Impulsive Control of Dynamical Networks

by

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Examining Committee Membership

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

Dynamical networks (DNs) consist of a large set of interconnected nodes with each node being a fundamental unit with detailed contents. A great number of natural and man-made networks such as social networks, food networks, neural networks, World Wide Web, electrical power grid, etc., can be effectively modeled by DNs. The main focus of the present thesis is on delay-dependent impulsive control of DNs. To study the impulsive control problem of DNs, we firstly construct stability results for general nonlinear time-delay systems with delayed impulses by using the method of Lyapunov functionals and Razumikhin technique.

Secondly, we study the consensus problem of multi-agent systems with both fixed and switching topologies. A hybrid consensus protocol is proposed to take into consideration of continuous-time communications among agents and delayed instant information exchanges on a sequence of discrete times. Then, a novel hybrid consensus protocol with dynamically changing interaction topologies is designed to take the time-delay into account in both the continuous-time communication among agents and the instant information exchange at discrete moments. We also study the consensus problem of networked multi-agent systems. Distributed delays are considered in both the agent dynamics and the proposed impulsive consensus protocols.

Lastly, stabilization and synchronization problems of DNs under pinning impulsive control are studied. A pinning algorithm is incorporated with the impulsive control method. We propose a delay-dependent pinning impulsive controller to investigate the synchronization of linear delay-free DNs on time scales. Then, we apply the pinning impulsive controller proposed for the delay-free networks to stabilize time-delay DNs. Results show that the delay-dependent pinning impulsive controller can successfully stabilize and synchronize DNs with/without time-delay. Moreover, we design a type of pinning impulsive controllers that relies only on the network states at history moments (not on the states at each impulsive instant). Sufficient conditions on stabilization of time-delay networks are obtained, and results show that the proposed pinning impulsive controller can effectively stabilize the network even though only time-delay states are available to the pinning controller at each impulsive instant. We further consider the pinning impulsive controllers with both discrete and distributed time-delay effects to synchronize the drive and response systems modeled by globally Lipschitz time-delay systems. As an extension study of pinning impulsive control approach, we investigate the synchronization problem of systems and networks governed by PDEs.

Acknowledgments

First and foremost, I would like to thank my supervisors, Prof. Xinzhi Liu and Prof. Wei-Chau Xie, for the guidance and support during my years at Waterloo. I would also like to thank my examining committee members, Prof. Sue Ann Campbell, Prof. Albert C.J. Luo, Prof. Kirsten Morris and Prof. Sherman Shen, for their time and invaluable feedback.

I would like to extend a thank you to my friends and members of my research group that I have met over the years for all the fun we have had, and for the useful feedback and insightful comments on my work.

Dedication

This thesis is dedicated to my parents, wife and daughter

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Chapter 1

Introduction

1.1 Motivations

Dynamical networks (DNs) consist of a large set of interconnected nodes with each node being a fundamental unit with detailed contents (see, e.g., [105, 114, 13, 96]). A great number of natural and man-made complex networks such as social networks, food networks, neural networks, World Wide Web, computer networks, electrical power grid, etc., can be effectively modeled by DNs. See the following two examples for illustrations.

Example 1.1.1 *(**Internet**[114]) Internet is a dynamical network of routers or domains which are called nodes of the network connected by physical links such as optical fibers. See Figure 1.1.*

Example 1.1.2 *(**Birds flocking**) Flocking is a collective animal behavior exhibited by many living beings such as birds, fish and bacteria [94]. Figure 1.2 shows the flocking habit of migratory birds. Each bird in Figure 1.2 behaves autonomously, and the flocking behavior of a group of birds can be modeled by a dynamical network.*

As illustrated in the above examples, DNs consist of three main attributes [26]:

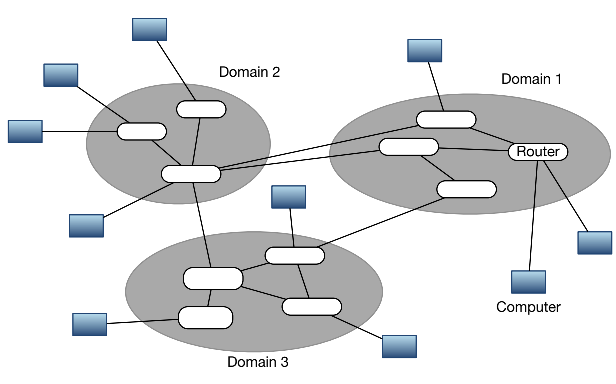


Figure 1.1: Network structures of Internet

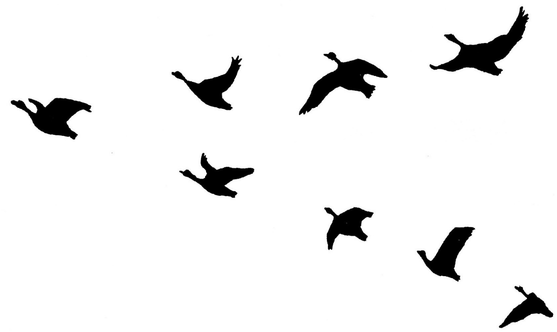


Figure 1.2: Flocking habit of migratory birds

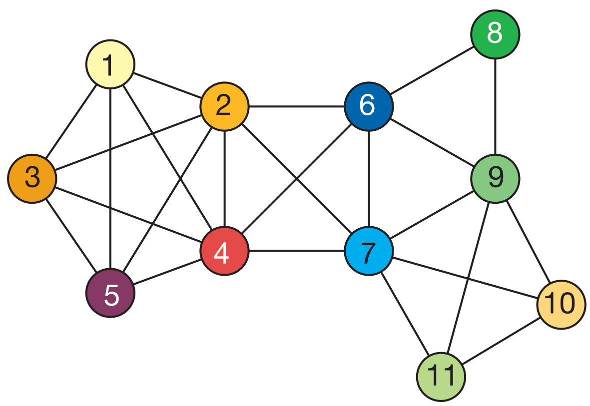


Figure 1.3: Structure of a DN

- a mathematical description of the dynamical behavior of each of the agents in the network;
- an interaction (or coupling) protocol used by agents to communicate with each other;
- a graph describing the network of interconnections between neighboring agents.

As an example, the structure of a DN is shown in Figure 1.3. Then, possible complications arise according to these three aspects, as described by Strogatz [105]:

- Node
    - Dynamical complexity: the nodes could be nonlinear dynamical systems. In a gene network or a Josephson junction array, the state of each node can vary in time in complicated ways.
    - Node diversity: there could be many different kinds of nodes. The biochemical network that controls cell division in mammals consists of a bewildering variety of substrates and enzymes.

- Interaction
  - Structural complexity: the wiring diagram could be an intricate tangle.
  - Network evolution: the wiring diagram could change over time. On WWW, pages and links are created and lost every minute.
- Graph
  - Connection diversity: the links between nodes could have different weights, directions and signs. Synapses in the nervous system can be strong or weak, inhibitory or excitatory.

Moreover, various complications can influence each other, which is called Meta-complication. All these complications bring difficulties to the study of control problems of DNs.

Due to the common existence in many evolution processes, control schemes, and physical systems, time-delay has been widely considered for dynamical systems. It was shown that the existence of time delay may cause divergence, oscillation, instability, and poor performance in various dynamical systems. Therefore, it is reasonable to study dynamical systems subject to time-delay. During the past decades, there has been extensive investigation on stability issues and control problems of delayed dynamical systems. In terms of DNs, time-delay might present in the intrinsic dynamic of the single node which means the dynamic of the isolated node is past dependent. The time lag could also exist at the communications or interactions among nodes. These time delays in DNs also increase the difficulties to investigate the dynamic properties of DNs.

Various control problems about DNs have arisen recently which will be introduced in Chapter 2, and a wide variety of conventional and novel control schemes have been proposed to achieve the desired control objectives, such as, the adaptive control (e.g., [131, 30]), pinning control (e.g., [79, 27]), impulsive control (e.g., [73, 100]), and hybrid control (e.g., [49, 62]), etc. Among these approaches, the impulsive control method appears to be an effective method to achieve the control goal of DNs. The main idea of this method is to control the states of the system by using only small impulses which are samples of the state variables at discrete moments. And then the impulsive control approach has its advantages in the following two aspects described in [123]:

- Based on the control mechanism, impulsive control can give an efficient way to deal with plants.
- In some applications, it is impossible to provide continuous control inputs. For example, a government can not change savings rates of its central bank everyday. A deep-space spacecraft can not leave its engine on continuously if it has only limited fuel supply.

On the other hand, time-delay is unavoidable in sampling and transmission of the impulsive information in dynamical systems. As one type of typical time-delay, distributed delays have been widely employed in biological and industrial systems to describe delays in the spread of disease, network connections, transportation, etc. However, it is practically needed to consider distributed delays when applying the impulsive controller to dynamical systems. For example, a deep-spaced spacecraft cannot leave its engine on continuously if it has only limited fuel supply, and then impulsive control is an ideal way to manipulate the spacecraft. Fuel economy is the fuel efficiency relationship between the distance traveled and the amount of fuel consumed by the vehicle, which is normally expressed in terms of volume of fuel to travel a unit distance, or the distance traveled to consume a unit volume of fuel. It is an important aspect which needs to be considered in the design of impulsive controller. Compared with the fuel economy at a specific time t or a history moment $t - d$ (instantaneous fuel economy), the fuel economy over the time period $[t - d, t]$ is a more accurate, accessible, and reliable data, which is directly related to the fuel economy of spacecraft's engine, and can be easily derived from central control system of the spacecraft. Therefore, when considering the fuel economy in the impulsive controller for the deep-spaced spacecraft, distributed delays are employed to describe the fuel consumptions over a particular time period.

Due to the wide applications of DNs, effectiveness of the impulsive control method, and ubiquitousness of delay effects, it is necessary and significant to study impulsive control problems of DNs with time-delay. The objective of this thesis is to construct sufficient conditions to design impulsive controllers to stabilize DNs with delays and furthermore to achieve the desired dynamical performance.

1.2 Thesis Organization

This thesis is organized into 7 chapters, which are listed and summarized as follows.

Chapter 1. Introduction

A background introduction of this thesis.

Chapter 2. Control Problems of Dynamical Networks

Chapter 2 will introduce the method of impulsive control to DNs, and control problems related to DNs, such as stabilization, synchronization, and consensus.

Chapter 3. Stability of Impulsive Systems with Time-Delay

In Chapter 3, we will introduce the mathematical background of impulsive systems with time-delay. We will introduce several global exponential stability results for time-delay systems with delayed impulses, by using the method of Lyapunov functionals and Razumikhin technique. We will also introduce an exponential stability result for locally Lipschitz time-delay systems with distributed-delay dependent impulses. Results in Section 3.2 were published in [84], and results in Section 3.3 will be submitted for publication.

Chapter 4. Consensus of Multi-Agent Systems

In Chapter 4, we will introduce consensus results of multi-agent systems via hybrid protocols with impulse delays, in addition to briefly introducing notions and results from matrix theory and graph theory. We will introduce consensus results of multi-agent systems via hybrid impulsive protocols with dynamically changing topologies and time-delay. We will also introduce results for consensus of networked multi-agent systems with distributed delays in both the agent dynamics and the impulsive protocols. Results in Section 4.2 were published in [86], results in Section 4.3 will be submitted, and results in Section 4.4 have been submitted for publication (see, [88]).

Chapter 5. Stabilization and Synchronization of Dynamical Networks

Chapter 5 will study synchronization of delay-free DNs and stabilization of neural networks with time-delay, and discrete delays will be considered in the pinning impulsive control approach, in addition to introducing the theory of time scales. We will also investigate synchronization problem of globally Lipschitz time-delay systems, and the distributed-delay effects will be considered in the impulsive controller. Results in Section 5.1, Subsection 5.2.1, and Subsection 5.2.2 were published in [82], [85], and [84], respectively. Results in Section 5.3 will be submitted for publication.

Chapter 6. Applications to Systems and Networks Governed by PDEs

In Chapter 6, we will apply the pinning impulsive control approach, introduced in Chapter 5, to control problems of systems and networks governed by partial differential equations. We will introduce pinning impulsive stabilization and synchronization of Gray-Scott model, which is a delay-free PDE system. We will also introduce pinning impulsive synchronization of neural networks with reaction-diffusion terms, which is a time-delay PDE model. Results in Section 6.1 were published in [83], and results in Section 6.2 were published in [87].

Chapter 7. Conclusions and Future Research

In this chapter, we will summarize the results present in the thesis, and highlight the contributions of this thesis. We will also discuss some future research directions along the line of this thesis.

1.3 Notation

The notation in this thesis is more or less standard, with a few exceptions. In this section, we describe the mostly commonly used notation in the thesis.

Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{R}$ the set of real numbers, $\mathbb{R}^+$ the set of nonnegative real numbers, and $\mathbb{R}^n$ the n-dimensional real space equipped with the Euclidean norm $\|\cdot\|$. $C^m(W)$ represents the set of continuous m-time differentiable real-valued functions on the domain W. $\#\mathcal{G}$ denotes the cardinality of set $\mathcal{G}$ (that is, the number of elements in set $\mathcal{G}$ if it is finite). For any matrix $A \in \mathbb{R}^{n\times n}$, let A^T denote the transpose of A, $\lambda_{max}(A)$ the largest eigenvalue of A, and $\|A\| = \sqrt{\lambda_{max}(A^T A)}$, i.e., the norm of A induced by the Euclidean norm. Denote $I \in \mathbb{R}^{n\times n}$ the $n \times n$ identity matrix.

For $a, b \in \mathbb{R}$ with $a < b$ and $S \subseteq \mathbb{R}^n$, we define

$$\begin{aligned}
\mathcal{PC}([a,b],S) &= \Big\{\psi : [a,b] \to S \Big| \psi(t) = \psi(t^+),\ \text{for any } t \in [a,b);\ \psi(t^-) \\
&\qquad \text{exists in } S,\ \text{for any } t \in (a,b];\ \psi(t^-) = \psi(t) \text{ for all but} \\
&\qquad \text{at most a finite number of points } t \in (a,b]\Big\}, \\
\mathcal{PC}([a,\infty),S) &= \Big\{\psi : [a,\infty) \to S \Big| \text{ for any } c > a, \psi|_{[a,c]} \in \mathcal{PC}([a,c],S)\Big\},
\end{aligned}$$

where $\psi(t^+)$ and $\psi(t^-)$ denote the right and left limit of function ψ at t, respectively. For a given constant $\tau > 0$, the linear space $\mathcal{PC}([-\tau,0],\mathbb{R}^n)$ is equipped with the norm defined by

$||\psi||_\tau = \sup_{s\in[-\tau,0]} ||\psi(s)||$, for $\psi \in \mathcal{PC}([-\tau, 0], \mathbb{R}^n)$. For constant $\rho > 0$, define $\mathcal{B}(\rho) = \{x \in \mathbb{R}^n \mid \|x\| \leq \rho\}$.

Let $\delta(\cdot)$ denote the **Dirac delta function** which is defined as a generalized function on the real line which is zero everywhere except at the origin, where it is infinite,

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

and which is also constrained to satisfy the identity

$$\int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1.$$

The **floor function** $\lfloor \chi \rfloor$ gives the largest integer less than χ.

Other symbols will be introduced in the thesis when needed. For example, the notations related to graph theory will be introduced in Section 4.1, and symbols related to the theory of time scales will be introduced in Subsection 5.1.1.

Chapter 2

Control Problems of Dynamical Networks

2.1 Network Model

Based on the introduction of DNs in Chapter 1, we will introduce the system model that will be investigated. The evolution of the nodes' states for the time-delay DN with N nodes can be described as follows:

$$\dot{x}_i = g_i(t, x_{it}) + \psi_i(t, \mathbf{x}_t), \;\; i = 1, 2, ..., N, \tag{2.1}$$

where $x_i \in \mathbb{R}^m$ denotes the states of the ith node, while x_{it} is defined as $x_{it}(s) = x_i(t+s)$, for $s \in [-\tau, 0]$ where $\tau > 0$ represents the time-delay in the DN and $\mathbf{x}_t = (x_{1t}^T, x_{2t}^T, ..., x_{Nt}^T)^T$. The function $g_i : \mathbb{R}^+ \times \mathcal{PC}([-\tau, 0], \mathbb{R}^m) \to \mathbb{R}^m$ denotes the intrinsic dynamics of the ith node, and $\psi_i : \mathbb{R}^+ \times \mathcal{PC}([-\tau, 0], \mathbb{R}^{mN}) \to \mathbb{R}^m$ describes the interactions of the ith node with other nodes.

Remark 2.1.1 *Various DNs can be written in the form of (2.1), e.g., BAM neural network [60], genetic regulatory network [17], Cohen-Grossberg neural network [127], Hopfield neural network [112], cellular neural network [24]. Therefore, results obtained from our research could be applied to control problems of these networks.*

If let $x = (x_1^T, x_2^T, ..., x_N^T)^T \in \mathbb{R}^{mN}$ and define a function $f : \mathbb{R}^+ \times \mathcal{PC}([-\tau, 0], \mathbb{R}^{mN}) \to \mathbb{R}^{mN}$ by $f = (g_1^T + \psi_1^T, g_2^T + \psi_2^T, ..., g_N^T + \psi_N^T)^T$. Then, the system (2.1) can be rewritten as the general form of functional differential equations:

$$\dot{x} = f(t, x_t), \tag{2.2}$$

where x_t is defined by $x_t(s) = x(t+s)$ for $s \in [-\tau, 0]$.

2.2 Impulsive Control Method

Consider the nonlinear system with time-delay

$$\begin{cases} \dot{x} = f(t, x_t), \\ y(t) = \psi(x(t)), \\ x_{t_0} = \phi, \end{cases} \tag{2.3}$$

where $f : \mathbb{R}^+ \times \mathcal{PC}([-\tau, 0], \mathbb{R}^N) \to \mathbb{R}^N$ and $\psi : \mathbb{R}^N \to \mathbb{R}^N$, $x \in \mathbb{R}^N$ is the state vector, x_t is a function defined by $x_t(s) = x(t+s)$ for $s \in [-\tau, 0]$, and τ represents the time delay in system (2.3). $y \in \mathbb{R}^l$ is the measured output vector, $\phi \in \mathcal{PC}([-\tau, 0], \mathbb{R}^N)$ is the initial function.

An impulsive control law of system (2.3) is given by a sequence $\{t_k, U_k(t_k, y(t_k))\}$, where

$$0 \le t_0 < t_1 < ... < t_k < ..., \quad \lim_{k\to\infty} t_k = \infty,$$

and $U_k : \mathbb{R}^+ \times \mathbb{R}^l \to \mathbb{R}^N$ denotes the control input at each impulsive instant t_k, $k = 1, 2, ...$ It works as follows. Let $x(t) = x(t, t_0, \phi)$ be a solution of system (2.3). The point $P_t(t, x(t))$ begins its motion from the initial point $P_{t_0}(t_0, x(t_0))$ with $x(t_0) = \phi(t_0)$ and moves along the state trajectory $\{(t, x(t)) : t \ge t_0 \text{ and } x(t) = x(t, t_0, \phi)\}$ until the time t_1 at which the point $P_{t_1}(t_1, x(t_1))$ is mapped into $P_{t_1}(t_1, x(t_1^+))$ immediately, where $x(t_1^+) = x(t_1) + U_1(t_1, y(t_1))$. Then the motion P_t continues to move further along the trajectory set $\{(t, x(t)) : t \ge t_1$ and $x(t) = x(t; t_1, x_{t_1})$ where $x_{t_1}(s) = x(t_1 + s)$ for $s \in [-\tau, 0]\}$ until the time t_2 when the point $P_{t_2}(t_2, x(t_2))$ is transferred to $P_{t_2}(t_2, x(t_2^+))$, where $x(t_2^+) = x(t_2) + U_2(t_2, y(t_2))$. This process continues as long as the solution of system (2.3) with initial condition $x_{t_k}(s) = x(t_k + s)$, $s \in [-\tau, 0]$ exists. According to the above control mechanism, the impulsive controller can be written in the following form of feedback controller:

$$u(t, y) = \sum_{k=1}^{\infty} U_k(t, y(t))\delta(t - t_k), \tag{2.4}$$

where $\delta(\cdot)$ is the Dirac delta function, then, the impulsive controlled system (2.3):

$$\begin{cases} \dot{x} = f(t, x_t) + u(t, y), \\ y(t) = \psi(x(t)), \\ x_{t_0} = \phi, \end{cases} \tag{2.5}$$

is in the form of an impulsive system

$$\begin{cases} \dot{x} = f(t, x_t), \ \ t \ne t_k, \\ \Delta x(t_k) = U_k(t_k, y(t_k)), \ \ k \in \mathbb{N}, \\ y(t) = \psi(x(t)), \\ x_{t_0} = \phi, \end{cases} \tag{2.6}$$

Where $\Delta x(t) = x(t^+) - x(t^-)$. See Subsection 4.4.1 for an example of getting (2.6) from (2.5) by using the property of the Dirac delta function. If we let $h_k(t, x) = U_k(t, \psi(x))$, then system (2.6) can be written as the following impulsive system

$$\begin{cases} \dot{x} = f(t, x_t), \ \ t \ne t_k, \\ \Delta x(t_k) = h_k(t_k, x(t_k)), \ \ k \in \mathbb{N}, \\ x_{t_0} = \phi. \end{cases} \tag{2.7}$$

It can be seen that no time-delay is considered in the impulses in (2.7). Similarly, if (2.4) is a delay-dependent feedback controller (i.e., $u(t, y) = \sum_{k=1}^{\infty} U_k(t, y(t-\tau))\delta(t - t_k)$, where τ is the impulse delay), then the impulsive control system can be described by the general form of impulsive functional differential equations:

$$\begin{cases} \dot{x} = f(t, x_t), \ \ t \ne t_k, \\ \Delta x(t) = I_k(t, x_t), \ \ t = t_k, \text{ and } k \in \mathbb{N}, \\ x_{t_0} = \phi, \end{cases} \tag{2.8}$$

where $I_k : \mathbb{R}^+ \times \mathcal{PC}([-\tau, 0], \mathbb{R}^N) \to \mathbb{R}^N$. Fundamental results about (2.8) will be introduced in Section 3.1. However, system (3.1) considered in Section 3.1 is slightly different from (2.8), and see Remark 3.1.1 for discussions. Lyapunov stability of the system (2.8) can be defined similarly to the definitions for nonlinear systems in [52]. Now we are in the position to state the impulsive stabilization problem: subject to the time-delay system (2.3), find an impulsive control law $\{t_k, U_k\}$ such that the trivial solution of the impulsive delay system (2.8) is stable.

Various stability criteria have been established by employing different kinds of methods, such as comparison principle [71], Lyapunov functional method [75] and Razumikhin technique [116]. The Lyapunov functional method and the Lyapunov-Razumikhin technique are two commonly used approaches to determine sufficient conditions for stability of delay systems. In terms of the Lyapunov functional method, it is a natural generalization of the second method of Lyapunov for ordinary differential equations. Compared with the Lyapunov functional method, the Lyapunov-Razumikhin technique is based on utilizing the changing rate of a function on $\mathbb{R}^N$ to investigate stabilities of delay systems. For many delay systems, it appears to be easier to use Razumikhin-type results to establish sufficient conditions for stability than to construct appropriate Lyapunov functionals [39]. It can been seen from [20] that, even for linear impulsive delay systems, the Lyapunov functionals are quite complicated. Hence, the construction of suitable Lyapunov functionals for large-scale dynamical networks will be more challenging. However, the well-known Razumikhin technique is based on Lyapunov functions, and doesn't require Lyapunov function to be decreasing on the whole state space. Furthermore, when dealing with impulsive stabilization, it is straightforward for the impulse to bring down the value of a Lyapunov function, whereas, the impulse cannot cut down the value of a functional instantaneously. These properties of Razumikhin technique sometimes make choosing a suitable Lyapunov function easier than constructing an appropriate Lyapunov functional. Therefore, it is worthwhile to investigate the stability of time-delay systems using Razumikhin technique.

Since the method of Lyapunov functionals is a natural generalization of the second method of Lyapunov for systems without time-delay, the reasoning of sufficient conditions based on the Lyapunov functional method is normally easier than the reasoning of stability criteria by the Lyapunov-Razumikhin technique. Furthermore, since the Lyapunov-Razumikhin technique can be considered as a particular case of the Lyapunov functional method ([69] and [55], Section 4.8, p.254), the approach of Lyapunov functionals is ordinarily more general than the Lyapunov-Razumikhin technique. However, sometimes it seems to be more difficult to construct suitable Lyapunov functionals for stability than using Lyapunov-Razumikhin technique to establish appropriate Lyapunov functions. Moreover, as far as impulsive effects are concerned, the Lyapunov functional method is more challenging due to the fact that an impulse occurs at a discrete time normally can not bring down the value of a functional.

On the other hand, one of the most significant problems in the stability analysis of dynamical systems is exponential stability. Since it has a more stringent requirement on convergence rate, the exponential stability criteria are more important than the general stability or asymptotic stability criteria in some practical applications, such as synchronization of dynamical networks [68] and stabilization of cellular neural networks [117]. Recently, many control problems of dynamical systems have been investigated via delayed impulses in recent years, such as stabilization of stochastic functional systems [18, 119], synchronization of dynamical networks [135], and stability analysis of nonlinear impulsive and switched time-delay systems with delayed impulses [37]. Though the study of dynamical systems subject to impulses with time-delay has drawn increasing research attention (see, e.g., [25, 54]), almost all the existing works focused on impulses with discrete delays. To our best knowledge, no work about impulses with distributed delays has been reported, and the method of Lyapunov functionals has not been applied to stability analysis of systems with delayed impulses. In Chapter 3, we will establish several exponential stability results for system (2.8) with delay-dependent impulses, by using the method of Lyapunov functionals and Razumikhin technique.

2.3 Stabilization

Stability is one of the essential properties of a dynamical system. Consider a power network with N identical or different nodes, each of which can generate power and consume power. Ideally, the power network is required to supply steady power which means the power network is designed to be stable. Actually, faults might happen in the power system. For instance, a fault in two power lines in Oregon led to blackouts in 11 US states and 2 Canadian provinces, leaving about 7 million customers without power for up to 16 hours on August 10, 1996 [105]. Since the fault could lead the network to be unstable, it is vital important to apply suitable controllers to stabilize the power network when it suffers unpredicted faults.

In terms of the mathematical model (2.1), stability of a DN can be defined similarly to the stability of a dynamical system. Recently, stability of DNs has received lots of research interests, and numerous stability results have been reported for various types of networks (see, e.g., [97, 28, 48]). Actually, stability of a DN can be understood as synchronization of the DN with its trivial state, or the consensus of the DN. Synchronization and consensus of the DN will be introduced in the following two sections, most of the the results discussed in which can be applied to the stability analysis of the corresponding networks.

2.4 Synchronization

Synchronization of a group of dynamical nodes in a complex network topology is one of the most interesting and significant collective behaviors in DNs (see, e.g., [45, 65, 66, 67, 68, 80, 91, 108, 109, 125, 134, 136]). Recently, synchronization has a wide application in the secure communication between agents of a DN. In terms of secure communication, an illustrative process is shown in Figure 2.1, which works as follows. At the transmitter, the plain-text is masked by certain algorithm and then the cipher-text is sent to the receiver. Because of the unknown masking algorithm, it is very difficult for the eavesdropper to distinguish the cipher-text from noise and get the original text. Once the receiver gets the masked text, the shared secret key can be used to recover the original message. Chaos systems are often applied to encrypt and decrypt the text because of its unpredicted behaviors and its sensitivity to the initial conditions. The initial condition can be used as the shared secret key in the secure communication. When the transmitter and the receiver synchronize with each other, the receiver can recover the cipher-text to get the plain-text. When the secure communication is applied to the agents in a network, the synchronization of all the nodes will guarantee that all the nodes can get the same original message.

Now we are in the position to give the formal definition of synchronization in a DN.

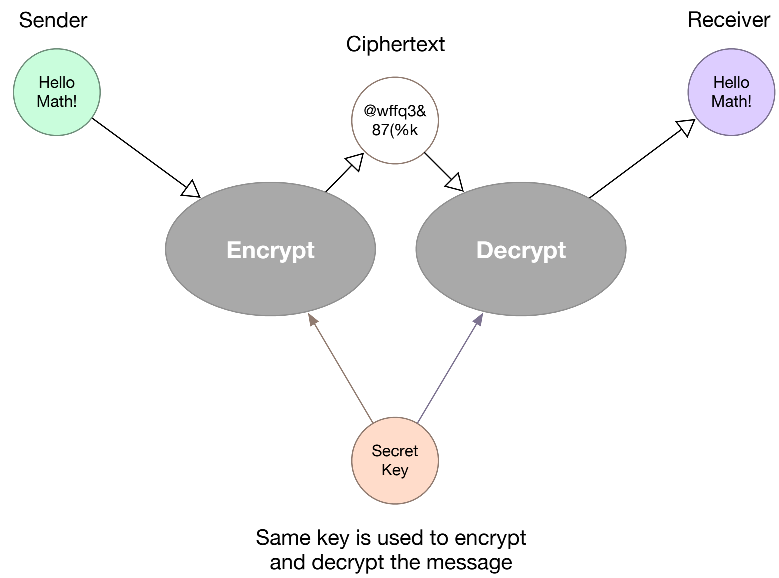


Figure 2.1: Symmetric-key cryptography

Definition 2.4.1 *DN (2.1) is said to achieve synchronization if*

$$\lim_{t\to\infty} ||x_i(t) - x_j(t)|| = 0, \quad i, j = 1, 2, ..., N.$$

Remark 2.4.1 *Definition 2.4.1 is the most intuitive one of synchronization which is normally called complete synchronization. Other forms of synchronization are possible, e.g., exponential synchronization (e.g., Definition 6.2.1), cluster synchronization [21], lag synchronization [35], and phase synchronization [2].*

It can be seen that the synchronization behavior of a DN is closely related to the dynamics of the individual nodes and the interconnections among them. A possible solution to this control problem is to add a control input to each of the network nodes. Due to the advantages of the impulsive control method, this approach has been successfully used for the synchronization of DNs (see, e.g., [68, 49]). In [68], impulsive synchronization of the general form of DNs has been studied. By linearization method, local synchronization criteria of impulsive synchronization were established, and then additional conditions on the dynamics of the isolated nodes were assumed to generalize these results to get the global synchronization criteria. Based on this innovative work, the impulsive synchronization approach has been applied to various networks, e.g., TS fuzzy DNs [128], switched neural networks [126], and cellular neural networks [29].

In the above mentioned results, appropriate Lyapunov functions or functionals were utilized to analyze the synchronization problems. The construction process of sufficient conditions for synchronization is as follows. The conditions for the Lyapunov candidates on the impulsive intervals are obtained, while the conditions on the Lyapunov candidates on the impulsive instants are established. Then conditions that connect the restrictions on these two sufficient conditions will be needed to balance the dynamics of the network and the control effects of the impulsive controllers at each impulsive instant. According to these three types of conditions, suitable impulsive controller can be designed to achieve synchronization of DNs.

The traditional method to synchronize a DN is to add a controller to each of the network nodes to tame the node dynamics to approach a desired synchronization trajectory. However, a DN is normally composed of a large number of high dimensional nodes, and it is expensive and infeasible to control all of them. Motivated by this practical consideration, the idea of controlling a small portion of nodes, named pinning control, was introduced in [113, 76], and many pinning algorithms have been reported for synchronization of DNs (see, e.g., [129, 115, 78, 107, 61]). Obviously, the pinning control method reduces the control cost to a certain extent by reducing the amount of controllers added to the nodes. It is worth noting that the cost of control can be further reduced by combining pinning control and impulsive control, i.e., adding the impulsive controllers to a small fraction of network nodes. Hence, the notion of pinning impulsive control has stimulated many interesting pinning impulsive control strategies for synchronization of DNs with and without delays (see, e.g., [67, 91, 46, 136]). To our best knowledge, no time-delay effects have been considered in pinning impulsive synchronization problems. In Chapter 5, we will discussion this type of stabilization and synchronization problems.

2.5 Consensus

Distributed coordination of multi-agent systems (networks of agents or dynamical systems) has been studied extensively due to its wide range of applications in many areas, such as spacecraft formation flying [15], multiple robot coordination [103], flocking [110], cooperative control of vehicle formations [31], and so on. As one of the basic collective behaviors, consensus problems naturally arise when a group of networked agents are seeking an agreement according to a certain quantity interest that depends on the state of all agents.

In recent years, various consensus algorithms have been proposed for the multi-agent systems (see, e.g., [33, 36, 42, 50, 63, 64, 90, 95, 98, 101, 120, 121]). The typical result about average consensus of multi-agent systems with fixed topology was provided by Olfati-Saber and Murray in [95], which has shown that if the interaction topology is strongly connected and balanced, then the consensus problem can be solved asymptotically. In [98], Ren and Beard constructed classical consensus criteria under dynamically changing topologies, and proved that if the union of the directed interaction graphs contains a spanning tree frequently enough, then the consensus can be achieved asymptotically. These verifiable conditions have established close relations between the connectivity of the interaction topologies and consensus behavior of the multi-agent systems. However, when the state of agents is subject to abrupt changes or instantaneous information exchanges with their neighbours (see, e.g., [72]), these typical results cannot be applied directly to this type of consensus problems. Therefore, the impulsive consensus method has been developed, and recently has attracted many researchers' interests (see, e.g., [36, 121, 33, 64, 42]).

The mechanism of impulsive consensus is based on the strategy of impulsive control which is to control the state of each agent by using only small samples of the state variables of the multi-agent system at a sequence of discrete moments. Up to now, much attention has been paid to the higher order multi-agent systems (see, e.g., [33, 64, 42]). Although there are many interesting results reported for the impulsive consensus of first-order multi-agent systems, this research area still worths further investigation, and many existing consensus results can be improved. In [36], a fundamental result has been derived for the impulsive consensus, which says that the graphs of continuous-time and impulsive-time topologies containing a spanning tree respectively is necessary to guarantee the consensus behaviour. However, inspired by the results in [98], the consensus criteria in [36] may require more graph connections among agents to achieve the consensus, and the recent results in [90] have shown that a spanning tree in the union of continuous-time and discrete-time topologies can guarantee the consensus of the multi-agent system. But, no delay has been considered in those impulsive-time topologies. To our best knowledge, very few work has been done on the consensus problem of multi-agent systems with delayed impulses. In [120], a hybrid impulsive consensus protocol is proposed to achieve the network consensus. However, the continuous-time and the discrete-time topologies are assumed to be the same, and also are required to share the same communication delay among agents. Furthermore, distributed delays have not been considered in the impulsive consensus protocols. In Chapter 4, we will focus on these consensus protocols.

Chapter 3

Stability of Impulsive Systems with Time-Delay

This chapter studies exponential stability of general nonlinear time-delay systems with delayed impulsive effects. Stability results are constructed by using the method of Lyapunov functionals and Razumikhin technique, respectively. Some results will be used in Chapters 4 and 5.

3.1 Impulsive Systems with Time-Delay

Consider the following impulsive system

$$\begin{cases} \dot{x}(t) = f(t, x_t), \ \ t \neq t_k, \\ \Delta x(t) = I_k(t, x_{t^-}), \ \ t = t_k, \ k \in \mathbb{N}, \\ x_{t_0} = \phi, \end{cases} \tag{3.1}$$

where $f, I_k : \mathbb{R}^+ \times \mathcal{PC}([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$, $0 \leq t_0 < t_1 < ... < t_k < ...$ with $\lim_{k\to\infty} t_k = \infty$, $\Delta x(t) = x(t^+) - x(t^-)$. Here, we assume $x(t)$ is right-continuous at each t_k, i.e., $x(t_k^+) = x(t_k)$. $x_t, x_{t^-} \in \mathcal{PC}([-\tau, 0], \mathbb{R}^n)$ are defined as $x_t(s) = x(t+s)$, $x_{t^-}(s) = x(t^- + s)$ for $s \in [-\tau, 0]$, respectively. The function $\phi \in \mathcal{PC}([-\tau, 0], \mathbb{R}^n)$ is the initial condition of the system.

Definition 3.1.1 *A function $x \in \mathcal{PC}([t_0 - \tau, t_0 + \alpha], \mathcal{D})$ where $\alpha > 0$ and $\mathcal{D} \subseteq \mathbb{R}^n$ is said to be a solution of (3.1) if*

(i) x is continuous at each $t \neq t_k$ in $(t_0, t_0 + \alpha]$;

(ii) the derivative of x exists and is continuous at all but at most a finite number of points t in $(t_0, t_0 + \alpha)$;

(iii) the right-hand derivative of x exists and satisfies the delay differential equation in (3.1) for all $t \in [t_0, t_0 + \alpha)$;

(iv) x satisfies the delay difference equation in (3.1) at each $t_k \in (t_0, t_0 + \alpha]$;

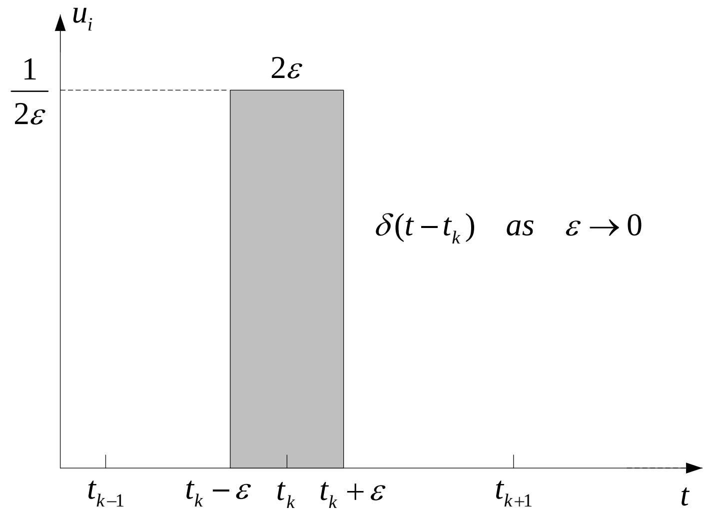


Figure 3.1: Modeling the control input with impulsive control

(v) x satisfies the initial condition in (3.1).

Remark 3.1.1 *Many evolution processes are subject to short-term perturbations whose duration is negligible in comparison with the duration of the precess (see, [58] and Figure 3.1 for demonstration). Impulsive differential equations are powerful tools to model this type of evolution processes. The term $\Delta x(t_k)$ in (3.1) characterizes the state jump in the process. $x(t_k^+)$ and $x(t_k^-)$ represent the state after and before the jump, respectively. In the literature of impulsive time-delay systems, system state $x(t)$ is commonly assumed to be right-continuous at each impulsive instant, i.e., $x(t_k^+) = x(t_k)$. One advantage of assuming right-continuous solutions is that the entire initial condition can be incorporated into the single function ϕ and we do not need to separately include a extra initial condition specifying the right limit of the solution at the initial time, i.e., $x(t_0^+)$ (see, [6]). Therefore, when considering the impulse effects as control inputs, we use system (3.1) to model the impulsive control system with time-delay. To make the impulsive control system (2.6) to be well-defined according to the fundamental theory of (3.1) introduced in this chapter, we may transform system (2.6) to the form of (3.1) by rewriting the impulsive controller (2.4) as follows:*

$$u(t, y) = \sum_{k=1}^{\infty} U_k(t, y(t))\delta(t - t_k^-). \tag{3.2}$$

See controllers (5.18) and (6.19) for examples.

In this chapter, it is assumed that $f(t, 0) \equiv 0$ and $I_k(t, 0) \equiv 0$ for all $t \geq t_0$ and $k \in \mathbb{N}$, then system (3.1) admits the trivial solution. Furthermore, we make the following assumptions on system (3.1).

(H_1) For each fixed $t \in \mathbb{R}^+$, $f(t, \psi)$ is a continuous function of ψ on $\mathcal{PC}([-\tau, 0], \mathcal{D})$.

(H_2) f is composite-PC, i.e., for each $t_0 \in \mathbb{R}^+$ and $\sigma > 0$, if $x \in \mathcal{PC}([t_o - \tau, t_0 + \sigma], \mathcal{D})$ and x is continuous at each $t \neq t_k$ in $(t_0, t_0 + \sigma]$, then the composite function h defined by $h(t) := f(t, x_t)$ is an element of the function set $\mathcal{PC}([t_0, t_0 + \sigma], \mathbb{R}^n)$.

(H_3) $f(t,\psi)$ is quasi-bounded on $\mathbb{R}^+ \times \mathcal{PC}([-\tau,0],\mathcal{D})$, i.e., for each $t_0 \in \mathbb{R}^+$ and $\sigma > 0$, and for each compact set $\mathcal{F} \subset \mathcal{D}$, there exists some $M > 0$ such that $|f(t,\psi)| \le M$ for all $(t_0,\psi) \in [t_0, t_0+\sigma] \times \mathcal{PC}([-\tau,0],\mathcal{F})$.

It is shown in [14] that under assumptions (H_1), (H_2) and (H_3), for any initial condition $\phi \in \mathcal{PC}([-\tau,0],\mathcal{D})$, system (3.1) admits a solution $x(t) := x(t,t_0,\phi)$ that exists on a maximal interval $[t_0-\tau, t_0+T)$ where $0 < T \le \infty$.

Before introducing the stability results, we will list the definition of exponential stability for system (3.1) and definitions related to the Lyapunov function and functional.

Definition 3.1.2 *The trivial solution of system (3.1) is said to be exponentially stable (ES), if there exist positive constants ρ_0, M and α such that*

$$\|x(t)\| \le M\|\phi\|_\tau e^{-\alpha(t-t_0)}, \quad t \ge t_0, \tag{3.3}$$

for any $\phi \in \mathcal{PC}([-\tau,0],\mathcal{B}(\rho_0))$. Furthermore, if (3.3) holds for any $\phi \in \mathcal{PC}([-\tau,0],\mathbb{R}^n)$, then the trivial solution of (3.1) is said to be globally exponentially stable (GES).

Definition 3.1.3 *Function $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ is said to belong to the class ν_0 if the following is true:*

1) *V is continuous in each of the sets $[t_{k-1},t_k) \times \mathbb{R}^n$, and for each $x \in \mathbb{R}^n$, $t \in [t_{k-1},t_k)$, and $k \in \mathbb{N}$, $\lim_{(t,y)\to(t_k^-,x)} V(t,y) = V(t_k^-,x)$ exists;*

2) *$V(t,x)$ is locally Lipschitz in all $x \in \mathbb{R}^n$, and for all $t \ge t_0$, $V(t,0) \equiv 0$.*

Definition 3.1.4 *Given a function $V \in \nu_0$, the upper right-hand derivative $D^+V(t,\psi(0))$ along the solution of system (3.1) is defined by*

$$D^+V(t,\psi(0)) = \limsup_{h\to 0} \frac{1}{h}[V(t+h,\psi(0)+hf(t,\psi)) - V(t,\psi(0))],$$

where $(t,\psi) \in [t_0,\infty) \times \mathcal{PC}([-\tau,0],\mathbb{R}^n)$.

Definition 3.1.5 *A functional $V : \mathbb{R}^+ \times \mathcal{PC}([-\tau,0],\mathbb{R}^n) \to \mathbb{R}^+$ belongs to ν_0^* if*

1) *V is continuous on $[t_{k-1},t_k) \times \mathcal{PC}([-\tau,0],\mathbb{R}^n)$, and $\lim_{(t,\psi)\to(t_k^-,\phi)} V(t,\psi) = V(t,\phi)$ exists, for all $\psi,\phi \in \mathcal{PC}([-\tau,0],\mathbb{R}^n)$ and $k \in \mathbb{N}$;*

2) *$V(t,\psi)$ is locally Lipschitz in ψ on each compact set in $\mathcal{PC}([-\tau,0],\mathbb{R}^n)$, and $V(t,0) \equiv 0$, for all $t \ge t_0$;*

3) *for any $x \in \mathcal{PC}([t_0-\tau,\infty),\mathbb{R}^n)$, $V(t,x_t)$ is continuous for $t \ge t_0$.*

Definition 3.1.6 *Given a functional $V \in \nu_0^*$, the upper right-hand derivative $D^+V(t,\psi)$ along the solution of system (3.1) is defined by*

$$D^+V(t,\psi) = \limsup_{h\to 0^+} \frac{1}{h}[V(t+h,x_{t+h}(t,\psi)) - V(t,\psi)],$$

for $(t,\psi) \in \mathbb{R}^+ \times \mathcal{PC}([-\tau,0],\mathbb{R}^n)$.

For simplicity, we may use V' to represent the upper right-hand derivative D^+V in later sections.

3.2 The Method of Lyapunov Functionals

In this section, we shall study the global exponential stability of system (3.1). Stability results will be stated and proved by employing the Lyapunov functional method. Our results show that delayed impulses play an important role in stabilizing the nonlinear delay systems.

Theorem 3.2.1 *Assume that there exist $V_1 \in \nu_0$, $V_2 \in \nu_0^*$, and constants $p, c, w_1, w_2, w_3, \rho_1 > 0$ and $\rho_2 \geq 0$, such that*

(i) $w_1||x||^p \leq V_1(t,x) \leq w_2||x||^p$, $0 \leq V_2(t,\psi) \leq w_3||\psi||_\tau^p$, *for* $t \in \mathbb{R}^+$, $x \in \mathbb{R}^n$, *and* $\psi \in \mathcal{PC}([-\tau,0],\mathbb{R}^n)$;

(ii) for $V(t,\psi) = V_1(t,\psi(0)) + V_2(t,\psi)$,

$$D^+V(t,\psi) \leq cV(t,\psi),$$

for $t \in [t_{k-1},t_k)$, $\psi \in \mathcal{PC}([-\tau,0],\mathbb{R}^n)$, *and* $k \in \mathbb{N}$;

(iii) for $k \in \mathbb{N}$ *and* $\psi \in \mathcal{PC}([-\tau,0],\mathbb{R}^n)$,

$$V_1(t_k,\psi(0)+I_k(t_k,\psi)) \leq \rho_1 V_1(t_k^-,\psi(0)) + \rho_2 \sup_{s\in[-\tau,0]}\{V_1(t_k^-+s,\psi(s))\};$$

(iv) $\ln(\rho + \frac{w_3}{w_1}) < -c\delta$, *where* $\rho = \rho_1 + \rho_2$ *and* $\delta = \sup_{k\in\mathbb{N}}\{t_{k+1}-t_k\}$.

Then the trivial solution of system (3.1) is GES.

Proof. Let $x(t) = x(t,t_0,\phi)$ be any solution of system (3.1) with initial condition $x_{t_0} = \phi$. Let $v_1(t) = V_1(t,x(t))$, $v_2(t) = V_2(t,x_t)$, and $v(t) = v_1(t)+v_2(t)$. From condition (iv), there exists a constant $\alpha > 0$ such that

$$\ln(\rho_1 + \rho_2 e^{\alpha\tau} + \frac{w_3}{w_1}e^{\alpha\tau}) = -(\alpha+c)\delta. \tag{3.4}$$

From condition (ii), we have

$$v(t) \leq v(t_{k-1})e^{c(t-t_{k-1})}, \text{ for } t \in [t_{k-1},t_k),\ k \in \mathbb{N}. \tag{3.5}$$

Since $\lim_{k\to\infty} t_k = \infty$, there exists an integer $i \geq 1$ such that $t_i - \tau \geq t_0$, and for $t \in [t_0,t_i)$, we have

$$v(t) = v(t)e^{\alpha(t-t_0)}e^{-\alpha(t-t_0)} \leq Me^{-\alpha(t-t_0)}, \tag{3.6}$$

where $M = e^{\alpha(t_i-t_0)}\sup_{t\in[t_0,t_i)}\{v(t)\}$.

We shall show

$$v(t) \leq Me^{-(\alpha+c)(t_{k+1}-t_0)}e^{c(t-t_0)}, \text{ for } t \in [t_k,t_{k+1}),\ k \geq i. \tag{3.7}$$

When $k = i$, we can get from condition (iii) and (3.6) that

$$v_1(t_i) \quad \leq \quad \rho_1 v_1(t_i^-) + \rho_2 \sup_{s\in[-\tau,0]}\{v_1(t_i^-+s)\}$$

$$\begin{aligned}
&\leq \rho_1 v(t_i^-) + \rho_2 \sup_{s\in[-\tau,0]} \{v(t_i^- + s)\} \\
&\leq \rho_1 M e^{-\alpha(t_i - t_0)} + \rho_2 M e^{-\alpha(t_i - \tau - t_0)} \\
&= (\rho_1 + \rho_2 e^{\alpha\tau}) M e^{-\alpha(t_i - t_0)}. \qquad (3.8)
\end{aligned}$$

By condition (i), (3.6), and the continuity of v_2, we get

$$\begin{aligned}
v_2(t_i) = v_2(t_i^-) &\leq w_3 ||x_{t_i^-}||_\tau^p = w_3 \sup_{s\in[-\tau,0]} ||x(t_i^- + s)||^p \\
&\leq \frac{w_3}{w_1} \sup_{s\in[-\tau,0]} \{v_1(t_i^- + s)\} \\
&\leq \frac{w_3}{w_1} M e^{-\alpha(t_i - \tau - t_0)} \\
&= \frac{w_3}{w_1} e^{\alpha\tau} M e^{-\alpha(t_i - t_0)}. \qquad (3.9)
\end{aligned}$$

Then, (3.8) and (3.9) imply that

$$\begin{aligned}
v(t_i) &= v_1(t_i) + v_2(t_i) \\
&\leq (\rho_1 + \rho_2 e^{\alpha\tau} + \frac{w_3}{w_1} e^{\alpha\tau}) M e^{-\alpha(t_i - t_0)} \\
&= e^{-(\alpha+c)\delta} M e^{-\alpha(t_i - t_0)} \\
&\leq e^{-(\alpha+c)(t_{i+1} - t_i)} M e^{-(\alpha+c)(t_i - t_0)} e^{c(t_i - t_0)} \\
&= M e^{-(\alpha+c)(t_{i+1} - t_0)} e^{c(t_i - t_0)},
\end{aligned}$$

i.e., (3.7) is satisfied for $t = t_i$. For $t \in (t_i, t_{i+1})$, we have

$$\begin{aligned}
v(t) &\leq v(t_i) e^{c(t - t_i)} \\
&\leq M e^{-(\alpha+c)(t_{i+1} - t_0)} e^{c(t - t_0)}, \qquad (3.10)
\end{aligned}$$

which implies (3.7) holds for $t \in (t_i, t_{i+1})$. Hence, (3.7) is true for $k = i$.

Next, suppose (3.7) is true for $k \leq j (j > i)$, and we shall prove (3.7) holds for $k = j + 1$.

When $t = t_{j+1}$, we estimate the upper bound of $v(t_{j+1}^- + s)$ for $s \in [-\tau, 0]$ by considering the following two cases:

Case 1: $t_{j+1}^- + s < t_i$ for some $s \in [-\tau, 0]$.

Then, (3.6) implies that

$$\begin{aligned}
v(t_{j+1}^- + s) &\leq M e^{-\alpha(t_{j+1} + s - t_0)} \\
&\leq M e^{-\alpha(t_{j+1} - \tau - t_0)} \\
&= M e^{-(\alpha+c)(t_{j+1} - t_0)} e^{c(t_{j+1} - t_0)} e^{\alpha\tau}.
\end{aligned}$$

Case 2: $t_{j+1}^- + s \geq t_i$ for some $s \in [-\tau, 0]$.

Then, there exists an integer $\hat{k} \geq i$ such that $t_{j+1}^- + s \in [t_{\hat{k}}, t_{\hat{k}+1})$. From (3.7), we have

$$\begin{aligned} v(t_{j+1}^- + s) &\leq Me^{-(\alpha+c)(t_{\hat{k}+1}-t_0)}e^{c(t_{j+1}+s-t_0)} \\ &\leq Me^{-(\alpha+c)(t_{j+1}+s-t_0)}e^{c(t_{j+1}+s-t_0)} \\ &= Me^{-\alpha(t_{j+1}+s-t_0)} \\ &\leq Me^{-\alpha(t_{j+1}-\tau-t_0)} \\ &= Me^{-(\alpha+c)(t_{j+1}-t_0)}e^{c(t_{j+1}-t_0)}e^{\alpha\tau}. \end{aligned}$$

Therefore, for all $s \in [-\tau, 0]$,

$$v(t_{j+1}^- + s) \leq Me^{-(\alpha+c)(t_{j+1}-t_0)}e^{c(t_{j+1}-t_0)}e^{\alpha\tau}. \tag{3.11}$$

Then, we can obtain from condition (iii) and (3.11) that

$$\begin{aligned} v_1(t_{j+1}) &\leq \rho_1 v_1(t_{j+1}^-) + \rho_2 \sup_{s\in[-\tau,0]}\{v_1(t_{j+1}^- + s)\} \\ &\leq \rho_1 v(t_{j+1}^-) + \rho_2 \sup_{s\in[-\tau,0]}\{v(t_{j+1}^- + s)\} \\ &\leq \rho_1 Me^{-(\alpha+c)(t_{j+1}-t_0)}e^{c(t_{j+1}-t_0)} + \rho_2 Me^{-(\alpha+c)(t_{j+1}-t_0)}e^{c(t_{j+1}-t_0)}e^{\alpha\tau} \\ &= (\rho_1 + \rho_2 e^{\alpha\tau})Me^{-(\alpha+c)(t_{j+1}-t_0)}e^{c(t_{j+1}-t_0)}. \end{aligned} \tag{3.12}$$

By condition (i), (3.11), and the continuity of v_2, we have

$$\begin{aligned} v_2(t_{j+1}) = v_2(t_{j+1}^-) &\leq w_3||x_{t_{j+1}^-}||_\tau^p = w_3 \sup_{s\in[-\tau,0]}||x(t_{j+1}^- + s)||^p \\ &\leq \frac{w_3}{w_1}\sup_{s\in[-\tau,0]}\{v_1(t_{j+1}^- + s)\} \\ &\leq \frac{w_3}{w_1}\sup_{s\in[-\tau,0]}\{v(t_{j+1}^- + s)\} \\ &\leq \frac{w_3}{w_1}e^{\alpha\tau}Me^{-(\alpha+c)(t_{j+1}-t_0)}e^{c(t_{j+1}-t_0)}. \end{aligned} \tag{3.13}$$

Then, (3.12) and (3.13) imply that

$$\begin{aligned} v(t_{j+1}) &= v_1(t_{j+1}) + v_2(t_{j+1}) \\ &\leq (\rho_1 + \rho_2 e^{\alpha\tau} + \frac{w_3}{w_1}e^{\alpha\tau})Me^{-(\alpha+c)(t_{j+1}-t_0)}e^{c(t_{j+1}-t_0)} \\ &= e^{-(\alpha+c)\delta}Me^{-(\alpha+c)(t_{j+1}-t_0)}e^{c(t_{j+1}-t_0)} \\ &\leq e^{-(\alpha+c)(t_{j+2}-t_{j+1})}Me^{-(\alpha+c)(t_{j+1}-t_0)}e^{c(t_{j+1}-t_0)} \end{aligned}$$

$$= \quad Me^{-(\alpha+c)(t_{j+2}-t_0)}e^{c(t_{j+1}-t_0)},$$

i.e., (3.7) is satisfied for $t = t_{j+1}$. For $t \in (t_{j+1}, t_{j+2})$, we have

$$\begin{aligned} v(t) &\leq v(t_{j+1})e^{c(t-t_{j+1})} \\ &\leq Me^{-(\alpha+c)(t_{j+2}-t_0)}e^{c(t-t_0)}, \end{aligned} \tag{3.14}$$

which implies (3.7) holds for $t \in (t_{j+1}, t_{j+2})$. Hence, (3.7) holds for $k = j+1$. Thus we conclude from mathematical induction that (3.7) is true for all $k \geq i$. Then,

$$\begin{aligned} v(t) &\leq Me^{-(\alpha+c)(t_{k+1}-t_0)}e^{c(t-t_0)} \\ &\leq Me^{-\alpha(t_{k+1}-t_0)} \\ &\leq Me^{-\alpha(t-t_0)}, \text{ for } t \in [t_k, t_{k+1}) \text{ and } k \geq i. \end{aligned} \tag{3.15}$$

Thus, from condition (i), (3.6), and (3.15), we have

$$||x(t)||^p \leq \frac{1}{w_1}v(t) \leq \frac{M}{w_1}e^{-\alpha(t-t_0)}, \text{ for } t \geq t_0,$$

i.e.,

$$||x(t)|| \leq \left(\frac{M}{w_1||\phi||_\tau^p}\right)^{1/p}||\phi||_\tau e^{-\frac{\alpha}{p}(t-t_0)}, \text{ for } t \geq t_0.$$

From condition (i), we have

$$\begin{aligned} M &= e^{\alpha(t_i-t_0)} \sup_{t\in[t_0,t_i)} \{v(t)\} \\ &\leq v(t_0)e^{(\alpha+c)(t_i-t_0)} \\ &\leq (w_2||x(t_0)||^p + w_3||\phi||_\tau^p)e^{(\alpha+c)i\delta} \\ &\leq \max\{w_2, w_3\}||\phi||_\tau^p e^{(\alpha+c)i\delta}, \end{aligned}$$

then,

$$\frac{M}{w_1||\phi||_\tau^p} \leq \frac{\max\{w_2, w_3\}}{c_1}e^{(\alpha+c)i\delta}.$$

Therefore,

$$||x(t)|| \leq \bar{M}||\phi||_\tau e^{-\frac{\alpha}{p}(t-t_0)}, \text{ for } t \geq t_0,$$

where $\bar{M} = \left(\frac{\max\{w_2,w_3\}}{w_1}e^{(\alpha+c)i\delta}\right)^{1/p} > 1$. This completes the proof. □

Remark 3.2.1 *It can be seen from the proof that the convergence rate of impulsive system (3.1) is $\frac{\alpha}{p}$, and α can be obtained by solving equation (3.4). It can also be observed from condition (iii) and (iv) that Theorem 3.2.1 throws uniform restrictions on each impulse, i.e., constants ρ_1 and ρ_2 are independent of $t_k (k \in \mathbb{N})$, and the upper bound δ of the length of each impulsive interval needs to satisfy the inequality in condition (iv). Actually, we can get nonuniform conditions for each impulse, which are stated in the following theorem.*

Theorem 3.2.2 *Assume that there exist $V_1 \in \nu_0$, $V_2 \in \nu_0^*$, and constants $\alpha, p, c, w_1, w_2, w_3, \rho_{1k} > 0$ and $\rho_{2k} \geq 0$ $(k \in \mathbb{N})$, such that*

(i) $w_1||x||^p \leq V_1(t,x) \leq w_2||x||^p$, $0 \leq V_2(t,\psi) \leq w_3||\psi||_\tau^p$, *for* $t \in \mathbb{R}^+$, $x \in \mathbb{R}^n$, *and* $\psi \in \mathcal{PC}([-\tau,0],\mathbb{R}^n)$;

(ii) for $V(t,\psi) = V_1(t,\psi(0)) + V_2(t,\psi)$,

$$D^+V(t,\psi) \leq cV(t,\psi),$$

for all $t \in [t_{k-1},t_k)$, $\psi \in \mathcal{PC}([-\tau,0],\mathbb{R}^n)$, *and* $k \in \mathbb{N}$;

(iii) for $k \in \mathbb{N}$ *and* $\psi \in \mathcal{PC}([-\tau,0],\mathbb{R}^n)$,

$$V_1(t_k,\psi(0)+I_k(t_k,\psi)) \leq \rho_{1k}V_1(t_k^-,\psi(0)) + \rho_{2k}\sup_{s\in[-\tau,0]}\{V_1(t_k^-+s,\psi(s))\};$$

(iv) $\ln(\rho_{1k} + \rho_{2k}e^{\alpha\tau} + \frac{w_3}{w_1}e^{\alpha\tau}) \leq -c(t_{k+1}-t_k)$ *for all* $k \in \mathbb{N}$.

Then the trivial solution of system (3.1) is GES.

Proof. Replace equation (3.4) by the inequalities in condition (iv) of Theorem 3.2.2, and the rest of the proof is similar to that of Theorem 3.2.1. Thus, the detail is omitted. □

Remark 3.2.2 *In our results, the Lyapunov functional V is divided into a function part V_1 and a functional part V_2, which has been widely used when studying the control problem of time-delay systems (see, e.g., [3]). Nevertheless, these two parts will play different roles in the stability analysis, when it comes to time-delay systems with impulses. The function V_1 plays an important role in describing the dynamic of impulsive behavior, while the functional V_2 is not affected by impulses. Since the constant c is positive, condition (ii) implies that the impulse-free nonlinear system can be unstable. Hence, Theorem 3.2.1 gives sufficient conditions to design suitable impulsive controllers to stabilize nonlinear delay systems. Furthermore, condition (iii) allows existence of time-delay in each impulse. In this sense, our result is more general than the results in [75]. Compared with the stability results in [37], our results are more general in the sense that the impulses $x(t) = g_k(x(t^-), x((t-d)^-))$ for $t = t_k$ in [37] are special cases of the impulses considered here $\Delta x(t) = I_k(t, x_{t^-})$ for $t = t_k$. Also, Theorem 3.2.1 is a global result for exponential stability of general nonlinear time-delay systems, while the results in [75] and [37] are sufficient conditions for local exponential stability of locally Lipschitz time-delay systems.*

As a simple illustration of our results, let us consider a linear impulsive differential equation with time-delay:

$$\begin{cases} \dot{x}(t) = ax(t) + bx(t-r), & t \neq t_k, \\ \Delta x(t_k) = \gamma_1 x(t_k^-) + \gamma_2 x(t_k - d), & k \in \mathbb{N}, \end{cases} \tag{3.16}$$

where $a, b, \gamma_1, \gamma_2 \in \mathbb{R}$ and $r = d = \tau > 0$.

