Contributions to the theory of radicals for noncommutative rings

by

Blake Madill

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Pure Mathematics

Waterloo, Ontario, Canada, 2017

© Blake Madill 2017

Examining Committee

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

Dr. Jason Bell (Supervisor)
Dr. David Riley (External Examiner)
Dr. Ross Willard (Internal Member)
Dr. Rahim Moosa (Internal Member)
Dr. Jeffrey Shallit (Internal-external Member)

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

I am the sole author of Chapters 1 and 2. Chapters 3 and 4 are partially from joint work with Jason Bell and Forte Shinko.

Abstract

We consider several radical classes of noncommutative rings. In particular, we provide new results regarding the radical theory of semigroup-graded rings, monomial algebras, and Ore extensions of derivation type. In Chapter 2 we prove that every ring graded by a torsion free nilpotent group has a homogeneous upper nilradical. Moreover, we show that a ring graded by unique-product semigroup is semiprime only when it has no nonzero nilpotent homogeneous ideals. This provides a graded-analogue of the classical result that says a ring is semiprime if and only if it has no nonzero nilpotent ideals. In Chapter 3, the class of iterative algebras are introduced. Iterative algebras serve as an interesting class of monomial algebras where ring theoretical information may be determined from combinatorial information of an associated right-infinite word. We use these monomial algebras to construct an example of a prime, semiprimitive, graded-nilpotent algebra of Gelfand-Kirillov dimension 2 which is finitely generated as a Lie algebra. In Chapter 4, the Jacobson radical of Ore extensions of derivation type is discussed. We show that if R is a locally nilpotent ring which satisfies a polynomial identity, then any Ore extension of R of derivation type will also be locally nilpotent. Finally, we show that the Jacobson radical of any Ore extension of derivation type of a polynomial identity ring has a nil coefficient ring.

Acknowledgments

First and foremost, I would like to thank my incredible advisor, Dr. Jason Bell. Your patience and dedication helped make this thesis a success. This degree would not have been possible if it was not for your enthusiasm, helpfulness, approachability, and brilliance. I firmly believe that you are the best supervisor I could have had throughout this journey. You gave me endless support when I asked for it; However, you also gave me the time and space I needed to solve problems and learn in my own time. I cannot thank you enough.

I would also like to sincerely thank my thesis examining committee: Dr. Ross Willard, Dr. Rahim Moosa, Dr. Jeffrey Shallit, and Dr. David Riley. Your edits and feedback greatly improved the quality of this thesis and for that I am extremely thankful.

I would also like to thank the amazing faculty and staff in the Department of Pure Mathematics. I truly believe that there is something extremely special in this department. The collegial atmosphere, the hilarious and friendly people, and the respect towards graduate students make this place one-of-a-kind! In particular, I would like to thank Dr. David McKinnon for being a part of my oral examination. Moreover, I would like to sincerely thank Dr. Wentang Kuo for supervising my master's degree and for first showing me the opportunities available to me here at Waterloo.

The administrative staff in Pure Math deserve all the recognition in the world. In no particular order, let me first acknowledge Ms. Pavlina Penk. Pavlina, you may want to grab a tissue before you read this. Your kindness, excellent sense of humour, and competence at your job are beyond words. I am going to miss our daily chats, our long reflections about life, and making you cry each time I tell a mediocre joke. Ms. Nancy Maloney, you have treated me like family throughout my entire Waterloo career. In fact, I am quite sure you know me better than many members of my actual family. I will miss hearing about your fun plans for the weekend and with your grandson, and, possibly most of all, your desserts at the department barbecue. Ms. Lis D'Alessio, each year of my stay in Waterloo you successfully coordinated a large group of occasionally disorganized mathematicians. I do not know how you did this, but it speaks volumes to your competence and skill. I will miss hearing your heart-warming laugh and seeing your welcoming smile. I would also like to thank Ms. Jackie Hilts for all of her hard work and patience. I am so happy that you get to be a permanent part of the Pure Math family. I sincerely wish I could take all four of you with me to my new job!

To my fellow graduate students, the lasting friendships I am taking away from my Waterloo experience are more important to me than all of the Ph.D.'s in the world. I came here for a degree, but meeting you all was the best part. I am not going to list all of you, partially because there are too many of you, and also because I am afraid of omitting someone. But, I want you to know how special each of you are to me. I would, however, like to give a special thank you to my officemate Jon Herman. I hope we have many more curling games in the future.

Finally, I would like to thank my parents Lyle and Laurie, and my soon-to-be wife Danielle. Mom and Dad, your love and support is beyond measure. You are the two coolest people I know. Danielle, above everything else, the most special part of the last three years was meeting you. Your love, kindness, encouragement, and friendship has made me the happiest man in the world. I would be terrified of this next big journey if it were not for you. I love you.

Dedication

This thesis is dedicated to my wonderful fiancée Danielle Fearon and to my amazing parents Lyle and Laurie.

Table of Contents

Chapter 1

Preliminaries

1.1 Radicals of rings

Let R denote an associative ring which is not necessarily commutative. For now, we shall also assume that R is unital. A central and classical problem in noncommutative ring theory is to be able to describe the structure of rings in the more straight-forward language of finite dimensional linear algebra. The beauty of such a classification is well-demonstrated in the case of semisimple rings. We refer the reader to [38] for a more in-depth look at semisimple rings. A ring R is said to be *semisimple* if R is isomorphic to a (finite) direct sum of minimal left ideals of R. The famous Artin-Wedderburn theorem says that any semisimple ring R is isomorphic to a finite product of matrix rings over division rings. That is, the structure of semisimple rings may be understood purely through matrix rings over division rings.

Now define the *Jacobson radical* of a ring R, denoted by $J(R)$, to be the intersection of all maximal left ideals of R. Although seemingly unrelated to semisimple rings, it is indeed the case that a ring is semisimple if and only if it has zero Jacobson radical and is left artinian (cf. [38]). It then follows from the Artin-Wedderburn theorem that if R is left artinian then $R/J(R)$ is a finite product of matrix rings over division rings. Note that for this to hold it is important that $J(R)$ be defined so that $J(R/J(R)) = (0)$. In a sense, what we have done is remove “bad” elements from R and, by considering $R/J(R)$ instead, we have a ring with nice linear algebraic structural properties.

It might also be reasonable for one to consider zero divisors to be “bad” elements of a ring. Even worse would be a ring which has nonzero nilpotent ideals. That is, a ring with a nonzero ideal I such that $I^n = (0)$ for some positive integer n. It is well-known that R has no nonzero nilpotent ideals if and only if the intersection of prime ideals of R is zero (cf. [38]). The intersection of prime ideals of R is called the *prime radical* (also known as the lower nilradical) of R. We denote it by $\mathrm{Nil}_*(R)$. Moreover, for any ring R it is the case that $\mathrm{Nil}_*(R/\mathrm{Nil}_*(R)) = (0)$. Similar to the above discussion, if one wishes to study the structure of a ring R it may be useful to instead consider $R/\mathrm{Nil}_*(R)$.

The idea of using well-behaved ideals of a ring (e.g., the Jacobson radical or the prime radical) to collect “bad” elements of a ring was first suggested by Wedderburn in 1908 via the notion of a *radical* of a ring. This technique was first used in [35] by Köthe in 1930. An excellent reference for the general theory of radicals is [18]. We shall now give a brief outline of this theory. At this point, we no longer insist that our rings be unital. With this in mind, we may consider ideals of a ring to be themselves rings. We use the notation $I \trianglelefteq R$ to mean I is a (two-sided) ideal of R.

Definition 1. A class of rings γ is said to be a radical class in the sense of Kurosh and Amitsur (or radical class, for short) if

1. γ is homomorphically closed;
2. For every ring R, the ideal $\gamma(R) := \sum\{I \trianglelefteq R : I \in \gamma\}$ is in γ;
3. $\gamma(R/\gamma(R)) = (0)$ for every ring R.

Definition 2. Let γ be a radical class of rings. The ring $\gamma(R)$ is called the γ-radical of R. Moreover, if R is a ring such that $\gamma(R) = R$ then R is said to be a γ-radical ring.

We now discuss the radical classes which have enjoyed the most popularity and which appear in this thesis.

1.1.1 The upper nilradical

A ring R is said to be a *nil* ring if for every $a \in R$ there exists an $n = n(a) \in \mathbb{N}$ such that $a^n = 0$. That is, every element of R is nilpotent. Now, consider the class $\mathcal{N}$ of nil rings. We first need some lemmas.

Lemma 1.1.1. *Let R be a ring and let I be an ideal of R. If I is nil and R/I is nil, then R is nil.*

Proof. Take some $a \in R$. Since R/I is nil there exists an n such that $a^n \in I$. Moreover, since I is nil there exists an m such that $(a^n)^m = 0$. Therefore $a^{nm} = 0$ and so R is nil. □

Lemma 1.1.2. *Let I and J be two nil ideals of a ring R. Then $I + J$ is also a nil ideal.*

Proof. Notice that $(I + J)/J \cong I/(I \cap J)$ is nil since I is nil. As J is also nil, it follows from the above lemma that $I + J$ is nil. □

It then inductively follows that any finite sum of nil ideals is also nil. We now prove that the class of nil rings is indeed a radical class of rings.

Proposition 1.1.1. *The class of nil rings is a radical class of rings.*

Proof. First assume that R and S are rings, R is a nil ring, and $f : R \to S$ is a surjective ring homomorphism. Let $b \in S$ so that there exists an $a \in R$ such that $f(a) = b$. Since R is nil there exists an n such that $a^n = 0$. We then easily see that $b^n = f(a)^n = f(a^n) = 0$ so that S is a nil ring and $\mathcal{N}$ is homomorphically closed.

Now let R be a ring and let $\mathcal{N}(R)$ denote the sum of nil ideals in R. It is straightforward to check that $\mathcal{N}(R)$ is an ideal of R. We need only show that $\mathcal{N}(R)$ is nil. Consider an element $a \in \mathcal{N}(R)$. By the definition of $\mathcal{N}(R)$ there exists finitely many ideals $I_1, I_2, \ldots, I_m$ in $\mathcal{N}$ such that $a \in I_1 + I_2 + \cdots + I_m$. By the above lemma and remark we see that a is then nilpotent. Therefore $\mathcal{N}(R) \in \mathcal{N}$, as required.

Finally, suppose that I is a nil ideal of $R/\mathcal{N}(R)$. By correspondence, there is then an ideal J of R such that $\mathcal{N}(R) \subseteq J$ and $I = J/\mathcal{N}(R)$. Since I is nil and $\mathcal{N}(R)$ is nil, from above, it again follows from the lemma that J is a nil ideal of R. Therefore $J \subseteq \mathcal{N}(R)$, $I = (0)$, and so $\mathcal{N}(R/\mathcal{N}(R)) = (0)$.

$\square$

To precisely correspond with our definition of a radical class, if we denote the class of nil rings by $\mathcal{N}$, then for any ring R we should denote the $\mathcal{N}$-radical of R by $\mathcal{N}(R)$. However, the notation most common in the literature is to denote this radical by $\mathrm{Nil}^*(R)$. Moreover, this radical is usually called the *upper nilradical* of R. This was the radical first used by Köthe in [35] while studying nil one-sided ideals of rings. We now pause to give some examples of the upper nilradical.

Example 1. Suppose R is a commutative ring. Let S denote the set of all nilpotent elements of R. We certainly have that $\mathrm{Nil}^*(R) \subseteq S$. Moreover, since R is commutative it is straightforward to check that S forms a nil ideal of R. It then follows that $S \subseteq \mathrm{Nil}^*(R)$ and so the upper nilradical of a commutative ring R is precisely the set of nilpotent elements of R.

Example 2. Of course, the above phenomenon need not hold for noncommutative rings. Consider $R = M_2(\mathbb{C})$ and let

$$a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then $a^2 = b^2 = 0$ but $a + b$ is invertible and hence is certainly not nilpotent.

Example 3. Consider the ring

$$R = \mathbb{C}\langle x, y\rangle/(x^2, xy).$$

We see that (x) is a nil ideal and so $(x) \subseteq \mathrm{Nil}^*(R)$. Furthermore, $R/(x) \cong \mathbb{C}[y]$ is a domain so that $\mathrm{Nil}^*(R/(x)) = (0)$. It then follows that $\mathrm{Nil}^*(R) \subseteq (x)$ and so $\mathrm{Nil}^*(R) = (x)$. Notice that in this case the upper nilradical is actually nilpotent. In particular, $\mathrm{Nil}^*(R)^2 = (0)$.

1.1.2 The Jacobson radical

The radical which has enjoyed the most popularity is undoubtably the Jacobson radical. Its uses in the study of semisimple modules has developed much of modern ring theory. Concretely, the Jacobson radical of a ring may be defined as follows. Recall that we denote the Jacobson radical of a ring R by $J(R)$.

Definition 3. Let R be a unital ring. Then the following are equivalent.

1. $r \in J(R)$;
2. $r \in M$ for every maximal left ideal M of R;
3. $r \in M$ for every maximal right ideal M of R;
4. $1 + xr$ is right-invertible for any $x \in R$;
5. $1 + rx$ is left-invertible for any $x \in \mathbb{R}$;
6. r right-annihilates any simple right R-module.
7. r left-annihilates any simple left R-module.

Proof. See Lemma 4.1 in [38]. □

We now define the class of rings which give rise to the Jacobson radical via Definition 1. Moreover, we extend the definition of the Jacobson radical to nonunital rings.

A ring R is said to be *right-quasi-regular* if for every $a \in R$ there exists a $b \in R$ such that

$$a + b + ab = 0.$$

Similarly, R is said to be *left-quasi-regular* if for every $a \in R$ there exists a $b \in R$ such that $a + b + ba = 0$. We say that a ring is *quasi-regular* if it is both right- and left-quasi-regular. We call the class of right-quasi-regular rings the Jacobson class and we denote it by $\mathcal{J}$. To be consistent with the literature, we call the $\mathcal{J}$-radical the Jacobson radical and we denote it by $J(R)$ instead of $\mathcal{J}(R)$.

Lemma 1.1.3. *Suppose that R is a quasi-regular ring and $a \in R$ so that there exists $b, c \in R$ with $a + c + ca = a + b + ab = 0$. Then $b = c$.*

Proof. We see that $ab + cb + cab = ca + cb + cab = 0$ and so $ab = ca$. The result follows. $\square$

A remarkable property of the Jacobson radical is that all right-quasi-regular right ideals of a ring are contained in the Jacobson radical. Furthermore, all left-quasi-regular left ideals of a ring are contained in its Jacobson radical. This greatly simplifies ones work when trying to prove containment in the Jacobson radical. The analogous statement involving the upper nilradical is still an open problem. This is the famous Köthe conjecture, which will be discussed in the next section. We summarize the above paragraph in the following results.

Lemma 1.1.4. *Let R be a ring and let I be a (left-)right-quasi-regular (left) right ideal. Then I is quasi-regular.*

Proof. Suppose that I is a right ideal which is right-quasi-regular and let $a \in I$. Then there exists $b \in I$ such that $a + b + ab = 0$. Since I is a right ideal, we have that $b \in I$. Hence there exists a $c \in I$ such that $b + c + bc = 0$. As in the proof of the above lemma we have that $c = a$. Therefore I is quasi-regular. $\square$

In particular, we could have defined the Jacobson radical class of rings to be the class of *quasi-regular* rings.

Proposition 1.1.2. *If R is a ring then all quasi-regular right (left) ideals are contained in $J(R)$.*

Proof. See Chapter 1 of [23]. $\square$

That is to say,

$$\{a \in R : aR \text{ quasi-regular}\} = \{a \in R : Ra \text{ quasi-regular}\} = J(R).$$

We remark that if R is a quasi-regular ring and we define the binary operation $a \circ b := a + b + ab$ for every $a, b \in R$ then $(R, \circ)$ is a group. In fact, R is quasi-regular if and only if $(R, \circ)$ is a group (see [18]).

If R is quasi-regular and $a, b \in R$ such that $a + b + ab = 0$ then we call b the *quasi-inverse* of a.

Proposition 1.1.3. *Any nil ring is quasi-regular. Hence* $\mathrm{Nil}^*(R) \subseteq J(R)$ *for any ring* R.

Proof. Suppose that R is nil. Take $a \in R$ so that $a^{n+1} = 0$ for some $n \in \mathbb{N}$. Consider

$$b = -a + a^2 - a^3 + \cdots + (-1)^n a^n.$$

Then $ab + b = -a$ and so $a + b + ab = 0$. Therefore R is quasi-regular. □

Lemma 1.1.5. *Let R be a ring and let I be an ideal of R. If R/I is quasi-regular and I is quasi-regular then R is quasi-regular.*

Proof. Take $a \in R$. Since R/I is quasi-regular there exists a $b \in R$ such that $a+b+ab \in I$. Moreover, since I is quasi-regular there exists $c \in I$ such that $a+b+ab+c+ac+bc+abc = 0$. From this we see that $a + (b + c + bc) + a(b + c + bc) = 0$ so that the quasi-inverse of a is $b + c + bc$. □

Lemma 1.1.6. *Let R be a ring. The sum of two quasi-regular ideals of R is again quasi-regular.*

Proof. As in the upper nilradical case, if I and J are quasi-regular ideals then $(I + J)/I$ and I are quasi-regular so that by the above lemma, $I + J$ is quasi-regular as well. □

It now follows that $J(R)$ is quasi-regular, as desired. Moreover, the property of being quasi-regular is clearly preserved through homomorphic images. Finally, if R is a ring and I is a quasi-regular ideal of $R/J(R)$ then there exists an ideal A of R containing $J(R)$ such that $I = A/J(R)$. Since $J(R)$ is quasi-regular and I is quasi-regular, we have by the above lemma that A is quasi-regular as well. Hence $A \subseteq J(R)$ and so $I = (0)$. Therefore $J(R/J(R)) = (0)$ for any ring R. In conclusion, we have the following proposition.

Proposition 1.1.4. *The Jacobson class of rings forms a radical class.*

We now take a moment to verify that this definition of the Jacobson radical is consistent with the one presented in Definition 3 for unital rings.

Proposition 1.1.5. *Let R be a unital ring. Then aR is quasi-regular if and only if $1 + ax$ is left-invertible for any $x \in R$.*

Proof. Let $a \in J(R)$ so that aR is quasi-regular. Take $x \in R$ so that $ax \in aR$. As aR is quasi-regular there exists a $b \in R$ such that $ax + b + bax = 0$. It then follows that $b(1 + ax) + ax = 0$. Now, $1 + ax, ax \in R(1 + ax)$ and so $1 \in R(1 + ax) = R$. Therefore there exists $r \in R$ such that $r(1 + ax) = 1$. The converse is similar. □

1.1.3 Köthe's conjecture

As discussed above, it is enough to understand quasi-regular one-sided ideals in order to understand the Jacobson radical. However, the analogous result for the upper nilradical is only conjectured to be true. That is, it was conjectured by Köthe in [35] that the upper nilradical of a ring is simply the sum of its nil right ideals. This conjecture is now known as *Köthe's conjecture*. With hope to prove this conjecture true, many equivalent formulations of the Köthe conjecture have been given. The following equivalent statements can be found in [3, 37]. An excellent survey of this famous conjecture is given by Smoktunowicz in [60].

Theorem 1.1.1. *The following statements are equivalent.*

1. *For any ring R, $\mathrm{Nil}^*(R) = \{a \in R : aR$ is nil $\}$;*

2. *For any ring R, $\mathrm{Nil}^*(R) = \{a \in R : Ra$ is nil $\}$;*

3. *The sum of two nil right ideals in any ring is nil;*

4. *For every nil ring R, $M_2(R)$ is nil;*

5. *For every nil ring R, $M_n(R)$ is nil for every $n \in \mathbb{N}$;*

6. *For every nil ring R, $J(R[x]) = R[x]$.*

For many nice classes of rings, the Köthe conjecture is known to be true. From our work in the previous section, we certainly know that the conjecture is true whenever the upper nilradical and Jacobson radical coincide. More specifically, the Köthe conjecture is true for all rings whose Jacobson radical is nil. This includes:

1. artinian rings (see [23]);
2. affine polynomial identity rings, including affine commutative rings (see [57]);
3. algebraic k-algebras, where k is a field;
4. k-algebras R, where k is a field, such that $\dim_k(R) < |k|$.

To see these last two points, suppose that R is a k-algebra such that $\dim_k(R) < |k|$. It then follows that for any $x \in J(R)\backslash\{0\}$, the set

$$\left\{(1-\alpha x)^{-1} : \alpha \in k\right\}$$

is linearly dependent. By considering a linear dependence and clearing denominators we see that x is algebraic over k. Therefore it is sufficient to show that $J(R)$ is nil whenever R is an algebraic algebra.

