Control of the Landau–Lifshitz Equation

A.N. Chow and K.A. Morris

Department of Applied Mathematics, University of Waterloo, Canada
a29chow@uwaterloo.ca, kmorris@uwaterloo.ca

Abstract

The Landau–Lifshitz equation describes the dynamics of magnetization inside a ferromagnet. This equation is nonlinear and has an infinite number of stable equilibria. It is desirable to control the system from one equilibrium to another. A control that moves the system from an arbitrary initial state, including an equilibrium point, to a specified equilibrium is presented. It is proven that the second point is an asymptotically stable equilibrium of the controlled system. The results are illustrated with some simulations.

Key words: Asymptotic stability, Equilibrium, Lyapunov function, Nonlinear control systems, Partial differential equations

1 Introduction

The Landau–Lifshitz equation is a partial differential equation (PDE), which describes the magnetic behaviour within ferromagnetic structures. This equation was originally developed to model the behaviour of domain walls, which separate magnetic regions within a ferromagnet [1]. Ferromagnets are often found in memory storage devices such as hard disks, credit cards or tape recordings. Each set of data stored in a memory device is uniquely assigned to a specific stable magnetic state of the ferromagnet, and hence it is desirable to control magnetization between different stable equilibria. This is difficult due to the presence of hysteresis in the Landau–Lifshitz equation. Hysteresis indicates the presence of multiple equilibria [2,3]. Because of this, a particular control can lead to different magnetizations; that is, the particular path of magnetization depends on the initial state of the system and looping in the input-output map is typical [2,3].

There is now an extensive body of results on control and stabilization of linear PDE's; see for instance the books [4–7] and the review paper [8]. Stability results for the Landau-Lifshitz equation are often based on linearization [9–12]. In these works, the spectral properties of the linear operator are determined. In [13], sufficient assumptions are made that simplify a general form of the Landau-Lifshitz equation into an ordinary differential equation; and based on this, the magnetization dynamics are shown to be stable.

The magnetic state of a ferromagnet can be changed by an applied magnetic field, which is viewed as the control. From this physically meaningful perspective, the control enters the Landau-Lifshitz equation nonlinearly. In [14], the Landau–Lifshitz equation is linearized and shown to have an unstable equilibrium; and to stabilize this equilibrium a control that is the average of the magnetization in one direction and zero in the other two directions is used. In [15,16], solutions to the Landau–Lifshitz equation are shown to be arbitrarily close to domain walls given a constant control. Experiments and numerical simulations demonstrating the control of domains walls in a nanowire are presented in [17,18].

In the next section, the uncontrolled Landau–Lifshitz equation is described. It is known to have multiple stable equilibria [19, Theorem 6.1.1]. In section 3, a control, acting as the applied magnetic field, is introduced into the Landau–Lifshitz equation nonlinearly. The control objective is to steer the system dynamics between stable equilibrium points. Results demonstrate the controlled Landau–Lifshitz equation is stable, and the linearized controlled Landau–Lifshitz equation is asymptotically stable. In Section 4, simulations for the full equation are presented.

2 Landau-Lifshitz Equation

Consider the magnetization

$$\mathbf{m}(x,t) = (m_1(x,t), m_2(x,t), m_3(x,t)),$$

at position $x \in [0, L]$ and time $t \geq 0$ in a long thin ferromagnetic material of length $L > 0$. If only the exchange energy term is considered, the magnetization is modelled by the one–dimensional (uncontrolled) Landau–Lifshitz equation [20],[19, Chapter 6]

$$\frac{\partial \mathbf{m}}{\partial t} = \mathbf{m} \times \mathbf{m}_{xx} - \nu \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx}) \tag{1a}$$
$$\mathbf{m}(x,0) = \mathbf{m}_0(x) \tag{1b}$$

where $\times$ denotes the cross product and $\nu \geq 0$ is the damping parameter, which depends on the type of ferromagnet. The term $\mathbf{m}_{xx}$ denotes magnetization differentiated with respect to x twice. The Landau–Lifshitz equation sometimes includes a parameter called the gyromagnetic ratio multiplying $\mathbf{m} \times \mathbf{m}_{xx}$. The gyromagnetic ratio has been set to 1 for simplicity. For more on the damping parameter and gyromagnetic ratio, see [21].

The Landau–Lifshitz equation is a coupled set of three nonlinear PDEs. It is assumed that there is no magnetic flux at the boundaries and so Neumann boundary conditions are appropriate:

$$\mathbf{m}_x(0,t) = \mathbf{m}_x(L,t) = \mathbf{0}. \tag{1c}$$

Existence and uniqueness of solutions to (1) with different degrees of regularity has been shown [22,23].

