

# On Perturbative Methods in Spectral Geometry

by

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A thesis  
presented to the University of Waterloo  
in fulfillment of the  
thesis requirement for the degree of  
Doctor of Philosophy  
in  
Applied Mathematics

Waterloo, Ontario, Canada, 2017

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## **Author's Declaration**

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

## Statement of Contributions

Most of the original results of this thesis were previously published in two papers [86, 87] co-authored with my supervisor Prof. Achim Kempf. I carried out all of the research leading to the publication of said papers and wrote their first drafts. Then, the papers were co-edited by me and my supervisor.

The results of [86] and [87] are reported in Chapter 3 and Section 5.3.2, respectively. In both cases, the presentation of the results is a significantly modified and expanded version of the one found in the papers.

Figures adapted from the aforementioned publications have the relevant papers cited in their captions.

## Abstract

The goal of spectral geometry is to establish how much information about the geometry of compact Riemannian manifolds is contained in the spectra of natural differential operators, especially Laplacians, defined on them. Ideally, one would like to be able to recover the Riemannian manifold, up to isometry, from the spectra of one or several such operators. This would be a very powerful result, as it would introduce an invariant way to describe the shape of Riemannian manifolds. The consequences of such a result would range from practical applications such as shape recognition to theoretical insights into quantum gravity.

However, the most general form of such statements is known to be false. There are a number of known counterexamples, that is isospectral but not isometric manifolds. Indeed, there are even techniques to construct such counterexamples. Nonetheless, it is believed that almost all Riemannian manifolds can be identified by their spectra. In other words, the counterexamples are expected to be exceedingly rare special cases. This has been shown to be the case in some restricted classes of manifolds. The proof in the general case has remained elusive.

The main goal of this thesis is to move towards such a proof by studying the structure of isospectral sets of metrics. The main tool we use for this purpose is perturbation theory, a method ubiquitous in physics, but strangely underused in spectral geometry. Consequently, a secondary goal of this work is to demonstrate the usefulness of perturbation theory to the study of spectral geometry. We begin by a numerical exploration of spectral geometry in a perturbative regime. Then, we show that sets of isospectral conformally equivalent metrics on boundaryless manifolds of dimension two contain no convex subsets. This is an entirely new type of result in spectral geometry. We argue that it could lead to a proof of the rarity of counterexamples in the program of identifying shapes by their spectra.

The thesis also includes reviews of the fundamentals of the spectral theory of Laplace-type operators, of major results in spectral geometry and of perturbation theory.

## Acknowledgements

I wish to thank my PhD supervisor Prof. Achim Kempf for the support and guidance he provided me during my studies.

I wish to thank the members of the Examining Committee, especially Profs. Spiro Karigiannis and Dmitry Jakobson, for their comments and suggestions.

I am grateful for the excellent courses given by Prof. Spiro Karigiannis, some of which I had the good fortune to follow.

I wish to thank my officemates, in no particular order, Robert Jonsson, Jason Pye, Daniel Hümmer, William Donnelly, David Layden, Guillaume Verdon-Akzam, Maria Pappageorgiou and Ding Jia for both useful discussions and great times.

I am grateful for the financial support I received from the Natural Sciences and Engineering Research Council (NSERC) Canada Postgraduate Scholarship (PGS) program.

The numerical experiments reported in this thesis were made possible by the facilities of the Shared Hierarchical Academic Research Computing Network (SHARCNET) and Compute/Calcul Canada.

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# Chapter 1

## Introduction

Consider the wave equation in some compact domain  $\Omega \subset \mathbb{R}^2$ ,

$$\frac{\partial^2}{\partial t^2}\psi + \Delta\psi = 0 . \tag{1.1}$$

This equation can be seen as the mathematical model for the vibration of the membrane of a drum. From usual separation of variables considerations [54], a general solution for this problem will involve solving the Laplacian eigenvalue problem:

$$\Delta\psi_n = \lambda_n\psi_n . \tag{1.2}$$

The eigenvalues  $\{\lambda_n\}_{n=0}^\infty$  depend on the geometry of  $\Omega$ , as well as on the boundary conditions imposed on  $\partial\Omega$ . In principle, one can always obtain the spectrum  $\{\lambda_n\}_{n=0}^\infty$  given  $\Omega$  and the boundary conditions.

One can then pose the inverse problem. Given the  $\{\lambda_n\}_{n=0}^\infty$ , can one find  $\Omega$ ? Since the eigenvalues  $\{\lambda_n\}_{n=0}^\infty$  represent the squares of the resonant frequencies of a drum, this question can be rephrased as “Can one hear the shape of a drum?” This formulation was popularized by Mark Kac’s paper of the same title [64] <sup>1</sup>.

