	215126.06
	9	6.46E-06	921.975	0.005956		5.491192	
	10	2.23E-07	1275.55	0.000284		0.362233	
	11	4.23E-09	1342.94	5.68E-06		0.007624	
	12	3.92E-11	1430.68	5.61E-08		8.03E-05	
	13	1.46E-13	1526.22	2.22E-10		3.39E-07	
	14	1.48E-16	1630.74	2.42E-13		3.94E-10	
	15	1.60E-20	1752.75	2.81E-17		4.92E-14	
	16	8.59E-10	399.303	3.43E-07		0.000137	
	17	5.98E-07	218.207	0.00013		0.028452	
	18	5.64E-05	478.394	0.026992		12.91268	
	19	1.57E-03	452.009	0.70846		320.2308	
	20	1.74E-02	444.209	7.71E+0		3426.645	
	21	0.089418	440.4181	39.38122		17344.2	
	22	0.232462	438.152	101.8538		44627.45	
R	23	0.31826	436.6302	138.9617	436.999	60674.87	190972.67
	24	0.232462	435.5278	101.2438		44094.48	
	25	0.089418	434.6846	38.86854		16895.56	
	26	1.74E-02	434.012	7.536954		3.27E+03	
	27	1.57E-03	433.4558	0.67938		294.4813	
	28	5.64E-05	432.9798	0.024429		10.57742	
	29	5.98E-07	432.5555	0.000258		0.111803	
	30	8.59E-10	432.1501	3.71E-07		0.00016	
Fixed Mean Values	31	N/A	436.6302				

Note: w_j denotes the Gauss Laguerre and Gauss Hermite weights.

Table 4.4. Statistical Moments of the response.

T_L	MDRM (31 Trials)	MCS (10^6 Simulations)	Relative Error (%)
First Moment	438.36	437.0078	0.309
Second Moment	215494.80	215210.1181	0.132
Standard Deviation	152.76	155.6737	1.87
COV	0.3485	0.3562	2.16

Note: Relative Error (%) $= \frac{|MCS-MDRM|}{MCS} \times 100$

Table 4.4 shows the agreeability of these two methods. The relative errors are all within 2% thus proving that the MDRM is a good alternative to the high computational cost of MCS.

The output responses obtained using MDRM are combined with the MaxEnt principle with fractional moment constraints, in order to estimate the response probability distribution. Table 4.5 provides the Lagrange multipliers (λ_i) and the fractional exponents (α_i) which are used to estimate the probability distribution of the response. The number of fractional moments used are m=2, m=3, and m=4. The estimated probability distribution of the time to leak is compared to the MCS (Figure 4.3) then the probability of failure is estimated by plotting the probability of exceedance (POE). From these two figures it is seen that MDRM provides highly accurate approximation for almost the entire range of the output response distribution (Figure 4.4).

Table 4.5. MaxEnt parameters for time to leak.

Fractional Moments	Entropy	i	0	1	2	3	4
		λ_i	22.6948	-2.4025	0.03409		
m=2	6.4310	α_i		0.4174	0.9859		
		$M_X^{\alpha_i}$		12.4727	401.8782		
		λ_i	5.4455	0.9562	27.2316	-1.3553	
m=3	6.4306	α_i		0.8193	-0.2335	0.7689	
		$M_X^{\alpha_i}$		144.6556	0.2463	106.2883	
		λ_i	20.02579	0.008981	0.00009737	1.2785	-2.3776

m=4	6.4306	α_i	1.1386	-2.1609	0.08132	0.3888
		$M_X^{\alpha_i}$	1028.4586	3.154E-06	1.6319	10.4808

Cubature was done using the 5 formulas stated earlier. With Formula 1, 8 points were used to determine the mean and standard deviation. Formula 2 could not be used because this problem does not have 3 or more random variables, this is the same reason Formula 5 could not be used as well. Formula 3 and 4 both used 9 points.

Data points were simulated using equation (1) and a probability paper plot was done. From the probability paper plot, it was determined that a Weibull distribution most accurately depicted the probability density function for equation (1). Once the means and standard deviations were found using the cubature method, the shape and scale factors were calculated and a MCS of 10^6 simulations was done to determine the PDF and POE of each formula (1, 3, and 4). Table 4.6 compares the means and standard deviations of the Cubature Formulas, MCS and MDRM.

Table 4.6. Means and Standard Deviations of the response.

T_L	MDRM (31 Trials)	MCS (10^6 Simulations)	Formula 1	Formula 3	Formula 4
Mean	438.36	437.0078	437.03	598.21	599.14
St. Dev	152.76	155.6737	155.67	234.39	243.69
M_RE(%)	0.31	NA	0.005	36.9	37
S_RE(%)	1.87	NA	0.002	50.6	56.5

Note: M_RE is the mean relative error and S_RE is the standard deviation relative error, compared to MCS, where Relative Error (%) $= \frac{|MCS-x|}{MCS} \times 100$

From Table 4.6 it is seen that Formula 1 gives the closest values to MCS and MDRM whereas Formulas 3 and 4 are not close at all. Formula 1 was selected as the "best" cubature formula in terms of being able to efficiently be able to achieve the statistical moments with the highest accuracy for multiple examples (Xu & Lu, 2017) and in this case gives values that are even more accurate than the MDRM. This is clearly seen here between the 3 formulas and also on the PDF and POE plots below, Figure 4.3 and 4.4.

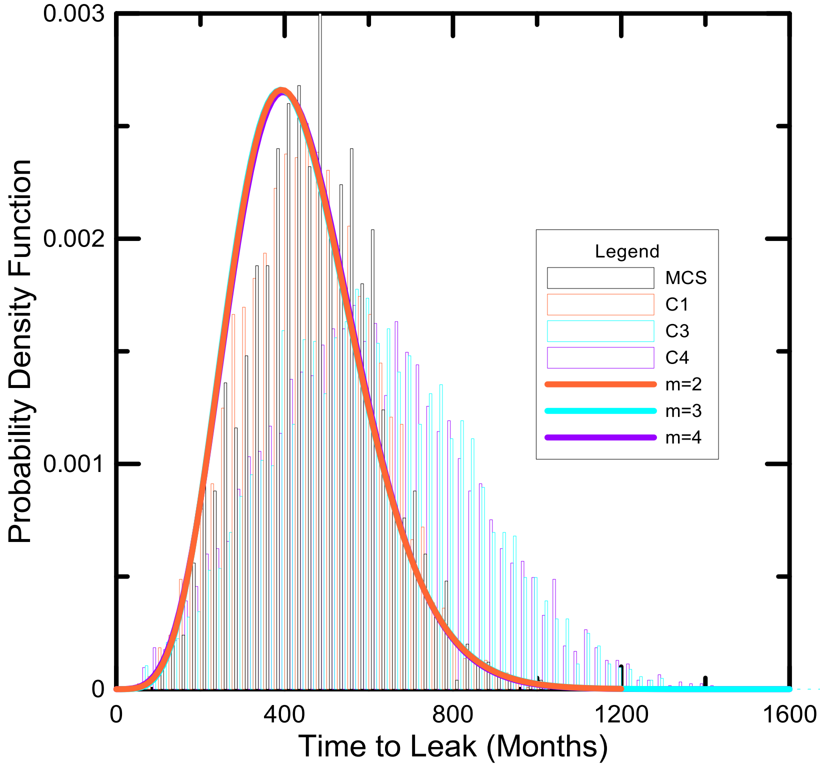

Figure 4.3. Probability Distribution of the response.

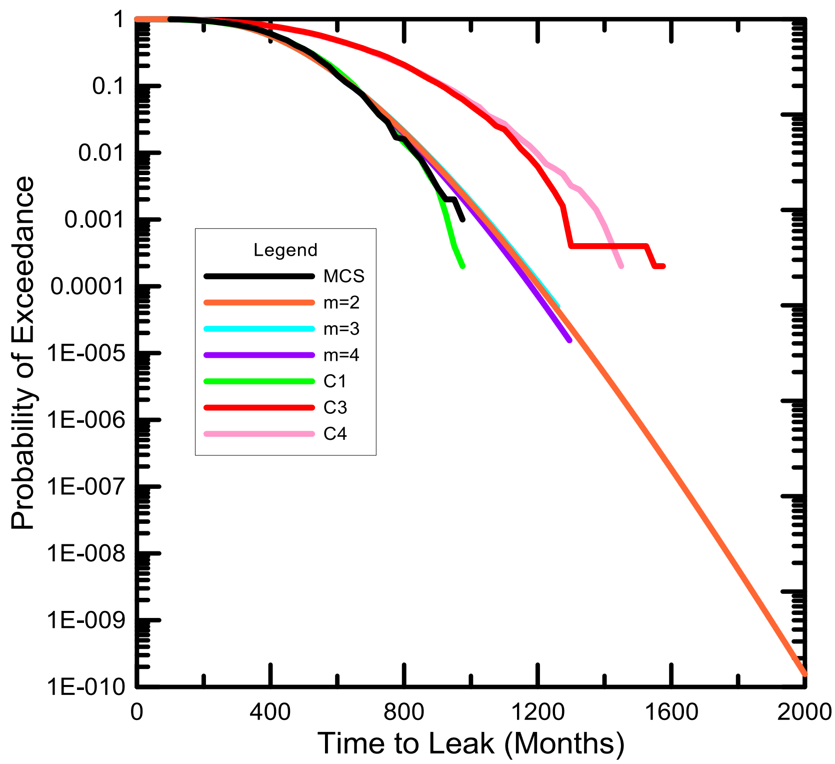

Figure 4.4. Probability of Exceedance (POE) of the response.

In this example there are only 3 random variables; with this low amount of random variables Cubature formulae provide a more accurate answer than the MDRM that is also achieved more efficiently. This is seen in the graphs above and in Table 4.6.

4.2.2 Observations

From this problem and the one solved in the previous chapter it is observed that Cubature formulae do not match up with efficiency and accuracy of MDRM as the number of random variables increase. The number of points needed for Cubature formulae increase exponentially as the number of random variables increase whereas the accuracy stays the same as compared to MDRM. Since this is the case the problems done from here on out will not be done using Cubature formulae, only MDRM and MCS.

4.3 Problem with Epistemic Variable

4.3.1 Time to Leak for a Pipe Problem with Epistemic Random Variable

To show the impact of the separation of variables, as aleatory and epistemic, the time to initiation (T_I) is subject to epistemic uncertainty while the crack growth rate (R) is subject to aleatory uncertainty. Figure 4.5 outlines the process required to solve a problem of this nature.

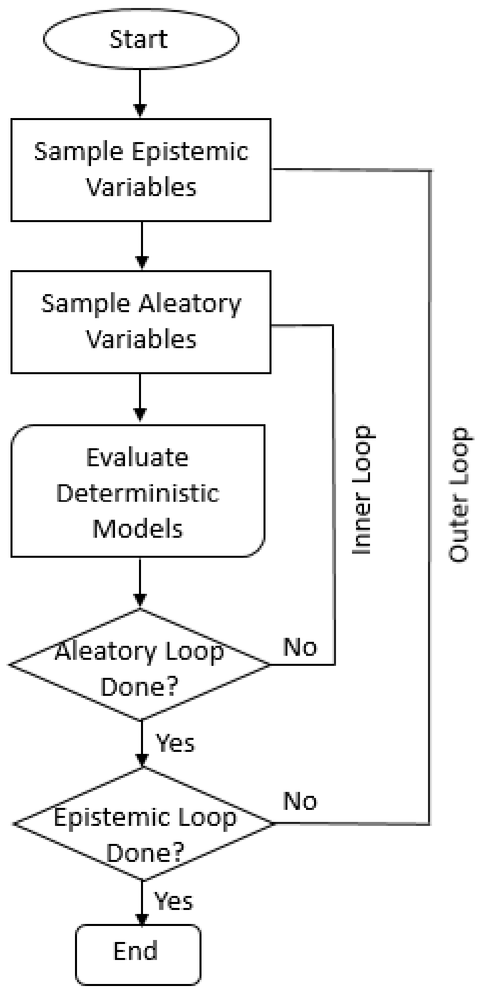

Figure 4.5. Two-staged nested Monte Carlo Simulation/ Multiplicative Dimensional Reduction Method approach involving separated aleatory and epistemic random variables

The time to initiation is sampled as part of the outer epistemic loop while the crack growth rate is part of the inner aleatory loop, this is a second order random variable probabilistic model definition. Table 4.7 defines each variable. The outcome of the second-order uncertainty is that the probability of time to leak will be random as well. The uncertainty in the probability of the time to leak will be further influenced by the degree of uncertainty of the distribution parameters (Duan, Wang & Kozluk, 2015).

This was problem was also solved using both Monte Carlo Simulation (MCS) and Multiplicative Dimensional Reduction Method (MDRM).

Table 4.7. Variables.

Variable	Type of Distribution	Parameters of Distribution
$\boldsymbol{\alpha}$	Normal	(3.5,0.5) – (mean, st.dev)
$\boldsymbol{T_I}$	Weibull	(α,480 months) – (shape, scale)
$\boldsymbol{W}$	Constant	40 mm
$\boldsymbol{R}$	Normal	(5 mm/month, 1 mm/month) – (mean, st.dev)

Using MCS, 100 simulations were taken for the outer loop and 10^4 simulations were taken for the inner loop. To solve this problem using MCS, the outer loop values are simulated first. Then by taking each of the 100 simulated values; 10^4 simulations of the time to initiation and crack growth rate are taken, and each corresponding time to leak is calculated. Figure 4.6 shows the cumulative distribution of time to leak, T_L.

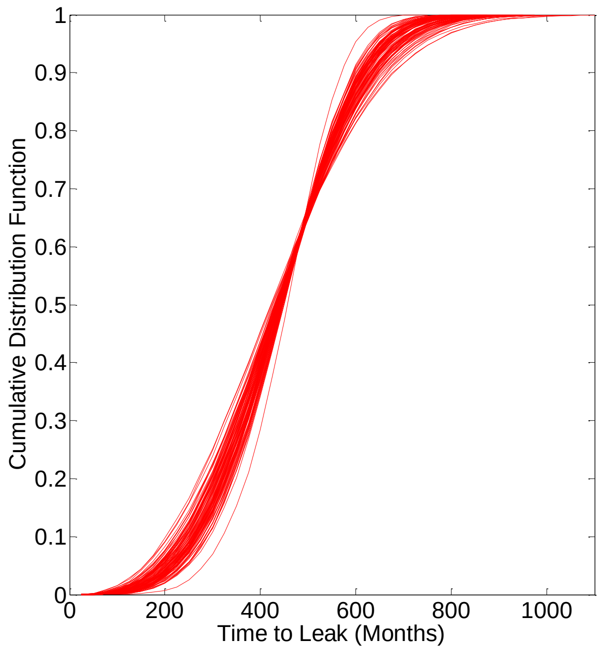

Figure 4.6. Cumulative distribution function of time to leak.

The MDRM was done using the fifteenth order (L=15) Gauss quadrature and considering two input random variables (n=2). Even though alpha is now a random variable it does not go into the input grid and therefore the number of random variables is still two. If for example there were 2 epistemic variables, then for each evaluation point (ex. $(2 \times 15) = 30$ hypothetical evaluation points, $\left((2 \times 15) + 1\right) \times 30 = 930$ hypothetical response evaluations for this case) the other random variable is held to its mean value. 15 input grids are generated to evaluate the response. The Gauss Hermite and Gauss Laguerre formulas are adopted since one random variable and alpha follows Normal distribution and the other follows Weibull distribution. In total there are $((2 \times 15) + 1) \times 15 = 465$ response evaluations. The first step is to generate the 15 alpha values. Then for each of the 15 alpha values (Table 4.8), an input grid is generated (Table 4.9). For each evaluation point, the other random variable is fixed to its mean value.

Table 4.8. Alpha values to be used.

	z_j	α
1	-6.36394788	0.318026
2	-5.19009359	0.904953
3	-4.19620771	1.401896
4	-3.28908242	1.855459
5	-2.43243682	2.283782
6	-1.60671006	2.696645
7	-0.79912906	3.100435
8	-2.32E-16	3.5
9	0.799129068	3.899565
10	1.606710069	4.303355
11	2.432436827	4.716218
12	3.289082424	5.144541
13	4.196207711	5.598104
14	5.190093591	6.095047
15	6.363947889	6.681974

Table 4.9. Input Grid for the response evaluation of alpha 4.

Random Variable	Trial	z_J	T_I (months)	R (mm/month)	W (mm)	T_L (months)
	1	0.093307812	133.6836444	5	40	141.6836
	2	0.49269174	327.7589841	5	40	335.759
	3	1.215595412	533.259154	5	40	541.2592
	4	2.269949526	746.6486117	5	40	754.6486
	5	3.667622722	966.9777806	5	40	974.9778
	6	5.425336627	1194.154981	5	40	1202.155
	7	7.565916227	1428.577772	5	40	1436.578
T_I	8	10.12022857	1671.047613	5	40	1641.721
	9	13.13028248	1922.805431	5	40	1712.383
	10	16.65440771	2185.670626	5	40	2235.435
	11	20.7764789	2462.334124	5	40	2485.713
	12	25.62389423	2756.962767	5	40	2772.542
	13	31.40751917	3076.577833	5	40	3088.366
	14	38.53068331	3434.884829	5	40	3444.407
	15	48.02608557	3867.887286	5	40	3875.887
	16	-6.363947889	426.2927152	-1.36	40	396.9661
	17	-5.190093591	426.2927152	-0.19	40	215.8701
	18	-4.196207711	426.2927152	0.80	40	476.0568
	19	-3.289082424	426.2927152	1.71	40	449.672

	20	-2.432436827	426.2927152	2.57	40	441.8717
	21	-1.606710069	426.2927152	3.39	40	438.0807
	22	-0.799129068	426.2927152	4.20	40	435.8146
R	23	-2.32E-16	426.2927152	5.00	40	434.2927
	24	0.799129068	426.2927152	5.80	40	433.1903
	25	1.606710069	426.2927152	6.61	40	432.3472
	26	2.432436827	426.2927152	7.43	40	431.6745
	27	3.289082424	426.2927152	8.29	40	431.1183
	28	4.196207711	426.2927152	9.20	40	430.6423
	29	5.190093591	426.2927152	10.19	40	430.2181
	30	6.363947889	426.2927152	11.36	40	429.8126
Fixed mean values	31	N/A	426.2927152	5.00	40	434.2927

Note: z_j denotes the Gauss Laguerre and Gauss Hermite points.

The next step is to calculate the mean (ρ_i) and the mean square (θ_i) of an i^{th} cut function is approximated as a weighted sum (Table 4.10). Then the MDRM approximation is used to calculate the statistical moment of the response function (Table 4.11).

Table 4.10. Output Grid for each cut function evaluation for alpha 4.

Random Variable	Trial	w_J	T_L	w_j x T_L	ρ_i	w_j x T_L^2	θ_i
	1	0.218234886	141.68364	30.9203		4380.90	
	2	0.342210178	335.75898	114.900		38578.7	
	3	2.63E-01	541.25915	142.366		77056.9	
	4	0.126425818	754.64861	95.4070		71998.8	
	5	4.02E-02	974.97778	39.2008		38219.9	
	6	0.008563878	1202.1549	10.2951		12376.3	
	7	1.21E-03	1436.5777	1.74175		2502.17	
T_I	8	1.12E-04	1641.7209	0.18333	435.0261	300.990	245434
	9	6.46E-06	1712.3827	0.01106		18.9421	
	10	2.23E-07	2235.4347	0.00049		1.11252	
	11	4.23E-09	2485.7133	1.05E-05		0.02612	
	12	3.92E-11	2772.5417	1.09E-07		0.00030	
	13	1.46E-13	3088.3658	4.5E-10		1.39E-06	
	14	1.48E-16	3444.4066	5.11E-13		1.76E-09	
	15	1.60E-20	3875.8872	6.2E-17		2.4E-13	
	16	8.59E-10	396.96608	3.41E-07		0.00013	

	17	5.98E-07	215.87005	0.00012		0.02784	
	18	5.64E-05	476.05681	0.02686		12.7868	
	19	1.57E-03	449.67198	7.05E-01		316.927	
	20	1.74E-02	441.87168	7.67E+0		3390.67	
	21	0.089417795	438.08068	39.1722		17160.5	
	22	0.232462294	435.81455	101.310		44152.5	
R	23	0.318259518	434.29271	138.217	434.6622	60026.9	188935
	24	0.232462294	433.19030	100.700		43622.4	
	25	0.089417795	432.347165	38.6595		16714.3	
	26	1.74E-02	431.674530	7.49636		3.24E+0	
	27	1.57E-03	431.118339	0.67571		291.313	
	28	5.64E-05	430.642334	0.02429		10.4635	
	29	5.98E-07	430.218096	0.00025		0.11059	
	30	8.59E-10	429.812618	3.69E-07		0.00015	
Fixed mean values	31	N/A	434.292715				

Note: w_j denotes the Gauss Laguerre and Gauss Hermite weights.

