Semicrossed Products, Dilations, and Jacobson Radicals

by

Jaspar Wiart

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Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

Dr. Kenneth Davidson (Supervisor)
Dr. Justin Peters (External Examiner)
Dr. Laurent Marcoux (Internal Member)
Dr. Nico Spronk (Internal Member)
Dr. David Siegel (Internal-External Member)

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

We compute the C*-envelope of the isometric semicrossed product $\mathcal{A}_R \times_\alpha^{\mathrm{is}} R^\times$ of a C*-algebra arising from number theory by the multiplicative semigroup of a number ring R, and prove that it is isomorphic to $\mathfrak{T}[R]$, the left regular representation of the $ax+b$-semigroup $R \rtimes R^\times$ of R on $\ell^2(R \rtimes R^\times)$. We do this by explicitly dilating an arbitrary representation of $\mathcal{A}_R \times_\alpha^{\mathrm{is}} R^\times$ to a representation of $\mathfrak{T}[R]$ and show that such representations are maximal.

We also study the Jacobson radical of the semicrossed product $\mathcal{A} \times_\alpha P$ when $\mathcal{A}$ is a simple C*-algebra and P is either a subsemigroup of an abelian group or a free semigroup. A full characterization of the Jacobson radical is obtained for a large subset of these semicrossed products and we apply our results to a number of examples.

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Chapter 1

Introduction

During my graduate studies at the University of Waterloo two research projects yielded results. This thesis contains those results along with some additional materials intended to help a strong undergraduate student or a student early in their graduate studies understand the details of the proofs. In both projects I studied the semicrossed product of a C*-algebra with a semigroup, but the similarities end there.

Chapter 2 contains some background on semicrossed products including a general construction. In Chapter 3 the reader and I will compute the C*-envelope of the semicrossed product of a C*-algebra arising from number theory with the multiplicative semigroup of the associated number ring. In chapter 4 we will characterize the Jacobson radical of several classes of semicrossed products of simple C*-algebras with either abelian semigroups or free semigroups.

In [4], Cuntz, Deninger, and Laca associated to a number ring R a C*-algebra which encodes the additive, multiplicative, and ideal structure of the ring. The C*-algebra is the Toeplitz algebra of the $ax+b$-semigroup of the number ring which they realized as a universal C*-algebra $\mathfrak{T}[R]$ defined by relations on a generating set of unitaries u^x, indexed by R, isometries s_a, indexed by the multiplicative semigroup $R^\times = R\backslash\{0\}$, and projections e_I, indexed by the ideals of the ring. This C*-algebra, together with a one-parameter group of automorphisms, forms a dynamical system with a KMS-structure, which they computed directly.

Only one of the relations defining $\mathfrak{T}[R]$ requires the use of an adjoint, $s_a e_I s_a^* = e_{aI}$. If we replace this relation by the nonself-adjoint analogue, $s_a e_I = e_{aI} s_a$, then the relations determine the isometric semicrossed product, $\mathcal{A}_R \times_\alpha^{\text{is}} R^\times$, of a certain semigroup dynamical system, whose underlying C*-algebra, $\mathcal{A}_R$, is a C*-subalgebra of $\mathfrak{T}[R]$, and is acted upon by $R^\times$.

It is easy to show that any representation of $\mathfrak{T}[R]$ is also a representation of $\mathcal{A}_R \times_\alpha^{\text{is}} R^\times$, however the converse is not true. The issue is that the relation $s_a e_I = e_{aI} s_a$ does not imply $s_a e_I s_a^* = e_{aI}$. The main result of Chapter 3 is that the C*-envelope of $\mathcal{A}_R \times_\alpha^{\text{is}} R^\times$ is isomorphic to $\mathfrak{T}[R]$, which we establish by showing that maximal representations of $\mathcal{A}_R \times_\alpha^{\text{is}} R^\times$ are representations of $\mathfrak{T}[R]$. To do this, given an arbitrary representation of $\mathcal{A}_R \times_\alpha^{\text{is}} R^\times$, we explicitly dilate it to a representation of $\mathfrak{T}[R]$, and then show that any such dilation is maximal.

We show in Section 3.2 that a representation $\pi \times S$ of $\mathcal{A}_R \times_\alpha^{\text{is}} R^\times$ on $\mathcal{B}(\mathcal{H})$ is a representation of $\mathfrak{T}[R]$ if and only if $\pi(e_{cR}) - S_c S_c^* = 0$ for all $c \in R^\times$. The dilation theorem of that section tells us how to dilate a representation when there exists some $c \in R^\times$ such that $\pi(e_{cR}) - S_c S_c^* \neq 0$ to a representation $\widetilde{\pi} \times \widetilde{S}$ satisfying $(\widetilde{\pi}(e_{cR}) - \widetilde{S}_c \widetilde{S}_c^*)|_{\mathcal{H}} = 0$. In Section 3.3 we obtain an explicit dilation $\widehat{\pi} \times \widehat{S}$ that satisfies $\widehat{\pi}(e_{cR}) - \widehat{S}_c \widehat{S}_c^* = 0$. Repeated application of this technique eventually yields a maximal representation.

In addition to the isometric semicrossed product, it is natural to consider the contractive semicrossed product. However a standard counterexample shows that $\mathcal{A}_R \times_\alpha R^\times$ is not isometrically isomorphic to $\mathcal{A}_R \times_\alpha^{\text{is}} R^\times$. We do not know the C*-envelope of $\mathcal{A}_R \times_\alpha R^\times$, but it is at least as complicated as the polydisk algebra.

A C*-dynamical system is a triple $(\mathcal{A}, \alpha, P)$ consisting of a C*-algebra $\mathcal{A}$, a semigroup P, and an action α of P on $\mathcal{A}$ by $*$-endomorphisms. The semicrossed product $\mathcal{A} \times_\alpha P$ of $\mathcal{A}$ by P is a universal operator algebra associated to a C*-dynamical system. In Chapter 4 we characterize the Jacobson radical of several classes of semicrossed products of simple C*-algebras by either semigroups contained in an abelian group or free semigroups.

A full characterization of the Jacobson radical when $\mathcal{A} = C_0(X)$ is a commutative C*-algebra and $P = \mathbb{Z}_+^n$ was achieved in [7]. In the case $n = 1$ the C*-dynamical system $(\mathcal{A}, \alpha, \mathbb{Z}_+)$ becomes a topological dynamical system (X, ϕ), where ϕ is a continuous surjection, and the Jacobson radical is generated in a certain way by functions that vanish on the recurrent points of (X, ϕ). For $n \geq 2$ their characterization uses a variation on recurrence. When $\mathcal{A}$ is simple, the notion of recurrent points does not seem to arise. However some form of recurrence will likely be needed in the non-simple case.

Our main results show that if $(\mathcal{A}, \alpha, P)$ is a C*-dynamical system where $\mathcal{A}$ is a simple unital C*-algebra, P is either a semigroup contained in an abelian group or a free semigroup, and either

(i) $\mathcal{A}$ is purely infinite (Theorem 4.1.6), or

(ii) there exists a a faithful conditional expectation $E_s : \alpha_s(1)\mathcal{A}\alpha_s(1) \to \alpha_s(\mathcal{A})$ for each $s \in P$ (Theorem 4.1.10),

then the Jacobson radical of $\mathcal{A} \times_\alpha P$ is generated by monomials $a \otimes e_s$ where $a \in \mathcal{A}(1 - \alpha_s(1))$ (equivalently monomials such that $(a \otimes e_s)x = 0$ for all $x \in \mathcal{A} \times_\alpha P$). These theorems yield a number of corollaries including the case where each α_s is an automorphism (Corollary 4.1.11) and the case where the range of each α_s is hereditary (Corollary 4.1.15). We also apply our results to several examples including some standard $*$-endomorphisms of the Cuntz algebra and various shifts on the CAR algebra.

One obstruction to the characterization of the Jacobson radical in the non-unital case is that it is not clear that for fixed $s \in P$ that the set $\{a \in \mathcal{A} : a \otimes e_s \in \operatorname{rad}(\mathcal{A} \times_\alpha P)\}$ is not all of $\mathcal{A}$. However in two cases we are able to say that the above set is either $\{a \in \mathcal{A} : a\alpha_s(\mathcal{A}) = \{0\}\}$ or all of $\mathcal{A}$. In Proposition 4.2.2 we show that this holds when $(\mathcal{A}, \alpha, P)$ is an automorphic C*-dynamical system where $\mathcal{A}$ is simple and P is either contained in an abelian group or a free semigroup. Because $\alpha_s(\mathcal{A}) = \mathcal{A}$, in this case we have that the set is either zero or all of $\mathcal{A}$. With the additional assumption that $P = \mathbb{Z}_+$ we get that the radical of $\mathcal{A} \times_\alpha \mathbb{Z}_+$ is either zero or the ideal generated by $\mathcal{A} \otimes e_1$ (Corollary 4.2.3). In Corollary 4.2.5 we see that the above also holds when $(\mathcal{A}, \alpha, P)$ is a C*-dynamical system where $\mathcal{A}$ is a simple separable C*-algebra, P is contained in an abelian group, and the range of each α_s is hereditary. These results agree with the unital case because the condition $a\alpha_s(\mathcal{A}) = \{0\}$ is the same as $a \in \mathcal{A}(1 - \alpha_s(1))$. As a final example we apply our results to the action obtained by conjugating the compact operators by the unilateral shift.

Chapter 2

Background

2.1 Semicrossed Products

The first dynamical systems studied are now referred to as classical dynamical systems. They consist of a locally compact Hausdorff space X and a proper continuous map σ from X to itself. We can reformulate this in terms of C*-algebras by encoding X in the commutative C*-algebra $C_0(X)$. When we do this the map σ induces a $*$-endomorphism α using the rule $\alpha(f) = f \circ \sigma$ for $f \in C_0(X)$. It is natural to ask how iterations of α evolve. This leads us to consider $*$-endomorphisms $\{\alpha^n : n \in \mathbb{Z}_+\}$, where α^0 denotes the identity. This set satisfies $\alpha^n \alpha^m = \alpha^{n+m}$ and therefore is a semigroup under composition that is isomorphic to $\mathbb{Z}_+$.

A *C*-dynamical system* is a triple $(\mathcal{A}, \alpha, P)$ consisting of a C*-algebra $\mathcal{A}$, a semigroup P, and a semigroup homomorphism $\alpha : P \to \mathrm{End}(\mathcal{A})$. We call α an *action* of P on $\mathcal{A}$ by $*$-endomorphisms and use the notation $s \mapsto \alpha_s$.

One class of C*-dynamical systems that has seen much attention are automorphic C*-dynamical systems, those in which α is an action of a group G on a C*-algebra $\mathcal{A}$ by $*$-automorphisms. In this case the C*-dynamical system can be encoded in a single C*-algebra. One way to do this is to find a covariant pair (π, U) which consists of a $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and a unitary representation $U : G \to \mathcal{B}(\mathcal{H})$ which are related by the covariance relation

$$\pi\alpha_g(a) = U_g \pi(A) U_g^* \text{ for } a \in \mathcal{A} \text{ and } g \in G.$$

While the C*-algebra generated by the image of $\mathcal{A}$ and G in $\mathcal{B}(\mathcal{H})$ encodes (some of) the structure of $(\mathcal{A}, \alpha, G)$, it obviously depends on the chosen covariant pair. We avoid this choice by constructing a C*-algebra that is universal with respect to all the covariant pairs of the system, which we call the crossed product $\mathcal{A} \rtimes_\alpha G$ of $\mathcal{A}$ by G.

We wish to use a similar construction when we have a semigroup P acting on a C*-algebra $\mathcal{A}$ by $*$-endomorphisms. The main obstruction is that the action need not be invertible. Because of this we might not be able to find any covariant pairs as defined above. We will therefore need a different definition of covariant pairs which will result in the finished product being a universal operator algebra instead of a universal C*-algebra.

The definition of the semicrossed product as an operator algebra goes back to Peters [18]. There he studied actions of a single $*$-endomorphism on an arbitrary C*-algebra which generates an action of $\mathbb{Z}_+$. He weakened the requirement that a covariant pair should contain a unitary representation and instead required covariant pairs (π, V) to contain an isometry (which generates an isometric representation of $\mathbb{Z}_+$) which together with the $*$-representation of $\mathcal{A}$ satisfied a covariance relation. He had two possible choices which came to be called the left covariance relation $\pi(a)V = V\pi\alpha(a)$, and the right covariance relation $V\pi(a) = \pi\alpha(a)V$. He observed that the left covariance relation had the property $\ker \pi \subseteq \ker \pi \circ \alpha$. More importantly he observed that left isometric covariant pairs always exist while the same is not true for right isometric covariant pairs. For these reasons he chose to define the semicrossed product using the left covariant relation.

The construction of the semicrossed product used by Peters generalizes nicely to arbitrary actions of abelian semigroups P on C*-algebras (or even non-selfadjoint operator algebras). Even though we often relax the requirement that a covariant pair contain an isometric representation of the semigroup and simply require that a covariant pair (π, T) contain a contractive representation of P, and that right contractive covariant pairs always exist, the use of the left covariance relation became the dominant choice. In addition to the historical reason, this is because for a long time the research was focused on abelian actions, and in that case his original choice was arguably superior. Unfortunately this is problematic when we have a non-abelian action because the left covariance relation requires an abelian semigroup in order to be associative, as we observe:

$$\pi(a)T_{st} = \pi(a)T_s T_t = T_s \pi\alpha_s(a) T_t = T_s T_t \pi\alpha_t\alpha_s(a) = T_{st}\pi\alpha_{ts}(a).$$

When we have a free semigroup action we can use the left free covariance relation

$$\pi(a)T_w = T_w \pi\alpha_{\overline{w}}(a),$$

where $\overline{w}$ denotes the reverse of the word w, but this is approach does not generalize to other kinds of non-abelian semigroups.

In Chapter 3 our goal is to prove that the universal C*-algebra $\mathfrak{T}[R]$ defined using relations on a generating set can be realized as the C*-envelope of a certain semicrossed product. The relations make it clear that the right semicrossed product is the correct choice in that case. In Chapter 4 we use the right semicrossed product for the convenience of being able to combine the abelian semigroup case with the free semigroup case.

2.2 Constructing a Semicrossed Product

A *semigroup* is a set P that is closed under an associative binary operation with identity e. We will restrict ourselves to two classes, namely semigroups that are contained in abelian groups and free semigroups (which are also contained in groups). Such semigroups satisfy left and right cancellation, that is the equalities $st = sr$ and $ts = rs$ both imply $t = r$ for all $s, t, r \in P$.

The *free semigroup* $\mathbb{F}_I^+$ *over the generating set* I is the set of (finite) words with alphabet I with multiplication defined by concatenation. The empty word e is the identity. The map $\ell : \mathbb{F}_I^+ \to \mathbb{Z}_+$, where $\mathbb{Z}_+$ is the semigroup of non-negative integers under addition, taking a word $w = i_1 i_2 \cdots i_k$ to k, the *length* of w, is a semigroup homomorphism.

To construct a semicrossed product we must first define covariant pairs. A *(right) covariant pair* (π, T) for $(\mathcal{A}, \alpha, P)$ is a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of $\mathcal{A}$ and a contractive representation $T : P \to \mathcal{B}(\mathcal{H})$ of P that together satisfy the covariance relation

$$T_s \pi(a) = \pi\alpha_s(a) T_s \text{ for all } a \in \mathcal{A} \text{ and } s \in P.$$

To construct a universal operator algebra with respect to the covariant pairs of $(\mathcal{A}, \alpha, P)$ we begin with the algebra $c_{00}(\mathcal{A}, \alpha, P)$ which is the vector space $\mathcal{A} \otimes c_{00}(P)$ with multiplication given by the rule

$$(a \otimes e_s)(b \otimes e_t) = (a\alpha_s(b)) \otimes e_{st} \text{ for all } s, t \in P \text{ and } a, b \in \mathcal{A}.$$

Each covariant pair gives rise to a representation $\pi \times T : c_{00}(\mathcal{A}, \alpha, P) \to \mathcal{B}(\mathcal{H})$ defined by $(\pi \times T)(a \otimes e_s) = \pi(a) T_s$, which together can be used to construct a family of matrix norms. For each $n \geq 1$ we define a norm on $M_n(c_{00}(\mathcal{A}, \alpha, P))$ by

$$\Big\| \sum_{s \in P} A_s \otimes e_s \Big\| = \sup \Big\{ \Big\| \sum_{s \in P} (I_n \otimes T_s) \pi^{(n)}(A_s) \Big\|_{\mathcal{B}(\mathcal{H}^{(n)})} : (\pi, T) \text{ a covariant pair} \Big\}$$

where $A_s \in M_n(\mathcal{A})$ and $A_s = 0$ except finitely often. We note that because the orbit representation in Example 2.2.1 is injective on $\mathcal{A} \otimes c_{00}(P)$, the formula above assigns zero only to the zero element, making it a norm. The *semicrossed product* $\mathcal{A} \times_\alpha P$ *of* $\mathcal{A}$ *by* P is the operator algebra completion of $c_{00}(\mathcal{A}, \alpha, P)$ with respect to the family of matrix norms given above.