More generally, analogous problems can be posed for the various differential operators that one can define on compact Riemannian manifolds. The study of the relationship between the spectra and the geometry of those manifolds is known as spectral geometry. By far, the favorite subject in spectral geometry is the spectrum of the Laplace-Beltrami

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<sup>1</sup>In [64], Kac attributes this picturesque phrasing to Lipman Bers.

operator, the generalization of the usual Laplacian from multivariate and vector calculus. Other operators, such as the Hodge Laplacian, are also commonly considered. The specific problem of recovering the shape of a manifold from spectrum alone is sometimes known as inverse spectral geometry. We will adopt this terminology.

The program of inverse spectral geometry makes some very tempting promises. Indeed, suppose that one is able to reconstruct Riemannian manifolds from the spectra of some Laplacians, up to isometry. This would provide a privileged description of shapes. Indeed, the eigenvalues are geometric invariants. This means that they depend on the Riemannian structure alone. This contrasts sharply with the usual descriptions of geometric shapes, which all depend on choices like coordinates or spatial orientations.

Having access to such methods would be of great benefit in a number of fields. For instance, in shape recognition, it has been proposed to use the Laplace-Beltrami spectrum as a shape identifier [94]. Inverse spectral methods could be used to design resonators for electromagnetic or acoustic waves. For example, inverse spectral ideas have been applied to the design of musical instruments [11].

The most tantalizing of the potential applications of inverse spectral geometry is the description of the shape of space-time in the context of (Euclidean) quantum gravity. A major hurdle to the formulation of a theory of quantum gravity lies in the fact that one does not have access to a unique description of possible configurations of space-time. It has been proposed that inverse spectral geometry could be the solution to that problem [69, 70]. Moreover, being at the interface of differential geometry and functional analysis, spectral geometry seems like a natural candidate for a mathematical framework unifying the very distinct worlds of general relativity and quantum theory. See [73, 37] for some applications of spectral geometric ideas to the study of theories of gravity.

There is, however, a problem with the inverse spectral geometric program. Indeed, there exist counterexamples to the reconstruction of shape from spectrum. In other words, there exist pairs, or even continuous families, of isospectral yet non-isometric manifolds. In fact, methods to construct such counterexamples have been devised [100]. Thus, in general, one cannot hear the shape of a drum [51]. Nonetheless, it is expected that the counterexamples are not generic. That is, it is believed that the manifolds that can indeed be determined from their spectra form a residual set in a natural topology<sup>2</sup>. This is known to hold on some special sets of manifolds [110, 111, 113]. In a general setting, however, this conjecture remains unproven.

It is thus of interest to gain a better understanding of the nature of isospectrality. This can be approached in two ways. The first is to understand the construction of coun-

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<sup>2</sup>Residual sets consist of generic points in a topological space. See Appendix A for a precise definition.

terexamples. The goal is to obtain a method that would produce all possible families of isospectral non-isometric manifolds. The second way is to study the properties of sets of isospectral manifolds. The present thesis adopts this second approach.

We choose a particular set of mathematical tools for our studies. In [69], it has been proposed that the main difficulty of spectral geometry lies in the high nonlinearity of the map between shape and spectrum. Consequently, we study the inverse spectral problem in the perturbative regime. Perturbation theory is a ubiquitous tool in theoretical physics, but is only rarely employed in spectral geometry (see [9, 12, 35] for exceptions). A secondary objective of this thesis is to demonstrate the usefulness of perturbation theory in spectral geometry.

We use explicit formulas for the first and second order eigenvalue perturbations in order to rule out the existence of certain types of families of isospectral conformally equivalent manifolds in dimension 2. We use this to obtain Theorems 5.10 and 5.12, the main results of this thesis. In brief, they state that isospectral sets of conformally equivalent Riemannian metrics in dimension 2 contain no convex subsets. To the best of our knowledge, this is the first result on the geometry (or linear structure) of isospectral sets. Previously known results dealt with their topology, namely their compactness [82, 18, 114].

We begin this thesis by a review of spectral geometry. Chapter 2 starts with the basic definitions of Laplace-type operators and covers their spectral theorem. Then, the heat and wave traces are defined and the geometric information that one can extract from them is discussed. This is followed by a review of positive and negative results in inverse spectral geometry. In particular, we review Sunada's method for constructing isospectral non-isometric manifolds. Finally, results regarding the compactness of sets of isospectral manifolds are surveyed.

In Chapter 3, we begin our exploration of perturbative approaches to spectral geometry. Specifically, we conduct a numerical experiment regarding the reconstruction of small changes in shape from small changes in spectrum. The success of this experiment provides motivation to pursue our perturbative program.

Chapter 4 is dedicated to the fundamentals of perturbation theory. The focus of the presentation is on the explicit formulas for eigenvalue corrections, rather than questions of existence and convergence of perturbation expansions. The eigenvalue corrections are first computed heuristically. A rigorous approach is then sketched, confirming the heuristic results.

In Chapter 5, we compute the corrections of the Laplace-Beltrami eigenvalues due to a perturbation of the Riemannian metric. Then, in Section 5.3, we use those formulas to establish the main result of this thesis: isospectral sets of conformally equivalent metrics

on surfaces contain no convex subsets. This is done in two steps. First, in Section 5.3.1, a proof strategy valid in a more general perturbative context is presented. Then, in Section 5.3.2, the strategy is specialized to the geometric case at hand and the main results are established.