Theorem 1. [19, Lemma 6.3.1] *If $||\mathbf{m}_0(x)||_2 = 1$, the solution, $\mathbf{m}$, to (1a) satisfies*

$$||\mathbf{m}(x,t)||_2 = 1 \tag{2}$$

where $||\cdot||_2$ is the Euclidean norm.

The following statement is a more restrictive version of the theorem stated in [22].

Theorem 2. *[22, Thm. 1.3,1.4]. If $\mathbf{m}_0 \in H_2(0,L)$, $\mathbf{m}_{0,x}(0) = \mathbf{m}_{0,x}(L) = \mathbf{0}$ and $\|\mathbf{m}_0\|_2 = 1$, then there exists a time $T^* > 0$ and an unique solution $\mathbf{m}$ of (1) such that for all $T < T^*$, $\mathbf{m} \in \mathcal{C}([0,T;H_2(0,L)) \cap \mathcal{L}_2(0,L;H_3(0,L))$.*

With more general initial conditions, solutions to (1) are defined on $\mathcal{L}_2^3 = \mathcal{L}_2([0,L];\mathbb{R}^3)$ with the usual inner–product and norm. The notation $\|\cdot\|_{\mathcal{L}_2^3}$ is used for the norm. Define the operator

$$f(\mathbf{m}) = \mathbf{m} \times \mathbf{m}_{xx} - \nu \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx}), \tag{3}$$

and its domain

$$D = \{\mathbf{m} \in \mathcal{L}_2^3 : \mathbf{m}_x \in \mathcal{L}_2^3,\ \mathbf{m}_{xx} \in \mathcal{L}_2^3,\ \mathbf{m}_x(0) = \mathbf{m}_x(L) = \mathbf{0}\}. \tag{4}$$

Theorem 3. [24, Theorem 4.7] *The operator $f(\mathbf{m})$ with domain D generates a nonlinear contraction semigroup on $\mathcal{L}_2^3$.*

Ferromagnets are magnetized to saturation [25, Section 4.1]; that is $||\mathbf{m}_0(x)||_2 = M_s$ where M_s is the magnetization saturation. In much of the literature, M_s is set to 1; see for example, [19, Section 6.3.1], [22,23,26]. This convention is used here. Physically, this means that at each point, x, the magnitude of $\mathbf{m}_0(x)$ equals the magnetization saturation. The initial condition $\mathbf{m}_0(x)$ is furthermore assumed to be real–valued, and hence $\mathbf{m}(x,t)$ for $t > 0$ is real–valued.

The set of equilibrium points of (1) is [19, Theorem 6.1.1]

$$E = \{\mathbf{a} = (a_1, a_2, a_3) : a_1, a_2, a_3 \text{ constants and } \mathbf{a}^{\mathrm{T}}\mathbf{a} = 1\}. \tag{5}$$

Theorem 4. [24, Theorem 4.11] *The equilibrium set in (5) is asymptotically stable in the $\mathcal{L}_2^3$–norm.*

3 Controller Design

In current applications, the control enters as an applied magnetic field [9,10,14–16]. More precisely, a control, $\mathbf{u}(t)$, is introduced into the Landau-Lifshitz equation (1a) as follows

$$\begin{aligned}\frac{\partial \mathbf{m}}{\partial t} =& \mathbf{m} \times (\mathbf{m}_{xx} + \mathbf{u}) - \nu \mathbf{m} \times (\mathbf{m} \times (\mathbf{m}_{xx} + \mathbf{u})) \\ =& \mathbf{m} \times \mathbf{m}_{xx} - \nu \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx}) \\ &+ \mathbf{m} \times \mathbf{u} - \nu \mathbf{m} \times (\mathbf{m} \times \mathbf{u}), \\ \mathbf{m}(x,0) =& \mathbf{m}_0(x).\end{aligned} \tag{6}$$

As for the uncontrolled system, the boundary conditions are $\mathbf{m}_x(0,t) = \mathbf{m}_x(L,t) = \mathbf{0}$. Equation (6) is the Landau-Lifshitz equation with a nonlinear control. Its existence and uniqueness results can be found in [22, Thm. 1.1,1.2] and is similar to Theorem 2.

As for the uncontrolled equation, since

$$\begin{aligned}\frac{1}{2}\frac{\partial ||\mathbf{m}(x,t)||_2}{\partial t} =& \mathbf{m}^{\mathrm{T}} \frac{\partial \mathbf{m}}{\partial t} \\ =& \mathbf{m}^{\mathrm{T}}(\mathbf{m} \times \mathbf{m}_{xx} - \nu \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx}) \\ &+ \mathbf{m} \times \mathbf{u} - \nu \mathbf{m} \times (\mathbf{m} \times \mathbf{u})) = 0,\end{aligned}$$

this implies $||\mathbf{m}||_2 = c$, where c is a constant. The convention is to take $c = 1$. It follows that any equilibrium point is trivially stable in the $\mathcal{L}_2$-norm.