Suppose R is an algebraic k-algebra where k is a field. Take $x \in J(R)$. Since R is algebraic, there exist $\alpha_i \in k$ such that

$$x^m + \alpha_{m-1}x^{m-1} + \cdots + \alpha_1 x + \alpha_0 = 0.$$

Moreover, since $J(R)$ is an ideal $\alpha_0 \in J(R)$. Since R is unital, we have $J(R) \neq R$ and so $\alpha_0 = 0$. Therefore there exists an $r < m$ such that

$$x^m + \alpha_{m-1}x^{m-1} + \cdots + \alpha_{m-r}x^{m-r} = 0,$$

where $\alpha_{m-r} \neq 0$. Note that if $r = 0$ then we are done, so we may assume $r > 0$. However, we then see that

$$\alpha_{m-r}x^{m-r}(1 + yx) = 0,$$

for some $y \in R$. Since $x \in J(R)$, $1 + yx$ is invertible. Therefore $x^{m-r} = 0$, as required.

For a large period of time Köthe's conjecture was widely believed to be true. However, recent examples constructed by Smoktunowicz give evidence that the conjecture may be false.

Theorem 1.1.2 (Smoktunowicz, 2000)**.** *For every countable field F there exists a nil F-algebra R such that $R[x]$ is not nil.*

We note that, had it been true that a ring R is nil if and only if $R[x]$ is nil, then Köthe's conjecture would be true. This follows from the characterization Krempa gave in [37] which is statement 6 in Theorem 1.1.1. Namely, R being nil implying $R[x]$ is nil would have guaranteed that $J(R[x]) = R[x]$.

1.1.4 Other radicals

In this section we briefly examine some other radicals which have enjoyed popularity throughout classical and modern ring theory. We begin with the *prime radical* of a ring, sometimes refereed to as the lower nilradical or Baer-McCoy radical. An excellent reference for the theory surrounding the prime radical is chapter 4 of [38]. We now outline the important theory of this radical.

Given a ring R, we defined the prime radical of R, denoted by $\text{Nil}_*(R)$, to be the intersection of all prime ideals of R. It is the smallest semiprime ideal of a ring. In the case of commutative rings, it is well-known that the prime radical of a ring is simply the set of nilpotent elements of the ring.

While the above definition is the one mostly commonly seen in the literature, we quickly discuss the class of rings which correspond to the prime radical. Namely, we discuss the class of *strongly nilpotent* rings.

Definition 4. Let R be a ring. An *m-sequence* in R is a sequence $(a_i)_{i=1}^{\infty}$ in R such that $a_{n+1} \in a_n R a_n$ for every $n \in \mathbb{N}$. An element a of R is said to be strongly nilpotent if every m-sequence in R beginning with a is eventually zero.

Theorem 1.1.3. *Let R be a ring. Then $a \in \mathrm{Nil}_*(R)$ if and only if a is strongly nilpotent.*

Proof. First suppose that a is strongly nilpotent. For contradiction suppose that there exists a prime ideal P such that $a \notin P$. Since P is prime, $aRa \nsubseteq P$. Therefore there exists an $a_2 \in R$ such that $aa_2a \notin P$. Similarly, we get an $a_3 \in R$ such that $(aa_2a)a_3(aa_2a) \notin P$. Continuing this way we get an m-sequence (a_i) starting with a such that every term in the sequence is not in P. However, since a is strongly nilpotent we must have that $0 \notin P$, a contradiction. Therefore $a \in \mathrm{Nil}_*(R)$.

Conversely, suppose $a \in \mathrm{Nil}_*(R)$ and let (a_i) be an m-sequence starting with a. Let $S = \{a_i\}$. For contradiction suppose that 0 does not occur in S. Let I be an ideal such that $I \cap S = \emptyset$. For instance, (0) satisfies this property. Furthermore, suppose I is chosen to be maximal with this property. Let J_1 and J_2 be ideals of R such that $J_1 J_2 \subseteq I$ but with $J_1 \nsubseteq I$ and $J_2 \nsubseteq I$. By maximality there exists $a_i \in I + J_1$ and $a_j \in I + J_2$. However, we then see that for large enough ℓ, there exists an $a_\ell \in I$, a contradiction. Therefore I must be a prime ideal. Since a is in every prime ideal, we must have that $a = a_1 \in I$, yielding yet another contradiction. Hence $0 \in S$ so that (a_i) is eventually zero. We conclude that a is strongly nilpotent. □

Let us say that a ring is strongly nilpotent if each of its elements is strongly nilpotent. Using the prime ideal characterization we easily see that the class of strongly nilpotent rings forms a radical class of rings.

Corollary 1.1.1. *The class of strongly nilpotent rings forms a radical class of rings. The radical associated to this class is the prime radical.*

One particular consequence of the strongly nilpotent characterization of the prime radical is that for any ring R, $\mathrm{Nil}_*(R)$ is nil. To see this, observe that for any $a \in \mathrm{Nil}_*(R)$,

$(a, a^3, a^7, \dots)$ is an m-sequence. Therefore $a^n = 0$ for some n. At this point we now know that for every ring R

$$\mathrm{Nil}_*(R) \subseteq \mathrm{Nil}^*(R) \subseteq J(R).$$

Recall an ideal P of a ring R is said to be *semiprime* if whenever $a \in R$ such that $aRa \subseteq P$ then $a \in P$ or $a \in P$. Moreover, we say that a ring is semiprime if (0) is a semiprime ideal of R. That is, when $a \in R$ such that $aRa = (0)$ then $a = 0$. When working with the prime radical, the following result is often useful.

Proposition 1.1.6 ([38], Proposition 10.16)**.** *For any ring R, the following are equivalent.*

1. *R is semiprime;*
2. *$\mathrm{Nil}_*(R) = (0)$;*
3. *R has no nonzero nilpotent ideal;*
4. *R has no nonzero nilpotent left (or right) ideal.*

The last radical which shall be mentioned in this thesis is the locally nilpotent radical, often referred to as the *Levitzki radical*.

Definition 5. A ring R is said to be locally nilpotent if for every finite subset $S \subseteq R$ there is a positive integer $N = N(S)$ such that $S^N = (0)$. That is, for every $a_1, a_2, \dots, a_N \in S$, $a_1 a_2 \cdots a_N = 0$.

Let us denote the class of locally nilpotent rings by $\mathcal{L}$. Using similar techniques as in the case of the upper nilradical, it can be shown that the class of locally nilpotent rings forms a radical class of rings.

Proposition 1.1.7. *The locally nilpotent class of rings $\mathcal{L}$ forms a radical class of rings.*

To be clear, given a ring R, the Levitzki radical of R, $\mathcal{L}(R)$, is simply the sum of all locally nilpotent ideals of R. It contains all nilpotent ideals of the ring. As $\mathcal{L}(R)$ is locally nilpotent, it is also nil. Therefore we have that

$$\mathcal{L}(R) \subseteq \text{Nil}^*(R) \subseteq J(R).$$

Less clear, however, is that the Levitzki radical contains the prime radical.

Proposition 1.1.8. *If R is a ring then $\text{Nil}_*(R) \subseteq \mathcal{L}(R)$. That is, the prime radical of R is locally nilpotent.*

Proof. By considering $R/\mathcal{L}(R)$ we may as well assume $\mathcal{L}(R) = (0)$. We must show that $\text{Nil}_*(R) = (0)$ as well. However, by the characterization in Proposition 1.1.6 we then have that R has no nonzero nilpotent ideal. Hence $\text{Nil}_*(R) = (0)$, as desired. □

Therefore we have the following useful containments:

$$\text{Nil}_*(R) \subseteq \mathcal{L}(R) \subseteq \text{Nil}^*(R) \subseteq J(R).$$

1.1.5 Honourable mentions

So far, we have considered classes of rings which are quasi-regular, nil, locally nilpotent, and strongly nilpotent. An obvious class which is missing here is the class of nilpotent rings. Recall that a ring R is said to be *nilpotent* if there exists a positive integer n such that $R^n = (0)$. The number n is called the *index of nilpotence* of the ring. At first glance, it seems as though the class of nilpotent rings should form a radical class of rings in a similar way to the Levitzki radical. However, when considering the Levitzki radical, the "finiteness" condition in the definition of locally nilpotent rings plays a crucial role in proving that sum of locally nilpotent ideals of a ring is again locally nilpotent. As for the upper nilradical, in order to show the infinite sum of nil ideals of a ring is again a nil ideal, it is extremely important that we are able to consider one element at a time, and consider the finite sum of nil ideals containing that element. In this thesis, the class of nilpotent rings shall serve as the first example of a class of rings which is not a radical class of rings.

As we will see, considering the sum of nilpotent ideals of a ring can still be useful. Given a ring R, let us denote the sum of nilpotent ideals of R by $W(R)$. This ideal is often referred to as the *Wedderburn radical*, even though it is not itself a radical. Consider the following example.

Example 4. Let $R = \mathbb{C}[x_i : i \in \mathbb{N}]$ and let $I = (x_i^{i+1} : i \in \mathbb{N})$. We then see that for every $n \in \mathbb{N}$, $(x_n)^{n+1} \subseteq I$. Note that we are using commutativity at this step. Let $S := R/I$. We then see that $(x_n) \subseteq W(S)$ for every $n \in \mathbb{N}$. Here, we are abusing notation and writing $(x_n) + I$ as (x_n). It then follows that $W(S) = (x_i : i \in \mathbb{N})$. However, we then observe that for any $n \in \mathbb{N}$, $x_1 x_2 x_3 \cdots x_n \neq 0$. Therefore $W(S)$ is not nilpotent, even though S is a commutative ring!

It should be noted that in the above example we still have that $W(S/W(S)) = (0)$. This always happens for commutative rings. To see this recall that if R is a commutative ring, then $\mathrm{Nil}_*(R)$ is precisely the set of nilpotent elements of R. Moreover, for every $a \in \mathrm{Nil}_*(R)$, (a) is a nilpotent ideal. Therefore $a \in W(R)$ and so $W(R) = \mathrm{Nil}_*(R)$. In particular, $\mathrm{Nil}_*(R/W(R)) = (0)$ and so $W(R/W(R)) = (0)$ as well. We record this as a proposition.

Proposition 1.1.9. *If R is a commutative ring then $W(R) = \mathrm{Nil}_*(R)$. Moreover,*

$$W(R/W(R)) = (0).$$

However, this need not hold for noncommutative rings.

Example 5. Consider the ring R constructed by taking $F = \mathbb{C}\langle y, x_i : i \in \mathbb{N}\rangle$ modulo the relations $x_i F x_i = (0)$, $[x_i, x_j] = 0$, and $y^2 = x_1$, for all $i, j \in \mathbb{N}$. We see that $W(R) = \sum_{i=1}^{\infty}(x_i)$. Consider

$$S := R/W(R) = \mathbb{C}[y]/(y^2).$$

Notice that $W(S) \neq 0$.

As every nilpotent ideal of a ring R is contained in each of its prime ideals, we have the following useful containments:

$$W(R) \subseteq \mathrm{Nil}_*(R) \subseteq \mathcal{L}(R) \subseteq \mathrm{Nil}^*(R) \subseteq J(R).$$

Moreover, by Proposition 1.1.6 we have the following result.

Proposition 1.1.10. *If R is a ring then $W(R) = (0)$ if and only if $\mathrm{Nil}_*(R) = (0)$.*

Proof. This follows from the facts that $W(R) \subseteq \mathrm{Nil}_*(R)$ and $\mathrm{Nil}_*(R) = (0)$ only when R has no nonzero nilpotent ideal. □

Due to this close connection between the Wedderburn radical and the prime radical, the Wedderburn radical is still a useful object to consider when studying the structure of a noncommutative ring.

It is clear that being nilpotent is a much stronger property for a ring to have than being nil. An interesting class of rings lying strictly between the classes of nilpotent and nil rings is the class of nil rings of bounded index. A ring R is said to be *nil of bounded index* if for all $r \in R$ there exists an $n \in \mathbb{N}$, not depending on r, such that $r^n = 0$. Clearly all rings which are nil of bounded index are also nil, as the name indicates. Moreover, every nilpotent ring is certainly nil of bounded index—just take the index of nilpotence for the ring. To see that these containments are strict, consider the following two examples.

Example 6. Let R be the positive part of $\mathbb{Z}_2[x_i : i \in \mathbb{N}]$ with respect to the usual grading. Let $I = (x_i^2 : i \in \mathbb{N})$. Then $S := R/I$ is nil of bounded index but not nilpotent.

Example 7. Let R again be the positive part of $\mathbb{C}[x_i : i \in \mathbb{N}]$. Let $I = (x_i^{i+1} : i \in N)$. Then $S := R/I$ is nil but not of bounded index.

Consider the class B of all rings which are nil of bounded index. For any ring R, let us define

$$B(R) := \{a \in R : RaR \text{ is nil of bounded index}\}.$$

Should this class of rings form a radical class, $B(R)$ would be our corresponding radical. We call $B(R)$ the *bounded nilradical* of R, even though it is also not a radical. Similar to the Wedderburn radical, the boundedness condition does not work well with infinite sums of rings which are nil of bounded index. An excellent survey on the bounded nilradical can be found in [45]. Consider the example below.

Example 8. Let R be the positive part of $\mathbb{C}[x_i : i \in \mathbb{N}]$ and let $I = (x_i^2 : i \in \mathbb{N})$. Then $S := R/I$ is nil but not of bounded index. Moreover, $B(S) = S$ which is not nil of bounded index.

As in the case of the Wedderburn radical, the bounded nilradical is still an interesting object of study due to its close relationship with the prime radical. This relationship will be further described in the next section. For the time being, let us observe the following useful results.

Proposition 1.1.11. *Let R be a ring and let $S := \{a \in R : aR$ is nil of bounded index$\}$. Then $B(R) = S$. That is, the bounded nilradical may be defined in terms of right ideals which are nil of bounded index.*

Proof. See [45]. □

Much as in the case of the Jacobson radical, this makes computations with the bounded nilradical much easier.

For any ring R we have the following containments:

$$W(R) \subseteq B(R) \subseteq \mathrm{Nil}_*(R) \subseteq \mathcal{L}(R) \subseteq \mathrm{Nil}^*(R) \subseteq J(R).$$

To see that $B(R) \subseteq \mathrm{Nil}_*(R)$ for any ring R, we consider the following characterization of the bounded nilradical, first proved by Klein in [34].

Proposition 1.1.12. *If R is a ring then*

$$B(R) = \{a \in R : \text{ any } m\text{-sequence starting with } a \text{ has bounded length}\}.$$

By the lentgh of an m-sequence we mean the number of its nonzero entries. With that being said, we can see that $B(R)$ is strongly nilpotent and so $B(R) \subseteq \mathrm{Nil}_*(R)$ for any ring R. Also, since $W(R) \subseteq B(R) \subseteq \mathrm{Nil}_*(R)$ we have the following.

Proposition 1.1.13. *If R is a ring then $B(R) = (0)$ if and only if $W(R) = (0)$ if and only if $\mathrm{Nil}_*(R) = (0)$.*

An even deeper connection between these radicals will be explained in the next section.

1.1.6 Radical constructions

In this section we examine a construction of Levitzki in [40] which demonstrates an even stronger relationship between the Wedderburn and prime radicals. This construction uses transfinite induction and relies on the fact that the class of all ordinals is too big to be a set (cf. [39]). Following [39], we consider the following construction.

Let R be a ring and let $W_1(R) = W(R)$, the sum of all nilpotent ideals of R. For each ordinal α we define W_α inductively according to the following construction.

1. If α is a limit ordinal then
$$W_\alpha(R) := \bigcup_{\beta<\alpha} W_\beta(R).$$
2. If α is a successor ordinal such that $\beta + 1 = \alpha$ then
$$W_\alpha(R) := \{a \in R : a + W_\beta(R) \in W(R/W_\beta(R))\}.$$

We note that each $W_\alpha(R)$ is an ideal of R. Moreover, we have the following ordering result.

Lemma 1.1.7. *If α and β are ordinals such that $\alpha < \beta$ then $W_\alpha(R) \subseteq W_\beta(R)$ for any ring R. Therefore $\{W_\alpha(R)\}$ forms a well-ordered set under inclusion.*

Proof. See [40]. $\square$

Now, if $W_\alpha(R) \neq W_{\alpha+1}(R)$ for all ordinals α then there is a one-to-one correspondence between a set (a subset of ideals of R) and the class of all ordinals, which is a contradiction. We then conclude that there must be an ordinal α such that $W_\alpha(R) = W_{\alpha+1}(R)$.

The following result describes how the prime radical may be constructed from the Wedderburn radical.

Proposition 1.1.14. *Let R be a ring and α be an ordinal such that $W_\alpha(R) = W_{\alpha+1}(R)$. Then $R/W_\alpha(R)$ has no nonzero nilpotent ideals and $W_\alpha(R) = \mathrm{Nil}_*(R)$.*

Proof. Begin by observing that

$$W_{\alpha+1}(R) := \{a \in R : a + W_\alpha(R) \in W(R/W_\alpha(R))\} = W_\alpha(R).$$

Therefore $W(R/W_\alpha(R)) = (0)$ and so $R/W_\alpha(R)$ has no nonzero nilpotent ideals. By the characterization in Proposition 1.1.6 we see that $\mathrm{Nil}_*(R) \subseteq W_\alpha(R)$. Using transfinite induction, it can be shown that $W_\beta(R) \subseteq \mathrm{Nil}_*(R)$ for every ordinal β. Therefore $W_\alpha(R) = \mathrm{Nil}_*(R)$, as desired. $\square$

Note that if $W(R) = (0)$ then $W_\alpha(R) = (0)$ for any ordinal α. From this it easily follows that $\mathrm{Nil}_*(R) = (0)$ when R has no nonzero nilpotent ideals.

1.2 Rings satisfying a polynomial identity

1.2.1 Preliminaries

A large portion of this thesis is dedicated to the study of *polynomial identity rings* or *algebras*. For this reason, in this section we give a brief introduction to polynomial identity (algebras) rings and their radical theory. Excellent references for this theory are [15, 57].

Recall that if R is a commutative ring then $A = R\langle x_1, \dots, x_n\rangle$ denotes the free R-algebra generated by $\{x_1, \dots, x_n\}$. Moreover, a polynomial $f(x_1, \dots, x_n) \in A$ is said to be *homogeneous* of degree d if it is a R-linear combination of monomials of length d. A homogeneous polynomial $f(x_1, \dots, x_n) \in A$ is said to be *multilinear* if each monomial of f has degree 1 in x_i for each $1 \le i \le n$.

Example 9. The polynomial

$$\mathfrak{S}_n(x_1, \dots, x_n) := \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)},$$

is a multilinear polynomial called the standard multilinear polynomial of degree n.

This brings us to the definition of a polynomial identity ring.

Definition 6. Suppose that k is a commutative ring and R is a k-algebra. We say that $0 \neq f(x_1, \dots, x_n) \in k\langle x_1, \dots, x_n\rangle$ is a polynomial identity (PI) for R if $f(a_1, \dots, a_n) = 0$ for all $a_i \in R$. We say that a polynomial identity for R is proper if at least one coefficient of a monomial of highest degree is 1. If a proper polynomial identity exists for R we call R a polynomial identity algebra (PI algebra, for short) over k. If $k = \mathbb{Z}$ and R is a PI k-algebra then we call R a PI ring.

Note that if $f(x_1, \dots, x_n)$ is a multilinear proper polynomial identity for some k-algebra then f is of the form

$$f(x_1, \dots, x_n) = x_1 x_2 \cdots x_n + \sum_{\sigma \in S_n \setminus \{1\}} c_\sigma x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)},$$

where $c_\sigma \in k$.

Consider a ring R of characteristic p. We do not want all such rings to be considered as polynomial identity rings just because they satisfy the identity $f(x) = px \in \mathbb{Z}\langle x\rangle$. This is why the polynomial identity is required to be proper in the above definition. Note that if k is a field then there is no need to insist that the polynomial identity is proper.

Example 10. We now give some examples and non-examples of PI algebras.

1. Every commutative ring satisfies the identity $xy - yx$ so that every commutative ring is a PI ring.
2. If k is a field then $M_2(k)$ satisfies $\mathfrak{S}_5(x_1, x_2, x_3, x_4, x_5)$. We will prove this shortly.
3. For any commutative ring k, the free algebra $k\langle x_1, x_2, \ldots, x_n\rangle$, where $n > 1$, is not a PI algebra.
4. The Weyl algebra $k\langle x, y\rangle/(xy - yx - 1)$ is not PI whenever k is a field of characteristic zero.