We dedicate Chapter 6 to some observations and conjectures regarding integrals of products of Laplace-Beltrami eigenfunctions. First, we speculate on the nature of the space of functions spanned by the squares of the eigenfunctions. Then, we obtain an infinite family of formulas similar to the Rayleigh quotient and numerically explore its domain of definition.

Finally, in Chapter 7, we conclude this thesis by a discussion of our results and directions for future research.

Unless specified otherwise, all manifolds are assumed smooth, connected, compact, oriented and without boundary. We use Einstein's summation convention throughout the thesis.

# Chapter 2

## Laplacians and Spectral Geometry

This chapter is an introduction to the field of spectral geometry. The discussion assumes the reader to be familiar with the fundamentals of differential and Riemannian geometry. For that, as well as for some advanced notions, we refer the reader to [40, 81].

In Section 2.1, we begin by introducing the notion of Laplace-type operator. The major examples of such operators are defined. In Section 2.2, we state the spectral theorem for Laplace-type operators and discuss four ways of proving it. Then, we define the heat and wave kernels and their traces. The wave and heat traces are spectral invariants and provide the main tools for extracting geometric information from spectra. The information that can be extracted in this manner is reviewed in Section 2.3. Section 2.4 is a survey of major results, both positive and negative, in spectral geometry. Finally, Section 2.5 reviews the known results regarding the compactness of sets of isospectral manifolds.

### 2.1 Laplace-type Operators

Let  $(\mathcal{M}, g)$  be a Riemannian manifold of dimension  $N$ . A second order differential operator  $D$  defined on  $(\mathcal{M}, g)$  is said to be of Laplace type if, locally, it can be written as

$$D = -g^{ij} \partial_i \partial_j + \text{lower order terms.} \quad (2.1)$$

More precisely, let  $D$  act on  $\Gamma(E)$ , the space of smooth sections of a vector bundle  $E$  over  $\mathcal{M}$ . Then,  $D$  is of Laplace type if  $\mathcal{M}$  can be covered by coordinate neighborhoods that

trivialize  $E$  in which  $D$  takes the form in Equation (2.1). Notice that at the center of a Riemann normal coordinate system, the above expression becomes

$$D = - \sum_{i=1}^N \partial_i^2 + \text{lower order terms.} \quad (2.2)$$

Up to a sign, this indeed looks like the usual Laplacian from vector calculus. The presence of the negative sign is not fundamental. However, this choice turns out to be useful, as it ensures that the eigenvalues of Laplace-type operators are unbounded towards  $\infty$  rather than  $-\infty$ .

In this section, we define the most common Laplace-type operators, namely the Laplace-Beltrami operator, Hodge Laplacian and the Covariant Laplacian. For further details, see standard texts on differential geometry, say [81].

### Laplace-Beltrami Operator

The Laplace-Beltrami operator is the generalization to Riemannian manifolds of the usual Laplacian. This operator is the main object of study in spectral geometry in general, and in this thesis in particular.

Recall that, in vector calculus, the usual Laplacian on functions is defined as the divergence of the gradient.

$$\Delta f = \mathbf{div}(\mathbf{grad}(f)) = \sum_{i=1}^N \partial_i^2 f \quad (2.3)$$

Except for the sign convention, the Laplace-Beltrami operator is defined analogously. Let  $f \in C^\infty(\mathcal{M})$ . Then, its gradient is defined to be vector field that is the metric dual of its exterior derivative. That is,  $\mathbf{grad}(f) \in \Gamma(T\mathcal{M})$  is the unique vector field such that, for all  $X \in \Gamma(T\mathcal{M})$ ,

$$g(\mathbf{grad}(f), X) = df(X) = Xf \quad (2.4)$$

In local coordinates, it takes the form

$$\mathbf{grad}(f)^i = g^{ij} \partial_j f . \quad (2.5)$$

The definition of the divergence is somewhat more complicated. Let  $dV_g$  denote the volume form induced by  $g$  and let  $\mathcal{L}_X$  denote the Lie derivative in the direction of the vector field  $X$ . Then,  $\mathbf{div}X$  is the unique smooth function such that

$$\mathcal{L}_X dV_g = (\mathbf{div}X)dV_g . \quad (2.6)$$

In local coordinates, let  $\det(g)$  denote the determinant of  $g_{ij}$ . The local expression for the divergence of  $X$  then becomes

$$\mathbf{div}X = \frac{1}{\sqrt{|\det(g)|}} \partial_i \left( \sqrt{|\det(g)|} X^i \right) . \quad (2.7)$$

Putting all of the above together, the definition of the Laplace-Beltrami operator is obtained:

$$\Delta f = -\mathbf{div}(\mathbf{grad}(f)) = -\frac{1}{\sqrt{|\det(g)|}} \partial_i \left( \sqrt{|\det(g)|} g^{ij} \partial_j f \right) \quad (2.8)$$

The coordinate expression for the Laplace-Beltrami operator is a key tool for our investigations of its perturbations in Chapter 5.