The goal is to choose a control so that the system governed by the Landau–Lifshitz equation moves from an arbitrary initial condition, possibly an equilibrium point, to a specified equilibrium point $\mathbf{r}$. The control needs to be chosen so that $\mathbf{r}$ becomes a stable equilibrium point of the controlled system. It can be shown that zero is an eigenvalue of the linearized uncontrolled Landau–Lifshitz equation [24, Chapter 4.3.2]. For finite-dimensional linear systems, simple proportional control of a system with a zero eigenvalue yields asymptotic tracking of a specified state and this motivates choosing the control

$$\mathbf{u} = k(\mathbf{r} - \mathbf{m}) \tag{7}$$

where $\mathbf{r} \in E$ is an equilibrium point of the uncontrolled equation (1) and k is a positive constant control parameter.

Theorem 5. *For any $\mathbf{r} \in E$ and any positive constant k with control defined in (7), $\mathbf{r}$ is a locally stable equilibrium point of (6) in the H_1-norm. That is, for any initial condition $\mathbf{m}_0(x) \in D$, where D is defined in (4), the H_1-norm of the error $\mathbf{m} - \mathbf{r}$ does not increase.*

PROOF. Let $B(\mathbf{r}, p) = \{\mathbf{m} \in \mathcal{L}_2^3 : ||\mathbf{m} - \mathbf{r}||_{\mathcal{L}_2^3} < p\} \subset D$ for some constant $0 < p < 2$. Note that since $p < 2$, then $-\mathbf{r} \notin B(\mathbf{r}, p)$. For any $\mathbf{m} \in B(\mathbf{r}, p)$, consider the H_1-norm of the error

$$V(\mathbf{m}) = k\,||\mathbf{m} - \mathbf{r}||^2_{\mathcal{L}_2^3} + ||\mathbf{m}_x||^2_{\mathcal{L}_2^3}.$$

Taking the derivative of V,

$$\begin{aligned}\frac{dV}{dt} &= \int_0^L k(\mathbf{m} - \mathbf{r})^{\mathrm{T}}\dot{\mathbf{m}}dx + \int_0^L \mathbf{m}_x^{\mathrm{T}}\dot{\mathbf{m}}_x dx\\ &= \int_0^L k(\mathbf{m} - \mathbf{r})^{\mathrm{T}}\dot{\mathbf{m}}dx - \int_0^L \mathbf{m}_{xx}^{\mathrm{T}}\dot{\mathbf{m}}dx\\ &= \int_0^L \left(k(\mathbf{m} - \mathbf{r})^{\mathrm{T}}\dot{\mathbf{m}} - \mathbf{m}_{xx}^{\mathrm{T}}\dot{\mathbf{m}}\right) dx. \end{aligned} \tag{8}$$

Let $\mathbf{h} = \mathbf{m} - \mathbf{r}$, then the integrand becomes

$$k\mathbf{h}^{\mathrm{T}}\dot{\mathbf{m}} - \mathbf{m}_{xx}^{\mathrm{T}}\dot{\mathbf{m}} \tag{9}$$

and equation (6) becomes

$$\dot{\mathbf{m}} = \mathbf{m} \times (\mathbf{m}_{xx} - k\mathbf{h}) - \nu\mathbf{m} \times (\mathbf{m} \times (\mathbf{m}_{xx} - k\mathbf{h}))$$

where the dot represents differentiation with respect to t. It follows that

$$\begin{aligned}\mathbf{h}^{\mathrm{T}}\dot{\mathbf{m}} =& \mathbf{h}^{\mathrm{T}}(\mathbf{m} \times \mathbf{m}_{xx}) - \nu(\mathbf{m} \times \mathbf{m}_{xx})^{\mathrm{T}}(\mathbf{h} \times \mathbf{m})\\ &- \nu k||\mathbf{m} \times \mathbf{h}||_2^2 \end{aligned} \tag{10}$$

and

$$\begin{aligned}\mathbf{m}_{xx}^{\mathrm{T}}\dot{\mathbf{m}} =& -k\mathbf{m}_{xx}^{\mathrm{T}}(\mathbf{m} \times \mathbf{h}) + \nu||\mathbf{m} \times \mathbf{m}_{xx}||_2^2\\ &+ \nu k(\mathbf{m} \times \mathbf{h})^{\mathrm{T}}(\mathbf{m}_{xx} \times \mathbf{m}). \end{aligned} \tag{11}$$