As one can see, PI algebras have nice structural properties resembling a generalization of commutativity. The class of PI algebras is a large class of algebras so that it is often of interest to examine the structure of PI algebras and extend results from commutative ring theory.

Let k be a field. We now set a temporary goal of proving why $M_2(k)$, for instance, satisfies the standard polynomial of degree 5. As it turns out, the only information about $M_2(k)$ being used here is the fact that it is finite dimensional over k.

Proposition 1.2.1. *Let k be a field and let R be an n-dimensional k-algebra. Then R satisfies the standard polynomial of degree $n + 1$.*

Proof. Let $\mathfrak{S}$ denote the standard polynomial of degree $n+1$. As $\mathfrak{S}$ is multilinear, it suffices to show that $\mathfrak{S}$ vanishes on a basis for R over k. Let $B = \{a_1, \ldots, a_n\}$ be a k-basis for R. Then for any $i_j \in \{1, 2, \ldots, n\}$ we see that two of the arguments of $\mathfrak{S}(a_{i_1}, a_{i_2}, \ldots, a_{i_{n+1}})$ must be equal so that $\mathfrak{S}(a_{i_1}, a_{i_2}, \ldots, a_{i_{n+1}}) = 0$ for all $a_{i_j} \in B$. $\square$

In particular, we then have the following corollary which achieves our above goal.

Corollary 1.2.1. *If k is a field then $M_n(k)$ satisfies $\mathfrak{S}_{n^2+1}(x_1, \ldots, x_{n^2+1})$.*

It should be noted that the degree of the polynomial identity in the corollary is not as small as possible. In fact, if k is a commutative ring then $M_n(k)$ satisfies $\mathfrak{S}_{2n}(x_1, \ldots, x_{2n})$ and no polynomial identity of smaller degree (see Theorems 2.7 and 2.8 in [15]).

In the proof of the above proposition, the multilinearity of the standard identity greatly simplified our computations. With that being said, a very useful fact in PI theory is the following reduction.

Proposition 1.2.2. *If A satisfies a proper polynomial identity of degree d then it satisfies a proper multilinear (homogeneous) polynomial identity of degree d.*

Proof. See Lemma 1.14 of [15]. □

We now outline two of the biggest results in PI theory (cf. [15]) which demonstrate the beautiful ring theory of polynomial identity algebras. First, recall that a ring R is said to be (left) *primitive* if it has a faithful simple left R-module.

Theorem 1.2.1 (Kaplansky's theorem)**.** *If R is a primitive PI algebra then R is isomorphic to $M_n(D)$, where D is a division algebra which is finite dimensional over its centre. That is, R is a central simple algebra which is finite dimensional over its centre.*

Theorem 1.2.2 (Posner's theorem)**.** *Let R be a semiprime PI algebra with centre Z. If I is a nonzero ideal of R then $I \cap Z \neq (0)$.*

1.2.2 Radicals of PI algebras

We now focus on the radical theory of PI algebras. In this section we assume throughout that k is a field. This theory will be used a great deal throughout this thesis. While information on this topic can be found in [15] and [23], the best reference for this material is Rowen's book [57]. As we shall see, polynomial identity algebras have a very nice radical structure. In particular, for any PI algebra R, $\mathrm{Nil}^*(R) = \mathcal{L}(R)$. As the Levitzki radical behaves more nicely than the upper nilradical with respect to one-sided ideals in general, we have that the upper nilradical of PI algebras has additional structure.

We now present a famous theorem resulting from the combined works of Braun, Kemer, and Razmyslov in [13], [33], and [56], respectively. This theorem describes the Jacobson radical of a finitely generated polynomial identity algebra.

Theorem 1.2.3 (Braun-Kemer-Razymoslov Theorem)**.** *Let k be a field. The Jacobson radical of a finitely generated PI k-algebra is nilpotent.*

With respect to the upper nilradical of PI rings, we have the following structural result.

Theorem 1.2.4 (cf. [57])**.** *Every nil, multiplicatively closed subset of a PI algebra is locally nilpotent.*

Clearly the sum of two locally nilpotent one-sided ideals is again locally nilpotent. In particular, the Köthe conjecture has a positive solution for the class of PI algebras!

Theorem 1.2.5. *Let R be a PI algebra. If I and J are nil right ideals of R then $I + J$ is also nil.*

1.3 Graded rings

Let R be an associative ring with unity. Many well-known rings have a natural notion of degree. Therefore, to describe radicals of such rings, we have an additional tool (the degree) at our disposal. For instance, when working with a polynomial ring, it is useful to know when an ideal is generated by homogeneous polynomials, or even better, monomials. We generalize this idea when considering semigroup-graded rings.

Definition 7. If X is a semigroup, we say that R is X-graded if there exists additive subgroups R_x of R such that

$$R = \bigoplus_{x \in X} R_x,$$

and such that $R_x R_y \subseteq R_{xy}$ for all $x, y \in X$. Each nonzero $r \in R$ can be written uniquely as $r = \sum r_x$, where $r_x \in R_x$, and $r_x = 0$ for almost all $x \in X$. The elements of each R_x

are called homogeneous elements. Moreover, we say that an ideal of R is homogeneous if it is generated by homogeneous elements.

For $r \in R$, define the support of r to be the set $\{x \in X : r_x \neq 0\}$. We then define the length of r, denoted $\ell(r)$, to be the number of elements in the support of r.

Example 11. Let R be a ring. Consider the polynomial ring $S = R[x_1, \ldots, x_m]$. Now, assign each x_i a degree $d_i \in \mathbb{Z}$ and define the degree of a monomial $x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ to be

$$\alpha_1 d_1 + \alpha_2 d_2 + \cdots + \alpha_m d_m.$$

Letting R_n denote the R-module spanned by the set of monomials of degree n, for each $n \in \mathbb{Z}$, we see that

$$S = \bigoplus_{n \in \mathbb{Z}} R_n,$$

is a natural $\mathbb{Z}$-grading on S. If we consider the grading of S given by monomial length (i.e. $d_i = 1$, $i = 1, 2, \ldots, m$) then $I = (x_1^2 x_2 + x_4^3, x_3^{19})$ is homogeneous whereas $J = (x_1 x_2 + x_3)$ is not.

Example 12. A similar grading to the above can be given to the free R-algebra generated by $\{x_1, \ldots, x_m\}$.

Example 13. Any ring R can be given a trivial X-grading, for any monoid X, by defining $R_1 := R$ and $R_x = (0)$ for any $x \in X \backslash \{1\}$. Here 1 denotes the monoid identity element of X.

Example 14. For any semigroup X and ring R the semigroup algebra $R[X]$ may be given an X-grading by defining

$$R_x := \{rx : r \in R\},$$

for each $x \in X$.

Example 15. Let R be a ring and let $S = R\langle x_1, \ldots, x_m \rangle$. By considering S as a semigroup algebra of the free semigroup $\langle x_1, \ldots, x_m \rangle$ we see that an ideal of S is homogeneous if and only if it generated by monomials!

In graded-ring radical theory, the goal is to prove that certain radicals are homogeneous, for then their structure may be better understood. A common style of question appearing in this area of study is the following.

For which semigroups X does any X-graded ring have a homogeneous C-radical?

A particular class of semigroups which has been useful in answering the above style of question is the class of unique-product semigroups.

1.3.1 Semigroups with the unique-product property

There has been extensive study on radicals of rings graded by unique-product semigroups (u.p.-semigroups) and two-unique-product semigroups (t.u.p.-semigroups). For a survey of these results, see [25] and [30]. These semigroups have been particularly useful with respect to the Kaplansky unit and zero divisor conjectures (see [54]).

Conjecture 1 (Kaplansky unit conjecture)**.** *If G is a torsion-free group and k is a field then the only units of the group ring $k[G]$ are of the form αg, where $\alpha \in k^{\times}$ and $g \in G$.*

Conjecture 2 (Kaplansky zero divisor conjecture)**.** *If G is a torsion-free group and k is a field then the group ring $k[G]$ is a domain.*

These famous conjectures are currently major open problems in ring theory. However, it is known that the unit conjecture implies the zero divisor conjecture (cf. [54]). The relevance of u.p.-semigroups to these conjectures is contained in the following result.

Theorem 1.3.1. *If X is a u.p.-semigroup then the semigroup algebra $K[X]$ is a domain. Moreover, if X is a t.u.p.-semigroup then $K[X]$ has only the trivial units.*

The definition of this important class of semigroups is as follows.

Definition 8. A semigroup X is said to be a *u.p.-semigroup* (resp., *t.u.p.-semigroup*) if for any two non-empty finite subsets $A, B \subseteq X$ such that $|A| + |B| \geq 3$, there exists one (resp., two) elements of X which can be uniquely written as ab, where $a \in A$ and $b \in B$.

Easily every t.u.p.-semigroup is a u.p.-semigroup, but it was shown by Krempa (cf. [47]) that not every u.p-semigroup is a t.u.p.-semigroup.

Example 16. Let T be the free semigroup generated by $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ subject to the following relations:

1. $x_1y_1 = x_2y_3$;
2. $x_1y_2 = x_3y_1$;
3. $x_1y_3 = x_2y_2$;
4. $x_3y_2 = x_2y_1$;

Then T is a u.p.-semigroup which is not a t.u.p.-semigroup.

However, it was shown in [64] that every u.p.-group is a t.u.p.-group. Furthermore, every u.p.-semigroup which can be embedded into a u.p.-group is a t.u.p.-semigroup (cf. [30]). It is easy to see that every u.p.-semigroup is torsion-free. However, not every torsion-free group is a u.p.-group (cf. [55]). Moreover, every u.p-semigroup is cancellative.

A semigroup X is said to be right-ordered if there exists a total ordering $\leq$ on X such that $a \leq b$ implies $ac \leq ac$ for all $a, b, c \in X$. It is well-known that every right-ordered semigroup is a t.u.p.-semigroup (cf. [30]). Since every infinite cyclic group and every free group are right-ordered, these are two common examples of u.p.-groups.

We now remark that torsion-free nilpotent groups are also u.p.-groups. To see this, let G be a torsion-free nilpotent group. By ([30, Theorem 31.5]) if G has a finite subnormal series

$$1 = G_0 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_n = G,$$

such that each G_{i+1}/G_i is torsion-free and abelian, then G is right-ordered. However, it is then sufficient to consider the upper central series of G, and so G is right-ordered and hence a u.p.-group. We make special note that if G is torsion-free and nilpotent then $G/Z(G)$, where $Z(G)$ is the centre of G, is also torsion-free and nilpotent, and hence right-ordered ([30, Lemma 30.16]).

We now consider a class of semigroups which have a nilpotent group of fractions. Using this fact, we may extend many results known about the homogeneity of radicals of rings graded by torsion-free nilpotent groups. The following definitions are due to [42]. Let X be a semigroup and let $x, y, w_1, w_2, w_3, \ldots \in X$. Consider the elements $x_0 = x$ and $y_0 = y$ and then inductively define

$$x_n = x_{n-1} w_n y_{n-1} \text{ and } y_n = y_{n-1} w_n x_{n-1},$$

for all $n \geq 1$. We say that X satisfies the identity $X_n = Y_n$ if $x_n = y_n$ for all $x, y, w_i \in X$. It was proved in [42] that if G is a group then $X_n = Y_n$ is satisfied by G if and only if G is a nilpotent group of index n. With this result in mind, we say that a semigroup X is weakly nilpotent of index n if X satisfies $X_n = Y_n$ and n is the least positive integer with this property. It is known that any cancellative weakly nilpotent semigroup of index n has a group of fractions which is nilpotent of index n [42].

Theorem 1.3.2 ([47], Theorem 6/Proposition 9)**.** *Let X be a cancellative weakly nilpotent semigroup so that X has a nilpotent group of fractions G. Then the following conditions are equivalent:*

1. *X is a u.p.-semigroup*
2. *X is a t.u.p.-semigroup*
3. *G is a u.p.-group*
4. *X is an ordered semigroup*
5. *X is torsion-free.*

1.4 Skew polynomial extensions

In this section we briefly introduce the work of Ore in [51] on the theory of noncommutative polynomials. For now, let R be a domain. Consider the additive abelian group of polynomials $R[x]$. Suppose we want to equip $R[x]$ with a multiplication so that it forms an associative ring with the following desirable polynomial-like properties:

1. If $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in R[x]$ and $r \in R$ then $rf(x) = ra_n x^n + \cdots + ra_1 x + ra_0$.
2. If $f(x), g(x) \in R[x]$ then $\deg(f(x)g(x)) = \deg(f(x)) + \deg(g(x))$.

A natural question to ask at this point is

"*How many ways could we define such a multiplication on R[x]?*"

By distributivity and associativity it suffices to define the multiplication xr for $r \in R$. By our degree requirement, we must have that

$$xr = \sigma(r)x + \delta(r),$$

for some $\sigma(r), \delta(r) \in R$. Using distributivity and associativity we must have that for all $a, b \in R$,

$$\sigma(a+b)x + \delta(a+b) = x(a+b) = xa + xb = (\sigma(a) + \sigma(b))x + \delta(a) + \delta(b)$$

and

$$\sigma(ab)x + \delta(ab) = xab = (\sigma(a)x + \delta(a))b = \sigma(a)\sigma(b)x + \sigma(a)\delta(b) + \delta(a)b.$$

It then follows that

1. $\sigma(a+b) = \sigma(a) + \sigma(b)$;
2. $\delta(a+b) = \delta(a) + \delta(b)$;

3. $\sigma(ab) = \sigma(a)\sigma(b)$;

4. $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$;

for all $a, b \in R$.

With this motivation in mind, we make the following definitions for all k-algebras, where k is a commutative domain.

Definition 9. Let k be a commutative domain and let R be a k-algebra. Let σ be a k-endomorphism of R.

1. We call a k-linear map $\delta : R \to R$ a σ-derivation of R if $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$. This condition is often called the Leibniz rule. In the case when $\sigma = 1$, we simply call δ a derivation of R.

2. If δ is a σ-derivation of R then we define the skew polynomial ring $R[x; \sigma; \delta]$ by starting with the abelian group $R[x]$ and defining a multiplication by

$$xr := \sigma(r)x + \delta(r),$$

   and then extending by linearity, associativity, and distributivity.

3. In the case when $\delta = 0$ we denote $R[x; \sigma; \delta]$ by $R[x; \sigma]$ and call it a skew polynomial ring of endomorphism type or Ore extension of endomorphism type.

4. In the case when $\sigma = 1$ we denote $R[x; \sigma; \delta]$ by $R[x; \delta]$ and call it a differential polynomial ring, or skew polynomial ring of derivation type, or Ore extension of derivation type.

Example 17. Any polynomial ring is a skew polynomial ring with $\sigma = 1$ and $\delta = 0$.

Example 18. Let k be a field and let δ denote the ordinary derivative on $R = k[y]$. Then $R[x; \delta]$ is a differential polynomial ring. Moreover, we see that this ring is precisely the ring $k\langle x, y\rangle$ modulo the relation $xy = yx + \delta(y) = yx + 1$. Therefore, this differential polynomial ring is the Weyl k-algebra.

Skew polynomial rings still have many nice polynomial-like properties.

1. A ring R is a domain if and only if $R[x;\sigma;\delta]$ is a domain, for every σ and δ.
2. (Skew Hilbert Basis Theorem) If R is a left Noetherian ring then $R[x;\sigma;\delta]$ is also left Noetherian.
3. If R is a division ring then $R[x;\sigma;\delta]$ is a principal ideal domain.

We are especially interested in the ideal theory of differential polynomial rings. Let R be a ring and let δ be a derivation on R. We say that an ideal I of R is a δ-ideal if $\delta(I) \subseteq I$. These ideals are particularly useful for the following result.

Proposition 1.4.1. *Let R be a k-algebra, δ a derivation on R, and I a δ-ideal of R. Then $I[x;\delta]$ is an ideal of $R[x;\delta]$.*

Example 19. Let R be a k-algebra and δ a derivation on R. Suppose I is an ideal of $R[x;\delta]$ Then $A = I \cap R$ is a δ-ideal of R. Clearly A is an ideal of R. Now take $a \in A$. Then

$$xa - ax = \delta(a) \in I,$$

so that $\delta(a) \in A$.

In particular, if C is any radical of R and δ is a derivation on R then $C(R[x;\delta]) \cap R$ is a δ-ideal of R. In particular, $(C(R[x;\delta]) \cap R)[x;\delta]$ is an ideal of $R[x;\delta]$. Specifying to the Jacobson radical and the upper nilradical, we see that $J(R[x;\delta]) \cap R$ is a quasi-regular ideal of R and $\mathrm{Nil}^*(R[x;\delta]) \cap R$ is a nil ideal of R. It then follows that

$$(J(R[x;\delta]) \cap R)[x;\delta] \subseteq J(R)[x;\delta]$$

and

$$(\mathrm{Nil}^*(R[x;\delta]) \cap R)[x;\delta] \subseteq \mathrm{Nil}^*(R)[x;\delta].$$

The reverse containments, however, are unclear and relate to the content of Chapter 4.

Chapter 2

Radicals of semigroup-graded rings

2.1 Preliminaries

In this chapter let R always denote a unital ring. Suppose X is a u.p-semigroup and R is an X-graded ring. It was shown in [27] that all minimal prime ideals of R are homogeneous. In particular, it is then clear that the prime radical of R is homogeneous. It was also shown that the Levitzki radical of R is homogeneous.

The Jacobson radical of semigroup-graded rings is much less understood. It is known that if G is a free group, a finitely generated torsion-free nilpotent group, a free solvable group, or a residually finite p-group for two distinct primes p, then $J(R)$ is homogeneous for any G-graded ring R [28]. It should be noted that each torsion-free group of the previously listed group types is a particular type of u.p.-group. If X is a commutative semigroup then it was shown in [32] that $J(R)$ is homogeneous for any X-graded ring R if and only if X can be embedded in a torsion-free abelian group. It was asked in [25] if $J(R)$ is homogeneous, where R is graded by a u.p.-semigroup.

Even less is known about the upper nilradical of R, where R is graded by some semigroup X. By [61] and [43] it is known that if X is an infinite cyclic group or a commutative and torsion-free group, then $\mathrm{Nil}^*(R)$ is homogeneous. Jespers asked [25] if

$\mathrm{Nil}^*(R)$ is homogeneous provided that X is a u.p.-semigroup. Moreover, Smoktunowicz asked [61]: Which groups G are such that every G-graded ring R has a homogeneous upper nilradical?

The main theorems of this chapter are as follows.

Theorem 2.1.1. *Let G be a torsion-free nilpotent group. Then for every G-graded ring R, $\mathrm{Nil}^*(R)$ is homogeneous.*

Theorem 2.1.2. *Let X be a semigroup such that any X-graded ring has a homogeneous prime radical. Let R be any X-graded ring. If $W(R)$ has no non-zero homogeneous elements then $\mathrm{Nil}_*(R) = (0)$. In particular, if $W(R)$ has no non-zero homogeneous elements then $W(R) = (0)$ as well.*

2.2 Rings graded by torsion-free nilpotent groups

In [28] it was shown that the Jacobson radical of any ring graded by a finitely generated torsion-free nilpotent group is homogeneous. We now provide an analogue of this result for the upper nilradical. Moreover, we relax the assumptions by not requiring that G be finitely generated.

Let X be a semigroup and let R be an X-graded ring. Let us define $\mathrm{Nil}^*_g(R)$ to be the sum of all nil homogeneous ideals of R. Note that $\mathrm{Nil}^*_g(R) \subseteq \mathrm{Nil}^*(R)$ and $\mathrm{Nil}^*_g(R) = \mathrm{Nil}^*(R)$ if and only if $\mathrm{Nil}^*(R)$ is homogeneous. To show that $\mathrm{Nil}^*(R)$ is homogeneous we can then show that $\mathrm{Nil}^*(R/\mathrm{Nil}^*_g(R)) = (0)$ so that $\mathrm{Nil}^*(R) \subseteq \mathrm{Nil}^*_g(R)$. In particular, it is sufficient to assume that $\mathrm{Nil}^*_g(R) = (0)$ and show that $\mathrm{Nil}^*(R) = (0)$.

Theorem 2.2.1. *Let G be a torsion-free nilpotent group. Then for every G-graded ring R, $\mathrm{Nil}^*(R)$ is homogeneous.*

Proof. Let $R = \bigoplus R_g$ be a G-graded ring. We prove this by induction on the index of nilpotence of G, N. Using the above remark, let us assume throughout the proof that $\mathrm{Nil}^*_g(R) = (0)$.