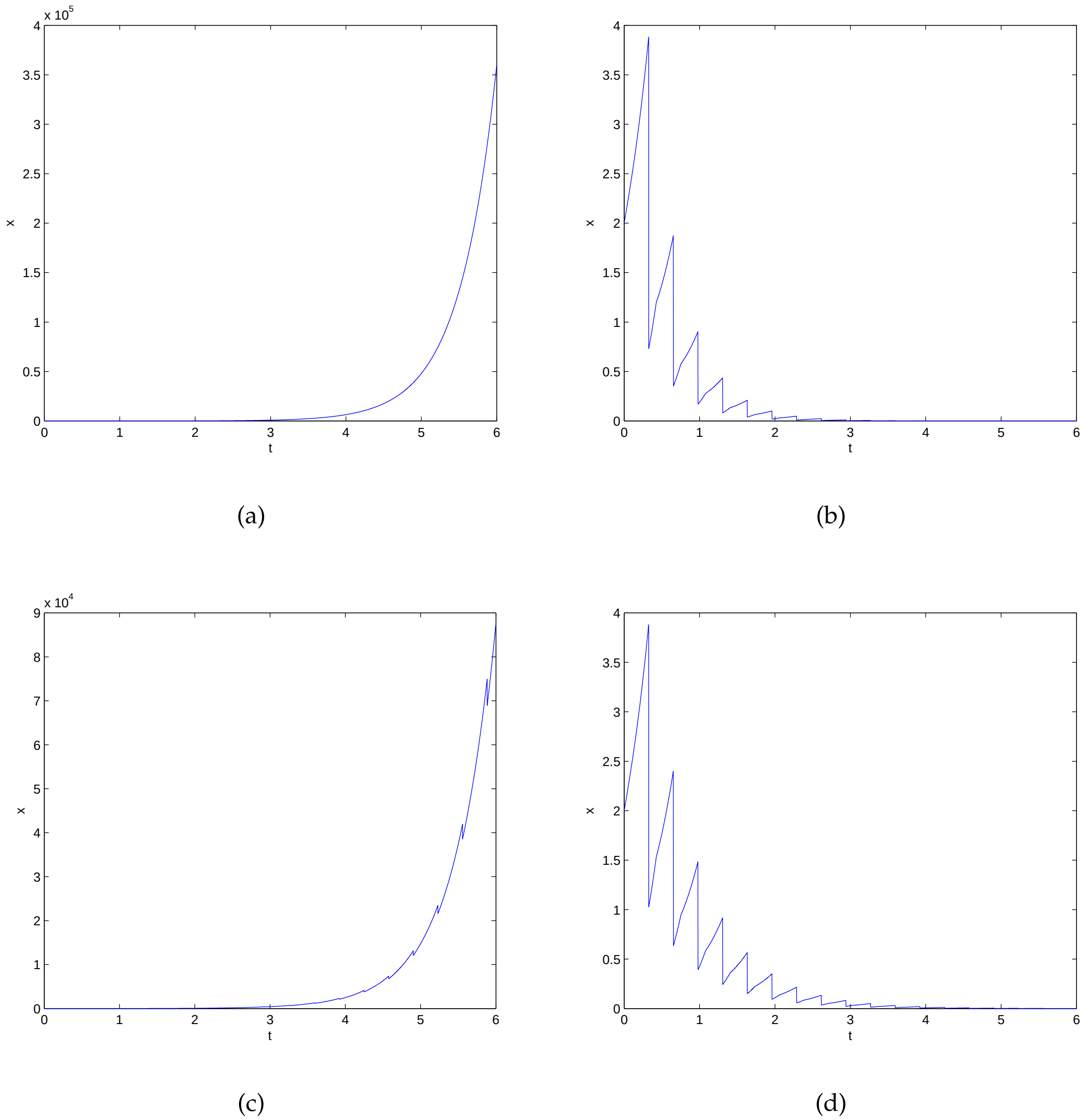


Figure 3.2: Numerical simulations of equation (3.16). In (a), no impulses are considered, i.e., $\gamma_1 = \gamma_2 = 0$, and it can be seen that the state x diverges as $t \to \infty$, while the state x converges with the proposed impulses in (b) with $\gamma_1 = -0.73$ and $\gamma_2 = -0.1$. In (c), impulses with $\gamma_1 = 0$ and $\gamma_2 = -0.1$ are considered, and it is shown that the state x diverges though the impulses slow down the divergence process. In (d), the solution of equation (3.16) with impulsive parameters $\gamma_1 = 0$ and $\gamma_2 = -0.9$ is simulated.

Choose $V_1(t,x) = x^2$, $V_2(t,x_t) = |b| \int_{t-\tau}^{t} x^2(s)\mathrm{d}s$, and $V(t,x_t) = V_1(t,x) + V_2(t,x_t)$, so condition (i) of Theorem 3.2.1 is satisfied with $w_1 = w_2 = 1$, $w_3 = |b|\tau$, and $p = 2$. The upper right hand derivative of V along equation (3.16) is

$$V'(t,x_t) \;=\; 2x(t)x'(t) + |b|x^2(t) - |b|x^2(t-\tau)$$

$$
\begin{aligned}
&= 2ax^2(t) + 2bx(t)x(t-\tau) + |b|x^2(t) - |b|x^2(t-\tau) \\
&\leq 2(a+|b|)V(t,x_t),
\end{aligned}
$$

then condition (ii) of Theorem 3.2.1 holds with $c = 2(a+|b|)$, if $a+|b| > 0$.

When $t = t_k$, we can get that

$$
\begin{aligned}
V_1(t_k, x(t_k)) &= x^2(t_k) = [(1+\gamma_1)x(t_k^-) + \gamma_2 x(t_k-\tau)]^2 \\
&= (1+\gamma_1)^2 x^2(t_k^-) + \gamma_2^2 x^2(t_k-\tau) + 2(1+\gamma_1)\gamma_2 x(t_k^-)x(t_k-\tau) \\
&\leq (1+\gamma_1)^2 x^2(t_k^-) + \gamma_2^2 x^2(t_k-\tau) \\
&\quad + \varepsilon(1+\gamma_1)^2 x^2(t_k^-) + \varepsilon^{-1}\gamma_2^2 x^2(t_k-\tau) \\
&= (1+\varepsilon)(1+\gamma_1)^2 x^2(t_k^-) + (1+\varepsilon^{-1})\gamma_2^2 x^2(t_k-\tau) \\
&= \rho_1 V_1(t_k^-, x(t_k^-)) + \rho_2 \sup_{s\in[-\tau,0]} \{V_1(t_k^- + s, x(t_k^- + s))\}, \qquad (3.17)
\end{aligned}
$$

where $\rho_1 = (1+\varepsilon)(1+\gamma_1)^2$, $\rho_2 = (1+\varepsilon^{-1})\gamma_2^2$, and $\varepsilon > 0$ is a constant. It can be seen that $\rho = \rho_1 + \rho_2$ can be minimized by choosing $\varepsilon = \frac{|\gamma_2|}{|1+\gamma_1|}$, then $\rho = (|1+\gamma_1| + |\gamma_2|)^2$. Thus, condition (iii) of Theorem 3.2.1 is satisfied.

Based on the above discussion and Theorem 3.2.1, we have that if $a+|b| > 0$ and $\ln[(|1+\gamma_1| + |\gamma_2|)^2 + |b|\tau] < -(a+|b|)\delta$, then the trivial solution of equation (3.16) is GES. The numerical simulation of the impulse-free delay differential equation with $a = 1.2$, $b = 1$, $\tau = 0.1$, and initial condition $t_0 = 0$, $\phi(s) = 2$ for $s \in [-\tau, 0]$ is shown in Fig. 3.2(a), while the simulation of the impulsive differential equation is given in Fig. 3.2(b) with $\gamma_1 = -0.73$, $\gamma_2 = -0.1$, and $t_k - t_{k-1} = 0.327$ for $k \in \mathbb{N}$.

If $\gamma_1 = 0$, then $\rho_1 = 1+\varepsilon > 1$. Hence, Theorem 3.2.1 cannot be used to analyze the stability of system (3.16) according to the estimation method used in (3.17). Simulation result in Fig. 3.2(c) implies that the impulses with $\gamma_2 = -0.1$ cannot stabilize the linear delay system. Compared with the simulation result in Fig 3.2(b), we can see that the linear part $\gamma_1 x(t_k^-)$ plays an important role in the stabilization process. However, if we replace the impulsive control gain considered in Fig. 3.2(c) with $\gamma_2 = -0.9$, then the numerical simulations in Fig. 3.2(d) show the corresponding impulsive system is stable. Therefore, in order to apply Theorem 3.2.1 to investigate the stability of time-delay systems with delayed impulses considered in Fig. 3.2(c) and 3.2(d), an estimation of the relation between $x(t_k^-)$ and $x(t_k - d)$ is necessary to guarantee $\rho_1 < 1$ when testifying condition (iii) of Theorem 3.2.1. This is the key point and main difficult to deal with systems subject to delayed impulses in the form of $\Delta x(t_k) = I_k(x(t_k - d))$. Details will be discussed in Chapter 4 about delayed impulsive control of DNs.

3.3 Razumikhin Technique

In this section, we will study exponential stability of system (3.1) by using Razumikhin technique and Lyapunov functions. The following theorem gives sufficient conditions for GES of system (3.1), which is a direct consequence of Theorem 3.1 in [18] for stochastic impulsive systems.

3.3.1 Results for General Nonlinear Systems

Theorem 3.3.1 *Assume that there exist $V \in \nu_0$, and constants $p, q, c, w_1, w_2, \rho_1 > 0$, and $\rho_2 \geq 0$, such that*

(i) $w_1||x||^p \leq V(t,x) \leq w_2||x||^p$, *for* $t \in \mathbb{R}^+$ *and* $x \in \mathbb{R}^n$;

(ii) *for* $t \in [t_{k-1}, t_k)$, $\psi \in \mathcal{PC}([-\tau, 0], \mathbb{R}^n)$, *and* $k \in \mathbb{N}$,

$$D^+V(t, \psi(0)) \leq cV(t, \psi(0)),$$

whenever $V(t+s, \psi(s)) < qV(t, \psi(0))$ *for all* $s \in [-\tau, 0]$;

(iii) *for* $k \in \mathbb{N}$ *and* $\psi \in \mathcal{PC}([-\tau, 0], \mathbb{R}^n)$,

$$V(t_k, \psi(0) + I_k(t_k, \psi)) \leq \rho_1 V(t_k^-, \psi(0)) + \rho_2 \sup_{s\in[-\tau,0]} \{V(t_k^- + s, \psi(s))\};$$

(iv) $q > \frac{1}{\rho_1+\rho_2} > e^{cd}$, *where* $d = \sup_{k\in\mathbb{N}}\{t_{k+1} - t_k\}$.

Then the trivial solution of system (3.1) is GES.

From Theorem 3.3.1, we can obtain the following result of generalized Halanay-type inequalities.

Theorem 3.3.2 *For constant α and non-negative constants β, ρ_1, and ρ_2, the function $v \in \mathcal{PC}([t_0 - \tau, \infty), \mathbb{R}^+)$ satisfies*

$$\begin{cases} v'(t) \leq \alpha v(t) + \beta \sup\limits_{s\in[-\tau,0]} v(t+s), \; t \neq t_k, \\ v(t_k) \leq \rho_1 v(t_k^-) + \rho_2 \sup\limits_{s\in[-\tau,0]} v(t_k^- + s), \; k \in \mathbb{N}, \\ v_{t_0} = \psi, \end{cases} \tag{3.18}$$

where $\psi \in \mathcal{PC}([-\tau, 0], \mathbb{R}^+)$. *If* $\alpha + \beta \geq 0$ *and*

$$\frac{1}{\rho_1 + \rho_2} > e^{(\alpha + \frac{\beta}{\rho_1+\rho_2})\sigma} > 1, \tag{3.19}$$

where $\sigma = \sup_{k\in\mathbb{N}}\{t_k - t_{k-1}\}$, *then*

$$\lim_{t\to\infty} v(t) = 0.$$

Proof. Since $\alpha + \beta \geq 0$ and $\rho_1, \rho_2 \geq 0$, inequality (3.19) implies that $(\alpha + \frac{\beta}{\rho_1+\rho_2})\sigma > 0$, and then $e^{(\alpha+\frac{\beta}{\rho_1+\rho_2})\sigma} > 1$ and $\rho_1 + \rho_2 < 1$. Thus, if (3.19) holds, then there exists a constant q such that

$$q > \frac{1}{\rho_1 + \rho_2} > e^{(\alpha+q\beta)\sigma} > e^{(\alpha+\frac{\beta}{\rho_1+\rho_2})\sigma}.$$

According to the above inequality, one can choose a small enough constant $\lambda > 0$ such that

$$q > \frac{e^{\lambda\tau}}{\rho_1 + \rho_2 e^{\lambda\tau}} > \frac{1}{\rho_1 + \rho_2 e^{\lambda\tau}} > e^{(\alpha+q\beta+\lambda)\sigma}.$$

If $v(t+s) \leq qv(t)$ for all $s \in [-\tau, 0]$, then (3.19) implies that

$$v'(t) \leq \alpha v(t) + \beta \sup_{s\in[-\tau,0]} v(t+s) \leq cv(t)$$

with constant $c = \alpha + q\beta$. Then, inequalities (3.19) are related to the Razumikhin-type conditions in Theorem 3.3.1. The rest proof is similar to that of Theorem 3.1 in [18], and thus omitted. □

Theorems 3.3.1 and 3.3.2 will be used to study consensus problem of multi-agent systems in Chapter 4.

3.3.2 Case Study: Nonlinear Systems with Distributed-Delay Dependent Impulses

Next, consider the following nonlinear time-delay system subject to distributed-delay dependent impulses, which is a special case of system (3.1):

$$\begin{cases} \dot{x}(t) = f(t, x_t), \ t \in [t_{k-1}, t_k), \\ \Delta x(t) = I_k(t, \int_{t-r_k}^{t} x(s)\mathrm{d}s), \ \ t = t_k, \ k \in \mathbb{N}, \\ x_{t_0} = \psi, \end{cases} \tag{3.20}$$

where $x \in \mathbb{R}^n$, $0 \leq t_0 < t_1 < ... < t_k < ...$ with $\lim_{t\to\infty} t_k = \infty$, and $\Delta x(t) = x(t^+) - x(t^-)$. Here, we assume that x is right-continuous at each $t = t_k$, i.e., $x(t_k^+) = x(t_k)$. $x_t \in \mathcal{PC}([-\tau, 0], \mathbb{R}^n)$ is defined as $x_t(s) = x(t+s)$ for $s \in [-\tau, 0]$, where τ denotes the time-delay in the continuous dynamics of system (3.20). $r_k \geq 0$ represents the distributed delay in the impulse satisfying $r_k \leq r \leq \tau$ for all $k \in \mathbb{N}$. Assume $f : \mathbb{R}^+ \times \mathcal{PC}([-\tau, 0], \mathcal{D}) \to \mathbb{R}^n$ and $I_k : \mathbb{R}^+ \times \mathcal{D} \to \mathbb{R}^n$, where $\mathcal{D} \subseteq \mathbb{R}^n$ is an open set, satisfy all the sufficient conditions introduced in Section 3.1 so that system (3.20) admits a solution $x(t) := x(t, t_0, \psi)$ that exists on a maximal interval $[t_0 - \tau, t_0 + T)$ where $0 < T \leq \infty$, and moreover, suppose $f(t, 0) = I_k(t, 0) = 0$ for all $k \in \mathbb{N}$. Next, we further assume that, for some $\rho > 0$ and $\mathcal{B}(\rho) \subseteq \mathcal{D}$,

(A_1) there exists a positive constant L_1 such that $\|f(t,\phi)\| \leq L_1\|\phi\|_\tau$ for any $(t, \phi) \in \mathbb{R}^+ \times \mathcal{PC}([-\tau, 0], \mathcal{B}(\rho))$;

(A_2) there exists a positive constant L_2 such that $\|I_k(t,y) - I_k(t,z)\| \leq L_2\|y-z\|$ for any $t \in \mathbb{R}^+$ and $y, z \in \mathcal{B}(\rho)$;

(A_3) there exist positive constants $\underline{\sigma}$ and $\overline{\sigma}$ such that $\underline{\sigma} \leq t_k - t_{k-1} \leq \overline{\sigma}$ for all $k \in \mathbb{N}$, i.e., all the impulsive intervals are uniformly bounded;

(A_4) there exists a nonnegative integer l such that $l\underline{\sigma} < r \leq (l+1)\underline{\sigma}$, i.e., there are at most l impulses on each interval $[t_k - r_k, t_k)$.

Remark 3.3.1 *Impulsive system (3.20) can be derived from the following control system*

$$\dot{x} = f(t, x_t) + u(t), \tag{3.21}$$

with impulsive controller (IC)

$$u(t) = \sum_{k=1}^{\infty} I_k(t, \int_{t-r_k}^{t} x(s)\mathrm{d}s)\delta(t - t_k) \tag{3.22}$$

where $\delta(\cdot)$ is the Delta Dirac function. Recent results about delay-dependent impulsive control of time-delay systems were reported in [73], and the following form of delay-dependent impulses was considered:

$$x(t_k) = \Gamma_k x(t_k - \varsigma_k), \tag{3.23}$$

where $\Gamma_k \in \mathbb{R}^n$ and ς_k denotes the discrete delay in the impulse. Rewrite (3.23) as $\Delta x(t_k) = -x(t_k^-) + \Gamma_k x(t_k - \varsigma_k)$, then the corresponding IC is

$$u(t) = \sum_{k=1}^{\infty} [-x(t) + \Gamma_k x(t - \varsigma_k)]\delta(t - t_k^-), \tag{3.24}$$

which depends not only on the states at a history instant (i.e., $x(t_k - \varsigma_k)$) but also on the states at the impulsive time (i.e., $x(t_k^-)$). Therefore, sufficient conditions obtained in [73] could guarantee the IC (3.24) to stabilize the time-delay system, but the authors cannot make conclusion that the delayed states contribute to the systems stability. However, it can be seen that IC (3.22) relies purely on the distributed-delay dependent states, i.e., the distributed delays in IC (3.24) play a key role in stabilization of the nonlinear system.

The objective of the following discussion is to use Lyapunov-Razumikhin method to establish exponential stability criteria for impulsive system (3.20). We first construct an exponential stability criterion for system (3.20).

Theorem 3.3.3 *Suppose assumptions (A_1)-(A_4) are satisfied, and there exist a function $V \in \nu_0$, and positive constants c_1, c_2, p, c, q, K_1, K_2 and ν such that*

(i) $c_1\|x\|^p \leq V(t,x) \leq c_2\|x\|^p$ *for all* $(t,x) \in [t_0 - \tau, \infty) \times \mathcal{B}(\rho)$;

(ii) $D^+V(t,\phi(0)) \leq cV(t,\phi(0))$ *for all* $t \geq t_0$, $t \neq t_k (k \in \mathbb{N})$ *and* $\phi \in \mathcal{PC}([-\tau, 0], \mathcal{B}(\rho))$, *whenever* $V(t+s, \phi(s)) \leq qV(t, \phi(0))$ *for all* $s \in [-\tau, 0]$;

(iii) $V(t, x+y) \leq K_1V(t,x) + K_2V(t,y)$ *for all* $t = t_k$ *and* $x, y \in \mathcal{B}(\rho)$ *satisfying* $x + y \in \mathcal{B}(\rho)$;

(iv) $V(t, x + I_k(t, r_k x)) \le \nu V(t^-, x)$ *for all* $t = t_k$ *and* $x \in \mathcal{B}(\frac{\rho}{1+rL_2})$*;*

(v) $q > \{K_1\nu + K_2\frac{c_2}{c_1}[r^2L_2(L_1 + lL_2)]^p\}^{-1} > e^{c\overline{\sigma}}$,

then the trivial solution of system (3.20) is ES.

Proof: Let $d_1 = K_1\nu$ and $d_2 = K_2\frac{c_2}{c_1}[r^2L_2(L_1 + lL_2)]^p$. From condition (v), we can find a small enough constant α such that

$$q > \frac{e^{\alpha\bar{\tau}}}{d_1 + d_2e^{\alpha\bar{\tau}}} > \frac{1}{d_1 + d_2e^{\alpha\bar{\tau}}} > e^{(c+\alpha)\bar{\sigma}}, \tag{3.25}$$

where $\bar{\tau} = \tau + r$. Set $\bar{q} = qe^{-\alpha\bar{\tau}}$, then (3.25) implies that $\bar{q} > e^{(c+\alpha)\overline{\sigma}} > 1$. Choose $M > 0$ such that $\bar{q}c_2 < M$, then let $\eta = \max\{r, 1 + rL_2, r^2L_2(L_1 + lL_2)\}$ and $\epsilon = (\eta\sqrt[p]{M/c_1})^{-1}\rho$. Suppose $x(t) = x(t, t_0, \psi)$ is a solution of (3.20) with $(t, \psi) \in \mathbb{R}^+ \times \mathcal{PC}([-\tau, 0], \mathcal{B}(\epsilon))$, and is exits on a maximal interval $[t_0 - \tau, t_0 + T)$ where $T > 0$.

Let $V(t) := V(t, x(t))$, and we will show that

$$V(t) \le M\|\psi\|_\tau^p e^{-\alpha(t-t_0)}, \text{ for } t \in [t_0, t_0 + T). \tag{3.26}$$

If (3.26) is true, then condition (i) implies

$$\|x(t)\| \le \sqrt[p]{M/c_1}\|\psi\|_\tau e^{-\frac{\alpha}{p}(t-t_0)} \le \rho,$$

i.e., $x(t) \in \mathcal{B}(\rho)$ for all $t \in [t_0 - \tau, t_0 + T)$. It then follows from the continuation theorem in [14] that $T = +\infty$, that is, the global existence of the solution $x(t)$. Therefore, it is sufficient to prove (3.26) is true for $t \ge t_0$, and then the global existence of $x(t)$ follows directly.

Set $Q(t) := e^{\alpha(t-t_0)}V(t)$, then we will prove

$$Q(t) < M\|\psi\|_\tau^p, \text{ for } t \ge t_0. \tag{3.27}$$

For $t \in [t_0 - \tau, t_0]$, we have

$$Q(t) \le V(t) \le c_2\|\psi\|_\tau^p < \frac{M}{\bar{q}}\|\psi\|_\tau^p < M\|\psi\|_\tau^p. \tag{3.28}$$

To prove (3.27), we first show that

$$Q(t) < M\|\psi\|_\tau^p, \text{ for } t \in [t_0, t_1). \tag{3.29}$$

We prove (3.29) by contradiction. Suppose (3.29) is not true, then there exists a $t^* \in [t_0, t_1)$ such that $Q(t^*) = M\|\psi\|_\tau^p$ and $Q(t) < M\|\psi\|_\tau^p$ for $t < t^*$. Note that $t^* \ne t_0$, since (3.28) implies $Q(t_0) < \frac{1}{\bar{q}}M\|\psi\|_\tau^p < M\|\psi\|_\tau^p$. Furthermore, there exists a $t^{**} \in (t_0, t^*)$ such that $Q(t^{**}) = \frac{1}{\bar{q}}M\|\psi\|_\tau^p$ and $Q(t) > \frac{1}{\bar{q}}M\|\psi\|_\tau^p$ for $t \in (t^{**}, t^*]$. Therefore, for $t \in [t^{**}, t^*]$, $t + s \le t^*$ and $Q(t + s) \le M\|\psi\|_\tau^p \le \bar{q}Q(t)$ for all $s \in [-\tau, 0]$, which implies that $V(t + s) \le \bar{q}e^{\alpha s}V(t) \le qV(t)$

for $s \in [-\tau, 0]$. For $t \leq t^*$, $Q(t) \leq M\|\psi\|_\tau^p$ implies $V(t) \leq M\|\psi\|_\tau^p e^{-\alpha(t-t_0)}$ and then $x(t) \in \mathcal{B}(\rho)$. From condition (ii), we can get

$$\begin{aligned} D^+Q(t) &= \alpha e^{\alpha(t-t_0)}V(t) + e^{\alpha(t-t_0)}D^+V(t) \\ &\leq \alpha e^{\alpha(t-t_0)}V(t) + ce^{\alpha(t-t_0)}V(t) \\ &= (\alpha + c)Q(t), \ \text{ for } t \in [t^{**}, t^*]. \end{aligned} \tag{3.30}$$

Then, it follows from (3.30) and (3.25) that

$$\begin{aligned} Q(t^*) &\leq Q(t^{**})e^{(\alpha+c)(t^*-t^{**})} \leq Q(t^{**})e^{(\alpha+c)\bar{\sigma}} \\ &= \frac{1}{\bar{q}}M\|\psi\|_\tau^p e^{(\alpha+c)\bar{\sigma}} < M\|\psi\|_\tau^p, \end{aligned}$$

which is a contradiction to the choice of t^*. Hence, (3.29) is true.

Now we assume that, for some $m \in \mathbb{N}$,

$$Q(t) < M\|\psi\|_\tau^p, \ \text{ for } t \in [t_0, t_m). \tag{3.31}$$

and then, we will show that

$$Q(t) < M\|\psi\|_\tau^p, \ \text{ for } t \in [t_m, t_{m+1}). \tag{3.32}$$

First, we claim that

$$Q(t_m) < (d_1 + d_2 e^{\alpha\bar{\tau}})M\|\psi\|_\tau^p. \tag{3.33}$$

For $t \in [t_m - r_m, t_m)$, integrating system (3.20) on both sides from t to t_m^- yields

$$x(t_m^-) - x(t) = \int_t^{t_m} f(s, x_s)\mathrm{d}s + \sum_{i=1}^{i_0} I_{m-i}, \tag{3.34}$$

where $i_0 := i_0(t)$ denotes the number of impulses on the interval $[t, t_m^-)$, and we use I_{m-i} to represent $I_{m-i}(t_{m-i}, \int_{t_{m-i}-r_{m-i}}^{t_{m-i}} x(s)\mathrm{d}s)$ for simplicity. Integrate both sides of (3.34) from $t_m - r_m$ to t_m^-:

$$\begin{aligned} &\Big\| r_m x(t_m^-) - \int_{t_m-r_m}^{t_m} x(s)\mathrm{d}s \Big\| \\ &= \Big\| \int_{t_m-r_m}^{t_m} \int_t^{t_m} f(s, x_s)\mathrm{d}s\mathrm{d}t + \int_{t_m-r_m}^{t_m} \Big(\sum_{i=1}^{i_0} I_{m-i} \Big)\mathrm{d}t \Big\| \\ &\leq \int_{t_m-r_m}^{t_m} \int_t^{t_m} \|f(s, x_s)\|\mathrm{d}s\mathrm{d}t + \int_{t_m-r_m}^{t_m} \Big(\sum_{i=1}^{i_0} \|I_{m-i}\| \Big)\mathrm{d}t. \end{aligned} \tag{3.35}$$

It can be seen from (3.31) and the definition of ϵ that both x and $\int_{t_{m-i}-r_{m-i}}^{t_{m-i}} x(s)\mathrm{d}s$ $(i = 1, 2, ..., i_0)$ for $t < t_m$ belong to $\mathcal{B}(\rho)$. It then follows from (3.35) and assumptions (A_1), (A_2), and (A_4) that

$$\Big\| r_m x(t_m^-) - \int_{t_m-r_m}^{t_m} x(s)\mathrm{d}s \Big\|$$

$$
\begin{aligned}
&\leq \int_{t_m-r_m}^{t_m}\int_t^{t_m} L_1\|x_s\|_\tau \mathrm{d}s\mathrm{d}t + \int_{t_m-r_m}^{t_m}\Big(\sum_{i=1}^{i_0} L_2\Big\|\int_{t_{m-i}-r_{m-i}}^{t_{m-i}} x(s)\mathrm{d}s\Big\|\Big)\mathrm{d}t \\
&\leq L_1 r_m \int_{t_m-r_m}^{t_m} \|x_s\|_\tau \mathrm{d}s + L_2 r_m \sum_{i=1}^{l}\int_{t_{m-i}-r_{m-i}}^{t_{m-i}} \|x(s)\|\mathrm{d}s \\
&\leq L_1 r_m \int_{t_m-r_m}^{t_m} \sup_{\theta\in[-\tau,0]} \|x(s+\theta)\|\mathrm{d}s + L_2 r_m \sum_{i=1}^{l} r_{m-i} \sup_{\theta\in[-r_{m-i},0]} \|x(t_{m-i}+\theta)\| \\
&\leq L_1 r_m^2 \sup_{\theta\in[-\tau-r_m,0]} \|x(t_m^-+\theta)\| + L_2 r_m r l \sup_{\theta\in[-2r,0]} \|x(t_m^-+\theta)\| \\
&\leq r^2(L_1+lL_2) \sup_{\theta\in[-\bar{\tau},0]} \|x(t_m^-+\theta)\|. \qquad (3.36)
\end{aligned}
$$

In the second inequality of (3.36), if $m-i<1$, we set $r_{m-i}=t_{m-i}=0$. Denote $\Delta I_m := I_m(t_m, \int_{t_m-r_m}^{t_m} x(s)\mathrm{d}s) - I_m(t_m, r_m x(t_m^-))$. It follows from (3.36) and assumption (A_2) that

$$
\begin{aligned}
\|\Delta I_m\|^p &\leq L_2^p\Big\|r_m x(t_m^-) - \int_{t_m-r_m}^{t_m} x(s)\mathrm{d}s\Big\|^p \\
&\leq [r^2 L_2(L_1+lL_2)]^p \sup_{\theta\in[-\bar{\tau},0]} \|x(t_m^-+\theta)\|^p \\
&\leq \frac{1}{c_1}[r^2 L_2(L_1+lL_2)]^p \sup_{\theta\in[-\bar{\tau},0]} V(t_m^-+\theta). \qquad (3.37)
\end{aligned}
$$

Then, from conditions (iii), (iv), (i), and the definition of ϵ, we have

$$
\begin{aligned}
V(t_m) &= V(t_m, x(t_m^-) + I_m(t_m, \int_{t_m-r_m}^{t_m} x(s)\mathrm{d}s)) \\
&= V(t_m, x(t_m^-) + I_m(t_m, r_m x(t_m^-)) + \Delta I_m) \\
&\leq K_1 V(t_m, x(t_m^-) + I_m(t_m, r_m x(t_m^-))) + K_2 V(t_m, \Delta I_m) \\
&\leq K_1 \nu V(t_m^-, x(t_m^-)) + K_2 c_2 \|\Delta I_m\|^p. \qquad (3.38)
\end{aligned}
$$

Using (3.37), (3.38), and (3.25), we obtain

$$
\begin{aligned}
Q(t_m) &= e^{\alpha(t_m-t_0)} V(t_m) \\
&\leq K_1 \nu Q(t_m^-) + K_2 c_2 e^{\alpha(t_m-t_0)} \|\Delta I_m\|^p \\
&\leq d_1 Q(t_m^-) + d_2 e^{\alpha\bar{\tau}} \sup_{\theta\in[-\bar{\tau},0]} Q(t_m^-+\theta) \\
&< (d_1 + d_2 e^{\alpha\bar{\tau}}) M \|\psi\|_\tau^p \\
&< M\|\psi\|_\tau^p, \qquad (3.39)
\end{aligned}
$$

that is, claim (3.33) is true.

Next, we will prove (3.32) by contradiction. Suppose (3.32) is not true, then there exists a $\bar{t} \in (t_m, t_{m+1})$ such that $Q(\bar{t}) = M\|\psi\|_\tau^p$ and $Q(t) < M\|\psi\|_\tau^p$ for $t < \bar{t}$. On the other hand, (3.39) implies that there exists a $\underline{t} \in (t_m, \bar{t})$ such that $Q(\underline{t}) = (d_1 + d_2 e^{\alpha\bar{\tau}})M\|\psi\|_\tau^p$ and $Q(t) > (d_1 + d_2 e^{\alpha\bar{\tau}})M\|\psi\|_\tau^p$ for $t \in (\underline{t}, \bar{t}]$. Hence, for $t \in [\underline{t}, \bar{t}]$, $t + s \leq \bar{t}$ and $Q(t+s) \leq M\|\psi\|_\tau^p \leq (d_1 + d_2 e^{\alpha\bar{\tau}})^{-1}Q(t)$ for all $s \in [-\tau, 0]$, which implies that $V(t+s) \leq e^{\alpha s}(d_1 + d_2 e^{\alpha\bar{\tau}})^{-1}V(t) \leq qV(t)$ for $s \in [-\tau, 0]$. Similar to the discussion of (3.30), we can get from condition (ii) and (3.25) that

$$
\begin{aligned}
Q(\bar{t}) &\leq Q(\underline{t})e^{(\alpha+c)(\bar{t}-\underline{t})} \\
&\leq e^{(\alpha+c)\bar{\sigma}}(d_1 + d_2 e^{\alpha\bar{\tau}})M\|\psi\|_\tau^p \\
&< M\|\psi\|_\tau^p,
\end{aligned}
$$

which is a contradiction to the choice of $\bar{t}$. Thus, (3.32) is true, i.e., $Q(t) < M\|\psi\|_\tau^p$ for $t \in [t_0, t_{m+1})$, and then we conclude from mathematical induction that (3.32) is true for all $m \in \mathbb{N}$. Therefore,

$$
\begin{aligned}
\|x(t)\| &\leq \sqrt[p]{V(t)/c_1} = \sqrt[p]{Q(t)e^{-\alpha(t-t_0)}/c_1} \\
&\leq \sqrt[p]{M/c_1}\|\psi\|_\tau e^{-\frac{\alpha}{p}(t-t_0)},
\end{aligned}
$$

for $t \geq t_0$, i.e., the trivial solution of system (3.20) is ES and the proof is complete. □

Remark 3.3.2 *The Razumikhin-type condition (ii) in Theorem 3.3.3 characterizes the changing rate of function V on each impulsive interval. The positive constant c implies that the delay-free system can be unstable. Therefore, Theorem 3.3.3 shows that an unstable time-delay system can be exponentially stabilized by distributed-delay dependent impulses. Conditions (iii) and (iv) are requirements on the Lyapunov function V at each impulsive instant. As pointed out in [25], for any positive definite matrix P, the Lyapunov function $V(t,x) = (x^T Px)^p$ satisfies condition (iii) with $V(t, x+y) \leq \max\{2^{\frac{p}{2}-1}, 1\}[(1+\varepsilon)^{\frac{p}{2}}V(t,x) + (1+\varepsilon^{-1})^{\frac{p}{2}}V(t,y)]$ for any $\varepsilon > 0$.*

Remark 3.3.3 *As a special case of system (3.1), Theorem 3.3.1 can be applied to analyze the stability property of system (3.20). But, with the locally Lipschitz conditions given in assumptions (A_1) and (A_2), Theorem 3.3.1 is not applicable to system (3.20), since it is a global result for ES. However, if f and I_k in (3.20) satisfy globally Lipschitz conditions, then Theorem 3.3.3 can be derived from Theorem 3.3.1 with the estimation techniques used in (3.35)-(3.38). These techniques will also be applied in Sections 4.4 and 5.3 when distributed delays are considered in the proposed impulsive controllers.*

Now consider the following linear impulsive system with time-delay

$$
\begin{cases}
\dot{x}(t) = Ax + Bx(t-\tau), \ t \in [t_{k-1}, t_k), \\
\Delta x(t) = E\displaystyle\int_{t-r}^{t} x(s)\mathrm{d}s, \ \ t = t_k, \ k \in \mathbb{N}, \\
x_{t_0} = \psi,
\end{cases}
\tag{3.40}
$$

where A, B, and E are $n \times n$ matrices, $0 < r \leq \tau$, $t_0 = 0$, and $t_k = k\sigma (k \in \mathbb{N})$ with $\sigma > 0$. $\psi \in \mathcal{PC}([-\tau, 0], \mathbb{R}^n)$ is the initial condition for system (3.40). It can be seen that conditions

(A_1)-(A_4) are satisfied for system (3.40) with $L_1 = \|A\| + \|B\|$, $L_2 = \|E\|$, $\sigma = \overline{\sigma} = \underline{\sigma}$, and $l = \lfloor \frac{r}{\sigma} \rfloor$ where the floor function gives the largest integer less than $\frac{r}{\sigma}$.

Next, we will apply Theorem 3.3.3 to establish a GES result for system (3.40).

Theorem 3.3.4 *If*

$$\sigma < \frac{-2\ln\mu}{\lambda_A + \frac{2\|B\|}{\mu}}, \tag{3.41}$$

where $\mu = \|I + rE\| + r^2\|E\|(\|A\| + \|B\| + l\|E\|)$ and $\lambda_A = \lambda_{max}(A^T + A)$, then the trivial solution of system (3.40) is GES.

Proof: Consider $V(x) := V(t,x) = x^T x$, then condition (i) of Theorem 3.3.3 is satisfied with $c_1 = c_2 = 1$ and $p = 2$.

For $t \neq t_k$ and any $\epsilon > 0$, we have

$$\begin{aligned} V'(x) &= [Ax + Bx(t-\tau)]^T x + x^T[Ax + Bx(t-\tau)] \\ &\leq \lambda_A x^T x + 2\|B\| \cdot \|x\| \cdot \|x(t-\tau)\| \\ &\leq \lambda_A V(x) + \|B\|(\epsilon\|x\|^2 + \epsilon^{-1}\|x(t-\tau)\|^2) \\ &\leq [\lambda_A + (\epsilon + \epsilon^{-1}q)\|B\|]V(x), \end{aligned} \tag{3.42}$$

whenever $V(x(t+s)) \leq qV(x(t))$ for $s \in [-\tau, 0]$. To get a less conservative estimation in (3.42), we can minimize the term $\epsilon + \epsilon^{-1}q$ for $\epsilon > 0$. Then, condition (ii) in Theorem 3.3.3 is satisfied with $c = \min_{\epsilon>0}\{\lambda_A + (\epsilon + \epsilon^{-1}q)\|B\|\} = \lambda_A + 2\sqrt{q}\|B\|$.

For $t = t_k$, we can conclude from Remark 3.3.2 that $V(x+y) \leq (1+\varepsilon)V(x) + (1+\varepsilon^{-1})V(y)$ for any $\varepsilon > 0$, then condition (iii) holds with $K_1 = 1+\varepsilon$ and $K_2 = 1+\varepsilon^{-1}$. Condition (iv) is satisfied with $\nu = \|I + rE\|^2$. Moreover, if $q > \frac{1}{\kappa} > e^{c\sigma}$ with $\kappa = K_1\nu + K_2\frac{c_2}{c_1}L_2^p(L_1r^2 + L_2r^2l)^p$, then Theorem 3.3.3 implies that system (3.40) is ES. However, $\kappa = (1+\varepsilon)\|I+rE\|^2 + (1+\varepsilon^{-1})\|E\|^2r^4(\|A\| + \|B\| + l\|E\|)^2$ depends on the positive parameter ε. To obtain a larger upper bound for σ, we minimize κ for $\varepsilon > 0$, then we have $\min_{\varepsilon>0}\kappa = \mu^2$.

On the other hand, (3.41) implies that there exist a q with $q > \frac{1}{\mu^2}$ and $q - \frac{1}{\mu^2}$ small enough such that $\sigma < \frac{-2\ln\mu}{c} < \frac{-2\ln\mu}{\lambda_A + 2\|B\|/\mu}$, i.e., $\frac{1}{\mu^2} > e^{c\sigma}$. With the choice of q, we have $q > \frac{1}{\mu^2} > e^{c\sigma}$. Hence, condition (v) holds.

Up to now, conditions (A_1)-(A_4) and all the conditions in Theorem 3.3.3 are satisfied with $\rho = \infty$, then we can conclude from Theorem 3.3.3 that systems (3.40) is GES. □

Remark 3.3.4 *Inequality (3.41) in Theorem 3.3.4 gives an upper bound of σ explicitly, which is a common condition for admissible impulsive sequences in most of the impulsive control literature. On the other hand, parameter μ in (3.41) depends on l. If σ is small enough (l large enough) such that $\mu \geq 1$, then (3.41) cannot be satisfied for any $\sigma > 0$. Therefore, a lower bound of σ is contained in (3.41) implicitly. See the following example for details about how to get all the possible values of σ for the stability of system (3.40) from Theorem 3.3.4.*

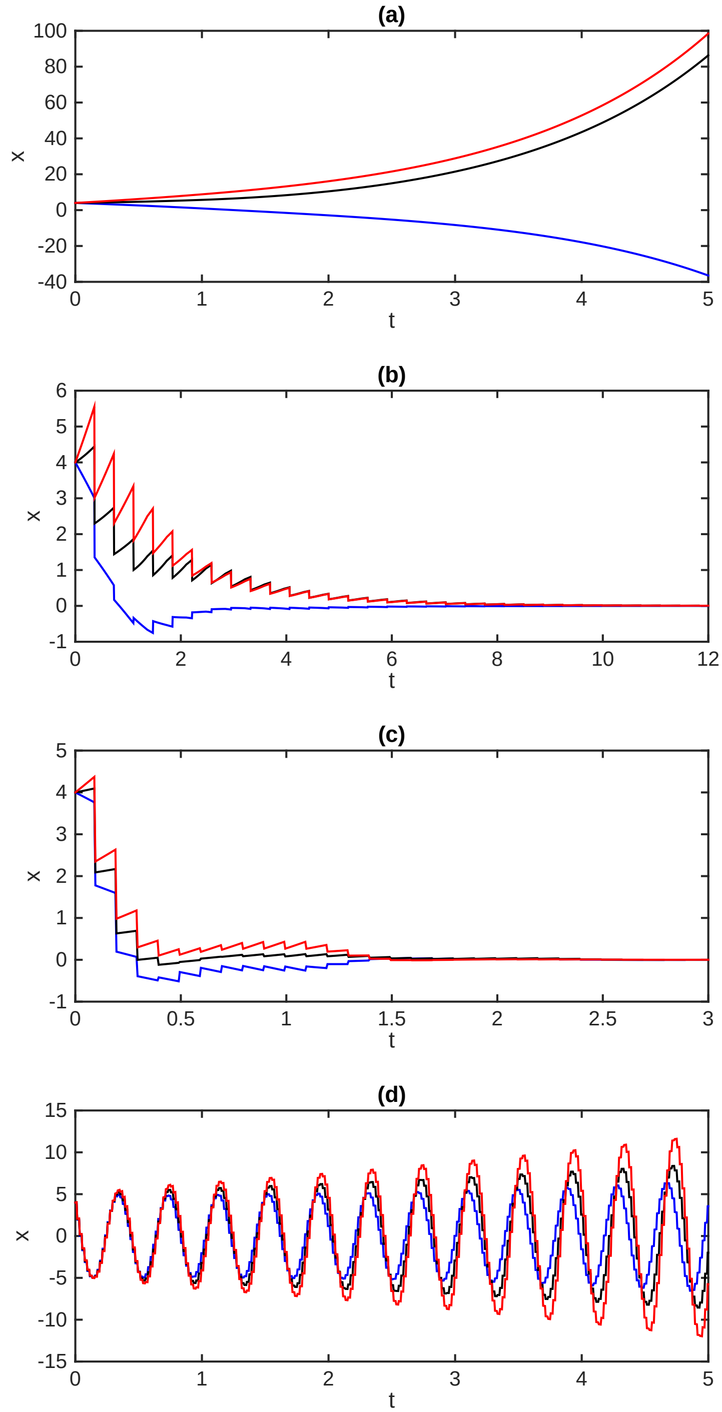


Figure 3.3: State trajectories of system (3.40) with different choices of σ. (a) System response without impulses ($\sigma = \infty$). (b) Impulsive stabilization with $\sigma = 0.37 \in \Omega$. (c) Impulsive stabilization with $\sigma = 0.1 \in \Omega$. (d) Impulsive control with $\sigma = 0.02 \notin \Omega$. Note that, for $\sigma = 0.02$ in (d), the distributed-delay dependent impulses fail to stabilize the time-delay system, which means that frequent impulses (small σ) may lead to the failure of the controller with distributed delays for stabilization of time-delay systems.

Consider linear impulsive system (3.40) with parameters given as follows:

$$A = \begin{bmatrix} -0.01 & -0.26 & -0.13 \\ -0.16 & 0.14 & 0.17 \\ 0.05 & 0.32 & 0.25 \end{bmatrix}, \; B = \begin{bmatrix} -0.31 & 0.08 & 0.14 \\ -0.28 & 0.13 & 0.25 \\ 0.11 & 0.16 & 0.13 \end{bmatrix}$$

$E = -2.5I$, $\tau = 1$, and $r = 0.2$. Then, $\lambda_A = 1.0029$, $\|A\| = 0.5412$, $\|B\| = 0.5028$, and $\|I + rE\| = 0.5$.

Next, we will use Theorem 3.3.4 to identify the admissible impulsive sequences for stabilization of system (3.40) with the given parameters:

- $l = 0$, that is, no impulse in the interval $[t_k - r, t_k)$. Then, $\mu = 0.6044$ and $\frac{-2\ln\mu}{\lambda_A + 2\|B\|/\mu}\big|_{l=0} = 0.3776$. Thus, $l = \lfloor \frac{r}{\sigma} \rfloor = 0$ implies that $0.2 = r \leq \sigma$, and (3.41) implies $\sigma < 0.3776$. We can conclude from Theorem 3.3.4 that system (3.40) is GES for any $\sigma \in [0.2, 0.3776)$.
- $l = 1$, that is, there is one impulse in $[t_k - r, t_k)$. Then, $\mu = 0.8544$ and $\frac{-2\ln\mu}{\lambda_A + 2\|B\|/\mu}\big|_{l=1} = 0.1444$. $l = \lfloor \frac{r}{\sigma} \rfloor = 1$ implies that $\sigma < r \leq 2\sigma$, then $r/2 \leq \sigma < r$. on the other hand, (3.41) implies $\sigma < 0.1444$. Therefore, system (3.40) is GES if $\sigma \in [r/2, r) \cap (0, 0.1444) = [0.1, 0.1444)$.
- If there are two impulses on $[t_k - r, t_k)$, then $l = 2$ and $\mu = 1.1044 > 1$. Hence, for any $l \geq 2$, we get $\mu > 1$, and (3.41) fails to hold for any $\sigma > 0$.

Based on the above analysis, the trivial solution of system (3.40) is GES if $\sigma \in \Omega := [0.1, 0.1444)$ $\cup [0.2, 0.3776)$. Numerical simulations of system (3.40) are shown in Fig. 3.3 with initial condition $\psi(s) = (4, 4, 4)^T$ for $s \in [-\tau, 0]$. We conclude the analysis with the following algorithm for the application of Theorem 3.3.4.

Algorithm 1. Computation the admissible set Ω for σ

---

require: r, E

$l = 0$, $bdd = 0$

$\mu = \|I + rE\| + r^2\|E\|(\|A\| + \|B\|)$

$\Omega = \Phi$ (initialed with empty set)

while $\mu < 1$ **do**

  $bdd \leftarrow \frac{-2\ln\mu}{\lambda_A + 2\|B\|/\mu}$

  if $l = 0$ **then**

    $\Omega \leftarrow [r, \infty) \cap (0, bdd)$

  else

    $\Omega \leftarrow \Omega \cup ([\frac{r}{l+1}, \frac{r}{l}) \cap (0, bdd))$

        end if

        $l \leftarrow l + 1$

        $\mu \leftarrow \|I + rE\| + r^2\|E\|(\|A\| + \|B\| + l\|E\|)$

end while

return Ω

If the set Ω obtained from Algorithm 1. is not empty, then Theorem 3.3.4 can be applied to design suitable impulsive sequences for stabilization of system (3.40) with given parameters r and E.

Chapter 4

Consensus of Multi-Agent Systems

This chapter studies the consensus problem of multi-agent systems with both fixed and switching topologies. In Section 4.2, a hybrid consensus protocol is proposed to take into consideration of continuous-time communications among agents and delayed instantaneous information exchanges on a sequence of discrete times. In Section 4.3, a novel hybrid consensus protocol with dynamically changing interaction topologies is designed to take the time-delay into account in both the continuous-time communication among agents and the instantaneous information exchange at discrete-time moments. Section 4.4 studies the consensus problem of networked multi-agent systems. Distributed delays are considered in both the agent dynamics and the proposed impulsive consensus protocols.

4.1 Network Topology

In this section, we introduce some preliminary notions in graph theory.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a **digraph** (or **directed graph**) of order n with the set of nodes $\mathcal{V} = \{v_1, v_2, ..., v_n\}$ and the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. An **edge** of $\mathcal{G}$ is denoted by (v_i, v_j) which means the node v_j can receive information from node v_i, and then v_i and v_j are called **parent** and **child nodes**, respectively. The index set of neighbors of node v_i is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} \mid (v_j, v_i) \in \mathcal{E}\}$. For a given matrix $A = [a_{ij}]_{n\times n}$, the digraph of A, denoted by $\mathcal{G}(A) = (\mathcal{V}, \mathcal{E}_A)$, is the directed graph of order n with the set of nodes $\mathcal{V}$ and the set of edges $\mathcal{E}_A \subseteq \mathcal{V} \times \mathcal{V}$ such that an edge (v_j, v_i) exists if and only if $a_{ij} \neq 0$.

Next, we will introduce some terminology for the digraph $\mathcal{G}$ (i.e., $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ or $\mathcal{G} = \mathcal{G}(A) = (\mathcal{V}, \mathcal{E}_A)$). A **directed path** of digraph $\mathcal{G}$ is a sequence of edges (v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), (v_{i_3}, v_{i_4}),... in digraph $\mathcal{G}$. A digraph $\mathcal{G}$ is called **strongly connected** if there is a directed path connecting any two arbitrary nodes in $\mathcal{G}$. A **directed tree** is a digraph such that there is only one root (that is, no edge points to this node) in it, and every node except the root has exactly one parent. A **spanning tree** of digraph $\mathcal{G}$ is a directed tree that connects all the nodes of $\mathcal{G}$. Let $\bar{\mathcal{G}} = \{\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_M\}$ denote the set of all possible digraphs defined for $\mathcal{V}$. Then the **union** of a group of digraphs $\{\mathcal{G}_{i_1}, \mathcal{G}_{i_2}, ..., \mathcal{G}_{i_m}\} \subseteq \bar{\mathcal{G}}$ is defined as a digraph with nodes given by the set $\mathcal{V}$, and the edge set is given by the union of the edge sets $\mathcal{G}_{i_j}$, $j = 1, 2, ..., m$.

A **weighted digraph** $\mathcal{G}_{\mathcal{A}} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ associated with a **weighted adjacency matrix** $\mathcal{A} = [\alpha_{ij}]_{n\times n}$ with nonnegative adjacency elements α_{ij} such that $(v_j, v_i) \in \mathcal{E}$

if and only if $\alpha_{ij} > 0$. Denote the set $\{1, 2, ..., n\}$ by $\mathcal{I}$. It is assumed that $\alpha_{ii} = 0$ for all $i \in \mathcal{I}$. The **graph Laplacian** $\mathcal{L}$ of the weighted digraph $\mathcal{G}_\mathcal{A}$ is defined by $\mathcal{L} := D - \mathcal{A}$ where $D = diag\{d_1, d_2, ..., d_n\}$ with element $d_i := \sum_{j \in \mathcal{N}_i} \alpha_{ij}$ which is called the **in-degree** of node v_i. A weighted digraph $\mathcal{G}_\mathcal{A}$ is said to be **balanced** if $\sum_{j=1, j\neq i}^{n} \alpha_{ij} = \sum_{j=1, j\neq i}^{n} \alpha_{ji}$ for all $i \in \mathcal{I}$. The following are equivalent (Theorem 1.37, [8]): (i) $\mathcal{G}$ is balanced; (ii) $\mathbf{1}^T\mathcal{L} = 0$, where $\mathbf{1} = (1, 1, ..., 1)^T$ is a $n \times 1$ vector; (iii) $\mathcal{L} + \mathcal{L}^T$ is positive semi-definite. If $\mathcal{G}$ is balanced and strongly connected, then 0 is a simple eigenvalue of $\mathcal{L} + \mathcal{L}^T$.

4.2 Hybrid Protocols with Impulse Delays

The purpose of this section is to study the consensus problem of multi-agent systems via hybrid consensus protocols with impulse delays. Following the idea utilized in [98], we aim to extend the results in [36] to the case of hybrid continuous-time and delayed impulsive consensus protocols, and then establish verifiable consensus results by using results from graph theory and matrix theory. The outline of this section is as follows. We introduce the hybrid consensus protocol in Subsection 4.2.1, and provide some lemmas in graph theory and matrix theory in Subsection 4.2.2. Consensus results are established for multi-agent systems with fixed topologies and switching topologies in Subsections 4.2.3 and 4.2.4, respectively. Subsection 4.2.5 contains the discussion of the obtained results, and highlights the contributions of these results by comparison with the existing ones. Simulations are presented at the end of Subsection 4.2.5 to demonstrate our theoretical results.

4.2.1 Consensus Protocols

Let $x_i \in \mathbb{R}$ denote the state of node v_i, and consider each node of a graph $\mathcal{G}$ to be a dynamic agent with integrator dynamics

$$\dot{x}_i(t) = u_i, \;\; i \in \mathcal{I}, \tag{4.1}$$

where u_i is a state feedback. We say u_i is a protocol with topology $\mathcal{G}$ if the state feedback u_i only depends on the information of v_i and its neighbors, i.e., $u_i = u_i(x_{i_1}, x_{i_2}, ..., x_{i_m})$ and the corresponding set of nodes $\{v_{i_1}, v_{i_2}, ..., v_{i_m}\}$ are all taken from the set $\{v_i\} \cup \mathcal{N}_i$.

We say a protocol solves the consensus problem if and only if

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0 \tag{4.2}$$

for any $i, j \in \mathcal{I}$. Furthermore, we say a protocol u_i solves the average-consensus problem if and only if

$$\lim_{t\to\infty} \|x_i(t) - Ave(x(0))\| = 0$$

for all $i \in \mathcal{I}$, where $Ave(x(0)) = \frac{1}{n}\sum_{j=1}^{n} x_j(0)$.

Remark 4.2.1 *It can be seen that a consensus problem is specified in terms of two events: 1. propose: agreement on the agent states as described in (4.2); 2. algorithm: an interaction rule that specifies the* *information exchange among agents. Definition 2.4.1 implies that synchronization problem shares the same control objective with the consensus problem. However, the controller design for realization of network synchronization does not necessarily depends on the interaction rule among the network nodes, while this interaction rule is essential for the protocol design of a consensus problem.*

We consider the following consensus protocol which is based on two interaction topologies $\mathcal{G}_{\mathcal{A}} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ and $\mathcal{G}_{\mathcal{A}'} = (\mathcal{V}, \mathcal{E}', \mathcal{A}')$:

$$u_i(t) = \sum_{v_j \in \mathcal{N}_i} \alpha_{ij}[x_j(t) - x_i(t)] + \sum_{k=1}^{\infty} \sum_{v_j \in \mathcal{N}_i'} \alpha_{ij}'[x_j(t-d) - x_i(t-d)]\delta(t - t_k), \tag{4.3}$$

where α_{ij} (or α_{ij}') is the (i,j)th entry of the weighted adjacent matrix $\mathcal{A}$ (or $\mathcal{A}'$), and $\mathcal{N}_i$ (or $\mathcal{N}_i'$) denotes the set of node v_i's neighbors in graph $\mathcal{G}_{\mathcal{A}}$ (or $\mathcal{G}_{\mathcal{A}'}$); $\delta(\cdot)$ denotes the Dirac delta function; t_k is called impulsive instant, and the time sequence $\{t_k\}$ satisfies $0 < t_1 < t_2 < ... < t_k < ...$ and $\lim_{k\to\infty} t_k = \infty$; $d \geq 0$ denotes the time-delay when processing the impulsive information according to graph $\mathcal{G}_{\mathcal{A}'}$.

By the definition of $\delta(\cdot)$, the collective dynamics of system (4.1) under consensus protocol (4.3) can be written as an impulsive system:

$$\begin{cases} \dot{x}_i(t) = \sum_{v_j \in \mathcal{N}_i} \alpha_{ij}[x_j(t) - x_i(t)], \ \ t \neq t_k, \\ \Delta x_i(t_k) = \sum_{v_j \in \mathcal{N}_i'} \alpha_{ij}'[x_j(t_k - d) - x_i(t_k - d)], \ \ k \in \mathbb{N}, \end{cases} \tag{4.4}$$

for $i \in \mathcal{I}$, where $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$; $x_i(t_k^+)$ and $x_i(t_k^-)$ represent the right and left limit of x_i at t_k, respectively. Without loss of generality, we assume that $x_i(t_k^+) = x_i(t_k)$, which implies that $x_i(t)$ is right continuous at each impulsive instant t_k. Throughout this section, we further assume that $t_1 - d \geq 0$, which implies that no information about the states before initial time $t_0 = 0$ is required, and then the initial conditions $x_i(0) = x_{i,0}$ for $i \in \mathcal{I}$ are sufficient for the evolution of system (4.4).

It can be seen that the consensus protocol (4.3) works as follows: on each impulsive interval (t_k, t_{k+1}), the interaction among nodes is connected according to the graph $\mathcal{G}_{\mathcal{A}}$, and at each impulsive instant t_k, the nodes exchange information instantaneously according to the topology of $\mathcal{G}_{\mathcal{A}'}$. The objective of this section is to determine sufficient conditions on the graphs $\mathcal{G}_{\mathcal{A}}$, $\mathcal{G}_{\mathcal{A}'}$ and the impulsive sequence $\{t_k\}$ to guarantee that the consensus protocol (4.3) solves the consensus problem.

4.2.2 Some Lemmas

A matrix $A = [a_{ij}]_{n\times n}$ is said to be **nonnegative** and denoted as $A \geq 0$, if all its entries are nonnegative. For the set of nonnegative matrices, we define an ordering as follows: if A and B are nonnegative matrices, then $A \geq B$ implies $A - B$ is a nonnegative matrix. A is a **stochastic matrix**, if A is nonnegative and all its row sums are 1. A stochastic matrix P is called **indecomposable and aperiodic (SIA)** if $\lim_{n\to\infty} P^n = \mathbf{1}y^T$, where $\mathbf{1} = (1, 1, ..., 1)^T$ is a $n \times 1$ vector, and y is some column vector.

Next, we will list some lemmas which will be used in the proof of our results.

Lemma 4.2.1 *[50] Let $m \geq 2$ be an integer and P_1, P_2, ..., P_m be nonnegative $n \times n$ matrices with positive diagonal elements, then*

$$P_1 P_2 ... P_m \geq \gamma(P_1 + P_2 + ... + P_m),$$

where $\gamma > 0$ can be specified from matrices P_i, $i = 1, 2, ..., m$.

Lemma 4.2.2 *[122] Let Γ be a compact set consisting of $n \times n$ SIA matrices with the property that for any nonnegative integer k and any A_1, A_2, ..., $A_k \in \Gamma$ (repetitions permitted), $\Pi_{i=1}^{k} A_i$ is SIA. Then, given any infinite sequence A_1, A_2, A_3, ... (repetitions permitted) of matrices from Γ, there exists a column vector ν such that $\lim_{l \to \infty} \Pi_{i=1}^{l} A_i = \mathbf{1}\nu^T$.*

Lemma 4.2.3 *[99] For any $t > 0$, $e^{-\mathcal{L}t}$ is a stochastic matrix with positive diagonal entries, where $\mathcal{L}$ is the graph Laplacian of graph $\mathcal{G}_{\mathcal{A}} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$.*

The last lemma is a direct conclusion of Corollary 3.5 and Lemma 3.7 in [98].

Lemma 4.2.4 *If $A = [a_{ij}]_{n \times n}$ is a stochastic matrix with positive diagonal elements, and the digraph associated with A has a spanning tree, then A is SIA.*

4.2.3 Consensus Problem with Fixed Topologies

In this subsection, consensus problem of multi-agent system (4.4) is studied with fixed topologies, i.e., both the weighted digraphs $\mathcal{G}_{\mathcal{A}}$ and $\mathcal{G}_{\mathcal{A}'}$ in protocol (4.3) are time-invariant. Let $x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$, then system (4.4) can be written in a compact form

$$\begin{cases} \dot{x}(t) = -\mathcal{L}x(t), \ \ t \neq t_k, \\ \Delta x(t_k) = -\mathcal{L}' x(t_k - d), \ \ k \in \mathbb{N}, \end{cases} \tag{4.5}$$

where $\mathcal{L}$ (or $\mathcal{L}'$) is the graph Laplacian of $\mathcal{G}_{\mathcal{A}}$ (or $\mathcal{G}_{\mathcal{A}'}$).

From Lemma 4.2.3, we know that, for $t > 0$, $e^{-\mathcal{L}t}$ is a stochastic matrix, which implies that $\mathcal{D} := e^{-\mathcal{L}d} - \mathcal{L}'$ is also a stochastic matrix for $t > 0$. To establish our main results, we make the following assumption:

(A1) $\mathcal{D}$ has positive diagonal entries.

Since $e^{-\mathcal{L}d}$ has positive diagonal entries, graph $\mathcal{G}_{\mathcal{A}'}$ with small enough in-degree d_i' ($i \in \mathcal{I}$) will make assumption (A1) hold.

Now we are in the position to introduce one of the main results.

Theorem 4.2.1 *Assume that impulsive intervals $(t_{k-1}, t_k]$ for $k \in \mathbb{N}$ are uniformly bounded, that is, there exist positive constants τ_{min} and τ_{max} such that $\tau_{min} \leq t_k - t_{k-1} \leq \tau_{max}$ for all $k \in \mathbb{N}$. Furthermore, suppose that $d < \tau_{min}$ and assumption (A1) holds. Then, protocol (4.3) solves the consensus problem if the union of graphs $\mathcal{G}_{\mathcal{A}}$ and $\mathcal{G}_{\mathcal{A}'}$ contains a spanning tree.*

proof For any $t > 0$, there exists a positive integer k such that $t \in [t_{k-1}, t_k)$. Then the solution of (4.5) with initial condition $x(0) = x_0$ can be obtained by induction:

$$x(t) = e^{-\mathcal{L}(t-t_{k-1})}\mathcal{D}e^{-\mathcal{L}(\tau_{k-1}-d)}...\mathcal{D}e^{-\mathcal{L}(\tau_1-d)}x_0, \tag{4.6}$$

for $t \in [t_{k-1}, t_k)$, where $\tau_k = t_k - t_{k-1}$ for $k \in \mathbb{N}$.

It can be seen that protocol (4.3) solves the consensus problem if and only if $x(t) \to \mathbf{1}\xi$ as $t \to \infty$, where $\xi \in \mathbb{R}$ is some constant. Next, we will show $x(t) \to \mathbf{1}\xi$ as $t \to \infty$ is equivalent to the union of graphs $\mathcal{G}_{\mathcal{A}}$ and $\mathcal{G}_{\mathcal{A}'}$ containing a spanning tree.

Lemma 4.2.3 implies $e^{-\mathcal{L}t}$ is a stochastic matrix with positive diagonal entries (SPD) for any $t > 0$. On the other hand, we can get from Lemma 4.2.1 that $\mathcal{D}e^{-\mathcal{L}t} \geq \gamma(\mathcal{D} + e^{-\mathcal{L}t})$ for $t > 0$, where γ is some positive constant. Moreover, we know that $\mathcal{D}$ is a stochastic matrix, and then from assumption (A1) we see that $\mathcal{D}$ has positive diagonal entries, i.e., $\mathcal{D}$ is SPD. Hence, the matrix $\mathcal{D}e^{-\mathcal{L}t}$ is a SPD when $t > 0$.

Next, we claim that, for $t > 0$, the graph of $\mathcal{D}e^{-\mathcal{L}t}$ has a spanning tree. Let $\eta = \max\{d_i\}$ and $\mathcal{M} = \eta I - \mathcal{L}$, then the (i, j)th $(i \neq j)$ entry of $\mathcal{M}$ is α_{ij} which implies the graph $\mathcal{G}_{\mathcal{A}}$ and the graph of $\mathcal{M}$ have the same edge set. Since $e^{-\mathcal{L}t} = e^{-\eta t}e^{\mathcal{M}t} \geq \rho\mathcal{M}$ for a given $t > 0$ and some $\rho > 0$, we know that the edge set of $\mathcal{G}_{\mathcal{A}}$ is a subset of the edge set of the graph associated with $e^{-\mathcal{L}t}$. On the other hand, the graph of $\mathcal{G}_{\mathcal{A}'}$ and the graph of matrix $\mathcal{L}'$ share the same edge set. Hence, the union of graphs of $\mathcal{G}_{\mathcal{A}}$ and $\mathcal{G}_{\mathcal{A}'}$ has a spanning tree implies that the union of graphs of matrices $e^{-\mathcal{L}d}$ and $\mathcal{L}'$ has a spanning tree, which implies the graph of $\mathcal{D} := e^{-\mathcal{L}d} - \mathcal{L}'$ has a spanning tree. Note that $\mathcal{D}e^{-\mathcal{L}t} \geq \gamma(\mathcal{D} + e^{-\mathcal{L}t})$ for $t > 0$, then the graph of $\mathcal{D}e^{-\mathcal{L}t}(t > 0)$ has a spanning tree. The claim is true.