To calculate the total mean of the response, the following equation is used:

Mean of means:

$$\mu_{\mu_Y} = E[\mu_Y] \approx m_0^{(1-m)} \times \prod_{i=1}^{m} \left[\sum_{j=1}^{L} w_j \times (\mu_{Yj}) \right] \tag{4.2}$$

where w_j denotes the Gauss Laguerre weights corresponding to the alpha values in this case, m is the number of distribution parameters that are random variables which in this problem is equal to one, m_0 is the mean value from the inner loop when all epistemic variables are held to their mean. Since m=1 in this example there is no need to calculate this mean value.

Mean of mean squares:

$$\mu_{\mu_{2Y}} = E[\mu_{2Y}] \approx m_0^{(1-m)} \times \prod_{i=1}^{m} \left[\sum_{j=1}^{L} w_j \times (\mu_{2Yj}) \right] \tag{4.3}$$

The variance then:

$$V_{\mu_Y} = \mu_{\mu_{2Y}} - \left(\mu_{\mu_Y}\right)^2 \tag{4.4}$$

Where the standard deviation can be calculated as the square root of the variance.

Table 4.11. Statistical Moments of the response.

T_L	MDRM (465 Trials)	MCS (10^5 Simulations)	Relative Error (%)
First Moment	441.7064	440.1863	0.35
Second Moment	213850.4087	213658.702	0.09
Standard Deviation	136.9154	139.8482	2.10
COV	0.3099	0.3177	2.46

Note: Relative Error (%) $= \frac{|MCS - MDRM|}{MCS} \times 100$

Table 4.11 shows the agreeability of these two methods. The relative errors are all within 2.5% thus proving that the MDRM matches the accuracy of MCS.

The output responses obtained using MDRM are combined with the MaxEnt principle with fractional moment constraints, in order to estimate the response probability distribution. Table 4.12 provides the Lagrange multipliers (λ_i) and the fractional exponents (α_i) which are used to estimate the probability distribution of the response for alpha 4. The number of fractional moments used are m=2, m=3, and m=4. Then the probability of failure is estimated by plotting the probability of exceedance (POE). From this figure it is seen that MDRM provides highly accurate approximation for almost the entire range of the output response distribution (Figure 4.7). Where the red plots are from MCS and the blue plots are from MDRM. The MDRM encapsulates all of the MCS plots.

Table 4.12. MaxEnt parameters for time to leak for alpha 4.

Fractional Moments	Entropy	i	0	1	2	3	4
		λ_i	-1.6972	0.006356	30.47323		
m=2	6.7984	α_i		0.9937	-0.2821		

		$M_X^{\alpha_i}$		0.006619	0.126		
		λ_i	4.4183	-0.1845	33.0421	0.1544	
m=3	6.7952	α_i		0.8826	-0.5117	0.9142	
		$M_X^{\alpha_i}$		0.08386	174.6405	0.9575	
		λ_i	706.024	1398.961	-1200.386	733.0901	-397.4088
m=4	6.7911	α_i		-1.2174	0.1214	0.1541	-0.3106
		$M_X^{\alpha_i}$		2.00E+13	2.79E-11	0.6162	5674.9473

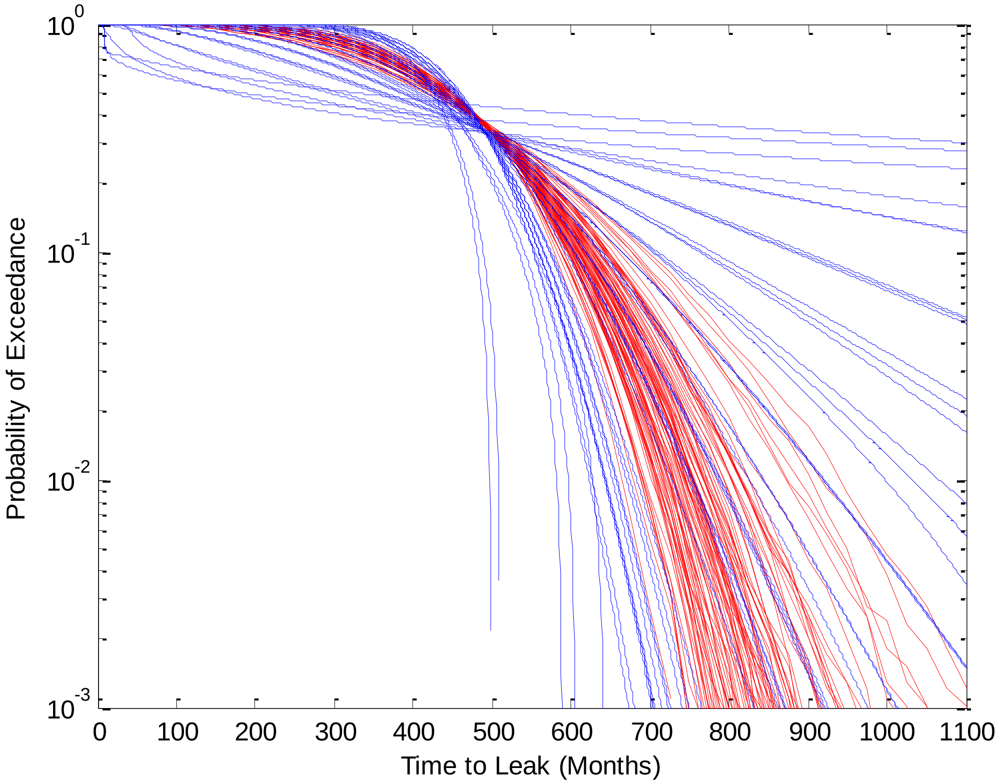

Figure 4.7. Probability of Exceedance (POE) of the response.

4.3.2 Observations

After solving this problem it was concluded that it is more efficient to just use MCS rather than MDRM when solving an equation based uncertainty analysis with an epistemic variable. The number of gauss points chosen for the epistemic variable (15 in this case) determines the number of times the MDRM has to be done. There was only one epistemic variable in this problem but with more, the number of times the MDRM would have had to be done would be multiplied by the number of epistemic variables.

4.4 Conclusion

In conclusion, after comparing all the results, it is seen that MDRM is the most logical and “best” method to use out of the three methods used in the comparison. MDRM provides the accuracy of MCS while decreasing the number of evaluation points significantly. While also being able to solve for the probability density function.

Cubature formulae are also very good but their downside is that as the number of random variables in a reliability problem increases the number of trials increase exponentially. Out of the 5 cubature formulae the ‘best’ (most efficient and accurate) is Formula 1, which rivals the MDRM when there are a low number of random variables (up to 3).

For MDRM use in equation based double loop problems (where the random variables are split between aleatory and epistemic), MDRM is not a good solution as the computational effort significantly increases based on the number of gauss quadrature points used for the aleatory random variable.

5 MDRM for Fire Resistant Design of Structures

5.1 Introduction

5.1.1 Reliability Studies for Fire Resistant Design of Structures

When preparing any sort of structural design is always essential that the designer has a sense of the level of risk associated with design they have come up with. However, current practices in the standard fire test fail to give any information about the reliability of the structure due to a fire. The only information given is the fire resistance under a standard fire. Prescriptive methodology is considered to be generally over conservative (Bailey, 2006); which results in a practice in which the structural reliability is indeterminate (Lange, Usmani and Torero, 2008) and inconsistent with the design for other hazards such as wind and earthquake (Ellingwood, 2005).

Research that led to the formulation of performance-based methods of structural fire design has provided an improved understanding of structural fire resistance (Jeffers et al, 2012). Apart from the philosophical basis for the reliability-based design methodology, the probabilistic treatment of structural performance in fire is a matter of practicality in understanding the structural responses observed in fire resistance tests. For example, in standard fire tests, a large amount of scatter can be observed in results from different testing facilities due to variations in heating conditions, material properties of the specimens, magnitudes of applied loads, and the degrees of restraint provided by the surrounding structure (Witteveen & Twilt, 1981). Furthermore, the fire resistance of steel structures is dependent on the level of fire protection that is present. The spray-applied fire resistant materials (SFRMs) have large variability's due to the nature of the materials, the manner in which they are applied in the fields and their adhesion and durability characteristics (Ryder, Wolin and Milke, 2002). There is a lot of uncertainty in construction as well. What an engineer has designed may not be what ends up being constructed due to this uncertainty. This will also affect the fire resistance of the steel.

The topic of structural reliability in fire is not new, but a review of literature reveals that the coverage of this topic is fairly incomplete and although progress has been made, previous work is limited to Monte Carlo simulations (Guo & Jeffers, 2015). The work described herein will mainly take inspiration from the work of Jeffers et al (2012). Jeffers et al (2012) utilized probabilistic methods to evaluate the fire resistance of structures given uncertainties in key model parameters. The proposed methodology accounted for the uncertainty stemming from the fire exposure and structural resistance parameters. The approach is capable of giving designers the ability to rationally evaluate the robustness provided by the various design options by providing a quantitative measure of the structure's reliability. (Jeffers et al, 2012) conducted an analysis of a protected steel beam given uncertainties in the fire load and structural resistance parameters via a sequentially coupled, stochastic finite element simulation embedded within a Monte Carlo simulation. This research demonstrates that a probabilistic treatment of the structural fire problem yields a wealth of data that may lead to a better understanding of the factors affecting structural fire resistance.

5.1.2 Objective

The work in this section includes the analysis of a protected steel beam given uncertainties in the fire load and structural resistance parameters using a performance-based design embedded within a Multiplicative Dimensional Reduction Method (MDRM) and comparing it to Monte Carlo simulation (MCS). MCS is overwhelmingly expensive for calculating failure probabilities that are relatively small (Madsen et al., 2006), and therefore researchers tend to be less inclined to using this method. This research demonstrates that the MDRM is a 'better' method (versus MCS) to use in terms of computational effort, while also demonstrating the importance of a reliability analysis; which is, that a reliability-based analysis of structural performance in fire provides data that enables risk-informed decision making, which is an essential component of performance design.

5.1.3 Organization

The organization of this chapter is as follows. Section 5.2 presents the performance based approach to solving a fire resistance problem that will be used. Section 5.3 presents the results that were obtained by using the performance based approach. This was done using MDRM and MCS and the results were compared. Conclusions are summarized in Section 5.4.

5.2 Performance Based Approach for Calculating Fire Resistance

The following steps must be taken in order to calculate the fire resistance time of a steel beam with loads acting all along the beam.

First the beam properties such as beam span (L), section modulus (Z_X), beam area (F), beam volume (V), steel yield strength (f_y), and the dead (G_k) and live loads (Q_k) acting on the beam must be known.

The calculations for the beam at room temperature must be done first to determine if the beam can support the loads that will be placed on it. Using the dead and live loads calculate the factored design load (w_c) and the bending moment (M^*_{cold}) caused by this factored design load (units in brackets):

$$w_c = 1.25G_k + 1.5Q_k \left(\frac{kN}{m}\right) \quad (5.1)$$

$$M^*_{cold} = \frac{w_c L^2}{8} \ (kNm) \quad (5.2)$$

Next calculate the bending strength (M_n) and design flexural strength (ϕM_n) of the beam (units in brackets):

$$M_n = Z_X f_y \ (kNm) \tag{5.3}$$

$$\phi M_n \tag{5.4}$$

$$\phi = 0.9$$

Now check if the beam will be able to support the loads by determining $M^*_{cold} < \phi M_n$. If this is the case then the beam is good, if not select another beam.

Now that the calculations at room temperature are done, the next step is to do the calculations for when the beam is under a fire load. First calculate the factored design load (w_f) and the bending moment (M^*_{fire}) caused by this factored design load (units in brackets):

$$w_f = G_k + 0.4Q_k \ (\frac{kN}{m}) \tag{5.5}$$

$$M^*_{fire} = \frac{w_f L^2}{8} \ (kNm) \tag{5.6}$$

Next calculate the load ratio (r_{load}) between the moment of the beam under fire and the bending strength:

$$r_{load} = \frac{M^*_{fire}}{M_n} \tag{5.7}$$

Using the load ratio calculate the limiting steel temperature ($T_{lim} -$ °C):

$$T_{lim} = 905 - 690 r_{load} \tag{5.8}$$

From here there are two different ways to continue this problem, one is if the beam is unprotected meaning no insulation on the beam to protect it from fire and the other is if the beam is protected by insulation. Calculations for both cases will be shown.

For unprotected steel beams the time to reach the limiting temperature (also can be referred to as the fire resistance) is calculated as follows:

$$t = 0.54 \frac{(T_{lim} - 50)}{(F/V)^{0.6}} \tag{5.9}$$

Where t is the time in minutes, and F/V is the area to volume ratio (m^{-1}). To determine the area to volume ratio first calculate the area of the cross section of the beam (m^2) then multiply this by 1m of length to get the volume (V). Next calculate the perimeter of the cross section and multiply it by 1m length to determine the surface area (F).

For protected steel beams, the properties of the insulation must be known. The properties needed are the thickness of the insulation($d_i - meters$), the thermal conductivity of the insulation($k_i - W/(mK)$), the moisture content of the insulation($m - \%$), and the density of the insulation($\rho_i - \frac{kg}{m^3}$).

The time to reach the limiting temperature is calculated as follows:

$$t = 40(T_{lim} - 140)\left[\frac{d_i/k_i}{F/V}\right]^{0.66} \tag{5.10}$$

Where t is time in minutes and F/V is still calculated the same as it was for unprotected steel beams. The insulation causes there to be a time delay which can be calculated as follows:

$$t_v = \frac{m\rho_i {d_i}^2}{5k_i} \tag{5.11}$$

The total time (also can be referred to as the fire resistance) is then:

$$t_{total} = t + t_v \tag{5.12}$$

All equations used in the performance based approach were provided by Dr. Venkatesh Kodur through his class notes.

5.3 Problem and Analysis

Firstly, a problem of a W360x900 beam that was unprotected was done. This example had 3 random variables which were the loads (dead and live) and the steel yield strength. These random variables did not cause the final fire resistance to vary much at all which renders the reliability analysis useless. Unless there were random variables for the dimensions of the beam itself (i.e. the thickness of the web, etc.), then there is no need to conduct a reliability analysis for an unprotected beam. The variables in Table 5.1 (other than variables for the spray applied fire resistant material) were used to solve this problem, which resulted in a mean of 69.3 and a standard deviation of 0.015 using MCS. The low standard deviation proves that there is no need to do a reliability analysis in this case since most of the values will be near the mean value of 69.3.

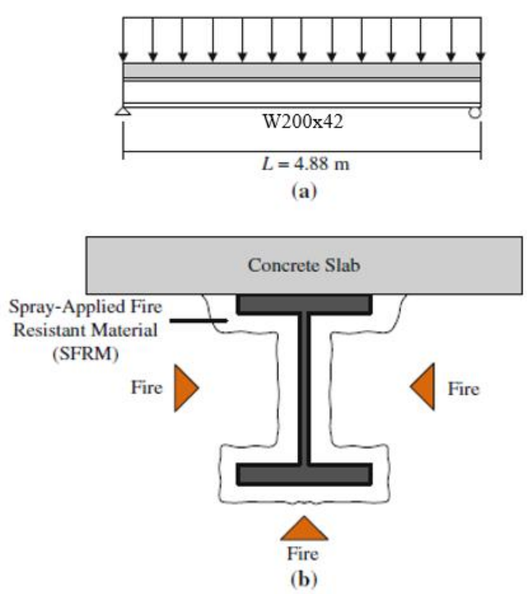

Figure 5.1. Protected steel beam exposed to fire: (a) loading, and (b) cross-section. (Jeffers et al, 2012)

Secondly, a problem of a W200x42 beam that is protected by spray applied fire resistant material (SFRM) which is seen in Figure 5.1 will be solved considering 6 random variables. This beam must achieve a 1-hr fire rating; the probability of failure will be determined. Table 5.1 defines each variable that will be used in this problem including the constants.

Table 5.1. Variables.

Variable	Type of Distribution	Parameters of Distribution
$\boldsymbol{G_k}$	Normal	(1.05 x nominal, 0.10) – (mean, COV)
	Nominal Value	5.15 kN/m
$\boldsymbol{Q_k}$	Weibull	(0.24 x nominal, 0.80) – (mean, COV)
	Nominal Value	3.65 kN/m
$\boldsymbol{f_y}$	Normal	(380 MPa, 0.08) – (mean, COV)
$\boldsymbol{d_i}$	Lognormal	(1.6 mm + nominal, 0.20) – (mean, COV)
	Nominal Value	11.1 mm
$\boldsymbol{k_i}$	Lognormal	(0.12, 0.24) – (mean, COV)
$\boldsymbol{\rho_i}$	Normal	(300, 0.29) – (mean, COV)
$\boldsymbol{L}$	Constant	4.88 m
$\boldsymbol{Z_X}$	Constant	445 x 10^3 mm^3
$\boldsymbol{m}$	Constant	15%

This problem was solved using Monte Carlo Simulation (MCS) and the Multiplicative Dimensional Reduction Method (MDRM).

Using MCS, 10^6 simulations were done for each random variable and each corresponding fire resistance was calculated. Figure 5.1 shows the cumulative distribution of fire resistance, t_{total}. The mean and standard deviation were calculated as 86.7014 and 20.6131 respectively.

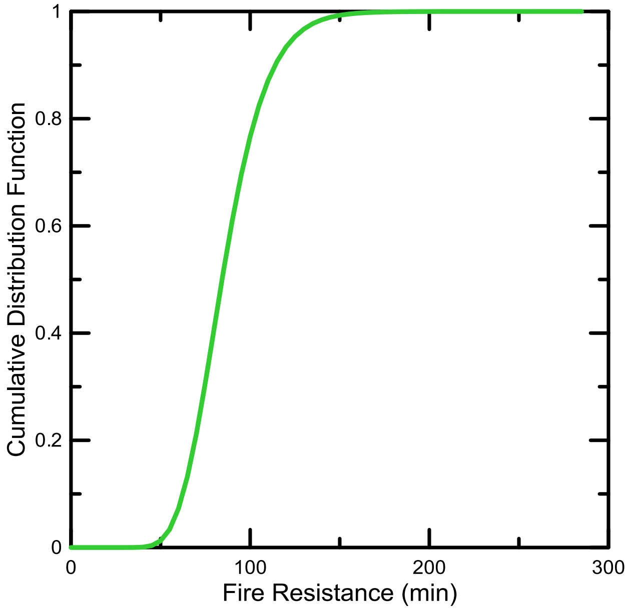

Figure 5.2. Cumulative distribution function of fire resistance.

The MDRM was done using the fifth order (L=5) Gauss quadrature and considering six input random variables (n=6). An input grid is generated to evaluate the response (Table 5.2). The Gauss Hermite and Gauss Laguerre formulas are adopted since 5 random variables follow Normal/Lognormal distribution and the other follows Weibull distribution. In total there are $(6 \times 5) + 1 = 31$ response evaluations. For each evaluation point, the other random variable is fixed to its mean value.

Table 5.2. Input Grid for the response evaluation.