Substituting (10) and (11) into equation (9) leads to

$$\begin{aligned}k\mathbf{h}^{\mathrm{T}}\dot{\mathbf{m}} - \mathbf{m}_{xx}^{\mathrm{T}}\dot{\mathbf{m}} =& 2\nu k(\mathbf{m} \times \mathbf{m}_{xx})^{\mathrm{T}}(\mathbf{m} \times \mathbf{h})\\ &- \nu k^2||\mathbf{m} \times \mathbf{h}||_2^2 - \nu||\mathbf{m} \times \mathbf{m}_{xx}||_2^2\\ =& -\nu||\mathbf{m} \times \mathbf{m}_{xx} - k\mathbf{m} \times \mathbf{h}||_2^2 \end{aligned}$$

Substituting this expression into equation (8) leads to

$$\frac{dV}{dt} = -\nu||\mathbf{m} \times (\mathbf{m}_{xx} + \mathbf{u})||^2_{\mathcal{L}_2^3} \leq 0.$$

Thus, the H_1-norm of the error does not increase. □

For any equilibrium point $\mathbf{r} \in E$ of (6) and $\mathbf{m} \in D$, let $\mathbf{m} = \mathbf{r} + \mathbf{v}$ where $\mathbf{v}$ is any admissible perturbation; that is, $\mathbf{v} \in D$ and $\|\mathbf{r} + \mathbf{v}\|_2 = 1$. The linearization of (6) at $\mathbf{r}$ with control defined in (7) is

$$\begin{aligned}\frac{\partial \mathbf{v}}{\partial t} =& \mathbf{r} \times \mathbf{v}_{xx} - \nu\mathbf{r} \times (\mathbf{r} \times \mathbf{v}_{xx})\\ &+ k\mathbf{v} \times \mathbf{r} - k\nu\mathbf{r} \times (\mathbf{v} \times \mathbf{r})\\ \mathbf{v}(0) =& \mathbf{v}_0. \end{aligned} \tag{12}$$

The following lemmas are needed in the proof of Theorem 8. The result in Lemma 6 is a consequence of the product rule.

Lemma 6. *For $\mathbf{m} \in \mathcal{L}_2^3$, the derivative of $\mathbf{g} = \mathbf{m} \times \mathbf{m}_x$ is $\mathbf{g}_x = \mathbf{m} \times \mathbf{m}_{xx}$.*

Lemma 7. *For $\mathbf{m} \in \mathcal{L}_2^3$ satisfying (1c),*

$$\int_0^L (\mathbf{m} - \mathbf{r})^{\mathrm{T}}(\mathbf{m} \times \mathbf{m}_{xx})dx = 0.$$

PROOF. Integrating by parts, and applying Lemma 6 and the boundary conditions (1c) implies that

$$\int_0^L (\mathbf{m} - \mathbf{r})^{\mathrm{T}}(\mathbf{m} \times \mathbf{m}_{xx})dx = -\int_0^L \mathbf{m}_x^{\mathrm{T}}(\mathbf{m} \times \mathbf{m}_x)dx.$$

From properties of cross products, $\mathbf{m}_x^{\mathrm{T}}(\mathbf{m} \times \mathbf{m}_x) = \mathbf{m}^{\mathrm{T}}(\mathbf{m}_x \times \mathbf{m}_x) = 0$, and hence the integral is zero. □

Theorem 8. *Let $\mathbf{r} \in E$ be an equilibrium point of (12). For any positive constant k, $\mathbf{r}$ is a locally asymptotically stable equilibrium of (12) in the $\mathcal{L}_2^3$-norm.*

PROOF. For an admissible $\mathbf{v}$ with $||\mathbf{v} - \mathbf{r}||_2 \leq 2$, consider the Lyapunov candidate

$$V(\mathbf{v}) = \frac{1}{2}\,||\mathbf{v} - \mathbf{r}||^2_{\mathcal{L}_2^3}.$$

It is clear that $V \geq 0$ for all $\mathbf{v} \in D$ and furthermore, $V(\mathbf{v}) = 0$ only when $\mathbf{v} = \mathbf{r}$. Therefore, $V(\mathbf{v}) > 0$ for all $\mathbf{v} \in D \backslash \{\mathbf{r}\}$.

Taking the derivative of $V(\mathbf{v})$ implies

$$\frac{dV}{dt} = \int_0^L (\mathbf{v}-\mathbf{r})^{\mathrm{T}} \dot{\mathbf{v}} dx$$

and substituting in (12) leads to

$$\begin{aligned}\frac{dV}{dt} = & \int_0^L (\mathbf{v}-\mathbf{r})^{\mathrm{T}} (\mathbf{r} \times \mathbf{v}_{xx}) dx \\ & - \nu \int_0^L (\mathbf{v}-\mathbf{r})^{\mathrm{T}} \left(\mathbf{r} \times (\mathbf{r} \times \mathbf{v}_{xx})\right) dx \\ & + k \int_0^L (\mathbf{v}-\mathbf{r})^{\mathrm{T}} (\mathbf{v} \times \mathbf{r}) dx \\ & - k\nu \int_0^L (\mathbf{v}-\mathbf{r})^{\mathrm{T}} \left(\mathbf{r} \times (\mathbf{v} \times \mathbf{r})\right) dx.\end{aligned}$$