Based on the above discussion, we have shown that, for $t > 0$, the matrix $\mathcal{D}e^{-\mathcal{L}t}$ is a SPD, and the graph of it has a spanning tree. From Lemma 4.2.4, one can get that, for $t > 0$, the matrix $\mathcal{D}e^{-\mathcal{L}t}$ is SIA.

Since the intervals $[t_{k-1}, t_k)$ for $k \in \mathbb{N}$ are uniformly bounded, define a matrix set $\Theta = \{\mathcal{D}e^{-\mathcal{L}t} \mid t \in [\tau_{min} - d, \tau_{max} - d]\}$, then Θ is compact and all of its elements are SIA matrices. Therefore, by Lemma 4.2.2, there exists a column vector c such that

$$\lim_{k\to\infty}\prod_{i=1}^{k-1}\mathcal{D}e^{-\mathcal{L}(\tau_i-d)} = \mathbf{1}c^T. \tag{4.7}$$

By Lemma 4.2.3, we can see that $e^{-\mathcal{L}(t-t_{k-1})}$ is a stochastic matrix, which implies that the row sums of it are all 1s. Then, $e^{-\mathcal{L}(t-t_{k-1})}\mathbf{1} = \mathbf{1}$. Therefore, we can obtain from (4.7) that

$$e^{-\mathcal{L}(t-t_{k-1})}\prod_{i=1}^{k-1}\mathcal{D}e^{-\mathcal{L}(\tau_i-d)} - \mathbf{1}c^T = e^{-\mathcal{L}(t-t_{k-1})}\Big(\prod_{i=1}^{k-1}\mathcal{D}e^{-\mathcal{L}(\tau_i-d)} - \mathbf{1}c^T\Big).$$

Moreover, $e^{-\mathcal{L}(t-t_{k-1})}$ is bounded for $t - t_{k-1} \in [\tau_{min}, \tau_{max}]$. From this and (4.7), it follows that

$$e^{-\mathcal{L}(t-t_{k-1})}\prod_{i=1}^{k-1}\mathcal{D}e^{-\mathcal{L}(\tau_i-d)} \to \mathbf{1}c^T$$

as $t \to \infty$. Hence, $\lim_{t\to\infty} x(t) = \mathbf{1}\xi$ with $\xi = c^T x_0$, which implies protocol (4.3) solves the consensus problem. □

Remark 4.2.2 *From the control point of view, it is practical to assume that $d < \tau_{min}$. For an impulsive protocol with large time-delay d, we can design the continuous-time consensus protocol with long enough time period of activations (that is, the continuous-time protocol works for a long enough time period) before the impulsive protocol is activated at each impulsive instant. Then, the condition $d < \tau_{min}$ is naturally satisfied.*

4.2.4 Consensus Problem with Switching Topologies

In practice, the links among agents may fail to work, and new links are created as time goes by, such as the hyper-links in the World Wide Web. To model the dynamic changing of the topology, we consider the consensus problem of multi-agent systems with switching in both the continuous-time topology and the impulsive-time topology.

Denote two finite index sets $P = \{1, 2, ..., p\}$ and $Q = \{1, 2, ..., q\}$, and two families of digraphs $\Omega = \{\mathcal{G}_i : i \in P\}$, $\Omega' = \{\mathcal{G}'_j : j \in Q\}$. Let $\sigma : \mathbb{R}^+ \to P$ be a piecewise constant and left-continuous function called 'continuous-time switching signal', and $s : \mathbb{N} \to Q$ be a constant function called 'discrete-time switching signal'. The collective behavior of system (4.5) can be written as an impulsive switching system

$$\begin{cases} \dot{x}(t) = -\mathcal{L}_{\sigma(t)} x(t), \ \ t \in [t_{k-1}, t_k), \\ \Delta x(t_k) = -\mathcal{L'}_{s(k)} x(t_k - d_{s(k)}), \ \ k \in \mathbb{N}, \end{cases} \tag{4.8}$$

where $\mathcal{L}_i$ (or $\mathcal{L}'_j$) is the graph Laplacian of $\mathcal{G}_i$ (or $\mathcal{G}'_j$) for $i \in P$ (or $j \in Q$); and $d_{s(k)} \geq 0$ denotes the delay when processing the impulsive information among agents according to graph $\mathcal{G}'_{s(k)}$ at impulse time $t = t_k$. If switchings only occur at impulsive instants (i.e., there is no switching on each impulsive interval), then $\sigma(t) = \sigma(t_k)$ for $t \in [t_{k-1}, t_k)$ and $k \in \mathbb{N}$, and system (4.8) reduces to the following system

$$\begin{cases} \dot{x}(t) = -\mathcal{L}_{\sigma(t_k)} x(t), \ \ t \in [t_{k-1}, t_k), \\ \Delta x(t_k) = -\mathcal{L'}_{s(k)} x(t_k - d_{s(k)}), \ \ k \in \mathbb{N}, \end{cases} \tag{4.9}$$

Next, we will construct a consensus criterion for system (4.9).

Theorem 4.2.2 *If the following conditions are satisfied:*

(i) matrix $\mathcal{D}_{ij} := e^{-\mathcal{L}_i d_j} - \mathcal{L}'_j$ has positive diagonal entries for any $i \in P$ and $j \in Q$;

(ii) there exist positive constants τ_{min} and τ_{max} such that $\tau_{min} \leq t_k - t_{k-1} \leq \tau_{max}$ and $d_{s(k)} < t_k - t_{k-1}$ for all $k \in \mathbb{N}$;

(iii) there exists a subsequence $\{t_{k_j}\} \subseteq \{t_k\}$ such that intervals $(t_{k_{j-1}}, t_{k_j}]$ for $j \in \mathbb{N}$ are uniformly bounded from above, and the union of graphs across each interval $(t_{k_{j-1}}, t_{k_j}]$ has a spanning tree,

then protocol (4.3) with switching topologies solves the consensus problem.

proof Condition (ii) implies that initial conditions $x(0) = x_0$ is well-defined for system (4.9), and it can be obtained from condition (iii) that there exists a positive constant T such that $t_{k_j} - t_{k_{j-1}} \leq T$ for all $j \in \mathbb{N}$. Then, for any interval $(t_{k_{j-1}}, t_{k_j}]$, the following matrix

$$\prod_{i=k_{j-1}+1}^{k_j} \mathcal{D}_{\sigma(t_i),s(i)} e^{-\mathcal{L}_{\sigma(t_i)}(\tau_i - d_{s(i)})} \tag{4.10}$$

is a product of finite number of matrices. Following the similar argument in Theorem 4.2.1, we have the matrix (4.10) is SIA, since the union of graphs across the interval $(t_{k_{j-1}}, t_{k_j}]$ has a spanning tree.

Next, define the following matrix set

$\Theta = \{\prod_{i=1}^{l} \mathcal{D}_{p_i,q_i} e^{-\mathcal{L}_{p_i}(\tau_i - d_{q_i})}$ | integer l satisfies $1 \leq l \leq T/\tau_{min}$; $\tau_i \in [\tau_{min}, \tau_{max}]$ for $i = 1, 2, ..., l$; the union of graphs $\mathcal{G}_{p_1}, \mathcal{G}_{p_2}, ..., \mathcal{G}_{p_l}$ and $\mathcal{G}'_{q_1}, \mathcal{G}'_{q_2}, ..., \mathcal{G}'_{q_l}$ has a spanning tree $\}$.

As discussed for matrix (4.10), we can see that Θ is a set of SIA matrices. Furthermore, since all τ_i's belong to a closed interval and $\sum_{i=1}^{l} \tau_i$ is bounded, the set Θ is compact.

For any $t > 0$, there exist nonnegative integers k and $\hat{j}$ such that $t \in (t_k, t_{k+1}] \subseteq (t_{k_{\hat{j}}}, t_{k_{\hat{j}+1}}]$, and then, for $t \in (t_k, t_{k+1}]$, the solution $x(t)$ can be obtained by mathematical induction, and then combine the matrices products according to each interval $(t_{k_j}, t_{k_{j+1}}]$ to get the compact form:

$$x(t) = \bar{\mathcal{H}}_{\hat{j}+1}(t) \mathcal{H}_{\hat{j}}(t) x_0,$$

where

$$\mathcal{H}_{\hat{j}}(t) = \sum_{j=0}^{\hat{j}-1} \prod_{i=k_j+1}^{k_{j+1}} \mathcal{D}_{\sigma(t_i),s(i)} e^{-\mathcal{L}_{\sigma(t_i)}(\tau_i - d_{s(i)})},$$

and

$$\bar{\mathcal{H}}_{\hat{j}+1}(t) = e^{-\mathcal{L}_{\sigma(t_{k+1})}(t - t_k)} \prod_{i=k_{\hat{j}}+1}^{k} \mathcal{D}_{\sigma(t_i),s(i)} e^{-\mathcal{L}_{\sigma(t_i)}(\tau_i - d_{s(i)})}.$$

It can be obtained from Lemma 4.2.2 that, there exists a column vector v such that

$$\lim_{\hat{j} \to \infty} \mathcal{H}_{\hat{j}}(t) = \mathbf{1} v^T, \tag{4.11}$$

since $\prod_{i=k_j+1}^{k_{j+1}} \mathcal{D}_{\sigma(t_i),s(i)} e^{-\mathcal{L}_{\sigma(t_i)}(\tau_i - d_{s(i)})} \in \Theta$ for $j \geq 0$. Moreover, $t_{k_{\hat{j}+1}} - t_{k_{\hat{j}}} \leq T$ implies that $\mathcal{H}_{\hat{j}+1}(t)$ is bounded. Note that $\mathcal{H}_{\hat{j}+1}(t)$ is also a stochastic matrix because it is a product of stochastic matrices.

Then,

$$x(t) - \mathbf{1} v^T x_0 = \mathcal{H}_{\hat{j}+1}(t) \mathcal{H}_{\hat{j}}(t) x_0 - \mathbf{1} v^T x_0 = \mathcal{H}_{\hat{j}+1}(t) \big(\mathcal{H}_{\hat{j}}(t) - \mathbf{1} v^T\big) x_0 \tag{4.12}$$

From (4.11) and (4.12), it follows that $\lim_{t \to \infty} x(t) = \mathbf{1} v^T x_0$, i.e., the protocol (4.3) solves the consensus problem. □

4.2.5 Discussion and Simulation Results

The contribution of this section can be clarified by comparison with the existing results in [98], [90], and [120].

Reference [98] shows that the consensus problem can be solved if the union of graphs has a spanning tree frequently enough. We generalize the result to the hybrid consensus protocol (4.3). Theorem 4.2.1 and Theorem 4.2.2 imply that the union of graphs of both continuous-time topologies and impulsive-time topologies across certain bounded time intervals having a spanning tree can guarantee the protocol solves the consensus problem. According to our results, many links in Reference [98] can be replaced by instantaneous connections. The advantage of this is to overcome the difficulties in construction of continuous-time links among certain nodes due to their special geography locations or connection cost considerations. Actually, the impulsive behaviour is an abrupt state jump which can be treated as discrete time dynamic. In this sense, our results unify the consensus results in [98] for both continuous and discrete multi-agent systems when $d = 0$. Furthermore, for the switching topology case, our results are less conservative than the results in [98] when stimulate the random switching of interaction graphs. The reason is that, the switching times are only required to be bounded in our results while the switching times in [98] belong to an infinite set generated by any finite set of positive numbers.

Compared with the results in [90], we have applied different theoretical methods to generalize these results to hybrid impulsive consensus with delayed impulsive protocols for the case of time-invariant topology. For the special case with $d = 0$ in protocol (4.3), our results are contained in [90]. Since no delay is considered in [90], in this sense our results are more general than those in [90]. Furthermore, our results imply that the results in [90] are robust to certain impulsive delays. In terms of the final equilibrium point, the leader following scenario can be achieved if the union of the graphs has a spanning tree and there exists only one node in the union graph as the root of the spanning tree; the average consensus scenario can be realized if the union of the graphs has a spanning tree and all the digraphs are balanced. We will illustrate these two scenarios by two numerical examples at the end of this section.

In [120], time delays have been considered in both the continuous and impulsive consensus protocols. The continuous-time and impulsive-time network topologies are assumed to share the same structure in [120], and both of these network topologies are required to be strongly connected. Although time delays are only studied in the impulsive topologies in this paper, we require much fewer connections between nodes in both of the continuous-time topologies and the impulsive-time topologies. Moreover, the results in [120] are not applicable to our consensus problem.

Next, we consider two examples both with 10 agents to illustrate our results and the above discussion. In the following digraphs, the solid lines represent the edges of digraphs at non-impulsive time, and the dash lines denote the edges of digraphs at impulsive instants; on each impulsive interval, all digraphs are assumed to have $0 - 1$ weights; at each impulsive instant, the digraph in Example 4.2.1 has equal weight 0.36, while, in Example 4.2.2, all the digraphs at every impulsive time are supposed to have identical weights 0.48.

Example 4.2.1 *Consider fixed network topologies given by Figure 4.1, then the union of the digraph at non-impulsive time and the digraph at impulsive instants across each impulsive interval has a spanning*

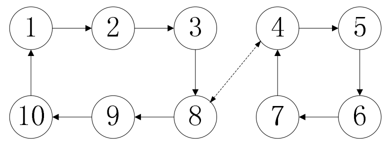


Figure 4.1: Fixed topologies.

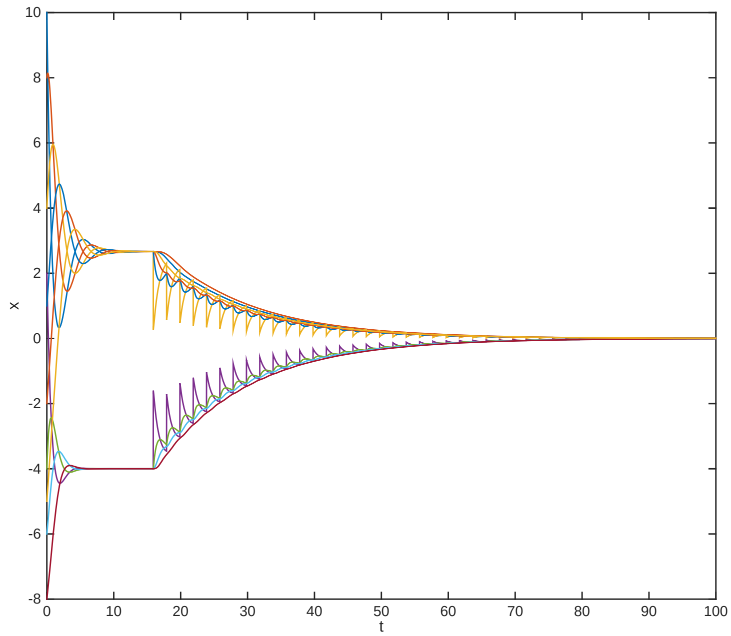


Figure 4.2: Average consensus.

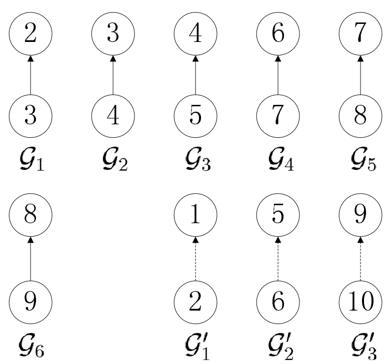


Figure 4.3: Switching topologies.

tree. In this example, choose $t_k = 2(k-1) + 14$ *for* $k \in \mathbb{N}$ *and* $d = 1$*, then assumption (A1) is satisfied which can be easily checked by using MATLAB. Therefore, Theorem 4.2.1 implies that the protocol (4.3) can solve the consensus problem. Moreover, both of these two digraphs are balanced. Thus, according to the previous discussion, the protocol (4.3) can solve the average consensus problem. The initial states are chosen as* $x(0) = [10, 8, 4, 2, -4, -6, -8, 1, -2, -5]^T$ *so that* $Ave(x(0)) = 0$*, and Figure 4.2 confirmed the average consensus of the multi-agent system.*

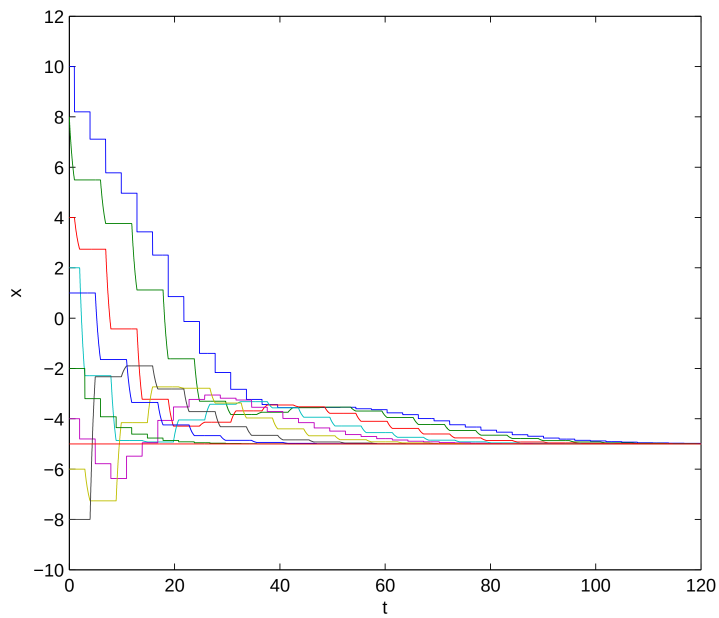


Figure 4.4: Leader following scenario.

It can be observed from Figure 4.1 that there are two subgraphs in the digraph of non-impulsive time, and both of them are strongly connected and balanced. According to the result in [95], average consensus will achieved in each subgraph. In the simulation, the first impulsive instant is chosen to be $t_1 = 14$ so that the average consensus of the two subgraphs can be observed, separately. For the digraph at the impulsive instants, there is information exchange between the 4th agent and the 8th agent, and the result in [98] implies that average consensus can be achieved between them for the special case $d = 0$. Therefore, the dynamic process of the consensus protocol (4.3) with the topologies given in this example can be concluded as follows: during each non-impulsive time period, the two subgraphs will tame the state of each agent approach to each other, according to the corresponding subgraph respectively; the digraph at each impulsive instant will reduce the difference between the average states of the two subgraphs even time-delay is considered in the impulses, and then the protocol solves the consensus problem.

Example 4.2.2 *Consider the network with delay-free dynamically changing topologies given by Figure 4.3, in which only the agents with information exchange are illustrated and the other agents are omitted. We assume that the impulse and switching occur simultaneously at each impulsive instant $t_k = k(k \in \mathbb{N})$, and the switchings happen in the order of the digraphs' sub-indices. Then the union of these graphs has a spanning tree, and Theorem 4.2.2 concludes that the consensus problem can be solved. Further observation will make it clear that no information flows into the 10th agent, which means the 10th agent is the parent node of the spanning tree. Hence, this consensus problem falls into the leader following scenario which is demonstrated by the simulation results in Figure 4.4 with the same initial conditions given in Example 4.2.1. At each time $t \geq t_0$, it can be seen that there exists only one edge in the graph, no matter the time t is on a impulsive interval or is the impulsive instant. Compared with Example 4.1 in [36], we require much less edges to solve the consensus problem.*

4.3 Hybrid Impulsive Protocols with Time-Delay

This section investigates consensus problems of multi-agent systems. A novel hybrid consensus protocol with dynamically changing interaction topologies is designed to take the time-delay into account in both the continuous-time communication among agents and the instant information exchange at discrete-time moments. Using a Halanay-type inequality, we establish sufficient conditions to guarantee the proposed consensus protocols lead to average-consensus. It is shown that the networked multi-agent system with time-delay can achieve average-consensus with appropriate network topologies, designed impulsive instants, and admissible time delays according to our consensus criteria. The rest of this section is organized as follows. In Subsection 4.3.1, we formulate the consensus problem and propose the hybrid impulsive consensus protocol. The consensus results for multi-agent systems with fixed and switching topologies are established, respectively, in Subsection 4.3.2. Two numerical examples are provided to demonstrate the theoretical results in Subsection 4.3.3. The detailed proofs of the main results are introduced in Subsection 4.3.4.

4.3.1 Consensus Protocols

We consider the following consensus protocol which is based on the dynamically changing digraph $\mathcal{G}_{\mathcal{A}}(t) = (\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t))$ and the fixed digraph $\mathcal{G}_{\mathcal{A}'} = (\mathcal{V}, \mathcal{E}', \mathcal{A}')$:

$$\begin{aligned} u_i(t) &= \sum_{v_j \in \mathcal{N}_i(t)} \alpha_{ij}(t)[x_j(t-r(t)) - x_i(t-r(t))] \\ &\quad + \sum_{k=1}^{\infty} \sum_{v_j \in \mathcal{N}'_i} \alpha'_{ij}[x_j(t-\tau_k) - x_i(t-\tau_k)]\delta(t-t_k), \end{aligned} \tag{4.13}$$

where r denotes the time-varying delay in the continuous-time consensus protocol satisfying $0 \le r(t) \le \bar{r}$ ($\bar{r}$ is a constant), and τ_k represents the time-delay in the discrete-time consensus protocol at time $t = t_k$ satisfying $0 \le \tau_k \le \bar{\tau}$ ($\bar{\tau}$ is a constant and $k \in \mathbb{N}$); $\alpha_{ij}(t)$ is the (i, j)th entry of the weighted adjacent matrix $\mathcal{A}(t)$ at time t, and $\mathcal{N}_i(t)$ denotes the set of node v_i's neighbors in graph $\mathcal{G}_{\mathcal{A}}$ at time t; α'_{ij} is the (i, j)th entry of the weighted adjacent matrix $\mathcal{A}'$ at time t_k, and $\mathcal{N}'_i$ denotes the set of node v_i's neighbors in graph $\mathcal{G}_{\mathcal{A}'}$ at time t_k; $\delta(\cdot)$ denotes the Dirac delta function; t_k is called impulsive instant, and the time sequence $\{t_k\}$ satisfies $0 < t_1 < t_2 < ... < t_k < ...$ and $\lim_{k\to\infty} t_k = \infty$.

By the definition of $\delta(\cdot)$, the collective dynamics of system (4.1) under consensus protocol (4.13) can be written as an impulsive system:

$$\begin{cases} \dot{x}_i(t) = \sum_{v_j \in \mathcal{N}_i(t)} \alpha_{ij}(t)[x_j(t-r(t)) - x_i(t-r(t))], \ \ t \neq t_k \\ \Delta x_i(t_k) = \sum_{v_j \in \mathcal{N}'_i} \alpha'_{ij}[x_j(t_k-\tau_k) - x_i(t_k-\tau_k)], \\ x_{it_0} = \phi_i \end{cases} \tag{4.14}$$

for $i \in \mathcal{I}$ and $k \in \mathbb{N}$, where $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$; $x_i(t_k^+)$ and $x_i(t_k^-)$ represent the right and left limit of x_i at t_k, respectively; we assume that $x_i(t_k^+) = x_i(t_k)$, which implies that $x_i(t)$

is right continuous at each impulsive instant t_k; x_{it_0} is defined by $x_{it_0}(s) = x_i(t_0 + s)$ for all $s \in [-\tau, 0]$ with $\tau = \max\{\bar{r}, \bar{\tau}\}$; $\phi_i \in \mathcal{PC}([-\tau, 0], \mathbb{R}^n)$ is the initial condition.

It can be seen that consensus protocol (4.13) works as follows: for $t \neq t_k$, the interaction among agents is described by the graph $\mathcal{G}_{\mathcal{A}}(t)$, and at each impulsive instant t_k, the nodes exchange information instantaneously according to the topology of $\mathcal{G}_{\mathcal{A}'}$. The objective of this section is to derive sufficient conditions on graphs $\mathcal{G}_{\mathcal{A}}(t)$, $\mathcal{G}_{\mathcal{A}'}$ and impulsive sequence $\{t_k\}$ to guarantee that consensus protocol (4.13) solves the average-consensus problem.

4.3.2 Consensus Results

In this subsection, the consensus properties of impulsive system (4.14) will be analyzed. For the sake of simplicity, the discussion throughout this section is based on the following assumptions:

(A_1) uniform impulses: $\sigma = t_k - t_{k-1}$ for all $k \in \mathbb{N}$.

(A_2) time-invariant impulse delays: $\tau_k = \bar{\tau}$ for all $k \in \mathbb{N}$.

(A_3) assume that all the weighting factors are uniformly upper bounded, i.e., there exists a constant $\bar{\alpha}$ such that $\alpha_{ij}(t) \leq \bar{\alpha}$ for all $t \geq t_0$.

Then, there are ζ impulses on time interval $(t_k - \bar{\tau}, t_k)$ for any $k \in \mathbb{N}$, that is, $\zeta = \lfloor \frac{\bar{\tau}}{\sigma} \rfloor$, where the floor function $\lfloor \chi \rfloor$ gives the largest integer less than χ. For non-uniform impulses and/or time-variant impulse delays, the number of impulses on each time interval $(t_k - \bar{\tau}, t_k)$ is not a fixed value. However, the analysis of consensus properties can be discussed similarly.

Networks with Fixed Topologies

We start by analyzing multi-agent systems with fixed topology, i.e., the weighted digraph $\mathcal{G}_{\mathcal{A}'}$ is time-invariant with $\mathcal{L}'$ as its Laplacian.

Theorem 4.3.1 *Suppose that $\mathcal{G}_{\mathcal{A}}(t)$ is balanced for all $t \geq t_0$ with $\mathcal{L}(t)$ as its Laplacian at time t, and $\mathcal{G}_{\mathcal{A}'}$ is strongly connected and balanced. Let $\lambda_2(\mathcal{L}'_s)$ denote the second smallest eigenvalue of $\mathcal{L}'_s = (\mathcal{L}' + \mathcal{L}'^T)/2$, and*

$$\rho_{min} := \left(\sqrt{1 - 2\lambda_2(\mathcal{L}'_s) + \|\mathcal{L}'\|^2} + nd\bar{\tau}\|\mathcal{L}'\| + \zeta\|\mathcal{L}'\|^2\right)^2$$

with $d = max_i\{\sup_{t\in[t_0,\infty)} d_i(t)\}$. If $\rho_{min} < 1$ and

$$\sigma < \begin{cases} \dfrac{-\sqrt{\rho_{min}}\ln(\rho_{min})}{2l}, & if\ e^{-2} < \rho_{min} < 1, \\ \dfrac{1}{el}, & if\ \rho_{min} \leq e^{-2}, \end{cases} \tag{4.15}$$

where $l = \sup_{t\in[t_0,\infty)} \|\mathcal{L}(t)\|$, then the consensus protocol (4.13) leads to the average-consensus for agents in (4.1).

If the digraph $\mathcal{G}_{\mathcal{A}}(t)$ is time-invariant, i.e., $\mathcal{G}_{\mathcal{A}} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. Then its Laplacian $\mathcal{L}$ is a constant matrix. We will have the following consensus results.

Theorem 4.3.2 *Suppose that $\mathcal{G}_{\mathcal{A}}$ is balanced, and $\mathcal{G}_{\mathcal{A}'}$ is strongly connected and balanced. Let $\lambda_2(\mathcal{L}'_s)$ denote the second smallest eigenvalue of $\mathcal{L}'_s = (\mathcal{L}' + \mathcal{L}'^T)/2$, and*

$$\rho_{min} := (\sqrt{1 - 2\lambda_2(\mathcal{L}'_s) + \|\mathcal{L}'\|^2} + \bar{\tau}\|\mathcal{L}'\|\|\mathcal{L}\| + \zeta\|\mathcal{L}'\|^2)^2.$$

If $\rho_{min} < 1$ and

$$\sigma < \begin{cases} \dfrac{-\sqrt{\rho_{min}}\ln(\rho_{min})}{2\|\mathcal{L}\|}, \; if \; e^{-2} < \rho_{min} < 1, \\ \dfrac{1}{e\|\mathcal{L}\|}, \; if \; \rho_{min} \le e^{-2}, \end{cases} \tag{4.16}$$

then consensus protocol (4.13) leads to the average-consensus for agents in (4.1).

To prove the above results, we introduce the following displacement vector

$$b(t) = x(t) - \mathbf{1}a(t),$$

where $x = (x_1, x_2, ..., x_n)^T$, $\mathbf{1}$ denotes the column n-vector with all ones, and $a(t) = Ave(x(t)) = \frac{1}{n}\sum_{j=1}^{n} x_j(t)$. For balanced graphs $\mathcal{G}_{\mathcal{A}}(t)$ and $\mathcal{G}_{\mathcal{A}'}$, we have $a'(t) = 0$, $\Delta a(t_k) = 0$ and $\mathcal{L}(t)a\mathbf{1} = \mathcal{L}'a\mathbf{1} = 0$, which imply that $a(t)$ is an invariant quantity for $t \ge 0$. Therefore, $b(t)$ evolves according to the following disagreement dynamics:

$$\begin{cases} \dot{b}(t) = -\mathcal{L}(t)b(t - r(t)), \; t \neq t_k, \\ \Delta b(t_k) = -\mathcal{L}'b(t_k - \bar{\tau}), \; k \in \mathbb{N}. \end{cases} \tag{4.17}$$

The consensus analysis is based on the Lyapunov function $V(t) = b^T(t)b(t)$ and a Halanay-type inequality.

It can be seen from Theorems 4.3.1 and 4.3.2 that, to guarantee the consensus, the length σ of each impulsive interval is closely related to the value of ρ_{min}. In the following discussion, we take Theorem 4.3.2 for example. If $\rho_{min} \le e^{-2}$, then the upper bound of σ is $\frac{1}{e\|\mathcal{L}\|}$. If $\rho_{min} \in (e^{-2}, 1)$, define a map $g(\rho) = \frac{-\sqrt{\rho}\ln(\rho)}{2\|\mathcal{L}\|}$, then we have $\dot{g}(\rho) = \frac{-(\ln\rho + 2)}{4\sqrt{\rho}\|L\|} < 0$ for $\rho \in (e^{-2}, 1)$. Therefore, $g(\rho)$ is strictly decreasing on $(e^{-2}, 1)$, i.e., smaller ρ_{min} implies larger upper bound of σ. On the other hand, the value of ρ_{min} depends on the impulsive delay size $\bar{\tau}$ and also the length σ of each impulsive interval. If $\rho_{min} < e^{-2}$, and increase the value of $\bar{\tau}$ such that the corresponding value of ρ_{min} still belongs to $(0, e^{-2})$, then the upper bound of σ remains unchanged. If increasing $\bar{\tau}$ leads to ρ_{min} greater than e^{-2}, then the increase of $\bar{\tau}$ implies decrease of the upper bound for σ which then may cause the increase of ζ. Based on the above discussion and Theorem 4.3.2, we can see that the relation between $\bar{\tau}$ and σ needs to be carefully examined according to (4.16) to guarantee the average consensus.

For a given value of $\bar{\tau}$, Algorithm 2 outlines a computation procedure to obtain the upper bound bdd_σ of the length σ of each impulsive interval from Theorem 4.3.2. If bdd_σ obtained from Algorithm 2 is positive, then any $\sigma < bdd\_\sigma$ can guarantee the proposed protocol (4.13) to solve the average-consensus problem. If the impulse delay $\bar{\tau}$ is not prescribed beforehand, suitable relations between $\bar{\tau}$ and σ can be constructed as follows: vary $\bar{\tau}$ from 0 to $\bar{\tau}_{max}$, where $\bar{\tau}_{max}$ is an estimation of the upper bound for feasible values of $\bar{\tau}$; for each value of $\bar{\tau}$, obtain a corresponding bdd_σ from Algorithm 2; varying $\bar{\tau}$ will then yield a sequence of pairs $\{(\bar{\tau}, bdd\_\sigma)\}$ which demonstrates the admissible relations between $\bar{\tau}$ and σ.

Algorithm 2. Computation the upper bound bdd_σ of σ

---

require: $\mathcal{L}$, $\mathcal{L}'$, $\bar{\tau}$

$\zeta = 0$, $bdd\_\sigma = 0$, $\sigma = 0$

$\rho_{min} = (\sqrt{1 - 2\lambda_2(\mathcal{L}'_s) + \|\mathcal{L}'\|^2} + \bar{\tau}\|\mathcal{L}'\|\|\mathcal{L}\| + \zeta\|\mathcal{L}'\|^2)^2$

while $\rho_{min} < 1$ **do**

    if $\rho_{min} \leq e^{-2}$ **then**

        $\sigma = \frac{1}{e||\mathcal{L}||}$

    else

        $\sigma = \frac{-\sqrt{\rho_{min}}\ln(\rho_{min})}{2\|\mathcal{L}\|}$

    end if

    if $d > (\zeta + 1)\sigma$ **then**

        $bdd_\sigma = 0$

    else if $d \leq \zeta\sigma$ **then**

        $bdd\_\sigma = \frac{d}{\zeta+1}$

    else

        $bdd\_\sigma = \sigma$

    end if

    $\zeta \leftarrow \zeta + 1$

    $\rho_{min} \leftarrow (\sqrt{1 - 2\lambda_2(\mathcal{L}'_s) + \|\mathcal{L}'\|^2} + \bar{\tau}\|\mathcal{L}'\|\|\mathcal{L}\| + \zeta\|\mathcal{L}'\|^2)^2$

end while

return bdd_σ

A similar consensus protocol has been considered in [120] with $\mathcal{G}_{\mathcal{A}} = \mathcal{G}_{\mathcal{A}'}$ and $r = \bar{\tau}$. However, there are many improvements in our consensus protocol (4.13) compared with that in [120]. First, different network topologies are considered in the continuous-time and the discrete-time consensus protocols. Second, the discrete-time network topology is required to be balanced and strongly connected, while the continuous-time topology is only assumed to be balanced. Moreover, the delays in the continuous and discrete protocols are considered to be distinct which is more general than that in [120].

Next, we consider a special case of consensus protocol (4.13), that is, $\mathcal{A} = 0$ (no continuous-time network topology). It can be seen from Theorem 4.3.1 that

$$\rho_{min} := (\sqrt{1 - 2\lambda_2(\mathcal{L}'_s) + \|\mathcal{L}'\|^2} + \zeta\|\mathcal{L}'\|^2)^2,$$

and if $\rho_{min} < 1$, then the consensus protocol (4.13) leads to the average-consensus for agents in (4.1). Actually, if $\rho_{min} < 1$, then

$$\sqrt{1 - 2\lambda_2(\mathcal{L}'_s) + \|\mathcal{L}'\|^2} + \zeta\|\mathcal{L}'\|^2 < 1,$$

i.e., $\zeta < \frac{1-\sqrt{1-2\lambda_2(\mathcal{L}'_s)+\|\mathcal{L}'\|^2}}{\|\mathcal{L}'\|^2}$. According to the definition of ζ, we have

$$\lfloor\frac{\bar{\tau}}{\sigma}\rfloor < \frac{1 - \sqrt{1 - 2\lambda_2(\mathcal{L}'_s) + \|\mathcal{L}'\|^2}}{\|\mathcal{L}'\|^2},$$

which gives the condition on the relation between $\bar{\tau}$, σ and $\mathcal{G}_{\mathcal{A}'}$ to guarantee the average-consensus. We conclude the above analysis by the following corollary.

Corollary 4.3.1 *Suppose that $\mathcal{G}_{\mathcal{A}'}$ is strongly connected and balanced and $\mathcal{A} = 0$ in (4.13). If*

$$\bar{\tau} < \left(\frac{1 - \sqrt{1 - 2\lambda_2(\mathcal{L}'_s) + \|\mathcal{L}'\|^2}}{\|\mathcal{L}'\|^2} + 1\right)\sigma, \tag{4.18}$$

then consensus protocol (4.13) leads to the average-consensus for agents in (4.1).

Networks with Switching Topologies

Next, we consider the consensus problem of multi-agent systems with switching in both the continuous-time topology and the impulsive-time topology.

Denote two finite index sets $P = \{1, 2, ..., p\}$, $Q = \{1, 2, ..., q\}$, and two families of time-invariant digraphs $\Omega = \{\mathcal{G}_i : i \in P\}$, $\Omega' = \{\mathcal{G}'_j : j \in Q\}$. Let $\eta : \mathbb{R}^+ \to P$ be a piecewise constant and right-continuous function called 'continuous-time switching signal', and $\omega : \mathbb{N} \to Q$ be a constant function called 'discrete-time switching signal'. Throughout this subsection, we assume that all the digraphs in Ω are balanced, and all the digraphs in Ω' are strongly connected and balanced, then the collective behavior of system (4.17) can be written as a switching impulsive system

$$\begin{cases} \dot{b}(t) = -\mathcal{L}_{\eta(t)} b(t - r(t)), \ \ t \in [t_k, t_{k+1}), \\ \Delta b(t_k) = -\mathcal{L}'_{\omega(k)} b(t_k - \bar{\tau}), \ \ k \in \mathbb{N}, \end{cases} \tag{4.19}$$

where $\mathcal{L}_i$ (or $\mathcal{L}'_j$) is the graph Laplacian of $\mathcal{G}_i$ (or $\mathcal{G}'_j$) for $i \in P$ (or $j \in Q$). If switchings only occur at impulsive instants (i.e., there is no switching on each impulsive interval), then $\eta(t) = \eta(t_k)$ for $t \in [t_k, t_{k+1})$ and $k \in \mathbb{N}$, and system (4.19) reduces to the following system

$$\begin{cases} \dot{b}(t) = -\mathcal{L}_{\eta(t_k)} b(t - r(t)), \ \ t \in [t_k, t_{k+1}), \\ \Delta b(t_k) = -\mathcal{L}'_{\omega(k)} b(t_k - \bar{\tau}), \ \ k \in \mathbb{N}. \end{cases} \tag{4.20}$$

Denote $l = \max_{i \in P} \|\mathcal{L}_i\|$, $l' = \max_{j \in Q} \|\mathcal{L}'_j\|$, and $l'_s = \min_{j \in Q} \lambda_2(\mathcal{L}'_{js})$, where $\lambda_2(\mathcal{L}'_{js})$ represents the second smallest eigenvalue of $\mathcal{L}'_{js} = (\mathcal{L}'_j + \mathcal{L}'^T_j)/2$, then define

$$\rho = (\sqrt{1 - 2l'_s + l'^2} + \bar{\tau} l' l + \zeta l'^2)^2.$$

In the following result, sufficient conditions are constructed for consensus of multi-agent systems with switching topologies.

Theorem 4.3.3 *If $\rho < 1$ and*

$$\sigma < \begin{cases} \dfrac{-\sqrt{\rho}\ln(\rho)}{2l}, \ \ \text{if } e^{-2} < \rho < 1, \\ \dfrac{1}{el'}, \ \ \text{if } \rho \le e^{-2}, \end{cases} \tag{4.21}$$

then consensus protocol (4.13) leads to the average-consensus for agents in (4.1) under arbitrary switching signals.

In Theorem 4.3.3, the digraphs in Ω are assumed to be time-invariant. However, if the digraphs in Ω are dynamically changing, i.e., $\Omega = \{\mathcal{G}_i(t) = (\mathcal{V}, \mathcal{E}_i(t), \mathcal{A}_i(t)) : i \in P\}$, then we can define a dynamically changing digraph $\mathcal{G}(t) = \mathcal{G}_{\eta(t)}(t)$, according to the continuous-time switching signal. If all the digraphs in Ω are balanced, then the digraph $\mathcal{G}(t)$ is balanced for all $t \geq t_0$. Hence, protocol (4.13) with switching topologies is a special case of protocol (4.13) with dynamically changing continuous-time topology and switching discrete-time topologies. The collective behavior of system (4.17) can then be written as follows

$$\begin{cases} \dot{b}(t) = -\mathcal{L}(t) b(t - r(t)), \ \ t \in [t_k, t_{k+1}), \\ \Delta b(t_k) = -\mathcal{L}'_{\omega(k)} b(t_k - \bar{\tau}), \ \ k \in \mathbb{N}, \end{cases} \tag{4.22}$$

where $\mathcal{L}(t)$ is the Laplacian of digraph $\mathcal{G}(t)$ at time t. We further assume that the weighting factors of $\mathcal{G}(t)$ are uniformly upper bounded, and denote $l = \sup_{t \in [t_0, \infty)} \|\mathcal{L}(t)\|$ and

$$\rho = (\sqrt{1 - 2l'_s + l^2} + nd\bar{\tau} l' + \zeta l'^2)^2,$$

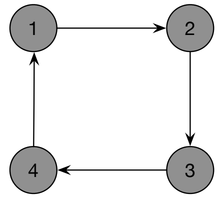


Figure 4.5: Discrete-time topology $\mathcal{G}_{\mathcal{A}'}$.

where l and l' are the same as those defined in Theorem 4.3.3, and d is the same as that defined in Theorem 4.3.1. Then, we can see that protocol (4.13) with dynamically changing continuous-time topology leads to the average-consensus for agents in (4.1) under arbitrary discrete-time switching signals. See Example 4.3.2 for demonstration.

4.3.3 Numerical Simulations

To demonstrate our consensus results, we present two examples of dynamical networks with four agents. In the first example, we consider the impulsive consensus protocol (with $\mathcal{A} = 0$).

Example 4.3.1 *Consider consensus protocol (4.13) with $\mathcal{A} = 0$ and digraph $\mathcal{G}_{\mathcal{A}'}$ shown in Figure 4.5 with $0 - 0.1035$ weights. It can be seen that $\mathcal{G}_{\mathcal{A}'}$ is balanced and strongly connected with $\|\mathcal{L}'\| = 0.2070$, $\lambda_2(\mathcal{L}'_s) = 0.1035$, and $\frac{1-\sqrt{1-2\lambda_2(\mathcal{L}'_s)+\|\mathcal{L}'\|^2}}{\|\mathcal{L}'\|^2} = 2.0013$. Then, Corollary 4.3.1 implies that $\bar{\tau} \leq 3.0013\sigma$ can guarantee the protocol leads to the average-consensus. Figure 4.6 demonstrates the consensus region which describes the feasible relations between $\bar{\tau}$ and σ. The initial conditions are chosen so that $Ave(x(0)) = 0$, $\bar{\tau} = 3$, and $\sigma = 1$, then the average-consensus is confirmed by simulation shown in Figure 4.7.*

In the next example, we consider a hybrid consensus protocol with switching topologies.

Example 4.3.2 *Consider a hybrid consensus protocol with switching topologies shown in Figure 4.8 with $\Omega = \{\mathcal{G}_1(t), \mathcal{G}_2(t)\}$ and $\Omega' = \{\mathcal{G}'_1, \mathcal{G}'_2\}$. Suppose the digraphs in Ω' have $0 - 0.25$ weights, digraph $\mathcal{G}_1(t)$ has $0 - 0.375\sin(t)$ weights, and digraph $\mathcal{G}_2(t)$ has $0 - 0.25\cos(t)$ weights. While the digraphs in Ω' are balanced and strongly connected, the 4th node in the digraphs of Ω is isolated. It can be calculated that $\|\mathcal{L}_1(t)\| = 0.75\sin(t)$, $\|\mathcal{L}_2(t)\| = 0.4430\cos(t)$, and $\|\mathcal{L}'_1\| = \|\mathcal{L}'_2\| = 0.5$, $\lambda_2(\mathcal{L}'_{1s}) = \lambda_2(\mathcal{L}'_{2s}) = 0.25$, then $l' = 0.5$, and $l'_s = 0.25$. Both $\mathcal{L}_1(t)$ and $\mathcal{L}_2(t)$ can be written as products of a trigonometric function and a constant Laplacian, then we can still use inequality (4.31) to replace the estimation of $\Upsilon_2^T \Upsilon_2$ in (4.29). Therefore, Theorem 4.3.2 is applicable to this example by replacing l with $l = \max_{t \geq t_0}\{\mathcal{L}_1(t), \mathcal{L}_2(t)\} = 0.75$.*

In this example, choose $r = 4.5$. Figure 4.9 illustrates the suitable relations between $\bar{\tau}$ and σ to guarantee the network consensus with switching topologies. Next, choose $\bar{\tau} = 0.1$, and $\sigma = 0.12$, then $\zeta = \lfloor \frac{\bar{\tau}}{\sigma} \rfloor = 0$ and $\rho = 0.8164 > e^{-2}$. Since $\frac{-\sqrt{\rho}\ln(\rho)}{2l} = 0.1222 > \sigma = 0.12$, we can conclude from Theorem 4.3.3 that protocol (4.13) with switching topologies leads to the average-consensus under arbitrary switching signals. This is confirmed by the simulation shown in Figure 4.10 with the following

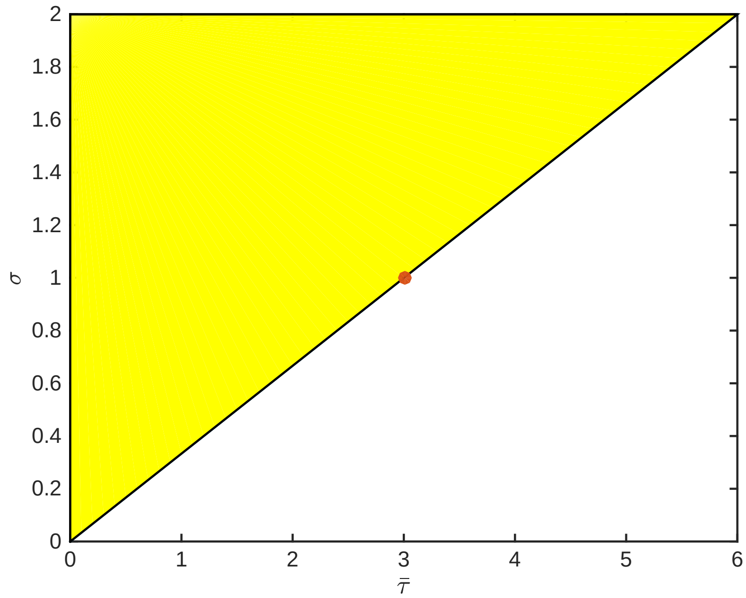


Figure 4.6: Consensus region for Example 4.3.1: if the point $(\bar{\tau}, \sigma)$ is in the yellow region, then protocol (4.13) solves the average-consensus problem. The red dot indicates the point $(\bar{\tau}, \sigma) = (3, 1)$, which is considered in the simulation of Figure 4.7.

continuous-time and discrete-time switching signals:

$$\eta(t) = \begin{cases} 1, \ if \ t \in \cup_{k=1}^{\infty}(3.6k - 1.8, 3.6k), \\ 2, \ if \ t \in \cup_{k=1}^{\infty}(3.6k, 3.6k + 1.8), \end{cases} \tag{4.23}$$

and

$$\omega(k) = \begin{cases} 1, \ mod(k, 30) < 15, \\ 2, \ otherwise, \end{cases} \tag{4.24}$$

where $mod(\cdot, \cdot)$ is the modulo operation which gives the remainder after division. Since the 4th agent in $\mathcal{G}_1$ and $\mathcal{G}_2$ is isolated, no switches between only $\mathcal{G}_1$ and $\mathcal{G}_2$ can achieve the network consensus. Therefore, the impulsive protocols play an important role in the consensus process: the topologies in $\mathcal{G}_1'$ and $\mathcal{G}_2'$ make the network topology to be strongly connected, and then the impulsive protocols realize the consensus convergence.

4.3.4 Proofs

Proof of Theorem 4.3.1

For $t \neq t_k$, take derivative of $V(t)$ along the trajectory of system (4.17), and apply the inequality $x^T y + y^T x \leq \epsilon x^T x + \epsilon^{-1} y^T y$ for any $\epsilon > 0$. Then we have

$$\dot{V}(t) \quad = \quad \dot{b}^T(t) b(t) + b(t) \dot{b}^T(t)$$

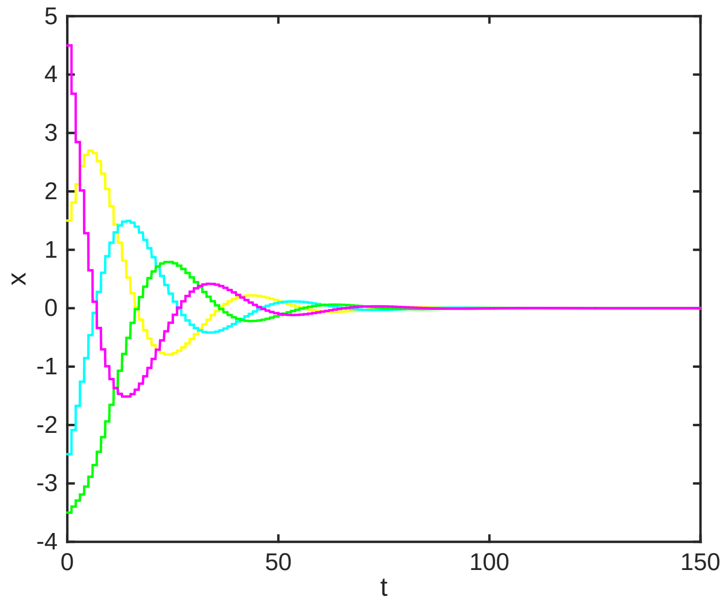
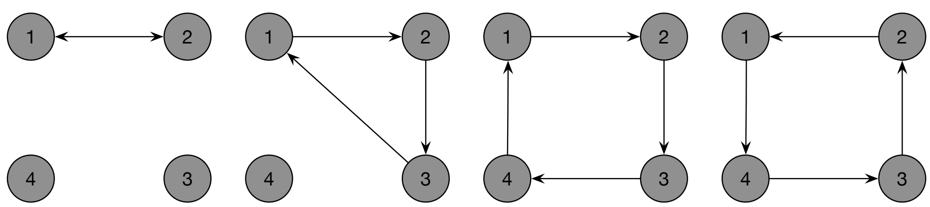


Figure 4.7: Consensus process of Example 4.3.1.



(a) $\mathcal{G}_1$ (b) $\mathcal{G}_2$ (c) $\mathcal{G}_1'$ (d) $\mathcal{G}_2'$

Figure 4.8: Switching topologies

$$\begin{aligned} &= -b^T(t-r(t))\mathcal{L}^T(t)b(t) - b^T(t)\mathcal{L}(t)b(t-r(t)) \\ &\leq \epsilon b^T(t)b(t) + \epsilon^{-1}b^T(t-r(t))\mathcal{L}^T(t)\mathcal{L}(t)b(t-r(t)) \\ &\leq \epsilon V(t) + \epsilon^{-1}\|\mathcal{L}(t)\|^2 V(t-r(t)) \\ &\leq \epsilon V(t) + \epsilon^{-1}l^2 V(t-r(t)) \end{aligned} \tag{4.25}$$

For $t = t_k$, in order to compare $V(t_k)$ with $V(t_k^-)$, we need to estimate the relation between $b(t_k^-)$ and $b(t_k - \bar{\tau})$. To do so, we will integrate both side of (4.17) from $t_k - d$ to t_k. From the definition of ζ, we can see that there are ζ impulses on the interval $(t_k - \bar{\tau}, t_k)$. Next, we conduct the integration process step by step:

S1. Integrating both side of (4.17) from $t_k - \bar{\tau}$ to $t_{k-\zeta}$ yields

$$b(t_{k-\zeta}^-) - b(t_k - \bar{\tau}) = \int_{t_k-\bar{\tau}}^{t_{k-\zeta}} -\mathcal{L}(t)b(t-r(t))\mathrm{d}t,$$

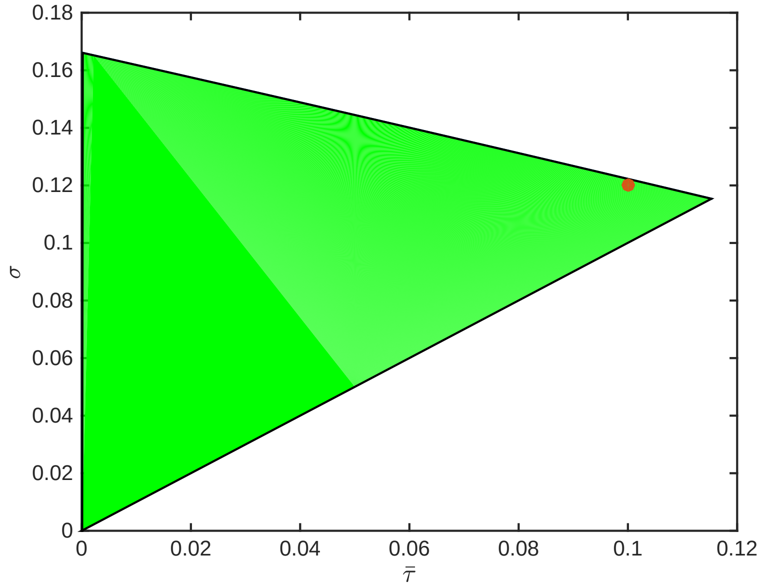


Figure 4.9: Consensus region for Example 4.3.2: if the point $(\bar{\tau}, \sigma)$ is in the green region, then protocol (4.13) solves the average-consensus problem. The red dot indicates the point $(\bar{\tau}, \sigma) = (0.1, 0.12)$, which is considered in the simulation of Figure 4.10.

$$b(t_{k-\zeta}) - b(t^{-}_{k-\zeta}) \;=\; -\mathcal{L}' b(t_{k-\zeta} - \bar{\tau}).$$

S2. Similarly, integrate both side of (4.17) from t_{k-j} to t_{k-j+1} for $j = \zeta, (\zeta - 1), ..., 3, 2$, then we have

$$\begin{aligned} b(t^{-}_{k-j+1}) - b(t_{k-j}) &= \int_{t_{k-j}}^{t_{k-j+1}} -\mathcal{L}(t) b(t - r(t)) \mathrm{d}t, \\ b(t_{k-j+1}) - b(t^{-}_{k-j+1}) &= -\mathcal{L}' b(t_{k-j+1} - \bar{\tau}), \end{aligned}$$

for $j = \zeta, (\zeta - 1), ..., 3, 2$.

S3. Integrating both side of (4.17) from t_{k-1} to t_k leads to

$$b(t^{-}_{k}) - b(t_{k-1}) = \int_{t_{k-1}}^{t_k} -\mathcal{L}(t) b(t - r(t)) \mathrm{d}t.$$

Adding up the equations in the above steps can obtain that

$$b(t_k - \bar{\tau}) = b(t^{-}_{k}) + \int_{t_k - \bar{\tau}}^{t_k} \mathcal{L}(t) b(t - r(t)) \mathrm{d}t + \mathcal{L}' \sum_{i=1}^{\zeta} b(t_{k-i} - \bar{\tau}),$$

then,

$$b(t_k) = b(t^{-}_{k}) - \mathcal{L}' b(t_k - \bar{\tau}) = \Upsilon_1 + \Upsilon_2 + \Upsilon_3, \tag{4.26}$$

where $\Upsilon_1 = (I - \mathcal{L}') b(t^{-}_{k})$, $\Upsilon_2 = -\mathcal{L}' \int_{t_k - \bar{\tau}}^{t_k} \mathcal{L}(t) b(t - r(t)) \mathrm{d}t$, and $\Upsilon_3 = -\mathcal{L}' \mathcal{L}' \sum_{i=1}^{\zeta} b(t_{k-i} - \bar{\tau})$.

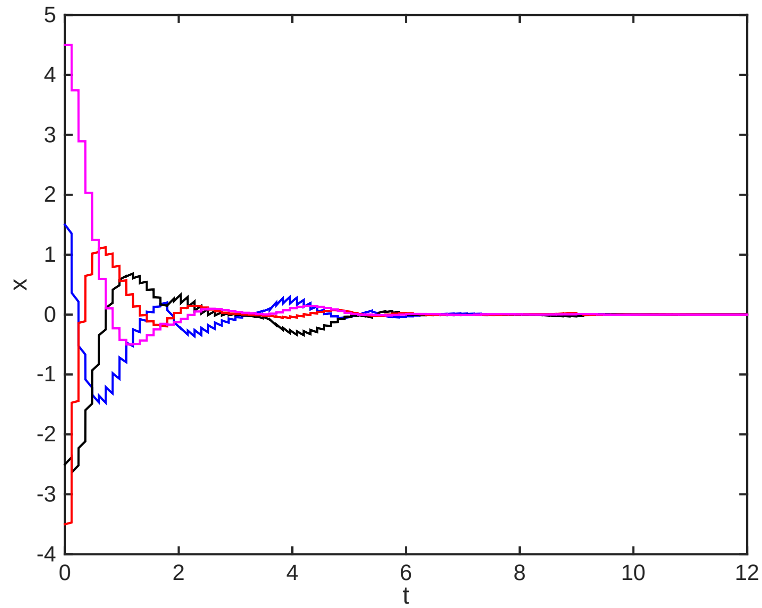


Figure 4.10: Consensus process of Example 4.3.2.

From (4.26) and applying Lemma 5.2.3, we have

$$\begin{aligned} V(t_k) &= b^T(t_k)b(t_k) \\ &= (Y_1+Y_2+Y_3)^T(Y_1+Y_2+Y_3) \\ &\leq (1+\varepsilon)Y_1^TY_1 + (1+\varepsilon^{-1})(1+\xi)Y_2^TY_2 \\ &\quad +(1+\varepsilon^{-1})(1+\xi^{-1})Y_3^TY_3. \end{aligned} \tag{4.27}$$

Since $\mathcal{L}'$ is balanced and strongly connected, we have, for $\mathcal{L}'_s = \frac{1}{2}(\mathcal{L}' + \mathcal{L}'^T)$,

$$0 = \lambda_1(\mathcal{L}'_s) < \lambda_2(\mathcal{L}'_s) \leq ... \leq \lambda_n(\mathcal{L}'_s).$$

Then,

$$\begin{aligned} Y_1^TY_1 &= b^T(t_k^-)(I - \mathcal{L}' - \mathcal{L}'^T + \mathcal{L}'^T\mathcal{L}')b(t_k^-) \\ &\leq (1 - 2\lambda_2(\mathcal{L}'_s) + \|\mathcal{L}'\|^2)b^T(t_k^-)b(t_k^-) \\ &= (1 - 2\lambda_2(\mathcal{L}'_s) + \|\mathcal{L}'\|^2)V(t_k^-). \end{aligned} \tag{4.28}$$

Applying the Schwartz's inequality yields

$$\begin{aligned} Y_2^TY_2 &\leq \|\mathcal{L}'\|^2\Big(\int_{t_k-\bar{\tau}}^{t_k} \mathcal{L}(t)b(t-r(t))\mathrm{d}t\Big)^T\Big(\int_{t_k-\bar{\tau}}^{t_k} \mathcal{L}(t)b(t-r(t))\mathrm{d}t\Big) \\ &= \|\mathcal{L}'\|^2\sum_{i=1}^{n}\Big[\int_{t_k-\bar{\tau}}^{t_k}\sum_{j=1}^{n} l_{ij}(t)b_j(t-r(t))\mathrm{d}t\Big]^2 \end{aligned}$$

$$
\begin{aligned}
&\leq \bar{\tau}\|\mathcal{L}'\|^2 \sum_{i=1}^{n} \int_{t_k-\bar{\tau}}^{t_k} \Big(\sum_{j=1}^{n} l_{ij}(t) b_j(t-r(t)) \Big)^2 \mathrm{d}t \\
&\leq n\bar{\tau}\|\mathcal{L}'\|^2 \sum_{i=1}^{n} \int_{t_k-\bar{\tau}}^{t_k} \sum_{j=1}^{n} l_{ij}^2(t) b_j^2(t-r(t)) \mathrm{d}t \\
&\leq n\bar{\tau}\|\mathcal{L}'\|^2 \sum_{i=1}^{n} \int_{t_k-\bar{\tau}}^{t_k} l_{ii}^2(t) \sum_{j=1}^{n} b_j^2(t-r(t)) \mathrm{d}t \\
&\leq nd^2\bar{\tau}\|\mathcal{L}'\|^2 \sum_{i=1}^{n} \int_{t_k-\bar{\tau}}^{t_k} V(t-r(t)) \mathrm{d}t \\
&\leq (nd\bar{\tau}\|\mathcal{L}'\|)^2 \sup_{s\in[-(\bar{\tau}+\bar{r}),0]} V(t_k^- + s),
\end{aligned} \tag{4.29}
$$

where $l_{ij}(t)$ denotes the (i,j)th entry of $\mathcal{L}(t)$, and then $d_i(t) = l_{ii}(t)$ which is the in-degree of node v_i at time t.

For Y_3, we have

$$
\begin{aligned}
Y_3^T Y_3 &\leq \|\mathcal{L}'\|^4 \sum_{i=1}^{\zeta} b^T(t_{k-i}-\bar{\tau}) \sum_{i=1}^{\zeta} b(t_{k-i}-\bar{\tau}) \\
&\leq \zeta\|\mathcal{L}'\|^4 \sum_{i=1}^{\zeta} b^T(t_{k-i}-\bar{\tau}) b(t_{k-i}-\bar{\tau}) \\
&= \zeta\|\mathcal{L}'\|^4 \sum_{i=1}^{\zeta} V(t_{k-i}-\bar{\tau}) \\
&\leq \zeta^2\|\mathcal{L}'\|^4 \sup_{s\in[-2\bar{\tau},0]} V(t_k^- + s).
\end{aligned} \tag{4.30}
$$

It then follows from (4.28), (4.29), (4.30) and (4.27) that (3.18) holds for $v(t) = V(t)$ and $\tau = \max\{\bar{\tau}+\bar{r}, 2\bar{\tau}\}$ in Lemma 3.3.2 with $\alpha = \epsilon$, $\beta = \epsilon^{-1}l^2$, $\rho_1 = (1+\varepsilon)(1-2\lambda_2(\mathcal{L}_s') + \|\mathcal{L}'\|^2)$, and $\rho_2 = (1+\varepsilon^{-1})[(1+\xi)(nd\bar{\tau}\|\mathcal{L}'\|)^2 + (1+\xi^{-1})\zeta^2\|\mathcal{L}'\|^4]$. Denote $\rho = \rho_1 + \rho_2$, then Lemma 3.3.2 implies that if $\sigma < \frac{\ln(1/\rho)}{\alpha+\beta/\rho}$ then $V(t)$ converges to zero as t goes to infinity, which means the average-consensus will be achieved.

It can be seen that α, β and ρ depend on the parameters ϵ, ε and ξ, respectively. Next, we will specify the values of ϵ, ε and ξ to maximize $\frac{\ln(1/\rho)}{\alpha+\beta/\rho}$ which is the upper bound of the length σ for each impulsive interval.