Random Variable	Trial	z_j	G_k	Q_k	f_y	d_i	ρ_i	k_i	t_{total}
	1	-2.857	3.862577	0.876	380000	0.0127	300	0.12	85.95088857
	2	-1.3556	4.674459	0.876	380000	0.0127	300	0.12	84.78017668
G_k	3	0	5.4075	0.876	380000	0.0127	300	0.12	83.72315187
	4	1.3556	6.140541	0.876	380000	0.0127	300	0.12	82.66612707
	5	2.857	6.952423	0.876	380000	0.0127	300	0.12	81.49541517
	6	0.26356	5.4075	0.451933	380000	0.0127	300	0.12	83.96774938
	7	1.4134	5.4075	1.201751	380000	0.0127	300	0.12	83.53526231
Q_k	8	3.5964	5.4075	2.070179	380000	0.0127	300	0.12	83.03436236
	9	7.0858	5.4075	3.072666	380000	0.0127	300	0.12	82.45613882
	10	12.641	5.4075	4.304354	380000	0.0127	300	0.12	81.74571455
	11	-2.857	5.4075	0.876	293147.2	0.0127	300	0.12	81.26324151
	12	-1.3556	5.4075	0.876	338789.8	0.0127	300	0.12	82.7132107
f_y	13	0	5.4075	0.876	380000	0.0127	300	0.12	83.72315187
	14	1.3556	5.4075	0.876	421210.2	0.0127	300	0.12	84.53547241
	15	2.857	5.4075	0.876	466852.8	0.0127	300	0.12	85.267784
	16	-2.857	5.4075	0.876	380000	0.0070723	300	0.12	52.94749539
	17	-1.3556	5.4075	0.876	380000	0.00952125	300	0.12	66.77819314
d_i	18	0	5.4075	0.876	380000	0.01245337	300	0.12	82.44003426
	19	1.3556	5.4075	0.876	380000	0.01628847	300	0.12	101.930905
	20	2.857	5.4075	0.876	380000	0.02192874	300	0.12	129.2606009
	21	-2.857	5.4075	0.876	380000	0.0127	51.441	0.12	82.72089984
	22	-1.3556	5.4075	0.876	380000	0.0127	182.0628	0.12	83.2475996
ρ_i	23	0	5.4075	0.876	380000	0.0127	300	0.12	83.72315187
	24	1.3556	5.4075	0.876	380000	0.0127	417.9372	0.12	84.19870415
	25	2.857	5.4075	0.876	380000	0.0127	548.559	0.12	84.7254039
	26	-2.857	5.4075	0.876	380000	0.0127	300	0.059346184	144.3458713

	27	-1.3556	5.4075	0.876	380000	0.0127	300	0.084664032	109.6505375
k_i	28	0	5.4075	0.876	380000	0.0127	300	0.116686476	85.55588052
	29	1.3556	5.4075	0.876	380000	0.0127	300	0.160820756	66.76100387
	30	2.857	5.4075	0.876	380000	0.0127	300	0.229428966	50.72781116
Fixed mean values	31		5.4075	0.876	380000	0.0127	300	0.12	83.72315187

Note: z_j denotes the Gauss Laguerre and Gauss Hermite points.

The next step is to calculate the mean (ρ_i) and the mean square (θ_i) of an i^{th} cut function is approximated as a weighted sum (Table 5.3). Then the MDRM approximation is used to calculate the statistical moment of the response function (Table 5.4). This table also shows the relative errors between the statistical moments obtained by MDRM and MCS which are very low showing a good agreement between the two methods.

Table 5.3. Output Grid for each cut function evaluation.

Random Variable	Trial	w_j	t_{total}	$w_j \times t_{total}$	ρ_i	$w_j \times t_{total}^2$	θ_i
	1	1.13E-02	85.95089	0.97		83.48	
	2	0.22208	84.78018	18.83		1596.24	
G_k	3	0.53333	83.72315	44.65	83.73	3738.41	7010.81
	4	0.22208	82.66613	18.36		1517.63	
	5	1.13E-02	81.49542	0.92		75.05	
	6	0.52176	83.96775	43.81		3678.71	
	7	0.39867	83.53526	33.30		2781.98	
Q_k	8	7.59E-02	83.03436	6.30	83.72	523.31	7008.70
	9	3.61E-03	82.45614	0.30		24.54	
	10	2.34E-05	81.74571	0.00		0.16	
	11	1.13E-02	81.26324	0.92		74.62	
	12	0.22208	82.71321	18.37		1519.35	
f_y	13	0.53333	83.72315	44.65	83.68	3738.41	7001.58
	14	0.22208	84.53547	18.77		1587.04	
	15	1.13E-02	85.26778	0.96		82.16	
	16	1.13E-02	52.9475	0.60		31.68	
	17	0.22208	66.77819	14.83		990.33	
d_i	18	0.53333	82.44003	43.97	83.49	3624.70	7142.90
	19	0.22208	101.9309	22.64		2307.39	
	20	1.13E-02	129.2606	1.46		188.80	
	21	1.13E-02	82.7209	0.93		77.32	
	22	0.22208	83.2476	18.49		1539.05	
ρ_i	23	0.53333	83.72315	44.65	83.73	3738.41	7010.32
	24	0.22208	84.1987	18.70		1574.42	
	25	1.13E-02	84.7254	0.96		81.12	
	26	1.13E-02	144.3459	1.63		235.44	
	27	0.22208	109.6505	24.35		2670.12	
k_i	28	0.53333	85.55588	45.63	87.01	3903.87	7828.33

	29	0.22208	66.761	14.83	989.82
	30	1.13E-02	50.72781	0.57	29.08
Fixed mean value	31		83.72315		

Note: w_j denotes the Gauss Laguerre and Gauss Hermite weights.

Table 5.4. Statistical Moments of the response.

t_{total}	**MDRM (31 Trials)**	**MCS (10^6 Simulations)**	**Relative Error (%)**
First Moment	86.7324	86.7014	0.0358
Second Moment	7969.4401	7942.0331	0.345
Standard Deviation	21.1404	20.6131	2.558
COV	0.2437	0.2377	2.524

Note: Relative Error (%) $= \frac{|MCS-MDRM|}{MCS} \times 100$

The output responses obtained using MDRM are combined with the MaxEnt principle with fractional moment constraints, in order to estimate the response probability distribution. Table 5.5 provides the Lagrange multipliers (λ_i) and the fractional exponents (α_i) which are used to estimate the probability distribution of the response. The number of fractional moments used are m=2, m=3, and m=4. The estimated probability distribution of the fire resistance is compared to the MCS (Figure 5.2). The probability distribution functions (PDF) match up accurately.

Then the probability of failure is estimated by plotting the probability of exceedance (POE). When two fractional moments (m=2) are used, the POE does not match up that well but as the number of fractional moments increase the POE converges and shows that MDRM provides very accurate approximation for almost the entire range of the output response distribution, which can be seen in Figure 5.3 below.

Table 5.5. MaxEnt parameters for failure pressure.

Fractional Moments	Entropy	i	0	1	2	3	4
		λ_i	705.9991	2328.6595	-2761.634		
m=2	4.1396	α_i		-0.2072	-0.1187		
		$M_X^{\alpha_i}$		1.8676	0.7455		
		λ_i	705.9865	-1168.1429	62.4452	585.2371	
m=3	4.1391	α_i		-0.01267	0.2164	-0.2017	
		$M_X^{\alpha_i}$		2.8492	1.6988	0.3411	
		λ_i	705.9864	712.1589	-3108.974	1586.1246	280.1337
m=4	4.1391	α_i		0.08442	-0.00833	-0.08365	-0.1205
		$M_X^{\alpha_i}$		0.7868	1.1397	1.6746	1.2813

The probabilities of failure, meaning the probability that this beam will last less than 60 minutes during a fire, from both MCS and MDRM can be seen in Table 5.6 below.

Table 5.6. Probability of failure; probability that the steel beam will fail before 60 minutes.

t_{total}	MDRM (%)	MCS (%)	Relative Error (%)
Probability of Failure	7.9654	7.2774	9.4541

Note: Relative Error (%) $= \frac{|MCS-MDRM|}{MCS} \times 100$

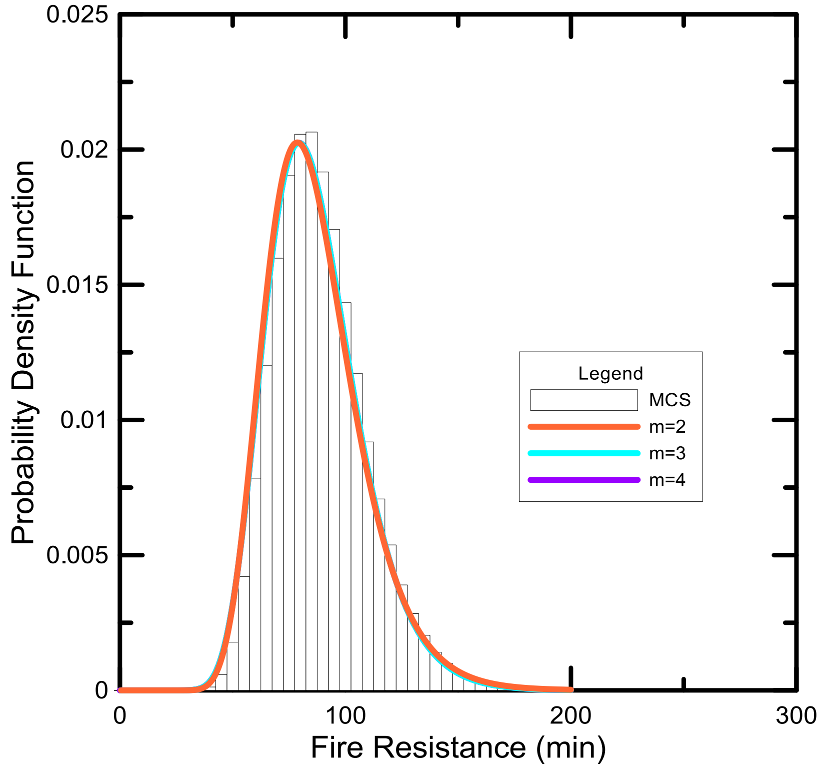

Figure 5.3. Probability Distribution of the response.

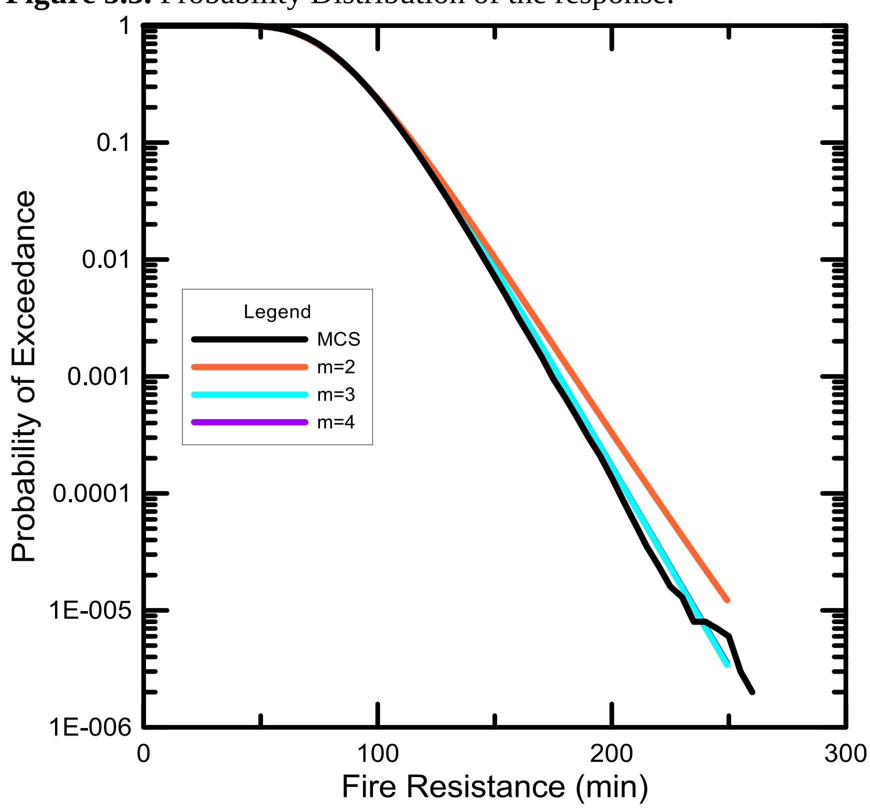

Figure 5.4. Probability of Exceedance (POE) of the response.

5.4 Conclusion

In this example MDRM only needs 31 trials compared to the 1,000,000 trials used for MCS. Even though only 31 trials were needed the accuracy did not suffer. For MDRM the mean and standard deviation are 86.7324 and 21.1404 respectively, and for MCS the mean and standard deviation are 86.7014 and 20.6131. Which give a relative error of 0.0358 and 2.558 for mean and standard deviation respectively which are very low numbers that show agreeability between the two methods.

The probability distribution function (PDF) graphs and the probability of exceedance (POE) graphs (Figure 5.2 and 5.3) also show a good agreement between the two methods. The PDF graphs match up very accurately The POE graphs provide a very accurate approximation for almost the entire range of the output response distribution as the number of fractional moments increase. When only two fractional moments are used there is not a good agreement between MCS and MDRM but as the number of fractional moments increase the graphs converge and show good agreement between MCS and MDRM.

The suitability of MDRM was determined by considering an application of a protected steel beam under natural fire. The 1-h rated beam was found to have a probability of failure of 7.9654% under natural fire exposure which indicates that the beam is likely to survive a fire for 1-h. However, discussion is needed amongst the fire safety engineering community to determine what an acceptable percentage for the probability of failure is so engineers can do reliability studies based on that number.

In summary, this study shows the importance of applying a reliability analysis of structural fire resistance which provides an enhanced understanding of the factors affecting the resistance of structures to fire and offers a means for improving the design to meet performance objectives. Which in this case is done by using the Multiplicative Dimensional Reduction Method.

6 MDRM for Geomechanics Problems

6.1 Introduction

6.1.1 Reliability Analysis for Geomechanics

Engineering design with a consideration of soil is a difficult task because there are many important sources of uncertainty that could lead to over- or under-designing. Measurements of soil properties along with detailed observation suggest considerable variability in soil properties, not only from site to site but even within samples at a single site (Christian & Baecher, 2003).

There are multiple different ways to apply a reliability analysis such as First-Order Reliability Method (FORM), Second Order Reliability Method (SORM), First Order Second Moment (FOSM), or Monte Carlo Simulation (MCS). MCS is perhaps the most well-known and easiest to implement. When the complete output distribution is of interest, MCS is the generic approach to solve the problem (Zhang, 2013).

However as discussed previously the fact that most reliability analysis methods are computationally expensive is the reason why there are not a lot of risk and reliability analyses done in the field of geomechanics. Thus, the main motivation behind this research is to use the Multiplicative Dimensional Reduction Method which is a computationally efficient, robust, and easy to implement method that can be compared to the accuracy of MCS. Using this method for geomechanics problems will provide evidence to support the claims of accuracy and computational efficiency and hopefully make it so more engineers do a reliability analysis instead of a deterministic one.

6.1.2 Objective

The work in this section includes 4 problems. These problems are based on solving equations and finite element analyses. The first two problems are equation based which require substituting random variables into the equation and the second two problems are finite element analyses which require substituting random variables into finite element models. These problems show the accuracy and computational efficiency of MDRM compared to MCS. MCS is overwhelmingly expensive for calculating failure probabilities that are relatively small (Madsen et al., 2006), and therefore researchers tend to be less inclined to using this method. This research demonstrates that the MDRM is a 'better' method (versus MCS) to use in terms of computational effort. It also demonstrates the importance of a reliability analysis. The importance is that a reliability-based analysis of soil integrity provides data that enables risk-informed decision making.

6.1.3 Organization

The organization of this chapter is as follows. Section 6.2 presents a step by step detailed calculation of MDRM for a 1D consolidation problem (settlement of foundation). This was compared to MCS. Section 6.3 presents a simple 1D consolidation problem (determining degree of consolidation) was is done using MDRM and MCS, and the results are compared. These first two sections deal with equation based problems. Section 6.4 presents a vertical drain problem. This problem was done using finite element analysis. The area of influence is modeled in ABAQUS and meshed. The mesh is then output into a FORTRAN program provided by Dipanjan Basu. This is done using MDRM and MCS and the results and computational effort are compared. Section 6.5 presents a concrete infinite beam on an elastic foundation problem. This problem is also done using a finite element analysis. The foundation is modeled using two different models, the two parameter Pasternak Model and Modified Vlasov Model. Each foundation model is done twice for a static and dynamic load. This is done using MDRM and MCS and the results and computational effort are compared. Finally, Section 6.6 discusses the conclusions that were made.

6.2 Detailed Calculation Steps of MDRM

To show a step by step detailed calculation using MDRM a simple 1D consolidation problem will be solved in which the settlement of the foundation will be determined. This problem will also compare the accuracy of MDRM to MCS. Computational efficiency will not be looked at in this example since only equations are used to solve the problem and not a finite element analysis. The problem is to determine the settlement of a flexible, circular foundation of diameter 2.0 m. The axial load is 1200 kN. The soil properties are taken as random variables which are defined in Table 6.1 below. The equations that will be used to solve this problem are:

$$q_b = \frac{Q}{\pi\left(\frac{B}{2}\right)^2} \tag{6.1}$$

$$q_b = \frac{1200}{\pi\left(\frac{2}{2}\right)^2}$$

$$q_b = 382\ kPa$$

and

$$w = \frac{q_b B(1-\mu^2)}{E} \tag{6.2}$$

Where Q is the axial load, B is the diameter of the foundation, q_b is a distributed load, μ is the Poisson's ratio and E is the elastic modulus of soil.

Table 6.1. Variables

Variable	Type of Distribution	Parameters of Distribution
$\boldsymbol{E}$	Lognormal	(10500 kPa, 30%) – (mean, coefficient of variance)
$\boldsymbol{\mu}$	Lognormal	(0.45, 7%) – (mean, coefficient of variance)

This problem was solved using Monte Carlo Simulation (MCS), and the Multiplicative Dimensional Reduction Method (MDRM).

Using MCS, 10000 simulations were done for each random variable and each settlement of the foundation was determined. The 10000 MCS values determined for E and μ were substituted into the settlement equation to solve for the statistical moments and the probability density function. Figure 6.1 shows the cumulative distribution of the settlement of the foundation. The mean and standard deviation were calculated as 0.06306 and 0.01905 respectively.

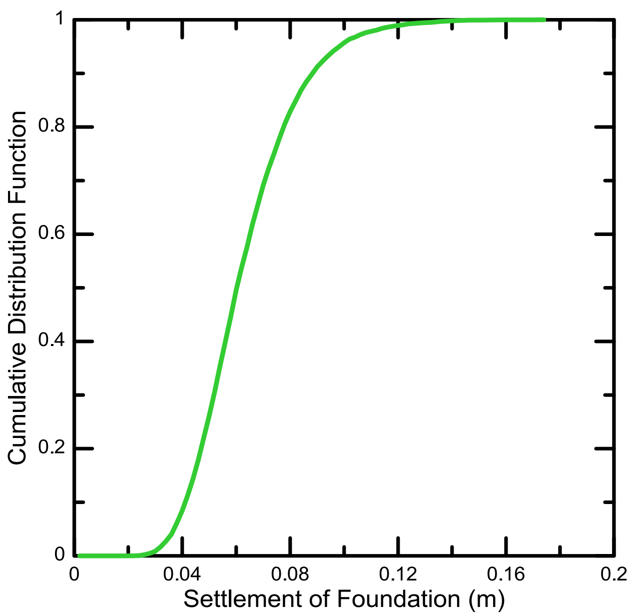

Figure 6.1. Cumulative distribution function of settlement of the foundation obtained by simulations

To use MDRM, the input grid must first be assembled. The first step to assembling the input grid is determining the gauss quadrature that will be used for each random variable according to the type of distribution. In this case both random variables are lognormal therefore the Probabilists' Gauss-Hermite integration formula, weights, and points will be used. Since random variables are of lognormal distribution the shape and scale parameters must be solved for using the following equations:

$$\zeta = \sqrt{\ln(1+\frac{\sigma^2}{\mu^2})} \text{ (shape parameter) and } \lambda = \ln(\mu) - \frac{1}{2}\zeta^2 \text{(scale parameter)}$$

In this case the shape and scale factors are $\zeta = 0.06991$ and $\lambda = -0.8009$ for poisons ratio and $\zeta = 0.29356$ and $\lambda = 9.216042$ for the elastic modulus of soil. For this example the MDRM was done using the fifth order (L=5) Gauss quadrature, therefore 5 values of x_j were found for each random variable using the following equation from Table 2.1:

$$x_j = \exp(\lambda + \zeta z_j)$$

Trial 1 for elastic modulus of soil for example will be done as follows:

$$x_1 = \exp(\lambda + \zeta z_1)$$

$$x_1 = \exp(9.216042 + 0.29356\,(-2.85697))$$

$$x_1 = 4347.4279$$

Trial 4 for poisons ratio for example will be done as follows:

$$x_4 = \exp(\lambda + \zeta z_4)$$

$$x_4 = \exp(-0.8009 + 0.06991\,(1.35563))$$

$$x_4 = 0.4935$$

This step will be done a total of 5 times for each random variable which will result in 10 trials plus one more trial at which the random variables will both be kept at their mean. In total, there are $(2 \times 5) + 1 = 11$ response evaluations. For each evaluation point, the other random variable is fixed to its mean value. The corresponding values of each trial will then be substituted back into the equation that is being solved to determine the response values of each trial. An example of this step is shown below and the input grid is shown in Table 6.2:

$$w = \frac{q_b B(1 - {\mu_1}^2)}{E_1}$$

$$w = \frac{383 \times 2(1 - 0.45^2)}{4347.4279}$$

$$w = 0.1401 \text{ m}$$

Table 6.2. Input Grid for the response evaluation.