It is straightforward to notice that the second order part of the above local expression is indeed of the form  $-g^{ij} \partial_i \partial_j$ . Thus, the Laplace-Beltrami operator is, as expected, of Laplace type.

## Hodge Laplacian

The Hodge Laplacian is a Laplace type operator acting on smooth differential forms. In terms of its importance to spectral geometry, it is a close second to the Laplace-Beltrami operator. In fact, the Hodge Laplacian operator on 0-forms coincides with the Laplace-Beltrami operator. While the main results of this thesis do not rely on the Hodge Laplacian, some of the results surveyed later in this chapter do. It is thus convenient to briefly discuss it here.

Let  $\Omega^p(\mathcal{M})$  denote the space of smooth sections of  $\Lambda^p(T^*\mathcal{M})$ .  $\Omega^p(\mathcal{M})$  is said to be the space of smooth  $p$ -forms of  $\mathcal{M}$ . The Hodge Laplacian defined below is a map  $\Delta : \Omega^p(\mathcal{M}) \rightarrow \Omega^p(\mathcal{M})$ . Nonetheless, it is convenient to discuss it on the entirety of the exterior algebra:



$$\Omega^\bullet(\mathcal{M}) = \bigoplus_{p=0}^N \Omega^p(\mathcal{M}) . \quad (2.9)$$

Recall that the exterior derivative is an anti-derivation  $d : \Omega^p(\mathcal{M}) \rightarrow \Omega^{p+1}(\mathcal{M})$ . Recall that the Riemannian metric  $g$  induces a  $L_2$  inner product on  $\Omega^p(\mathcal{M})$  and the Hodge star operator  $*$  :  $\Omega^p(\mathcal{M}) \rightarrow \Omega^{N-p}(\mathcal{M})$ . It is well-known that the formal dual  $\delta : \Omega^{p+1}(\mathcal{M}) \rightarrow \Omega^p(\mathcal{M})$  of  $d$  is

$$\delta = (-1)^{Np+N+1} * d * . \quad (2.10)$$

The Hodge Laplacian is then defined to be

$$\Delta = \delta d + d \delta . \quad (2.11)$$

It can be shown (see [81]) that this reduces to the Laplace-Beltrami operator on  $\Omega^0(\mathcal{M}) = C^\infty(\mathcal{M})$ .

Depending on the context, the Hodge Laplacian can be studied as acting on either one of the  $\Omega^p(\mathcal{M})$  or on all of the exterior algebra  $\Omega^\bullet(\mathcal{M})$ .

Showing that the Hodge Laplacian is of Laplace type is quite a bit more involved than in the case of the Laplace-Beltrami operator and will thus not be discussed here. The interested reader can consult [81].

## Connection Laplacian and the Weitzenböck Remainder

The connection Laplacian is in a sense the most general form of Laplacian, in a sense specified by Theorem 2.1 below. This can be used to show properties common to all such operators, see [14] for an example of such an approach. For the details of the construction sketched below, see [81].

Let  $E$  be a smooth vector bundle over  $\mathcal{M}$  equipped with a linear connection  $\nabla$ . Recall that it is a linear map  $\nabla : \Gamma(E) \rightarrow \Gamma(T^*(\mathcal{M}) \otimes E)$  satisfying  $\nabla(fu) = df \otimes u + f\nabla u$  for all  $f \in C^\infty(\mathcal{M})$  and  $u \in \Gamma(E)$ . Let  $E$  be equipped with a fiber metric, that is a fiberwise positive definite inner product. Together with the Riemannian metric  $g$ , this induces a fiber metric on  $T^*(\mathcal{M}) \otimes E$ . Integrating those inner products with respect to the volume form  $dV_g$  yields  $L_2$  inner products on sections of  $E$  and  $T^*(\mathcal{M}) \otimes E$ . Using those  $L_2$  inner

products one can define a formal adjoint to  $\nabla$ , denoted  $\nabla^*$ . Then, the connection Laplacian  $\Delta : \Gamma(E) \rightarrow \Gamma(E)$  is defined as

$$\Delta = \nabla^* \nabla . \tag{2.12}$$

It is no coincidence that this looks like the divergence of the gradient definition of the Laplace-Beltrami operator. Indeed, the connection  $\nabla$  and its adjoint  $\nabla^*$  are the appropriate replacements for the gradient and divergence operators, respectively. Note, however, that they are not *generalizations* of the gradient and divergence operators. As an illustration of this, consider the trivial connection on  $C^\infty(\mathcal{M})$ . There, the connection is simply the exterior derivative  $d$ . Its dual is, of course,  $\delta$ . Together, this yields the Hodge Laplacian on 0-forms, which is the same as the Laplace-Beltrami operator. Nonetheless,  $d \neq \mathbf{grad}$  and  $\delta \neq \mathbf{div}$ .