For any given $\rho \in (0,1)$, to maximize $\frac{\ln(1/\rho)}{\alpha+\beta/\rho}$ is equivalent to minimize $\alpha + \beta/\rho = \epsilon + \frac{1}{\rho}\epsilon^{-1}l^2$ for $\epsilon > 0$. Define the map $H(\epsilon) := \epsilon + \frac{1}{\rho}\epsilon^{-1}l^2$, then $\dot{H}(\epsilon) = 1 - \frac{l^2}{\epsilon^2\rho}$, which implies that, for $\epsilon^* = \frac{l}{\sqrt{\rho}}$, $\dot{H}(\epsilon^*) = 0$ and $H(\epsilon^*) = \frac{2l}{\sqrt{\rho}}$. Thus, for given $\rho > 0$, the maximum of $\frac{\ln(1/\rho)}{\alpha+\beta/\rho}$ is $\frac{-\sqrt{\rho}\ln(\rho)}{2l}$. Next, define a function $G(\rho) := \frac{-\sqrt{\rho}\ln(\rho)}{2l}$ for $\rho \in (0,1)$. Then, $G(\rho) > 0$ and $\dot{G}(\rho) = -\frac{2+\ln(\rho)}{2\sqrt{\rho}}$. Thus, $\dot{G}(e^{-2}) = 0$ and

$$
\dot{G}(\rho) \begin{cases} > 0, \ if \ \rho < e^{-2} \\ < 0, \ if \ \rho > e^{-2}. \end{cases}
$$

It can be seen that $\min_{\varepsilon,\xi>0}\{\rho_1+\rho_2\}=\rho_{min}$, and if $\rho_{min}>e^{-2}$ then

$$\rho \geq \rho_{min} \text{ and } \max_{\rho\in[\rho_{min},1)}\{G(\rho)\} = \frac{-\sqrt{\rho_{min}}\ln(\rho_{min})}{2l}.$$

On the other hand, if $\rho_{min}<e^{-2}$, there exist ε and ξ such that $\rho=e^{-2}$, and then

$$\max_{\rho\in[\rho_{min},1)}\{G(\rho)\} = \frac{1}{el}.$$

Hence, we have

$$\max_{\epsilon,\varepsilon,\xi>0}\left\{\frac{\ln(\frac{1}{\rho})}{\alpha+\frac{\beta}{\rho}}\right\} = \begin{cases} \dfrac{-\sqrt{\rho_{min}}\ln(\rho_{min})}{2l}, \; if\ \rho_{min}>e^{-2}; \\ \dfrac{1}{el}, \; if\ \rho_{min}\leq e^{-2}, \end{cases}$$

which completes the proof. □

Proof of Theorem 4.3.2

Since the Laplacian $\mathcal{L}$ is a constant matrix, the prove is similar to the proof of Theorem 4.3.1 with (4.25) and (4.29) replaced with the following inequalities, respectively.

$$\begin{aligned} V(t) &= \dot{b}^T(t)b(t) + b(t)\dot{b}^T(t) \\ &= -b^T(t-r(t))\mathcal{L}^T b(t) - b^T(t)\mathcal{L}b(t-r(t)) \\ &\leq \epsilon b^T(t)b(t) + \epsilon^{-1}b^T(t-r(t))\mathcal{L}^T\mathcal{L}b(t-r(t)) \\ &\leq \epsilon V(t) + \epsilon^{-1}\|\mathcal{L}\|^2 V(t-r(t)), \end{aligned}$$

and

$$\begin{aligned} \mathrm{Y}_2^T\mathrm{Y}_2 &\leq \|\mathcal{L}'\|^2\|\mathcal{L}\|^2\int_{t_k-\bar{\tau}}^{t_k} b^T(t-r(t))\mathrm{d}t\int_{t_k-\bar{\tau}}^{t_k} b(t-r(t))\mathrm{d}t \\ &= \|\mathcal{L}'\|^2\|\mathcal{L}\|^2\sum_{i=1}^{n}\Big[\int_{t_k-\bar{\tau}}^{t_k} b_i(t-r(t))\mathrm{d}t\Big]^2 \\ &\leq \bar{\tau}\|\mathcal{L}'\|^2\|\mathcal{L}\|^2\sum_{i=1}^{n}\int_{t_k-\bar{\tau}}^{t_k} b_i^2(t-r(t))\mathrm{d}t \\ &= \bar{\tau}\|\mathcal{L}'\|^2\|\mathcal{L}\|^2\int_{t_k-\bar{\tau}}^{t_k}\sum_{i=1}^{n} b_i^2(t-r(t))\mathrm{d}t \\ &= \bar{\tau}\|\mathcal{L}'\|^2\|\mathcal{L}\|^2\int_{t_k-\bar{\tau}}^{t_k} V(t-r(t))\mathrm{d}t \\ &\leq \bar{\tau}^2\|\mathcal{L}'\|^2\|\mathcal{L}\|^2 \sup_{s\in[-(\bar{\tau}+\bar{r}),0]} V(t_k^-+s). \end{aligned} \tag{4.31}$$

The rest of the proof is omitted. □

Proof of Theorem 4.3.3

Choose Lyapunov candidate $V(t) = b^T(t)b(t)$, and repeat the argument in the proof for Theorem 4.3.1, and then we can get that (3.18) holds for $v(t) = V(t)$ and $\tau = \max\{\bar{\tau} + \bar{r}, 2\bar{\tau}\}$ in Theorem 3.3.2 with $\alpha = \epsilon$, $\beta = \epsilon^{-1}l^2$, $\rho_1 = (1+\varepsilon)(1 - 2l'_s + l'^2)$, and $\rho_2 = (1+\varepsilon^{-1})[(1+\xi)(\bar{\tau}l'l)^2 + (1+\xi^{-1})\zeta^2 l'^4]$. The rest of the proof is essentially the same as that in the proof for Theorem 4.3.1, and thus omitted. □

4.4 Impulsive Protocols with Distributed Delays

A networked multi-agent system (NMAS) is a dynamical system consisting of a group of interacting agents, which have their own dynamics, distributed over a network. This section studies the impulsive consensus problem of NMASs with distributed delays in both agent dynamics and impulsive protocols. The objective is to construct sufficient conditions to guarantee the proposed impulsive consensus protocol leads to the consensus of NMASs with distributed delays. The rest of this section is organized as follows. In Subsection 4.4.1, we formulate the consensus problem, and propose a impulsive consensus protocol with distributed delays. Consensus results are established in Subsection 4.4.2 for networks with fixed and switching topologies, respectively. Numerical simulations are provided in Subsection 4.4.3 to demonstrate these theoretical results. Subsection 4.4.4 discussed the detailed proof of our consensus results.

4.4.1 Problem Formulations and Consensus Protocols

Consider a NMAS composed of N agents, where the dynamics of the ith agent are described by a linear system with distributed delay as follows:

$$\begin{cases} \dot{x}_i(t) = Ax_i(t) + B\int_{t-r}^{t} x_i(s)\mathrm{d}s + u_i(t), \\ x_{i,t_0} = \phi_i, \end{cases} \tag{4.32}$$

where $i \in \mathcal{I} := \{1, 2, ..., N\}$, $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^n$ are the state and control input of agent i; A and B are $n \times n$ constant matrices; r is the system delay; x_{i,t_0} is defined as $x_{i,t_0}(s) = x_i(t_0 + s)$ for $s \in [-r, 0]$; $\phi_i \in \mathcal{PC}([-r, 0], \mathbb{R}^n)$ is the initial function.

The control input is designed as the following impulsive controller with distributed delays which is based on digraph $\mathcal{G}_{\mathcal{A}} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$:

$$u_i(t) = \sum_{k=1}^{\infty} \sum_{v_j \in \mathcal{N}_i} \alpha_{ij} \int_{t-d}^{t} [x_j(s) - x_i(s)]\mathrm{d}s\delta(t - t_k), \tag{4.33}$$

where the time sequence $\{t_k\}$ satisfies $\{t_k\} \subseteq \mathbb{R}$, $0 \le t_1 < t_2 < ... < t_k < ...$, and $\lim_{t\to\infty} t_k = \infty$; $\delta(\cdot)$ is the Dirac Delta function; d represents the delay size in each impulse. Throughout this section, we assume that $t_1 - d \ge t_0 - r$, which is straightforward since controller u_i can only obtain information provided by the agent dynamics, and the initial function ϕ_i should be independent of the delays in the designed controller u_i. It is worth noting that controller u_i

only depends on the states of the ith agent v_i and its neighbors, then impulsive controller u_i is called a protocol with topology $\mathcal{G}_{\mathcal{A}}$. Furthermore, we assume that digraph $\mathcal{G}_{\mathcal{A}}$ is strongly connected and balanced.

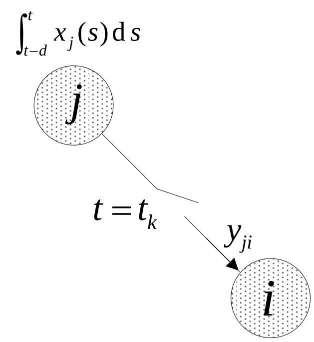


Figure 4.11: Accumulated information measurements at impulsive instants.

The control mechanism of (4.33) works as follows. On each impulsive interval (t_k, t_{k+1}), there are no connections among agents, and each agent's states evolve according to its own dynamics. At impulsive instant t_k, the ith agent receive information from all of its neighbors instantly, that is, the switch in Figure 4.11 on the edge (v_j, v_i) is turned on and then off in a very short time of period, during which information is transfered from agent j to agent i. Since the time between on and off mode of the switch is tiny, we model the short time information exchange by instantaneous information delivery, i.e., agent i receives information from agent j instantly at time t_k. The information y_{ji} (shown in Figure 4.11) transfered to agent i is the accumulated information of agent j, that is, $\int_{t_k-d}^{t_k} x_j(s)\mathrm{d}s$. Different from the impulsive protocols in [36, 63, 120], our consensus protocol does not require each agent's states to be available at specific times (e.g., t_k or $t_k - d$).

For $t = t_k$ and positive constant ϵ satisfying $\epsilon < \min\{t_k - t_{k-1}, t_{k+1} - t_k\}$, it can be seen that there is only one impulse on time period $(t_k - \epsilon, t_k + \epsilon)$. With the proposed impulsive protocol (4.33), integrating both sides of (4.32) from $t_k - \epsilon$ to $t_k + \epsilon$ yields

$$\begin{aligned}\int_{t_k-\epsilon}^{t_k+\epsilon} \dot{x}_i(s)\mathrm{d}s &= \int_{t_k-\epsilon}^{t_k+\epsilon} \big(Ax_i(t) + B\int_{t-r}^{t} x_i(s)\mathrm{d}s\big)\mathrm{d}t \\ &\quad + \int_{t_k-\epsilon}^{t_k+\epsilon} \big(\sum_{v_j\in\mathcal{N}_i} \alpha_{ij}\int_{t-d}^{t}[x_j(s) - x_i(s)]\mathrm{d}s\delta(t-t_k)\big)\mathrm{d}t,\end{aligned}$$

which implies

$$\begin{aligned}x_i(t_k+\epsilon) - x_i(t_k-\epsilon) &= \int_{t_k-\epsilon}^{t_k+\epsilon} \big(Ax_i(t) + B\int_{t-r}^{t} x_i(s)\mathrm{d}s\big)\mathrm{d}t \\ &\quad + \sum_{v_j\in\mathcal{N}_i} \alpha_{ij}\int_{t_k-d}^{t_k}[x_j(s) - x_i(s)]\mathrm{d}s.\end{aligned}$$

Let $\epsilon \to 0^+$, then we have

$$x_i(t_k^+) - x_i(t_k^-) = \sum_{v_j\in\mathcal{N}_i} \alpha_{ij}\int_{t_k-d}^{t_k}[x_j(s) - x_i(s)]\mathrm{d}s,$$

where $x_i(t_k^+)$ and $x_i(t_k^-)$ denote the right and left limit of x_i at t_k. Denote $\Delta x_i(t_k) := x_i(t_k^+) - x_i(t_k^-)$, then system (4.32) can be rewritten as an impulsive system:

$$\begin{cases} \dot{x}_i(t) = Ax_i(t) + B\int_{t-r}^{t} x_i(s)\mathrm{d}s, \ \ t \neq t_k, \\ \Delta x_i(t_k) = \sum_{v_j \in \mathcal{N}_i} \alpha_{ij} \int_{t_k-d}^{t_k} [x_j(s) - x_i(s)]\mathrm{d}s, \ \ k \in \mathbb{N}, \\ x_{i,t_0} = \phi_i. \end{cases} \tag{4.34}$$

Throughout this paper, we suppose x_i is right continuous at t_k, i.e., $x_i(t_k^+) = x_i(t_k)$.

Definition 4.4.1 *We say protocol (4.33) leads to the consensus of NMAS (4.32) (or protocol (4.33) solves the consensus problem), if*

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0, \ \ \forall\, i, j \in \mathcal{I}.$$

Our objective is to construct sufficient conditions to guarantee the proposed protocol u_i solves the consensus problem.

4.4.2 Consensus Results

Throughout this section, we assume that the length of each impulsive interval is fixed, i.e., $\sigma = t_k - t_{k-1}$ for all $k \in \mathbb{N}$. Then, there are $\bar{\zeta}$ impulses on time interval $(t_k - d, t_k)$ for any $k \in \mathbb{N}$, that is, $\bar{\zeta} = \lfloor \frac{d}{\sigma} \rfloor$, where the floor function $\lfloor \chi \rfloor$ gives the largest integer less than χ.

Networks with Fixed Topology

We start by analyzing NMAS (4.34) with fixed topology, i.e., the weighted digraph $\mathcal{G}_{\mathcal{A}}$ is time-invariant with $\mathcal{L}$ as its Laplacian.

Theorem 4.4.1 *Suppose that $\mathcal{G}_{\mathcal{A}}$ is balanced and strongly connected. Let $\lambda_2(\hat{\mathcal{L}})$ denotes the second smallest eigenvalue of $\hat{\mathcal{L}} = (\mathcal{L} + \mathcal{L}^T)/2$ and*

$$\rho_{min} = \Big(\sqrt{1 - 2d\lambda_2(\hat{\mathcal{L}}) + d^2\|\mathcal{L}\|^2} + d^2\|\mathcal{L}\|(\|A\| + r\|B\|) + d^2\|\mathcal{L}\|^2 \sqrt{\sigma \sum_{m=1}^{\bar{\zeta}} m^2} \Big)^2.$$

Let $a = \lambda_{max}(A + A^T)$ and $b = 2r\|B\|$, and assume $\rho_{min} < 1$. Then, consensus protocol (4.33) leads to the consensus for agents in (4.32) if either of the following conditions are satisfied:

i) $a + b = 0$ *and* $\sigma < 1/b$.

ii) $a+b>0$ and

$$\sigma < \begin{cases} -\dfrac{\ln \rho_{min}}{a+b/\sqrt{\rho_{min}}}, & \textit{if } \rho_{min} > \rho^*, \\ -\dfrac{\ln \rho^*}{a+b/\sqrt{\rho^*}}, & \textit{if } \rho_{min} \le \rho^*, \end{cases}$$

where $\rho = \rho^*$ is the unique solution of algebraic equation:

$$2(a\sqrt{\rho}+b) + b\ln\rho = 0, \textit{ for } \rho \in (0,1).$$

To prove this theorem, we shall transform the consensus problem of NMAS into a stability analysis problem of an impulsive system.

Denote I_n (or I_N) the $n\times n$ (or $N\times N$) identity matrix. Let $\bar{x}(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t) = \frac{1}{N}(\mathbf{1}^T \otimes I_n)x$, where $\mathbf{1} = (1,1,...,1)^T \in \mathbb{R}^N$, $x = (x_1^T, x_2^T, ..., x_N^T)^T$, and $\otimes$ represents the Kronecker product. Then, for $t = t_k$, we have

$$\begin{aligned} \bar{x}(t_k^+) &= \frac{1}{N}(\mathbf{1}^T \otimes I_n)x(t_k^+) \\ &= \frac{1}{N}(\mathbf{1}^T \otimes I_n)\Big(x(t_k^-) - (\mathcal{L}\otimes I_n)\int_{t_k-d}^{t_k} x(s)\mathrm{d}s\Big) \\ &= \frac{1}{N}(\mathbf{1}^T \otimes I_n)x(t_k^-) - \frac{1}{N}(\mathbf{1}^T \otimes I_n)(\mathcal{L}\otimes I_n)\int_{t_k-d}^{t_k} x(s)\mathrm{d}s \\ &= \bar{x}(t_k^-), \end{aligned}$$

since $(\mathbf{1}^T \otimes I_n)(\mathcal{L}\otimes I_n) = (\mathbf{1}^T\mathcal{L})\otimes I_n$, and $\mathbf{1}^T\mathcal{L}$ is a zero vector. It can be seen that the dynamics of $\bar{x}$ satisfies the following equations:

$$\begin{cases} \dot{\bar{x}}(t) = A\bar{x}(t) + B\displaystyle\int_{t-r}^{t} \bar{x}(s)\mathrm{d}s, \ t \neq t_k, \\ \bar{x}(t_k^+) = \bar{x}(t_k^-), \ k \in \mathbb{N}. \end{cases} \tag{4.35}$$

Denote $e_i = x_i - \bar{x}$ and $e = (e_1^T, e_2^T, ..., e_N^T)^T$, then, from (4.35), we can get the dynamics of e_i described as follows:

$$\begin{cases} \dot{e}_i(t) = Ae_i(t) + B\displaystyle\int_{t-r}^{t} e_i(s)\mathrm{d}s, \ \ t \neq t_k, \\ \Delta e_i(t_k) = \displaystyle\sum_{v_j\in\mathcal{N}_i} \alpha_{ij}\int_{t_k-d}^{t_k} [e_j(s)-e_i(s)]\mathrm{d}s, \ \ k\in\mathbb{N}, \\ e_{i,t_0} = \varphi_i, \end{cases} \tag{4.36}$$

where $\varphi_i(s) = \phi_i(s) - \frac{1}{N}\sum_{j=1}^{N}\phi_j(s)$ for $s\in[-r,0]$. Then the dynamics of the NMAS error state $e = (e_1^T, e_2^T, ..., e_N^T)^T$ can be described by the following compact form of impulsive system

$$\begin{cases} \dot{e}(t) = \bar{A}e(t) + \bar{B}\displaystyle\int_{t-r}^{t} e(s)\mathrm{d}s, \ \ t \neq t_k, \\ \Delta e(t_k) = -\bar{\mathcal{L}}\displaystyle\int_{t_k-d}^{t_k} e(s)\mathrm{d}s, \ \ k\in\mathbb{N}, \\ e_{t_0} = \varphi, \end{cases} \tag{4.37}$$

where $\bar{A} = I_N \otimes A$, $\bar{B} = I_N \otimes B$, $\bar{\mathcal{L}} = \mathcal{L} \otimes I_n$, and $\varphi = (\varphi_1^T, \varphi_2^T, ..., \varphi_N^T)^T \in \mathcal{PC}([-r, 0], \mathbb{R}^{nN})$.

It can be seen that if the trivial solution of impulsive system (4.37) is globally asymptotically stable, then $\lim_{t\to\infty} \|e(t)\| = 0$, which implies $\lim_{t\to\infty} \|e_i(t)\| = 0$, and then

$$\begin{aligned} \lim_{t\to\infty} \|x_i(t) - x_j(t)\| &= \lim_{t\to\infty} \|x_i(t) - \bar{x}(t) - (x_j(t) - \bar{x}(t))\| \\ &= \lim_{t\to\infty} \|e_i(t) - e_j(t)\| = 0, \end{aligned}$$

that is, the consensus is reached. However, we apply a global exponential stability result to derive this consensus criterion, and the detailed proof for Theorem 4.4.1 is included in Subsection 4.4.4.

Networks with Switching Topologies

To model the dynamic changing of the topology structures, we consider the consensus problem of NMASs with switching in the topology of impulsive protocol (4.33).

Denote a finite index set $Q = \{1, 2, ..., q\}$ with $q \in \mathbb{N}$, and a family of weighted digraphs $\Omega = \{\mathcal{G}_i : i \in Q\}$. Let $\omega : \mathbb{N} \to Q$ be a constant function called 'discrete-time switching signal'. Throughout this subsection, we assume that all the digraphs in Ω are strongly connected and balanced, then the collective behavior of system (4.37) can be written as the following impulsive system

$$\begin{cases} \dot{e}(t) = \bar{A}e(t) + \bar{B}\displaystyle\int_{t-r}^{t} e(s)\mathrm{d}s, & t \neq t_k, \\ \Delta e(t_k) = -\bar{\mathcal{L}}_{\omega(k)}\displaystyle\int_{t_k-d}^{t_k} e(s)\mathrm{d}s, & k \in \mathbb{N}, \\ e_{t_0} = \varphi, \end{cases} \tag{4.38}$$

where $\bar{\mathcal{L}}_i = \mathcal{L}_i \otimes I_n$, and $\mathcal{L}_i$ is the graph Laplacian of $\mathcal{G}_i$ for $i \in Q$.

Denote

$$l = \max_{i\in Q} \|\mathcal{L}_i\|, \; \hat{l} = \min_{i\in Q} \lambda_2(\hat{\mathcal{L}}_i),$$

where $\lambda_2(\hat{\mathcal{L}}_i)$ represents the second smallest eigenvalue of $\hat{\mathcal{L}}_i = (\mathcal{L}_i + \mathcal{L}_i^T)/2$, then define

$$\rho_s = \left(\sqrt{1 - 2d\hat{l} + d^2 l^2} + d^2 l(\|A\| + r\|B\|) + d^2 l^2 \sqrt{\sigma \sum_{m=1}^{\bar{\zeta}} m^2} \right)^2.$$

Theorem 4.4.2 *Suppose $\rho_s < 1$, then consensus protocol (4.33) leads to the consensus for agents in (4.32) under arbitrary switching signals if either of the following conditions are satisfied:*

i) $a + b = 0$ and $\sigma < 1/b$.

ii) $a+b>0$ *and*

$$\sigma < \begin{cases} -\dfrac{\ln \rho_s}{a+b/\sqrt{\rho_s}}, & \text{if } \rho_s > \rho^*, \\ -\dfrac{\ln \rho^*}{a+b/\sqrt{\rho^*}}, & \text{if } \rho_s \le \rho^*, \end{cases}$$

where ρ^* *is the same as that defined in Theorem 4.4.1.*

4.4.3 Numerical Simulations

Consider dynamical networks with four agents. Figure 4.12 shows two topologies denoted by $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Both of the digraphs in the figure have $0-10$ weights, and they are also strongly connected and balanced. It can be calculated that $\lambda_2(\hat{\mathcal{L}}_1) = \lambda_2(\hat{\mathcal{L}}_2) = 10$, and $\|\mathcal{L}_1\| = \|\mathcal{L}_2\| = 20$. Furthermore, consider

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 & 0.4 \\ -0.4 & 0.6 \end{bmatrix},$$

and $r=0.1$, then $\|A\| + r\|B\| = 2.9661$. It can be seen that $a+b = 5.4601 > 0$. For impulses, we consider $d = 0.02$, and impulsive sequence $\{t_k\}$ is chosen as $\sigma = t_k - t_{k-1} = 0.01$ for all $k \in \mathbb{N}$. From the definition of $\bar{\zeta}$, we have $\bar{\zeta} = 1$, that is, there is only one impulse on each interval $(t_k - d, t_k)$.

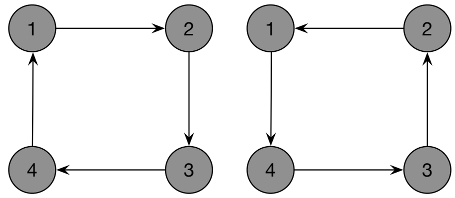
With the above given parameters, we can calculate that $\rho_{min} = 0.938$, and then all the conditions of Theorem 4.4.1 are satisfied. For the switching scenario, $\rho = \rho_{min}$ and conditions of Theorem 4.4.2 holds. We simulate three different situations with initial functions chosen as $\phi(s) = (\phi_1^T, \phi_2^T, \phi_3^T, \phi_4^T)^T = (1,3,6,4,7,2,2,1)^T$ for all $s \in [-r, 0]$.

First, we consider the impulsive consensus protocol with fixed topologies $\mathcal{G}_1$ and $\mathcal{G}_2$, which are illustrated in Figure 4.13(a) and 4.13(b), respectively. It is shown in these figures that the states of each agent tend to converge to each other, and consensus is achieved with the phase portraits of the agent error states converging to zero in anticlockwise (or clockwise) directions according to the network topologies $\mathcal{G}_1$ (or $\mathcal{G}_2$).

Next, we consider the situation where the network topologies are switching between $\mathcal{G}_1$ and $\mathcal{G}_2$ with periodic switching signal given in Figure 4.14. It can be seen that the agent states converge to zero in a anticlockwise-clockwise direction switching mode shown in Figure 4.13(c), the reason for which is that digraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ share the same connection structure but the information transfers among agents in reverse directions. Figure 4.13(d) shows that the state trajectories of system (4.37), which clearly demonstrates that the agent error states converge to zero, and consensus is reached.

4.4.4 Proofs

In this subsection, we will present the proofs for the main results which rely on a Razumikhin-type stability result for nonlinear impulsive functional differential equations.



(a) $\mathcal{G}_1$ (b) $\mathcal{G}_2$

Figure 4.12: Network topologies with 4 agents.

Proof of Theorem 4.4.1

Rewrite impulsive system (4.37) into the following form of feedback control system:

$$\dot{e}(t) = \bar{A}e(t) + \bar{B}\int_{t-r}^{t} e(s)\mathrm{d}s + \sum_{k=1}^{\infty} -\bar{\mathcal{L}}\int_{t-d}^{t} e(s)\mathrm{d}s\delta(t-t_k), \text{ for } t \geq t_0. \tag{4.39}$$

For $t \in [t_k - d, t_k)$, integrating both side of (4.39) from t to t_k^- yields

$$e(t_k^-) - e(t) = \int_{t}^{t_k}[\bar{A}e(s) + \bar{B}\int_{-r}^{0} e(s+\theta)\mathrm{d}\theta]\mathrm{d}s - \bar{\mathcal{L}}\sum_{m=1}^{\zeta(t)}\int_{t_{k-m}-d}^{t_{k-m}} e(s)\mathrm{d}s, \tag{4.40}$$

where ζ denotes the number of impulses on (t, t_k). Therefore, ζ depends on t, and is a piecewise constant function defined as follows:

$$\zeta(t) = \begin{cases} 0, & \text{if } t \in [t_{k-1}, t_k), \\ 1, & \text{if } t \in [t_{k-2}, t_{k-1}), \\ \vdots, & \vdots \\ \vdots, & \vdots \\ \bar{\zeta}-1, & \text{if } t \in [t_{k-\bar{\zeta}}, t_{k-\bar{\zeta}+1}), \\ \bar{\zeta}, & \text{if } t \in [t_k - d, t_{k-\bar{\zeta}}), \end{cases} \tag{4.41}$$

which is illustrated in Figure 4.15.

Next, integrate both side of (4.40) from $t_k - d$ to t_k^- to get that

$$\begin{aligned} de(t_k^-) - \int_{t_k-d}^{t_k} e(s)\mathrm{d}s &= \int_{t_k-d}^{t_k}\left(\int_{t}^{t_k}[\bar{A}e(s) + \bar{B}\int_{-r}^{0} e(s+\theta)\mathrm{d}\theta]\mathrm{d}s\right)\mathrm{d}t \\ &\quad -\bar{\mathcal{L}}\int_{t_k-d}^{t_k}\left(\sum_{m=1}^{\zeta(t)}\int_{t_{k-m}-d}^{t_{k-m}} e(s)\mathrm{d}s\right)\mathrm{d}t. \end{aligned} \tag{4.42}$$

For $t = t_k$, we can obtain from (4.37) that

$$e(t_k) = e(t_k^-) - \bar{\mathcal{L}}\int_{t_k-d}^{t_k} e(s)\mathrm{d}s. \tag{4.43}$$

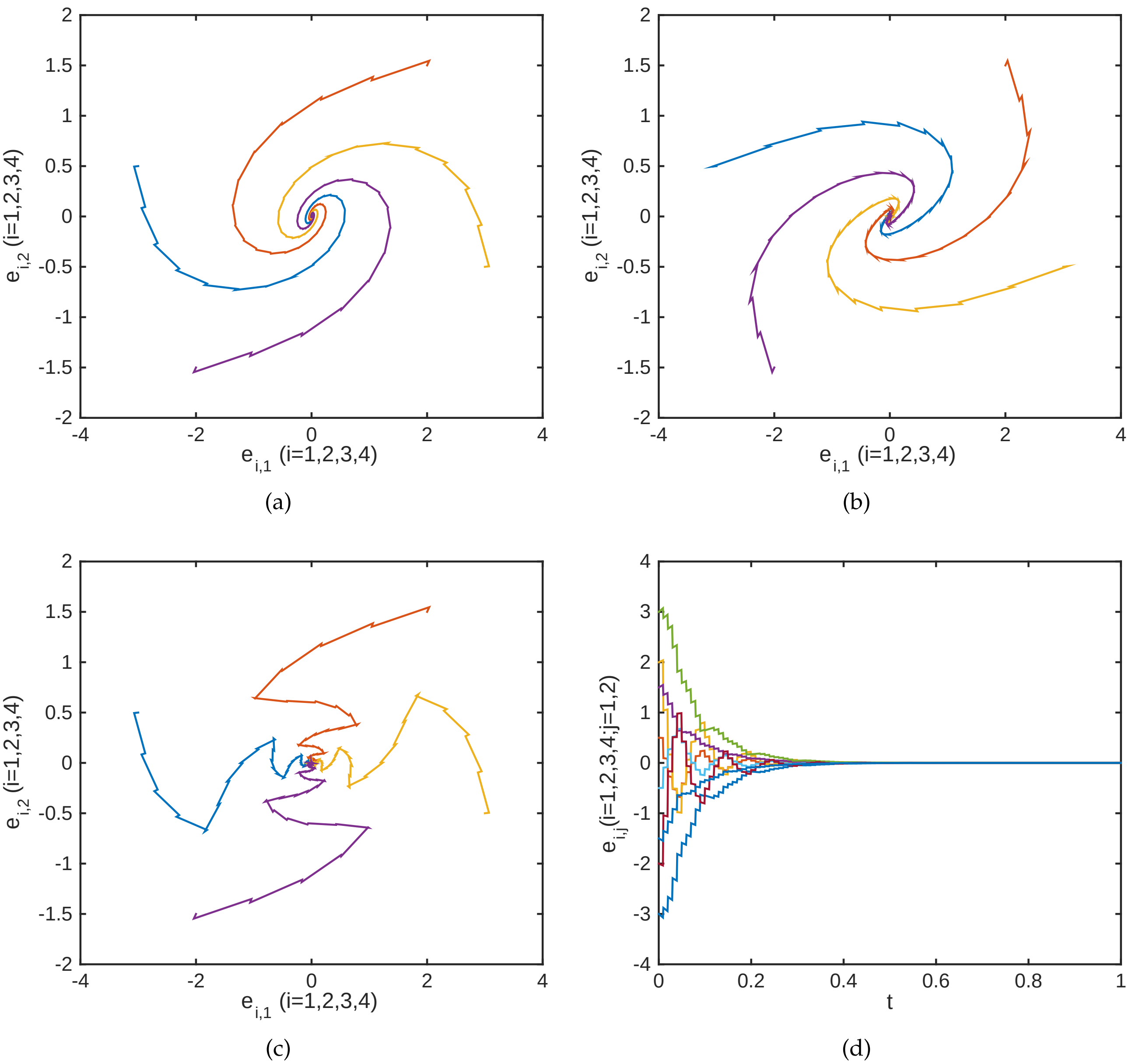


Figure 4.13: Consensus processes: (a) impulsive consensus with digraph $\mathcal{G}_1$; (b) impulsive consensus with digraph $\mathcal{G}_2$; (c) impulsive consensus with switchings between digraphs $\mathcal{G}_1$ and $\mathcal{G}_2$; (d) state trajectories of the error states $e_{i,j}$ $(i = 1, 2, 3, 4$ and $j = 1, 2)$.

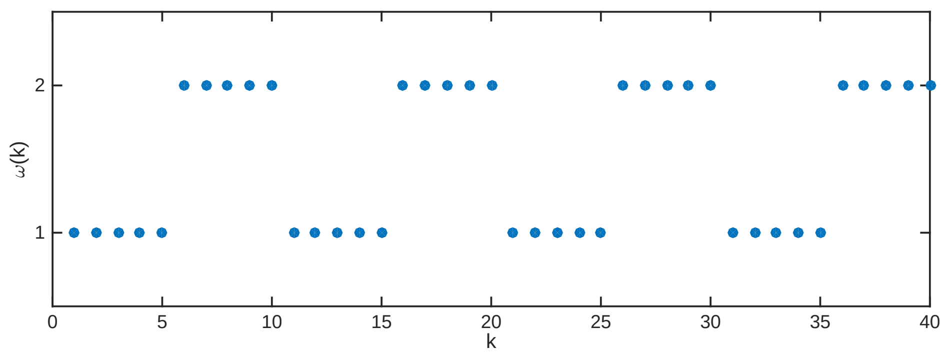


Figure 4.14: Periodic switching signal $\omega(k)$ for $k \in \mathbb{N}$.

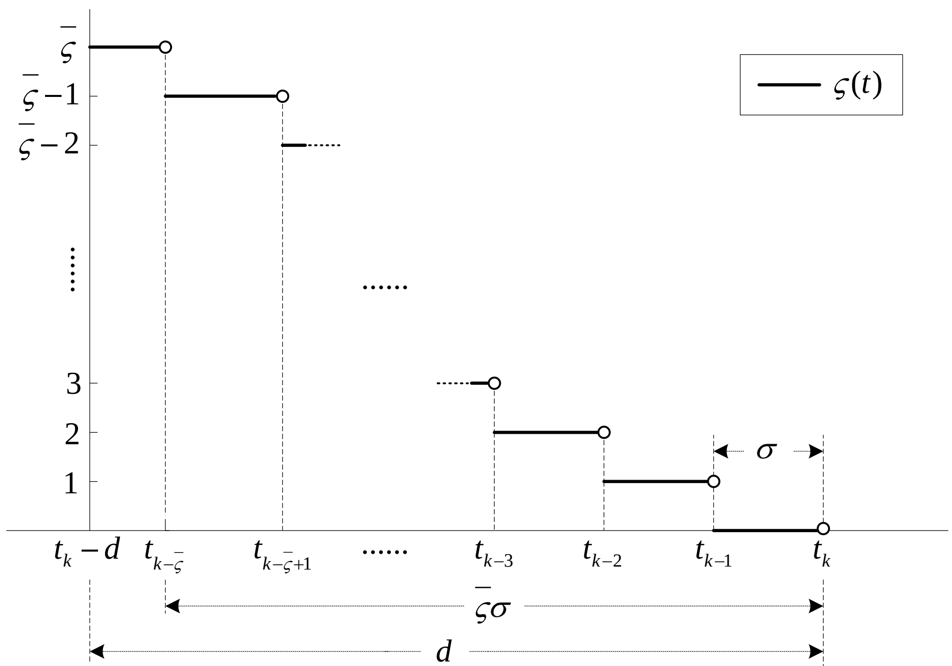


Figure 4.15: Illustration of function $\zeta(t)$ for $t \in [t_k - d, t_k)$ and relations between time $t_k - d$ and different impulsive instants.

The estimation of the integral $\int_{t_k-d}^{t_k} e(s)\mathrm{d}s$ can be derived from (4.42), and then substitution of it into (4.43) gives

$$e(t_k) = \Upsilon_1 + \Upsilon_2 + \Upsilon_3, \tag{4.44}$$

with

$$\begin{aligned} \Upsilon_1 &= (I - d\bar{\mathcal{L}})e(t_k^-), \\ \Upsilon_2 &= \bar{\mathcal{L}} \int_{t_k-d}^{t_k} \left(\int_t^{t_k} [\bar{A}e(s) + \bar{B} \int_{-r}^{0} e(s+\theta)\mathrm{d}\theta]\mathrm{d}s \right) \mathrm{d}t, \end{aligned}$$

$$Y_3 = -\bar{\mathcal{L}}^2 \int_{t_k-d}^{t_k} \Big(\sum_{m=1}^{\zeta(t)} \int_{t_{k-m}-d}^{t_{k-m}} e(s) \mathrm{d}s \Big) \mathrm{d}t,$$

where I is the $nN \times nN$ identity matrix.

Choose the Lyapunov function $V(t, e(t)) = e^T(t)e(t)$, and let $v(t) = V(t, e(t))$, then condition (i) of Theorem 3.3.1 is satisfied with $w_1 = w_2 = 1$ and $p = 2$. Applying the inequality $(x+y+z)^T(x+y+z) \le (1+\varepsilon)x^Tx + (1+\varepsilon^{-1})(1+\xi)y^Ty + (1+\varepsilon^{-1})(1+\xi^{-1})z^Tz$ for any $\varepsilon, \xi > 0$, we have

$$\begin{aligned} v(t_k) &= e^T(t_k)e(t_k) \\ &= (Y_1 + Y_2 + Y_3)^T(Y_1 + Y_2 + Y_3) \\ &\le (1+\varepsilon)Y_1^TY_1 + (1+\varepsilon^{-1})(1+\xi)Y_2^TY_2 \\ &\quad +(1+\varepsilon^{-1})(1+\xi^{-1})Y_3^TY_3. \end{aligned} \tag{4.45}$$

Since $\mathcal{L}$ is balanced and strongly connected, we have, for $\hat{\mathcal{L}} = \frac{1}{2}(\mathcal{L} + \mathcal{L}^T)$,

$$0 = \lambda_1(\hat{\mathcal{L}}) < \lambda_2(\hat{\mathcal{L}}) \le ... \le \lambda_N(\hat{\mathcal{L}}).$$

Then, from the properties of Kronecker product, we have

$$\begin{aligned} Y_1^TY_1 &= e^T(t_k^-)(I - d\bar{\mathcal{L}} - d\bar{\mathcal{L}}^T + d^2\bar{\mathcal{L}}^T\bar{\mathcal{L}})e(t_k^-) \\ &= (1 - 2d\lambda_2(\hat{\mathcal{L}}) + d^2\|\mathcal{L}\|^2)v(t_k^-). \end{aligned} \tag{4.46}$$

Applying the Schwartz's inequality twice yields

$$\begin{aligned} Y_2^TY_2 &\le d\int_{t_k-d}^{t_k} \Big(\bar{\mathcal{L}} \int_t^{t_k} [\bar{A}e(s) + \bar{B}\int_{-r}^0 e(s+\theta)\mathrm{d}\theta]\mathrm{d}s\Big)^T \\ &\qquad \Big(\bar{\mathcal{L}} \int_t^{t_k} [\bar{A}e(s) + \bar{B}\int_{-r}^0 e(s+\theta)\mathrm{d}\theta]\mathrm{d}s\Big)\mathrm{d}t \\ &\le d\int_{t_k-d}^{t_k} (t_k - t)\Big(\int_t^{t_k} [\bar{\mathcal{L}}\bar{A}e(s) + \bar{\mathcal{L}}\bar{B}\int_{-r}^0 e(s+\theta)\mathrm{d}\theta]^T \\ &\qquad [\bar{\mathcal{L}}\bar{A}e(s) + \bar{\mathcal{L}}\bar{B}\int_{-r}^0 e(s+\theta)\mathrm{d}\theta]\mathrm{d}s\Big)\mathrm{d}t \\ &\le d^2\int_{t_k-d}^{t_k} \Big(\int_t^{t_k} [\bar{\mathcal{L}}\bar{A}e(s) + \bar{\mathcal{L}}\bar{B}\int_{-r}^0 e(s+\theta)\mathrm{d}\theta]^T \\ &\qquad [\bar{\mathcal{L}}\bar{A}e(s) + \bar{\mathcal{L}}\bar{B}\int_{-r}^0 e(s+\theta)\mathrm{d}\theta]\mathrm{d}s\Big)\mathrm{d}t \\ &\le d^3\int_{t_k-d}^{t_k} [\bar{\mathcal{L}}\bar{A}e(s) + \bar{\mathcal{L}}\bar{B}\int_{-r}^0 e(s+\theta)\mathrm{d}\theta]^T[\bar{\mathcal{L}}\bar{A}e(s) + \bar{\mathcal{L}}\bar{B}\int_{-r}^0 e(s+\theta)\mathrm{d}\theta]\mathrm{d}s \\ &\le d^3\int_{t_k-d}^{t_k} \Big((1+\kappa)e^T(s)[(\mathcal{L}^T\mathcal{L}) \otimes (A^TA)]e(s) \end{aligned}$$

$$
\begin{aligned}
&+ (1+\kappa^{-1})\int_{-r}^{0} e^T(s+\theta)\mathrm{d}\theta[(\mathcal{L}^T\mathcal{L})\otimes(B^TB)]\int_{-r}^{0} e(s+\theta)\mathrm{d}\theta\Big)\mathrm{d}s \\
\leq\;& d^3\|\mathcal{L}\|^2\int_{t_k-d}^{t_k}[(1+\kappa)\|A\|^2 v(s)+(1+\kappa^{-1})r\|B\|^2\int_{-r}^{0} v(s+\theta)\mathrm{d}\theta]\mathrm{d}s \\
\leq\;& d^4\|\mathcal{L}\|^2\Big[(1+\kappa)\|A\|^2\sup_{s\in[-d,0]}\{v(t_k^-+s)\}+(1+\kappa^{-1})r^2\|B\|^2\sup_{s\in[-r-d,0]}\{v(t_k^-+s)\}\Big] \\
\leq\;& d^4\|\mathcal{L}\|^2[(1+\kappa)\|A\|^2+(1+\kappa^{-1})r^2\|B\|^2]\sup_{s\in[-\tau_1,0]}\{v(t_k^-+s)\}, \qquad (4.47)
\end{aligned}
$$

where $\tau_1 = d+r$ and $\kappa > 0$. To minimize the right-hand side of inequality (4.47), choose $\kappa = \frac{r\|B\|}{\|A\|}$, then we have

$$
\mathrm{Y}_2^T\mathrm{Y}_2 \leq d^4\|\mathcal{L}\|^2(\|A\|+r\|B\|)^2\sup_{s\in[-\tau_1,0]}\{v(t_k^-+s)\}. \qquad (4.48)
$$

For Y_3, we have

$$
\begin{aligned}
\mathrm{Y}_3^T\mathrm{Y}_3 \;\leq\;& d\|\mathcal{L}\|^4\int_{t_k-d}^{t_k}\Big(\sum_{m=1}^{\zeta(t)}\int_{t_{k-m}-d}^{t_{k-m}} e(s)\mathrm{d}s\Big)^T\Big(\sum_{m=1}^{\zeta(t)}\int_{t_{k-m}-d}^{t_{k-m}} e(s)\mathrm{d}s\Big)\mathrm{d}t \\
\leq\;& d\|\mathcal{L}\|^4\int_{t_k-d}^{t_k}\zeta(t)\sum_{m=1}^{\zeta(t)}\Big(\int_{t_{k-m}-d}^{t_{k-m}} e(s)\mathrm{d}s\Big)^T\Big(\int_{t_{k-m}-d}^{t_{k-m}} e(s)\mathrm{d}s\Big)\mathrm{d}t \\
\leq\;& d^2\|\mathcal{L}\|^4\int_{t_k-d}^{t_k}\zeta(t)\Big(\sum_{m=1}^{\zeta(t)}\int_{t_{k-m}-d}^{t_{k-m}} e^T(s)e(s)\mathrm{d}s\Big)\mathrm{d}t \\
\leq\;& d^3\|\mathcal{L}\|^4\int_{t_k-d}^{t_k}\zeta(t)\Big(\sum_{m=1}^{\zeta(t)}\sup_{s\in[-d,0]}\{v(t_{k-m}^-+s)\}\Big)\mathrm{d}t \\
\leq\;& d^3\|\mathcal{L}\|^4\sup_{s\in[-2d,0]}\{v(t_k^-+s)\}\int_{t_k-d}^{t_k}\zeta^2(t)\mathrm{d}t \\
\leq\;& d^4\|\mathcal{L}\|^4\Big(\sigma\sum_{m=1}^{\bar{\zeta}} m^2\Big)\sup_{s\in[-\tau_2,0]}\{v(t_k^-+s)\}, \qquad (4.49)
\end{aligned}
$$

where $\tau_2 = 2d$. By the definition of ζ in (4.41) and its illustration in Figure 4.15, the estimation the integral regarding to $\zeta(t)$ in the derivation of (4.49) is as follows:

$$
\begin{aligned}
\int_{t_k-d}^{t_k}\zeta^2(t)\mathrm{d}t \;=\;& \sigma(1+4+...+(\bar{\zeta}-1)^2)+(d-\bar{\zeta}\sigma)\bar{\zeta}^2 \\
\leq\;& \sigma(1+4+...+(\bar{\zeta}-1)^2)+\sigma\bar{\zeta}^2 \\
=\;& \sigma\sum_{m=1}^{\bar{\zeta}} m^2.
\end{aligned}
$$

From (4.46), (4.48), (4.49), and (4.45), we can obtain that

$$
v(t_k) \;\leq\; (1+\varepsilon)\gamma_1 v(t_k^-)+(1+\varepsilon^{-1})(1+\xi)\gamma_2\sup_{s\in[-\tau_1,0]}\{v(t_k^-+s)\}
$$

$$
\begin{aligned}
& + (1+\varepsilon^{-1})(1+\xi^{-1})\gamma_3 \sup_{s\in[-\tau_2,0]}\{v(t_k^- + s)\} \\
\leq \;& \rho_1 v(t_k^-) + \rho_2 \sup_{s\in[-\tau,0]}\{v(t_k^- + s)\},
\end{aligned} \tag{4.50}
$$

where $\tau = \max\{\tau_1, \tau_2\}$, and

$$
\begin{aligned}
\rho_1 &= (1+\varepsilon)\gamma_1, \\
\rho_2 &= (1+\varepsilon^{-1})[(1+\xi)\gamma_2 + (1+\xi^{-1})\gamma_3],
\end{aligned}
$$

with

$$
\begin{aligned}
\gamma_1 &= 1 - 2d\lambda_2(\hat{\mathcal{L}}) + d^2\|\mathcal{L}\|^2, \\
\gamma_2 &= d^4\|\mathcal{L}\|^2(\|A\| + r\|B\|)^2, \\
\gamma_3 &= d^4\|\mathcal{L}\|^4\Big(\sigma\sum_{m=1}^{\bar{\zeta}} m^2\Big).
\end{aligned}
$$

For $t \neq t_k$, take the derivative of $v(t)$ along the trajectory of system (4.37), apply the inequality $2x^Ty \leq \epsilon x^Tx + \epsilon^{-1}y^Ty$ for any $\epsilon > 0$, and use the properties of the Kronecker product, then we have

$$
\begin{aligned}
\dot{v}(t) &\leq [\bar{A}e(t) + \bar{B}\int_{t-r}^{t} e(s)\mathrm{d}s]^T e(t) + e^T(t)[\bar{A}e(t) + \bar{B}\int_{t-r}^{t} e(s)\mathrm{d}s] \\
&= e^T(t)[I_N \otimes (A + A^T)]e(t) + 2e^T(t)[I_N \otimes B]\int_{t-r}^{t} e(s)\mathrm{d}s \\
&\leq e^T(t)[I_N \otimes (A + A^T)]e(t) + \epsilon e^T(t)[I_N \otimes (BB^T)]e(t) \\
&\quad + \epsilon^{-1}\int_{t-r}^{t} e^T(s)\mathrm{d}s \int_{t-r}^{t} e(s)\mathrm{d}s \\
&\leq e^T(t)[I_N \otimes (A + A^T + \epsilon BB^T)]e(t) + \epsilon^{-1} r\int_{t-r}^{t} e^T(s)e(s)\mathrm{d}s \\
&\leq \alpha v(t) + \beta \sup_{s\in[-r,0]}\{v(t+s)\} \\
&\leq \alpha v(t) + \beta \sup_{s\in[-\tau,0]}\{v(t+s)\}, \text{ for } t \neq t_k,
\end{aligned} \tag{4.51}
$$

where $\alpha = \lambda_{max}(A + A^T) + \epsilon\|B\|^2$ and $\beta = \epsilon^{-1}r^2$. Since $a + b = \lambda_{max}(A + A^T) + 2r\|B\| \geq 0$, we have $\alpha + \beta \geq 0$.

We conclude the above discussion with the following inequalities:

$$
\dot{v}(t) \leq \alpha v(t) + \beta \sup_{s\in[-\tau,0]}\{v(t+s)\}, \; t \neq t_k, \tag{4.52a}
$$

$$
v(t_k) \leq \rho_1 v(t_k^-) + \rho_2 \sup_{s\in[-\tau,0]}\{v(t_k^- + s)\}, \; k \in \mathbb{N}, \tag{4.52b}
$$

with $\alpha+\beta \geq 0$, and $\rho_1, \rho_2 > 0$. If $\rho_1+\rho_2 < 1$ and

$$\frac{1}{\rho_1+\rho_2} > e^{(\alpha+\frac{\beta}{\rho_1+\rho_2})\sigma}, \tag{4.53}$$

then there exists a constant q such that

$$q > \frac{1}{\rho_1+\rho_2} > e^{(\alpha+q\beta)\sigma} > e^{(\alpha+\frac{\beta}{\rho_1+\rho_2})\sigma}.$$

For $t \neq t_k$, if $v(t+s) \leq qv(t)$ for all $s \in [-\tau, 0]$, then (4.52a) implies that $v'(t) \leq \alpha v(t) + \beta \sup_{s\in[-\tau,0]}\{v(t+s)\} \leq cv(t)$, with constant $c = \alpha + q\beta > 0$. Thus, inequalities (4.52a) are related to the Razumikhin-type condition (ii) in Theorem 3.3.1. And then, all the conditions of Theorem 3.3.1 are satisfied. Before making conclusion from Theorem 3.3.1, we need to clarify that the delay is τ in (4.52a) and (4.52b), which is greater than the system delays r and d in (4.37) since $\tau = \max\{d+r, 2d\}$. However, the stability result introduced in Theorem 3.3.1 is still valid for system (4.37) with estimations (4.52a) and (4.52b). The reason is that system (4.37) is actually a particular case of system (3.1) with delay size τ: let $y = e$, then

- for $t \neq t_k$,

$$\begin{aligned} y' &= \bar{A}y + \bar{B}\int_{t-r}^{t} y(s)\mathrm{d}s \\ &= \bar{A}y + \bar{B}\int_{-r}^{0} y(t+s)\mathrm{d}s \\ &= \bar{A}y + \bar{B}\int_{-r}^{0} y_t(s)\mathrm{d}s \\ &\stackrel{def}{=} f(t, y_t); \end{aligned}$$

- for $t = t_k$,

$$\begin{aligned} \Delta y(t_k) &= -\bar{\mathcal{L}}\int_{t_k-d}^{t_k} y(s)\mathrm{d}s \\ &= -\bar{\mathcal{L}}\int_{-d}^{0} y(t_k^- + s)\mathrm{d}s \\ &= -\bar{\mathcal{L}}\int_{-d}^{0} y_{t_k^-}(s)\mathrm{d}s \\ &\stackrel{def}{=} I_k(t_k, y_{t_k^-}); \end{aligned}$$

- the initial function ψ can be defined as

$$\psi(s) \stackrel{def}{=} \begin{cases} \varphi(s), & \text{if } s \in [-r, 0], \\ 0, & \text{if } s \in [-\tau, -r). \end{cases}$$

Therefore, we can conclude that, if $\rho_1 + \rho_2 < 1$ and (4.53) hold, then system (4.37) is GES, which implies that protocol (4.33) leads to the consensus for agents in (4.32).

Nevertheless, there are three positive constants ε, ξ, and ϵ to be determined in (4.53). Denote $\rho = \rho_1 + \rho_2$, then (4.53) implies that $\sigma < \frac{\ln(1/\rho)}{\alpha+\beta/\rho}$. For given dynamics of each agent in (4.32) and network topology in (4.33), we will specify values of ε, ξ, and ϵ by maximizing $\frac{\ln(1/\rho)}{\alpha+\beta/\rho}$ which is the upper bound of the length σ for the impulsive interval.

For any $\rho \in (0,1)$, to maximize $\frac{\ln(1/\rho)}{\alpha+\beta/\rho}$ is equivalent to minimize $\alpha + \beta/\rho = \lambda_{max}(A + A^T) + \epsilon\|B\|^2 + \epsilon^{-1}r^2/\rho$ for $\epsilon > 0$. Define the map $H(\epsilon) := \lambda_{max}(A + A^T) + \epsilon\|B\|^2 + \epsilon^{-1}r^2/\rho$, then $H'(\epsilon) = \|B\|^2 - \epsilon^{-2}r^2/\rho$, which implies that, for $\epsilon^* = \frac{r}{\sqrt{\rho}\|B\|}$, $H'(\epsilon^*) = 0$ and $H(\epsilon^*) = \lambda_{max}(A + A^T) + \frac{2r\|B\|}{\sqrt{\rho}}$. Hence, for given $\rho \in (0,1)$, we have

$$\max_{\epsilon>0}\left\{\frac{\ln(1/\rho)}{\alpha+\beta/\rho}\right\} = \frac{\ln(1/\rho)}{a+b/\sqrt{\rho}},$$

where $a = \lambda_{max}(A + A^T)$ and $b = 2r\|B\|$.

Next, define a function $G(\rho) := \frac{\ln(1/\rho)}{a+b/\sqrt{\rho}}$, then $G'(\rho) = -\frac{2(a\sqrt{\rho}+b)+b\ln\rho}{2\sqrt{\rho}(a\sqrt{\rho}+b)^2}$. Define function $F(\rho) := -2(a\sqrt{\rho} + b) - b\ln\rho$, then $F'(\rho) = -\frac{a\sqrt{\rho}+b}{\rho} < 0$ for $\rho \in (0,1)$.

If $a + b = 0$, then $F(1) = 0$ and we have that $G'(\rho) > 0$ for $\rho \in (0,1)$, that is, $G(\rho)$ is strictly increasing on $(0,1)$. Moreover, we can yield from $\rho_{min} < 1$ that there exist $\varepsilon, \xi > 0$ such that $\rho_1 + \rho_2 = \bar{\rho}$ for any $\bar{\rho} \in (\rho_{min}, 1)$. Therefore, $\sigma < \sup_{\rho\in[\rho_{min},1)}\{G(\rho)\} = \lim_{\rho\to 1^-} G(\rho) = 1/b$.

On the other hand, if $a + b > 0$, then $F(1) < 0$. Since there exists a small enough $\hat{\rho} \in (0,1)$ such that $F(\hat{\rho}) > 0$, and F is strictly monotone on $(0,1)$, there exists a unique solution $\rho = \rho^*$ of the following algebraic equation:

$$2(a\sqrt{\rho} + b) + b\ln\rho = 0, \text{ for } \rho \in (0,1), \tag{4.54}$$

then we have

$$G'(\rho) = \begin{cases} > 0, & \text{if } \rho \in (0, \rho^*), \\ < 0, & \text{if } \rho \in (\rho^*, 1). \end{cases}$$

It can be seen that

$$\min_{\varepsilon,\xi>0}\{\rho_1 + \rho_2\} = (\sqrt{\gamma_1} + \sqrt{\gamma_2} + \sqrt{\gamma_3})^2 = \rho_{min}.$$

If $\rho_{min} > \rho^*$, then $\rho > \rho^*$ for any $\varepsilon, \xi > 0$. Thus, $\max_{\rho\in[\rho_{min},1)}\{G(\rho)\} = -\frac{\ln\rho_{min}}{a+b/\sqrt{\rho_{min}}}$. If $\rho_{min} \le \rho^*$, then there exist positive ε and ξ such that $\rho = \rho^*$, and then $\max_{\rho\in[\rho_{min},1)}\{G(\rho)\} = -\frac{\ln\rho^*}{a+b/\sqrt{\rho^*}}$.

Concluding the above selection process yields

$$\sigma < \max_{\epsilon,\varepsilon,\xi>0}\left\{\frac{\ln(1/\rho)}{\alpha+\beta/\rho}\right\} = \begin{cases} -\dfrac{\ln\rho_{min}}{a+b/\sqrt{\rho_{min}}}, & \text{if } \rho_{min} > \rho^*, \\ -\dfrac{\ln\rho^*}{a+b/\sqrt{\rho^*}}, & \text{if } \rho_{min} \le \rho^*. \end{cases}$$

The proof is completed. □

Remark 4.4.1 *If $a + b = \lambda_{max}(A + A^T) + 2r\|B\| < 0$, then there exists a constant $\epsilon > 0$ such that $\lambda_{max}(A + A^T) + \epsilon\|B\| + \epsilon^{-1}r^2 < 0$. From (4.51), we have*

$$\dot{v}(t) \leq \alpha v(t) + \beta \sup_{s\in[-r,0]} \{v(t+s)\},$$

with $\alpha + \beta < 0$. By the Hanalay inequality (Lemma on page 378, [38]), we have $v(t)$ will converge to zero exponentially as $t \to \infty$, when $u_i(t) \equiv 0$ for all $i \in \mathcal{I}$ and $t \geq t_0$. This means that each agent system is exponentially stable. The consensus will be achieved even no control input is added to this isolated network. Therefore, in Theorem 4.4.1, we have only studied the case of $a + b \geq 0$.

Proof of Theorem 4.4.2

Choose Lyapunov function $v(t) = e^T(t)e(t)$, and repeat the similar argument as presented in the previous subsection, then we can get that (4.52b) holds for $\rho_1 = (1+\varepsilon)(1 - 2d\hat{l} + d^2l^2)$ and $\rho_2 = (1+\varepsilon^{-1})[(1+\xi)d^4l^2(\|A\| + r\|B\|)^2 + (1+\xi^{-1})d^4l^4\sigma\sum_{m=1}^{\bar{\zeta}} m^2]$. The rest of the proof is essentially the same as the proof of Theorem 4.4.1, and thus omitted. □

Chapter 5

Stabilization and Synchronization of Dynamical Networks

This chapter studies stabilization and synchronization problems of DNs under pinning impulsive control. Throughout this chapter, a pinning algorithm is incorporated with the impulsive control approach. In Section 5.1, we propose a delay-dependent pinning impulsive controller to investigate the synchronization of linear delay-free DNs on time scales. Then, in Subsection 5.2.2, we apply the pinning impulsive controller proposed in Section 5.1 to stabilize time-delay DNs. Results in these two sections show that the delay-dependent pinning impulsive controller can successfully stabilize and synchronize DNs with/without time-delay. However, the pinning impulsive controller depends on the network states at both impulsive instants and history times, that is, the contributions of time-delay states to the stabilization or synchronization processes can not be observed explicitly. Therefore, in Subsection 5.2.3, we design a type of pinning impulsive controls relies only on the network states at history moments (not on the states at each impulsive instant). Results show that the proposed pinning impulsive controller can effectively stabilize the network even though only states at history moments are available to the pinning controller at each impulsive instants. In Section 5.1 and 5.2, only discrete delays are considered in the impulsive controllers. We further consider pinning impulsive controllers with both discrete and distributed time-delay effects, in Section 5.3, to synchronize the drive and response systems modeled by globally Lipschitz time-delay systems. All the theoretical results are illustrated by numerical simulations, accordingly.

5.1 Synchronization of Delay-Free Dynamical Networks

During the past decades, the method of impulsive control has been successfully used for synchronization of both continuous and discrete DNs (see, e.g., [68, 80, 134]). It is clear to see that the continuous and discrete networks are normally investigated separately, and the results concerning discrete DNs are carried quite easily from the corresponding results of their continuous counterparts. Therefore, it is natural to consider whether it is possible to provide a framework to study both the continuous and discrete DNs simultaneously. On the other hand, from the modeling and numerical points of view (see, e.g., [4, 104]), it is more realistic to model a network by DN which incorporates both continuous and discrete times. The recently developed theory of time scales, which was initialed by Stefan Hilger in his Ph.D. thesis in 1988, offers the desired unified method. The purpose of this theory is to unify the existing theory of continuous and discrete dynamical systems, and extend these theories to dynamical systems on generalized hybrid (continuous/discrete) domains. The theory of time scales has gained much attention and is undergoing rapid development in diverse areas (see, e.g., [10], [89], [104]).

Recently, neural networks on time scales have attracted increasing interest, and stability and synchronization of different kinds of DNs on time scales have been studied (see, e.g., [45, 70]). In this section, we investigate the synchronization problem of DNs on time scales. A pinning impulsive control scheme that takes into account of time-delay effects is designed to achieve synchronization of DNs on time scales with the state of an isolated node. Based on the theory of time scales and the direct Lyapunov method, a synchronization criterion is established for linear DNs on general time scales. Our result shows that, by impulsive control a small portion of nodes, the consensus of DNs on time scales can be achieved. According to our pinning impulsive control scheme, different numbers of nodes will be selected at each impulsive instant and time-delay is considered in the pinning impulses. The modeling framework developed in this section is a unification and generalization of many existing continuous-time and discrete-time DN models, while the pinning impulsive control scheme is an extension of the existing control scheme for synchronization of continuous-time DNs. Moreover, the idea of studying dynamical systems on time scales provide a unified approach to investigate continuous-time system and its discrete-time counterpart simultaneously.

The outline of this section is as follows. In Subsection 5.1.1, we introduce some basic knowledge for the theory of time scales. In Subsection 5.1.2, we formulate the problem of synchronization for linear DNs on time scales, and propose the pinning delayed-impulsive control strategy. In Subsection 5.1.3, an impulsive synchronization criterion is established for linear DNs on general time scales. In Subsection 5.1.4, numerical simulations are given to illustrate the effectiveness of the proposed control algorithm.

5.1.1 Preliminaries on Time Scales

In this subsection, we recall some basic definitions and properties of time scales which are used in what follows. Let $\mathbb{T}$ be a **time scale** (an arbitrary nonempty closed subset of the real number set $\mathbb{R}$). We assume that $\mathbb{T}$ is a topological space with relative topology induced from $\mathbb{R}$. If $a, b \in \mathbb{T}$, we then define the **interval** $[a, b]$ in $\mathbb{T}$ by $[a, b] := \{t \in \mathbb{T} : a \leq t \leq b\}$. Open intervals and half-open intervals etc. are defined accordingly.

Definition 5.1.1 *The mappings $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined as*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

and

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$

are called forward and backward jump operators, respectively.

A non-maximal element $t \in \mathbb{T}$ is **right-scattered** if $\sigma(t) > t$ and **right-dense (rd)** if $\sigma(t) = t$. A non-minimal element $t \in \mathbb{T}$ is **left-scattered** if $\rho(t) < t$ and **left-dense** if $\rho(t) = t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$, otherwise, $\mathbb{T}^k = \mathbb{T}$. The **graininess function** $\mu : \mathbb{T} \to \mathbb{R}^+$ is defined by $\mu(t) = \sigma(t) - t$.

Definition 5.1.2 *For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of $y(t)$, $y^\Delta(t)$, to be the number (when it exists) with the property that for any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \bigcap \mathbb{T}$ for some $\delta > 0$) such that*

$$|y(\sigma(t)) - y(s) - y^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in U$.

