The Liftable Mapping Class Group

by

Tyrone Ghaswala

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Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

External Examiner	Dan Margalit Professor Georgia Institute of Technology
Co-supervisor	David McKinnon Professor University of Waterloo
Co-supervisor	B. Doug Park Professor University of Waterloo
Internal Member	Rahim Moosa Professor University of Waterloo
Internal Member	Matthew Satriano Assistant Professor University of Waterloo
Internal-external Member	Jim Geelen Professor University of Waterloo

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Statement of Contributions

I am the sole author of Chapter 1, Chapter 2, Chapter 5, Chapter 6, and Appendix A. Chapter 3 and Chapter 4 are from joint work with Rebecca Winarski. The results from Chapter 6 are to appear in joint work with Alan McLeay.

Abstract

Broadly, this thesis lies at the interface of mapping class groups and covering spaces. The foundations of this area were laid down in the early 1970s by Birman and Hilden. Building on these foundations, there has been a plethora of results, especially in the context of a particular family of branched covering spaces over the sphere. We call these covers the hyperelliptic covering spaces.

One of the reasons the hyperelliptic covers provide such fertile ground for research is that every homeomorphism of a marked sphere lifts to a homeomorphism of the covering space. Rephrasing this, the liftable mapping class group coincides with the entire mapping class group of a marked sphere. Since the mapping class group of a marked sphere is well understood, this understanding can be lifted to help understand a particular subgroup of the mapping class group of the covering space.

However, for a general covering space, the liftable mapping class group does not coincide with the mapping class group of the base space. Instead, it is a finite index subgroup. This thesis is devoted to studying the liftable mapping class group in contexts other than the hyperelliptic covers.

Chapter 2 provides the necessary preliminaries for the rest of the thesis. Chapter 3 classifies cyclic branched covers of the sphere with the property that the liftable mapping class group coincides with the mapping class group of the marked sphere. Chapter 4 studies the liftable mapping class group for a family of cyclic branched covers over the sphere, called balanced superelliptic covers. We find an explicit finite presentation for the liftable mapping class groups corresponding to the balanced superelliptic covers, compute the indexes of the liftable mapping class groups, and compute their abelianizations.

In Chapter 5 we study an infinite family of cyclic branched covers over a torus. The liftable mapping class groups corresponding to this family are all subgroups of the mapping class group of a twice marked torus. We prove that the intersection of any two of these liftable mapping class groups is also a liftable mapping class group, and the subgroup generated by any two is again a liftable mapping class group. In a few special cases we find a finite generating set, and for some of those, an explicit finite presentation.

Finally, in Chapter 6 we study the liftable mapping class group for covers of surfaces with boundary. Given a covering space of a surface with boundary, we characterize the corresponding liftable mapping class group by the action of its members on a particular fundamental groupoid of the base surface.

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Dedication

To my parents, who had the unfortunate task of raising a mathematician.

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Chapter 1

Introduction

The mapping class group of a surface of genus g, is a group of symmetries of the surface Σ_g. More formally, it is the group of orientation-preserving homeomorphisms of Σ_g up to isotopy. If the surface has boundary $\partial\Sigma_g$ or marked points $\mathcal{B}$, then the homeomorphisms must preserve the set of marked points and fix the boundary pointwise. The mapping class group will be denoted $\mathrm{Mod}(\Sigma_g, \partial\Sigma_g, \mathcal{B})$. The $\partial\Sigma$ and $\mathcal{B}$ may be omitted if they are empty or if we do not require them to be fixed.

The study of mapping class groups began in the 1920s with the work of Dehn [19] and Nielsen [39, 40, 41]. Today, mapping class groups are ubiquitous in mathematics. For a few examples among many, they appear in 4-manifold topology through Lefschetz fibrations, and 3-manifold topology through Heegaard splittings and mapping tori. Mapping class groups are closely related to automorphism groups of free groups and right-angled Artin groups, and are a generalization of the braid group. The mapping class group is also the fundamental group of the moduli space of a Riemann surface, providing a bridge to algebraic and complex geometry.

Broadly, this thesis studies the interplay of mapping class groups and finite-sheeted, regular covering spaces. This particular story begins with the search for a presentation of the mapping class group of a genus 2 surface, which was arrived at in the work of Birman and Hilden [7].

Mapping Class Groups and Covering Spaces

The key insight of Birman and Hilden was to use covering spaces to relate the mapping class group of the base surface to that of the covering surface. In [7], they study a particular family of branched covers over the sphere which we will call *hyperelliptic covers*. The hyperelliptic covers are constructed as follows. For each genus $g \geq 1$ there is a 2-sheeted

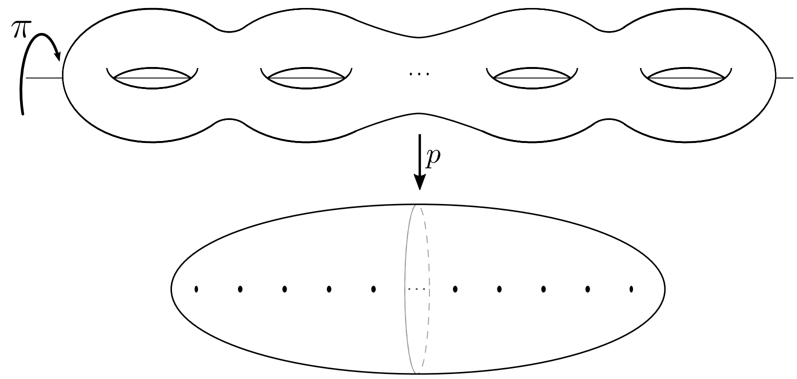


Figure 1.1: A hyperelliptic involution on Σ_g and the hyperelliptic cover $p_g : \Sigma_g \to \Sigma_0$.

branched cover $p_g : \Sigma_g \to \Sigma_0$ of the sphere Σ_0 by a surface Σ_g of genus g, branched at $2g+2$ points $\mathcal{B} \subset \Sigma_0$. The nontrivial deck transformation is a rotation by π about the axis shown in Figure 1.1. Such a rotation is called a *hyperelliptic involution* and is denoted by ι. See Figure 1.1 for an illustration of the hyperelliptic covers. For a more formal construction of the hyperelliptic covers, see the construction of the balanced superelliptic covers when $k = 2$ in Section 4.1.

Birman and Hilden proved that for $g \geq 2$, there is a surjective homomorphism $\mathrm{SMod}_{p_g}(\Sigma_g) \to \mathrm{Mod}(\Sigma_0, \mathcal{B})$ with kernel $\langle \iota \rangle$. Here, $\mathrm{SMod}_{p_g}(\Sigma_g)$ is the *hyperelliptic mapping class group*, and is equal to the centralizer of ι in $\mathrm{Mod}(\Sigma_g)$. When $g = 2$, $\langle \iota \rangle$ is equal to the center of $\mathrm{Mod}(\Sigma_g)$, so $\mathrm{Mod}(\Sigma_2)/\langle \iota \rangle$ is isomorphic to $\mathrm{Mod}(\Sigma_0, \mathcal{B})$. At the time, presentations for $\mathrm{Mod}(\Sigma_0, \mathcal{B})$ and $\langle \iota \rangle$ were known (the latter is of course isomorphic to $\mathbb{Z}/2\mathbb{Z}$). As a result, the first presentation for $\mathrm{Mod}(\Sigma_2)$ was obtained (for example, by applying Lemma 2.4.2 below).

The results of Birman and Hilden were subsequently generalized. Assume the genus of $\widetilde{\Sigma}$ is at least 2. Let $p : \widetilde{\Sigma} \to \Sigma$ be a regular, finite-sheeted, possibly branched covering space with deck group D, branched at finitely many points $\mathcal{B} \subset \Sigma$. Let $\mathrm{SMod}_p(\widetilde{\Sigma})$ be the subgroup of $\mathrm{Mod}(\widetilde{\Sigma})$ consisting of isotopy classes of fiber-preserving homeomorphisms, called the *symmetric mapping class group*. Let $\mathrm{LMod}_p(\Sigma, \mathcal{B})$ be the subgroup of $\mathrm{Mod}(\Sigma, \mathcal{B})$ consisting of isotopy classes of homeomorphisms that lift to homeomorphisms of $\widetilde{\Sigma}$, called the *liftable mapping class group*.

Projecting fiber-preserving homeomorphisms of $\widetilde{\Sigma}$ to homeomorphisms of Σ induces a surjective homomorphism $\mathrm{SMod}_p(\widetilde{\Sigma}) \to \mathrm{LMod}_p(\Sigma, \mathcal{B})$ with kernel D. This result is known as the *Birman-Hilden theorem*. Birman and Hilden proved the result in [9] for solvable covers, and for unbranched covers in [8]. MacLachlan and Harvey proved it for all finite-sheeted regular covers in [34]. A version of the Birman-Hilden theorem was proved by Winarski in [47] for possibly irregular, fully-ramified covers, and by Aramayona, Leininger and Souto in [4] for irregular unbranched covers.

For the hyperelliptic covers, the isomorphism $\mathrm{SMod}_{p_g}(\Sigma_g)/\langle\iota\rangle \cong \mathrm{Mod}(\Sigma_0, \mathcal{B})$ has been successfully exploited. For example, Bigelow and Budney [5] prove that the hyperelliptic mapping class group, and in particular $\mathrm{Mod}(\Sigma_2)$, is linear. Brendle, Margalit, and Putman [14] find an explicit generating set for the kernel of the integral Burau representation. Stukow classifies all the conjugacy classes of maximal finite subgroups of the hyperelliptic mapping class group in [44], and proves that the hyperelliptic mapping class group is generated by two torsion elements in [45]. The homology and cohomology of the hyperelliptic mapping class group has been extensively studied in the works of Bödigheimer-Cohen-Peim [11], Cohen [17], Gries [25], and Kawazumi [31, 32] to name a few. The list goes on (see the works of A'Campo [1], Ahara-Takasawa [2], Brendle-Childers-Margalit [13], Calegari-Monden-Sato [16], Endo [21], Hain [26], Kasagawa [30], and Morifuji [38] to name a few more). All of these results rely in some way on the Birman-Hilden theorem for hyperelliptic covers.

One of the reasons the hyperelliptic covers have been fertile ground for research is that $\mathrm{LMod}_p(\Sigma_0, \mathcal{B}) = \mathrm{Mod}(\Sigma_0, \mathcal{B})$. That is, every homeomorphism of Σ_0 fixing the branch points lifts to a homeomorphism of Σ_g. Since $\mathrm{Mod}(\Sigma_0, \mathcal{B})$ is well understood, the Birman-Hilden theorem can be used to understand the hyperelliptic mapping class group. In general, the liftable mapping class group is a finite index subgroup of the mapping class group. This thesis is concerned with understanding the liftable mapping class group for finite-sheeted, regular, possibly branched covers.

Layout of the thesis

Chapter 2 is an overview of the preliminaries required for the rest of the thesis. Sections 2.1 and 2.2 cover the basic theory about lifting and projecting homeomorphisms. Section 2.3 introduces mapping class groups, including a section on the Birman-Hilden theory. Finally, relevant results about group presentations relating to short exact sequences and a review of the Reidemeister-Schreier rewriting process are presented in Section 2.4.

Chapters 3 and 4 are from joint work with Rebecca Winarski, and are essentially the papers [24] and [23] respectively. Chapter 3 classifies cyclic branched covers of the sphere with the property that every homeomorphism lifts. As a result Theorem 3.2.4 is proved, which is a correction to Theorem 5 in [9] (see the erratum [10]).

In Chapter 4 we study the liftable mapping class group corresponding to a family of cyclic branched covers of the sphere called the *balanced superelliptic covers*. The balanced superelliptic covers are a natural generalization of the hyperelliptic covers, and provide examples of covers where the liftable mapping class group is a proper subgroup of the mapping class group. We compute the index $[\mathrm{Mod}(\Sigma_0, \mathcal{B}) : \mathrm{LMod}_p(\Sigma_0, \mathcal{B})]$, find an explicit finite presentation for $\mathrm{LMod}_p(\Sigma_0, \mathcal{B})$, and compute the abelianization $H_1(\mathrm{LMod}_p(\Sigma_0, \mathcal{B}); \mathbb{Z})$. As a corollary, we show that the abelianization of the symmetric mapping class group, called the *balanced superelliptic mapping class group*, is a non-cyclic finite group.

A family of cyclic branched covers over a torus branched at two points is the focus of Chapter 5. The family is indexed by integers $k \geq 2$, giving rise to a family of liftable subgroups $\mathrm{LMod}_k(\Sigma_1, \mathcal{B}) < \mathrm{Mod}(\Sigma_1, \mathcal{B})$. Using the Reidemeister-Schreier rewriting process and soliciting the help of the Sage code in Appendix A, we arrive at a finite presentation for $\mathrm{LMod}_k(\Sigma_1, \mathcal{B})$ for $k = 2, 3, 4$ and a finite generating set for $k = 5, 6$. We also prove that the way the liftable subgroups live in $\mathrm{Mod}(\Sigma_1, \mathcal{B})$ is well behaved. Indeed, for integers $k, l \geq 2$,

$$\mathrm{LMod}_k(\Sigma_1, \mathcal{B}) \cap \mathrm{LMod}_l(\Sigma_1, \mathcal{B}) = \mathrm{LMod}_{\mathrm{lcm}(k,l)}(\Sigma_1, \mathcal{B})$$

and

$$\mathrm{LMod}_k(\Sigma_1, \mathcal{B})\, \mathrm{LMod}_l(\Sigma_1, \mathcal{B}) = \mathrm{LMod}_{\mathrm{gcd}(k,l)}(\Sigma_1, \mathcal{B}).$$

Finally, Chapter 6 looks at the liftable mapping class group for surfaces with boundary. The chapter begins by surveying the relevant algebraic properties of groupoids. The main result characterizes the liftable mapping class group by the action of its members on the fundamental groupoid of the base surface. As a corollary we prove that, like the case for surfaces without boundary, the liftable mapping class group is a finite index subgroup.

Chapter 2

Preliminaries

2.1 Notation and conventions for homeomorphisms of surfaces

Let Σ be a compact orientable 2-manifold of genus g with boundary $\partial\Sigma$ and finitely many marked points $\mathcal{B} \subset \Sigma \backslash \partial\Sigma$. Let $\mathrm{Homeo}(\Sigma)$ be the homeomorphism group of Σ. Throughout this thesis, the focus is on various subgroups of $\mathrm{Homeo}(\Sigma)$. If $\partial\Sigma$ appears in the argument of Homeo, then we require homeomorphisms to fix $\partial\Sigma$ pointwise. If $\mathcal{B}$ appears, then we require homeomorphisms to fix $\mathcal{B}$ setwise. If a $+$ appears as a superscript, then we require homeomorphisms to preserve the orientation of Σ. For example,

$$\begin{aligned}
\mathrm{Homeo}^+(\Sigma) &= \{f \in \mathrm{Homeo}(\Sigma) : f \text{ is orientation preserving}\} \\
\mathrm{Homeo}^+(\Sigma, \mathcal{B}) &= \{f \in \mathrm{Homeo}^+(\Sigma) : f(\mathcal{B}) = \mathcal{B}\} \\
\mathrm{Homeo}(\Sigma, \partial\Sigma, \mathcal{B}) &= \{f \in \mathrm{Homeo}(\Sigma) : f(\mathcal{B}) = \mathcal{B}, f(x) = x \text{ for all } x \in \partial\Sigma\}.
\end{aligned}$$

If $\partial\Sigma \neq \emptyset$ and f is a homeomorphism fixing $\partial\Sigma$ pointwise, then f must be orientation preserving. That is, $\mathrm{Homeo}^+(\Sigma, \partial\Sigma) = \mathrm{Homeo}(\Sigma, \partial\Sigma)$.

Occasionally, it may be easier to remove the marked points and consider the resulting surface with punctures $\Sigma^\circ = \Sigma \setminus \mathcal{B}$. Since Σ is Hausdorff and $\Sigma^\circ \subset \Sigma$ is dense, any homeomorphism in $\mathrm{Homeo}(\Sigma^\circ)$ extends uniquely to a homeomorphism in $\mathrm{Homeo}(\Sigma, \mathcal{B})$. Conversely, every homeomorphism in $\mathrm{Homeo}(\Sigma, \mathcal{B})$ restricts to a homeomorphism of Σ°. Therefore, an isomorphism $\mathrm{Homeo}(\Sigma, \mathcal{B}) \cong \mathrm{Homeo}(\Sigma^\circ)$ arises by restricting homeomorphisms. We will go back and forth between punctures and marked points as is convenient.

If it is necessary to keep track of the genus g, the number of punctures n, and the number of boundary components b of Σ, we will denote the surface Σ by $\Sigma_{g,n}^b$. If b or n are

0, then they will be omitted. If $|\mathcal{B}| = m$ and we wish to keep track of the number of marked points, then we will write $\mathcal{B} = \mathcal{B}(m)$. So, for example, $\mathrm{Homeo}^+(\Sigma_{g,n}) \cong \mathrm{Homeo}^+(\Sigma_g, \mathcal{B}(n))$.

As a general rule, the group multiplication in homeomorphism groups and mapping class groups will be performed right to left, so fg means perform g, then f. The exception to this will be in Chapter 4, where the focus is on group presentations.

2.2 Lifting and projecting homeomorphisms

Let $p : \widetilde{\Sigma} \to \Sigma$ be a regular, finite-sheeted, possibly branched covering space branched at finitely many points $\mathcal{B} \subset \Sigma \setminus \partial\Sigma$ with deck group $D < \mathrm{Homeo}^+(\widetilde{\Sigma})$.

We say a homeomorphism $f \in \mathrm{Homeo}^+(\widetilde{\Sigma})$ is *fiber-preserving* if whenever $p(x) = p(y)$, $pf(x) = pf(y)$. Let $\mathrm{SHomeo}_p^+(\widetilde{\Sigma})$ be the subgroup consisting of fiber-preserving homeomorphisms. There is a homomorphism $\Pi : \mathrm{SHomeo}_p^+(\widetilde{\Sigma}) \to \mathrm{Homeo}^+(\Sigma, \mathcal{B})$ given by $\Pi(\tilde{f})(x) = p\tilde{f}(\tilde{x})$ for any $\tilde{x} \in p^{-1}(x)$. It follows that $\ker(\Pi) = D$. If $\Pi(\tilde{f}) = f$, then the square

$$\begin{tikzcd} \widetilde{\Sigma} \arrow[r, "\tilde{f}"] \arrow[d, "p"] & \widetilde{\Sigma} \arrow[d, "p"] \\ \Sigma \arrow[r, "f"] & \Sigma \end{tikzcd}$$

commutes, that is $p\tilde{f} = fp$. It is true that $\mathrm{SHomeo}_p^+(\widetilde{\Sigma})$ is the normalizer of D in $\mathrm{Homeo}^+(\widetilde{\Sigma})$, and the justification is left to the reader.

We say a homeomorphism $f \in \mathrm{Homeo}^+(\Sigma, \mathcal{B})$ *lifts* if there exists a homeomorphism $\tilde{f} \in \mathrm{Homeo}^+(\widetilde{\Sigma})$ such that $p\tilde{f} = fp$. Note that such a $\tilde{f}$ must necessarily be in $\mathrm{SHomeo}_p^+(\widetilde{\Sigma})$. Let $\mathrm{LHomeo}_p^+(\Sigma, \mathcal{B})$ be the subgroup of $\mathrm{Homeo}^+(\Sigma, \mathcal{B})$ consisting of homeomorphisms that lift. Then the image of Π is $\mathrm{LHomeo}_p^+(\Sigma, \mathcal{B})$ and $\mathrm{SHomeo}_p^+(\widetilde{\Sigma})/D \cong \mathrm{LHomeo}_p^+(\Sigma, \mathcal{B})$.

2.2.1 Lifting and projecting with boundary

Suppose $\widetilde{\Sigma}$ and Σ have non-empty boundary $\partial\widetilde{\Sigma}$ and $\partial\Sigma$ respectively. We wish to lift and project homeomorphisms that preserve the boundary pointwise, to homeomorphisms that preserve the boundary pointwise.

Let $\mathrm{SHomeo}_p(\widetilde{\Sigma}, \partial\widetilde{\Sigma}) = \mathrm{SHomeo}_p^+(\widetilde{\Sigma}) \cap \mathrm{Homeo}(\widetilde{\Sigma}, \partial\widetilde{\Sigma})$.

Proposition 2.2.1. $\mathrm{SHomeo}_p(\widetilde{\Sigma}, \partial\widetilde{\Sigma}) = C \cap \mathrm{Homeo}(\widetilde{\Sigma}, \partial\widetilde{\Sigma})$ *where C is the centralizer of the deck group D in* $\mathrm{Homeo}^+(\widetilde{\Sigma})$.

Proof. It suffices to show that any homeomorphism $\tilde{f}$ that fixes the boundary and is in the normalizer of D is in the centralizer of D. Let $x \in \partial\widetilde{\Sigma}$ and let $d \in D$. Then $d(x) \in \partial\widetilde{\Sigma}$. Since $\tilde{f}$ fixes the boundary pointwise we have $\tilde{f}^{-1}d\tilde{f}(x) = \tilde{f}^{-1}d(x) = d(x)$. Since $\tilde{f}$ is in the normalizer of D, $\tilde{f}^{-1}d\tilde{f} \in D$. Since the deck group acts freely on $\partial\widetilde{\Sigma}$, $\tilde{f}^{-1}d\tilde{f} = d$, completing the proof. ■

Any fiber-preserving homeomorphism of $\widetilde{\Sigma}$ that fixes $\partial\widetilde{\Sigma}$ pointwise must project to a homeomorphism of Σ that fixes $\partial\Sigma$ pointwise. Since the only element of D that fixes $\partial\widetilde{\Sigma}$ pointwise is the identity, restricting the domain of Π gives us an injective homomorphism $\Pi : \mathrm{SHomeo}_p(\widetilde{\Sigma}, \partial\widetilde{\Sigma}) \to \mathrm{LHomeo}_p^+(\Sigma, \mathcal{B}) \cap \mathrm{Homeo}(\Sigma, \partial\Sigma)$.

Define $\mathrm{LHomeo}_p(\Sigma, \partial\Sigma, \mathcal{B}) = \Pi(\mathrm{SHomeo}_p(\widetilde{\Sigma}, \partial\widetilde{\Sigma}))$. That is, the set of homeomorphisms of Σ fixing $\partial\Sigma$ pointwise that lift to a homeomorphism $\tilde{f}$ of $\widetilde{\Sigma}$ fixing $\partial\widetilde{\Sigma}$ pointwise.

While it is tempting to define $\mathrm{LHomeo}_p(\Sigma, \partial\Sigma, \mathcal{B})$ as $\mathrm{LHomeo}_p^+(\Sigma, \mathcal{B}) \cap \mathrm{Homeo}(\Sigma, \partial\Sigma)$, in Chapter 6 we will see that in general we only have the inclusion

$$\mathrm{LHomeo}_p(\Sigma, \partial\Sigma, \mathcal{B}) < \mathrm{LHomeo}_p^+(\Sigma, \mathcal{B}) \cap \mathrm{Homeo}(\Sigma, \partial\Sigma).$$

Indeed, there may be a homeomorphism of Σ fixing $\partial\Sigma$ pointwise that lifts to a homeomorphism of $\widetilde{\Sigma}$, but none of the lifts fix $\partial\widetilde{\Sigma}$ pointwise.

2.2.2 Branched versus unbranched covers

The theme running through this thesis is identifying when homeomorphisms lift in a branched covering space. To do this, it is helpful to consider the unbranched covering space obtained by deleting the branch points and their preimages. Let $p : \widetilde{\Sigma} \to \Sigma$ be a finite-sheeted, branched cover branched at $\mathcal{B} \subset \Sigma$. Let $\widetilde{\Sigma}^\circ = \widetilde{\Sigma} \setminus p^{-1}(\mathcal{B})$ and $\Sigma^\circ = \Sigma \setminus \mathcal{B}$. The restriction of p to $\widetilde{\Sigma}^\circ$ results in an unbranched cover $p^\circ : \widetilde{\Sigma}^\circ \to \Sigma^\circ$.

Conversely, suppose $p : \widetilde{\Sigma} \to \Sigma$ is a finite-sheeted unbranched cover, and Σ has at least one puncture. That is, there exists a surface Σ' and a non-empty finite set $\mathcal{B} \subset \Sigma' \setminus \partial\Sigma'$ such that $\Sigma = \Sigma' \setminus \mathcal{B}$. Then p can be completed to a (possibly branched) cover $\overline{p} : \overline{\widetilde{\Sigma}} \to \overline{\Sigma}$ by filling in the punctures of Σ with marked points $\mathcal{B} \subset \overline{\Sigma}$. To determine whether or not $q \in \mathcal{B}$ is a branch point of $\overline{p}$, assume for simplicity that p is a regular cover with finite abelian deck group A. Let $x \in H_1(\Sigma; \mathbb{Z})$ be the homology class of a loop surrounding only the puncture corresponding to q counterclockwise. The cover p is determined by the kernel of a surjective homomorphism $\varphi : H_1(\Sigma; \mathbb{Z}) \to A$. The number of preimages of q under $\overline{p}$ is given by $|\overline{p}^{-1}(q)| = [A : \langle\varphi(x)\rangle]$. In particular, q is a branch point if and only if $\varphi(x) \neq 0$. For the formal definition of filling in branch points, and for proofs of the claims made in this paragraph, see [46, §4.2].

As in Section 2.1 there is an isomorphism $\mathrm{Homeo}^+(\Sigma, \mathcal{B}) \cong \mathrm{Homeo}^+(\Sigma^\circ)$ given by restricting a homeomorphism of Σ to a homeomorphism of Σ°. Similarly, since any fiber-preserving homeomorphism of $\widetilde{\Sigma}$ must preserve the set $p^{-1}(\mathcal{B})$, there is an isomorphism $\mathrm{SHomeo}_p^+(\widetilde{\Sigma}) \cong \mathrm{SHomeo}_p^+(\widetilde{\Sigma}^\circ)$.

It follows that a homeomorphism is in $\mathrm{LHomeo}_p^+(\Sigma, \mathcal{B})$ if and only if its restriction to Σ° is in $\mathrm{LHomeo}_p^+(\Sigma^\circ)$. The same holds if $\partial\Sigma \neq \emptyset$: a homeomorphism is in $\mathrm{LHomeo}_p(\Sigma, \partial\Sigma, \mathcal{B})$ if and only if its restriction is in $\mathrm{LHomeo}_p(\Sigma^\circ, \partial\Sigma^\circ)$.

This observation motivates the general approach to identifying liftable homeomorphisms. If we want to identify liftable homeomorphisms for a branched cover, it suffices to identify liftable homeomorphisms for the associated unbranched cover.

2.3 Mapping class groups

The mapping class group is the main object of study in this thesis. In this section we introduce the basic definitions and results surrounding mapping class groups. For an introduction to the subject along with proofs of the statements below, see Farb and Margalit's book [22] or Ivanov's survey [29].

2.3.1 Basic definitions

Recall that two homeomorphisms $f_0, f_1 : X \to Y$ between topological spaces are *isotopic* if there exists a continuous map $H : X \times [0,1] \to Y$ such that for all $x \in X$, $H(x, 0) = f_0(x)$, $H(x, 1) = f_1(x)$ and $H(-, t) : X \to Y$ is a homeomorphism for all $t \in [0, 1]$.

Let Σ be a genus g surface with possibly empty boundary $\partial\Sigma$ and finitely many marked points $\mathcal{B} \subset \Sigma \setminus \partial\Sigma$. Define $\mathrm{Homeo}_0(\Sigma, \partial\Sigma, \mathcal{B}) = \{f \in \mathrm{Homeo}(\Sigma, \partial\Sigma, \mathcal{B}) : f \simeq \mathrm{id}\}$. Here, $f \simeq \mathrm{id}$ means f is isotopic to the identity via an isotopy that fixes $\partial\Sigma$ pointwise and $\mathcal{B}$ as a set. Since $\mathcal{B}$ is discrete, any isotopy fixing $\mathcal{B}$ as a set must fix $\mathcal{B}$ pointwise.

Definition. The *mapping class group* $\mathrm{Mod}(\Sigma, \partial\Sigma, \mathcal{B})$ is defined to be

$$\mathrm{Mod}(\Sigma, \partial\Sigma, \mathcal{B}) := \mathrm{Homeo}^+(\Sigma, \partial\Sigma, \mathcal{B}) / \mathrm{Homeo}_0(\Sigma, \partial\Sigma, \mathcal{B}).$$

In this thesis, the mapping class group will always consist of isotopy classes of orientation-preserving homeomorphisms. As in the notational conventions for homeomorphism groups, the entries $\mathcal{B}$ and $\partial\Sigma$ in $\mathrm{Mod}(\Sigma, \partial\Sigma, \mathcal{B})$ are optional. If $\mathcal{B}$ appears, elements of the mapping class group are isotopy classes of homeomorphisms fixing $\mathcal{B}$ as a set, where the isotopies must also fix $\mathcal{B}$. If $\partial\Sigma$ appears, elements of the mapping class group are isotopy classes of homeomorphisms fixing $\partial\Sigma$ pointwise, and the isotopies must also fix $\partial\Sigma$ pointwise. For a homeomorphism f, let $[f]$ denote its isotopy class in the mapping class group.

As was the case with the homeomorphism groups, marked points and punctures contain the same topological information. Indeed, $\mathrm{Mod}(\Sigma^b_{g,n}, \partial\Sigma^b_{g,n}) \cong \mathrm{Mod}(\Sigma^b_{g,0}, \partial\Sigma^b_{g,0}, \mathcal{B})$ if $|\mathcal{B}| = n$.

The subgroup of the mapping class group consisting of isotopy classes of homeomorphisms that fix $\mathcal{B}$ pointwise, called the pure mapping class group, will play an important role in Chapter 4. For simplicity, the next definition and the following short exact sequence are stated without insisting $\partial\Sigma$ is fixed pointwise. However, they hold if we require $\partial\Sigma$ to be fixed.

Definition. Let $\mathcal{B} \neq \emptyset$. The *pure mapping class group* is defined as

$$\mathrm{PMod}(\Sigma, \mathcal{B}) := \{[f] \in \mathrm{Mod}(\Sigma, \mathcal{B}) : f(x) = x \text{ for all } x \in \mathcal{B}\}.$$

If $|\mathcal{B}| = n$, there is a short exact sequence

$$1 \longrightarrow \mathrm{PMod}(\Sigma, \mathcal{B}) \longrightarrow \mathrm{Mod}(\Sigma, \mathcal{B}) \xrightarrow{\phi} S_n \longrightarrow 1$$

where S_n is the symmetric group on n letters and ϕ is given by the action of $\mathrm{Mod}(\Sigma, \mathcal{B})$ on $\mathcal{B}$. The map ϕ is surjective since any transposition can be obtained by taking a half twist about an arc joining two marked points (see Section 2.3.2).

Throughout the thesis we may abuse notation and identify an element $f \in \mathrm{Mod}(\Sigma)$ with a representative homeomorphism.

2.3.2 Dehn twists and half twists

In this section we define two special elements of the mapping class group: Dehn twists and half twists. Dehn twists are determined by an isotopy class of a simple closed curve, and half twists are determined by an isotopy class of a simple arc.

A *simple closed curve* α on a surface Σ is an embedding $\alpha : S^1 \to \Sigma \setminus (\mathcal{B} \cup \partial\Sigma)$. A *simple arc* δ is an embedding $\delta : [0,1] \to \Sigma \setminus \partial\Sigma$ such that $\delta^{-1}(\mathcal{B}) = \{0,1\}$. We usually identify simple closed curves and simple arcs with their images in Σ.

We will often identify a simple closed curve or simple arc α with its isotopy class $[\alpha]$. If α and β are isotopic we write $\alpha \simeq \beta$. We insist that isotopies cannot pass through marked points. For arcs, we also insist isotopies fix the endpoints. In particular, if δ and μ are isotopic simple arcs, then $\delta(0) = \mu(0)$ and $\delta(1) = \mu(1)$.

We now define Dehn twists and half Dehn twists. For a picture of both types of mapping class group elements, see Figure 2.1.

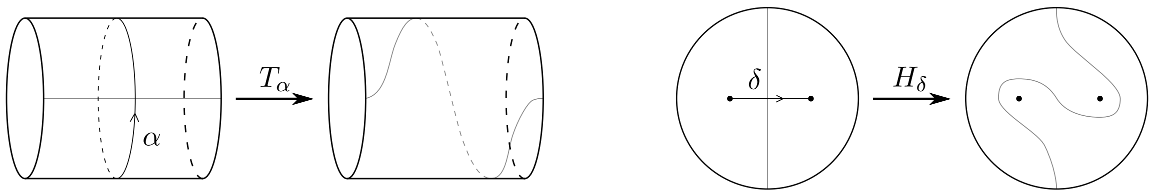


Figure 2.1: A Dehn twist about the curve α and a half Dehn twist about the arc δ.

On an oriented surface Σ, a regular neighbourhood of a simple closed curve is homeomorphic to an annulus $A = S^1 \times I$. Give A coordinates (s,t) where $s = e^{i\theta}$ with $\theta \in [0, 2\pi]$ and $t \in [0,1]$. Define a homeomorphism T on A by $T(s,t) = (se^{-2\pi i t}, t)$.

Definition. Let N be a regular neighbourhood of a simple closed curve α on an oriented surface Σ. Choose an orientation-preserving homeomorphism $\phi : N \to A$. Define the *Dehn twist about* α, denoted $T_\alpha \in \mathrm{Mod}(\Sigma)$, as the isotopy class of the homeomorphism defined by $\phi^{-1}T\phi$ on N and the identity on $\Sigma \setminus N$.

The element $T_\alpha \in \mathrm{Mod}(\Sigma)$ is independent of the choice of N, homeomorphism ϕ, and isotopy class representative of $[\alpha]$. If a is the isotopy class of α, we may also write T_a as the Dehn twist about α. Perhaps surprisingly, T_α is independent of the orientation of α. That is, if $-\alpha$ is a curve isotopic to α with the opposite orientation, then $T_\alpha = T_{-\alpha}$. Here are some important facts about Dehn twists:

- $T_\alpha = T_\beta$ if and only if $\alpha \simeq \beta$ or $\alpha \simeq -\beta$.
- For any homeomorphism f, $[f]T_\alpha[f]^{-1} = T_{f(\alpha)}$.
- $T_\alpha(\alpha) \simeq \alpha$.
- If α does not bound a disk or a once marked disk, then T_α has infinite order.

Similar to a Dehn twist, we define a half Dehn twist about a particular isotopy class of an arc. A regular neighbourhood of an arc is homeomorphic to the unit disc $D^2 \subset \mathbb{C}$. Define a homeomorphism H on D^2 by $H(z) = e^{-2\pi i|z|}z$.

Definition. Let δ be a simple arc on an oriented surface Σ such that $\delta(0), \delta(1) \in \mathcal{B}$. Let F be a regular neighbourhood of δ such that $F \cap \mathcal{B} = \{\delta(0), \delta(1)\}$. Choose an orientation-preserving homeomorphism $\phi : F \to D^2$ such that $F\delta(t) = t - \frac{1}{2}$. Define the *half Dehn twist* or simply the *half twist about* δ, denoted $H_\delta \in \mathrm{Mod}(\Sigma, \mathcal{B})$, as the isotopy class of a homeomorphism defined by $\phi^{-1}H\phi$ on F and the identity on $\Sigma \setminus F$.

As is the case with Dehn twists, the element $H_\delta \in \mathrm{Mod}(\Sigma, \mathcal{B})$ is independent of choice of F, homeomorphism ϕ, and isotopy class representative of $[\delta]$. Again, if $\mu = -\delta$, then $H_\mu = H_\delta$. Here are some important facts about half twists:

- For any homeomorphism f, $[f]H_\delta[f]^{-1} = H_{f(\delta)}$.
- $H_\delta(\delta) \simeq -\delta$.
- H_δ switches the marked points $\delta(0)$ and $\delta(1)$.
- $H_\delta^2 = T_\alpha$ where α is a curve isotopic to the boundary of a regular neighbourhood of δ.

Since both Dehn twists are independent of the orientation of the curve defining the twist, we may define a Dehn twist by specifying (the isotopy class of) an unoriented curve. The same goes for arcs and half twists.

Dehn twists and half twists play an important role in the theory of mapping class groups. In particular, $\mathrm{PMod}(\Sigma, \partial\Sigma, \mathcal{B})$ is finitely generated by Dehn twists [18]. Since finitely many transpositions generate the symmetric group $S_{|\mathcal{B}|}$, and each half twist induces a transposition that switches the endpoints of the arc, $\mathrm{Mod}(\Sigma, \partial\Sigma, \mathcal{B})$ is finitely generated by Dehn twists and half twists.