It is clear from the definition that $\mathcal{A} \times_\alpha P$ has the universal property that each covariant pair (π, T) gives rise to a completely contractive representation, which we also denote by $\pi \times T$, on $\mathcal{A} \times_\alpha P$ extending the representation on $c_{00}(\mathcal{A}, \alpha, P)$.

The following example shows that for C*-dynamical systems (A, α, P) where P has the right cancellation property, covariant pairs always exist and $\mathcal{A} \times_\alpha P$ contains a faithful copy of $\mathcal{A}$.

Example 2.2.1 (The Orbit Representation)**.** Let $(\mathcal{A}, \alpha, P)$ be a C*-dynamical system and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a faithful representation of $\mathcal{A}$. Let $\widetilde{\mathcal{H}} = \mathcal{H} \otimes \ell^2(P)$ and define $\widetilde{\pi} : \mathcal{A} \to \mathcal{B}(\widetilde{\mathcal{H}})$ and $T : P \to \mathcal{B}(\widetilde{\mathcal{H}})$ by

$$\begin{aligned} \widetilde{\pi}(a)(\xi \otimes \delta_t) &= (\pi\alpha_t(a)\xi) \otimes \delta_t \text{ and} \\ T_s(\xi \otimes \delta_t) &= \begin{cases} \xi \otimes \delta_r & \text{if } t = rs \text{ for some } r \in P, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Note that T_s is well-defined because right cancellation holds in P. We claim that $(\widetilde{\pi}, T)$ is a covariant pair for $(\mathcal{A}, \alpha, P)$. It is clear that $\widetilde{\pi}$ is a (faithful) representation of $\mathcal{A}$ and that T_s is a co-isometry, and is therefore contractive, for each $s \in P$. We verify that T is a semigroup homomorphism

$$\begin{aligned} T_{s_1}T_{s_2}(\xi \otimes \delta_t) &= \begin{cases} T_{s_1}(\xi \otimes \delta_{r_2}) & \text{if } t = r_2 s_2 \text{ for some } r_2 \in P, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \xi \otimes \delta_{r_1} & \text{if } t = r_1 s_1 s_2 \text{ for some } r_1 \in P, \\ 0 & \text{otherwise} \end{cases} \\ &= T_{s_1 s_2}(\xi \otimes \delta_t) \end{aligned}$$

and that the covariance relation is satisfied

$$\begin{aligned} T_s\widetilde{\pi}(a)(\xi \otimes \delta_t) &= \begin{cases} (\pi\alpha_{rs}(a)\xi) \otimes \delta_r & \text{if } t = rs \text{ for some } r \in P, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \widetilde{\pi}\alpha_s(a)(\xi \otimes \delta_r) & \text{if } t = rs \text{ for some } r \in P, \\ 0 & \text{otherwise} \end{cases} \\ &= \widetilde{\pi}\alpha_s(a)T_s(\xi \otimes \delta_t). \end{aligned}$$

2.3 Other Semicrossed Products

The construction of the semicrossed product is a special case of a general construction outlined in [6, Section 2.1]. We start with an algebra A and a collection $\mathcal{F}$ of homomorphisms of A into $\mathcal{B}(\mathcal{H})$ that is

(i) closed under arbitrary direct sums,

(ii) closed under restriction to reducing subspaces, and

(iii) closed under unitary equivalence.

Such a collection is called a *family* of representations. Using $\mathcal{F}$ we get a family of matrix seminorms, from which we can complete A (or a quotient of it) to get the *enveloping operator algebra of* A *with respect to* $\mathcal{F}$ denoted $\widetilde{A}$. This operator algebra has the property that every element of $\mathcal{F}$ extends uniquely to a completely contractive representation of $\widetilde{A}$.

In our case, by restricting the covariant pairs in the supremum formula for the matrix norms to certain families of covariant pairs we get other semicrossed products that are universal with respect to those families. For example

Definition 2.3.1. (i) the *(right) unitary semicrossed product* $\mathcal{A} \times_{\alpha}^{\mathrm{un}} P$ of $\mathcal{A}$ by P is obtained by completing $c_{00}(\mathcal{A}, \alpha, P)$ with respect to covariant pairs with a unitary representation of P (called *(right) unitary covariant pairs*),

(ii) the *(right) isometric semicrossed product* $\mathcal{A} \times_{\alpha}^{\mathrm{is}} P$ of $\mathcal{A}$ by P is obtained by completing $c_{00}(\mathcal{A}, \alpha, P)$ with respect to (right) isometric covariant pairs, and

(iii) the *(right) co-isometric semicrossed product* $\mathcal{A} \times_{\alpha}^{\mathrm{co}} P$ of $\mathcal{A}$ by P is obtained by completing $c_{00}(\mathcal{A}, \alpha, P)$ with respect to (right) co-isometric covariant pairs.

We note that the first two might not exist.

Remark 2.3.2. Because our analysis of the Jacobson radical in Chapter 4 is mostly algebraic in nature and when we do estimate the norms of elements, we only test monomials, our results hold for the more general semicrossed products defined above, if they exist.

When α is an action of an abelian semigroup on a C*-algebra $\mathcal{A}$, the *left semicrossed product* of $\mathcal{A}$ by P is constructed as follows. We begin with the algebra $c_{00}(P, \alpha, \mathcal{A})$ which is the vector space $c_{00}(P) \otimes \mathcal{A}$ with multiplication defined by

$$(e_s \otimes a)(e_t \otimes b) = e_{s+t} \otimes (\alpha_t(a)b) \text{ for all } s, t \in P \text{ and } a, b \in \mathcal{A}.$$

We note that this rule is associative because P is abelian. Each left covariant pair (π, T) gives rise to a representation $T \times \pi$ defined by $(T \times \pi)(e_s \otimes a) = T_s \pi(a)$. For each $n \geq 1$ we defined a norm on $M_n(c_{00}(\mathcal{A}, \alpha, P))$ by

$$\Big\| \sum_{s \in P} e_s \otimes A_s \Big\| = \sup \Big\{ \Big\| \sum_{s \in P} (T_s \otimes I_n) \pi^{(n)}(A_s) \Big\|_{\mathcal{B}(\mathcal{H}^{(n)})} : (\pi, T) \text{ a left covariant pair} \Big\}$$

where $A_s \in M_n(\mathcal{A})$ and $A_s = 0$ except finitely often. The *left semicrossed product of* $\mathcal{A}$ *by* P is the operator algebra completion of $c_{00}(P) \otimes \mathcal{A}$ with respect to the family of matrix norms. The *left unitary/isometric semicrossed product of* $\mathcal{A}$ *by* P are defined similarly.

Remark 2.3.3. With minor changes reflecting that multiplication in the left semicrossed product is dual to that of the right semicrossed product, our statements and proofs in Chapter 4 can be reformulated to handle the left semicrossed product and the other variations described in [6, Section 3.1].

2.4 Dilations and C*-envelopes of Semicrossed Products

A *representation* of an operator algebra $\mathcal{A}$ is a completely contractive homomorphism $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. A *dilation* $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ of ρ is a representation of $\mathcal{A}$ such that $\mathcal{H} \subseteq \mathcal{K}$ and $P_{\mathcal{H}} \pi(a)|_{\mathcal{H}} = \rho(a)$ for all $a \in \mathcal{A}$. Every dilation can be represented by an upper triangular matrix of the form

$$\begin{pmatrix} * & * & * \\ 0 & \rho(a) & * \\ 0 & 0 & * \end{pmatrix}.$$

A *maximal representation* of $\mathcal{A}$ is a representation ρ of $\mathcal{A}$ that has the property that any dilation π of ρ is of the form $\pi = \rho \oplus \varphi$.

In his early papers Arveson noticed that an operator algebra can be embedded in a variety of C*-algebras. More precisely, an operator algebra may admit more than one *C*-cover*, i.e. pairs $(\mathcal{C}, j)$ consisting of a C*-algebra $\mathcal{C}$ and a completely isometric homomorphism $j : \mathcal{A} \to \mathcal{C} = C^*(j(\mathcal{A}))$. The C*-envelope of $\mathcal{A}$ is the unique minimal C*-cover, denoted $(C^*_{\text{env}}(\mathcal{A}), \iota)$ or just C^*_{env}. By minimal we mean that if $(\mathcal{C}, j)$ is any C*-cover, then there exists a unique $*$-epimorphism $\Phi : \mathcal{C} \to C^*_{\text{env}}(\mathcal{A})$ making the following diagram commute

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "j"] \arrow[dr, "\iota"] & \mathcal{C} \arrow[d, dashed, "\Phi"] \\
 & C^*_{\mathrm{env}}(\mathcal{A})
\end{tikzcd}$$

Although Arveson calculated the C*-envelope for a large family of examples, the proof of its existence (due to Hamana) took ten years. The connection to dilations was found more than twenty years after that by Dritschel and McCullough [9]. They showed that the universal C*-algebra for the maximal representations of an operator algebra $\mathcal{A}$ is the the C*-envelope and that every completely contractive representation of $\mathcal{A}$ can be dilated to a maximal one. Because of this result we can compute the C*-envelope of an operator algebra by dilating an arbitrary representation to maximal ones.

Typically we seek a nice description of the C*-envelope of an operator algebra. For a semicrossed product this usually means finding an automorphic C*-dynamical system $(\mathcal{B}, \beta, G)$ such that $C^*_{\mathrm{env}}(\mathcal{A} \times_\alpha P)$ is either $\mathcal{B} \rtimes_\beta G$ or a full corner of it. The first result of this form was due to Muhly and Solel [14] and acts as a prototype.

Theorem 2.4.1. *Let $\mathcal{A}$ be a unital C*-algebra and let α be a $*$-automorphism. Then*

$$C^*_{env}(\mathcal{A} \times_\alpha \mathbb{Z}_+) \simeq \mathcal{A} \rtimes_\alpha \mathbb{Z}.$$

C*-dynamical systems over $\mathbb{Z}_+$ are generated by a single $*$-endomorphism. When dilating representations of their semicrossed products we only need to deal with a single contraction or isometry that generates the representation of $\mathbb{Z}_+$. It is not surprising then that these semicrossed products have a nice C*-envelope. In [13] Kakariadis and Katsoulis show that given any C*-dynamical system $(\mathcal{A}, \alpha, \mathbb{Z}_+)$ over $\mathbb{Z}_+$ we can construct an automorphic C*-dynamical system $(\mathcal{B}, \beta, \mathbb{Z})$ such that $\mathcal{A} \times_\alpha \mathbb{Z}_+$ is a full corner of $\mathcal{B} \rtimes_\beta \mathbb{Z}$.

Getting such unconditional results for other semigroups is not generally possible. This is due to the fact that one cannot always dilate three commuting contractions to three commuting unitaries. To get a nice dilation theory we must impose restriction on the semigroup or use other semicrossed products. For a detailed summary see [13].

2.5 The Jacobson Radical of Banach Algebras

The goal of this section is to define the Jacobson radical of a Banach algebra as the intersection of the kernels of its irreducible representations and to state a theorem that gives some well-known alternative characterizations. We will follow Bonsall and Duncan [2]. Throughout this section A will denote a (possibly non-unital) Banach algebra.

A left ideal I of A is called *modular* if it has a right modular unit, that is an element $e \in A$ satisfying $A(1-e) \subseteq I$. A modular left ideal is called maximal if it is not contained in any other proper left ideal of A. One can prove every modular left ideal is contained in a maximal modular left ideal by applying Zorn's Lemma. We will see that maximal modular left ideals appear as the annihilators of elements of irreducible left A-modules.

A *representation* of A is a homomorphism $\pi : A \to \mathcal{L}(X)$ from A into the set of linear maps on a complex vector space. We will consider (X, π) as a left A-module using the convention

$$ax = \pi(a)x \text{ for all } a \in A \text{ and } x \in X.$$

We say that a representation π is *trivial* if it is the zero map and X is one dimensional. If π is non-trivial and (X, π) has no proper left A-submodules we say that π is *irreducible* (or X is *irreducible*) The following two propositions relate maximal left ideals and irreducible left A-modules. They appear in [2] as Proposition 24.4 and Proposition 24.5 respectively.

Proposition 2.5.1. *Let X be an irreducible left A-module. If $x_0 \in X \setminus \{0\}$,*

(i) then x_0 is a cyclic vector (i.e. $Ax_0 = X$).

(ii) Each element $e \in \{e \in A : ex_0 = x_0\}$ is a right modular unit for the left ideal $\ker(x_0) = \{a \in A : ax_0 = 0\}$.

(iii) The ideal $\ker(x_0)$ is maximal.

(iv) The kernel $\ker(\pi)$ of π is the intersection of maximal modular left ideals

$$\ker(\pi) = \bigcap_{x_0 \in X \setminus \{0\}} \ker(x_0).$$

Proposition 2.5.2. *Let $I \subseteq A$ be a maximal modular left ideal. Then there exists an irreducible left A-module X and an element $x_0 \in X \setminus \{0\}$ such that $I = \ker(x_0)$.*

Definition 2.5.3. The *Jacobson radical* $\operatorname{rad}(A)$ of a Banach algebra is the intersection of the kernels of all the irreducible representations of A. If A has no irreducible representations the convention is to put $\operatorname{rad}(A) = A$ and we call A *radical.* When $\operatorname{rad}(A) = \{0\}$ we say A is *semi-simple.*

We give a few of the many different characterizations of $\operatorname{rad}(A)$, for the proof see Propositions 24.14 and 25.1 in [2].

Proposition 2.5.4. *Let A be a Banach algebra.*

(i) $\operatorname{rad}(A)$ *is the intersection of the maximal modular left ideals of A.*

(ii) $\operatorname{rad}(A) = \{a \in A : \lim_{n\to\infty} \|(ab)^n\|^{1/n} = 0$ *for all* $b \in A\}$.

(iii) $\operatorname{rad}(A) = \{a \in A : \lim_{n\to\infty} \|(ba)^n\|^{1/n} = 0$ *for all* $b \in A\}$.

We say that an element $a \in A$ is *quasi-nilpotent* if its spectral radius is zero, which is equivalent to $\lim_{n\to\infty} \|a^n\|^{1/n} = 0$. In Chapter 4 we will use the quasi-nilpotence characterization to compute the Jacobson radical of certain semicrossed products. We will also need two more facts, both of which follow easily from that characterization.

Corollary 2.5.5. *Let A be a Banach algebra.*

(i) The elements of $\operatorname{rad}(A)$ *are quasi-nilpotent.*

(ii) The Jacobson radical is an automorphism invariant ideal.

Chapter 3

Dilations From Number Theory

3.1 Preliminaries

A *number field* K is a finite field extension of $\mathbb{Q}$. An *algebraic integer* is the root of a monic polynomial with integer coefficients. The set of all algebraic integers, $\mathbb{A}$, is countable. The *ring of integers* of a number field K is the ring $R = K \cap \mathbb{A}$. A *number ring* R is the ring of integers in a number field. Examples of number rings include $\mathbb{Z}$, $\mathbb{Z}[i]$, and $\mathbb{Z}[\zeta_n]$ where ζ_n is a primitive nth root of unity. Number rings are Dedekind domains, thus every ideal in R factors uniquely as a product of prime ideals. In general, number rings are not principal ideal domains.