A function $f : \mathbb{T} \to \mathbb{R}$ is **rd-continuous** provided it is continuous at right-dense points in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R})$. If f is **continuous** at each right-dense point and each left-dense point, f is said to be continuous function on $\mathbb{T}$.

Definition 5.1.3 *Let $f \in C_{rd}$. A function $g : \mathbb{T} \to \mathbb{R}$ is called the anti-derivative of f on $\mathbb{T}$ if it is differentiable on $\mathbb{T}$ and satisfies $g^\Delta(t) = f(t)$ for $t \in \mathbb{T}$. In this case, we define*

$$\int_a^t f(s)\Delta s = g(t) - g(a),$$

where $t, a \in \mathbb{T}$.

We say that a function $p : \mathbb{T} \to \mathbb{R}$ is **regressive** provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$ holds. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted in this paper by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$, and the set of all positively regressive elements of $\mathcal{R}$ is denoted by $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Definition 5.1.4 *If $p \in \mathcal{R}$, then we define the exponential function on time scale $\mathbb{T}$ by $e_p(t,s) = \exp\big(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\big)$, for $t, s \in \mathbb{T}$, where the cylinder transformation*

$$\xi_h(z) = \begin{cases} \dfrac{Log(1+hz)}{h}, & h \neq 0 \\ z, & h = 0 \end{cases}$$

where Log is the natural logarithm function.

Remark 5.1.1 *Let $\alpha \in \mathcal{R}$ be constant. If $\mathbb{T} = \mathbb{Z}$, then $e_\alpha(t, t_0) = (1+\alpha)^{t-t_0}$ for all $t \in \mathbb{T}$. If $\mathbb{T} = \mathbb{R}$, then $e_\alpha(t, t_0) = e^{\alpha(t-t_0)}$ for all $t \in \mathbb{T}$. If $\alpha \geq 0$, then $e_\alpha(t,s) \geq 1$ for $t \geq s$ and $t, s \in \mathbb{T}$. Moreover, for $t, s, r \in \mathbb{T}$, $e_\alpha(t,s) = \frac{1}{e_\alpha(s,t)}$ and $e_\alpha(t,r)e_\alpha(r,s) = e_\alpha(t,s)$, which will be used in the proof of main result in this paper.*

In the sequel, we present two lemmas from [16] which will be essential to prove our main result.

Lemma 5.1.1 *If $f \in C_{rd}$ and $t \in \mathbb{T}^k$, then*

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t).$$

Remark 5.1.2 *If $p \in \mathcal{R}$ and $t \in \mathbb{T}^k$, then, from Definition 5.1.4 and Remark 5.1.1, we have*

$$e_p(\sigma(t), t) = 1 + \mu(t)p(t).$$

Lemma 5.1.2 *Let $f \in C_{rd}$ and $p \in \mathcal{R}^+$. Then, for all $t \in \mathbb{T}$, inequality $y^\Delta(t) \leq p(t)y(t) + f(t)$ implies that*

$$y(t) \leq y(t_0)e_p(t, t_0) + \int_{t_0}^{t} e_p(t, \sigma(\tau))f(\tau)\Delta\tau.$$

5.1.2 Problem Formulation

Consider the linear DN of N identical nodes (n-dimensional dynamic systems) on time scale $\mathbb{T}$

$$x_i^\Delta = Ax_i + c\sum_{j=1}^{N} g_{ij}x_j, \quad i = 1, 2, ..., N, \tag{5.1}$$

where $x_i = (x_{i1}, x_{i2}, ..., x_{in})^T \in \mathbb{R}^n$ is the state vector of the ith node, A is a $n \times n$ matrix, c is the coupling strength of the network, the coupling configuration matrix $G = (g_{ij})_{N\times N}$ represents the connection topology of the network and is defined as follows: if there is a connection between the ith node and the jth node ($i \neq j$), then $g_{ij} = g_{ji} = 1$, otherwise, $g_{ij} = g_{ji} = 0$, and the diagonal elements are defined as $g_{ii} = -\sum_{j=1, j\neq i}^{N} g_{ij}$.

Clearly, the isolated node of network (5.1) is in the form of

$$y^\Delta = Ay, \quad t \in \mathbb{T}. \tag{5.2}$$

Let $s(t)$ be the state of an isolated node: $s^\Delta = As$. Our goal is to design a pinning impulsive control scheme to achieve the synchronization among the node states $x_i(t)$ and the objective state $s(t)$, namely, $\lim_{t\to\infty} \|x_i(t) - s(t)\| = 0$, for all $i = 1, 2, ..., N$.

Consider the DN (5.1) under the feedback control,

$$x_i^\Delta = Ax_i + c\sum_{j=1}^{N} g_{ij}x_j + u_i(t, x_i, s), \tag{5.3}$$

for $i \in \mathcal{I} := \{1, 2, ..., N\}$, where $\{u_i(t, x_i, s),\ i \in \mathcal{I}\}$ is the pinning impulsive controller designed as follows

$$u_i = \begin{cases} \sum_{k=1}^{\infty}[q_{1k}y_i(t) + q_{2k}y_i(t - \tau_k)]\delta(t - t_k),\ i \in \mathfrak{D}_k \subseteq \mathcal{I}, \\ 0,\ i \notin \mathfrak{D}_k. \end{cases}$$

Here, the constant q_{1k} and q_{2k} are the impulsive control gains to be determined, and $\delta(\cdot)$ is the Dirac delta function. The impulsive instant sequence $\{t_k\}$ satisfies $\{t_k\} \subset \mathbb{T}$, $0 = t_0 < t_1 < t_2 < ... < t_k < ...$, and $\lim_{t\to\infty} t_k = \infty$. $\tau_k \geq 0$ denotes the time delay in the pinning impulsive controller u_i at time t_k, and there exists a constant $\tau > 0$ such that $\tau_k \leq \tau$ for all $k \in \mathbb{N}$. The time sequence $\{t_k - \tau_k\}$ satisfies $t_1 - \tau \geq t_0$ and $t_k - \tau_k \in \mathbb{T}$. $y_i(t) = x_i(t) - s(t)$ is the error state of the ith node at time t, and l_k denotes the number of nodes to be controlled at each impulsive instant t_k. The index set $\mathfrak{D}_k = \{p_1, p_2, ..., p_{l_k}\} \subseteq \mathcal{I}$ is defined as follows: $p_i \neq p_j$ if $i \neq j$; at the impulsive instant t_k, $\|y_i(t_k)\| \geq \|y_j(t_k)\|$ if $i \in \mathfrak{D}_k$ and $j \in \mathcal{I}/\mathfrak{D}_k$. Then, we have $\sharp\mathfrak{D}_k = l_k$. The pinning impulsive control mechanism can be explained as follows: at each impulsive instant, we only control l_k nodes that have larger deviations with the trivial state than the rest $n - l_k$ nodes. Throughout the rest of this thesis, we will study various pinning impulsive control problems evolving around this type of impulsive pinning algorithm.

Remark 5.1.3 *Definitions of $\mathfrak{D}_k$ and $\sharp\mathfrak{D}_k$ are borrowed from [67]. However, our control scheme is more general than the control schemes in [67], since the number l_k of nodes controlled at different impulsive instants are different and the existence of time delay in the pinning controller. Recently, many results about pinning impulsive control of diverse dynamical networks have been reported in the literature (see, [46, 67, 66, 108, 136]). However, the results in [46, 136] have some essential errors, see Remark 3.11 of [125] for details. The results in [66] is not applicable to synchronize the dynamical networks without impulsive effects, and in the results of [108], the pinning adaptive controller played a key role in the synchronization process. It is worth noting that no time delay is considered in the above mentioned pinning impulsive control algorithms. Moreover, it is well known that the existence of time delay is a double-edged sword to the dynamic performance of systems. Therefore, it is worthwhile to study systems subject to delayed impulses. See the numerical example in Subsection 5.1.4 for detailed discussion of stabilizing delayed impulses and delayed impulsive perturbations.*

By the properties of the Dirac delta function $\delta(\cdot)$, system (5.3) can be rewritten as the following impulsive system,

$$\begin{cases} x_i^{\Delta} = Ax_i + c\sum_{j=1}^{N} g_{ij}x_j, \ t \neq t_k, \\ \Delta x_i(t_k) = q_{1k}y_i(t_k) + q_{2k}y_i(t_k - \tau_k), \ i \in \mathfrak{D}_k, \end{cases} \tag{5.4}$$

where $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$, $x_i(t_k^+)$ and $x_i(t_k^-)$ denote the right and left limit of x_i at t_k, respectively. In this section, we assume that $x_i(t_k^-) = x_i(t_k)$. Without loss of generality, in the following subsection, all the impulsive instants t_k are assumed to be right-dense on time scale $\mathbb{T}$.

Since the synchronization error is defined as $y_i(t) := x_i(t) - s(t)$, we have the following error system

$$\begin{cases} y_i^{\Delta} = Ay_i + c\sum_{j=1}^{N} g_{ij}y_j, \ t \neq t_k, \\ \Delta y_i(t_k) = q_{1k}y_i(t_k) + q_{2k}y_i(t_k - \tau_k), \ i \in \mathfrak{D}_k, \end{cases} \tag{5.5}$$

Hence, DN (5.1) can achieve synchronization with $s(t)$ if and only if $\|y_i\| \to \infty$ as $t \to \infty$.

5.1.3 Synchronization Results

In this subsection, we shall establish an impulsive synchronization criterion for linear DN (5.1).

Theorem 5.1.1 *Assume that there exist constants $a > 0$, $\varepsilon_k > 0$ and α_{ik} ($i = 1, 2, ..., N$, $k \in \mathbb{N}$) such that*

(i) $H_i^T + H_i + \mu H_i^T H_i \leq \alpha_{ik} I_n$ *for* $t \in (t_k, t_{k+1})$, *where* $H_i = A + c\lambda_i I_n$ *and* $\lambda_1, \lambda_2, ..., \lambda_N$ *are eigenvalues of matrix* G;

(ii) $(\rho_{1k} + \rho_{2k} e^{a\tau_k}) e^{a(t_{k+1}-t_k)} e_{\alpha_k}(t_{k+1}, t_k) \leq 1$, *where* $\rho_{1k} = 1 - \frac{l_k}{N}[1 - (1+\varepsilon_k)(1+q_{1k})^2]$, $\rho_{2k} = (1+\varepsilon_k^{-1}) q_{2k}^2$, *and* $\alpha_k = \max\{0, \alpha_{1k}, \alpha_{2k}, ..., \alpha_{Nk}\}$.

Then, DN (5.1) can achieve synchronization with $s(t)$.

Proof: Define $y(t) = (y_1^T(t), y_2^T(t), ..., y_N^T(t))^T$, then the error system (5.5) can be rewritten as follows

$$\begin{cases} y^{\Delta}(t) = (I_N \otimes A)y(t) + c(G \otimes I_n)y(t), \ t \neq t_k, \\ \Delta y_i(t_k) = q_{1k} y_i(t_k) + q_{2k} y_i(t_k - \tau_k), \ i \in \mathfrak{D}_k, \end{cases} \tag{5.6}$$

where $\otimes$ is the Kronecker product. By matrix decomposition theory, there exists an orthogonal matrix $\mathcal{U} = (v_1, v_2, ..., v_N) \in \mathbb{R}^{N \times N}$ such that $G = \mathcal{U}\Lambda\mathcal{U}^T$ where $\Lambda = \mathrm{Diag}\{\lambda_1, \lambda_2, ...\lambda_N\}$ and $v_i \in \mathbb{R}^N$. Let $\delta_i = (v_i^T \otimes I_n)y$ and $\delta = (\delta_1^T, \delta_2^T, ..., \delta_N^T)^T$, then $\delta = (\mathcal{U}^T \otimes I_n)y$, i.e., $y = (\mathcal{U}^T \otimes I_n)^{-1}\delta = (\mathcal{U} \otimes I_n)\delta$. From (5.6), we have, for $t \neq t_k$,

$$\begin{aligned} (\mathcal{U} \otimes I_n)\delta^{\Delta} &= (I_N \otimes A)(\mathcal{U} \otimes I_n)\delta + c(G \otimes I_n)(\mathcal{U} \otimes I_n)\delta \\ &= (\mathcal{U} \otimes A)\delta + c(\mathcal{U}\Lambda \otimes I_n)\delta. \end{aligned}$$

Multiply both side by $(\mathcal{U} \otimes I_n)^{-1}$ yields

$$\begin{aligned} \delta^{\Delta} &= (\mathcal{U}^T \otimes I_n)(\mathcal{U} \otimes A)\delta + c(\mathcal{U}^T \otimes I_n)(\mathcal{U}\Lambda \otimes I_n)\delta \\ &= (I_N \otimes A)\delta + c(\Lambda \otimes I_n)\delta, \end{aligned}$$

which implies $\delta_i^{\Delta} = (A + c\lambda_i I_n)\delta_i$, for $t \neq t_k$ and $i = 1, 2, ..., N$. Consider the Lyapunov function $V(t) = y^T(t)y(t)$. By the definition of δ, we can obtain that

$$\begin{aligned} V(t) &= \delta^T(t)(\mathcal{U} \otimes I_n)^T(\mathcal{U} \otimes I_n)\delta(t) \\ &= \delta^T(t)(\mathcal{U}^T \otimes I_n)(\mathcal{U} \otimes I_n)\delta(t) \\ &= \delta^T(t)(\mathcal{U}^T\mathcal{U} \otimes I_n)\delta(t) \\ &= \delta^T(t)\delta(t). \end{aligned}$$

For $t \neq t_k$, by calculating the Δ-derivative of $V(t)$ along the trajectories of the system (5.6), we can get

$$V^{\Delta}(t) = \sum_{i=1}^{N} [(\delta_i^T)^{\Delta}\delta_i + (\delta_i^{\sigma})^T \delta_i^{\Delta}], \ \text{where } \delta_i^{\sigma} = \delta_i(\sigma(t))$$

$$
\begin{aligned}
&= \sum_{i=1}^{N}[(\delta_i^T)^\Delta\delta_i + (\delta_i + \mu\delta_i^\Delta)^T\delta_i^\Delta] \\
&= \sum_{i=1}^{N}\delta_i^T[(A + c\lambda_i I_n)^T + (A + c\lambda_i I_n) + \mu(t)(A + c\lambda_i I_n)^T(A + c\lambda_i I_n)]\delta_i \\
&= \sum_{i=1}^{N}\delta_i^T(H_i^T + H_i + \mu H_i^T H_i)\delta_i \\
&\leq \sum_{i=1}^{N}\alpha_{ik}\delta_i^T\delta_i \leq \alpha_k V(t). \qquad (5.7)
\end{aligned}
$$

Since $1 + \mu(t)\alpha_k > 0$ for $t \in (t_k, t_{k+1})$, we have, by Lemma 2.2 and (5.7),

$$V(t) \leq V(t_k^+)e_{\alpha_k}(t, t_k). \qquad (5.8)$$

Next, we shall show

$$V(t_{k+1}) \leq V(t_k^+)e_{\alpha_k}(t_{k+1}, t_k). \qquad (5.9)$$

If t_{k+1} is left-dense, then, by the continuity of $V(t)$ and $e_{\alpha_k}(t, t_k)$, we have

$$V(t_{k+1}) = \lim_{t\to t_{k+1}^-} V(t) \leq \lim_{t\to t_{k+1}^-} V(t_k^+)e_{\alpha_k}(t, t_k) = V(t_k^+)e_{\alpha_k}(t_{k+1}, t_k).$$

If t_{k+1} is left-scattered, then

$$
\begin{aligned}
V(t_{k+1}) &= V(\rho(t_{k+1})) + \mu(\rho(t_{k+1}))V^\Delta(\rho(t_{k+1})) \\
&\leq V(\rho(t_{k+1})) + \mu(\rho(t_{k+1}))\alpha_k V(\rho(t_{k+1})) \\
&= [1 + \mu(\rho(t_{k+1}))\alpha_k]V(\rho(t_{k+1})) \\
&= e_{\alpha_k}(t_{k+1}, \rho(t_{k+1}))V(\rho(t_{k+1})) \\
&\leq e_{\alpha_k}(t_{k+1}, \rho(t_{k+1}))V(t_k^+)e_{\alpha_k}(\rho(t_{k+1}), t_k) \\
&= V(t_k^+)e_{\alpha_k}(t_{k+1}, t_k).
\end{aligned}
$$

Thus, (5.9) is proved. For $k \in \mathbb{N}$, we have

$$
\begin{aligned}
(1 - \rho_{1k})\sum_{i\notin\mathfrak{D}_k} y_i^T(t_k)y_i(t_k) &\leq (1 - \rho_{1k})(N - l_k)\min_{i\in\mathfrak{D}_k}\{y_i^T(t_k)y_i(t_k)\} \\
&= l_k[\rho_{1k} - (1 + \varepsilon_k)(1 + q_{1k})^2]\min_{i\in\mathfrak{D}_k}\{y_i^T(t_k)y_i(t_k)\} \\
&\leq [\rho_{1k} - (1 + \varepsilon_k)(1 + q_{1k})^2]\sum_{i\in\mathfrak{D}_k} y_i^T(t_k)y_i(t_k), \qquad (5.10)
\end{aligned}
$$

then,

$$V(t_k^+) = \sum_{i\in\mathfrak{D}_k} y_i^T(t_k^+)y_i(t_k^+) + \sum_{i\notin\mathfrak{D}_k} y_i^T(t_k^+)y_i(t_k^+)$$

$$
\begin{aligned}
&= \sum_{i\notin\mathfrak{D}_k} y_i^T(t_k)y_i(t_k) + \sum_{i\in\mathfrak{D}_k}\Big[(1-q_{1k})^2 y_i^T(t_k)y_i(t_k) \\
&\quad + q_{2k}^2 y_i^T(t_k-\tau_k)y_i(t_k-\tau_k) + 2(1-q_{1k})q_{2k}y_i^T(t_k)y_i(t_k-\tau_k)\Big] \\
&\le (1+\varepsilon_k)(1+q_{1k})^2 \sum_{i\in\mathfrak{D}_k} y_i^T(t_k)y_i(t_k) \\
&\quad + (1+\varepsilon_k^{-1})q_{2k}^2 \sum_{i\in\mathfrak{D}_k} y_i^T(t_k-\tau_k)y_i(t_k-\tau_k) + \sum_{i\notin\mathfrak{D}_k} y_i^T(t_k)y_i(t_k) \\
&\le \rho_{1k}V(t_k) + \rho_{2k}V(t_k-\tau_k). \qquad (5.11)
\end{aligned}
$$

Since $t_1 - \tau \ge t_0$, we have

$$V(t) \le Me^{-a(t-t_0)},\ t\in[t_0,t_1], \qquad (5.12)$$

where $M = e^{a(t_1-t_0)}\sup_{t\in[t_0,t_1]}\{V(t)\}$. In the following, we shall show that for $k\ge 1$

$$V(t) \le M\frac{e^{-a(t_{k+1}-t_0)}}{e_{\alpha_k}(t_{k+1},t)},\ t\in(t_k,t_{k+1}]. \qquad (5.13)$$

For $t = t_1$, we can get from condition (ii) that

$$
\begin{aligned}
V(t_1^+) &\le \rho_{11}V(t_1) + \rho_{21}V(t_1-\tau_1) \\
&\le \rho_{11}Me^{-a(t_1-t_0)} + \rho_{21}Me^{-a(t_1-\tau_1-t_0)} \\
&= M(\rho_{11}+\rho_{21}e^{a\tau_1})e^{-a(t_1-t_0)} \\
&\le M\frac{e^{-a(t_2-t_0)}}{e_{\alpha_1}(t_2,t_1)}, \qquad (5.14)
\end{aligned}
$$

then, from (5.8) and Remark 5.1.1, we have

$$V(t) \le V(t_1^+)e_{\alpha_1}(t,t_1) \le M\frac{e^{-a(t_2-t_0)}}{e_{\alpha_1}(t_2,t)},\ \text{for } t\in(t_1,t_2],$$

which implies (5.13) is true for $k=1$. Next, suppose (5.13) is true for $k\le j\,(j>1)$, and we shall prove (5.13) holds for $k=j+1$. For $t=t_{j+1}$, we estimate the upper bound of $V(t_{j+1}-\tau_{j+1})$ by considering the following two cases:

- $t_{j+1}-\tau_{j+1} \le t_1$, then

$$V(t_{j+1}-\tau_{j+1}) \le Me^{-a(t_{j+1}-\tau_{j+1}-t_0)}. \qquad (5.15)$$

- $t_{j+1}-\tau_{j+1} > t_1$, then there exists an integer $\hat{k}\ge 1$ such that $t_{j+1}-\tau_{j+1}\in(t_{\hat{k}},t_{\hat{k}+1}]$, and then

$$V(t_{j+1}-\tau_{j+1}) \le M\frac{e^{-a(t_{\hat{k}+1}-t_0)}}{e_{\alpha_{\hat{k}}}(t_{\hat{k}+1},t_{j+1}-\tau_{j+1})} \le Me^{-a(t_{j+1}-\tau_{j+1}-t_0)}, \qquad (5.16)$$

  since $\alpha_{\hat{k}}\ge 0$ and $e_{\alpha_{\hat{k}}}(t_{\hat{k}+1},t_{j+1}-\tau_{j+1})\ge 1$.

From (5.15) and (5.16), we obtain

$$\begin{aligned} V(t_{j+1}^+) &\le \rho_{1,j+1}V(t_{j+1}) + \rho_{2,j+1}V(t_{j+1}-\tau_{j+1}) \\ &\le M(\rho_{1,j+1} + \rho_{2,j+1}e^{a\tau_{j+1}})e^{-a(t_{j+1}-t_0)} \\ &\le M\frac{e^{-a(t_{j+2}-t_0)}}{e_{\alpha_{j+1}}(t_{j+2}, t_{j+1})}, \end{aligned}$$

and then, for $t \in (t_{j+1}, t_{j+2}]$,

$$\begin{aligned} V(t) &\le V(t_{j+1}^+)e_{\alpha_{j+1}}(t, t_{j+1}) \\ &\le Me^{-a(t_{j+2}-t_0)}\frac{e_{\alpha_{j+1}}(t, t_{j+1})}{e_{\alpha_{j+1}}(t_{j+2}, t_{j+1})} \\ &= M\frac{e^{-a(t_{j+2}-t_0)}}{e_{\alpha_{j+1}}(t_{j+2}, t)}, \end{aligned}$$

which implies (5.13) is true for $k = j+1$. Thus, we conclude from mathematical induction that (5.13) is true for all $k \ge 1$. Then, for $t \in (t_k, t_{k+1}](k \ge 1)$,

$$V(t) \le M\frac{e^{-a(t_{k+1}-t_0)}}{e_{\alpha_{k+1}}(t_{k+1}, t)} \le Me^{-a(t_{k+1}-t_0)} \le Me^{-a(t-t_0)},$$

which implies $V(t) \to 0$ as $t \to \infty$, i.e., $\lim_{t\to\infty} ||y(t)|| = 0$. □

Remark 5.1.4 *By minimizing the term $\rho_{1k} + \rho_{2k}e^{a\tau_k}$, the constant ε_k can be specified to make Theorem 5.1.1 less conservative. Define $h_k(\varepsilon) = \frac{l_k}{N}\varepsilon(1+q_{1k})^2 + \varepsilon^{-1}q_{2k}^2e^{a\tau_k}$, then, for $\varepsilon = \frac{|q_{2k}|}{|1+q_{1k}|}\sqrt{\frac{N}{l_k}}e^{\frac{1}{2}a\tau_k}$, we have $h_k'(\varepsilon) = 0$, i.e., $h_k(\varepsilon)$ attains its minimum for $\varepsilon > 0$. Hence,*

$$\min_{\varepsilon_k>0}\{\rho_{1k} + \rho_{2k}e^{a\tau_k}\} = 1 - \frac{l_k}{N} + [\sqrt{\frac{l_k}{N}}|1+q_{1k}| + |q_{2k}|e^{\frac{1}{2}a\tau_k}]^2.$$

Since l_k denotes the number of nodes to be controlled at impulsive instant t_k, l_k/N represents the proportion of the impulsively controlled nodes at $t = t_k$. It can be seen from Theorem 5.1.1 that the proportion l_k/N depends not only on the control parameters of the pinning controller but also the structure of the time scale $\mathbb{T}$, because exponential function $e_{\alpha_k}(t_{k+1}, t_k)$ is closely related to the graininess function of time scale $\mathbb{T}$.

Remark 5.1.5 *For the right-scattered case of t_k, our results are applicable by defining $x_i(t_k^+)$ to be the state after the impulse and $x_i(t_k)$ to be the state before the impulse, that is, the impulse is defined as a state update at each impulsive instant (as discussed in [81]). Then conditions (i) and (ii) of Theorem 5.1.1 on t are restricted to $t_k^+ \le t < t_k$. Therefore, Theorem 5.1.1 can be used as a synchronization criterion for the discrete DNs. Though the synchronization criterion in Theorem 5.1.1 is established for DNs on general time scales, for specific time scales we can get some verifiable sufficient conditions. For example,*

- $\mathbb{T} = \mathbb{N}$, *then* $\mu \equiv 1$, $H_i + H_i^T + \mu H_i^T H_i = H_i + H_i^T + H_i^T H_i$ *and*

$$e_{\alpha_k}(t_{k+1}, t_k) = (1 + \alpha_k)^{t_{k+1} - t_k};$$

- $\mathbb{T} = \mathbb{R}$, *then* $\mu \equiv 0$, $H_i + H_i^T + \mu H_i^T H_i = H_i + H_i^T$ *and*

$$e_{\alpha_k}(t_{k+1}, t_k) = e^{\alpha_k (t_{k+1} - t_k)}.$$

For time scales with bounded graininess functions ($\mu(t) \leq \bar{\mu}$ for all $t \in \mathbb{T}$), $H_i + H_i^T + \mu H_i^T H_i \leq H_i + H_i^T + \bar{\mu} H_i^T H_i$ and the exponential function $e_{\alpha_k}(t_{k+1}, t_k)$ can be calculated according to the structure of the time scale.

5.1.4 Numerical Simulations

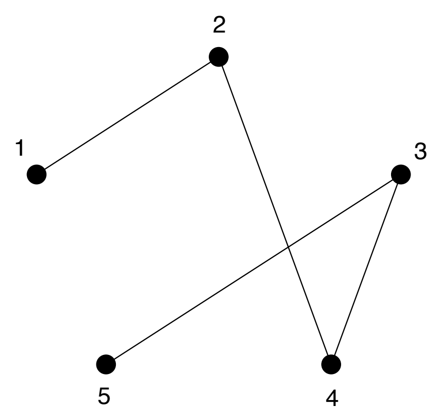


Figure 5.1: Network topology of linear DN (5.1).

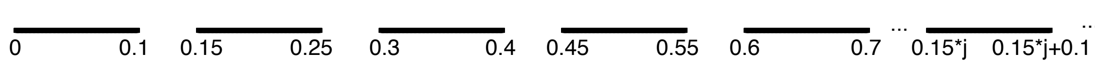
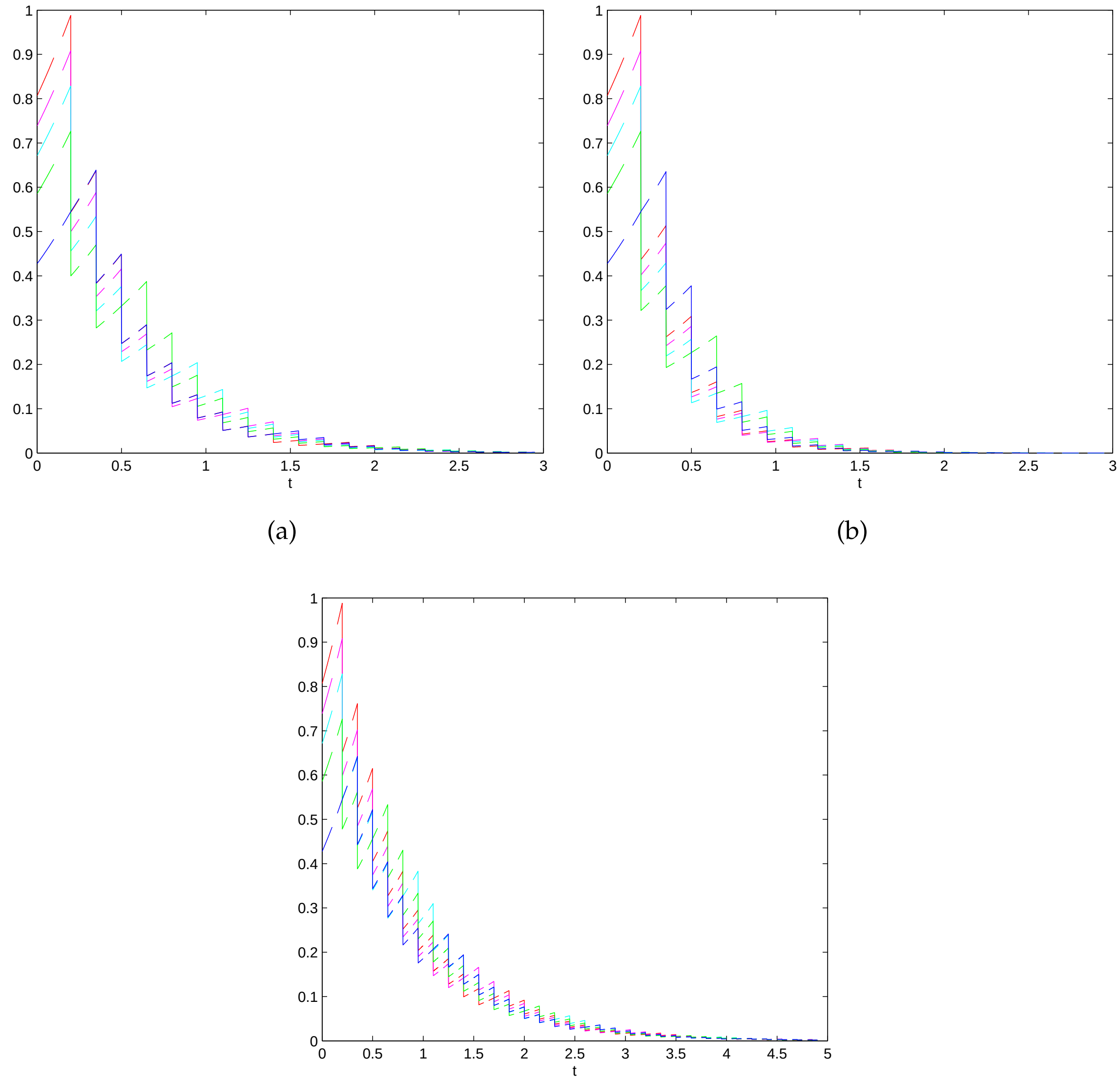


Figure 5.2: Demonstration of time scale $\mathbb{T}$.

In this subsection, we present a numerical example to illustrate the proposed result. Consider the linear DN (5.1) on time scale $\mathbb{T}$ with $n = 2$, $c = 0.1$,

$$A = \begin{bmatrix} -1.2 & 0.1 \\ -0.2 & 1.1 \end{bmatrix},$$

$$G = \begin{bmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 & -2 \end{bmatrix},$$



(a) (b) (c)

Figure 5.3: Numerical simulations of $||y_i||$ $(i = 1, 2, ..., 5)$ with different pinning impulsive controllers.

and the time scale

$$\mathbb{T} = \bigcup_{j=0}^{\infty} [\frac{3}{20}j, \frac{3}{20}j + \frac{1}{10}].$$

(See Figure 5.1 for the network topology and Figure 5.2 for the demonstration of the given time scale.) Then, $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = -2$, and $\lambda_4 = \lambda_5 = -3$. The graininess function of $\mathbb{T}$ is given by

$$\mu(t) = \begin{cases} 0, \ t \in \bigcup_{j=0}^{\infty} [\frac{3}{20}j, \frac{3}{20}j + \frac{1}{10}), \\ 0.05, \ t = \frac{3}{20}j + \frac{1}{10}, \ j \in \mathbb{Z}^+, \end{cases}$$

which implies that $\mu(t) \leq 0.05$ for all $t \in \mathbb{T}$. Thus, for any $t \in \mathbb{T}$, $\lambda_{max}(H_i^T + H_i + \mu(t)H_i^T H_i) \leq 2.52$, then we can choose $\alpha_{ik} \equiv \alpha = 2.52$. In order to observe the pinning impulsive control process clearly, in this example, we consider the network with $N = 5$ identical nodes, and the impulsive sequence is chosen as follows: $t_k = 0.15k + 0.05$, for $k \in \mathbb{N}$. According to the structure of the time scale $\mathbb{T}$, we have the following estimation of the exponential function

$$\begin{aligned} e_\alpha(t_{k+1}, t_k) &= e_\alpha(t_{k+1}, t_{k+1} - 0.05)e_\alpha(t_{k+1} - 0.05, t_k + 0.05)e_\alpha(t_k + 0.05, t_k) \\ &= e^{0.05\alpha}(1 + \mu\alpha)^{\frac{(t_{k+1}-0.05)-(t_k+0.05)}{0.05}} e^{0.05\alpha} \\ &= (1 + 0.05\alpha)e^{0.1\alpha} \approx 1.45. \end{aligned}$$

In the following simulations, let $l_{2k-1} = 4$ and $l_{2k} = 5$ for $k \in \mathbb{N}$, that is, controlling 4 nodes at each odd impulsive instant, and controlling all of the nodes at each even impulsive instant. Next, we consider the pinning impulsive controller with $q_{1,2k-1} = -0.45$, $q_{1,2k} = -0.4$, $\tau_k = 0.1$, and three different types of control gains q_{2k}:

(a) $q_{2,k} \equiv 0$ for $k \in \mathbb{N}$, i.e., there is no delay in the pinning impulsive controller;

(b) $q_{2,2k-1} = -0.12$ and $q_{2,2k} = -0.1$ for $k \in \mathbb{N}$;

(c) $q_{2,2k-1} = 0.12$ and $q_{2,2k} = 0.1$ for $k \in \mathbb{N}$.

It can be checked that conditions in Theorem 5.1.1 are satisfied for the first three cases with $a = 0.01$. In (a), no delay is considered in the pinning impulsive controller, Figure 5.3(a) shows that the synchronization of DN can be realized. In (b), time delay exists in the pinning controller. Compared with (a), the existence of delay in the pinning controller contribute to the synchronization of the DN. See Figure 5.3(b) for illustration. On the other hand, the existence of time delay could be a perturbation to the synchronization process. Hence, in (c), we consider a pinning impulsive controller with delayed impulsive perturbations, and Figure 5.3(c) shows that the the existence of delay in the pinning controller slows down the convergence rate of the synchronization.

5.2 Stabilization of Neural Networks with Time-Delay

Neural networks (NNs) are a family of statistical learning models inspired by the central nervous systems of animals (see, [11]). NNs are generally presented as systems of densely interconnected simple elements which model the biological neurons, and send (or receive) messages to (or from) each other. In recent decades, the research on NNs has attracted the attention of numerous researchers. This mainly due to their broad applications in many areas including image processing and pattern recognition (see, e.g., [24, 47]), data fusion [22], odor classification [5], and solving partial differential equations [44].

This section studies impulsive stabilization problem of time-delay neural networks. Discrete time-delay effects are considered in the impulsive controllers. In Subsection 5.2.2, a pinning impulsive controller is proposed with delay effects. The impulsive controller depends on not only the network states at each impulsive instant but also the states at history time.

Sufficient conditions for the stabilization are constructed by using a Razumikhin-type stability result. In Subsection 5.2.3, the proposed pinning impulsive controller relies only on the network states at history instants, that is, the time-delay states play an key role in the stabilization process. Stabilization result is obtained by using the Lyapunov functional method. Numerical examples are provided to demonstrate the theoretical results.

5.2.1 Neural Network Model and Preliminaries

Consider the following time-delay neural network (DNN):

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-r)) + J_i, \tag{5.17}$$

for $i \in \mathcal{I} := \{1, 2, ..., n\}$, where $x_i \in \mathbb{R}$ is the state of the ith neuron; n denotes the number of neurons in DNN (5.17); $f_j(x_j(t))$ denotes the output of the jth neuron at time t; constants a_{ij} and b_{ij} represent the strengths of connectivity between neurons i and j at time t and $t-r$, respectively; r corresponds to the transmission delay when processing information from the jth neuron; constant J_i denotes the external bias or input from the outside of the network to the ith neuron; constant c_i denotes the rate with which the ith neuron will reset its potential when disconnected with the other neurons of the network and external input.

Throughout this section, we assume that $f_i(0) = J_i = 0$ for all $i \in \mathcal{I}$. Here we have assumed $J_i = 0$ for all $i \in \mathcal{I}$. Actually, for nontrivial constant external input J_i, stability analysis of the equilibrium of DNN (5.17) can be studied similarly by change of variables. Based on our assumptions, system (5.17) admits the trivial solution.

The objective is to design the following delay-dependent pinning impulsive controller to exponentially stabilize DNN (5.17):

$$U_i(t, x_i) = \begin{cases} \sum_{k=1}^{\infty} I(x_i(t), x_i(t-d))\delta(t - t_k^-), \ i \in \mathfrak{D}_k^l, \\ 0, \ i \notin \mathfrak{D}_k^l, \end{cases} \tag{5.18}$$

for $i \in \mathcal{I}$, where $I : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $d > 0$ denotes the time delay in controller (5.18); the impulsive instant sequence $\{t_k\}$ satisfies $\{t_k\} \subseteq \mathbb{R}$, $0 \le t_0 < t_1 < ... < t_k < ...$, and $\lim_{k\to\infty} t_k = \infty$; $\delta(\cdot)$ is the Dirac Delta function. Let l denote the number of neurons to be pinned at each impulsive instant, and the index set $\mathfrak{D}_k^l = \{p_1, p_2, ..., p_l\} \subseteq \mathcal{I}$ is defined as follows: $p_i \neq p_j$ if $i \neq j$; at the impulsive instant t_k, $\|x_i(t_k^-)\| \ge \|x_j(t_k^-)\|$ if $i \in \mathfrak{D}_k^l$ and $j \in \mathcal{I}/\mathfrak{D}_k^l$. The definition of $\mathfrak{D}_k^l$ is similar to definition of $\mathfrak{D}_k$ in Section 5.1. The difference is that the same number of neurons are controlled at different impulsive instants (i.e., $l_k = l$).

The closed-loop system can be written in the following form of nonlinear differential equations:

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-r)) + U_i, \tag{5.19}$$

for $i = 1, 2, ..., n$. Furthermore, we can rewrite system (5.19) into a matrix-form impulsive system:

$$\begin{cases} \dot{x}(t) = -Cx(t) + Af(x(t)) + Bf(x(t-r)), \ t \in [t_{k-1}, t_k), \\ \Delta x_i(t_k) = I(x_i(t_k^-), x_i(t_k - d)), \ \ i \in \mathfrak{D}_k^l, \ k \in \mathbb{N}, \\ x_{t_0} = \phi, \end{cases} \tag{5.20}$$

where $C = diag\{c_1, c_2, ..., c_n\}$, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, $f(x) = (f_1(x_1), f_2(x_2), ..., f_n(x_n))^T$ and x_{t_0} is defined by $x_{t_0}(s) = x(t_0 + s)$ for $s \in [-\tau, 0]$ and $\tau = \max\{r, d\}$; $\phi = (\phi_1, \phi_2, ..., \phi_n)^T$ is the initial function with $\phi_i \in \mathcal{PC}([-\tau, 0], \mathbb{R})$. Then the pinning impulsive stabilization problem of DNN (5.17) is transformed into the stability problem of impulsive system (5.20).

The following lemmas will be used in the proof of the main result.

Lemma 5.2.1 *For $x, y \in \mathbb{R}$, the following inequality holds*

$$2xy \leq \varepsilon x^2 + \varepsilon^{-1} y^2,$$

for any $\varepsilon > 0$.

Lemma 5.2.2 *For $x, y, z \in \mathbb{R}$, the following inequality holds*

$$(x + y + z)^2 \leq (1 + \varepsilon)x^2 + (1 + \varepsilon^{-1})(1 + \xi)y^2 + (1 + \varepsilon^{-1})(1 + \xi^{-1})z^2,$$

for any $\varepsilon, \xi > 0$.

Lemma 5.2.3 *For $\varepsilon, \xi > 0$, and given constants $x, y, z \in \mathbb{R}$, define function*

$$H(\varepsilon, \xi) := (1 + \varepsilon)x^2 + (1 + \varepsilon^{-1})(1 + \xi)y^2 + (1 + \varepsilon^{-1})(1 + \xi^{-1})z^2,$$

then function H attains its minimum $H_{min} = (|x| + |y| + |z|)^2$ at $(\varepsilon, \xi) = (\frac{|y|+|z|}{|x|}, \frac{|z|}{|y|})$.

Remark 5.2.1 *Applying Lemma 5.2.1 twice, Lemma 5.2.2 can be proved. Lemma 5.2.3 can be easily obtained by using the extreme value theory of multivariate functions. Hence, the detailed proofs for Lemma 5.2.2 and 5.2.3 are omitted. The above lemmas will be used to reduce the conservatism of the sufficient conditions of our results.*

5.2.2 Delay-Dependent Impulsive Control

In this subsection, we consider the pinning impulsive controller with $I(x_i(t), x_i(t-d)) = \gamma_1 x_i(t) + \gamma_2 x_i(t-d)$, that is,

$$U_i(t, x_i) = \begin{cases} \sum_{k=1}^{\infty} [\gamma_1 x_i(t) + \gamma_2 x_i(t-d)]\delta(t - t_k^-), \ i \in \mathfrak{D}_k^l, \\ \qquad\qquad 0, \ i \notin \mathfrak{D}_k^l, \end{cases} \tag{5.21}$$

where γ_1 and γ_2 are impulsive control gains to be determined. Then, the impulsive controlled DNN (5.17) can be written in the form of an impulsive system:

$$\begin{cases} \dot{x}(t) = -Cx(t) + Af(x(t)) + Bf(x(t-r)), \ t \in [t_{k-1}, t_k), \\ \Delta x_i(t_k) = \gamma_1 x_i(t_k^-) + \gamma_2 x_i(t_k - d), \ \ i \in \mathfrak{D}_k^l, \ k \in \mathbb{N}, \\ x_{t_0} = \phi. \end{cases} \tag{5.22}$$

Throughout this subsection, we make the following assumption:

(A) there exists a constant L such that $||f(u) - f(v)|| \leq L||u - v||$ for all $u, v \in \mathbb{R}^n$.

The Lipschtiz condition on the nonlinear activation function has been widely considered due to its significance in the application of NNs (see e.g., [24, 106, 124]). Next, we will use a Razumikhin-type stability criterion to construct verifiable conditions for the GES of impulsive DNN (5.22).

Theorem 5.2.1 *If the following inequality is satisfied*

$$\rho e^{cd} < 1, \tag{5.23}$$

where $\rho = 1 - \frac{l}{n} + \left(\sqrt{\frac{l}{n}}|1+\gamma_1| + |\gamma_2|\right)^2$, $c = -2\min_i\{c_i\} + 2L(||A|| + \frac{||B||}{\sqrt{\rho}}) > 0$, *and* $d = \sup_{k\in\mathbb{N}}\{t_{k+1} - t_k\}$, *then the trivial solution of system (5.22) is GES.*

Proof: Consider the Lyapunov function $V(x) = x^T x$. Note that condition (i) of Theorem 3.3.1 is satisfied with $w_1 = w_2 = 1$ and $p = 2$. Taking the time-derivative along solutions of (5.22)

$$\begin{aligned} \dot{V}(x) &= 2x^T(t)\dot{x}(t) \\ &= 2x^T(t)\Big[-Cx(t) + Af(x(t)) + Bf(x(t-r))\Big] \\ &\leq \big(-2c_{\min} + 2||A||L\big)V(x(t)) + 2||B||L||x(t)||||x(t-r)|| \\ &\leq \big(-2c_{\min} + 2||A||L + ||B||L\varepsilon^{-1}\big)V(x(t)) + \varepsilon||B||LV(x(t-r)), \end{aligned} \tag{5.24}$$

where $c_{\min} = \min_i\{c_i\}$ and constant $\varepsilon > 0$. It can be seen from (5.23) that there exists a constant $q > 0$ such that

$$q > \frac{1}{\rho} > e^{\bar{c}d}, \tag{5.25}$$

where $\bar{c} = -2\min_i\{c_i\} + 2L(||A|| + \sqrt{q}||B||)$. If $V(x(t+s)) < qV(x(t))$ for all $s \in [-\tau, 0]$, then we can obtain from (5.24) that

$$\dot{V}(x) \leq \big[-2c_{\min} + 2||A||L + ||B||L(\varepsilon^{-1} + q\varepsilon)\big]V(x(t)). \tag{5.26}$$

Define h as a function of ε: $h(\varepsilon) = -2c_{\min} + 2||A||L + ||B||L(\varepsilon^{-1} + q\varepsilon)$. Then, for $\varepsilon > 0$, function h attains its minimum value $\bar{c}$ at $\varepsilon = \frac{1}{\sqrt{q}}$ (that is, $h'(\varepsilon) = 0$ at $\varepsilon = \frac{1}{\sqrt{q}}$). Therefore, we can get from (5.26) that $\dot{V}(x) \leq \bar{c}V(x)$.

Given a constant $\xi > 0$, letting $\bar{\gamma}_1 = 1 - \frac{l}{n}[1 - (1+\xi)(1+\gamma_1)^2]$, then

$$\begin{aligned}(1-\bar{\gamma}_1)\sum_{i\notin\mathfrak{D}_k^l} x_i^2(t_k^-) &\leq (1-\bar{\gamma}_1)(n-l)\min_{i\in\mathfrak{D}_k^l}\{x_i^2(t_k^-)\} \\ &= l[\bar{\gamma}_1 - (1+\xi)(1+\gamma_1)^2]\min_{i\in\mathfrak{D}_k^l}\{x_i^2(t_k^-)\} \\ &\leq [\bar{\gamma}_1 - (1+\xi)(1+\gamma_1)^2]\sum_{i\in\mathfrak{D}_k^l} x_i^2(t_k^-),\end{aligned}$$

i.e.,

$$(1+\xi)(1+\gamma_1)^2\sum_{i\in\mathfrak{D}_k^l} x_i^2(t_k^-) + \sum_{i\notin\mathfrak{D}_k^l} x_i^2(t_k^-) \leq \bar{\gamma}_1\sum_{i=1}^n x_i^2(t_k^-).$$

Then, for $t = t_k$, we have

$$\begin{aligned}V(x(t_k)) &= \sum_{i\in\mathfrak{D}_k^l} x_i^2(t_k) + \sum_{i\notin\mathfrak{D}_k^l} x_i^2(t_k) \\ &= \sum_{i\in\mathfrak{D}_k^l}[(1+\gamma_1)x_i(t_k^-) + \gamma_2 x_i(t_k-d)]^2 + \sum_{i\notin\mathfrak{D}_k^l} x_i^2(t_k) \\ &\leq \sum_{i\in\mathfrak{D}_k^l}[(1+\xi)(1+\gamma_1)^2 x_i^2(t_k^-) + (1+\xi^{-1})\gamma_2^2 x_i^2(t_k-d)] + \sum_{i\notin\mathfrak{D}_k^l} x_i^2(t_k) \\ &\leq (1+\xi)(1+\gamma_1)^2\sum_{i\in\mathfrak{D}_k^l} x_i^2(t_k^-) + \sum_{i\notin\mathfrak{D}_k^l} x_i^2(t_k) + (1+\xi^{-1})\gamma_2^2\sum_{i=1}^n x_i^2(t_k-d) \\ &\leq \bar{\gamma}_1 V(x(t_k^-)) + (1+\xi^{-1})\gamma_2^2 V(x(t_k-d)) \\ &\leq \rho_1 V(x(t_k^-)) + \rho_2\sup_{s\in[-\tau,0]}\{V(x(t_k^-+s))\},\end{aligned}$$

where $\rho_1 = \bar{\gamma}_1$, $\rho_2 = (1+\xi^{-1})\gamma_2^2$, and constant $\xi > 0$ to be determined to minimize the value of $\rho_1 + \rho_2$.

Let $\bar{h}(\xi) = \frac{l}{n}(1+\gamma_1)^2\xi + \gamma_2^2\xi^{-1}$, then, for $\xi = |\frac{\gamma_2}{1+\gamma_1}|\sqrt{\frac{n}{l}}$, we have $\bar{h}'(\xi) = 0$, i.e., $\bar{h}(\xi)$ attains its minimum for $\xi > 0$. Hence,

$$\rho = \min_{\xi>0}\{\rho_1+\rho_2\} = 1 - \frac{l}{n} + \left(\sqrt{\frac{l}{n}}|1+\gamma_1| + |\gamma_2|\right)^2.$$

Based on the above discussion, we can conclude that all the conditions of Theorem 3.3.1 are satisfied. Thus, the trivial solution of system (5.22) is GES. □

Remark 5.2.2 *Parameter ρ is related to impulsive control gains γ_1, γ_2 and the ratio l/n. It can be seen from (5.23) that, the fewer units are controlled at impulsive instants, the more frequently the impulsive controllers need to be added to the network.*

Numerical Simulations

Next, we will consider an example to demonstrate our theoretical result. In order to observe the pinning control process clearly, we will investigate a DNN with only two units in the following example.

Example 5.2.1 *Consider DNN (5.17) with $n = 2$, $c_1 = c_2 = -1$, $r = 1$,*

$$A = \begin{bmatrix} 2 & -0.1 \\ -5 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -2.5 \end{bmatrix},$$

and $f(x) = (f_1(x_1), f_2(x_2))^T$ with $f_1(\cdot) = f_2(\cdot) = \tanh(\cdot)$. The chaotic attractor of DNN (5.17) is shown in Figure 5.4.

We consider two types of impulsive controllers:

1) $l = 1$, i.e., impulsive control one unit at each impulsive instant. Let $t_k - t_{k-1} = 0.03$, $d = 1$, $\gamma_1 = -0.868$, and $\gamma_2 = 0.2$, then (5.23) is satisfied. Thus, Theorem 5.2.1 implies that the trivial solution of (5.22) is GES. See Figure 5.5 for numerical simulations.

2) $l = 2$, i.e., impulsive control two units at each impulsive instant. Let $t_k - t_{k-1} = 0.08$, and d, γ_1, γ_2 are the same as those in the first scenario, then (5.23) is satisfied and Theorem 5.2.1 implies that the trivial solution of (5.22) is GES. Numerical results are shown in Figure 5.6.

   The initial data in Figure 5.5 and Figure 5.6 is chosen the same as that in Figure 5.4, and the red dot denotes the state x at initial time $t = 0$. The vertical (or horizontal) lines in Figure 5.5(a) represent the state jump of x_2 (or x_1) while the other state is unchanged. Since both units are controlled in Figure 5.6(a), no vertical and horizontal lines can be observed. It can be seen from Figure 5.5 that different unit may be controlled at different impulsive instants. This is consistent with our pinning algorithm of controlling the unit which has the largest state deviation with the equilibrium. However, it is more practical to control one specific unit at all impulsive instants. Next, we apply the pinning impulsive controller to the first and second unit at all impulsive instants respectively, and numerical results are shown in Figure 5.7(a) and 5.7(b). The impulsive control gains γ_1, γ_2, and the impulsive sequence $\{t_k\}$ are chosen the same as those in Figure 5.5. Figure 5.7 implies that stabilization cannot be realized via this type of pinning strategy with the given parameters, and more strict conditions may be required to guarantee the stability which will be investigated in our future research.

5.2.3 Control via Delayed Impulses

In this subsection, we consider the pinning impulsive controller with $I(x_i(t), x_i(t-d)) = qx_i(t-d)$, that is,

$$U_i(t, x_i) = \begin{cases} \sum_{k=1}^{\infty} qx_i(t-d)\delta(t-t_k^-), \ i \in \mathfrak{D}_k^l, \\ \qquad\qquad 0, \ i \notin \mathfrak{D}_k^l, \end{cases} \tag{5.27}$$

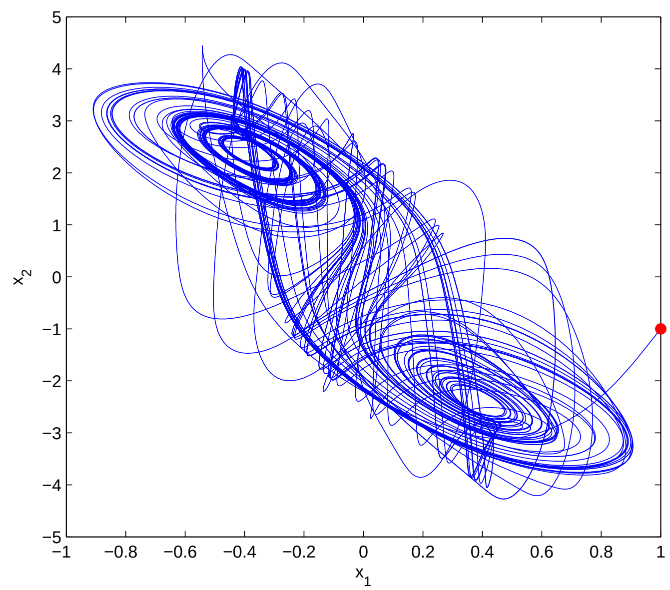
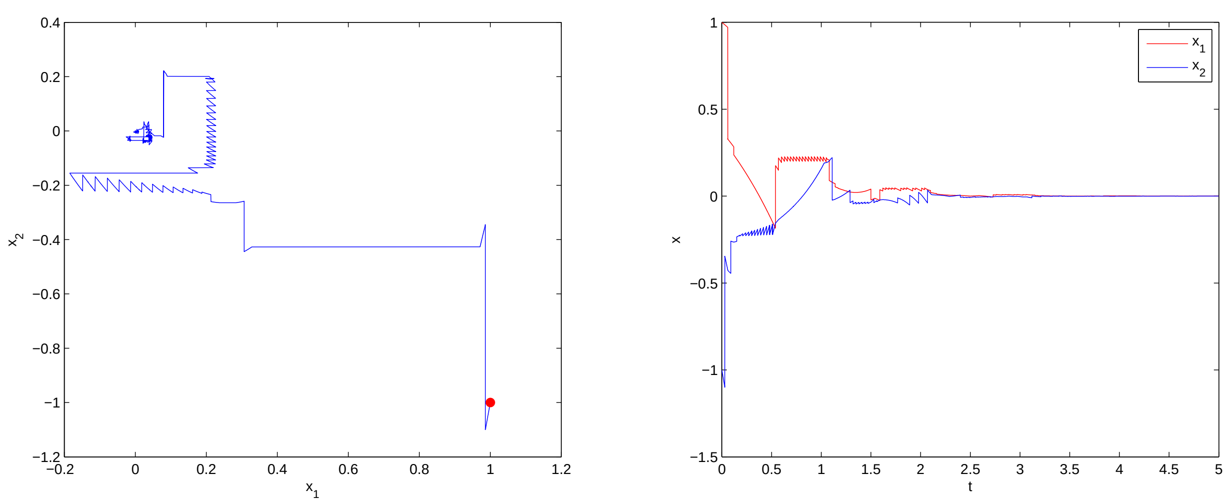


Figure 5.4: Chaotic behavior of DNN (5.17) with the parameters given in Example 5.2.1. The initial data for this simulation is $\phi(s) = [1, -1]^T$ for $s \in [-r, 0]$, and the red dot denotes the state x at the initial time $t = 0$.



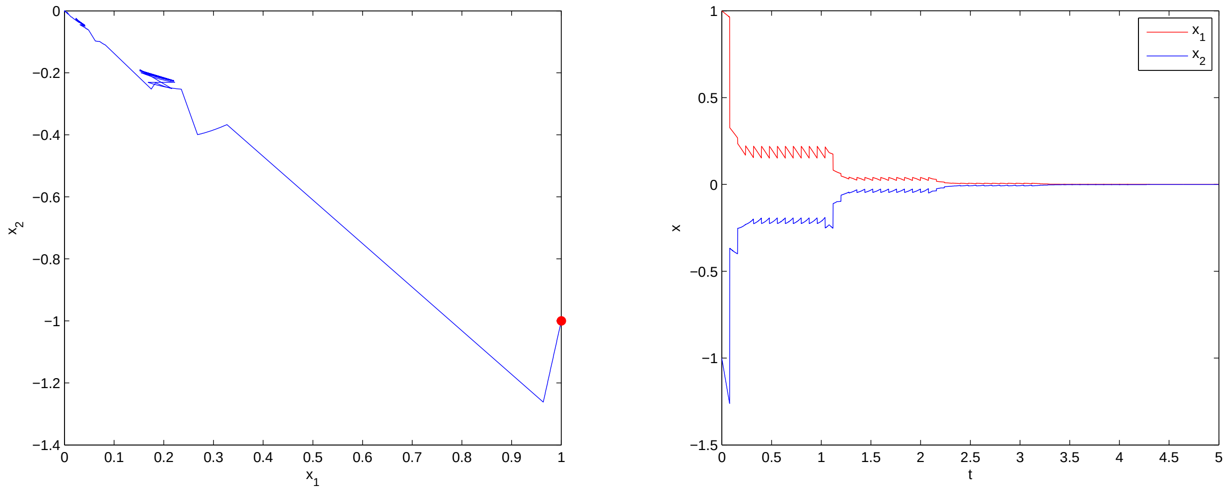
(a) Phase portrait

(b) State trajectories

Figure 5.5: Impulsive control one unit of DNN (5.17) at each impulsive instant

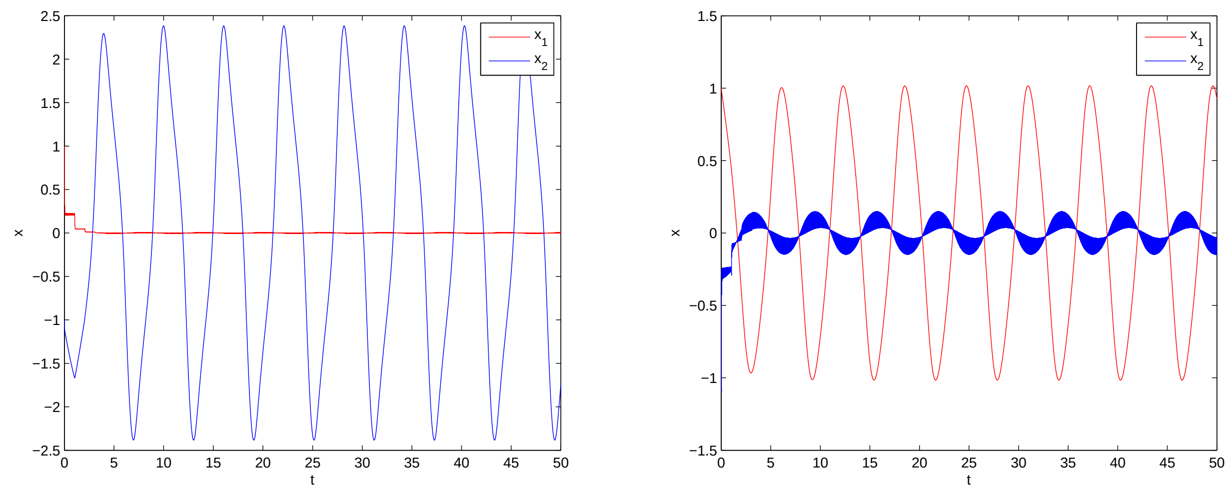
where $q \in (-1, 0)$ is the impulsive control gain to be determined. Then, the impulsive controlled DNN (5.17) can be written in the form of an impulsive system:

$$\begin{cases} \dot{x}(t) = -Cx(t) + Af(x(t)) + Bf(x(t-r)),\ t \in [t_{k-1}, t_k), \\ \Delta x_i(t_k) = q x_i(t_k - d),\ \ i \in \mathfrak{D}_k^l,\ k \in \mathbb{N}, \\ x_{t_0} = \phi. \end{cases} \tag{5.28}$$



(a) Phase portrait (b) State trajectories

Figure 5.6: Impulsive control both units of DNN (5.17) at each impulsive instant.



(a) Control the first unit (b) Control the second unit

Figure 5.7: Impulsive control one specific unit of DNN (5.17) through all the impulsive instants.

Remark 5.2.3 *The pinning algorithm introduced in [67] can be treated as a particular case of our pinning delayed-impulsive control strategy (i.e., $d = 0$). It is worth noting that the existence of time delay in controller (5.18) brings dramatic difficulties to estimate the relation between the states $x_i(t_k^-)$ and $x_i(t_k - d)$, and then guarantee the delayed impulses contribute to the stabilization process of DNNs. Though, Section 5.1 and Subsection 5.2.2 have considered the delay state $x_i(t_k - d)$ in the pinning impulsive controller, the controller depends on both the state $x_i(t_k)$ and $x_i(t_k - d)$, and there is no theoretical analysis of how the delay state $x_i(t_k - d)$ affects the pinning control process. Actually, results in Section 5.1 have shown that the delay states can either contribute to the stability of the system or* *act as disturbances to the dynamical system. To our best knowledge, this is the first time that a pinning impulsive controller is proposed with delayed impulse effects which depend only on the delay state $x_i(t_k - d)$. The detailed discussion of the delay effects on the stabilization process of DNNs can be found in the following discussion.*

Throughout this subsection, we make the following assumption:

(B) there exists a constant L_i such that $||f_i(u) - f_i(v)|| \leq L_i||u - v||$ for all $u, v \in \mathbb{R}$.

Next, we will use Theorem 3.2.1 to construct verifiable conditions for the GES of impulsive DNN (5.28). For convenience, we define the following notations:

$$
\begin{aligned}
c_{min} &= \min_i\{c_i\}, \\
c_{max} &= \max_i\{c_i\}, \\
L &= \max_i\{L_i\}, \\
\lambda &= c_{max} + \sqrt{ln}\max_{i,j}\{|a_{ij}|L_j\} + \sqrt{ln}\max_{i,j}\{|b_{ij}|L_j\}, \\
\varsigma &= \lfloor \frac{d}{\sigma} \rfloor,
\end{aligned}
$$

where $\sigma = \sup_{k\in\mathbb{N}}\{t_k - t_{k-1}\}$, and $\lfloor \cdot \rfloor$ is the floor function.

Theorem 5.2.2 *If there exists a constant $\varepsilon > 0$ such that*

$$
\ln(\rho + \varepsilon L r) < -c\sigma, \tag{5.29}
$$

where

$$
\begin{aligned}
\rho &= 1 - \frac{l}{n} + \left(\sqrt{\frac{l}{n}}(1+q) - qd\lambda + q^2\varsigma\right)^2 \\
c &= -2c_{min} + 2||A||L + \varepsilon^{-1}||B||^2L + \varepsilon L \\
\sigma &= t_k - t_{k-1} \text{ for all } k \in \mathbb{N},
\end{aligned}
$$

then the trivial solution of system (5.28) is GES.

Proof. Choose Lyapunov functional $V(t, x_t) = V_1(t, x) + V_2(t, x_t)$ with

$$
\begin{aligned}
V_1(t, x) &= x^T x, \\
V_2(t, x_t) &= \varepsilon L \int_{t-r}^{t} x^T(s)x(s)\mathrm{d}s.
\end{aligned}
$$

Then, condition (i) of Theorem 3.2.1 is satisfied with $w_1 = w_2 = 1$, $w_3 = \varepsilon L r$, and $p = 2$.