Random Variable	Trial	$\boldsymbol{E}$	$\boldsymbol{\mu}$	Settlement of Foundation (m)
	1	4347.427999	0.45	0.140149532
	2	6755.357941	0.45	0.090193592
$\boldsymbol{E}$	3	10057.17599	0.45	0.060582613
	4	14972.82452	0.45	0.040693057
	5	23265.8917	0.45	0.026188122
	6	10500	0.36762298	0.062928377
	7	10500	0.40831033	0.060631235
$\boldsymbol{\mu}$	8	10500	0.44890154	0.058099465
	9	10500	0.49352802	0.055039295
	10	10500	0.54815014	0.050899239
Fixed mean value	11	10500	0.45	0.058027619

Note: z_j denotes the Gauss Hermite points.

The next step is to calculate the mean (ρ_i) and the mean square (θ_i) of an i^{th} cut function which is approximated as a weighted sum. To do this the response values must first be multiplied by the corresponding Gaussian weights and the response values squared must also be multiplied by the Gauussian weights which are shown in Table 6.3. Using these values the first moment and second moments for each variable can be calculated. The first moment of the modulus of elasticity of soil can be calculated as follows using equation 2.12:

$$\rho_E = \sum_{j=1}^{5} w_j \, h(E_j, \mu_0)$$

$$\rho_E = 0.00157766 + 0.02003019 + 0.03231052 + 0.00903711 + 0.0002948$$

$$\rho_E = 0.063250295$$

Similarly, the second moment of the poisons ratio can be calculated as follows using equation 2.13:

$$\theta_\mu = \sum_{j=1}^{5} w_j \, [h(E_0, \mu_j)]^2$$

$$\theta_\mu = (4.45775\text{E} - 05) + 0.000816399 + 0.001800281 + 0.000672752 + (2.91639\text{E} - 05)$$

$$\theta_\mu = 0.003363173$$

These values as well as all the other moments can be found in Table 6.3.

Table 6.3. Output Grid for each cut function evaluation.

Random Variable	Trial	w_j	Settlement (w)	$w_j \times w$	ρ_i	$w_j \times w^2$	θ_i
	1	1.13E-02	0.140149532	0.00157766		0.000221109	
	2	0.22208	0.090193592	0.02003019		0.001806595	
E	3	0.53333	0.060582613	0.03231052	0.063250295	0.001957456	0.004360628
	4	0.22208	0.040693057	0.00903711		0.000367748	
	5	1.13E-02	0.026188122	0.0002948		7.72025E-06	
	6	1.13E-02	0.062928377	0.00070838		4.45775E-05	
	7	0.22208	0.060631235	0.01346498		0.000816399	
μ	8	0.53333	0.058099465	0.03098619	0.057955656	0.001800281	0.003363173
	9	0.22208	0.055039295	0.01222313		0.000672752	
	10	1.13E-02	0.050899239	0.00057297		2.91639E-05	
Fixed Mean Value	11		0.058027619				

Note: w_j denotes the Gauss Hermite weights.

Then the MDRM approximation is used to calculate the statistical moments of the response function. The mean of the response is calculated as follows using equation 2.14:

$$\mu_Y = E[Y] \approx \; h_0^{(1-n)} \times \prod_{i=1}^{n} \rho_i$$

$$\mu_Y = 0.058027619^{(1-2)} \times 0.063250295 \times 0.057955656$$

$$\mu_Y = 0.063171855$$

The mean square of the response is calculated as follows using equation 2.15:

$$\mu_{2Y} = E[Y^2] \approx \; h_0^{(2-2n)} \times \prod_{i=1}^{n} \theta_i$$

$$\mu_{2Y} = \; 0.058027619^{(2-2(2))} \times 0.004360628 \times 0.003363173$$

$$\mu_{2Y} = 0.004355407$$

Using the mean and mean square the variance can then be calculated using equation 2.16:

$$V_Y = \mu_{2Y} - (\mu_Y)^2$$

$$V_Y = 0.004355407 - (0.063171855)^2$$

$$V_Y = 0.000364724$$

The standard deviation can then be calculated by taking the square root of the variance. Table 6.4 shows all the statistical moments of the response from both MDRM and MCS. This table also shows the relative errors between the two methods which are very low showing a good agreement between the two methods.

Table 6.4. Statistical Moments of the response.

Settlement of Foundation (m)	MDRM (11 Trials)	MCS (10000 Simulations)	Relative Error (%)
First Moment	0.063171855	0.063069361	0.16251105
Second Moment	0.004355407	0.004340762	0.337377552
Standard Deviation	0.019097745	0.019053035	0.234658499
COV	0.302314138	0.302096537	0.072030391

Note: Relative Error (%) $= \frac{|MCS - MDRM|}{MCS} \times 100$

The output responses obtained using MDRM are combined with the MaxEnt principle with fractional moment constraints, in order to estimate the response probability distribution. The MaxEnt code must be used here to solve for the Lagrange multipliers (λ_i) and the fractional exponents (α_i). The program should be ran about 100 times or more, in this example the program was ran 100 times. Each time the fractional exponent must be randomized to find the minimum function evaluation. The fractional exponents and Lagrange multipliers for the lowest function evaluation must then be saved. Using these fractional exponents and Lagrange multipliers the fractional moments, λ_0, and estimated PDF can be calculated. The fractional moment for m=2 and i=1 can be calculated as follows using equation 2.20:

$$M_Y^{\alpha_1} = E[Y^{\alpha_1}] \approx h_0^{\alpha_1(1-n)} \prod_{i=1}^{n} \sum_{j=1}^{L} w_j \left[h_i(x_j)\right]^{\alpha_1}$$

$$M_Y^{\alpha_1} = 0.058027619^{-1.3151(1-2)}$$

$$\times [1.13\text{E} - 02(0.140149532)^{-1.3151} + 0.22208(0.090193592)^{-1.3151} + 0.53333(0.060582613)^{-1.3151} + 0.22208(0.040693057)^{-1.3151} + 1.13\text{E} - 02(0.026188122)^{-1.3151}]$$

$$\times [1.13\text{E} - 02(0.062928377)^{-1.3151} + 0.22208(0.060631235)^{-1.3151} + 0.53333(0.058099465)^{-1.3151} + 0.22208(0.055039295)^{-1.3151} + 1.13\text{E} - 02(0.050899239)^{-1.3151}]$$

$$M_Y^{\alpha_1} = 0.0002096$$

λ_0 can then be solved substituting the fractional exponents and Lagrange multipliers and then performing the following integral using equation 2.22:

$$\lambda_0 = \ln\left[\int_0^\infty \exp(-\sum_{i=1}^{m} \lambda_i\, y^{\alpha_i}) dy\right]$$

Using the fractional exponents, Lagrange multipliers and λ_0 the estimated PDF can be obtained by expanding the following equation and substituting a range of values, *y*, so a graph can be plotted using equation 2.23:

$$\hat{f}_Y(y) = \exp\left(-\sum_{i=0}^{m} \lambda_i\, y^{\alpha_i}\right)$$

Table 6.5 provides the Lagrange multipliers (λ_i) and the fractional exponents (α_i) which are used to estimate the probability distribution of the response. The number of fractional moments used are m=2, m=3, and m=4.

Table 6.5. MaxEnt parameters for failure pressure.

Fractional Moments	Entropy	i	0	1	2	3	4
		λ_i	-9.5702	6.98E-02	125.1533		
m=2	-2.6043	α_i		-1.3151	1.2561		
		$M_X^{\alpha_i}$		2.10E-04	3.74E-04		
		λ_i	81.23629	279.8922	-289.151	1.07E-04	
m=3	-2.6051	α_i		0.4681	0.2113	-2.6841	
		$M_X^{\alpha_i}$		1.43E-04	1.20E+03	4.38E-02	
		λ_i	-13.1159	2723.707	6.86E-01	-1593.96	229.9049
m=4	-2.6032	α_i		5.3209	-0.7794	3.5828	1.4461
		$M_X^{\alpha_i}$		2.18E+02	1.34E-04	3.80E-01	1.41E-05

The estimated probability distribution of the settlement of the foundation is compared to the MCS (Figure 6.2). The probability distribution functions (PDF) match up spot on and there are no differences in the PDF plots between the two methods.

Then the probability of failure is estimated by plotting the probability of exceedance (POE) (Figure 6.3). The POE matches up very well. The MDRM plots converge to the MCS plot, this can be seen as the fractional moment increase it gets closer to the MCS plot.

In conclusion it can be seen that MDRM provides the same accuracy of MCS in a lesser amount of trials/simulations.

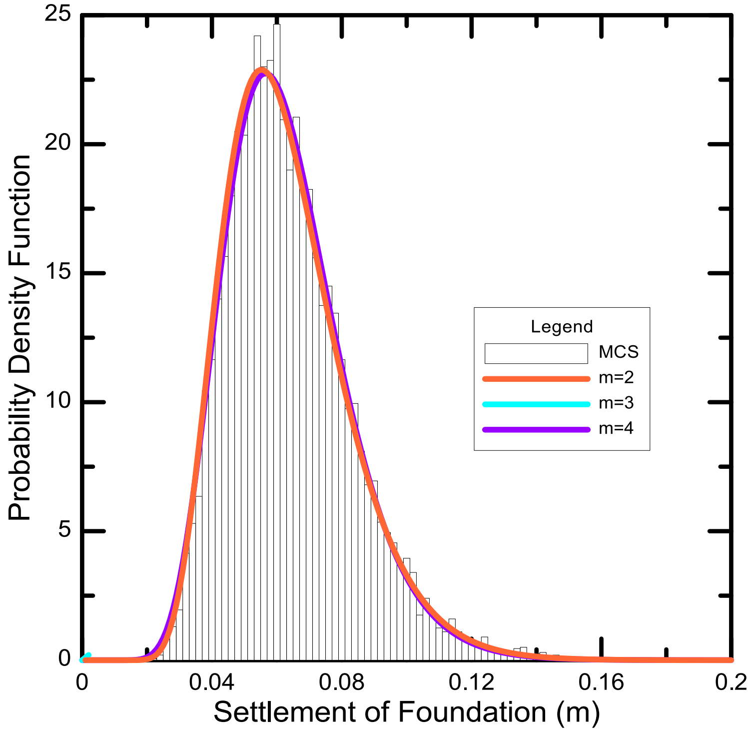

Figure 6.2. Probability Density Function of the response.

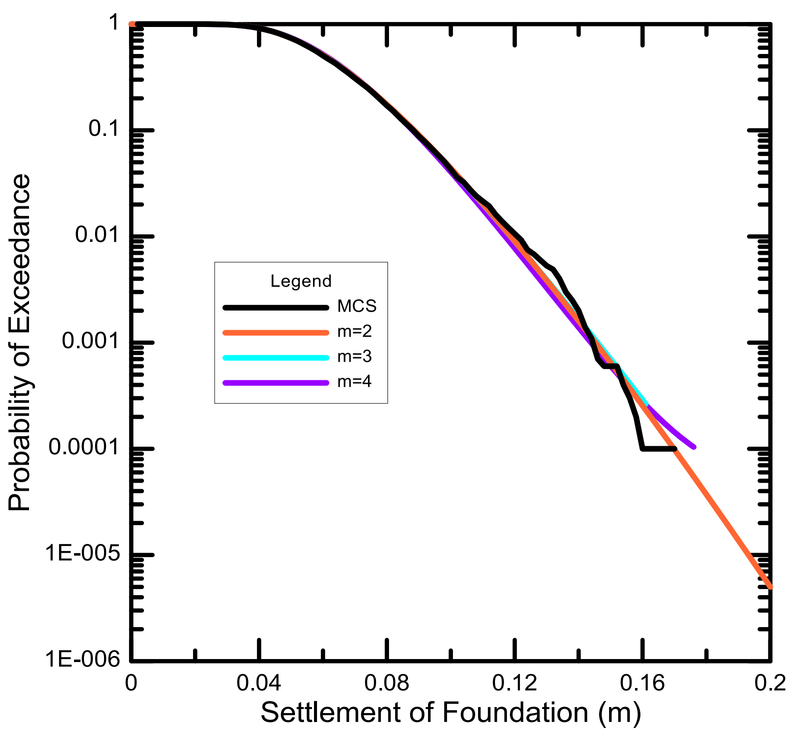

Figure 6.3. Probability of Exceedance (POE) of the response.

6.3 Simple 1D Consolidation Problem

Another equation based example is done to provide further evidence that the Multiplicative Dimensional Reduction Method provides similar accuracy to that of Monte Carlo Simulation. This example is a simple 1D consolidation problem will be solved in which the consolidation of soil after 2 years will be determined. Computational efficiency will not be looked at in this example since only equations are used to solve the problem and not a finite element analysis. The problem will be solved based on the following Fourier series solution:

$$u(z,t) = \sum_{n=1}^{\infty} \frac{2u_i}{n\pi}(1-\cos n\pi)\sin\left(\frac{n\pi z}{2H_{dr}}\right)\exp\left(\frac{-n^2\pi^2 c_v t}{4{H_{dr}}^2}\right) \tag{6.3}$$

From this Fourier series solution, the following equation for time factor can be derived, which will be used in calculations:

$$T = \frac{c_v t}{{H_{dr}}^2} \tag{6.4}$$

The equation used for degree of consolidation can be derived from the following equation:

$$U(T) = 1 - \frac{\int_0^{H_{dr}} u(z,t)\,dz}{H_{dr}u_i} \tag{6.5}$$

The equation for degree of consolidation is:

$$U = \sqrt{\frac{4T}{\pi}} \tag{6.6}$$

Where T is the time factor, c_v is the coefficient of consolidation in m^2/years, t is the time in years, H_{dr} is the maximum vertical distance the water has to travel in m, and U is the degree of consolidation.

For this problem two random variables were taken which are c_v and the height of the soil layer, H. These random variables along with the constants are defined in Table 6.6.

Table 6.6. Variables

Variable	Type of Distribution	Parameters of Distribution
$\boldsymbol{H}$	Normal	(3, 0.15306) – (mean, standard deviation)
$\boldsymbol{c_v}$	Lognormal	(0.2, 0.1) – (mean, standard deviation)
$\boldsymbol{t}$	Constant	2 years

This problem was solved using Monte Carlo Simulation (MCS), and the Multiplicative Dimensional Reduction Method (MDRM).

Using MCS, 10000 simulations were done for each random variable and each degree of consolidation after 2 years was determined. The 10000 MCS values determined for H and c_v, substituted into the time factor and degree of consolidation equations to determine the moments, and the probability density function. Figure 6.4 shows the cumulative distribution of the degree of consolidation after 2 years. The mean and standard deviation were calculated as 0.4626 and 0.1143 respectively.

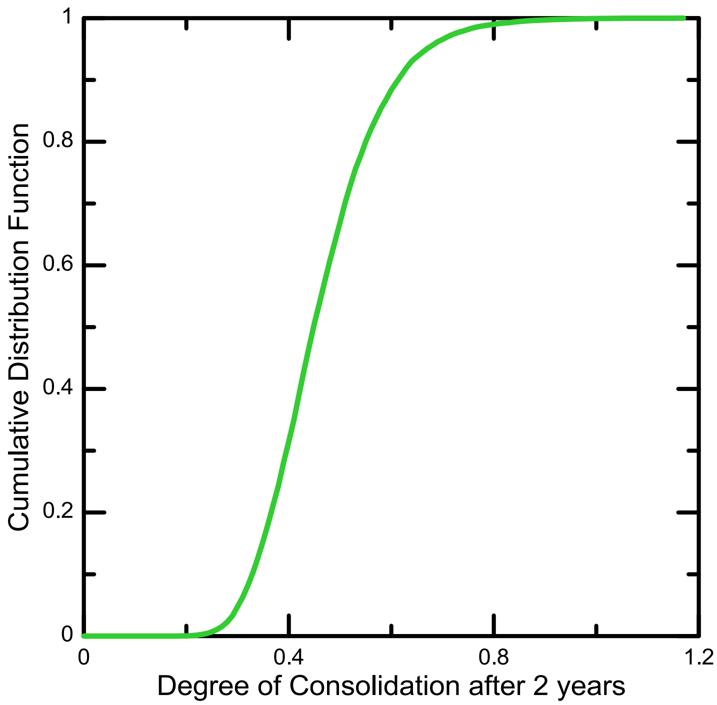

Figure 6.4. Cumulative distribution function of degree of consolidation after 2 years.

The MDRM was done using the fifth order (L=5) Gauss quadrature and considering two input random variables (n=2). An input grid is generated to evaluate the response (Table 4). The Gauss Hermite formulas are adopted since the random variables follow normal and lognormal distribution. In total there are $(2 \times 5) + 1 = 11$ response evaluations. For each evaluation point, the other random variable is fixed to its mean value. The values for H and c_v shown below in Table 6.7 are then taken and substituted into the time factor and degree of consolidation equations to determine the results.

Table 6.7. Input Grid for the response evaluation.

Random Variable	Trial	c_v	H	Degree of Consolidation after 2 yrs
	1	0.046393244	3	0.22914278
	2	0.09429105	3	0.3266736
c_v	3	0.178885438	3	0.44995211
	4	0.339374733	3	0.61975288

	5	0.689755597	3	0.8835404
	6	0.2	2.56269605	0.55695224
	7	0.2	2.79250639	0.51111765
H	8	0.2	3	0.47576643
	9	0.2	3.20749361	0.44498897
	10	0.2	3.43730395	0.41523802
Fixed mean value	11	0.2	3	0.47576643

Note: z_j denotes the Gauss Hermite points.

The next step is to calculate the mean (ρ_i) and the mean square (θ_i) of an i^{th} cut function is approximated as a weighted sum (Table 6.8). Then the MDRM approximation is used to calculate the statistical moment of the response function (Table 6.9). This table also shows the relative errors between the statistical moments obtained by MDRM and MCS which are very low showing a good agreement between the two methods.

Table 6.8. Output Grid for each cut function evaluation.

Random Variable	**Trial**	w_j	**U after 2 years**	$w_j \times U$	ρ_i	$w_j \times U^2$	θ_i
	1	1.13E-02	0.22914278	2.58E-03		5.91E-04	
	2	0.22208	0.3266736	7.25E-02		2.37E-02	
c_v	3	0.53333	0.44995211	2.40E-01	0.462680824	1.08E-01	0.226354031
	4	0.22208	0.61975288	1.38E-01		8.53E-02	
	5	1.13E-02	0.8835404	9.95E-03		8.79E-03	
	6	1.13E-02	0.55695224	6.27E-03		3.49E-03	
	7	0.22208	0.51111765	1.14E-01		5.80E-02	
H	8	0.53333	0.47576643	2.54E-01	0.477016615	1.21E-01	0.228145722
	9	0.22208	0.44498897	9.88E-02		4.40E-02	
	10	1.13E-02	0.41523802	4.67E-03		1.94E-03	
Fixed Mean Value	11		0.47576643				

Note: w_j denotes the Gauss Hermite weights.

Table 6.9. Statistical Moments of the response.

Degree of Consolidation after 2 yrs	MDRM (11 Trials)	MCS (10000 Simulations)	Relative Error (%)
First Moment	0.463897	0.462612	0.277595071
Second Moment	0.228146	0.227083	0.46794171
Standard Deviation	0.11378	0.114338	0.487661837
COV	0.245271	0.247157	0.763138473

Note: Relative Error (%) $= \frac{|MCS-MDRM|}{MCS} \times 100$

The output responses obtained using MDRM are combined with the MaxEnt principle with fractional moment constraints, in order to estimate the response probability distribution. Table 6.10 provides the Lagrange multipliers (λ_i) and the fractional exponents (α_i) which are used to estimate the probability distribution of the response. The number of fractional moments used are m=2, m=3, and m=4.

Table 6.10. MaxEnt parameters for failure pressure.