The position of the connection Laplacian as the general form of Laplace type operator is established by the following theorem.

**Theorem 2.1** (Weitzenböck Remainder). *Let  $E$  be a smooth vector bundle over  $(\mathcal{M}, g)$  equipped with a fiber metric and let  $L : \Gamma(E) \rightarrow \Gamma(E)$  be a Laplace-type operator on  $E$ . Then, there exists a unique metric-compatible connection  $\nabla$  on  $E$  as well as a unique bundle endomorphism  $\mathcal{R} : E \rightarrow E$  such that*

$$L = \nabla^* \nabla + \mathcal{R} , \tag{2.13}$$

where the adjoint is taken with respect to the  $L_2$  inner product induced by  $g$  and the fiber metric on  $E$ .  $\mathcal{R}$  is known as the Weitzenböck remainder.

*Proof.* See [81]. □

The above theorem gives a complete description of an arbitrary Laplace-type operator. It thus can be used to obtain general statements about all such operators.

## 2.2 Spectral Theorem and Spectral Invariants

The first goal of this section is to state the spectral theorem for Laplace-type operators. Without this theorem, the program of reconstructing shape from spectrum would fail to make sense, as there would be no spectrum to speak of. The second goal is to introduce

the heat and wave traces, two important tools in spectral geometry. These objects depend only on the spectrum; they are *spectral invariants*. As will be discussed later this chapter, the study of the asymptotics of these invariants is the main way to extract geometric information from the spectrum.

### 2.2.1 Spectral Theorem for Laplace-Type Operators

Let  $(\mathcal{M}, g)$  denote an oriented compact Riemannian manifold without boundary and let  $E$  be a vector bundle over it equipped with a fiber metric. Then, the following theorem holds.

**Theorem 2.2** (Laplace-Type Spectral Theorem). *Let  $L : \Gamma(E) \rightarrow \Gamma(E)$  be a symmetric<sup>1</sup> Laplace-type operator. Then  $L$  has a spectrum consisting of discrete eigenvalues  $\{\lambda_n\}_{n=0}^{\infty}$  corresponding to eigensections  $\{\psi_n\}_{n=0}^{\infty}$  with the following properties:*

- (i) *The eigensections  $\{\psi_n\}_{n=0}^{\infty}$  form an orthonormal basis for  $L_2(E)$ .*
- (ii) *The eigenvalues  $\{\lambda_n\}_{n=0}^{\infty}$  are real.*
- (iii) *The eigenspaces of  $L$  have finite dimension.*
- (iv) *The spectrum has no finite accumulation point.*
- (v) *The eigensections  $\{\psi_n\}_{n=0}^{\infty}$  are smooth.*

There exists a multitude of ways of proving this theorem, especially the less general versions dealing with specific Laplacians, rather than all symmetric Laplace-type operators. We do not pursue a rigorous proof of it here. Instead, we briefly sketch three ways one can approach it. A fourth approach is also mentioned. References containing the full proofs are provided.

*Proof 1: Compact Resolvent.* The first approach relies on the spectral theorem for compact normal operators (see [79] for the theorem). Since Laplace-type operators are unbounded, this theorem cannot be directly applied to them. However, for  $\lambda$  in the resolvent set of  $L$ ,  $R(\lambda, L) = (\lambda - L)^{-1}$  can be shown to be compact [74]. This can be done by showing that  $R(\lambda, L)$  can be written as an integral operator with a sufficiently well-behaved integral kernel. Then, the spectrum of  $R(\lambda, L)$  is  $\{(\lambda - \lambda_n)^{-1}\}_{n=0}^{\infty}$  and the eigensections are the same

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<sup>1</sup>See Appendix B for a discussion of the difference between symmetric and self-adjoint operators.

as those of  $L$ . In sum, the properties (i) through (iv) of the theorem directly follow from the corresponding properties of the eigenvalues and eigenvectors of compact self-adjoint operators, with property (iv) following from the fact that the spectrum of a compact self-adjoint operator can only have  $\lambda = 0$  as an accumulation point.

Property (v), however, cannot be deduced from a standard spectral theorem. An additional ingredient is needed to be able to discuss differentiability. That ingredient is the notion of elliptic regularity. Elliptic regularity is a property of elliptic operators, a category in which all Laplace-type operators fall. Elliptic regularity states that the equation  $Du = v$  only has smooth solutions  $u$  if  $D$  is elliptic and  $v$  smooth [81]. In particular, this means that the eigenvalue equation  $(D - \lambda_n \mathbb{1})\psi_n = 0$  only admits smooth solutions, as 0 is a smooth section. The proof of elliptic regularity requires significant machinery that we do not wish to introduce here. The interested reader is invited to consult the excellent [74], as well as other sources [81, 42]. The author's Master's thesis contains an overview of the subject [85].  $\square$