Let R be a number ring and let $R^\times = R \setminus \{0\}$ denote the multiplicative semigroup of R. The $ax+b$-semigroup $R \rtimes R^\times$ of R is the semigroup with elements $R \times R^\times$ and multiplication given by

$$(x,a)(y,b) = (x+ay, ab).$$

The Toeplitz algebra $\mathfrak{T}_{R\rtimes R^\times}$ of this semigroup is the C*-algebra generated by the left-regular representation of $R \rtimes R^\times$ on $\ell^2(R \rtimes R^\times)$. Explicitly, it is the C*-algebra generated by the isometries $T_{(x,a)}$, $(x,a) \in R \rtimes R^\times$, which act on the standard orthonormal basis $\{\xi_{(y,b)} : (y,b) \in R \rtimes R^\times\}$ according to

$$T_{(x,a)}(\xi_{(y,b)}) = \xi_{(x,a)(y,b)}.$$

Contained in $\mathfrak{T}_{R\rtimes R^\times}$ is a family of projections $e_{(x,I)}$, $x \in R$ and I an ideal of R, corresponding to cosets $(x+I)$ of ideals. These projections are characterized by their action on the basis

$$e_{(x,I)}(\xi_{(y,b)}) = \begin{cases} \xi_{(y,b)} & \text{if } y + bR \subseteq x + I, \\ 0 & \text{otherwise,} \end{cases}$$

and they multiply according to the rule

$$e_{(x,I)}e_{(y,J)} = \begin{cases} 0 & \text{if } (x+I) \cap (y+J) = \emptyset \\ e_{(z,I\cap J)} & \text{for any } z \in (x+I) \cap (y+J) \neq \emptyset. \end{cases}$$

When $I = aR$ is a principal ideal, the projection $e_{(x,aR)} = T_{(x,a)}T^*_{(x,a)}$ is just the range projection of $T_{(x,a)}$. When I is not principal, we use the fact that I can be written in the form $\frac{a}{b}R \cap R$ for some $a, b \in R^\times$ [4, Lemma 4.15] and write

$$e_{(x,I)} = T_{(x,1)}T^*_{(0,b)}T_{(0,a)}T^*_{(0,a)}T_{(0,b)}T_{(-x,1)}.$$

Cuntz, Deninger, and Laca showed in [4] that $\mathfrak{T}_{R\rtimes R^\times}$ is isomorphic to the universal C*-algebra $\mathfrak{T}[R]$ generated by elements u^x, $x \in R$, s_a, $a \in R^\times$, e_I, I a non-zero ideal in R, satisfying the following relations

- Ta: The u^x are unitary and satisfy $u^x u^y = u^{x+y}$, the s_a are isometries and satisfy $s_a s_b = s_{ab}$. Moreover $s_a u^x = u^{ax} s_a$ for all $x \in R$, $a \in R^\times$.
- Tb: The e_I are projections and satisfy $e_{I\cap J} = e_I e_J$, $e_R = 1$.
- Tc: We have $s_a e_I s_a^* = e_{aI}$.
- Td: For $x \in I$ one has $u^x e_I = e_I u^x$, for $x \notin I$ one has $e_I u^x e_I = 0$.

The relation Ta simply says that the map $(x, a) \mapsto u^x s_a$ is an isometric representation of $R \rtimes R^\times$. The other three relations recover the structure of the projections $e_{(x,I)}$: Tc gives us that $e_I = s_b^* s_a s_a^* s_b$ for any $a, b \in R^\times$ such that $I = \frac{a}{b}R \cap R$, and Tb together with Td tell us that the family of projections $e_I^x = u^x e_I u^{-x}$ multiply in the same way as the $e_{(x,I)} \in \mathfrak{T}_{R\rtimes R^\times}$.

Definition 3.1.1. The dynamical system that we are interested in consists of the C*-subalgebra $\mathcal{A}_R$ of $\mathfrak{T}[R]$ generated by the elements u^x and e_I, with an action of $R^\times$ given by $\alpha_a(u^x) = u^{ax}$ and $\alpha_a(e_I) = e_{aI}$.

3.2 Dilating Representations

In this section we will prove that a representation of $\mathcal{A}_R \times_\alpha^{\text{is}} R^\times$ is maximal if and only if it is also a representation of $\mathfrak{T}[R]$. We will also explicitly dilate a non-maximal representation.

Let $\pi \times S : \mathcal{A}_R \times_\alpha^{\text{is}} R^\times \to \mathcal{B}(\mathcal{H})$ be a covariant representation. Let $U^x = \pi(u^x)$, $x \in R$, and $E_I = \pi(e_I)$, I an ideal of R. Then U^x, S_a, and E_I satisfy the following relations:

Ca: The U^x are unitary and satisfy $U^x U^y = U^{x+y}$, the S_a are isometries and satisfy $S_a S_b = S_{ab}$. Moreover $S_a U^x = U^{ax} S_a$ for all $x \in R$, $a \in R^\times$.

Cb: The E_I are projections and satisfy $E_{I\cap J} = E_I E_J$, $E_R = 1$.

Cc: We have $S_a E_I = E_{aI} S_a$.

Cd: For $x \in I$ one has $U^x E_I = E_I U^x$, for $x \notin I$ one has $E_I U^x E_I = 0$.

If we are given a collection of elements U^x, $x \in R$, S_a, $a \in R^\times$, E_I, I a non-zero ideal in R, satisfying the above relations, then the assignment $\pi(u^x) = U^x$ and $\pi(e_I) = E_I$ gives us a covariant representation of $\mathcal{A}_R \times_\alpha^{\text{is}} R^\times$. For convenience we will consider covariant representations of $\mathcal{A}_R \times_\alpha^{\text{is}} R^\times$ as C*-subalgebras of $\mathcal{B}(\mathcal{H})$ generated by elements U^x, $x \in R$, S_a, $a \in R^\times$, E_I, I a non-zero ideal in R satisfying Ca-Cd. If in addition to satisfying Ca-Cd, the generators also satisfy Tc, then they are also a representation of $\mathfrak{T}[R]$. As we will see in Example 3.2.3 this is not always the case.

Before we show how to dilate a representation of $\mathcal{A}_R \times_\alpha^{\text{is}} R^\times$ that is not maximal , we will first characterize representations of $\mathcal{A}_R \times_\alpha^{\text{is}} R^\times$ that are also representations of $\mathfrak{T}[R]$. It will turn out that the maximal representations are those that are also representations of $\mathfrak{T}[R]$.

Proposition 3.2.1. *Let U^x, S_a, and E_I be an isometric covariant representation of the isometric semicrossed product $\mathcal{A}_R \times_\alpha^{\text{is}} R^\times$ on $\mathcal{B}(\mathcal{H})$. Then the map $\varphi : \mathfrak{T}[R] \to \mathcal{B}(\mathcal{H})$ defined by*

$$\varphi(u^x) = U^x,\ \varphi(s_a) = S_a,\ \varphi(e_I) = E_I,$$

is a homomorphism if and only if $E_{aR} = S_a S_a^*$ for all $a \in R^\times$.

Proof. $(\Rightarrow)$ The relation Tc implies $E_{aR} = S_a S_a^*$ for all $a \in R^\times$.

$(\Leftarrow)$ We only need to check that Tc is satisfied. Given $a \in R^\times$ and an ideal I of R we have

$$S_a E_I S_a^* = E_{aI} S_a S_a^* = E_{aI} E_{aR} = E_{aI \cap aR} = E_{aI}.$$

Thus the representation satisfies Tc. $\square$

Theorem 3.2.2. *Let* U^x*,* $x \in R$*,* S_a*,* $a \in R^\times$*,* E_I*,* I *a non-zero ideal in* R*, be elements in some* $\mathcal{B}(\mathcal{H})$ *that satisfy Ca-Cd. Suppose that there exists some element* $c \in R^\times$ *such that* $\mathcal{L} = (E_{cR} - S_c S_c^*)\mathcal{H} \neq \{0\}$*. Let* $\mathcal{K} \cong \mathcal{L}$*, and let* $T : \mathcal{K} \to \mathcal{L}$ *be a surjective isometry. Then the bounded linear operators* $\widetilde{U}^x$*,* $\widetilde{S}_a$*, and* $\widetilde{E}_I$ *acting on* $\widetilde{\mathcal{H}} := \mathcal{H} \oplus \mathcal{K}$ *according to*

$$\widetilde{U}^x = \begin{pmatrix} U^x & 0 \\ 0 & T^* U^{cx} T \end{pmatrix},$$
$$\widetilde{S}_a = \begin{pmatrix} S_a & S_c^* S_a T \\ 0 & T^* S_a T \end{pmatrix}, \text{ and}$$
$$\widetilde{E}_I = \begin{pmatrix} E_I & 0 \\ 0 & T^* E_{cI} T \end{pmatrix}.$$

dilate the representation.

Most of the rest of the section will be devoted to proving the above theorem. To do this we must show that the dilation satisfies Ca-Cd. But first we present an example.

Example 3.2.3. Let $\mathcal{H} = \ell^2(\mathbb{Z} \times \mathbb{Z}^\times)$ and let $\{\xi_{(y,b)} : (y,b) \in \mathbb{Z} \times \mathbb{Z}^\times\}$ be the standard orthonormal basis. Define bounded linear operators U^x, $x \in \mathbb{Z}$, S_a, $a \in \mathbb{Z}^\times$, and $E_{2^k n\mathbb{Z}}$, $k \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$ odd, on $\mathcal{H}$ by their action on the basis elements $\xi_{(y,b)}$, $(y,b) \in \mathbb{Z} \times \mathbb{Z}^\times$:

$$\begin{aligned} U^x \xi_{(y,b)} &= \xi_{(x+y,b)} \\ S_a \xi_{(y,b)} &= \xi_{(ay,ab)} \\ E_{2^k n\mathbb{Z}} \xi_{(y,b)} &= \begin{cases} \xi_{(y,b)} & \text{if } 2^k n | y \text{ and } n | b, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

It is easy to check that U^x, S_a, and $E_{2^k n\mathbb{Z}}$ satisfy Ca-Cd and that $E_{2\mathbb{Z}} - S_2 S_2^* \neq 0$.

The representation $U^x = T_{(x,1)}$ and $S_a = T_{(0,a)}$ is just the Toeplitz representation of $\mathbb{Z} \rtimes \mathbb{Z}^\times$. In that representation $e_{(0,a\mathbb{Z})} = S_a S_a^*$ is the orthogonal projection onto the subspace generated by the basis vectors $\xi_{(y,b)}$ where y and b are divisible by a and

$$S_a^* \xi_{(y,b)} = \begin{cases} \xi_{(y/a,b/a)} & \text{if } a \mid y \text{ and } a \mid b, \\ 0 & \text{otherwise,} \end{cases}$$

can be thought of as division by a in $\ell^2(\mathbb{Z} \rtimes \mathbb{Z}^\times)$ (when possible). In the representation defined above we can think of $E_{2\mathbb{Z}}$ as the orthogonal projection onto the subspace of $\mathcal{H}$ generated by the basis vectors that the representation thinks should be divisible by two, and $S_2S_2^*$ as the orthogonal projection onto the subspace generated by the basis vectors that are actually divisible by two. The problem is that these two subspaces do not agree. To fix the problem we need to define division on the subspace

$$\mathcal{L} = (E_{2\mathbb{Z}} - S_2S_2^*)\mathcal{H} = \overline{\operatorname{span}}\{\xi_{(2y,b)} : y, b \in \mathbb{Z} \text{ and } b \text{ odd}\},$$

generated by the basis vectors that should be divisible by two but are not. Consider the basis vector $\xi_{(2,1)}$ in $\mathcal{L}$. Because $E_{2\mathbb{Z}}\xi_{(2,1)} = \xi_{(2,1)}$, we should be able to divide $\xi_{(2,1)}$ by two, but we cannot because $S_2^*\xi_{(2,1)} = 0$. To dilate the representation we define a new Hilbert space $\widetilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{K}$, where

$$\mathcal{K} = \overline{\operatorname{span}}\{\xi_{(y,b/2)} : y, b \in \mathbb{Z} \text{ and } b \text{ odd}\} \cong \mathcal{L},$$

which contains $\xi_{(1,1/2)}$, and define $\widetilde{S}_2$ on $\widetilde{\mathcal{H}}$ in such a way that $\widetilde{S}_2^*\xi_{(2,1)} = \xi_{(1,1/2)}$. Explicitly $\widetilde{S}_2 = S_2$ on $\mathcal{H}$ and $\widetilde{S}_2\xi_{(y,b/2)} = \xi_{(2y,b)}$ on $\mathcal{K}$.

Next we need to know how the $\widetilde{U}^x$, $\widetilde{S}_a$, and $\widetilde{E}_{2^kn\mathbb{Z}}$ should act on $\mathcal{K}$. As an example consider $\widetilde{U}^x\xi_{(y,b/2)}$. Because the dilation should satisfy Ca, we have that $\widetilde{U}^x = \widetilde{S}_2^*\widetilde{S}_2\widetilde{U}^x = \widetilde{S}_2^*\widetilde{U}^{2x}\widetilde{S}_2$. Therefore to compute $\widetilde{U}^x\xi_{(y,b/2)}$ we can first multiply $\xi_{(y,b/2)}$ by 2 and apply U^{2x} before dividing by 2 to get

$$\widetilde{U}^x\xi_{(y,b/2)} = \widetilde{S}_2^*\widetilde{U}^{2x}\widetilde{S}_2\xi_{(y,b/2)} = \widetilde{S}_2^*\widetilde{U}^{2x}\xi_{(2y,b)} = \widetilde{S}_2^*\xi_{(2y+2x,b)} = \xi_{(y+x,b/2)}.$$

We can use the same trick to compute $\widetilde{S}_a$ and $\widetilde{E}_{2^kn\mathbb{Z}}$

Using the notation of the theorem let $T : \mathcal{K} \to \mathcal{L}$ be the isometry $\widetilde{S}_2|_{\mathcal{K}}$. We define $\widetilde{U}^x$, $\widetilde{S}_a$ and $\widetilde{E}_{2^kn\mathbb{Z}}$ to be U^x, S_a, and $E_{2^kn\mathbb{Z}}$ on $\mathcal{H}$ and $T^*U^{2x}T = \widetilde{S}_2^*U^{2x}\widetilde{S}_2$, $\widetilde{S}_2^*S_a\widetilde{S}_2$, and $T^*E_{2^{k+1}n\mathbb{Z}}T = \widetilde{S}_2^*E_{2^{k+1}n\mathbb{Z}}\widetilde{S}_2$ or explicitly

$$\begin{aligned}
\widetilde{U}^x\xi_{(y,b/2)} &= \xi_{(x+y,b/2)}, \\
\widetilde{S}_a\xi_{(y,b/2)} &= \xi_{(ay,ab/2)}, \text{ and} \\
\widetilde{E}_{2^kn\mathbb{Z}}\xi_{(y,b/2)} &= \begin{cases} \xi_{(y,b/2)} & \text{if } 2^kn|y \text{ and } n|b, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

on $\mathcal{K}$. As we will see in the calculation after Lemma 3.2.10, this dilation fixes the issue on $\mathcal{H}$, i.e. $(\widetilde{E}_2 - \widetilde{S}_2\widetilde{S}_2^*)|_{\mathcal{H}} = 0$, but on $\mathcal{K}$ we still have a problem, i.e. $(\widetilde{E}_2 - \widetilde{S}_2\widetilde{S}_2^*)|_{\mathcal{K}} \neq 0$.

For the remainder of the section we will use the notation in Theorem 3.2.3. Before we prove that the dilation satisfies Ca-Cd we need to establish a few useful identities.

Lemma 3.2.4. *We have* $TT^* = E_{cR} - S_c S_c^*$ *and* $S_c^* T = 0$.

Proof. Technically T is a map from $\mathcal{K}$ into $\mathcal{H}$, so that $T^* : \mathcal{H} \to \mathcal{K}$ maps $\mathcal{H}$ onto $\mathcal{K}$. This way the range projection $TT^* = E_{cR} - S_c S_c^*$ is the projection from $\mathcal{H}$ onto $\mathcal{L}$. The second identity follows from the fact that S_c^* is zero on $\mathcal{L}$, the range of T. $\square$

Lemma 3.2.5. *For any* $x \in R$ *and ideal* I *of* R, U^{cx} *and* E_{cI} *commute with* TT^*.

Proof. We use that fact that $TT^* = E_{cR} - S_c S_c^*$ together with the relations to compute

$$\begin{aligned} TT^* U^{cx} &= E_{cR} U^{cx} - S_c S_c^* U^{cx} = U^{cx} E_{cR} - S_c (U^{-cx} S_c)^* \\ &= U^{cx} E_{cR} - S_c (S_c U^{-x})^* = U^{cx} E_{cR} - S_c U^x S_c^* \\ &= U^{cx}(E_{cR} - S_c S_c^*) = U^{cx} TT^*, \end{aligned}$$

and

$$\begin{aligned} TT^* E_{cI} &= E_{cR} E_{cI} - S_c S_c^* E_{cI} = E_{cI} E_{cR} - S_c E_I S_c^* \\ &= E_{cI}(E_{cR} - S_c S_c^*) = E_{cI} TT^*. \end{aligned}$$

$\square$

Lemma 3.2.6. *If* $\xi \in \mathcal{H}$ *satisfies* $E_{cR}\xi = \xi$, *then* $E_{cR} S_a \xi = S_a \xi$ *for all* $a \in R$. *In particular* $E_{cR} S_a T = S_a T$ *for all* $a \in R$.