If $N = 1$ then G is abelian and so this follows by [43]. We give their elegant argument now, as it will be referenced later in the proof. Assume for contradiction that there exists an $a = a_{n_1} + \cdots + a_{n_k} \in \mathrm{Nil}^*(R)$ such that a is nonzero and of minimal length (at least 2) in $\mathrm{Nil}^*(R)$, where $a_{n_i} \in R_{n_i}$ and $n_i \neq n_j$ for $i \neq j$. Let $r_g \in R_g$ be homogeneous, for some $g \in G$. We see that $ar_g a_{n_k} - a_{n_k} r_g a$ has fewer homogeneous components than a and so by minimality and our assumption we have that $ar_g a_{n_k} - a_{n_k} r_g a = 0$. Here the assumption that G is abelian is used, since we use the fact that $n_k g n_i = n_i g n_k$ for every $i = 1, 2, \ldots, k$. But since this holds for every homogeneous element, we have that $ara_{n_k} - a_{n_k} ra = 0$ for every $r \in R$. Now, since $a_{n_k} \notin \mathrm{Nil}^*(R)$ there exists $p_i, q_i \in R$ such that $\alpha := \sum_{i=1}^{b} p_i a_{n_k} q_i$ is not nilpotent. However, since $a \in \mathrm{Nil}^*(R)$, $\left(\sum_{i=1}^{b} p_i a q_i\right)^m = 0$ for some $m \in \mathbb{N}$. We observe that

$$\begin{aligned}(\alpha^m RaR)^m \subseteq \alpha^m (RaR)^m &= \left(\sum p_i a_{n_k} q_i\right)^m (RaR)^m \\ &\subseteq \left(\sum p_i a q_i\right)^m (Ra_{n_k} R)^m = 0.\end{aligned}$$

Now, if P is a minimal prime ideal of R we must have that $\alpha^m \in P$ or $RaR \subseteq P$. If $RaR \subseteq P$ then $a \in P$. As noted in Chapter 1, G is a u.p.-group. Then by [25], P is homogeneous and so $a_{n_k} \in P$. Therefore, in any case, we have that $\alpha^m \in P$. But then α^m is in the prime radical and so α^m is nilpotent. Hence α is nilpotent, which is a contradiction. Therefore no such a exists and so $\mathrm{Nil}^*(R) = (0)$, as required.

Assume that every H-graded ring has a homogeneous upper nilradical provided that H is torsion-free and has index of nilpotence less than N. Now let G be a torsion-free nilpotent group with index of nilpotence N. Then $G/Z(G)$ has index of nilpotence $N - 1$ and there is a natural $G/Z(G)$-grading of R. Moreover, by the remark made in chapter 1, $G/Z(G)$ is torsion-free. As above, assume $a \in \mathrm{Nil}^*(R)$ is nonzero of minimal length. By induction and minimality of k, we must have that each $n_1, n_2, \ldots, n_k$ are in the same coset of $Z(G)$. Say $n_i = g z_i$ for some $g \in G$ and $z_i \in Z(G)$, for each $1 \leq i \leq k$. However, since $n_i h n_j = n_j h n_i$, for every $1 \leq i, j \leq k$, $h \in G$, we get as above that $ara_{n_k} - a_{n_k} ra = 0$ for every $r \in R$. Moreover, every torsion-free nilpotent group is a u.p.-group, and so we may reproduce the abelian-case argument to get that $\mathrm{Nil}^*(R) = (0)$. $\square$

Corollary 2.2.1. *Let X be a torsion-free, cancellative, and weakly nilpotent semigroup. Then any X-graded ring will have a homogeneous upper nilradical.*

Proof. Suppose X is a torsion-free, cancellative, and weakly nilpotent semigroup and let $R = \bigoplus_{x \in X} R_x$ be an X-graded ring. By Theorem 1.3.2 we see that X is a u.p.-semigroup with a nilpotent group of fractions G which also satisfies the unique-product property. In particular, G is a torsion-free nilpotent group. We consider R as a G-graded ring by setting $R_g = R_x$ if $g = x \in X$ and $R_g = \{0\}$, otherwise. It then follows from Theorem 2.2.1 that $\mathrm{Nil}^*(R)$ is homogeneous with respect to the G-grading and is therefore also homogeneous with respect to the original X-grading. $\square$

2.3 Nilpotent homogeneous ideals

Recall the following construction of the prime radical of a ring R which was outlined in §1.1.6. Define $W_1(R) := W(R)$, and for any ordinal α inductively define

$$W_\alpha(R) := \{a \in R : a + W_\beta(R) \in W(R/W_\beta(R)\},$$

provided α is the successor ordinal of β and

$$W_\alpha(R) := \bigcup_{\beta<\alpha} W_\beta(R),$$

provided α is a limit ordinal. Levitzki [40] proved that this sequence stabilizes to the prime radical of R.

We now use this radical construction to provide a graded analogue of Proposition 1.1.6.

Theorem 2.3.1. *Let X be a semigroup such that any X-graded ring has a homogeneous prime radical. Let R be any X-graded ring. If $W(R)$ has no non-zero homogeneous elements then $\mathrm{Nil}_*(R) = (0)$. In particular, if $W(R)$ has no non-zero homogeneous elements then $W(R) = (0)$ as well.*

Proof. Suppose R is an X-graded ring such that $W(R)$ has no nonzero homogeneous elements. Suppose $\mathrm{Nil}_*(R) \neq (0)$. By assumption, $\mathrm{Nil}_*(R)$ has a non-zero homogeneous element. By the construction of Levitzki we have that there exists an ordinal α such that $W_\alpha(R)$ has a nonzero homogeneous element. Since membership well-orders the class of ordinals, we may assume α is the smallest ordinal such that $W_\alpha(R)$ contains a nonzero homogeneous element. By assumption, $\alpha > 1$. Say $0 \neq a \in W_\alpha(R)$ is homogeneous. If α is a limit ordinal than we see that

$$a \in \bigcup_{\beta<\alpha} W_\beta(R),$$

so that $a \in W_\beta(R)$ for some ordinal $\beta < \alpha$. This is a contradiction so that α must be a successor ordinal. Say $\beta + 1 = \alpha$ for some ordinal β. By minimality of α we have that $\overline{a} := a + W_\beta(R) \neq 0$. Moreover, $\overline{a} \in W(R/W_\beta(R))$. Since $(\overline{a}) \subseteq W(R/W_\beta(R))$ is finitely generated we have that $(\overline{a})$ is nilpotent. Therefore there exists an $n \in \mathbb{N}$ such that $(a)^n \subseteq W_\beta(R)$. Since $(a)^n$ is homogeneous and $W_\beta(R)$ has no non-zero homogeneous elements we must in fact have that $(a)^n = (0)$. Therefore $a \in W_1(R) = W(R)$, which is a contradiction. Therefore $\mathrm{Nil}_*(R) = (0)$ and so $W(R) = (0)$ as well. $\square$

We then get the following immediate corollaries.

Corollary 2.3.1. *Let X be a semigroup such that any X-graded ring has a homogeneous prime radical. Let R be any X-graded ring. If I is an ideal of R such that $W(R) \subseteq I$ then if I has no nonzero homogeneous elements then* $\mathrm{Nil}_*(R) = (0)$.

Corollary 2.3.2. *Let X be a semigroup such that any X-graded ring has a homogeneous prime radical. Let R be any X-graded ring. If $B(R)$ has no nonzero homogeneous elements then $B(R) = \mathrm{Nil}_*(R) = (0)$.*

By [25] we may take X to be any u.p.-semigroup in the above results.

Chapter 3

Iterative Algebras

In this chapter we introduce a new class of *monomial algebras*, which we call *iterative algebras*, to solve various problems involving graded-nilpotent rings. This chapter is based on the content of [6]. The author's main contribution to this chapter is the original construction of an algebra R which is just infinite, graded-nilpotent, semiprimitive, and has quadratic growth. Working with Dr. Jason Bell, this result was strengthened to the following.

Theorem 3.0.1. *There exists an algebra R such that R is just infinite, graded-nilpotent, has quadratic growth, has trivial Jacobson radical, and is finitely generated as a Lie algebra.*

This theorem answers several questions of Greenfeld, Leroy, Smoktunowicz, and Ziembowski [21].

We first recall the definition of a monomial algebra.

Definition 10. Let k be field. We say that an algebra A is a monomial algebra if there exists an $m \in \mathbb{N}$ such that

$$A \cong k\langle x_1, x_2, \dots, x_m \rangle / I,$$

where I is an ideal generated by monomials.

We note that every monomial algebra may be graded by length of monomial. An ideal of a monomial algebra is said to be a *monomial ideal* if it is generated by monomials. Notice that every monomial ideal is a homogeneous ideal with respect to the usual grading, but the converse is not true.

Example 20. Consider $A = \mathbb{C}\langle x_1, x_2\rangle/(x_1^2x_2)$. Then A is a monomial algebra which is $\mathbb{N}$-graded by length. We observe that $(x_1 + x_2)$ is a homogeneous ideal of A which is not a monomial ideal.

The structure of monomial algebras has been studied extensively in [4, 8]. Monomial algebras serve as an important class of $\mathbb{Z}$-graded rings. As monomial algebras have more refined structure than graded rings, in general, they serve as a useful tool for example or counterexample constructions. With regards to the Jacobson radical, we begin by making the following observation.

Proposition 3.0.1. *Let A be a monomial k-algebra, where k is a field. Then $J(A)$ is a monomial ideal. That is, $J(A)$ is generated by monomials in A.*

Proof. Suppose $A = k\{x_1, \dots, x_m\}/I$, for some monomial ideal I of the free algebra $k\{x_1 \dots, x_m\}$. Then A is naturally graded by the free group $F = \langle x_1, \dots, x_m\rangle$. Namely,

$$A = \bigoplus_{x\in F} A_x,$$

where $A_x = \{\alpha x : \alpha \in k\}$. By [28], $J(A)$ is a homogeneous ideal with respect to the F-grading on A. However, the homogeneous elements of A with respect to the F-grading are precisely the monomials of A. Therefore $J(A)$ is a monomial ideal. $\square$

Moreover, we have the following result of Beidar and Fong (see [4]).

Theorem 3.0.2 ([4], Theorem 1)**.** *The Jacobson radical of a monomial algebra over a field is locally nilpotent.*

In particular, the Levitzki radical, upper nilradical, and Jacobson radical of a monomial algebra coincide. In order to introduce further results in the structural theory of monomial algebras, we must first discuss some preliminary notions from combinatorics on words.

3.1 Combinatorics on words

A large part of this chapter on monomial algebras involves tools from combinatorics on words. There is a very rich and well-developed theory of combinatorics on words for which we now give a brief introduction. For a more in-depth look, we refer the reader to [1]. This theory has many applications to algebra, combinatorics, number theory, and computer science.

Combinatorics on words is the study of finite and infinite sequences, called *words*, with entries from a finite set, called the *alphabet*. Unlike in real analysis, one is not interested in sequences for their convergence-like properties. Instead, one is interested in studying patterns within the sequence and patterns which the sequence avoids.

3.1.1 Finite and right-infinite words

Definition 11. Let Σ be a non-empty finite set, which we call an *alphabet*. A (finite) word w over Σ is simply a finite sequence

$$w = a_1 a_2 \cdots a_n,$$

where each $a_i \in \Sigma$. We define the *length* of w, denoted by $|w|$, to be n. We denote the set of finite words over Σ by Σ^*. This includes the *empty word* ε. A right-infinite word w over Σ is a infinite sequence

$$w = a_1 a_2 a_2 \cdots ,$$

where $a_i \in \Sigma$ for every $i \in \mathbb{N}$. We denote the set of right-infinite words over Σ by $\Sigma^{\mathbb{N}}$.

Definition 12. Let w be a word, finite or infinite, over the alphabet Σ. We say that a finite word u is a subword of w if there exists a finite word x and a possibly infinite word y such that

$$w = xuy.$$

If w is a word and u is a non-empty word over Σ then any subword of w of the form uu is called a square. Any subword of w of the form uuu is called a cube. In general, any subword of w of the form u^k is called a k-power in w.

Example 21. Words (finite or right-infinite) are called binary if they are over an alphabet of size 2. Usually, the alphabet is taken to be $\Sigma = \{0, 1\}$. For instance, $u = 0110 \in \Sigma^*$ and 11 is a square in u. Moreover,

$$w = 01001000100001 \cdots$$

is a right-infinite binary word.

A special type of subword which we shall make use of is known as a *prefix*.

Definition 13. Let w be a right-infinite word. A (finite) word u is said to be a prefix of w if there exists a right-infinite word w' such that $w = uw'$. Similarly, a finite word is said to be a prefix of a word v if there is a (possibly empty) word v' such that $v = uv'$.

Throughout this thesis we will make reference to some special types of right-infinite words.

Definition 14. Let w be a right-infinite word over the alphabet Σ.

1. We say that w is *periodic* if there exists a non-empty word v such that $w = v^\omega := vvvv\cdots$.
2. We say that w is *eventually periodic* if there exists a non-empty word v and a word u (possibly empty) such that $w = uv^\omega := uvvvv\cdots$.
3. We say that w is *recurrent* if every subword of w occurs infinitely many times in w.

4. We say that w is *uniformly recurrent* if for every subword v of w there exists an $N = N(v) \in \mathbb{N}$ such that every subword of w of length N contains v as a subword.

   Easily, one has that

$$\{ \text{ periodic words } \} \subseteq \{ \text{ uniformly recurrent words } \} \subseteq \{ \text{ recurrent words } \}.$$

Example 22. It is easily seen that $w = 0111\cdots$ is an eventually periodic word which is not recurrent.

Example 23. An example of a recurrent word which is not uniformly recurrent is

$$w = v_1 0 v_2 00 v_3 000 v_4 0000 \cdots ,$$

where $v_1, v_2, \ldots$ is an enumeration of all finite words over $\{0, 1\}$.

The following propositions give some useful equivalent statements of being recurrent or uniformly recurrent.

Proposition 3.1.1. *Let w be a right-infinite word. Then w is recurrent if and only if every subword of w occurs at least twice.*

Proposition 3.1.2. *Let w be a right-infinite word. Then w is uniformly recurrent if and only if every subword of w occurs infinitely many times with bounded gaps.*

Throughout subsequent chapters, we shall hereby make the following technical assumption:

When we say that w is a right-infinite word over Σ we assume that every letter of Σ occurs in w.

3.1.2 Morphisms and pure morphic words

Let Σ be an alphabet. Observe that the set of finite words Σ^* is precisely the free monoid generated by Σ. Therefore if Γ is another alphabet, then it makes sense to define a *morphism* $\varphi : \Sigma^* \to \Gamma^*$ to simply be a monoid morphism. For instance, if $a, b \in \Sigma$ then $\varphi(ab) = \varphi(a)\varphi(b)$.

Example 24. Let $\Sigma = \{0, 1\}$ and $\Gamma = \{a, b, c\}$. Consider the word $0110 \in \Sigma^*$ and the morphism $\varphi : \Sigma^* \to \Gamma^*$ given by

$$0 \mapsto abc$$

$$1 \mapsto a^2.$$

We then see that

$$\varphi(0110) = \varphi(0)\varphi(1)\varphi(1)\varphi(0) = abca^5bc.$$

Example 25. Let $\Sigma = \{a, b, c\}$ and define a morphism $\varphi : \Sigma^* \to \Sigma^*$ by $\varphi(a) = abc, \varphi(b) = b^2$, and $\varphi(c) = a$. Observe that

$$\varphi^2(a) = abcb^2a,$$

$$\varphi^3(a) = abcb^2ab^4abc,$$

and

$$\varphi^4(a) = abcb^2ab^4abcb^8abcb^2a.$$

Continuing this way we see that $\varphi^n(a)$ is a prefix of $\varphi^m(a)$ whenever $n \leq m$. Moreover, it makes sense to consider the right-infinite word obtained by repeatedly applying our morphism to a. This happens because $\varphi(a)$ begins with a and the length of $\varphi^n(a)$ increased with n.

This brings us to two definitions which play a central role in the construction of iterative algebras.

Definition 15. Let Σ be an alphabet and let $\varphi : \Sigma^* \to \Sigma^*$ be a morphism. We say that φ is *prolongable* on $a \in \Sigma$ if $\varphi(a) = av$, where $v \in \Sigma^*$ such that $\varphi^i(v) \neq \varepsilon$ for all $i \in \mathbb{N}$. We denote the right-infinite word obtained by repeatedly applying this morphism to a by $\varphi^\omega(a)$.

Definition 16. We say that a right-infinite word w over the alphabet Σ is *pure morphic* if there exists a morphism $\varphi : \Sigma^* \to \Sigma^*$, prolongable on some $a \in \Sigma$, such that $w = \varphi^\omega(a)$.

Let $\varphi : \Sigma^* \to \Sigma^*$ be a morphism which is prolongable on $a \in \Sigma$. Say $\varphi(a) = av$, where $v \in \Sigma^*$. We see that

$$\varphi^\omega(a) = av\varphi(v)\varphi^2(v)\varphi^3(v)\cdots.$$

Moreover, if we extend φ to $\Sigma^\mathbb{N}$ letter-by-letter then we see that $\varphi(\varphi^\omega(a)) = \varphi^\omega(a)$. In fact, $\varphi^\omega(a)$ is the unique right-infinite fixed point of φ starting with the letter a (cf. [1]). We now define some special classes of pure morphic words.

Definition 17. Let $w = \varphi^\omega(a)$ be a pure morphic word over the alphabet Σ.

(i) We say that w is a *d-uniform* pure morphic word if $|\varphi(b)| = d$ for every letter $b \in \Sigma$.

(ii) We say that w is *primitive* if for every $b \in \Sigma$ there exists $m = m(b) \in \mathbb{N}$ such that a appears in $\varphi^m(b)$.

The motivation for the name "primitive" comes from the following object. Given a pure morphic word on an alphabet $\Sigma = \{x_1, \ldots, x_m\}$, we can associate an $m \times m$ matrix, $M(w)$, called the *incidence matrix* of w. The (i, j)-entry of $M(w)$ is defined to be the number of occurrences of x_i in $\varphi(x_j)$. Given a word $u \in \Sigma^*$, we can then define an $m \times 1$ integer vector $\theta(u)$ whose j-th coordinate is the number of occurrences of x_j in u. By [1, Proposition 8.2.2] we have that

$$\theta(\varphi^n(u)) = M(w)^n\theta(u). \tag{3.1}$$

Now, recall that a matrix $A \in M_n(\mathbb{R})$ is said to be primitive if for some $k \in \mathbb{N}$, A^k is positive. Otherwise stated, every entry of A^k is positive. We then have the following proposition (cf. [12]) which explains our choice of notation.

Proposition 3.1.3. *Let $w = \varphi^\omega(x_1)$ be a pure morphic word over the alphabet $\{x_1, \ldots, x_n\}$. Then w is primitive if and only if $M(w)$ is a primitive matrix.*

Proof. Suppose $M(w)$ is a primitive matrix such that $M(w)^m$ is positive for some $m \in \mathbb{N}$. Then for any letter x_j we have that

$$\theta(\varphi^m(x_j)) = M(w)^m e_j,$$

where e_j denotes the standard j-th basic vector. As $M(w)^m$ is positive, we see that every entry of $\theta(\varphi^m(x_j))$ is positive. In particular, x_1 occurs in $\varphi^m(x_j)$ for every letter x_j. Hence w is primitive. Similarly, if x_1 occurs in every $\varphi^m(x_j)$, $1 \leq j \leq n$, then $M(w)^m$ is positive. Therefore $M(w)$ is primitive. □

It is often the case that pure morphic words are primitive exactly when they are uniformly recurrent. Let us say that a morphism $\varphi : \Sigma^* \to \Sigma^*$ is *growing* if $|\varphi^n(x)| \to \infty$ as $n \to \infty$ for every $x \in \Sigma$.

Example 26. The morphism φ sending $0 \mapsto 01$ and $1 \mapsto 1$ is not growing. Moreover,

$$w = \varphi^\omega(0) = 01111\cdots,$$

is not uniformly recurrent and not primitive.

Proposition 3.1.4 (cf. [12])**.** *Let $w = \varphi^\omega(a)$ be a pure morphic word generated by a growing morphism φ. Then w is primitive if and only if w is uniformly recurrent.*

Proof. Suppose w is uniformly recurrent. In particular, there exists a positive integer N such that the first letter a of w occurs in every subword of w of length N. Then for any letter $b \in \Sigma$, $|\varphi^m(b)| > N$ for m sufficiently large since φ is growing. Therefore a occurs in $\varphi^m(b)$ and so w is primitive.