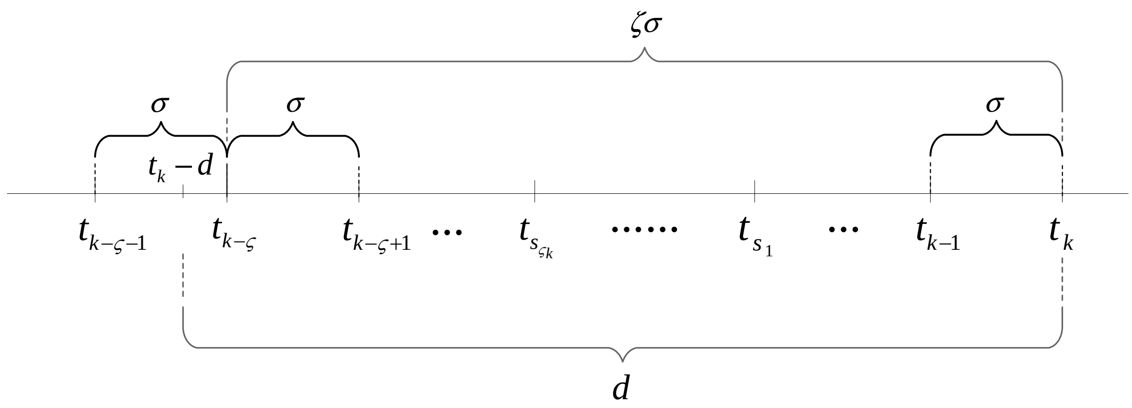


Figure 5.8: Schematic figure describing the different impulsive instants and time differences.

For $t \in [t_{k-1}, t_k)$, differentiate V_1 and V_2 along the solution of (5.28), then we can get

$$\begin{aligned} V_1'(t,x) &= 2\dot{x}^T x \\ &= -2x^T Cx + 2x^T Af(x) + 2x^T Bf(x(t-r)) \\ &\leq -2c_{min}x^T x + 2||A||Lx^T x + 2||B||L||x|| \cdot ||x(t-r)|| \\ &\leq (-2c_{min} + 2||A||L + \varepsilon^{-1}||B||^2 L)x^T x + \varepsilon L x^T(t-r)x(t-r), \end{aligned}$$

and

$$V_2'(t,x_t) = \varepsilon L x^T x - \varepsilon L x^T(t-r)x(t-r).$$

Then,

$$\begin{aligned} V'(t,x_t) &\leq (-2c_{min} + 2||A||L + \varepsilon^{-1}||B||^2 L + \varepsilon L)x^T x \\ &= cV_1(t,x) \leq cV(t,x_t), \end{aligned}$$

which implies condition (ii) of Theorem 3.2.1 is satisfied.

Next, we will show condition (iii) of Theorem 3.2.1 holds. Integrating both sides of (5.19) from $t_k - d$ to t_k, then

$$\begin{aligned} x_i(t_k^-) - x_i(t_k - d) &= \int_{t_k-d}^{t_k} -c_i x_i(s) + \sum_{j=1}^{n} a_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(s-r))\mathrm{d}s \\ &\quad + \sum_{m=1}^{\varsigma_k} q x_i(t_{k-m} - d), \end{aligned} \tag{5.30}$$

where ς_k denotes the number of impulses added to the ith neuron during the time period $(t_k - d, t_k)$.

According to the pinning strategy introduced in (5.27), we have $\varsigma_k \leq \varsigma$, since some impulses may be added to other neurons than the ith neuron during this time period. See Fig. 5.8 for illustrations. In the figure, we assume that the ith neuron is controlled at $t = t_k^-$, i.e., $i \in \mathfrak{D}_k^l$. The set $\{t_{s_j}\}_{j=1}^{\varsigma_k}$ denotes the collection of the impulsive moments when the ith neuron is pinned on time interval $(t_k - d, t_k)$. It can be seen from the definition of ς that ς represents the number of impulses that the DNN (5.17) subject to on each time interval $(t_k - d, t_k)$ for $k \in \mathbb{N}$. Therefore, $\{t_{s_j}\}_{j=1}^{\varsigma_k}$ is a subset of the set $\{t_j\}_{j=k-\varsigma}^{k-1}$, which implies $\varsigma_k \leq \varsigma$. Only when the ith neuron subjects to all the impulses on the time interval $(t_k - d, t_k)$, we have $\{t_{s_j}\}_{j=1}^{\varsigma_k} = \{t_j\}_{j=k-\varsigma}^{k-1}$, i.e., $\varsigma_k = \varsigma$.

From (5.30) and the second equation of (5.28), we have that, for $i \in \mathfrak{D}_k^l$,

$$\begin{aligned} x_i(t_k) &= x_i(t_k^-) + qx_i(t_k - d) \\ &= (1+q)x_i(t_k^-) - q\int_{t_k-d}^{t_k} -c_ix_i(s) + \sum_{j=1}^n a_{ij}f_j(x_j(s)) + \sum_{j=1}^n b_{ij}f_j(x_j(s-r))ds \\ &\quad - q^2\sum_{m=1}^{\varsigma_k} x_i(t_{k-m} - d). \end{aligned}$$

Let

$$\begin{aligned} \Upsilon_{i1} &= (1+q)x_i(t_k^-), \\ \Upsilon_{i2} &= -q\int_{t_k-d}^{t_k} -c_ix_i(s) + \sum_{j=1}^n a_{ij}f_j(x_j(s)) + \sum_{j=1}^n b_{ij}f_j(x_j(s-r))ds, \\ \Upsilon_{i3} &= -q^2\sum_{m=1}^{\varsigma_k} x_i(t_{k-m} - d). \end{aligned}$$

Then, by Lemma 5.2.2, we have

$$\begin{aligned} \sum_{i\in\mathfrak{D}_k^l} x_i^2(t_k) &= \sum_{i\in\mathfrak{D}_k^l} \{\Upsilon_{i1} + \Upsilon_{i2} + \Upsilon_{i3}\}^2 \\ &\leq (1+\varepsilon_1)\sum_{i\in\mathfrak{D}_k^l}\Upsilon_{i1}^2 + (1+\varepsilon_1^{-1})(1+\xi_1)\sum_{i\in\mathfrak{D}_k^l}\Upsilon_{i2}^2 + (1+\varepsilon_1^{-1})(1+\xi_1^{-1})\sum_{i\in\mathfrak{D}_k^l}\Upsilon_{i3}^2, \end{aligned} \tag{5.31}$$

for any $\varepsilon_1, \xi_1 > 0$.

Applying Lemma 5.2.2 and Schwartz's inequality to the second term of the right hand side of (5.31), we have

$$\begin{aligned} \sum_{i\in\mathfrak{D}_k^l}\Upsilon_{i2}^2 &\leq q^2\sum_{i\in\mathfrak{D}_k^l}\left\{\int_{t_k-d}^{t_k} -c_ix_i(s) + \sum_{j=1}^n a_{ij}f_j(x_j(s)) + \sum_{j=1}^n b_{ij}f_j(x_j(s-r))\mathrm{d}s\right\}^2 \\ &\leq q^2d\sum_{i\in\mathfrak{D}_k^l}\int_{t_k-d}^{t_k}\Big(-c_ix_i(s) + \sum_{j=1}^n a_{ij}f_j(x_j(s)) + \sum_{j=1}^n b_{ij}f_j(x_j(s-r))\Big)^2\mathrm{d}s \\ &\leq q^2d\int_{t_k-d}^{t_k}(1+\varepsilon_2)\sum_{i\in\mathfrak{D}_k^l} c_i^2x_i^2(s) \end{aligned}$$

$$
\begin{aligned}
&\quad + (1+\varepsilon_2^{-1})(1+\xi_2)\sum_{i\in\mathfrak{D}_k^l}\Big(\sum_{j=1}^{n} a_{ij}f_j(x_j(s))\Big)^2 \\
&\quad + (1+\varepsilon_2^{-1})(1+\xi_2^{-1})\sum_{i\in\mathfrak{D}_k^l}\Big(\sum_{j=1}^{n} b_{ij}f_j(x_j(s-r))\Big)^2 \mathrm{d}s \\
\leq\;& q^2 d\int_{t_k-d}^{t_k}(1+\varepsilon_2)c_{max}^2\sum_{i\in\mathfrak{D}_k^l} x_i^2(s) \\
&\quad + (1+\varepsilon_2^{-1})(1+\xi_2)n\sum_{i\in\mathfrak{D}_k^l}\sum_{j=1}^{n} a_{ij}^2 f_j^2(x_j(s)) \\
&\quad + (1+\varepsilon_2^{-1})(1+\xi_2^{-1})n\sum_{i\in\mathfrak{D}_k^l}\sum_{j=1}^{n} b_{ij}^2 f_j^2(x_j(s-r))\mathrm{d}s \\
\leq\;& q^2 d\int_{t_k-d}^{t_k}(1+\varepsilon_2)c_{max}^2\sum_{i=1}^{n} x_i^2(s) \\
&\quad + (1+\varepsilon_2^{-1})(1+\xi_2)n\sum_{i\in\mathfrak{D}_k^l}\sum_{j=1}^{n} a_{ij}^2 L_j^2 x_j^2(s) \\
&\quad + (1+\varepsilon_2^{-1})(1+\xi_2^{-1})n\sum_{i\in\mathfrak{D}_k^l}\sum_{j=1}^{n} b_{ij}^2 L_j^2 x_j^2(s-r)\mathrm{d}s \\
\leq\;& q^2 d\int_{t_k-d}^{t_k}(1+\varepsilon_2)c_{max}^2\sum_{i=1}^{n} x_i^2(s) \\
&\quad + (1+\varepsilon_2^{-1})(1+\xi_2)nl\max_{i,j}\{a_{ij}^2 L_j^2\}\sum_{i=1}^{n} x_i^2(s) \\
&\quad + (1+\varepsilon_2^{-1})(1+\xi_2^{-1})nl\max_{i,j}\{b_{ij}^2 L_j^2\}\sum_{i=1}^{n} x_i^2(s-r)\mathrm{d}s \\
\leq\;& q^2 d^2\Big((1+\varepsilon_2)c_{max}^2 + (1+\varepsilon_2^{-1})(1+\xi_2)nl\max_{i,j}\{a_{ij}^2 L_j^2\} \\
&\quad + (1+\varepsilon_2^{-1})(1+\xi_2^{-1})nl\max_{i,j}\{b_{ij}^2 L_j^2\}\Big)\sup_{s\in[-r-d,0]} V_1(t_k^- + s, x(t_k^- + s)) \\
=\;& q^2 d^2 \lambda^2 \sup_{s\in[-r-d,0]} V_1(t_k^- + s, x(t_k^- + s)), \qquad (5.32)
\end{aligned}
$$

with $(\varepsilon_2,\xi_2) = \Big(\frac{\sqrt{nl}(\max_{i,j}\{|a_{ij}|L_j\}+\max_{i,j}\{|b_{ij}|L_j\})}{c_{max}}, \frac{\max_{i,j}\{|b_{ij}|L_j\}}{\max_{i,j}\{|a_{ij}|L_j\}}\Big)$.

For the third term of the right hand side of (5.31), we have

$$
\begin{aligned}
\sum_{i\in\mathfrak{D}_k^l}\Upsilon_{i3}^2 &= q^4\sum_{i\in\mathfrak{D}_k^l}\big(\sum_{m=1}^{\varsigma_k} x_i(t_{k-m}-d)\big)^2 \\
&\leq q^4\varsigma_k\sum_{i\in\mathfrak{D}_k^l}\sum_{m=1}^{\varsigma_k} x_i^2(t_{k-m}-d) \\
&\leq q^4\varsigma_k\sum_{m=1}^{\varsigma_k}\sum_{i=1}^{n} x_i^2(t_{k-m}-d)
\end{aligned}
$$

$$\leq \quad q^4\varsigma^2 \sup_{s\in[-2d,0]} V_1(t_k^- + s, x(t_k^- + s)). \tag{5.33}$$

From (5.31), (5.32), and (5.33), we can conclude that

$$\sum_{i\in\mathfrak{D}_k^l} x_i^2(t_k) \leq \rho_1' \sum_{i\in\mathfrak{D}_k^l} x_i^2(t_k^-) + \rho_2 \sup_{s\in[-\tau_1-d,0]} V_1(t_k^- + s, x(t_k^- + s)), \tag{5.34}$$

where

$$\begin{aligned}
\tau_1 &= \max\{r, d\}, \\
\rho_1' &= (1+\varepsilon_1)(1+q)^2, \\
\rho_2 &= (1+\varepsilon_1^{-1})(1+\xi_1)q^2d^2\lambda^2 + (1+\varepsilon_1^{-1})(1+\xi_1^{-1})q^4\varsigma^2.
\end{aligned}$$

Let $\rho_1 = 1 - \frac{l}{n}(1-\rho_1')$, then

$$\begin{aligned}
(1-\rho_1)\sum_{i\notin\mathfrak{D}_k^l} x_i^2(t_k^-) &\leq (1-\rho_1)(n-l)\min_{i\in\mathfrak{D}_k^l}\{x_i^2(t_k^-)\} \\
&= l(\rho_1-\rho_1')\min_{i\in\mathfrak{D}_k^l}\{x_i^2(t_k^-)\} \\
&\leq (\rho_1-\rho_1')\sum_{i\in\mathfrak{D}_k^l} x_i^2(t_k^-),
\end{aligned}$$

i.e.,

$$\rho_1'\sum_{i\in\mathfrak{D}_k^l} x_i^2(t_k^-) + \sum_{i\notin\mathfrak{D}_k^l} x_i^2(t_k^-) \leq \rho_1\sum_{i=1}^n x_i^2(t_k^-).$$

Then, for $t = t_k$, we have

$$\begin{aligned}
V_1(t_k, x(t_k)) &= \sum_{i\in\mathfrak{D}_k^l} x_i^2(t_k) + \sum_{i\notin\mathfrak{D}_k^l} x_i^2(t_k) \\
&= \rho_1'\sum_{i\in\mathfrak{D}_k^l} x_i^2(t_k^-) + \rho_2 \sup_{s\in[-\tau_1-d,0]} V_1(t_k^- + s, x(t_k^- + s)) + \sum_{i\notin\mathfrak{D}_k^l} x_i^2(t_k) \\
&\leq \rho_1 V_1(t_k^-, x(t_k^-)) + \rho_2 \sup_{s\in[-\tau_1-d,0]} \{V_1(t_k^- + s, x(t_k^- + s))\},
\end{aligned} \tag{5.35}$$

which implies condition (iii) of Theorem 3.2.1 is satisfied.

Applying Lemma 5.2.3, we have

$$\begin{aligned}
\min_{\varepsilon_1,\xi_1>0}\{\rho_1+\rho_2\} &= \min_{\varepsilon_1,\xi_1>0}\Big\{1 - \frac{l}{n} + (1+\varepsilon_1)\frac{l}{n}(1+q)^2 + (1+\varepsilon_1^{-1})(1+\xi_1)q^2d^2\lambda^2 \\
&\qquad + (1+\varepsilon_1^{-1})(1+\xi_1^{-1})q^4\varsigma^2\Big\} \\
&= \rho,
\end{aligned}$$

with $(\varepsilon_1, \xi_1) = (\frac{q^2\varsigma - qd\lambda}{1+q}, \frac{-q\varsigma}{d\lambda})$.

With inequality (5.29), we can see that condition (iv) of Theorem 3.2.1 holds. Actually, there is a slight difference between (5.35) and the inequality in condition (iii) of Theorem 3.2.1: in equality (5.35), the last term is defined on interval $[-\tau_1 - d, 0]$, while the interval in condition (iii) of Theorem 3.2.1 is $[-\tau, 0]$. Simply replace (3.4) by the following inequality

$$\ln(\rho_1 + \rho_2 e^{\alpha(\tau_1 + d)} + \frac{w_3}{w_1} e^{\alpha\tau}) = -(\alpha + c)\sigma,$$

and we can see that Theorem 3.2.1 is still true. Therefore, we can conclude from Theorem 3.2.1 that the trivial solution of (5.28) is GES. □

Remark 5.2.4 *It can be seen that DNN (5.17) can be stabilized by pinning control l neurons of the network at each impulsive instant, and the number l is closely related to the length δ of each impulsive interval. Inequality (5.29) implies that the less neurons are controlled at each impulsive instant, the smaller the length of each impulsive interval is required. It can also be observed that a positive constant ε is introduced via the Lyapunov functional part V_2. For large time-delay (e.g., $r > \frac{1}{L}$), we can pick up small enough $\varepsilon > 0$ so that $\varepsilon r L < 1$ and make inequality (5.29) to be satisfied. Therefore, Theorem 5.2.2 is applicable to NNs with large time-delay size. It is also worth noting that Lemmas 5.2.2 and 5.2.3 are applied in the proof to reduce the conservatism of the sufficient conditions in Theorem 5.2.2.*

Remark 5.2.5 *Theorem 5.2.2 gives sufficient conditions to design suitable pinning impulsive controller (5.27) with uniform impulsive interval and control gain. However, the nonuniform impulsive controller can also be designed according to Theorem 3.2.2. Moreover, nonlinear impulsive controller $U_i(t, x_i) = \sum_{k=1}^{\infty} I_k(t, x_i(t-d))\delta(t - t_k^-)$ for $i \in \mathfrak{D}_k^l$ can be investigated according to Theorem 3.2.1 and 3.2.2, if there exist positive constants q_k such that the function $I_k : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ satisfies the following inequality for $k \in \mathbb{N}$*

$$|I_k(t, x)| \leq q_k |x|, \text{ for any } t \in \mathbb{R}^+, \text{ and } x \in \mathbb{R}.$$

The detailed discussions are omitted, since they are identical to the proof of Theorem 5.2.2.

Remark 5.2.6 *In this section, we have successfully applied Theorem 3.2.1 and 3.2.2 to study the pinning delayed-impulsive control of DNNs. Actually, since Theorem 3.2.1 and 3.2.2 are stability results for general nonlinear impulsive systems with delays, these sufficient conditions can be used to study stabilization and synchronization problems of various dynamical time-delay systems, such as chaotic systems [9], BAM neural networks [7], Hopfield neural networks [112]. Similar discussions of impulsive discrete-time systems with time-delay can also be investigated by employing the techniques in [81, 130]. Furthermore, according to the pinning control algorithm introduced in subsection 5.2.1, different neurons will be controlled at different impulsive instants. Therefore, our results do not require the network to be well connected (i.e., no isolated clusters exist in the network [76]), which is essential in many existing pinning control strategies (see, e.g., [113, 76, 46, 136, 125]).*

In what follows, we will consider three particular cases of our pinning controller (5.27).

For $l = n$, all the neurons will be controlled at each impulsive instant. Then pinning controller (5.27) reduces to the following delayed impulsive controller:

$$U(t, x) = \sum_{k=1}^{\infty} q x(t-d)\delta(t - t_k^-), \tag{5.36}$$

where $U(t,x) = (U_1(t,x_1), U_2(t,x_2), ..., U_n(t,x_n))^T$. For controller (5.36), we have the following stabilization result.

Corollary 5.2.1 *Suppose inequality (5.29) holds with parameter ρ replaced by*

$$\rho := \rho_\alpha = (1 + q - qd\lambda + q^2\varsigma)^2,$$

then the trivial solution of system (5.19) is GES.

In the previous discussion, we have assumed that $d > 0$. Actually, for $d = 0$, controller (5.27) reduces to the following pinning impulsive controller (delay-free):

$$U_i(t,x_i) = \begin{cases} \sum_{k=1}^{\infty} qx_i(t)\delta(t - t_k^-), \ i \in \mathfrak{D}_k^l \\ 0, \ i \notin \mathfrak{D}_k^l, \end{cases} \tag{5.37}$$

for $i = 1,2,...,n$. Then, for each neuron, $\varsigma_k = 0$. Hence, we can get the following stabilization result for controller (5.37) with $\varsigma = 0$.

Corollary 5.2.2 *Suppose inequality (5.29) holds with parameter ρ replaced by*

$$\rho := \rho_\beta = 1 - \frac{l}{n} + \frac{l}{n}(1+q)^2,$$

then the trivial solution of system (5.19) is GES.

Furthermore, for both $l = n$ and $d = 0$, we can get from (5.27) the standard linear impulsive feedback controller:

$$U(t,x) = \sum_{k=1}^{\infty} qx(t)\delta(t - t_k^-), \tag{5.38}$$

and the corresponding stabilization criterion which can be easily derived from Corollary 5.2.1 or Corollary 5.2.2.

Corollary 5.2.3 *Suppose inequality (5.29) holds with parameter ρ replaced by*

$$\rho := \rho_\gamma = (1+q)^2,$$

then the trivial solution of system (5.19) is GES.

Comparing Corollary 5.2.3 with Corollary 5.2.1 and 5.2.2 can help us to understand the effects that the time delay in the impulses and the ratio l/n plays on the stabilization process, respectively.

The main difference between Corollary 5.2.1 and Corollary 5.2.3 lies in the two terms $qd\lambda$ and $q^2\varsigma$ in parameter ρ which are both related to the impulse delay, and are all original from (5.30) in the estimation of the relation between states $x(t_k^-)$ and $x(t_k - d)$. $qd\lambda$ depends on the continuous dynamic of DNN (5.17), while $q^2\varsigma$ corresponds to the number of impulses on time interval $(t_k - d, t_k)$. Therefore, for fixed impulsive control gain, increasing the impulse delay size will reduce the length of impulsive interval. However, the permissible impulse delay d is required to be bounded in Corollary 5.2.1. To guarantee inequality (5.29) is true in Corollary 5.2.1, parameter ρ must be less than 1. Then, we can see from ρ_α that impulse delay d is bounded by $1/\lambda$, and satisfies $\varsigma < \frac{1-d\lambda}{-q}$. Intuitively, it is difficult to estimate the relation between states $x(t_k^-)$ and $x(t_k - d)$ precisely for large delay size d (e.g. chaotic systems), and then it is not practical to use the state $x(t_k - d)$ as impulsive feedback signal to stabilize the system, which is in accordance with our theoretical analysis.

Next, we will compare Corollary 5.2.3 with Corollary 5.2.1 to demonstrate how the pinning algorithm affects the design of impulsive controllers. Define a function $F(\omega) = 1 + \omega[(1+q)^2 - 1]$, then $F(\frac{l}{n}) = \rho_\beta$ and $F(1) = \rho_\gamma$. Since $F'(\omega) < 0$, we have $F(\frac{l}{n}) > F(1)$ (i.e., $\rho_\beta > \rho_\gamma$) for $l < n$. Therefore, it can be seen from inequality (5.29) that, with the same impulsive control gain, reducing the number of neurons to be pinned will lead to increasing the frequency that the impulses added to the network.

Numerical Simulations

Next, we consider DNN (5.17) with parameters given in Example 5.2.1. It has been shown in [74] that DNN (5.17) with the given parameters has a chaotic attractor, see Figure 5.4 for illustration. We consider two types of impulsive controllers:

1) controller (5.27) with $l = 1$, i.e., impulsive control one neuron at each impulsive instant. Let $\sigma = 0.004$, $d = 0.002$, and $q = -0.43$, then inequality (5.29) is satisfied with $\varepsilon = 0.125$. Therefore, we can conclude from Theorem 5.2.2 that the given DNN (5.17) can be exponentially stabilized by the pinning controller (5.27) with control gain $q = -0.43$. Numerical simulations can be found in Figure 5.9.

2) $l = 2$, i.e., impulsive control both neurons at each impulsive instant. Let $\sigma = 0.01$, $d = 0.02$, and $q = -0.58$, then inequality (5.29) is satisfied with $\varepsilon = 0.25$. Therefore, we can conclude from Theorem 5.2.2 that the impulsive DNN (5.28) is exponentially stable. See Figure 5.10 for simulation results.

The initial data in Figure 5.9 and Figure 5.10 is chosen the same as that in Figure 5.9, and the red dot denotes the state x at initial time $t_0 = 0$. In Figure 5.9 and Figure 5.10, sub-figures (a) illustrate the phase portrait of DNN (5.17) under the above two distinct impulsive controllers. In order to observe the pinning impulsive effects and demonstrate the stabilization process, sub-figures (b), (c), and (d) are provided with the state trajectories of impulsive DNN (5.20). The vertical (or horizontal) lines in Figure 5.9(a) represent the state jump of x_2 (or x_1) while the other state is unchanged. Since both neurons are controlled in Figure 5.10(a), no vertical and horizontal lines can be observed. It can be seen from Figure 5.9 that different neuron may be controlled at different impulsive instant. This is consistent with our pinning algorithm of control the neuron which has the larger state deviation with the equilibrium.

Finally, for the delayed impulsive controller (5.36), we compare our results with those in [25] and [54]. As discussed in Example 2 of [25], we assume that $q = -0.8$ and $\sigma = 0.01$.

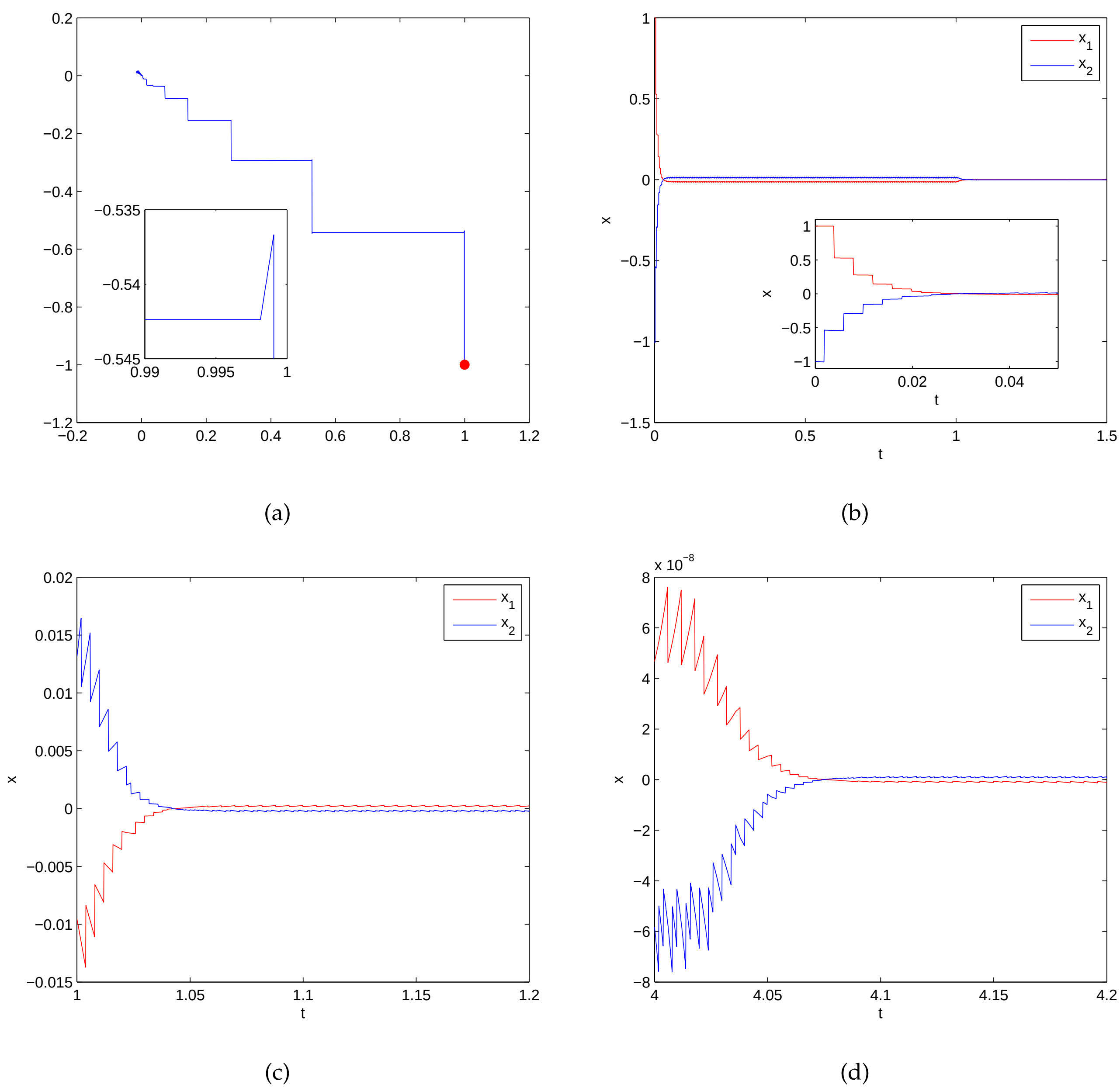


Figure 5.9: Impulsive control one neuron of DNN (5.17) at each impulsive instant. In this simulation, the length of the impulsive interval is so small that the phase portrait of the network states on each impulsive interval can be barely observed. Hence, a small figure is presented in sub-figure (a) to demonstrate the phase portrait of the network states on the second impulsive interval $[t_1, t_2)$. Small serrations in the trajectories of x_1 and x_2 can be clearly seen in sub-figure (b) for $t < 1$, which can be explained by the existence of time-delay in DNN (5.17) with delay size $r = 1$. Therefore, the form of serration can also be affected by the network initial data. In sub-figure (b), the small figure is given to illustrate the pinning algorithm introduced in (5.27).

Then, from [25], we know that the upper bound of d is 0.0194 according to Corollary 2 of [25], while the upper bound of d is 0.0199 by Corollary 1 of [54]. However, it is worth noting that the impulse delay in the simulation of Figure 5.10 is $d = 0.02$, which is larger than the upper bound

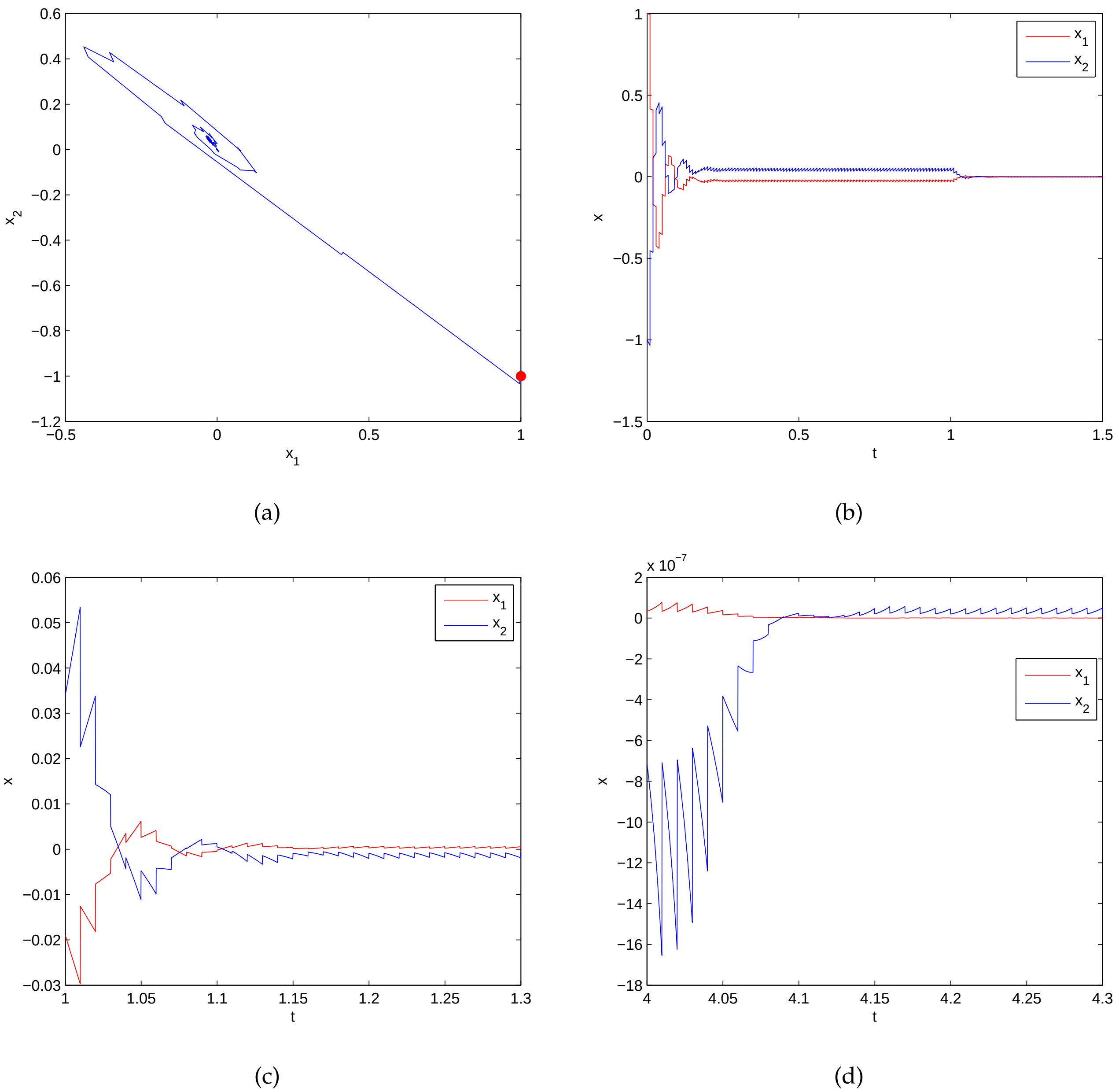


Figure 5.10: Impulsive control both neurons of DNN (5.17) at each impulsive instant. Similar serrations can also be observed in sub-figure (b), since the same initial data and time-delay are considered in Example 5.2.1.

of d in both [25] and [54]. Moreover, in our simulation, the impulsive control strength can be reduced to $|q| = 0.58$ which is smaller than $|q| = 0.8$. Hence, our results not only provide new criteria to design pinning delayed impulsive controller to stabilize the DNN (5.17), but also are less conservative than the stabilization results constructed in previous literatures [25] and [54] for delayed impulsive controller (5.36).

5.3 Synchronization of Nonlinear Time-Delay Systems

This section investigate the synchronization of globally Lipschitz time-delay systems using pinning impulsive control. We propose a novel class of pinning impulsive controllers that takes into account of both discrete and distributed delays. Verifiable synchronization conditions for pinning impulsive controller with discrete delay, distributed delay and both of these two types of delays are established using a Halanay-type inequality, respectively. The theoretical results provide insight into the feasible relation between the impulse delays and impulse frequency to guarantee the synchronization of drive and response systems via impulsive control a small portion of the system states. The findings are illustrated by stability analysis of a linear impulsive time-delay system and synchronization control of a nonlinear chaotic time-delay system with numerical simulations.

5.3.1 Problem Formulation

In this subsection, we formulate the general synchronization synthesis problem. Consider the drive system

$$\begin{cases} \dot{x}(t) = g(t, x_t), \\ x_{t_0} = \phi_1, \end{cases} \tag{5.39}$$

with $x \in \mathbb{R}^n$, $\phi_1 \in \mathcal{PC}([-\tau, 0], \mathbb{R}^n)$, $g = (g_1, g_2, ..., g_n)^T$ and $g_i \in \mathbb{R} \times \mathcal{PC}([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}$, and a response system

$$\begin{cases} \dot{y}(t) = g(t, y_t) + u(t), \\ y_{t_0} = \phi_2, \end{cases} \tag{5.40}$$

where $y \in \mathbb{R}^n$, $\phi_2 \in \mathcal{PC}([-\tau, 0], \mathbb{R}^n)$, and $u(t) := u(t, x, y)$ is the control input to be designed to synchronize these two systems, i.e., $\lim_{t\to\infty} \|y(t) - x(t)\| = 0$. For the nonlinear functionals g, we make the following globally Lipschitz assumptions: for any $\psi_1, \psi_2 \in \mathcal{PC}([-\tau, 0], \mathbb{R}^n)$,

(A_1) $|g_i(t, \psi_1) - g_i(t, \psi_2)| \leq L_i \|\psi_1 - \psi_2\|_\tau$, for $i = 1, 2, ..., n$;

(A_2) $\|g(t, \psi_1) - g(t, \psi_2)\| \leq K \|\psi_1 - \psi_2\|_\tau$.

Clearly, (A_1) implies (A_2) with $K = \sqrt{\sum_{i=1}^n L_i^2}$. However, for some functionals (for example, $g_i(t, \psi) = \psi_i(0)$), we can derive the Lipschitz constant K smaller than $\sqrt{\sum_{i=1}^n L_i^2}$. Hence, we write assumption (A_2) separately.

Construct the pinning impulsive controller as $u = (u_1, u_2, ..., u_n)^T$,

$$u_i(t) = \begin{cases} \sum_{k=1}^{\infty} -\big(q_1 e_i(t - d_1) + q_2 \int_{t-d_2}^{t} e_i(s)\mathrm{d}s\big)\delta(t - t_k), \text{ if } i \in \mathfrak{D}_k^l, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0, \text{ otherwise}, \end{cases} \tag{5.41}$$

where $e_i = y_i - x_i$, impulse times $t_k = t_0 + k\sigma$ with $\sigma > 0$ and $k \in \mathbb{N}$; q_1 and q_2 are impulsive control gains; d_1 and d_2 are discrete and distributed delays in the controller, respectively; $\delta(\cdot)$ is the Dirac delta function; the index set $\mathfrak{D}_k^l = \{p_1, p_2, ..., p_l\} \subseteq \mathcal{I} : \{1, 2, ..., n\}$ is defined as follows: $p_i \neq p_j$ if $i \neq j$; at the impulsive instant t_k, $\|e_i(t_k^-)\| \geq \|e_j(t_k^-)\|$ if $i \in \mathfrak{D}_k^l$ and $j \in \mathcal{I}/\mathfrak{D}_k^l$, that is, l states are controlled at each impulsive instant. Accordingly, under controller (5.41), the closed-loop response system becomes an impulsive system:

$$\begin{cases} \dot{y}(t) = g(t, y_t),\ t \neq t_k, \\ \Delta y_i(t_k) = -q_1 e_i(t_k - d_1) - q_2 \displaystyle\int_{t_k - d_2}^{t_k} e_i(s)\mathrm{d}s,\ k \in \mathbb{N} \text{ and } i \in \mathfrak{D}_k^l, \\ y_{t_0} = \phi_2, \end{cases} \tag{5.42}$$

and then, the synchronization error $e := y - x = (e_1, e_2, ..., e_n)^T$ is governed the error system:

$$\begin{cases} \dot{e}(t) = f(t, e_t),\ t \neq t_k, \\ \Delta e_i(t_k) = -q_1 e_i(t_k - d_1) - q_2 \displaystyle\int_{t_k - d_2}^{t_k} e_i(s)\mathrm{d}s,\ k \in \mathbb{N} \text{ and } i \in \mathfrak{D}_k^l, \\ e_{t_0} = \phi, \end{cases} \tag{5.43}$$

where $f(t, e_t) = g(t, y_t) - g(t, x_t)$ and $\phi = \phi_2 - \phi_1$. The objective is to find admissible relations among the length of impulsive interval σ, impulsive control gains q_1 and q_2, and impulse delays d_1 and d_2 to guarantee $e(t) \to 0$ as $t \to \infty$.

5.3.2 Synchronization Criteria

Synchronization results are established by considering the following three scenarios.

Case I: Impulses with only Discrete Delays (i.e., $q_1 \neq 0$ and $q_2 = 0$)

The error system (5.43) can be written as follows:

$$\begin{cases} \dot{e}(t) = f(t, e_t),\ t \neq t_k, \\ \Delta e_i(t_k) = -q_1 e_i(t_k - d_1),\ k \in \mathbb{N} \text{ and } i \in \mathfrak{D}_k^l, \\ e_{t_0} = \phi. \end{cases} \tag{5.44}$$

Let $L = \max_i\{L_i\}$ and denote

$$\rho_{min} := 1 - \frac{l}{n} + \left(\sqrt{\frac{l}{n}}(1 - q_1) + \sqrt{l} q_1 d_1 L + q_1^2 \zeta\right)^2,$$

with $\zeta = \lfloor \frac{d_1}{\sigma} \rfloor$ (that is, the number of impulses on interval $[t_k - d_1, t_k)$), and then, we can generalize the results obtained in Section 5.2 to obtain the following synchronization result for system (5.44).

Theorem 5.3.1 *If*

$$\sigma < \begin{cases} G(\rho^*),\ \text{if } \rho_{min} \in (0, \rho^*], \\ G(\rho_{min}),\ \text{if } \rho_{min} \in (\rho^*, 1), \end{cases} \tag{5.45}$$

with $\rho^* = e^{-2}$ and $G(\rho) := \frac{-\sqrt{\rho}\ln(\rho)}{2K}$, then (5.44) achieves synchronization.

Case II: Impulses with only Distributed Delays (i.e., $q_1 = 0$ and $q_2 \neq 0$)

The error system (5.43) can be written as follows:

$$\begin{cases} \dot{e}(t) = f(t, e_t), \ t \neq t_k, \\ \Delta e_i(t_k) = -q_2 \displaystyle\int_{t_k - d_2}^{t_k} e_i(s) \mathrm{d}s, \ k \in \mathbb{N} \text{ and } i \in \mathfrak{D}_k^l, \\ e_{t_0} = \phi. \end{cases} \tag{5.46}$$

It is worth noting that, to our best knowledge, this is the first time to consider distributed delays in the pinning impulsive controller.

Theorem 5.3.2 *If inequality (5.45) is satisfied with ρ_{min} replaced with*

$$\rho_{min} := 1 - \frac{l}{n} + \left(\sqrt{\frac{l}{n}}(1 - q_2 d_2) + \sqrt{l q_2 d_2^2 L + q_2^2 d_2^2 \varsigma} \right)^2,$$

where $\varsigma = \lfloor \frac{d_2}{\sigma} \rfloor$ (that is, the number of impulses on interval $[t_k - d_2, t_k)$), then (5.44) achieves synchronization.

Case III: Impulses with Both Discrete and Distributed Delays (i.e., $q_1 \neq 0$ and $q_2 \neq 0$)

This scenario is a generalization and combination of results in Case I and II.

Theorem 5.3.3 *If inequality (5.45) is satisfied with ρ_{min} replaced with*

$$\rho_{min} := 1 - \frac{l}{n} + \left(\sqrt{\frac{l}{n}}(1 - q_1 - q_2 d_2) + \sqrt{l q_1 d_1 L + q_1^2 \zeta} + \sqrt{l q_2 d_2^2 L + q_2^2 d_2^2 \varsigma} \right)^2,$$

where ζ and ς are the same as those defined in Theorem 5.3.1 and 5.3.2, respectively, then (5.44) achieves synchronization.

5.3.3 Simulation Results

In this subsection, we consider two examples to illustrate our theoretical results. In the first example, we will study the exponential stability of a linear impulsive system with delays:

Example 5.3.1 *Consider the following linear scalar system with discrete delays in both the continuous and discrete dynamics:*

$$\begin{cases} \dot{x}(t) = Kx(t - \tau), \ t \neq t_k, \\ \Delta x(t_k) = -q_1 x(t_k - d) - q_2 \displaystyle\int_{t_k - d}^{t_k} x(s) \mathrm{d}s, k \in \mathbb{N}, \\ x_{t_0} = \phi, \end{cases} \tag{5.47}$$

where $x \in \mathbb{R}$, $K = 0.25$, $\tau = 1$ and $\phi(s) = 2$ for $s \in [-\tau, 0]$.

It can be seen that assumptions (A_1) and (A_2) are satisfied with $L = K = 0.25$. It is shown in Figure 5.12(a) that if $q_1 = q_2 = 0$ (or $\sigma = \infty$) the impulse-free system is unstable. In the first simulation, let $q_1 = 0.8$ and $q_2 = 0$, that is, only discrete delays are considered in the impulses. Since the state of system (5.47) is scalar, we have $n = l = 1$. Theorem 5.3.1 implies that if $\rho_{min} < 1$ and

$$\sigma < \begin{cases} e^{-1}/K, & \rho_{min} \leq e^{-2}, \\ G(\rho_{min}), & \rho_{min} > e^{-2}, \end{cases} \tag{5.48}$$

where $\rho_{min} = (1 - q + qdK + \zeta q^2)^2$, then the trivial solution of (5.47) is GES. Therefore, with the given system parameters, the stability regions are illustrated in Figure 5.11, which gives the relation between d and σ to guarantee the exponential stability of system (5.47).

If $\zeta = 0$, then there is no impulse on each impulsive interval $(t_k - d, t_k)$, i.e., $d \leq \sigma$. The regions in green and blue in Figure 5.11 demonstrate the feasible relation between σ and d when $\zeta = 0$. If $\zeta = 1$, then there is one impulse on the interval $(t_k - d, t_k)$, i.e., $\sigma < d \leq 2\sigma$. The red region in Figure 5.11 describes the stability region for this scenario. For $\zeta \geq 2$, it can be calculated that $\rho_{min} > 1$, which implies that conditions of Theorem 5.3.1 can not be satisfied. Although only sufficient conditions are derived in our results, they are less conservative than the existing results. For system (5.47), the condition $\sigma < G(\rho_{min})$ is equivalent to the condition of Corollary 1 in [25], which is related to the blue region in Figure 5.11. In our result, we have improved this condition for small value of ρ_{min} (e.g., $\rho_{min} < e^{-2}$), and the stability region is enlarged with the green part.

Next, we simulate the state trajectory of system (5.47) with parameters pair (d, σ) selected from different regions shown in Figure 5.11. For the point $(d, \sigma) = (0.2, 1.4)$ in the green region, it is shown in Figure 5.12(b) that the system can be stabilized by the delayed impulses. For $(d, \sigma) = (0.3, 0.16)$ in the red region, stabilization process is illustrated in Figure 5.12(c). Normally, if no delays exist in the impulsive controller, increasing the acting frequency of impulsive controller will accelerate the stabilization process of dynamical systems. When it comes to impulses with time delays, reducing the length of impulsive intervals may lead to the instability of the system. Choose $(d, \sigma) = (0.3, 0.12)$ in the white region of Figure 5.11. It can be seen that we reduce the impulsive interval length σ (considered in Figure 5.12(c)) from 0.16 to 0.12. Figure 5.12(d) shows that impulsive system (5.47) is unstable. Simulation results shown in Figures 5.12(c) and 5.12(d) inspire that, for given q_1 and d, there should be a lower bound of σ to guarantee the stability of system (5.47). Actually, for $q_1 = 0.8$ and given $d > 0$, the boundaries of stability region shown in Figure 5.11 represent the upper and lower bounds of σ. The upper bound is shown explicitly in (5.48), and the lower bound of σ is restricted in $\rho_{min} < 1$ and (5.48) implicitly, since ρ_{min} is closely related to ζ which depends on σ.

In the following simulations, let $(d, \sigma) = (1.39, 1.4)$. It is shown in Figure 5.13(a) that system (5.47) can not be stabilized by impulses with control gain $q_1 = 0.4$ and discrete delays. However, for $q_1 = 0$ and $q_2 = 0.4$, all the conditions of Theorem 5.3.1 are satisfied, and the stabilization process are shown in Figure 5.13(b) for impulses with distributed delays. In the last simulation of Example 5.47, we consider an impulsive controller with $q_1 = q_2 = 0.4$ which is a combination of controllers considered in Figures 5.13(a) and 5.13(b). Theorem 5.3.3 implies that impulsive control system (5.47), and simulation results are shown in Figure 5.13(c). It can be observed that the impulses with both discrete and distributed delays stabilize the system faster than impulses with only distributed delays. The reason is that both the impulses with

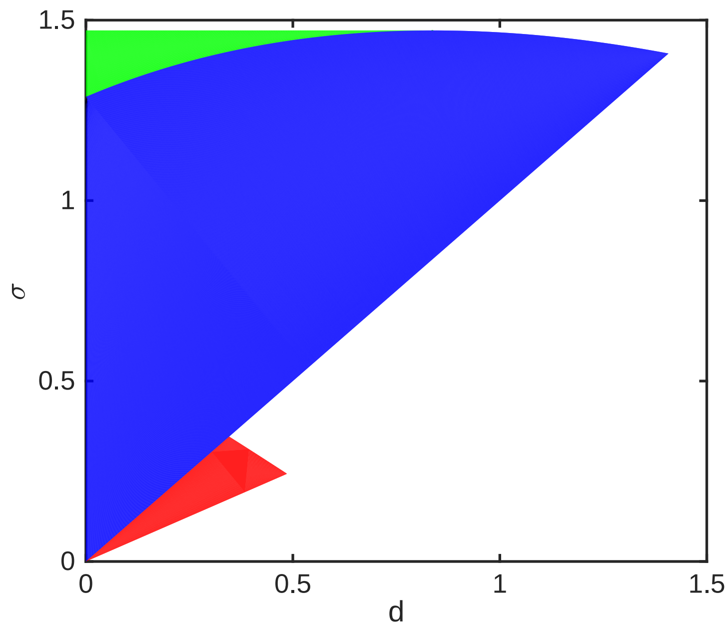


Figure 5.11: Stability region: admissible relations between d and σ for stabilization.

discrete delays and the impulses with distributed delays contribute to the system stabilization which can be seen from Figures 5.13(a) and 5.13(b).

In the next example, we will study the synchronization problem via pinning impulses with distributed delays.

Example 5.3.2 *Consider the drive system (5.39) modeled by the following time-delay system:*

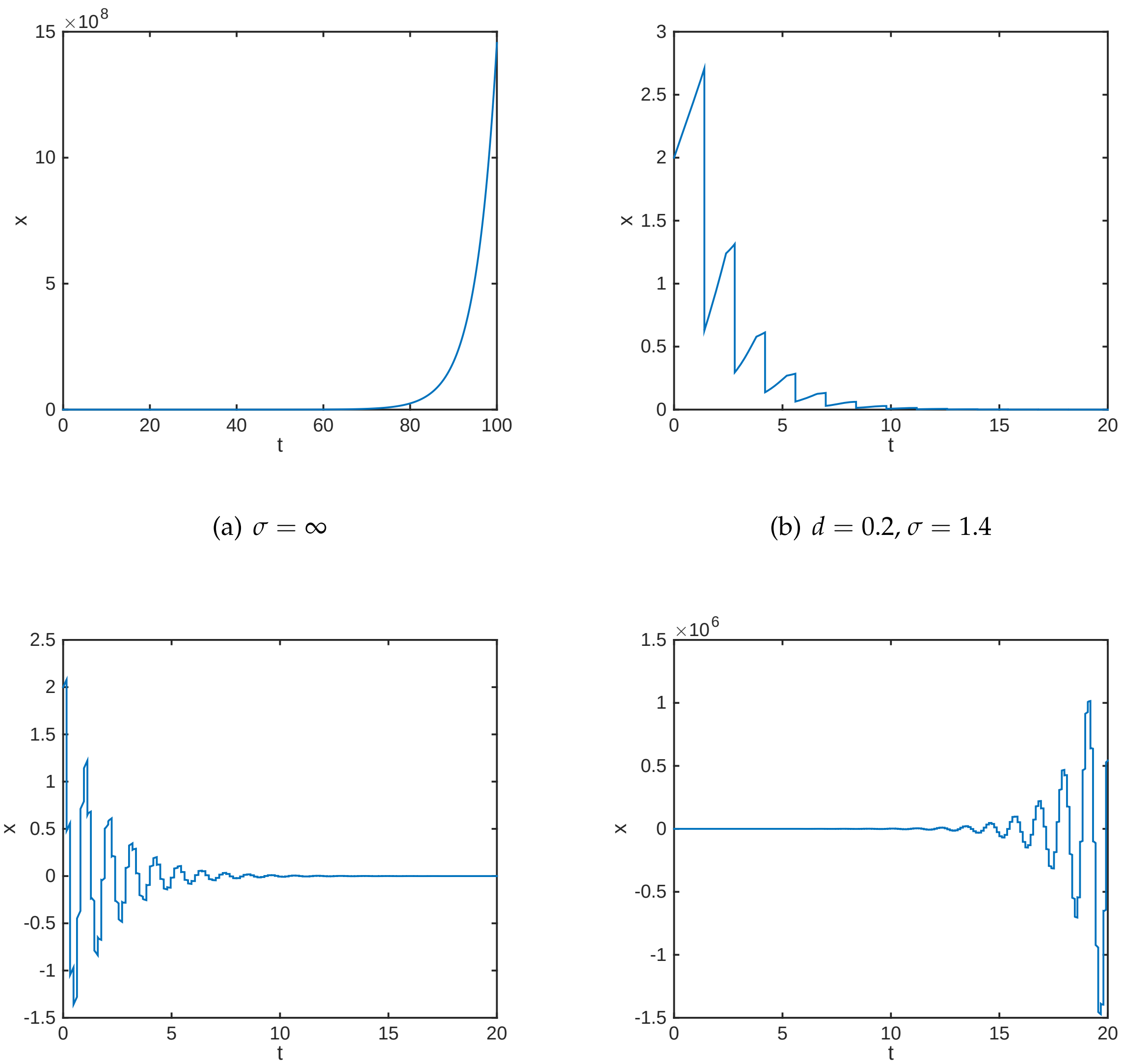
$$\begin{cases} \dot{x}_1(t) = -x_1(t) + a_1 h(x_2(t)) - b_1 h(x_2(t-\tau)), \\ \dot{x}_2(t) = -x_2(t) + a_2 h(x_1(t)) - b_2 h(x_1(t-\tau)), \end{cases} \tag{5.49}$$

where $a_1 = 1$, $b_1 = -1.9$, $a_2 = 1.71$, $b_2 = -1.037$, $\tau = 1$ *and* $h(\chi) = \sin(2.81\chi)$.

It is shown in [133] that system (5.49) exhibits chaotic behaviors with the above given parameters, which is illustrated in Figure 5.14. Consider response system with the following impulsive controller with distributed delays:

$$u_i(t) = \begin{cases} \sum_{k=1}^{\infty} -q \int_{t-d}^{t} e_i(s) \mathrm{d}s \delta(t - t_k), \ i \in \mathfrak{D}_k^l, \\ \qquad\qquad\qquad\qquad 0, \ \text{otherwise}. \end{cases} \tag{5.50}$$

It can be calculated that assumptions (A_1) and (A_2) hold with $L_1 = 8.2101$, $L_2 = 7.7836$ and $K = 11.3133$. In the first simulation, we consider the full-state controller (i.e., $l = n = 2$) with $q = 17$, $d = 0.02$ and $\sigma = 0.0202$. Theorem 5.3.2 implies that synchronization can be achieved. Numerical results are shown in Figure 5.15(a). Pinning impulsive controller (i.e., l=1) is considered in the next simulation with $q = 31.82$, $d = 0.01$ and $\sigma = 0.0114$. Then all



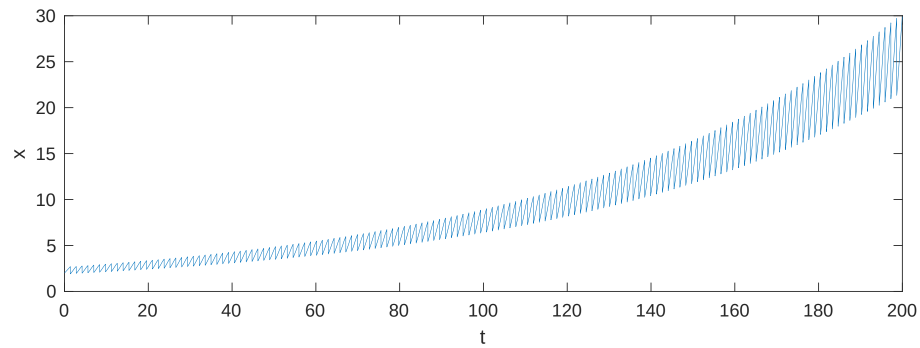
(a) $\sigma = \infty$

(b) $d = 0.2, \sigma = 1.4$

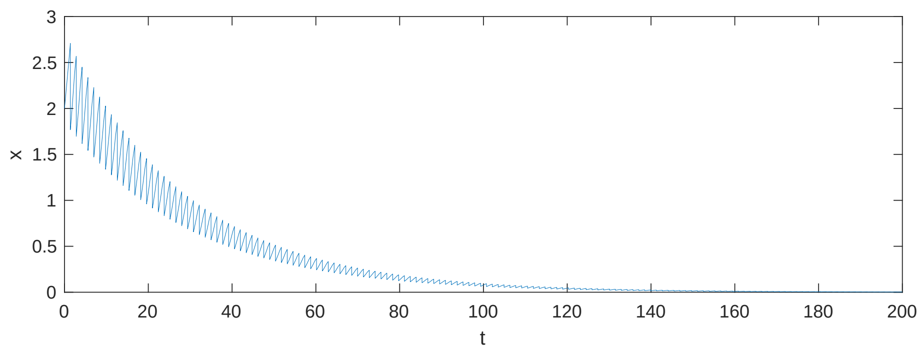
(c) $d = 0.3, \sigma = 0.16$

(d) $d = 0.3, \sigma = 0.12$

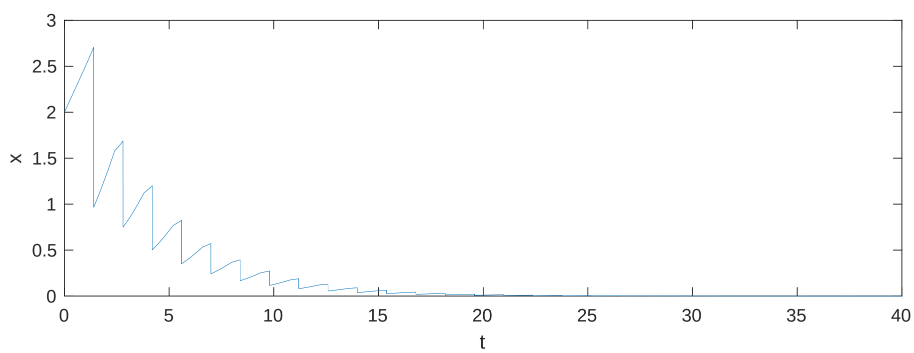
Figure 5.12: State trajectory of system (5.47) with $q_1 = 0.8$ and $q_2 = 0$.



(a) $q_1 = 0.4$ and $q_2 = 0$



(b) $q_1 = 0$ and $q_2 = 0.4$



(c) $q_1 = q_2 = 0.4$

Figure 5.13: State trajectory of system (5.47) with $d = 1.39$ and $\sigma = 1.4$.

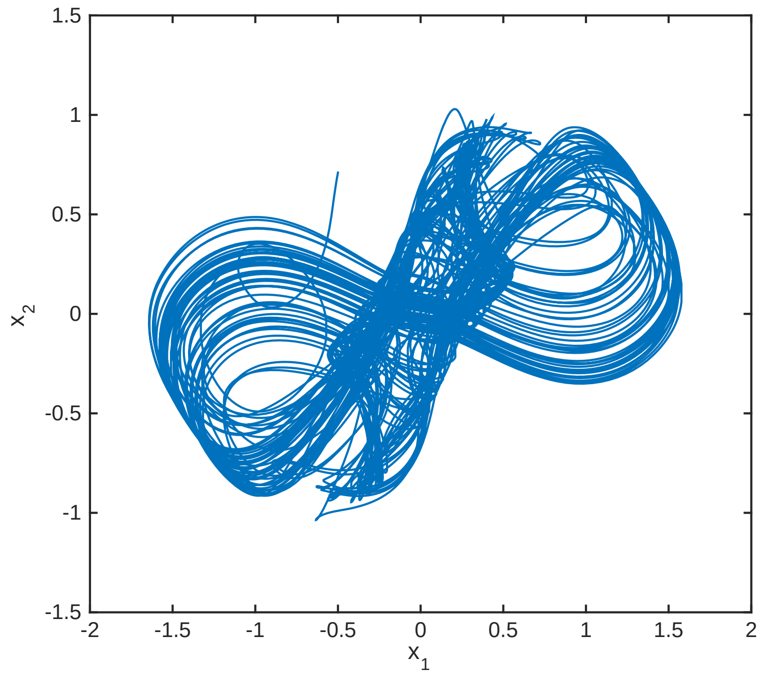


Figure 5.14: Chaotic attractor of system (5.49) with initial condition $x_{t_0}(s) = (-0.5, 0.71)^T$ for all $s \in [-\tau, 0]$.

the conditions of Theorem 5.3.2 are satisfied, which implies the drive and response systems are synchronized. This is confirmed by simulation as shown in Figure 5.15(b). It can be clearly observed from this figure that more impulses are added to one state of response system than the other one when $t < 1$.

