Resolution of Ties in Parametric Quadratic Programming

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

We consider the convex parametric quadratic programming problem when the end of the parametric interval is caused by a multiplicity of possibilities ("ties"). In such cases, there is no clear way for the proper active set to be determined for the parametric analysis to continue. In this thesis, we show that the proper active set may be determined in general by solving a certain non-parametric quadratic programming problem. We simplify the parametric quadratic programming problem with a parameter both in the linear part of the objective function and in the right-hand side of the constraints to a quadratic programming without a parameter. We break the analysis into three parts. We first study the parametric quadratic programming problem with a parameter only in the linear part of the objective function, and then a parameter only in the right-hand side of the constraints. Each of these special cases is transformed into a quadratic programming problem having no parameters. A similar approach is then applied to the parametric quadratic programming problem having a parameter both in the linear part of the objective function and in the right-hand side of the constraints.

Acknowledgments

I would like to express my deep thanks my supervisor and friend, Professor Michael J. Best, for introducing me to this subject, for his excellent guidance on my research and writing, for his continuous encouragement and support without which this work would have hardly been done.

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Finally, to my mon and dad, I dedicate this thesis and degree to you both.

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Chapter 1

Introduction

The general parametric quadratic programming (PQP) problem is

$$\min\{(c + tq)'x + \frac{1}{2}x'Cx \mid Ax \le b + tp\}, \tag{1.1}$$

where c and q are given n-vectors, b and p are given m-vectors, C is a given $(n,\ n)$ symmetric positive semi-definite matrix, $A = [a_1, ..., a_m]'$ is a given $(m,\ n)$ matrix, where a_i is an n-vector, $i = 1, ..., m$, and x is an n-vector whose optimal value is to be determined. Throughout this thesis, prime($'$) denotes transposition. All vectors are column vectors unless primed. For quick reading, the end of a proof will be denoted by a hollow box ($\square$) and the end of an example will be denoted by a diamond ($\diamond$).

The optimality conditions [1] for (1.1) are

$$Ax \le b + tp, \tag{1.2}$$

$$-(c + tq) - Cx = A'u,\ u \ge 0, \tag{1.3}$$

$$u'(Ax - b - tp) = 0. \tag{1.4}$$

We refer to (1.2), (1.3) and (1.4) as primal feasibility, dual feasibility and complementary slackness, respectively. For a convex parametric QP problem, these conditions are necessary and sufficient for optimality.

It is known [1] that both the optimal solution and the associated multiplier vector for (1.1) are piecewise linear functions of t in a finite set of intervals $t_0 \leq t \leq t_1,\ t_1 \leq t \leq t_2, ...,\ t_{v-1} \leq t \leq t_v$, and $t_0 < t_1 < t_2 < ... < t_v$. Each interval corresponds to a different set of the active constraints. At the end of each interval, either some previously inactive constraints become active, or some previously active constraints become inactive, or both. Each t_i corresponds to a "corner" point. The optimal solution and the associated multiplier vector are of the form

$$x(t) = \begin{cases} h_{10} + th_{20} & 0 \leq t \leq t_1 \\ h_{11} + th_{21} & t_1 \leq t \leq t_2 \\ ... & ... \\ h_{1j} + th_{2j} & t_j \leq t \leq t_{j+1} \\ ... & ... \\ h_{1,v-1} + th_{2,v-1} & t_{v-1} \leq t \leq t_v, \end{cases}$$

and

$$u(t) = \begin{cases} u_{10} + tu_{20} & 0 \leq t \leq t_1 \\ u_{11} + tu_{21} & t_1 \leq t \leq t_2 \\ ... & ... \\ u_{1j} + tu_{2j} & t_j \leq t \leq t_{j+1} \\ ... & ... \\ u_{1,v-1} + tu_{2,v-1} & t_{v-1} \leq t \leq t_v, \end{cases}$$

where h_{1j} and h_{2j} $(j = 0, ..., v-1)$ are n-vectors, u_{1j} and u_{2j} $(j = 0, ..., v-1)$ are m-vectors.

In each interval j, $x(t)$ and $u(t)$ must satisfy the primal and dual feasibility. The first restriction that $x(t)$ is feasible implies $t_{j+1} \leq \hat{t}_{j+1}$, where

$$\begin{aligned} \hat{t}_{j+1} &= \min\{\frac{b_i - a_i' h_{1j}}{a_i' h_{2j} - p_i} \mid \text{all } i = 1, ..., m \text{ with } a_i' h_{2j} > p_i\}, \\ &= \frac{b_l - a_l' h_{1j}}{a_l' h_{2j} - p_l}. \end{aligned} \tag{1.5}$$

The second restriction that $u(t)$ is non-negative implies $t_{j+1} \leq \tilde{t}_{j+1}$, where

$$\begin{aligned} \tilde{t}_{j+1} &= \min\{\frac{-(u_{1j})_i}{(u_{2j})_i} \mid \text{all } i = 1, ..., m \text{ with } (u_{2j})_i < 0\}, \\ &= -\frac{(u_{1j})_k}{(u_{2j})_k}. \end{aligned} \tag{1.6}$$

We have used $(u_{1j})_i$ to denote the i-th component of u_{1j}, and $(u_{2j})_i$ to denote the i-th component of u_{2j}. We use the convention $\hat{t}_{j+1} = +\infty$ to mean that $a_i' h_{2j} \leq p_i$ for all $i = 1, ..., m$. These two restrictions above give the upperbound of the interval; *i.e.*, $t_{j+1} = \min\{\hat{t}_{j+1},\ \tilde{t}_{j+1}\}$.

Definition 1.1

(a) The problem (1.1) has **primal ties** *at the corner point t_{j+1}, if $\hat{t}_{j+1} < \tilde{t}_{j+1}$ and the minimum in (1.5) is obtained for at least two distinct indices.*

(b) The problem (1.1) has **dual ties** *at the corner point t_{j+1}, if $\tilde{t}_{j+1} < \hat{t}_{j+1}$ and the minimum in (1.6) is obtained for at least two distinct indices.*

(c) The problem (1.1) has **primal-dual ties** *at the corner point t_{j+1} if $\hat{t}_{j+1} = \tilde{t}_{j+1}$.*

(d) The problem has **ties** *at the corner point t_{j+1} if it has primal ties, dual ties or primal-dual ties.*

If there are no ties at all the corner points, the PQP problem (1.1) can be solved by using Best's method [2].

For the remainder of this chapter, we will present some basic properties of the PQP plus a number of examples which illustrate the types of problems which can arise when ties do occur. We begin with an example of a PQP which has no ties at the corner point.

Example 1.1

$$\begin{array}{llcll} \text{minimize :} & (-2+t)x_1 - 2x_2 + x_1^2 + \frac{1}{2}x_2^2 & & & \\ \text{subject to :} & x_1 & \leq & 1, & (1) \\ & \quad\quad x_2 & \leq & 1, & (2) \\ & x_1 & \geq & 0, & (3) \\ & \quad\quad x_2 & \geq & 0. & (4) \end{array}$$

For every t with $t \leq 0$, the optimal solution is $x(t) = (1,\ 1)'$. There are no ties at $t = 0$. The first two constraints are active at $t = 0$. The first constraint becomes inactive and the second constraint remains active when t increases a small amount from 0. The optimal solution is

$$x(t) = \begin{bmatrix} 1 - \frac{1}{2}t \\ 1 \end{bmatrix},$$

for every t with $0 \leq t \leq 2$.

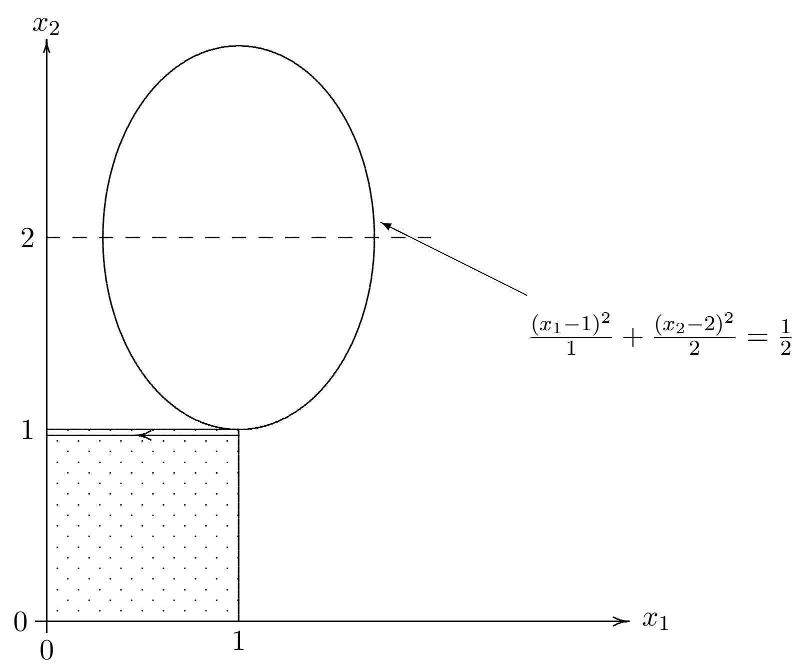

Figure 1.1: An example of no ties with t in the linear part of the objective function.

The geometry of this example is illustrated in Figure 1.1, where the feasible region is shaded.

The objective function is an ellipse, its center moves along the line $x_2 = 2$ from the point $(1,2)'$ to the point $(0,2)'$, as t increases from 0 to 2. The optimal solution moves along the line $x_2 = 1$ from the point $(1,1)'$ to the point $(0,1)'$. For every t with $t \geq 2$, the optimal solution is $(0,1)'$. ◇

Best [2] solves the PQP problem under the assumption that ties do not occur at the corner points. He gives an algorithm which requires solving linear equations with the coefficient matrix

$$H_j = \begin{bmatrix} C & A_j' \\ A_j & 0 \end{bmatrix},$$

where A_j' is the matrix of gradients of all the constraints active at iteration j.

Best proves one of the properties of $H(A)$ as follows [2]:
Let $H(A) = \begin{bmatrix} C & A' \\ A & 0 \end{bmatrix}$, where A is any $(m,\, n)$ matrix. Suppose A has full row rank. Then $H(A)$ is nonsingular only if $s'Cs > 0$ for all non-zero s such that $As = 0$.

Suppose (1.1) has an optimal solution $x(t_j)$ for $t = t_j$. Suppose A_j' is the matrix of gradients of all the constraints active at $x(t_j)$; b_j and p_j are vectors whose components are associated with the rows of A_j, respectively. Assume A_j has full row rank and $s'Cs > 0$ for all $s \neq 0$ with $A_j s = 0$. Then, $H_j = \begin{bmatrix} C & A_j' \\ A_j & 0 \end{bmatrix}$ is non-singular. The optimality conditions assert that the optimal solution $x(t_j)$ and the associated multiplier vector $v(t_j)$ are uniquely determined by the linear equations

$$H_j \begin{bmatrix} x(t_j) \\ v(t_j) \end{bmatrix} = \begin{bmatrix} -c \\ b_j \end{bmatrix} + t_j \begin{bmatrix} -q \\ p_j \end{bmatrix}.$$

The full (m-dimensional) vector of multipliers, $u(t_j)$, is obtained from $v(t_j)$ by assigning zero to those components of $u(t_j)$ associated with constraints inactive at $x(t_j)$, and the appropriately indexed components of $v(t_j)$, otherwise.

Now suppose t increases from t_j. Let $x(t)$ denote the optimal solution and $v(t)$ denote the multiplier vector whose components are associated with the active constraints as functions of the parameter t. Provided there are no changes in the active set, $x(t)$ and $v(t)$ are uniquely determined by the linear equations

$$H_j \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} -c \\ b_j \end{bmatrix} + t \begin{bmatrix} -q \\ p_j \end{bmatrix}.$$

The solution can conveniently be obtained by solving two sets of linear equations:

$$H_j \begin{bmatrix} h_{1j} \\ v_{1j} \end{bmatrix} = \begin{bmatrix} -c \\ b_j \end{bmatrix}, \tag{1.7}$$

$$H_j \begin{bmatrix} h_{2j} \\ v_{2j} \end{bmatrix} = \begin{bmatrix} -q \\ p_j \end{bmatrix}, \tag{1.8}$$

which both have the coefficient matrix H_j. Having solved these for h_{1j}, h_{2j}, v_{1j} and v_{2j}, the optimal solution for (1.1) is

$$x(t) = h_{1j} + th_{2j}, \tag{1.9}$$

and the associated multiplier vector is

$$v(t) = v_{1j} + tv_{2j}. \tag{1.10}$$

The full vector of multipliers $u(t)$ may be obtained from $v(t)$ and the set of the constraints inactive at $x(t_j)$. We write $u(t)$ as

$$u(t) = u_{1j} + tu_{2j}. \tag{1.11}$$

We set $t_{j+1} = \min\{\hat{t}_{j+1},\ \tilde{t}_{j+1}\}$, where

$$\begin{aligned} \hat{t}_{j+1} &= \min\{\frac{b_i - a_i' h_{1j}}{a_i' h_{2j} - p_i} \mid \text{all } i = 1, ..., m \text{ with } a_i' h_{2j} > p_i\}, \\ &= \frac{b_l - a_l' h_{1j}}{a_l' h_{2j} - p_l}, \end{aligned} \tag{1.12}$$

$$\begin{aligned} \tilde{t}_{j+1} &= \min\{\frac{-(u_{1j})_i}{(u_{2j})_i} \mid \text{all } i = 1, ..., m \text{ with } (u_{2j})_i < 0\}, \\ &= -\frac{(u_{1j})_k}{(u_{2j})_k}. \end{aligned} \tag{1.13}$$

Then the optimal solution is given by (1.9) and the associated multiplier vector is given by (1.11), for every t with $t_j \le t \le t_{j+1}$.

The algorithm leaves a specified method by which the linear equations (1.7) and (1.8) are solved. Possible ways of solving them are factorizations of H_j, submatrices of H_j, or partitions of H_j^{-1}. It shows updating formulae for the new factors when A_j is modified by the addition, deletion, or exchange of a row. At the end of each parametric interval, the active set changes by either adding, deleting or exchanging a constraint, with the assumption that no ties occur. The method terminates when either the optimal solution has been obtained for all values of the parameter, or, a further increase in the parameter results in either the feasible region being null or the objective function being unbounded from below. It uses the linear equation solving method associated with a particular quadratic programming algorithm to provide a natural extension of that method for the solution of the PQP problem (1.1).

Ritter [2] gives a more general method for the PQP problem with ties. In his method, he solves the similar linear equations with the same coefficient matrix H_j in each iteration j. If at a corner point t_j, there are ties, then the method chooses an $\varepsilon > 0$ sufficiently small, and solves the problem from $t = t_j + \varepsilon \le t_{j+1}$, which has no ties. The difficulty of this approach is that t_{j+1} is not known before we determined the optimal solution for the interval j. Therefore we do not know how small to make ε such that $t \le t_{j+1}$.

Perold [4] does not consider the possibility of "ties" when describing a parametric algorithm for large-scale mean-variance portfolio optimization problems.

In practice, PQP problems can be quite large. For example, portfolio optimization problems may have many thousands of variables. There may exist ties at the corner points. If there are ties, it is not easy to decide which constraints become active and which constraints become inactive in the next interval of t.

Suppose $t = t_j$ is a corner point, and assume that there are ties at t_j. There may be many subsets of linearly independent gradients of the active constraints, and it is hard to find which subset will remain active when t increases a small amount from t_j.

Arseneau [5] develops an PQP algorithm in which "ties" may occur. The algorithm solves a convex quadratic programming problem where both the objective function and the constraints involve a small and positive scalar ε if the corner point t_j has "ties". However, ε is only used to symbolically determine the current active set. A numerical value for ε is never used. The algorithm is modified from algorithm in [2]. This modified QP algorithm give an efficient way to solve the PQP problem with "ties", however, it is complex and needs several assumptions.

Berkelaar, Roos and Terlaky [8] introduce an algorithm for solving a parametric QP with the perturbation either in the linear part of the objective function or in the right-hand side of the constraints. They use the optimal set and optimal partition approach to solve the problem when degeneracy occurs. It is an algorithm using primal and dual optimal solutions. Terlaky and his students Hadigheh, Romanko [9] extended the algorithm for solving the convex quadratic optimization with the perturbation with in the linear part of the objective function and in the right-hand side of the constraints.

The convex parametric quadratic programming problem can also be solved by a different method of solving a parametric LCP (linear complementarity programming) problem [10]. In order to solve the parametric PQP problem of the following form:

$$\begin{array}{rrcl} \text{minimize :} & \multicolumn{3}{l}{(c+\lambda c^*)'x + \frac{1}{2}x'Dx} \\ \text{subject to :} \quad A & x & \geq & b+\lambda b^*, \\ & x & \geq & 0, \end{array}$$

where D is a symmetric positive semi-definite matrix and λ is the parameter, we can solve a parametric LCP problem

$$\begin{array}{lcl} w - Mz & = & q + \lambda q^*, \\ w, z & \geq & 0, \end{array}$$

where

$$M = \begin{bmatrix} D & -A' \\ A & 0 \end{bmatrix}, \; q = \begin{bmatrix} c \\ -b \end{bmatrix}, \; q^* = \begin{bmatrix} c^* \\ -b^* \end{bmatrix}.$$

Since M is a positive semi-definite matrix, this parametric LCP can be solved by the algorithm given in Murty's book [10].

The contribution of this thesis is to provide solutions for (1.1) in the presence of ties, by simplifying the parametric QP problem into a related QP problem without the parameter. The results depend on a number of special cases which will be analyzed separately. In the rest of this chapter, we will give a series of numerical examples which will illustrate the nature of the problem.

The following is an example of a PQP problem having a tie at the corner point with the parameter t only in the linear part of the objective function. The problem will be solved in Section 2.1 by the method proposed in this thesis.