Proof. It easy to check that for all $a \in R^\times$, $I = \frac{c}{a} R \cap R$ is an ideal of R, and that $aI = cR \cap aR$. This fact, together with Cc, gives us the identity

$$S_a E_{\frac{c}{a} R \cap R} = E_{cR \cap aR} S_a.$$

Using the identity and the relations we see that

$$\begin{aligned} E_{cR} S_a \xi &= E_{cR} S_a E_R \xi = E_{cR} E_{aR} S_a \xi \\ &= E_{cR \cap aR} S_a \xi = S_a E_{\frac{c}{a} R \cap R} \xi = S_a \xi. \end{aligned}$$

The last equality holds since $cR \subseteq \frac{c}{a} R \cap R$ implies $E_{cR} \leq E_{\frac{c}{a} R \cap R}$, and so $E_{\frac{c}{a} R \cap R} \xi = \xi$. $\square$

The next four lemmas prove that our dilation does indeed satisfy Ca-Cd.

Lemma 3.2.7. *The dilation in Theorem 3.2.3 satisfies Ca. That is the* $\widetilde{U}^x$ *are unitaries that satisfy* $\widetilde{U}^x \widetilde{U}^y = \widetilde{U}^{x+y}$, *the* $\widetilde{S}_a$ *are isometries that satisfy* $\widetilde{S}_a \widetilde{S}_b = \widetilde{S}_{ab}$. *Moreover for all* $x \in R$ *and* $a \in R^\times$ *we have* $\widetilde{S}_a \widetilde{U}^x = \widetilde{U}^{ax} \widetilde{S}_a$.

Proof. First, for all $x, y \in R$ we have

$$\begin{aligned}
\widetilde{U}^x\widetilde{U}^y &= \begin{pmatrix} U^x & 0 \\ 0 & T^*U^{cx}T \end{pmatrix}\begin{pmatrix} U^y & 0 \\ 0 & T^*U^{cy}T \end{pmatrix} = \begin{pmatrix} U^xU^y & 0 \\ 0 & T^*U^{cx}TT^*U^{cy}T \end{pmatrix} \\
&= \begin{pmatrix} U^{x+y} & 0 \\ 0 & T^*TT^*U^{cx+cy}T \end{pmatrix} = \widetilde{U}^{x+y},
\end{aligned}$$

where the second to last equality follows from Lemma 3.2.5. Since T is an isometry, $\widetilde{U}^0 = 1$, and we can conclude that the $\widetilde{U}^x$ are unitaries that satisfy $\widetilde{U}^x\widetilde{U}^y = \widetilde{U}^{x+y}$.

The $\widetilde{S}_a$ are isometries because

$$\begin{aligned}
\widetilde{S}_a^*\widetilde{S}_a &= \begin{pmatrix} S_a^* & 0 \\ T^*S_a^*S_c & T^*S_a^*T \end{pmatrix}\begin{pmatrix} S_a & S_c^*S_aT \\ 0 & T^*S_aT \end{pmatrix} \\
&= \begin{pmatrix} S_a^*S_a & S_a^*S_c^*S_aT \\ T^*S_a^*S_cS_a & T^*S_a^*S_cS_c^*S_aT + T^*S_a^*TT^*S_aT \end{pmatrix} \\
&= \begin{pmatrix} 1 & S_c^*S_a^*S_aT \\ T^*S_a^*S_aS_c & T^*S_a^*S_cS_c^*S_aT + T^*S_a^*(E_{cR} - S_cS_c^*)S_aT \end{pmatrix} \\
&= \begin{pmatrix} 1 & S_c^*T \\ T^*S_c & T^*S_a^*E_{cR}S_aT \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & T^*S_a^*S_aT \end{pmatrix} = 1,
\end{aligned}$$

where the second to last equality follows from Lemma 3.2.6 and the fact that $S_c^*T = 0$. Next for all $a, b \in R^\times$

$$\begin{aligned}
\widetilde{S}_a\widetilde{S}_b &= \begin{pmatrix} S_a & S_c^*S_aT \\ 0 & T^*S_aT \end{pmatrix}\begin{pmatrix} S_b & S_c^*S_bT \\ 0 & T^*S_bT \end{pmatrix} \\
&= \begin{pmatrix} S_aS_b & S_aS_c^*S_bT + S_c^*S_aTT^*S_bT \\ 0 & T^*S_aTT^*S_bT \end{pmatrix} \\
&= \begin{pmatrix} S_{ab} & S_c^*S_cS_aS_c^*S_bT + S_c^*S_a(E_{cR} - S_cS_c^*)S_bT \\ 0 & T^*S_a(E_{cR} - S_cS_c^*)S_bT \end{pmatrix} \\
&= \begin{pmatrix} S_{ab} & S_c^*S_aS_cS_c^*S_bT + S_c^*S_a(E_{cR} - S_cS_c^*)S_bT \\ 0 & T^*S_aE_{cR}S_bT - T^*S_aS_cS_c^*S_bT \end{pmatrix} \\
&= \begin{pmatrix} S_{ab} & S_c^*S_aE_{cR}S_bT \\ 0 & T^*S_aE_{cR}S_bT - T^*S_cS_aS_c^*S_bT \end{pmatrix} \\
&= \begin{pmatrix} S_{ab} & S_c^*S_aS_bT \\ 0 & T^*S_aS_bT \end{pmatrix} = \widetilde{S}_{ab},
\end{aligned}$$

where the second to last equality uses Lemma 3.2.6 and the fact that $T^*S_c = 0$. Thus the $\widetilde{S}_a$ are isometries that satisfy $\widetilde{S}_a\widetilde{S}_b = \widetilde{S}_{ab}$.

We now verify the last statement

$$\begin{aligned}
\widetilde{S}_a\widetilde{U}^x &= \begin{pmatrix} S_a & S_c^*S_aT \\ 0 & T^*S_aT \end{pmatrix}\begin{pmatrix} U^x & 0 \\ 0 & T^*U^{cx}T \end{pmatrix} \\
&= \begin{pmatrix} S_aU^x & S_c^*S_aTT^*U^{cx}T \\ 0 & T^*S_aTT^*U^{cx}T \end{pmatrix} \\
&= \begin{pmatrix} S_aU^x & S_c^*S_aU^{cx}TT^*T \\ 0 & T^*S_aU^{cx}TT^*T \end{pmatrix} \\
&= \begin{pmatrix} U^{ax}S_a & S_c^*U^{acx}S_aT \\ 0 & T^*TT^*U^{acx}S_aT \end{pmatrix} \\
&= \begin{pmatrix} U^{ax}S_a & U^{ax}S_c^*S_aT \\ 0 & T^*U^{acx}TT^*S_aT \end{pmatrix} \\
&= \begin{pmatrix} U^{ax} & 0 \\ 0 & T^*U^{acx}T \end{pmatrix}\begin{pmatrix} S_a & S_c^*S_aT \\ 0 & T^*S_aT \end{pmatrix} = \widetilde{U}^{ax}\widetilde{S}_a,
\end{aligned}$$

where the third and fifth equality use Lemma 3.2.5. ☐

Lemma 3.2.8. *The dilation in Theorem 3.2.3 satisfies Cb. That is the $\widetilde{E}_I$ are projections that satisfy $\widetilde{E}_I\widetilde{E}_J = \widetilde{E}_{I\cap J}$ and $E_R = 1$.*

Proof. Let I and J be ideals of R. Since $\widetilde{E}_I$ is self-adjoint, and $I \cap I = I$, the fact that $\widetilde{E}_I$ is a projection will follow from the multiplication relation, which we now verify:

$$\begin{aligned}
\widetilde{E}_I\widetilde{E}_J &= \begin{pmatrix} E_I & 0 \\ 0 & T^*E_{cI}T \end{pmatrix}\begin{pmatrix} E_J & 0 \\ 0 & T^*E_{cJ}T \end{pmatrix} \\
&= \begin{pmatrix} E_IE_J & 0 \\ 0 & T^*E_{cI}TT^*E_{cJ}T \end{pmatrix} \\
&= \begin{pmatrix} E_{I\cap J} & 0 \\ 0 & T^*E_{cI\cap cJ}TT^*T \end{pmatrix} \\
&= \begin{pmatrix} E_{I\cap J} & 0 \\ 0 & T^*E_{c(I\cap J)}T \end{pmatrix} = \widetilde{E}_{I\cap J},
\end{aligned}$$

where the third equality follows from Lemma 3.2.5. Because E_{cR} is the identity on the range of T, we have $\widetilde{E}_R = 1$, as required. ☐

Lemma 3.2.9. *The dilation in Theorem 3.2.3 satisfies Cc. That is for all $a \in R^\times$ and ideals I in R, we have $\widetilde{S}_a\widetilde{E}_I = \widetilde{E}_{aI}\widetilde{S}_a$.*

Proof. We compute

$$\begin{aligned}
\widetilde{S}_a\widetilde{E}_I &= \begin{pmatrix} S_a & S_c^*S_aT \\ 0 & T^*S_aT \end{pmatrix}\begin{pmatrix} E_I & 0 \\ 0 & T^*E_{cI}T \end{pmatrix} \\
&= \begin{pmatrix} S_aE_I & S_c^*S_aTT^*E_{cI}T \\ 0 & T^*S_aTT^*E_{cI}T \end{pmatrix} \\
&= \begin{pmatrix} S_aE_I & S_c^*S_aE_{cI}T \\ 0 & T^*S_aE_{cI}T \end{pmatrix} \\
&= \begin{pmatrix} E_{aI}S_a & S_c^*E_{acI}S_aT \\ 0 & T^*E_{acI}S_aT \end{pmatrix} \\
&= \begin{pmatrix} E_{aI}S_a & E_{aI}S_c^*S_aT \\ 0 & T^*E_{acI}TT^*S_aT \end{pmatrix} \\
&= \begin{pmatrix} E_{aI} & 0 \\ 0 & T^*E_{acI}T \end{pmatrix}\begin{pmatrix} S_a & S_c^*S_aT \\ 0 & T^*S_aT \end{pmatrix} = \widetilde{E}_{aI}\widetilde{S}_a,
\end{aligned}$$

where the third and fifth equality follow from Lemma 3.2.5. □

Lemma 3.2.10. *The dilation in Theorem 3.2.3 satisfies Cd. That is if $x \in I$ then $\widetilde{U}^x\widetilde{E}_I = \widetilde{E}_I\widetilde{U}^x$. If $x \notin I$ then $\widetilde{E}_I\widetilde{U}^x\widetilde{E}_I = 0$.*

Proof. If $x \in I$, then

$$\begin{aligned}
\widetilde{U}^x\widetilde{E}_I &= \begin{pmatrix} U^x & 0 \\ 0 & T^*U^{cx}T \end{pmatrix}\begin{pmatrix} E_I & 0 \\ 0 & T^*E_{cI}T \end{pmatrix} \\
&= \begin{pmatrix} U^xE_I & 0 \\ 0 & T^*U^{cx}TT^*E_{cI}T \end{pmatrix} \\
&= \begin{pmatrix} U^xE_I & 0 \\ 0 & T^*U^{cx}E_{cI}T \end{pmatrix} \\
&= \begin{pmatrix} E_IU^x & 0 \\ 0 & T^*E_{cI}U^{cx}T \end{pmatrix} \\
&= \begin{pmatrix} E_IU^x & 0 \\ 0 & T^*E_{cI}T^*TU^{cx}T \end{pmatrix} \\
&= \begin{pmatrix} E_I & 0 \\ 0 & T^*E_{cI}T \end{pmatrix}\begin{pmatrix} U^x & 0 \\ 0 & T^*U^{cx}T \end{pmatrix} = \widetilde{E}_I\widetilde{U}^x,
\end{aligned}$$

where the third and fifth equality follow from Lemma 3.2.5, and the fourth equality follows because $x \in I$ if and only if $cx \in cI$. The proof that $\widetilde{E}_I \widetilde{U}^x \widetilde{E}_I = 0$ when $x \notin I$ is similar. $\square$

We have now shown that $\widetilde{U}^x$, $\widetilde{S}_a$, and $\widetilde{E}_I$ satisfy the relations, which completes the proof of Theorem 3.2.2.

Although the dilation of Theorem 3.2.2 satisfies $(\widetilde{E}_{cR} - \widetilde{S}_c \widetilde{S}_c^*)|_{\mathcal{H}} = 0$, we do not have $\widetilde{E}_{cR} - \widetilde{S}_c \widetilde{S}_c^* = 0$. Indeed

$$\begin{aligned}
\widetilde{E}_{cR} - \widetilde{S}_c \widetilde{S}_c^* &= \begin{pmatrix} E_{cR} & 0 \\ 0 & T^* E_{c^2R} T \end{pmatrix} - \begin{pmatrix} S_c & S_c^* S_c T \\ 0 & T^* S_c T \end{pmatrix} \begin{pmatrix} S_c^* & 0 \\ T^* S_c^* S_c & T^* S_c^* T \end{pmatrix} \\
&= \begin{pmatrix} E_{cR} & 0 \\ 0 & T^* E_{c^2R} T \end{pmatrix} - \begin{pmatrix} S_c & T \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S_c^* & 0 \\ T^* & 0 \end{pmatrix} \\
&= \begin{pmatrix} E_{cR} & 0 \\ 0 & T^* E_{c^2R} T \end{pmatrix} - \begin{pmatrix} S_c S_c^* + T T^* & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} E_{cR} & 0 \\ 0 & T^* E_{c^2R} T \end{pmatrix} - \begin{pmatrix} S_c S_c^* + E_{cR} - S_c S_c^* & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & T^* E_{c^2R} T \end{pmatrix} = T^* E_{c^2R} T.
\end{aligned}$$

In the next section we will repeat this dilation until we obtain a representation $\widehat{U}^x$, $\widehat{S}_a$, and $\widehat{E}_I$ on some $\widehat{\mathcal{H}}$ that satisfies $\widehat{E}_{cR} - \widehat{S}_c \widehat{S}_c^* = 0$. To get a representation of $\mathfrak{T}[R]$ we the repeat the process for every $d \in R^\times$ that does not satisfy $\widehat{E}_{dR} - \widehat{S}_d \widehat{S}_d^* = 0$. Since $R^\times$ is countable, there can be at most countably many such d. The following lemma tells us that the dilation of Theorem 3.2.2 does not produce anymore such d.

Lemma 3.2.11. *If $E_{aR} - S_a S_a^* = 0$, then $\widetilde{E}_{aR} - \widetilde{S}_a \widetilde{S}_a^* = 0$.*

Proof. We first compute the following identity

$$\begin{aligned}
S_a T T^* S_a^* S_c &= S_a (E_{cR} - S_c S_c^*) S_a^* S_c = (E_{acR} S_a S_a^* - S_c S_a S_a^* S_c^*) S_c \\
&= (E_{acR} E_{aR} - S_c E_{aR} S_c^*) S_c = (E_{acR} - S_c E_{aR} S_c^*) S_c \\
&= S_c E_{aR} - S_c E_{aR} = 0.
\end{aligned}$$

Note that this calculation also shows that $S_a T T^* S_a^* = E_{acR} - S_c E_{aR} S_c^*$. Using these identities we compute

$$\begin{aligned}
\widetilde{S}_a\widetilde{S}_a^* &= \begin{pmatrix} S_a & S_c^*S_aT \\ 0 & T^*S_aT \end{pmatrix}\begin{pmatrix} S_a^* & 0 \\ T^*S_a^*S_c & T^*S_a^*T \end{pmatrix} \\
&= \begin{pmatrix} S_aS_a^* + S_c^*S_aTT^*S_a^*S_c & S_c^*S_aTT^*S_a^*T \\ T^*S_aTT^*S_a^*S_c & T^*S_aTT^*S_a^*T \end{pmatrix} \\
&= \begin{pmatrix} S_aS_a^* & 0 \\ 0 & T^*(E_{acR} - S_cE_{aR}S_c^*)T \end{pmatrix} \\
&= \begin{pmatrix} E_{aR} & 0 \\ 0 & T^*E_{acR}T \end{pmatrix} = \widetilde{E}_{aR}.
\end{aligned}$$

□

3.3 Maximal Representations

In Section 3.2 we saw that a representation of $\mathcal{A}_R \times_\alpha^{\text{is}} R^\times$ is maximal only if it is also a representation of $\mathfrak{T}[R]$, and how to dilate a representation of $\mathcal{A}_R \times_\alpha^{\text{is}} R^\times$ that is not a representation of $\mathfrak{T}[R]$. In this section we will show explicitly how to dilate such a representation to a representation of $\mathfrak{T}[R]$.