Fractional Moments	Entropy	i	0	1	2	3	4
		λ_i	-11.2039	1.5641	14.8481		
m=2	-0.7982	α_i		-1.3151	1.2561		
		$M_X^{\alpha_i}$		0.106699958	0.122852389		
		λ_i	128.3307	165.6812	-288.819	0.03882	
m=3	-0.7993	α_i		0.4681	0.2113	-2.6841	
		$M_X^{\alpha_i}$		0.09723	8.2904	0.4202	
		λ_i	-7.5808	1.6903	12.3769	-7.9791	6.526
m=4	-0.7984	α_i		-3.9118	1.3176	-3.735	-3.6569
		$M_X^{\alpha_i}$		0.636994	4.899731	19.81892	0.393433

The estimated probability distribution of the degree of consolidation after 2 years is compared to the MCS (Figure 6.5). The probability distribution functions (PDF) match up spot on and there are no differences in the PDF plots between the two methods.

Then the probability of failure is estimated by plotting the probability of exceedance (POE) (Figure 6.6). The POE matches up very well. The MDRM plots converge to the MCS plot, this can be seen as the fractional moment increase it gets closer to the MCS plot.

In conclusion it can be seen that MDRM provides the same accuracy of MCS in a lesser amount of trials/simulations.

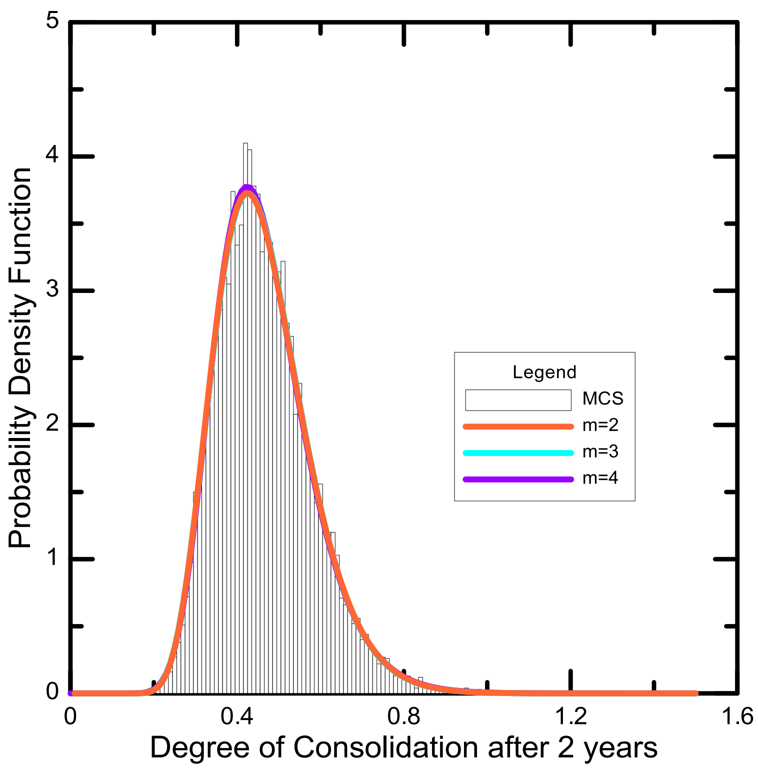

Figure 6.5. Probability Density Function of the response.

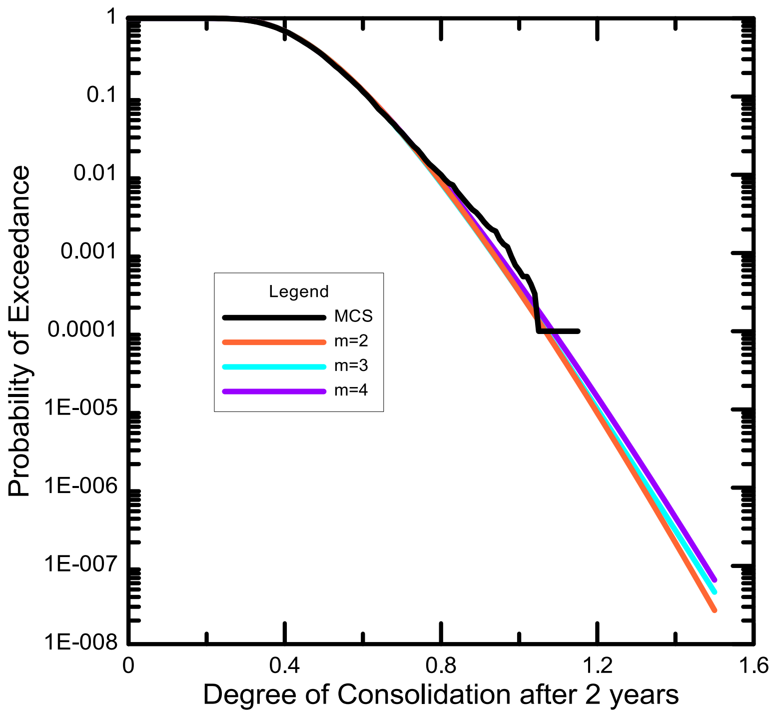

Figure 6.6. Probability of Exceedance (POE) of the response.

6.4 Vertical Drains

Techniques such as preloading are needed to increase the strength and stiffness of soils due the fact that thick deposits of soft, saturated clay have low shear strength, high compressibility and low hydraulic conductivity (Prezzi & Basu, 2007). Installation of vertical drains are combined with preloading to speed up the consolidation process and hence increase the strength gain rate (Holtz, 1987; Balasubramaniam, Alfaro and Bergado, 1993). The prefabricated vertical drains (PVDs) are installed at regular intervals in a square, rectangular or triangular pattern (Anderson and Bergado, 1996). The area of influence of PVDs installed in triangular and square patterns is shown below in Figure 6.7 (left) and (right). PVDs are installed using closed-ended mandrels; the installation of the PVDs significantly disturbs the surrounding soil which creates a disturbed zone around the PVD (Prezzi & Basu, 2007). The disturbed zone consists of two zones: the smear zone and the transition zone (Anderson and Bergado, 1996; Miura, Park and Madhav, 1993). Since the mandrel displaces and drags down the surrounding soil during PVD installation, the size of the mandrel cross section determines the size of the smear and transition zones (Prezzi & Basu, 2007), this can be seen in Figure 6.8.

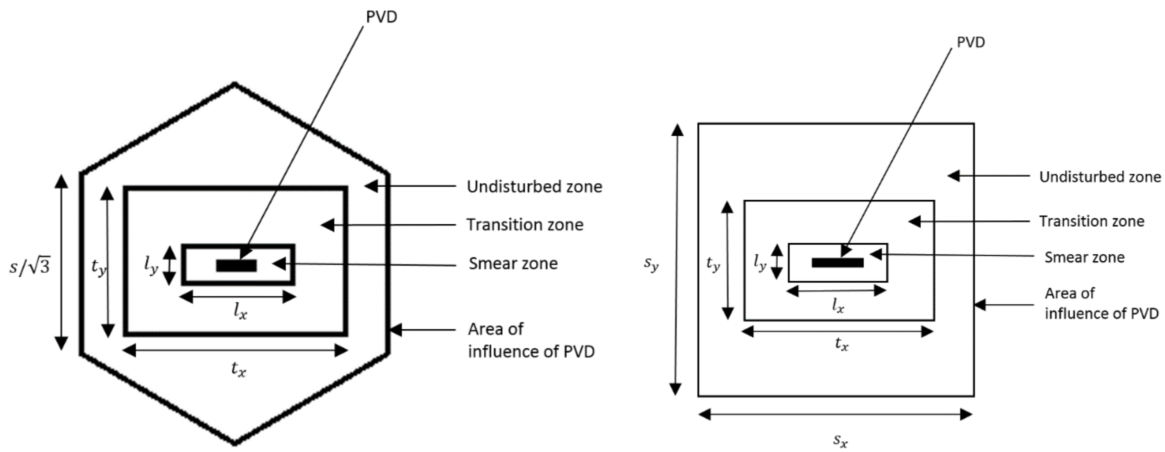

Figure 6.7. Area of influence of PVD of triangular spacing (left) and square spacing (right).

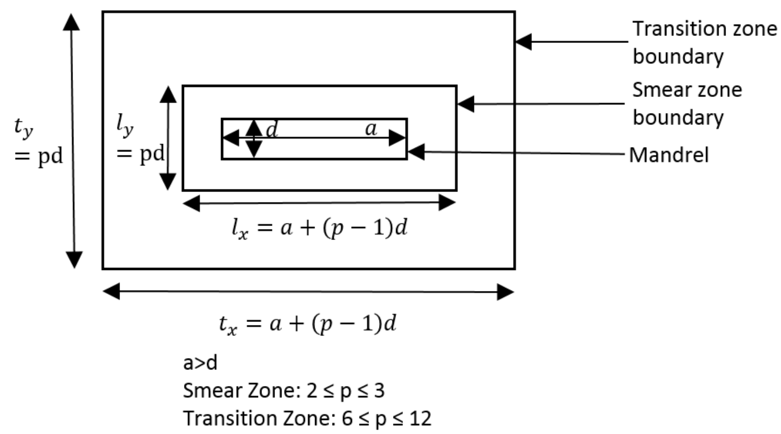

Figure 6.8. Dimensions of smear and transition zones in terms of mandrel size.

The degree of disturbance in the smear zone is described in terms of the ratio k_{hs}/k_{ho}, where k_{hs} is the hydraulic conductivity in the smear zone for horizontal flow and k_{ho} is the in situ hydraulic conductivity for the horizontal flow (Madhav, Prezzi and Basu, 2009).

A reliability analysis is done to determine the time factor at 90% consolidation using PVDs in square and triangular pattern. To solve this problem the method from (Prezzi & Basu, 2007; Madhav, Prezzi and Basu, 2009) is used. A quadrant of the total area of influence of a PVD is modeled and meshed in ABAQUS (since the total are of influence of a PVD is symmetrical only one quadrant has to be modeled). Next the connectivity matrix of the elements and the coordinates of all the nodes are output from ABAQUS and then input into the code provided by Dr. Dipanjan Basu. The code is ran and the time factor at 90% consolidation is taken as the output. This code employs a finite element analysis to solve the problem. Two random variables are assumed those being the spacing and the degree of disturbance in the smear zone ($k_{smearratio}$). All variables used for both patterns are shown in the Table 6.11 below.

Table 6.11. Variables

Variable	Type of Distribution	Parameters of Distribution
$\boldsymbol{Spacing}$	Normal	(1, 0.10204) – (mean, standard deviation)
$\boldsymbol{k_{smearratio}}$	Lognormal	(0.35, 0.12755) – (mean, standard deviation)

The PVD size is taken as 100 mm X 4 mm. The mandrel size is taken as 125 mm X 50 mm (*a* X *d*). The smear zone is therefore taken as 175 mm X 100 mm ($l_x \times l_y$) where *p* is 2. The transition zone is therefore taken as 525 mm X 450 mm ($t_x \times t_y$) where *p* is 9. The boundary condition where the drain is located is a dirichlet boundary where u=0.

6.4.1 Triangular Pattern

The triangular pattern is modeled in ABAQUS by taking one quadrant of Figure 6.7 (left).

This problem was solved using Monte Carlo Simulation (MCS), and the Multiplicative Dimensional Reduction Method (MDRM).

Using MCS, 1000 simulations were done for each random variable and each corresponding time factor (TF) at 90% consolidation was determined. One MCS value of spacing and one MCS value of $k_{smearratio}$ were used in the method explained above 1000 times. Figure 6.9 shows the cumulative distribution of the time factor at 90% consolidation. The mean and standard deviation were calculated as 1.5964 and 0.4795 respectively.

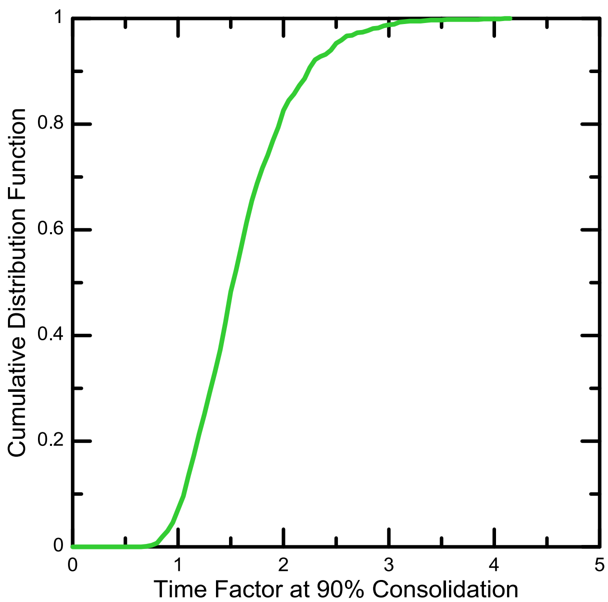

Figure 6.9. Cumulative distribution function of time factor at 90% consolidation.

The MDRM was done using the fifth order (L=5) Gauss quadrature and considering two input random variables (n=2). An input grid is generated to evaluate the response (Table 6.12). The Gauss Hermite formulas are adopted since the random variables follow normal and lognormal distribution. In total there are $(2 \times 5) + 1 = 11$ response evaluations. For each evaluation point, the other random variable is fixed to its mean value. These 11 response evaluations are then used in the method explained above.

Table 6.12. Input Grid for the response evaluation.

Random Variable	Trial	$k_{smearratio}$	$Spacing$	Time Factor @ 90% Consolidation
	1	0.119901116	1	3.46623961
	2	0.20374465	1	2.249047331
$k_{smearratio}$	3	0.328842933	1	1.52981011
	4	0.530750987	1	1.040849378
	5	0.901890477	1	0.701820931
	6	0.35	0.70846403	1.320787826
	7	0.35	0.86167093	1.395962018
$Spacing$	8	0.35	1	1.455066313
	9	0.35	1.13832907	1.499681671
	10	0.35	1.29153597	1.541008111
Fixed mean value	11	0.35	1	1.455066313

Note: z_j denotes the Gauss Hermite points.

The next step is to calculate the mean (ρ_i) and the mean square (θ_i) of an i^{th} cut function is approximated as a weighted sum (Table 6.13). Then the MDRM approximation is used to calculate the statistical moment of the response function (Table 6.14). This table also shows the relative errors between the statistical moments obtained by MDRM and MCS which are very low showing a good agreement between the two methods.

Table 6.13. Output Grid for each cut function evaluation.

Random Variable	Trial	w_j	TF @ 90% Consolidation	$w_j \times TF$	ρ_i	$w_j \times TF^2$	θ_i
	1	1.13E-02	3.46623961	3.90E-02		0.135250795	
	2	0.22208	2.249047331	0.499468431		1.123328142	
$k_{smearratio}$	3	0.53333	1.52981011	0.815893626	1.59343374	1.248162318	2.752880158
	4	0.22208	1.040849378	0.23115183		0.240594238	
	5	1.13E-02	0.701820931	0.007900398		0.005544665	
	6	1.13E-02	1.320787826	0.014868109		0.019637617	
	7	0.22208	1.395962018	0.310015245		0.432769507	
$Spacing$	8	0.53333	1.455066313	0.776030517	1.4513103	1.129175863	2.107782992
	9	0.22208	1.499681671	0.333049305		0.499467939	
	10	1.13E-02	1.541008111	0.017347128		0.026732065	
Fixed Mean Value	11		1.455066313				

Note: w_j denotes the Gauss Hermite weights.

Table 6.14. Statistical Moments of the response.

Time Factor at 90% Consolidation	MDRM (11 Trials)	MCS (1000 Simulations)	Relative Error (%)
First Moment	1.589321	1.596464	0.447467824
Second Moment	2.740612	2.778651	1.368973749
Standard Deviation	0.463328	0.479535	3.379704986
COV	0.291526	0.300373	2.945416955

Note: Relative Error (%) $= \frac{|MCS-MDRM|}{MCS} \times 100$

The output responses obtained using MDRM are combined with the MaxEnt principle with fractional moment constraints, in order to estimate the response probability distribution. Table 6.15 provides the Lagrange multipliers (λ_i) and the fractional exponents (α_i) which are used to estimate the probability distribution of the response. The number of fractional moments used are m=2, m=3, and m=4.

Table 6.15. MaxEnt parameters for failure pressure.

Fractional Moments	Entropy	i	0	1	2	3	4
		λ_i	-6.9398	5.35E+00	2.3371		
m=2	0.58519	α_i		-1.3151	1.2561		
		$M_X^{\alpha_i}$		2.7104	2.2629		
		λ_i	-12.6714	1.70E-07	7.7814	5.6248	
m=3	0.5850	α_i		-2.7381	-1.1538	0.8298	
		$M_X^{\alpha_i}$		1.9074	0.4814	1.1843	
		λ_i	-6.5882	6.9676	4.08E-01	7.5366	-7.5838
m=4	0.58486	α_i		-1.1669	1.1727	0.7549	0.3005
		$M_X^{\alpha_i}$		14.1052	0.3257	2.4957	4.6067

The estimated probability distribution of the displacement at midspan is compared to the MCS (Figure 6.10). The probability distribution functions (PDF) match up spot on and there are no differences between the two methods.

Then the probability of failure is estimated by plotting the probability of exceedance (POE) (Figure 6.11). The POE matches up very well. The MDRM plots converge to the MCS plot, this can be seen as the fractional moment increase it gets closer to the MCS plot.

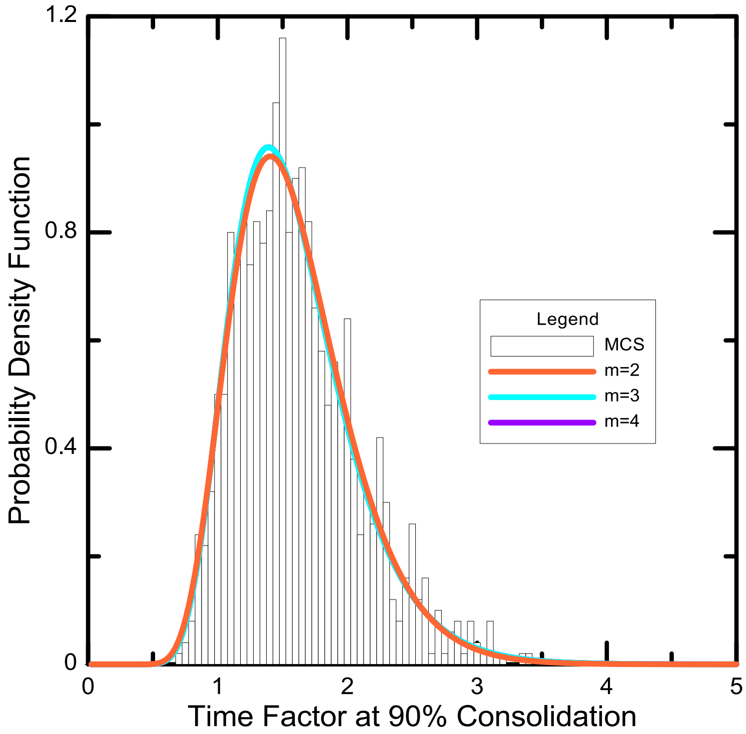

Figure 6.10. Probability Density Function of the response.

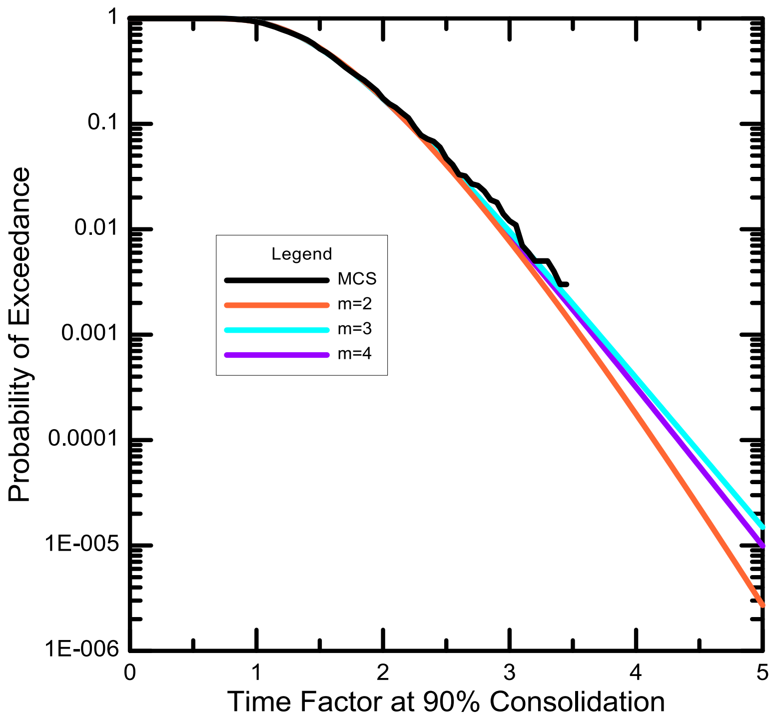

Figure 6.11. Probability of Exceedance (POE) of the response.