Example 1.2

$$\begin{array}{rrcrcl}
\text{minimize :} & \multicolumn{5}{l}{-\frac{10}{3}x_1 + (-\frac{8}{3}+t)x_2 + \frac{1}{2}x_1^2 + x_2^2} \\
\text{subject to :} & x_1 & + & 2x_2 & \leq & 2, \quad (1) \\
 & 2x_1 & + & x_2 & \leq & 2, \quad (2) \\
 & x_1 & + & x_2 & \leq & \frac{4}{3}, \quad (3) \\
 & x_1 & & & \geq & 0, \quad (4) \\
 & & & x_2 & \geq & 0. \quad (5)
\end{array}$$

When $-\frac{4}{3} \leq t \leq 0$, the optimal solution is $x_0 = (\frac{2}{3},\ \frac{2}{3})'$, the first three constraints are active at x_0 and their gradients are linearly dependent. The multiplier vector for the first three constraints is

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -t \\ \frac{4}{3}+t \\ -t \end{bmatrix},$$

for every t with $-\frac{4}{3} \leq t \leq 0$. When t increases from negative to zero, u_1 and u_3 become zero simultaneously, therefore, there are dual ties at the corner point $t = 0$. When t increases a small amount from zero, both the first and the third constraints become inactive and only the second constraint remains active. The optimal solution is

$$x(t) = \begin{bmatrix} \frac{2}{3} + \frac{2}{9}t \\ \frac{2}{3} - \frac{4}{9}t \end{bmatrix},$$

for every t with $0 \leq t \leq \frac{3}{2}$.

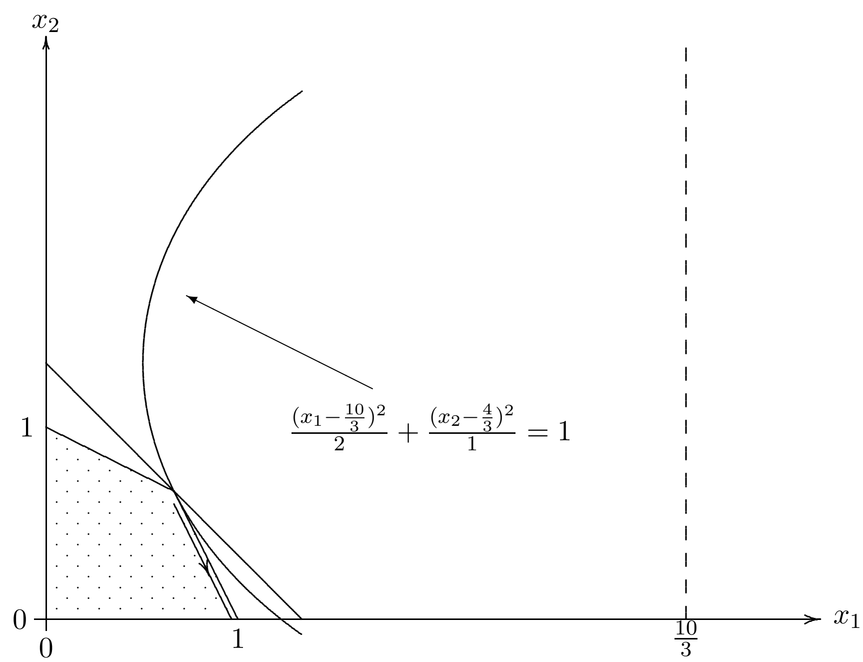

Figure 1.2: An example of a tie with t in the linear part of the objective function.

The geometry of this example is illustrated in Figure 1.2, where the feasible region is shaded. The objective function is an ellipse, its center moves down the line $x_1 = \frac{10}{3}$ from the point $(\frac{10}{3}, \frac{4}{3})'$, as t increases from 0. The optimal solution moves along the line $2x_1 + x_2 = 2$ from the point $(\frac{2}{3}, \frac{2}{3})'$ to the point $(1, 0)'$ as t increases from 0 to $\frac{3}{2}$. For every t with $t \geq \frac{3}{2}$, the optimal solution is $(1, 0)'$, and is a constant. ◇

The following example is a special case of several constraints becoming active simultaneously (ties). This problem will be solved in Section 2.3 by the method proposed in this thesis.

Example 1.3

$$\min\{tq'x \mid Ax \le b\}. \tag{1.14}$$

Assume the feasible region for (1.14) is non-null. The geometry of this type of problem is illustrated in Figure 1.3, where the feasible region is shaded. When $t = 0$, (1.14) becomes

$$\min\{0 \mid Ax \le b\}.$$

The phenomenon being illustrated here is alternate optimal solutions. When $t = 0$, any feasible solution is also optimal. Let y^- denote the optimal solution for (1.14) when $t < 0$, and let y^+ denote the optimal solution for (1.14) when $t > 0$. Let x_0 be an interior point in the feasible region, then x_0 is optimal for (1.14) for $t = 0$.

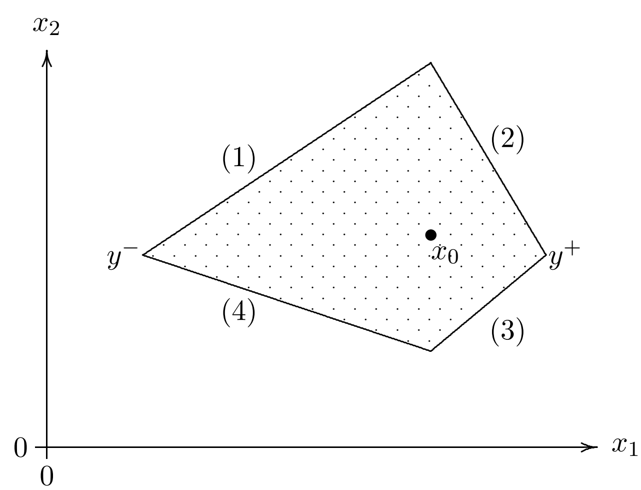

Figure 1.3: A special case of parametric programming problem.

In this example, when $t < 0$, constraints (1) and (4) in Figure 1.3 are active at the optimal solution y^-; when $t = 0$, there are no constraints active at the optimal solution x_0; when $t > 0$, constraints (2) and (3) are active at the optimal solution y^+. There are ties at the corner point $t = 0$ because when t increases from negative to zero, the multipliers corresponding to constraints (1) and (4) become zero simultaneously. This example also illustrates that x_0 and y^+ cannot be connected with a linear function of t. ◇

The following is an example of a PQP problem having a tie at a corner point with the parameter t only in the right-hand side of the constraints. The problem will be solved in Section 3.1 by the method proposed in this thesis.

Example 1.4

$$\begin{array}{lrcrcll} \text{minimize :} & -2x_1 & - & 2x_2 & + & \frac{1}{2}x_1^2 + x_2^2 & \\ \text{subject to :} & x_1 & & & \leq & 1, & (1) \\ & & & x_2 & \leq & 1, & (2) \\ & x_1 & + & x_2 & \leq & 2-t, & (3) \\ & x_1 & + & 2x_2 & \leq & 3-\frac{1}{2}t, & (4) \\ & x_1 & & & \geq & 0, & (5) \\ & & & x_2 & \geq & 0. & (6) \end{array}$$

When $t \leq 0$, the optimal solution is $x(t) = (1,\ 1)'$. The first two constraints are active at x_0 when $t < 0$. The third and the fourth constraints become active simultaneously when $t = 0$, so there are primal ties at $t = 0$. When t increases a small amount from zero, the second and the fourth constraints become inactive, and both the first and the third constraints remain active. The optimal solution is

$$x(t) = \begin{bmatrix} 1 \\ 1-t \end{bmatrix},$$

for every t with $0 \leq t \leq \frac{1}{2}$.

The geometry of this example is illustrated in Figure 1.4, where the feasible region for $t = 0$ is shaded. The objective function is an ellipse. The optimal solution moves along the line $x_1 = 1$ from the point $(1, 1)'$ to the point $(1, \frac{1}{2})'$, as t increases from 0 to $\frac{1}{2}$.

For every t with $\frac{1}{2} \leq t \leq 2$, the optimal solution is $x(t) = \begin{bmatrix} \frac{2}{3}(2-t) \\ \frac{1}{3}(2-t) \end{bmatrix}$. For every t with $t > 2$, the problem is infeasible. ◇

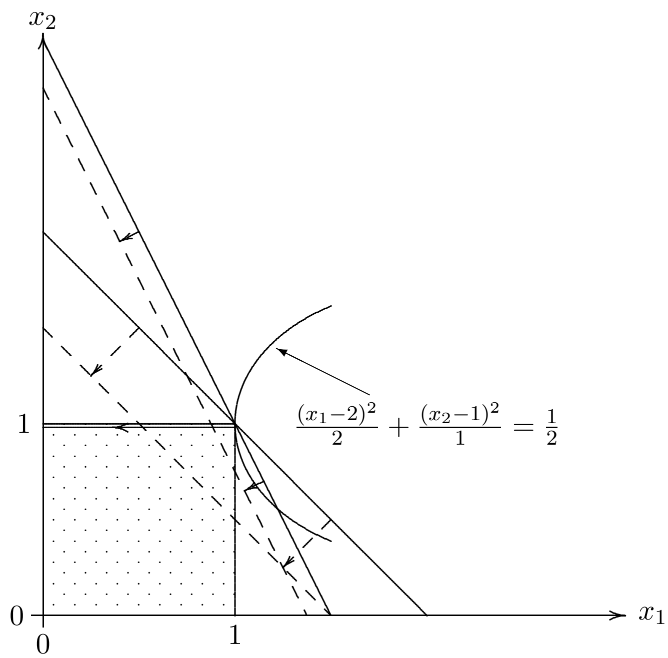

Figure 1.4: An example of a tie with t in the right-hand side of the constraints.

The following is an example of a PQP problem having a tie at a corner point with the parameter t both in the linear part of the objective function and in the right-hand side of the constraints. The problem will be solved in Section 4.1 by the method proposed in this thesis.

Example 1.5

$$
\begin{array}{lrcrcll}
\text{minimize}: & -2x_1 + (-2+t)x_2 + \frac{1}{2}x_1^2 + x_2^2 & & & & & \\
\text{subject to}: & x_1 & & & \leq & 1-t, & (1) \\
 & & & x_2 & \leq & 1, & (2) \\
 & x_1 & + & x_2 & \leq & 2-t, & (3) \\
 & x_1 & + & 2x_2 & \leq & 3-\frac{1}{2}t, & (4) \\
 & x_1 & & & \geq & 0, & (5) \\
 & & & x_2 & \geq & 0. & (6)
\end{array}
$$

For every t with $-\frac{1}{5} \leq t \leq 0$, the optimal solution is $x(t) = (1-t,\ 1+\frac{1}{4}t)'$. The first and the fourth constraints are active when $-\frac{1}{5} < t \leq 0$. The second and the third constraints become active simultaneously when $t = 0$, so there are primal ties at $t = 0$. When t increases a small amount from zero, the second, the third and the fourth constraints all become inactive and only the first constraint remains active. The optimal solution is

$$x(t) = \begin{bmatrix} 1-t \\ 1-\frac{1}{2}t \end{bmatrix},$$

for every t with $0 \leq t \leq 1$.

The geometry of this example is illustrated in Figure 1.5, where the feasible region for $t = 0$ is shaded. The objective function is an ellipse, its center moves down along the line $x_1 = 2$ from the point $(2,1)'$, as t increases from zero. The optimal solution moves along the line $x_1 - 2x_2 = -1$ from the point $(1,1)'$ to the point $(0, \frac{1}{2})'$, as t increases from 0 to 1. When $t > 1$, the problem is infeasible. ◇

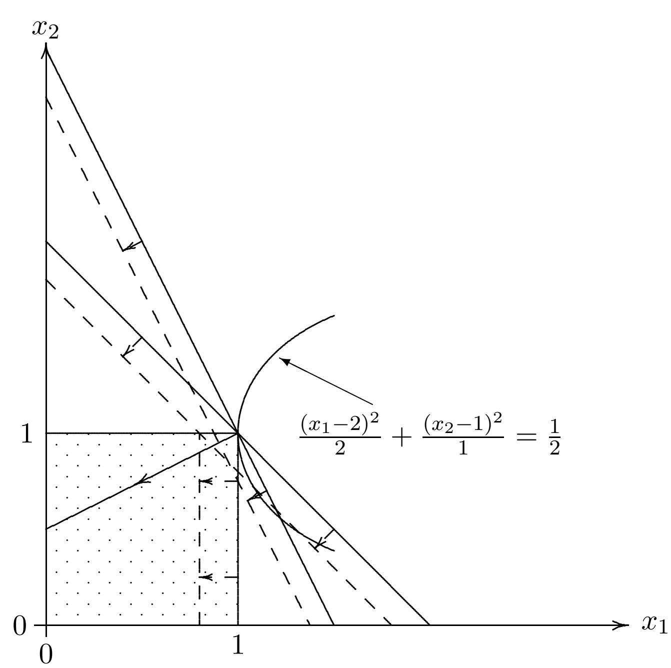

Figure 1.5: An example of a tie with t both in linear part of the objective function and in the right-hand side of the constraints.

In the following part, some definitions, notations and lemmas will be introduced. They will be used in the later chapters.

Notation 1.1 *Let x_0 be an optimal solution and u_0 be an associated multiplier vector for (1.1) for $t = 0$. Let A_0' be the matrix of gradients of all the constraints active at x_0, let b_0 be the vector whose components are those b_i associated with the rows of A_0; i.e., $A_0 x_0 = b_0$.*

Definition 1.2 *The optimal solution for (1.1) for some $t > 0$, $x(t)$, is a* **diminishment** *of x_0 if the set of the constraints active at $x(t)$ is a subset of or equals to the set of those constraints active at x_0.*

The following result shows that the optimal solution and the associated multiplier vector for a general parametric QP problem are linear functions of the parameter provided that the active constraints remain unchanged.

Lemma 1.1 *Let x_1 and x_2 be optimal solutions for (1.1) for $t = t_1$ and $t = t_2$, respectively, and suppose $t_1 < t_2$. Let u_1 and u_2 be associated multiplier vectors for x_1 and x_2, respectively. Assume that x_1 and x_2 have the same active constraints. Let A_0' be the matrix of gradients of all the constraints active at x_1, let b_0 and p_0 be the vectors whose components are those b_i and p_i associated with the rows of A_0, respectively. So, $A_0 x_1 = b_0 + t_1 p_0$, $A_0 x_2 = b_0 + t_2 p_0$. Then*

$$x^*(t) = x_1 + \frac{x_2 - x_1}{t_2 - t_1}(t - t_1)$$

is an optimal solution for (1.1) with an associated multiplier vector

$$u^*(t) = u_1 + \frac{u_2 - u_1}{t_2 - t_1}(t - t_1),$$

for every t with $t_1 \le t \le t_2$.

Proof

Since u_1 and u_2 are multiplier vectors, we have $u_1 \geq 0,\ u_2 \geq 0$. It is easy to see that $u^*(t) \geq 0$ for every t with $t_1 \leq t \leq t_2$. Let A_1' be the matrix of gradients of all the constraints inactive at x_1, and let b_1 and p_1 be the vectors whose components are those b_i and p_i associated with the rows of A_1. Then $A_1x_1 < b_1 + t_1p_1,\ A_1x_2 < b_1 + t_2p_1$. We have

$$A_0x^*(t) = A_0x_1 + A_0(x_2 - x_1)\frac{t - t_1}{t_2 - t_1} = A_0x_1\frac{t_2 - t}{t_2 - t_1} + A_0x_2\frac{t - t_1}{t_2 - t_1} = b_0 + tp_0, \tag{1.15}$$

$$A_1x^*(t) = A_1x_1 + A_1(x_2 - x_1)\frac{t - t_1}{t_2 - t_1} = A_1x_1\frac{t_2 - t}{t_2 - t_1} + A_1x_2\frac{t - t_1}{t_2 - t_1} < b_1 + tp_1. \tag{1.16}$$

So A_0' is the matrix of gradients of all the constraints active at $x^*(t)$, for every t with $t_1 \leq t \leq t_2$. For $t = t_1$ and $t = t_2$, the optimality conditions for (1.1) assert that

$$-c - t_1q - Cx_1 = A_0'u_1, \tag{1.17}$$

$$-c - t_2q - Cx_2 = A_0'u_2. \tag{1.18}$$

Subtract (1.17) from (1.18), then multiply both sides by $\frac{t-t_1}{t_2-t_1}$,

$$-\ (t - t_1)q - \frac{t - t_1}{t_2 - t_1}C(x_2 - x_1) = \frac{t - t_1}{t_2 - t_1}A_0'(u_2 - u_1). \tag{1.19}$$

Add (1.17) to (1.19),

$$-c - tq - Cx_1 - \frac{t - t_1}{t_2 - t_1}C(x_2 - x_1) = A_0'u_1 + \frac{t - t_1}{t_2 - t_1}A_0'(u_2 - u_1).$$

That is,

$$-c - tq - Cx^*(t) = A_0'u^*(t).$$

From (1.15) and (1.16),

$$Ax^*(t) \leq b + tp,$$

for every t with $t_1 \leq t \leq t_2$. Thus,

$$\left.\begin{array}{l} Ax^*(t) \leq b + tp, \\ -c - tq - Cx^*(t) = A_0'u^*(t),\ u^*(t) \geq 0, \\ A_0x^*(t) = b_0 + tp_0. \end{array}\right\}$$

So, $x^*(t)$ and $u^*(t)$ satisfy the optimality conditions for (1.1). Therefore, $x^*(t)$ is an optimal solution for (1.1) and $u^*(t)$ is an associated multiplier vector, for every t with $t_1 \le t \le t_2$. □

From Lemma 1.1, we know that if the active constraints for the optimal solutions for some t_1 and t_2 are coincident, the optimal solution is a linear function of t, for every t with $t_1 \le t \le t_2$. In fact, when t changes a small amount, the active constraints of the optimal solutions may not remain coincident. When t increases, sometimes there are originally inactive constraints becoming active; sometimes all the inactive constraints remains inactive and there may be some active constraints becoming inactive. In this chapter, we mainly study the latter case: there are no inactive constraints becoming active, but there may be some active constraints becoming inactive. In this case, the optimal solution (when t only increases a small amount) is also a linear function of t.

Instead of studying t from t_j to t_{j+1}, in this thesis, we always let t begin from $t_0 = 0$. We can do that because at each interval $t_j \le t \le t_{j+1}$, we can let $t = t - t_j$, then t begins from zero.

For t beginning at 0, we have an n-vector h_0 such that $x(t) = x_0 + th_0$ is an optimal solution for (1.1), for every t with $0 \le t \le \bar{t}$, for some $\bar{t} > 0$.

Lemma 1.2 *Let $x(t) = x_0 + th_0$ be an optimal solution for (1.1) and $u(t) = \bar{u}_0 + tu_1$ be an associated multiplier vector, for every t with $0 < t \le \bar{t}$, where $\bar{t}$ is some positive number. Suppose that $x(t)$ is a diminishment of x_0, for every t with $0 < t < \bar{t}$. Then $\bar{u}_0$ is an associated multiplier vector for x_0 for (1.1).*

Proof

Let t^0 satisfy $0 < t^0 < \bar{t}$.