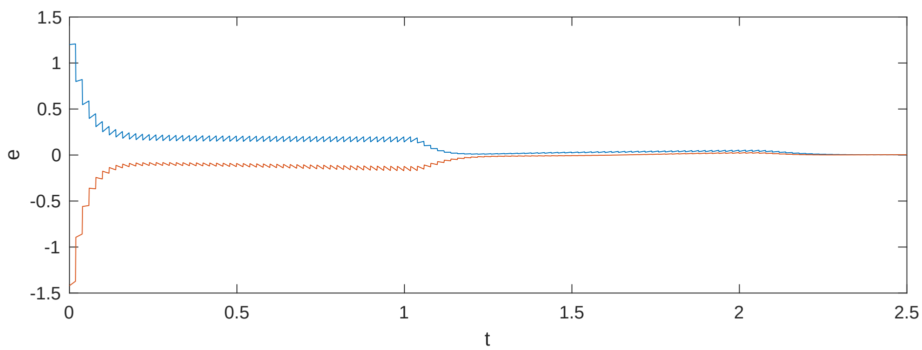
5.3.4 Proofs

Proof of Theorem 5.3.1

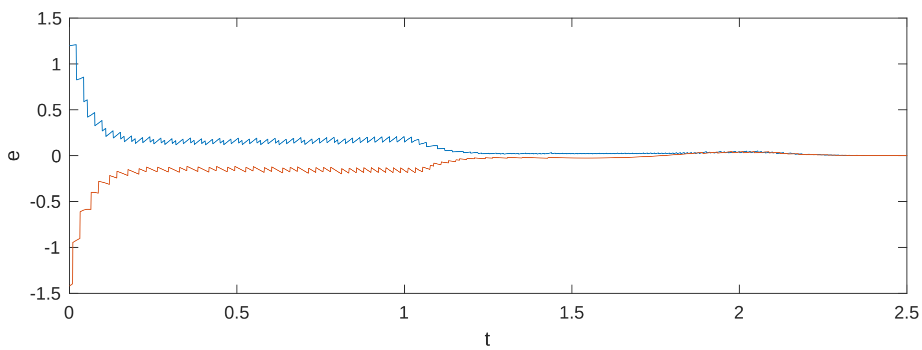
For $t = t_k$ and $i \in \mathfrak{D}_k^l$, $e_i(t_k) = e_i(t_k^-) - q_1 e_i(t_k - d_1)$. First, we will estimate the relation between states $e_i(t_k^-)$ and $e_i(t_k - d_1)$. Integrating both sides of the system equation of the ith state in (5.44) from $t_k - d_1$ to t_k^- yields

$$e_i(t_k^-) - e_i(t_k - d_1) = \int_{t_k - d_1}^{t_k} f_i(t, e_t)\mathrm{d}t - \sum_{m=1}^{\zeta_{i,k}} q_1 e_i(t_{k-m} - d_1), \tag{5.51}$$

where $\zeta_{i,k}$ denotes the number of impulses activated on the ith state during time period $(t_k - d_1, t_k)$. If $d_1 \le \sigma$, then $\zeta_{i,k} = \zeta = 0$. If $d_1 > \sigma$, then $\zeta \ge 1$, $\zeta_{i,k} \ge 0$ and $t_{k-1} \in (t_k - d, t_k)$. According to our pinning algorithm, different states might be controlled at different impulsive instant. This means that, at time $t = t_{k-1}$, the ith state of the response system may not be controlled by the pinning impulsive controller, i.e., $\zeta_{i,k} \le \zeta$. Hence, $0 \le \zeta_{i,k} \le \zeta$ for all $i \in \mathfrak{D}_k^l$



(a) Full-state control



(b) Pinning control

Figure 5.15: Simulations of error states in Example 5.3.2 with initial condition for the drive system as given in Figure 5.14 and initial condition for the response system $y_{t_0}(s) = (0.7, -0.71)^T$ for all $s \in [-\tau, 0]$.

and $k \in \mathbb{N}$. From (5.44) and (5.51), we have $e_i(t_k) = Y_1 + Y_2 + Y_3$ with

$$\begin{aligned}
Y_1 &= (1-q_1)e_i(t_k^-), \\
Y_2 &= q_1 \int_{t_k-d_1}^{t_k} f_i(t, e_t)\mathrm{d}t, \\
Y_3 &= -q_1^2 \sum_{m=1}^{\zeta_{i,k}} e_i(t_{k-m} - d_1).
\end{aligned} \tag{5.52}$$

Choose the Lyapunov function $v(t) = e^T(t)e(t)$, then

$$\begin{aligned}
Y_2^2 &= q_1^2 \Big(\int_{t_k-d_1}^{t_k} f_i(t, e_t)\mathrm{d}t \Big)^2 \\
&\le q_1^2 d_1 \int_{t_k-d_1}^{t_k} f_i^2(t, e_t)\mathrm{d}t \\
&\le q_1^2 d_1 \int_{t_k-d_1}^{t_k} L_i^2 \|e_t\|_\tau^2 \mathrm{d}t \\
&\le q_1^2 d_1 \int_{t_k-d_1}^{t_k} L_i^2 \sup_{s\in[-\tau,0]} \{\|e(t+s)\|^2\}\mathrm{d}t \\
&\le q_1^2 d_1^2 L_i^2 \sup_{s\in[-\tau-d_1,0]} \{\|e(t_k^- + s)\|^2\} \\
&= q_1^2 d_1^2 L_i^2 \sup_{s\in[-\tau-d_1,0]} \{v(t_k^- + s)\}, \\
Y_3^2 &= q_1^4 \Big(\sum_{m=1}^{\zeta_{i,k}} e_i(t_{k-m} - d_1) \Big)^2 \\
&\le q_1^4 \zeta_{i,k} \sum_{m=1}^{\zeta_{i,k}} e_i^2(t_{k-m} - d_1)
\end{aligned} \tag{5.53}$$

For any $\varepsilon, \xi > 0$, we have

$$\begin{aligned}
\sum_{i\in\mathfrak{D}_k^l} e_i^2(t_k) &= \sum_{i\in\mathfrak{D}_k^l} (Y_1 + Y_2 + Y_3)^2 \\
&\le \sum_{i\in\mathfrak{D}_k^l} \{(1+\varepsilon)Y_1^2 + (1+\varepsilon^{-1})[(1+\xi)Y_2^2 + (1+\xi^{-1})Y_3^2]\} \\
&\le (1+\varepsilon)(1-q_1)^2 \sum_{i\in\mathfrak{D}_k^l} e_i^2(t_k^-) \\
&\quad + (1+\varepsilon^{-1})(1+\xi)\Big(\sum_{i\in\mathfrak{D}_k^l} L_i^2 \Big) q_1^2 d_1^2 \sup_{s\in[-\tau-d_1]} \{v(t_k^- + s)\} \\
&\quad + (1+\varepsilon^{-1})(1+\xi^{-1}) q_1^4 \sum_{i\in\mathfrak{D}_k^l} \zeta_{i,k} \sum_{m=1}^{\zeta_{i,k}} e_i^2(t_{k-m} - d_1)
\end{aligned} \tag{5.54}$$

For the last term on the right-hand side of the above inequality, we have the following estimation:

$$\begin{aligned}\sum_{i\in\mathfrak{D}_k^l}\zeta_{i,k}\sum_{m=1}^{\zeta_{i,k}}e_i^2(t_{k-m}-d_1) &\le \zeta\sum_{m=1}^{\zeta_{i,k}}\sum_{i\in\mathfrak{D}_k^l}e_i^2(t_{k-m}-d_1)\\ &\le \zeta\sum_{m=1}^{\zeta_{i,k}}\sum_{i=1}^{n}e_i^2(t_{k-m}-d_1)\\ &= \zeta\sum_{m=1}^{\zeta_{i,k}}v(t_{k-m}-d_1)\\ &\le \zeta^2\sup_{s\in[-2d_1,0]}\{v(t_k^-+s)\}. \end{aligned}\tag{5.55}$$

Hence, we obtain from (5.54) and (5.55) that

$$\sum_{i\in\mathfrak{D}_k^l}e_i^2(t_k)\le\rho_1'\sum_{i\in\mathfrak{D}_k^l}e_i^2(t_k^-)+\rho_2\sup_{s\in[-r,0]}\{v(t_k^-+s)\}\tag{5.56}$$

where

$$\begin{aligned} r &= \max\{\tau+d_1,2d_1\},\\ \rho_1' &= (1+\varepsilon)(1-q_1)^2,\\ \rho_2 &= (1+\varepsilon^{-1})[(1+\xi)lq_1^2d_1^2L^2+(1+\xi^{-1})q_1^4\zeta^2]. \end{aligned}$$

Let $\rho_1=1-\frac{l}{n}(1-\rho_1')$, then

$$\begin{aligned}(1-\rho_1)\sum_{i\notin\mathfrak{D}_k^l}e_i^2(t_k) &\le (1-\rho_1)(n-l)\min_{i\in\mathfrak{D}_k^l}\{e_i^2(t_k^-)\}\\ &= l(\rho_1-\rho_1')\min_{i\in\mathfrak{D}_k^l}\{e_i^2(t_k^-)\}\\ &\le (\rho_1-\rho_1')\sum_{i\in\mathfrak{D}_k^l}e_i^2(t_k^-),\end{aligned}$$

which implies that

$$\rho_1'\sum_{i\in\mathfrak{D}_k^l}e_i^2(t_k^-)+\sum_{i\notin\mathfrak{D}_k^l}e_i^2(t_k^-)\le\rho_1\sum_{i=1}^{n}e_i^2(t_k^-).$$

Then, for $t=t_k$, we have

$$\begin{aligned} v(t_k) &= \sum_{i\in\mathfrak{D}_k^l}e_i^2(t_k^-)+\sum_{i\notin\mathfrak{D}_k^l}e_i^2(t_k^-)\\ &\le \rho_1'\sum_{i\in\mathfrak{D}_k^l}e_i^2(t_k^-)+\sum_{i\notin\mathfrak{D}_k^l}e_i^2(t_k^-)+\rho_2\sup_{s\in[-r,0]}\{v(t_k^-+s)\}\end{aligned}$$

$$\leq \rho_1 v(t_k^-) + \rho_2 \sup_{s\in[-r,0]} \{v(t_k^- + s)\}. \tag{5.57}$$

On the other hand, for $t \neq t_k$ and $\epsilon > 0$, we get

$$\begin{aligned}
\dot{v}(t) &= f^T(t, e_t)e(t) + e^T(t)f(t, e_t) \\
&\leq \epsilon e^T(t)e(t) + \epsilon^{-1} f^T(t, e_t) f(t, e_t) \\
&\leq \epsilon v(t) + \epsilon^{-1} K^2 \|e_t\|_\tau^2 \\
&\leq \epsilon v(t) + \epsilon^{-1} K^2 \sup_{s\in[-\tau,0]} \{v(t+s)\} \\
&\leq \alpha v(t) + \beta \sup_{s\in[-r,0]} \{v(t+s)\}
\end{aligned} \tag{5.58}$$

with $\alpha = \epsilon$ and $\beta = \epsilon^{-1}K^2$.

To apply Theorem 3.3.2, define

$$\psi(s) = \begin{cases} \phi(s), & \text{if } s \in [-\tau, 0], \\ 0, & \text{if } s \in [-r, -\tau), \end{cases}$$

then, $\psi \in \mathcal{PC}([-r, 0], \mathbb{R}^n)$. Since $\alpha, \beta > 0$, we can conclude from Theorem 3.3.2 that if $\rho_1 + \rho_2 < 1$ and $\frac{1}{\rho_1+\rho_2} > e^{(\alpha + \frac{\beta}{\rho_1+\rho_2})\sigma}$ then $v(t) \to 0$ as $t \to \infty$, i.e., synchronization between drive and response systems can be achieved.

Denote $\rho = \rho_1 + \rho_2 < 1$, then (3.19) implies $\sigma < \frac{\ln(1/\rho)}{\alpha+\beta/\rho}$. Next, we will determine the values of ϵ, ε and ξ in α, β, ρ_1 and ρ_2 by maximizing the upper bound of σ, that is, $\frac{\ln(1/\rho)}{\alpha+\beta/\rho}$.

For any $\rho \in (0, 1)$, to maximize $\frac{\ln(1/\rho)}{\alpha+\beta/\rho}$ is equivalent to minimize $\alpha + \beta/\rho = \epsilon + \epsilon^{-1}K^2/\rho$ for $\epsilon > 0$. Define the map $H(\epsilon) := \epsilon + \epsilon^{-1}K^2/\rho$, then $H'(\epsilon) = 1 - \epsilon^{-2}K^2/\rho$, which implies that, for $\epsilon^* = \frac{K}{\sqrt{\rho}}$, $H'(\epsilon^*) = 0$ and $H(\epsilon^*) = \frac{2K}{\sqrt{\rho}}$. Hence, for given $\rho \in (0, 1)$, we have

$$\max_{\epsilon>0} \left\{ \frac{\ln(1/\rho)}{\alpha + \beta/\rho} \right\} = \frac{\sqrt{\rho}\ln(1/\rho)}{2K} = G(\rho).$$

For function $G(\rho)$ with $\rho \in (0, 1)$, we have $G'(\rho) = -\frac{2+\ln\rho}{4\sqrt{\rho}K}$ and $G'(\rho^*) = 0$. Then,

$$G'(\rho) \begin{cases} > 0, & \text{if } \rho \in (0, \rho^*), \\ < 0, & \text{if } \rho \in (\rho^*, 1), \end{cases}$$

i.e., $G(\rho)$ is strictly increasing for $\rho \in (0, \rho^*)$, and strictly decreasing for $\rho \in (\rho^*, 1)$. From Lemma 5.2.3, we have $\rho_{min} = \min_{\varepsilon,\xi>0}\{\rho_1 + \rho_2\}$. If $\rho_{min} \leq \rho^*$, then there exist $\varepsilon, \xi > 0$ such that $(\rho_1 + \rho_2)|_{(\varepsilon,\xi)} = \rho^*$, and $\sigma < \max_{\rho\in[\rho_{min},1)}\{G(\rho)\} = \max_{\rho\in[\rho_{min},\rho^*]}\{G(\rho)\} = G(\rho^*)$. If $\rho_{min} > \rho^*$, then $\sigma < \max_{\rho\in[\rho_{min},1)}\{G(\rho)\} = G(\rho_{min})$. Therefore,

$$\sigma < \max_{\epsilon,\varepsilon,\xi>0} \left\{ \frac{\ln(1/\rho)}{\alpha + \beta/\rho} \right\} = \begin{cases} G(\rho^*), & \text{if } \rho \in (0, \rho^*], \\ G(\rho_{min}), & \text{if } \rho \in (\rho^*, 1). \end{cases}$$

The proof is complete. □

Proof of Theorem 5.3.2

For $i \in \mathfrak{D}_k^l$ and $t = t_k$, we will estimate the relation between $x_i(t_k^-)$ and $\int_{t_k-d_2}^{t_k} x_i(s)\mathrm{d}s$. For $t \in [t_k - d_2, t_k)$, integrating both sides of the system equation of the ith state in (5.46) from t to t_k^- yields

$$e_i(t_k^-) - e_i(t) = \int_t^{t_k} f_i(s, e_s)\mathrm{d}s - q_2 \sum_{m=1}^{\varsigma_i(t)} \int_{-d_2}^{0} e_i(t_{k-m} + s)\mathrm{d}s, \tag{5.59}$$

where $\varsigma_i(t)$ denotes the number of impulses activated onto the ith state during time period (t, t_k). Similar to the discussion of $\zeta_{i,k}$ in the previous subsection, it can be seen that ς_i depends not only on t but also on the pinning algorithm, and $0 \leq \varsigma_i(t) \leq \varsigma$ for $t \in [t_k - d_2, t_k)$. Next, integrate both sides of (5.59) from $t_k - d_2$ to t_k^-, and then we get

$$d_2 e_i(t_k^-) - \int_{t_k-d_2}^{t_k} e_i(t)\mathrm{d}t = \int_{t_k-d_2}^{t_k} \int_t^{t_k} f_i(s, e_s)\mathrm{d}s\mathrm{d}t - q_2 \int_{t_k-d_2}^{t_k} \sum_{m=1}^{\varsigma_i(t)} \int_{-d_2}^{0} e_i(t_{k-m} + s)\mathrm{d}s\mathrm{d}t,$$

which implies

$$e_i(t_k) \;=\; e_i(t_k^-) - q_2 \int_{t_k-d_2}^{t_k} e_i(s)\mathrm{d}s = \Gamma_1 + \Gamma_2 + \Gamma_3,$$

with

$$\begin{aligned}
\Gamma_1 &= (1 - q_2 d_2) e_i(t_k^-), \\
\Gamma_2 &= q_2 \int_{t_k-d_2}^{t_k} \int_t^{t_k} f_i(s, e_s)\mathrm{d}s\mathrm{d}t, \\
\Gamma_3 &= -q_2^2 \int_{t_k-d_2}^{t_k} \sum_{m=1}^{\varsigma_i(t)} \int_{-d_2}^{0} e_i(t_{k-m} + s)\mathrm{d}s\mathrm{d}t.
\end{aligned}$$

Choose the Lyapunov function $v(t) = e^T(t)e(t)$, then

$$\begin{aligned}
\Gamma_2^2 &= q_2^2 \Big(\int_{t_k-d_2}^{t_k} \int_t^{t_k} f_i(s, e_s)\mathrm{d}s\mathrm{d}t \Big)^2 \\
&\leq q_2^2 d_2 \int_{t_k-d_2}^{t_k} \Big(\int_t^{t_k} f_i(s, e_s)\mathrm{d}s \Big)^2 \mathrm{d}t \\
&\leq q_2^2 d_2 \int_{t_k-d_2}^{t_k} (t_k - t) \int_t^{t_k} f_i^2(s, e_s)\mathrm{d}s\mathrm{d}t \\
&\leq q_2^2 d_2^2 \int_{t_k-d_2}^{t_k} \int_t^{t_k} f_i^2(s, e_s)\mathrm{d}s\mathrm{d}t \\
&\leq q_2^2 d_2^2 \int_{t_k-d_2}^{t_k} \int_t^{t_k} L_i^2 \|e_s\|_{-\tau}^2 \mathrm{d}s\mathrm{d}t \\
&\leq q_2^2 d_2^3 L_i^2 \int_{t_k-d_2}^{t_k} \sup_{s\in[-\tau,0]} \{\|e(t+s)\|^2\}\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&\leq q_2^2 d_2^4 L_i^2 \sup_{s\in[-\tau-d_2,0]} \{v(t_k^- + s)\}, \\
\Gamma_3^2 &= q_2^4 \Big(\int_{t_k-d_2}^{t_k} \sum_{m=1}^{\varsigma_i(t)} \int_{-d_2}^{0} e_i(t_{k-m}+s)\mathrm{d}s\mathrm{d}t \Big)^2 \\
&\leq q_2^4 d_2 \int_{t_k-d_2}^{t_k} \Big(\sum_{m=1}^{\varsigma_i(t)} \int_{-d_2}^{0} e_i(t_{k-m}+s)\mathrm{d}s \Big)^2 \mathrm{d}t \\
&\leq q_2^4 d_2 \int_{t_k-d_2}^{t_k} \varsigma_i(t) \sum_{m=1}^{\varsigma_i(t)} \int_{-d_2}^{0} \Big(e_i(t_{k-m}+s)\mathrm{d}s \Big)^2 \mathrm{d}t \\
&\leq q_2^4 d_2^2 \int_{t_k-d_2}^{t_k} \Big(\varsigma_i(t) \sum_{m=1}^{\varsigma_i(t)} \int_{-d_2}^{0} e_i^2(t_{k-m}+s)\mathrm{d}s \Big) \mathrm{d}t.
\end{aligned}
$$

Then, for any $\varepsilon, \xi > 0$,

$$
\begin{aligned}
\sum_{i\in\mathfrak{D}_k^l} e_i^2(t_k) \leq\ & (1+\varepsilon)(1-q_2d_2)^2 \sum_{i\in\mathfrak{D}_k^l} e_i^2(t_k^-) \\
&+ (1+\varepsilon^{-1})(1+\xi) q_2^2 d_2^4 \Big(\sum_{i\in\mathfrak{D}_k^l} L_i^2 \Big) \sup_{s\in[-\tau-d_2,0]} \{v(t_k^- + s)\} \\
&+ (1+\varepsilon^{-1})(1+\xi^{-1}) q_2^4 d_2^2 \sum_{i\in\mathfrak{D}_k^l} \int_{t_k-d_2}^{t_k} \Big(\varsigma_i(t) \sum_{m=1}^{\varsigma_i(t)} \int_{-d_2}^{0} e_i^2(t_{k-m}+s)\mathrm{d}s \Big) \mathrm{d}t.
\end{aligned}
$$

For the last term on the right-hand side of the above inequality, we have

$$
\begin{aligned}
\sum_{i\in\mathfrak{D}_k^l} \int_{t_k-d_2}^{t_k} \Big(\varsigma_i(t) \sum_{m=1}^{\varsigma_i(t)} \int_{-d_2}^{0} e_i^2(t_{k-m}+s)\mathrm{d}s \Big) \mathrm{d}t &\leq \sum_{i=1}^{n} \int_{t_k-d_2}^{t_k} \Big(\varsigma \sum_{m=1}^{\varsigma} \int_{-d_2}^{0} e_i^2(t_{k-m}+s)\mathrm{d}s \Big) \mathrm{d}t \\
&\leq d_2 \varsigma \sum_{m=1}^{\varsigma} \int_{-d_2}^{0} v(t_{k-m}+s)\mathrm{d}s \\
&\leq d_2^2 \varsigma^2 \sup_{s\in[-2d_2,0]} \{v(t_k^- + s)\}.
\end{aligned}
$$

Therefore, we have (5.56) satisfied with

$$
\begin{aligned}
r &= \max\{\tau + d_2, 2d_2\}, \\
\rho_1' &= (1+\varepsilon)(1-q_2d_2)^2, \\
\rho_2 &= (1+\varepsilon^{-1})[(1+\xi) l q_2^2 d_2^4 L^2 + (1+\xi^{-1}) q_2^4 d_2^4 \zeta^2].
\end{aligned}
$$

Similarly, let $\rho_1 = 1 - \frac{l}{n}(1-\rho_1')$, and then we have (5.57) satisfied. The rest of the proof is essentially the same as the proof of Theorem 5.3.1. □

Proof of Theorem 5.3.3

For $i \in \mathfrak{D}_k^l$ and $t = t_k$, we can get from (5.51) and (5.59) that $e_i(t_k) = \Xi_1 + \Xi_2 + \Xi_3$ with $\Xi_j = \Upsilon_j + \Gamma_j$ for $j = 1, 2, 3$. For $\xi_1, \xi_2 > 0$ and Lyapunov function $v(t) = e^T(t)e(t)$, we have

$$\begin{aligned}
\sum_{i\in\mathfrak{D}_k^l} \Xi_2^2 &= \sum_{i\in\mathfrak{D}_k^l} (\Upsilon_2 + \Gamma_2)^2 \\
&\leq \sum_{i\in\mathfrak{D}_k^l} (1+\xi_1)\Upsilon_2^2 + (1+\xi_1^{-1})\Gamma_2^2 \\
&\leq \sum_{i\in\mathfrak{D}_k^l} [(1+\xi_1)q_1^2 d_1^2 + (1+\xi_1^{-1})q_2^2 d_2^4] L_i^2 \sup_{s\in[-r,0]} \{v(t_k^- + s)\} \\
&\leq [(1+\xi_1)q_1^2 d_1^2 + (1+\xi_1^{-1})q_2^2 d_2^4] l L^2 \sup_{s\in[-r,0]} \{v(t_k^- + s)\} \\
\sum_{i\in\mathfrak{D}_k^l} \Xi_3^2 &= \sum_{i\in\mathfrak{D}_k^l} (\Upsilon_3 + \Gamma_3)^2 \\
&\leq \sum_{i\in\mathfrak{D}_k^l} \Big[(1+\xi_2)\Upsilon_3^2 + (1+\xi_2^{-1})\Gamma_3^2\Big] \\
&\leq \sum_{i\in\mathfrak{D}_k^l} \Big[(1+\xi_2)q_1^4 \zeta_k \sum_{m=1}^{\zeta_k} e_i^2(t_{k-m} - d_1) \\
&\qquad + (1+\xi_2^{-1})q_2^4 d_2^2 \int_{t_k - d_2}^{t_k} \Big(\varsigma_i(t) \sum_{m=1}^{\varsigma_i(t)} \int_{-d_2}^{0} e_i^2(t_{k-m} + s)\mathrm{d}s\Big)\mathrm{d}t\Big] \\
&\leq [(1+\xi_2)q_1^4 \zeta^2 + (1+\xi_2^{-1})q_2^4 d_2^4 \varsigma^2] \sup_{s\in[-r,0]} \{v(t_k^- + s)\}.
\end{aligned}$$

Here, $r = \max\{\tau + d_1, \tau + d_2, 2d_1, 2d_2\}$. Then,

$$\begin{aligned}
\sum_{i\in\mathfrak{D}_k^l} e_i^2(t_k) &= \sum_{i\in\mathfrak{D}_k^l} (\Xi_1 + \Xi_2 + \Xi_3)^2 \\
&\leq (1+\varepsilon) \sum_{i\in\mathfrak{D}_k^l} \Xi_1^2 + (1+\varepsilon^{-1})\Big[(1+\xi) \sum_{i\in\mathfrak{D}_k^l} \Xi_2^2 + (1+\xi^{-1}) \sum_{i\in\mathfrak{D}_k^l} \Xi_3^2\Big] \\
&\leq \rho_1' \sum_{i\in\mathfrak{D}_k^l} e_i^2(t_k^-) + \rho_2 \sup_{s\in[-r,0]} \{v(t_k^- + s)\},
\end{aligned}$$

where

$$\begin{aligned}
\rho_1' &= (1+\varepsilon)(1 - q_1 - q_2 d_2)^2, \\
\rho_2 &= (1+\varepsilon^{-1})\Big\{(1+\xi)[(1+\xi_1)q_1^2 d_1^2 + (1+\xi_1^{-1})q_2^2 d_2^4] l L^2 \\
&\qquad + (1+\xi^{-1})[(1+\xi_2)q_1^4 \zeta^2 + (1+\xi_2^{-1})q_2^4 d_2^4 \varsigma^2]\Big\}.
\end{aligned}$$

The rest proof is similar to the proof of Theorem 5.3.1, and thus omitted. □

Chapter 6

Applications to Systems and Networks Governed by PDEs

In this chapter, we apply the pinning algorithm discussed in Chapter 5 to the impulsive control problems of systems and networks governed by partial differential equations (PDEs). Section 6.1 studies the pinning impulsive stabilization and synchronization of spatiotemporal chaos in Gray-Scott model, which is a delay-free dynamical system modeled by PDEs. Section 6.2 extends the study to synchronization of reaction-diffusion neural networks with time-varying delays, which is described by time-delay PDEs.

6.1 Stabilization and Synchronization of Gray-Scott Model

This section investigates the impulsive control and synchronization problem of spatiotemporal chaos in Gray-Scott model. Based on the Lyapunov function method, a class of pinning impulsive controller is designed to stabilize and synchronize the spatiotemporal chaos in Gray-Scott model.

6.1.1 Introduction of Gray-Scott Model

Gray-Scott model is one of the typical reaction-diffusion systems which has a wide variety of spatiotemporal structures[34]:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = -u_1 u_2^2 + a(1-u_1) + d_1 \nabla^2 u_1 \\ \dfrac{\partial u_2}{\partial t} = u_1 u_2^2 - (a+b)u_2 + d_2 \nabla^2 u_2 \end{cases} \tag{6.1}$$

where u_1 and u_2 are the concentrations of chemical species U_1 and U_2, respectively, a is the inflow rate, $a+b$ is the removal rate of U_2 from the reaction, and d_1 and d_2 are the diffusion coefficients of the two species, for more details about the Gray-Scott model refer to [51] and [93].

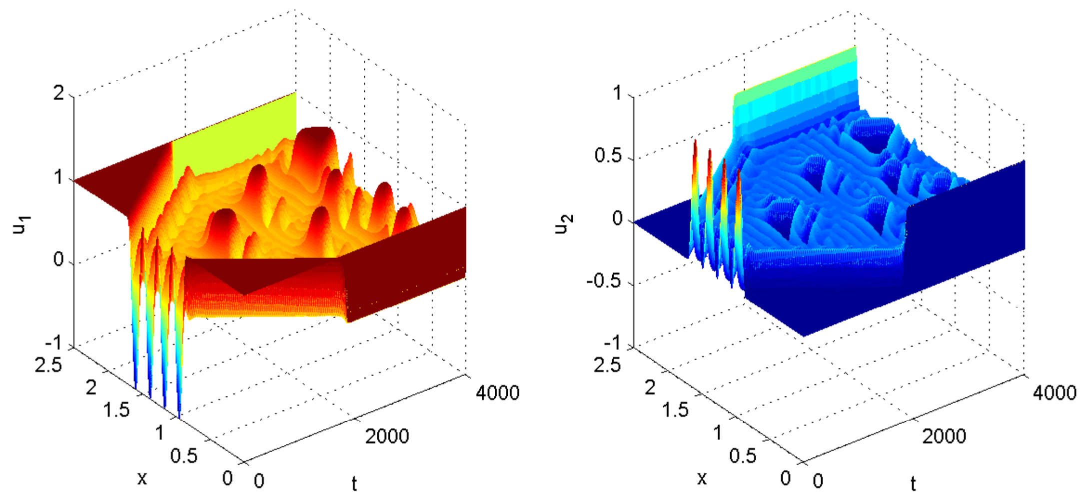


Figure 6.1: Spatiotemporal evolutions of u_1 and u_2.

In this section, we consider the one-dimensional version of Gray-Scott model with $a = 0.028$, $b = 0.053$, $d_1 = 2 \times 10^{-5}$ and $d_2 = 10^{-5}$. Since $E_0 = (1, 0)$ is a trivial steady state, it is necessary to add certain perturbation to it to obtain non-trivial pattern from the initial state $(1, 0)$. The initial conditions are chosen to be $(u_1(0, x), u_2(0, x))^T = (1, 0)^T$ with strong perturbations in the center region, and the periodic boundary conditions are given by $u_1(t, 0) = u_1(t, L) = 1$ and $u_2(t, 0) = u_2(t, L) = 0$. The spatiotemporal chaotic evolutions of the 1-D system (6.1) are shown in Figure 6.1.

6.1.2 Impulsive Synchronization of Gray-Scott Model

In this subsection, we shall discuss the impulsive synchronization of one dimensional Gray-Scott model with another identical system starting from different initial states.

Let the following one-dimensional Gray-Scott model serve as the drive system:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = -u_1 u_2^2 + a(1 - u_1) + d_1 \dfrac{\partial^2 u_1}{\partial x^2} \\ \dfrac{\partial u_2}{\partial t} = u_1 u_2^2 - (a + b) u_2 + d_2 \dfrac{\partial^2 u_2}{\partial x^2} \\ \mathbf{u}(0, x) = \mathbf{u}_0(x), \;\; x \in [0, L], \\ \mathbf{u}(t, 0) = \mathbf{u}(t, L) = \mathbf{h}, \;\; t \in \mathbb{R}^+, \end{cases} \tag{6.2}$$

where $\mathbf{u}(t,x) = (u_1(t,x), u_2(t,x))^T$, and the response system is given by

$$\begin{cases} \dfrac{\partial v_1}{\partial t} = -v_1 v_2^2 + a(1-v_1) + d_1 \dfrac{\partial^2 v_1}{\partial x^2},\ t \neq t_k, \\ \dfrac{\partial v_2}{\partial t} = v_1 v_2^2 - (a+b)v_2 + d_2 \dfrac{\partial^2 v_2}{\partial x^2},\ t \neq t_k, \\ \Delta \mathbf{v}(t,x) = I_k(\mathbf{e}(t,x)),\ \ t = t_k,\ x \in [0,L],\ k \in \mathbb{N} \\ \mathbf{v}(0,x) = \mathbf{v}_0(x),\ \ x \in [0,L], \\ \mathbf{v}(t,0) = \mathbf{v}(t,L) = \mathbf{h},\ \ t \in \mathbb{R}^+, \end{cases} \tag{6.3}$$

where a, b, d_1 and d_2 are chosen as in the previous section, $L = 2.5$, $\mathbf{v}(t,x) = (v_1(t,x), v_2(t,x))^T$, $\mathbf{u}_0$ and $\mathbf{v}_0$ are different initial conditions, $\mathbf{h}(t) = (h_1, h_2)^T$ is the periodic boundary condition for both systems with constants $h_1, h_2 \geq 0$. Since the Gray-Scott model exhibits chaotic behaviors, the same Gray-Scott systems will evolve differently if they have different initial conditions, and states $\mathbf{u}$ and $\mathbf{v}$ are uniformly bounded which is a very important property that will be used in the proof of our main results.

Remark 6.1.1 *Theorem 1 in [43] implies that system (6.2) admits an unique global (classical) solution. The case for system (6.3) with impulses is essentially the same, by an argument using the method of steps over all the impulse intervals.*

In (6.3), $I_k : \mathbb{R}^2 \to \mathbb{R}^2$, $\Delta \mathbf{v}(t,x) = \mathbf{v}(t^+,x) - \mathbf{v}(t^-,x)$ where $\mathbf{v}(t^+,x)$ and $\mathbf{v}(t^-,x)$ denote the right limit and left limit of $\mathbf{v}(t,x)$ at t, respectively. $\mathbf{e}(t,x) = \mathbf{u}(t,x) - \mathbf{v}(t,x)$ denotes the error state of the drive system and response system. The sequence $\{t_k\}$ satisfies $0 = t_0 < t_1 < t_2 < \ldots t_n < \ldots$, and $\lim_{n\to\infty} t_n = \infty$.

According to (6.2) and (6.3), the error system will be given

$$\begin{cases} \dfrac{\partial e_1}{\partial t} = -u_1 u_2^2 + v_1 v_2^2 - a e_1 + d_1 \dfrac{\partial^2 e_1}{\partial x^2},\ t \neq t_k, \\ \dfrac{\partial e_2}{\partial t} = u_1 u_2^2 - v_1 v_2^2 - (a+b)e_2 + d_2 \dfrac{\partial^2 e_2}{\partial x^2},\ t \neq t_k, \\ \Delta \mathbf{e}(t,x) = I_k(\mathbf{e}(t,x)),\ \ t = t_k,\ x \in [0,L],\ k \in \mathbb{N} \\ \mathbf{e}(0,x) = \mathbf{e}_0(x),\ \ x \in [0,L], \\ \mathbf{e}(t,0) = \mathbf{e}(t,L) = \mathbf{0},\ \ t \in \mathbb{R}^+, \end{cases} \tag{6.4}$$

where $\mathbf{e}_0(x) = \mathbf{u}_0(x) - \mathbf{v}_0(x)$.

Definition 6.1.1 *Suppose that $\boldsymbol{u}(t,x) : \mathbb{R}^+ \times [0,L] \to \mathbb{R}^m$ for some $m > 0$, where $\boldsymbol{u}$ is of class $\mathcal{L}_2[0,L]$ with respect to x. Then $\|\cdot\|_2$ is defined by*

$$\|\boldsymbol{u}(t,\cdot)\|_2 := \Big[\int_0^L \|\boldsymbol{u}(t,x)\|^2 dx \Big]^{1/2},$$

where $\|\cdot\|$ is the Euclidean norm.

Definition 6.1.2 *We say that synchronization of the drive system (6.2) and the response system (6.3) are achieved under impulsive controller* $\{t_k, I_k\}$ *if*

$$\lim_{t\to\infty} \|\boldsymbol{u}(t,\cdot) - \boldsymbol{v}(t,\cdot)\|_2 = 0.$$

We can see that to explore the synchronization of the two systems (6.2) and (6.3) is equivalent to investigating the attractive property of the error states:

$$\lim_{t\to\infty} \|\mathbf{e}(t,\cdot)\|_2 = 0.$$

In order to force the response system (6.3) to synchronize with the drive system (6.2), we design the following impulsive controller:

$$I_k(\mathbf{e}(t_k, x)) = \begin{cases} -qe_i(t_k, x), & i = \mathfrak{D}_k, \\ 0, & i \neq \mathfrak{D}_k, \end{cases} \tag{6.5}$$

where the constant $q \in (0, 1]$ is the impulsive strength to be designed, and the index $\mathfrak{D}_k$ is defined as follows: for the impulsive instant t_k, one can reorder the error states $e_1(t_k, x)$ and $e_2(t_k, x)$ such that $\|e_{j_1}(t_k, x)\|_2 \geq \|e_{j_2}(t_k, x)\|_2$, then the index $\mathfrak{D}_k$ is defined as $\mathfrak{D}_k = j_1$. We can see that the controller is only added to one state of the response system (6.3) at each impulsive instant.

In [53], sufficient conditions about uniform impulsive controller is designed, which requires an upper bound for each impulsive interval. In order to improve these sufficient conditions, we introduce the following definition.

Definition 6.1.3 *([65]**Average Impulsive Interval**) The average impulsive interval of impulsive sequence* $\zeta = \{t_k\}$ *is less than* T_a*, if there exist a positive integer* N_0 *and a positive number* T_a*, so that* $N_\zeta(T, t) \geq \frac{T-t}{T_a} - N_0, \quad \forall T \geq t \geq 0$*, where* $N_\zeta(T, t)$ *denotes the number of impulsive times of the impulsive sequence* ζ *in the time interval* (t, T)*.*

According to this definition, there is no requirement on the upper bound of each impulsive interval, which is necessary for the impulsive control scheme in [53]. Before introducing the main results about the synchronization problem, we need the following lemma from [53].

Lemma 6.1.1 *Let* $f(u_1, u_2) := u_1 u_2$ *be defined over the set* $S = \{(u_1, u_2)^T \in \mathbb{R}^2 : 0 \leq |u_1| \leq \beta_1$ *and* $0 \leq |u_2| \leq \beta_2\}$*. Then the function* f *satisfies Lipschitz condition on* S *with Lipschitz constant given by* $L_0 := \beta_2\sqrt{\beta_1^2 + 4\beta_1^2}$*. In other words, for every* $(u_1, u_2)^T, (v_1, v_2)^T \in S$*, we have*

$$|f(u_1, u_2) - f(v_1, v_2)| \leq L_0 \|(u_1 - v_1, u_2 - v_2)\|.$$

Now we are in the position to introduce the main result to guarantee the synchronization of the drive system (6.2) and the response system (6.3).

Theorem 6.1.1 *Suppose the average impulsive interval of the impulsive sequence $\zeta = \{t_k\}$ is less than T_a. Let $\rho = 1 - q(2-q)/2$, and $\beta = 4\beta_2\sqrt{\beta_1^2 + 4\beta_2^2} - 2a$, where*

$$\beta_i := \max\Big\{ \sup_{t\in\mathbb{R}^+} |u_i(t,x)|, \sup_{t\in\mathbb{R}^+} |v_i(t,x)| \Big\}$$

for $i = 1,2$. If $\frac{\ln\rho}{T_a} + \beta < 0$, then the synchronization of the drive system (6.2) and the response system (6.3) is achieved.

Proof. Consider the following Lyapunov function (or energy function)

$$V(t) := \frac{1}{2}\int_0^L \mathbf{e}^T(t,x)\mathbf{e}(t,x)\mathrm{d}x = \frac{1}{2}\int_0^L \big(e_1^2(t,x) + e_1^2(t,x)\big)\mathrm{d}x.$$

For $t \in (t_{k-1}, t_k]$, $k \in \mathbb{N}$, we have, by (6.4) and Lemma 6.1.1,

$$\begin{aligned}
\dot{V}(t) &= \int_0^L \left(e_1\frac{\partial e_1}{\partial t} + e_2\frac{\partial e_2}{\partial t}\right)\mathrm{d}x \\
&= \int_0^L \Bigg[-(u_1u_2^2 - v_1v_2^2)e_1 - ae_1^2 + d_1e_1\frac{\partial^2 e_1}{\partial x^2} \\
&\quad + (u_1u_2^2 - v_1v_2^2)e_2 - (a+b)e_2 + d_2e_2\frac{\partial^2 e_2}{\partial x^2}\Bigg]\mathrm{d}x \\
&\le \int_0^L \Big[|u_1u_2^2 - v_1v_2^2|(|e_1| + |e_2|)\Big]\mathrm{d}x + \int_0^L \big[-ae_1^2 - (a+b)e_2^2\big]\mathrm{d}x \\
&\quad + \int_0^L \left[d_1e_1\frac{\partial^2 e_1}{\partial x^2} + d_2e_2\frac{\partial^2 e_2}{\partial x^2}\right]\mathrm{d}x \\
&\le 2\beta_2\sqrt{\beta_1^2 + 4\beta_2^2}\int_0^L \|\mathbf{e}\|^2\mathrm{d}x - \int_0^L \big[ae_1^2 + (a+b)e_2^2\big]\mathrm{d}x \\
&\quad + \int_0^L \left[d_1e_1\frac{\partial^2 e_1}{\partial x^2} + d_2e_2\frac{\partial^2 e_2}{\partial x^2}\right]\mathrm{d}x \\
&\le \left(2\beta_2\sqrt{\beta_1^2 + 4\beta_2^2} - a\right)\int_0^L \|\mathbf{e}\|^2\mathrm{d}x + \int_0^L \left[d_1e_1\frac{\partial^2 e_1}{\partial x^2} + d_2e_2\frac{\partial^2 e_2}{\partial x^2}\right]\mathrm{d}x
\end{aligned} \tag{6.6}$$

Applying integration by parts to the second term of (6.6), we have, by the periodic boundary condition,

$$\int_0^L e_i\frac{\partial^2 e_i}{\partial x^2}\mathrm{d}x = e_i\frac{\partial e_i}{\partial x}\Big|_0^L - \int_0^L \left(\frac{\partial e_i}{\partial x}\right)^2\mathrm{d}x = -\int_0^L \left(\frac{\partial e_i}{\partial x}\right)^2\mathrm{d}x \le 0, \quad i = 1,2,$$

Thus, for $t \in (t_{k-1}, t_k]$

$$\dot{V}(t) \le \left(2\beta_2\sqrt{\beta_1^2 + 4\beta_2^2} - a\right)\int_0^L \|\mathbf{e}\|^2\mathrm{d}x = \beta V(t). \tag{6.7}$$

Since $q \in (0,1)$, we have $\rho \in (0,1)$ and $\rho - (1-q)^2 = 1-\rho$. Setting $i = \mathfrak{D}_k$ and $j \neq \mathfrak{D}_k$, we get

$$\begin{aligned}(1-\rho)\int_0^L e_j^2(t_k,x)\mathrm{d}x &\leq (1-\rho)\int_0^L e_i^2(t_k,x)\mathrm{d}x \\ &= [\rho-(1-q)^2]\int_0^L e_i^2(t_k,x)\mathrm{d}x\end{aligned}$$

i.e.,

$$\begin{aligned}(1-q)^2\int_0^L e_i^2(t_k,x)\mathrm{d}x + \int_0^L e_j^2(t_k,x)\mathrm{d}x &\leq \rho\int_0^L \Big(e_1^2(t_k,x)+e_2^2(t_k,x)\Big)\mathrm{d}x \\ &= \rho\int_0^L \|\mathbf{e}(t_k,x)\|^2\mathrm{d}x\end{aligned}$$

Then, for any $k \in \mathbb{N}$, we yield

$$\begin{aligned}V(t_k^+) &= \frac{1}{2}\int_0^L \Big(e_1^2(t_k^+,x)+e_2^2(t_k^+,x)\Big)\mathrm{d}x \\ &= \frac{1}{2}\int_0^L e_i^2(t_k^+,x)\mathrm{d}x + \frac{1}{2}\int_0^L e_j^2(t_k^+,x)\mathrm{d}x \\ &= \frac{1}{2}\int_0^L (1-q)^2 e_i^2(t_k,x)\mathrm{d}x + \frac{1}{2}\int_0^L e_j^2(t_k,x)\mathrm{d}x \\ &\leq \frac{1}{2}\rho\int_0^L \|\mathbf{e}(t_k,x)\|^2\mathrm{d}x \\ &\leq \rho V(t_k).\end{aligned} \tag{6.8}$$

By (6.7) and (6.8), we have the following inequality system:

$$\begin{cases} \dot{V}(t) \leq & \beta V(t), \quad t \neq t_k, \\ V(t_k^+) \leq & \rho V(t_k), \quad k \in \mathbb{N}, \\ V(t_0) = & \frac{1}{2}\int_0^L \|\mathbf{e}_0(x)\|^2\mathrm{d}x \end{cases} \tag{6.9}$$

According to (6.9), we have, for any $t \in \mathbb{R}^+$,

$$V(t) \leq V(t_0)e^{\beta(t-t_0)}\rho^{N_\zeta(t,t_0)}, \tag{6.10}$$

where N_ζ denotes the number of impulsive instants in the time interval (t_0, t).

Since the average impulsive interval is less than T_a, it follows from the Definition 6.1.3 and (6.10) that

$$\begin{aligned}V(t) &\leq V(t_0)e^{\beta(t-t_0)}\rho^{\frac{t-t_0}{T_a}-N_0} \\ &= V(t_0)\rho^{-N_0}e^{(\frac{\ln\rho}{T_a}+\beta)(t-t_0)}\end{aligned} \tag{6.11}$$

Since $\frac{\ln\rho}{T_a}+\beta < 0$, we have

$$\lim_{t\to\infty} V(t) = 0,$$

i.e., $\lim_{t\to\infty}\|\mathbf{e}(t,\cdot)\|_2 = 0$, which implies that the synchronization of the drive system (6.2) and the response system (6.3) is achieved. □

Remark 6.1.2 *Based on Lyapunov function method, the impulsive synchronization criterion has been established. Compared with the existing result in [53], there are two improvements of this criterion: we derived an upper bound for the average impulsive interval which is less conservative than the criterion in [53] since there is no strict restriction on the upper bound of each impulsive interval; pinning impulsive controller is designed which is added to one state of the Gray-Scott model at each impulsive instant. Let T_a be the upper bound of each impulsive interval and control all the states of the system (6.3) at each time, then the Theorem 6.1.1 will reduce to a special case of the result in [53].*

Remark 6.1.3 *The criterion presented in Theorem 6.1.1 is closely related to the system parameters, the average impulsive interval T_a and the impulsive strength q. From Theorem 6.1.1, we can get the upper bound for the average impulsive interval:*

$$T_a < \frac{1}{\beta} \ln \left(1 - q(2-q)/2\right). \tag{6.12}$$

However, the criterion in Theorem 6.1.1 is a sufficient condition, which means that the synchronization of the drive system (6.2) and system (6.3) can be realized even if (6.12) does not hold.

6.1.3 Impulsive Stabilization of Gray-Scott Model

Since $E_0 = (1,0)^T$ is a trivial state of the Gray-Scott model, if choose $\mathbf{u}_0(x) = \mathbf{h}(t) = (1,0)^T$, then we have, from system (6.2), $(u_1(t,x), u_2(t,x))^T \equiv (1,0)^T$.

Therefore, the synchronization problem of the drive system (6.2) and the response system (6.3) reduces to the stability problem of the equilibrium E_0 of the following impulsive partial differential system:

$$\begin{cases} \dfrac{\partial v_1}{\partial t} = -v_1 v_2^2 + a(1-v_1) + d_1 \dfrac{\partial^2 v_1}{\partial x^2},\ t \neq t_k, \\ \dfrac{\partial v_2}{\partial t} = v_1 v_2^2 - (a+b)v_2 + d_2 \dfrac{\partial^2 v_2}{\partial x^2},\ t \neq t_k, \\ \Delta \mathbf{v}(t,x) = I_k(\mathbf{v}(t,x)),\ \ t = t_k,\ x \in [0,L],\ k \in \mathbb{N}, \\ \mathbf{v}(0,x) = \mathbf{v}_0(x),\ \ x \in [0,L], \\ \mathbf{v}(t,0) = \mathbf{v}(t,L) = \mathbf{h},\ \ t \in \mathbb{R}^+, \end{cases} \tag{6.13}$$

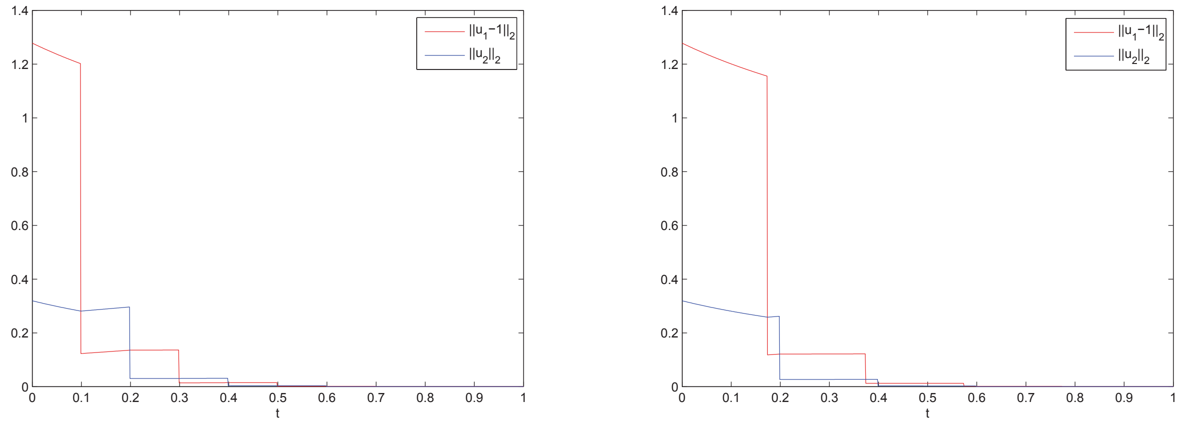
where $\mathbf{h} = (h_1, h_2)^T = (1,0)^T$.

The impulsive controller is designed as follows:

$$I_k(\mathbf{v}(t_k,x)) = \begin{cases} -q(h_i - v_i(t_k,x)),\ i = \mathfrak{D}_k, \\ \qquad\qquad\qquad\quad 0,\ i \neq \mathfrak{D}_k, \end{cases} \tag{6.14}$$

where the index $\mathfrak{D}_k$ is defined the same as in the controller (6.5) with $\mathbf{e}(t,x) = \mathbf{h} - \mathbf{v}(t,x)$.

We have the following stability result about the impulsive system (6.13), the proof of which is similar as the proof of Theorem 6.1.1, and thus omitted.



(a) (b)

Figure 6.2: State trajectories of the error system (6.4) : (a) uniform impulsive intervals; (b) nonuniform impulsive intervals.

Theorem 6.1.2 *Suppose the average impulsive interval of the impulsive sequence $\zeta = \{t_k\}$ is less than T_a. Let $\rho = 1 - q(2-q)/2$ and $\beta = 4\beta_2\sqrt{\beta_1^2 + 4\beta_2^2} - 2a$, where $\beta_1 := \max\left\{\sup_{t\in\mathbb{R}^+}|v_1(t,x)|, 1\right\}$ and $\beta_2 := \sup_{t\in\mathbb{R}^+}|v_2(t,x)|$. If $\frac{\ln\rho}{T_a} + \beta < 0$, then the equilibrium E_0 of the impulsive system (6.13) is globally asymptotically stable.*

Remark 6.1.4 *From Theorem 6.1.2, we see that based on the Lyapunov function method the states of the one-dimensional Gray-Scott model are driven to the equilibrium $E_0 = (1,0)^T$ effectively by a pinning impulsive controller. Actually, from the proof of Theorem 6.1.1, we can see that the equilibrium E_0 of the impulsive system (6.13) is globally exponentially stable with the convergence rate $-\frac{1}{2}(\frac{\ln\rho}{T_a} + \beta)$. In the following numerical simulations, we choose $T_a = 0.1$ and $q = 0.78$ which implies that all the conditions of Theorem 6.1.2 are satisfied. Uniform impulsive intervals are chosen in Figure 6.2(a) with $t_{k+1} - t_k = 0.1$, while $t_{2k} - t_{2k-1} = 0.04$ and $t_{2k+1} - t_{2k} = 0.16 > T_a$ are selected in Figure 6.2(b). We can see from Figure 6.2 that the equilibrium E_0 of system (6.13) is asymptotically stable.*

6.2 Synchronization of Reaction-Diffusion Neural Networks with Time-Delay

In practice, reaction-diffusions are inevitable in some applications of neural networks due to, for example, the noneven electromagnetic field in which electrons are moving [59] and the diffusion effects in biological systems (see, e.g., [92], [111]). Therefore, it is necessary to consider the state activations that vary in both space and time, leading to neural networks in the form of partial differential equations. In recent years, impulsive partial differential equations have received a great deal of attentions (see, e.g., [40], [102], [19], [12]), and impulsive control and stabilization has been shown to be a powerful tool in applications of various neural networks with reaction-diffusions (see, [125], [59], [41], [1]). Due to the advantages of the pinning impulsive control and the existence of time-delay, the study of pinning impulsive control and synchronization of reaction-diffusion neural networks with delays is an interesting and challenging research area yet to be fully developed. However, to the best of our knowledge, little work has been done on this topic, and the existing results in [125] is inconvincible as mentioned in Remark 6.2.8. Moreover, the pinning impulsive control schemes for delayed dynamical networks obtained in [109] and [77] can not be applied and extended directly to the synchronization problem of neural networks with both reaction-diffusion terms and time-varying delays.

In order to fill the research gap discussed above, this section studies the pinning impulsive synchronization problem of reaction-diffusion neural networks with time-varying delays. There are several difficulties to conduct this research. First, to apply pinning impulsive control method, it is necessary and difficult to select appropriate neurons to control at each impulsive instant (see Example 6.2.1 for detailed discussion). Second, for networks with delays, it is practically needed to establish sufficient conditions to guarantee the synchronization of networks with various delay sizes. The Lyapunov-Krasovskii functional method is one of the main approach to study the stability and synchronization of dynamical networks with time-varying delays. However, the previous two difficulties tighten the restriction on constructing feasible functional candidates when applying the method of Lyapunov-Krasovskii functionals. In this section, we introduce a type of Lyapunov-Krasovskii functionals and a pinning algorithm to overcome the above difficulties, and sufficient conditions are derived to guarantee the synchronization of neural networks with small and large time-delay, respectively. The pinning algorithm in this section is more general that the one in [67] and [91] for synchronization of stochastic dynamical networks, since we can control different amount of nodes at distinct impulsive instants while the number of nodes to be controlled is fixed for all impulsive times in [67] and [91]. The Lyapunov-Krasovskii functional candidate is divided into a function part and a functional part. The function part plays an important role to carry over the pinning algorithm and handle the effects that impulses act on the Lyapunov-Krasovskii functional. The idea of constructing Lyapunov-Krasovskii functional candidate and the mathematical analysis approach used in this section can also be applied to extend our pinning impulsive control scheme to control problems of various dynamical systems with time-delay.

The rest of this section is organized as follows. In Subsection 6.2.1, the control problem of the reaction-diffusion neural networks with time-varying delays is formulated, and the pinning algorithm on selecting neurons to add the impulsive controllers is introduced. The main synchronization results are presented in Subsection 6.2.2 with some discussions. Numerical simulations and further discussions are conducted in Subsection 6.2.3. The detailed proofs of the main results are given in Subsection 6.2.4.

6.2.1 Network Model and Problem Formulation

Consider the following reaction-diffusion neural network with time-varying delays:

$$\begin{aligned}\frac{\partial u_i(t,x)}{\partial t} &= \sum_{l=1}^{m}\frac{\partial}{\partial x_l}\left(d_{il}\frac{\partial u_i(t,x)}{\partial x_l}\right) - c_i u_i(t,x) + \sum_{j=1}^{n} a_{ij} f_j(u_j(t,x)) \\ &\quad + \sum_{j=1}^{n} b_{ij} f_j(u_j(t-\tau_{ij}(t),x)) + J_i, \quad i=1,2,...,n, \qquad (6.15)\end{aligned}$$

where $x = (x_1, x_2, ..., x_m)^T \in \Omega \subset \mathbb{R}^m$ is the space variable with $\Omega = \{x = (x_1, x_2, ..., x_m)^T : |x_k| \leq h_k, k = 1, 2, ..., m\}$, and h_k $(k = 1, 2, ..., m)$ are positive constants; $u_i(t, x)$ denotes the state of the ith neuron at time t and in space x; the activation function $f_j(u_j(t, x))$ stands for the output of the jth neuron at time t and in space x. J_i, c_i, a_{ij} and b_{ij} are constants: J_i is the external bias or input to the ith neuron; $c_i > 0$ represents the rate with which the ith neuron will reset its potential to the resting state when disconnected from the network and under external input J_i; a_{ij} and b_{ij} are the connection weights between neurons. τ_{ij} denotes the transmission time-varying delay from the jth neuron to the ith neuron; $d_{il} > 0$ is the transmission diffusion coefficient along the ith neuron.

Throughout this section, we make the following assumptions on time-varying delays and activation functions:

(A_1) There exist positive constants τ and δ_{ij} such that

$$0 \leq \tau_{ij}(t) \leq \tau \quad \text{and} \quad \dot{\tau}_{ij}(t) \leq \delta_{ij} < 1,$$

for all $i, j \in \{1, 2, ..., n\}$.

(A_2) There exists a constant L_i such that

$$|f_i(u) - f_i(v)| \leq L_i |u - v|,$$

for all $u, v \in \mathbb{R}$ and $i = 1, 2, ..., n$.

Remark 6.2.1 *(A_1) implies that the time-delay in network (6.15) is bounded and $\frac{d(t-\tau_{ij}(t))}{dt} > 0$, i.e., $t - \tau_{ij}(t)$ is increasing. Intuitively, as time t increases, the delay dependence of the state $u_i(t, x)$ is increasing. Thus, it is straightforward to make this assumption. In terms of the activation function, various neural networks possess the properties concluded in assumption (A_2) (see e.g., bidirectional memory networks [32] and BAM networks [23]).*

The Dirichlet boundary condition of system (6.15) is given by

$$u_i(t, x) = 0, \tag{6.16}$$

for $(t, x) \in [t_0 - \tau, +\infty) \times \partial\Omega$ and $i = 1, 2, ..., n$, where t_0 is the initial time, and $\partial\Omega$ denotes the boundary of Ω. The initial value of system (6.15) is given as follows:

$$u_i(t_0 + s, x) = \phi_i(s, x), \tag{6.17}$$

for $(s, x) \in [-\tau, 0] \times \Omega$ and $i = 1, 2, ..., n$, where $\phi = (\phi_1, \phi_2, ...\phi_n)^T \in C([-\tau, 0] \times \Omega, \mathbb{R}^n)$. For $\phi \in C([-\tau, 0] \times \Omega, \mathbb{R}^n)$ and a given $s \in [-\tau, 0]$, define the following norm:

$$\|\phi(s, \cdot)\|_2 = \left(\sum_{i=1}^{n} \int_\Omega \phi_i^2(s, x) \mathrm{d}x \right)^{1/2}.$$

Similarly, for $u = (u_1, u_2, ..., u_n)^T \in C([0, \infty) \times \Omega, \mathbb{R}^n)$ and a given $t \geq 0$, we define the following norm:

$$\|u(t, \cdot)\|_2 = \left(\sum_{i=1}^{n} \int_\Omega u_i^2(t, x) \mathrm{d}x \right)^{1/2}.$$

Let us introduce a drive system in the form of (6.15) with Dirichlet boundary condition (6.16) and initial condition (6.17), and a response system in the form of

$$\begin{cases} \dfrac{\partial v_i(t,x)}{\partial t} = \displaystyle\sum_{l=1}^{m} \frac{\partial}{\partial x_l}\Big(d_{il}\frac{\partial v_i(t,x)}{\partial x_l}\Big) - c_i v_i(t,x) + \sum_{j=1}^{n} a_{ij} f_j(v_j(t,x)) \\ \qquad\qquad + \displaystyle\sum_{j=1}^{n} b_{ij} f_j(v_j(t-\tau_{ij}(t),x)) + J_i + U_i(t,x), \\ v_i(t_0+s,x) = \varphi_i(s,x),\ (s,x) \in [-\tau,0]\times\Omega, \\ v_i(t,x) = 0,\ (t,x)\in[t_0-\tau,\infty)\times\partial\Omega, \end{cases} \tag{6.18}$$

where U_i is the control input to the ith neuron yet to be designed, and $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n)^T \in C([-\tau,0]\times\Omega, \mathbb{R}^n)$. Without loss of generality, we assume that $\varphi(s,x) \not\equiv \phi(s,x)$ for $(s,x) \in [-\tau,0]\times\Omega$ so that the synchronization behavior can be observed.

Definition 6.2.1 *Drive system (6.15) and response system (6.18) are said to be exponentially synchronized under controller $U(t,x)$, if there exist constants $\mu > 0$ and $M \geq 1$ such that*

$$\|u(t,\cdot) - v(t,\cdot)\|_2 \leq Me^{-\mu(t-t_0)} \sup_{s\in[-\tau,0]} \|\varphi(s,\cdot) - \phi(s,\cdot)\|_2,$$

for all $t \geq t_0$, where $U = (U_1, U_2, ..., U_n)^T$, $u = (u_1, u_2, ..., u_n)^T$, and $v = (v_1, v_2, ..., v_n)^T$. The constant μ is called the synchronization rate.

The objective of this section is to exponentially synchronize system (6.18) with (6.15) by designing a suitable impulsive controller $U(t,x)$. In order to force the trajectory of network (6.18) to approach the trajectory of network (6.15) exponentially, we design the following pinning impulsive controller

$$U_i(t,x) = \begin{cases} \displaystyle\sum_{k=1}^{\infty} -q_k e_i(t,x)\delta(t-t_k^-),\ i \in \mathcal{D}_k, \\ \qquad\qquad\qquad 0,\ i \notin \mathcal{D}_k, \end{cases} \tag{6.19}$$

where $i = 1,2,...,n$, and $q_k \in (0,1)$ is the impulsive control gain to be determined. The impulsive instant sequence $\{t_k\}$ satisfies $\{t_k\} \subset \mathbb{R}$, $0 \leq t_0 < t_1 < t_2 < ... < t_k < ....$, and $\lim_{k\to\infty} t_k = \infty$. $\delta(\cdot)$ is the Dirac delta function. The error state $e_i(t,x) = v_i(t,x) - u_i(t,x)$ represents the state difference of the two networks (6.15) and (6.18). l_k denotes the number of neurons to be pinned at each impulsive instant. Similar to the definition introduced in Section 5.1, we have the index set $\mathcal{D}_k = \{p_1, p_2, ..., p_{l_k}\} \subseteq \mathcal{I} := \{1,2,...,n\}$ defined as follows: $p_i \neq p_j$ if $i \neq j$; at the impulsive instant t_k, $\|e_i(t_k^-,\cdot)\|_2 \geq \|e_j(t_k^-,\cdot)\|_2$ if $i \in \mathcal{D}_k$ and $j \in \mathcal{I}/\mathcal{D}_k$. Then, we have $\sharp\mathcal{D}_k = l_k$ where $0 < l_k \leq n$. The pinning impulsive synchronization mechanism is illustrated in Figure 6.3.

The response system (6.18) with the pinning impulsive controller (6.19) can be rewritten in

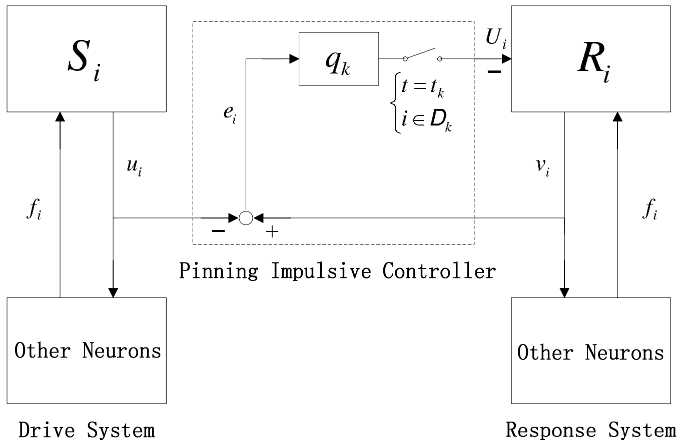


Figure 6.3: Pinning impulsive synchronization mechanism. S_i and R_i represent the ith neurons of network (6.15) and (6.18), respectively. f_i depicts all the activation functions corresponding to the ith neuron of the network.

the form of an impulsive system

$$\begin{cases} \dfrac{\partial v_i(t,x)}{\partial t} = \displaystyle\sum_{l=1}^{m} \frac{\partial}{\partial x_l}\Big(d_{il}\frac{\partial v_i(t,x)}{\partial x_l}\Big) - c_i v_i(t,x) + \sum_{j=1}^{n} a_{ij} f_j(v_j(t,x)) \\ \qquad\qquad + \displaystyle\sum_{j=1}^{n} b_{ij} f_j(v_j(t-\tau_{ij}(t),x)) + J_i,\ t \neq t_k, \\ \Delta v_i(t_k,x) = -q_k e_i(t_k^-,x),\ i \in \mathcal{D}_k,\ \sharp\mathcal{D}_k = l_k,\ k \in \mathbb{N}, \\ v_i(t_0+s,x) = \varphi_i(s,x),\ (s,x) \in [-\tau,0]\times\Omega, \\ v_i(t,x) = 0,\ (t,x) \in [t_0-\tau,\infty)\times\partial\Omega, \end{cases} \tag{6.20}$$

where $\Delta v_i(t_k,x) = v_i(t_k^+,x) - v_i(t_k^-,x)$, $v_i(t_k^+,x)$ and $v_i(t_k^-,x)$ denote the right and left limit of v_i at t_k, respectively.