Now suppose that w is primitive. Notice that it suffices to show that a occurs infinitely many times in w with bounded gaps. To see this, suppose a occurs with bounded gaps and take any subword v of w. Note that v is a subword of $\varphi^m(a)$ for m sufficiently large. As $\varphi^m(w) = w$, we have that $\varphi^m(a)$, and hence v, occurs in w with bounded gaps. Now, if arbitrarily long subwords of w do not contain the letter a then arbitrarily long subwords of the form $\varphi^m(b)$, where b is a letter and $m \in \mathbb{N}$, do not contain the letter a. This contradicts that w is primitive. $\square$

Example 27. Define $\varphi : \{a, b, c\}^* \to \{a, b, c\}^*$ by $a \mapsto abc$, $b \mapsto b$, and $c \mapsto a$. Then

$$w := \varphi^\omega(a) = abcbababcbabcba \cdots .$$

is uniformly recurrent but not primitive.

3.1.3 Subword complexity

Let w be a right-infinite word. In the next chapter we will be considering all subwords of w as a basis for a particular algebra. For dimension purposes, we will need to investigate counting all subwords of length n of w. This brings us to the following definition.

Definition 18. Let w be a right-infinite word over a finite alphabet Σ. For any non-negative integer n we define $\rho_w(n)$ to the the number of distinct subwords of w of length n. The function

$$\rho_w : \mathbb{N}_{\geq 0} \to \mathbb{N}$$

is known as the *subword complexity function* of w.

Note that the empty word ε is the unique word in Σ^* of length 0 so that for any right-infinite word w, $\rho_w(0) = 1$. Moreover, by our assumption we have that $\rho_w(1) = |\Sigma|$. Let us now consider two drastically different subword complexity functions associated to binary words.

Example 28. Let $w_1 := 010101010101 \cdots$. Observe that $\rho_{w_1}(1) = 2, \rho(2) = 2$, and $\rho_{w_1}(n) = 2$ for any $n \geq 3$. We then see that

$$\rho_{w_1} = O(1).$$

Now let $v_1, v_2, v_3, \cdots$ be an enumeration of the words in $\{0,1\}^*$ and consider the right-infinite word

$$w_2 := v_1 v_2 v_3 \cdots .$$

We then see that every possible binary word is a subword of w_2. Therefore $\rho_{w_2}(n) = 2^n$ for every $n \in \mathbb{N}$. In conclusion, ρ_{w_2} grows exponentially while ρ_{w_1} is bounded.

The behaviour of ρ_{w_1} can be explained through the following proposition.

Proposition 3.1.5. *Let $w = vu^\omega$ be an eventually periodic word. Suppose $|u| = n \geq 1$ and that no power of u is a power of a shorter word. Then ρ_w is bounded and $\rho_w(n) = |vu|$ for all $n \geq |vu|$. Moreover, for any right infinite word x, either ρ_x is strictly increasing or x is eventually periodic.*

Proof. See Proposition 4.2.2 and Theorem 4.3.1 of [12]. □

In general, computing the subword complexity of an arbitrary right-infinite word can be extremely difficult. However, the behaviour of ρ_w is much more predictable when w is a pure morphic word. For this, we follow the work found in Chapter 4 of [12]. We recall that given two maps $f, g : \mathbb{N}_{\geq 0} \to \mathbb{R}_+$, we say that $f(n) = \Theta(g(n))$ if there exist positive constants C_1 and C_2 such that

$$C_1 g(n) \leq f(n) \leq C_2 g(n)$$

for all n sufficiently large.

The subword complexity function of pure morphic words is described through the following famous result of Pansiot in [52].

Theorem 3.1.1. *If w is a pure morphic word then ρ_w is either*

$$O(1), \Theta(n), \Theta(n \log\log n), \Theta(n \log n), \textit{ or } \Theta(n^2).$$

At this point, it is natural to ask what properties of the morphism corresponding to a pure morphic word determine which class its subword complexity function belongs to. To answer this question, consider the following technical definitions.

Definition 19. Let $\varphi : \Sigma^* \to \Sigma^*$ be a growing morphism. We say that φ is

1. *quasi-uniform* if there exists an $\alpha \in \mathbb{R}$, $\alpha \geq 1$, such that $|\varphi^n(a)| = \Theta(\alpha^n)$ for any $a \in \Sigma$;

2. *polynomially diverging* if there exists an $\alpha \in \mathbb{R}$, $\alpha > 1$, and a nonzero function $f : \Sigma \to \mathbb{N}$ such that $|\varphi^n(a)| = \Theta(n^{f(a)}\alpha^n)$ for any $a \in \Sigma$;

3. *exponentially diverging* if there exist $a, b \in \Sigma$, $p, q \in \mathbb{N}$, and distinct $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1, \alpha_2 > 1$, such that $|\varphi^n(a)| = \Theta(n^p \alpha_1^n)$ and $|\varphi^n(b)| = \Theta(n^q \alpha_2^n)$.

Example 29. Any d-uniform morphism is quasi-uniform.

Example 30 (Example 4.7.42 of [12])**.** For $r \in \mathbb{N}$ define $\varphi_r : \{0,1\}^* \to \{0,1\}^*$ by $\varphi_r(0) = 010^r$ and $\varphi_r(1) = 11$. For $r = 1$ we see that

$$|\varphi_1^n(0)| = \left(\frac{1}{2}n + 1\right) 2^n$$

and

$$|\varphi_1^n(1)| = 2^n.$$

It is then clear that $|\varphi_1^n(a)| = \Theta(n2^n)$ and so φ_1 is polynomially diverging. For any $r \geq 2$ we have that

$$|\varphi_r^n(0)| = \frac{r}{r-1}(r+1)^n - \frac{1}{r-1}2^n,$$

and so φ_r is exponentially diverging.

It is important to note that every growing morphism fits into one of the above three categories (cf. [12], Theorem 4.7.43).

Theorem 3.1.2. *Let φ be a growing morphism and let $w = \varphi^\omega(a)$ be a pure morphic word.*

1. *If φ is quasi-uniform then $\rho_w(n) = O(n)$.*

2. *If φ is polynomially diverging then $\rho_w(n) = \Theta(n \log\log n)$.*

3. *If φ is exponentially diverging then $\rho_w(n) = \Theta(n \log n)$.*

Note that by Proposition 3.1.5, If φ is quasi-uniform then $\rho_w(n)$ is either $\Theta(n)$ or $O(1)$.

Corollary 3.1.1. *If w is a pure-morphic word generated by a d-uniform morphism then $\rho_w(n) = O(n)$.*

As for the case of morphisms which are not growing, the following theorems (cf. [12]) describe the situation. Let us define a word $v \in \Sigma^*$ to be *bounded*, relative to a morphism $\varphi : \Sigma^* \to \Sigma^*$, if $|\varphi^n(v)| = O(1)$ as n goes to infinity.

Theorem 3.1.3. *If w is a pure morphic word with finitely many bounded subwords then ρ_w is either*

$$O(1), \Theta(n), \Theta(n \log\log n), \text{ or } \Theta(n \log n).$$

Theorem 3.1.4. *Let w be a pure morphic word. If w is not eventually periodic and infinitely many subwords of w are bounded then $\rho_w(n) = \Theta(n^2)$.*

Example 31. Let φ be a binary morphism defined by $\varphi(0) = 001$ and $\varphi(1) = 1$. Then $w := \varphi^\omega(0)$ is not eventually periodic since 01^n0 is a subword of w for every $n \in \mathbb{N}$. By the above theorem, we have that $\rho_w(n) = \Theta(n^2)$.

3.2 Combinatorial monomial algebras

We now define some classes of monomial algebras which shall be used extensively throughout this chapter. These are new notions which were first introduced in [6].

Definition 20. Let k be a field, let $\Sigma = \{x_1, \ldots, x_m\}$ be a finite alphabet, and let w be a right-infinite word over Σ. We define the monomial algebra A_w associated with w to be the quotient $k\langle x_1, \ldots, x_m\rangle/I$, where I is the ideal generated by all finite words over Σ that do not appear as a subword of w. In the case when w is a pure morphic word, we call A_w the *iterative algebra* associated with w.

These algebras form a subclass of the monomial algebras studied in [8, Chapter 3].

Example 32. Consider the right-infinite binary word

$$w = abaabaaabaaaabaaaaab \cdots$$

We then see that $b^2 = 0$ in A_w. Moreover, $J(A_w) = \mathrm{Nil}^*(A_w) = (b)$ and $A_w/J(A_w) \cong k[x]$.

Example 33. Consider the famous Thue-Morse word w, which may be defined as the pure morphic word generated by the morphism $T : \{a, b\}^* \to \{a, b\}^*$, prolongable on a, such that $T(a) = ab$ and $T(b) = ba$. That is, $w = T^\omega(a)$. We see that

$$w = abbabaabbaababba \cdots .$$

It is well-known (cf. [1]) that w is cube-free. Therefore, every nonempty monomial v in A_w is nilpotent of index 3.

We note that monomial algebras of the form A_w are infinite-dimensional because w has arbitrarily long subwords. A natural question is:

Is every monomial algebra of the form A_w, for some right-infinite word w?

The answer to this question is "no". In particular, by the above remark no finite-dimensional monomial algebra can be of the form A_w. Furthermore, consider the following example.

Example 34. Let $A = \mathbb{C}\langle a, b\rangle/I$, where I is the ideal generated by all br, where r is a homogeneous element in the positive part of A. Then b is nonzero in A, but $bv = 0$ for every nonempty word $v \in \{a, b\}^*$. Therefore A cannot be of the form A_w for any right-infinite word w.

However, the answer to this question is "yes" for a large class of well-behaved monomial algebras. In particular, every prime monomial algebra looks like A_w, for some right-infinite word w. Recall that a ring R is prime if whenever $a, b \in R$ such that $aRb = (0)$ then $a = 0$ or $b = 0$.

Proposition 3.2.1. *If A is a prime monomial algebra then there exists a right-infinite word w such that $A = A_w$.*

Proof. Let $v_1, v_2, \ldots$ be an enumeration of the nonzero words/monomials in A. Since A is prime there exists a monomial u_1 in A such that $v_1u_1v_2 \neq 0$. Similarly, there exists a monomial $u_2 \in A$ such that $v_1u_1v_2u_2v_3 \neq 0$. Continuing in this way we have the right-infinite word

$$w = v_1u_1v_2u_2v_3u_3 \cdots .$$

Therefore $A = A_w$, as required. $\square$

The benefit of being able to describe a monomial algebra as a monomial algebra of the form A_w is that much of the combinatorial information about w, which may be easily computed, can give you deep ring-theoretical results about monomial algebras. We now survey some of the recent results where combinatorial monomial algebras have been used.

1. In [8], Belov showed that $J(A_w)$ is the ideal of A_w generated by subwords of w which occur with a bounded gap in arbitrarily long subwords of w.

2. In [7], Bell and Smoktunowicz proved that prime monomial algebras of quadratic growth have finitely many prime ideals of co-GKdimension 1. Moreover, they proved that all prime ideals of co-GKdimension 1 are monomial ideals.

3. Again in [7], Sturmian words (right-infinite words w such that $\rho_w(n) = n + 1$) were used to construct the first example of a finitely generated prime monomial algebra of GK dimension 2 and with unbounded matrix images.

4. In [50], Okninski used monomial algebras of the form A_w to show that every prime monomial algebra either satisfies a polynomial identity, is primitive, or has a nonzero locally nilpotent Jacobson radical.

We now begin our investigation of iterative algebras and how they may be used to study graded-nilpotent rings.

3.2.1 Gelfand-Kirillov dimension

In this section, we discuss the growth of iterative algebras, showing that their Gelfand-Kirillov dimension is either 1, 2, or 3.

Let k be a field and let A be a finitely generated k-algebra. The *Gelfand-Kirillov* (GK) dimension of A, denoted $\mathrm{GKdim}(A)$, is given by

$$\mathrm{GKdim}(A) := \limsup_{n\to\infty} \frac{\log\, \dim(V^n)}{\log\, n},$$

where V is a finite-dimensional subspace of A that contains 1 and generates A as a k-algebra. As A is assumed to be finitely generated, certainly such a V exists. Remarkably, this quantity does not depend upon the choice of subspace V. For more information on Gelfand-Kirillov dimension, we refer the reader to the book of Krause and Lenagan [36]. We note that being GK dimension one is equivalent to $\dim(V^n) = \Theta(n)$ (i.e., linear growth) by Bergman's gap theorem [36, Theorem 2.5] and that an important subclass of algebras of GK dimension two are given by those of *quadratic growth*. An algebra is said to be of quadratic growth if it contains a generating subspace V such that $\dim(V^n) = \Theta(n^2)$.

In the case of combinatorial monomial algebras, the GK dimension of A_w may be calculated using the subword complexity function of w, $\rho_w(n)$. This is particularly easy to do for iterative algebras. In fact, this is one of the main reasons iterative algebras were introduced.

Recall that for a pure morphic word w, $p_w(n)$ is either

$$O(1), \Theta(n), \Theta(n \log\log n), \Theta(n \log n), \text{ or } \Theta(n^2).$$

Moreover, each of these possibilities can be realized by some purely morphic word and $p_w(n) = O(1)$ if and only if w is eventually periodic.

If A_w is the combinatorial monomial algebra corresponding to the word w, and if $V = k + \sum_{a\in\Sigma} ka$ in A_w, then V^n has a basis consisting of subwords of w of length at most n. In particular, $\dim(V^n) = \sum_{j=0}^{n} p_w(j)$. Now suppose that w is a pure morphic word. If $p_w(n) = \Theta(n), \Theta(n \log n)$, or $\Theta(n \log\log n)$, then $\sum_{j=0}^{n} p_w(j) = \Theta(n^2), \Theta(n^2 \log n)$, or $\Theta(n^2 \log\log n)$, and so

$$\frac{\log\, \dim(V^n)}{\log\, n} \to 2$$

as $n \to \infty$. On the other hand, if $p_w(n) = \Theta(n^2)$ then

$$\frac{\log\, \dim(V^n)}{\log\, n} \to 3$$

as $n \to \infty$. Finally, if $p_w(n) = \mathrm{O}(1)$ then $\dim(V^n)$ grows at most linearly with n and since A_w is infinite-dimensional, we see that the GK dimension of A_w is one in this case. Putting these observations together we obtain the following result.

Theorem 3.2.1. *Let k be a field and let A_w be an iterative k-algebra. Then* $\mathrm{GKdim}(A_w) \in \{1, 2, 3\}$ *and the Gelfand-Kirillov dimension is equal to one if and only if w is eventually periodic.*

3.2.2 Combinatorial correspondences to ring theory

The benefit of considering combinatorial monomial algebras is that interesting ring theoretical information may be obtained by studying some combinatorics of the associated right-infinite word w. We remind the reader that a k-algebra A is said to be *just infinite* if $\dim_k(A) = \infty$ but for all nonzero $I \trianglelefteq R$, $\dim_k(A/I) < \infty$.

Theorem 3.2.2. *Let k be a field and let w be a right-infinite word over the alphabet Σ. Let $A = A_w$ be the associated combinatorial monomial k-algebra. We have the following correspondences.*

1. *A is semiprime if and only if w is recurrent;*
2. *A is just infinite if and only if w is uniformly recurrent;*
3. *A is PI if and only if w is eventually periodic;*
4. *A is noetherian if and only if w is eventually periodic.*

Proof. Let A be as above.

1. Suppose A is semiprime. Then for any subword v of w, (v) is a nonzero ideal of A which is not nilpotent. Therefore, for any $n \in \mathbb{N}$, there exists subwords $w_1, \ldots, w_n$ of w such that
$$vw_1vw_2v \cdots vw_n \neq 0.$$
Therefore v occurs infinitely often in w and so w is recurrent.

   Now suppose that w is recurrent. As above, A has no nonzero nilpotent monomial ideals. Let X be the free semigroup generated by Σ so that X is a u.p.-semigroup. By Theorem 2.3.1 we have that $\mathrm{Nil}_*(A) = (0)$ and so A is semiprime.

2. By [8, Theorem 3.2], A_w is just infinite if and only if w is a uniformly recurrent word.

3. Note that if w is eventually periodic, then A has Gelfand-Kirillov dimension one by Theorem 3.2.1 and hence is PI [58]. Conversely, suppose that A satisfies a polynomial identity. If we take the ideal I of A generated by the images of all subwords of w that only appear finitely many times in w then by construction the algebra $B = A/I$ is a prime monomial algebra and satisfies a polynomial identity. It is straightforward to show that there is some recurrent word w' such that the images of the subwords of w' in B form a basis for B. By [8, Remark, page 3523], we then have that w' is periodic since B satisfies a polynomial identity. We let $q(n)$ denote the collection of subwords

of w of length n that appear infinitely often in w. Then by construction $q(n)$ is precisely the number of distinct subwords of w' of length n and so $q(n) = \mathrm{O}(1)$. In particular, there is some d such that $q(d) \le d$. Let $v_1, \ldots, v_m$ denote the distinct subwords of w of length d. By assumption, at most d of these words occur infinitely often and so we may write $w = vw'$ where v is finite and w' has at most d subwords of length d. We then have that w' is eventually periodic [1, Theorem 10.2.6] and so w is eventually periodic.

4. Finally, by [8, Corollary 5.40], A_w is noetherian if and only if A_w has GK dimension one. But this occurs if and only if w is eventually periodic by Theorem 3.2.1.

□

Specializing to iterative algebras, we get the following correspondences.

Theorem 3.2.3. *Let k be a field, let $w = \varphi^\omega(a)$ be a pure morphic word over an alphabet Σ, and let A_w be the iterative k-algebra associated with w. The following results hold.*

1. *A_w is semiprime if and only if the first letter of w occurs at least twice in w;*
2. *A_w is just infinite if and only if w is uniformly recurrent;*
3. *A_w satisfies a polynomial identity if and only if w is eventually periodic;*
4. *A_w is noetherian if and only if w is eventually periodic;*

Proof. The only thing that is left to prove is that a pure morphic word $w = \varphi^\omega(a)$ is recurrent if and only if a occurs at least twice w. Clearly, if w is recurrent then a occurs infinitely many times in w. So suppose a occurs infinitely many times in w. For any $n \in \mathbb{N}$ we see that $\varphi^n(w) = w$ and so $\varphi^n(a)$ occurs infinitely many times in w. Since each subword of w occurs in $\varphi^n(a)$, for some n depending on the subword, we see that w is recurrent.

□

3.3 Graded-nilpotent rings

In this section, we give an example of an iterative algebra, which answers several questions of Greenfeld, Leroy, Smoktunowicz, and Ziembowski [21]. (In particular, Question 31 and 36 have the answer 'no' and Question 32 has the answer 'yes'.) The questions we consider deal with *graded-nilpotent* algebras. These are graded algebras A with the property that if S is a subset of A consisting of homogeneous elements of the same degree, then there is some natural number $N = N(S)$ such that $S^N = (0)$. We will show how one can use iterative algebras to construct such rings. Iterative algebras are unital and thus will not be graded-nilpotent without some alteration. However, by taking the positive part of an iterative algebra then in certain cases one can find a grading that gives a graded-nilpotent algebra. Questions 31 and 32 of [21] ask respectively whether a graded-nilpotent algebra must be Jacobson radical and whether it can have GK dimension two; Question 36 asks whether a graded-nilpotent algebra that is finitely generated as a Lie algebra must be nilpotent. We give answers to these questions with a single example. Specifically, we construct a graded-nilpotent algebra of quadratic growth that is finitely generated as a Lie algebra and whose Jacobson radical is trivial.

Let $\Sigma = \{x_1, \ldots, x_6, y_1, \ldots, y_6\}$ and let $\varphi : \Sigma^* \to \Sigma^*$ be the morphism given by

$$\begin{array}{lll} x_1 \mapsto x_1x_2y_1y_2 & x_2 \mapsto x_1x_3y_1y_3 & x_3 \mapsto x_1x_4y_1y_4 \\ x_4 \mapsto x_1x_5y_1y_5 & x_5 \mapsto x_1x_6y_1y_6 & x_6 \mapsto x_2x_3y_2y_3 \\ y_1 \mapsto x_2x_4y_2y_5 & y_2 \mapsto x_2x_5y_3y_4 & y_3 \mapsto x_2x_6y_2y_6 \\ y_4 \mapsto x_3x_4y_3y_5 & y_5 \mapsto x_3x_5y_3y_6 & y_6 \mapsto x_3x_6y_4y_5. \end{array}$$

Let w be the unique right infinite word whose first letter is x_1 and that is a fixed point of φ. We note that w is 4-uniform.