2.3.3 The capping homomorphism

The short exact sequence associated to the capping homomorphism will play an important role in arriving at a presentation for $\mathrm{PMod}(\Sigma_{0,n})$ in Section 4.4.1. For a complete exposition and proof, see [22, §3.6].

Suppose Σ' is a closed subsurface of Σ such that $\Sigma \setminus \Sigma'$ is a disk with one marked point x and boundary β. We say that Σ is obtained from Σ' by *capping* Σ' at β. Note that Σ has one more marked point than Σ', and one less boundary component. Let $\mathcal{B}' \subset \Sigma'$ be finitely many (possibly zero) marked points.

Since representatives for mapping classes must fix boundary components pointwise, the inclusion $\Sigma' \hookrightarrow \Sigma$ induces a homomorphism

$$Cap : \mathrm{PMod}(\Sigma', \partial\Sigma', \mathcal{B}') \to \mathrm{PMod}(\Sigma, \partial\Sigma, \mathcal{B}' \cup \{x\}).$$

We call this homomorphism the *capping homomorphism.*

Theorem 2.3.1 (The Capping Homomorphism)**.** *There is a short exact sequence*

$$1 \longrightarrow \langle T_\beta \rangle \longrightarrow \mathrm{PMod}(\Sigma', \partial\Sigma', \mathcal{B}') \xrightarrow{Cap} \mathrm{PMod}(\Sigma, \partial\Sigma, \mathcal{B}' \cup \{x\}) \longrightarrow 1$$

where β is a curve isotopic to the boundary component being capped.

The capping homomorphism is also well defined on the mapping class group, not just the pure mapping class group. In this case however, it is not surjective.

2.3.4 The Birman-Hilden theory

Given a branched or unbranched covering space $p : \widetilde{\Sigma} \to \Sigma$, it is natural to ask how the mapping class groups of $\widetilde{\Sigma}$ and Σ are related. What is now known as the Birman-Hilden theory deals with this question. For a wonderful survey by Margalit and Winarski see [36].

Let $p : \Sigma_g \to \Sigma_h$ be a finite-sheeted, regular, possibly branched covering space with deck group $D < \mathrm{Homeo}^+(\Sigma_g)$, branched at finitely many $\mathcal{B} \subset \Sigma_h$. Suppose $g \geq 2$.

Abusing notation, denote both the quotient maps by $\mathfrak{P} : \mathrm{Homeo}^+(\Sigma_g) \to \mathrm{Mod}(\Sigma_g)$ and $\mathfrak{P} : \mathrm{Homeo}^+(\Sigma_h, \mathcal{B}) \to \mathrm{Mod}(\Sigma_h, \mathcal{B})$. As in Section 2.2, $\mathrm{SHomeo}^+_p(\Sigma_g)$ denotes the group of fiber-preserving homeomorphisms, and $\mathrm{LHomeo}^+_p(\Sigma_h, \mathcal{B})$ denotes the group of liftable homeomorphisms.

Definition. Define the *liftable mapping class group* as

$$\mathrm{LMod}_p(\Sigma_h, \mathcal{B}) = \mathfrak{P}(\mathrm{LHomeo}^+_p(\Sigma_h, \mathcal{B})).$$

Define the *symmetric mapping class group* as

$$\mathrm{SMod}_p(\Sigma_g) = \mathfrak{P}(\mathrm{SHomeo}^+_p(\Sigma_g)).$$

Recall that there is a short exact sequence

$$1 \longrightarrow D \longrightarrow \mathrm{SHomeo}^+_p(\Sigma_g) \xrightarrow{\Pi} \mathrm{LHomeo}^+_p(\Sigma_h, \mathcal{B}) \longrightarrow 1$$

where Π is defined in Section 2.2. Furthermore, $\mathrm{SHomeo}^+_p(\Sigma_g)$ is the normalizer of D in $\mathrm{Homeo}^+(\Sigma_g)$. At first glance, there is no reason to expect that there is a corresponding short exact sequence for mapping class groups. Remarkably, there is.

Since $g \geq 2$ and D is finite, $\mathfrak{P}$ restricted to D is injective (see [22, Section 7.1.2]). We will identify D with its image in $\mathrm{Mod}(\Sigma_g)$. The next theorem was proved by Birman and Hilden for the hyperelliptic covers in [7], for unbranched covers in [8], and for branched covers with solvable deck group in [9]. Maclachlan and Harvey proved Theorem 2.3.2 as is stated here in [34].

Theorem 2.3.2 (The Birman-Hilden theorem)**.** *Let $g \geq 2$ and let $p : \Sigma_g \to \Sigma_h$ be a finite-sheeted, regular, possibly branched covering space, branched at finitely many points $\mathcal{B} \subset \Sigma_h$. Let D be the deck group. There is a short exact sequence*

$$1 \longrightarrow D \longrightarrow \mathrm{SMod}_p(\Sigma_g) \xrightarrow{\overline{\Pi}} \mathrm{LMod}_p(\Sigma_h, \mathcal{B}) \longrightarrow 1$$

where $\overline{\Pi}([f]) = [\Pi(f)]$ *for any* $f \in \mathrm{SHomeo}_p^+(\Sigma_g)$. *Furthermore,* $\mathrm{SMod}_p(\Sigma_g)$ *is the normalizer of* D *in* $\mathrm{Mod}(\Sigma_g)$.

The next theorem was proved by Birman and Hilden as an important ingredient in the proof of Theorem 2.3.2. On the other hand, Machlachlan and Harvey deduce the next theorem as a corollary of Theorem 2.3.2. Regardless of your point of view, it would be an injustice not to include the statement.

We say symmetric homeomorphisms $f_0, f_1 \in \mathrm{SHomeo}_p^+(\Sigma_g)$ are *symmetrically isotopic* if there is an isotopy $H : \Sigma_g \times I \to \Sigma_g$ such that $H(-,0) = f_0$, $H(-,1) = f_1$ and $H(-,t) \in \mathrm{SHomeo}_p^+(\Sigma_g)$ for all $t \in I$.

Theorem 2.3.3. *Let* $g \geq 2$ *and let* $p : \Sigma_g \to \Sigma_h$ *be a finite-sheeted, regular, possibly branched covering space, branched at finitely many points* $\mathcal{B} \subset \Sigma_h$. *If* $f_0, f_1 \in \mathrm{SHomeo}_p^+(\Sigma_g)$ *are isotopic, then they are symmetrically isotopic.*

Theorem 2.3.2 was initially proved to arrive at a finite presentation for $\mathrm{Mod}(\Sigma_2)$. Let $p_2 : \Sigma_2 \to \Sigma_0$ be the hyperelliptic cover (see Chapter 1), which is branched at 6 points $\mathcal{B} \subset \Sigma_0$. In this case, $\mathrm{SMod}_{p_2}(\Sigma_2) = \mathrm{Mod}(\Sigma_2)$ and $\mathrm{LMod}_p(\Sigma_0, \mathcal{B}) = \mathrm{Mod}(\Sigma_0, \mathcal{B})$. By Theorem 2.3.2 there is a short exact sequence

$$1 \longrightarrow \langle \iota \rangle \longrightarrow \mathrm{Mod}(\Sigma_2) \longrightarrow \mathrm{Mod}(\Sigma_0, \mathcal{B}) \longrightarrow 0.$$

A presentation for $\mathrm{Mod}(\Sigma_0, \mathcal{B})$ was known at the time, as was a presentation for $\langle \iota \rangle \cong \mathbb{Z}/2\mathbb{Z}$. Using Lemma 2.4.2 below, Birman and Hilden were then able to arrive at a presentation for $\mathrm{Mod}(\Sigma_2)$ [7, Theorem 8].

The Birman-Hilden theory with boundary

The Birman-Hilden theorem stated above is for surfaces without boundary. However, there is a corresponding Birman-Hilden theorem for surfaces with boundary, which can be arrived at using the techniques from [9]. An alternate proof due to Alan McLeay will appear in upcoming joint work by myself and McLeay. His proof is of the same flavour as Winarski's proof of the Birman-Hilden theorem for a particular family of irregular branched covers in [47].

Suppose $\widetilde{\Sigma}$ and Σ have non-empty boundary. Let $p : \widetilde{\Sigma} \to \Sigma$ be a finite-sheeted, regular, possibly branched covering space, branched at finitely many points $\mathcal{B} \subset \Sigma \setminus \partial\Sigma$.

Recall the definitions of $\mathrm{LHomeo}_p(\Sigma, \partial\Sigma, \mathcal{B})$ and $\mathrm{SHomeo}_p(\widetilde{\Sigma}, \partial\widetilde{\Sigma})$ from Section 2.2.1. As above, define the *liftable mapping class group* and *symmetric mapping class group* as

$$\mathrm{LMod}_p(\Sigma, \partial\Sigma, \mathcal{B}) = \mathfrak{P}(\mathrm{LHomeo}_p(\Sigma, \partial\Sigma, \mathcal{B})) \text{ and } \mathrm{SMod}_p(\widetilde{\Sigma}, \partial\widetilde{\Sigma}) = \mathfrak{P}(\mathrm{SHomeo}_p(\widetilde{\Sigma}, \partial\widetilde{\Sigma}))$$

respectively.

Recall the isomorphism $\Pi : \mathrm{SHomeo}_p(\widetilde{\Sigma}, \partial\widetilde{\Sigma}) \to \mathrm{LHomeo}_p(\Sigma, \partial\Sigma, \mathcal{B})$ from Section 2.2.1 that arises by projecting homeomorphisms. Again, there is little reason to expect that such an isomorphism transfers over to the setting of mapping class groups.

Theorem 2.3.4 (The Birman-Hilden theorem with boundary)**.** *The map*

$$\overline{\Pi} : \mathrm{SMod}_p(\widetilde{\Sigma}, \partial\widetilde{\Sigma}) \to \mathrm{LMod}_p(\Sigma, \partial\Sigma, \mathcal{B})$$

given by $\overline{\Pi}([f]) = [\Pi(f)]$ *is an isomorphism.*

There are a few key differences with the theorem for closed surfaces. First, there is no restriction on the genus of the surfaces involved. Second, there is no nontrivial kernel for the map $\overline{\Pi}$ since no nontrivial element of the deck group fixes the $\partial\widetilde{\Sigma}$ pointwise.

We conclude this section with the following observation. Since homotopies always lift in covering spaces [12, p. 140], if a homeomorphism lifts, then so does every isotopic homeomorphism. Using the notation above, this means $\mathfrak{P}^{-1}(\mathrm{LMod}_p(\Sigma_h, \mathcal{B})) = \mathrm{LHomeo}^+_p(\Sigma_h, \mathcal{B})$ and $\mathfrak{P}^{-1}(\mathrm{LMod}_p(\Sigma, \partial\Sigma, \mathcal{B})) = \mathrm{LHomeo}_p(\Sigma, \partial\Sigma, \mathcal{B})$. However, this is not the case for the symmetric mapping class group. That is, there exist representatives for elements of the symmetric mapping class group that are not symmetric homeomorphisms.

2.3.5 A lifting criterion for abelian covers

Suppose $\widetilde{\Sigma}$ and Σ do not have boundary. Let $p : \widetilde{\Sigma} \to \Sigma$ be a regular, finite-sheeted, possibly branched covering space branched at finitely many points $\mathcal{B} \subset \Sigma$. Furthermore, assume the deck group A is abelian. Let $p^\circ : \widetilde{\Sigma}^\circ \to \Sigma^\circ$ be the associated unbranched cover. Then p° is determined by the kernel of a surjective homomorphism $\overline{\varphi} : \pi_1(\Sigma^\circ, x) \to A$. Let $\Phi : \pi_1(\Sigma^\circ, x) \to H_1(\Sigma^\circ; \mathbb{Z})$ be the Hurewicz homomorphism (see [12, p. 250]). Since A is abelian, $\overline{\varphi} = \varphi\Phi$ for a unique surjective homomorphism $\varphi : H_1(\Sigma^\circ; \mathbb{Z}) \to A$. Therefore p° is determined by the homomorphism φ. Conversely, two surjective homomorphisms $\varphi, \phi : H_1(\Sigma^\circ; \mathbb{Z}) \to A$ determine the same cover if and only if there is an autmorphism $\psi \in \mathrm{Aut}(A)$ such that $\varphi = \psi\phi$.

For a mapping class $f \in \mathrm{Mod}(\Sigma, \mathcal{B})$, let $f_* \in \mathrm{Aut}(H_1(\Sigma^\circ; \mathbb{Z}))$ be the automorphism induced by the action of any representative of f on $H_1(\Sigma^\circ; \mathbb{Z})$. The following lemma is an application of the lifting criterion [27, Proposition 1.33], and will play an important role in the rest of this thesis.

Lemma 2.3.5. *Let* $p : \widetilde{\Sigma} \to \Sigma$ *be a regular, finite-sheeted, possibly branched covering space branched at finitely many points* $\mathcal{B} \subset \Sigma$ *with abelian deck group* A*. Let* $\varphi : H_1(\Sigma^\circ; \mathbb{Z}) \to A$ *be a surjective homomorphism determining the cover. Then* $f \in \mathrm{LMod}_p(\Sigma, \mathcal{B})$ *if and only if* $f_*(\ker\varphi) < \ker\varphi$*.*

Since A is finite, the condition $f_*(\ker\varphi) < \ker\varphi$ is equivalent to $f_*(\ker\varphi) = \ker\varphi$.

2.4 Combinatorial group theory

Group presentations play a central role in Chapters 4 and 5. In this section we survey three results that will be heavily relied upon.

To set notation, suppose S is an alphabet, that is a list of symbols. Denote by S^* the set of words in the alphabet. More precisely, S^* is the set of finite strings in the symbols $S \cup \{s^{-1} : s \in S\}$. Denote the empty word by $1 \in S^*$.

Given a presentation of a group, we may abuse notation and identify a word in the generating symbols with the group element it represents.

2.4.1 Presentations from short exact sequences

To obtain various group presentations in Chapters 4 and 5, we use two well-known results concerning short exact sequences and group presentations.

Lemma 2.4.1. *Let*

$$1 \longrightarrow K \xrightarrow{\alpha} G \xrightarrow{\pi} H \longrightarrow 1$$

be a short exact sequence of groups. Let $\langle S \mid R\rangle$ be a presentation for G where each symbol $s \in S$ denotes a generator $g_s \in G$. Let K be normally generated by $\{k_\beta\} \subset K$ and for each β, let $w_\beta \in S^*$ denote $\alpha(k_\beta)$. Then H admits the presentation $\langle S \mid R \cup \{w_\beta\}\rangle$ where $s \in S$ denotes $\pi(g_s)$.

A proof of Lemma 2.4.1 can be found in [35, §2.1]

For Lemma 2.4.2, let

$$1 \longrightarrow K \xrightarrow{\alpha} G \xrightarrow{\pi} H \longrightarrow 1$$

be a short exact sequence of groups. Suppose K and H admit presentations $\langle S_K \mid R_k\rangle$ and $\langle S_H \mid R_H\rangle$ respectively.

For each $s \in S_H$, let $h_s \in H$ be the corresponding generator. For each $s \in S_H$, choose $g_s \in \pi^{-1}(h_s)$. Let $\widetilde{S}_H = \{\tilde{s} : s \in S_H\}$ and assign the symbol $\tilde{s}$ to g_s. For each $t \in S_K$, let $k_t \in K$ be the corresponding generator. Let $\widetilde{S}_K = \{\tilde{t} : t \in S_k\}$ and assign $\tilde{t}$ to $\alpha(k_t)$.

Let $r = s_1^{\epsilon_1} \cdots s_m^{\epsilon_m} \in R_H$ where $s_i \in S_H$ and $\epsilon_i \in \{\pm 1\}$. Let $\tilde{r} = \tilde{s}_1^{\epsilon_1} \cdots \tilde{s}_m^{\epsilon_m} \in \widetilde{S}_H^*$. Then $\tilde{r}$ denotes an element $g \in G$ such that $\pi(g) = e$, where e is the identity in H. By exactness, $g \in \alpha(K)$, so it is represented by some word $w_r \in \widetilde{S}_K^*$. Define the set of words

$$R_1 := \{\tilde{r} w_r^{-1} : r \in R_H\}.$$

For every $s \in S_H$ and $t \in S_K$, $g_s\alpha(k_t)g_s^{-1} \in \alpha(K)$. Let $v_{s,t} \in \widetilde{S}_K^*$ be a word denoting $g_s\alpha(k_t)g_s^{-1}$. Define the set of words

$$R_2 := \{\tilde{s}\tilde{t}\tilde{s}^{-1}v_{s,t}^{-1} : \tilde{t} \in \widetilde{S}_K, \tilde{s} \in \widetilde{S}_H\}.$$

For each $r = t_1^{\eta_1} \cdots t_n^{\eta_n} \in R_K$, let $\tilde{r} = \tilde{t}_1^{\eta_1} \cdots \tilde{t}_n^{\eta_n}$. Define $\widetilde{R}_K := \{\tilde{r} : r \in R_K\}$.

Lemma 2.4.2. *Let*

$$1 \longrightarrow K \xrightarrow{\alpha} G \xrightarrow{\pi} H \longrightarrow 1$$

be a short exact sequence of groups. Then G admits the presentation

$$G \cong \langle \widetilde{S}_K \cup \widetilde{S}_H \mid R_1 \cup R_2 \cup \widetilde{R}_K \rangle.$$

where $\widetilde{S}_K, \widetilde{S}_H, R_1, R_2$, and $\widetilde{R}_K$ are defined as above.

This lemma is well known, however a proof is hard to come by in the literature. As such, one is included here for completeness.

Proof. To set notation, for words v, u we write $v \sim u$ if u can be obtained from v by adding or deleting relators. For a word $v = t_1^{\eta_1} \cdots t_n^{\eta_n} \in S_K^*$, let $\tilde{v} = \tilde{t}_1^{\eta_1} \cdots \tilde{t}_n^{\eta_n} \in \widetilde{S}_K^*$. Similarly for words in S_H^* and $\widetilde{S}_H^*$.

By the definitions of R_1, R_2 and $\widetilde{R}_K$, each word in $R_1 \cup R_2 \cup \widetilde{R}_K$ is a relator for G. It remains to show that if $w \in (\widetilde{S}_K \cup \widetilde{S}_H)^*$ denotes the identity $e \in G$, then $w \sim 1$ through relators in $R_1 \cup R_2 \cup \widetilde{R}_K$.

Using the relators from R_2 we have that $w \sim \widetilde{v}_k \widetilde{v}_H$ for some $\widetilde{v}_K \in \widetilde{S}_K^*$ and $\widetilde{v}_H \in \widetilde{S}_H^*$. By exactness, $v_H \in S_H^*$ denotes the identity $e \in H$. Therefore there are words $\left\{v_H^{(0)}, v_H^{(1)}, \dots, v_H^{(l)}\right\} \subset S_H^*$ such that

$$v_H = v_H^{(0)} \sim v_H^{(1)} \sim \cdots \sim v_H^{(l)} = 1$$

through relators in R_H.

Suppose $v_H^{(i+1)}$ is derived from $v_H^{(i)}$ by inserting or deleting a relator $r_i \in R_H$ or its inverse r_i^{-1}. Insert $w_{r_i}\tilde{r}_i^{-1}$ or $\tilde{r}_i w_{r_i}^{-1} \in \widetilde{R}_1$ in the corresponding position in $\widetilde{v}_H^{(i)}$ and use relators from R_2 to group letters in $\widetilde{S}_H$ to the right. Thus $\widetilde{v}_H^{(i)}$ is replaced by $u'\widetilde{v}_H^{(i+1)}$ for some $u' \in \widetilde{S}_K^*$.

Therefore, using relators from $R_1 \cup R_2$ we have $w \sim \widetilde{v}_K \widetilde{v}_H \sim \widetilde{u}_K$ where $\widetilde{u}_K \in \widetilde{S}_K^*$. Since $\widetilde{u}_K$ denotes the identity, so does u_K and therefore $u_K \sim 1$ through relators from R_K. Using the corresponding relators in $\widetilde{R}_K$, we have $\widetilde{u}_K \sim 1$, completing the proof. ■

2.4.2 The Reidemeister-Schreier rewriting process

The Reidemeister-Schreier rewriting process is an algorithm that, given a presentation for a group G, produces a presentation for a subgroup H of G. Furthermore, if G is finitely presented and H is finite index in G, then the presentation produced for H is finite. The algorithm will be used in Chapter 5, and we describe the algorithm here without proof. For a full proof see [35, §2.3].

Fix a presentation $G = \langle \{a_\alpha : \alpha \in I\} \mid \{R_\beta : \beta \in J\} \rangle$.

Definition. A *right coset representative system for G mod H* is a subset $\overline{A} \subset \{a_\alpha : \alpha \in I\}^*$ such that $1 \in \overline{A}$ and for each right coset of Hg in G, there exists a unique word $W \in \overline{A}$ such that W defines an element in Hg. A right coset representative system is called a *Schreier system for G mod H* if whenever $W \in \overline{A}$, every initial segment of W is in $\overline{A}$.

It is a convenient fact that a Schreier system always exists. Given a word $W \in \{a_\alpha : \alpha \in I\}^*$, let $\overline{W} \in \overline{A}$ be the unique right coset representative of W in H.

Definition. Let $\overline{A} \subset \{a_\alpha\}^*$ be a Schreier system for G mod H. Introduce the symbols $\{C_{K,a_\alpha} : \alpha \in I, K \in \overline{A}\}$. Let $U = a_{\alpha_1}^{\epsilon_1} \cdots a_{\alpha_r}^{\epsilon_r}$, $\epsilon_i \in \{\pm 1\}$ be an arbitrary word such that $U \in H$. Define $\tau(U) := C_{K_1,a_{\alpha_1}}^{\epsilon_1} \cdots C_{K_r,a_{\alpha_r}}^{\epsilon_r}$ where

$$K_j = \begin{cases} \overline{W_j} & \text{if } \epsilon_j = 1 \\ \overline{W_j a_{\alpha_j}^{-1}} & \text{if } \epsilon_j = -1 \end{cases}$$

and $W_j = a_{\alpha_1}^{\epsilon_1} \cdots a_{\alpha_{j-1}}^{\epsilon_{j-1}}$. The function τ is called a *Reidemeister-Schreier rewriting process for G mod H*.

If W and V are words in the same alphabet that are freely equal (that is, W can be obtained from V by inserting and deleting substrings of the form xx^{-1} or $x^{-1}x$) we write $W \approx V$. We are now ready to write down a presentation for the subgroup H.

Theorem 2.4.3 (The Reidemeister-Schreier rewriting process)**.** *Let τ be a Reidemeister-Schreier rewriting process for G mod H. Then H admits the presentation*

$$\langle \{C_{K,a_\alpha} : \alpha \in I, K \in \overline{A}\} \mid \Upsilon_1 \cup \Upsilon_2 \rangle$$

where $C_{K,a_\alpha} = K a_\alpha (\overline{K a_\alpha})^{-1}$ and

$$\Upsilon_1 = \{C_{M,a_\alpha} : \alpha \in I, M \in \overline{A}, M a_\alpha \approx \overline{M a_\alpha}\}$$
$$\Upsilon_2 = \{\tau(K R_\beta K^{-1}) : K \in \overline{A}, \beta \in J\}.$$

Chapter 3

Cyclic Branched Covers of Spheres

The first part of this chapter provides two different ways of viewing a cyclic branched cover over a sphere. We will see that admissible tuples provide a convenient combinatorial way of identifying a cyclic branched cover. Then a plane curve description will be given for each cover in the form of a superelliptic curve. The first viewpoint holds more generally for abelian covers, and will be presented as such.

The second part classifies the cyclic branched covers of the sphere with the property that every homeomorphism of the sphere lifts to a homeomorphism of the covering space.

3.1 Admissible tuples and superelliptic curves

Here we present two ways to view a cyclic branched cover over a sphere. The first is via admissible tuples.

Let $p : \Sigma \to \Sigma_0$ be a finite-sheeted, regular, branched cover over a sphere Σ_0 with abelian deck group A. Let $\mathcal{B} \subset \Sigma_0$ be the branch points and let $|\mathcal{B}| = k$. Any such cover is determined by the associated unbranched cover $p : \Sigma^\circ \to \Sigma_0^\circ$. Since the deck group is abelian, the unbranched cover is determined by the kernel of a surjective homomorphism $\phi : H_1(\Sigma_0^\circ; \mathbb{Z}) \to A$. The homomorphism ϕ is unique up to an automorphism of A.

The next definition will be useful to classify surjective homomorphisms from $H_1(\Sigma_0^\circ; \mathbb{Z})$ to A.

Definition. Let A be a finite abelian group. An *admissible k-tuple* is a tuple

$$(a_1, \ldots, a_k) \in (A \setminus \{0\})^k$$

such that $\sum_{i=1}^k a_i = 0$ and $\{a_1, \ldots, a_k\}$ is a generating set for A.

Fix an enumeration of the punctures. Let x_i be the homology class of a loop surrounding only the ith puncture oriented counterclockwise. The homology classes $\{x_i\}$ generate $H_1(\Sigma_0^\circ;\mathbb{Z})$. An admissible k-tuple $(a_1,\ldots,a_k)$ defines a surjective homomorphism $\phi : H_1(\Sigma_0^\circ;\mathbb{Z}) \to A$ by $\phi(x_i) = a_i$, and therefore determines a regular cover of Σ_0° with deck group A. Another admissible k-tuple $(a'_1,\ldots,a'_k)$ determines an equivalent covering space if and only if there is an automorphism $\psi \in \mathrm{Aut}(A)$ such that $\psi(a_i) = a'_i$ for all i.

Conversely, a surjective homomorphism $\phi : H_1(\Sigma_0^\circ;\mathbb{Z}) \to A$ is determined by the admissible k-tuple $(a_1,\ldots,a_k)$ where $a_i = \phi(x_i)$. Therefore there is a one to one correspondence between branched covers of a sphere with k points and deck group A (up to equivalence) and admissible k-tuples (up to an automorphism of A).

Every fininte-sheeted cyclic branched cover over the sphere may also be viewed as a superelliptic curve. Recall that a *cyclic* cover is a regular covering space with a cyclic deck group. We now provide an outline of how a finite-sheeted cyclic cover of the sphere can be viewed as a superelliptic curve. See [43, Chapter 2] for more details.

Choose distinct points $z_1,\ldots,z_t \in \mathbb{C}$. Any cyclic branched cover of the sphere can be modeled by an irreducible plane curve C of the form:

$$y^n = (x - z_1)^{a_1} \cdots (x - z_t)^{a_t} \tag{3.1}$$

for some $n \geq 2$ and integers $1 \leq a_i \leq n-1$. Indeed, let $\widetilde{C}$ be the normalization of the projective closure of the affine curve C. Projection onto the x axis gives a n-sheeted cyclic branched covering $\widetilde{C} \to \mathbb{P}^1$, branched at each z_i and possibly at infinity. The cyclic deck group is generated by the map $(x,y) \mapsto (x, \zeta_n y)$ where ζ_n is a primitive nth root of unity. There is branching at infinity if and only if $\sum_{i=1}^t a_i \not\equiv 0 \mod n$.

A cyclic branched covering space defined by (3.1) has deck group $A \cong \mathbb{Z}/n\mathbb{Z}$. Such a cover is defined by the admissible t-tuple $(a_1,\ldots,a_t)$ if there is no branching at infinity. If there is a branch point at infinity, then the cover is defined by the admissible $(t+1)$-tuple $(a_1,\ldots,a_t,-\sum_{i=1}^t a_i)$. The irreducibility of C ensures that the $\{a_i\}$ form a generating set for A.

3.2 Lifting homeomorphisms

The objective of this section is to identify which admissible tuples give rise to cyclic branched covers over the sphere with the property that every homeomorphism of the sphere lifts to a homeomorphism of the covering surface.

Let $p : \Sigma \to \Sigma_0$ be a finite-sheeted, regular, branched cover over a sphere Σ_0 with cyclic deck group. Let $\mathcal{B} \subset \Sigma_0$ be the branch points and let $|\mathcal{B}| = k$. Let $\Sigma_0^\circ = \Sigma_0 \setminus \mathcal{B}$.

As above, fix an enumeration of the punctures of Σ_0°. Let x_i be the homology class of a loop surrounding the ith puncture that is oriented counterclockwise. Each $x_i \in H_1(\Sigma_0^\circ;\mathbb{Z})$ is supported on a neighborhood of the ith puncture.

Let f be a homeomorphism of Σ_0°. The automorphism $f_* \in \mathrm{Aut}(H_1(\Sigma_0^\circ,\mathbb{Z}))$ is determined by the permutation f induces on the punctures of Σ_0°. Indeed, let $\sigma \in S_k$ be the permutation induced by f. If f is orientation preserving, then $f_*(x_i) = x_{\sigma(i)}$. If f is orientation reversing, then $f_*(x_i) = -x_{\sigma(i)}$.

The immediate goal is to prove Theorem 3.2.3 below. The path to this proof will traverse through Lemmas 3.2.1 and 3.2.2.

Lemma 3.2.1. *Let A be a finite abelian group and $(a_1, \dots, a_k)$ an admissible k-tuple. Let $\Sigma^\circ \to \Sigma_0^\circ$ be the covering space defined by this tuple. Let f be a homeomorphism of Σ_0°, and let $\sigma \in S_k$ be the permutation of the punctures induced by f. The homeomorphism f lifts if and only if there is an automorphism $\psi \in \mathrm{Aut}(A)$ such that $\psi(a_i) = a_{\sigma(i)}$ for all i.*

Proof. Let $\phi : H_1(\Sigma_0^\circ;\mathbb{Z}) \to A$ be the homomorphism defining the cover, which is defined by $\phi(x_i) = a_i$. Let f_* be the automorphism of $H_1(\Sigma_0^\circ;\mathbb{Z})$ induced by f. We use the following facts from group theory:

1. The equality $f_*(\ker(\phi)) = \ker(\phi)$ holds if and only if $\ker(\phi f_*) = \ker(\phi)$.
2. Let $f, g : G \to A$ be surjective homomorphisms. Then $\ker(f) = \ker(g)$ if and only if $f = \xi g$ for some $\xi \in \mathrm{Aut}(A)$.

By Lemma 2.3.5, the homeomorphism f lifts if and only if $f_*(\ker(\phi)) = \ker(\phi)$. Therefore by fact 1, f lifts if and only if $\ker(\phi f_*) = \ker(\phi)$. By fact 2, f lifts if and only if there exists an automorphism $\psi \in \mathrm{Aut}(A)$ such that $\phi f_* = \psi\phi$. If f is orientation-preserving, then $a_{\sigma(i)} = \phi f_*(x_i) = \psi\phi(x_i) = \psi(a_i)$ for all i and the result follows.

The map $a \mapsto -a$ is an automorphism of an abelian group. If f is orientation-reserving then we compose $a \mapsto -a$ with the automorphism ψ in the orientation-preserving case. ■

Lemma 3.2.2. *Let A be a finite cyclic group, and let $(a_1, \dots, a_k)$ be an admissible k-tuple. For all permutations $\sigma \in S_k$, there exists $\psi \in \mathrm{Aut}(A)$ such that $\psi(a_i) = a_{\sigma(i)}$ for all i if and only if one of the following is true:*

- *There is an isomorphism $\delta : A \to \mathbb{Z}/n\mathbb{Z}$ with $k \equiv 0 \mod n$ such that $\delta(a_i) = 1$ for all i.*
- *$k = 2$ and there is an isomorphism $\delta : A \to \mathbb{Z}/n\mathbb{Z}$ for some $n \geq 3$ such that $\delta(a_1) = 1$ and $\delta(a_2) = -1$.*

Proof. For the forward direction, suppose all the a_i are equal. Then each a_i is a generator of A and there is an isomorphism $\delta : A \to \mathbb{Z}/n\mathbb{Z}$ such that $\delta(a_i) = 1$ for all i. The condition that $\sum_{i=1}^{k} a_i = 0$ implies that $k \equiv 0 \mod n$.

Suppose now that the a_i are not all equal. Then they must all be distinct. Indeed, assume to the contrary that there exist three elements a_p, a_q, a_r of the admissible k-tuple such that $a_p = a_q \neq a_r$. Let $\sigma \in S_k$ be the transposition that switches q and r. By assumption, there exists $\psi \in \mathrm{Aut}(A)$ such that $\psi(a_i) = a_{\sigma(i)}$ for all i. Therefore $a_p = \psi(a_p) = \psi(a_q) = a_r$, which is a contradiction.

We may therefore assume the a_i are distinct. Then there is a subgroup of $\mathrm{Aut}(A)$ isomorphic to the symmetric group S_k. Since the automorphism group of a cyclic group is abelian, it must be that $k = 2$. Since $k = 2$, we have that $a_1 = -a_2$ with a_1 a generator of A. Since a_1 and a_2 are distinct, $|A| \geq 3$. Therefore the map $\delta : A \to \mathbb{Z}/n\mathbb{Z}$ with $\delta(a_1) = 1$ and $\delta(a_2) = -1$ is an isomorphism when $n = |A|$.

For the converse, we must write down an appropriate automorphism for each permutation $\sigma \in S_k$. In the case that $\delta(a_i) = 1$ for all i, the identity automorphism suffices for all permutations. In the case where $k = 2$, $\delta(a_1) = 1$ and $\delta(a_2) = -1$, the automorphism $a \mapsto -a$ of A suffices for the nontrivial permutation. ■

Theorem 3.2.3. *Let A be a finite cyclic group, and $(a_1, \ldots, a_k)$ an admissible k-tuple. Let $\Sigma \to \Sigma_0$ be the cyclic branched cover of the sphere with deck group A and k branch points defined by the admissible k-tuple. Every homeomorphism of Σ_0 lifts if and only if one of the following is true:*

- *There is an isomorphism $\delta : A \to \mathbb{Z}/n\mathbb{Z}$ with $k \equiv 0 \mod n$ such that $\delta(a_i) = 1$ for all i.*
- *$k = 2$ and there is an isomorphism $\delta : A \to \mathbb{Z}/n\mathbb{Z}$ for some $n \geq 3$ such that $\delta(a_1) = 1$ and $\delta(a_2) = -1$.*

Proof. Any permutation of the branch points can be induced by a homeomorphism of the sphere. Therefore, the result follows by combining Lemmas 3.2.1 and 3.2.2. ■

We now apply Theorem 3.2.3 to mapping class groups using the Birman-Hilden theorem. Let $p : \Sigma \to \Sigma_0$ be an n-sheeted cyclic branched cover of a sphere branched at k points $\mathcal{B} \subset \Sigma_0$. If the genus of Σ is at least 2, then $\mathrm{SMod}_p(\Sigma)/D \cong \mathrm{LMod}_p(\Sigma_0, \mathcal{B})$ (see Theorem 2.3.2). Recall that $\mathrm{SMod}_p(\Sigma)$ is the normalizer of the deck group $D \cong \mathbb{Z}/n\mathbb{Z}$ in $\mathrm{Mod}(\Sigma)$ and $\mathrm{LMod}_p(\Sigma_0, \mathcal{B}) < \mathrm{Mod}(\Sigma_0, \mathcal{B})$ is the liftable mapping class group.

There is an error in the statement of Theorem 5 in [9] (see the erratum [10]). Theorem 5 in [9] relies on a lemma that incorrectly claims that every cyclic branched cover over the sphere has the property that every homeomorphism lifts. However, as we have seen in

Theorem 3.2.3, this is not the case. The next theorem provides a correction to Theorem 5 in [9].

Theorem 3.2.4. *Let A be a finite cyclic group, and $(a_1, \ldots, a_k)$ an admissible k-tuple. Let $p : \Sigma \to \Sigma_0$ be the cyclic branched cover of the sphere with deck group A and k branch points defined by the admissible k-tuple. The quotient $\mathrm{SMod}_p(\Sigma)/A$ is isomorphic to $\mathrm{Mod}(\Sigma_0, \mathcal{B})$ if there is an isomorphism $\delta : A \to \mathbb{Z}/n\mathbb{Z}$ with $k \equiv 0 \mod n$ such that $\delta(a_i) = 1$ for all i, and (n, k) is not equal to $(2, 2)$, $(2, 4)$, or $(3, 3)$.*

Proof. If a homeomorphism lifts, then so do all isotopic homeomorphisms (see the comment at the end of Section 2.3.4). Therefore, we have that $\mathrm{LMod}_p(\Sigma_0, \mathcal{B}) = \mathrm{Mod}(\Sigma_0, \mathcal{B})$ if and only if $(a_1, \ldots, a_k)$ satisfies one of the two conditions in the statement of Theorem 3.2.3. To satisfy the conditions of the Birman-Hilden theorem (Theorem 2.3.2), it suffices to determine when the genus of the covering surface is at least 2. Let I_{a_i} be the index of $\langle a_i \rangle$ in A. In the cases we are considering, $I_{a_i} = 1$ for all i so each branch point has exactly one preimage under the covering map $p : \Sigma \to \Sigma_0$ (see Section 2.2.2). Let g be the genus of Σ. The Riemann-Hurwitz formula implies $g = \frac{1}{2}(k-2)(n-1)$.