The first integral is zero, and to show this the proof is similar to the one in Lemma 7. Applying integrating by parts, Lemma 6 and the boundary conditions (1c) implies that

$$\int_0^L (\mathbf{v}-\mathbf{r})^{\mathrm{T}} (\mathbf{r} \times \mathbf{v}_{xx}) dx = -\int_0^L \mathbf{v}_x^{\mathrm{T}} (\mathbf{v} \times \mathbf{v}_x) dx,$$

which is equal to zero from properties of cross products. The second integral can be written as

$$\begin{aligned}& \int_0^L (\mathbf{v}-\mathbf{r})^{\mathrm{T}} \left(\mathbf{r} \times (\mathbf{r} \times \mathbf{v}_{xx})\right) dx \\ & = \int_0^L (\mathbf{r} \times \mathbf{v}_{xx})^{\mathrm{T}} \left((\mathbf{v}-\mathbf{r}) \times \mathbf{r}\right) dx \\ & = \int_0^L (\mathbf{r} \times \mathbf{v}_{xx})^{\mathrm{T}} (\mathbf{v} \times \mathbf{r}) dx,\end{aligned}$$

then integrating by parts, and applying Lemma 6 and the boundary conditions leads to

$$\int_0^L (\mathbf{r} \times \mathbf{v}_{xx})^{\mathrm{T}} (\mathbf{v} \times \mathbf{r}) dx \quad = -\int_0^L (\mathbf{v}_x \times \mathbf{r})^{\mathrm{T}} (\mathbf{r} \times \mathbf{v}_x).$$

Therefore,

$$\int_0^L (\mathbf{v}-\mathbf{r})^{\mathrm{T}} \left(\mathbf{r} \times (\mathbf{r} \times \mathbf{v}_{xx})\right) dx = ||\mathbf{v}_x \times \mathbf{r}||^2_{\mathcal{L}_2^3}.$$

Letting $\mathbf{h} = \mathbf{v} - \mathbf{r}$, it follows that

$$\begin{aligned}\frac{dV}{dt} = & -\nu ||\mathbf{v}_x \times \mathbf{r}||^2_{\mathcal{L}_2^3} + k \int_0^L (\mathbf{v}-\mathbf{r})^{\mathrm{T}} (\mathbf{v} \times \mathbf{r}) dx \\ & - k\nu \int_0^L (\mathbf{v}-\mathbf{r})^{\mathrm{T}} \left(\mathbf{r} \times (\mathbf{v} \times \mathbf{r})\right) dx \\ = & -\nu ||\mathbf{v}_x \times \mathbf{r}||^2_{\mathcal{L}_2^3} + k \int_0^L \mathbf{h}^{\mathrm{T}} (\mathbf{h} \times \mathbf{r}) dx \\ & - k\nu \int_0^L \mathbf{h}^{\mathrm{T}} \left(\mathbf{r} \times (\mathbf{h} \times \mathbf{r})\right) dx.\end{aligned}$$

The first integral is zero since $\mathbf{h}^{\mathrm{T}}(\mathbf{h} \times \mathbf{r}) = \mathbf{r}^{\mathrm{T}}(\mathbf{h} \times \mathbf{h}) = 0$, and the last integral can be simplified using the fact that

$$\mathbf{h}^{\mathrm{T}} \left(\mathbf{r} \times (\mathbf{h} \times \mathbf{r})\right) = (\mathbf{h} \times \mathbf{r})^{\mathrm{T}} (\mathbf{h} \times \mathbf{r}).$$

Therefore,

$$\frac{dV}{dt} = -\nu ||\mathbf{v}_x \times \mathbf{r}||^2_{\mathcal{L}_2^3} - k\nu ||\mathbf{h} \times \mathbf{r}||^2_{\mathcal{L}_2^3}.$$

For $k > 0$,

$$\frac{dV}{dt} = -\nu \left(||\mathbf{v}_x \times \mathbf{r}||^2_{\mathcal{L}_2^3} + k ||\mathbf{h} \times \mathbf{r}||^2_{\mathcal{L}_2^3} \right) \leq 0$$

and furthermore, $dV/dt = 0$ if and only if $\mathbf{v}_x \times \mathbf{r} = \mathbf{0}$ and $\mathbf{h} \times \mathbf{r} = \mathbf{v} \times \mathbf{r} = \mathbf{0}$. This is true only if $\mathbf{v} = \alpha \mathbf{r}$ where α is any scalar. Since $||\mathbf{r} + \mathbf{v}||_2 = 1$ and $||\mathbf{v} - \mathbf{r}||_2 \leq 2$, then $\alpha = 0$.