6.4.1.1 Computational Time

The difference in computational time is the main advantage of MDRM. For the analysis of time factor at 90% consolidation due to vertical drains, simulation of 1000 iterations of this process takes 66.7 hours on a personal computer with Intel I5-4690 4th Generation Processor and 8GB of RAM. MDRM approximation based on 11 finite element analyses takes 19 minutes and MaxEnt method requires 3.3 minutes. Thus the total time taken by MDRM is 22.3 minutes which is only 0.56% of the time taken by MCS. The reason why it takes so long to do the MCS is that it is required to change the FE model each time due to the random variability of the spacing 1000 times versus only having to change it 11 times for MDRM.

6.4.2 Square Pattern

The square pattern is modeled in ABAQUS by taking one quadrant of Figure 6.7 (right).

This problem was solved using Monte Carlo Simulation (MCS), and the Multiplicative Dimensional Reduction Method (MDRM).

Using MCS, 1000 simulations were done for each random variable and each corresponding time factor (TF) at 90% consolidation was determined. One MCS value of spacing and one MCS value of $k_{smearratio}$ were used in the method explained above 1000 times. Figure 6.12 shows the cumulative distribution of the time factor at 90% consolidation. The mean and standard deviation were calculated as 1.6878 and 0.5116 respectively.

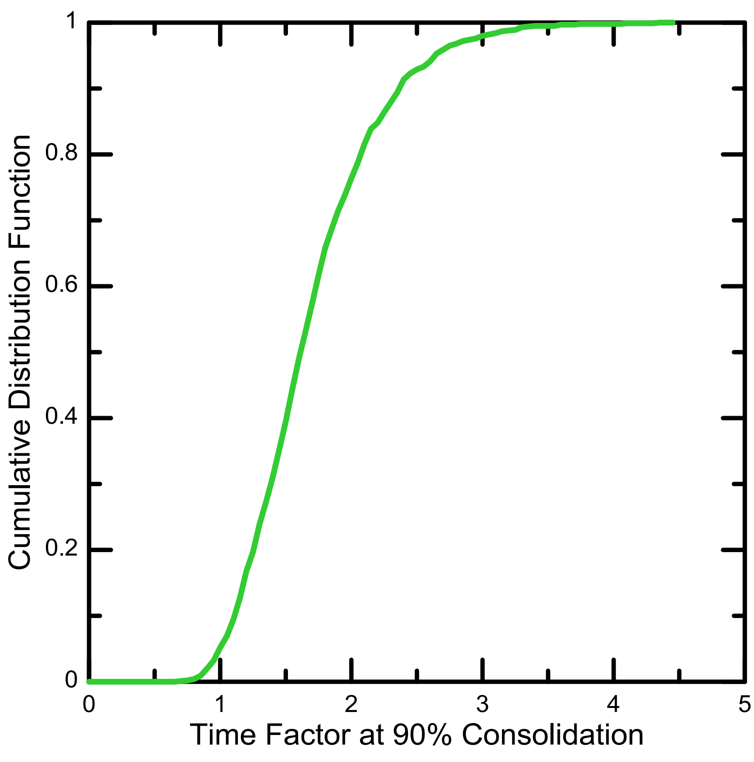

Figure 6.12. Cumulative distribution function of time factor at 90% consolidation.

The MDRM was done using the fifth order (L=5) Gauss quadrature and considering two input random variables (n=2). An input grid is generated to evaluate the response (Table 6.16). The Gauss Hermite formulas are adopted since the random variables follow normal and lognormal distribution. In total there are $(2 \times 5) + 1 = 11$ response evaluations. For each evaluation point, the other random variable is fixed to its mean value. These 11 response evaluations are then used in the method explained above.

Table 6.16. Input Grid for the response evaluation.

Random Variable	Trial	$k_{smearratio}$	$Spacing$	Time Factor @ 90% Consolidation
	1	0.119901116	1	3.675785887
	2	0.20374465	1	2.379341088
$k_{smearratio}$	3	0.328842933	1	1.613944511
	4	0.530750987	1	1.094331675
	5	0.901890477	1	0.73277109
	6	0.35	0.70846403	1.400554796
	7	0.35	0.86167093	1.480486878
$Spacing$	8	0.35	1	1.534455825
	9	0.35	1.13832907	1.580517773
	10	0.35	1.29153597	1.618017173
Fixed Mean Value	11	0.35	1	1.534455825

Note: z_j denotes the Gauss Hermite points.

The next step is to calculate the mean (ρ_i) and the mean square (θ_i) of an i^{th} cut function is approximated as a weighted sum (Table 6.17). Then the MDRM approximation is used to calculate the statistical moment of the response function (Table 6.18). This table also shows the relative errors between the statistical moments obtained by MDRM and MCS which are very low showing a good agreement between the two methods.

Table 6.17. Output Grid for each cut function evaluation.

Random Variable	Trial	w_j	TF @ 90% Consolidation	$w_j \times TF$	ρ_i	$w_j \times TF^2$	θ_i
	1	1.13E-02	3.675785887	4.14E-02		0.152097851	
	2	0.22208	2.379341088	0.528404069		1.257253512	
$k_{smearratio}$	3	0.53333	1.613944511	0.860765026	1.681825399	1.389226989	3.070577365
	4	0.22208	1.094331675	0.243029178		0.265954528	
	5	1.13E-02	0.73277109	0.008248804		0.006044485	
	6	1.13E-02	1.400554796	0.015766045		0.02208121	
	7	0.22208	1.480486878	0.328786526		0.486764137	
$Spacing$	8	0.53333	1.534455825	0.818371325	1.532139303	1.255754648	2.348834522
	9	0.22208	1.580517773	0.351001387		0.554763931	
	10	1.13E-02	1.618017173	0.018214019		0.029470596	
Fixed Mean Value	11		1.534455825				

Note: w_j denotes the Gauss Hermite weights.

Table 6.18. Statistical Moments of the response.

Time Factor at 90% Consolidation	MDRM (11 Trials)	MCS (1000 Simulations)	Relative Error (%)
First Moment	1.679286	1.687858	0.507846486
Second Moment	3.063118	3.110617	1.526999277
Standard Deviation	0.493067	0.511617	3.625754186
COV	0.293617	0.303116	3.133822709

Note: Relative Error (%) $= \frac{|MCS-MDRM|}{MCS} \times 100$

The output responses obtained using MDRM are combined with the MaxEnt principle with fractional moment constraints, in order to estimate the response probability distribution. Table 6.19 provides the Lagrange multipliers (λ_i) and the fractional exponents (α_i) which are used to estimate the probability distribution of the response. The number of fractional moments used are m=2, m=3, and m=4.

Table 6.19. MaxEnt parameters for failure pressure.

Fractional Moments	Entropy	i	0	1	2	3	4
		λ_i	-6.7922	2.34E+00	5.472684636		
m=2	0.64701	α_i		1.1947	-1.3936		
		$M_X^{\alpha_i}$		8.1071	0.7406		
		λ_i	91.1141	-123.0873	32.3623	0.6149	
m=3	0.64637	α_i		0.1699	0.5669	-3.4351	
		$M_X^{\alpha_i}$		14.0842	1.6025	0.5044	
		λ_i	-2.7193	3.3673	4.57E-01	0.1385	-0.2073
m=4	0.64662	α_i		-1.7205	3.0296	4.9947	4.8097
		$M_X^{\alpha_i}$		7.6822	6.5451	1.3302	0.4057

The estimated probability distribution of the displacement at midspan is compared to the MCS (Figure 6.13). The probability distribution functions (PDF) match up spot on and there are no differences between the two methods.

Then the probability of failure is estimated by plotting the probability of exceedance (POE) (Figure 6.14). The POE matches up very well. The MDRM plots converge to the MCS plot, this can be seen as the fractional moment increase it gets closer to the MCS plot.

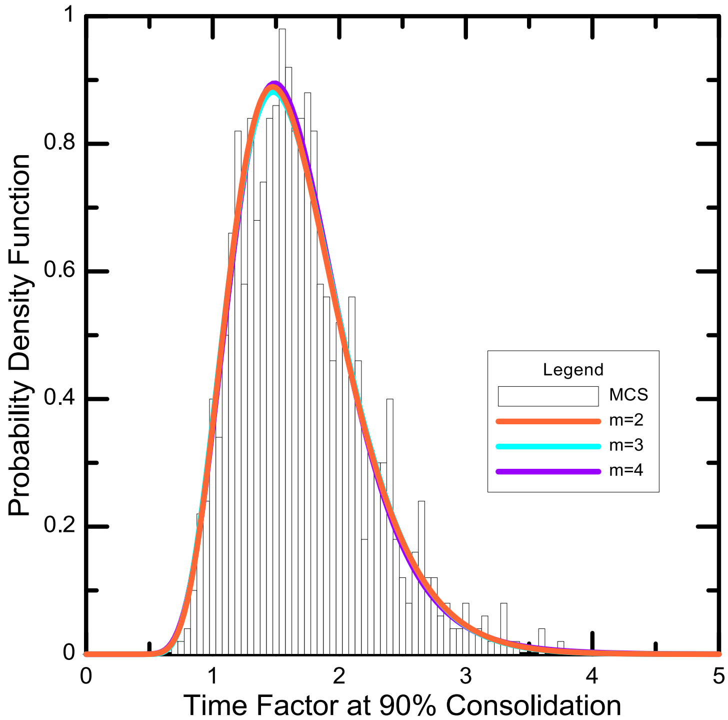

Figure 6.13. Probability Density Function of the response.

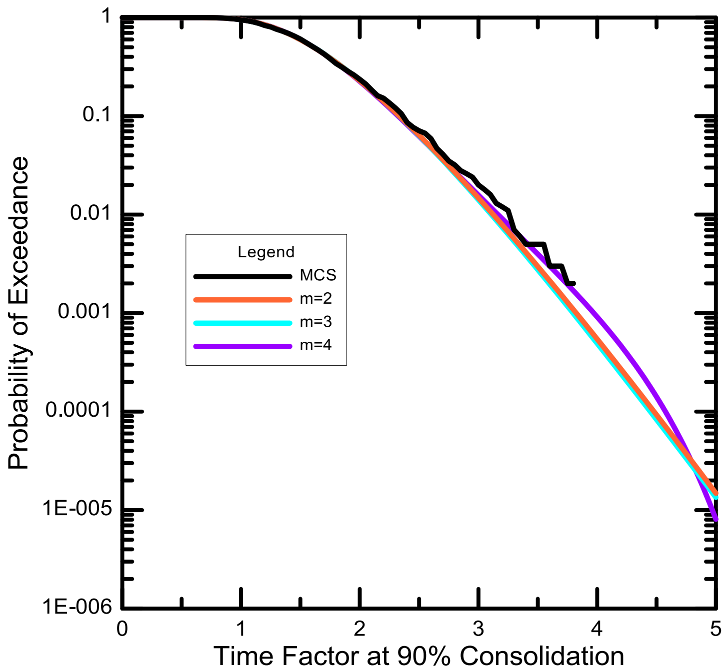

Figure 6.14. Probability of Exceedance (POE) of the response.

6.4.2.1 Computational Time

The difference in computational time is the main advantage of MDRM. For the analysis of time factor at 90% consolidation due to vertical drains, simulation of 1000 iterations of this process takes 58.3 hours on a personal computer with Intel I5-4690 4th Generation Processor and 8GB of RAM. MDRM approximation based on 11 finite element analyses takes 15 minutes and MaxEnt method requires 3.23 minutes. Thus the total time taken by MDRM is 18.23 minutes which is only 0.52% of the time taken by MCS. The reason why it takes so long to do the MCS is that it is required to change the FE model each time due to the random variability of the spacing 1000 times versus only having to change it 11 times for MDRM.

6.5 Concrete Infinite Beam on an Elastic Foundation

A problem of a concrete infinite beam on an elastic foundation will be solved to provide another example of the computational efficiency provided by MDRM. A MATLAB computer program that uses a finite element analysis, developed by a colleague, Hesham Elhuni, was used to solve this problem.

Random variables were inputted into this program and probability density functions and probability of exceedance graphs were generated using both Monte Carlo Simulation and Multiplicative Dimensional Reduction Method to compare results and computational effort.

This program solves for the displacement of a concrete infinite beam on an elastic foundation. There are two foundation models used, the Modified Vlasov Model (continuum model) and the two parameter Pasternak Foundation Model (discrete model which is done by modelling the soil using springs). These two models are used twice once as a static problem and once as a dynamic problem.

The differential equation that is solved by this MATLAB computer program by a finite element analysis is:

$$EI\frac{d^4w}{dx^4} - 2t_s\frac{d^2w}{dx^2} + kw + \rho\frac{d^2w}{dt^2} + c\frac{dw}{dt} = P\delta(x - vt) \tag{6.7}$$

Where w is the transverse deflection of beam (m), E is the modulus of elasticity of the beam (N/m^2), I is the moment of inertia of the beam (m^4), ρ is the mass per unit length of the beam-foundation system contributing in vibration (kg/m), c is the coefficient of viscous damping of the system per unit length of the beam (N-sec/m^2), P is the applied force (N). If a dynamic problem is being solved (moving load) then the P is the applied moving concentrated force (N) and a dirac delta function (δ) is added. Where v is the velocity of the moving load (m/sec), x is the horizontal distance measured from left side of the beam (m), and t is the time (sec) (Basu & Elhuni, 2017). When solving the two parameter Pasternak Model:

$$t_s = t_s \tag{6.8}$$

$$k = k_s \tag{6.9}$$

Where t_s is the shear parameter of soil (N) and k_s is the Winkler spring constant (N/m^2) (Basu & Elhuni, 2017). Figure 6.15 shows the two parameter Pasternak Model. When solving using the Modified Vlasov Model:

$$2t_s = \int_0^H \frac{E_s b}{2(1+\mu_s)} \emptyset^2 dz \tag{6.10}$$

$$k = \int_0^H \frac{E_s b(1-\mu_s)}{(1+\mu_s)(1-2\mu_s)} \left(\frac{d\emptyset}{dz}\right)^2 dz \tag{6.11}$$

Where E_s is the modulus of elasticity of the soil, H and b are the height and width of the soil model respectively, μ_s is the Poisson's ratio of the soil, and $\emptyset(z)$ is a function assumed by Vlasov for the vertical displacement of the soil. Additional simplifications and derivations of the above two formulas are found in (Girija Vallabhan and Das, 1991). Figure 6.16 shows the Modified Vlasov Model.

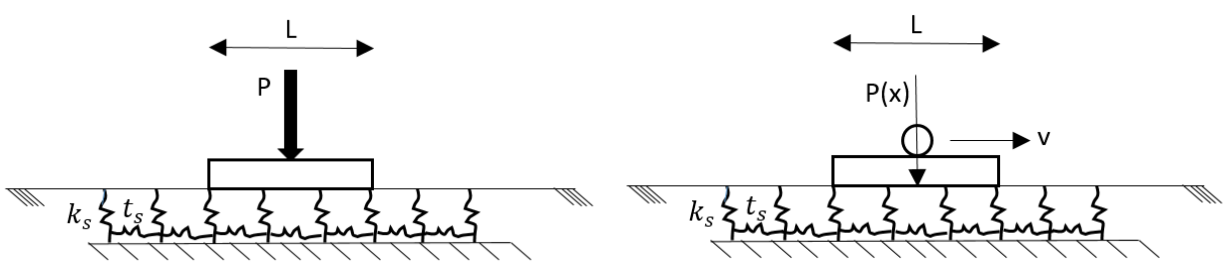

Figure 6.15. Two Parameter Pasternak Model: static load (left) and dynamic load (right)

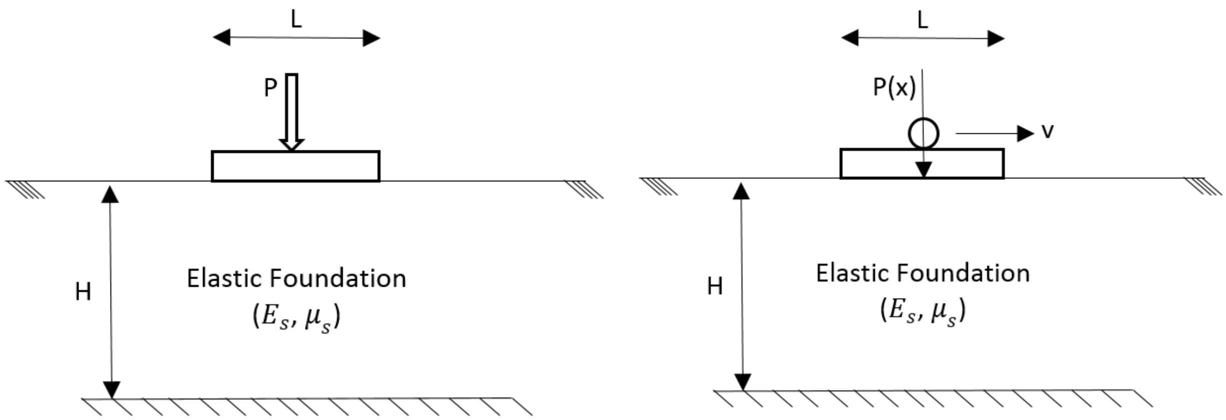

Figure 6.16. Modified Vlasov Model: static load (left) and dynamic load (right)

6.5.1 Static Continuum Problem

This problem will be solved considering that the elastic modulus of soil (E_s) and the Poisson's ratio (μ_s) are random variables using a Modified Vlasov Model as seen in Figure 6.16 (left). The beam was considered to be 20 m in length, with a 100 kN static load at the midpoint. 20 m of soil was considered on both sides of the beam. Table 6.20 defines each random variable used in this problem.

Table 6.20. Variables.

Variable	Type of Distribution	Parameters of Distribution
$\boldsymbol{E_s}$	Lognormal	(40, 30%) – (mean, coefficient of variance)
$\boldsymbol{\mu_s}$	Lognormal	(0.4, 7%) – (mean, coefficient of variance)

This problem was solved using Monte Carlo Simulation (MCS), and the Multiplicative Dimensional Reduction Method (MDRM).

Using MCS, 1000 simulations were done for each random variable and each corresponding midspan displacement was calculated. Figure 6.17 shows the cumulative distribution of the displacement at midspan. The mean and standard deviation were calculated as 0.000753 and 0.000221 respectively.

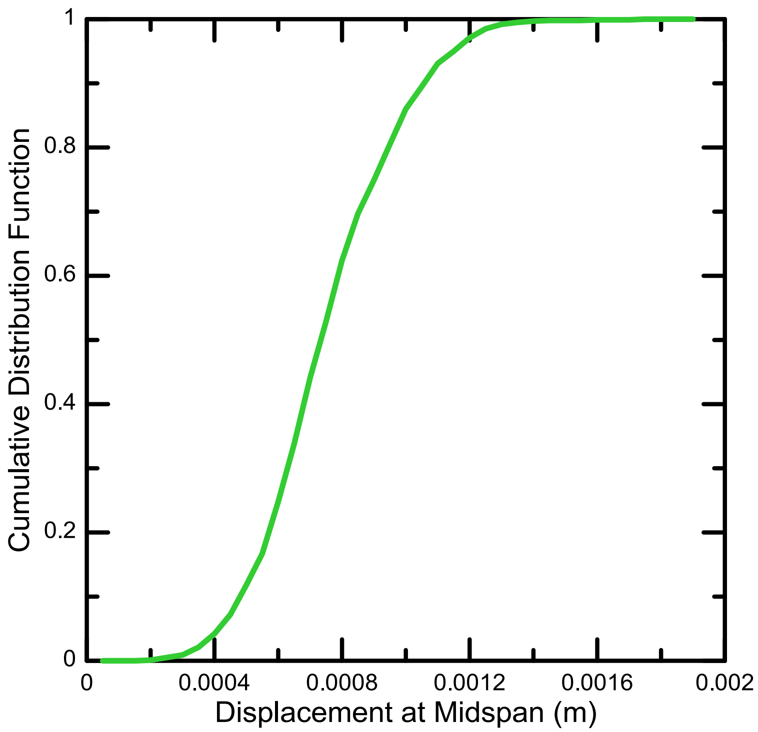

Figure 6.17. Cumulative distribution function of displacement at midspan.