*Proof 2: Heat Equation.* An alternative proof strategy is to study solutions of the following equation on  $\mathcal{M} \times (0, \infty)$ :

$$\frac{\partial}{\partial t}\psi(x, t) + L\psi(x, t) = 0 . \quad (2.14)$$

This is known as the heat equation for  $L$ . For an initial condition  $\psi(x, 0)$ , solutions  $\psi(x, t)$  can be written as

$$\psi(x, t) = e^{-tL}\psi(x, 0) . \quad (2.15)$$

Here,  $e^{-tL}$  is known as the heat operator of  $L$ . Let  $dV_g(y)$  denote the volume form induced by  $g$  at  $y \in \mathcal{M}$ . Let  $E_x$  and  $E_y$  denote the fibers of  $E$  over  $x \in \mathcal{M}$  and  $y \in \mathcal{M}$ , respectively. The heat operator can be shown to be of the form

$$e^{-tL}\psi = \int_{\mathcal{M}} h(x, y, t)\psi(y)dV_g(y) , \quad (2.16)$$

where  $h(x, y, t) : E_y \rightarrow E_x$  is a linear map. Moreover,  $h(x, y, t)$  is smooth on  $\mathcal{M} \times \mathcal{M} \times (0, \infty)$ . The quantity  $h(x, y, t)$  is known as the *heat kernel* of  $L$ . We will return to this important object shortly. In particular, certain proofs of its existence yield some of the best known results of spectral geometry.

The heat operator  $e^{-tL}$  can be shown to be compact and self-adjoint on  $L_2(E)$ . By the standard spectral theorem for such operators [79], properties (i) through (iv) follow. The spectrum of  $e^{-tL}$  is  $\{e^{-t\lambda_n}\}_{n=0}^{\infty}$  and the eigensections are the same as those of  $L$ .

Property (v) can be deduced from the smoothness of  $h(x, y, t)$ . Formally, for  $f \in L_2(E)$  and  $m \in \mathbb{N}$ ,

$$(\partial_i)^m e^{-tL} f = \int_{\mathcal{M}} (\partial_i^m h(x, y, t)) f(y) dV_g(y) . \quad (2.17)$$

This argument can be made rigorous, which we do not pursue here. Thus,  $e^{-tL} f$  has partial derivatives of all orders. This implies that the eigensections of  $e^{-tL}$  are smooth. As those eigensections match those of  $L$ , property (v) is proven.

A complete proof using this approach can be found in [96] for the Laplace-Beltrami operator and Hodge Laplacian. The Laplace-Beltrami case is also treated in [25].  $\square$

*Proof 3: Weak Formulation.* Another proof strategy is to replace the eigenvalue problem by a bilinear form, which is known as the *weak formulation* of the problem. For  $f \in L_2(E)$ , consider the following problem:

$$L\psi = f . \quad (2.18)$$

Let  $\phi$  be an arbitrary test section of  $E$ . The precise space in which it lies will be specified below. Let  $(\cdot, \cdot)$  denote the appropriate fiberwise inner products on  $E$  and  $E \otimes T^*\mathcal{M}$ . Then, one can take the  $L_2(E)$  inner product of the problem with  $\phi$ .

$$\int_{\mathcal{M}} (\phi, L\psi) dV_g = \int_{\mathcal{M}} (\phi, f) dV_g \quad (2.19)$$

Express  $L = \nabla^* \nabla + \mathcal{R}$ , as dictated by Theorem 2.1. Note that  $\mathcal{R}$  is a symmetric operator since  $L$  and  $\nabla^* \nabla$  are symmetric. Then, the expression can be rewritten as

$$\int_{\mathcal{M}} (\nabla \phi, \nabla \psi) dV_g + \int_{\mathcal{M}} (\phi, \mathcal{R}\psi) dV_g = \int_{\mathcal{M}} (\phi, f) dV_g . \quad (2.20)$$

Require that this expression holds for all  $\phi$  in an appropriate Hilbert space (see below). This is said to be the weak formulation of Equation (2.18). Notice that the left-hand side of the equation is a bilinear form, while the right-hand side is a linear functional. Denote

the left hand side  $W(\psi, \phi)$ . Consider the above equation on the Sobolev space  $H^1(E)$ . This space can be defined as the completion of  $\Gamma(E)$  with respect to the norm induced by the following inner product:

$$\langle \phi, \psi \rangle_{H^1} = \int_{\mathcal{M}} (\phi, \psi) dV_g + \int_{\mathcal{M}} (\nabla \phi, \nabla \psi) dV_g . \quad (2.21)$$

$H^1(E)$  is a Hilbert space. In that space, one can show that  $W(\psi, \phi)$  is symmetric and continuous ( $\exists C > 0$  such that  $|W(\psi, \phi)| \leq C \|\psi\|_{H^1} \|\phi\|_{H^1}$ ). Furthermore, assume that  $W(\psi, \phi)$  is coercive ( $\exists \alpha > 0$  such that  $W(\psi, \psi) \geq \alpha \|\psi\|_{H^1}^2$ ). This last assumption is a condition on  $\mathcal{R}$ . It is sufficient that  $(\phi, \mathcal{R}\psi) \geq 0$  for all  $\psi$  and  $\phi$ . In particular,  $\mathcal{R} = 0$  yields the desired result. Note that  $\mathcal{R} = 0$  is satisfied in the case of the Laplace-Beltrami operator. This is the only loss of generality in the present proof.