It follows that $\mathbf{r}$ is a locally asymptotically stable equilibrium point of (12). □

4 Example

Simulations illustrating the stabilization of the Landau-Lifshitz equation were done using a Galerkin approximation with 12 linear spline elements. For the following simulations, the parameters are $\nu = 0.02$ and $L = 1$ with initial condition $\mathbf{m_0}(x) = (\sin(2\pi x), \cos(2\pi x), 0)$. Figure 1 illustrates that the solution to the uncontrolled Landau–Lifshitz equation settles to $\mathbf{r_0} = (0, -0.6, 0)$.

Stabilization of the Landau–Lifshitz equation with nonlinear control (6) is illustrated in Figure 2 with the second equilibrium point chosen to be $\mathbf{r}_1 = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$. The control parameter is $k = 10$. It is clear from the figure that the system converges to the specified equilibrium point, $\mathbf{r}_1$. The control can also be applied after the dynamics have settled to $\mathbf{r}_0$ as shown in Figure 3. In Figure 4, the dynamics settle (without a control) to $\mathbf{r}_0$, then the control is applied in succession twice, which forces the system from $\mathbf{r}_0$ to $\mathbf{r}_1$, and then finally to $\mathbf{r}_2 = (0, 1, 0)$.

5 Conclusion

The Landau-Lifshitz equation is a nonlinear system of PDEs with multiple equilibrium points. In applications, the control enters through an applied field and the control enters nonlinearly. The presence of a 0 eigenvalue in the linearized equation suggested that a simple feedback proportional control could be used to steer the system to an arbitrary equilibrium point. It was shown that the controlled Landau-Lifshitz equation with a nonlinear control has a stable equilibrium point and the linearization has an asymptotically stable equilibrium point.

Simulations indicate that proportional control also stabilizes the fully nonlinear model. This suggests the nonlinear equation has an asymptotically stable equilibrium point. Proving this remains an open research problem. This would be a significant contribution as the Landau–Lifshitz equation is not quasi-linear, which means linearization is not guaranteed to predict stability of the nonlinear equation [27,28].

References

[1] L. Landau and E. Lifshitz, "On the theory of the dispersion of magnetic permeability in ferromagnetic bodies," *Ukrainian Journal of Physics*, vol. 53, no. Special Issue, pp. 14–22, 2008.

[2] A. Chow and K. A. Morris, "Hysteresis in the Landau–Lifshitz equation," in *Proceedings of the American Control Conference*, 2014.

[3] K. Morris, "What is hysteresis?," *Applied Mechanics Reviews*, vol. 64, no. 5, 2011.

[4] A. Bensoussan, *Representation and Control of Infinite Dimensional Systems*. Birkhauser, 2007.

[5] R. Curtain and H. Zwart, *An introduction to Infinite-Dimensional Linear Systems Theory*, vol. 21 of *Texts in Applied Mathematics*. Springer-Verlag, 1995.

[6] I. Lasiecka and R. Triggiani, *Control Theory for Partial Differential Equations: Continuous and Approximation Theories*, vol. I. Cambridge University Press, 2000.

[7] I. Lasiecka and R. Triggiani, *Control Theory for Partial Differential Equations: Continuous and Approximation Theories*, vol. II. Cambridge University Press, 2000.

[8] K. A. Morris, "Control of systems governed by partial differential equations," in *Control Handbook* (W. Levine, ed.), IEEE, 2010.

[9] G. Carbou and S. Labbé, "Stability for walls in ferromagnetic nanowire," in *Numerical mathematics and advanced applications*, pp. 539–546, Berlin: Springer, 2006.

[10] G. Carbou and S. Labbé, "Stability for static walls in ferromagnetic nanowires," *Discrete Contin. Dyn. Syst. Ser. B*, vol. 6, no. 2, pp. 273–290 (electronic), 2006.

[11] R. Jizzini, "Optimal stability criterion for a wall in a ferromagnetic wire in a magnetic field," *Journal of Differential Equations*, vol. 250, pp. 3349–3361, APR 15 2011.

[12] S. Labbe, Y. Privat, and E. Trelat, "Stability properties of steady-states for a network of ferromagnetic nanowires," *J. Differ. Equ. (USA)*, vol. 253, no. 6, pp. 1709 – 28, 2012.

[13] I. D. Mayergoyz, C. Serpico, and G. Bertotti, "On stability of magnetization dynamics in nanoparticles," *IEEE Transactions on Magnetics*, vol. 46, no. 6, pp. 1718 – 1721, 2010.

[14] G. Carbou and S. Labbe, "Stabilization of walls for nano-wires of finite length," *ESAIM, Control Optim. Calc. Var. (France)*, vol. 18, no. 1, pp. 1 – 21, 2012.