If $k = 2$, $g = 0$ ruling out the case when $k = 2$. If $n = 2$, $g \geq 2$ if and only if $k \geq 6$, ruling out the covers corresponding to $n = 2$ and $k = 2$ or 4. If $n = 3$, $g \geq 2$ if and only if $k \geq 4$ ruling out the cover corresponding to $n = 3$ and $k = 3$. If $n \geq 4$ and $k \equiv 0 \mod n$, then $g \geq 2$ completing the proof. ■

We now rephrase Theorem 3.2.3 in the language of superelliptic curves.

Corollary 3.2.5. *Let $\widetilde{C} \to \mathbb{P}^1$ be a cyclic branched cover defined by an irreducible superelliptic curve as in equation (3.1). Then every homeomorphism of $\mathbb{P}^1$ lifts if and only if one of the following is true.*

- *$a_1 = \cdots = a_t$ and $t \equiv 0$ or $-1 \mod n$,*
- *$n \geq 3$ and $t = 1$, or*
- *$n \geq 3$, $t = 2$ and $a_1 \equiv -a_2 \mod n$.*

Chapter 4

Balanced Superelliptic Covers

The balanced superelliptic covers are a family of cyclic branched covers over the sphere. They are indexed by pairs of integers $g, k \geq 2$ such that $k-1$ divides g. Let the liftable mapping class group corresponding to the balanced superelliptic cover indexed by the pair of integers g, k be denoted by $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$. The covers are denoted $p_{g,k} : \Sigma_g \to \Sigma_0$ and will be formally constructed below.

We study this family of covers for a variety of reasons. First, when $k = 2$, the balanced superelliptic covers coincide with the hyperelliptic covers defined in Chapter 1. In particular, $\mathrm{LMod}_{g,2}(\Sigma_0, \mathcal{B}) = \mathrm{Mod}(\Sigma_0, \mathcal{B})$ (see Corollary 4.3.3 below for a new proof). As a result, the balanced superelliptic covers provide a natural generalization of the hyperelliptic covers. Second, when $k > 2$, $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$ is a proper subgroup of $\mathrm{Mod}(\Sigma_0, \mathcal{B})$. Therefore the covers $p_{g,k}$ for $k > 2$ are a family of counterexamples to Lemma 5.1 of [9] (see the erratum [10]). A correction is provided in Theorem 3.2.4. Lastly, the covers can be embedded in $\mathbb{R}^3$ so that the deck group is generated by a rotation about the z-axis. This viewpoint should provide insight into the balanced superelliptic mapping class groups $\mathrm{SMod}_{g,k}(\Sigma_g)$.

4.1 Constructing the covers

Choose a pair of integers $g, k \geq 2$ such that $k-1$ divides g, and let $n = g/(k-1)$. Embed Σ_g in $\mathbb{R}^3$ so it is invariant under a rotation by $2\pi/k$ about the z-axis as we describe below.

The intersection of Σ_g with the plane $z = a$ is:

- Empty for $a < 0$ and $a > 2n + 1$
- A point at the origin for $a = 0$ and $a = 2n + 1$

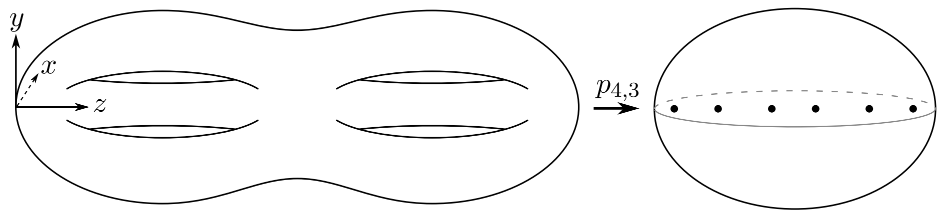


Figure 4.1: The embedding of Σ_4 in $\mathbb{R}^3$ when $k = 3$ and the balanced superelliptic cover $p_{4,3} : \Sigma_4 \to \Sigma_0$ branched at 6 points.

- Homeomorphic to a circle for $2m < a < 2m + 1$ with $m \in \{0, \ldots, n\}$
- A rose with k petals for $a \in \{1, \ldots, 2n\}$
- k disjoint simple closed curves invariant under a rotation of $2\pi/k$ about the z-axis for $2m - 1 < a < 2m$ with $m \in \{1, \ldots, n\}$. In the special case $a = 2m - \frac{1}{2}$, endow the plane $z = a$ with polar coordinates (r, θ). Then we have k disjoint circles with centers on the rays $\theta = 2\pi d/k$, $d \in \{0, \ldots, k - 1\}$.

See Figure 4.1 for the embedding when $g = 4$ and $k = 3$.

Consider a homeomorphism $\zeta : \Sigma_g \to \Sigma_g$ of order k given by rotation about the z-axis by $2\pi/k$. The homeomorphism ζ fixes $2n + 2$ points, which are the points of intersection of Σ_g with the z-axis. Define an equivalence relation on Σ_g given by $x \sim y$ if and only if $\zeta^q(x) = y$ for some q. The resulting surface $\Sigma_g/\sim$ is homeomorphic to a closed sphere Σ_0. The quotient map $p_{g,k} : \Sigma_g \to \Sigma_0$ is a k-sheeted cyclic branched covering map with $2n + 2$ branch points $\mathcal{B} \subset \Sigma_0$. The branch points are the images of the points fixed by ζ. The deck group of $p_{g,k}$ is a cyclic group of order k generated by ζ. When $k = 2$, ζ is a hyperelliptic involution.

4.2 The admissible tuples

Let $p_{g,k} : \Sigma_g \to \Sigma_0$ be a balanced superelliptic covering space. The branch points are the images under $p_{g,k}$ of the intersection of Σ_g with the z-axis. Let $\mathcal{B} = \{q_1, \ldots, q_{2n+2}\}$ be the branch points where $q_i = p_{g,k}((0, 0, i - 1))$. That is, enumerate the branch points in increasing order of the z-coordinates of their preimages under $p_{g,k}$. Fix this enumeration for the rest of the chapter.

The deck group of $p_{g,k} : \Sigma_g \to \Sigma_0$ is isomorphic to $\mathbb{Z}/k\mathbb{Z}$. Therefore each balanced superelliptic cover corresponds to an admissble $(2n + 2)$-tuple with entries in $\mathbb{Z}/k\mathbb{Z}$. The aim of this section is to prove that the balanced superelliptic covers correspond to the admissible $(2n+2)$-tuples $(1, -1, 1, -1, \dots, 1, -1)$.

4.2.1 Lifting curves

For any covering space $p : \widetilde{X} \to X$, we say a curve $c : S^1 \to X$ *lifts* if there exists a curve $\tilde{c} : S^1 \to \widetilde{X}$ such that $p\tilde{c} = c$. If $p : \widetilde{X} \to X$ is a regular covering space with abelian deck group A, then p is determined by the kernel of a surjective homomorphism $\varphi : H_1(X; \mathbb{Z}) \to A$. With this setup we have the following result, which is an application of the lifting criterion [27, Proposition 1.33].

Lemma 4.2.1. *A curve $c : S^1 \to X$ lifts if and only if $[c] \in \ker \varphi < H_1(X; \mathbb{Z})$.*

Recall from Section 3.1 that an admissible k-tuple determines a branched cover over a sphere by defining a surjective homomorphism $\varphi : H_1(\Sigma_0^\circ; \mathbb{Z}) \to A$ where A is the deck group.

Let $p_{g,k} : \Sigma_g \to \Sigma_0$ be a balanced superelliptic cover. Let $\varphi : H_1(\Sigma_0^\circ; \mathbb{Z}) \to \mathbb{Z}/k\mathbb{Z}$ be the surjective homomorphism defined by the admissible $(2n+2)$-tuple $(1, -1, 1, -1, \dots, 1, -1)$. In order to show $p_{g,k}$ corresponds to the admissible $(2n+2)$-tuple $(1, -1, 1, -1, \dots, 1, -1)$, it suffices to show a curve $\gamma : S^1 \to \Sigma_0^\circ$ lifts if and only if $[\gamma] \in \ker \varphi$ (see Section 3.1).

With this in mind, we now focus our attention on characterizing when a curve lifts in a balanced superelliptic covering space.

An important collection of arcs. Fix a pair of integers $g, k \geq 2$ such that $k - 1 \mid g$, and consider the surface Σ_g embedded in $\mathbb{R}^3$ as described above. Using cylindrical coordinates in $\mathbb{R}^3$, let $P_{\theta_0} = \{(r, \theta_0, z) \in \mathbb{R}^3 : r \geq 0\}$ be a closed half plane. The intersection of Σ_g and $P_{\pi/k}$ is a collection of $n+1$ arcs where $n = g/(k-1)$. Call these arcs $\beta_1, \dots, \beta_{n+1}$. Orient each β_i so that $\beta_i(0) = (0, 0, 2i - 2)$ and $\beta_i(1) = (0, 0, 2i - 1)$.

Consider the balanced superelliptic covering map $p_{g,k}$ as defined above. For each i with $1 \leq i \leq n+1$, let $\alpha_i = p_{g,k}\beta_i$. Notice that $\alpha_i(0) = q_{2i-1}$ and $\alpha_i(1) = q_{2i}$ for all i, where $q_i \in \Sigma_0$ is the ith branch point. Let α be the union of the arcs α_i. Let $[\alpha]$ denote the relative homology class of α in $H_1(\Sigma_0, \mathcal{B}; \mathbb{Z})$. Figure 4.2 shows the embeddings of the arcs $\beta_1, \beta_2, \beta_3 \in \Sigma_4$ and $\alpha_1, \alpha_2, \alpha_3 \in \Sigma_0$ for the 3-sheeted balanced superelliptic cover of Σ_4 over Σ_0.

Intersection data with α will characterize when a curve lifts.

An intersection form for punctured surfaces. The next lemma is well known and and a proof sketch is included for completeness. In Lemma 4.2.2, we abuse notation and identify curves in $\Sigma_{0,m}$ with their image in Σ_0 under the inclusion $\Sigma_{0,m} \hookrightarrow \Sigma_0$.

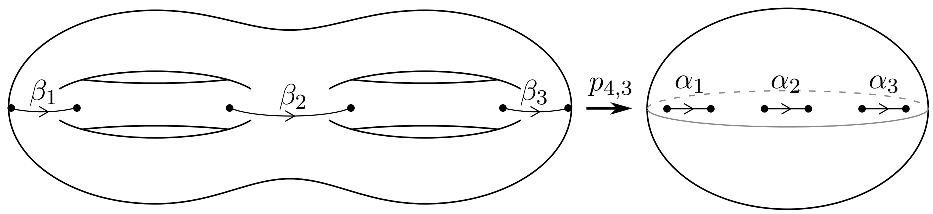


Figure 4.2: The arcs $\beta_1, \beta_2, \beta_3 \in \Sigma_4$ and the arcs $\alpha_1, \alpha_2, \alpha_3 \in \Sigma_0$.

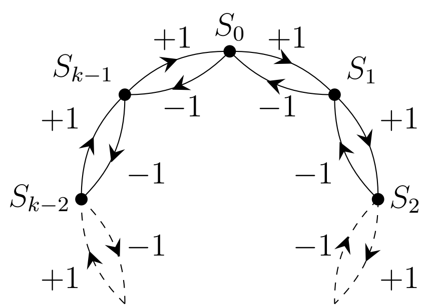
Lemma 4.2.2. *Let Σ_g be a closed surface and $\mathcal{B} \subset \Sigma_g$ a finite set of points. Let $\Sigma_g^\circ = \Sigma_g \backslash \mathcal{B}$. There exists a homomorphism*

$$\bullet : H_1(\Sigma_g^\circ; \mathbb{Z}) \otimes H_1(\Sigma_g, \mathcal{B}; \mathbb{Z}) \to \mathbb{Z}$$

given by $c \bullet b = \hat{i}(\gamma, \beta)$. Here γ is a union of curves on Σ_g° such that $[\gamma] = c$, β is a union of curves and arcs on Σ_g such that $[\beta] = b$ and γ and β are in general position. The integer $\hat{i}(\gamma, \beta)$ is the algebraic intersection number of γ and β.

Proof sketch. Let Σ_g' be the genus g surface with $m = |\mathcal{B}|$ boundary components obtained by deleting m open disks from Σ_g, each containing a single marked point. Let $B_1, \ldots, B_m$ be the boundary components. Define the equivalence relation $\sim$ on Σ_g' by $x \sim y$ if and only if $x, y \in B_i$ for some i. Let $\rho : \Sigma_g' \to \Sigma_g' / \sim$ be the quotient map. Note that $\Sigma_g' / \sim$ is homeomorphic to Σ_g, and $\rho(B_i)$ is a single point for all i. Let $\imath : \Sigma_g' \hookrightarrow \Sigma_g^\circ$ be the inclusion map. The induced maps on homology $\rho_* : H_1(\Sigma_g', \partial\Sigma_g'; \mathbb{Z}) \to H_1(\Sigma_g, \mathcal{B}; \mathbb{Z})$ and $\imath_* : H_1(\Sigma_g'; \mathbb{Z}) \to H_1(\Sigma_g^\circ; \mathbb{Z})$ are both isomorphisms. Composing the isomorphisms ρ_* and $\imath_*$ with the standard homology intersection product $\bullet : H_1(\Sigma_g'; \mathbb{Z}) \otimes H_1(\Sigma_g', \partial\Sigma_g'; \mathbb{Z}) \to \mathbb{Z}$ (see [12, §VI.11]) completes the proof. ∎

Lemma 4.2.3. *Let G be the weighted digraph*



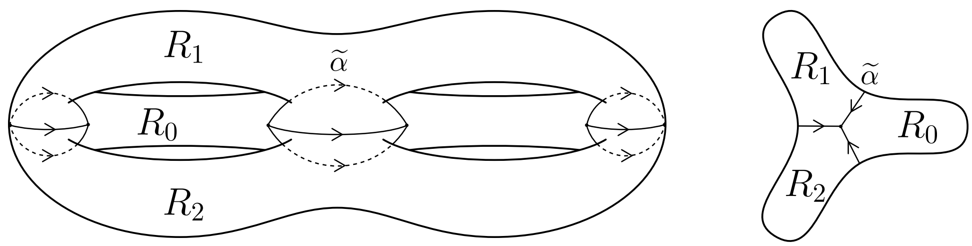


Figure 4.3: The subsurfaces R_0, R_1, and R_2 corresponding to $g = 4, k = 3$. The 9 arcs on the left image make up $\widetilde{\alpha} = p_{3,4}^{-1}(\alpha)$. The left image is from the negative x-axis in $\mathbb{R}^3$, and the right is from the positive z-axis.

and let Γ be a finite walk on G beginning at S_0. Define the weight of Γ, which we denote $w(\Gamma)$, as the sum of the weights of the edges traversed in the walk. Then Γ terminates at S_i if and only if $w(\Gamma) \equiv i \mod k$.

Proof. Induct on the length of the walk. ■

In order to apply Lemma 4.2.3, we use the union of arcs α in Σ_0 defined above and their preimages $p_{g,k}^{-1}(\alpha) \subset \Sigma_g$.

The full preimage $\widetilde{\alpha} = p_{g,k}^{-1}(\alpha)$ is a collection of $k(n+1)$ oriented arcs in Σ_g. The union of arcs $\widetilde{\alpha}$ consists of the orbits of β_i under the action of the deck group.

The surface $\Sigma_g \setminus \{\widetilde{\alpha}\}$ is a union of k subsurfaces of Σ_g. The subsurfaces are cyclically permuted by the action of the deck group $D \cong \mathbb{Z}/k\mathbb{Z}$. Label the subsurfaces as follows. Choose one subsurface and label it R_0. Then every subsurface is of the form $d(R_0)$ for some $d \in D$. Label $d(R_0)$ by R_d. See Figure 4.3 for a picture of $\widetilde{\alpha}$ and the subsurfaces R_0, R_1, and R_2 in the case $g = 4$ and $k = 3$.

Lemma 4.2.4. *Let $p_{g,k}^{\circ} : \Sigma_g^{\circ} \to \Sigma_0^{\circ}$ be the associated unbranched cover of the balanced superelliptic cover $p_{g,k}$. If a curve γ in Σ_0° lifts, then $[\gamma] \bullet [\alpha] \equiv 0 \mod k$.*

Proof. Identify γ with its image under the inclusion $\Sigma_0^{\circ} \hookrightarrow \Sigma_0$. By adjusting γ by isotopy, we may assume γ and α intersect transversally at finitely many points. Parametrize γ by $\gamma : [0,1]/\{0,1\} \to \Sigma_0^{\circ}$. Then there are real numbers $0 < t_1 < \cdots < t_r < 1$ such that $\gamma \cap \alpha = \{\gamma(t_1), \ldots, \gamma(t_r)\}$. Choose a lift $\tilde{\gamma}$ of γ such that $\tilde{\gamma}(0) \in R_0$. Then $\tilde{\gamma} \cap \widetilde{\alpha} = \{\tilde{\gamma}(t_1), \ldots, \tilde{\gamma}(t_r)\}$. As with γ, we are identifying $\tilde{\gamma}$ with its image under the inclusion $\Sigma_g^{\circ} \hookrightarrow \Sigma_g$.

Choose $\varepsilon > 0$ such that $\tilde{\gamma}([t_i - \varepsilon, t_i + \varepsilon]) \cap \alpha = \{\tilde{\gamma}(t_i)\}$ for all i. Suppose $\tilde{\gamma}(t_i - \varepsilon) \in R_j$. Then by the way $\widetilde{\alpha}$ is constructed, $\tilde{\gamma}(t_i + \varepsilon) \in R_{j\pm1}$. The point $\tilde{\gamma}(t_i)$ contributes $+1$ or

-1 to the algebraic intersection number $\hat{i}(\tilde{\gamma}, \widetilde{\alpha})$. By relabelling R_l by R_{-l} for all $l \in \mathbb{Z}/k\mathbb{Z}$ if necessary, we may assume that if $\tilde{\gamma}(t_i + \varepsilon) \in R_{j+1}$, then the intersection point $\tilde{\gamma}(t)$ contributes $+1$ to $\hat{i}(\tilde{\gamma}, \widetilde{\alpha})$. It follows that if $\tilde{\gamma}(t_i + \varepsilon) \in R_{j-1}$, then $\tilde{\gamma}(t)$ contributes -1 to $\hat{i}(\tilde{\gamma}, \widetilde{\alpha})$.

Without loss of generality, assume $p^\circ_{g,k}$ is locally orientation preserving. If $\tilde{\gamma}(t_i)$ contributes ± 1 to $\hat{i}(\tilde{\gamma}, \widetilde{\alpha})$, then $p^\circ_{g,k}\tilde{\gamma}(t_i) = \gamma(t_i)$ contributes ± 1 to $\hat{i}(p^\circ_{g,k}\tilde{\gamma}, p_{g,k}\widetilde{\alpha}) = \hat{i}(\gamma, \alpha)$. Therefore $\hat{i}(\tilde{\gamma}, \widetilde{\alpha}) = \hat{i}(\gamma, \alpha)$.

We now construct a walk $\Gamma_{\tilde{\gamma}}$ on the weighted digraph G from Lemma 4.2.3 associated to $\tilde{\gamma}$. Let S_0 be the first vertex of $\Gamma_{\tilde{\gamma}}$. For each intersection point $\tilde{\gamma}(t_i) \in \tilde{\gamma} \cap \widetilde{\alpha}$, add another vertex as follows. Suppose S_m is the ith vertex. If $\tilde{\gamma}(t_i)$ contributes ± 1 to $\hat{i}(\tilde{\gamma}, \widetilde{\alpha})$, then set $S_{m\pm 1}$ to be the $i+1$st vertex in $\Gamma_{\tilde{\gamma}}$. By the weighting of the edges in G, if $\tilde{\gamma}(t_i)$ contributes ± 1 to $\hat{i}(\tilde{\gamma}, \widetilde{\alpha})$, then moving from the ith vertex to the $i+1$st vertex in $\Gamma_{\tilde{\gamma}}$ contributes ± 1 to the weight $w(\Gamma_{\tilde{\gamma}})$. Therefore $w(\Gamma_{\tilde{\gamma}}) = \hat{i}(\tilde{\gamma}, \widetilde{\alpha})$.

Building $\Gamma_{\tilde{\gamma}}$ in this fashion gives a walk with $r+1$ vertices. Furthermore, if $t_i < s < t_{i+1}$ and $\tilde{\gamma}(s) \in R_j$, the $i+1$st vertex is S_j. Since $\tilde{\gamma}(0) = \tilde{\gamma}(1) \in R_0$, $\Gamma_{\tilde{\gamma}}$ is a walk that starts and ends at the vertex S_0. By Lemma 4.2.3, $w(\Gamma_{\tilde{\gamma}}) \equiv 0 \mod k$. The proof is completed by observing $w(\Gamma_{\tilde{\gamma}}) = \hat{i}(\tilde{\gamma}, \widetilde{\alpha}) = \hat{i}(\gamma, \alpha)$. ∎

We are now ready to prove Lemma 4.2.5, which is Lemma 4.2.4 and its converse.

Lemma 4.2.5 (A lifting criterion for curves)**.** *Let $p^\circ_{g,k} : \Sigma^\circ_g \to \Sigma^\circ_0$ be the unbranched balanced superelliptic covering space. Let γ be a curve on Σ°_0. Then γ lifts if and only if $[\gamma] \bullet [\alpha] \equiv 0 \mod k$.*

Proof. Let $- \bullet [\alpha] : H_1(\Sigma^\circ_0; \mathbb{Z}) \to \mathbb{Z}$ be the homomorphism from Lemma 4.2.2, and let $\pi : \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$ be the natural projection map. Let $\phi = \pi \circ (- \bullet [\alpha]) : H_1(\Sigma^\circ_0; \mathbb{Z}) \to \mathbb{Z}/k\mathbb{Z}$. The homomorphism ϕ is surjective since there is a curve γ such that $\hat{i}(\gamma, \alpha) = 1$.

Let $K = \{[\gamma] \in H_1(\Sigma^\circ_0; \mathbb{Z}) : \gamma \text{ lifts}\}$. Since $\overline{p}_{g,k}$ is a k-sheeted cover, the index of K in $H_1(\Sigma^\circ_0; \mathbb{Z})$ is k. By Lemma 4.2.4, $K < \ker \phi$. However, both K and $\ker \phi$ are index k in $H_1(\Sigma^\circ_0; \mathbb{Z})$ so they are equal.

Unwrapping definitions we have γ lifts if and only if $[\gamma] \in K$ and $[\gamma] \in \ker \phi$ if and only if $[\gamma] \bullet [\alpha] \equiv 0 \mod k$, completing the proof. ∎

Proposition 4.2.6. *The balanced superelliptic cover $p_{g,k}$ corresponds to the admissible $(2n+2)$-tuple $(1, -1, 1, -1, \ldots, 1, -1)$ with entries in $\mathbb{Z}/k\mathbb{Z}$.*

Proof. Let x_i be homology class of a loop surrounding only the ith puncture counterclockwise. Recall the surjective homomorphism $\varphi : H_1(\Sigma^\circ_0; \mathbb{Z}) \to \mathbb{Z}/k\mathbb{Z}$ defined by the admissible tuple $(1,-1,1,-1,\ldots,1,-1)$ is given by

$$\varphi(x_i) = \begin{cases} 1 & \text{if } i \text{ is odd,} \\ -1 & \text{if } i \text{ is even.} \end{cases}$$

It suffices to show a curve γ lifts if and only if $[\gamma] \in \ker \varphi$. As in the proof of Lemma 4.2.5, let $\phi = \pi \circ (- \bullet [\alpha]) : H_1(\Sigma_0^\circ; \mathbb{Z}) \to \mathbb{Z}/k\mathbb{Z}$ where $\pi : \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$ is the natural projection map and $\bullet$ is the intersection product from Lemma 4.2.2.

Since x_i is the homology class of a loop surrounding only the ith puncture counterclockwise we have

$$\phi(x_i) = \begin{cases} 1 & \text{if } i \text{ is odd,} \\ -1 & \text{if } i \text{ is even.} \end{cases}$$

Therefore $\varphi = \phi$. By Lemma 4.2.5, γ lifts if and only if $[\gamma] \in \ker \phi$, completing the proof. ∎

4.3 Lifting homeomorphisms

Identifying an admissible tuple for each balanced superelliptic cover now allows us to use Lemma 3.2.1 to characterize when a homeomorphism of Σ_0° lifts.

Parity of a permutation. Fix an integer $m \geq 2$. Let τ be a permutation in S_m. We say that τ *preserves parity* if $\tau(q) = q \mod 2$ for all $q \in \{1, \cdots, m\}$. We say that τ *reverses parity* if $\tau(q) \neq q \mod 2$ for all $q \in \{1, \cdots, m\}$.

Let S_{2l} be the symmetric group on the set $\{1, \ldots, 2l\}$. Let $W_{2l} < S_{2l}$ be the subgroup consisting of permutations that either preserve parity, or reverse parity. Then

$$W_{2l} \cong (S_l \times S_l) \rtimes \mathbb{Z}/2\mathbb{Z}$$

where $\mathbb{Z}/2\mathbb{Z}$ acts on $S_l \times S_l$ by switching the coordinates.

Lemma 4.3.1. *Let $p_{g,k} : \Sigma_g \to \Sigma_0$ be a balanced superelliptic cover. Let $f \in \mathrm{Homeo}^+(\Sigma_0, \mathcal{B})$ and let $\sigma \in S_{2n+2}$ be the permutation induced by f on the branch points. If $k = 2$, f always lifts. If $k > 2$, f lifts if and only if $\sigma \in W_{2n+2}$.*

Proof. By Proposition 4.2.6, the admissible tuple corresponding to $p_{g,k}$ is the $2n+2$-tuple $(a_1, \ldots, a_{2n+2})$ with entries in $\mathbb{Z}/k\mathbb{Z}$ where

$$a_i = \begin{cases} 1 & \text{if } i \text{ is odd} \\ -1 & \text{if } i \text{ is even.} \end{cases}$$

By Lemma 3.2.1, f lifts if and only if there is an automorphism $\psi \in \operatorname{Aut}(\mathbb{Z}/k\mathbb{Z})$ such that $\psi(a_i) = a_{\sigma(i)}$ for all i.

For $k = 2$, the admissible tuple is $(1, 1, \ldots, 1)$. For any $\sigma \in S_{2n+2}$, $\mathrm{id}_{\mathbb{Z}/k\mathbb{Z}}(a_i) = a_{\sigma(i)}$ for all i so every homeomorphism lifts.

Suppose $k > 2$. If σ is parity preserving then $\mathrm{id}_{\mathbb{Z}/k\mathbb{Z}}(a_i) = a_{\sigma(i)}$ for all i. If σ is parity reversing, let ψ be the automorphism defined by $\psi(g) = -g$. Then $\psi(a_i) = a_{\sigma(i)}$ for all i. Therefore if $\sigma \in W_{2n+2}$, f lifts. If $\sigma \notin W_{2n+2}$, there are odd indices i and j such that $a_{\sigma(i)} = 1$ and $a_{\sigma(j)} = -1$. Since there is no automorphism $\varphi \in \operatorname{Aut}(\mathbb{Z}/k\mathbb{Z})$ such that $\varphi(1) = 1$ and $\varphi(1) = -1$, f does not lift. ■

Proposition 4.3.2. *Let $k > 2$ and let $p_{g,k} : \Sigma_g \to \Sigma_0$ be a balanced superelliptic covering map. There is a short exact sequence*

$$1 \longrightarrow \operatorname{PMod}(\Sigma_0, \mathcal{B}) \longrightarrow \operatorname{LMod}_{g,k}(\Sigma_0, \mathcal{B}) \xrightarrow{\Psi} W_{2n+2} \to 1$$

where Ψ is given by the action of $\operatorname{LMod}_{g,k}(\Sigma_0, \mathcal{B})$ on the branch points.

Proof. By Lemma 4.3.1 there is a surjective homomorphism $\Psi : \operatorname{LMod}_{g,k}(\Sigma_0, \mathcal{B}) \to W_{2n+2}$. The kernel is exactly the mapping classes with representatives that act trivially on the branch points, which is the pure mapping class group $\operatorname{PMod}(\Sigma_0, \mathcal{B})$. ■

Proposition 4.3.2 gives us the following result. The case $k = 2$ has already been proven by Birman and Hilden [7] using different methods.

Corollary 4.3.3. *For $k = 2$, $\operatorname{LMod}_{g,k}(\Sigma_0, \mathcal{B}) = \operatorname{Mod}(\Sigma_0, \mathcal{B})$. For $k > 2$, the index $[\operatorname{Mod}(\Sigma_0, \mathcal{B}) : \operatorname{LMod}_{g,k}(\Sigma_0, \mathcal{B})]$ is $\frac{(2n+2)!}{2((n+1)!)^2}$.*

Proof. If $k = 2$, the result follows from Lemma 4.3.1. For $k > 2$,

$$\begin{aligned}
&[\operatorname{Mod}(\Sigma_0, \mathcal{B}) : \operatorname{LMod}_{g,k}(\Sigma_0, \mathcal{B})] \\
&\quad = [\operatorname{Mod}(\Sigma_0, \mathcal{B}) / \operatorname{PMod}(\Sigma_0, \mathcal{B}) : \operatorname{LMod}_{g,k}(\Sigma_0, \mathcal{B}) / \operatorname{PMod}(\Sigma_0, \mathcal{B})] \\
&\quad = [S_{2n+2} : W_{2n+2}] .
\end{aligned}$$

Observing that $|W_{2n+2}| = 2((n+1)!)^2$ completes the proof. ■

Proposition 4.3.2 reveals a somewhat surprising phenomenon regarding the liftable mapping class groups. Fix an integer $n \geq 1$. Then for each integer $k \geq 2$, there is a balanced superelliptic cover $p_{g,k} : \Sigma_g \to \Sigma_0$ branched at $2n+2$ points, where $g = n(k-1)$. Therefore there is a family of subgroups $\{\operatorname{LMod}_{n(k-1),k}(\Sigma_0, \mathcal{B}(2n+2)) : k \geq 2\}$ of $\operatorname{Mod}(\Sigma_0, \mathcal{B}(2n+2))$. These subgroups exhibit the following stability property.

Corollary 4.3.4. *Fix an integer $n \geq 1$. Then*

$$\mathrm{LMod}_{n(k_1-1),k_1}(\Sigma_0, \mathcal{B}(2n+2)) = \mathrm{LMod}_{n(k_2-1),k_2}(\Sigma_0, \mathcal{B}(2n+2))$$

for any integers $k_1, k_2 > 2$.

By Lemma 4.3.1, $\mathrm{LMod}_{n,2}(\Sigma_0, \mathcal{B}(2n+2)) = \mathrm{Mod}(\Sigma_0, \mathcal{B}(2n+2))$.

4.4 Presentations of $\mathrm{PMod}(\Sigma_0, \mathcal{B}(m))$ and W_{2n+2}

For the remainder of this chapter, all operations in mapping class groups will be performed left to right. That is, the mapping class $[f][g]$ is the isotopy class of the homeomorphism obtained by performing f, then g.

As in Section 4.3.2, $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B}(2n+2))$ can be written as a group extension of W_{2n+2} by the pure mapping class group $\mathrm{PMod}(\Sigma_0, \mathcal{B}(2n+2))$. A presentation of the pure mapping class group $\mathrm{PMod}(\Sigma_0, \mathcal{B}(2n+2))$ is found in Lemma 4.4.1. A presentation of W_{2n+2} is found in Lemma 4.4.2.

4.4.1 A presentation of $\mathrm{PMod}(\Sigma_0, \mathcal{B}(m))$

Let D be a disk with m marked points $\mathcal{B}(m) \subset D \setminus \partial D$. Then $\mathrm{Mod}(D, \partial D, \mathcal{B}(m)) \cong B_m$ and $\mathrm{PMod}(D, \partial D, \mathcal{B}(m)) \cong PB_m$ where B_m and PB_m are the braid group and pure braid group on m strands respectively. For a survey of the braid group see [6].

Number the marked points from 1 to m and let $\{\sigma_1, \ldots, \sigma_{m-1}\}$ be the standard braid generators. The arc about which σ_i is a half twist is shown in Figure 4.4. The pure braid group, denoted PB_m is generated by elements $A_{i,j}$ with $1 \leq i < j \leq m$ of the form:

$$A_{i,j} = (\sigma_{j-1} \cdots \sigma_{i+1})\sigma_i^2(\sigma_{j-1} \cdots \sigma_{i+1})^{-1}.$$

The curve about which $A_{i,j}$ is a Dehn twist is shown in Figure 4.4.

Lemma 4.4.1. *The group $\mathrm{PMod}(\Sigma_0, \mathcal{B}(m))$ is generated by $A_{i,j}$ for $1 \leq i < j \leq m-1$ and has relations:*

1. $[A_{p,q}, A_{r,s}] = 1$ *where* $p < q < r < s$
2. $[A_{p,s}, A_{q,r}] = 1$ *where* $p < q < r < s$
3. $A_{p,r}A_{q,r}A_{p,q} = A_{q,r}A_{p,q}A_{p,r} = A_{p,q}A_{p,r}A_{q,r}$ *where* $p < q < r$
4. $\left[A_{r,s}A_{p,r}A_{r,s}^{-1}, A_{q,s}\right] = 1$ *where* $p < q < r < s$

5. $(A_{1,2}A_{1,3}\cdots A_{1,m-1})\cdots(A_{m-3,n-2}A_{m-3,n-1})(A_{m-2,m-1})=1$

Proof. Let D be a disk with $m-1$ marked points. Capping the boundary of D we get a sphere Σ_0 with m marked points. By Theorem 2.3.1 we have the short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow PB_{m-1} \xrightarrow{Cap} \mathrm{PMod}(\Sigma_0, \mathcal{B}(m)) \longrightarrow 1. \tag{4.1}$$

Here $\mathbb{Z}$ is generated by the Dehn twist about a curve homotopic to the boundary of D_{m-1}, which we will denote T_β. From [22, p. 250] we have

$$T_\beta = (A_{1,2}A_{1,3}\cdots A_{1,m})\cdots(A_{m-3,m-2}A_{m-3,m-1})(A_{m-2,m-1}).$$

Using the presentation for PB_m in Margalit–McCammond [37, Theorem 2.3] and Lemma 2.4.1, we obtain the desired presentation. ■

4.4.2 A presentation of W_{2n+2}

As in Section 4.3, W_{2n+2} is the subgroup of the symmetric group S_{2n+2} given by all permutations of $\{1,\ldots,2n+2\}$ that either preserve or reverse parity.

The symmetric group S_m admits the presentation:

$$S_m = \left\langle \tau_1,\ldots,\tau_{m-1} \mid \begin{cases} \tau_i^2 = 1 & \text{for all } i \in \{1,\ldots,m-1\} \\ \tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1} & \text{for all } i \in \{1,\ldots,m-2\} \\ [\tau_i,\tau_j] = 1 & \text{for } |i-j|>1 \end{cases} \right\rangle \tag{4.2}$$

where τ_i is the transposition $(i\ \ i+1)$.

Lemma 4.4.2. *Let S_{2n+2} be the symmetric group on $\{1,\ldots,2n+2\}$. Let $x_i = (2i-1\ \ 2i+1)$, $y_i = (2i\ \ 2i+2)$, and $z = (1\ \ 2)(3\ \ 4)\cdots(2n+1\ \ 2n+2)$. Then W_{2n+2} admits a presentation with generators $\{x_1,\ldots,x_n,y_1,\ldots,y_n,z\}$ and relations*

1. $[x_i,y_j]=1$ *for all* $i,j\in\{1,\ldots,n\}$,
2. $x_i^2=1$ *and* $y_i^2=1$ *for all* $i\in\{1,\ldots,n\}$,
3. $x_ix_{i+1}x_i = x_{i+1}x_ix_{i+1}$ *and* $y_iy_{i+1}y_i = y_{i+1}y_iy_{i+1}$ *for all* $i\in\{1,\ldots,n-1\}$,
4. $[x_i,x_j]=1$ *and* $[y_i,y_j]=1$ *for all* $|i-j|\geq 2$,
5. $z^2=1$, *and*
6. $zx_iz^{-1}=y_i$ *for all* $i\in\{1,\ldots,n\}$.

Proof. We have the short exact sequence

$$1 \longrightarrow S_{n+1} \times S_{n+1} \xrightarrow{\alpha} W_{2n+2} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.$$

The homomorphism α maps the first coordinate in $S_{n+1} \times S_{n+1}$ to permutations of the subset $\{1, 3, \ldots, 2n+1\}$ and the second coordinate to permutations of $\{2, 4, \ldots, 2n+2\}$. The map π is given by $\pi(\sigma) = 0$ if σ is parity preserving and $\pi(\sigma) = 1$ if σ is parity reversing.