Let U^x, $x \in R$, S_a, $a \in R^\times$, E_I, I a non-zero ideal in R, be elements in some $\mathcal{B}(\mathcal{H})$ that satisfy Ca-Cd. Assume that this is not a representation of $\mathfrak{T}[R]$. Then there exists some $c \in R^\times$ such that $\mathcal{L}_0 = (E_{cR} - S_cS_c^*)\mathcal{H} \neq \{0\}$. Let $\mathcal{K}_1 \cong \mathcal{L}_0$, let $T_1 : \mathcal{K}_1 \to \mathcal{L}_0$ be a surjective isometry, and $V_1 = T_1$. Then by Theorem 3.2.3, the bounded linear operators $\widetilde{U}^x_{(1)}$, $\widetilde{S}^{(1)}_a$, and $\widetilde{E}^{(1)}_I$, acting on $\mathcal{H}_1 = \mathcal{H}_0 \oplus \mathcal{K}_1$ given by

$$\begin{aligned}
\widetilde{U}^x_{(1)} &= \begin{pmatrix} U^x & 0 \\ 0 & V_1^*U^{cx}V_1 \end{pmatrix}, \\
\widetilde{S}^{(1)}_a &= \begin{pmatrix} S_a & S_c^*S_aV_1 \\ 0 & V_1^*S_aV_1 \end{pmatrix}, \text{ and} \\
\widetilde{E}^{(1)}_I &= \begin{pmatrix} E_I & 0 \\ 0 & V_1^*E_{cI}V_1 \end{pmatrix}
\end{aligned}$$

dilate the representation.

Let $\mathcal{L}_n = (\widetilde{E}^{(n)}_{cR} - \widetilde{S}^{(n)}_c\widetilde{S}^{(n)}_c)\mathcal{H}_n$, $\mathcal{K}_{n+1} \cong \mathcal{L}_n$, $T_{n+1} : \mathcal{K}_{n+1} \to \mathcal{L}_n$ be a surjective isometry, and $V_{n+1} = V_n T_{n+1} = T_1 T_2 \cdots T_{n+1}$. Let

$$\widetilde{U}^x_{(n+1)} = \begin{pmatrix} U^x & & & & \\ & V_1^* U^{cx} V_1 & & & \\ & & V_2^* U^{c^2 x} V_2 & & \\ & & & \ddots & \\ & & & & V_{n+1}^* U^{c^{n+1} x} V_{n+1} \end{pmatrix}, \text{ and}$$

$$\widetilde{E}^{(n+1)}_I = \begin{pmatrix} E_I & & & & \\ & V_1^* E_{cI} V_1 & & & \\ & & V_2^* E_{c^2 I} V_2 & & \\ & & & \ddots & \\ & & & & V_{n+1}^* E_{c^{n+1} I} V_{n+1} \end{pmatrix},$$

be diagonal matrices, and

$$\widetilde{S}^{(n+1)}_a = \begin{pmatrix} S_a & S_c^* S_a V_1 & (S_c^*)^2 S_a V_2 & \cdots & (S_c^*)^{n+1} S_a V_{n+1} \\ & V_1^* S_a V_1 & V_1^* S_c^* S_a V_2 & \cdots & V_1^* (S_c^*)^n S_a V_{n+1} \\ & & V_2^* S_a V_2 & \cdots & V_2^* (S_c^*)^{n-1} S_a V_{n+1} \\ & & & \ddots & \vdots \\ & & & & V_{n+1}^* S_a V_{n+1} \end{pmatrix}$$

be upper triangular matrices acting on the Hilbert space $\mathcal{H}_{n+1} = \mathcal{H}_0 \oplus \mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_{n+1}$. To sumarize

(i) $\mathcal{L}_n \subseteq \mathcal{K}_n \subseteq \mathcal{H}_n = \mathcal{H}_{n-1} \oplus \mathcal{K}_n$,

(ii) $\widetilde{U}^x_{(n)}, \widetilde{S}^{(n)}_a, \widetilde{E}^{(n)}_I \in \mathcal{B}(\mathcal{H}_n)$,

(iii) $T_n : \mathcal{K}_n \to \mathcal{L}_{n-1}$, and

(iv) $V_n = T_1 T_2 \cdots T_n : \mathcal{K}_n \to \mathcal{L}_0$.

Remark 3.3.1. We will see in the proof of Theorem 3.3.3 that $\mathcal{L}_n = (V_n^* E_{c^{n+1} R} V_n)\mathcal{K}_n$.

Lemma 3.3.2. *Using the above notation,* $\widetilde{U}^x_{(n)}$, $\widetilde{S}^{(n)}_a$, *and* $\widetilde{E}^{(n)}_I$ *is the representation obtained after* n *applications of the dilation of Theorem 3.2.2.*

Proof. We proceed by induction. The case $n = 1$ is done, so suppose that $\widetilde{U}^x_{(n)}$, $\widetilde{S}^{(n)}_a$, $\widetilde{E}^{(n)}_I$ is the representation on the Hilbert space $\mathcal{H}_n$ obtained by n applications of the dilation of Theorem 3.2.2. We need to show that

$$\widetilde{U}^x_{(n+1)} = \begin{pmatrix} \widetilde{U}^x_{(n)} & 0 \\ 0 & T^*_{n+1}\widetilde{U}^{cx}_{(n)}T_{n+1} \end{pmatrix}, \tag{3.1}$$

$$\widetilde{E}^{(n+1)}_I = \begin{pmatrix} \widetilde{E}^{(n)}_I & 0 \\ 0 & T^*_{n+1}\widetilde{E}^{(n)}_{cI}T_{n+1} \end{pmatrix}, \tag{3.2}$$

$$\widetilde{S}^{(n+1)}_a = \begin{pmatrix} \widetilde{S}^{(n)}_a & \widetilde{S}^{(n)*}_c\widetilde{S}^{(n)}_a T_{n+1} \\ 0 & T^*_{n+1}\widetilde{S}^{(n)}_a T_{n+1} \end{pmatrix}. \tag{3.3}$$

The first equality follows from the fact that

$$T^*_{n+1}\widetilde{U}^{cx}_{(n)}T_{n+1} =$$

$$= \begin{pmatrix} 0 & \cdots & 0 & T^*_{n+1} \end{pmatrix} \begin{pmatrix} U^{cx} & & & \\ & V_1^*U^{c^2x}V_1 & & \\ & & \ddots & \\ & & & V_n^*U^{c^{n+1}x}V_n \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ T_{n+1} \end{pmatrix}$$

$$= \begin{pmatrix} V^*_{n+1}U^{c^{n+1}x}V_{n+1} \end{pmatrix}.$$

Similarly, we have $T^*_{n+1}\widetilde{E}^{(n)}_{cI}T_{n+1} = V^*_{n+1}E_{c^nI}V_{n+1}$, from which (3.2) follows. Finally, (3.3) holds because $T^*_{n+1}\widetilde{S}^{(n)}_a T_{n+1} = V^*_{n+1}S_aV_{n+1}$ and

$$\widetilde{S}^{(n)*}_c\widetilde{S}^{(n)}_a T_{n+1} =$$

$$= \begin{pmatrix} S_c^* & 0 & \cdots & 0 \\ T_1^* & 0 & \cdots & 0 \\ & \ddots & & \vdots \\ & & T_n^* & 0 \end{pmatrix} \begin{pmatrix} S_a & S_c^*S_aV_1 & \cdots & (S_c^*)^{n-1}S_aV_n \\ & V_1^*S_aV_1 & \cdots & V_1^*(S_c^*)^{n-2}S_aV_n \\ & & \ddots & \vdots \\ & & & V_n^*S_aV_n \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ T_{n+1} \end{pmatrix}$$

$$= \begin{pmatrix} (S_c^*)^nS_aV_{n+1} \\ V_1^*(S_c^*)^{n-1}S_aV_{n+1} \\ \vdots \\ V^*_{n+1}S_aV_{n+1} \end{pmatrix},$$

where in the first equality we use that fact that $S_c^* T_1 = S_c^* V_i = 0$ for $i \geq 1$. $\square$

Theorem 3.3.3. *Let U^x, $x \in R$, S_a, $a \in R^\times$, E_I, I a non-zero ideal in R, be elements in some $\mathcal{B}(\mathcal{H})$ that satisfy Ca-Cd. Suppose that there exists some element $c \in R^\times$ such thatthat $\mathcal{L} = (E_{cR} - S_c S_c^*)\mathcal{H} \neq \{0\}$. Using the above notation, let $\widehat{\mathcal{H}}$ be the inductive limit of the directed system $\{\mathcal{H}_n\}_{n\geq 1}$, and let $\widehat{U}^x$, $\widehat{S}_a$, and $\widehat{E}_I$ be the inductive limits of the directed systems $\{\widetilde{U}^x_{(n)}\}_{n\geq 1}$, $\{\widetilde{S}_a^{(n)}\}_{n\geq 1}$, and $\{\widetilde{E}_I^{(n)}\}_{n\geq 1}$ respectively. Then $\widehat{U}^x$, $\widehat{S}_a$, and $\widehat{E}_I$ dilate U^x, S_a, and E_I. Moreover $\widehat{S}_c \widehat{S}_c^* = \widehat{E}_{cR}$.*

Proof. It is clear that $\widehat{U}^x$, $\widehat{S}_a$, and $\widehat{E}_I$ dilate U^x, S_a, and E_I. To verify the last statement we first need to compute

$$T_{n+1}T_{n+1}^* = \widetilde{E}_{cR}^{(n)} - \widetilde{S}_c^{(n)}\widetilde{S}_c^{(n)} = T_n^* \widetilde{E}_{c^2R}^{(n-1)} T_n = V_n^* E_{c^{n+1}R} V_n,$$

where the first equality is due to Lemma 3.2.4, the second equality comes from the calculation after Lemma 3.2.10, and the third equality follows from (3.2) in the proof of the previous theorem. The last statement now becomes

$$\begin{aligned}
\widehat{S}_c\widehat{S}_c^* &= \begin{pmatrix} S_c & T_1 & & & \\ & & T_2 & & \\ & & & T_3 & \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} S_c^* & & & \\ T_1^* & & & \\ & T_2^* & & \\ & & & \ddots \end{pmatrix} \\
&= \begin{pmatrix} S_cS_c^* + T_1T_1^* & & & \\ & T_2T_2^* & & \\ & & T_3T_3^* & \\ & & & \ddots \end{pmatrix} \\
&= \begin{pmatrix} E_{cR} & & & \\ & V_1^*E_{c^2R}V_1 & & \\ & & V_2^*E_{c^3R}V_2 & \\ & & & \ddots \end{pmatrix} \\
&= \widehat{E}_{cR}.
\end{aligned}$$

$\square$

Corollary 3.3.4. *Any representation U^x, S_a, E_I, of $\mathcal{A}_R \times_\alpha^{is} R^\times$ can be dilated to a representation of $\mathfrak{T}[R]$.*

Proof. Since $R^\times$ is countable, there can be at most countably many $c \in R^\times$ for which $E_{cR} - S_cS_c^* \neq 0$. Let $C_0 = \{c_1, c_2, \dots\}$ be the set of all such c indexed by $\mathbb{N}$. By Theorem

3.3.3, we can find a dilation $\widehat{U}^x_{(1)}$, $\widehat{S}^{(1)}_a$, $\widehat{E}^{(1)}_I$ such that $\widehat{E}^{(1)}_{c_1R} - \widehat{S}^{(1)}_{c_1}\widehat{S}^{(1)}_{c_1} = 0$. It follows from Lemma 3.2.11 that the set C_1 of elements $c \in R^\times$ such that $\widehat{E}^{(1)}_{cR} - \widehat{S}^{(1)}_c\widehat{S}^{(1)}_c \neq 0$ is strictly contained in C_0. Let c_{i_2} be the next element in C_0 that is also in C_1. Then by Theorem 3.3.3 we can find a dilation $\widehat{U}^x_{(2)}$, $\widehat{S}^{(2)}_a$, $\widehat{E}^{(2)}_I$ of $\widehat{U}^x_{(1)}$, $\widehat{S}^{(1)}_a$, $\widehat{E}^{(1)}_I$ that satisfies $\widehat{E}^{(2)}_{c_{i_2}R} - \widehat{S}^{(2)}_{c_{i_2}}\widehat{S}^{(2)}_{c_{i_2}} = 0$. Continuing in this way we get a directed system of representations $\{\widehat{U}^x_{(n)}\}_{n\geq 1}$, $\{\widehat{S}^{(n)}_a\}_{n\geq 1}$, $\{\widehat{E}^{(n)}_I\}_{n\geq 1}$ satisfying $\bigcap_{n\geq 1} C_n = \emptyset$. The inductive limit $\widehat{U}^x$, $\widehat{S}_a$, $\widehat{E}_I$ of the directed system dilates U^x, S_a, E_I and is a representation of $\mathfrak{T}[R]$. ☐

Corollary 3.3.5. *A representation U^x, S_a, E_I, of $\mathcal{A}_R \times^{is}_\alpha R^\times$ is maximal if it is also a representation of $\mathfrak{T}[R]$.*

Proof. Suppose U^x, S_a, and E_I satisfy Ta-Td. By [9], we can dilate U^x, S_a, and E_I to a maximal representation $\overline{U}^x$, $\overline{S}_a$, $\overline{E}_I$, of $\mathcal{A}_R \times^{\text{is}}_\alpha R^\times$. This maximal representation must satisfy Ta-Td, because otherwise by Theorem 3.2.2 we could find a non-trivial dilation. Since any dilation of a unitary or a projection must be trivial, E_{aR} must be of the form

$$\overline{E}_{aR} = \begin{pmatrix} E_{aR} & 0 \\ 0 & F \end{pmatrix} = \begin{pmatrix} S_aS_a^* & 0 \\ 0 & F \end{pmatrix}.$$

Since $\overline{S}_a$ and S_a are isometries, $\overline{S}_a$ must be of the form

$$\overline{S}_a = \begin{pmatrix} S_a & A \\ 0 & B \end{pmatrix}.$$

Using the fact that the dilation satisfies Tc, we must have

$$\overline{S}_a\overline{S}_a^* = \begin{pmatrix} S_aS_a^* + AA^* & AB^* \\ BA^* & BB^* \end{pmatrix} = \begin{pmatrix} S_aS_a^* & 0 \\ 0 & F \end{pmatrix} = \overline{E}_{aR},$$

which implies $A = 0$. Thus $\overline{S}_a$ is a trivial dilation of S_a, and U^x, S_a, E_I is a maximal representation of $\mathcal{A}_R \times^{\text{is}}_\alpha R^\times$. ☐

Having characterized the maximal representations of the isometric semicrossed product, we are now ready for our main result.

Corollary 3.3.6. *The C^*-envelope of $\mathcal{A}_R \times^{is}_\alpha R^\times$ is isomorphic to $\mathfrak{T}[R]$.*

Proof. Let $\varphi : \mathfrak{T}[R] \to \mathcal{B}(\mathcal{H})$ be a a faithful representation of $\mathfrak{T}[R]$. Then $U^x = \varphi(u^x)$, $x \in R$, $S_a = \varphi(s_a)$, $a \in R^\times$, and $E_I = \varphi(e_I)$, I an ideal of R satisfy Ta-Td, and therefore also satisfy Ca-Cd. Thus we have a maximal representation of $\mathcal{A}_R \times^{\text{is}}_\alpha R^\times$. Since $\mathfrak{T}[R]$ is isomorphic to the C*-algebra generated by the representation, we have $C^*_{\text{env}}(\mathcal{A}_R \times^{\text{is}}_\alpha R^\times) \cong \mathfrak{T}[R]$. ☐

3.4 Contractive Representations

For some dynamical systems, the contractive semicrossed product is isomorphic to the isometric semicrossed product. This occurs when all the contractive covariant representations of the system dilate to isometric covariant representations. As is well known, in general it is not possible to dilate three or more commuting contractions to commuting isometries [17, Example 5.7]. Since it is necessary for three commuting contraction to satisfy the generalized von Neumann inequality if they are to dilate to three commuting isometries, to show that three commuting contractions do not dilate it is sufficient to show that the generalized von Neumann inequality fails for the contractions. Using this fact we can construct a contractive representation of $\mathcal{A}_R \times_\alpha R^\times$ that does not dilate to an isometric representation, thus showing that $\mathcal{A}_R \times_\alpha R^\times$ is not isomorphic to $\mathcal{A}_R \times_\alpha^{\text{is}} R^\times$.