[15] G. Carbou, S. Labbé, and E. Trélat, "Control of travelling walls in a ferromagnetic nanowire," *Discrete and Continuous Dynamical Systems. Series S*, vol. 1, no. 1, pp. 51–59, 2008.

[16] G. Carbou, S. Labbé, and E. Trélat, "Smooth control of nanowires by means of a magnetic field," *Communications on Pure and Applied Analysis*, vol. 8, no. 3, pp. 871–879, 2009.

[17] S. Noh, Y. Miyamoto, M. Okuda, N. Hayashi, and Y. K. Kim, "Control of magnetic domains in co/pd multilayered nanowires with perpendicular magnetic anisotropy," *Journal of Nanoscience and Nanotechnology*, vol. 12, no. 1, pp. 428 – 432, 2012.

[18] R. Wieser, E. Y. Vedmedenko, and R. Wiesendanger, "Indirect control of antiferromagnetic domain walls with spin current," *Physical Review Letters*, vol. 106, no. 6, 2011.

[19] B. Guo and S. Ding, *Landau-Lifshitz Equations*, vol. 1 of *Frontier Of Research with the Chinese Academy of Sciences*. World Scientific, 2008.

[20] W. Brown, *Micromagnetics*. No. 18 in Interscience Tracts on Physics and Astronomy, Wiley, 1963.

[21] T. Gilbert, "A phenomenological theory of damping in ferromagnetic materials," *IEEE Transactions on Magnetics*, vol. 40, no. 6, pp. 3443 – 3449, 2004.

[22] G. Carbou and P. Fabrie, "Regular solutions for Landau-Lifschitz equation in a bounded domain," *Differential Integral Equations*, vol. 14, no. 2, pp. 213–229, 2001.

[23] F. Alouges and A. Soyeur, "On global weak solutions for Landau-Lifshitz equations: existence and nonuniqueness," *Nonlinear Anal.*, vol. 18, no. 11, pp. 1071–1084, 1992.

[24] A. Chow, *Control of hysteresis in the Landau-Lifshitz equation*. PhD thesis, University of Waterloo, 2013.

[25] B. Cullity and C. Graham, *Introduction to Magnetic Materials*. Wiley, second ed., 2009.

[26] M. Lakshmanan, "The fascinating world of the Landau-Lifshitz-Gilbert equation: an overview," *Phil. Tran. Royal Society A*, vol. 369, pp. 1280–1300, 2011.

[27] R. A. Jamal and K. A. Morris, "Linearized stability of partial differential equations with application to stabilization of the Kuramoto-Sivashinsky equation." submitted.

[28] R. al Jamal, A. Chow, and K. A. Morris, "Linearized stability analysis of nonlinear partial differential equations," in *Proceedings of the 21st International Symposium on Mathematical Theory of Networks and Systems*, 2014.

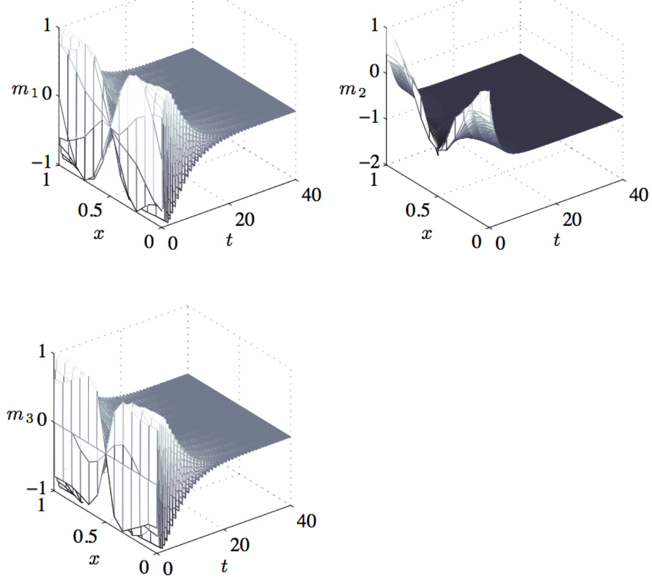

Fig. 1. Magnetization in the uncontrolled Landau–Lifshitz equation moves from initial condition $\mathbf{m_0}(x)$, to the equilibrium $\mathbf{r_0} = (0, -0.6, 0)$.