The desired presentation is obtained by using the presentation (4.2) for S_{n+1} and Lemma 2.4.2. ∎

4.5 Presentation of $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$

In this section we compute a presentation for $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$, which is given in Theorem 4.5.7. Throughout this section, let $p_{g,k} : \Sigma_g \to \Sigma_0$ be the balanced superelliptic cover. Let $\mathcal{B} \subset \Sigma_0$ be the set of $2n+2$ branch points where $n = g/(k-1)$.

We apply Lemma 2.4.2 to the short exact sequence

$$1 \longrightarrow \mathrm{PMod}(\Sigma_0, \mathcal{B}) \xrightarrow{\iota} \mathrm{LMod}(\Sigma_0, \mathcal{B}) \xrightarrow{\Psi} W_{2n+2} \longrightarrow 1$$

from Proposition 4.3.2. Recall from Lemma 2.4.2 that $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$ admits the presentation

$$\langle \widetilde{S}_K \cup \widetilde{S}_H \mid R_1 \cup R_2 \cup \widetilde{R}_K \rangle$$

where the generating symbols $\widetilde{S}_K, \widetilde{S}_H$ and the sets of relators $R_1, R_2,$ and $\widetilde{R}_K$ are defined immediately prior to the statement of Lemma 2.4.2. The generators $\widetilde{S}_K \cup \widetilde{S}_H$ are arrived at in Lemma 4.5.1. The set of relators R_1 is constructed in Lemma 4.5.3. The set of relators R_2 is derived in three steps in Section 4.5.2.

4.5.1 Lifts of generators and relations

Let σ_i be the half twist that exchanges the ith and $i+1$st branch points about the arc in Σ_0 as in image Figure 4.4. The mapping class group $\mathrm{Mod}(\Sigma_0, \mathcal{B})$ admits the presentation [22, p. 123]

$$\left\langle \sigma_1, \ldots, \sigma_{2n+1} \;\middle|\; \begin{cases} [\sigma_i, \sigma_j] = 1 & |i-j| > 1, \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & i \in \{1, \ldots, 2n\}, \\ (\sigma_1 \sigma_2 \cdots \sigma_{2n+1})^{2n+2} = 1, & \\ (\sigma_1 \cdots \sigma_{2n+1} \sigma_{2n+1} \cdots \sigma_1) = 1 & \end{cases} \right\rangle.$$

Since $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$ is a subgroup of $\mathrm{Mod}(\Sigma_0, \mathcal{B})$, the generators of $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$ will be defined in terms of the σ_i.

Lemma 4.5.1. *The group* $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$ *is generated by*

$$\{A_{i,j} : 1 \leq i < j \leq 2n+1\} \cup \{a_i : i \in \{1, \ldots, n\}\} \cup \{b_i : i \in \{1, \ldots, n\}\} \cup \{c\}$$

where

$$\begin{aligned} A_{i,j} &= (\sigma_{j-1}\sigma_{j-2} \cdots \sigma_{i+1})\sigma_i^2(\sigma_{j-1}\sigma_{j-2} \cdots \sigma_{i+1})^{-1} \\ a_i &= \sigma_{2i}\sigma_{2i-1}\sigma_{2i}^{-1} \\ b_i &= \sigma_{2i+1}\sigma_{2i}\sigma_{2i+1}^{-1} \\ c &= \sigma_1\sigma_3 \cdots \sigma_{2n+1}. \end{aligned}$$

Proof. The set $\{A_{i,j} : 1 \leq i < j \leq 2n+1\}$ is the image under ι of the generators of $\mathrm{PMod}(\Sigma_0, \mathcal{B})$ from Lemma 4.4.1. We have $\Psi(c) = z$, $\Psi(a_i) = x_i$ and $\Psi(b_i) = y_i$ for all i, where x_i, y_i, and c are the generators for W_{2n+2} from Lemma 4.4.2. Therefore, using the notation from Lemma 2.4.2, $\widetilde{S}_K = \{A_{i,j} : 1 \leq i < j \leq 2n+1\}$ and

$$\widetilde{S}_H = \{a_i : i \in \{1, \ldots, n\}\} \cup \{b_i : i \in \{1, \ldots, n\}\} \cup \{c\},$$

completing the proof. ■

The generators $A_{i,j}$, a_i, and b_i are all shown in Figure 4.4. The elements a_i exchange consecutive odd marked points and the elements b_i exchange consecutive even marked points. The generator c is the composition of half twists about the arcs on the right side of Figure 4.4, and c switches each odd marked point with an even marked point.

Although the elements $A_{i,2n+2} = (\sigma_{2n+1} \cdots \sigma_{i+1})\sigma_i^2(\sigma_{2n+1} \cdots \sigma_{i+1})^{-1}$ are in $\mathrm{PMod}(\Sigma_0, \mathcal{B})$, they are not part of the generating set from Lemma 4.5.1. However, it will be useful to use the elements $A_{i,2n+2}$ in the set of relations for our final presentation. The next lemma rewrites the elements $A_{i,2n+2}$ as words in the generators $A_{i,j}$ with $1 \leq i < j \leq 2n+1$.

Lemma 4.5.2. *Fix* $\ell \in \{1, \ldots, 2n+1\}$. *Define*

$$\overline{A}_{i,j} := \begin{cases} A_{i,j} & \text{if } j < \ell \\ A_{\ell,j+1}^{-1} A_{i,j+1} A_{\ell,j+1} & \text{if } i < \ell \leq j \\ A_{i+1,j+1} & \text{if } \ell \leq i. \end{cases}$$

Then $A_{\ell,2n+2} = (\overline{A}_{1,2} \cdots \overline{A}_{1,2n})(\overline{A}_{2,3} \cdots \overline{A}_{2,2n}) \cdots (\overline{A}_{2n-1,2n})$.

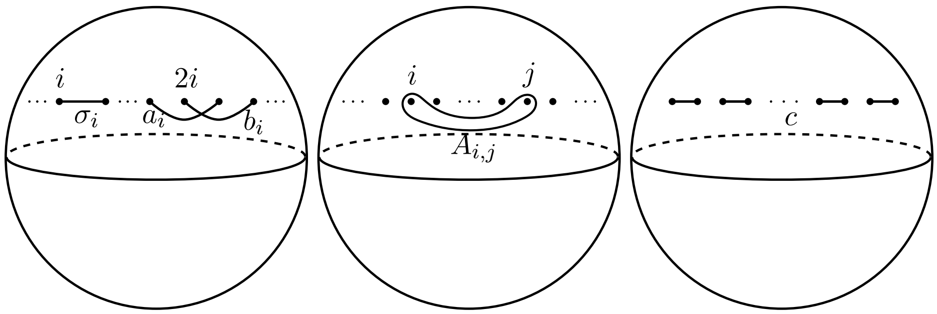


Figure 4.4: *Left to right*: the arcs about which σ_i, a_i, and b_i are half twists, the curve about which $A_{i,j}$ is a Dehn twist, and the collection of arcs about which c is a composition of half twists. The labels above the marked points are to indicate the enumeration.

Proof. Let $\gamma_{i,j}$ be the curve about which $A_{i,j}$ is a Dehn twist. Then $\gamma_{2n+1,2n+2}$ bounds a disk containing the first $2n$ marked points. Therefore

$$A_{2n+1,2n+2} = (A_{1,2} \cdots A_{1,2n})(A_{2,3} \cdots A_{2,2n}) \cdots (A_{2n-2,2n-1}A_{2n-2,2n})(A_{2n-1,2n}). \tag{4.3}$$

The right hand side of equation (4.3) is a Dehn twist about the boundary of a disk containing $2n$ marked points, as seen in [22, p. 260]. Let $Q_\ell = \sigma_{2n}\sigma_{2n-1} \cdots \sigma_\ell$. The desired relations will be obtained from equation (4.3) by conjugating by Q_ℓ.

We have $Q_\ell(\gamma_{2n+1,2n+2}) \simeq \gamma_{\ell,2n+2}$ so $Q_\ell^{-1} A_{2n+1,2n+2} Q_\ell = A_{\ell,2n+2}$ (see Section 2.3.2). It suffices to show $Q_\ell^{-1} A_{i,j} Q_\ell = \overline{A}_{i,j}$ for $1 \leq i < j \leq 2n$. We have

$$Q_\ell(\gamma_{i,j}) \simeq \begin{cases} \gamma_{i,j} & \text{if } j < \ell \\ A_{\ell,j+1}(\gamma_{i,j+1}) & \text{if } i < \ell \leq j \\ \gamma_{i+1,j+1} & \text{if } \ell \leq i. \end{cases}$$

Therefore $Q_\ell^{-1} A_{i,j} Q_\ell = \overline{A}_{i,j}$ completing the proof. ■

We are now ready to derive the relators R_1 from Lemma 2.4.2. To ease notation, let

$$C_{i,j} = \begin{cases} A_{2i-1,2i}^{-1} A_{2i+1,2i+2}^{-1} A_{2i-1,2i+2} A_{2i,2i+1} & \text{if } i = j \\ A_{2i+1,2i+2}^{-1} A_{2i,2i+3}^{-1} A_{2i+2,2i+3} A_{2i,2i+1} & \text{if } i = j+1 \\ 1 & \text{otherwise} \end{cases} \tag{4.4}$$

for $1 \leq i \leq j \leq n$.

Lemma 4.5.3. *Let $A_{i,j}, a_i, b_i$, and c be the generators defined in Lemma 4.5.1. The following relations hold:*

Commutator relations

1. $[a_i, b_j] = C_{i,j}$ *where* $C_{i,j}$ *is given by (4.4)*

Braid relations

2. $a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}$ *and* $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ *for* $i \in \{1, \ldots, n-1\}$
3. $[a_i, a_j] = [b_i, b_j] = 1$ *if* $|j - i| > 1$

Half twists squared are Dehn twists

4. $a_i^2 = A_{2i-1,2i+1}$ *and* $b_i^2 = A_{2i,2i+2}$ *for* $i \in \{1, \ldots, n\}$.
5. $c^2 = A_{1,2} A_{3,4} \cdots A_{2n+1,2n+2}$

Parity Flip

6. $c a_i c^{-1} b_i^{-1} = 1$.

Proof. Since the σ_i satisfy the braid relations, we may use any solution to the word problem for the braid group to reduce a word in the σ_i to the empty word. Our weapon of choice will be Dehornoy's handle reduction [20]. We will underline the handles as we perform the reduction.

For relation 1 it suffices to show $[a_i, b_j] C_{i,j}^{-1}$ is the identity. If $i = j$

$$
\begin{aligned}
& a_i b_i a_i^{-1} b_i^{-1} A_{2i,2i+1}^{-1} A_{2i-1,2i+2}^{-1} A_{2i+1,2i+2} A_{2i-1,2i} \\
&= (\sigma_{2i}\sigma_{2i-1}\underline{\sigma_{2i}^{-1})(\sigma_{2i+1}\sigma_{2i}}\sigma_{2i+1}^{-1})(\sigma_{2i}\sigma_{2i-1}^{-1}\sigma_{2i}^{-1})(\sigma_{2i+1}\sigma_{2i}^{-1}\sigma_{2i+1}^{-1}) \\
&\qquad (\sigma_{2i}^{-1}\underline{\sigma_{2i}^{-1})(\sigma_{2i+1}\sigma_{2i}}\sigma_{2i-1}^{-1}\underline{\sigma_{2i-1}^{-1}\sigma_{2i}^{-1}\sigma_{2i+1}^{-1})(\sigma_{2i+1}^{2})(\sigma_{2i-1}}\sigma_{2i-1}) \\
&= \sigma_{2i}\underline{\sigma_{2i-1}\sigma_{2i+1}\sigma_{2i}\sigma_{2i+1}^{-1}\sigma_{2i+1}^{-1}\sigma_{2i}\sigma_{2i-1}^{-1}}\sigma_{2i}^{-1}\sigma_{2i+1}\sigma_{2i}^{-1}\sigma_{2i+1}^{-1} \\
&\qquad \underline{\sigma_{2i}^{-1}\sigma_{2i+1}\sigma_{2i}}\sigma_{2i+1}^{-1}\sigma_{2i-1}^{-1}\sigma_{2i}\underline{\sigma_{2i-1}^{-1}\sigma_{2i}^{-1}\sigma_{2i+1}\sigma_{2i-1}} \\
&= \underline{\sigma_{2i}\sigma_{2i+1}\sigma_{2i}^{-1}}\sigma_{2i-1}\underline{\sigma_{2i}\sigma_{2i+1}^{-1}\sigma_{2i+1}^{-1}\sigma_{2i}^{-1}}\ \underline{\sigma_{2i-1}\sigma_{2i+1}^{-1}\sigma_{2i-1}^{-1}}\sigma_{2i}\sigma_{2i}\sigma_{2i-1}^{-1}\sigma_{2i}^{-1}\sigma_{2i+1} \\
&= \sigma_{2i+1}^{-1}\sigma_{2i}\sigma_{2i+1}\underline{\sigma_{2i-1}\sigma_{2i+1}^{-1}\sigma_{2i-1}^{-1}}\sigma_{2i}^{-1}\sigma_{2i+1} \\
&= 1
\end{aligned}
$$

A similar computation estabilshes the relation if $i = j + 1$.

For the cases where $i \notin \{j, j+1\}$ we note $|2j - 2i| \geq 2$ and $|(2i-1) - (2j+1)| \geq 2$. Therefore $a_i = \sigma_{2i}\sigma_{2i-1}\sigma_{2i}^{-1}$ commutes with $b_j = \sigma_{2j+1}\sigma_{2j}\sigma_{2j+1}^{-1}$, establishing relation 1. Similarly, $[a_i, a_j] = [b_i, b_j] = 1$ if $|j - i| > 1$, establishing relation 3.

For relation 2 we have

$$\begin{aligned}
&a_i a_{i+1} a_i a_{i+1}^{-1} a_i^{-1} a_{i+1}^{-1} \\
&= (\sigma_{2i}\sigma_{2i-1}\sigma_{2i}^{-1})(\sigma_{2i+2}\sigma_{2i+1}\sigma_{2i+2}^{-1})(\sigma_{2i}\sigma_{2i-1}\sigma_{2i}^{-1}) \\
&\qquad (\sigma_{2i+2}\sigma_{2i+1}^{-1}\sigma_{2i+2}^{-1})(\sigma_{2i}\sigma_{2i-1}^{-1}\sigma_{2i}^{-1})(\sigma_{2i+2}\sigma_{2i+1}^{-1}\sigma_{2i+2}^{-1}) \\
&= \sigma_{2i+2}(\sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}\sigma_{2i}\sigma_{2i+1}^{-1})\sigma_{2i-1}\sigma_{2i}^{-1}\sigma_{2i+2}^{-1} \\
&\qquad \sigma_{2i+2}\sigma_{2i+1}^{-1}(\sigma_{2i-1}^{-1}\sigma_{2i}^{-1}\sigma_{2i-1})(\sigma_{2i+2}^{-1}\sigma_{2i+2})\sigma_{2i+1}^{-1}\sigma_{2i+2}^{-1} \\
&= \sigma_{2i+2}\sigma_{2i}\sigma_{2i+1}(\sigma_{2i}\sigma_{2i-1}\sigma_{2i})(\sigma_{2i}^{-1}\sigma_{2i+1}^{-1}\sigma_{2i}^{-1})\sigma_{2i-1}^{-1}\sigma_{2i}^{-1}\sigma_{2i-1}\sigma_{2i+1}^{-1}\sigma_{2i+2}^{-1} \\
&= \sigma_{2i+2}(\sigma_{2i}\sigma_{2i+1}\sigma_{2i})\sigma_{2i-1}\sigma_{2i+1}^{-1}\sigma_{2i}^{-1}\sigma_{2i-1}^{-1}\sigma_{2i}^{-1}\sigma_{2i-1}\sigma_{2i+1}^{-1}\sigma_{2i+2}^{-1} \\
&= \sigma_{2i+2}\sigma_{2i}\sigma_{2i+1}\sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}^{-1}(\sigma_{2i-1}^{-1}\sigma_{2i}^{-1}\sigma_{2i-1}^{-1})\sigma_{2i-1}\sigma_{2i+1}^{-1}\sigma_{2i+2}^{-1} \\
&= \sigma_{2i+2}\sigma_{2i}\sigma_{2i+1}(\sigma_{2i+1}^{-1}\sigma_{2i}^{-1}\sigma_{2i+1})\sigma_{2i+1}^{-1}\sigma_{2i+2}^{-1} \\
&= 1
\end{aligned}$$

The same proof can be used for the relation $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ by decreasing all indices by 1.

For relation 4 we see $a_i^2 = \sigma_{2i}\sigma_{2i-1}^2\sigma_{2i}^{-1} = A_{2i-1,2i+1}$. Similarly, for relation 5 we have $b_i^2 = A_{2i,2i+2}$. For relation 6 we have

$$\begin{aligned}
ca_ic^{-1} &= (\sigma_1\sigma_3\cdots\sigma_{2i-1}\sigma_{2i+1}\cdots\sigma_{2n+1})(\sigma_{2i}\sigma_{2i-1}\sigma_{2i}^{-1})(\sigma_1^{-1}\sigma_3^{-1}\cdots\sigma_{2i-1}^{-1}\sigma_{2i+1}^{-1}\cdots\sigma_{2n+1}^{-1}) \\
&= \sigma_{2i+1}\sigma_{2i-1}\sigma_{2i}\sigma_{2i-1}\sigma_{2i}^{-1}\sigma_{2i-1}^{-1}\sigma_{2i+1}^{-1} \\
&= \sigma_{2i+1}\sigma_{2i}\sigma_{2i+1}^{-1} \\
&= b_i
\end{aligned}$$

which completes the proof. ■

Topological interpretation. Although the proof of Lemma 4.5.3 is purely algebraic, there are topological interpretations of most of the relations. Let α_i be the arc about which a_i is a half twist, and β_i the arc about which b_i is a half twist.

When $i \neq j, j+1$, α_i and β_j can be modified by homotopy to be disjoint so the relations $[a_i, b_j] = 1$ in 1 hold. The homeomorphisms $\{a_i\}$ are supported on a closed neighborhood of the union $\alpha_1 \cup \cdots \cup \alpha_n$, which is an embedded disk D_{n+1} with $n+1$ marked points. The mapping class group of D_{n+1} is isomorphic to the braid group B_{n+1}. Embedding D_{n+1} in Σ_0 with $2n+2$ marked points induces a homomorphism $\phi : B_{n+1} \to \mathrm{Mod}(\Sigma_0, \mathcal{B}(2n+2))$. The homomorphism ϕ maps the standard braid generators to the a_i, and so the braid relations 2 and 3 hold. The same applies to the b_i.

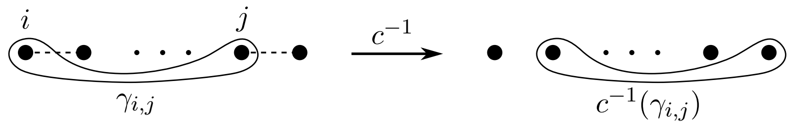


Figure 4.5: *Both i and j are odd.* The curve $\gamma_{i,j}$ and its image under c^{-1}, which is a product of half twists about the dashed arcs.

Relations 4 and 5 reflect the fact that squaring a half twist about an arc is equal to a Dehn twist about a curve surrounding the arc. Recall from Section 2.3.2 that if τ_δ is a half twist about an arc δ in Σ_0 and f is a mapping class, then $f^{-1}\tau_\gamma f = \tau_{f(\gamma)}$ (the homeomorphisms are applied left to right). We realize $ca_ic^{-1} = b_i$ in relation 6 by applying the homeomorphism c^{-1} to the arc α_i.

4.5.2 Conjugation relations

We now shift our attention to finding the relations that make up R_2 from Lemma 2.4.2.

First we consider conjugation of the generators of $\mathrm{PMod}(\Sigma_0, \mathcal{B})$ by c. Let

$$X_{i,j} = \begin{cases} A_{i,j} & \text{for odd } i,\ j = i+1 \\ A_{i+1,j+1} & \text{for odd } i, j \\ (A_{i-1,j}A_{i-1,i}^{-1})^{-1}A_{i-1,j-1}(A_{i-1,j}A_{i-1,i}^{-1}) & \text{for even } i, j \\ A_{i,j+1}^{-1}A_{i-1,j+1}A_{i,j+1} & \text{for even } i, \text{ odd } j \\ A_{j-1,j}A_{i+1,j-1}A_{j-1,j}^{-1} & \text{otherwise.} \end{cases} \tag{4.5}$$

Lemma 4.5.4. *For $1 \le i < j \le 2n+1$, let $A_{i,j}$ and c be as above. Then*

$$cA_{i,j}c^{-1} = X_{i,j}$$

where the $X_{i,j}$ are as in (4.5).

Proof. Let $\gamma_{i,j}$ be the simple closed curve in Σ_0 about which $A_{i,j}$ is a Dehn twist, so $T_{\gamma_{i,j}} = A_{i,j}$. Recall that $cT_{\gamma_{i,j}}c^{-1} = T_{c^{-1}(\gamma_{i,j})}$ (where we maintain our convention that we read products from left to right). Therefore it suffices to show that $c^{-1}(\gamma_{i,j})$ is the curve about which $X_{i,j}$ is a twist.

We first note that when i is odd and $j = i+1$ the curve $\gamma_{i,j}$ is disjoint from the arcs that define c, therefore $cA_{i,j}c^{-1} = A_{i,j}$. We then consider the remaining cases.

i and j are both odd. The curves $c^{-1}(\gamma_{i,j})$ and $\gamma_{i+1,j+1}$ are shown to be isotopic in Figure 4.5. Therefore $cA_{i,j}c^{-1} = A_{i+1,j+1}$.

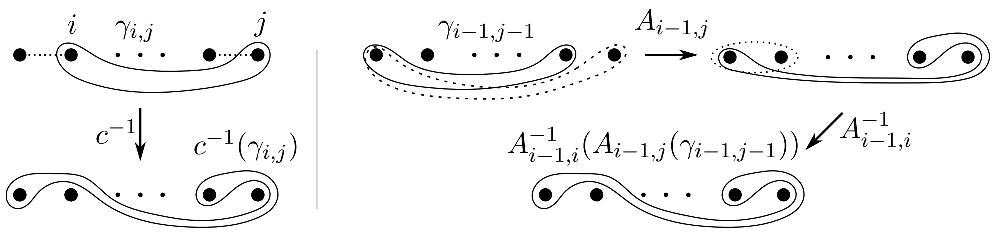


Figure 4.6: *Both i and j are even.* The left figure shows $\gamma_{i,j}$ and its image under c^{-1}, which is a product of half twists about the dashed arcs. Starting at the top left and going clockwise, the right figure shows $\gamma_{i-1,j-1}$, $A_{i-1,j}(\gamma_{i-1,j-1})$, and $A^{-1}_{i-1,i}(A_{i-1,j}(\gamma_{i-1,j-1}))$. The dashed curves indicate the curves about which $A_{i-1,j}$ and $A^{-1}_{i-1,i}$ are Dehn twists.

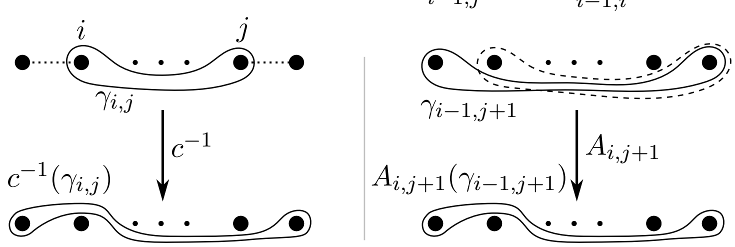


Figure 4.7: *Even i and odd j.* The left figure shows $\gamma_{i,j}$ and its image under c^{-1}, which is a product of half twists about the dashed arcs. The right figure shows $\gamma_{i-1,j+1}$ and its image under $A_{i,j+1}$, which is a Dehn twist about the dashed curve.

i and j are both even. The curves $c^{-1}(\gamma_{i,j})$ and $A^{-1}_{i-1,i}(A_{i-1,j}(\gamma_{i-1,j-1}))$ (with composition applied as indicated) are shown to be isotopic in Figure 4.6. Therefore

$$cA_{i,j}c^{-1} = (A_{i-1,j}A^{-1}_{i-1,i})^{-1}A_{i-1,j-1}(A_{i-1,j}A^{-1}_{i-1,i}).$$

i is even and j is odd. The curves $c^{-1}(\gamma_{i,j})$ and $A_{i,j+1}(\gamma_{i-1,j+1})$ are shown to be isotopic in Figure 4.7. Therefore $cA_{i,j}c^{-1} = A^{-1}_{i,j+1}A_{i-1,j+1}A_{i,j+1}$.

i is odd and j is even $j \neq i+1$. The curves $c^{-1}(\gamma_{i,j})$ and $A^{-1}_{j-1,j}(\gamma_{i+1,j-1})$ are shown to be isotopic in Figure 4.8. Therefore $cA_{i,j}c^{-1} = A_{j-1,j}A_{i+1,j-1}A^{-1}_{j-1,j}$. ■

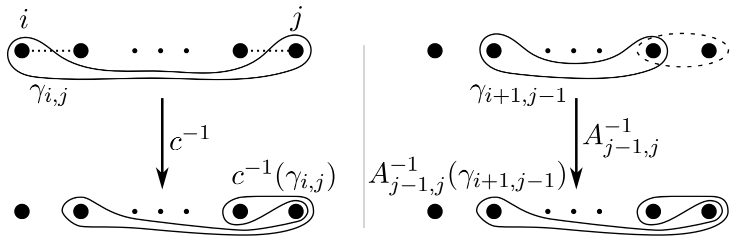


Figure 4.8: *Odd i and even j, $j \neq i+1$.* The left figure shows $\gamma_{i,j}$ and its image under c^{-1}, which is a product of half twists about the dashed arcs. The right figure shows $\gamma_{i+1,j-1}$ and its image under $A^{-1}_{j-1,j}$, which is an inverse Dehn twist about the dashed curve.

Next we consider conjugation of the elements $A_{i,j}$ by the generators a_ℓ. Let

$$Y_{i,j,\ell} = \begin{cases} A_{i,j} & \text{if } i < 2\ell-1, j > 2\ell+1, \\ A_{i,j} & \text{if } i,j > 2\ell+1 \text{ or } i,j < 2\ell-1 \\ A_{i,j+2} & \text{if } i < 2\ell-1, j = 2\ell-1 \\ (A^{-1}_{i,j-1}A_{i,j+1})^{-1}A_{i,j}(A^{-1}_{i,j-1}A_{i,j+1}) & \text{if } i < 2\ell-1, j = 2\ell \\ A^{-1}_{i,j}A_{i,j-2}A_{i,j} & \text{if } i < 2\ell-1, j = 2\ell+1 \\ A_{i,j+1}A_{j,j+1}A^{-1}_{i,j+1} & \text{if } i = 2\ell-1, j = 2\ell \\ A_{i,j} & \text{if } i = 2\ell-1, j = 2\ell+1 \\ A_{i+2,j} & \text{if } i = 2\ell-1, j > 2\ell+1 \\ A_{i-1,j-1} & \text{if } i = 2\ell, j = 2\ell+1 \\ (A^{-1}_{i,i+1}A_{i-1,i})^{-1}A_{i,j}(A^{-1}_{i,i+1}A_{i-1,i}) & \text{if } i = 2\ell, j > 2\ell+1 \\ A^{-1}_{i,j}A_{i-2,j}A_{i,j} & \text{if } i = 2\ell+1, j > 2\ell+1 \end{cases} \tag{4.6}$$

Lemma 4.5.5. *For $1 \leq i < j \leq 2n+1$ and $\ell \in \{1, \ldots, n\}$, let $A_{i,j}$ and a_ℓ be as above. Then*

$$a_\ell A_{i,j} a_\ell^{-1} = Y_{i,j,\ell}$$

where the $Y_{i,j,\ell}$ are as in (4.6).

Proof. Recall that $a_\ell = \sigma_{2\ell}\sigma_{2\ell-1}\sigma^{-1}_{2\ell}$ and

$$A_{i,j} = (\sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1})\sigma_i^2(\sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1})^{-1}.$$

The transpositions σ_p and σ_q commute if $|p-q| \geq 2$. Therefore a_ℓ and $A_{i,j}$ commute if both $(2\ell-1)-(j-1) \geq 2$ and $2\ell-(i+1) \geq 2$, if $i,j > 2\ell+1$, or if $i,j < 2\ell-1$.

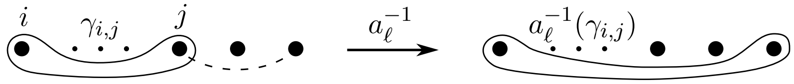


Figure 4.9: $i < 2\ell - 1, j = 2\ell - 1$. The curve $\gamma_{i,j}$ and its image under a_ℓ^{-1}, which is a counterclockwise half twist about the dashed arc.

Therefore in the first two cases of (4.6), a_ℓ and $A_{i,j}$ commute. In the remaining cases, at least one of i and j is equal to $2\ell - 1, 2\ell$, or $2\ell + 1$.

If $i = 2\ell - 1$ and $j = i + 2$, then $A_{i,j} = \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1} = A_{i,i+2}$ and

$$\begin{aligned} a_\ell A_{i,j} a_\ell^{-1} &= (\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1})\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}(\sigma_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1}) \\ &= \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1} \\ &= A_{i,j} \end{aligned}$$

If $i = 2\ell$ and $j = i + 1$, then $a_\ell = \sigma_i\sigma_{i-1}\sigma_i^{-1}$ and $A_{i,j} = \sigma_i^2$. Then:

$$\begin{aligned} a_\ell A_{i,j} a_\ell^{-1} &= (\sigma_i\sigma_{i-1}\sigma_i^{-1})\sigma_i^2(\sigma_i\sigma_{i-1}^{-1}\sigma_i^{-1}) \\ &= (\sigma_{i-1}\sigma_i\sigma_{i-1})(\sigma_{i-1}^{-1}\sigma_i^{-1}\sigma_{i-1}) \\ &= \sigma_{i-1}^2 \\ &= A_{i-1,j-1} \end{aligned}$$

If $i = 2\ell - 1$ and $j = i + 1$, then $A_{i,j} = \sigma_i^2$ and $a_\ell = \sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}$.

$$\begin{aligned} a_\ell A_{i,j} a_\ell^{-1} &= (\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1})\sigma_i^2(\sigma_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1}) \\ &= \sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i(\sigma_{i+1}\sigma_{i+1}^{-1})\sigma_i\sigma_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1} \\ &= \sigma_{i+1}\sigma_i(\sigma_i\sigma_{i+1}\sigma_i^{-1})(\sigma_i\sigma_{i+1}\sigma_i^{-1})\sigma_i^{-1}\sigma_{i+1}^{-1} \\ &= (\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1})\sigma_{i+1}^2(\sigma_{i+1}\sigma_i^{-2}\sigma_{i+1}^{-1}) \\ &= A_{i,j+1}A_{j,j+1}A_{i,j+1}^{-1} \end{aligned}$$

We prove the remaining cases topologically. As above, let $\gamma_{i,j}$ be a curve such that $T_{\gamma_{i,j}} = A_{i,j}$. As in the proof of Lemma 4.5.4, it suffices to prove that $Y_{i,j,\ell} = T_{a_\ell^{-1}(\gamma_{i,j})}$ for various relationships between i, j and ℓ.

The case where $i < 2\ell - 1$ **and** $j = 2\ell - 1$**.** The curves $a_\ell^{-1}(\gamma_{i,j})$ and $\gamma_{i,j+2}$ are shown to be isotopic in Figure 4.9. Therefore $a_\ell A_{i,j} a_\ell^{-1} = A_{i,j+2}$.

The case where $i < 2\ell - 1$ **and** $j = 2\ell$**.** The curves $a_\ell^{-1}(\gamma_{i,j})$ and $A_{i,j+1}(A_{i,j-1}^{-1}(\gamma_{i,j}))$ are shown to be isotopic in Figure 4.10. Therefore $a_\ell A_{i,j} a_\ell^{-1} = (A_{i,j-1}^{-1}A_{i,j+1})^{-1}A_{i,j}(A_{i,j-1}^{-1}A_{i,j+1})$.

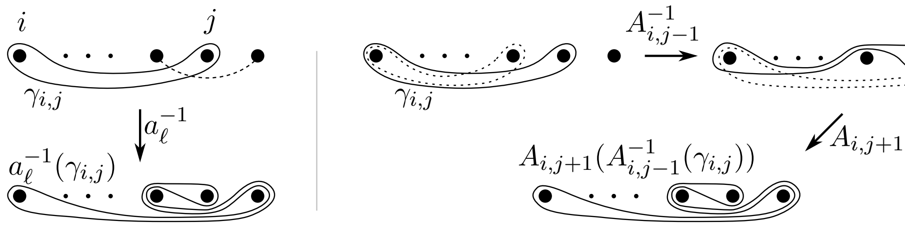


Figure 4.10: $i < 2\ell - 1, j = 2\ell$. The left figure shows $\gamma_{i,j}$ and its image under a_ℓ^{-1}, which is a half twist about the dashed arc. Starting at the top left and going clockwise, the right figure shows $\gamma_{i,j}$, $A_{i,j-1}^{-1}(\gamma_{i,j})$ and $A_{i,j+1}(A_{i,j-1}^{-1}(\gamma_{i,j}))$. The dashed curves indicate the curves about which $A_{i,j-1}^{-1}$ and $A_{i,j+1}$ are Dehn twists.

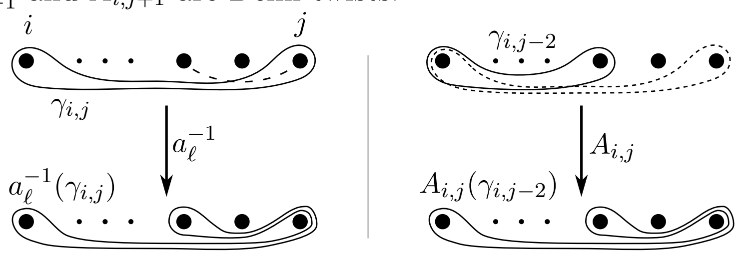


Figure 4.11: $i < 2\ell - 1, j = 2\ell + 1$. The left figure shows $\gamma_{i,j}$ and its image under a_ℓ^{-1}, which is a half twist about the dashed arc. The right figure shows $\gamma_{i,j-2}$ and its image under $A_{i,j}$, which is a Dehn twist about the dashed curve.

The case where $i < 2\ell - 1$ and $j = 2\ell + 1$. The curves $a_\ell^{-1}(\gamma_{i,j})$ and $A_{i,j}(\gamma_{i,j-2})$ are shown to be isotopic in Figure 4.11. Therefore $a_\ell A_{i,j} a_\ell^{-1} = A_{i,j}^{-1} A_{i,j-2} A_{i,j}$.

The case where $i = 2\ell - 1$ and $j > 2\ell + 1$. The curves $a_\ell^{-1}(\gamma_{i,j})$ and $\gamma_{i+2,j}$ are shown to be isotopic in Figure 4.12. Therefore $a_\ell A_{i,j} a_\ell^{-1} = A_{i+2,j}$.

The case where $i = 2\ell$ and $j > 2\ell + 1$. The curves $a_\ell^{-1}(\gamma_{i,j})$ and $A_{i-1,i}(A_{i,i+1}^{-1}(\gamma_{i,j}))$ are shown to be isotopic in Figure 4.13. Therefore

$$a_\ell A_{i,j} a_\ell^{-1} = (A_{i,i+1}^{-1} A_{i-1,i})^{-1} A_{i,j} (A_{i,i+1}^{-1} A_{i-1,i}).$$

The case where $i = 2\ell + 1$ and $j > 2\ell + 1$. The curves $a_\ell^{-1}(\gamma_{i,j})$ and $A_{i,j}(\gamma_{i-2,j})$ are shown to be isotopic in Figure 4.14. Therefore $a_\ell A_{i,j} a_\ell^{-1} = A_{i,j}^{-1} A_{i-2,j} A_{i,j}$. ■

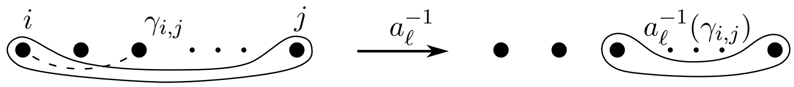


Figure 4.12: $i = 2\ell - 1, j > 2\ell + 1$. The curve $\gamma_{i,j}$ and its image under a_ℓ^{-1}, which is an inverse half twist about the dashed arc.