When $t = t^0$, $x(t^0) = x_0 + t^0 h_0$, $u(t^0) = \bar{u}_0 + t^0 u_1$, from the optimality conditions,

$$-c - t^0 q - Cx_0 - t^0 Ch_0 = A_0'\bar{u}_0 + t^0 A_0' u_1. \tag{1.20}$$

When $t = \frac{t^0}{2} < \bar{t}$, $x(\frac{t^0}{2}) = x_0 + \frac{t^0}{2}h_0$, $u(\frac{t^0}{2}) = \bar{u}_0 + \frac{t^0}{2}u_1$, from the optimality conditions,

$$-c - \frac{t^0}{2}q - Cx_0 - \frac{t^0}{2}Ch_0 = A_0'\bar{u}_0 + \frac{t^0}{2}A_0'u_1. \quad (1.21)$$

Multiply (1.21) by 2,

$$-2c - t^0 q - 2Cx_0 - t^0 Ch_0 = 2A_0'\bar{u}_0 + t^0 A_0'u_1. \quad (1.22)$$

Subtracting (1.20) from (1.22) gives

$$-c - Cx_0 = A_0'\bar{u}_0.$$

Since $u(t) = \bar{u}_0 + tu_1 \geq 0$, for every t with $0 < t \leq \bar{t}$, we have $\bar{u}_0 \geq 0$. Thus,

$$\left.\begin{aligned} &Ax_0 \leq b, \\ &-c - Cx_0 = A_0'\bar{u}_0,\ \ \bar{u}_0 \geq 0, \\ &A_0x_0 = b_0. \end{aligned}\right\}$$

Thus, $\bar{u}_0$ is an associated multiplier vector for x_0 for (1.1) as required. □

From the above analysis, we can introduce the following notation.

Notation 1.2 *Let $x(t) = x_0 + th_0$ denote an optimal solution for (1.1) with an associated multiplier vector $u(t) = u_0 + tu_1$, for every t with $0 \leq t \leq \bar{t}$, where x_0 is an optimal solution for (1.1) for $t_0 = 0$, and u_0 is an associated multiplier vector for x_0.*

Lemma 1.3 *Assume that (1.1) has optimal solutions, for every t with $0 \leq t \leq \hat{t}$. There exists an optimal solution $x(t) = x_0 + th_0$ for (1.1) being a diminishment of x_0, for every t with $0 < t < \bar{t}$, where $0 < \bar{t} \leq \hat{t}$.*

Proof

Let H be the set of all the indices of the gradients of the constraints inactive at x_0. For every

$i \in H$, $a_i'x_0 < b_i$. If $a_i'h_0 \le p_i$, for every t with $0 \le t < \hat{t}$,

$$a_i'x(t) = a_i'(x_0 + th_0) = a_i'x_0 + ta_i'h_0 < b_i + tp_i.$$

If $a_i'h_0 > p_i$, let $c_i = b_i - a_i'x_0 > 0$, there always exists a $\bar{t}_i > 0$ such that $\bar{t}_i(a_i'h_0 - p_i) \le c_i$, then for every t with $0 \le t < \min\{\bar{t}_i,\ \hat{t}\}$,

$$a_i'x(t) = a_i'(x_0 + th_0) < a_i'x_0 + \bar{t}_i a_i'h_0 = b_i - c_i + ta_i'h_0 \le b_i + tp_i.$$

Let $\bar{t} = \min\{\hat{t},\ \min\{\bar{t}_i \mid a_i'h_0 > 0\}\} > 0$. Then for every t with $0 \le t < \bar{t}$ and every $i \in H$, we have $a_i'x(t) < b_i + tp_i$, and this completes the proof. □

Definition 1.3 *We call $(h_0,\ \bar{t})$ an* **optimal continuation** *of x_0 for (1.1), where $\bar{t} > 0$, if $x(t) = x_0 + th_0$ is optimal for (1.1) with an associated multiplier vector $u(t) = u_0 + tu_1$ for every t with $0 \le t < \bar{t}$.*

Remark 1.1 *An optimal continuation depends on a specified active set. For some active sets, an optimal continuation may not exist.*

From Best's algorithm, $\bar{t} = \min\{\hat{t},\ \tilde{t}\}$, where

$$\begin{aligned} \hat{t} &= \min\{\frac{b_i - a_i'x_0}{a_i'h_0 - p_i} \mid \text{all } i = 1, ..., m \text{ with } a_i'h_0 > p_i\}, \\ &= \frac{b_l - a_l'x_0}{a_l'h_0 - p_l}, \qquad (1.23) \\ \tilde{t} &= \min\{\frac{-(u_0)_i}{(u_1)_i} \mid \text{all } i = 1, ..., m \text{ with } (u_1)_i < 0\}, \\ &= -\frac{(u_0)_k}{(u_1)_k}. \qquad (1.24) \end{aligned}$$

Remark 1.2 *For $t = \bar{t}$, $x(\bar{t}) = x_0 + \bar{t}h_0$ is also an optimal solution for (1.1). However, we do not consider it in the optimal continuation because $x(\bar{t})$ is not a diminishment of x_0. In the proofs of this thesis, we mainly base on the "diminishment".*

Many proofs in the next three chapters involve the following property of the convex quadratic programming problem.

Lemma 1.4 *If the convex quadratic programming problem*

$$\min\{c'x + \frac{1}{2}x'Cx \mid Ax \leq b\} \tag{1.25}$$

is unbounded from below, then for any feasible solution x_1 for (1.25), there exists a vector s such that $x_1 - \sigma s$ is feasible for (1.25), for every positive scalar σ, and $s'Cs = 0$, $c's > 0$.

Proof

Since (1.25) is unbounded from below, for any feasible solution x_1, there exists a vector s such that $x_1 - \sigma s$ is also feasible for (1.25), for every positive scalar σ, and

$$c'(x_1 - \sigma s) + \frac{1}{2}(x_1 - \sigma s)'C(x_1 - \sigma s) \rightarrow -\infty, \text{ as } \sigma \rightarrow +\infty.$$

The objective function for (1.25) for $x_1 - \sigma s$ is

$$c'(x_1 - \sigma s) + \frac{1}{2}(x_1 - \sigma s)'C(x_1 - \sigma s) = c'x_1 + \frac{1}{2}x_1'Cx_1 - \sigma c's - \sigma x_1'Cs + \frac{1}{2}\sigma^2 s'Cs. \tag{1.26}$$

If $s'Cs \neq 0$, then $s'Cs > 0$, since C is positive semidefinite. Then (1.26) is bounded from below, for $\sigma > 0$, which is a contradiction. Therefore, $s'Cs = 0$, and this implies $Cs = 0$ since C is positive semidefinite. So the right-hand side of (1.26) becomes

$$c'x_1 + \frac{1}{2}x_1'Cx_1 - \sigma c's.$$

Since this must decrease to negative infinity as σ increases to positive infinity. Thus, $c's > 0$, and it completes the proof. □

Chapter 2

A Parameter only in the Objective Function

Before studying (1.1), we analyze a simpler case in which the parameter is only in the linear part of the objective function. We solve it by solving a related QP problem which has no parameter.

2.1 Solution of the PQP Problem with a Parameter in the Linear Part of the Objective Function by Solving a Related QP Problem without the Parameter

Consider the following PQP problem

$$\min\{(c+tq)'x + \frac{1}{2}x'Cx \mid Ax \leq b\}. \tag{2.1}$$

Assumption 2.1 *There exists a $\hat{t} > 0$ such that (2.1) has an optimal solution for every t with*

$0 \le t < \hat{t}$.

Before introducing the theorems, we need the following lemma first.

Lemma 2.1 *If there is a* $t_1 > 0$ *such that (2.1) has no optimal solution, for every* t *with* $0 < t \le t_1$, *then (2.1) is unbounded from below, for every* t *with* $t > 0$.

Proof

Since the feasible region of (2.1) doesn't change as t changes, (2.1) is always feasible. Since (2.1) has no optimal solution for every t with $0 < t \le t_1$, (2.1) is unbounded from below for every t with $0 < t \le t_1$. Let $f_t(x)$ denote the objective function of (2.1) with the subscript t denoting explicit dependence on t. Then for a feasible solution x_1 for (2.1), there exists an n-vector s such that $x_1 - \sigma s$ is feasible for (2.1), for every positive scalar σ, and the objective function $f_t(x)$ satisfies

$$f_t(x_1 - \sigma s) \to -\infty, \text{ as } \sigma \to +\infty,$$

for every t with $0 < t \le t_1$. Therefore, s satisfies $(c + t_1 q)'s > 0$ and $s'Cs = 0$. Combining these with the feasibility restriction, s needs to satisfy

$$As \ge 0, \; s'Cs = 0, \; t_1 q's > -c's.$$

Since (2.1) has an optimal solution for $t = 0$, we have $c's \le 0$. So, $t_1 q's > -c's \ge 0$. Then for every t with $t > t_1$, we have $A(x_1 - \sigma s) \le b$, $Cs = 0$ and $(c + tq)'s > 0$, for every positive scalar σ. Therefore,

$$(c+tq)'(x_1 - \sigma s) + \frac{1}{2}(x_1 - \sigma s)'C(x_1 - \sigma s) = (c+tq)'x_1 + \frac{1}{2}x_1'Cx_1 - \sigma(c+tq)'s \to -\infty, \text{ as } \sigma \to +\infty.$$

Thus, (2.1) is unbounded from below, for every t with $t > 0$. □

Remark 2.1 *If there are alternate optimal solutions for (2.1) when* $t = 0$, *then* x_0 *may have no optimal continuation.*

In order to illustrate Remark 2.1 , recall (1.14) in Example 1.3 that we introduced in Chapter 1. In (1.14), x_0 is an optimal solution for $t = 0$, the optimal solution "jumps" to y^+ from x_0 as t increases from zero to positive, and x_0 and y^+ cannot be connected with a linear function of t. Thus, x_0 has no optimal continuation in this case. However, there does exist an optimal solution for $t = 0$ for which there is an optimal continuation, namely y^+.

The following two theorems show how to get h_0^* in the optimal continuation of x_0 for (2.1).

Theorem 2.1 *Let Assumption 2.1 be satisfied. Suppose* $(h_0^*,\ \bar{t})$ *is an optimal continuation of* x_0 *for (2.1). In addition, suppose the optimal solution* $x(t) = x_0 + th_0^*$ *is a diminishment of* x_0*, for every* t *with* $0 < t < \bar{t}$*. Then* h_0^* *is an optimal solution for*

$$\min\{q'h_0 + \frac{1}{2}h_0'Ch_0 \mid A_0h_0 \leq 0,\ (c + Cx_0)'h_0 = 0\}. \tag{2.2}$$

Proof

The optimality conditions for (2.1) when $t = 0$ assert that

$$\left.\begin{aligned} &A_0x_0 = b_0, \\ &-c - Cx_0 = A_0'u_0, \quad u_0 \geq 0, \end{aligned}\right\} \tag{2.3}$$

where u_0 is a multiplier vector for x_0 whose components are associated with the rows of A_0. The optimality conditions when $0 < t < \bar{t}$ assert that

$$\left.\begin{aligned} &Ax(t) \leq b, \\ &-c - tq - Cx(t) = A'u, \quad u \geq 0, \\ &u'(Ax(t) - b) = 0. \end{aligned}\right\} \tag{2.4}$$

Since the optimal solution $x(t)$ is a diminishment of x_0, for every t with $0 < t < \bar{t}$, the matrix of gradients of all the constraints active at $x(t)$ is a submatrix of A_0'. We can simplify (2.4) to

$$\left.\begin{aligned} &A_0 x(t) \le b_0, \\ &-c - tq - Cx(t) = A_0' u, \quad u \ge 0, \\ &u'(A_0 x(t) - b_0) = 0, \end{aligned}\right\} \tag{2.5}$$

where $u = u(t)$ is a multiplier vector for $x(t)$ whose components are those u_i associated with the rows of A_0. Some components of u may be zero corresponding to constraints active at x_0 but inactive at $x(t)$. Substitute $x(t) = x_0 + th_0^*$ and $u = u_0 + tu_1$ into (2.5), and with (2.3), we have

$$\left.\begin{aligned} &A_0 h_0^* \le 0, \ (c + Cx_0)'h_0^* = 0, \\ &-c - tq - Cx_0 - tCh_0^* = A_0' u, \quad u \ge 0, \\ &u' A_0 h_0^* = 0. \end{aligned}\right\} \tag{2.6}$$

The optimality conditions for the problem

$$\min\{(c + Cx_0)'h_0 + tq'h_0 + \frac{t}{2} h_0' C h_0 \mid A_0 h_0 \le 0, \ (c + Cx_0)'h_0 = 0\} \tag{2.7}$$

are

$$\left.\begin{aligned} &A_0 h_0 \le 0, \ (c + Cx_0)'h_0 = 0, \\ &-c - tq - Cx_0 - tCh_0 = A_0' v + (c + Cx_0)w, \quad v \ge 0, \\ &v' A_0 h_0 = 0. \end{aligned}\right\} \tag{2.8}$$

Define $h_0 = h_0^*$, $v = u$, and $w = 0$. Then from (2.6), h_0, v and w satisfy (2.8). So, $h_0 = h_0^*$ is optimal for (2.7). Using the primal constraint $(c + Cx_0)'h_0 = 0$, the objective function for (2.7) can be simplified. Thus $h_0 = h_0^*$ is also optimal for

$$\min\{tq'h_0 + \frac{t}{2} h_0' C h_0 \mid A_0 h_0 \le 0, \ (c + Cx_0)'h_0 = 0\}.$$

For $t > 0$, we have shown that $h_0 = h_0^*$ is optimal for

$$\min\{q'h_0 + \frac{1}{2} h_0' C h_0 \mid A_0 h_0 \le 0, \ (c + Cx_0)'h_0 = 0\}$$

as required. □

The importance of the optimal problem (2.2) is illustrated in the following theorem.

Theorem 2.2 *Let Assumption 2.1 be satisfied. Suppose h_0^* is an optimal solution for (2.2), and suppose that w_1 and w_2 are multipliers associated with the constraints $A_0h_0 \le 0$ and $(c+Cx_0)'h_0 = 0$ in (2.2), respectively. Then $(h_0^*,\ \bar{t})$ is an optimal continuation of x_0 for (2.1), and $v(t) = u_0 + t(w_1 - w_2u_0)$ is a multiplier vector for $x(t) = x_0 + th_0^*$ whose components are associated with the rows of A_0, for every t with $0 \le t < \bar{t}$, where $\bar{t} = \min\{\hat{t},\ \tilde{t}\} > 0$, and*

$$\hat{t} = \min\{\frac{b_i - a_i'x_0}{a_i'h_0^*} \mid \textit{all } i = 1, ..., m \textit{ with } a_i'h_0^* > 0\}, \tag{2.9}$$

$$\tilde{t} = \min\{\frac{-(u_0)_i}{(w_1 - w_2u_0)_i} \mid \textit{all } i = 1, ..., m \textit{ with } (w_1 - w_2u_0)_i < 0\}. \tag{2.10}$$

The full (m-dimensional) vector of multipliers, $u(t)$, is obtained from $v(t)$ by assigning zero to those components of $u(t)$ associated with constraints inactive at x_0 and the appropriately indexed components of $v(t)$, otherwise.

Proof

Let A_1' be the matrix of gradients of all the constraints inactive at x_0, let b_1 be the vector whose components are those b_i associated with the rows of A_1; *i.e.*, $A_1x_0 < b_1$. Similar to the proof of Lemma 1.3, there exists a $\bar{t}_1 > 0$ such that $A_1(x_0 + th_0^*) < b_1$, for every t with $0 < t < \bar{t}_1$.

Since h_0^* is optimal for (2.2), h_0^* and the associated multipliers w_1 and w_2 satisfy the optimality conditions

$$A_0h_0^* \le 0,\ (c + Cx_0)'h_0^* = 0, \tag{2.11}$$

$$-q - Ch_0^* = A_0'w_1 + (c + Cx_0)w_2,\ w_1 \ge 0, \tag{2.12}$$

$$w_1'A_0h_0^* = 0, \tag{2.13}$$

where w_2 is a scalar. For $t > 0$, (2.12) can also be written as

$$-tq - tCh_0^* = A_0'(tw_1) + (c + Cx_0)(tw_2),\ tw_1 \geq 0, \tag{2.14}$$

or

$$-c - Cx_0 - tq - tCh_0^* = A_0'(tw_1) + (c + Cx_0)(tw_2 - 1),\ tw_1 \geq 0. \tag{2.15}$$

Since x_0 is an optimal solution for (2.1) for $t = 0$ and A_0' is the matrix of gradients of all the constraints active at x_0, there exists an multiplier vector u_0 whose components are associated with the constraints active at x_0, satisfying

$$-c - Cx_0 = A_0'u_0,\ u_0 \geq 0. \tag{2.16}$$

From (2.15) and (2.16),

$$\begin{aligned} -(c + Cx_0) - tq - tCh_0^* &= A_0'(tw_1) - (tw_2 - 1)A_0'u_0, \\ &= A_0'[tw_1 - (tw_2 - 1)u_0]. \end{aligned} \tag{2.17}$$

There exists a $\bar{t}_2 > 0$ such that $tw_2 \leq 1$; *i.e.*, $tw_2 - 1 \leq 0$, for every t with $0 < t < \bar{t}_2$. Since $(tw_1) \geq 0,\ u_0 \geq 0$ and $tw_2 - 1 \leq 0$, then for every t with $0 < t \leq \bar{t}_2$,

$$tw_1 - (tw_2 - 1)u_0 \geq 0. \tag{2.18}$$

From (2.16) and the second constraint in (2.2), we have

$$u_0'A_0h_0^* = -(c + Cx_0)'h_0^* = 0,$$

together with (2.13), it follows

$$[tw_1 - (tw_2 - 1)u_0]'A_0h_0^* = 0. \tag{2.19}$$

Then from (2.17), (2.18) and (2.19), we have

$$\left.\begin{aligned} &-c - Cx_0 - tq - tCh_0^* = A_0'[tw_1 - (tw_2 - 1)u_0], \quad tw_1 - (tw_2 - 1)u_0 \geq 0, \\ &(tw_1 - (tw_2 - 1)u_0)'A_0h_0^* = 0. \end{aligned}\right\} \tag{2.20}$$

Let $v = tw_1 - (tw_2 - 1)u_0$, combine (2.11) and (2.20),

$$\left.\begin{aligned} &A_0 h_0^* \leq 0, \\ &-c - tq - C(x_0 + th_0^*) = A_0' v, \quad v \geq 0, \\ &v'(A_0 t h_0^*) = 0. \end{aligned}\right\} \tag{2.21}$$

Let $\bar{t} = \min\{\bar{t}_1,\ \bar{t}_2\} > 0$. Since $A_0 x_0 = b_0$ and $A_1(x_0 + th_0^*) < b_1$, for every t with $0 < t < \bar{t}_1$, it follows that for every t with $0 < t < \bar{t}$,

$$\left.\begin{aligned} &A(x_0 + th_0^*) \leq b, \\ &-c - tq - C(x_0 + th_0^*) = A_0' v, \quad v \geq 0, \\ &v'(A_0(x_0 + th_0^*) - b_0) = 0. \end{aligned}\right\} \tag{2.22}$$

Thus, $x(t) = x_0 + th_0^*$ and the multiplier vector $v = u_0 + t(w_1 - w_2 u_0)$ whose components are associated with the rows of A_0 satisfy the optimality conditions for (2.1), for every t with $0 \leq t < \bar{t}$. So $x(t) = x_0 + th_0^*$ is optimal for (2.1), for every t with $0 \leq t < \bar{t}$. Therefore, $(h_0^*,\ \bar{t})$ is an optimal continuation of x_0 for (2.1) as required.