Example 3.4.1. Let $A_2, A_3, A_5 \in \mathcal{B}(\mathcal{K})$ be three commuting contractions such that there exists a polynomial $q \in \mathbb{C}[z_1, z_2, z_3]$ such that

$$\|q\|_\infty < \|q(A_2, A_3, A_5)\|.$$

Let $A_{-1} = -I_\mathcal{K}$ and $A_p = I_\mathcal{K}$ for all primes $p > 5$. Then the collection $\{A_p : p \text{ prime}\} \cup \{A_{-1}\}$ is a family of commuting contractions on $\mathcal{K}$. Define $A_a = A_{-1}^i A_{p_1}^{f_1} \cdots A_{p_k}^{f_k}$ for each $a = (-1)^i p_1^{f_1} \cdots p_k^{f_k} \in \mathbb{Z}^\times$.

Let $\mathcal{H} = \bigoplus_{n\in\mathbb{Z}} \mathcal{K}$, and let P_n be the orthogonal projection onto the n^{th} coordinate of $\mathcal{H}$. For $x \in \mathbb{Z}$, $a \in \mathbb{Z}^\times$, and I an ideal of $\mathbb{Z}$, define

$$\begin{aligned} U^x(\eta_y)_{y\in\mathbb{Z}} &= (\eta_{x+y})_{y\in\mathbb{Z}}, \\ S_a(\eta_y)_{y\in Z} &= (\eta_{ay})_{y\in\mathbb{Z}}, \text{ and} \\ E_I &= \sum_{n\in I} P_n, \end{aligned}$$

where $(\eta_y)_{y\in\mathbb{Z}} \in \mathcal{H}$. We claim that U^x, S_a, and E_I satisfy Ca-Cd.

First it is clear that U^x is a unitary representation of $\mathbb{Z}$, that S_a is an isometric representation of $\mathbb{Z}^\times$, and that Cb holds. An easy calculation shows that

$$S_a U^x(\eta_y)_{y\in\mathbb{Z}} = (\eta_{ax+ay})_{y\in\mathbb{Z}} = U^{ax} S_a(\eta_y)_{y\in\mathbb{Z}},$$

which proves Ca holds. Next observe that for all $n, x \in \mathbb{Z}$ and $a \in \mathbb{Z}^\times$, we have $U^x P_n = P_{x+n} U^x$ and $S_a P_n = P_{an} S_a$. It follows that

$$S_a E_I = S_a\big(\sum_{n\in I} P_n\big) = \big(\sum_{n\in I} P_{an}\big) S_a = \big(\sum_{n\in aI} P_n\big) S_a = E_{aI} S_a,$$

$$U^x E_I = \big(\sum_{n\in I} P_{x+n}\big)U^x = \big(\sum_{n\in x+I} P_n\big)U^x = \big(\sum_{n\in I} P_n\big)U^x = E_{aI}U^x,$$

when $x \in I$, because $I = x + I$, and that

$$E_I U^x E_I = \big(\sum_{n\in I} P_n\big)\big(\sum_{m\in x+I} P_m\big)U^x = 0$$

when $x \notin I$, because the P_n's are mutually orthogonal and $I \cap (x + I) = \emptyset$. Thus Cc and Cd hold, which proves the claim.

Let $B_a = \bigoplus_{n\in\mathbb{Z}} A_a$. Then B_a is a contractive representation of $\mathbb{Z}^\times$ that commutes with all the U^x, S_a, and E_I. Thus we may define a contractive covariant representation (π, T) as follows:

$$\pi(u^x) = U^x, \quad \pi(e_I) = E_I, \quad T_a = S_a B_a.$$

We cannot dilate this to an isometric covariant representation because the three variable von Neumann inequality fails for T_2, T_3, and T_5:

$$\|q(T_1, T_2, T_3)\| \geq \|q(T_2, T_3, T_5)|_{P_0\mathcal{H}}\| = \|q(A_2, A_3, A_5)\| > \|q\|_\infty,$$

since $T_a|_{P_0\mathcal{H}} = A_a$ for all $a \in R^\times$.

Chapter 4

The Jacobson Radical of Certain Semicrossed Products

4.1 The Unital Case

Our main results show that under certain assumptions on a C*-dynamical system $(\mathcal{A}, \alpha, P)$, the radical of the semicrossed product $\mathcal{A}\times_\alpha P$ is generated by the monomials $a\otimes e_s$ satisfying $(a \otimes e_s)x = 0$ for all $x \in \mathcal{A} \times_\alpha P$.

When P is abelian there are conditional expectations from $\mathcal{A} \times_\alpha P$ onto the monomials that leave the radical invariant. This tells us that the radical is generated by its monomials and it makes sense to consider the set $\mathcal{J}_s = \{a \in \mathcal{A} : a \otimes e_s \in \operatorname{rad}(\mathcal{A} \times_\alpha P)\}$ consisting of the coefficients of the s-monomials in $\operatorname{rad}(\mathcal{A} \times_\alpha P)$. These sets turn out to be $\mathcal{A}$–$\alpha_s(\mathcal{A})$ bimodules, and those bimodules are well behaved in simple C*-algebras. For example $\mathcal{J}_s \cap \alpha_s(\mathcal{A})$ is an ideal of $\alpha_s(\mathcal{A})$, and when $\mathcal{A}$ is simple the intersection must be either $\{0\}$ or all of $\alpha_s(\mathcal{A})$. The case when P is a free semigroup is more complicated because we do not have such conditional expectations. To get around this we will need to define the s-Fourier coefficient of an element in $\mathcal{A} \times_\alpha P$.

First the case when P is contained in an abelian group G. Given a covariant pair (π, T) of $(\mathcal{A}, \alpha, P)$ and a member $\widehat{g}$ of the dual $\widehat{G}$ we get another covariant pair $(\pi, \widehat{g}T)$ by setting

$$\widehat{g}T_s = \langle \widehat{g}, s\rangle T_s.$$

By applying the universal property of the semicrossed product $\mathcal{A} \times_\alpha P$, we can construct a continuous action $\gamma : \widehat{G} \to \operatorname{Aut}(\mathcal{A} \times_\alpha P)$ of $\widehat{G}$ on $\mathcal{A} \times_\alpha P$ by automorphisms defined on the generators to be $\gamma_{\widehat{g}}(a_t \otimes e_t) = \langle \widehat{g}, t\rangle a_t \otimes e_t$. This action yields a conditional expectation $F_s : \mathcal{A} \times_\alpha P \to \mathcal{A} \otimes e_s$ given by the formula

$$F_s(x) = \int_{\widehat{G}} \overline{\langle \widehat{g}, s\rangle} \gamma_{\widehat{g}}(x) d\mu = a_s \otimes e_s,$$

where μ is the Haar measure, $x \in \mathcal{A} \times_\alpha P$, and $a_s \in \mathcal{A}$ is called the *s-Fourier coefficient of x*. We note that on the monomials this formula becomes

$$F_s(a \otimes e_t) = \begin{cases} a \otimes e_t & \text{if } t = s, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\operatorname{rad}(\mathcal{A} \times_\alpha P)$ is an automorphism invariant ideal, $x \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$ implies $F_s(x) \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$.

Now let P be a free semigroup and fix $s \in P$. Using a similar argument as in the abelian case, we get a continuous action $\gamma : \mathbb{T} \to \operatorname{Aut}(\mathcal{A} \times_\alpha P)$ of the dual of $\mathbb{Z}$ on $\mathcal{A} \times_\alpha P$ by automorphisms defined on the generators by $\gamma_z(a_t \otimes e_t) = z^{\ell(t)} a_t \otimes e_t$. This action gives us a conditional expectation $F_{\ell(s)} : \mathcal{A} \times_\alpha P \to \mathcal{A} \times_\alpha P$, similar to the one above, defined by the formula

$$F_{\ell(s)}(x) = \int_{\mathbb{T}} \overline{z}^{\ell(s)} \gamma_z(x) dm(z),$$

where m is normalized Lebesgue measure. On the monomials this formula becomes

$$F_{\ell(s)}(a \otimes e_t) = \begin{cases} a \otimes e_t & \text{if } \ell(t) = \ell(s), \\ 0 & \text{otherwise.} \end{cases}$$

As above $F_{\ell(s)}(x) \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$ whenever $x \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$.

Let $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a faithful representation, and let $\widetilde{\mathcal{H}} = \mathcal{H} \otimes \ell^2(P)$, $\widetilde{\pi} : \mathcal{A} \to \mathcal{B}(\widetilde{\mathcal{H}})$, and $T : P \to \mathcal{B}(\widetilde{\mathcal{H}})$ to be the covariant pair defined in Example 2.2.1. Recall that T_t is the co-isometry defined by the formula

$$T_t(\xi \otimes \delta_r) = \begin{cases} \xi \otimes \delta_{r_1} & \text{if } r = r_1 t \text{ for some } r_1 \in P, \\ 0 & \text{otherwise,} \end{cases}$$

and that T_t^* is the isometry $T_t^*(\xi \otimes \delta_r) = \xi \otimes \delta_{rt}$. Observe that the isometries corresponding to words of the same length $\{T_t^* : \ell(t) = \ell(s)\}$ have orthogonal ranges. It follows that if $y = \sum b_t \otimes e_t \in \operatorname{span}\{a \otimes e_t : \ell(t) = \ell(s)\}$ then $(\widetilde{\pi} \times T)(y) T_s^* = \widetilde{\pi}(b_s)$. We define the *$s$-Fourier coefficient of x* to be the unique $a_s \in \mathcal{A}$ that satisfies $(\widetilde{\pi} \times T) \circ F_{\ell(s)}(x) T_s^* = \widetilde{\pi}(a_s)$.

Together these few paragraphs prove the following lemma.

Lemma 4.1.1. *Let* $(\mathcal{A}, \alpha, P)$ *be a C*-dynamical system. If* $x \in \mathcal{A} \times_\alpha P$ *and* $s \in P$*, then* $\|x\| \geq \|a_s\|$*, where* $a_s \in \mathcal{A}$ *is the* s*-Fourier coefficient of* x*.*

Definition 4.1.2. Let $(\mathcal{A}, \alpha, P)$ be a C*-dynamical system. For each $s \in P$ define $\mathcal{J}_s \subseteq \mathcal{A}$ to be the set of s-Fourier coefficients of the elements in $\operatorname{rad}(\mathcal{A} \times_\alpha P)$.

It turns out that the above sets are very well behaved. The following lemma shows $\mathcal{J}_s$ is an $\mathcal{A}$–$\alpha_s(\mathcal{A})$ bimodule. In particular each $\mathcal{J}_s$ is a left ideal in $\mathcal{A}$ and $\mathcal{J}_s \cap \alpha_s(\mathcal{A})$ is a two-sided ideal in $\alpha_s(\mathcal{A})$.

Lemma 4.1.3. *Let* $(\mathcal{A}, \alpha, P)$ *be a C*-dynamical system. For all* $s \in P$*, the set* $\mathcal{J}_s$ *is an* $\mathcal{A}$–$\alpha_s(\mathcal{A})$ *bimodule.*

Proof. Let $x \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$ with s-Fourier coefficient $a_s \in \mathcal{J}_s$. The case when P is abelian is easy because $F_s(x) = a_s \otimes e_s \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$. We simply use the fact that the radical is an ideal to get

$$(b \otimes e_e)(a \otimes e_s)(c \otimes e_e) = b a_s \alpha_s(c) \otimes e_s \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$$

for all $b, c \in \mathcal{A}$, whence $b a_s \alpha_s(c) \in \mathcal{J}_s$.

Now let P be a free semigroup. For all $a, b, c \in \mathcal{A}$ and $t \in P$ with $\ell(t) = \ell(s)$ we have

$$(b \otimes e_e)(a \otimes e_t)(c \otimes e_e) = b a \alpha_s(c) \otimes e_t \in \operatorname{span}\{a \otimes e_t : \ell(t) = \ell(s)\}.$$

It follows that $F_{\ell(s)}\big((b \otimes e_e) y (c \otimes e_e)\big) = (b \otimes e_e) y (c \otimes e_e)$ whenever $y \in \operatorname{span}\{a \otimes e_t : \ell(t) = \ell(s)\}$. Passing to limits gives $F_{\ell(s)}\big((b \otimes e_e) F_{\ell(s)}(x)(c \otimes e_e)\big) = (b \otimes e_e) F_{\ell(s)}(x)(c \otimes e_e)$, which is in $\operatorname{rad}(\mathcal{A} \times_\alpha P)$ because the radical is invariant under $F_{\ell(s)}$. Now the calculation

$$\begin{aligned}
(\widetilde{\pi} \times T) \circ F_{\ell(s)}\big((b \otimes e_e) F_{\ell(s)}(x)(c \otimes e_e)\big) T_s^* &= (\widetilde{\pi} \times T)\big((b \otimes e_e) F_{\ell(s)}(x)(c \otimes e_e)\big) T_s^* \\
&= \widetilde{\pi}(b)\big((\widetilde{\pi} \times T) \circ F_{\ell(s)}(x) T_s^*\big) \widetilde{\pi}\alpha_s(c) \\
&= \widetilde{\pi}(b) \widetilde{\pi}(a_s) \widetilde{\pi}\alpha_s(c) = \widetilde{\pi}(b a_s \alpha_s(c)),
\end{aligned}$$

which uses the covariance relation $(T_s \pi(c)^*)^* = (\pi\alpha_s(c)^* T_s)^*$ in the second equality, shows that the s-Fourier coefficient of the product $(b \otimes e_e) F_{\ell(s)}(x)(c \otimes e_e)$ is $b a_s \alpha_s(c) \in \mathcal{J}_s$. □

The following lemma says that if $a \in \mathcal{J}_s$, then $a \otimes e_s$ is quasi-nilpotent.

Lemma 4.1.4. *Let* $(\mathcal{A}, \alpha, P)$ *be a C*-dynamical system where* P *is either contained in an abelian group or a free semigroup. If* $a \in \mathcal{J}_s$*, then*

$$\lim_{n \to \infty} \|a \alpha_s(a) \cdots \alpha_{s^{(n-1)}}(a)\|^{1/n} = 0. \tag{4.1}$$

Proof. Let $a \in \mathcal{J}_s$. When P is abelian we may use the conditional expectation F_s to show that $a \otimes e_s \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$. The n^{th} power of $a \otimes e_s$ is $a\alpha_s(a) \cdots \alpha_{s^{n-1}}(a) \otimes e_{s^n}$. We apply Lemma 4.1.1 to get the limit in the statement as a lower bound on the spectral radius of $a \otimes e_s$, which must be zero.

When P is a free semigroup we use the conditional expectation $F_{\ell(s)}$ to find an element $x \in \operatorname{span}\{b \otimes e_t : \ell(t) = \ell(s)\}$, with s-Fourier coefficient a, in the radical. Observing that for all $b_1, b_2 \in \mathcal{A}$ and $t_1, t_2 \in P$,

$$(b_1 \otimes e_{t_1})(b_2 \otimes e_{t_2}) = b_1\alpha_{t_1}(b_2) \otimes e_{t_1t_2} \in \operatorname{span}\{b \otimes e_t : \ell(t) = \ell(t_1t_2)\},$$

we see that $y^n \in \operatorname{span}\{b \otimes e_t : \ell(t) = \ell(s^n)\}$ whenever $y \in \operatorname{span}\{b \otimes e_t : \ell(t) = \ell(s)\}$. Passing to limits yeilds $x^n \in \overline{\operatorname{span}}\{b \otimes e_t : \ell(t) = \ell(s^n)\}$. Proceeding by induction on n, assume that the s^n-Fourier coefficient of x^n is $a\alpha_s(a) \cdots \alpha_{s^{(n-1)}}(a)$. Then the calculation

$$\begin{aligned}
(\widetilde{\pi} \times T) \circ F_{\ell(s^{n+1})}(x^n x)T^*_{s^{n+1}} &= (\widetilde{\pi} \times T)(x^n)(\widetilde{\pi} \times T)(x)T^*_s T^*_{s^n} \\
&= (\widetilde{\pi} \times T)(x^n)\widetilde{\pi}(a)T^*_{s^n} \\
&= (\widetilde{\pi} \times T)(x^n)T^*_{s^n}\widetilde{\pi}\alpha_{s^n}(a) \\
&= \widetilde{\pi}(a\alpha_s(a) \cdots \alpha_{s^{(n-1)}}(a))\widetilde{\pi}\alpha_{s^n}(a)
\end{aligned}$$

shows that the s^{n+1}-Fourier coefficient of x^{n+1} is $a\alpha_s(a) \cdots \alpha_{s^n}(a)$. As above we apply Lemma 4.1.1 to complete the proof. ☐

Lemma 4.1.4 gives us a way to show when an $a \in \mathcal{A}$ is not in $\mathcal{J}_s$. The next lemma makes use of the fact that any monomial $a \otimes e_s$ that satisfies $(a \otimes e_s)x = 0$ for all $x \in \mathcal{A} \times_\alpha P$ is in $\operatorname{rad}(\mathcal{A} \times_\alpha P)$ to show in particular that $\mathcal{J}_s$ is non-zero whenever α_s is a non-unital $*$-endomorphism of a unital C*-algebra.