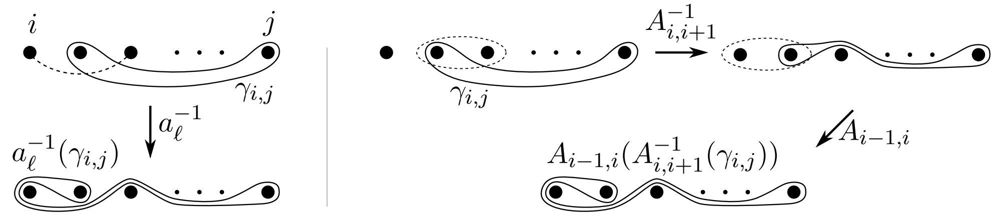


Figure 4.13: $i = 2\ell, j > 2\ell + 1$. The left figure shows $\gamma_{i,j}$ and its image under a_ℓ^{-1}, which is a half twist about the dashed arc. Starting at the top left and going clockwise, the right figure shows $\gamma_{i,j}$, $A_{i,i+1}^{-1}(\gamma_{i,j})$ and $A_{i-1,i}(A_{i,i+1}^{-1}(\gamma_{i,j}))$. The dashed curves indicate the curves about which $A_{i,i+1}^{-1}$ and $A_{i-1,i}$ are Dehn twists.

Next we consider conjugation of the elements $A_{i,j}$ by the generators b_ℓ. Let

$$Z_{i,j,\ell} = \begin{cases} A_{i,j} & \text{if } i < 2\ell, j > 2\ell+2 \\ A_{i,j} & \text{if } i,j > 2\ell+2 \text{ or } i,j < 2\ell \\ A_{i,j+2} & \text{if } i < 2\ell, j = 2\ell \\ (A_{i,j-1}^{-1}A_{i,j+1})^{-1}A_{i,j}(A_{i,j-1}^{-1}A_{i,j+1}) & \text{if } i < 2\ell, j = 2\ell+1 \\ A_{i,j}^{-1}A_{i,j-2}A_{i,j} & \text{if } i < 2\ell, j = 2\ell+2 \\ A_{i,j+1}A_{j,j+1}A_{i,j+1}^{-1} & \text{if } i = 2\ell, j = 2\ell+1 \\ A_{i,j} & \text{if } i = 2\ell, j = 2\ell+2 \\ A_{i+2,j} & \text{if } i = 2\ell, j > 2\ell+2 \\ A_{i-1,j-1} & \text{if } i = 2\ell+1, j = 2\ell+2 \\ (A_{i,i+1}^{-1}A_{i-1,i})^{-1}A_{i,j}(A_{i,i+1}^{-1}A_{i-1,i}) & \text{if } i = 2\ell+1, j > 2\ell+2 \\ A_{i,j}^{-1}A_{i-2,j}A_{i,j} & \text{if } i = 2\ell+2, j > 2\ell+2. \end{cases} \tag{4.7}$$

Lemma 4.5.6. *For $1 \le i < j \le 2n+1$ and $\ell \in \{1, \dots, n\}$, let $A_{i,j}$ and b_ℓ be as above. Then*

$$b_\ell A_{i,j} b_\ell^{-1} = Z_{i,j,\ell}$$

where the $Z_{i,j,\ell}$ are as in (4.7).

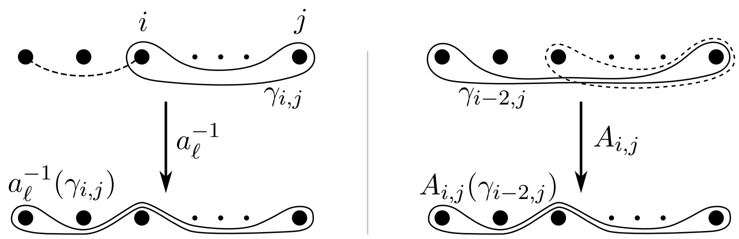


Figure 4.14: $i = 2\ell + 1, j > 2\ell + 1$. The left figure shows $\gamma_{i,j}$ and its image under a_ℓ^{-1}, which is a half twist about the dashed arc. The right figure shows $\gamma_{i-2,j}$ and its image under $A_{i,j}$, which is a Dehn twist about the dashed curve.

The proof of Lemma 4.5.6 is the same as the proof of Lemma 4.5.5 with an increase in index by 1.

4.5.3 Proof of the presentation

We are now ready to write down a presentation for $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$.

Theorem 4.5.7. *Let $p_{g,k} : \Sigma_g \to \Sigma_0$ be a balanced superelliptic cover with $k \geq 3$. The liftable mapping class group $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$ is generated by*

$$\begin{aligned} A_{i,j} &= (\sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1})\sigma_i^2(\sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1})^{-1}, 1 \leq i < j \leq 2n+1 \\ c &= \sigma_1\sigma_3\cdots\sigma_{2n-1}\sigma_{2n+1} \\ a_i &= \sigma_{2i}\sigma_{2i-1}\sigma_{2i}^{-1},\ i \in \{1,\ldots,n\} \\ b_i &= \sigma_{2i+1}\sigma_{2i}\sigma_{2i+1}^{-1},\ i \in \{1,\ldots,n\}. \end{aligned}$$

For $\ell \in \{1,\ldots,2n+1\}$, let $A_{\ell,2n+2}$ be defined as in Lemma 4.5.2. Then $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$ has defining relations:

Commutator relations

1. $[A_{i,j}, A_{p,q}] = 1$ *where* $1 \leq i < j < p < q \leq 2n+1$.
2. $[A_{i,q}, A_{j,p}] = 1$ *where* $1 \leq i < j < p < q \leq 2n+1$.
3. $\left[A_{p,q}A_{i,p}A_{p,q}^{-1}, A_{j,q}\right] = 1$ *where* $1 \leq i < j < p < q \leq 2n+1$.
4. $[a_i, b_j] = C_{i,j}$ *where $C_{i,j}$ are as in (4.4).*

Braid relations

5. $A_{i,p}A_{j,p}A_{i,j} = A_{j,p}A_{i,j}A_{i,p} = A_{i,j}A_{i,p}A_{j,p}$ *where* $1 \leq i < j < p \leq 2n+1$.

6. $a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}$ *and* $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ *for* $i \in \{1, \ldots, n-1\}$.
7. $[a_i, a_j] = [b_i, b_j] = 1$ *if* $|j - i| > 1$.

Subsurface support

8. $(A_{1,2}A_{1,3} \cdots A_{1,m-1}) \cdots (A_{m-3,n-2}A_{m-3,n-1})(A_{m-2,m-1}) = 1$ *for* $m = 2n + 2$.

Half twists squared are Dehn twists

9. $a_i^2 = A_{2i-1,2i+1}$ *and* $b_i^2 = A_{2i,2i+2}$ *for* $i \in \{1, \ldots, n\}$.
10. $c^2 = A_{1,2}A_{3,4} \cdots A_{2n+1,2n+2}$.

Parity Flip

11. $ca_ic^{-1}b_i^{-1} = 1$

Conjugation relations

12. $cA_{i,j}c^{-1} = X_{i,j}$ *where the* $X_{i,j}$ *are as in (4.5).*
13. $a_\ell A_{i,j} a_\ell^{-1} = Y_{i,j,\ell}$ *where the* $Y_{i,j,\ell}$ *are as in (4.6).*
14. $b_\ell A_{i,j} b_\ell^{-1} = Z_{i,j,\ell}$ *where the* $Z_{i,j,\ell}$ *are as in (4.7).*

Proof. The proposed generators are proven to be generators in Lemma 4.5.1. Using the notation from Lemma 2.4.2, $\widetilde{R}_K$ consists of the relations 1, 2, 3, 5, and 8 by Lemma 4.4.1, R_1 consists of the relations 4, 6, 7, 9, 10, and 11 by Lemma 4.5.3, and R_2 consists of the relations 12, 13, and 14 as by Lemmas 4.5.4, 4.5.5, and 4.5.6. The result follows by applying Lemma 2.4.2. ■

4.6 Abelianization

In this section we will compute the abelianization of $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$ in Theorem 4.6.6, and compute the first Betti number of the balanced superelliptic mapping class group $\mathrm{SMod}_{g,k}(\Sigma_g)$ in Theorem 4.6.7. Recall that for any group G, $H_1(G;\mathbb{Z}) \cong G/[G,G]$. For this section, fix $k \geq 3$ and let $p_{g,k} : \Sigma_g \to \Sigma_0$ be the balanced superelliptic cover. Recall that there are $2n+2$ branch points where $n = g/(k-1)$. By Corollary 4.3.4, $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$ depends only on n. For ease of notation, let $G_n = \mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$ for the remainder of this section. Let $\phi : G_n \to G_n/[G_n, G_n]$ be the abelianization map. Note that if $a, b \in G_n$ are in the same conjugacy class of G_n, then $\phi(a) = \phi(b)$.

A presentation for $G_n/[G_n, G_n]$ is given by taking a presentation for G_n and adding the set of all commutators to the set of defining relators. We begin with the presentation given in Theorem 4.5.7.

So we do not have to deal with the symbols $A_{\ell,2n+2}$ separately, we will add the symbols $A_{\ell,2n+2}$ for $\ell \in \{1, \ldots, 2n+1\}$ as generators along with the relations

$$A_{\ell,2n+2} = (\overline{A}_{1,2} \cdots \overline{A}_{1,2n})(\overline{A}_{2,3} \cdots \overline{A}_{2,2n}) \cdots (\overline{A}_{2n-2,2n-1}\overline{A}_{2n-2,2n})(\overline{A}_{2n-1,2n})$$

where the $\overline{A}_{i,j}$ are as in Lemma 4.5.2.

Lemma 4.6.1. *If $j - i \equiv t - s \mod 2$, then $A_{i,j}$ is conjugate to $A_{s,t}$ in G_n.*

Proof. We consider two cases: either $j - i \equiv t - s \equiv 0 \mod 2$ or $j - i \equiv t - s \equiv 1 \mod 2$.

Case 1: $j - i \equiv t - s \equiv 0 \mod 2$.
Let i and j be even. Recall the conjugation relations

$$b_\ell A_{i,j} b_\ell^{-1} = A_{i,j}^{-1} A_{i,j-2} A_{i,j}$$

for $i < 2\ell$ and $j = 2n + 2$, and

$$b_\ell A_{i,j} b_\ell^{-1} = A_{i,j+2}$$

for $i < 2\ell$ and $j = 2\ell$. Therefore for any fixed even i, all generators $A_{i,j}$ with even j are in the same conjugacy class of G_n. We also have the conjugation relations

$$b_\ell A_{i,j} b_\ell^{-1} = A_{i+2,j}$$

for $i = 2\ell$ and $j > 2\ell + 2$, and

$$b_\ell A_{i,j} b_\ell^{-1} = A_{i,j}^{-1} A_{i-2,j} A_{i,j}$$

for $i = 2\ell+2$ and $j > 2\ell+2$. Therefore for any fixed even j, all the $A_{i,j}$ such that i is even are in the same conjugacy class of G_n. Then by varying j, we conclude that if i, j, s, t are all even, then $A_{i,j}$ and $A_{s,t}$ are conjugate.

Similarly we can consider the conjugacy relations $a_\ell A_{i,j} a_\ell^{-1} = Y_{i,j,\ell}$ to conclude that if i, j, s, t are all odd, then $A_{i,j}$ is conjugate to $A_{s,t}$ in G_n.

Observe that $cA_{1,3}c^{-1} = A_{2,4}$. We may finally conclude that if $j - i \equiv t - s \equiv 0 \mod 2$, then $A_{i,j}$ is conjugate to $A_{s,t}$ in G_n.

Case 2: $j - i \equiv t - s \equiv 1 \mod 2$**.**
Similar to case 1, we use relations from the family of relations $a_\ell A_{i,j} a_\ell^{-1} = Y_{i,j,\ell}$ to conclude that for any fixed even i, all the $A_{i,j}$ for any odd j are in the same conjugacy class of G. Using relations of the form $b_\ell A_{i,j} b_\ell^{-1} = Z_{i,j,\ell}$ gives us that for any fixed odd j, all the $A_{i,j}$

for any even i are in the same conjugacy class of G_n. Therefore if i and s are even and j and t are odd, then $A_{i,j}$ and $A_{s,t}$ are conjugate in G_n.

Similarly, if i and s are odd and j and t are even, then $A_{i,j}$ and $A_{s,t}$ are conjugate in G_n.

Finally, the relation $cA_{2,3}c^{-1} = A_{2,4}^{-1}A_{1,4}A_{2,4}$ allows us to conclude that if $j-i \equiv t-s \equiv 1 \mod 2$, then $A_{i,j}$ is conjugate to $A_{s,t}$ in G_n, completing the proof. ∎

From now on, let $A = \phi(A_{1,2})$ and $B = \phi(A_{1,3})$.

Lemma 4.6.2. *For each $\ell \in \{1, \ldots, 2n+1\}$, consider the relation*

$$A_{\ell,2n+2} = (\overline{A}_{1,2} \cdots \overline{A}_{1,2n})(\overline{A}_{2,3} \cdots \overline{A}_{2,2n}) \cdots (\overline{A}_{2n-2,2n-1}\overline{A}_{2n-2,2n})(\overline{A}_{2n-1,2n})$$

where the $\overline{A}_{i,j}$ are as in Lemma 4.5.2. Applying ϕ to each of these relations gives the relation $B^{n^2-n} = A^{1-n^2}$ in $G_n/[G_n, G_n]$.

Proof. Fix $\ell \in \{1, \ldots, 2n+1\}$ and let

$$\begin{aligned} \overline{W} &= (\overline{A}_{1,2} \cdots \overline{A}_{1,2n})(\overline{A}_{2,3} \cdots \overline{A}_{2,2n}) \cdots (\overline{A}_{2n-2,2n-1}\overline{A}_{2n-2,2n})(\overline{A}_{2n-1,2n}) \\ W &= (A_{1,2} \cdots A_{1,2n+1})(A_{2,3} \cdots A_{2,2n+1}) \cdots (A_{2n-1,2n}A_{2n-1,2n+1})(A_{2n,2n+1}) \\ L &= \prod_{\substack{1 \le i < j \le 2n+1 \\ i=\ell \text{ or } j=\ell}} A_{i,j}. \end{aligned}$$

Observe that $\phi(\overline{W}) = \phi(W)\phi(L)^{-1}$. By Lemma 4.6.1 we have

$$\begin{aligned} \phi(W) &= ((AB)^n)((AB)^{n-1}A)((AB)^{n-1}) \cdots (AB)(A) \\ &= A^{2n}A^{2(n-1)} \cdots A^2 B^n B^{2(n-1)} B^{2(n-2)} \cdots B^2 \\ &= A^{n(n+1)} B^{n^2} \end{aligned}$$

since $\sum_{i=1}^{n-1} 2i = n(n-1)$.

If ℓ is even, $\phi(L) = A^{n+1}B^{n-1}$. Applying ϕ to the relation above gives

$$B = \phi(W) = A^{n(n+1)}B^{n^2}A^{-n-1}B^{1-n}.$$

This rearranges to $B^{n^2-n} = A^{1-n^2}$.

If ℓ is odd, $\phi(L) = A^nB^n$. Applying ϕ to the relation above gives $B^{n^2-n} = A^{1-n^2}$. ∎

Lemma 4.6.3. *In the abelianization of G_n, $B^{n^2} = A^{-n^2-1}$.*

Proof. Consider the subsurface support relation,

$$(A_{1,2}\cdots A_{1,2n+1})(A_{2,3}\cdots A_{2,2n+1})\cdots(A_{2n-1,2n}A_{2n-1,2n+1})(A_{2n,2n+1})=1.$$

Applying ϕ to both sides gives $1=A^{n(n+1)}B^{n^2}$ by the computation of $\phi(W)$ in the proof of Lemma 4.6.2. ∎

Lemma 4.6.4. *For all $1\leq i,j\leq n$, $\phi(a_i)=\phi(b_j)$.*

Proof. By Lemma 4.5.3, we have the braid relations $(a_{i+1}^{-1}a_i)a_{i+1}(a_{i+1}^{-1}a_i)^{-1}=a_i$ and $(b_{i+1}^{-1}b_i)b_{i+1}(b_{i+1}^{-1}b_i)^{-1}=b_i$ for all $i\in\{1,\ldots,n-1\}$. Hence $\phi(a_i)=\phi(a_j)$ and $\phi(b_i)=\phi(b_j)$ for all $i,j\in\{1,\ldots,n-1\}$. The parity flip relation $ca_1c^{-1}=b_1$ allows us to deduce that a_i and b_j are conjugate for all $1\leq i,j\leq n$ and $\phi(a_i)=\phi(b_j)$. ∎

Lemma 4.6.5. *The abelianization $G_n/[G_n,G_n]$ admits the presentation*

$$\langle a,d,A,B\mid B^{n^2-n}=A^{1-n^2},B^{n^2}=A^{-n^2-1},a^2=B,d^2=A^{n+1},\mathcal{T}\rangle$$

where $a=\phi(a_1)$, $d=\phi(c)$, $A=\phi(A_{1,2})$, $B=\phi(A_{1,3})$, and $\mathcal{T}$ is the set of all commutators.

Proof. Lemmas 4.6.1 and 4.6.4 show that the elements $\phi(a_1),\phi(c),\phi(A_{1,2})$ and $\phi(A_{1,3})$ form a generating set for $G_n/[G_n,G_n]$.

Lemmas 4.6.2 and 4.6.3 show that the relations $B^{n^2-n}=A^{1-n^2}$ and $B^{n^2}=A^{-n^2-1}$ hold in $G_n/[G_n,G_n]$. Applying ϕ to the relation $a_1^2=A_{1,3}$ shows that $a^2=B$. Applying ϕ to the relation $c^2=A_{1,2}A_{3,4}\cdots A_{2n+1,2n+2}$ gives the relation $d^2=A^{n+1}$.

Lemma 4.6.2 shows that for all $\ell\in\{1,\ldots,2n+1\}$, the relation

$$A_{\ell,2n+2}=(\overline{A}_{1,2}\cdots\overline{A}_{1,2n})(\overline{A}_{2,3}\cdots\overline{A}_{2,2n})\cdots(\overline{A}_{2n-2,2n-1}\overline{A}_{2n-2,2n})(\overline{A}_{2n-1,2n})$$

is derivable from $\mathcal{T}$ and $B^{n^2-n}=A^{1-n^2}$.

It remains to show that in the abelianization, the relations from the presentation of G_n in Theorem 4.5.7 can be derived from the proposed defining relations.

The commutator relations 1-4 of Theorem 4.5.7 all map to the identity under ϕ. The braid relations 5 and 7 of Theorem 4.5.7 are derivable from $\mathcal{T}$. The braid relation 6 is also derivable from $\mathcal{T}$ since all relations in this family take the form $a=a$ in the abelianization. Relation 8 is derivable from $B^{n^2}=A^{-n^2-1}$ by Lemma 4.6.3. Relations 9 and 10 are derivable from $a^2=B$ and $d^2=A^{n+1}$ respectively. The image $\phi(ca_ic^{-1}b_i^{-1})$ is the identity by Lemma 4.6.4. Finally, the conjugation relations 12-14 are all of the form $A=A$ or $B=B$ in the abelianization, so they are all derivable from $\mathcal{T}$. ∎

We now have everything needed to prove Theorem 4.6.6.

Theorem 4.6.6. *Let $k \geq 3$. Then*

$$H_1(\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B}); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/(n(n-1)^2)\mathbb{Z} & \text{if } n \text{ is odd} \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/(2n(n-1)^2)\mathbb{Z} & \text{if } n \text{ is even.} \end{cases}$$

Proof. Recall $H_1(\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B}); \mathbb{Z}) = G_n/[G_n, G_n]$. We will start with the presentation from Lemma 4.6.5 and perform Tietze transformations to simplify it.

Starting with $B^{n^2-n} = A^{1-n^2}$, we may substitute in the relation $B^{n^2} = A^{-n^2-1}$ to obtain $A^2 = B^{-n}$. Thus we may add the relation $A^2 = B^{-n}$ to the set of defining relations. Observe $B^{n^2-n} = A^{1-n^2}$ is derivable from $A^2 = B^{-n}$ and $B^{n^2} = A^{-n^2-1}$ so we may delete the relation $B^{n^2-n} = A^{1-n^2}$.

Similarly, we may add the relation $A^{(n-1)^2} = 1$ and delete the relation $B^{n^2} = A^{-n^2-1}$. Deleting the generator B and replacing it with a^2 then gives the presentation

$$G_n/[G_n, G_n] \cong \langle a, d, A \mid A^2 = a^{-2n}, A^{(n-1)^2} = 1, d^2 = A^{n+1}, \mathcal{T} \rangle. \tag{4.8}$$

This presentation has presentation matrix $\begin{bmatrix} 2n & 0 & 2 \\ 0 & 0 & (n-1)^2 \\ 0 & 2 & -1-n \end{bmatrix}$.

If n is odd, this matrix has Smith normal form $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & n(n-1)^2 \end{bmatrix}$. Therefore

$$H_1(\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B}); \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/(n(n-1)^2)\mathbb{Z}.$$

If n is even, the presentation matrix has Smith normal form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2n(n-1)^2 \end{bmatrix}$, so

$$H_1(\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B}); \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/(2n(n-1)^2)\mathbb{Z}. \qquad \blacksquare$$

We now apply Theorem 4.6.6 to compute the first Betti number of the balanced superelliptic mapping class group $\mathrm{SMod}_{g,k}(\Sigma_g)$. The first Betti number of a group G is the rank of the abelian group $H_1(G; \mathbb{Z}) = G/[G, G]$.

Let $D < \mathrm{Mod}(\Sigma_g)$ be the image of the deck group of the balanced superelliptic cover $p_{g,k}$. Recall $D \cong \mathbb{Z}/k\mathbb{Z}$.

Theorem 4.6.7. *Let $k > 2$. The abelianization of the balanced superelliptic mapping class group $H_1(\mathrm{SMod}_{g,k}(\Sigma_g); \mathbb{Z})$ is a finite, non-cyclic group. In particular, the first Betti number of $\mathrm{SMod}_{g,k}(\Sigma_g)$ is 0.*

Proof. The Birman-Hilden theorem (Theorem 2.3.2) gives a short exact sequence

$$1 \longrightarrow \mathbb{Z}/k\mathbb{Z} \longrightarrow \mathrm{SMod}_{g,k}(\Sigma_g) \longrightarrow \mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B}) \longrightarrow 1.$$

Since the abelianization functor is right exact, we have the exact sequence

$$\mathbb{Z}/k\mathbb{Z} \longrightarrow H_1(\mathrm{SMod}_{g,k}(\Sigma_g);\mathbb{Z}) \longrightarrow H_1(\mathrm{LMod}_{g,k}(\Sigma_0,\mathcal{B});\mathbb{Z}) \longrightarrow 1.$$

Since $\mathbb{Z}/k\mathbb{Z}$ and $H_1(\mathrm{LMod}_{g,k}(\Sigma_0,\mathcal{B});\mathbb{Z})$ are both finite, so is $H_1(\mathrm{SMod}_{g,k}(\Sigma_g);\mathbb{Z})$. Moreover, since $H_1(\mathrm{LMod}_{g,k}(\Sigma_0,\mathcal{B});\mathbb{Z})$ is not cyclic, neither is $H_1(\mathrm{SMod}_{g,k}(\Sigma_g);\mathbb{Z})$. ■

To complete the story, one may use the presentation of the hyperelliptic mapping class group $\mathrm{SMod}_{g,2}(\Sigma_g)$ in [7, Theorem 8] to deduce that

$$H_1(\mathrm{SMod}_{g,2}(\Sigma_g);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/(2g+1)\mathbb{Z} & \text{if } g \text{ is odd} \\ \mathbb{Z}/(4g+2)\mathbb{Z} & \text{if } g \text{ is even.} \end{cases}$$

In particular, the first Betti number of $\mathrm{SMod}_{g,2}(\Sigma_g)$ is also 0.

Chapter 5

A Family of Cyclic Branched Covers Over the Torus

The thesis so far has dealt exclusively with cyclic branched covers over the sphere. A family of cyclic branched covers over the torus branched at two points will be investigated in this chapter.

Consider a torus with two punctures and let $x, y, z \in H_1(\Sigma_{1,2}; \mathbb{Z})$ be the homology classes of the loops α, β, δ respectively from Figure 5.1. The set $\{x, y, z\}$ forms a basis for $H_1(\Sigma_{1,2}; \mathbb{Z}) \cong \mathbb{Z}^{\oplus 3}$.

For any integer $k \geq 2$ define the surjective homomorphism $\varphi_k : H_1(\Sigma_{1,2}; \mathbb{Z}) \to \mathbb{Z}/k\mathbb{Z}$ by $\varphi_k(c_1 x + c_2 y + c_3 z) = c_3 \mod k$. Therefore φ_k determines a cyclic cover $p_k : \widetilde{\Sigma} \to \Sigma_{1,2}$. Let z' be the homology class of δ' in Figure 5.1. Then $z' = -z$, so $\varphi(z) = 1$ and $\varphi(z') = -1$. Since z and z' are the homology classes of loops surrounding only the punctures, p_k can be completed to a branched cover branched at two points $\mathcal{B} \subset \Sigma_1$, each with one preimage (see Section 2.2.2).

Using the Riemann-Hurwitz formula, we can conclude the genus of $\widetilde{\Sigma}$ is k. Abusing notation, from now on we denote the unbranched covers and associated branched covers by $p_k : \Sigma_{k,2} \to \Sigma_{1,2}$ and $p_k : \Sigma_k \to \Sigma_1$ respectively. For a picture of the covers p_2, p_3, and p_4, see Figure 5.2.

Let $\mathrm{LMod}_k(\Sigma_1, \mathcal{B}) < \mathrm{Mod}(\Sigma_1, \mathcal{B})$ be the liftable mapping class group corresponding to the branched cover $p_k : \Sigma_k \to \Sigma_1$. The goal of this chapter is to find presentations for the groups $\mathrm{LMod}_k(\Sigma_1, \mathcal{B})$ by using the Reidemeister-Schreier rewriting process from Section 2.4.2. While the goal as stated will unfortunately not be achieved, a finite presentation for $\mathrm{LMod}_k(\Sigma_1, \mathcal{B})$ is obtained for $k = 2, 3, 4$, and a finite generating set is obtained for $k = 5, 6$.

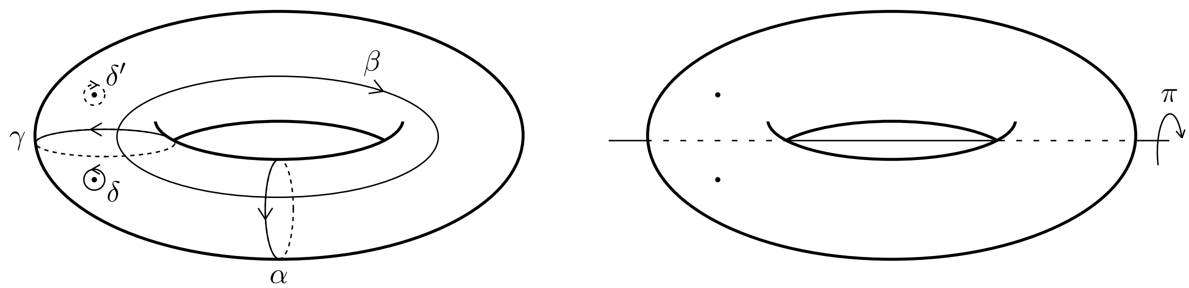


Figure 5.1: The curves $\alpha, \beta, \gamma, \delta$ and δ' on the left. The hyperelliptic involution ι on the right. In both images, the puncture at the top is on the back side of the torus.

5.1 Preparing the Reidemeister-Schreier rewriting process

In order to apply the Reidemeister-Schreier rewriting process, we need a finite presentation for $\mathrm{Mod}(\Sigma_1, \mathcal{B})$, a Schreier system of right coset representatives for $\mathrm{LMod}_k(\Sigma_1, \mathcal{B})$, and a way of identifying the coset representative corresponding to any element of $\mathrm{Mod}(\Sigma_1, \mathcal{B})$. We begin with a presentation for $\mathrm{Mod}(\Sigma_1, \mathcal{B})$.

5.1.1 A presentation for $\mathrm{Mod}(\Sigma_1, \mathcal{B})$

Let $\iota \in \mathrm{Mod}(\Sigma_1, \mathcal{B})$ be the hyperelliptic involution shown in Figure 5.1. Note that ι switches the two marked points $\mathcal{B} \subset \Sigma_1$.

Proposition 5.1.1. $\mathrm{Mod}(\Sigma_{1,2})$ *admits the presentation*

$$\langle a, b, c, d \mid aba = bab, bcb = cbc, [a, c], [a, d], [b, d], [c, d], d^2, (abc)^4 \rangle$$

where $d = \iota$ and $a = T_\alpha$, $b = T_\beta$ and $c = T_\gamma$ where $\alpha, \beta,$ and γ are the curves from Figure 5.1.

Proof. Consider the short exact sequence

$$1 \to \mathrm{PMod}(\Sigma_1, \mathcal{B}) \to \mathrm{Mod}(\Sigma_1, \mathcal{B}) \xrightarrow{\phi} S_2 \longrightarrow 1$$

given by the action of $\mathrm{Mod}(\Sigma_1, \mathcal{B})$ on the two marked points. We will apply Lemma 2.4.2 to this short exact sequence.

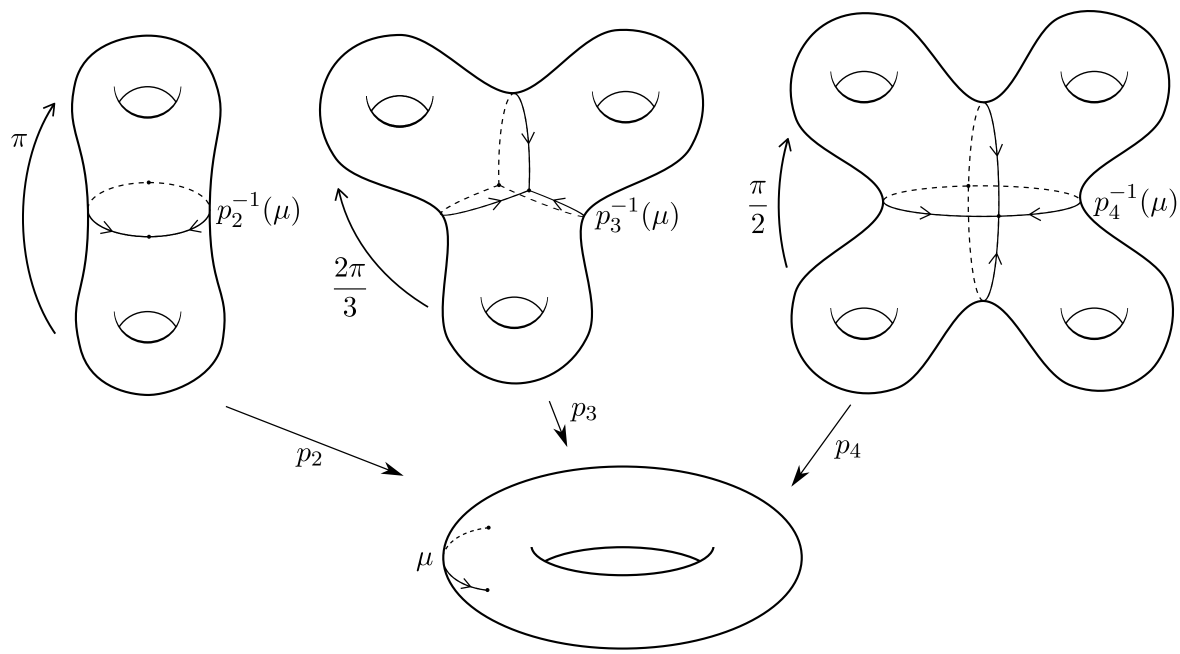


Figure 5.2: The arc μ and its preimages under p_2, p_3, and p_4. The rotations are generators of the deck groups.

The group S_2 admits the presentation $\langle d \mid d^2 \rangle$, and $\mathrm{PMod}(\Sigma_{1,2})$ admits the presentation

$$\langle a, b, c \mid ac = ca, aba = bab, bcb = cbc, (abc)^4 \rangle$$

where $a = T_\alpha$, $b = T_\beta$, and $c = T_\gamma$ are Dehn twists about the curves α, β, and γ from Figure 5.1 respectively (see [42, Theorem 3.2.1]).

To apply Lemma 2.4.2, let $\tilde{d} = \iota$ and notice $\phi(\tilde{d}) = d$. Since $\iota^2 = 1$, in the notation from Lemma 2.4.2 we have $R_1 = \{\tilde{d}^2\}$. Since ι preserves the unoriented isotopy classes of α,β, and γ, we have

$$R_2 = \{\tilde{d}a\tilde{d}^{-1}a^{-1}, \tilde{d}b\tilde{d}^{-1}b^{-1}, \tilde{d}c\tilde{d}^{-1}c^{-1}\}.$$

Applying Lemma 2.4.2 and replacing $\tilde{d}$ by d we see $\mathrm{Mod}(\Sigma_{1,2})$ admits the presentation

$$\langle a, b, c, d \mid aba = bab, bcb = cbc, [a, c], [a, d], [b, d], [c, d], d^2, (abc)^4 \rangle$$

completing the proof. ■

Those familiar with the braid group will notice that the presentation for $\mathrm{PMod}(\Sigma_1, \mathcal{B})$ shows that $\mathrm{PMod}(\Sigma_1, \mathcal{B}) \cong B_4/Z(B_4)$ where B_4 is the braid group on 4 strands and $Z(B_4)$ is the center of B_4. It is worth noting that the presentation in the statement of Proposition 5.1.1 is a presentation for the direct product $\mathrm{PMod}(\Sigma_1, \mathcal{B}) \times S_2 \cong B_4/Z(B_4) \times S_2$.

5.1.2 A Schreier system

Our focus now shifts to finding a Schreier system of right coset representatives of $\mathrm{Mod}(\Sigma_1, \mathcal{B})$ mod $\mathrm{LMod}_k(\Sigma_1, \mathcal{B})$. To do this, we look at the action of $\mathrm{Mod}(\Sigma_1, \mathcal{B})$ on $H_1(\Sigma_{1,2}; \mathbb{Z})$.

Let $\{x, y, z\} \subset H_1(\Sigma_{1,2}; \mathbb{Z})$ be the basis defined above. Let $\Psi : \mathrm{Mod}(\Sigma_1, \mathcal{B}) \to \mathrm{GL}_3(\mathbb{Z})$ be the homomorphism given by the action of $\mathrm{Mod}(\Sigma_1, \mathcal{B})$ with respect to this basis.

Lemma 5.1.2. *Let α, β, γ be the curves from Figure 5.1, and μ the arc in Figure 5.2. Let τ be the half Dehn twist about μ. Then*

$$\Psi(T_\alpha) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \Psi(T_\beta) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\Psi(T_\gamma) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad \Psi(\tau) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Proof. For any $f \in \mathrm{Mod}(\Sigma_1, \mathcal{B})$, let $f_* \in \mathrm{Aut}(H_1(\Sigma_{1,2}; \mathbb{Z}))$ be the induced automorphism of any representative of f.

Since $[\alpha] = x$ and $[\beta] = y$, $(T_\alpha)_*(x) = x$ and $(T_\beta)_*(y) = y$. Since δ is disjoint from α, β, and γ, $(T_\alpha)_*(z) = (T_\beta)_*(z) = (T_\gamma)_*(z) = z$. Since μ is disjoint from α and β, $\tau_*(x) = x$ and $\tau_*(y) = y$. Since α and γ are disjoint, $(T_\gamma)_*(x) = x$. For any homeomorphism f such that $[f] = \tau$, the curve $f(\delta)$ is homotopic to δ'. Since $[\delta] = -[\delta']$ in $H_1(\Sigma_{1,2}; \mathbb{Z})$, $\tau_*(z) = -z$. It remains to show $(T_\alpha)_*(y) = x + y$, $(T_\beta)_*(x) = x - y$ and $(T_\gamma)_*(y) = x + y + z$.