Then a straightforward computer computation shows that the incidence matrix $M(w)$ has characteristic polynomial

$$P_w(x) := x^{12} - x^{11} - 8x^{10} - 16x^9 - 2x^8 + 5x^7 + 5x^6 + 21x^5 + 31x^4 - 10x^3 - 8x^2. \quad (3.2)$$

We now put a grading on A_w by declaring that x_1 has degree one and that all other letters in Σ have degree two. We let W_n denote the degree of $\varphi^n(x_1)$. We then have that

$$W_n = \mathbf{u}M(w)^n\theta(x_1), \tag{3.3}$$

where $\mathbf{u} = [1, 2, \ldots, 2]$. A straightforward computer calculation shows that for $n = 0, 1, \ldots, 7$, we get the sequence of values

$$1, 7, 30, 120, 483, 1935, 7733, 30945 \tag{3.4}$$

for W_n.

Proposition 3.3.1. *Let d be a positive integer. Then with the grading described above, the algebra A_w has the property that the homogeneous piece of degree d is nilpotent.*

Proof. We let $\mathcal{S} = \{s_0, s_1, s_2, \ldots\}$ denote the subset of $\mathbb{N}_0$ constructed as follows. We define $s_0 = 0$ and for $i \geq 1$, we define $s_i = s_{i-1} + \deg(a_i)$, where $a_i \in \Sigma$ is the i-th letter of w. Then since $w = x_1x_2y_1y_2x_1x_3y_1y_3\cdots$, we see that $\mathcal{S} = \{0, 1, 3, 5, 7, 8, 10, 12, 14, \ldots\}$. We now let $(A_w)_d$ denote the homogeneous piece of A_w of degree d, in which x_1 has degree one and all other elements of Σ have degree two. Since $(A_w)_d$ is spanned by subwords of w of degree d, we see that if $(A_w)_d$ is not nilpotent, then for every $r \geq 1$ there must exist some subword of w of the form $u_1u_2\cdots u_r$ where $u_1, \ldots, u_r$ are words of degree d. In particular, $\mathcal{S}$ must contain an arithmetic progression of the form $i, i+d, i+2d, \ldots, i+(r-1)d$ for every $r \geq 1$.

We now show that this cannot occur. We suppose, in order to obtain a contradiction, that there is some $d \geq 1$ such that $\mathcal{S}$ contains arbitrarily long arithmetic progressions of the form

$$\{i, i+d, i+2d, \ldots, i+bd\},$$

where $i, b \in \mathbb{N}$. Then for every $n \geq 2$ there exists a subword $u = u_1u_2\cdots u_r$ of w such that each u_i is a subword of w of degree d and $r > 4^{n+1}$. Now u has length at least 4^{n+1} and since w is a fixed point of φ, we then see that u must have a subword of the form $\varphi^n(a)$ for some $a \in \Sigma$. It follows that there exist natural numbers i and j with $i > 1$ and $i + j < r$ such that $\varphi^n(a) = vu_i \cdots u_{i+j}v'$, where v is a suffix of u_{i-1} and v' is a prefix of u_{i+j}. Every $a \in \Sigma$ has the property that $\varphi^2(a)$ begins with x_1 and so $\varphi^n(a)$ begins with $\varphi^{n-2}(x_1)$. In particular, since $\varphi^{n-2}(x_1)$ is a prefix of w of length 4^{n-2} and it begins $vu_iu_{i+1}\cdots$, we see that $\mathcal{S}$ contains a progression of the form $t, t+d, t+2d, \ldots t+sd$ with $t < d$ and $t+(s+1)d$ strictly greater than the length of $\varphi^{n-2}(x_1)$. By assumption, $\mathcal{S}$ contains arbitrarily long arithmetic progressions of length d and so there are infinitely many natural numbers n for which $\varphi^n(x_1)$ contains a progression of the form $t, t+d, t+2d, \ldots t+sd$ for some $t < d$ and $t+(s+1)d$ strictly greater than the length of $\varphi^n(x_1)$. In particular, there is some fixed $t < d$ for which there are infinitely many natural numbers n with this property. Thus we see $\mathcal{S}$ contains arbitrarily long arithmetic progressions of the form $t, t+d, t+2d, \ldots, t+rd$ for some fixed $t < d$. It follows that $\mathcal{S}$ contains an infinite arithmetic progression $t, t+d, t+2d, \ldots$.

Now

$$\varphi^{n+2}(x_1) = \varphi^{n+1}(x_1x_2y_1y_2) = \varphi^{n+1}(x_1)\varphi^n(x_1x_3y_1y_3)\varphi^{n+1}(y_1y_2).$$

Thus $\varphi^{n+1}(x_1)\varphi^n(x_1)$ is a prefix of w for every $n \geq 1$.

Without loss of generality $d \in \mathbb{N}$ is minimal with respect to having the property that $\mathcal{S}$ has an infinite arithmetic progression of the form $a + d\mathbb{N}$, where $a < d$. Define

$$T := \{i : 0 \leq i < d, \{i + dn \colon n \geq 0\} \subseteq \mathcal{S}\}.$$

We write $T = \{i_1, i_2, \ldots, i_p\}$, where $i_1 < i_2 < \cdots < i_p$. Notice that there exists a positive integer N such that if $j \in \{0, \ldots, d-1\} \setminus T$ then there is some m, depending upon j, such that $j + md \leq N$ and $j + md \notin \mathcal{S}$.

Now let $a_j := i_{j+1} - i_j$ for $j = 1, \ldots, p$, where we take $i_{p+1} = i_1 + d$. Consider

$$\mathbf{a} := (a_1, a_2, \ldots, a_p) \in \mathbb{N}^p.$$

Let $\sigma = (1, 2, 3, \ldots, p) \in S_p$, the symmetric group on p letters. For $\pi \in S_p$, we let $\pi(\mathbf{a})$ denote $(a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(p)})$.

We claim that no non-trivial cyclic permutation of $\mathbf{a}$ can be equal to $\mathbf{a}$. Assume, in order to obtain a contradiction, that $\sigma^m(a) = a$ for some $m \in \{1, \ldots, p-1\}$. Let $\pi = \sigma^m$ so that we then have that $a_i = a_{\pi(i)}$. We see that, by definition, $\mathcal{S}$ contains the set

$$\{i_1, i_2, \ldots, i_p, i_1 + d, i_2 + d, \ldots, i_p + d, i_1 + 2d, \ldots\}$$

Moreover, the differences between successive terms of this sequence are given by the sequence

$$(a_1, a_2, \ldots, a_p, a_1, a_2, \ldots, a_p, a_1, \ldots).$$

By repeatedly applying the identity $\sigma^m(a) = a$, we have that

$$(a_1, a_2, \ldots, a_p, a_1, a_2, \ldots, a_p, a_1, \ldots) = (a_1, a_2, \ldots, a_m, a_1, a_2, \ldots, a_m, a_1, \ldots).$$

Therefore $\mathcal{S}$ contains an infinite arithmetic progression of the form $j + (a_1 + \cdots + a_m)\mathbb{N}$, and $a_1 + \cdots + a_m < d$. Moreover, by the argument we used earlier, where we observed that $\varphi^n(x_1)$ occurs infinitely often in w for every n, we see that we can take $j < a_1 + \cdots + a_m$. But this contradicts the minimality of d. Therefore $\sigma^m(a) \neq a$ for any $m \in \{1, \ldots, p-1\}$, as claimed.

Now for every $n \geq 2$, we have $\varphi^{n+1}(x_1)\varphi^n(x_1)$ is a prefix of w. Moreover, by assumption the infinite arithmetic progressions in $\mathcal{S}$ with difference d all appear in the subset

$$X := \{i_1, i_2, \ldots, i_p, i_1 + d, i_2 + d, \ldots\}.$$

Let W_m denote the degree of $\varphi^m(x_1)$ for each $m \geq 0$. Then given a positive integer $n > N$, there exists a unique $s \in \{1, \ldots, p\}$ and a unique $r \geq 0$ such that $i_s + dr$ is the largest positive integer in the set $\{i_q + d\ell \colon 1 \leq q \leq p, \ell \geq 0\}$ that is less than W_n. Since $\varphi^n(x_1)\varphi^{n-1}(x_1)$ is a prefix of w, we see that the part of the set

$$\{i_1, i_2, \ldots, i_p, i_1 + d, i_2 + d \ldots, i_s + dr, W_n + i_1, W_n + i_2, \ldots, W_n + i_p, W_n + i_1 + d, \ldots\}$$

in $[0, W_n + W_{n-1}]$ is entirely contained in $\mathcal{S}$.

For $j = 1, \ldots, p$, define i_{s+j} to be $i_{s+j-p} + d$ if $s + j > p$. Then by definition of T, we have that $\{i_{s+1} + dr, i_{s+2} + dr, \ldots, i_{s+p} + dr, i_{s+1} + d(r+1), \ldots\} \cap [0, W_n + W_{n-1}] \subseteq \mathcal{S}$ and so subtracting W_n, the degree of $\varphi^n(x_1)$, and using the fact that the prefix $\varphi^n(x_1)$ in w is then followed by $\varphi^{n-1}(x_1)$ in w, we see that

$$\{i_{s+1} + dr - W_n, i_{s+2} + dr - W_n, \ldots\} \cap [0, W_{n-1}] \subseteq \mathcal{S}.$$

Since $n-1 \geq N$ and $i_{s+j} + dr - W_n \in \{0, \ldots, d\}$ for $j = 0, \ldots, p$, we see from the definition of N that $i_{s+j} + dr - W_n = i_j$ for $j = 1, \ldots, p$. Notice also that $(i_{s+j} + dr - W_n) - (i_{s+j-1} + dr - W_n) = a_{s+j}$ for $j = 1, \ldots, p$ and since $i_{s+j} + dr - W_n = i_j$, we see that $(i_{s+j} + dr - W_n) - (i_{s+j-1} + dr - W_n) = a_j$ for $j = 1, \ldots, p$. Since no non-trivial cyclic permutation of $\mathbf{a}$ is equal to $\mathbf{a}$, we see that $s = p$ and so $i_1 + dr + d - W_n = i_1$. In particular, $W_n \equiv 0 \pmod{d}$ for all $n > N$.

We now show that there is no $d > 1$ such that $d | W_n$ for all sufficiently large n, and we will then obtain the desired result since $\mathcal{S}$ cannot contain an infinite arithmetic progression with difference one.

Using Equation (3.2) and the Cayley-Hamilton theorem, we have that W_n satisfies the recurrence

$$\begin{aligned} &W_n - W_{n-1} - 8W_{n-2} - 16W_{n-3} - 2W_{n-4} + 5W_{n-5} + 5W_{n-6} + 21W_{n-7} \\ &+31W_{n-8} - 10W_{n-9} - 8W_{n-10} = 0 \end{aligned} \tag{3.5}$$

for $n \geq 12$. In particular, $W_n \equiv W_{n-1} + W_{n-5} + W_{n-6} + W_{n-7} + W_{n-8} \pmod{2}$ for all $n \geq 12$. We can now show that there are infinitely many n for which W_n is not divisible by 2. To see this, suppose that this were not the case. Then there would exist some largest natural number ℓ such that W_ℓ is odd. From Item (3.4), we see that $\ell \geq 4$ and so $\ell + 8 \geq 12$. But now Equation (3.5) gives that $W_{\ell+8} + W_{\ell+7} + W_{\ell+3} + W_{\ell+2} + W_{\ell+1} \equiv W_\ell \pmod{2}$, which is a contradiction since the left-hand side is even and the right-hand side is odd. It follows that d must be odd. Now suppose that some odd prime p divides d. Then we have $p | W_n$ for all n sufficiently large. Let ℓ be the largest natural number for which p does not divide W_ℓ. Then since $\gcd(W_5, W_6) = 1$, we see that $\ell \geq 5$. But now, since $\ell + 10 \geq 12$, Equation (3.5) gives that $8W_\ell$ is a $\mathbb{Z}$-linear combination of elements of the form $W_{\ell+i}$ with $i = 1, \ldots, 10$ and so we see that if $p|W_n$ for $n > \ell$ then $p|8W_\ell$. Since p is odd and p does not divide W_ℓ we get a contradiction. It follows that $d = 1$ and so $\mathcal{S}$ must contain the progression $\{0, 1, 2, 3, \ldots\}$, but this is clearly false. The result follows. □

We next show that the algebra A_w is finitely generated as a Lie algebra.

Lemma 3.3.1. *Let v be a subword of w of length at least two. Then some cyclic permutation of v is not a subword of w.*

Proof. Let d denote the length of v. We prove this by induction on d. Our base cases are when $d = 2, 3, 4$. We consider each of these cases separately. In each case, we suppose towards a contradiction that there exists a word v of length d all of whose cyclic permutations are subwords of w and we derive a contradiction.

Case I: d=2. For the case when $d = 2$, observe that a subword of w of length two is either of the form x_ix_j with $i < j$ or y_iy_j with $i < j$ or of the form x_ky_ℓ or $y_\ell x_k$. It is immediate that if our subword of length two is of the form x_ix_j or y_iy_j then $i < j$ and so x_jx_i and y_jy_i cannot be subwords of w in this case. Thus for the case when $d = 2$, it only remains to show that if x_iy_j is a subword of w then y_jx_i cannot be. Observe that any subword of w of length two is either a subword of $\varphi(a)$ for some $a \in \Sigma$ or it is a subword of $\varphi(ab)$, with $a, b \in \Sigma$, consisting of the last letter of $\varphi(a)$ followed by the first letter of $\varphi(b)$. In the case that we have a word of the form x_iy_j, then we see that it must be the second and third letters of $\varphi(a)$ for some $a \in \Sigma$. We are also assuming that y_jx_i is a subword of w. In this case, we have that there are letters $b, c \in \Sigma$ such that bc is a subword of w and y_j is the last letter of $\varphi(b)$ and x_i is the first letter of $\varphi(c)$.

Since the second letter of any $\varphi(a)$ must be in $\{x_2, x_3, x_4, x_5, x_6\}$, we see that $i \neq 1$. Similarly, the first letter of any $\varphi(c)$ must be in $\{x_1, x_2, x_3\}$ and so $i \neq 4, 5, 6$. Thus $i \in \{2, 3\}$. Notice that the last letter of $\varphi(b)$ can never be y_1 and so $j \in \{2, 3, 4, 5, 6\}$. By assumption, there exists some $a \in \Sigma$ such that x_iy_j is a subword of $\varphi(a)$ with $i \in \{2, 3\}$ and $j > 1$. Looking at the map φ, we see that the only possibility is $i = 3$, $j = 2$ coming from $a = x_6$. But then $y_j x_i = y_2 x_3$. By assumption, there exist b and c in Σ such that bc is a subword of w and such that y_2 is the last letter of $\varphi(b)$ and x_3 is the first letter of $\varphi(c)$. But x_1 is the only letter in Σ with the property that applying φ to it gives a four-letter word ending in y_2. Thus $b = x_1$. Similarly, since $\varphi(c)$ begins with x_3 we see $c \in \{y_4, y_5, y_6\}$. But this then means that $x_1 y_k$ is a subword of w for some $k \geq 4$, which is impossible since x_1 is always followed by an element from $\{x_2, \ldots, x_6\}$ in w.

Case II: d=3. Since we can never have three consecutive letters from $\{x_1, \ldots, x_6\}$ occurring in w, we see that there are at most two letters from $x_1, \ldots, x_6$ in v. Similarly, there are at most two letters from $y_1, \ldots, y_6$ occurring in v. Thus either v has exactly two letters from $\{x_1, \ldots, x_6\}$ and one letter from $\{y_1, \ldots, y_6\}$ or it has exactly two letters from $\{y_1, \ldots, y_6\}$ and one letter from $\{x_1, \ldots, x_6\}$. We consider the first case, as the other case is identical. In the first case, v has some cyclic permutation of the form $x_i y_j x_k$, which cannot be a subword of w since we must always have a block of two consecutive y_j's between blocks of x_i's.

Case III: d=4. In this case, we can argue as in Case II to show that some cyclic permutation of v is of the form $x_i x_j y_k y_\ell$. Any subword of w of the form $x_i x_j y_k y_\ell$ must be $\varphi(a)$ for some $a \in \Sigma$. In particular, $i < j$ and $k < \ell$. But by assumption $y_k y_\ell x_i x_j$ is also a subword of w and so there exist $b, c \in \Sigma$ such that bc is a subword of w, $y_k y_\ell$ are the last two letters of $\varphi(b)$ and $x_i x_j$ are the last two letters of $\varphi(c)$. But the first two letters of $\varphi(d)$ for $d \in \Sigma$ completely determine d, and similarly for the last two letters. In particular, $b = c = a$ and so $bc = a^2$ is a subword of w. But we now see this is impossible from Case I.

We now complete the induction argument. Suppose that $d \geq 5$ is such that if $m \in \{2, \ldots, d-1\}$ then any subword u of w of length m has the property that some cyclic permutation of u is not a subword of w. Let v be a word of length d. Then arguing as in Case II, we have that some cyclic permutation v' of v is of the form $x_{i_1} x_{i_2} y_{j_1} y_{j_2} x_{i_3} \cdots$. Notice that if d is 1 or 2 mod 4, then v' ends with some x_k and so if we move this x_k to the beginning of v' then we get a cyclic permutation of v with three consecutive letters $x_k x_{i_1} x_{i_2}$, which cannot be a subword of w. If d is 3 mod 4, then the last two letters of v' are of the form $x_i y_j$ and so if we shift these two letters to the beginning of v' we see that v has a cyclic permutation with three consecutive letters $x_i y_j x_{i_1}$, which cannot be a subword of w. Thus we have $d = 4m$ with $m \geq 2$. Also $v' = \varphi(a_1) \cdots \varphi(a_m)$ for some $a_1, \ldots, a_m \in \Sigma$. But then if $u = a_1 \ldots a_m$ and u' is a cyclic permutation of u then $\varphi(u')$ is a cyclic permutation of $\varphi(u) = v'$. Moroever, since $\varphi(u')$ begins with two consecutive letters from x_i and x_j, we see that $\varphi(u')$ is a subword of w if and only if u' is a subword of w. In particular, if all cyclic permutations of v are subwords of w then all cyclic permutations of u are subwords of w. But u has length $m \in \{2, \ldots, d-1\}$ and so we see that this cannot occur by the induction hypothesis. The result now follows. $\square$

Proposition 3.3.2. *The algebra A_w is finitely generated as a Lie algebra.*

Proof. Let B denote the Lie subalgebra of A_w that is generated by the elements

$$\{1, x_1, \ldots, x_6, y_1, \ldots, y_6\}.$$

Since A_w is spanned by the images of all subwords of w, it suffices to show that the image of every subword of w is in B. Suppose that u is a subword of w whose image is not in B. We may assume that we pick u of minimal length d with respect to having this property. Since all subwords of w of length ≤ 1 have images in B, d must be at least two. By Lemma 3.3.1, u has some cyclic permutation that is not a subword of w. In particular, we may decompose $u = ab$ so that a and b are subwords of w but such that ba is not a subword of w; i.e., ba has zero image in A_w. Then $u = [a, b] \in A_w$. But now a and b have length less than d and so by minimality of d, we have that the images of a and b are in B and so the image of $u = [a, b]$ is in also in B, a contradiction. The result follows. $\square$

We need one last result to obtain our example.

Lemma 3.3.2. *The word w is uniformly recurrent.*

Proof. It is straightforward to check that φ is a growing morphism. Moreover, x_1 is a subword of $\varphi^2(a)$ for every $a \in \Sigma$. By Proposition 3.1.4 w is uniformly recurrent, as required. $\square$

Putting these results together, we obtain the following result, which answers questions 31, 32, and 36 of [21].

Theorem 3.3.1. *Let R denote the positive part of the algebra A_w with the grading described above. Then R is just infinite, graded-nilpotent, has quadratic growth, has trivial Jacobson radical, and is finitely generated as a Lie algebra.*

Proof. The fact that R is graded-nilpotent follows from Proposition 3.3.1. To show the remaining claims, we first show that A_w does not satisfy a polynomial identity. To see this, by Theorem 3.2.3 it suffices to show that w is not eventually periodic. However, by Lemma 3.3.1, w is square-free, and so w cannot be eventually periodic. Thus A_w is not PI.

Since w is not eventually periodic, we now see that R has quadratic growth since w is 4-uniform. By Lemma 3.3.2 and Theorem 3.2.3 (2), we see that A_w is just infinite and hence R is just infinite.