Remark 6.2.2 *It is shown in [118] that, under assumptions (A_1) and (A_2), system (6.15) with the Dirichlet boundary condition (6.16) and initial condition (6.17) admits a unique global solution. The existence of solution to system (6.20) can be guaranteed by the existence results of reaction-diffusion equations in [57] and the method of steps, since discrete delays are considered in the system.*

Throughout this section, we always assume that $v_i(t,x)$ is right continuous at $t_k (k \in \mathbb{N})$, i.e., $\lim_{t\to t_k^+} v_i(t,x) = v_i(t_k,x)$ for all $x \in \Omega$. Then, by introducing the error state e_i, we have the following error system

$$\begin{cases} \dfrac{\partial e_i(t,x)}{\partial t} = \displaystyle\sum_{l=1}^{m} \frac{\partial}{\partial x_l}\Big(d_{il}\frac{\partial e_i(t,x)}{\partial x_l}\Big) - c_i e_i(t,x) + \sum_{j=1}^{n} a_{ij} \widehat{f}_j(e_j(t,x)) \\ \qquad\qquad + \displaystyle\sum_{j=1}^{n} b_{ij} \widehat{f}_j(e_j(t-\tau_{ij}(t),x)),\ t \neq t_k, \\ \Delta e_i(t_k,x) = -q_k e_i(t_k^-,x),\ i \in \mathcal{D}_k,\ \sharp\mathcal{D}_k = l_k, k \in \mathbb{N}, \\ e_i(t_0+s,x) = \varphi_i(s,x) - \phi_i(s,x),\ (s,x) \in [-\tau,0]\times\Omega, \\ e_i(t,x) = 0,\ (t,x) \in [t_0-\tau,\infty)\times\partial\Omega, \end{cases} \tag{6.21}$$

where $\widehat{f}_j(e_j(\cdot,x)) = f_j(v_j(\cdot,x)) - f_j(u_j(\cdot,x))$ for $j = 1,2,...,n$. It can be seen from (6.21) and Definition 6.2.1 that if $\|e(t,\cdot)\|_2$ converges to zero exponentially as $t \to \infty$, then the drive system (6.15) and the response system (6.18) can be exponentially synchronized.

Remark 6.2.3 *Designing an appropriate pinning impulsive controller contains four aspects:*

- *How many neurons need to be pinned?*
- *Which neurons need to be selected?*
- *When does the impulsive control input need to be added to the neurons?*
- *How strong does the impulsive control gain need to be?*

The pinning impulsive control scheme (6.19) is inspired by the idea in [67] and [91] for nonlinear networks without delays and reaction-diffusions. The difficulties of applying the pinning impulsive control scheme (6.19) to networks with time-delay and reaction-diffusion terms will be discussed in detail in Remark 6.2.6. Up to now, we have designed only the control strategy about which neurons need to be controlled at each impulsive instant by control scheme (6.19), i.e., controlling l_k neurons that have larger state difference than the other $n - l_k$ neurons. Other aspects of designing a suitable pinning impulsive controller will be investigated in Subsection 6.2.3. Sufficient conditions on suitable relations among the impulsive instant t_k, impulsive control gain q_k, and the number of neurons to be pinned l_k will be established. Furthermore, different neurons may be selected to pin at different impulsive instant according to our pinning control mechanism (6.19). Therefore, we do not need the assumption about the connectivity of the network which is necessary in [125]. However, there is a fatal error in [125], which will be discussed in Remark 6.2.8.

6.2.2 Synchronization Results

In this subsection, exponential synchronization criteria for reaction-diffusion neural networks with time-varying delays are established, and these results will also be discussed. For convenience, we introduce the following notations. Let

$$\rho_k = 1 - \frac{l_k}{n} q_k(2 - q_k), \quad k \in \mathbb{N},$$

$$\lambda = \max_{1\le j\le n} \left\{ \sum_{i=1}^{n} \frac{\xi_{ij}^{-1}|b_{ij}|L_j}{1-\delta_{ij}} \right\},$$

$$c = \max_{1\le i\le n} \left\{ -2c_i - \frac{\pi^2}{2}\sum_{l=1}^{m}\frac{d_{il}}{h_l^2} + \sum_{j=1}^{n}\left(\varepsilon_{ij}|a_{ij}|L_j + \xi_{ij}|b_{ij}|L_j + \varepsilon_{ji}^{-1}|a_{ji}|L_i + \frac{\xi_{ji}^{-1}|b_{ji}|L_i}{1-\delta_{ji}}\right) \right\},$$

where ε_{ij} and ξ_{ij} $(i,j = 1,2,...,n)$ are positive real numbers.

Now we are in the position to state our main results, proofs of which will be presented in Subsection 6.2.4.

Theorem 6.2.1 *Suppose assumptions (A_1) and (A_2) hold, $c > 0$ and $\tau \leq t_k - t_{k-1}$ for all $k \in \mathbb{N}$. Moreover, if there exist positive constants ξ_{ij}, ε_{ij} $(i, j = 1, 2, ..., n)$, and α such that*

$$\ln(\rho_k + \lambda\tau) \leq -(\alpha + c)(t_{k+1} - t_k), \quad k \in \mathbb{N}, \tag{6.22}$$

then the drive system (6.15) and the response system (6.18) are exponentially synchronized by the pinning impulsive controller (6.19).

Remark 6.2.4 *It can be seen from the proof of Theorem 6.2.1 that the synchronization rate is $\frac{\alpha}{2}$ which is closely related to the system parameters λ, c, controller parameters ρ_k and the length of impulsive intervals $t_{k+1} - t_k$ $(k \in \mathbb{N})$. Since ρ_k depends on the impulsive control gain q_k and the number of neurons to be pinned at each impulsive instant t_k, inequality (6.22) gives us a guide line of balancing the values of q_k, l_k and $t_{k+1} - t_k$ to obtain a suitable pinning impulsive controller. However, the condition $\tau \leq t_k - t_{k-1}$ implies that the size of time-delay τ is a lower bound of the impulsive intervals. Therefore, large time-delay in network (6.15) may render Theorem 6.2.1 invalid. In order to resolve this issue, we have the following result.*

Theorem 6.2.2 *Suppose assumptions (A_1) and (A_2) hold, and $c > 0$. Furthermore, if there exist positive constants ξ_{ij}, ε_{ij} $(i, j = 1, 2, ..., n)$, and α such that*

$$\ln(\rho_k + \lambda\tau e^{\alpha\tau}) \leq -(\alpha + c)(t_{k+1} - t_k), \quad k \in \mathbb{N}, \tag{6.23}$$

then the drive system (6.15) and the response system (6.18) can be exponentially synchronized by the pinning impulsive controller (6.19).

Remark 6.2.5 *In this result, we do not need the condition $\tau \leq t_k - t_{k-1}$, i.e., the length of impulsive interval $t_k - t_{k-1}$ can be less that the size of time-delay. Therefore, Theorem 6.2.2 is applicable to networks with relatively large delays. However, it can be seen that the exponential term $e^{\alpha\tau}$ on the left-hand side of (6.23) makes (6.23) supply a more conservative condition on the choice of ρ_k and $t_{k+1} - t_k$ than condition (6.22) does for networks with small enough delays. Therefore, Theorem 6.2.1 and 6.2.2 give us sufficient conditions to design appropriate pinning impulsive controllers to synchronize networks (6.15) and (6.18) with small or large time-delay. Illustrative examples are presented in Subsection 6.2.3.*

Remark 6.2.6 *The Lyapunov-Krasovskii functional candidates in proofs of Subsection 6.2.4 are divided into two parts: a function part and a functional part. This kind of structure has been widely used in literature when stability of dynamical systems with delays is investigated (see, e.g., [125], [41], [137]). However, no pinning impulsive synchronization result has been reported for delayed networks by the method of Lyapunov-Krasovskii functionals. In this section, there are two reasons to consider this type of Lyapunov-Krasovskii functionals. First, it is straightforward for an impulse to alter the value of a function instantaneously. Thus, the value of the function part can be effectively reduced by the impulse, whereas the functional part is not affected by the impulse (see [69] for a similar discussion of impulsive delay systems). However, this fact brings dramatic difficulties to the theoretic reasoning of our main results. Second, the quadratic form of the function part makes it possible for us to generalize the pinning impulsive control strategy in [67] for networks without time-delay to synchronization problems of reaction-diffusion networks with time-varying delays.*

Remark 6.2.7 *Another commonly considered boundary condition for reaction-diffusion neural networks is the Neumann boundary condition*

$$\frac{\partial u_i(t,x)}{\partial \widehat{\mathbf{n}}} = \left(\frac{\partial u_i(t,x)}{\partial x_1}, \frac{\partial u_i(t,x)}{\partial x_2}, ..., \frac{\partial u_i(t,x)}{\partial x_m}\right)^T \cdot \widehat{\mathbf{n}} = 0,$$

for $(t,x) \in [t_0 - \tau, \infty) \times \partial\Omega$, *where* $\widehat{\mathbf{n}}$ *is the outward unit normal vector of* $\partial\Omega$, *and the dot is the inner product. Clearly, Lemma 6.2.1 in Subsection 6.2.4 is not valid for the Neumann boundary condition. However, our method is still applicable to the synchronization problem of reaction-diffusion neural networks with Neumann boundary conditions. For instance, (6.31) can be replaced by the following estimation:*

$$\int_\Omega e_i(t,x) \sum_{l=1}^{m} \frac{\partial}{\partial x_l}\left(d_{il}\frac{\partial e_i(t,x)}{\partial x_l}\right)\mathrm{d}x \leq 0.$$

Then replacing c by

$$c = \max_{1\leq i\leq n}\left\{ -2c_i + \sum_{j=1}^{n}\left(\varepsilon_{ij}|a_{ij}|L_j + \xi_{ij}|b_{ij}|L_j + \varepsilon_{ji}^{-1}|a_{ji}|L_i + \frac{\xi_{ji}^{-1}|b_{ji}|L_i}{1-d_{ji}}\right)\right\},$$

which is independent of the reaction-diffusion coefficients d_{il}, *Theorem 6.2.1 and 6.2.2 can be applied to design suitable pinning impulsive controllers to achieve synchronization of delayed reaction-diffusion networks (6.15) and (6.18) with Neumann boundary conditions. Furthermore, the technique, which is used in this section for designing the Lyapunov-Krasovskii functionals combined with the pinning impulsive control strategy, is also applicable for neural networks with distributed delays and various types of neural networks, such as BAM neural networks, stochastic neural networks, fuzzy neural networks, and discrete-time neural networks.*

Remark 6.2.8 *The pinning impulsive synchronization of reaction-diffusion neural networks has recently been studied in [125]. By using Lyapunov function, Halanay-type inequality, and comparison method, sufficient conditions have been obtained to design appropriate impulsive controllers to pin the same neurons at each impulsive instant, which is different from our pinning impulsive mechanism. Moreover, there is a fatal error in the proof of the main result in [125]. In the proof of Theorem 6.2.1 in [125], the estimation of (3.32) is based on the assumption that (3.14) and (3.15) are true. However, (3.13) does not imply that (3.14) and (3.15) hold, and there is no condition in Theorem 3.1 of [125] to guarantee (3.14) and (3.15) are true. Hence, the proof for Theorem 3.1 in [125] is not sufficient, and the corresponding results lack theoretical support.*

Remark 6.2.9 *In Theorems 6.2.1 and 6.2.2, two types of assistant parameters* ξ_{ij} *and* ε_{ij} *are used to reduce the conservatism of estimations in (6.22) and (6.23). These parameters make Theorems 6.2.1 and 6.2.2 flexible in dealing with networks with various system coefficients. For instance, to use Theorem 6.2.1 or Theorem 6.2.2, the parameter* λ *in (6.22) or (6.23) is required to make* $\lambda\tau$ *or* $\lambda\tau e^{\alpha\tau}$ *less than 1. For networks with large delay size, parameters* ξ_{ij} *can be chose to make* λ *small enough, then Theorem 6.2.1 or 6.2.2 can be applied to design suitable pinning impulsive controllers to realize network synchronization. For more details, see examples in Subsection 6.2.3.*

If the uniform impulsive controller is considered in (6.19), i.e., $l_k = l$, $q_k = q$, and $t_k - t_{k-1} = T$ for all $k \in \mathbb{N}$, then we have the following synchronization result.

Theorem 6.2.3 *Suppose assumptions (A_1) and (A_2) hold, and $c > 0$. Furthermore, if there exist positive constants ξ_{ij}, ε_{ij} $(i, j = 1, 2, ..., n)$ such that*

$$\ln(\rho + \lambda\tau) < -cT, \tag{6.24}$$

where $\rho = 1 - \frac{l}{n}q(2-q)$, then the drive system (6.15) and the response system (6.18) can be exponentially synchronized by the pinning impulsive controller (6.19).

Remark 6.2.10 *It can be observed from (6.24) that there exists a positive constant α such that*

$$\ln(\rho + \lambda\tau) = -(\alpha + c)T, \tag{6.25}$$

or

$$\ln(\rho + \lambda\tau e^{\alpha\tau}) = -(\alpha + c)T, \tag{6.26}$$

which implies (6.22) or (6.23) is satisfied. Therefore, Theorem 6.2.3 can be proved. Moreover, if $\tau \leq T$, then the convergence rate α can be estimated by (6.25). Otherwise, α can be obtained by solving (6.26).

6.2.3 Numerical Simulations and Discussions

In this section, we present two examples to demonstrate our main results. In order to clearly observe the pinning impulsive control process in the simulation results, we will consider neural networks with only two neurons, i.e., $n = 2$. In the first example, we consider the synchronization problem of reaction-diffusion neural networks with time-invariant delays.

Example 6.2.1 *Consider a delay reaction-diffusion neural network described by (6.15) with Dirichlet boundary condition (6.16) and initial condition (6.17), where $t_0 = 0$, $m = 1$, $n = 2$, $\Omega = [-4, 4]$, $d_1 = d_2 = 0.1$, $c_1 = c_2 = 1$, $\tau_{11} = \tau_{22} = 1$, $\tau_{12} = \tau_{21} = 0.5$, $f_1(\cdot) = f_2(\cdot) = \tanh(\cdot)$, $J_1 = J_2 = 0$, and*

$$[a_{ij}]_{2\times 2} = \begin{bmatrix} 2 & -0.1 \\ -5 & 3 \end{bmatrix}, \quad [b_{ij}]_{2\times 2} = \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -2.5 \end{bmatrix}.$$

It can be easily verified that assumption (A_1) is satisfied with $\tau = 1$ and $\delta_{ij} = 0$ for $i, j = 1, 2$, and assumption (A_2) is satisfied with $L_1 = L_2 = 1$. The chaotic behavior of neural network (6.15) with the given initial data are shown in Fig. 6.4.

For

$$[\xi_{ij}]_{2\times 2} = \begin{bmatrix} 10 & 1 \\ 1 & 10 \end{bmatrix}, \text{ and } [\varepsilon_{ij}]_{2\times 2} = \begin{bmatrix} 1 & 1 \\ 0.36 & 1 \end{bmatrix},$$

one can get the following estimations: $\lambda = 0.36$, $c_1 = 31.4264$, and $c_2 = 31.4375$. Then, $c = 31.4375$. In this example, we first consider the impulsive controller (6.19) with $l_k = n = 2$, $t_k - t_{k-1} = 0.02$, and $q_k = 0.59$ for all $k \in \mathbb{N}$; then, (6.23) is satisfied with $\alpha = 0.01$. Therefore, we can conclude from Theorem 6.2.2 that the drive system (6.15) can be exponentially synchronized with the response system (6.18) under impulsive controller (6.19). Figure 6.5 shows that trajectories of the synchronization error states $e_i(t, x)$, $i = 1, 2$. It can be seen that there are visible serrations in the trajectories of e_i when t is less than 1, which can be clearly observed in

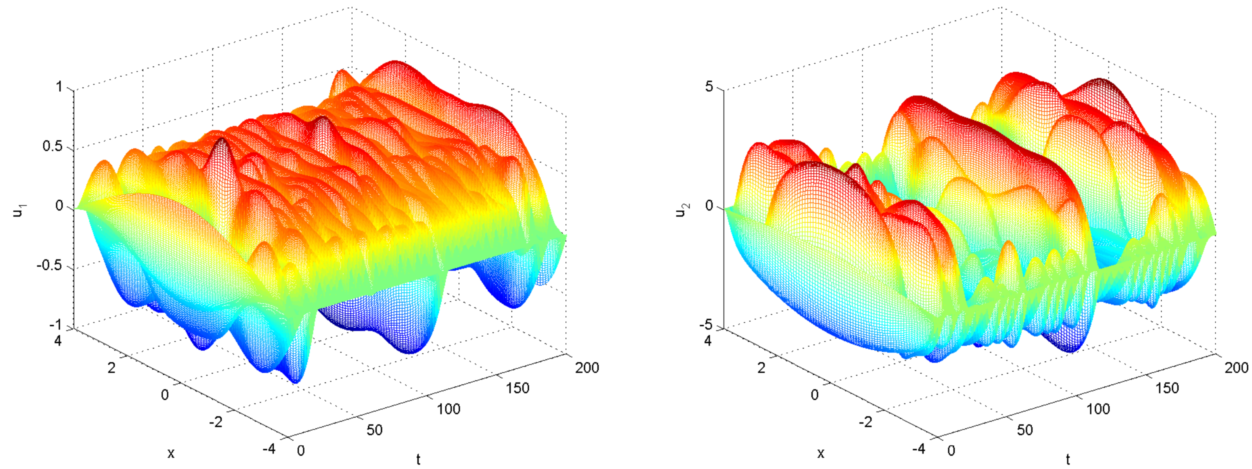


Figure 6.4: State trajectories of network (6.15) with the parameters given in Example 6.2.1 and initial data $\phi_1(s,x) = 0.5\cos(\frac{\pi x}{8})$, $\phi_2(s,x) = 0.4\cos(\frac{\pi x}{8})$, for $s \in [-\tau, 0]$ and $x \in \Omega$. The spatio-temporal chaotic behavior can be clearly observed in the above figures.

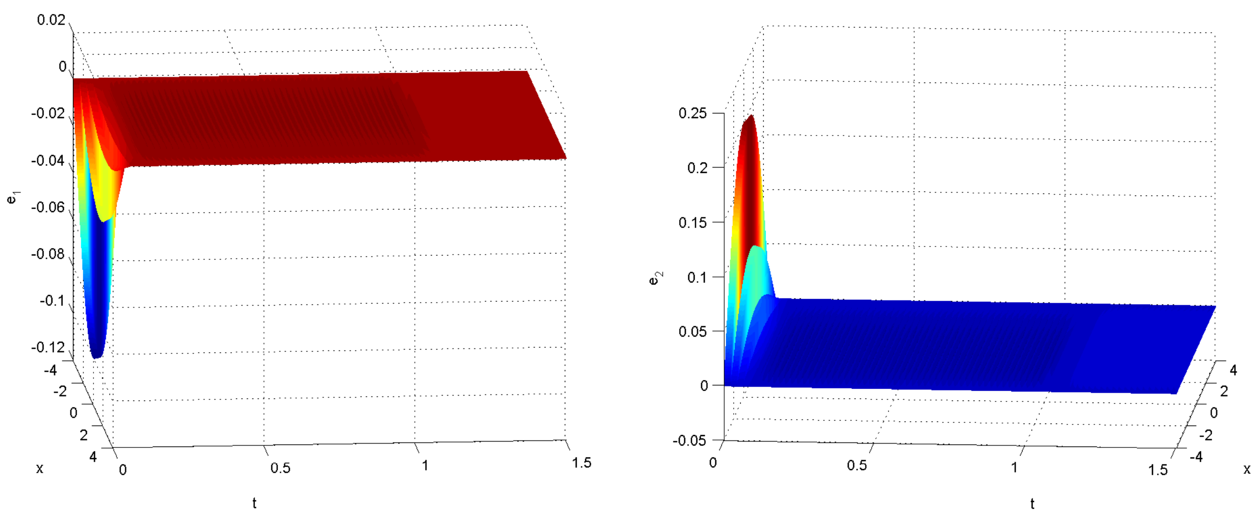


Figure 6.5: Synchronization errors e_1 and e_2. The initial data of the drive system (6.15) is chosen the same as that in Figure 6.4, and the initial conditions of the response system (6.18) are given by $\varphi_1(s,x) = 0.4\cos(\frac{\pi x}{8})$, $\varphi_2(s,x) = 0.6\cos(\frac{\pi x}{8})$, for $s \in [-\tau, 0]$ and $x \in \Omega$. It can be seen that the synchronization between the drive and response systems can be realized.

Figure 6.6 for $\|e_i\|_2$ $(i = 1, 2)$. This phenomena can be explained by the existence of time-delay with delay size 1 in the network, which verifies our discussion of the impact of the delay size on the synchronization rate in Remark 6.2.5.

Next, consider a pinning impulsive controller with $l_k = 1$, $t_k - t_{k-1} = 0.004$, and $q_k = 0.9$ for $k \in \mathbb{N}$, i.e., control one neuron at each impulsive instant. Then, all the conditions of Theorem 6.2.2 are satisfied, and numerical simulations are shown in Figure 6.7(a). Compared with the pinning impulsive control method in [125], we need to select a small fraction of neurons to pin at each impulsive instant according to our pinning algorithm, while no conditions are required

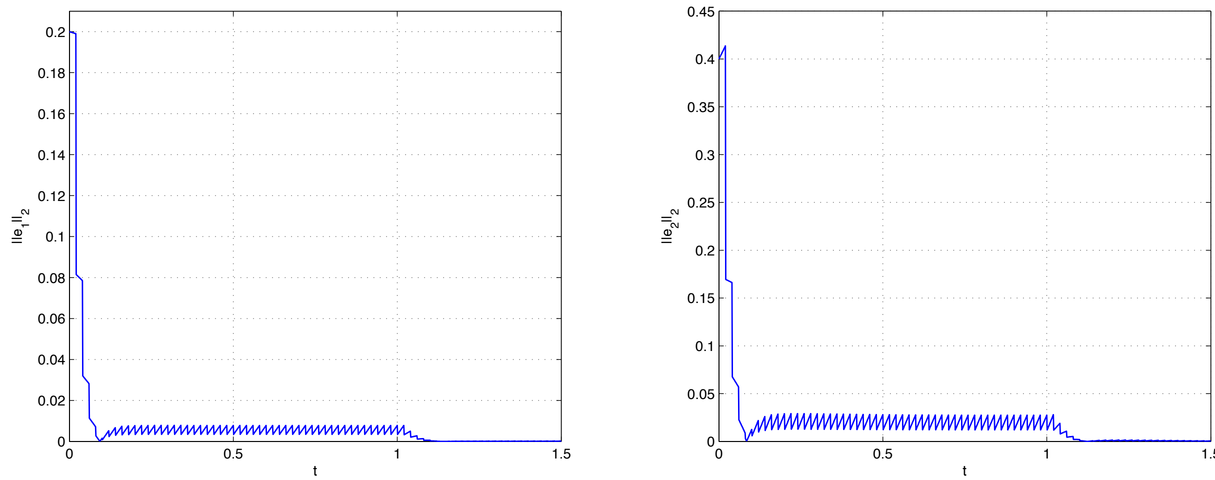
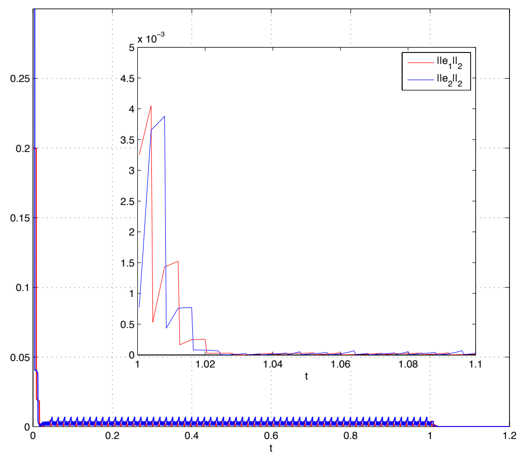


Figure 6.6: Synchronization errors e_1 and e_2 in norm. The effects of time-delay in the synchronization process can be clearly observed.

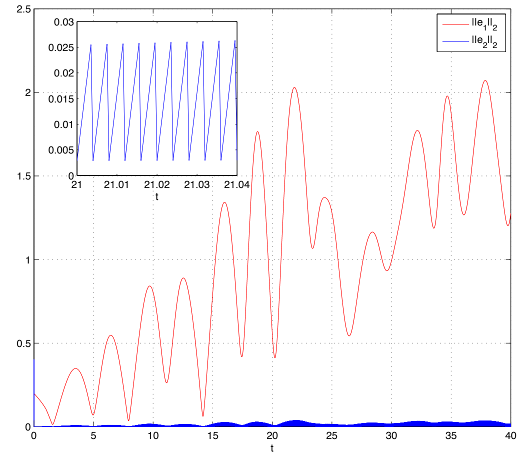
on how to select neurons to control in [125]. Actually, selection of suitable neurons to pin is necessary when applying pinning impulsive control approach. From the simulation results in Figure 6.7(b) and Figure 6.7(c), we can conclude that the pinning neurons need to be carefully selected to achieve the network synchronization, which verifies our theoretical analysis of the deficiency of the results in [125]. It is worth noting that the pinning strategy introduced in this section is one of the feasible selection methods of the pinning neurons to realize the network synchronization. In this sense, we have overcome the deficiency of the main result in [125].

Compared with the full-state impulsive controller, fewer neurons are required to be controlled at each impulsive instant. Moreover, by comparison of the numerical results shown in Figure 6.7(b) and Figure 6.7(c), we can see that our pinning algorithm is more efficient in synchronizing the networks. Though, the pinning method considered in Figure 6.7(c) (pinning a specific number of neurons at all the impulsive time) requires less information of the neurons' states, our simulations have shown that we need to look more deep into the dynamics of each isolated neurons, the network topology and their relations to figure out how to select appropriate neurons to stimulate, and, moreover, this pinning approach only applies to a specific class of networks that need to be classified, since even a simple linear system may not be stabilizable via this pinning impulsive control method. For example, consider the linear system $\dot{y} = Ay$, where $y = [y_1, y_2]^T \in \mathbb{R}^2$ and $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. No matter how frequently the impulsive controller is added to state y_1, state y_2 will blow up (that is, as y_1 approaches zero, the linear part $\dot{y}_2 = 2y_2$ will dominate the evolution of state y_2). Our future research will focus on figuring out conditions on the network dynamics and topologies to guarantee the validity of this pinning impulsive control method.

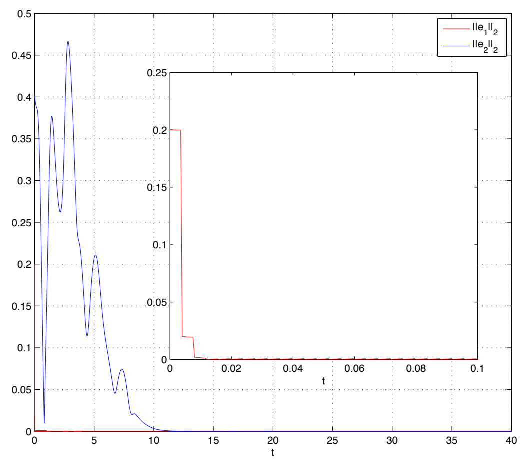
If the initial conditions of the drive system (6.15) are the same as its equilibrium, then system (6.15) reduces to a system with constant states. Thereafter, the synchronization problem of (6.15) and (6.18) reduces to the stabilization problem of system (6.15). When $d_{il} = 0$ ($i = 1, 2, ..., n$ and $l = 1, 2, ..., m$), our results can be applied to delayed neural networks without reaction-diffusion terms. In order to show the effectiveness of Theorem 6.2.1, we consider the



(a)



(b)



(c)

Figure 6.7: Synchronization processes via different pinning impulsive controllers. For these three sub-figures, the impulsive time sequence and impulsive control gains are chosen to be the same: $t_k - t_{k-1} = 0.004$ and $q_k = 0.9$ for all $k \in \mathbb{N}$. Different pinning algorithms are introduced as follows. **(a)**: pinning control the response system according to the pinning strategy in (6.19) with $l_k = 1$; **(b)**: impulsive control the second neuron of the response system at each impulsive instant; **(c)**: impulsive control the first neuron of the response system. It can be seen that synchronization can be achieved in (a) and (c), and the synchronization time in (c) is greater than 10 unit of time which is dramatically lager than that in (a). However, the synchronization can not be achieved in (b).

following example with small delay size.

Example 6.2.2 *Consider the following delayed neural network*

$$\dot{u}_i(t) \quad = \quad -c_i u_i(t) + \sum_{j=1}^{n} a_{ij} f(u_j(t)) + \sum_{j=1}^{n} b_{ij} f(u_j(t-\tau(t))) + J_i, \; i = 1, 2, \tag{6.27}$$

where $c_1 = 1$, $c_2 = 0.5$, $J_1 = J_2 = 2$, $f(u) = \frac{1}{2}(|u+1| - |u-1|)$ *for* $u \in \mathbb{R}$, $\tau(t) = \frac{0.01e^t}{1+e^t}$ *for* $t \geq 0$, *and*

$$[a_{ij}]_{2\times 2} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 1 \end{bmatrix}, \quad [b_{ij}]_{2\times 2} = \begin{bmatrix} -1.5 & -1.5 \\ -1 & -0.5 \end{bmatrix}.$$

By direct computation, we know that $u^* = (0.5, 4.5)^T$ is the unique equilibrium of system (6.27), and f satisfies assumption (A_2) with $L = 1$. For the time-delay $\tau(t)$, we have $\tau(t) \leq 0.01$ and $\dot{\tau}(t) \leq 0.0025$. Then $\tau = 0.01$ and $\delta = \delta_{ij} = 0.0025$. For $\xi_{ij} = \varepsilon_{ij} = 1$ $(i,j = 1,2)$, we have $\lambda = 2.5253$ and $c = \max\{c_1, c_2\} = 5.5253$. Design three different impulsive controllers in the form of (6.19) with $e_1(t) = u_1(t) - 0.5$ and $e_2(t) = u_2(t) - 4.5$, and then numerical results are shown in Figure 6.8.

6.2.4 Proofs

In this section, we present the proofs for the main results in Subsection 6.2.2, which are based on the Lyapunov-Krasovskii functional method and a Poincare-type inequality presented in the following lemma (see [137]).

Lemma 6.2.1 *Let* $w(x) = w(x_1, x_2, ..., x_m)$ *be a real-valued function defined on* Ω. *If* $w(x) \in C^1(\Omega)$ *and* $w(x)\mid_{\partial\Omega} = 0$, *then*

$$\int_\Omega w^2(x)dx \leq \left(\frac{2}{\pi}\right)^2 h_i^2 \int_\Omega \left|\frac{\partial w(x)}{\partial x_i}\right|^2 \mathrm{d}x.$$

Proof of Theorem 6.2.1

Consider the following Lyapunov-Krasovskii functional:

$$V(t) = V_1(t) + V_2(t),$$

where

$$V_1(t) = \sum_{i=1}^n \int_\Omega e_i^2(t,x)\mathrm{d}x,$$

and

$$V_2(t) = \sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} \int_{t-\tau_{ij}(t)}^t \Big(\int_\Omega e_j^2(s,x)\mathrm{d}x\Big)\mathrm{d}s,$$

where $\gamma_{ij} = \frac{\xi_{ij}^{-1}|b_{ij}|L_j}{1-\delta_{ij}}$ $(i,j = 1,2,...,n)$. The proof is divided into the following three steps.

Step 1: estimation of $\dot{V}(t)$ on each impulsive interval and $V(t)$ at each impulsive instant.

First, differentiate $V(t)$ along the trajectory of the error system (6.21) for $t \in [t_{k-1}, t_k)$. For $V_1(t)$,

$$\dot{V}_1(t) = \sum_{i=1}^n \int_\Omega 2e_i(t,x)\frac{\partial e_i(t,x)}{\partial t}\mathrm{d}x$$

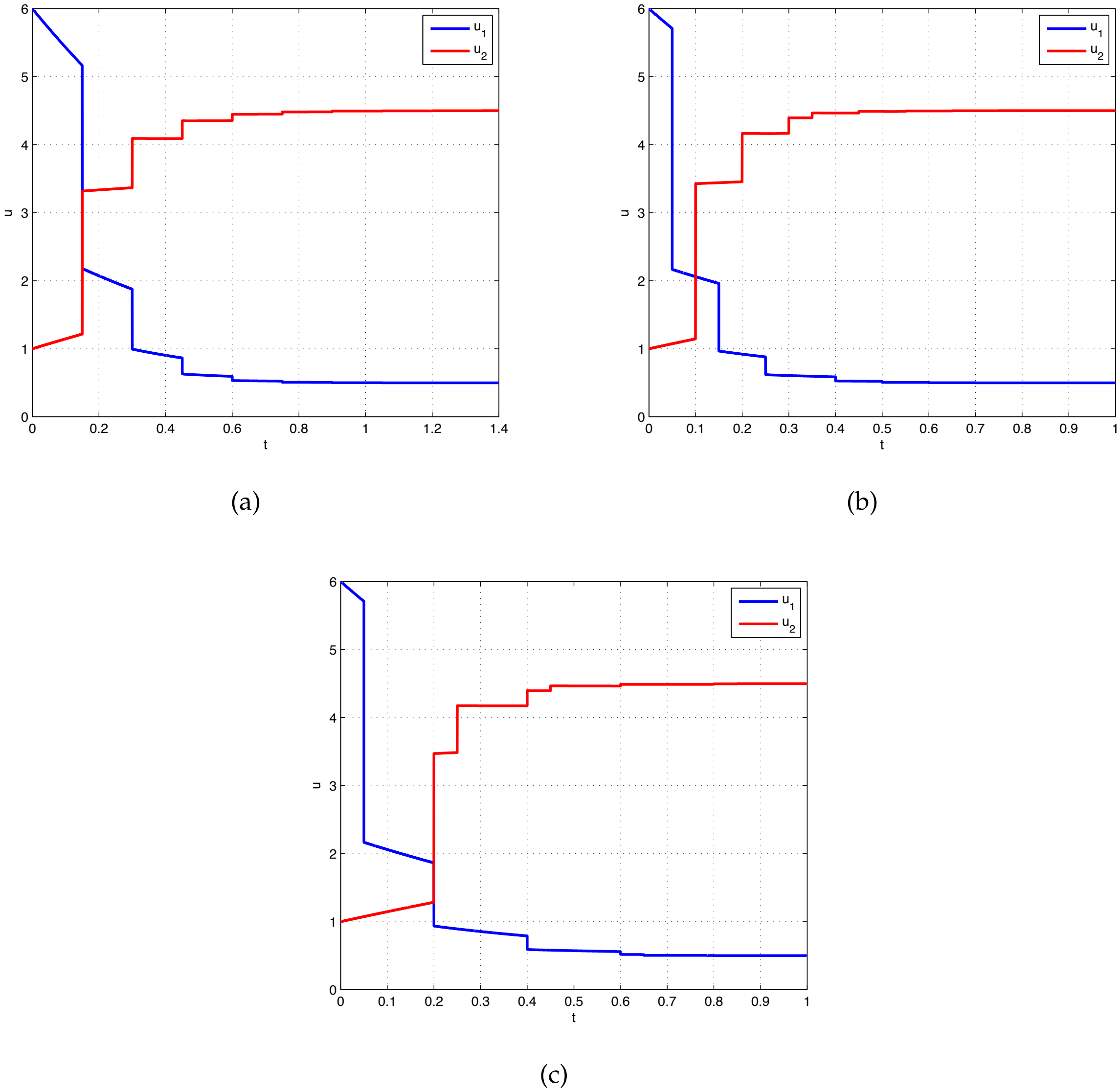


Figure 6.8: Simulation results of Example 6.2.2. Parameters of the three simulations are given as follows. **(a)**: $t_k - t_{k-1} = 0.15$, $q_k = 0.64$ and $l_k = 2$ ($k \in \mathbb{N}$), i.e., impulsive control of both neurons of network (6.27); **(b)**: $t_k - t_{k-1} = 0.05$, $q_k = 0.68$, $l_k = 1$ ($k \in \mathbb{N}$), i.e., impulsive control of one neuron at each impulsive instant; **(c)**: $t_{2k-1} - t_{2k-2} = 0.05$ and $t_{2k} - t_{2k-1} = 0.15$, $q_{2k-1} = 0.64$ and $q_{2k} = 0.68$, $l_{2k-1} = 1$ and $l_{2k} = 2$ ($k \in \mathbb{N}$), i.e., impulsive control of one neuron at each odd impulsive instant and two neurons at each even impulsive instant. All sufficient conditions of Theorem 6.2.1 are satisfied with $\alpha = 0.01$, and simulation results imply that the equilibrium of system (6.27) can be exponentially stabilized.

$$
\begin{aligned}
= \quad & \sum_{i=1}^{n} \int_{\Omega} \Bigg\{ 2 \sum_{l=1}^{m} e_i(t,x) \frac{\partial}{\partial x_l} \left(d_{il} \frac{\partial e_i(t,x)}{\partial x_l} \right) - 2c_i e_i^2(t,x) + 2 \sum_{j=1}^{n} a_{ij} e_i(t,x) \hat{f}_j(e_j(t,x)) \\
& + 2 \sum_{j=1}^{n} b_{ij} e_i(t,x) \hat{f}_j(e_j(t - \tau_{ij}(t), x)) \Bigg\} \mathrm{d}x. \qquad (6.28)
\end{aligned}
$$

Then, we have the following inequalities:

$$
\begin{aligned}
2\sum_{j=1}^{n} a_{ij}e_i(t,x)\hat{f}_j(e_j(t,x)) &\leq 2\sum_{j=1}^{n} |a_{ij}||e_i(t,x)||\hat{f}_j(e_j(t,x))| \\
&\leq \sum_{j=1}^{n} 2|a_{ij}|L_j|e_i(t,x)||e_j(t,x)| \\
&\leq \sum_{j=1}^{n} |a_{ij}|L_j(\varepsilon_{ij}e_i^2(t,x)+\varepsilon_{ij}^{-1}e_j^2(t,x)), \qquad (6.29)
\end{aligned}
$$

and

$$
\begin{aligned}
2\sum_{j=1}^{n} b_{ij}e_i(t,x)\hat{f}_j(e_j(t-\tau_{ij}(t),x)) &\leq 2\sum_{j=1}^{n} |b_{ij}||e_i(t,x)||\hat{f}_j(e_j(t-\tau_{ij}(t),x))| \\
&\leq \sum_{j=1}^{n} 2|b_{ij}|L_j|e_i(t,x)||e_j(t-\tau_{ij}(t),x)| \\
&\leq \sum_{j=1}^{n} |b_{ij}|L_j(\xi_{ij}e_i^2(t,x)+\xi_{ij}^{-1}e_j^2(t-\tau_{ij}(t),x)). \qquad (6.30)
\end{aligned}
$$

By the Dirichlet boundary condition (6.16), Divergence theorem, and Lemma 6.2.1, we get

$$
\begin{aligned}
&\int_\Omega \sum_{l=1}^{m} e_i(t,x)\frac{\partial}{\partial x_l}\left(d_{il}\frac{\partial e_i(t,x)}{\partial x_l}\right)\mathrm{d}x \\
= &\int_\Omega \sum_{l=1}^{m} \frac{\partial}{\partial x_l}\left(e_i(t,x)d_{il}\frac{\partial e_i(t,x)}{\partial x_l}\right)\mathrm{d}x - \int_\Omega \sum_{l=1}^{m} d_{il}\left(\frac{\partial e_i(t,x)}{\partial x_l}\right)^2\mathrm{d}x \\
= &\int_{\partial\Omega} \left(e_i(t,x)d_{il}\frac{\partial e_i(t,x)}{\partial x_l}\right)_{l=1}^{m}\cdot \widehat{\mathbf{n}}\mathrm{d}s - \int_\Omega \sum_{l=1}^{m} d_{il}\left(\frac{\partial e_i(t,x)}{\partial x_l}\right)^2\mathrm{d}x \\
= &-\int_\Omega \sum_{l=1}^{m} d_{il}\left(\frac{\partial e_i(t,x)}{\partial x_l}\right)^2\mathrm{d}x \\
= &-\sum_{l=1}^{m}\left[\frac{d_{il}}{h_l^2}h_l^2\int_\Omega\left(\frac{\partial e_i(t,x)}{\partial x_l}\right)^2\mathrm{d}x\right] \\
\leq &-(\frac{\pi}{2})^2\left(\sum_{l=1}^{m}\frac{d_{il}}{h_l^2}\right)\int_\Omega e_i^2(t,x)\mathrm{d}x. \qquad (6.31)
\end{aligned}
$$

For $V_2(t)$, we have

$$
\begin{aligned}
\dot{V}_2(t) &= \sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{ij}\left\{\int_\Omega e_j^2(t,x)\mathrm{d}x - \int_\Omega e_j^2(t-\tau_{ij}(t),x)(1-\dot{\tau}_{ij}(t))\mathrm{d}x\right\} \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{ij}\int_\Omega e_j^2(t,x)\mathrm{d}x - \sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{ij}(1-\delta_{ij})\int_\Omega e_j^2(t-\tau_{ij}(t),x)\mathrm{d}x. \qquad (6.32)
\end{aligned}
$$

Then, from (6.28) to (6.32), we get, for $t\in[t_{k-1},t_k)$,

$$
\dot{V}(t) = \dot{V}_1(t)+\dot{V}_2(t)
$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{n}\Big\{-\frac{\pi^2}{2}\sum_{l=1}^{m}\frac{d_{il}}{h_l^2}\int_\Omega e_i^2(t,x)\mathrm{d}x - 2c_i\int_\Omega e_i^2(t,x)\mathrm{d}x + \sum_{j=1}^{n}\varepsilon_{ij}|a_{ij}|L_j\int_\Omega e_i^2(t,x)\mathrm{d}x \\
&\quad + \sum_{j=1}^{n}\varepsilon_{ij}^{-1}|a_{ij}|L_j\int_\Omega e_j^2(t,x)\mathrm{d}x + \sum_{j=1}^{n}\xi_{ij}|b_{ij}|L_j\int_\Omega e_i^2(t,x)\mathrm{d}x + \sum_{j=1}^{n}\gamma_{ij}\int_\Omega e_j^2(t,x)\mathrm{d}x\Big\} \\
&= cV_1(t) \\
&\leq cV(t). \qquad (6.33)
\end{aligned}
$$

Next, we investigate the impact that the impulses play on the Lyapunov-Krasovskii functional. For $k \in \mathbb{N}$, define $\beta_k = \min_{j\in\mathcal{D}_k}\{\int_\Omega e_j^2(t_k^-,x)\mathrm{d}x\}$. From the pinning algorithm introduced with impulsive controller (6.19), one can have that, for any $i \notin \mathcal{D}_k$, and $j \in \mathcal{D}_k$,

$$
\int_\Omega e_i^2(t_k^-,x)\mathrm{d}x \leq \int_\Omega e_j^2(t_k^-,x)\mathrm{d}x,
$$

which implies that, for $i \notin \mathcal{D}_k$,

$$
\int_\Omega e_i^2(t_k^-,x)\mathrm{d}x \leq \min_{j\in\mathcal{D}_k}\{\int_\Omega e_j^2(t_k^-,x)\mathrm{d}x\} = \beta_k. \qquad (6.34)
$$

Hence, by (6.34), we obtain

$$
\begin{aligned}
(1-\rho_k)\sum_{i\notin\mathcal{D}_k}\int_\Omega e_i^2(t_k^-,x)\mathrm{d}x &\leq (1-\rho_k)(n-l_k)\beta_k \\
&= [\rho_k-(1-q_k)^2]l_k\beta_k \\
&\leq [\rho_k-(1-q_k)^2]\sum_{i\in\mathcal{D}_k}\int_\Omega e_i^2(t_k^-,x)\mathrm{d}x,
\end{aligned}
$$

which yields

$$
\begin{aligned}
V_1(t_k) &= \sum_{i=1}^{n}\int_\Omega e_i^2(t_k,x)\mathrm{d}x \\
&= \sum_{i\in\mathcal{D}_k}\int_\Omega e_i^2(t_k,x)\mathrm{d}x + \sum_{i\notin\mathcal{D}_k}\int_\Omega e_i^2(t_k,x)\mathrm{d}x \\
&= \sum_{i\in\mathcal{D}_k}\int_\Omega (1-q_k)^2 e_i^2(t_k^-,x)\mathrm{d}x + \sum_{i\notin\mathcal{D}_k}\int_\Omega e_i^2(t_k^-,x)\mathrm{d}x \\
&\leq \rho_k\sum_{i\in\mathcal{D}_k}\int_\Omega e_i^2(t_k^-,x)\mathrm{d}x + \rho_k\sum_{i\notin\mathcal{D}_k}\int_\Omega e_i^2(t_k^-,x)\mathrm{d}x \\
&= \rho_k V_1(t_k^-), \;\; k\in\mathbb{N}. \qquad (6.35)
\end{aligned}
$$

For $V_2(t)$, we get

$$
V_2(t_k) = V_2(t_k^-), \quad k \in \mathbb{N}. \qquad (6.36)
$$

Now, we conclude from (6.33), (6.35), and (6.36) with the following inequalities:

$$\dot{V}(t) \le cV(t), \quad t \in [t_{k-1}, t_k), \tag{6.37a}$$
$$V_1(t_k) \le \rho_k V_1(t_k^-), \tag{6.37b}$$
$$V_2(t_k) = V_2(t_k^-), \quad k \in \mathbb{N}. \tag{6.37c}$$

Step 2: mathematical induction.

We claim that

$$V(t) \le Me^{-(\alpha+c)(t_k-t_0)}e^{c(t-t_0)},\ t \in [t_{k-1}, t_k), \tag{6.38}$$

for all $k \in \mathbb{N}$, where $M = V(t_0)e^{(\alpha+c)(t_1-t_0)}$. We use mathematical induction to show that claim (6.38) is true. For $t \in [t_0, t_1)$, we can get from (6.37a) that

$$\begin{aligned} V(t) &\le V(t_0)e^{c(t-t_0)} \\ &= V(t_0)e^{(\alpha+c)(t_1-t_0)}e^{-(\alpha+c)(t_1-t_0)}e^{c(t-t_0)} \\ &= Me^{-(\alpha+c)(t_1-t_0)}e^{c(t-t_0)}. \end{aligned} \tag{6.39}$$

Then, (6.37b) and (6.39) imply that

$$V_1(t_1) \le \rho_1 V_1(t_1^-) \le \rho_1 V(t_1^-) \le \rho_1 Me^{-(\alpha+c)(t_1-t_0)}e^{c(t_1-t_0)}, \tag{6.40}$$

and we can obtain from (6.37c) that

$$\begin{aligned} V_2(t_1) &= V_2(t_1^-) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{ij}\int_{t_1-\tau_{ij}(t_1)}^{t_1}\Big(\int_\Omega e_j^2(s,x)dx\Big)ds \\ &\le \sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{ij}\int_{t_1-\tau}^{t_1}\Big(\int_\Omega e_j^2(s,x)dx\Big)ds \\ &\le \max_{1\le j\le n}\Big\{\sum_{i=1}^{n}\gamma_{ij}\Big\}\sum_{j=1}^{n}\int_{t_1-\tau}^{t_1}\Big(\int_\Omega e_j^2(s,x)dx\Big)ds \\ &= \lambda\int_{t_1-\tau}^{t_1}\|e(s,\cdot)\|_2^2 ds \\ &\le \lambda\tau \sup_{s\in[-\tau,0]}\{\|e(t_1^-+s,\cdot)\|_2^2\}, \end{aligned} \tag{6.41}$$

and, from the condition $\tau \le t_1 - t_0$ and (6.39), we get

$$\sup_{s\in[\tau,0]}\{\|e(t_1^-+s,\cdot)\|_2^2\} = \sup_{s\in[\tau,0]}\{V_1(t_1^-+s)\} \le Me^{-(\alpha+c)(t_1-t_0)}e^{c(t_1-t_0)}. \tag{6.42}$$

Thus,

$$V_2(t_1) \le \lambda\tau Me^{-(\alpha+c)(t_1-t_0)}e^{c(t_1-t_0)}. \tag{6.43}$$

It follows from (6.40), (6.43), and (6.22) that

$$\begin{aligned} V(t_1) &= V_1(t_1) + V_2(t_1) \\ &\le (\rho_1 + \lambda\tau) M e^{-(\alpha+c)(t_1-t_0)} e^{c(t_1-t_0)} \\ &\le M e^{-(\alpha+c)(t_2-t_0)} e^{c(t_1-t_0)}. \end{aligned}$$

Then, for $t \in [t_1, t_2)$, we have

$$\begin{aligned} V(t) &\le V(t_1) e^{c(t-t_1)} \\ &\le M e^{-(\alpha+c)(t_2-t_0)} e^{c(t-t_0)}, \end{aligned} \tag{6.44}$$

i.e., claim (6.38) holds for $k = 2$.

Next, we suppose (6.38) holds for $k = j$ ($j > 2$), i.e.,

$$V(t) \le M e^{-(\alpha+c)(t_j-t_0)} e^{c(t-t_0)}, \quad t \in [t_{j-1}, t_j).$$

We shall show that (6.38) holds for $k = j + 1$. As discussed in (6.40), (6.41), and (6.42), we can get

$$\begin{aligned} V(t_j) &= V_1(t_j) + V_2(t_j) \\ &\le (\rho_j + \lambda\tau) M e^{-(\alpha+c)(t_j-t_0)} e^{c(t_j-t_0)} \\ &\le M e^{-(\alpha+c)(t_{j+1}-t_0)} e^{c(t_j-t_0)}, \end{aligned}$$

then, for $t \in [t_j, t_{j+1})$,

$$\begin{aligned} V(t) &\le V(t_j) e^{c(t-t_j)} \\ &\le M e^{-(\alpha+c)(t_{j+1}-t_0)} e^{c(t-t_0)}, \end{aligned}$$

which implies that (6.38) holds for all $k \in \mathbb{N}$.

Step 3: convergence estimation.

From (6.38), we have, for $t \in [t_{k-1}, t_k)$,

$$\begin{aligned} V(t) &\le M e^{-(\alpha+c)(t_k-t_0)} e^{c(t-t_0)} \\ &\le M e^{-\alpha(t_k-t_0)} \\ &\le M e^{-\alpha(t-t_0)} \\ &= V(t_0) e^{(\alpha+c)(t_1-t_0)} e^{-\alpha(t-t_0)} \\ &\le (1 + \lambda\tau) \sup_{s \in [t_0-\tau, t_0]} \{\|e(s,\cdot)\|_2^2\} e^{(\alpha+c)(t_1-t_0)} e^{-\alpha(t-t_0)}, \end{aligned}$$

i.e.,

$$\|e(t,\cdot)\|_2 = \sqrt{V_1(t)} \le \sqrt{V(t)} \le \bar{M} \sup_{s\in[t_0-\tau,t_0]} \{\|e(s,\cdot)\|_2\} e^{-\frac{\alpha}{2}(t-t_0)},$$

for $t \ge t_0$, where $\bar{M} = \left[(1+\lambda\tau)e^{(\alpha+c)(t_1-t_0)}\right]^{1/2} > 1$. This completes the proof. □

Proof of Theorem 6.2.2

Consider the same Lyapunov-Krasovskii functional given in previous proof. Base on the discussion in the proof of Theorem 6.2.1, we have that (6.37a), (6.37b), and (6.37c) are true. Since $\lim_{k\to\infty} t_k = \infty$, there exists an integer $i \ge 1$ such that $t_i - \tau \ge t_0$, and for $t \in [t_0, t_i)$, we have

$$V(t) = V(t)e^{\alpha(t-t_0)}e^{-\alpha(t-t_0)} \le Me^{-\alpha(t-t_0)}, \tag{6.45}$$

where $M = \sup_{t\in[t_0-\tau,t_i)} \{V(t)\} e^{\alpha(t_i-t_0)}$.

Next, we shall show that

$$V(t) \le Me^{-(\alpha+c)(t_{k+1}-t_0)}e^{c(t-t_0)}, \tag{6.46}$$

for $t \in [t_k, t_{k+1})$, and $k \ge i$. When $k = i$, we obtain from (6.45) that $t_0 \le t_i + s \le t_i$ for $s \in [-\tau, 0]$, and

$$\begin{aligned}
\sup_{s\in[\tau,0]} \{\|e(t_i^- + s,\cdot)\|_2^2\} &= \sup_{s\in[\tau,0]} \{V_1(t_i^- + s)\} \\
&\le Me^{-\alpha(t_i-\tau-t_0)} \\
&= Me^{-(\alpha+c)(t_i-t_0)}e^{c(t_i-t_0)}e^{\alpha\tau}.
\end{aligned} \tag{6.47}$$

Similar to the estimations of $V_1(t_i)$ and $V_2(t_i)$ in (6.40) and (6.41), we have

$$\begin{aligned}
V(t_i) &= V_1(t_i) + V_2(t_i) \\
&\le \rho_i V_1(t_i^-) + V_2(t_i^-) \\
&\le \rho_i Me^{-\alpha(t_i-t_0)} + \lambda\tau e^{\alpha\tau} Me^{-(\alpha+c)(t_i-t_0)}e^{c(t_i-t_0)} \\
&= (\rho_i + \lambda\tau e^{\alpha\tau})Me^{-(\alpha+c)(t_i-t_0)}e^{c(t_i-t_0)} \\
&\le Me^{-(\alpha+c)(t_{i+1}-t_0)}e^{c(t_i-t_0)},
\end{aligned}$$

which, along with (6.37a), implies

$$V(t) \le Me^{-(\alpha+c)(t_{i+1}-t_0)}e^{c(t-t_0)} \text{ for } t \in [t_i, t_{i+1}). \tag{6.48}$$

Similar to the proof of Theorem 6.2.1, we can conclude by mathematical induction that (6.46) holds for all $k \ge i$. Then, for $t \in [t_k, t_{k+1})$ $(k \ge i)$, we have

$$V(t) \le Me^{-(\alpha+c)(t_k-t_0)}e^{c(t-t_0)} \le Me^{-\alpha(t_k-t_0)} \le Me^{-\alpha(t-t_0)}. \tag{6.49}$$

Hence, from (6.45) and (6.49), we can see that

$$V(t) \leq Me^{-\alpha(t-t_0)}, \quad t \geq t_0.$$

It follows that

$$\|e(t,\cdot)\|_2^2 \leq V(t) \leq Me^{-\alpha(t-t_0)}, \quad t \geq t_0,$$

i.e.,

$$\|e(t,\cdot)\|_2 \leq \bar{M} \sup_{s\in[t_0-\tau,t_0]} \left\{\|e(s,\cdot)\|_2\right\} e^{-\frac{\alpha}{2}(t-t_0)},\ t \geq t_0,$$

where $\bar{M} = \frac{\sqrt{M}}{\sup_{s\in[t_0-\tau,t_0]}\left\{\|e(s,\cdot)\|_2\right\}} > 1$. Therefore, the proof is complete. □

Remark 6.2.11 *The main difference between this proof and proof of Theorem 6.2.1 lies in the estimation of (6.47). Without the condition on the relation between the delay size and the length of impulsive interval, the impulsive interval that time $t_i - \tau$ belongs to can not be exactly located. Therefore, different estimation techniques are used in (6.42) and (6.47), respectively.*

Chapter 7

Conclusions and Future Research

In the present thesis, we have investigated impulsive control problems of dynamical networks. Stabilization, consensus, and synchronization issues related to networks have been studied. Particular emphasis has been given to dynamical networks with time-delay and subject to delay-dependent impulsive effects. In this chapter, we highlight the contributions of this thesis, and suggest possible future research related to topics we have considered in the thesis.

7.1 Stability Analysis

In Chapter 3, we have considered time-delay systems with delay-dependent impulses. By using the method of Lyapunov functionals and Razumikhin technique, global exponential stability results have been obtained for general nonlinear time-delay systems with delayed impulses. An exponential stability result for a class of locally Lipschitz time-delay systems subject to distributed-delay dependent impulses has been established by the Razumikhin technique. Linear impulsive systems with time-delay have been investigated with numerical simulations to demonstrate our theoretical results.

In this chapter, we have focused on impulsive stabilization of time-delay systems, that is, the delay-dependent impulses stabilize the time-delay systems. However, impulse is a double-edge sword, i.e., it could also destroy the stability of a delay system or lead to poor performance. Therefore, future research could be done on stability analysis of systems subject to delay-dependent impulsive perturbations (see, for example [132]). For systems with distributed-delay dependent impulses, future research can be directed to establish stability criteria by using the method of Lyapunov functionals.

7.2 Impulsive Consensus

Chapter 4 has studied the consensus problem of multi-agent systems with both fixed and switching topologies. A hybrid consensus protocol has been proposed to take into consideration of continuous-time communications among agents and delayed instant information exchanges on a sequence of discrete times. Based on the proposed algorithms, the multi-agent systems with the hybrid consensus protocols are described in the form of impulsive systems or impulsive switching systems. By employing results from matrix theory and algebraic graph theory, some sufficient conditions for the consensus of multi-agent systems with fixed and switching topologies have been established, respectively. Our results show that, for small impulse delays, the hybrid consensus protocols can solve the consensus problem if the union of continuous-time and impulsive-time interaction digraphs contains a spanning tree frequently enough.

By taking into account of time-delay, a new type of hybrid impulsive consensus protocols with dynamically changing interaction topologies has been proposed. Sufficient conditions on the relation among network topologies, the delay size, and the length of impulsive interval have been established to guarantee the average-consensus via the proposed consensus protocols. It is worth noting that only the discrete-time delay has been considered in the impulsive consensus protocols, and the impulsive intervals have been assumed to have equal length in the theoretical analysis. However, for more general hybrid consensus protocols with time-variant delays and nonuniform impulsive intervals, sufficient conditions on average-consensus of the corresponding networked multi-agent systems can be established similarly, according to the theoretical method introduced in Section 4.3.

We have also investigated the impulsive consensus of networked multi-agent systems. An impulsive consensus protocol with distributed delays has been designed. By comparing the agent states at the impulse instant and the distributed-delayed states and applying a Razumikhin type stability result, we have obtained sufficient conditions under which the proposed consensus protocol leads to the network consensus. The sufficient conditions provide the relation among the length of each impulsive interval, the impulse delay size, and the graph Laplacians to guarantee the network consensus. Although only distributed delays have been considered in agent dynamics, the technique used in Section 4.4 is applicable to the scenario of agents with discrete and/or time-variant delays.

For the impulsive consensus protocols, Section 4.3 has assumed that all the impulse-time digraphs are balanced and strongly connected. However, it has been shown in [90] that, for the hybrid consensus protocol (4.13) without time-delay (i.e., $\bar{\tau} = r = 0$), if the union of graphs $\mathcal{G}_{\mathcal{A}}$ and $\mathcal{G}_{\mathcal{A}'}$ are balanced and strongly connected, the average-consensus can be guaranteed. Therefore, the results in [90] inspire us to generalize these results to the time-delay scenario.

7.3 Pinning Impulsive Control

Chapter 5 has incorporated a pinning algorithm with the impulsive control approach. We have introduced the dynamical networks on time scales and studied the synchronization problem of linear networks in Section 5.1. A pinning delayed-impulsive controller has been designed to achieve the synchronization of dynamical networks on time scales. Some sufficient conditions have been established which guarantee the synchronization of linear networks on time scales. In this section, we have focused on the theoretical analysis of synchronization of linear networks on time scales. However, dynamical networks on time scales have tremendous application potential. Actually, network (5.1) can be used to model the opinion formation process of a group of people over a specific working schedule which can be represented by a time scale. System (5.2) denotes the opinion evolution of a single person, and the connection topology describes how the people in the group communicate with each other. Then the synchronization can be explained as a common opinion formation among these people. As discussed in Remark 5.1.4, the structure of time scales will effect the synchronization process of a network. This implies that the opinion formation indeed depends on the communication and working schedule, which then helps to arrange a suitable working schedule for the opinion formation purpose. For the future work, we will study more practical applications of dynamical networks on time scales and the corresponding control problems.

Section 5.2 has studied impulsive stabilization problem of neural networks with time-delay. We have successfully applied the pinning impulsive controller proposed in Section 5.1 for the linear delay-free networks to stabilize the networks with time-delay. We have also proposed a pinning impulsive controller depending only on the network states at history moments which is different from the one designed in Section 5.1. It has been shown that the global exponential stabilization of delayed neural networks can be effectively realized by controlling a small portion of neurons in the networks via delayed impulses, and, for fixed impulsive control gain, increasing the impulse delay or decreasing the number of neurons to be pinned at the impulsive moments will lead to high frequency of impulses added the corresponding neurons. Numerical examples have been provided to illustrate the theoretical results, which demonstrate that our results are less conservative than the results reported in the existing literature when the proposed pinning controller reduces to the full-state impulsive controller.

We have investigated the synchronization of globally Lipschitz time-delay systems using impulsive control in Section 5.3. We have proposed a novel class of pinning impulsive controllers that takes into account of both discrete and distributed delays. Verifiable synchronization conditions for pinning impulsive controller with discrete delay, distributed delay and both of these two type delays have been established using a Halanay-type inequality, respectively. The theoretical results provide insight into the feasible relation between the impulse delays and impulse frequency to guarantee the synchronization of drive and response systems via impulsive control a small portion of the system states. The findings have been illustrated by stability analysis of a linear impulsive time-delay system and synchronization control of a nonlinear chaotic time-delay system with numerical simulations.

Throughout Chapters 5 and 6, one pinning algorithm has been considered, which implies that different units of a network (or states of a system) might be controlled at distinct impulsive instants. For networks with large amount of nodes, this pinning algorithm will lead to huge computational work when comparing the network states at each impulsive instants. For future research, it would be interesting and challenging to study sufficient conditions on network topology to guarantee the synchronization of networks by pinning the same nodes at the impulsive instants (see discussions with numerical simulations in Section 5.2 and Section 6.2). Furthermore, when individual networks are connected by means of additional links among them, networks of networks arise (see [56]). Then, it may be possible to extend our approach to synchronize this type of generalized networks.

7.4 Systems Governed by PDEs

In Chapter 6, the pinning impulsive controller considered in Chapter 5 has been successfully applied to stabilize and synchronize systems and networks modeled by PDEs. Impulsive control and synchronization of spatiotemporal chaos of Gray-Scott model, a system governed by delay-free PDEs, have been studied in Section 6.1. In Section 6.2, the exponential synchronization of reaction-diffusion neural networks with time-varying delays has been studied. A pinning impulsive control algorithm proposed for dynamical networks without time-delay has been successfully generalized to control neural networks with both reaction-diffusion terms and time-varying delays. In order to overcome the difficulty of utilizing this pinning algorithm to control networks with time-delay, a Lyapunov-Krasovskii functional with two parts (a function part and a functional part) has been constructed. The function part is chosen as a quadratic form to carry over the pinning algorithm in [67] to neural networks with time-delay and handle the impulsive effects. Two sets of sufficient conditions have been derived to design suitable pinning impulsive controllers to synchronize the delayed reaction-diffusion neural networks with small and large delay size, respectively.

When processing the impulsive information in the controller, it is natural and practical to consider the time-delay effects in the pinning impulsive controller as discussed in Chapter 5 for networks modeled by ODEs. Future work could be done on synchronization of networks governed by PDEs via pinning impulsive controller with delay effects.

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