Since $x(t) = x_0 + th_0^*$ is an optimal solution for (2.1), if $a_i' h_0^* > 0$, then $a_i' x_0 < b_i$. From (2.9), $\hat{t} > 0$. Since $v = u_0 + t(w_1 - w_2 u_0) \geq 0$, if $(w_1 - w_2 u_0)_i < 0$, then $(u_0)_i > 0$. Thus, from (2.10), $\tilde{t} > 0$. Therefore, $\bar{t} = \min\{\hat{t},\ \tilde{t}\} > 0$. □

Note: If the constraint $a_i' x \leq b_i$ active at x_0 remains active at $x(t)$ for every t with $0 < t < \bar{t}$, then $a_i' h_0 = 0$.

We illustrate Theorems 2.1 and 2.2 by applying them to Example 1.2. In Example 1.2, $c = (-\frac{10}{3},\ -\frac{8}{3})'$, $q = (0,\ 1)'$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $x_0 = (\frac{2}{3},\ \frac{2}{3})'$, and $c + Cx_0 = (-\frac{8}{3},\ -\frac{4}{3})'$. Since the first three constraints are active at x_0, we have $A_0 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$. The optimal problem (2.2), in this case becomes

$$\begin{array}{lrcrcl}
\text{minimize :} & & h_2 + \frac{1}{2}h_1^2 + h_2^2 & & & \\
\text{subject to :} & h_1 & + & 2h_2 & \leq & 0, \quad (1) \\
 & 2h_1 & + & h_2 & \leq & 0, \quad (2) \\
 & h_1 & + & h_1 & \leq & 0, \quad (3) \\
 & -\ \frac{8}{3}h_1 & - & \frac{4}{3}h_2 & = & 0. \quad (4)
\end{array} \tag{2.23}$$

The optimal solution for (2.23) is

$$h^* = \begin{bmatrix} \frac{2}{9} \\ -\frac{4}{9} \end{bmatrix}.$$

The geometry of the problem (2.23) is shown in Figure 2.1. The half-line beginning at α and going towards β is the feasible region of (2.23). The level sets of the objective function are ellipses centered at $(0,\ -\frac{1}{2})'$.

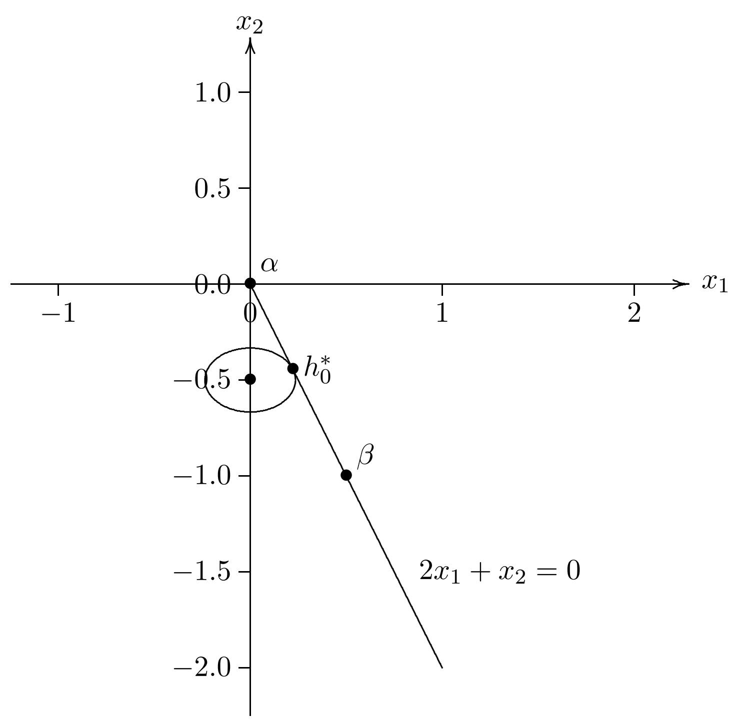

Figure 2.1: The related QP problem for Example 1.2.

Let $w_1 = (w_{11},\ w_{12},\ w_{13})'$ be a multiplier vector for the first three constraints, and w_2 be a multiplier for the fourth constraint in (2.23). Then, we have

$$w_{11} = w_{13} = 0,$$

and w_{12}, w_2 satisfy

$$w_{12} - \frac{4}{3}w_2 = -\frac{1}{9}.$$

Since $u_0 = (0,\ \frac{4}{3},\ 0)'$, it follows that

$$w_1 - u_0 w_2 = \begin{bmatrix} 0 \\ -\frac{1}{9} \\ 0 \end{bmatrix},$$

thus

$$v(t) = \begin{bmatrix} 0 \\ \frac{4}{3} \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -\frac{1}{9} \\ 0 \end{bmatrix}.$$

From Theorem 2.2,

$$x(t) = x_0 + th_0^* = \begin{bmatrix} \frac{2}{3} + \frac{2}{9}t \\ \frac{2}{3} - \frac{4}{9}t \end{bmatrix},$$

is an optimal solution for the problem in Example 1.2, for every t with $0 < t \leq \bar{t}$, where $\bar{t}$ is solved as following by applying (2.9) and (2.10):

$$\hat{t}_1 = \min\{-, -, -, -, \frac{\frac{2}{3}}{\frac{4}{9}}\} = \frac{3}{2},$$

$$\tilde{t}_1 = \min\{-, \frac{-\frac{4}{3}}{-\frac{1}{9}}, -, -, -\} = 12,$$

from which

$$\bar{t} = \min\{\frac{3}{2},\ 12\} = \frac{3}{2}.$$

One might wonder in (2.2) if the constraint $(c + Cx_0)'h = 0$ is really necessary. If we remove it from the previous problem, (2.23) becomes

$$\begin{array}{llcccl} \text{minimize :} & h_2 + \frac{1}{2}h_1^2 + h_2^2 & & & & \\ \text{subject to :} & h_1 & + & 2h_2 & \leq & 0, \\ & 2h_1 & + & h_2 & \leq & 0, \\ & h_1 & + & h_1 & \geq & 0. \end{array} \tag{2.24}$$

The optimal solution is

$$h^* = \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix}.$$

Because the objective function for the problem in Example 1.2 is strictly convex, the optimal solution for it is uniquely determined. Then h_0^* for the present problem is different from what obtained from (2.23), and is therefore incorrect. Thus, the constraint $(c + Cx_0)'h = 0$ is essential in (2.2). ◇

2.2 The Boundedness of the Problem (2.2) in Theorem 2.1

It is possible that when $t = 0$ the optimal solutions for (2.1) are not unique. In this case, if we cannot choose a proper x_0, the problem (2.2) may be unbounded from below. (The problem (2.2) is always feasible because $h_0 = 0$ is its feasible solution.) Refer back to the discussion and Figure 1.3 following Example 1.3.

What is the implication of (2.2) being unbounded from below? Theorem 2.3 below will give the answer. Before introducing the theorem, we first need two lemmas.

Lemma 2.2 *If (2.2) is unbounded from below, then the following problem*

$$\min\{-q's \mid Cs = 0,\ c's = 0,\ A_0 s \geq 0\} \tag{2.25}$$

is unbounded from below.

Proof

Since (2.2) is unbounded from below, then for a feasible solution h_1 for (2.2), there exists an s such that $h_1 - \sigma s$ is also feasible for (2.2), for any positive scalar σ, and s satisfies $q's > 0$ and $s'Cs = 0$. From the feasibility of $h_1 - \sigma s$, we can get $A_0 s \geq 0$ and $(c + Cx_0)'s = 0$. Because C is positive semi-definite, $s'Cs = 0$ implies $Cs = 0$. Furthermore, $Cs = 0$ and $(c + Cx_0)'s = 0$ imply $c's = 0$. Thus, s satisfies $-q's < 0$, $Cs = 0$, $c's = 0$, $A_0 s \geq 0$.

Therefore, s is feasible for (2.25) and $-q's < 0$. For every positive scalar σ, σs is also feasible for (2.25), and

$$-q'(\sigma s) \to -\infty, \text{ as } \sigma \to +\infty.$$

Thus (2.25) is unbounded from below. □

Lemma 2.3 *If (2.25) has an optimal solution $s^* = 0$, then (2.2) is bounded from below and thus has an optimal solution.*

Proof

Assume on the contrary that (2.2) is unbounded from below. Then from Lemma 2.2, (2.25) is also unbounded from below. This contradicts that (2.25) has an optimal solution $s^* = 0$. This contradiction establishes that (2.2) is indeed bounded from below. □

Consider the problem

$$\min\{-q's \mid c's = 0, \ Cs = 0, \ A_0 s \geq 0, \ As \geq Ax_0 - b\}. \tag{2.26}$$

It is feasible, because $s = 0$ is its feasible solution.

Theorem 2.3 *Assume (2.2) is unbounded from below. Assume (2.26) is bounded from below and has an optimal solution* s*. Then,* $s \neq 0$*. Let* $x_1 = x_0 - s$*. Let* A_1' *be the matrix of gradients of all the constraints active at* x_1 *in (2.1), and let* b_1 *be the vector whose components are those* b_i *associated with the rows of* A_1*; i.e.,* $A_1 x_1 = b_1$*. Then* x_1 *is also optimal for (2.1) for* $t = 0$*, and moreover, the problem*

$$\min\{q'h_1 + \frac{1}{2}h_1'Ch_1 \mid A_1h_1 \leq 0,\ (c + Cx_1)'h_1 = 0\} \tag{2.27}$$

has a finite optimal solution.

Proof

We first show that if (2.26) has an optimal solution s, then

$$s \neq 0. \tag{2.28}$$

Otherwise, if $s = 0$ is an optimal solution for (2.26), the optimality conditions assert

$$q = cu_1 + Cu_2 - A_0'u_3 - A'u_4,\ u_3, u_4 \geq 0, \tag{2.29}$$

$$u_4'(Ax_0 - b) = 0. \tag{2.30}$$

Since A_0' is the matrix of gradients of all the constraints active at x_0, (2.29) and (2.30) can be simplified to

$$q = cu_1 + Cu_2 - A_0'u_3 - A_0'\bar{u}_4 = cu_1 + Cu_2 - A_0'(u_3 + \bar{u}_4),\ u_3, \bar{u}_4 \geq 0, \tag{2.31}$$

$$A_0x_0 - b_0 = 0, \tag{2.32}$$

where $\bar{u}_4$ is the multiplier vector whose components are those $(u_4)_i$ associated with the rows of A_0. Then $s = 0$, u_1, u_2 and $u_3 + \bar{u}_4$ satisfy the optimality conditions for (2.25), which are

$$\left.\begin{array}{l} Cs = 0,\ c's = 0,\ A_0s \geq 0, \\ q = cu_1 + Cu_2 - A_0'(u_3 + \bar{u}_4),\ u_3 + \bar{u}_4 \geq 0, \\ (u_3 + \bar{u}_4)'A_0s = 0. \end{array}\right\}$$

Thus, $s = 0$ being an optimal solution for (2.25), together with Lemma 2.3, contradicts that (2.2) is unbounded from below. Thus, if (2.26) has an optimal solution s, then $s \neq 0$, which verifies (2.28).

Now we will prove that x_1 is also optimal for (2.1) for $t = 0$, and (2.27) has a finite optimal solution. From the fourth constraint of (2.26), $As \geq Ax_0 - b$, we have

$$A(x_0 - s) \leq b,$$

which means

$$Ax_1 \leq b.$$

From the first and second constraints of (2.26), $c's = 0,\ Cs = 0$, the objective function for $x = x_1$ is

$$c'x_1 + \frac{1}{2}x_1'Cx_1 = c'(x_0 - s) + \frac{1}{2}(x_0 - s)'C(x_0 - s) = c'x_0 + \frac{1}{2}x_0'Cx_0.$$

Thus, x_1 is also an optimal solution for (2.1) for $t = 0$.

Since s is an optimal solution for (2.26), the optimality conditions give us

$$\left.\begin{aligned} &q = Cu + cv - A_0'w_0 - A'w_1,\ w_0, w_1 \geq 0, \\ &w_0'A_0 s = 0,\ w_1'(Ax_0 - b - As) = 0. \end{aligned}\right\} \tag{2.33}$$

Since A_1 is the matrix of gradients of all the constraints active at x_1, $A_1x_1 = b_1$; *i.e.*, $A_1(x_0 - s) = b_1$, (2.33) can be simplified to

$$\left.\begin{aligned} &q = Cu + cv - A_0'w_0 - A_1'\bar{w}_1,\ w_0, \bar{w}_1 \geq 0, \\ &w_0'A_0 s = 0,\ A_1 s = A_1 x_0 - b_1, \end{aligned}\right\} \tag{2.34}$$

where $\bar{w}_1$ is the multiplier vector whose components are those $(w_1)_i$ associated with the rows of A_1. From $w_0'A_0s = 0$, we know that if $a_i's = (A_0s)_i \neq 0$, then $(w_0)_i = 0$. Let A_2' be the matrix of all the a_i in A_0 satisfying $a_i's = 0$; *i.e.*, $A_2s = 0$. Let b_2 be the vector whose components are those b_i

associated with the rows of A_2. Since A_2 is a submatrix of A_0, $A_2x_0 = b_2$. We have $A_2(x_0 - s) = b_2$; *i.e.*, $A_2x_1 = b_2$. Thus, A_2 is also a submatrix of A_1. So, (2.34) is equivalent to

$$\left.\begin{aligned} & q = Cu + cv - A_2'\bar{w}_0 - A_1'\bar{w}_1 = Cu + cv - A_1'w, \quad \bar{w}_0, \bar{w}_1, w \geq 0, \\ & A_2s = 0,\ A_1s = A_1x_0 - b_1, \end{aligned}\right\}$$

where w is a vector whose components are associated with the rows of A_1. Therefore, $s_1 = 0$ and $u,\ v,\ w$ satisfy

$$\left.\begin{aligned} & Cs_1 = 0,\ c's_1 = 0,\ A_1s_1 \geq 0, \\ & q = Cu + cv - A_1'w,\ w \geq 0, \\ & w'A_1s_1 = 0, \end{aligned}\right\}$$

which are precisely the optimality conditions for

$$\min\{-q's \mid Cs = 0,\ c's = 0,\ A_1s \geq 0\}. \tag{2.35}$$

Thus, $s_1 = 0$ is optimal for (2.35). Therefore from Lemma 2.3, (2.27) has a finite optimal solution. □

Theorem 2.4 *If (2.26) is unbounded from below, then (2.1) is also unbounded from below, for every t with $t > 0$.*

Proof

If (2.26) is unbounded from below, then for a feasible solution s for (2.26), there exists a vector d such that $s - \sigma d$ is feasible for (2.26), for every positive scalar σ, and $q'd < 0$. That is, $q'd < 0,\ c'd = 0,\ Cd = 0$, and $Ad \leq 0$.

Then, $x_0 + \sigma d$ satisfies

$$A(x_0 + \sigma d) = Ax_0 + \sigma Ad \leq b,$$

and

$$(c + tq)'(x_0 + \sigma d) + \frac{1}{2}(x_0 + \sigma d)'C(x_0 + \sigma d) = (c + tq)'x_0 + \frac{1}{2}x_0'Cx_0 + \sigma t q'd \to -\infty,\ \text{as } \sigma \to +\infty,$$

for every t with $t > 0$. Thus, (2.1) is unbounded from below, for every t with $t > 0$. □

2.3 Example 1.3 Continued

Consider (1.14) in Example 1.3. If we begin with $t = 0$ and an interior point optimal solution x_0, (2.2) in this problem becomes

$$\min\{q'h_0 \mid h_0 \text{ has no constraints}\}.$$

It is unbounded if $q \neq 0$. Then we consider (2.26), which in this problem is

$$\min\{-q's \mid As \geq Ax_0 - b\}. \tag{2.36}$$

Its optimal solution is same as

$$\min\{q'(x_0 - s) \mid A(x_0 - s) \leq b\}.$$

Suppose s_0 is an optimal solution for (2.36), and let $x_1 = x_0 - s_0$. Then, $x_1 = x_0 - s_0$ is optimal for

$$\min\{q'x_1 \mid Ax_1 \leq b\}. \tag{2.37}$$

So for every t with $t > 0$, x_1 is also optimal for

$$\min\{tq'x_1 \mid Ax_1 \leq b\},$$

which is precisely (1.14), and x_1 is precisely the point y^+ in Figure 1.3.

Assume A_1' is the matrix of the gradients of all the constraints active at x_1. From the optimality conditions for (2.37), we have $-q = A_1'u_1,\ u_1 \geq 0$. In this example, (2.27) becomes

$$\min\{q'h_1 \mid A_1h_1 \leq 0\}. \tag{2.38}$$

Since $-q = A_1' u_1$ and $u_1 \geq 0$, it follows that $h_1 = 0$ satisfies the optimality conditions for (2.38), which are

$$\left.\begin{array}{l} A_1 h_1 \leq 0, \\ -q = A_1' u_1, \ u_1 \geq 0, \\ u_1' A_1 h_1 = 0. \end{array}\right\}$$

Thus, $h_1 = 0$ is an optimal solution for (2.38), the multiplier vector u_1 satisfies $-q = A_1' u_1$. When $t = 0$, the multiplier for $x_1 = x_0 - s_0$ is $u_0 = 0$. When $t > 0$, x_1 and tu_1 satisfy $-q = A_1'(tu_1)$, $(tu_1) \geq 0$. Thus, $v(t) = u_0 + tu_1 = tu_1$ is the multiplier for $x(t) = x_1$, for every t with $t > 0$. $\diamond$

Chapter 3

A Parameter only in the Constraints

In this chapter, we will study another simple case – the PQP problem with the parameter t only in the right-hand side of the constraints.