The MDRM was done using the fifth order (L=5) Gauss quadrature and considering two input random variables (n=2). An input grid is generated to evaluate the response (Table 6.21). The Gauss Hermite formulas are adopted since the random variables follow lognormal distribution. In total there are $(2 \times 5) + 1 = 11$ response evaluations. For each evaluation point, the other random variable is fixed to its mean value. Each response evaluation is then substituted into the MATLAB computer program to determine the displacement at midspan.

Table 6.21. Input Grid for the response evaluation.

Random Variable	Trial	E_s	μ_s	Displacement at Midspan (m)
	1	16561630.47	0.4	0.00145803
	2	25734696.92	0.4	0.001023777
E_s	3	38313051.41	0.4	0.000743463
	4	57039331.49	0.4	0.000540816
	5	88631968.37	0.4	0.000381271
	6	40000000	0.32677598	0.00095386
	7	40000000	0.36294252	0.000852139
μ_s	8	40000000	0.39902359	0.000722201
	9	40000000	0.43869157	0.000533294
	10	40000000	0.48724457	0.00018069
Fixed mean value	11	40000000	0.4	0.000718202

Note: z_j denotes the Gauss Hermite points.

The next step is to calculate the mean (ρ_i) and the mean square (θ_i) of an i^{th} cut function is approximated as a weighted sum (Table 6.22). Then the MDRM approximation is used to calculate the statistical moment of the response function (Table 6.23). This table also shows the relative errors between the statistical moments obtained by MDRM and MCS which are very low showing a good agreement between the two methods.

Table 6.22. Output Grid for each cut function evaluation.

Random Variable	Trial	w_j	Displacement at Midspan (m)	$w_j \times disp$	ρ_i	$w_j \times disp^2$	θ_i
	1	1.13E-02	0.00145803	1.64E-05		2.39E-08	
	2	0.22208	0.00102378	0.00022736		2.32766E-07	
E_s	3	0.53333	0.00074346	0.000396511	0.00076468	2.94792E-07	6.18079E-07
	4	0.22208	0.00054082	0.000120104		6.49545E-08	
	5	1.13E-02	0.00038127	4.29196E-06		1.6364E-09	
	6	1.13E-02	0.00095386	1.07376E-05		1.02422E-08	
	7	0.22208	0.00085214	0.000189243		1.61261E-07	
μ_s	8	0.53333	0.0007222	0.000385171	0.00070562	2.78171E-07	5.13203E-07
	9	0.22208	0.00053329	0.000118434		6.31602E-08	
	10	1.13E-02	0.00018069	2.03402E-06		3.67528E-10	
Fixed Mean Value	11		0.0007182				

Note: w_j denotes the Gauss Hermite weights.

Table 6.23. Statistical Moments of the response.

Displacement at Midspan (m)	MDRM (11 Trials)	MCS (1000 Simulations)	Relative Error (%)
First Moment	0.000751285	0.00075181	0.069356
Second Moment	6.14949E-07	6.14951E-07	0.783699
Standard Deviation	0.000224768	0.00022302	0.000179043
COV	0.299178516	0.296646203	0.853647282

Note: Relative Error (%) $= \frac{|MCS-MDRM|}{MCS} \times 100$

The output responses obtained using MDRM are combined with the MaxEnt principle with fractional moment constraints, in order to estimate the response probability distribution. Table 6.24 provides the Lagrange multipliers (λ_i) and the fractional exponents (α_i) which are used to estimate the probability distribution of the response. The number of fractional moments used are m=2, m=3, and m=4.

Table 6.24. MaxEnt parameters for failure pressure.

Fractional Moments	Entropy	i	0	1	2	3	4
		λ_i	2.5426	535879.33	-436558.88		
m=2	-6.99816	α_i		1.0315	0.9987		
		$M_X^{\alpha_i}$		1.94E-05	6.03E-12		
		λ_i	3.0734	1869471.1	114325.1	-35224.8	
m=3	-6.99807	α_i		2.2499	1.0741	0.8836	
		$M_X^{\alpha_i}$		1.32E-12	4.78E+01	1.78E-10	
		λ_i	13.7829	-2856.51	4.79E+04	176122.4	-139.237
m=4	-6.99806	α_i		0.80006	4.0995	1.36006	0.2556
		$M_X^{\alpha_i}$		2.84E+10	6.27E-14	2.25E+15	2.44E+08

The estimated probability distribution of the displacement at midspan is compared to the MCS (Figure 6.18). The probability distribution functions (PDF) match up very accurately.

Then the probability of failure is estimated by plotting the probability of exceedance (POE) (Figure 6.19). The POE matches up very accurately and as the number of fractional moments increase the MDRM (m=4) matches closer to the MCS.

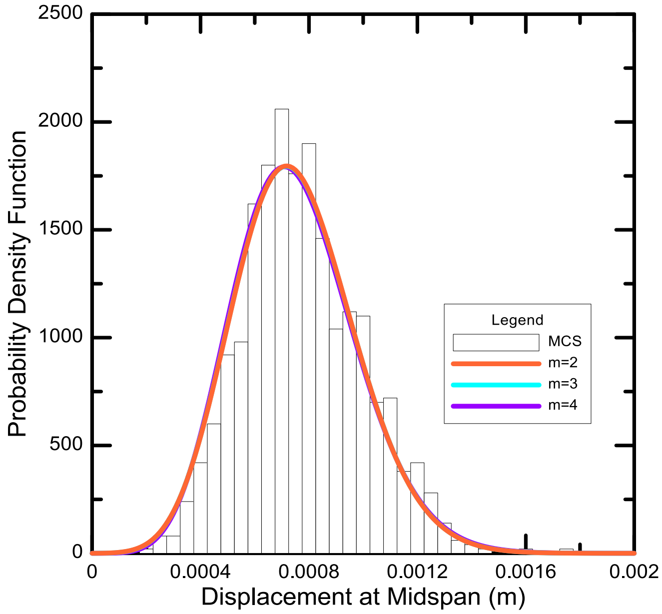

Figure 6.18. Probability Density Function of the response.

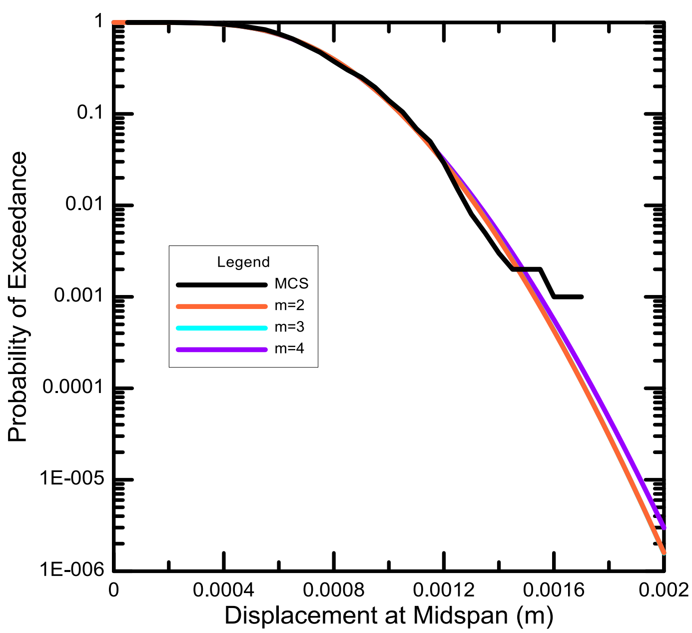

Figure 6.19. Probability of Exceedance (POE) of the response.

6.5.1.1 Computational Time

The difference in computational time is the main advantage of MDRM. For the analysis of a concrete infinite beam on an elastic foundation, simulation of 1000 iterations of the computer program takes 2.75 hours on a personal computer with Intel I5-4690 4th Generation Processor and 8GB of RAM. MDRM approximation based on 11 finite element analyses takes 1.5 minutes and MaxEnt method requires 4.8 minutes. Thus the total time taken by MDRM is 6.3 minutes which is only 3.8% of the time taken by MCS.

6.5.2 Static Discrete Problem

This problem will be solved considering that the compressive spring constant (k_s) and the shear parameter (t_s) are random variables using a two parameter Pasternak Model as seen in Figure 6.15 (left). These two variables come up in the Pasternak foundation model (two parameter foundation model) which uses springs to model soil. The beam was considered to be 5 m in length, with a 10 kN static load at the midpoint. Soil was not considered on either side of the beam. Table 6.25 defines each random variable used in this problem.

Table 6.25. Variables.

Variable	Type of Distribution	Parameters of Distribution
$\boldsymbol{k_s}$	Lognormal	(1140000, 30%) – (mean, coefficient of variance)
$\boldsymbol{t_s}$	Lognormal	(161993.8, 30%) – (mean, coefficient of variance)

This problem was solved using Monte Carlo Simulation (MCS), and the Multiplicative Dimensional Reduction Method (MDRM).

Using MCS, 1000 simulations were done for each random variable and each corresponding midspan displacement was calculated. Figure 6.20 shows the cumulative distribution of the displacement at midspan. The mean and standard deviation were calculated as 0.0005698 and 2.46754E-05 respectively.

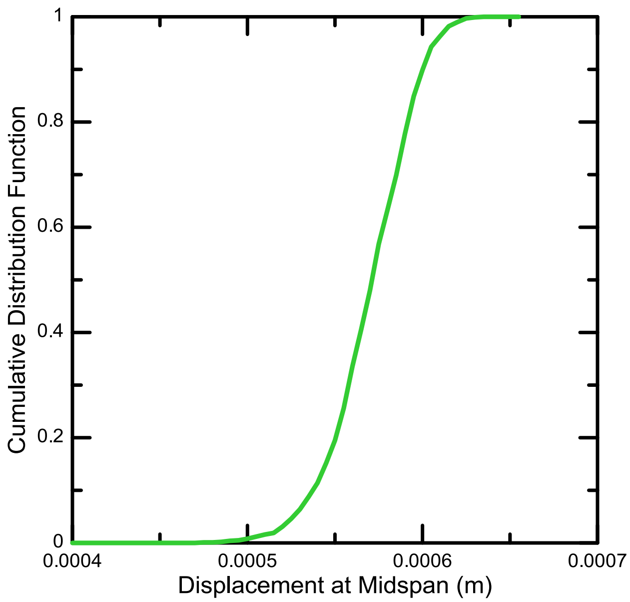

Figure 6.20. Cumulative distribution function of displacement midspan (m).

The MDRM was done using the fifth order (L=5) Gauss quadrature and considering two input random variables (n=2). The Gauss Hermite formulas are adopted since the random variables follow lognormal distribution. In total, there are $(2 \times 5) + 1 = 11$ response evaluations. For each evaluation point, the other random variable is fixed to its mean value. Each response evaluation is then substituted into the MATLAB computer program to determine the displacement at midspan. MDRM approximation is used to calculate the statistical moment of the response function (Table 6.26). This table also shows the relative errors between the statistical moments obtained by MDRM and MCS which are very low showing a good agreement between the two methods.

Table 6.26. Statistical Moments of the response.

Displacement at Midspan (m)	MDRM (11 Trials)	MCS (1000 Simulations)	Relative Error (%)
First Moment	0.000569871	0.0005698	0.012572977

Second Moment	3.25427E-07	3.25281E-07	0.044944468
Standard Deviation	2.59504E-05	2.46754E-05	5.167169212
COV	0.045537252	0.043305318	5.15394823

Note: Relative Error (%) $= \frac{|MCS-MDRM|}{MCS} \times 100$

The output responses obtained using MDRM are combined with the MaxEnt principle with fractional moment constraints, to estimate the response probability distribution. Table 6.27 provides the Lagrange multipliers (λ_i) and the fractional exponents (α_i) which are used to estimate the probability distribution of the response. The number of fractional moments used are m=2, m=3, and m=4.

Table 6.27. MaxEnt parameters for failure pressure.

Fractional Moments	**Entropy**	**i**	**0**	**1**	**2**	**3**	**4**
		λ_i	19.4899	-9.21E+14	1.29E+17		
m=2	-9.1575	α_i		3.9091	4.5929		
		$M_X^{\alpha_i}$		4.99E-11	2.74E-10		
		λ_i	61.49108	3.646E+12	-2.7E+09	2.27E+12	
m=3	-9.1486	α_i		3.1998122	2.183589	4.686493	
		$M_X^{\alpha_i}$		1.60E-11	8.05E+07	2.09E-04	
		λ_i	561.8688	7833.243	-2.16E+05	1193946	141438.8
m=4	-9.1360	α_i		0.2557	0.5439	0.8657	0.9454
		$M_X^{\alpha_i}$		4.22E-01	1.62E+01	1.47E+02	0.1603

The estimated probability distribution of the displacement at midspan is compared to the MCS (Figure 6.21). The probability distribution functions (PDF) match up very accurately. Then the probability of failure is estimated by plotting the probability of exceedance (POE) (Figure 6.22). The POE using MDRM and MCS both match up accurately.

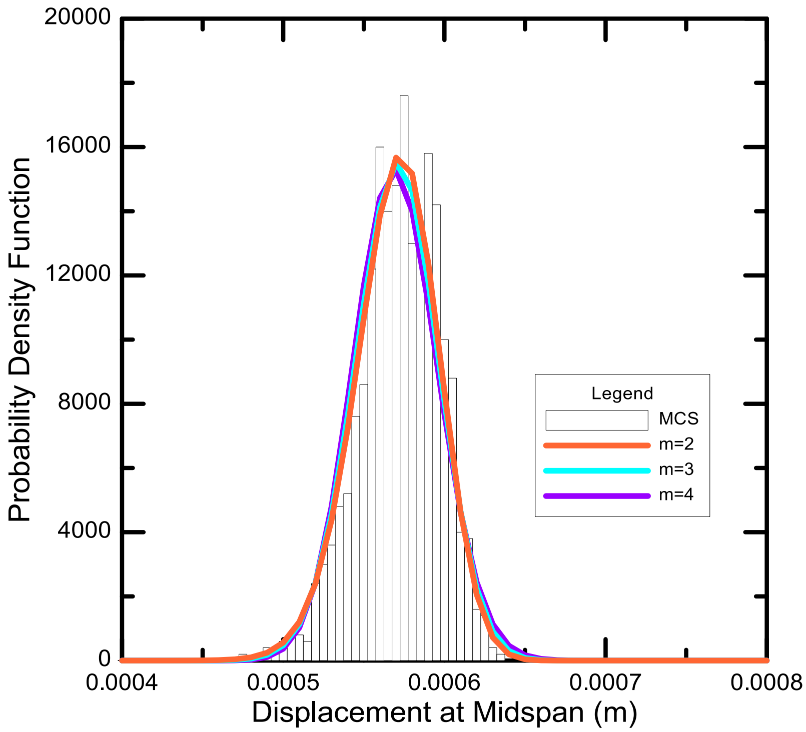

Figure 6.21. Probability Density Function of the response.

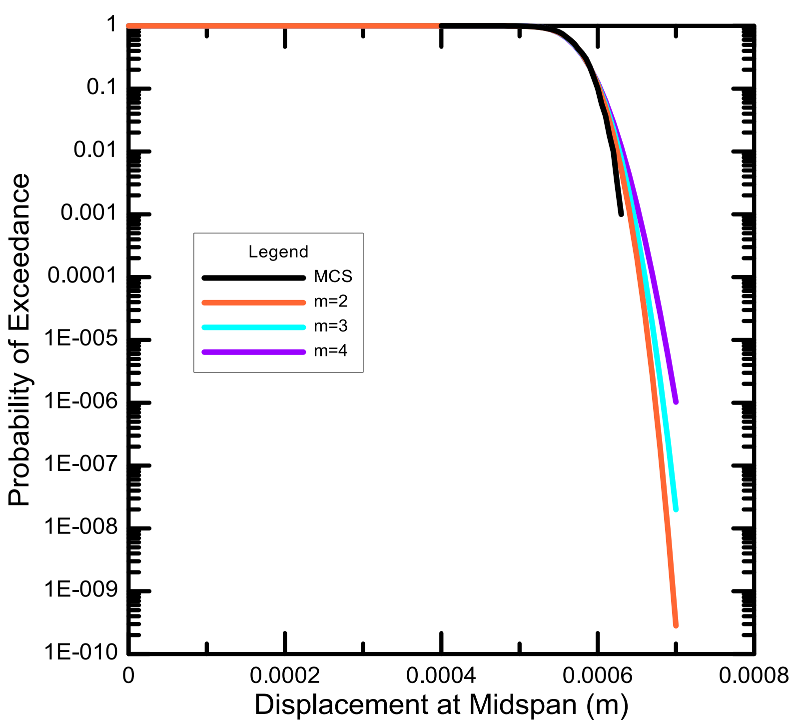

Figure 6.22. Probability of Exceedance (POE) of the response.

6.5.2.1 Computational Time

For the analysis of a concrete infinite beam on an elastic foundation, simulation of 1000 iterations of the computer program takes 50 minutes on a personal computer with Intel I5-4690 4th Generation Processor and 8GB of RAM. MDRM approximation based on 11 finite element analyses takes 0.5 minutes and MaxEnt method requires 5.5 minutes. Thus, the total time taken by MDRM is 6 minutes which is only 12% of the time taken by MCS.

6.5.3 Dynamic Continuum Problem

This problem will be solved considering that the elastic modulus of soil (E_s) and the Poisson's ratio (μ_s) are random variables using a Modified Vlasov Model as seen in Figure 6.16 (right). The beam was considered to be 5 m in length, with a 10 kN dynamic load starting at the left side and moving across the beam. 5 m of soil was considered on both sides of the beam. Table 6.28 defines each random variable used in this problem.

Table 6.28. Variables.

Variable	Type of Distribution	Parameters of Distribution
$\boldsymbol{E_s}$	Lognormal	(20, 30%) – (mean, coefficient of variance)
$\boldsymbol{\mu_s}$	Lognormal	(0.25, 7%) – (mean, coefficient of variance)

This problem was solved using Monte Carlo Simulation (MCS), and the Multiplicative Dimensional Reduction Method (MDRM).

Using MCS, 1000 simulations were done for each random variable and each corresponding displacement at the second node was calculated. Figure 6.23 shows the cumulative distribution of the displacement at the second node. The mean and standard deviation were calculated as 0.001104 and 4.107E-05 respectively.

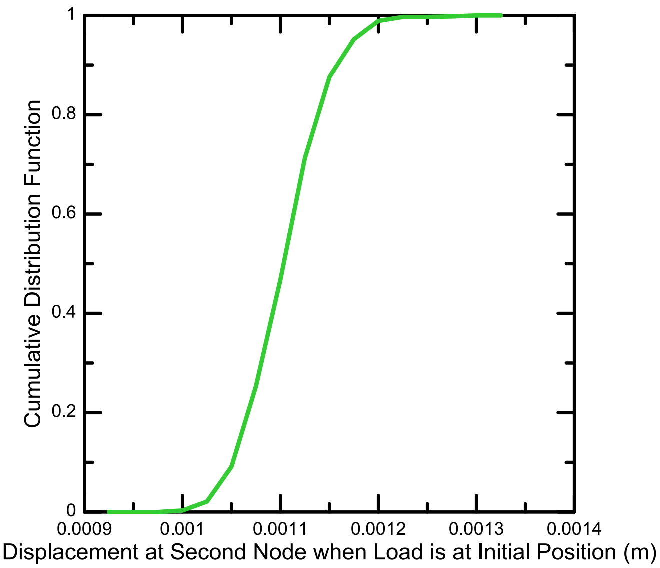

Figure 6.23. Cumulative distribution function of displacement at second node when load is at initial position (m).

The MDRM was done using the fifth order (L=5) Gauss quadrature and considering two input random variables (n=2). The Gauss Hermite formulas are adopted since the random variables follow lognormal distribution. In total there are $(2 \times 5) + 1 = 11$ response evaluations. For each evaluation point, the other random variable is fixed to its mean value. Each response evaluation is then substituted into the MATLAB computer program to determine the displacement at the second node. MDRM approximation is used to calculate the statistical moment of the response function (Table 6.29). This table also shows the relative errors between the statistical moments obtained by MDRM and MCS which are very low showing a good agreement between the two methods.

Table 6.29. Statistical Moments of the response.

Displacement at Node 2 (m)	MDRM (11 Trials)	MCS (1000 Simulations)	Relative Error (%)
First Moment	0.001103477	0.001103781	0.027511876
Second Moment	1.21934E-06	1.22002E-06	0.055696176
Standard Deviation	4.09579E-05	4.10703E-05	0.273797569
COV	0.037117088	0.037208753	0.246353469

Note: Relative Error (%) $= \frac{|MCS-MDRM|}{MCS} \times 100$

The output responses obtained using MDRM are combined with the MaxEnt principle with fractional moment constraints, in order to estimate the response probability distribution. Table 6.30 provides the Lagrange multipliers (λ_i) and the fractional exponents (α_i) which are used to estimate the probability distribution of the response. The number of fractional moments used are m=2, m=3, and m=4.