Since w is uniformly recurrent and w is not eventually periodic, Corollary 3.11 and Proposition 3.8 of [8] then give that the Jacobson radical of A_w is zero. Finally, R is finitely generated as a Lie algebra by Proposition 3.3.2. This completes the proof. □

In fact, since A_w is just infinite, it is prime and so by a result of Okniński [49], the algebra A_w must be primitive, since it is not PI and has zero Jacobson radical.

Chapter 4

Skew polynomial extensions of derivation type

In this chapter we explore the radical theory of differential polynomial rings. The main results of this chapter are as follows.

Theorem 4.0.1. *Let R be a locally nilpotent ring satisfying a polynomial identity and let δ be a derivation of R. Then $R[x;\delta]$ is locally nilpotent.*

Theorem 4.0.2. *Let R be a polynomial identity ring and let δ be a derivation of R. Then $S := J(R[x;\delta]) \cap R$ is a nil δ-ideal of R.*

Theorem 4.0.2 is original work of the author of this thesis. Theorem 4.0.2 is a result from joint work with Dr. Jason Bell and Mr. Forte Shinko. The author's most significant contributions regarding Theorem 4.0.2 are Lemma 4.3.1 and the techniques used in the proof of Theorem 4.0.2 involving k-valid words and $\mathbf{b}$-bounded sequences. Sections 4.2 and 4.3 are based on the work in [5] and Section 4.4 is based on the work in [41].

4.1 Radical properties of skew polynomial extensions of derivation type

Let R be a ring and let δ be a derivation of R. Recall that we define the differential polynomial ring $R[x;\delta]$ to be the set all polynomials of the form $a_nx^n + \cdots + a_1x + a_0$ with $n \geq 0$, $a_0, \ldots, a_n \in R$ equipped with usual polynomial addition and multiplication given by $xa = ax + \delta(a)$ for $a \in R$. There has been a lot of interest in studying the Jacobson radical of the ring $R[x;\delta]$ [2, 17, 29]. In the case when δ is the zero derivation, $R[x;\delta] = R[x]$. It was shown in [2] that $J(R[x]) = N[x]$, where N is a nil ideal of R given by $N = J(R[x]) \cap R$. Extending this work, Ferrero, Kishimoto, and Motose [17] proved the following result.

Theorem 4.1.1 ([17], Theorems 3.2 and 3.3)**.** *Let R be a ring and let δ be any derivation on R. Then, $J(R[x;\delta]) = (J(R[x;\delta]) \cap R)[x;\delta]$. Moreover, if R is commutative then $J(R[x;\delta]) \cap R$ is a nil ideal of R.*

Moreover, they proved the analogue of the above theorem for the prime radical.

Theorem 4.1.2 ([17], Corollary 2.2)**.** *If R is a ring and δ is any derivation on R then* $\mathrm{Nil}_*(R[x;\delta]) = (\mathrm{Nil}_*(R[x;\delta]) \cap R)[x;\delta]$.

In Section 4.4 we extend Theorem 4.1.1 to the class of polynomial identity rings. Namely, we show that if R is a PI ring and δ is any derivation on R then $J(R[x;\delta]) \cap R$ is a nil ideal of R. Very recently, Smoktunowicz constructed an example in [62] which shows that $J(R[x;\delta]) \cap R$ need not be nil in general.

A natural question stemming from the above considerations is with regards to the behaviour of the locally nilpotent (Levitzki) radical of differential polynomial rings. Recently, Smoktunowicz and Ziembowski negatively answered a question of Sheshtakov [63, Question 1.1], by constructing an example of a locally nilpotent ring R such that $R[x;\delta]$ is not equal to its own Jacobson radical. In then follows that their example $R[x;\delta]$ was not locally nilpotent. They asked [63, Page 2] whether Sheshtakov's question has an affirmative answer if one assumes, in addition, that R satisfies a polynomial identity. We answer this question in the affirmative by showing that if R is a locally nilpotent PI ring and δ is any derivation on R then $R[x;\delta]$ is locally nilpotent. In particular, $R[x;\delta]$ is Jacobson radical.

We note that our theorem need not hold if we instead consider a skew polynomial extension of R of automorphism type.

Example 35. Let R be the positive part of $\mathbb{C}[t_n\colon n \in \mathbb{Z}]$ modulo the relations $t_n^2 = 0$ for every $n \in \mathbb{N}$. Moreover, let σ be the automorphism of R given by $\sigma(t_i) = t_{i+1}$. Then R is commutative and locally nilpotent but $t_0 x \in R[x;\sigma]$ is not nilpotent.

4.2 Relevant combinatorics on words

In this section, we give some results on combinatorics on words that will be useful to us. We begin by recalling some of the basic notions we will use.

Let A be a (not necessarily finite) alphabet. For any word $u \in A^*$, we let $u_i \in A$ denote the i-th letter of u. We will be interested in the case when $A = \mathbb{N} := \{0, 1, \ldots\}$.

Let S_n be the symmetric group on n letters. Define weight: $\mathbb{N}^* \to \mathbb{N}$ as follows. If $u \in \mathbb{N}^*$ is a word of length n then we define

$$\mathrm{weight}(u) := \min\left\{\sum_{i=1}^{n}(n+1-i)u_{\sigma(i)} \mid \sigma \in S_n\right\}.$$

For a natural number k and a word $u \in \mathbb{N}^*$ of length n, we say that u is *k-valid* if $\mathrm{weight}(u) \le k\binom{n+1}{2}$. Roughly speaking, this says that the average letter of u is not too large compared to k. Let $\mathbf{b} = (b_0, b_1, b_2, \ldots)$ be a sequence of natural numbers. We say that $u \in \mathbb{N}^*$ is $\mathbf{b}$-bounded if for each $m \in \mathbb{N}$, every subword of length b_m contains at least one letter greater than m.

Let $(B, <)$ be a poset. We place a partial order $\prec$ on B^* as follows. Let $u, v \in B^*$. Then u and v are incomparable if one is a prefix of the other. Otherwise, we compare them lexicographically using the order from B. We will be most interested in this when B is the natural numbers and we will make use of this induced order on $\mathbb{N}$.

We say that a finite sequence of words $\{v_i\}_{i=1}^d \subseteq B^+$ is *d-decreasing* if

$$v_1 \succ v_2 \succ \cdots \succ v_d.$$

We say that a word $u \in B^*$ has a d-decreasing subword if we can express $u = vw_1w_2\cdots w_d x$ where $\{w_i\}_{i=1}^d$ is a d-decreasing subsequence. We observe that every word trivially contains a 0-decreasing subsequence.

Proposition 4.2.1. *Let $\mathbf{b} = (b_0, b_1, \ldots)$ be a sequence of natural numbers, let d and k be positive integers, and let $\varepsilon \in (0,1]$. Then there exist natural number constants $M = M(d, \mathbf{b}, k, \varepsilon)$ and $N = N(d, \mathbf{b}, k, \varepsilon)$ such that if $u \in \mathbb{N}^*$ is a k-valid, $\mathbf{b}$-bounded word of length $n \geq N$, then the subword of u consisting of the last $\lfloor \varepsilon n \rfloor$ letters contains a d-decreasing subsequence $\{w_i\}_{i=1}^d$ where the first letter of w_i is less than M for $i = 1, \ldots, d$.*

Proof. We proceed by induction on d. The case when $d = 0$ is vacuous and we may take $M = M(0, b, k, \varepsilon) = N(0, b, k, \varepsilon) = 1$.

Suppose now that the proposition is true for all nonnegative integers $\leq d$. We take

$$M_1 = M(d, \mathbf{b}, k, \varepsilon/2) \qquad \text{and} \qquad N_1 = N(d, \mathbf{b}, k, \varepsilon/2). \tag{4.1}$$

We pick a positive integer M_2 satisfying:

(i) $M_2 > M_1$;

(ii) $M_2 > 8b_{M_1}^2 k\varepsilon^{-2}$.

We have

$$M_2 \binom{(\varepsilon n/2 - 1)/b_{M_1}}{2} \sim M_2 \varepsilon^2 b_{M_1}^{-2} n^2/8 \geq kn^2.$$

It follows that there is a natural number $N_2 > N_1$ such that whenever $n > N_2$ we have

$$M_2\binom{(\varepsilon n/2-1)/b_{M_1}}{2} > k\binom{n+1}{2}. \tag{4.2}$$

Let $u \in \mathbb{N}^*$ be a k-valid, $\mathbf{b}$-bounded word of length $n \geq N_2$. We write $u = vwx$, where wx is of length $\lfloor \varepsilon n \rfloor$ and x is of length $\lfloor \varepsilon n/2 \rfloor$. We decompose w into subwords of length b_{M_1} as follows. We write $w = y_1 \cdots y_j y_{j+1}$ where each of $y_1, \ldots, y_j$ has length b_{M_1} and y_{j+1} has length less than b_{M_1} (possibly zero). By construction,

$$j = \left\lfloor \frac{\lfloor \varepsilon n \rfloor - \lfloor \varepsilon n/2 \rfloor}{b_{M_1}} \right\rfloor. \tag{4.3}$$

Since y_i has length b_{M_1} for $i \in \{1, \ldots, j\}$, it must contain a letter a_i with $a_i \geq M_1$.

We claim that there exists some $i \in \{1, \ldots, j\}$ such that $a_i < M_2$. To see this, suppose that this is not the case. Then u contains at least j letters that are each at least M_2. Since u contains j letters that are at least M_2, we have

$$\text{weight}(u) \geq jM_2 + (j-1)M_2 + \cdots + M_2 = M_2\binom{j+1}{2} \geq M_2\binom{(\varepsilon n/2-1)/b_{M_1}}{2}.$$

But Equation (4.2) gives that this contradicts the fact that u is a k-valid word.

We conclude that w must contain a letter a with $M_1 \leq a < M_2$. We write $w = bc$ where c is a word whose first letter is a.

By the inductive hypothesis, we can write $x = pv_1 \cdots v_d q$ where $v_1 \succ \cdots \succ v_d$ and the first letter of v_i is strictly less than M_1 for $i \in \{1, \ldots, d\}$. We then have that $wx = bcpv_1 \cdots v_d q$, and by construction

$$cp \succ v_1 \succ \cdots \succ v_d$$

is a $(d+1)$-decreasing subsequence where the first letter of each word in the sequence is less than M_2. The result now follows taking

$$N(d+1, \mathbf{b}, k, \varepsilon) = N_2 \qquad \text{and} \qquad M(d+1, \mathbf{b}, k, \varepsilon) = M_2.$$

$\square$

Corollary 4.2.1. *Let $\mathbf{b} = (b_0, b_1, \ldots)$ be a sequence of natural numbers and let d and k be positive integers. Then there exists a natural number $N = N(d, \mathbf{b}, k)$ such that if $u \in \mathbb{N}^*$ is a k-valid, $\mathbf{b}$-bounded word of length $n \geq N$, then u contains a d-decreasing subword.*

Proof. We take $N(d, \mathbf{b}, k) = N(d, \mathbf{b}, k, 1)$ from Proposition 4.2.1. $\square$

4.3 Differential polynomial extensions of locally nilpotent rings

We begin with a simple lemma that will allow us to apply our combinatorial results from the preceding section.

Lemma 4.3.1. *Let R be a ring, let $T = \{a_1, \ldots, a_m\}$ be a finite subset of R, and let δ be a derivation of R. If n and k are natural numbers and $p_1, \ldots, p_{n+1} \leq k$ then the product*

$$a_{i_0} x^{p_1} a_{i_1} x^{p_2} \cdots a_{i_n} x^{p_{n+1}}$$

in $R[x; \delta]$ can be written as a $\mathbb{Z}$-linear combination of of elements of the form

$$a_{i_0} \delta^{j_1}(a_{i_1}) \delta^{j_2}(a_{i_2}) \cdots \delta^{j_n}(a_{i_n}) x^M,$$

where M is a nonnegative integer and $j_1 j_2 \cdots j_n \in \mathbb{N}^*$ is k-valid.

Proof. Using the formula

$$x^d a = \sum_{j=0}^{d} \binom{d}{j} \delta^j(a) x^{d-j}, \tag{4.4}$$

for $d \geq 0$ and $a \in R$, it is straightforward to see that

$$a_{i_0} x^{p_1} a_{i_1} x^{p_2} a_{i_3} x^{p_2} \cdots a_{i_n} x^{p_{n+1}}$$

can be expressed as a $\mathbb{Z}$-linear combination of elements of the form

$$a_{i_0} \delta^{j_1}(a_{i_1}) \cdots \delta^{j_n}(a_{i_n}) x^{p_1+p_2+\cdots+p_n+p_{n+1}-j_1-\cdots-j_n}$$

where we have $j_i \leq p_1 + \cdots + p_i - j_1 - \cdots - j_{i-1}$ for $i = 1, \ldots, n$. In particular, we have

$$j_1 + \cdots + j_i \leq p_1 + \cdots + p_i \leq ki$$

for $i = 1, \ldots, n$. Summing over all i then gives

$$\sum_{i=1}^{n} (n+1-i) j_i \leq \sum_{i=1}^{n} ki = k \binom{n+1}{2}.$$

Thus the word $j_1 j_2 \cdots j_n \in \mathbb{N}^*$ is necessarily k-valid. The result follows. $\square$

Theorem 4.3.1. *Let R be a locally nilpotent ring satisfying a polynomial identity and let δ be a derivation of R. Then $R[x;\delta]$ is locally nilpotent.*

Proof. Let $S = \{p_1(x), \ldots, p_m(x)\}$ be a finite subset of $R[x;\delta]$. We wish to show that there is a natural number $N = N(S)$ such that $S^{N+1} = 0$; i.e., every product of $N+1$ elements of S is equal to zero. Then there is a finite subset $T = \{a_1, \ldots, a_t\}$ of R and a natural number k such that $S \subseteq T + Tx + \cdots + Tx^k$. Let $S_0 = T \cup Tx \cup \cdots \cup Tx^k$. Then every element of S^n can be expressed as a sum of elements of the form S_0^n and hence it is sufficient to show that there is a natural number N such that $S_0^{N+1} = 0$.

To show that $S_0^{N+1} = 0$, it is enough to show that

$$Tx^{p_1} Tx^{p_2} \cdots Tx^{p_{N+1}} = 0$$

for every sequence $(p_1, \ldots, p_{N+1}) \in \{0, \ldots, k\}^{N+1}$. For each $n \geq 0$, we let $T_n = T \cup \delta(T) \cup$

$\cdots \cup \delta^n(T) \subseteq R$. Then since R is locally nilpotent, there exists a natural number b_n such that $T_n^{b_n} = 0$. We let $\mathbf{b} = (b_0, b_1, b_2, \ldots)$.

By Lemma 4.3.1, if $(p_1, \ldots, p_{N+1}) \in \{0, \ldots, k\}^{N+1}$ then we have that

$$Tx^{p_1}Tx^{p_2}\cdots Tx^{p_{N+1}}$$

can be written as a $\mathbb{Z}$-linear combination of elements of the form

$$a_{i_0}\delta^{j_1}(a_{i_1})\cdots\delta^{j_N}(a_{i_N})x^M$$

where M is a nonnegative integer and $j_1j_2\cdots j_N$ is a word that is k-valid. Moreover, whenever $j_1j_2\cdots j_N$ is not $\mathbf{b}$-bounded we trivially have

$$a_{i_0}\delta^{j_1}(a_{i_1})\cdots\delta^{j_N}(a_{i_N}) = 0,$$

since it necessarily contains a factor from $T_n^{b_n}$ for some $n \geq 0$. In particular, it is sufficient to show that there is some natural number N such that all elements of R of the form

$$a_{i_0}\delta^{j_1}(a_{i_1})\cdots\delta^{j_N}(a_{i_N}),$$

with $j_1j_2j_3\cdots j_N \in \mathbb{N}^*$ a k-valid and $\mathbf{b}$-bounded word, are zero.

Let d be the PI degree of R and let $N = N(d, \mathbf{b}, k)$ be as in the statement of Corollary 4.2.1. We claim that whenever $j_1j_2j_3\cdots j_N$ a k-valid and $\mathbf{b}$-bounded word we have $a_{i_0}\delta^{j_1}(a_{i_1})\cdots\delta^{j_N}(a_{i_N}) = 0$. To see this, suppose towards a contradiction that this is not the case and let $j_1\cdots j_N$ be the lexicographically smallest (i.e., the smallest word with respect to $\prec$) k-valid and $\mathbf{b}$-bounded word of length N such that $a_{i_0}\delta^{j_1}(a_{i_1})\cdots\delta^{j_N}(a_{i_N})$ is not zero.

Given a subword $y = j_sj_{s+1}\cdots j_{s+r}$ of $j_1\cdots j_N$, we define

$$f(y) = \delta^{j_s}(a_{i_s})\cdots\delta^{j_{s+r}}(a_{i_{s+r}}) \in R.$$

By Corollary 4.2.1, we can write $j_1 \cdots j_N = uw_1w_2\cdots w_d v$ with

$$w_1 \succ w_2 \succ \cdots \succ w_d.$$

Furthermore, we have that R satisfies a homogeneous multilinear polynomial identity of degree d:

$$x_1 \cdots x_d = \sum_{\substack{\sigma \in S_d \\ \sigma \neq \mathrm{id}}} c_\sigma x_{\sigma(1)} \cdots x_{\sigma(d)}$$

with $c_\sigma \in \mathbb{Z}$ for each $\sigma \in S_d \setminus \{\mathrm{id}\}$. Taking $x_i = f(w_i)$ for $i = 1, \ldots, d$ we see that

$$f(w_1)f(w_2)\cdots f(w_d) \;=\; \sum_{\substack{\sigma \in S_d \\ \sigma \neq \mathrm{id}}} c_\sigma f(w_{\sigma(1)}) \cdots f(w_{\sigma(d)}).$$

Hence

$$a_{i_0}\delta^{j_1}(a_{i_1}) \cdots \delta^{j_N}(a_{i_N}) = \sum_{\substack{\sigma \in S_d \\ \sigma \neq \mathrm{id}}} c_\sigma a_{i_0} f(u) f(w_{\sigma(1)}) \cdots f(w_{\sigma(d)}) f(v). \tag{4.5}$$

By construction, for $\sigma \in S_d$ with $\sigma \neq \mathrm{id}$ we have $a_{i_0} f(u) f(w_{\sigma(1)}) \cdots f(w_{\sigma(d)}) f(v)$ is an element of the form $a_{i_0}\delta^{j_{\tau(1)}}(a_{i_{\tau(1)}}) \cdots \delta^{j_{\tau(N)}}(a_{i_{\tau(N)}})$ with $\tau \in S_N$ such that $j_{\tau(1)} \cdots j_{\tau(N)}$ is lexicographically less than $j_1 \cdots j_N$. We note that, by definition, permutations of k-valid words are again k-valid. Thus if we also have that if $j_{\tau(1)} \cdots j_{\tau(N)}$ is $\mathbf{b}$-bounded then we must have $a_{i_0}\delta^{j_{\tau(1)}}(a_{i_{\tau(1)}}) \cdots \delta^{j_{\tau(N)}}(a_{i_{\tau(N)}}) = 0$ by minimality of $j_1 \cdots j_N$. On the other hand, if $j_{\tau(1)} \cdots j_{\tau(N)}$ is not $\mathbf{b}$-bounded then $a_{i_0}\delta^{j_{\tau(1)}}(a_{i_{\tau(1)}}) \cdots \delta^{j_{\tau(N)}}(a_{i_{\tau(N)}})$ contains a factor that lies in $T_n^{b_n}$ for some $n \geq 0$ and hence it is zero. Thus we have shown that in either case we have $a_{i_0}\delta^{j_{\tau(1)}}(a_{i_{\tau(1)}}) \cdots \delta^{j_{\tau(N)}}(a_{i_{\tau(N)}})$ is zero for all applicable τ, and so from Equation (4.5) we see

$$a_{i_0}\delta^{j_1}(a_{i_1}) \cdots \delta^{j_N}(a_{i_N}) = 0,$$

a contradiction. It follows that $S^{N+1} = 0$. $\square$

4.4 On the Jacobson radical of differential polynomial extensions of polynomial identity rings

In this section we extend the work of Ferrero, Kishimoto, and Motose in [17] by extending Theorem 4.1.1 to polynomial identity rings, of which commutative rings are a special case.