Consider the following PQP problem

$$\min\{c'x + \frac{1}{2}x'Cx \mid Ax \leq b + tp\}. \tag{3.1}$$

From the QP duality, the dual of (3.1) is

$$\max\{c'x + \frac{1}{2}x'Cx + u'(Ax - b - tp) \mid Cx + A'u = -c,\ u \geq 0\}. \tag{3.2}$$

It is a PQP problem with the parameter t only in the linear part of the objective function, thus we could solve it using the method introduced in Chapter 2. Assume $(x^*(t),\ u^*(t))'$ is an optimal solution for (3.2), for every t with $t \in G$, where G is the region of t on which (3.2) has optimal solutions. If C is positive definite, then (3.1) is strictly convex. From Strict Converse Duality Theorem[1], $x^*(t)$ is an optimal solution for (3.1), for every t with $t \in G$. However, if C is positive semi-definite, that is, (3.1) is not strictly convex, then $x^*(t)$ may not be an optimal solution for (3.1), and it is hard to find an optimal solution for (3.1) from its dual.

In this chapter, we will develop another way to solve the PQP problem with the parameter only in the right-hand side of the constraints, with C being positive semi-definite.

3.1 Solution of the PQP Problem with a Parameter in the Right-Hand Side of the Constraints by Solving a Related QP Problem without the Parameter

Assumption 3.1 *There exists a $\hat{t} > 0$ such that (3.1) has an optimal solution for every t with $0 \le t < \hat{t}$.*

To determine the feasibility of (3.1) for $t > 0$, we can solve the $(n+1)$-variable linear programming problem

$$\max\{t \mid Ax - tp \le b\}, \tag{3.3}$$

where both x and t are variables in the LP. If the optimal solution for (3.3) is zero, then there exists no $t > 0$ such that $Ax \le b + tp$ has a solution, therefore, (3.1) is infeasible for every t with $t > 0$. If (3.3) has an optimal solution $\hat{t} > 0$, then (3.1) is feasible for every t with $0 \le t \le \hat{t}$, and infeasible for every t with $t > \hat{t}$. If (3.3) is unbounded from above, then (3.1) is feasible for every t with $t \ge 0$.

The following two theorems show how to obtain h_0^* in the optimal continuation of x_0 for (3.1).

Recall the notation in Chapter 1. x_0 is an optimal solution for (3.1) for $t = 0$. A_0' is the matrix of gradients of all the constraints active at x_0. Let p_0 be the vector whose components are those p_i associated with the rows of A_0.

Theorem 3.1 *Let Assumption 3.1 be satisfied. Suppose $(h_0^*,\ \bar{t})$ is an optimal continuation of x_0*

for (3.1). In addition, suppose the optimal solution $x(t) = x_0 + th_0^*$ for (3.1) is a diminishment of x_0, for every t with $0 < t < \bar{t}$. Then there exists a vector u_0 such that $\begin{bmatrix} h_0^* \\ u_0 \end{bmatrix}$ is an optimal solution for the problem

$$\min\{-p_0'u_0 + \frac{1}{2}h_0'Ch_0 \mid A_0h_0 \leq p_0,\ (c + Cx_0)'h_0 + p_0'u_0 = 0,\ -c - Cx_0 = A_0'u_0,\ u_0 \geq 0\}. \quad (3.4)$$

Proof

The outline of the proof is as follows. We first prove that $\begin{bmatrix} h_0^* \\ u_0 \end{bmatrix}$ is an optimal solution for (3.10) (see below). Then we change (3.10) to its equivalent form (3.12) (see below), and using the optimality conditions for (3.12), prove that $\begin{bmatrix} h_0^* \\ u_0 \end{bmatrix}$ is an optimal solution for (3.4).

Since $(h_0^*,\ \bar{t})$ is an optimal continuation of x_0 for (3.1); *i.e.*, $x(t) = x_0 + th_0^*$ is an optimal solution for (3.1), for every t with $0 \leq t < \bar{t}$. The optimality conditions for (3.1) assert

$$\left.\begin{aligned} &A(x_0 + th_0^*) \leq b + tp, \\ &-c - C(x_0 + th_0^*) = A'u,\ u \geq 0, \\ &u'[A(x_0 + th_0^*) - (b + tp)] = 0, \end{aligned}\right\}$$

where $u = u(t)$. These are equivalent to

$$\left.\begin{aligned} &A(x_0 + th_0^*) \leq b + tp, \\ &-c - Cx_0 - tCh_0^* = A'u,\ u \geq 0, \\ &u'[(Ax_0 - b) + t(Ah_0^* - p)] = 0. \end{aligned}\right\} \quad (3.5)$$

Since $x(t)$ is a diminishment of x_0, for every t with $0 < t < \bar{t}$, all the constraints active at $x(t)$ are also active at x_0. Consequently, the matrix of the gradients of all the constraints active at $x(t)$ is a submatrix of A_0'. Thus, (3.5) can be simplified to

$$\left.\begin{aligned} &A_0h_0^* \leq p_0, \\ &-c - Cx_0 - tCh_0^* = A_0'u,\ u \geq 0, \\ &u'(A_0h_0^* - p_0) = 0, \end{aligned}\right\} \quad (3.6)$$

where $u = u_0 + tu_1$ is a multiplier vector whose components are associated with the rows of A_0, and u_0 is a multiplier vector for x_0 whose components are associated with the rows of A_0, so u_0 satisfies the optimality conditions for $t = 0$,

$$-c - Cx_0 = A_0' u_0, \;\; u_0 \geq 0. \tag{3.7}$$

Since $(u_0 + tu_1)'(A_0 h_0^* - p_0) = 0$, for every t with $0 < t < \bar{t}$, u_0 must satisfy

$$u_0'(A_0 h_0^* - p_0) = 0. \tag{3.8}$$

Because $-c - Cx_0 = A_0' u_0$, $u_0'(A_0 h_0^* - p_0) = 0$ can also be written as

$$(c + Cx_0)' h_0^* + p_0' u_0 = 0. \tag{3.9}$$

Combining (3.6), (3.7) and (3.9), we get

$$\left.\begin{array}{l} A_0 h_0^* \leq p_0, \;\; (c + Cx_0)' h_0^* + p_0' u_0 = 0, \;\; -c - Cx_0 = A_0' u_0, \;\; u_0 \geq 0, \\ -c - Cx_0 - tCh_0^* = A_0' u, \;\; u \geq 0, \\ u'(A_0 h_0^* - p_0) = 0. \end{array}\right\}$$

Therefore, $\begin{bmatrix} h_0^* \\ u_0 \end{bmatrix}$ is an optimal solution for the problem

$$\min\{(c + Cx_0)' h_0 + \frac{1}{2} t h_0' C h_0 \mid A_0 h_0 \leq p_0, \;\; (c + Cx_0)' h_0 + p_0' u_0 = 0, \;\; -c - Cx_0 = A_0' u_0, \;\; u_0 \geq 0\}, \tag{3.10}$$

because $\begin{bmatrix} h_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} h_0^* \\ u_0 \end{bmatrix}$ and $v_1 = u, \;\; v_2, v_3, v_4 = 0$ satisfy the optimality conditions

$$\left.\begin{array}{l} A_0 h_0 \leq p_0, \;\; (c + Cx_0)' h_0 + p_0' u_0 = 0, \;\; -c - Cx_0 = A_0' u_0, \;\; u_0 \geq 0, \\ \begin{bmatrix} -(c + Cx_0) - tCh_0 \\ 0 \end{bmatrix} = \begin{bmatrix} A_0' \\ 0 \end{bmatrix} v_1 + \begin{bmatrix} c + Cx_0 \\ p_0 \end{bmatrix} v_2 + \begin{bmatrix} 0 \\ A_0 \end{bmatrix} v_3 + \begin{bmatrix} 0 \\ -I \end{bmatrix} v_4, \;\; v_1, v_4 \geq 0, \\ v_1'(A_0 h_0 - p_0) = 0, \\ v_4' u_0 = 0. \end{array}\right\} \tag{3.11}$$

From the constraint $(c+Cx_0)'h_0 + p_0'u_0 = 0$ in (3.10), (3.10) is equivalent to

$$\min\{-p_0'u_0 + \frac{1}{2}th_0'Ch_0 \mid A_0h_0 \le p_0,\ (c+Cx_0)'h_0 + p_0'u_0 = 0,\ -c - Cx_0 = A_0'u_0,\ u_0 \ge 0\}. \quad (3.12)$$

The optimality conditions for (3.12) are

$$\left.\begin{array}{l} A_0h_0 \le p_0,\ (c+Cx_0)'h_0 + p_0'u_0 = 0,\ -c - Cx_0 = A_0'u_0,\ u_0 \ge 0, \\ \begin{bmatrix} -tCh_0 \\ p_0 \end{bmatrix} = \begin{bmatrix} A_0' \\ 0 \end{bmatrix} w_1 + \begin{bmatrix} c+Cx_0 \\ p_0 \end{bmatrix} w_2 + \begin{bmatrix} 0 \\ A_0 \end{bmatrix} w_3 + \begin{bmatrix} 0 \\ -I \end{bmatrix} w_4,\ w_1, w_4 \ge 0, \\ w_1'(A_0h_0 - p_0) = 0, \\ w_4'u_0 = 0. \end{array}\right\} \quad (3.13)$$

From (3.11), it follows that $\begin{bmatrix} h_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} h_0^* \\ u_0 \end{bmatrix}$ and $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} u \\ 1 \\ 0 \\ 0 \end{bmatrix}$ satisfy (3.13). Therefore, $\begin{bmatrix} h_0^* \\ u_0 \end{bmatrix}$ satisfies

$$\begin{bmatrix} -Ch_0^* \\ p_0 \end{bmatrix} = \begin{bmatrix} A_0' \\ 0 \end{bmatrix} \frac{u}{t} + \begin{bmatrix} c+Cx_0 \\ p_0 \end{bmatrix} \frac{1}{t} + \begin{bmatrix} 0 \\ A_0 \end{bmatrix} \alpha_1 + \begin{bmatrix} 0 \\ -I \end{bmatrix} \alpha_2,$$

where

$$\alpha_1 = (1 - \frac{1}{t})h_0^*,$$

and

$$\alpha_2 = (1 - \frac{1}{t})(A_0h_0^* - p_0).$$

Since $A_0h_0^* \le p_0$, and for every t with $0 < t < 1$, $1 - \frac{1}{t} < 0$, thus,

$$\alpha_2 = (1 - \frac{1}{t})(A_0h_0^* - p_0) \ge 0.$$

Since $u \ge 0,\ t \ge 0$,

$$\frac{u}{t} \ge 0.$$

Since $u'(A_0h_0^* - p_0) = 0$, it follows that

$$(\frac{u}{t})'(A_0h_0^* - p_0) = 0.$$

Since we have $u_0'(A_0h_0^* - p_0) = 0$ from (3.8),

$$\alpha_2' u_0 = (1 - \frac{1}{t})(A_0h_0^* - p_0)'u_0 = 0.$$

Thus, $\begin{bmatrix} h_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} h_0^* \\ u_0 \end{bmatrix}$ and $w_1 = \frac{u}{t}$, $w_2 = \frac{1}{t}$, $w_3 = \alpha_1$, $w_4 = \alpha_2$ satisfy

$$\left.\begin{array}{l} A_0h_0 \le p_0,\ (c + Cx_0)'h_0 + p_0'u_0 = 0,\ -c - Cx_0 = A_0'u_0,\ u_0 \ge 0, \\ \begin{bmatrix} -Ch_0 \\ p_0 \end{bmatrix} = \begin{bmatrix} A_0' \\ 0 \end{bmatrix} w_1 + \begin{bmatrix} c + Cx_0 \\ p_0 \end{bmatrix} w_2 + \begin{bmatrix} 0 \\ A_0 \end{bmatrix} w_3 + \begin{bmatrix} 0 \\ -I \end{bmatrix} w_4,\ w_1, w_4 \ge 0, \\ w_1'(A_0h_0 - p_0) = 0, \\ w_4'u_0 = 0, \end{array}\right\}$$

which are precisely the optimality conditions for (3.4). Thus, we have $\begin{bmatrix} h_0^* \\ u_0 \end{bmatrix}$ is an optimal solution for (3.4) as required. □

The importance of the optimal problem (3.4) is illustrated in the following theorem.

Theorem 3.2 *Let Assumption 3.1 be satisfied. Suppose* $\begin{bmatrix} h_0^* \\ u_0 \end{bmatrix}$ *is an optimal solution for (3.4), and suppose that* w_1, w_2, w_3 *and* w_4 *are multipliers associated with the constraints* $A_0h_0 \le p_0$, $(c + Cx_0)'h_0 + p_0'u_0 = 0$, $-c - Cx_0 = A_0'u_0$ *and* $u_0 \ge 0$, *respectively. Then* $(h_0^*,\ \bar{t})$ *is an optimal continuation of* x_0 *for (3.1), and* $v(t) = u_0 + t(w_1 - w_2u_0)$ *is an associated multiplier vector for* $x(t) = x_0 + th_0^*$, *for every* t *with* $0 \le t < \bar{t}$, *where* $\bar{t} = \min\{\hat{t},\ \tilde{t}\} > 0$, *and*

$$\hat{t} = \min\{\frac{b_i - a_i'x_0}{a_i'h_0^* - p_i} \mid \text{all } i = 1, ..., m \text{ with } a_i'h_0^* > p_i\}, \tag{3.14}$$

$$\tilde{t} = \min\{\frac{-(u_0)_i}{(w_1 - w_2u_0)_i} \mid \text{all } i = 1, ..., m \text{ with } (w_1 - w_2u_0)_i < 0\}. \tag{3.15}$$

The full (m-dimensional) vector of multipliers, $u(t)$, is obtained from $v(t)$ by assigning zero to those components of $u(t)$ associated with constraints inactive at x_0 and the appropriately indexed components of $v(t)$, otherwise.

Proof

Let A_1' be the matrix of the gradients of all the constraints inactive at x_0 for (3.1), let b_1 be the vector whose components are those b_i associated with the rows of A_1. Then $A_1 x_0 < b_1$. Similar to the proof of Lemma 1.3, there is a $\bar{t}_1 > 0$, such that $A_1(x_0 + th_0^*) < b_1 + tp_1$ for every t with $0 \leq t < \bar{t}_1$.

Since $\begin{bmatrix} h_0^* \\ u_0 \end{bmatrix}$ is an optimal solution for (3.4), the optimality conditions assert that

$$\begin{bmatrix} -Ch_0^* \\ p_0 \end{bmatrix} = \begin{bmatrix} A_0' \\ 0 \end{bmatrix} w_1 + \begin{bmatrix} c + Cx_0 \\ p_0 \end{bmatrix} w_2 + \begin{bmatrix} 0 \\ A_0 \end{bmatrix} w_3 + \begin{bmatrix} 0 \\ -I \end{bmatrix} w_4, \quad w_1, w_4 \geq 0, \tag{3.16}$$

$$w_1'(A_0 h_0^* - p_0) = 0. \tag{3.17}$$

Multiplying both sides of (3.16) and (3.17) by t gives

$$\begin{bmatrix} -tCh_0^* \\ tp_0 \end{bmatrix} = \begin{bmatrix} A_0' \\ 0 \end{bmatrix} (tw_1) + \begin{bmatrix} c + Cx_0 \\ p_0 \end{bmatrix} (tw_2) + \begin{bmatrix} 0 \\ A_0 \end{bmatrix} (tw_3) + \begin{bmatrix} 0 \\ -I \end{bmatrix} (tw_4), \quad (tw_1), (tw_4) \geq 0, \tag{3.18}$$

$$(tw_1)'(A_0 h_0^* - p_0) = 0. \tag{3.19}$$

From (3.18), it follows

$$-tCh_0^* = A_0'(tw_1) + (c + Cx_0)(tw_2),$$

and this is equivalent to

$$-(c + Cx_0) - tCh_0^* = A_0'(tw_1) + (c + Cx_0)(tw_2 - 1).$$

From the optimality conditions for (3.1) when $t = 0$, $-(c + Cx_0) = A_0' u_0$, so

$$-(c + Cx_0) - tCh_0^* = A_0'(tw_1 - (tw_2 - 1)u_0).$$

The second and the third constraints of (3.4) give

$$u_0'(A_0 h_0^* - p_0) = 0,$$

together with (3.19), we have

$$(tw_1 - (tw_2 - 1)u_0)'(A_0 h_0^* - p_0) = 0.$$

Let $v(t) = tw_1 - (tw_2 - 1)u_0$. For $\bar{t}_2 > 0$ given small enough, $tw_2 \leq 1$; *i.e.*, $tw_2 - 1 \leq 0$, for every t with $0 \leq t \leq \bar{t}_2$. Thus, $v(t) \geq 0$ since $w_1, u_0 \geq 0$. Since $A_0 x_0 = b_0$, we have

$$\left.\begin{aligned} &tA_0 h_0^* \leq tp_0, \\ &-c - C(x_0 + th_0^*) = A_0' v(t),\ v(t) \geq 0, \\ &v(t)'((A_0 x_0 - b_0) + t(Ah_0^* - p_0)) = 0, \end{aligned}\right\}$$

and since $A_1(x_0 + th_0^*) < b_1 + tp_1$, it follows

$$\left.\begin{aligned} &A(x_0 + th_0^*) \leq b + tp, \\ &-c - C(x_0 + th_0^*) = A_0' v(t),\ v(t) \geq 0, \\ &v(t)'(A_0(x_0 + th_0^*) - (b_0 + tp_0)) = 0. \end{aligned}\right\}$$

Let $\bar{t} = \min\{\bar{t}_1,\ \bar{t}_2\} > 0$. Then $x(t) = x_0 + th_0^*$ and the associated multiplier $v(t) = u_0 + t(w_1 - w_2 u_0)$ satisfy the optimality conditions for (3.1), for every t with $0 \leq t < \bar{t}$. Thus $x(t) = x_0 + th_0^*$ is optimal for (3.1), for every t with $0 \leq t < \bar{t}$. Therefore, $(h_0^*,\ \bar{t})$ is an optimal continuation of x_0 for (3.1) as required.