Under the assumptions of continuity and coercivity, the Lax-Milgram theorem (see [15]) tells that the weak formulation of Equation (2.20) has a unique solution  $\psi \in H^1(E)$ . This fact can be used to define the following operator:

$$\begin{aligned} T : L_2(E) &\rightarrow H^1(E) \\ f &\mapsto \psi . \end{aligned} \quad (2.22)$$

$T$  can be shown to be linear, symmetric and bounded. Since the injection  $H^1(E) \hookrightarrow L_2(E)$  is compact [15, 81, 74, 42],  $T$  can be seen as a compact self-adjoint operator  $T : L_2(E) \rightarrow L_2(E)$ . By the spectral theorem for compact self-adjoint operators (see [79]),  $T$  has a basis of orthonormal eigensections  $\{\psi_n\}_{n=0}^{\infty}$  corresponding to eigenvalues  $\{\mu_n\}_{n=0}^{\infty}$ . Applying the weak formulation to the case  $f = \psi_n$  yields

$$W(\phi, T\psi_n) = \int_{\mathcal{M}} (\phi, \psi_n) dV_g , \quad \forall \phi \in H^1(E). \quad (2.23)$$

In other words,

$$W(\phi, \psi_n) = \begin{cases} \frac{1}{\mu_n} \int_{\mathcal{M}} (\phi, \psi_n) dV_g & , \mu_n \neq 0 \\ 0 & , \mu_n = 0 \end{cases} , \quad \forall \phi \in H^1(E). \quad (2.24)$$

Consequently, the eigenbasis  $\{\psi_n\}_{n=0}^{\infty}$  is the weak eigenbasis of  $L$  with eigenvalues given by

$$\lambda_n = \begin{cases} \frac{1}{\mu_n} & , \mu_n \neq 0 \\ 0 & , \mu_n = 0 . \end{cases} \quad (2.25)$$

This establishes properties (i) to (iv). In order to show that the eigensections  $\{\psi_n\}_{n=0}^\infty$  are indeed smooth, and thus that they are strong solutions to the original problem, one needs to invoke elliptic regularity, as before. That established property (v).

A complete version of this proof for the case of the Laplace-Beltrami operator on domains in  $\mathbb{R}^n$  can be found in [15].  $\square$

*Proof 4: Pseudo-Differential Operators.* This last approach discussed here is the deepest and most complex one of the four. Indeed, it is too complex to satisfactorily sketch here. This approach is best suited to prove the analogous theorem for all elliptic differential operators and then specialize it to Laplace-type operators. The key advantage of this approach is the generality of the obtained results.

We strongly encourage the interested reader to consult Chapter III of [74]. It is a largely self-contained and quite clear introduction to the subject. A more general, but harder to follow treatment can be found in Chapter 1 of [42]. It makes for an excellent reference. Chapter 2 of the author's Master's thesis [85] can also be used as an introduction to the subject, combining elements of the above texts with some alternative viewpoints found in [81]. Note that the approach in [81] is comparable to that in [74] and [42], but never introduces pseudo-differential operators, preferring to work with differential operators instead. It can thus be considered somewhat simpler.  $\square$

The above results also holds in the case where the boundary  $\partial\mathcal{M}$  of  $\mathcal{M}$  is nonempty, provided that the boundary is sufficiently smooth and that appropriate boundary conditions are imposed. For our purposes, it is sufficient to know that the spectral theorem holds for manifolds with smooth boundary equipped with Dirichlet or Neumann boundary conditions. Recall that Dirichlet boundary conditions require the sections of  $E$  to vanish on  $\partial\mathcal{M}$ , while Neumann boundary conditions require the same of their normal derivative at the boundary. See [42] for a proof of this statement. Manifolds with less common boundary conditions are studied in [43]. A simple treatment of the Dirichlet problem for the Laplace-Beltrami operator on a manifold with boundary can be found in [25]. A reader interested in the importance of boundary conditions is invited to consult [2] on the subject of self-adjoint extensions of unbounded, densely defined, symmetric operators.