Theorem 4.4.1. *Let R be a polynomial identity ring and let δ be a derivation of R. Then $S := J(R[x;\delta]) \cap R$ is a nil δ-ideal of R.*

As a direct consequence of our main result, we also get a result of Tsai, Wu, and Chuang [65], which says that if R is a PI ring with zero upper nilradical then $R[x;\delta]$ is semiprimitive. Namely, if we additionally make the assumption that $\mathrm{Nil}^*(R) = (0)$ then we get that $S = (0)$ and so $J(R[x;\delta]) = (0)$. We note that in general one cannot deduce the work of Tsai, Wu, and Chuang [65] from our result because the upper nilradical of a ring is not in general closed under the action of a derivation. Thus one cannot hope to obtain our result as a consequence by passing to a semiprimitive homomorphic image.

We note that other work on this topic has been done by other authors [9, 10, 29, 44]. In fact, a similar result to Theorem 4.4.1 can be found in [10].

Theorem 4.4.2 ([10], Corollary 3.5.)**.** *Let L be a Lie algebra over a field K which acts as K-derivations on a K-algebra R. This action determines a crossed product $R * U(L)$ where $U(L)$ is the enveloping algebra of L. Assume that $L \neq 0$ and that R is either right Noetherian, a PI algebra, or a ring with no nilpotent elements. Then $J(R * U(L)) = N * U(L)$ where N is the largest L-invariant nil ideal of R. Furthermore $J(R * U(L))$ is nil in this case.*

While the above result is in a way more general than our result, we do not require that R be a PI algebra over a field. Our result concerns PI rings and the proof given is purely ring theoretical.

Proof of Theorem 4.4.1. It is clear that S is an ideal of R. Now let $a \in S$. Then $xa, ax \in J(R[x;\delta])$ and $xa = ax + \delta(a)$. Therefore $\delta(a) \in J(R[x;\delta])$ and so $\delta(a) \in S$. This shows that S is a δ-ideal.

It is left to show that S is nil. To do this we consider the ring $L := (S+N)/N$, where $N = \mathrm{Nil}^*(R)$. We begin by noting that L is an ideal of R/N, which is a polynomial identity ring with zero upper nilradical.

If L is the zero ideal of R/N we have that $S \subseteq N$ and so S is nil. Suppose, towards a contradiction, that L is not the zero ideal. Thus by Posner's theorem we have that L intersects the centre of R/N nontrivially. Let $a \in S$ such that $a + N \neq N$ and $a + N$ is in the centre of R/N. We have that $xa \in J(R[x;D])$ and so by quasi-regularity there exists $f(x) \in J(R[x;\delta])$ such that

$$f(x) + xa - f(x)xa = 0, \tag{4.6}$$

and

$$f(x) + xa - xaf(x) = 0. \tag{4.7}$$

Write

$$f(x) = \sum_{i=0}^{n} x^i b_i.$$

From (4.7) we immediately get that $b_0 = 0$. Now, from (4.6) we see that

$$\begin{aligned}\sum_{i=0}^{n} x^i b_i + xa - \sum_{i=0}^{n} \big(x^i b_i xa\big) &= \sum_{i=0}^{n} x^i b_i + xa - \sum_{i=0}^{n} \big(x^i (xb_i - \delta(b_i))a\big) \\ &= \sum_{i=0}^{n} x^i b_i + xa - \sum_{i=0}^{n} x^{i+1} b_i a + \sum_{i=1}^{n} x^i \delta(b_i) a = 0.\end{aligned}$$

Upon equating coefficients we get that

$$b_n a = 0; \tag{4.8}$$

$$b_i - b_{i-1}a + \delta(b_i)a = 0, \text{ for } i = 2, \ldots, n; \tag{4.9}$$

$$b_1 + a + \delta(b_1)a = 0. \tag{4.10}$$

We have $J(R[x;\delta]) = S[x;\delta]$ (see [17]) and so $f(x) \in S[x;\delta]$. Therefore each $b_i \in S$.

We claim that $b_{n-j+1}a^j \in N$ for $j = 1, \ldots, n$. We prove this by induction on j. If $j = 1$ we clearly have that $0 = b_n a \in N$. Now assume the result is true for $j < k$ with $k \in \{2, \ldots, n\}$ and consider the case when $j = k$. In particular, we have that $b_{n-k+2}a^{k-1} \in N$. Now, post-multiplying (4.9), taking $i = n-k+2$, by a^{k-1} we have by the induction hypothesis that

$$\begin{aligned} b_{n-k+2}a^{k-1} &- b_{n-k+1}a^k + \delta(b_{n-k+2})a^k \\ &\equiv -b_{n-k+1}a^k + \delta(b_{n-k+2})a^k \pmod{N} \\ &\equiv 0 \pmod{N}. \end{aligned} \tag{4.11}$$

To prove the claim it is left to show that $\delta(b_{n-k+2})a^k \in N$. From (4.6) and (4.7) we also get that $xaf(x) = f(x)xa$. Recall that we have the identity

$$ax^r = \sum_{\ell=0}^{r} (-1)^\ell \binom{r}{\ell} x^{r-\ell}\delta^\ell(a),$$

for any $r \in \mathbb{N}$. But then we have that

$$\begin{aligned} 0 = f(x)xa - xaf(x) &= \sum_{i=0}^{n} x^{i+1}b_i a + \sum_{i=1}^{n} x^i\delta(b_i)a - \sum_{i=0}^{n} xax^ib_i \\ &= \sum_{i=0}^{n} x^{i+1}b_i a + \sum_{i=1}^{n} x^i\delta(b_i)a - \sum_{i=0}^{n}\sum_{\ell=0}^{i} (-1)^\ell \binom{i}{\ell} x^{i-\ell+1}\delta^\ell(a)b_i. \end{aligned} \tag{4.12}$$

Equating coefficients of x^{n-k+2} in (4.12) we have that

$$\begin{aligned} b_{n-k+1}a &+ \delta(b_{n-k+2})a + (-1)^k \binom{n}{k-1} \delta^{k-1}(a)b_n \\ &+ (-1)^{k-1}\binom{n-1}{k-2}\delta^{k-2}(a)b_{n-1} \\ &+ \cdots + \binom{n-k+2}{1}\delta(a)b_{n-k+2} - ab_{n-k+1} = 0. \end{aligned} \tag{4.13}$$

Notice that since a is central mod N, the first and last term of the left-hand-side of (4.13) are identical mod N. Also, by the induction hypothesis $b_n a^{k-1}, b_{n-1}a^{k-1}, \ldots, b_{n-k+2}a^{k-1} \in N$. Therefore by post-multiplying (4.13) by a^{k-1} and considering this equation modulo N we have

$$\delta(b_{n-k+2})a^k \in N.$$

Using this and (4.11) we get that $b_{n-k+1}a^k \in N$. The claim is then true by induction.

By the above claim with $j = n$ we have that $b_1 a^n \in N$ and so by (4.10)

$$0 \equiv b_1 a^n + a^{n+1} + \delta(b_1)a^{n+1} \equiv a^{n+1} + \delta(b_1)a^{n+1} \pmod{N}. \tag{4.14}$$

To see that $\delta(b_1)a^{n+1} \in N$ we equate coefficients of x in (4.12). From doing so we get that

$$\delta(b_1)a + (-1)^{n+1}\delta^n(a)b_n + (-1)^n\delta^{n-1}(a)b_{n-1} + \cdots + \delta(a)b_1 = 0.$$

Post-multiplying this by a^{n+1} and using our claim we observe that $\delta(b_1)a^{n+1} \in N$. Combining this with (4.14) we get that $a^{n+1} \in N$ and so a is nilpotent. But then, using the fact that a is central mod N, we see that $(a\mathbb{Z} + aR + N)/N$ is a nil ideal of R/N. Since N is a nil ideal of R, we see that $a\mathbb{Z} + aR + N$ is also a nil ideal of R. But N is the upper nilradical and so we see that $a \in N$ and so we have that $a + N = N$, which is a contradiction. We conclude that $L = (0)$ and so S is nil. □

We get the following immediate corollary, which is part of the main result of [65].

Corollary 4.4.1. *Let R be a PI ring with $\mathrm{Nil}^*(R) = (0)$. Then $R[x;\delta]$ is semiprimitive for any derivation δ of R.*

Chapter 5

Future Directions

While work has been done in this thesis to extend what is known regarding the homogeneity of the Jacobson radical and upper nilradical of rings graded by unique-product semigroups, the two biggest open questions in the area remain open.

Question 1. *If X is a unique-product semigroup then does any X-graded ring have a homogeneous Jacobson radical?*

Question 2. *If X is a unique-product semigroup then does any X-graded ring have a homogeneous upper nilradical?*

There has been much work done on how combinatorial information of a right-infinite word w translates to ring-theoretical information of A_w. It would be interesting to investigate which Hopf algebra structures may be placed on the algebra A_w. For instance, it would be interesting to investigate which right-infinite words admit a cancellation-free formula for all possible antipodes on A.

Question 3. *Given a right-infinite word w over a finite alphabet, when can A_w be endowed with the structure of a Hopf algebra?*

In [24], Holdaway and Smith were able to describe the module structure of finitely presented monomial algebras through a categorical equivalence involving quiver algebras.

It would be interesting to investigate if something similar to this technique could be applied to iterative algebras.

Question 4. *Let C denote the category of $\mathbb{Z}$-graded right A_w-modules modulo the full subcategory consisting of modules that are the sum of finite dimensional submodules. Can one give a concrete description of C for an iterative algebra A_w?*

It would also be interesting to investigate if the possible subword complexities of pure morphic words demonstrated in [52] could be described/determined through ring-theoretical information about the associated iterative algebras.

Question 5. *Can one give purely ring-theoretic characterizations that determine which of the five possible growth types an iterative algebra has?*

Given a right-infinite word w, consider the power series

$$H(t) = \sum_{n=0}^{\infty} \rho_w(n) t^n.$$

It would be interesting to investigate when the power series $H(t)$ is simply a rational function in disguise. Without providing the intermediate definitions, we are actually asking the following question.

Question 6. *Let A_w be an iterative algebra. Can one characterize the words for which A_w has rational Hilbert series with the standard grading? Can one characterize the words for which A_w is finitely presented?*

Finally, after considering the behaviour of differential polynomial extensions of locally nilpotent rings, it would be very interesting to consider the behaviour of differential polynomial extensions over strongly nilpotent rings. The following question was asked in [46].

Question 7. *If R is a strongly nilpotent ring and δ is a derivation on R then is $R[x; \delta]$ Jacobson radical?*

References

[1] Jean-Paul Allouche and Jeffrey Shallit, *Automatic sequences*, Cambridge University Press, Cambridge, 2003. Theory, applications, generalizations.

[2] S. A. Amitsur, *Radicals of polynomial rings*, Canad. J. Math. **8** (1956), 355–361.

[3] ———, *Nil radicals. Historical notes and some new results* (1973), 47–65. Colloq. Math. Soc. János Bolyai, Vol. 6.

[4] K. I. Beidar and Y. Fong, *On radicals of monomial algebras*, Comm. Algebra **26** (1998), no. 12, 3913–3919.

[5] Jason P. Bell and Blake W. Madill, *Iterative algebras*, Algebr. Represent. Theory **18** (2015), no. 6, 1533–1546.

[6] Jason P. Bell, Blake W. Madill, and Forte Shinko, *Differential polynomial rings over rings satisfying a polynomial identity*, J. Algebra **423** (2015), 28–36.

[7] Jason P. Bell and Agata Smoktunowicz, *The prime spectrum of algebras of quadratic growth*, J. Algebra **319** (2008), no. 1, 414–431.

[8] A. Ya. Belov, V. V. Borisenko, and V. N. Latyshev, *Monomial algebras*, J. Math. Sci. (New York) **87** (1997), no. 3, 3463–3575. Algebra, 4.

[9] Jeffrey Bergen and Piotr Grzeszczuk, *Jacobson radicals of ring extensions*, J. Pure Appl. Algebra **216** (2012), no. 12, 2601–2607.

[10] Jeffrey Bergen, S. Montgomery, and D. S. Passman, *Radicals of crossed products of enveloping algebras*, Israel J. Math. **59** (1987), no. 2, 167–184.

[11] G. M. Bergman, *On Jacobson radicals of graded rings*, Unpublished work, (1975), available at: http://math.berkeley.edu/∼gbergman/papers/unpub/.

[12] Valérie Berthé and Michel Rigo (eds.), *Combinatorics, automata and number theory*, Encyclopedia of Mathematics and its Applications, vol. 135, Cambridge University Press, Cambridge, 2010.

[13] Amiram Braun, *The nilpotency of the radical in a finitely generated PI ring*, J. Algebra **89** (1984), no. 2, 375–396.

[14] Bertrand Deroin, Andrés Navas, and Cristóbal Rivas, *Groups, orders, and dynamics*, arXiv preprint arXiv:1408.5805 (2014).

[15] Vesselin Drensky and Edward Formanek, *Polynomial identity rings*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, Basel, 2004.

[16] Fabien Durand, *Decidability of uniform recurrence of morphic sequences*, Internat. J. Found. Comput. Sci. **24** (2013), no. 1, 123–146.

[17] Miguel Ferrero, Kazuo Kishimoto, and Kaoru and Motose, *On radicals of skew polynomial rings of derivation type*, J. London Math. Soc. (2) **28** (1983), no. 1, 8–16.

[18] B. J. Gardner and R. Wiegandt, *Radical theory of rings*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 261, Marcel Dekker, Inc., New York, 2004.

[19] A. Giambruno, S. Mishchenko, and M. Zaicev, *Codimensions of algebras and growth functions*, Adv. Math. **217** (2008), no. 3, 1027–1052.

[20] K. R. Goodearl and J. T. Stafford, *The graded version of Goldie's theorem*, Algebra and its applications (Athens, OH, 1999), 2000, pp. 237–240.

[21] Be'eri Greenfeld, André Leroy, Agata Smoktunowicz, and Michał Ziembowski, *Chains of prime ideals and primitivity of $\mathbb{Z}$-graded algebras*, Algebr. Represent. Theory **18** (2015), no. 3, 777–800.

[22] Tero Harju and Matti Linna, *On the periodicity of morphisms on free monoids*, RAIRO Inform. Théor. Appl. **20** (1986), no. 1, 47–54.

[23] I. N. Herstein, *Noncommutative rings*, The Carus Mathematical Monographs, No. 15, Published by The Mathematical Association of America; distributed by John Wiley & Sons, Inc., New York, 1968.

[24] Cody Holdaway and S. Paul Smith, *An equivalence of categories for graded modules over monomial algebras and path algebras of quivers*, J. Algebra **353** (2012), 249–260.

[25] E. Jespers, *Radicals of graded rings*, Theory of radicals (Szekszárd, 1991), 1993, pp. 109–130.

[26] Eric Jespers, *On radicals of graded rings and applications to semigroup rings*, Comm. Algebra **13** (1985), no. 11, 2457–2472.

[27] Eric Jespers, Jan Krempa, and Edmund R. Puczył owski, *On radicals of graded rings*, Comm. Algebra **10** (1982), no. 17, 1849–1854.

[28] Eric Jespers and Edmund R. Puczył owski, *The Jacobson and Brown-McCoy radicals of rings graded by free groups*, Comm. Algebra **19** (1991), no. 2, 551–558.

[29] D. A. Jordan, *Noetherian Ore extensions and Jacobson rings*, J. London Math. Soc. (2) **10** (1975), 281–291.

[30] G. Karpilovsky, *The Jacobson radical of classical rings*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 53, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1991.

[31] ______, *The Jacobson radical of monoid-graded algebras*, Tsukuba J. Math. **16** (1992), no. 1, 19–52.

[32] A. V. Kelarev, *On the Jacobson radical of graded rings*, Comment. Math. Univ. Carolin. **33** (1992), no. 1, 21–24.

[33] A. R. Kemer, *Capelli identities and nilpotency of the radical of finitely generated PI-algebra*, Dokl. Akad. Nauk SSSR **255** (1980), no. 4, 793–797.

[34] Abraham A. Klein, *Rings of bounded index*, Comm. Algebra **12** (1984), no. 1-2, 9–21.

[35] Gottfried Köthe, *Die Struktur der Ringe, deren Restklassenring nach dem Radikal vollständig reduzibel ist*, Math. Z. **32** (1930), no. 1, 161–186.

[36] Günter R. Krause and Thomas H. Lenagan, *Growth of algebras and Gelfand-Kirillov dimension*, Revised, Graduate Studies in Mathematics, vol. 22, American Mathematical Society, Providence, RI, 2000.

[37] Jan Krempa, *Logical connections between some open problems concerning nil rings*, Fund. Math. **76** (1972), no. 2, 121–130.

[38] T. Y. Lam, *A first course in noncommutative rings*, Second, Graduate Texts in Mathematics, vol. 131, Springer-Verlag, New York, 2001.

[39] W. G. Leavitt, *General radical theory in rings*, Rocky Mountain J. Math. **3** (1973), 431–439.

[40] Jakob Levitzki, *On the radical of a general ring*, Bull. Amer. Math. Soc. **49** (1943), 462–466.

[41] Blake W. Madill, *On the Jacobson radical of skew polynomial extensions of rings satisfying a polynomial identity*, Comm. Algebra **44** (2016), no. 3, 913–918.

[42] A. I. Malcev, *Nilpotent semigroups*, Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki **4** (1953), 107–111.

[43] Ryszard Mazurek, Pace P. Nielsen, and Michał Ziembowski, *The upper nilradical and Jacobson radical of semigroup graded rings*, J. Pure Appl. Algebra **219** (2015), no. 4, 1082–1094.

[44] Alireza Nasr-Isfahani, *Jacobson radicals of skew polynomial rings of derivation type*, Canad. Math. Bull. **57** (2014), no. 3, 609–613.

[45] Pace P. Nielsen, *Bootstrapping the bounded nilradical*, J. Pure Appl. Algebra **217** (2013), no. 9, 1711–1715.

[46] Pace P. Nielsen and Michał Ziembowski, *Derivations and bounded nilpotence index*, Internat. J. Algebra Comput. **25** (2015), no. 3, 433–438.

[47] Jan Okniński, *Semigroup algebras*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 138, Marcel Dekker, Inc., New York, 1991.

[48] ———, *Algebras of cancellative semigroups*, Bull. Austral. Math. Soc. **49** (1994), no. 1, 165–170.

[49] ———, *Structure of prime finitely presented monomial algebras*, J. Algebra **320** (2008), no. 8, 3199–3205.

[50] ———, *Trichotomy for finitely generated monomial algebras*, J. Algebra **417** (2014), 145–147.

[51] Oystein Ore, *Theory of non-commutative polynomials*, Ann. of Math. (2) **34** (1933), no. 3, 480–508.

[52] Jean-Jacques Pansiot, *Complexité des facteurs des mots infinis engendrés par morphismes itérés*, Automata, languages and programming (Antwerp, 1984), 1984, pp. 380–389.

[53] ———, *Decidability of periodicity for infinite words*, RAIRO Inform. Théor. Appl. **20** (1986), no. 1, 43–46.

[54] Donald S. Passman, *The algebraic structure of group rings*, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1977.

[55] S. David Promislow, *A simple example of a torsion-free, nonunique product group*, Bull. London Math. Soc. **20** (1988), no. 4, 302–304.

[56] Ju. P. Razmyslov, *The Jacobson Radical in PI-algebras*, Algebra i Logika **13** (1974), 337–360, 365.

[57] Louis Halle Rowen, *Polynomial identities in ring theory*, Pure and Applied Mathematics, vol. 84, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980.

[58] L. W. Small, J. T. Stafford, and R. B. Warfield Jr., *Affine algebras of Gelfand-Kirillov dimension one are PI*, Math. Proc. Cambridge Philos. Soc. **97** (1985), no. 3, 407–414.

[59] L. W. Small and R. B. Warfield Jr., *Prime affine algebras of Gelfand-Kirillov dimension one*, J. Algebra **91** (1984), no. 2, 386–389.

[60] Agata Smoktunowicz, *On some results related to Köthe's conjecture*, Serdica Math. J. **27** (2001), no. 2, 159–170.

[61] ———, *A note on nil Jacobson radicals in graded rings*, J. Algebra Appl. **13** (2014), no. 4, 1350121, 8.

[62] ______, *How far can we go with Amitsur's theorem in differential polynomial rings?*, Israel J. Math. **219** (2017), no. 2, 555–608.

[63] Agata Smoktunowicz and Michał Ziembowski, *Differential polynomial rings over locally nilpotent rings need not be Jacobson radical*, J. Algebra **412** (2014), 207–217.

[64] Andrzej Strojnowski, *A note on u.p. groups*, Comm. Algebra **8** (1980), no. 3, 231–234.

[65] Yuan-Tsung Tsai, Tsu-Yang Wu, and Chen-Lian Chuang, *Jacobson radicals of Ore extensions of derivation type*, Comm. Algebra **35** (2007), no. 3, 975–982.