Since $x(t) = x_0 + th_0^*$ is an optimal solution for (3.1), if $a_i' h_0^* > p_i$, then $a_i' x_0 < b_i$. From (3.14), $\hat{t} > 0$. Since $v = u_0 + t(w_1 - w_2 u_0) \geq 0$, if $(w_1 - w_2 u_0)_i < 0$, then $(u_0)_i > 0$. From (3.15), $\tilde{t} > 0$. Therefore, $\bar{t} = \min\{\hat{t},\ \tilde{t}\} > 0$. □

Recall Example 1.4 in Chapter 1. The first four constraints are active at $x_0 = (1,\ 1)'$. Let $u_0 = (v_1, v_2, v_3, v_4)'$ be an multiplier vector for x_0 whose components are associated with the first four constraints. Then we can get an optimal continuation $h_0^* = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$ of x_0 by solving (3.4), which in this problem is

$$
\begin{array}{llcccccccccccl}
\text{minimize :} & \multicolumn{13}{l}{\frac{1}{2}h_1^2 + h_2^2 + v_3 + \frac{1}{2}v_4} \\
\text{subject to :} & h_1 & & & & & & & & & & & \leq & 0, \\
& & & h_2 & & & & & & & & & \leq & 0, \\
& h_1 & + & h_2 & & & & & & & & & \leq & -1, \\
& h_1 & + & 2h_2 & & & & & & & & & \leq & -\frac{1}{2}, \\
& -h_1 & & & & & & - & v_3 & - & \frac{1}{2}v_4 & & = & 0, \\
& & & & v_1 & & & + & v_3 & + & v_4 & & = & 1, \\
& & & & & & v_2 & + & v_3 & + & 2v_4 & & = & 0, \\
& & & & v_1 & & & & & & & & \geq & 0, \\
& & & & & & v_2 & & & & & & \geq & 0, \\
& & & & & & & & v_3 & & & & \geq & 0, \\
& & & & & & & & & & v_4 & & \geq & 0.
\end{array}
$$

The optimal solution is

$$h_0^* = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

and

$$u_0 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

From Theorem 3.2, the optimal solution for the problem of Example 1.4 is

$$x(t) = x_0 t h_0^* = \begin{bmatrix} 1 \\ 1-t \end{bmatrix},$$

with the multiplier vector

$$v(t) = u_0 + t(w_1 - w_2 u_0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 2 \\ 0 \end{bmatrix},$$

whose components are associated with the first four constraints, for every t with $0 < t < \bar{t}$.

Again from Theorem 3.2, the upper limit $\bar{t}$ is determined by applying (3.14) and (3.15):

$$\hat{t}_1 = \min\{-, -, -, -, -, \frac{1}{1}\} = 1,$$

$$\tilde{t}_1 = \min\{-\frac{1}{-2}, -, -, -\} = \frac{1}{2},$$

from which

$$\bar{t} = \min\{1,\ \frac{1}{2}\} = \frac{1}{2}.$$

Therefore,

$$x(t) = \begin{bmatrix} 1 \\ 1 - t \end{bmatrix}$$

is optimal for the problem, for every t with $0 \le t \le \frac{1}{2}$, in agreement with our geometric determination of the optimal solution in Example 1.4. $\diamond$

3.2 Reduction of Theorem 3.2 to the "No Ties" Case

In this section, we will show that (3.4) can be simplified to known results for the "no ties" case.

Consider the problem

$$\min\{c'x + \frac{1}{2}x'Cx \mid Ax \le b + tp\}. \tag{3.20}$$

Let x_0 be an optimal solution for (3.20) for $t = 0$, let A_0' be the matrix of gradients of all the constraints active at x_0, and let b_0 and p_0 be the vectors whose components are those b_i and p_i

associated with the rows of A_0, respectively. Suppose that there exists a $\bar{t} > 0$ such that an optimal solution $x(t) = x_0 + th_0^*$ for (3.20) has the same active constraints as those for x_0, for every t with $0 < t < \bar{t}$, and $u = u_0 + tu_1$ is an associated multiplier vector. Assume A_0 has full row rank and $H_0 = \begin{bmatrix} C & A_0' \\ A_0 & 0 \end{bmatrix}$ is nonsingular. Then, $s'Cs > 0$ for all $s \neq 0$, $A_0 s = 0$.

From the optimality conditions for (3.20) for $t = 0$,

$$\begin{bmatrix} C & A_0' \\ A_0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} -c \\ b_0 \end{bmatrix}. \tag{3.21}$$

From the optimality conditions for (3.20) for $t > 0$, we have

$$A_0(x_0 + th_0^*) = b_0 + tp_0 \Rightarrow A_0 h_0^* = p_0,$$

and

$$-c - C(x_0 + th_0^*) = A_0'(u_0 + tu_1). \tag{3.22}$$

From (3.21), we have $-c - Cx_0 = A_0'u_0$. Thus (3.22) implies $-Ch_0^* = A_0'u_1$. So we get

$$\begin{bmatrix} C & A_0' \\ A_0 & 0 \end{bmatrix} \begin{bmatrix} h_0^* \\ u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ p_0 \end{bmatrix}. \tag{3.23}$$

Since $\begin{bmatrix} C & A_0' \\ A_0 & 0 \end{bmatrix}$ is nonsingular, $\begin{bmatrix} h_0^* \\ u_1 \end{bmatrix}$ and $\begin{bmatrix} x_0 \\ u_0 \end{bmatrix}$ are uniquely determined by (3.23) and (3.21). Indeed, this is the identical solution obtained by Best in the “no ties” case.

Under the same “no ties” assumption, (3.4) can be simplified. Since A_0 has full row rank, which means that the active constraints at x_0 are linear independent, we know that u_0 is unique. So we can take out the third and the fourth constraints without changing the problem. The second constraint $(c + Cx_0)'h_0 + u_0'p_0 = 0$ can be written as $u_0'(A_0 h_0 - p_0) = 0$. Also because of the uniqueness of u_0, the term $-p_0'u_0$ in the objective function is a constant. Thus, the optimal solution h_0^* for (3.4) is also optimal for

$$\min\{\frac{1}{2}h_0'Ch_0 \mid A_0 h_0 \leq p_0,\ u_0'(A_0 h_0 - p_0) = 0\}. \tag{3.24}$$

From the optimality conditions, the optimal solution h_0^* for (3.24) satisfies

$$-Ch_0^* = A_0'v_1 + A_0'u_0v_2,\ v_1 \geq 0,$$

$$v_1'(A_0h_0^* - p_0) = 0,$$

where v_2 is a scalar. Let $v = v_1 + v_2u_0$. Since $u_0'(A_0h_0^* - p_0) = 0$, we have

$$-Ch_0^* = A_0'v, \tag{3.25}$$

$$v'(A_0h_0^* - p_0) = 0. \tag{3.26}$$

Since $\begin{bmatrix} C & A_0' \\ A_0 & 0 \end{bmatrix}$ is nonsingular, we can get a unique solution $\begin{bmatrix} h_0 \\ v \end{bmatrix}$ from

$$\begin{bmatrix} C & A_0' \\ A_0 & 0 \end{bmatrix} \begin{bmatrix} h_0 \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ p_0 \end{bmatrix}. \tag{3.27}$$

The solution $\begin{bmatrix} h_0 \\ v \end{bmatrix}$ for (3.27) satisfies (3.25) and (3.26), so it is an optimal solution for (3.24). Thus the optimal solution that has same active constraints as x_0 is uniquely determined by (3.27). This verifies that we will get the correct optimal solution using the result in Theorem 3.2 in Section 3.1 for the "no ties" case.

3.3 Feasibility and Boundedness of the Problem (3.4) in Theorem 3.1

Assume (3.1) is feasible for every t with $0 < t \leq \bar{t}$ throughout this section. The critical problem (3.4) may in general be infeasible, feasible and bounded, or feasible and unbounded. In this section, we will show that it is always feasible and bounded.

Consider the feasible region of (3.4), namely

$$S \equiv \{ \begin{bmatrix} h_0 \\ u_0 \end{bmatrix} \mid A_0h_0 \leq p_0,\ (c + Cx_0)'h_0 + p_0'u_0 = 0,\ -c - Cx_0 = A_0'u_0,\ u_0 \geq 0\},$$

$$= \{ \begin{bmatrix} h_0 \\ u_0 \end{bmatrix} \mid A_0 h_0 \le p_0,\ u_0'(A_0 h_0 - p_0) = 0,\ -c - Cx_0 = A_0' u_0,\ u_0 \ge 0 \}.$$

The optimality conditions for the problem

$$\min\{(c + Cx_0)' h_0 \mid A_0 h_0 \le p_0\}, \tag{3.28}$$

imply that if (3.28) has an optimal solution, then the set S is not empty.

Lemma 3.1 *(3.28) is feasible.*

Proof

We know that x_0 is optimal for $\min\{c'x + \frac{1}{2}x'Cx \mid Ax \le b\}$ and $A_0 x_0 = b_0$. Assumption 3.1 implies that (3.1) is feasible for every t with $0 < t < \bar{t}$. Then, there exists an x_1 such that $A_0 x_1 \le b_0 + t p_0$, for some t_0 satisfying $0 < t_0 < \bar{t}$. Then,

$$A_0(x_1 - x_0) \le b_0 + t_0 p_0 - b_0,$$

$$\Rightarrow A_0(x_1 - x_0) \le t_0 p_0,$$

$$\Rightarrow A_0 \frac{x_1 - x_0}{t_0} \le p_0.$$

Thus, $\frac{x_1 - x_0}{t_0}$ is a feasible solution for (3.28). So (3.28) is feasible. □

Lemma 3.2 *(3.28) is bounded.*

Proof

Assume on the contrary that (3.28) is unbounded. Then for a feasible solution h_1 for (3.28), there exists an s_1 such that $h_1 - \sigma s_1$ is feasible, for every positive scalar σ, and

$$(c + Cx_0)'(h_1 - \sigma s_1) \to -\infty, \text{ as } \sigma \to +\infty.$$

So we have

$$\left.\begin{aligned} &(c + Cx_0)' s_1 > 0, \\ &A_0 s_1 \geq 0. \end{aligned}\right\} \qquad (3.29)$$

From the optimality conditions for the original problem (3.1), when $t = 0$, we have

$$-c - Cx_0 = A_0' u_0, \;\; u_0 \geq 0.$$

Together with (3.29), it follows

$$(c + Cx_0)' s_1 = -(A_0' u_0)' s_1 = -u_0' (A_0 s_1) \leq 0.$$

This is in contradiction to $(c + Cx_0)' s_1 > 0$. So, (3.28) is bounded. □

Theorem 3.3 *(3.4) is feasible.*

From Lemma 3.1 and Lemma 3.2, (3.28) is feasible and bounded, which means (3.28) has an optimal solution. So the set S is not empty. Therefore, (3.4) is feasible.

To study the boundedness of (3.4), we rewrite (3.4) as

$$\text{minimize}: \quad [\,0 \quad -p_0'\,]\begin{bmatrix} h_0 \\ u_0 \end{bmatrix} + \tfrac{1}{2}\,[\,h_0' \quad u_0'\,]\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} h_0 \\ u_0 \end{bmatrix}$$

$$\text{subject to}: \quad [\,A_0 \quad 0\,]\begin{bmatrix} h_0 \\ u_0 \end{bmatrix} \le p_0,$$

$$[\,0 \quad A_0'\,]\begin{bmatrix} h_0 \\ u_0 \end{bmatrix} = -c - Cx_0, \tag{3.30}$$

$$[\,(c + Cx_0)' \quad p_0'\,]\begin{bmatrix} h_0 \\ u_0 \end{bmatrix} = 0,$$

$$[\,0 \quad -I\,]\begin{bmatrix} h_0 \\ u_0 \end{bmatrix} \le 0.$$

Theorem 3.4 *(3.30); i.e., (3.4) is bounded.*

Proof

Assume on the contrary that (3.30) is unbounded. Then for a feasible solution $\begin{bmatrix} h_0 \\ u_0 \end{bmatrix}$ for (3.30), such that there exists a vector $\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$, satisfying

$$[\,0 \quad -p_0'\,]\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} > 0,$$

$$[\,s_1' \quad s_2'\,]\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = 0,$$

and $\begin{bmatrix} h_0 \\ u_0 \end{bmatrix} - \sigma \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$ is feasible, for every positive scalar σ. Thus we have

$$\left.\begin{array}{l} -p_0' s_2 > 0, \\ s_1' C s_1 = 0 \Rightarrow C s_1 = 0, \\ A_0 s_1 \geq 0, \\ A_0' s_2 = 0, \\ (c + C x_0)' s_1 + p_0' s_2 = 0 \Rightarrow c' s_1 + p_0' s_2 = 0, \\ s_2 \leq 0. \end{array}\right\}$$

From $-p_0' s_2 > 0$ and $c' s_1 + p_0' s_2 = 0$, we get $c' s_1 > 0$. Since $A_0 s_1 \geq 0$, $A(x_0 - \sigma s_1) \leq b$, for σ small and positive. we have

$$\begin{aligned} & c'(x_0 - \sigma s_1) + \frac{1}{2}(x_0 - \sigma s_1)' C (x_0 - \sigma s_1) \\ = {} & c' x_0 + \frac{1}{2} x_0' C x_0 - \sigma c' s_1 < c' x_0 + \frac{1}{2} x_0' C x_0. \end{aligned}$$

This is in contradiction to x_0 being an optimal solution for (3.1) for $t = 0$. Therefore, (3.30) is bounded. □

3.4 The Boundedness of the Original Problem (3.1)

Lemma 3.3 *If (3.1) has an optimal solution x_0 when $t = 0$, and it is feasible for every t with $t > 0$, then it is also bounded from below for every t with $t > 0$.*

Proof

Assume on the contrary that (3.1) is unbounded for some $t = t_1 > 0$. Then for a feasible solution x_1 for $t = t_1$, there exists a vector s such that $x_1 - \sigma s$ is feasible for (3.1) for $t = t_1$, for every positive scalar σ, and $c's > 0,\ s'Cs = 0$. From the feasibility of x_1 and $x_1 - \sigma s$, we have $As \geq 0$. Let $x_2(\sigma) = x_0 - \sigma s$. Then $x_2(\sigma)$ is feasible for (3.1) for $t = 0$, for every positive σ. The objective function

$$c'x_2(\sigma) + \frac{1}{2}x_2(\sigma)'Cx_2(\sigma) = c'x_0 + \frac{1}{2}x_0'Cx_0 - \sigma c's \rightarrow -\infty, \text{ as } \sigma \rightarrow +\infty.$$

This contradicts that (3.1) has an optimal solution x_0 when $t = 0$. Thus we get the result as required. □

Chapter 4

The General Parametric QP Problem

In this chapter, we will study the general parametric QP problem with a parameter both in the linear part of the objective function and in the right-hand side of the constraints.

4.1 Solution of the General PQP Problem by Solving a Related QP Problem without the Parameter

Consider the following PQP problem

$$\min\{(c+tq)'x + \frac{1}{2}x'Cx \mid Ax \le b + tp\}. \tag{4.1}$$

Assumption 4.1 *There exists a $\hat{t} > 0$ such that (4.1) has an optimal solution for every t with $0 \le t < \hat{t}$.*

Recall the notation in Chapter 1. x_0 is an optimal solution for (4.1) for $t = 0$. A_0' is the matrix of the gradients of all the constraints active at x_0. Let p_0 be the vector whose components are those p_i associated with the rows of A_0.

Theorem 4.1 *Let Assumption 4.1 be satisfied. Suppose* $(h_0^*,\ \bar{t})$ *is an optimal continuation of* x_0 *for (4.1). In addition, suppose the optimal solution* $x(t) = x_0 + th_0^*$ *is a diminishment of* x_0*, for every* t *with* $0 < t < \bar{t}$*. Then there exists a vector* u_0 *such that* $\begin{bmatrix} h_0^* \\ u_0 \end{bmatrix}$ *is an optimal solution for the problem*

$$\min\{-p_0'u_0+q'h_0+\frac{1}{2}h_0'Ch_0 \mid A_0h_0 \le p_0,\ (c+Cx_0)'h_0+p_0'u_0 = 0,\ -c-Cx_0 = A_0'u_0,\ u_0 \ge 0\}. \quad (4.2)$$

The proof of the theorem is similar to the proof of Theorem 3.1.