Lemma 4.1.5. *Let $(\mathcal{A}, \alpha, P)$ be a C*-dynamical system where $\mathcal{A}$ is a unital C*-algebra and P is either contained in an abelian group or a free semigroup. If $p_s = \alpha_s(1)$, then $a(1 - p_s) \otimes e_s \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$ for all $a \in \mathcal{A}$. In particular $\mathcal{A}(1 - p_s) \subseteq \mathcal{J}_s$.*

Proof. For all $a \in \mathcal{A}(1 - p_s)$ and finite sums $\sum_{t \in P} a_t \otimes e_t \in \mathcal{A} \times_\alpha P$ the product

$$a \otimes e_s\Big(\sum_{t \in P} a_t \otimes e_t\Big) = \sum_{t \in P} a\alpha_s(a_t) \otimes e_{st}$$

is zero because

$$a\alpha_s(a_t) = a(1 - p_s)p_s\alpha_s(a_t) = 0.$$

Passing to limits we see that $(a \otimes e_s)x = 0$ for all $x \in \mathcal{A} \times_\alpha P$. Since $a \otimes e_s$ satisfies the condition in the quasi-nilpotence characterization of the radical, that element must be in $\operatorname{rad}(\mathcal{A} \times_\alpha P)$. ☐

The obvious question raised by the above lemma is: does $\mathcal{J}_s = \mathcal{A}(1-p_s)$? Although we cannot give a general answer, in the unital case we can prove this equality for two large sets of examples. The first is when $\mathcal{A}$ is a purely infinite simple unital C*-algebra.

Theorem 4.1.6. *Let $(\mathcal{A}, \alpha, P)$ be a C*-dynamical system where $\mathcal{A}$ is a purely infinite simple unital C*-algebra and P is either a subsemigroup of an abelian group or a free semigroup. Then $\operatorname{rad}(\mathcal{A} \times_\alpha P)$ is generated by monomials of the form $a(1-p_s) \otimes e_s$, where $a \in \mathcal{A}$ and $p_s = \alpha_s(1)$.*

Proof. Fix $s \in P$. The projection p_s decomposes $\mathcal{A}$ as $\mathcal{A}p_s \oplus \mathcal{A}(1-p_s)$. Since we already know from Lemma 4.1.5 that $a(1-p_s) \otimes e_s \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$ for all $a \in \mathcal{A}$, it remains to show that $\mathcal{J}_s \cap \mathcal{A}p_s$ is zero. Suppose that $a \in \mathcal{J}_s \cap \mathcal{A}p_s$ is non-zero. Then since $\mathcal{A}$ is a purely infinite simple unital C*-algebra there exist $b, c \in \mathcal{A}$ such that $bac = 1$ [8, Theorem V.5.5]. But then $bap_s = ba \in \mathcal{J}_s$ is an element of $\mathcal{J}_s$ that does not satisfy (4.1). Indeed estimating

$$\begin{aligned}
\|ba\alpha_s(ba)\cdots\alpha_{s^n}(ba)\| &\geq \|ba\alpha_s(ba)\cdots\alpha_{s^{n-1}}(ba)\alpha_{s^n}(bac)\|\|c\|^{-1} \\
&= \|ba\alpha_s(ba)\cdots\alpha_{s^{n-1}}(bap_s)p_{s^n}\|\|c\|^{-1} \\
&= \|ba\alpha_s(ba)\cdots\alpha_{s^{n-1}}(ba)\|\|c\|^{-1},
\end{aligned}$$

we see by induction that

$$\lim_{n\to\infty} \|ba\alpha_s(ba)\cdots\alpha_{s^{n-1}}(ba)\|^{1/n} \geq \lim_{n\to\infty} \|c\|^{-1} > 0. \qquad \square$$

Example 4.1.7. Let $\mathcal{O}_n$ be the Cuntz algebra on $2 \leq n \leq \infty$ generators, that is the universal C*-algebra generated by isometries $\{s_i\}_{i=1}^n$ satisfying

$$\begin{aligned}
&\sum_{i=1}^{n} s_i s_i^* = 1 \text{ when } n < \infty, \text{ or} \\
&\sum_{i=1}^{r} s_i s_i^* \leq 1 \text{ for all } r \in \mathbb{N} \text{ when } n = \infty.
\end{aligned}$$

It is well known that $\mathcal{O}_n$ is a purely infinite simple unital C*-algebra. We associate an isometry $s_w \in \mathcal{O}_n$ to each word $w = i_1 i_2 \cdots i_k \in P$ where P is the free semigroup on the generating set $\{1, \ldots, n\}$ when n is finite, or $\mathbb{N}$ when n is infinite, with the convention that $s_e = 1$. Observing that $s_{w_1} s_{w_2} = s_{w_1 w_2}$, we get an action of P on $\mathcal{O}_n$ by setting $\alpha_w(a) = s_w a s_w^*$ for each $w \in P$. The C*-dynamical system $(\mathcal{O}_n, \alpha, P)$ satisfies the hypotheses of Theorem 4.1.6 and we conclude that $\operatorname{rad}(\mathcal{O}_n \times_\alpha P)$ is generated by monomials of the form $a(1 - s_w s_w^*) \otimes e_w$.

The following is an immediate corollary to Theorem 4.1.6.

Corollary 4.1.8. *Let* $(\mathcal{A}, \alpha, P)$ *be a C*-dynamical system where* $\mathcal{A}$ *is a purely infinite simple unital C*-algebra and* P *is either contained in an abelian group or a free semigroup. If each* $*$*-endomorphism* α_s *is unital then* $\mathcal{A} \times_\alpha P$ *is semi-simple.*

Example 4.1.9. Let $\mathcal{O}_n$ be the Cuntz algebra with $2 \leq n < \infty$ generators $\{s_i\}_{i=1}^n$. Define a unital $*$-endomorphism $\alpha : \mathcal{O}_n \to \mathcal{O}_n$ by

$$\alpha(a) = \sum_{i=1}^{n} s_i a s_i^*.$$

Setting $\alpha_n = \alpha^n$ we get a unital action of $\mathbb{Z}_+$ on $\mathcal{O}_n$. Since $(\mathcal{O}_n, \alpha, \mathbb{Z}_+)$ satisfies the hypotheses of the above corollary we conclude that $\mathcal{O}_n \times_\alpha \mathbb{Z}_+$ is semi-simple.

Our first theorem assumed a restriction on the C*-algebra. Our second theorem will instead impose a restriction on the action of P on $\mathcal{A}$.

Theorem 4.1.10. *Let* $(\mathcal{A}, \alpha, P)$ *be a C*-dynamical system where* $\mathcal{A}$ *is a simple unital C*-algebra and* P *is either contained in an abelian group or a free semigroup. Suppose that for all* $s \in P$ *there exists a faithful conditional expectation* $E_s : p_s \mathcal{A} p_s \to \alpha_s(\mathcal{A})$ *where* $p_s = \alpha_s(1)$*. Then* $\mathrm{rad}(\mathcal{A} \times_\alpha P)$ *is generated by monomials of the form* $a(1-p_s) \otimes e_s$*, where* $a \in \mathcal{A}$*.*

Proof. Fix $s \in P$. The projection p_s decomposes $\mathcal{A}$ as $\mathcal{A}p_s \oplus \mathcal{A}(1-p_s)$. By Lemma 4.1.5 we already know that $a(1-p_s) \otimes e_s \in \mathrm{rad}(\mathcal{A} \times_\alpha P)$ for all $a \in \mathcal{A}$. Since every $a \in \mathcal{J}_s \cap \mathcal{A}p_s$ satisfies $a^*a \in \mathcal{J}_s \cap p_s\mathcal{A}p_s$, it suffices to show that $\mathcal{J}_s \cap p_s\mathcal{A}p_s$ is zero.

Because the conditional expectation E_s is a completely positive unital map that fixes $\alpha_s(\mathcal{A})$, we may apply the well-known characterization of the multiplicative domain [17, Theorem 3.18] to see that E_s is an $\alpha_s(\mathcal{A})$-bimodule map. Using the bimodule property of Lemma 4.1.3 we observe that, since $\mathcal{J}_s \cap p_s\mathcal{A}p_s$ is an $\alpha_s(\mathcal{A})$-bimodule, it must be mapped to an $\alpha_s(\mathcal{A})$-bimodule in $\alpha_s(\mathcal{A})$. It follows that $E_s(\mathcal{J}_s \cap p_s\mathcal{A}p_s)$ is a two-sided ideal in $\alpha_s(\mathcal{A})$ which is non-zero because E_s is faithful. By simplicity $E_s(\mathcal{J}_s \cap p_s\mathcal{A}p_s) = \alpha_s(\mathcal{A})$, and we can find $a \in \mathcal{J}_s \cap p_s\mathcal{A}p_s$ such that $E_s(a) = p_s$. But then

$$\begin{aligned} \|a\alpha_s(a)\cdots\alpha_{s^n}(a)\| &\geq \|\alpha_s^{-1}E_s(a\alpha_s(a)\cdots\alpha_{s^n}(a))\| \\ &= \|\alpha_s^{-1}(E_s(a)\alpha_s(a)\cdots\alpha_{s^n}(a))\| \\ &= \|\alpha_s^{-1}(p_s\alpha_s(a)\cdots\alpha_{s^n}(a))\| \\ &= \|a\alpha_s(a)\cdots\alpha_{s^{n-1}}(a)\|, \end{aligned}$$

and we see by induction that $\lim_{n\to\infty} \|a\alpha_s(a)\cdots\alpha_{s^{n-1}}(a)\|^{1/n} \geq 1$. This contradicts (4.1) and we conclude that $\mathcal{J}_s \cap \mathcal{A}p_s$ is zero. □

The following corollary is immediate.

Corollary 4.1.11. *Let $(\mathcal{A}, \alpha, P)$ be a C^*-dynamical system where $\mathcal{A}$ is a simple unital C^*-algebra and P is either contained in an abelian group or a free semigroup. If every α_s is an automorphism, then $\mathcal{A} \times_\alpha P$ is semi-simple.*

Corollary 4.1.12. *Let $(\mathcal{A}, \alpha, P)$ be a C^*-dynamical system where $\mathcal{A}$ is a simple unital C^*-algebra and P is either a subsemigroup of an abelian group or a free semigroup. Suppose that for all $s \in P$ there exists a faithful conditional expectation $E_s : \mathcal{A} \to \alpha_s(\mathcal{A})$. Then each α_s is unital and $\mathcal{A} \times_\alpha P$ is semi-simple.*

Proof. Fix $s \in P$, let $E_s : \mathcal{A} \to \alpha_s(\mathcal{A})$ be a faithful conditional expectation, and let $p_s = \alpha_s(1)$. Since $p_s \leq 1$ and E_s is positive

$$p_s \leq E_s(1).$$

Conjugating the above inequality by $E_s(1)^{1/2}$ gives $E_s(1) \leq E_s(1)^2$. Applying the modified Schwarz inequality for 2-positive maps [17, pg 40],

$$E_s(a)^* E_s(a) \leq \|E_s(1)\| E_s(a^*a),$$

to $a = 1$ gives the reverse inequality $E_s(1)^2 \leq \|E_s(1)\|E_s(1) \leq E_s(1)$. This shows that $E_s(1)$ is a projection in $\alpha_s(\mathcal{A})$ that dominates p_s, which implies $E_s(1) = p_s$.

Using the well known characterization of the multiplicative domain for completely positive unital maps we see that E_s is an $\alpha_s(\mathcal{A})$-bimodule map. It follows that $E_s(1-p_s)p_s = 0$ which can only happen if $E_s(1-p_s) = 0$. Since E_s was assumed to be faithful we have $p_s = 1$. The result now follows from Theorem 4.1.10. □

Example 4.1.13 (The Shift on the CAR Algebra)**.** Let $\mathcal{A} = \bigotimes_{n\geq 1} M_2$ be the CAR algebra expressed as a tensor product. Extend the map

$$\alpha : a_1 \otimes a_2 \otimes \cdots \mapsto 1_{M_2} \otimes a_1 \otimes a_2 \otimes \cdots$$

defined on the elementary tensors to get a unital $*$-endomorphism $\alpha : \mathcal{A} \to \mathcal{A}$ which we call the shift. By setting $\alpha_n = \alpha^n$ we get a unital action of $\mathbb{Z}_+$ on $\mathcal{A}$. Identifying $\mathcal{A} \cong M_2 \otimes \mathcal{A}$ and $\alpha_1(\mathcal{A}) \cong \mathbb{C}1_{M_2} \otimes \mathcal{A}$ we can define a faithful conditional expectation $E_1 : \mathcal{A} \to \alpha_1(\mathcal{A})$ by

$$E_1(a \otimes b) = \operatorname{tr}(a)1_{M_2} \otimes b,$$

where $\mathrm{tr} : M_2 \to \mathbb{C}$ is the unique tracial state on M_2. One can easily check that for $n \geq 2$

$$E_n = \alpha_{n-1} \underbrace{(E_1\alpha^{-1}) \cdots (E_1\alpha^{-1})}_{(n-1)\text{-times}} E_1$$

is a faithful conditional expectation from $\mathcal{A}$ onto $\alpha_n(\mathcal{A})$. Thus $(\mathcal{A}, \alpha, \mathbb{Z}_+)$ satisfies the hypothesis of Corollary 4.1.12 and we conclude that $\mathcal{A} \times_\alpha \mathbb{Z}_+$ is semi-simple.

We say that a conditional expectation $E : \mathcal{A} \to \mathcal{B}$ is *finite index* if there exists a *quasi-basis*, i.e. a set $\{(u_i, u_i^*)\}_{i=1}^n \subseteq \mathcal{A} \times \mathcal{A}$ such that

$$a = \sum_{i=1}^n u_i E(u_i^* a) = \sum_{i=1}^n E(au_i)u_i^*.$$

When a quasi-basis exists we define the index of E to be

$$\mathrm{Ind}(E) = \sum_{i=1}^n u_i u_i^*.$$

It is well known that $\mathrm{Ind}(E)$ does not depend on the choice of quasi-basis [19, Proposition 1.2.8]. We call a $*$-endomorphism α *finite index* if there exists a finite index conditional expectation from $\mathcal{A}$ onto the range of α. Such $*$-endomorphisms were considered by Exel in [10].

Corollary 4.1.14. *Let $(\mathcal{A}, \alpha, P)$ be a C*-dynamical system where $\mathcal{A}$ is a simple unital C*-algebra and P is either contained in an abelian group or a free semigroup. Suppose that for each $s \in P$ there exists a finite index conditional expectation $E_s : p_s\mathcal{A}p_s \to \alpha_s(\mathcal{A})$ where $p_s = \alpha_s(1)$. Then $\mathrm{rad}(\mathcal{A} \times_\alpha P)$ is generated by monomials of the form $a(1 - p_s) \otimes e_s$, where $a \in \mathcal{A}$. Moreover if each α_s is finite index, then $\mathcal{A} \times_\alpha P$ is semi-simple.*

Proof. Let $E_s : p_s\mathcal{A}p_s \to \alpha_s(\mathcal{A})$ be a finite index conditional expectation. By [19, Proposition 2.6.2], for all positive $a \in \mathcal{A}$ we have $E_s(a) \geq \|\mathrm{Ind}(E_s)\|^{-1}a$. It follows that E_s is faithful and we may apply Theorem 4.1.10 and Corollary 4.1.12. □

We get another special case of Theorem 4.1.10 when each $*$-endomorphism has hereditary range, a condition which has been considered before in [11, 16]. This corollary follows from the fact that $\alpha_s(\mathcal{A})$ is hereditary if and only if $\alpha_s(\mathcal{A}) = p_s\mathcal{A}p_s$.