Figure 5.3A shows three representatives for the elements $\overline{\alpha}, \overline{\beta}, \overline{\delta} \in \pi_1(\Sigma_{1,2}, x_0)$. Let $\Phi : \pi_1(\Sigma_{1,2}, x_0) \to H_1(\Sigma_{1,2}; \mathbb{Z})$ be the Hurewicz homomorphism. Then $\Phi(\overline{\alpha}) = x$, $\Phi(\overline{\beta}) = y$, and $\Phi(\overline{\delta}) = z$. Figure 5.3B shows $T_\alpha(\beta)$, which is homotopic to any representative of $\overline{\alpha}\overline{\beta} \in \pi_1(\Sigma_{1,2}, x_0)$. Therefore $(T_\alpha)_*(y) = x+y$. Figure 5.3C shows $T_\beta(\alpha)$, which is homotopic to any representative of $\overline{\alpha}\overline{\beta}^{-1} \in \pi_1(\Sigma_{1,2}, x_0)$. Therefore $(T_\beta)(x) = x - y$. Finally, the curve $T_\gamma(\beta)$ in Figure 5.3D is homotopic to any representative of $\overline{\alpha}\overline{\beta}\overline{\delta} \in \pi_1(\Sigma_{1,2}, x_0)$. Therefore $(T_\gamma)_*(y) = x + y + z$ completing the proof. ■

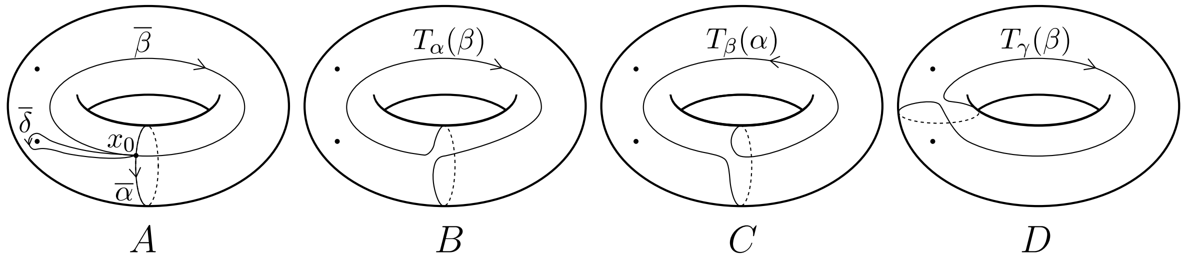


Figure 5.3: *Left to right*: representatives of $\overline{\alpha}, \overline{\beta}, \overline{\delta} \in \pi_1(\Sigma_{1,2}, x_0)$, the curve $T_\alpha(\beta)$, the curve $T_\beta(\alpha)$, and the curve $T_\gamma(\beta)$.

Lemma 5.1.3. *The image of* Ψ *is given by*

$$\Psi(\mathrm{Mod}(\Sigma_1, \mathcal{B})) = \left\{ \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ v_1 & v_2 & \pm 1 \end{bmatrix} : \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}), v_1, v_2 \in \mathbb{Z} \right\}.$$

Proof. Let α, β, γ be the curves from Figure 5.1, and μ the arc in Figure 5.2. Then $\mathrm{Mod}(\Sigma_1, \mathcal{B})$ is generated by $\{T_\alpha, T_\beta, T_\gamma, \tau\}$. For notational convenience, let $Q = \Psi(T_\alpha)$, $R = \Psi(T_\beta)$, $S = \Psi(T_\gamma)$, and $T = \Psi(\tau)$ be the matrices from Lemma 5.1.2. Let

$$G := \left\{ \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ v_1 & v_2 & \pm 1 \end{bmatrix} : \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}), v_1, v_2 \in \mathbb{Z} \right\} < \mathrm{GL}_3(\mathbb{Z}).$$

It suffices to show G is generated by Q, R, S, and T.

Let $B = \left[\begin{smallmatrix} A & 0 \\ \mathbf{v} & \pm 1 \end{smallmatrix}\right] \in G$ with $A \in \mathrm{SL}_2(\mathbb{Z})$ and $\mathbf{v} = (v_1, v_2) \in \mathbb{Z} \oplus \mathbb{Z}$. We will write B as a word in $\{Q, R, S, T\}$ by starting with the identity and multiplying on the left using the following procedure.

1. If the bottom right entry of B is -1, start with T, otherwise skip this step.
2. Let $k = \gcd(v_1, v_2)$ and multiply on the left by $S^k = \left[\begin{smallmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{smallmatrix}\right]$. This gives us a word representing $\left[\begin{smallmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & k & \pm 1 \end{smallmatrix}\right]$.
3. Since $\mathrm{SL}_2(\mathbb{Z})$ is generated by $\{\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right]\}$, there exists a word in Q and R representing $\left[\begin{smallmatrix} Z & 0 \\ 0 & 1 \end{smallmatrix}\right]$ for any $Z \in \mathrm{SL}_2(\mathbb{Z})$. Since $\gcd(0, k) = \gcd(v_1, v_2) = k$, there exists a $Y \in \mathrm{SL}_2(\mathbb{Z})$ such that $\begin{bmatrix} 0 & k \end{bmatrix} Y = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$. Multiply on the left by the word in Q and R representing $\left[\begin{smallmatrix} Y & 0 \\ 0 & 1 \end{smallmatrix}\right]$. This leaves us with a word representing $\left[\begin{smallmatrix} X & 0 \\ \mathbf{v} & \pm 1 \end{smallmatrix}\right]$ for some $X \in \mathrm{SL}_2(\mathbb{Z})$.

4. Finally, since $AX^{-1} \in \mathrm{SL}_2(\mathbb{Z})$, there exists a word in Q and R representing $\left[\begin{smallmatrix} AX^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Multiply on the left by this word. The result is the desired element $B \in G$. ■

Lemma 5.1.4. *A mapping class $f \in \mathrm{Mod}(\Sigma_1, \mathcal{B})$ is in $\mathrm{LMod}_k(\Sigma_1, \mathcal{B})$ if and only if*

$$\Psi(f) \in \left\{ \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ v_1 & v_2 & \pm 1 \end{bmatrix} : \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}), v_1 \equiv v_2 \equiv 0 \mod k \right\}.$$

Proof. With respect to the basis $\{x, y, z\}$ of $H_1(\Sigma_{1,2}; \mathbb{Z})$ we have

$$\ker \varphi_k = \left\{ \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \in H_1(\Sigma_{1,2}; \mathbb{Z}) : c_3 \equiv 0 \mod k \right\}.$$

Let $c = c_1 x + c_2 y + c_3 z \in \ker \varphi_k$. By Lemma 2.3.5, it suffices to show

$$\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ v_1 & v_2 & \pm 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} A_{11}c_1 + A_{12}c_2 \\ A_{21}c_1 + A_{22}c_2 \\ v_1 c_1 + v_2 c_2 \pm c_3 \end{bmatrix} \in \ker \varphi_k$$

if and only if $v_1 \equiv v_2 \equiv 0 \mod k$. If $v_1 \equiv v_2 \equiv 0 \mod k$, then $v_1 c_1 + v_2 c_2 \pm c_3 \equiv 0 \mod k$. Conversely, suppose $v_1 \not\equiv 0 \mod k$. Consider $\left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right] \in \ker \varphi_k$. Then

$$\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ v_1 & v_2 & \pm 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} A_{11} \\ A_{21} \\ v_1 \end{bmatrix} \notin \ker \varphi_k.$$

If $v_2 \not\equiv 0 \mod k$, then the same argument with $\left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right] \in \ker \varphi_k$ completes the proof. ■

Lemma 5.1.5. *Let $f, g \in \mathrm{Mod}(\Sigma_1, \mathcal{B})$ with $\Psi(f) = \left[\begin{smallmatrix} A_1 & 0 \\ \mathbf{v}_1 & \epsilon_1 \end{smallmatrix}\right]$ and $\Psi(g) = \left[\begin{smallmatrix} A_2 & 0 \\ \mathbf{v}_2 & \epsilon_2 \end{smallmatrix}\right]$ where $A_i \in \mathrm{SL}_2(\mathbb{Z})$, $\mathbf{v}_i \in \mathbb{Z} \oplus \mathbb{Z}$, and $\epsilon_i = \pm 1$. Then f and g are in the same right coset of $\mathrm{LMod}_k(\Sigma_1, \mathcal{B})$ if and only if $\mathbf{v}_1 \equiv \mathbf{v}_2 \mod k$ when $\epsilon_1 = \epsilon_2$ and $\mathbf{v}_1 \equiv -\mathbf{v}_2 \mod k$ when $\epsilon_1 = -\epsilon_2$.*

Proof. As above, let τ be the half twist about the arc μ from Figure 5.2. By Lemma 5.1.4, $\tau \in \mathrm{LMod}_k(\Sigma_1, \mathcal{B})$ since $\Psi(\tau) = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right]$. Therefore for any $f \in \mathrm{Mod}(\Sigma_1, \mathcal{B})$, τf and f are in the same right coset of $\mathrm{LMod}_k(\Sigma_1, \mathcal{B})$. Since multiplying a matrix on the left by $\Psi(\tau)$ simply negates the third row, it suffices to prove the result in the case where $\epsilon_1 = \epsilon_2 = 1$.

Since $\left[\begin{smallmatrix} A & 0 \\ \mathbf{v} & 1 \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} A^{-1} & 0 \\ -\mathbf{v}A^{-1} & 1 \end{smallmatrix}\right]$ we have $\Psi(fg^{-1}) = \left[\begin{smallmatrix} A_1 A_2^{-1} & 0 \\ (\mathbf{v}_1 - \mathbf{v}_2)A_2^{-1} & 1 \end{smallmatrix}\right]$. Since $\det(A_2) = 1$, A_2 is invertible over $\mathbb{Z}/k\mathbb{Z}$. By Lemma 5.1.4, $fg^{-1} \in \mathrm{LMod}_k(\Sigma_1, \mathcal{B})$ if and only if $\mathbf{v}_1 \equiv \mathbf{v}_2 \mod k$, completing the proof. ■

We have the following immediate corollary of Lemma 5.1.5.

Corollary 5.1.6. *The index of* $\mathrm{LMod}_k(\Sigma_1, \mathcal{B})$ *in* $\mathrm{Mod}(\Sigma_1, \mathcal{B})$ *is equal to* k^2*.*

Lemma 5.1.5 not only allows us to find a Schreier system of right coset representatives, but also provides a tool for identifying to which coset an element of $\mathrm{Mod}(\Sigma_1, \mathcal{B})$ belongs.

To proceed, we will focus on the case $k = 3$. Let a, b, c, d be the generating symbols from Proposition 5.1.1.

Proposition 5.1.7. *The set* $\mathcal{S} = \{1, c, c^{-1}, cb, cb^{-1}, c^{-1}b, c^{-1}b^{-1}, cba, cb^{-1}c\}$ *is a Schreier system of right coset representatives of* $\mathrm{Mod}(\Sigma_1, \mathcal{B})$ mod $\mathrm{LMod}_3(\Sigma_1, \mathcal{B})$*.*

Proof. Recall $a = T_\alpha$, $b = T_\beta$, and $c = T_\gamma$ where α, β, and γ are the curves defined in Figure 5.1. Let $q : \mathrm{GL}_3(\mathbb{Z}) \to \mathrm{GL}_3(\mathbb{Z}/3\mathbb{Z})$ be the quotient map. By Lemma 5.1.2 we have

$$q\Psi(1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad q\Psi(cb) = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \quad q\Psi(c^{-1}b^{-1}) = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

$$q\Psi(c) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad q\Psi(cb^{-1}) = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad q\Psi(cba) = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$q\Psi(c^{-1}) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \quad q\Psi(c^{-1}b) = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \quad q\Psi(cb^{-1}c) = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

By Lemma 5.1.5, $\mathcal{S}$ is a complete set of right coset representatives of $\mathrm{LMod}_3(\Sigma_1, \mathcal{B})$. Observing that any initial segment of a word in $\mathcal{S}$ is again in $\mathcal{S}$ completes the proof. ∎

We are now ready to apply the Reidemeister-Schreier rewriting process to arrive at a presentation for $\mathrm{LMod}_3(\Sigma_1, \mathcal{B})$.

5.1.3 Presentations and generating sets

Applying Theorem 2.4.3 using Lemma 5.1.5 and Proposition 5.1.7 we get the following presentation for $\mathrm{LMod}_3(\Sigma_1, \mathcal{B})$.

Theorem 5.1.8. *The liftable mapping class group* $\mathrm{LMod}_3(\Sigma_1, \mathcal{B})$ *admits the presentation*

$$\langle A, B, C, D \mid [A, C], [A, D], [B, D], [C, D], \\ ABA = BAB, BCBCBC = CBCBCB, D^2, (BACBC)^4 \rangle$$

where $A = T_\alpha$*,*$B = T_\beta$*,*$C = T_\gamma^3$*, and* $D = \iota$*.*

The proof relies on Sage code to perform the Reidemeister-Schreier rewriting process, as well as to help perform some of the Tietze transformations to arrive at the desired presentation. Since it is lengthy and computationally dense, the proof is deferred to Appendix A.

Using the same method as in the $k = 3$ case, presentations for the cases $k = 2$ and $k = 4$ are obtained, and finite generating sets are obtained for $k = 5$ and $k = 6$. Unfortunately the current method of arriving at presentations using the Reidemeister-Schreier rewriting process does not appear to lend itself well to finding a general form for presentations for all k. The next paragraph summarizes the results for $k = 2, 4, 5, 6$, which were arrived at using similar methods to the $k = 3$ case.

The liftable mapping class group $\mathrm{LMod}_2(\Sigma_1, \mathcal{B})$ admits the presentation

$$\langle A, B, C, D \mid [A, D], [A, C], [C, D], [B, D], D^2, ABA = BAB, BCBC = CBCB, (BAC)^3\rangle$$

where $A = T_\alpha$, $B = T_\beta$, $C = T_\gamma^2$, and $D = \iota$.

$\mathrm{LMod}_4(\Sigma_1, \mathcal{B})$ admits the presentation

$$\begin{aligned}\langle A, B, C, D, H \mid &[A, D], [A, C], [C, D], [B, D], [H, D], D^2, ABA = BAB,\\ &HAHA = AHAH, CBH = HCB, HCBCB = BHCBC, (BAHC)^3\rangle\end{aligned}$$

where $A = T_\alpha$, $B = T_\beta$, $C = T_\gamma^4$, $H = T^2_{T^2_\gamma(\beta)}$ and $D = \iota$.

$\mathrm{LMod}_5(\Sigma_1, \mathcal{B})$ and $\mathrm{LMod}_6(\Sigma_1, \mathcal{B})$ are generated by

$$\{T_\alpha, T_\beta, T_\gamma^5, \iota, T_\gamma T_\beta^3 T_\gamma T_\beta^{-2} T_\gamma\} \quad \text{and} \quad \{T_\alpha, T_\beta, T_\gamma^6, \iota, T^3_{T^2_\gamma(\beta)}, T^2_{T^3_\gamma(\beta)}\}$$

respectively.

5.1.4 Abelianization

Using the presentation from Theorem 5.1.8 we may now compute the abelianization of $\mathrm{LMod}_3(\Sigma_1, \mathcal{B})$. Recall that for any group G, $H_1(G; \mathbb{Z}) \cong G/[G, G]$.

Theorem 5.1.9. *$H_1(\mathrm{LMod}_3(\Sigma_1, \mathcal{B}); \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}$.*

Proof. In the following computations, let $\mathcal{T}$ be the set of all commutators. By Theorem 5.1.8, $H_1(\mathrm{LMod}_3(\Sigma_1, \mathcal{B}); \mathbb{Z})$ admits the presentation

$$\langle A, B, C, D \mid D^2, A = B, (AB^2C^2)^4, \mathcal{T}\rangle \cong \langle A, C, D \mid D^2, A^{12}C^8, \mathcal{T}\rangle.$$

The presentation matrix for this presentation is $\left[\begin{smallmatrix} 2 & 0 & 0 \\ 0 & 4 & 8 \end{smallmatrix}\right]$, which has Smith normal form $\left[\begin{smallmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \end{smallmatrix}\right]$. Therefore $H_1(\mathrm{LMod}_3(\Sigma_1, \mathcal{B}); \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}$. ∎

By similar computations to those in the proof of Theorem 5.1.9 we have

$$H_1(\mathrm{LMod}_k(\Sigma_1,\mathcal{B});\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z} & \text{if } k=2 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & \text{if } k=4. \end{cases}$$

We have the following corollary. Recall that the first Betti number of a group G, denoted $b_1(G)$, is the rank of the abelian group $H_1(G;\mathbb{Z})$. Let $\mathrm{SMod}_k(\Sigma_k)$ be the symmetric mapping class group corresponding to the cyclic branched cover $p_k : \Sigma_k \to \Sigma_1$.

Corollary 5.1.10. *The first Betti number of* $\mathrm{SMod}_3(\Sigma_3)$ *is* 1.

Proof. By the Birman-Hilden theorem (Theorem 2.3.2) there is a short exact sequence

$$1 \longrightarrow \mathbb{Z}/k\mathbb{Z} \longrightarrow \mathrm{SMod}_k(\Sigma_k) \longrightarrow \mathrm{LMod}_k(\Sigma_1,\mathcal{B}) \longrightarrow 1.$$

Since the abelianization functor is right exact we have

$$\mathbb{Z}/k\mathbb{Z} \longrightarrow H_1(\mathrm{SMod}_k(\Sigma_k);\mathbb{Z}) \longrightarrow H_1(\mathrm{LMod}_k(\Sigma_1,\mathcal{B});\mathbb{Z}) \longrightarrow 1.$$

Since $\otimes\mathbb{Q}$ is an exact functor we have $H_1(\mathrm{SMod}_k(\Sigma_k);\mathbb{Z})\otimes\mathbb{Q} \cong H_1(\mathrm{LMod}_k(\Sigma_1,\mathcal{B});\mathbb{Z})\otimes\mathbb{Q}$. Therefore $b_1(\mathrm{SMod}_k(\Sigma_k)) = b_1(\mathrm{LMod}_k(\Sigma_1,\mathcal{B}))$. Computing $b_1(\mathrm{LMod}_3(\Sigma_1,\mathcal{B}))$ by Theorem 5.1.9 completes the proof. ∎

Again, performing similar computations for $k=2$ and $k=4$ we get

$$b_1(\mathrm{SMod}_k(\Sigma_k)) = \begin{cases} 1 & \text{if } k=2 \\ 2 & \text{if } k=4. \end{cases}$$

5.2 Interactions between the liftable mapping class groups

In the previous section, we constructed a family of finite index subgroups

$$\{\mathrm{LMod}_k(\Sigma_1,\mathcal{B}) : k \geq 2\}$$

of $\mathrm{Mod}(\Sigma_1,\mathcal{B})$. A natural question to ask is how do these subgroups interact with each other.

To state the main result of this section, we need the following standard result from group theory. For subgroups $H, K < G$, denote $HK := \{hk : h \in H, k \in K\}$. In general HK is not a group.

Lemma 5.2.1. *Let $H, K < G$ be subgroups of finite index. Then*

$$[G : H \cap K] \leq [G : H][G : K]$$

with equality if and only if $G = HK$.

For the statement of the next theorem, let $\mathrm{LMod}_1(\Sigma_1, \mathcal{B}) = \mathrm{Mod}(\Sigma_1, \mathcal{B})$.

Theorem 5.2.2. *For integers $k, l \geq 2$, $\mathrm{LMod}_k(\Sigma_1, \mathcal{B}) \cap \mathrm{LMod}_l(\Sigma_1, \mathcal{B}) = \mathrm{LMod}_{\mathrm{lcm}(k,l)}(\Sigma_1, \mathcal{B})$ and $\mathrm{LMod}_k(\Sigma_1, \mathcal{B}) \, \mathrm{LMod}_l(\Sigma_1, \mathcal{B}) = \mathrm{LMod}_{\gcd(k,l)}(\Sigma_1, \mathcal{B})$.*

Proof. For ease of notation, let $G_k = \mathrm{LMod}_k(\Sigma_1, \mathcal{B})$.

Let $\{x, y, z\}$ be the basis for $H_1(\Sigma_{1,2}; \mathbb{Z})$ defined at the beginning of this chapter. Let $\Psi : \mathrm{Mod}(\Sigma_1, \mathcal{B}) \to \mathrm{GL}_3(\mathbb{Z})$ be the homomorphism given by the action of $\mathrm{Mod}(\Sigma_1, \mathcal{B})$ on $H_1(\Sigma_{1,2}; \mathbb{Z})$ with respect to $\{x, y, z\}$. Let $f \in \mathrm{Mod}(\Sigma_1, \mathcal{B})$, and let $\Psi(f) = \left[\begin{smallmatrix} A & 0 \\ \mathbf{v} & \pm 1 \end{smallmatrix}\right]$ where $A \in \mathrm{SL}_2(\mathbb{Z})$ and $\mathbf{v} = (v_1, v_2) \in \mathbb{Z} \oplus \mathbb{Z}$ by Lemma 5.1.3. Then $f \in G_k \cap G_l$ if and only if $v_1 \equiv v_2 \equiv 0 \mod k$ and $v_1 \equiv v_2 \equiv 0 \mod l$ by Lemma 5.1.4. This is true if and only if $v_1 \equiv v_2 \equiv 0 \mod \mathrm{lcm}(k, l)$. Therefore by Lemma 5.1.4 we can conclude $G_k \cap G_l = G_{\mathrm{lcm}(k,l)}$.

Since $\gcd(k, l)$ divides k and l, it follows that G_k and G_l are both subgroups of $G_{\gcd(k,l)}$. By Lemma 5.2.1, it suffices to show $[G_{\gcd(k,l)} : G_{\mathrm{lcm}(k,l)}] = [G_{\gcd(k,l)} : G_k][G_{\gcd(k,l)} : G_l]$. By Corollary 5.1.6 we have

$$\begin{aligned} [G_{\gcd(k,l)} : G_{\mathrm{lcm}(k,l)}] &= \frac{\mathrm{lcm}(k,l)^2}{\gcd(k,l)^2} \\ &= \frac{k^2}{\gcd(k,l)^2} \frac{l^2}{\gcd(k,l)^2} \\ &= [G_{\gcd(k,l)} : G_k][G_{\gcd(k,l)} : G_l] \end{aligned}$$

since $\mathrm{lcm}(k, l) \gcd(k, l) = kl$. ■

Chapter 6

Lifting Mapping Classes on Surfaces with Boundary

So far we have only dealt with liftable mapping class groups coming from covers of closed surfaces. We now shift our focus to the case where the surfaces have non-empty boundary. Assume $\partial\widetilde{\Sigma} \neq \emptyset$ and $\partial\Sigma \neq \emptyset$. Let $p : \widetilde{\Sigma} \to \Sigma$ be a finite-sheeted, regular, possibly branched covering space with finite deck group D, branched at $\mathcal{B} \subset \Sigma \setminus \partial\Sigma$.

The primary goal of this chapter is to prove the analogous result to Lemma 2.3.5 for surfaces with boundary in Theorem 6.2.4. As a consequence we prove that $\mathrm{LMod}_p(\Sigma, \partial\Sigma, \mathcal{B})$ is a finite index subgroup of $\mathrm{Mod}(\Sigma, \partial\Sigma, \mathcal{B})$ in Theorem 6.2.9.

To characterize homeomorphisms that lift, it will once again be useful to look at the corresponding unbranched covering $p^\circ : \widetilde{\Sigma}^\circ \to \Sigma^\circ$.

If we were not concerned with fixing boundary components, then a homeomorphism f of Σ° lifts if and only if $f_* p^\circ_* \pi_1(\widetilde{\Sigma}^\circ, \tilde{x}) = p^\circ_* \pi_1(\widetilde{\Sigma}^\circ, \widetilde{f(x)})$ for any $x \in \Sigma^\circ$, $\tilde{x} \in p^{-1}(x)$ and $\widetilde{f(x)} \in p^{-1}(f(x))$. However, the following example shows that such a characterization in terms of the fundamental group is inadequate when boundaries are to be fixed.

Consider the 2-sheeted unbranched cover of an annulus S by an annulus $\widetilde{S}$, and choose a basepoint $x \in \partial S$. Then any homeomorphism fixing ∂S pointwise acts trivially on $\pi_1(S, x)$ and therefore lifts to a homeomorphism of $\widetilde{S}$. However, a Dehn twist on the annulus S does not have a lift that fixes $\partial\widetilde{S}$ pointwise, whereas the square of a Dehn twist does. Indeed, the square of a Dehn twist on S lifts to a Dehn twist on $\widetilde{S}$ (see Example 6.2.7 below).

To characterize when a homeomorphism lifts to a homeomorphism that fixes boundary components, we must instead look at the action of a homeomorphism on the fundamental groupoid $\pi_1(\Sigma^\circ, A)$ for a specific choice of basepoints $A \subset \Sigma^\circ$.

6.1 Groupoids

Here we survey the relevant results about groupoids. See the books [28] and [15] for more details.

A *groupoid* is a small category where every morphism is an isomorphism. Equivalently, a groupoid $\mathcal{G}$ is a disjoint collection of sets $\{G_{ij}\}_{i,j\in I}$ together with an associative partial operation $\cdot : G_{ij} \times G_{jk} \to G_{ik}$ such that

- For each $i \in I$ there is an identity $e_i \in G_{ii}$ such that $e_i f = f$ and $g e_i = g$ for all f, g such that the products $e_i f$ and $g e_i$ are defined, and
- For each $g \in G_{ij}$ there is an inverse $g^{-1} \in G_{ji}$ such that $gg^{-1} = e_i$ and $g^{-1}g = e_j$.

We will call I the *object set* of $\mathcal{G}$. If $|I| = 1$ (equivalently if the category has one object), then $\mathcal{G}$ is a group.

A groupoid is *connected* if $G_{ij} \neq \emptyset$ for all $i, j \in I$. Notice that G_{ii} is a group for all $i \in I$, and if $\mathcal{G}$ is connected then $G_{ii} \cong G_{jj}$ for all $i, j \in I$. The groups G_{ii} will be called *vertex groups*. From now on we will assume $\mathcal{G}$ is a connected groupoid.

Fix an $i_0 \in I$. For each $i \in I$ choose an element $\iota_i \in G_{i_0 i}$ with $\iota_{i_0} = e_{i_0}$. Then $\mathcal{G}$ is generated by the vertex group $G_{i_0 i_0}$ and $\{\iota_i\}_{i\in I}$. In fact, every element in G_{ij} is uniquely written as $\iota_i^{-1} g \iota_j$ for some $g \in G_{i_0 i_0}$. We call $\{\iota_i\}_{i\in I}$ a *star* based at i_0.

A *subgroupoid* $\mathcal{H} < \mathcal{G}$ is a collection of subsets $\{H_{ij} \subset G_{ij}\}_{i,j\in J}$ for some non-empty $J \subset I$ such that $\mathcal{H}$ is a groupoid with the operation from $\mathcal{G}$. A subgroupoid is *wide* if $J = I$. A subgroupoid $\mathcal{H} < \mathcal{G}$ is *normal* if $f^{-1} H_{ii} f \subset H_{jj}$ for all $f \in G_{ij}$. It follows that normal subgroupoids of connected groupoids are wide, and $h \mapsto f^{-1} h f$ is an isomorphism of groups $H_{ii} \cong H_{jj}$.

In this chapter we will be interested in connected normal subgroupoids of connected groupoids. Let $\mathcal{H}$ be a connected normal subgroupoid of $\mathcal{G}$. Construct the *quotient groupoid* $\mathcal{G}/\mathcal{H}$ to be a groupoid with one object, or a group, as follows. Put an equivalence relation $\sim$ on $\mathcal{G}$ by $a \sim b$ if there exists $x, y \in \mathcal{H}$ such that $a = xby$. The equivalence classes will be called the *cosets* of $\mathcal{H}$ in $\mathcal{G}$, and these are the elements of $\mathcal{G}/\mathcal{H}$. Define an operation on the cosets by $[a][b] = [axb]$ for some $x \in \mathcal{H}$. This is a well defined group operation on $\mathcal{G}/\mathcal{H}$.

Although we will not need it, the quotient groupoid can be defined for disconnected normal subgroupoids of connected groupoids. The only difference is that there is one object for each connected component of $\mathcal{H}$ (see [28, Chapter 12]).

6.1.1 Automorphisms of groupoids

Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids with object sets I and J respectively. A *morphism* $\Phi : \mathcal{G} \to \mathcal{H}$ is a functor from $\mathcal{G}$ to $\mathcal{H}$. Explicitly, Φ is a function $\phi : I \to J$ together with functions $\phi_{ij} : G_{ij} \to H_{\phi(i)\phi(j)}$ for all $i, j \in I$ such that $\phi_{ij}(a)\phi_{jk}(b) = \phi_{ik}(ab)$ for all $a \in G_{ij}$, $b \in G_{jk}$. It follows that $\phi_{ii}(e_i) = e_{\phi(i)}$ and $\phi_{ji}(g^{-1}) = \phi_{ij}(g)^{-1}$ for all $i, j \in I$ an $g \in G_{ij}$. We may suppress the subscripts and simply write $\phi_{ij}(g)$ as $\Phi(g)$.

An *automorphism* of $\mathcal{G}$ is a morphism $\Phi : \mathcal{G} \to \mathcal{G}$ with a two-sided inverse. The set of automorphisms of $\mathcal{G}$ forms a group under composition, denoted $\mathrm{Aut}(\mathcal{G})$.

We now restrict our attention to connected groupoids with finite object set. Let G be a group and consider the semi-direct product $G^n \rtimes \mathrm{Aut}(G)$. To set notation, the group operation on $G^n \rtimes \mathrm{Aut}(G)$ is given by

$$((g_1, \ldots, g_n), \psi)((h_1, \ldots, h_n), \varphi) = ((\psi(h_1)g_1, \ldots \psi(h_n)g_n), \psi\varphi)$$

for all $g_i, h_i \in G$ and $\psi, \varphi \in \mathrm{Aut}(G)$.

Define the *pure automorphism group* of $\mathcal{G}$ by

$$\mathrm{PAut}(\mathcal{G}) := \{\Phi \in \mathrm{Aut}(\mathcal{G}) : \phi(i) = i \text{ for all } i \in I\}.$$

Let $\mathcal{G}$ be a connected groupoid with object set $I = \{0, 1, \ldots, n\}$. Let $G = G_{00}$ be the vertex group at $0 \in I$. Choose a star $\{\iota_i\}_{i \in I} \subset \mathcal{G}$ based at $0 \in I$ and let $g_0 = e_0 \in G$.

Lemma 6.1.1. *The map $\theta : G^n \rtimes \mathrm{Aut}(G) \longrightarrow \mathrm{PAut}(\mathcal{G})$ given by*

$$\theta(((g_1, \ldots, g_n), \psi))(\iota_i^{-1} a \iota_j) = \iota_i^{-1} g_i^{-1} \psi(a) g_j \iota_j$$

is an isomorphism.

Lemma 6.1.1 is proved in [3, §3]. Note that the isomorphism θ depends on the choice of star.

Let $\mathcal{H} < \mathcal{G}$ be a normal subgroupoid. If $\Phi \in \mathrm{Aut}(\mathcal{G})$ is such that $\Phi(\mathcal{H}) \subset \mathcal{H}$, then Φ induces an automorphism $\overline{\Phi} \in \mathrm{Aut}(\mathcal{G}/\mathcal{H})$ by $\overline{\Phi}([a]) = [\Phi(a)]$. Define the subgroup $\mathrm{LAut}_{\mathcal{H}}(\mathcal{G}) < \mathrm{PAut}(\mathcal{G})$ by

$$\mathrm{LAut}_{\mathcal{H}}(\mathcal{G}) = \{\Phi \in \mathrm{PAut}(\mathcal{G}) : \Phi(\mathcal{H}) = \mathcal{H} \text{ and } \overline{\Phi} = \mathrm{id} \in \mathrm{Aut}(\mathcal{G}/\mathcal{H})\}.$$

Our goal is to prove, with certain restrictions on $\mathcal{G}$ and $\mathcal{H}$, that $\mathrm{LAut}_{\mathcal{H}}(\mathcal{G})$ is finite index in $\mathrm{PAut}(\mathcal{G})$.

The next lemma requires the following setup. Let $\mathcal{G}$ be a connected groupoid with object set $I = \{0, 1, \ldots, n\}$ and let $G = G_{00}$ be the vertex group at $0 \in I$. Let $\mathcal{H}$ be a connected normal subgroupoid with vertex group $H = H_{00}$, which is a normal subgroup of G.

Define the subgroup $K < G^n \rtimes \operatorname{Aut}(G)$ by

$$K = \{((g_1, \dots, g_n), \psi) \in G^n \rtimes \operatorname{Aut}(G) : \psi \in \operatorname{LAut}_H(G), g_i \in H \text{ for all } i\}.$$

Here, $\operatorname{LAut}_H(G)$ is defined by considering a group as a groupoid with one object.

Lemma 6.1.2. *Choose a star $\mathcal{S} = \{\iota_i\}_{i\in I} \subset \mathcal{H}$ based at $0 \in I$. Consider the isomorphism $\theta : G^n \rtimes \operatorname{Aut}(G) \to \operatorname{PAut}(\mathcal{G})$ from Lemma 6.1.1 defined by $\mathcal{S}$. Then $\theta(K) = \operatorname{LAut}_{\mathcal{H}}(\mathcal{G})$.*

Proof. Let $k = ((h_1, \dots, h_n), \psi) \in K$, let $h_0 = e_0 \in H$ and let $\iota_i^{-1} g \iota_j$ be an arbitrary element in $\mathcal{G}$. Then $\theta(k)(\iota_i^{-1} g \iota_j) = \iota_i^{-1} h_i^{-1} \psi(g) h_j \iota_j$. Since $\psi(g) \in H$ if and only if $g \in H$, $\theta(k)(\iota_i^{-1} g \iota_j) \in \mathcal{H}$ if and only if $g \in H$. Therefore $\theta(k)(\mathcal{H}) = \mathcal{H}$. In $\mathcal{G}/\mathcal{H}$, $[\iota_i^{-1} g \iota_j] = [g]$ for any $g \in G$. Therefore $\overline{\theta(k)}([\iota_i^{-1} g \iota_j]) = \overline{\theta(k)}([g]) = [\psi(g)]$. Since $\psi \in \operatorname{LAut}_H(G)$, $[\psi(g)] = [g]$ implying $\theta(k) \in \operatorname{LAut}_{\mathcal{H}}(\mathcal{G})$.

Conversely, suppose $k = ((g_1, \dots, g_n), \psi) \in G^n \rtimes \operatorname{Aut}(G)$ is such that $\theta(k) \in \operatorname{LAut}_{\mathcal{H}}(\mathcal{G})$. We have $\overline{\theta(k)}([g]) = [\psi(g)] = [g]$ for all $g \in G$, so $\psi \in \operatorname{LAut}_H(G)$. Let $h\iota_j$ be an arbitrary element of H_{0j}. Then $\theta(k)(h\iota_j) = \psi(h) g_j \iota_j$. For $\theta(k)(h\iota_j)$ to be in $\mathcal{H}$, we must have $g_j \in H$. Therefore $k \in K$, completing the proof. ∎

Lemma 6.1.3. *Let $\mathcal{G}$ be a connected groupoid with object set $I = \{0, 1, \dots, n\}$ and $\mathcal{H}$ a connected normal subgroupoid. Let $G = G_{00}$ and $H = H_{00}$ as above. Suppose G is finitely generated and H is finite index in G. Then $\operatorname{LAut}_{\mathcal{H}}(\mathcal{G})$ is finite index in $\operatorname{PAut}(\mathcal{G})$.*

Proof. By Lemma 6.1.2, it suffices to show that K is finite index in $G^n \rtimes \operatorname{Aut}(G)$. It is easily checked that $((g_1, \dots, g_n), \psi)$ and $((h_1, \dots, h_n), \varphi)$ are in the same right coset of K if and only if $[h_i] = [g_i]$ in G/H for all i and ψ and φ are in the same right coset of $\operatorname{LAut}_H(G)$ in $\operatorname{Aut}(G)$. The result then follows from the fact that if G is finitely generated and H is finite index in G, $\operatorname{LAut}_H(G)$ is finite index in $\operatorname{Aut}(G)$. ∎

6.1.2 The fundamental groupoid

The groupoid that will arise in the next section is the fundamental groupoid of a surface. We will briefly state the definition here and state some properties without proof that will be useful later on. For a full treatment, see [15, Chapter 6] or [28, Chapter 6].

Let X be a topological space and $A \subset X$ a subset. As a set, the fundamental groupoid $\pi_1(X, A)$ is the set of homotopy classes of paths $\sigma : [0, 1] \to X$ relative to the endpoints.

The paths must satisfy $\sigma(0), \sigma(1) \in A$. A partial operation is defined on $\pi_1(X, A)$ as follows. If $\sigma(1) = \tau(0)$, then $[\sigma][\tau] = [\gamma]$ where

$$\gamma(t) = \begin{cases} \sigma(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \tau(2t-1) & \text{if } \frac{1}{2} < t \leq 1. \end{cases}$$

Intuitively, two paths can be concatenated if the endpoint of one meets the beginning of the next. With this partial operation, $\pi_1(X, A)$ forms a groupoid with object set A. The identities are the constant paths and the vertex group at $a \in A$ is the fundamental group $\pi_1(X, a)$. It is helpful to think of the fundamental groupoid as a fundamental group with multiple basepoints.