Proof

Since $(h_0^*,\ \bar{t})$ is an optimal continuation of x_0 for (4.1), $x(t) = x_0 + th_0^*$ is an optimal solution for (4.1), for every t with $0 < t < \bar{t}$. The optimality conditions for (4.1) assert,

$$\left.\begin{aligned} &A(x_0 + th_0^*) \le b + tp, \\ &-c - tq - C(x_0 + th_0^*) = A'u, \quad u \ge 0, \\ &u'[A(x_0 + th_0^*) - (b + tp)] = 0, \end{aligned}\right\}$$

where $u = u(t)$. These are equivalent to

$$\left.\begin{aligned} &A(x_0 + th_0^*) \le b + tp, \\ &-c - tq - Cx_0 - tCh_0^* = A'u, \quad u \ge 0, \\ &u'[(Ax_0 - b) + t(Ah_0^* - p)] = 0. \end{aligned}\right\} \quad (4.3)$$

Since $x(t)$ is a diminishment of x_0, for every t with $0 < t < \bar{t}$, all the constraints active at $x(t)$ are also active at x_0. So the matrix of the gradients of all the constraints active at $x(t)$ is a submatrix of A_0'. Thus, (4.3) can be simplified to

$$\left.\begin{aligned} &A_0 h_0^* \leq p_0, \\ &-c - Cx_0 - tq - tCh_0^* = A_0' u, \quad u \geq 0, \\ &u'(A_0 h_0^* - p_0) = 0, \end{aligned}\right\} \tag{4.4}$$

where $u = u_0 + tu_1$ is a multiplier vector for $x(t)$ whose components are associated with the rows of A_0, and u_0 is a multiplier vector for x_0 whose components are also associated with the rows of A_0. So u_0 satisfies the optimality conditions for $t = 0$, which are

$$-c - Cx_0 = A_0' u_0, \; u_0 \geq 0. \tag{4.5}$$

From (4.5), it follows that

$$(c + Cx_0)' h_0^* + p_0' u_0 = 0. \tag{4.6}$$

Combining (4.4), (4.5) and (4.6), we get

$$\left.\begin{aligned} &A_0 h_0^* \leq p_0, \; (c + Cx_0)' h_0^* + p_0' u_0 = 0, \; -c - Cx_0 = A_0' u_0, \; u_0 \geq 0, \\ &-c - Cx_0 - tq - tCh_0^* = A_0' u, \; u \geq 0, \\ &u'(A_0 h_0^* - p_0) = 0. \end{aligned}\right\}$$

Therefore, $\begin{bmatrix} h_0^* \\ u_0 \end{bmatrix}$ is an optimal solution for the problem

$$\min\{(c+Cx_0)'h_0 + tq'h_0 + \frac{1}{2} t h_0' C h_0 \mid A_0 h_0 \leq p_0, \; (c+Cx_0)'h_0 + p_0' u_0 = 0, \; -c - Cx_0 = A_0' u_0, \; u_0 \geq 0\}, \tag{4.7}$$

because $\begin{bmatrix} h_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} h_0^* \\ u_0 \end{bmatrix}$ and $v_1 = u, \; v_2, v_3, v_4 = 0$ satisfy the optimality conditions for (4.7), which are

$$\left.\begin{aligned} &A_0 h_0 \leq p_0, \; (c + Cx_0)' h_0 + p_0' u_0 = 0, \; -c - Cx_0 = A_0' u_0, \; u_0 \geq 0, \\ &\begin{bmatrix} -(c + Cx_0) - t(q + Ch_0) \\ 0 \end{bmatrix} = \begin{bmatrix} A_0' \\ 0 \end{bmatrix} v_1 + \begin{bmatrix} c + Cx_0 \\ p_0 \end{bmatrix} v_2 + \begin{bmatrix} 0 \\ A_0 \end{bmatrix} v_3 + \begin{bmatrix} 0 \\ -I \end{bmatrix} v_4, \; v_1, v_4 \geq 0, \\ &v_1'(A_0 h_0 - p_0) = 0, \\ &v_4' u_0 = 0. \end{aligned}\right\} \tag{4.8}$$

From the second constraint of (4.7), (4.7) is equivalent to

$$\min\{-p_0'u_0 + tq'h_0 + \frac{1}{2}th_0'Ch_0 \mid A_0h_0 \le p_0,\ (c+Cx_0)'h_0 + p_0'u_0 = 0,\ -c - Cx_0 = A_0'u_0,\ u_0 \ge 0\}. \tag{4.9}$$

The optimality conditions for (4.9) are

$$\left.\begin{array}{l} A_0h_0 \le p_0,\ (c+Cx_0)'h_0 + p_0'u_0 = 0,\ -c - Cx_0 = A_0'u_0,\ u_0 \ge 0, \\ \begin{bmatrix} -t(q+Ch_0) \\ p_0 \end{bmatrix} = \begin{bmatrix} A_0' \\ 0 \end{bmatrix} w_1 + \begin{bmatrix} c+Cx_0 \\ p_0 \end{bmatrix} w_2 + \begin{bmatrix} 0 \\ A_0 \end{bmatrix} w_3 + \begin{bmatrix} 0 \\ -I \end{bmatrix} w_4,\ w_1, w_4 \ge 0, \\ w_1'(A_0h_0 - p_0) = 0, \\ w_4'u_0 = 0. \end{array}\right\} \tag{4.10}$$

From (4.8), it follows that $\begin{bmatrix} h_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} h_0^* \\ u_0 \end{bmatrix}$ and $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} u \\ 1 \\ 0 \\ 0 \end{bmatrix}$ satisfy (4.10).

Therefore, $\begin{bmatrix} h_0^* \\ u_0 \end{bmatrix}$ satisfies

$$\begin{bmatrix} -q - Ch_0^* \\ p_0 \end{bmatrix} = \begin{bmatrix} A_0' \\ 0 \end{bmatrix} \frac{u}{t} + \begin{bmatrix} c+Cx_0 \\ p_0 \end{bmatrix} \frac{1}{t} + \begin{bmatrix} 0 \\ A_0 \end{bmatrix} \alpha_1 + \begin{bmatrix} 0 \\ -I \end{bmatrix} \alpha_2,$$

where

$$\alpha_1 = (1 - \frac{1}{t})h_0^*,$$

and

$$\alpha_2 = (1 - \frac{1}{t})(A_0h_0^* - p_0).$$

Since $A_0h_0^* \le p_0$, and for every t with $0 < t < 1,\ 1 - \frac{1}{t} < 0$. Thus,

$$\alpha_2 = (1 - \frac{1}{t})(A_0h_0^* - p_0) \ge 0,$$

and

$$\frac{u}{t} \ge 0.$$

Since $u'(A_0h_0^* - p_0) = 0$,

$$(\frac{u}{t})'(A_0h_0^* - p_0) = 0.$$

Since $u = u_0 + tu_1$, we can write the third equation in (4.4), $u'(A_0h_0^* - p_0) = 0$, as $(u_0 + tu_1)'(A_0h_0^* - p_0) = 0$, for every t with $0 < t < \bar{t}$. It follows that $u_0'(A_0h_0^* - p_0) = 0$. So we have

$$\alpha_2' u_0 = (1 - \frac{1}{t})(A_0h_0^* - p_0)'u_0 = 0.$$

Thus, $\begin{bmatrix} h_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} h_0^* \\ u_0 \end{bmatrix}$ and $w_1 = \frac{u}{t}$, $w_2 = \frac{1}{t}$, $w_3 = \alpha_1$, $w_4 = \alpha_2$ satisfy

$$\left.\begin{array}{l} A_0h_0 \le p_0,\ (c + Cx_0)'h_0 + p_0'u_0 = 0,\ -c - Cx_0 = A_0'u_0,\ u_0 \ge 0, \\ \begin{bmatrix} -q - Ch_0 \\ p_0 \end{bmatrix} = \begin{bmatrix} A_0' \\ 0 \end{bmatrix} w_1 + \begin{bmatrix} c + Cx_0 \\ p_0 \end{bmatrix} w_2 + \begin{bmatrix} 0 \\ A_0 \end{bmatrix} w_3 + \begin{bmatrix} 0 \\ -I \end{bmatrix} w_4,\ w_1, w_4 \ge 0, \\ w_1'(A_0h_0 - p_0) = 0, \\ w_4'u_0 = 0, \end{array}\right\}$$

which are precisely the optimality conditions for (4.2). Thus, $\begin{bmatrix} h_0^* \\ u_0 \end{bmatrix}$ is an optimal solution for (4.2) as required. □

The importance of the optimal problem (4.2) is illustrated in the following theorem.

Theorem 4.2 *Let Assumption 4.1 be satisfied. Suppose* $\begin{bmatrix} h_0^* \\ u_0 \end{bmatrix}$ *is an optimal solution for (4.2), and suppose that* w_1*,* w_2*,* w_3 *and* w_4 *are multipliers associated with the constraints* $A_0h_0 \le p_0$*,* $(c + Cx_0)'h_0 + p_0'u_0 = 0$*,* $-c - Cx_0 = A_0'u_0$ *and* $u_0 \ge 0$*, respectively. Then* $(h_0^*,\ \bar{t})$ *is an optimal continuation of* x_0 *for (4.1), and* $v(t) = u_0 + t(w_1 - w_2u_0)$ *is an associated multiplier vector for* $x(t) = x_0 + th_0^*$*, for every* t *with* $0 \le t < \bar{t}$*, where* $\bar{t} = \min\{\hat{t},\ \tilde{t}\} > 0$*, and*

$$\hat{t} = \min\{\frac{b_i - a_i'x_0}{a_i'h_0^* - p_i} \mid \text{all } i = 1, ..., m \text{ with } a_i'h_0^* > p_i\}, \tag{4.11}$$

$$\tilde{t} = \min\{\frac{-(u_0)_i}{(w_1 - w_2u_0)_i} \mid \text{all } i = 1, ..., m \text{ with } (w_1 - w_2u_0)_i < 0\}. \tag{4.12}$$

The full (m-dimensional) vector of multipliers, $u(t)$, is obtained from $v(t)$ by assigning zero to those components of $u(t)$ associated with constraints inactive at x_0 and the appropriately indexed components of $v(t)$, otherwise.

Proof

Let A_1' be the matrix of the gradients of all the constraints inactive at x_0 for (4.1), let b_1 be the vector whose components are those b_i associated with the rows of A_1. Then $A_1 x_0 < b_1$. Similar to the proof of Lemma 1.3, there exists a $\bar{t}_1 > 0$, such that $A_1(x_0 + th_0^*) < b_1 + tp_1$, for every t with $0 \leq t < \bar{t}_1$.

Since $\begin{bmatrix} h_0^* \\ u_0 \end{bmatrix}$ is an optimal solution for (4.2), the optimality conditions assert that

$$\begin{bmatrix} -q - Ch_0^* \\ p_0 \end{bmatrix} = \begin{bmatrix} A_0' \\ 0 \end{bmatrix} w_1 + \begin{bmatrix} c + Cx_0 \\ p_0 \end{bmatrix} w_2 + \begin{bmatrix} 0 \\ A_0 \end{bmatrix} w_3 + \begin{bmatrix} 0 \\ -I \end{bmatrix} w_4, \quad w_1, w_4 \geq 0, \tag{4.13}$$

$$w_1'(A_0 h_0^* - p_0) = 0. \tag{4.14}$$

Multiply both sides of (4.13) and (4.14) by t,

$$\begin{bmatrix} -tq - tCh_0^* \\ tp_0 \end{bmatrix} = \begin{bmatrix} A_0' \\ 0 \end{bmatrix} (tw_1) + \begin{bmatrix} c + Cx_0 \\ p_0 \end{bmatrix} (tw_2) + \begin{bmatrix} 0 \\ A_0 \end{bmatrix} (tw_3) + \begin{bmatrix} 0 \\ -I \end{bmatrix} (tw_4), \quad (tw_1), (tw_4) \geq 0, \tag{4.15}$$

$$(tw_1)'(A_0 h_0^* - p_0) = 0. \tag{4.16}$$

From (4.15), we have

$$-tq - tCh_0^* = A_0'(tw_1) + (c + Cx_0)(tw_2),$$

and this is equivalent to

$$-(c + Cx_0) - t(q + Ch_0^*) = A_0' tw_1 + (c + Cx_0)(tw_2 - 1).$$

From the optimality conditions for (4.1) when $t = 0$,

$$-(c + Cx_0) = A_0' u_0, \; u_0 \geq 0,$$

we have,

$$-(c + Cx_0) - t(q + Ch_0^*) = A_0'(tw_1 - u_0(tw_2 - 1)).$$

The second and the third constraints of (4.2) give

$$u_0'(A_0 h_0^* - p_0) = 0,$$

together with (4.16), we have

$$(tw_1 - (tw_2 - 1)u_0)'(A_0 h_0^* - p_0) = 0.$$

Let $v(t) = tw_1 - u_0(tw_2 - 1)$. For $\bar{t}_2$ is given small enough, we have $tw_2 \leq 1$; *i.e.*, $tw_2 - 1 \leq 0$, for every t with $0 \leq t \leq \bar{t}_2$. Thus, $v(t) \geq 0$. Since $A_0 x_0 = b_0$, it follows from above that

$$\left.\begin{aligned} & tA_0 h_0^* \leq tp_0, \\ & -c - tq - C(x_0 + th_0^*) = A_0' v(t),\ v(t) \geq 0, \\ & v(t)'((A_0 x_0 - b_0) + t(Ah_0^* - p_0)) = 0. \end{aligned}\right\}$$

Furthermore, since $A_1(x_0 + th_0^*) < b_1 + tp_1$, for every t with $0 \leq t < \bar{t}_1$, we have

$$\left.\begin{aligned} & A(x_0 + th_0^*) \leq b + tp, \\ & -c - tq - C(x_0 + th_0^*) = A_0' v(t),\ v(t) \geq 0, \\ & v(t)'(A_0(x_0 + th_0^*) - (b_0 + tp_0)) = 0. \end{aligned}\right\}$$

Let $\bar{t} = \min\{\bar{t}_1,\ \bar{t}_2\} > 0$. Then $x(t) = x_0 + th_0^*$ and the associated multiplier $v(t) = u_0 + t(w_1 - w_2 u_0)$ satisfy the optimality conditions for (4.1), for every t with $0 \leq t < \bar{t}$. Thus, $x(t) = x_0 + th_0^*$ is optimal for (4.1), for every t with $0 \leq t < \bar{t}$. Therefore, we have $(h_0^*,\ \bar{t})$ is an optimal continuation of x_0 for (4.1) as required.

Since $x(t) = x_0 + th_0^*$ is an optimal solution for (4.1), if $a_i' h_0^* > p_i$, then $a_i' x_0 < b_i$. From (4.11), $\hat{t} > 0$. Since $v = u_0 + t(w_1 - w_2 u_0) \geq 0$, if $(w_1 - w_2 u_0)_i < 0$, then $(u_0)_i > 0$. From (4.12), $\tilde{t} > 0$. Therefore, $\bar{t} = \min\{\hat{t},\ \tilde{t}\} > 0$. □

Recall Example 1.5 in Chapter 1. In the problem of Example 1.5, the first four constraints are active at $x_0 = (1,\ 1)'$. Let $u_0 = (v_1,\ v_2,\ v_3,\ v_4)'$ be an associated multiplier vector for x_0 whose components are associated with the first four constraints. Then, we can get an optimal continuation $h_0^* = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$ of x_0 by solving (4.2), which in this problem is

$$
\begin{array}{lrcrcrcrcrcl}
\text{minimize :} & \multicolumn{11}{l}{h_2 + \frac{1}{2}h_1^2 + h_2^2 + v_1 + v_3 + \frac{1}{2}v_4} \\
\text{subject to :} & h_1 & & & & & & & & & \le & -1, \\
 & & & h_2 & & & & & & & \le & 0, \\
 & h_1 & + & h_2 & & & & & & & \le & -1, \\
 & h_1 & + & 2h_2 & & & & & & & \le & -\frac{1}{2}, \\
 & -h_1 & & & - & v_1 & & - & v_3 & - \ \frac{1}{2}v_4 & = & 0, \\
 & & & & & v_1 & & + & v_3 & + \ v_4 & = & 1, \\
 & & & & & & v_2 & + & v_3 & + \ 2v_4 & = & 0, \\
 & & & & & v_1 & & & & & \ge & 0, \\
 & & & & & & v_2 & & & & \ge & 0, \\
 & & & & & & & & v_3 & & \ge & 0, \\
 & & & & & & & & & v_4 & \ge & 0.
\end{array}
$$

The optimal solution is

$$h_0^* = \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix},$$

and

$$u_0 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

From Theorem 4.2, the optimal solution for the problem of Example 1.5 is

$$x(t) = x_0 + t h_0^* = \begin{bmatrix} 1 - t \\ 1 - \frac{1}{2}t \end{bmatrix},$$

with the multiplier vector

$$v(t) = u_0 + t(w_1 - w_2 u_0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

whose components are associated with the first four constraints, for every t with $0 < t < \bar{t}$. From Theorem 4.2, the upper limit $\bar{t}$ is determined by applying (4.11) and (4.12):

$$\hat{t}_1 = \min\{-, -, -, -, \frac{1}{1}, \frac{1}{\frac{1}{2}}\} = 1,$$

$$\tilde{t}_1 = \min\{-, -, -, -\} = +\infty,$$

from which

$$\bar{t} = \min\{1,\ +\infty\} = 1.$$

Therefore,

$$x(t) = \begin{bmatrix} 1 - t \\ 1 - \frac{1}{2}t \end{bmatrix}$$

is optimal for the problem, for every t with $0 \le t \le 1$, in agreement with our geometric determination of the optimal solution in Example 1.5. $\diamond$

4.2 Feasibility of the Problem (4.2) in Theorem 4.1

In this section, we will show that the critical problem (4.2) is feasible. Let S be the feasible region of (4.2), namely,

$$S \equiv \{ \begin{bmatrix} h_0 \\ u_0 \end{bmatrix} \mid A_0 h_0 \le p_0,\ (c + C x_0)' h_0 + p_0' u_0 = 0,\ -c - C x_0 = A_0' u_0,\ u_0 \ge 0\},$$

$$= \{ \begin{bmatrix} h_0 \\ u_0 \end{bmatrix} \mid A_0 h_0 \le p_0,\ u_0'(A_0 h_0 - p_0) = 0,\ -c - C x_0 = A_0' u_0,\ u_0 \ge 0\}.$$

The optimality conditions for the problem

$$\min\{(c + C x_0)' h_0 \mid A_0 h_0 \le p_0\}, \tag{4.17}$$

imply that if (4.17) has an optimal solution, then the set S is not empty.

Theorem 4.3 *Assume that (4.1) is feasible for every* t *with* $0 < t \leq \bar{t}$. *Then S is not empty. So (4.2) is feasible.*

Proof

From the analysis of above, we only need to show that (4.17) has an optimal solution, that is, (4.17) is feasible and bounded.

We know that x_0 is optimal for $\min\{c'x + \frac{1}{2}x'Cx \mid Ax \leq b\}$ and $A_0x_0 = b_0$. Assumption 4.1 implies that (4.1) is feasible, for every t with $0 < t < \hat{t}$. Let x_1 be a feasible solution for (4.1) for $t = t_0$, where t_0 satisfies $0 < t_0 < \hat{t}$. Then, $A_0x_1 \leq b_0 + t_0p_0$. It follows that

$$A_0(x_1 - x_0) \leq b_0 + t_0p_0 - b_0,$$

$$\Rightarrow A_0(x_1 - x_0) \leq t_0p_0,$$

$$\Rightarrow A_0\frac{x_1 - x_0}{t_0} \leq p_0.$$

Thus, $\frac{x_1 - x_0}{t_0}$ is a feasible solution for (4.17), so (4.17) is feasible.

Assume on the contrary that (4.17) is unbounded. Then for a feasible solution h_1 for (4.17), there exists an s_1 such that $h_1 - \sigma s_1$ is feasible, for every positive scalar σ, and

$$(c + Cx_0)'(h_1 - \sigma s_1) \to -\infty, \text{ as } \sigma \to +\infty.$$

Thus, we have

$$\left.\begin{aligned} &(c + Cx_0)'s_1 > 0, \\ &A_0s_1 \geq 0. \end{aligned}\right\} \tag{4.18}$$

In the original problem (4.1), when $t = 0$, the optimality conditions assert that

$$-c - Cx_0 = A_0'u_0,\ u_0 \geq 0,$$

Together with (4.18), it follows that

$$(c + Cx_0)'s_1 = -(A_0'u_0)'s_1 = -u_0'(A_0s_1) \leq 0.$$

This contradicts $(c + Cx_0)'s_1 > 0$. So, (4.17) is bounded.