Corollary 4.1.15. *Let $(\mathcal{A}, \alpha, P)$ be a C*-dynamical system where $\mathcal{A}$ is a simple unital C*-algebra and P is either contained in an abelian group or a free semigroup. Suppose that the range of α_s is hereditary in $\mathcal{A}$ for all $s \in P$. Then* $\operatorname{rad}(\mathcal{A} \times_\alpha P)$ *is generated by monomials of the form $a(1-p_s) \otimes e_s$, where $a \in \mathcal{A}$ and $p_s = \alpha_s(1)$.*

Proof. It is clear that $\alpha_s(\mathcal{A}) \subseteq p_s \mathcal{A} p_s$ for each $s \in P$. For the reverse inclusion observe that for $0 \leq a \in p_s \mathcal{A} p_s$ we have $0 \leq a = p_s a p_s \leq \|a\| p_s \in \alpha_s(\mathcal{A})$. Since $\alpha_s(\mathcal{A})$ is hereditary, $a \in \alpha_s(\mathcal{A})$. We may now apply Theorem 4.1.10 because we have $\alpha_s(\mathcal{A}) = p_s \mathcal{A} p_s$. ☐

Example 4.1.16. Let $\mathcal{A}$ be a simple unital C*-algebra that contains an isometry s. Define a $*$-endomorphism $\alpha : \mathcal{A} \to \mathcal{A}$ by $\alpha(a) = sas^*$. We get an action of $\mathbb{Z}_+$ on $\mathcal{A}$ by setting $\alpha_n = \alpha^n$. The range of α_n is hereditary because $\alpha_n(\mathcal{A}) = p_n \mathcal{A} p_n$, where $p_n = \alpha_n(1) = s^n (s^*)^n$. By Corollary 4.1.15 we conclude that $\operatorname{rad}(\mathcal{A} \times_\alpha \mathbb{Z}_+)$ is generated by monomials of the form $a(1-p_n) \otimes e_n$.

Example 4.1.17 (Non-Commuting Non-Unital Shifts on the CAR Algebra)**.** Let $\mathcal{A} = \bigotimes_{n \geq 1} M_2$ be the CAR algebra and

$$q_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \text{ and } q_2 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right].$$

We define two non-unital shifts, $\alpha_1 : \mathcal{A} \to \mathcal{A}$ and $\alpha_2 : \mathcal{A} \to \mathcal{A}$, on the elementary tensors by

$$\begin{aligned} \alpha_1 &: a_1 \otimes a_2 \otimes \cdots \mapsto q_1 \otimes a_1 \otimes a_2 \otimes \cdots, \text{ and} \\ \alpha_2 &: a_1 \otimes a_2 \otimes \cdots \mapsto q_2 \otimes a_1 \otimes a_2 \otimes \cdots \end{aligned}$$

which extend to $*$-endomorphisms on $\mathcal{A}$. Let $\mathbb{F}_2^+$ be the free semigroup on the generating set $\{1, 2\}$. To get an action α of $\mathbb{F}_2^+$ on $\mathcal{A}$ we set

$$\alpha_w = \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k},$$

where $w = i_1 i_2 \cdots i_k \in \mathbb{F}_2^+$. The range of each α_w is hereditary for each $w \in \mathbb{F}_2^+$ because $\alpha_w(\mathcal{A}) = p_w \mathcal{A} p_w$, where

$$p_w = \alpha_w(1) = q_{i_1} \otimes q_{i_2} \otimes \cdots \otimes q_{i_k} \otimes 1.$$

Thus by Corollary 4.1.15 we conclude that $\operatorname{rad}(\mathcal{A} \times_\alpha \mathbb{F}_2^+)$ is generated by monomials of the form $a(1-p_w) \otimes e_w$.

4.2 The Non-Unital Case

The main obstruction in obtaining a characterization of the radical in the non-unital simple case is that without a unit it is not obvious that $\mathcal{J}_s \cap \alpha_s(\mathcal{A}) = \{0\}$ or even $\mathcal{J}_s \neq \mathcal{A}$. Because of this, in this section, we must assume that $\mathcal{J}_s \cap \alpha_s(\mathcal{A}) = \{0\}$, which in the abelian semigroup case is equivalent to $\mathcal{J}_s \neq \mathcal{A}$. Even with that assumption the proofs of Theorems 4.1.6 and 4.1.10 do not generalize. Theorem 4.1.6 used the fact that for each non-zero element a in a purely infinite simple unital C*-algebra $\mathcal{A}$ there exist $b, c \in \mathcal{A}$ such that $bac = 1$, and the characterization of the multiplicative domain of the conditional expectation used in the proof of Theorem 4.1.10 required that the map was unital. We will however be able to obtain non-unital versions of Corollaries 4.1.11 and 4.1.15.

Lemma 4.2.1. *Let $(\mathcal{A}, \alpha, P)$ be a C*-dynamical system where P is contained in an abelian group. Let $s \in P$. Then*

- *(i) we have $\mathcal{J}_s \subseteq \mathcal{J}_{st}$ for all $t \in P$, and*
- *(ii) if $\alpha_s(\mathcal{A}) \subseteq \mathcal{J}_s$, then $\mathcal{J}_s = \mathcal{A}$.*

Proof. (i) Let $a \in \mathcal{J}_s$ and $t \in P$. For all finite sums $\sum_{r\in P} a_r \otimes e_r \in \mathcal{A} \times_\alpha P$,

$$a \otimes e_{st}\Big(\sum_{r\in P} a_r \otimes e_r\Big) = a \otimes e_s\Big(\sum_{r\in P} \alpha_t(a_r) \otimes e_{tr}\Big)$$

is quasi-nilpotent because $a \otimes e_s \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$. Passing to limits we see that $(a \otimes e_{st})x$ is quasi-nilpotent for all $x \in \mathcal{A} \times_\alpha P$. It follows that $a \otimes e_{st} \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$ and $a \in \mathcal{J}_{st}$.

(ii) Suppose $\alpha_s(\mathcal{A}) \subseteq \mathcal{J}_s$, let $a \in \mathcal{A}$, and let $\sum_{t\in P} a_t \otimes e_t \in \mathcal{A} \times_\alpha P$ be a finite sum. From (i) and the fact that the radical of $\mathcal{A} \times_\alpha P$ is generated by its monomials we have $\sum_{t\in P} \alpha_s(a_t) \otimes e_{st} \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$. It follows that

$$a \otimes e_s\Big(\sum_{t\in P} a_t \otimes e_t\Big) = a \otimes e_e\Big(\sum_{t\in P} \alpha_s(a_t) \otimes e_{st}\Big)$$

is quasi-nilpotent. Passing to limits we see $(a \otimes e_s)x$ is quasi-nilpotent for all $x \in \mathcal{A} \times_\alpha P$, whence $a \otimes e_s \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$. ☐

Proposition 4.2.2. *Let $(\mathcal{A}, \alpha, \mathbb{Z}_+)$ be a C*-dynamical system where $\mathcal{A}$ is a simple C*-algebra and α is an action of either a semigroup contained in an abelian group or a free semigroup on $\mathcal{A}$ by $*$-automorphisms. Then $\mathcal{J}_s$ equals either $\mathcal{A}$ or $\{0\}$ for each $s \in P$.*

Proof. Because $\mathcal{J}_s$ is an $\mathcal{A}$–$\alpha_s(\mathcal{A})$ bimodule and $\alpha_s(\mathcal{A}) = \mathcal{A}$, $\mathcal{J}_s$ is an ideal of the simple C*-algebra $\mathcal{A}$. It follows that $\mathcal{J}_s$ is either $\mathcal{A}$ or $\{0\}$. □

Corollary 4.2.3. *Let $(\mathcal{A}, \alpha, \mathbb{Z}_+)$ be a C*-dynamical system where $\mathcal{A}$ is a simple C*-algebra and α is an action of $\mathbb{Z}_+$ on $\mathcal{A}$ by $*$-automorphisms. Then $\operatorname{rad}(\mathcal{A} \times_\alpha P)$ is either zero or the ideal generated by $\mathcal{A} \otimes e_1$.*

Proof. If $a \otimes e_0 \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$, then $(a^* \otimes e_0)(a \otimes e_0) = a^*a \otimes e_0$ should be quasi-nilpotent. We apply the C*-identity to the spectral radius formula

$$\lim_{n\to\infty} \|(a^*a \otimes e_0)^n\|^{1/n} = \lim_{n\to\infty} \|(a^*a)^n\|^{1/n} = \|a\|$$

to show that a must be zero and $\operatorname{rad}(\mathcal{A} \times_\alpha P)$ is contained in the ideal generated by $\mathcal{A} \otimes e_1$.

Suppose that $\mathcal{A} \times_\alpha P$ is not semi-simple. Then by the previous proposition there is some $n \geq 1$ for which $\mathcal{J}_n = \mathcal{A}$. By Lemma 4.2.1.(i) we have $\mathcal{A} = \mathcal{J}_n \subseteq \mathcal{J}_{n+k}$ for all $k \in \mathbb{Z}_+$. Observe that for all $a \in \mathcal{A}$ and finite sums $\sum_{k=0}^{m} a_k \otimes e_k \in \mathcal{A} \times_\alpha P$ we can write

$$\Big((a \otimes e_1) \sum_{k=0}^{m} a_k \otimes e_k\Big)^n = \sum_{k=n}^{n(m+1)} b_k \otimes e_k \in \operatorname{rad}(\mathcal{A} \times_\alpha P),$$

for some $b_k \in \mathcal{A}$. Passing to limits we see that $((a \otimes e_1)x)^n \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$ for all $x \in \mathcal{A} \times_\alpha P$, therefore $(a \otimes e_1)x$ is quasi-nilpotent and $a \otimes e_1 \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$. □

Proposition 4.2.4. *Let $(\mathcal{A}, \alpha, P)$ be a C*-dynamical system where $\mathcal{A}$ is a simple C*-algebra and P is either a semigroup contained in an abelian group or a free semigroup. Suppose that for each $s \in P$ there exists $0 < b_s \in \alpha_s(\mathcal{A})$ such that $\alpha_s(\mathcal{A}) = \overline{b_s \mathcal{A} b_s}$. If*

- *(i) P is abelian and $\mathcal{J}_s \neq \mathcal{A}$, or*
- *(ii) P is free and $\mathcal{J}_s \cap \alpha_s(\mathcal{A}) = \{0\}$,*

then

$$\mathcal{J}_s = \{a \in \mathcal{A} : a\alpha_s(\mathcal{A}) = \{0\}\} = \{a \in \mathcal{A} : ab_s = 0\}.$$

Proof. The equality

$$\{a \in \mathcal{A} : ab_s = 0\} = \{a \in \mathcal{A} : a\alpha_s(\mathcal{A}) = \{0\}\}$$

is easy and the containment

$$\mathcal{J}_s \supseteq \{a \in \mathcal{A} : a\alpha_s(\mathcal{A}) = \{0\}\}$$

is clear because an argument similar to the one in the proof of Lemma 4.1.5 shows that $a\alpha_s(\mathcal{A}) = \{0\}$ implies $(a \otimes e_s)x = 0$ for all $x \in \mathcal{A} \times_\alpha P$. For the reverse inclusion suppose that $\mathcal{J}_s \cap \alpha_s(\mathcal{A}) = \{0\}$, which by Lemma 4.2.1 is equivalent to $\mathcal{J}_s \neq \mathcal{A}$ in the abelian semigroup case. The bimodule property of $\mathcal{J}_s$ guarantees $0 \leq b_s a^* a b_s \in \mathcal{J}_s \cap \alpha_s(\mathcal{A}) = \{0\}$ for all $a \in \mathcal{J}_s$. It follows that $ab_s = 0$ for all $a \in \mathcal{J}_s$. □

Corollary 4.2.5. *Let $(\mathcal{A}, \alpha, P)$ be a C*-dynamical system where $\mathcal{A}$ is a simple separable C*-algebra and P is either a semigroup contained in an abelian group or a free semigroup. Suppose that for each $s \in P$ the range of α_s is hereditary in $\mathcal{A}$. If*

(i) P is abelian and $\mathcal{J}_s \neq \mathcal{A}$, or

(ii) P is free and $\mathcal{J}_s \cap \alpha_s(\mathcal{A}) = \{0\}$,

then

$$\mathcal{J}_s = \{a \in \mathcal{A} : a\alpha_s(\mathcal{A}) = \{0\}\} = \{a \in \mathcal{A} : ab_s = 0\},$$

where $0 < b_s \in \alpha_s(\mathcal{A})$ is an element that satisfies $\alpha_s(\mathcal{A}) = \overline{b_s \mathcal{A} b_s}$.

Proof. Recall that if $\mathcal{B}$ is a separable hereditary C*-subalgebra of a C*-algebra $\mathcal{A}$, then there exists $0 \leq b \in \mathcal{B}$ such that $\mathcal{B}$ is the closure of $b\mathcal{A}b$ [15, Theorem 3.2.5]. Therefore there exists $0 < b_s \in \alpha_s(\mathcal{A})$ such that $\alpha_s(\mathcal{A})$ is the closure of $b_s \mathcal{A} b_s$ and we can apply the previous proposition. □

Example 4.2.6 (The Unilateral Shift and the Compacts)**.** Let $\mathcal{K}$ be the compact operators on $\ell^2(\mathbb{Z}_+) = \overline{\operatorname{span}}\{\xi_i : i \geq 0\}$, let $S \in \mathcal{B}(\ell^2(\mathbb{Z}_+))$ be the unilateral shift, and let $S_n = S^n$. Since $\mathcal{K}$ is an ideal of $\mathcal{B}(\ell^2(\mathbb{Z}_+))$, we can define an action of $\mathbb{Z}_+$ on $\mathcal{K}$ by setting $\alpha_n(K) = S_n K S_n^*$ for all $K \in \mathcal{K}$. Corollary 4.2.5 applies because the range $\alpha_n(\mathcal{K}) = S_n S_n^* \mathcal{K} S_n S_n^*$ of each α_n is hereditary. To compute the radical we need only demonstrate $\mathcal{J}_n \neq \mathcal{K}$ for each $n \in \mathbb{Z}_+$.

We claim that $S_n^* P_n \notin \mathcal{J}_n$, where P_n is the orthogonal projection onto $\mathbb{C}\xi_n$. To see this first note that for all $k \geq 1$

$$\begin{aligned} \alpha_{(k-1)n}(S_n^* P_n))\alpha_{kn}(S_n^* P_n) &= S_{(k-1)n}(S_n^* P_n) S_{(k-1)n}^* \cdot S_{kn}(S_n^* P_n) S_{kn}^* \\ &= \alpha_{(k-1)n}(S_n^* P_n) S_n^* \end{aligned}$$

and then estimate

$$\begin{aligned}
\lim_{k\to\infty} \|S_n^* P_n \alpha_n(S_n^* P_n) \cdots \alpha_{(k-1)n}(S_n^* P_n)\|^{1/k} &= \lim_{k\to\infty} \|S_n^* P_n S_{(k-1)n}^*\|^{1/k} \\
&\geq \lim_{k\to\infty} \|S_n^* P_n S_{(k-1)n}^* \xi_{kn}\|^{1/k} \\
&= 1,
\end{aligned}$$

which by Lemma 4.1.4 tells us that $S_n^* P_n \notin \mathcal{J}_n$. We conclude by Corollary 4.2.5 that $\mathcal{J}_n = \{K \in \mathcal{K} : K\alpha_n(\mathcal{K}) = \{0\}\}$. Exploiting the fact that $\mathcal{K} \subseteq \mathcal{B}(\ell^2(\mathbb{Z}_+))$ is an ideal, we can write $\mathcal{J}_n = \mathcal{K}(I - S_n S_n^*)$, where $I \in \mathcal{B}(\ell^2(\mathbb{Z}_+))$ is the identity. It follows that $\mathrm{rad}(\mathcal{A} \times_\alpha P)$ is generated by monomials of the form $K(I - S_n S_n^*) \otimes e_n$, which mirrors the characterization of the radical in Corollary 4.1.15.

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