Table 6.30. MaxEnt parameters for failure pressure.

Fractional Moments	**Entropy**	**i**	**0**	**1**	**2**	**3**	**4**
		λ_i	700.5949	-2.61E+01	0.00050099		
m=2	-8.6941	α_i		-0.5321	-1.9365		
		$M_X^{\alpha_i}$		7.68E-12	3.29E-02		
		λ_i	493.8763	44.4784	-66.7168	1.21E-13	
m=3	-8.6937	α_i		0.6882	-0.3061	-4.8809	
		$M_X^{\alpha_i}$		5.14E-07	9.68E-01	7.16E+00	
		λ_i	700.5962	1010.034	1.81E+03	-2078.56	566.92
m=4	-8.6933	α_i		0.9243	-0.6284	-0.6139	-0.3167
		$M_X^{\alpha_i}$		3.16E-01	1.87E+01	1.15E-02	67.2909

The estimated probability distribution of the displacement at node 2 is compared to the MCS (Figure 6.24). The probability distribution functions (PDF) using MDRM and MCS both matchup accurately.

Then the probability of failure is estimated by plotting the probability of exceedance (POE) (Figure 6.25). The POE matches up very accurately. As the number of fractional moments increased, the POE converged better and better to the MCS POE.

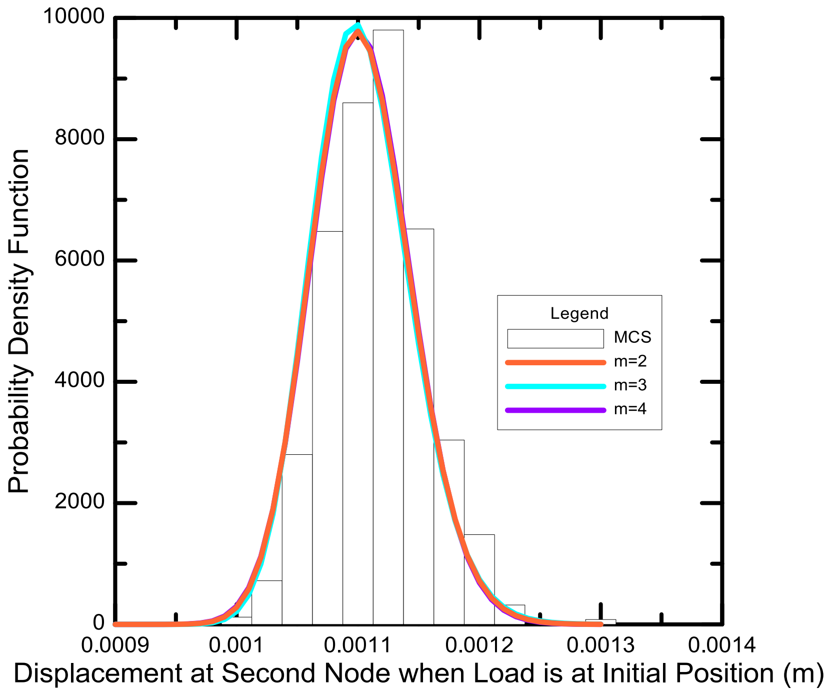

Figure 6.24. Probability Density Function of the response.

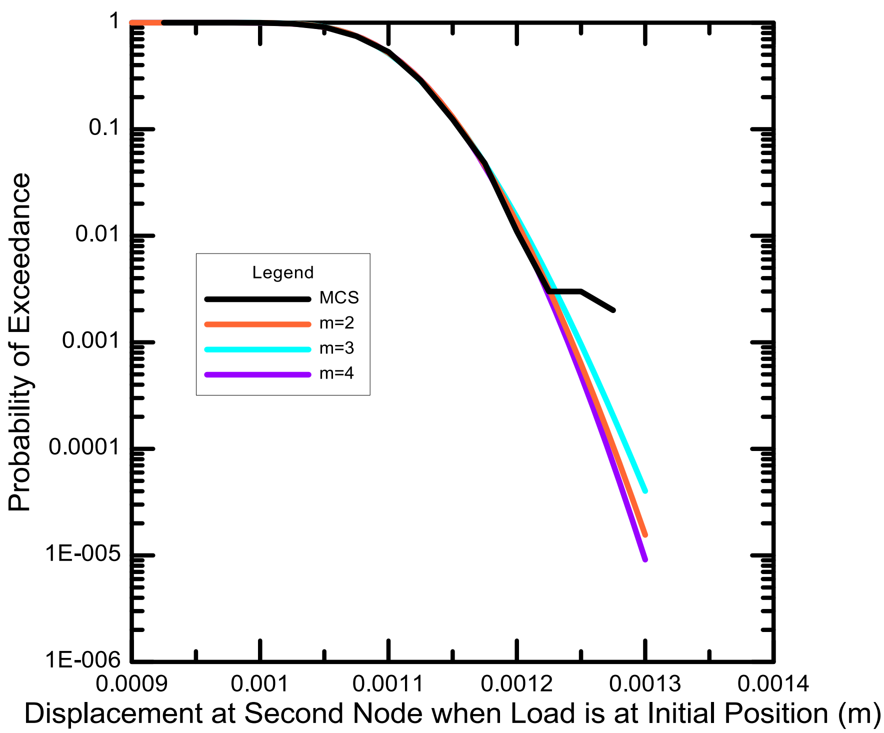

Figure 6.25. Probability of Exceedance (POE) of the response.

6.5.3.1 Computational Time

For the analysis of a concrete infinite beam on an elastic foundation, simulation of 1000 iterations of the computer program takes 21 hours on a personal computer with Intel I5-4690 4th Generation Processor and 8GB of RAM. MDRM approximation based on 11 finite element analyses takes 11.6 minutes and MaxEnt method requires 5.7 minutes. Thus the total time taken by MDRM is 17.5 minutes which is only 1.4% of the time taken by MCS.

6.5.4 Dynamic Discrete Problem

This problem will be solved considering that the compressive spring constant (k_s) and the shear parameter (t_s) are random variables using a two parameter Pasternak Model as seen in Figure 6.15 (right). The beam was considered to be 10 m in length, with a 100 kN dynamic load starting at the left side and moving across the beam. Soil was not considered on either side of the beam. Table 6.31 defines each random variable used in this problem.

Table 6.31. Variables.

Variable	Type of Distribution	Parameters of Distribution
$\boldsymbol{k_s}$	Lognormal	(1140000, 80%) – (mean, coefficient of variance)
$\boldsymbol{t_s}$	Lognormal	(161993.8, 30%) – (mean, coefficient of variance)

The coefficient of variance (COV) of k_s was increased significantly for this problem because when a lower value of COV was used the program did not give final displacements that varied which in turn did not give good results.

This problem was solved using Monte Carlo Simulation (MCS), and the Multiplicative Dimensional Reduction Method (MDRM).

Using MCS, 1000 simulations were done for each random variable and each corresponding midspan displacement was calculated when the load is at midspan. Figure 6.26 shows the cumulative distribution of the displacement at midspan. The mean and standard deviation were calculated as 0.003308 and 0.000222856 respectively.

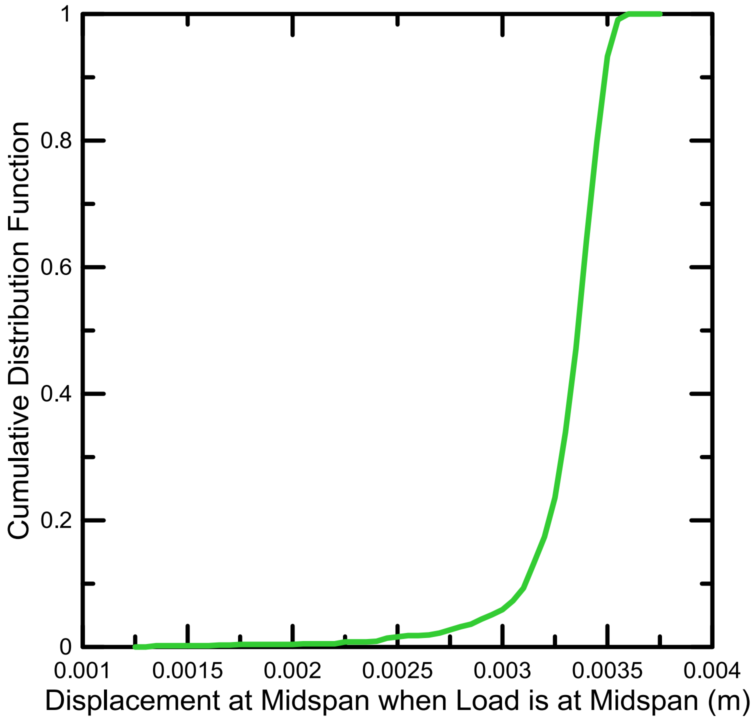

Figure 6.26. Cumulative distribution function of displacement at midspan when load is at midspan (m).

The MDRM was done using the tenth order (L=10) Gauss quadrature and considering two input random variables (n=2). A tenth order Gauss quadrature was used in this case because 5 quadrature points did not show the full range of values that could occur with the random variables used. The variance of the results using MDRM was used as a guideline to determine that a tenth order Gauss quadrature was needed. The Gauss Hermite formulas are adopted since the random variables follow lognormal distribution. In total there are $(2 \times 10) + 1 = 21$ response evaluations. For each evaluation point, the other random variable is fixed to its mean value. MDRM approximation is used to calculate the statistical moment of the response function (Table 6.32). This table also shows the relative errors between the statistical moments obtained by MDRM and MCS which are very low showing a good agreement between the two methods.

Table 6.32. Statistical Moments of the response.

Displacement at Midspan (m)	MDRM (21 Trials)	MCS (1000 Simulations)	Relative Error (%)
First Moment	0.003317	0.003308	0.26254
Second Moment	1.10454E-05	1.0995E-05	0.458412477
Standard Deviation	0.000206204	0.000222856	7.472123149
COV	0.062164528	0.067361034	7.714409506

Note: $\text{Relative Error (\%)} = \frac{|MCS-MDRM|}{MCS} \times 100$

The output responses obtained using MDRM are combined with the MaxEnt principle with fractional moment constraints, in order to estimate the response probability distribution. Table 6.33 provides the Lagrange multipliers (λ_i) and the fractional exponents (α_i) which are used to estimate the probability distribution of the response. The number of fractional moments used are m=2, m=3, and m=4.

Table 6.33. MaxEnt parameters for failure pressure.

Fractional Moments	Entropy	i	0	1	2	3	4
		λ_i	11.8013	-2.61E+14	3.91E+14		
m=2	-7.2671	α_i		4.575	4.6488		
		$M_X^{\alpha_i}$		1.35E-08	4.95E-08		
		λ_i	16.2638	-1.10E+11	5.25E+12	1.47E+13	
m=3	-7.2374	α_i		3.6044	4.344	4.8439	
		$M_X^{\alpha_i}$		1.10E-12	2.68E+02	1.02E-03	
		λ_i	111.9253	-7069.74	3.90E+09	8.48E+09	9.43E+09
m=4	-7.1064	α_i		0.6768	3.8786	3.4294	3.9879
		$M_X^{\alpha_i}$		2.10E-02	2.46E-10	3.19E-09	1.32E-10

The estimated probability distribution of the displacement at midspan is compared to the MCS (Figure 6.27).

The probability distribution functions (PDF) do not match up that well. The MCS and MDRM graphs start of the same but then due to the fact that k_s does not affect the results as intended. In fact this variable does not really affect the results at all which results in the poor results that were achieved. Then the probability of failure is estimated by plotting the probability of exceedance (POE) (Figure 6.28) which also shows a poor match between the two methods due to the same reasoning as why the PDF plots did not match.

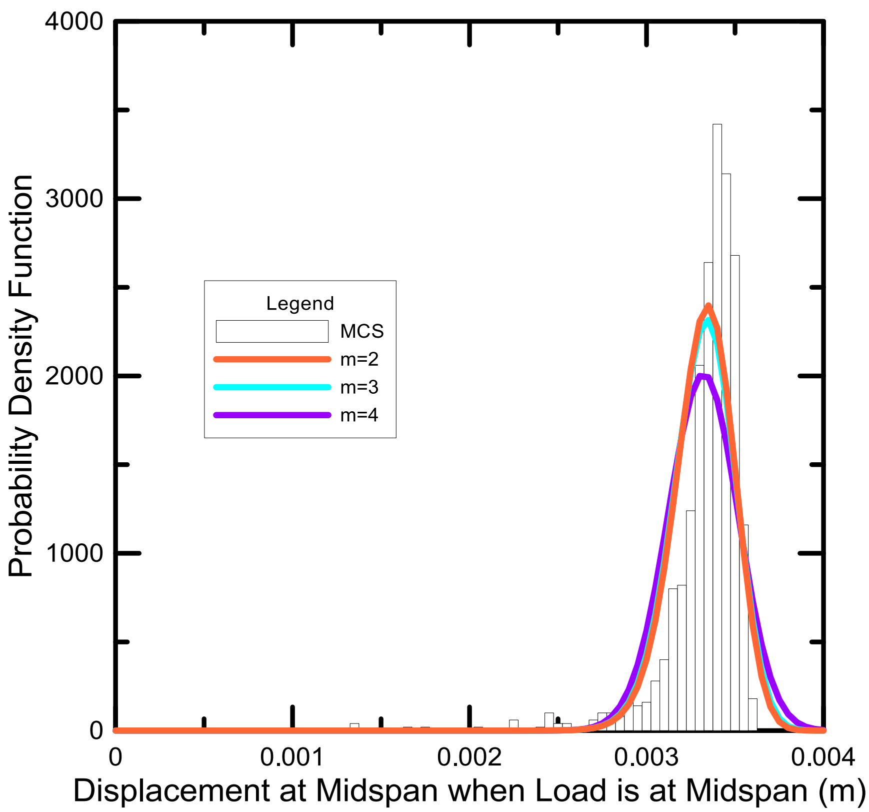

Figure 6.27. Probability Density Function of the response.

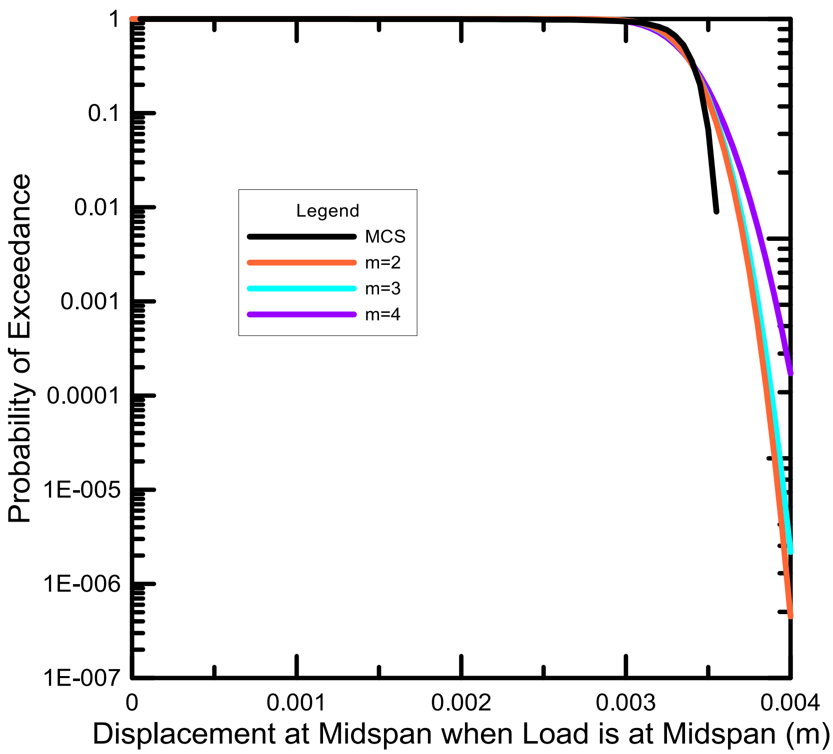

Figure 6.28. Probability of Exceedance (POE) of the response.

6.5.4.1 Computational Time

For the analysis of a concrete infinite beam on an elastic foundation, simulation of 1000 iterations of the computer program takes 6.2 hours on a personal computer with Intel I5-4690 4th Generation Processor and 8GB of RAM. MDRM approximation based on 21 finite element analyses takes 6.3 minutes and MaxEnt method requires 4.2 minutes. Thus the total time taken by MDRM is 10.5 minutes which is only 2.8% of the time taken by MCS.

6.6 Conclusions

Traditional reliability methods are very time consuming and often times make engineers not likely to use them. This chapter studies the Multiplicative Dimensional Reduction Method in order show that a reliability method exists that has the same accuracy of traditional methods while providing much greater computational efficiency.

Through the last two examples in this paper the computationally efficiency of MDRM is on full display. Without any drop in accuracy MDRM gives results much quicker (0.5% to 12% of time taken to complete the problem using MCS) depending on the type of problem. The first two examples showed that using MDRM with equations does not help. Instead MCS is more computationally efficient when solving an equation based analysis, however the results between MCS and MDRM are the same and the number of response evaluations using MDRM are significantly less which helps when employing a finite element analysis. MDRM is especially useful in the field of geomechanics as a lot of uncertainties exist. With most of the focus being on finite element modelling and analyses, pairing it up with a reliability analysis using MDRM offers great insight to engineers looking to design with the utmost respect to safety in mind.

7 Conclusions and Recommendations

7.1 Summary

Chapter 3 presented a model verification to ensure that the code developed to use the multiplicative dimensional reduction method was implemented correctly. A simple example was done to compare the results of MDRM, MCS and Cubature methods.

Chapter 4 presented two problems. The first problem was done considering all random variables were not designated to be epistemic or aleatory. This problem was done using MDRM, Cubature and MCS methods. It was determined that Cubature methods are not as accurate and required more trials than MDRM therefore Cubature method was not used after this. The second problem was done considering one random variable was epistemic and another was aleatory. This problem was done using MDRM and MCS and the results were compared.

Chapter 5 presented a fire resistance problem. A performance based approach was explained and used. This problem was done using MDRM and MCS and the results were compared.

Chapter 6 presented four geomechanics problems. The first set two problems were equation based 1D consolidation problems. The first problem was done to show a step by step method of how to use MDRM. The results of MDRM were compared to MCS. The second problem was also done using both MDRM and MCS and the results were compared. The second set of two problems were finite element analyses. The first problem was of vertical drains. For this problem a finite element model was created in ABAQUS and meshed. This mesh was then output into a FORTRAN code provided by Dipanjan Basu. This problem was done using MDRM and MCS and the results were compared. The second problem was of a concrete infinite beam on an elastic foundation. Two foundation models were used (two parameter Pasternak Model and Modified Vlasov Model). Each foundation model was done twice once for a static load and once for a dynamic load.

7.2 Conclusions

The main conclusion drawn from this research is that MDRM is efficient and it achieves accurate results (compared to MCS) in a considerably less number of trials/simulations. In terms of equation based problems MDRM is not worth implementing as MCS can be used and will lead to quicker results. However, most analysis done in engineering is finite element analyses. This is when MDRM provides considerable computational efficiency. Methods such as Cubature formulae were also compared but did not offer the same amount of accuracy and efficiency as MDRM did.

Even when solving an equation based problem an incentive of using MDRM over MCS is that MDRM provides the statistical moments, probability distribution, and if needed, the sensitivity coefficients, which are related to the response of interest. MDRM paired with the maximum entropy principle provides the probability distribution, therefore the probability of failure can be calculated from that. Lastly, MDRM is very easy to be implemented but maybe not as easy for the code to be developed. However, if a finite element analysis is to be done using a reliability method, the time taken to develop a MDRM code is well worth it in order to get quick and precise answers that cannot be had using other reliability methods.

7.3 Recommendations for Future Research

Future research can be extended more into the field of geomechanics and fire resistant design of structures, considering a finite element analysis is done. Both these areas lack the use of reliability analysis due to the high number of random variables and the stigma around traditional reliability methods that tend to be computationally expensive. More problems done in these fields using

MDRM can lead to further use of reliability methods in these fields. Future research can also be done for uncertainty problems with an epistemic variable. This paper covered an equation based problem but a finite element analysis could be researched and that might make MDRM more useful for an uncertainty problem with an epistemic variable.

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