Therefore, (4.17) is feasible and bounded, thus has an optimal solution, then we have (4.2) is feasible as required. □

4.3 The Boundedness of the Problem (4.2) in Theorem 4.1

The problem (4.2) maybe unbounded and have no optimal solution, which means that x_0 has no optimal continuation. Then we want to find another optimal solution x_1 for (4.1) for $t = 0$, such that x_1 has an optimal continuation.

In this section, we show how to decide whether (4.2) unbounded or not, and prove that such x_1 above always exists if Assumption 4.1 satisfies, and also give a way to find x_1.

The following theorem gives to a way to check the boundedness of (4.2) by checking a simpler optimal problem.

Theorem 4.4 *Let Assumption 4.1 be satisfied. Then, (4.2) is unbounded if and only if the problem*

$$\min\{-q's_1 \mid Cs_1 = 0,\ c's_1 = 0,\ A_0s_1 \geq 0\} \tag{4.19}$$

is unbounded from below.

Proof

Rewrite (4.2) as

$$\text{minimize}: \quad \begin{bmatrix} q' & -p_0' \end{bmatrix} \begin{bmatrix} h_0 \\ u_0 \end{bmatrix} + \tfrac{1}{2} \begin{bmatrix} h_0' & u_0' \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} h_0 \\ u_0 \end{bmatrix}$$

$$\text{subject to}: \quad \begin{bmatrix} A_0 & 0 \end{bmatrix} \begin{bmatrix} h_0 \\ u_0 \end{bmatrix} \leq p_0,$$

$$\begin{bmatrix} 0 & A_0' \end{bmatrix} \begin{bmatrix} h_0 \\ u_0 \end{bmatrix} = -c - Cx_0, \tag{4.20}$$

$$\begin{bmatrix} (c + Cx_0)' & p_0' \end{bmatrix} \begin{bmatrix} h_0 \\ u_0 \end{bmatrix} = 0,$$

$$\begin{bmatrix} 0 & -I \end{bmatrix} \begin{bmatrix} h_0 \\ u_0 \end{bmatrix} \leq 0.$$

If (4.2) is unbounded, then there exists a vector $\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$, such that

$$\begin{bmatrix} q' & -p_0' \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} > 0,$$

$$\begin{bmatrix} s_1' & s_2' \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = 0,$$

and $\begin{bmatrix} h_0 \\ u_0 \end{bmatrix} - \sigma \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$ is feasible, for every positive scalar σ. Thus, we have

$$\left.\begin{array}{l} q's_1 - p_0's_2 > 0, \\ s_1'Cs_1 = 0 \Rightarrow Cs_1 = 0, \\ A_0s_1 \geq 0, \\ A_0's_2 = 0, \\ (c + Cx_0)'s_1 + p_0's_2 = 0 \Rightarrow c's_1 + p_0's_2 = 0, \\ s_2 \leq 0. \end{array}\right\}$$

The optimal conditions for (4.1) when $t = 0$ assert that $-c - Cx_0 = A_0'u_0,\ u_0 \geq 0$. Together with $A_0s_1 \geq 0$, $Cs_1 = 0$, we have

$$c's_1 = (c + Cx_0)'s_1 = (-A_0'u_0)'s_1 = -u_0'(A_0s_1) \leq 0.$$

That is, $c's_1 \leq 0$.

If $c's_1 < 0$, from $c's_1 + p_0's_2 = 0$, we have $p_0's_2 > 0$. Since Assumption 4.1 satisfies, (4.1) has optimal solutions when $0 \leq t < \hat{t}$. Let $x(t_0) = x_0 + t_0h_0$ be an optimal solution for (4.1), for $t = t_0$ with $0 \leq t_0 < \hat{t}$, then $x(t_0) = x_0 + t_0h_0$ is also an feasible solution for (4.1) for $t = t_0$; *i.e.*, $A_0(x_0 + t_0h_0) \leq b_0 + t_0p_0$. So $A_0x_0 = b_0$ implies $A_0h_0 \leq p_0$. Furthermore, since $s_2 \leq 0$, we have

$$h_0'A_0's_2 \geq p_0's_2 > 0. \tag{4.21}$$

But since $A_0's_2 = 0$, the left-hand side of (4.21) equals to zero. It is a contradiction. Thus, we have $c's_1 = 0$ and $p_0's_2 = 0$.

Since $p_0's_2 = 0$ and $q's_1 - p_0's_2 > 0$, it follows that $q's_1 > 0$; *i.e.*, $-q's_1 < 0$. Since s_1 satisfies $Cs_1 = 0,\ c's_1 = 0,\ A_0s_1 \geq 0$ and $-q's_1 < 0$. For every positive scalar σ, σs_1 also satisfies $C(\sigma s_1) = 0,\ c'(\sigma s_1) = 0,\ A_0(\sigma s_1) \geq 0$, and

$$-q'(\sigma s_1) \to -\infty, \text{ as } \sigma \to +\infty.$$

Thus (4.19) is unbounded.

On the other hand, if (4.19) is unbounded from below, then there exists an s_1 such that

$$q's_1 > 0,\ Cs_1 = 0,\ c's_1 = 0,\ A_0s_1 \geq 0.$$

Then let $s_2 = 0$, and deserve that

$$[\, q' \quad -p_0' \,] \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} > 0,$$

and

$$[\, s_1' \quad s_2' \,] \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = 0.$$

Furthermore, $\begin{bmatrix} h_0 \\ u_0 \end{bmatrix} - \sigma \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$ is feasible, for every positive scalar $\sigma > 0$. Thus, (4.2) is unbounded from below. □

From the theorem above, it is straightforward to deduce the following lemma.

Lemma 4.1 *If (4.19) has an optimal solution $s_1^* = 0$, then (4.2) is bounded and thus has an optimal solution.*

Consider the problem

$$\min\{-q's \mid c's = 0,\ Cs = 0,\ A_0s \geq 0,\ As \geq Ax_0 - b\}. \tag{4.22}$$

It is feasible since $s_0 = 0$ is a feasible solution.

Theorem 4.5 *Suppose that (4.2) is unbounded from below. Assume (4.22) has an optimal solution s. Then $s \neq 0$. Let $x_1 = x_0 - s$. Let A_1' be the matrix of gradients of all the constraints active at x_1, let b_1 be the vector whose components are those b_i associated with the rows of A_1; i.e., $A_1x_1 = b_1$. Let p_1 be the vector whose components are those p_i associated with the rows of A_1. Then x_1 is also an optimal solution for problem (4.1) when $t = 0$, and moreover, the problem*

$$\min\{-p_1'u_0 + q'h_0 + \frac{1}{2}h_0'Ch_0 \mid A_1h_0 \leq p_1,\ (c + Cx_0)'h_0 + p_1'u_0 = 0,\ -c - Cx_1 = A_1'u_0,\ u_0 \geq 0\} \tag{4.23}$$

has a finite optimal solution.

Proof

We first show that if (4.22) has an optimal solution s, then

$$s \neq 0. \tag{4.24}$$

Otherwise, if $s = 0$ is an optimal solution for (4.22), the optimality conditions assert that

$$q = cu_1 + Cu_2 - A_0'u_3 - A'u_4, \quad u_3, u_4 \geq 0, \tag{4.25}$$

and

$$u_4'(Ax_0 - b) = 0, \tag{4.26}$$

Since A_0' is the matrix of gradients of all the constraints active at x_0, (4.25) and (4.26) can be simplified to

$$q = cu_1 + Cu_2 - A_0'u_3 - A_0'\bar{u}_4 = cu_1 + Cu_2 - A_0'(u_3 + \bar{u}_4), \quad u_3, \bar{u}_4 \geq 0, \tag{4.27}$$

$$A_0x_0 - b_0 = 0, \tag{4.28}$$

where $\bar{u}_4$ is the multiplier vector whose components are those $(u_4)_i$ associated with the rows of A_0. Then $s = 0$, u_1, u_2 and $u_3 + \bar{u}_4$ satisfy the optimality conditions for (4.19), which are

$$\left.\begin{array}{l} Cs = 0,\ c's = 0,\ A_0s \geq 0, \\[2ex] q = cu_1 + Cu_2 - A_0'(u_3 + \bar{u}_4),\ u_3 + \bar{u}_4 \geq 0, \\[2ex] (u_3 + \bar{u}_4)'A_0s = 0. \end{array}\right\}$$

Thus, $s = 0$ being an optimal solution for (4.19), together with Lemma 4.1, contradicts that (4.2) is unbounded from below. Thus, if (4.22) has an optimal solution s, then $s \neq 0$, which verifies (4.24).

Now we will prove that x_1 is also optimal for (4.1) for $t = 0$, and (4.23) has a finite optimal solution. From the fourth constraint of (4.22), $As \geq Ax_0 - b$, we have $A(x_0 - s) \leq b$, which means

$$Ax_1 \leq b.$$

From the first and second constraints of (4.22), $c's = 0,\ Cs = 0$, the objective function for x_1 is

$$c'x_1 + \frac{1}{2}x_1'Cx_1 = c'(x_0 - s) + \frac{1}{2}(x_0 - s)'C(x_0 - s) = c'x_0 + \frac{1}{2}x_0'Cx_0.$$

Thus, x_1 is also optimal for (4.1) for $t = 0$.

Since s is an optimal solution for (4.22), the optimality conditions give us:

$$\left.\begin{aligned} &q = Cu + cv - A_0'w_0 - A'w_1,\ w_0, w_1 \geq 0, \\ &w_0'A_0s = 0,\ w_1'(Ax_0 - b - As) = 0. \end{aligned}\right\} \tag{4.29}$$

Since A_1 is the matrix of gradients of all the constraints active at x_1, $A_1x_1 = b_1$; *i.e.*, $A_1(x_0-s) = b_1$, (4.29) can be simplified to

$$\left.\begin{aligned} &q = Cu + cv - A_0'w_0 - A_1'\bar{w}_1, \quad w_0, \bar{w}_1 \geq 0, \\ &w_0'A_0s = 0,\ A_1s = A_1x_0 - b_1, \end{aligned}\right\}$$

where $\bar{w}_1$ is a multiplier vector whose components are associated with the rows of A_1. From $w_0'A_0s = 0$, we know that if $a_i's = (A_0s)_i \neq 0$, then $(w_0)_i = 0$. Let A_2' be the matrix of all the a_i in A_0 such that $a_i's = 0$; *i.e.*, $A_2s = 0$. Let b_2 be the vector whose components are associated with the rows of A_2. Since A_2 is a submatrix of A_0, we have $A_2x_0 = b_2$. Then, $A_2(x_0 - s) = b_2$; *i.e.*, $A_2x_1 = b_2$. Thus, A_2 is also a submatrix of A_1. So,

$$\left.\begin{aligned} &q = Cu + cv - A_2'\bar{w}_0 - A_1'\bar{w}_1 = Cu + cu - A_1'w, \quad \bar{w}_0, \bar{w}_1, w \geq 0, \\ &A_2s = 0,\ A_1s = A_1x_0 - b_1, \end{aligned}\right\}$$

where w is a vector whose components are associated with the rows of A_1. Therefore, $s_1 = 0$ and w satisfy

$$\left.\begin{aligned} &Cs_1 = 0,\ c's_1 = 0,\ A_1s_1 \geq 0, \\ &q = Cu + cv - A_1'w,\ w \geq 0, \\ &w'A_1s_1 = 0, \end{aligned}\right\}$$

which are the optimality conditions for

$$\min\{-q's_1 \mid Cs_1 = 0,\ c's_1 = 0,\ A_1 s_1 \geq 0\}. \tag{4.30}$$

So $s_1 = 0$ is optimal for (4.30).

Then, from Lemma 4.1, (4.23) has a finite optimal solution. □

Theorem 4.6 *If (4.22) is unbounded from below, then (4.1) is either infeasible or unbounded from below, for every t with $t > 0$.*

Proof

If (4.22) is unbounded from below, then for a feasible solution s for (4.22), there exists a vector d such that $s - \sigma d$ feasible for (4.22), for every positive scalar σ, and $q'd < 0$. So d satisfies

$$q'd < 0,\ c'd = 0,\ Cd = 0,\ Ad \leq 0.$$

If for a $t > 0$, (4.1) is feasible. Let $\bar{x}(t)$ be a feasible solution for it. Then $A\bar{x}(t) \leq b + tp$. Since $Ad \leq 0$, we have

$$A(\bar{x}(t) + \sigma d) = A\bar{x}(t) + \sigma Ad \leq b + tp.$$

Furthermore,

$$(c+tq)'(\bar{x}(t)+\sigma d)+\frac{1}{2}(\bar{x}(t)+\sigma d)'C(\bar{x}(t)+\sigma d) = (c+tq)'\bar{x}(t)+\frac{1}{2}\bar{x}(t)'C\bar{x}(t)+\sigma t q'd \to -\infty, \text{ as } \sigma \to +\infty,$$

since $q'd < 0$. Thus, (4.2) is either infeasible or unbounded from below for every t with $t > 0$. □

Chapter 5

Concluding Remarks

We want to solve the general parametric quadratic programming problem (4.1). Assume it has an optimal solution x_0 for $t = 0$. First we study the feasibility of (4.1) for $t > 0$ by checking the optimal solution for (3.3). If (3.3) has an optimal solution $\hat{t} > 0$, then (4.1) is feasible for every t with $0 \leq t \leq \hat{t}$. Then we solve a non-parametric quadratic programming problem (4.2). We prove that (4.2) is feasible. If (4.2) is bounded and thus has an optimal solution h_0^*, then $x(t) = x_0 + th_0^*$ is an optimal solution for (4.1), for every t with $0 \leq t < \bar{t}$, where $\bar{t}$ can be solved from (4.11) and (4.12), and $\bar{t}$ is a "corner" point for the parametric QP. If (4.2) is unbounded from below, then we consider the LP problem (4.22). If (4.22) has an optimal solution s, then let $x_1 = x_0 - s$, and solve (4.23) for h_0^*. Then $x(t) = x_1 + th_0^*$ is an optimal solution for (4.1), for every $0 \leq t \leq \bar{t}$. If (4.22) is unbounded from below, then (4.1) is unbounded from below, for every t with $0 < t \leq \hat{t}$.

Appendix

Throughout this thesis, we have shown that difficulties arising from ties in a PQP can be resolved by solving an appropriate QP. It is possible that the resulting QP may have degenerate points, thus creating further difficulties. However, we argue here that such degenerate points are a consequence of the linear constraints in the model problem and can be resolved by solving an LP. The use of Bland's rules in solving the LP [11] guarantees that the LP and thus the QP can be solved in a finite number of steps.

Consider general convex QP problem

$$\min\{c'x + \frac{1}{2}x'Cx \mid Ax \leq b\}. \tag{1}$$

Let $f(x) = c'x + \frac{1}{2}x'Cx$. Suppose x_0 is a quasi-stationary point determined by an algorithm. Suppose that x_0 is degenerate; i.e., the gradients of those constraints active at x_0 are linearly dependent. Let A_0' be the matrix of gradients of all the constraints active at x_0 and let b_0 be the vector whose components are those b_i associated with the rows of A_0. We can consider the following LP problem

$$\min\{-(c + Cx_0)'s_0 \mid A_0 s_0 \geq 0\}. \tag{2}$$

Theorem 1 *The problem (2) either has an optimal solution $s_0 = 0$ or is unbounded from below. If (2) has an optimal solution then x_0 is optimal for the original QP problem. If (2) is unbounded from below, let s_0 be a feasible solution such that $(c + Cx_0)'s_0 > 0$ and let*

$$\hat{\sigma} = \max\{\sigma \mid A(x_0 - \sigma s_0) \leq b\},$$

$$\tilde{\sigma} = \begin{cases} (s_0'Cs_0)^{-1}(c + Cx_0)'s_0, & s_0'Cs_0 > 0, \\ +\infty, & s_0'Cs_0 = 0, \end{cases}$$

and $\sigma = \min\{\hat{\sigma}, \tilde{\sigma}\}$.

Then $\sigma > 0$. If $\sigma = +\infty$, then (1) is unbounded from below. If $\sigma < +\infty$, then $x_0 - \sigma s_0$ is feasible for (1) and $f(x_0 - \sigma s_0) < f(x_0)$.

Proof

The problem (2) is feasible since $s_0 = 0$ is a feasible solution. If (2) has an optimal solution, then the optimal solution is $s_0 = 0$, otherwise (2) is unbounded from below. From the optimality conditions for (2), there exists a $u_0 \geq 0$ such that $c + Cx_0 = -A_0' s_0$, thus x_0 is optimal for (1).

If (2) is unbounded from below, then there exist a feasible solution s_0 such that $(c+Cx_0)'s_0 > 0$. If $\hat{\sigma} = +\infty$, and $s_0' C s_0 = 0$; *i.e.*, $Cs_0 = 0$, then $c's_0 > 0$. Thus,

$$f(x_0 - \sigma s_0) = c'(x_0 - \sigma s_0) + \frac{1}{2}(x_0 - \sigma s_0)'C(x_0 - \sigma s_0) = c'x_0 + \frac{1}{2}x_0' C x_0 - \sigma c' s_0 \to -\infty, \text{ as } \sigma \to +\infty.$$

If $\hat{\sigma} < +\infty$ and $\tilde{\sigma} = +\infty$, then

$$f(x_0 - \sigma s_0) - f(x_0) = -\sigma c' s_0 < 0.$$

If $\tilde{\sigma} < +\infty$, then

$$f(x_0 - \sigma s_0) - f(x_0) = -\sigma(c+Cx_0)s_0 + \frac{\sigma^2}{2}s_0' C s_0 \leq -\sigma(c+Cx_0)'s + \frac{\sigma}{2}(c+Cx_0)'s_0 = -\frac{\sigma}{2}(c+Cx_0)'s_0 < 0.$$

Therefore, $f(x_0 - \sigma s_0) < f(x_0)$. □

Theorem 1 shows that when a degenerate quasi stationary point is determined by an active set QP algorithm, solving the indicated LP using Bland's rules will determine in a finite number of steps that either the current point is optimal or will construct a search direction which will give a strict decrease in the objective function.

Theorem 1 is apparently well known and was communicated to the author by M. J. Best [1].

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