Like the fundamental group, the fundamental groupoid provides a functor from the category of pairs of topological spaces to groupoids. In particular, if $f : X \to Y$ is a continuous map such that $f(A) \subset B$, then there is an induced groupoid morphism $f_* : \pi_1(X, A) \to \pi_1(Y, B)$. Furthermore, if $f : X \to X$ is a homeomorphism such that $f(A) = A$, then $f_* : \pi_1(X, A) \to \pi_1(X, A)$ is an automorphism of the groupoid $\pi_1(X, A)$.

The last point is especially important. In the next section we will characterize homeomorphisms that lift in a particular way by the induced automorphism on the fundamental groupoid.

6.2 Fundamental groupoids and lifting mapping classes

As hinted at above, to characterize when a homeomorphism lifts to a homeomorphism that fixes boundary components, we must look at the action of a homeomorphism on the fundamental groupoid $\pi_1(\Sigma^\circ, A)$ for a specific choice of basepoints $A \subset \Sigma^\circ$.

Suppose $\partial\Sigma^\circ$ has $n+1$ components. Let $A = \{x_0, x_1, \ldots, x_n\} \subset \partial\Sigma^\circ$ be such that each component contains exactly one of the x_i. For each x_i, choose a point $\tilde{x}_i \in p^{-1}(x_i)$ and let $\widetilde{A} = \{\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_n\} \subset \partial\widetilde{\Sigma}^\circ$. It follows from the unique path lifting property for covering spaces [27, p. 60] that the induced groupoid morphism $p_* : \pi_1(\widetilde{\Sigma}^\circ, \widetilde{A}) \to \pi_1(\Sigma^\circ, A)$ is injective. Since $p : \widetilde{\Sigma}^\circ \to \Sigma^\circ$ is a regular cover, $p_*\pi_1(\widetilde{\Sigma}^\circ, \widetilde{A})$ is a normal subgroupoid of $\pi_1(\Sigma^\circ, A)$. For the remainder of this section let $\mathcal{G} = \pi_1(\Sigma^\circ, A)$ and $\mathcal{H} = p_*\pi_1(\widetilde{\Sigma}^\circ, \widetilde{A})$.

For the next lemma, recall from Section 2.2.1 the definitions of $\mathrm{SHomeo}_p(\widetilde{\Sigma}^\circ, \partial\widetilde{\Sigma}^\circ)$, $\mathrm{LHomeo}_p(\Sigma^\circ, \partial\Sigma^\circ)$, and the homomorphism $\Pi : \mathrm{SHomeo}_p(\widetilde{\Sigma}^\circ, \partial\widetilde{\Sigma}^\circ) \to \mathrm{LHomeo}_p(\Sigma^\circ, \partial\Sigma^\circ)$.

Lemma 6.2.1. *Let $\tilde{f} \in \mathrm{SHomeo}_p(\widetilde{\Sigma}^\circ, \partial\widetilde{\Sigma}^\circ)$ and let $f = \Pi(\tilde{f}) \in \mathrm{LHomeo}_p(\Sigma^\circ, \partial\Sigma^\circ)$. Then $f_*(\mathcal{H}) = \mathcal{H}$ and $f_* = id \in \mathrm{Aut}(\mathcal{G}/\mathcal{H})$.*

Proof. Since $\tilde{f}$ is a homeomorphism that fixes $\widetilde{A}$ we have $\tilde{f}_*\pi_1(\widetilde{\Sigma}^\circ, \widetilde{A}) = \pi_1(\widetilde{\Sigma}^\circ, \widetilde{A})$. Then $f_*p_*\pi_1(\widetilde{\Sigma}^\circ, \widetilde{A}) = p_*\tilde{f}_*\pi_1(\widetilde{\Sigma}^\circ, \widetilde{A}) = p_*\tilde{\pi}_1(\widetilde{\Sigma}^\circ, \widetilde{A})$ so $f_*\mathcal{H} = \mathcal{H}$.

Now let $B = p^{-1}(A)$. Let $[\sigma] \in \mathcal{G}$ for some path $\sigma : [0,1] \to \Sigma^\circ$ such that $\sigma(0) = x_i$ and $\sigma(1) = x_j$. Let $\tilde{\sigma}$ be the unique path in $\widetilde{\Sigma}^\circ$ such that $p(\tilde{\sigma}) = \sigma$ and $\tilde{\sigma}(0) = \tilde{x}_i$. Then $\tilde{\sigma}(1) \in B$. Since $B \subset \partial\widetilde{\Sigma}^\circ$, $\tilde{f}\tilde{\sigma}(1) = \tilde{\sigma}(1)$. Therefore $[\tilde{\sigma}][\tilde{f}\tilde{\sigma}]^{-1}$ is defined in $\pi_1(\widetilde{\Sigma}^\circ, B)$. Then $p_*([\tilde{\sigma}][\tilde{f}\tilde{\sigma}]^{-1}) = [p\tilde{\sigma}][p\tilde{f}\tilde{\sigma}]^{-1} = [\sigma][fp\tilde{\sigma}]^{-1} = [\sigma][f(\sigma)]^{-1}$. However, $[\tilde{\sigma}][\tilde{f}\tilde{\sigma}]^{-1} \in \pi_1(\widetilde{\Sigma}^\circ, \widetilde{A})$ so $[\sigma][f(\sigma)]^{-1} \in \mathcal{H}$. We finally conclude that $[\sigma]$ and $[f(\sigma)]$ are in the same coset of $\mathcal{H}$ in $\mathcal{G}$, completing the proof. ∎

Lemma 6.2.1 is one direction of Theorem 6.2.4. Lemmas 6.2.2 and 6.2.3 prove the other direction.

Lemma 6.2.2. *Let $f \in \mathrm{Homeo}(\Sigma^\circ, \partial\Sigma^\circ)$ be such that $f_*(\mathcal{H}) = \mathcal{H}$ and $f_* = id \in \mathrm{Aut}(\mathcal{G}/\mathcal{H})$. Then there is a lift $\tilde{f}$ of f such that $\tilde{f}(\tilde{x}_0) = \tilde{x}_0$ and $\tilde{f}^{-1}d\tilde{f} = d$ for all $d \in D$.*

Proof. Since $f_*(\mathcal{H}) = \mathcal{H}$, $f_*p_*\pi_1(\widetilde{\Sigma}^\circ, \tilde{x}_0) = p_*\pi_1(\widetilde{\Sigma}^\circ, \tilde{x}_0)$. Therefore there is a lift $\tilde{g}$ of f such that $\tilde{g}(\tilde{x}_0) \in p^{-1}(x_0)$. Since D acts transitively on the fiber $p^{-1}(x_0)$, there is some $d \in D$ such that $d\tilde{g}(\tilde{x}_0) = \tilde{x}_0$. Therefore $\tilde{f} = d\tilde{g}$ is a lift of f such that $\tilde{f}(\tilde{x}_0) = \tilde{x}_0$.

Let $d \in D$. To see $\tilde{f}^{-1}d\tilde{f} = d$, we will first show $\tilde{f}d(\tilde{x}_0) = d\tilde{f}(\tilde{x}_0)$. Let $\tilde{\delta}$ be a path in $\widetilde{\Sigma}^\circ$ such that $\tilde{\delta}(0) = \tilde{x}_0$ and $\tilde{\delta}(1) = d(\tilde{x}_0)$. Let $\delta = p\tilde{\delta}$. Then $[\delta] \in \pi_1(\Sigma^\circ, x_0)$.

By assumption, $[\delta]$ and $[f\delta]$ are in the same coset of $\mathcal{H}$ in $\mathcal{G}$, so in particular they are in the same coset of $p_*\pi_1(\widetilde{\Sigma}^\circ, \tilde{x}_0)$ in $\pi_1(\Sigma^\circ, x_0)$. This implies that if $\tilde{\gamma}$ is a lift of $f\delta$ such that $\tilde{\gamma}(0) = \tilde{x}_0$, then $\tilde{\gamma}(1) = \tilde{\delta}(1)$. However, $p\tilde{f}\tilde{\delta} = f\delta$ and $\tilde{f}\tilde{\delta}(0) = \tilde{x}_0$ so $\tilde{f}\tilde{\delta} = \tilde{\gamma}$. Therefore $\tilde{f}d(\tilde{x}_0) = \tilde{f}\tilde{\delta}(1) = \tilde{\delta}(1) = d(\tilde{x}_0)$. Since $\tilde{f}(\tilde{x}_0) = \tilde{x}_0$ we have $d\tilde{f}(\tilde{x}_0) = d(\tilde{x}_0) = \tilde{f}d(\tilde{x}_0)$.

We now have $d\tilde{f}(\tilde{x}_0) = \tilde{f}d\tilde{f}^{-1}\tilde{f}(\tilde{x}_0)$. Since $\tilde{f}d\tilde{f}^{-1} \in D$ and the deck group acts freely on $\widetilde{\Sigma}^\circ$, $\tilde{f}d\tilde{f}^{-1} = d$, completing the proof. ∎

Lemma 6.2.3. *Let $f \in \mathrm{Homeo}(\Sigma^\circ, \partial\Sigma^\circ)$ be such that $f_*(\mathcal{H}) = \mathcal{H}$ and $f_* = id \in \mathrm{Aut}(\mathcal{G}/\mathcal{H})$. Then $f \in \mathrm{LHomeo}_p(\Sigma^\circ, \partial\Sigma^\circ)$.*

Proof. Let $\tilde{f}$ be the lift of f such that $\tilde{f}(\tilde{x}_0) = \tilde{x}_0$ ensured by Lemma 6.2.2. Let $\tilde{y} \in \partial\widetilde{\Sigma}^\circ$. It suffices to show $\tilde{f}(\tilde{y}) = \tilde{y}$.

Let $y = p(\tilde{y}) \in \partial\Sigma^\circ$. Then y is in the same component of $\partial\Sigma^\circ$ as x_i for some i. We will first show $\tilde{f}(\tilde{x}_i) = \tilde{x}_i$.

Let $\tilde{\delta}$ be a path in $\widetilde{\Sigma}^\circ$ such that $\tilde{\delta}(0) = \tilde{x}_0$ and $\tilde{\delta}(1) = \tilde{x}_i$. Let $\delta = p\tilde{\delta}$. Since $[\delta] \in \mathcal{H}$, $[f\delta] \in \mathcal{H}$. Since $p\tilde{f}\tilde{\delta} = fp\tilde{\delta} = f\delta$, we have that $\tilde{f}\tilde{\delta}$ is the unique lift of $f\delta$ such that $\tilde{f}\tilde{\delta}(0) = \tilde{x}_0$. Since $[f\delta] \in \mathcal{H}$, there is a lift $\tilde{\gamma}$ of $f\delta$ such that $\tilde{\gamma}(0) = \tilde{x}_0$ and $\tilde{\gamma}(1) = \tilde{x}_i$. Therefore $\tilde{f}\tilde{\delta}(1) = \tilde{x}_i$ so $\tilde{f}(\tilde{x}_i) = \tilde{x}_i$.

Now let σ be a path in $\partial\Sigma^\circ$ such that $\sigma(0) = x_i$ and $\sigma(1) = y$. Then $f\sigma = \sigma$. Let $\tilde{\sigma}$ be the lift of σ such that $\tilde{\sigma}(0) = \tilde{x}_i$. Then $\tilde{f}\tilde{\sigma}$ is a lift of σ. Furthermore, since $\tilde{f}(\tilde{x}_i) = \tilde{x}_i$, $\tilde{f}\tilde{\sigma}(0) = \tilde{\sigma}(0)$ so $\tilde{f}\tilde{\sigma} = \tilde{\sigma}$. Since $\tilde{\sigma}$ is a lift of σ and D acts transitively on $p^{-1}(y)$, there is some $d \in D$ such that $\tilde{\sigma}(1) = d(\tilde{y})$. Therefore $\tilde{f}d(\tilde{y}) = d(\tilde{y})$. By Lemma 6.2.2, $d\tilde{f}(\tilde{y}) = d(\tilde{y})$ so $\tilde{f}(\tilde{y}) = \tilde{y}$, completing the proof. ■

To state the following theorem, we return to the setting of the original branched cover $p : \widetilde{\Sigma} \to \Sigma$ branched at $\mathcal{B} \subset \Sigma \backslash \partial\Sigma$. If $f \in \mathrm{Mod}(\Sigma, \partial\Sigma, \mathcal{B})$ we will abuse notation and denote by $f_* \in \mathrm{Aut}(\pi_1(\Sigma^\circ, A))$ the automorphism induced by a representative homeomorphism for f. The abuse of notation is legal since any representative homeomorphism for f fixes A pointwise, and isotopic homeomorphisms induce the same groupoid automorphism.

Theorem 6.2.4. *The liftable mapping class group is given by*

$$\mathrm{LMod}_p(\Sigma, \partial\Sigma, \mathcal{B}) = \{f \in \mathrm{Mod}(\Sigma, \partial\Sigma, \mathcal{B}) : f_*(\mathcal{H}) = \mathcal{H}, f_* = id \in \mathrm{Aut}(\mathcal{G}/\mathcal{H})\}.$$

Proof. Recall that $f \in \mathrm{LHomeo}_p(\Sigma, \partial\Sigma, \mathcal{B})$ if and only if the restriction of f to Σ° is in $\mathrm{LHomeo}_p(\Sigma^\circ, \partial\Sigma^\circ)$. Combining Lemmas 6.2.1 and 6.2.3 completes the proof. ■

If Σ has one boundary component we get the following well-known corollary.

Corollary 6.2.5. *Suppose Σ has one boundary component. Choose a basepoint $x \in \partial\Sigma^\circ$ and $\tilde{x} \in p^{-1}(x)$. Then*

$$\mathrm{LMod}_p(\Sigma, \partial\Sigma, \mathcal{B}) = \{[f] \in \mathrm{Mod}(\Sigma, \partial\Sigma, \mathcal{B}) : qf_* = q\}$$

where $q : \pi_1(\Sigma^\circ, x) \to \pi_1(\Sigma^\circ, x)/p_*\pi_1(\widetilde{\Sigma}^\circ, \tilde{x})$ is the quotient map and f_* is the induced map on $\pi_1(\Sigma^\circ, x)$.

Proof. The condition $qf_* = q$ is equivalent to f_* acting trivially on the cosets of $p_*\pi_1(\widetilde{\Sigma}^\circ, \tilde{x})$ in $\pi_1(\Sigma^\circ, x)$. The result then follows from Theorem 6.2.4. ■

6.2.1 Identifying liftable mapping classes

The next proposition gives a direct way to check whether or not an element of $\mathrm{Mod}(\Sigma, \partial\Sigma, \mathcal{B})$ is in $\mathrm{LMod}_p(\Sigma, \partial\Sigma, \mathcal{B})$.

Choose a point $x_0 \in \partial\Sigma^\circ$ and a lift $\tilde{x}_0 \in p^{-1}(x_0)$. Choose a generating set $\{\gamma_1, \ldots, \gamma_m\}$ of $\pi_1(\Sigma^\circ, x_0)$. Since the cover is regular, $\pi_1(\Sigma^\circ, x_0)/p_*\pi_1(\widetilde{\Sigma}^\circ, \tilde{x}_0) \cong D$. Choose an isomorphism and let $q : \pi_1(\Sigma^\circ, x_0) \to D$ be the quotient map.

Suppose there are $n+1$ components of $\partial\Sigma^\circ$. Enumerate the components not containing x_0 from 1 to n. For each $i \in \{1,\dots,n\}$, choose an arc $\sigma_i : [0,1] \to \Sigma^\circ$ such that $\sigma_i(0) = x_0$ and $\sigma_i(1)$ is in the ith boundary component. Let $x_i = \sigma_i(1) \in \partial\Sigma^\circ$.

Let $A = \{x_0, x_1, \dots, x_n\} \subset \partial\Sigma^\circ$. Then the γ_i and $[\sigma_j]$ are all elements of $\pi_1(\Sigma^\circ, A)$. Given an element $[f] \in \mathrm{Mod}(\Sigma^\circ, \partial\Sigma^\circ)$, $f_*[\sigma_j] = a_j[\sigma_j]$ for some $a_j \in \pi_1(\Sigma^\circ, x_0)$.

Proposition 6.2.6. *A mapping class $[f]$ is in $\mathrm{LMod}_p(\Sigma, \partial\Sigma, \mathcal{B})$ if and only if $qf_*(\gamma_i) = q(\gamma_i)$ for all i and $a_j \in \ker q$ for all j.*

Proof. Choose a lift $\tilde{x}_0 \in p^{-1}(x_0)$. For all i choose lifts $\tilde{\sigma}_i$ of σ_i such that $\tilde{\sigma}_i(0) = \tilde{x}_0$. Let $\tilde{x}_i = \tilde{\sigma}_i(1)$ and let $\widetilde{A} = \{\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_n\}$. Let $\mathcal{G} = \pi_1(\Sigma^\circ, A)$ and $\mathcal{H} = p_*\pi_1(\widetilde{\Sigma}^\circ, \widetilde{A})$. Then by Theorem 6.2.4 $[f] \in \mathrm{LMod}_p(\Sigma, \partial\Sigma, \mathcal{B})$ if and only if $f_* \in \mathrm{LAut}_{\mathcal{H}}(\mathcal{G})$ where $\mathrm{LAut}_{\mathcal{H}}(\mathcal{G})$ is defined in Section 6.1.1.

The condition $qf_*(\gamma_i) = q(\gamma_i)$ for all i is equivalent to f_* acting trivially on the cosets of $p_*\pi_1(\widetilde{\Sigma}^\circ, \tilde{x}_0)$ in $\pi_1(\Sigma^\circ, x_0)$. The condition $a_j \in \ker q$ implies $a_j \in p_*\pi_1(\widetilde{\Sigma}^\circ, \tilde{x}_0)$ for all i. The result follows from observing that $\{[\sigma_1], \dots, [\sigma_n]\}$ is a star in $\mathcal{H}$ and applying Lemma 6.1.2. ∎

We will now apply Proposition 6.2.6 to two well-known situations.

Example 6.2.7. Consider the k-sheeted cyclic unbranched cover $p_k : \widetilde{A} \to A$ of an annulus by an annulus. Choose points $x_0, x_1 \in \partial A$, with x_0 in one boundary component and x_1 in the other. Choose an arc σ such that $\sigma(0) = x_0$, $\sigma(1) = x_1$. Let γ be a generator of $\pi_1(A, x_0)$.

The cover p_k is determined by the surjective homomorphism $q : \pi_1(A, x_0) \to \mathbb{Z}/k\mathbb{Z}$ given by $q(\gamma) = 1$, so $\ker q = \langle \gamma^k \rangle$.

The mapping class group $\mathrm{Mod}(A, \partial A)$ is an infinite cyclic group generated by a Dehn twist T. Then $T_*(\gamma) = \gamma$ and $T_*([\sigma]) = \gamma^{\pm 1}[\sigma]$ (the exponent of γ depends on the choice of generator γ of $\pi_1(A, x_0)$).

Applying Proposition 6.2.6, we see T^n lifts if and only if $n \equiv 0 \mod k$. Interestingly, T^k lifts to a Dehn twist on $\widetilde{A}$, so $\mathrm{SMod}_{p_k}(\widetilde{A}, \partial\widetilde{A}) = \mathrm{Mod}(\widetilde{A}, \partial\widetilde{A})$.

Example 6.2.8. Choose integers $n, k \geq 2$. Let D be a disk and choose n points $\mathcal{B} \subset D \backslash \partial D$. Let $D_n = D \backslash \mathcal{B}$. Enumerate the deleted points and let $x_i \in H_1(D_n; \mathbb{Z})$ be the homology class of a loop surrounding only the ith puncture counterclockwise. Let $\varphi : H_1(D_n; \mathbb{Z}) \to \mathbb{Z}/k\mathbb{Z}$ be the surjective homomorphism given by $\varphi(x_i) = 1$ for all i. Let $p^\circ : \Sigma^\circ \to D_n$ be the cyclic cover determined by $\ker(\varphi)$ and let $p : \Sigma \to D$ be the associated branched cover, branched at $\mathcal{B} \subset D$.

We have $\mathrm{Mod}(D, \partial D, \mathcal{B}) \cong B_n$, where B_n is the braid group on n strands [22, §9.1]. For any $f \in \mathrm{Mod}(D, \partial D, \mathcal{B})$, let $f_* \in \mathrm{Aut}(H_1(D_n; \mathbb{Z}))$ be the action of f on $H_1(D_n; \mathbb{Z})$ and let $\sigma_f \in S_n$ be the induced permutation on the marked points. Then $f_*(x_i) = x_{\sigma_f(i)}$ for all i. Therefore $\varphi f_*(x_i) = \varphi\left(x_{\sigma_f(i)}\right) = \varphi(x_i)$ for all i, so $\varphi f_* = \varphi$. Applying Corollary 6.2.5 we see $\mathrm{LMod}_p(D, \partial D, \mathcal{B}) = \mathrm{Mod}(D, \partial D, \mathcal{B})$.

Applying the Birman-Hilden theorem with boundary (Theorem 2.3.4), we get an isomorphism $B_n \cong \mathrm{SMod}_p(\Sigma, \partial\Sigma)$. An argument using the Riemann-Hurwitz formula shows that Σ has $\gcd(n,k)$ boundary components and genus $g = \frac{1}{2}(1+(1-n)(1-k)-\gcd(n,k))$. Therefore there is an injection $B_n \hookrightarrow \mathrm{Mod}\left(\Sigma_g^{\gcd(n,k)}, \partial\Sigma_g^{\gcd(n,k)}\right)$.

When $k = 2$ the embedding coincides with the usual embedding of the braid group sending each standard braid generator to a Dehn twist [22, §9.4]. An investigation when $k \geq 3$ will appear in upcoming joint work with Alan McLeay.

6.2.2 The index of the liftable mapping class group

In the case when $p : \widetilde{\Sigma} \to \Sigma$ is a finite-sheeted, regular branched cover between closed surfaces, it is known that the liftable mapping class group $\mathrm{LMod}_p(\Sigma, \mathcal{B})$ is finite index in $\mathrm{Mod}(\Sigma, \mathcal{B})$. In this section we prove the analogous result when $\widetilde{\Sigma}$ and Σ have non-empty boundary.

Theorem 6.2.9. *The index of* $\mathrm{LMod}_p(\Sigma, \partial\Sigma, \mathcal{B})$ *in* $\mathrm{Mod}(\Sigma, \partial\Sigma, \mathcal{B})$ *is finite.*

Proof. Let $\Psi : \mathrm{Mod}(\Sigma, \partial\Sigma, \mathcal{B}) \to \mathrm{PAut}(\mathcal{G})$ be the homomorphism given by the action of $\mathrm{Mod}(\Sigma, \partial\Sigma, \mathcal{B})$ on the fundamental groupoid $\mathcal{G}$. An application of the Alexander method [22, §2.3] shows that Ψ is injective [33, Theorem 3.1.1]. By Theorem 6.2.4,

$$\Psi(\mathrm{LMod}_p(\Sigma, \partial\Sigma, \mathcal{B})) \subset \mathrm{LAut}_{\mathcal{H}}(\mathcal{G}).$$

We have

$$\begin{aligned}
[\mathrm{Mod}(\Sigma, \partial\Sigma, \mathcal{B}), \mathrm{LMod}_p(\Sigma, \partial\Sigma, \mathcal{B})] &= [\Psi(\mathrm{Mod}(\Sigma, \partial\Sigma, \mathcal{B})), \Psi(\mathrm{LMod}_p(\Sigma, \partial\Sigma, \mathcal{B}))] \\
&\leq [\mathrm{PAut}(\mathcal{G}) : \mathrm{LAut}_{\mathcal{H}}(\mathcal{G})] \\
&< \infty.
\end{aligned}$$

The equality follows from the injectivity of Ψ. The first inequality follows from the following fact from group theory: if K, H are subgroups of G, then $[H : K \cap H] \leq [G : K]$. The fact that $\mathrm{LAut}_{\mathcal{H}}(\mathcal{G})$ is finite index in $\mathrm{PAut}(\mathcal{G})$ is Lemma 6.1.3, completing the proof. ∎

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APPENDICES

Appendix A

Proof of Theorem 5.1.8

In this appendix we provide a proof for Theorem 5.1.8. The proof relies heavily on code written in Sage, which has been included below in Section A.1.

Proof of Theorem 5.1.8. Applying the Reidemeister-Schreier rewriting process we obtain the generators

$$\{C_{K,\alpha} : K \in \mathcal{S}, \alpha \in \{a, b, c, d\}\}$$

and relators

$$\begin{aligned}&\{C_{K,\alpha} : K \in \mathcal{S}, \alpha \in \{a, b, c, d\}, K\alpha \approx \overline{K\alpha}\} \cup \Lambda \\ &= \{C_{1,c}, C_{c,b}, C_{c^{-1}b}, C_{c^{-1},c}, C_{cb^{-1},b}, C_{cb^{-1},c}, C_{cb,a}, C_{c^{-1}b^{-1},b}\} \cup \Lambda\end{aligned}$$

where Λ is defined below and $\overline{K\alpha}$ is the element in $\mathcal{S}$ that represents the same coset as $K\alpha$. Here $\approx$ means freely equal.

The relators in Λ are given by output 0 from the Sage code:

$C_{1,a}C_{1,b}C_{1,a}C^{-1}_{1,b}C^{-1}_{1,a}C^{-1}_{1,b}$
$C_{1,b}C_{1,c}C_{c,b}C^{-1}_{cb,c}C^{-1}_{c,b}C^{-1}_{1,c}$
$C_{1,a}C_{1,c}C^{-1}_{c,a}C^{-1}_{1,c}$
$C_{1,a}C_{1,d}C^{-1}_{1,a}C^{-1}_{1,d}$
$C_{1,b}C_{1,d}C^{-1}_{1,b}C^{-1}_{1,d}$
$C_{1,c}C_{c,d}C^{-1}_{1,c}C^{-1}_{1,d}$
$C_{1,d}C_{1,d}$
$C_{1,a}C_{1,b}C_{1,c}C_{c,a}C_{c,b}C_{cb,c}C_{cb,a}C_{cba,b}C_{cba,c}C_{cba,a}C_{c'b',b}C_{c',c}$
$C_{1,c}C_{c,a}C_{c,b}C_{cb,a}C^{-1}_{cba,b}C^{-1}_{cb,a}C^{-1}_{c,b}C^{-1}_{1,c}$
$C_{1,c}C_{c,b}C_{cb,c}C_{cb,b}C^{-1}_{c'b,c}C^{-1}_{c',b}C^{-1}_{c,c}C^{-1}_{1,c}$
$C_{1,c}C_{c,a}C_{c,c}C^{-1}_{c',a}C^{-1}_{c,c}C^{-1}_{1,c}$

$C_{1,c}C_{c,a}C_{c,d}C^{-1}_{c,a}C^{-1}_{c,d}C^{-1}_{1,c}$
$C_{1,c}C_{c,b}C_{cb,d}C^{-1}_{c,b}C^{-1}_{c,d}C^{-1}_{1,c}$
$C_{1,c}C_{c,c}C_{c',d}C^{-1}_{c,c}C^{-1}_{c,d}C^{-1}_{1,c}$
$C_{1,c}C_{c,d}C_{c,d}C^{-1}_{1,c}$
$C_{1,c}C_{c,a}C_{c,b}C_{cb,c}C_{cb,a}C_{cba,b}C_{cba,c}C_{cba,a}C_{c'b',b}C_{c',c}C_{1,a}C_{1,b}C_{1,c}C^{-1}_{1,c}$
$C^{-1}_{c',c}C_{c',a}C_{c',b}C_{c'b,a}C^{-1}_{cb'c,b}C^{-1}_{c'b,a}C^{-1}_{c',b}C_{c',c}$
$C^{-1}_{c',c}C_{c',b}C_{c'b,c}C_{cb',b}C^{-1}_{1,c}C^{-1}_{1,b}C^{-1}_{c',c}C_{c',c}$
$C^{-1}_{c',c}C_{c',a}C_{c',c}C^{-1}_{1,a}C^{-1}_{c',c}C_{c',c}$
$C^{-1}_{c',c}C_{c',a}C_{c',d}C^{-1}_{c',a}C^{-1}_{c',d}C_{c',c}$
$C^{-1}_{c',c}C_{c',b}C_{c'b,d}C^{-1}_{c',b}C^{-1}_{c',d}C_{c',c}$
$C^{-1}_{c',c}C_{c',c}C_{1,d}C^{-1}_{c',c}C^{-1}_{c',d}C_{c',c}$
$C^{-1}_{c',c}C_{c',d}C_{c',d}C_{c',c}$
$C^{-1}_{c',c}C_{c',a}C_{c',b}C_{c'b,c}C_{cb',a}C_{c'b,b}C_{c'b',c}C_{c'b',a}C_{cb,b}C_{cb',c}C_{cb'c,a}C_{cb',b}C_{c,c}C_{c',c}$
$C_{1,c}C^{-1}_{cb',b}C_{cb',c}C_{cb'c,a}C_{cb',b}C_{c,a}C^{-1}_{cb',b}C^{-1}_{cb'c,a}C^{-1}_{cb'c,b}C^{-1}_{cb',c}C_{cb',b}C^{-1}_{1,c}$
$C_{1,c}C^{-1}_{cb',b}C_{cb',c}C_{cb'c,b}C_{cb'c,c}C_{c'b,b}C^{-1}_{c'b',c}C^{-1}_{c'b,b}C^{-1}_{cb'c,c}C^{-1}_{cb',c}C_{cb',b}C^{-1}_{1,c}$
$C_{1,c}C^{-1}_{cb',b}C_{cb',c}C_{cb'c,a}C_{cb',c}C^{-1}_{c'b,a}C^{-1}_{cb'c,c}C^{-1}_{cb',c}C_{cb',b}C^{-1}_{1,c}$
$C_{1,c}C^{-1}_{cb',b}C_{cb',c}C_{cb'c,a}C_{cb',d}C^{-1}_{cb'c,a}C^{-1}_{cb'c,d}C^{-1}_{cb',c}C_{cb',b}C^{-1}_{1,c}$
$C_{1,c}C^{-1}_{cb',b}C_{cb',c}C_{cb'c,b}C_{cb'c,d}C^{-1}_{cb'c,b}C^{-1}_{cb'c,d}C^{-1}_{cb',c}C_{cb',b}C^{-1}_{1,c}$
$C_{1,c}C^{-1}_{cb',b}C_{cb',c}C_{cb'c,c}C_{c'b,d}C^{-1}_{cb'c,c}C^{-1}_{cb'c,d}C^{-1}_{cb',c}C_{cb',b}C^{-1}_{1,c}$
$C_{1,c}C^{-1}_{cb',b}C_{cb',c}C_{cb'c,d}C_{cb'c,d}C^{-1}_{cb',c}C_{cb',b}C^{-1}_{1,c}$
$C_{1,c}C^{-1}_{cb',b}C_{cb',c}C_{cb'c,a}C_{cb',b}C_{c,c}C_{c',a}C_{c',b}C_{c'b,c}C_{cb',a}C_{c'b,b}C_{c'b',c}C_{c'b',a}C_{cb,b}C_{cb',c}C^{-1}_{cb',c}C_{cb',b}C^{-1}_{1,c}$
$C_{1,c}C^{-1}_{cb',b}C_{cb',a}C_{c'b,b}C_{c'b',a}C^{-1}_{c,b}C^{-1}_{c,a}C^{-1}_{cb',b}C_{cb',b}C^{-1}_{1,c}$
$C_{1,c}C^{-1}_{cb',b}C_{cb',b}C_{c,c}C_{c',b}C^{-1}_{cb'c,c}C^{-1}_{cb'c,b}C^{-1}_{cb',c}C_{cb',b}C^{-1}_{1,c}$
$C_{1,c}C^{-1}_{cb',b}C_{cb',a}C_{c'b,c}C^{-1}_{cb'c,a}C^{-1}_{cb',c}C_{cb',b}C^{-1}_{1,c}$
$C_{1,c}C^{-1}_{cb',b}C_{cb',a}C_{c'b,d}C^{-1}_{cb',a}C^{-1}_{cb',d}C_{cb',b}C^{-1}_{1,c}$
$C_{1,c}C^{-1}_{cb',b}C_{cb',b}C_{c,d}C^{-1}_{cb',b}C^{-1}_{cb',d}C_{cb',b}C^{-1}_{1,c}$
$C_{1,c}C^{-1}_{cb',b}C_{cb',c}C_{cb'c,d}C^{-1}_{cb',c}C^{-1}_{cb',d}C_{cb',b}C^{-1}_{1,c}$
$C_{1,c}C^{-1}_{cb',b}C_{cb',d}C_{cb',d}C_{cb',b}C^{-1}_{1,c}$
$C_{1,c}C^{-1}_{cb',b}C_{cb',a}C_{c'b,b}C_{c'b',c}C_{c'b',a}C_{cb,b}C_{cb',c}C_{cb'c,a}C_{cb',b}C_{c,c}C_{c',a}C_{c',b}C_{c'b,c}C_{cb',b}C^{-1}_{1,c}$
$C^{-1}_{c',c}C_{c',b}C_{c'b,a}C_{cb'c,b}C_{cb'c,a}C^{-1}_{cb,b}C^{-1}_{c'b',a}C^{-1}_{c'b,b}C^{-1}_{c',b}C_{c',c}$
$C^{-1}_{c',c}C_{c',b}C_{c'b,b}C_{c'b',c}C_{c'b',b}C^{-1}_{c,c}C^{-1}_{cb',b}C^{-1}_{c'b,c}C^{-1}_{c',b}C_{c',c}$
$C^{-1}_{c',c}C_{c',b}C_{c'b,a}C_{cb'c,c}C^{-1}_{cb',a}C^{-1}_{c'b,c}C^{-1}_{c',b}C_{c',c}$
$C^{-1}_{c',c}C_{c',b}C_{c'b,a}C_{cb'c,d}C^{-1}_{c'b,a}C^{-1}_{c'b,d}C^{-1}_{c',b}C_{c',c}$
$C^{-1}_{c',c}C_{c',b}C_{c'b,b}C_{c'b',d}C^{-1}_{c'b,b}C^{-1}_{c'b,d}C^{-1}_{c',b}C_{c',c}$