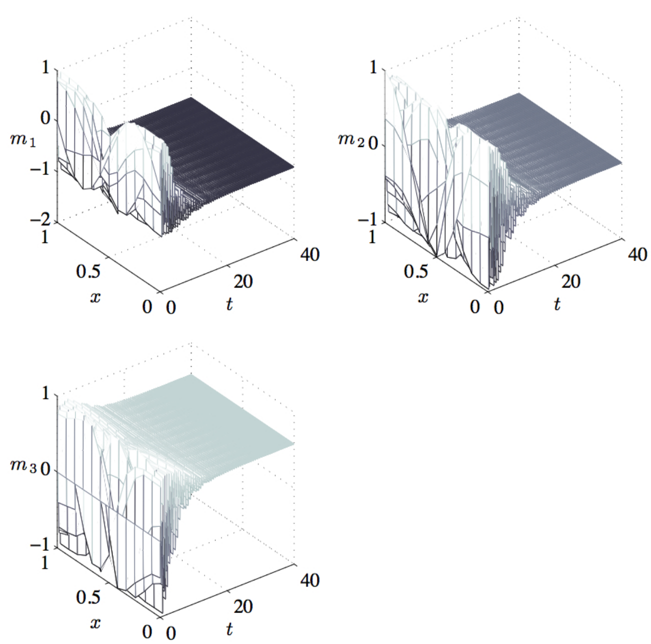

Fig. 2. Magnetization in the Landau–Lifshitz equation with nonlinear control moves from the initial condition $\mathbf{m_0}(x)$ to the specified equilibrium $\mathbf{r_1} = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$ with control parameter $k = 10$.

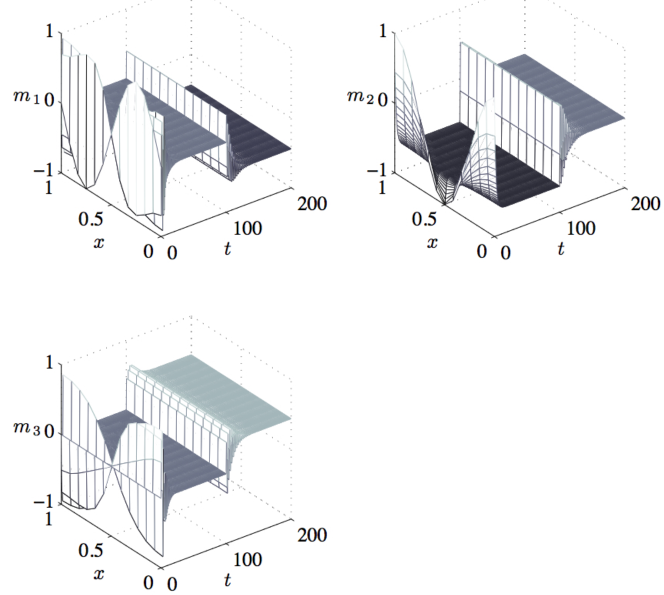

Fig. 3. Magnetization in the Landau–Lifshitz equation moves from the initial condition $\mathbf{m_0}(x)$ to the equilibrium $\mathbf{r_0} = (0, -0.6, 0)$ without a control. The control, $\mathbf{u}$ with $k = 10$ is then applied to the equation nonlinearly and steers the dynamics to the specified equilibrium $\mathbf{r_1} = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$.

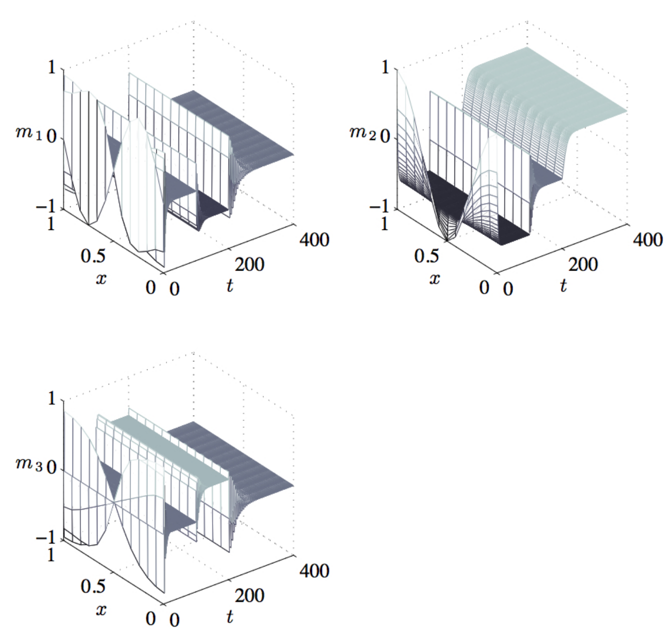

Fig. 4. Magnetization in the Landau–Lifshitz equation moves from the initial condition $\mathbf{m_0}(x)$ to the equilibrium $\mathbf{r_0} = (0, -0.6, 0)$ without a control. The control $\mathbf{u}$ with $k = 10$ is then applied to the equation nonlinearly and steers the dynamics to the specified equilibrium $\mathbf{r_1} = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$, and then to another equilibrium, $\mathbf{r_2} = (0, 1, 0)$.