$C_{c',c}^{-1}C_{c',b}C_{c'b,c}C_{cb',d}C_{c'b,c}^{-1}C_{c'b,d}^{-1}C_{c',b}^{-1}C_{c',c}$
$C_{c',c}^{-1}C_{c',b}C_{c'b,d}C_{c'b,d}C_{c',b}^{-1}C_{c',c}$
$C_{c',c}^{-1}C_{c',b}C_{c'b,a}C_{cb'c,b}C_{cb'c,c}C_{c'b,a}C_{cb'c,b}C_{cb'c,c}C_{c'b,a}C_{cb'c,b}C_{cb'c,c}C_{c'b,a}C_{cb'c,b}C_{cb'c,c}C_{c',b}^{-1}C_{c',c}$
$C_{1,c}C_{c,b}C_{cb,a}C_{cba,a}C_{c'b',b}C_{c',a}C_{c'b',b}^{-1}C_{cba,a}^{-1}C_{cba,b}^{-1}C_{cb,a}^{-1}C_{c,b}^{-1}C_{1,c}^{-1}$
$C_{1,c}C_{c,b}C_{cb,a}C_{cba,b}C_{cba,c}C_{cba,b}C_{cba,c}^{-1}C_{cba,b}^{-1}C_{cba,c}^{-1}C_{cb,a}^{-1}C_{c,b}^{-1}C_{1,c}^{-1}$
$C_{1,c}C_{c,b}C_{cb,a}C_{cba,a}C_{c'b',c}C_{cba,a}^{-1}C_{cba,c}^{-1}C_{cb,a}^{-1}C_{c,b}^{-1}C_{1,c}^{-1}$
$C_{1,c}C_{c,b}C_{cb,a}C_{cba,a}C_{c'b',d}C_{cba,a}^{-1}C_{cba,d}^{-1}C_{cb,a}^{-1}C_{c,b}^{-1}C_{1,c}^{-1}$
$C_{1,c}C_{c,b}C_{cb,a}C_{cba,b}C_{cba,d}C_{cba,b}^{-1}C_{cba,d}^{-1}C_{cb,a}^{-1}C_{c,b}^{-1}C_{1,c}^{-1}$
$C_{1,c}C_{c,b}C_{cb,a}C_{cba,c}C_{cba,d}C_{cba,c}^{-1}C_{cba,d}^{-1}C_{cb,a}^{-1}C_{c,b}^{-1}C_{1,c}^{-1}$
$C_{1,c}C_{c,b}C_{cb,a}C_{cba,d}C_{cba,d}C_{cb,a}^{-1}C_{c,b}^{-1}C_{1,c}^{-1}$
$C_{1,c}C_{c,b}C_{cb,a}C_{cba,a}C_{c'b',b}C_{c',c}C_{1,a}C_{1,b}C_{1,c}C_{c,a}C_{c,b}C_{cb,c}C_{cb,a}C_{cba,b}C_{cba,c}C_{cb,a}^{-1}C_{c,b}^{-1}C_{1,c}^{-1}$
$C_{1,c}C_{c,b}C_{cb,a}C_{cba,b}C_{cba,a}C_{c'b,b}^{-1}C_{cb',a}^{-1}C_{cb,b}^{-1}C_{c,b}^{-1}C_{1,c}^{-1}$
$C_{1,c}C_{c,b}C_{cb,b}C_{cb',c}C_{cb'c,b}C_{cb',c}^{-1}C_{cb,b}^{-1}C_{cb,c}^{-1}C_{c,b}^{-1}C_{1,c}^{-1}$
$C_{1,c}C_{c,b}C_{cb,a}C_{cba,c}C_{cb,a}^{-1}C_{cb,c}^{-1}C_{c,b}^{-1}C_{1,c}^{-1}$
$C_{1,c}C_{c,b}C_{cb,a}C_{cba,d}C_{cb,a}^{-1}C_{cb,d}^{-1}C_{c,b}^{-1}C_{1,c}^{-1}$
$C_{1,c}C_{c,b}C_{cb,b}C_{cb',d}C_{cb,b}^{-1}C_{cb,d}^{-1}C_{c,b}^{-1}C_{1,c}^{-1}$
$C_{1,c}C_{c,b}C_{cb,c}C_{cb,d}C_{cb,c}^{-1}C_{cb,d}^{-1}C_{c,b}^{-1}C_{1,c}^{-1}$
$C_{1,c}C_{c,b}C_{cb,d}C_{cb,d}C_{c,b}^{-1}C_{1,c}^{-1}$
$C_{1,c}C_{c,b}C_{cb,a}C_{cba,b}C_{cba,c}C_{cba,a}C_{c'b',b}C_{c',c}C_{1,a}C_{1,b}C_{1,c}C_{c,a}C_{c,b}C_{cb,c}C_{c,b}^{-1}C_{1,c}^{-1}$
$C_{c',c}^{-1}C_{c'b',b}^{-1}C_{c'b',a}C_{cb,b}C_{cb',a}C_{c',b}^{-1}C_{c',a}^{-1}C_{c'b',b}^{-1}C_{c'b',b}C_{c',c}$
$C_{c',c}^{-1}C_{c'b',b}^{-1}C_{c'b',b}C_{c',c}C_{1,b}C_{c',c}^{-1}C_{c'b',b}^{-1}C_{c'b',c}^{-1}C_{c'b',b}C_{c',c}$
$C_{c',c}^{-1}C_{c'b',b}^{-1}C_{c'b',a}C_{cb,c}C_{c'b',a}^{-1}C_{c'b',c}^{-1}C_{c'b',b}C_{c',c}$
$C_{c',c}^{-1}C_{c'b',b}^{-1}C_{c'b',a}C_{cb,d}C_{c'b',a}^{-1}C_{c'b',d}^{-1}C_{c'b',b}C_{c',c}$
$C_{c',c}^{-1}C_{c'b',b}^{-1}C_{c'b',b}C_{c',d}C_{c'b',b}^{-1}C_{c'b',d}^{-1}C_{c'b',b}C_{c',c}$
$C_{c',c}^{-1}C_{c'b',b}^{-1}C_{c'b',c}C_{c'b',d}C_{c'b',c}^{-1}C_{c'b',d}^{-1}C_{c'b',b}C_{c',c}$
$C_{c',c}^{-1}C_{c'b',b}^{-1}C_{c'b',d}C_{c'b',d}C_{c'b',b}C_{c',c}$
$C_{c',c}^{-1}C_{c'b',b}^{-1}C_{c'b',a}C_{cb,b}C_{cb',c}C_{cb'c,a}C_{cb',b}C_{c,c}C_{c',a}C_{c',b}C_{c'b,c}C_{cb',a}C_{c'b,b}C_{c'b',c}C_{c'b',b}C_{c',c}.$

Deleting all the symbols which are relators we are left with generators

$$\{C_{K,\alpha} : K \in \mathcal{S}, \alpha \in \{a,b,c,d\}, K\alpha \not\approx \overline{K\alpha}\}$$

and relators given by output 1 from the Sage code:

$C_{1,a}C_{1,b}C_{1,a}C_{1,b}^{-1}C_{1,a}^{-1}C_{1,b}^{-1}$
$C_{1,b}C_{cb,c}^{-1}$

$C_{1,a}C_{c,a}^{-1}$
$C_{1,a}C_{1,d}C_{1,a}^{-1}C_{1,d}^{-1}$
$C_{1,b}C_{1,d}C_{1,b}^{-1}C_{1,d}^{-1}$
$C_{c,d}C_{1,d}^{-1}$
$C_{1,d}C_{1,d}$
$C_{1,a}C_{1,b}C_{c,a}C_{cb,c}C_{cba,b}C_{cba,c}C_{cba,a}$
$C_{c,a}C_{cba,b}^{-1}$
$C_{cb,c}C_{cb,b}C_{c'b,c}^{-1}C_{c,c}^{-1}$
$C_{c,a}C_{c,c}C_{c',a}^{-1}C_{c,c}^{-1}$
$C_{c,a}C_{c,d}C_{c,a}^{-1}C_{c,d}^{-1}$
$C_{cb,d}C_{c,d}^{-1}$
$C_{c,c}C_{c',d}C_{c,c}^{-1}C_{c,d}^{-1}$
$C_{c,d}C_{c,d}$
$C_{c,a}C_{cb,c}C_{cba,b}C_{cba,c}C_{cba,a}C_{1,a}C_{1,b}$
$C_{c',a}C_{c'b,a}C_{cb'c,b}^{-1}C_{c'b,a}^{-1}$
$C_{c'b,c}C_{1,b}^{-1}$
$C_{c',a}C_{1,a}^{-1}$
$C_{c',a}C_{c',d}C_{c',a}^{-1}C_{c',d}^{-1}$
$C_{c'b,d}C_{c',d}^{-1}$
$C_{1,d}C_{c',d}^{-1}$
$C_{c',d}C_{c',d}$
$C_{c',a}C_{c'b,c}C_{cb',a}C_{c'b,b}C_{c'b',c}C_{c'b',a}C_{cb,b}C_{cb'c,a}C_{c,c}$
$C_{cb'c,a}C_{c,a}C_{cb'c,a}^{-1}C_{cb'c,b}^{-1}$
$C_{cb'c,b}C_{cb'c,c}C_{c'b,b}C_{c'b',c}^{-1}C_{c'b,b}^{-1}C_{cb'c,c}^{-1}$
$C_{cb'c,a}C_{c'b,a}^{-1}C_{cb'c,c}^{-1}$
$C_{cb'c,a}C_{cb',d}C_{cb'c,a}^{-1}C_{cb'c,d}^{-1}$
$C_{cb'c,b}C_{cb'c,d}C_{cb'c,b}^{-1}C_{cb'c,d}^{-1}$
$C_{cb'c,c}C_{c'b,d}C_{cb'c,c}^{-1}C_{cb'c,d}^{-1}$
$C_{cb'c,d}C_{cb'c,d}$
$C_{cb'c,a}C_{c,c}C_{c',a}C_{c'b,c}C_{cb',a}C_{c'b,b}C_{c'b',c}C_{c'b',a}C_{cb,b}$
$C_{cb',a}C_{c'b,b}C_{c'b',a}C_{c,a}^{-1}$
$C_{c,c}C_{cb'c,c}^{-1}C_{cb'c,b}^{-1}$
$C_{cb',a}C_{c'b,c}C_{cb'c,a}^{-1}$
$C_{cb',a}C_{c'b,d}C_{cb',a}^{-1}C_{cb',d}^{-1}$
$C_{c,d}C_{cb',d}^{-1}$

$C_{cb'c,d}C^{-1}_{cb',d}$
$C_{cb',d}C_{cb',d}$
$C_{cb',a}C_{c'b,b}C_{c'b',c}C_{c'b',a}C_{cb,b}C_{cb'c,a}C_{c,c}C_{c',a}C_{c'b,c}$
$C_{c'b,a}C_{cb'c,b}C_{cb'c,a}C^{-1}_{cb,b}C^{-1}_{c'b',a}C^{-1}_{c'b,b}$
$C_{c'b,b}C_{c'b',c}C^{-1}_{c,c}C^{-1}_{c'b,c}$
$C_{c'b,a}C_{cb'c,c}C^{-1}_{cb',a}C^{-1}_{c'b,c}$
$C_{c'b,a}C_{cb'c,d}C^{-1}_{c'b,a}C^{-1}_{c'b,d}$
$C_{c'b,b}C_{c'b',d}C^{-1}_{c'b,b}C^{-1}_{c'b,d}$
$C_{c'b,c}C_{cb',d}C^{-1}_{c'b,c}C^{-1}_{c'b,d}$
$C_{c'b,d}C_{c'b,d}$
$C_{c'b,a}C_{cb'c,b}C_{cb'c,c}C_{c'b,a}C_{cb'c,b}C_{cb'c,c}C_{c'b,a}C_{cb'c,b}C_{cb'c,c}C_{c'b,a}C_{cb'c,b}C_{cb'c,c}$
$C_{cba,a}C_{c',a}C^{-1}_{cba,a}C^{-1}_{cba,b}$
$C_{cba,b}C_{cba,c}C_{cba,b}C^{-1}_{cba,c}C^{-1}_{cba,b}C^{-1}_{cba,c}$
$C_{cba,a}C_{c'b',c}C^{-1}_{cba,a}C^{-1}_{cba,c}$
$C_{cba,a}C_{c'b',d}C^{-1}_{cba,a}C^{-1}_{cba,d}$
$C_{cba,b}C_{cba,d}C^{-1}_{cba,b}C^{-1}_{cba,d}$
$C_{cba,c}C_{cba,d}C^{-1}_{cba,c}C^{-1}_{cba,d}$
$C_{cba,d}C_{cba,d}$
$C_{cba,a}C_{1,a}C_{1,b}C_{c,a}C_{cb,c}C_{cba,b}C_{cba,c}$
$C_{cba,b}C_{cba,a}C^{-1}_{c'b,b}C^{-1}_{cb',a}C^{-1}_{cb,b}$
$C_{cb,b}C_{cb'c,b}C^{-1}_{cb,b}C^{-1}_{cb,c}$
$C_{cba,c}C^{-1}_{cb,c}$
$C_{cba,d}C^{-1}_{cb,d}$
$C_{cb,b}C_{cb',d}C^{-1}_{cb,b}C^{-1}_{cb,d}$
$C_{cb,c}C_{cb,d}C^{-1}_{cb,c}C^{-1}_{cb,d}$
$C_{cb,d}C_{cb,d}$
$C_{cba,b}C_{cba,c}C_{cba,a}C_{1,a}C_{1,b}C_{c,a}C_{cb,c}$
$C_{c'b',a}C_{cb,b}C_{cb',a}C^{-1}_{c',a}$
$C_{1,b}C^{-1}_{c'b',c}$
$C_{c'b',a}C_{cb,c}C^{-1}_{c'b',a}C^{-1}_{c'b',c}$
$C_{c'b',a}C_{cb,d}C^{-1}_{c'b',a}C^{-1}_{c'b',d}$
$C_{c',d}C^{-1}_{c'b',d}$
$C_{c'b',c}C_{c'b',d}C^{-1}_{c'b',c}C^{-1}_{c'b',d}$
$C_{c'b',d}C_{c'b',d}$
$C_{c'b',a}C_{cb,b}C_{cb'c,a}C_{c,c}C_{c',a}C_{c'b,c}C_{cb',a}C_{c'b,b}C_{c'b',c}.$

There are several relators which tell us that two generating symbols define the same element in $\mathrm{LMod}(\Sigma_{1,2})$, for example the second relator $C_{1,b}C_{cb,c}^{-1}$ allows us to replace all occurences of $C_{cb,c}$ with $C_{1,b}$ and delete the former as a generating symbol. Doing this for all such relators we can remove the generators on the left in the table below, and replace them with the symbols on the right.

$$\begin{aligned}\{C_{cba,b}, C_{c^{-1},a}, C_{c,a}\} &\longmapsto C_{1,a}\\ \{C_{cba,c}, C_{cb,c}, C_{c^{-1}b,c}, C_{c^{-1}b^{-1},c}\} &\longmapsto C_{1,b}\\ \{C_{K,d} : K \in \mathcal{S} \setminus \{1\}\} &\longmapsto C_{1,d}.\end{aligned}$$

Making these substitutions we are left with the generators

$$\{C_{1,a}, C_{1,b}, C_{c,c}, C_{1,d}, C_{cb^{-1}c,a}, C_{cb^{-1}c,b}, C_{cb^{-1}c,c}, C_{cb^{-1},a}, C_{c^{-1}b,a}, C_{c^{-1}b,b}, C_{cba,a}, C_{cb,b}, C_{c^{-1}b^{-1},a}\}.$$

To make this easier to read, we now replace these 13 generating symbols with the letters

$$\{A, B, C, D, E, F, G, H, I, J, K, L, M\}$$

respectively.

Making both these sets of substitutions in the relators, freely reducing, removing duplicate relators and removing empty relators, we are left with the following set of relators given by output 3 of the Sage code. We have numbered the relators so we can keep track of them as they are manipulated.

1. $ABAB^{-1}A^{-1}B^{-1}$
2. $ADA^{-1}D^{-1}$
3. $BDB^{-1}D^{-1}$
4. DD
5. $ABABABK$
6. $BLB^{-1}C^{-1}$
7. $ACA^{-1}C^{-1}$
8. $CDC^{-1}D^{-1}$
9. $ABABKAB$
10. $AIF^{-1}I^{-1}$
11. $ABHJBMLEC$

12. $EAE^{-1}F^{-1}$
13. $FGJB^{-1}J^{-1}G^{-1}$
14. $EI^{-1}G^{-1}$
15. $EDE^{-1}D^{-1}$
16. $FDF^{-1}D^{-1}$
17. $GDG^{-1}D^{-1}$
18. $ECABHJBML$
19. $HJMA^{-1}$
20. $CG^{-1}F^{-1}$
21. HBE^{-1}
22. $HDH^{-1}D^{-1}$
23. $HJBMLECAB$
24. $IFEL^{-1}M^{-1}J^{-1}$
25. $JBC^{-1}B^{-1}$
26. $IGH^{-1}B^{-1}$
27. $IDI^{-1}D^{-1}$
28. $JDJ^{-1}D^{-1}$
29. $IFGIFGIFGIFG$
30. $KAK^{-1}A^{-1}$
31. $KBK^{-1}B^{-1}$
32. $KDK^{-1}D^{-1}$
33. $KABABAB$
34. $AKJ^{-1}H^{-1}L^{-1}$
35. $LFL^{-1}B^{-1}$
36. $LDL^{-1}D^{-1}$
37. $ABKABAB$
38. $MLHA^{-1}$
39. $MBM^{-1}B^{-1}$

40. $MDM^{-1}D^{-1}$
41. $MLECABHJB$

We can now eliminate all but four of the generating symbols by rewriting various relators as follows:

14. $I = G^{-1}E$
20. $G = F^{-1}C$
12. $F = EAE^{-1}$
38. $M = AH^{-1}L^{-1}$
21. $E = HB$
34. $H = L^{-1}AKJ^{-1}$
25. $J = BCB^{-1}$
6. $L = B^{-1}CB$
5. $K = (BABABA)^{-1}$.

Notice that the last substitution is not exactly as it appears in relator 5. However, by 1 we have that $ABABAB = BABABA$ so this substitution for K is still valid.

Performing these substitutions in the order listed, freely reducing, and eliminating empty relators leaves us with generators $\{A, B, C, D\}$ and relators given by Sage output 4:

1. $ABAB^{-1}A^{-1}B^{-1}$
2. $ADA^{-1}D^{-1}$
3. $BDB^{-1}D^{-1}$
4. DD
5. $ABABABA^{-1}B^{-1}A^{-1}B^{-1}A^{-1}B^{-1}$
7. $ACA^{-1}C^{-1}$
8. $CDC^{-1}D^{-1}$
9. $ABABA^{-1}B^{-1}A^{-1}B^{-1}A^{-1}B^{-1}AB$
10. $AC^{-1}B^{-1}C^{-1}A^{-1}B^{-1}A^{-1}C^{-1}AB^{-1}C^{-1}A^{-1}B^{-1}A^{-1}C^{-1}A^{-1}$
$CABACBA^{-1}CABACBC$

11. $AC^{-1}A^{-1}C$
13. $CBCB^{-1}C^{-1}B^{-1}C^{-1}B^{-1}C^{-1}A^{-1}B^{-1}A^{-1}C^{-1}ACABACB$
15. $[(BACACB)^{-1}, D]$
16. $[B^{-1}C^{-1}A^{-1}B^{-1}A^{-1}C^{-1}ACABACB, D]$
17. $[B^{-1}C^{-1}A^{-1}B^{-1}A^{-1}C^{-1}A^{-1}CABACBC, D]$
18. $B^{-1}C^{-1}A^{-1}B^{-1}C^{-1}A^{-1}CABACB$
19. $B^{-1}C^{-1}A^{-1}B^{-1}A^{-1}B^{-1}ABCABABA^{-1}$
22. $[(BCABACB)^{-1}, D]$
23. $B^{-1}C^{-1}A^{-1}CAB$
24. $C^{-1}B^{-1}C^{-1}A^{-1}B^{-1}A^{-1}C^{-1}AB^{-1}C^{-1}A^{-1}B^{-1}$
$A^{-1}C^{-1}AB^{-1}C^{-1}A^{-1}B^{-1}A^{-1}C^{-1}B^{-1}A^{-1}BC^{-1}B^{-1}$
26. $C^{-1}B^{-1}C^{-1}A^{-1}B^{-1}A^{-1}C^{-1}AB^{-1}C^{-1}A^{-1}B^{-1}A^{-1}C^{-1}A^{-1}CABACBCBCABAC$
27. $[(A^{-1}CABACBC)^{-1}, D]$
28. $[BCB^{-1}, D]$
29. $C^{-1}B^{-1}C^{-1}A^{-1}B^{-1}A^{-1}C^{-1}AB^{-1}C^{-1}A^{-1}B^{-1}$
$A^{-1}C^{-1}AB^{-1}C^{-1}A^{-1}B^{-1}A^{-1}C^{-1}AB^{-1}C^{-1}A^{-1}B^{-1}A^{-1}C^{-1}AC$
30. $A^{-1}B^{-1}A^{-1}B^{-1}A^{-1}B^{-1}ABABAB$
31. $A^{-1}B^{-1}A^{-1}B^{-1}A^{-1}BABABAB^{-1}$
32. $[(BABABA)^{-1}, D]$
33. $A^{-1}B^{-1}A^{-1}B^{-1}A^{-1}B^{-1}ABABAB$
35. $B^{-1}A^{-1}B^{-1}A^{-1}C^{-1}ACABA$
36. $[B^{-1}CB, D]$
37. $ABA^{-1}B^{-1}A^{-1}B^{-1}A^{-1}B^{-1}ABAB$
39. $ABCABABA^{-1}B^{-1}A^{-1}C^{-1}B^{-1}A^{-1}B^{-1}$
40. $[ABCABAB, D]$
41. $ABCAC^{-1}A^{-1}B^{-1}A^{-1}$

We will now eliminate relators by showing they are derivable from the relators we want to end up with.

Note that the commutators 15, 16, 17, 22, 27, 28, 32, 36, and 40 can all be derived from 2, 3, and 8 so they can be eliminated. Now consider relator 19, which we can derive using only relators 1 and 7 as follows. The numbers above the equivalence symbols indicate which relator we are using to perform the manipulation.

$$\begin{aligned} B^{-1}C^{-1}A^{-1}B^{-1}A^{-1}B^{-1}ABCABABA^{-1} &\overset{7}{\sim} B^{-1}C^{-1}A^{-1}B^{-1}A^{-1}B^{-1}ABACBABA^{-1} \\ &\overset{1}{\sim} B^{-1}C^{-1}A^{-1}CBABA^{-1} \\ &\overset{1}{\sim} B^{-1}C^{-1}A^{-1}CABAA^{-1} \\ &\overset{7}{\sim} B^{-1}BAA^{-1} \\ &\sim 1. \end{aligned}$$

Similarly, using only 1 and 7 we can derive relators 5, 9, 11, 18, 23, 30, 31, 33, 35, 37, 39, and 41, so these can also be eliminated. Using 7, we can now rewrite 29 as

$$(C^{-1}A^{-1}CABACBA^{-1}CABACBA^{-1}CABACBA^{-1}CABACBC)^{-1} \sim (BACBC)^{-4},$$

so we can replace 29 with the relator $(BACBC)^4$. Similarly we can rewrite 13 using 1 and 7 as

$$\begin{aligned} &CBCB^{-1}C^{-1}B^{-1}C^{-1}B^{-1}C^{-1}A^{-1}B^{-1}A^{-1}C^{-1}ACABACB \\ &\sim CBCB^{-1}C^{-1}B^{-1}C^{-1}B^{-1}C^{-1}A^{-1}B^{-1}ABACB \\ &\sim CBCB^{-1}C^{-1}B^{-1}C^{-1}B^{-1}C^{-1}BCB, \end{aligned}$$

so 13 can be replaced with $(BC)^3(CB)^{-3}$.

This leaves us with relators

1. $ABAB^{-1}A^{-1}B^{-1}$
2. $ADA^{-1}D^{-1}$
3. $BDB^{-1}D^{-1}$
4. DD
7. $ACA^{-1}C^{-1}$
8. $CDC^{-1}D^{-1}$
10. $AC^{-1}B^{-1}C^{-1}A^{-1}B^{-1}A^{-1}C^{-1}AB^{-1}C^{-1}A^{-1}B^{-1}A^{-1}C^{-1}A^{-1}$ $CABACBA^{-1}CABACBC$
13. $(BC)^3(CB)^{-3}$

24. $C^{-1}B^{-1}C^{-1}A^{-1}B^{-1}A^{-1}C^{-1}AB^{-1}C^{-1}A^{-1}B^{-1}$
$A^{-1}C^{-1}AB^{-1}C^{-1}A^{-1}B^{-1}A^{-1}C^{-1}B^{-1}A^{-1}BC^{-1}B^{-1}$

26. $C^{-1}B^{-1}C^{-1}A^{-1}B^{-1}A^{-1}C^{-1}AB^{-1}C^{-1}A^{-1}B^{-1}A^{-1}C^{-1}A^{-1}CABACBCBCABAC$

29. $(BACBC)^4$

It remains to show 10, 24, and 26 can be derived from the others. For 24 we have

$$\begin{aligned}&(BCB^{-1}ABCABACBA^{-1}CABACBA^{-1}CABACBC)^{-1}\\ &\overset{1,7}{\sim} (BCABA^{-1}CA(BACBC)^3)^{-1}\\ &\overset{7}{\sim} (BACBC)^{-4}\\ &\overset{29}{\sim} 1.\end{aligned}$$

For 10 we have

$$\begin{aligned}&AC^{-1}B^{-1}C^{-1}A^{-1}B^{-1}A^{-1}C^{-1}AB^{-1}C^{-1}A^{-1}B^{-1}A^{-1}C^{-1}A^{-1}CABACBA^{-1}CABACBC\\ &\overset{7}{\sim} AC^{-1}B^{-1}A^{-1}C^{-1}B^{-1}C^{-1}B^{-1}C^{-1}A^{-1}B^{-1}A^{-1}BACBCBCABC\\ &\overset{1}{\sim} AC^{-1}B^{-1}A^{-1}C^{-1}B^{-1}C^{-1}B^{-1}C^{-1}B^{-1}CBCBCABC\\ &\overset{13}{\sim} AC^{-1}B^{-1}A^{-1}B^{-1}C^{-1}B^{-1}C^{-1}B^{-1}C^{-1}CBCBCABC\\ &\sim AC^{-1}B^{-1}A^{-1}B^{-1}ABC\\ &\overset{1,7}{\sim} 1.\end{aligned}$$

Similarly using only 1,7, and 13 we can derive 26. These transformations show us that we can remove relators 10, 24, and 26, leaving us with the presentation

$$\begin{aligned}\langle A, B, C, D \mid\ &[A, C], [A, D], [B, D], [C, D],\\ &ABA = BAB, BCBCBC = CBCBCB, D^2, (BACBC)^4\rangle.\end{aligned}$$

The final thing to do is work out which elements of $\mathrm{Mod}(\Sigma_{1,2})$ A, B, C, and D represent. Recall these symbols are just labels for the symbols $C_{1,a}, C_{1,b}, C_{c,c}$, and $C_{1,d}$ respectively. The rewriting process tells us that the symbol $C_{K,\alpha}$ represents the element represented by the word $K\alpha\overline{K\alpha}^{-1}$ in the original presentation.

Using this, we see A, B, C, and D represent the words a, b, c^3, and d in the presentation for $\mathrm{Mod}(\Sigma_{1,2})$ given by Proposition 5.1.1, which are the elements T_α, T_β, T_γ^3, and the hyperelliptic involution ι respectively. ■

A.1 Sage code

This is the code that helped perform the Reidemeister-Schreier rewriting process and the subsequent Tietze transformations in the proof of Theorem 5.1.8. Words are entered as strings with inverses denoted by a ′. For example, the word $ab^{-1}ca^{-1}$ would be entered as the string "$ab'ca'$".

```
generators = ['a','b','c','d']
inverses = [x + "'" for x in generators]
relators = ["abab'a'b'","bcbc'b'c'","aca'c'","ada'd'","bdb'd'",
"cdc'd'","dd","abcabcabcabc"]
coset_reps = ['',"c","c'","cb'c","cb'","c'b","cba","cb","c'b'"]

#Takes a word (as a string) and outputs a list of letters (eg
#"abc'" to ['a','b',"c'"]).
def word_to_letters(x):
    final = []
    L = list(x)
    while len(L) > 0:
        if L[1 % len(L)] == "'":
            final.append(L[0] + L[1])
            L.remove(L[0])
            L.remove(L[0])
        else:
            final.append(L[0])
            L.remove(L[0])
    return final

#Does the opposite of word_to_letters.
def letters_to_word(x):
    final = ''
    for letter in x:
        final = final + letter
    return final

#Takes a word and outputs a list, eg "ab'c" to [['a',1],
#['b',-1],['c',1]].
def word_to_list(x):
```

```
    L = word_to_letters(x)
    final = []
    for let in L:
        if let[-1] == "'":
            final.append([let[:-1],-1])
        else:
            final.append([let,1])
    return final

#does the opposite of word_to_list.
def list_to_word(x):
    final = ""
    for let in x:
        if let[1] == 1:
            final = final + let[0]
        else:
            final = final + let[0] + "'"
    return final

#Checks whether or not a letter (as a string) is an inverse.
def is_inverse(x):
    if x in inverses:
        return True
    else:
        return False

#Inverts a list, eg [['a',1],['b',-1]] to [['b',1],['a',-1]].
def invert_list(x):
    return [[let[0],-let[1]] for let in reversed(x)]

#Inverts a word, eg "abc'd'a" to "a'dcb'a'".
def invert_word(x):
    return list_to_word(invert_list(word_to_list(x)))

#Conjugates the first word by the second.
def conjugate(x,y):
    return y + x + invert_word(y)
```

```
#Freely reduces a word in list form.
def freely_reduce(rel):
    reducedrel = [x for x in rel]
    flag = 0
    while flag == 0:
        flag = 1
        for let in range(len(reducedrel)-1):
            if reducedrel[let][0] == reducedrel[let + 1][0] and
            reducedrel[let][1] + reducedrel[let+1][1] == 0:
                flag = 0
                del reducedrel[let]
                del reducedrel[let]
                break
    return reducedrel

#Replaces a letter with a string in a word, the word is a list,
#the letter and replacement word are strings.
def replace_letter(rel,letter,replacement):
    replacement_list = word_to_list(replacement)
    final = []
    for let in rel:
        if let[0] == letter:
            if let[1] == 1:
                for x in replacement_list:
                    final.append(x)
            else:
                for x in invert_list(replacement_list):
                    final.append(x)
        else:
            final.append(let)
    return final

#Makes a copy of a list of words (in list form).  It will help
#preserve the steps in the working out.
def copy_relators(rels):
    final1 = []
    for rel in rels:
        final2 = []
```

```
        for let in rel:
            final3 = [x for x in let]
            final2.append(final3)
        final1.append(final2)
    return final1


#Preliminary definitions and functions to define schreier_rep.
######################

a = matrix(GF(3),[[1,1,0],[0,1,0],[0,0,1]])
b = matrix(GF(3),[[1,0,0],[-1,1,0],[0,0,1]])
c = matrix(GF(3),[[1,1,0],[0,1,0],[0,1,1]])
d = matrix(GF(3),[[-1,0,0],[0,-1,0],[0,0,-1]])

alpha = {'a':a,'b':b,'c':c,"a'":a^(-1),"b'":b^(-1),"c'":c^(-1),
'd':d,"d'":d}
cosetdict = {(0,0):"",(0,1):'c',(0,2):"c'",(1,0):"cb'c",
(1,1):"cb'",(1,2):"c'b",(2,0):"cba",(2,1):'cb',(2,2):"c'b'"}

#This takes in a word and outputs the matrix it represents.
def word_to_matrix(x):
    letters = word_to_letters(x)
    final = matrix(GF(3),[[1,0,0],[0,1,0],[0,0,1]])
    for letter in letters:
        final = final*alpha[letter]
    return final

#This takes in a matrix and returns the coset it belongs to as
#a tuple.
def coset(x):
    if x[2,2] == -1:
        return tuple([-x[2,0],-x[2,1]])
    else:
        return tuple([x[2,0],x[2,1]])

########################
```

```
#This takes in a word as a string and returns its coset
#representative (as a string).
def schreier_rep(x):
    return cosetdict[coset(word_to_matrix(x))]

#This rewrites a word and outputs the c-symbols as a list.
def rewrite(x):
    final = []
    L = word_to_letters(x)
    for y in range(len(L)):
        if is_inverse(L[y]):
            symbol = [(schreier_rep(letters_to_word(L[0:y+1])),
            L[y][:1]),-1]
        else:
            symbol = [(schreier_rep(letters_to_word(L[0:y])),
            L[y]),1]
        final.append(symbol)
    return final
```

```
#Inputs a set of generators in the Schreier form and outputs
#LaTeX code.
def schreier_rels_to_tex(x):
    print "\\noindent $"
    for rel in x:
        texrel = ''
        for y in rel:
            if y[1] == 1:
                if y[0][0] == '':
                    texrel = texrel + "C_{1," + y[0][1] + "}"
                else:
                    texrel = texrel + "C_{" + y[0][0] + ","
                    + y[0][1] + "}"
            else:
                if y[0][0] == '':
                    texrel = texrel + "C_{1," + y[0][1] +
                    "}^{-1}"
                else:
                    texrel = texrel + "C_{" + y[0][0] + ","
```

```
                        + y[0][1] + "}^{-1}"
        print texrel + "\\\\\n\\\\"
    print "$"

#Inputs a list of generators in word/list form and outputs
#LaTeX code.
def list_rels_to_tex(x):
    print "\\noindent $"
    for rel in x:
        texrel = ''
        for y in rel:
            if y[1] == 1:
                texrel = texrel + y[0]
            else:
                texrel = texrel + y[0] + '^{-1}'
        print texrel + "\\\\\n\\\\"
    print "$"
```

```
#This section outputs the relators from the Reidemeister-
#Schreier rewriting process, without any simplification.

SR_relators = []

#Outputs the relators from the Schreier-Reidemeister rewriting
#process.
for x in coset_reps:
    for y in relators:
        SR_relators.append(rewrite(conjugate(y,x)))

#output 0
schreier_rels_to_tex(SR_relators)
```

```
#This section optimizes the relators by deleting any trivial
#generators.

trivial_generators = []
```

```
#Outputs the set of tuples (a,K) such that $C_{a,K}$ is a
#generator that defines the identity.
for y in coset_reps:
    for x in generators:
        test = y + x + invert_word(schreier_rep(y + x))
        if freely_reduce(word_to_list(test)) == []:
            trivial_generators.append((y,x))

#Deletes a trivial generator from a word in list form.
def delete_trivial_generators(x):
    final = []
    for y in x:
        if y[0] in trivial_generators:
            pass
        else:
            final.append(y)
    return final

#This is the new set of relators.
SR_relators1 = [delete_trivial_generators(x) for x in
copy_relators(SR_relators)]

#output 1
schreier_rels_to_tex(SR_relators1)
```

```
#This section optimises the relators by dealing with relations
#of the form $ab^{-1}$, which allow us to delete $b$ and
#replace it by $a$ wherever it appears.

#This is a list of all relators of the form we want.
short_relators = [x for x in copy_relators(SR_relators1) if
len(x) == 2 and x[0][1] + x[1][1] == 0]

#This is a list of pairs of relators that are equal
equal_generators = [[x[0][0],x[1][0]] for x in short_relators]

#This function takes in a list of pairs defining an equivalence
#relation and outputs the equivalence classes.
```

```
def equivalence_classes(pairs):
    equiv = [x for x in pairs]
    flag = 0
    while flag == 0:
        flag = 1
        for pair1 in equiv:
            for pair2 in equiv[equiv.index(pair1)+1:]:
                if len([x for x in pair1 if x in pair2]) > 0:
                    inindex = equiv.index(pair1)
                    outindex = equiv.index(pair2)
                    equiv.insert(inindex,list(set(pair1 +
                     pair2)))
                    del equiv[inindex + 1]
                    del equiv[outindex]
                    flag = 0
                    break
            if flag == 0:
                break
    return equiv

#These are the equivalence classes of generators.
generator_classes = [sorted(x) for x in
equivalence_classes(equal_generators)]

#This generates a dictionary which associates every generator
#in the list generator_classes with a specific representative,
#which is the first element in the equivalence class.
generator_classes_dict = {}
for cl in generator_classes:
    for gen in cl[1:]:
        generator_classes_dict[gen] = cl[0]

SR_relators2 = copy_relators(SR_relators1)
for rel in SR_relators2:
    for let in rel:
        if let[0] in generator_classes_dict:
            newgen = generator_classes_dict[let[0]]
            let.insert(0,newgen)
```

```
            del let[1]

#output 2
schreier_rels_to_tex(SR_relators2)
```

```
#This section replaces the $C$-symbols with letters.

#This creates a dictionary associating a tuple (corresponding
#to a $C$-symbol), with a letter.
new_letters = ['A','B','C','D','E','F','G','H','I','J','K','L',
'M']
old_tuples = [('','a'),('','b'),('c','c'),('','d'),
("cb'c",'a'),("cb'c",'b'),("cb'c",'c'),("cb'",'a'),("c'b",'a'),
("c'b",'b'),('cba','a'),('cb','b'),("c'b'",'a')]
tuples_to_letters = dict(zip(old_tuples,new_letters))

#This rewrites the relators in terms of the new letters defined
#above.
new_relators3 = copy_relators(SR_relators2)

for rel in new_relators3:
    for let in rel:
        newlet = tuples_to_letters[let[0]]
        let.insert(0,newlet)
        del let[1]

#This freely reduces all the words, removes empty relators and
#removes duplicate relators.

new_relators3a = [freely_reduce(x) for x in
copy_relators(new_relators3) if len(freely_reduce(x)) > 0]

new_relators3b = []
for rel in copy_relators(new_relators3a):
    if rel not in new_relators3b:
        new_relators3b.append(rel)

#output 3
```

```
list_rels_to_tex(new_relators3b)
```

```
#This section eliminates all but 4 symbols.  It then freely
#reduces every word and eliminates empty words.

new_relators4a = [replace_letter(x,'I',"G'E") for x in
copy_relators(new_relators3b)]

new_relators4b = [replace_letter(x,'G',"F'C") for x in
copy_relators(new_relators4a)]

new_relators4c = [replace_letter(x,'F',"EAE'") for x in
copy_relators(new_relators4b)]

new_relators4d = [replace_letter(x,'M',"AH'L'") for x in
copy_relators(new_relators4c)]

new_relators4e = [replace_letter(x,'E',"HB") for x in
copy_relators(new_relators4d)]

new_relators4f = [replace_letter(x,'H',"L'AKJ'") for x in
copy_relators(new_relators4e)]

new_relators4g = [replace_letter(x,'J',"BCB'") for x in
copy_relators(new_relators4f)]

new_relators4h = [replace_letter(x,'L',"B'CB") for x in
copy_relators(new_relators4g)]

new_relators4i = [replace_letter(x,'K',"A'B'A'B'A'B'") for x in
copy_relators(new_relators4h)]

new_relators4 = [freely_reduce(x) for x in
copy_relators(new_relators4i)]

#output 4
list_rels_to_tex(new_relators4)
```