## 2.2.2 Heat Kernel and Heat Trace

The notion of heat kernel has already been mentioned in the second proof of Theorem 2.2. There, it was defined as family of linear maps  $h(x, y, t) : E_y \rightarrow E_x$  smooth on  $\mathcal{M} \times \mathcal{M} \times (0, \infty)$  that expresses  $e^{-tL}$  as an integral operator:

$$(e^{-tL}\psi)(x) = \int_{\mathcal{M}} h(x, y, t)\psi(y)dV_g(y) . \quad (2.26)$$

Once one has access to the spectral theorem for Laplace-type operators, the heat kernel can be expressed as

$$h(x, y, t) = \sum_{n=0}^{\infty} e^{-t\lambda_n}\psi_n(x) \otimes \psi_n^*(y) . \quad (2.27)$$

Let  $f \in \Gamma(E)$ . In order to see that the heat kernel in Equation (2.27) indeed yields the desired result, we begin by writing  $f = \sum_{i=0}^{\infty} c_i\psi_i$ . One can then compute the action of  $e^{-tL}$  on  $f$ .

$$\begin{aligned} (e^{-tL}f)(x) &= \int_{\mathcal{M}} \left( \sum_{n=0}^{\infty} e^{-t\lambda_n}\psi_n(x) \otimes \psi_n^*(y) \sum_{i=0}^{\infty} c_i\psi_i(y) \right) dV_g(y) \\ &= \sum_{n=0}^{\infty} e^{-t\lambda_n}\psi_n(x) \sum_{i=0}^{\infty} c_i \int_{\mathcal{M}} (\psi_n(y), \psi_i(y))dV_g(y) \\ &= \sum_{n=0}^{\infty} e^{-t\lambda_n}\psi_n(x) \sum_{i=0}^{\infty} c_i \langle \psi_n, \psi_i \rangle_{L_2} \\ &= \sum_{n=0}^{\infty} e^{-t\lambda_n}c_n\psi_n(x) \end{aligned} \quad (2.28)$$

This is precisely the expected result. The expression of the heat kernel of Equation (2.27) is thus valid.

The convergence of the series for  $h(x, y, t)$  can be argued as follows.  $L$  can be shown to be bounded below [42]. Then, one can order the eigenvalues such that  $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots$ . In this ordering, it can be shown that  $\lambda_n$  grows fast enough with  $n$  to ensure the convergence of the series [42, 74].



This construction of the heat kernel is unavailable for the second proof of Theorem 2.2, as it uses the existence of the heat kernel. Consequently, one has to obtain the heat kernel through other means. When the studied operator is known, rather than abstract, the heat kernel can be computed explicitly on certain manifolds. Especially useful is the kernel on  $\mathbb{R}^N$  equipped with the standard euclidean metric. Indeed, in Riemann normal coordinates around a point, it can be used as a first approximation of the heat kernel on a Riemannian manifold.

Let  $L_x$  denote the operator  $L$  with partial derivatives taken with respect to the  $x$  variable. The computation of its heat kernel on  $\mathbb{R}^N$  relies on the following properties of the heat kernel:

$$\begin{aligned} \left(\frac{\partial}{\partial t} + L_x\right)h(x, y, t) &= 0 \\ \lim_{t \rightarrow 0^+} \int_{\mathcal{M}} h(x, y, t)\psi(x)dV_g(x) &= \psi(y) . \end{aligned} \tag{2.29}$$

This establishes the heat kernel as the *fundamental solution* or *Green's function* of the heat equation for  $L$ . The second property can be rewritten as

$$\lim_{t \rightarrow 0^+} h(x, y, t) = \delta_y(x)\mathbb{1}_E , \tag{2.30}$$

where  $\delta_y(x)$  is the Dirac delta function centered at  $y \in \mathcal{M}$  and  $\mathbb{1}_E$  is the identity operator on  $E$ . This form enables one to use standard Fourier transform techniques to solve this partial differential equation on  $\mathbb{R}^N$ . For instance, the heat kernel of the Laplace-Beltrami operator on  $\mathbb{R}^N$  is given by

$$h(x, y, t) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{1}{4t}\|x-y\|^2} . \tag{2.31}$$

The heat kernel on a Riemannian manifold  $(\mathcal{M}, g)$  is then obtained by successive approximations with the euclidean heat kernel used as the first approximation. The computation is quite technical and will not be presented here. For the Laplace-Beltrami operator, the construction is carried out in [25] and [96]. See also [43] for the general case of Laplace-type operators on manifolds with or without boundary. That last text treats a variety of unusual boundary conditions.

Roughly speaking, one considers  $x$  to be in a small neighborhood of a fixed  $y \in \mathcal{M}$ . In other words, the computations are valid near the diagonal  $x = y$ . In that neighborhood,

it is assumed that the heat kernel is a power series in  $t$  with coefficients  $u_k(x, y)$  and the euclidean heat kernel as the leading term. Requiring that the power series satisfies the heat equation yields equations for all the coefficients. Once solved, a series expression for  $h(x, y, t)$  near the diagonal is obtained. This series can be used to argue the existence of  $h(x, y, t)$  globally, but doesn't directly provide a global expression for it. This is of course sufficient to complete the second proof of Theorem 2.2.

Once the existence of the heat kernel has been established, one can define a new quantity  $H(t)$ , called the *heat trace* of  $L$ . While the full heat kernel  $h(x, y, t)$  is a family of linear maps  $E_y \rightarrow E_x$ , the diagonal  $h(x, x, t)$  of the heat kernel is a pointwise linear map  $E_x \rightarrow E_x$ . Consequently, one can define the pointwise trace  $\text{Tr}(h(x, x, t))$ . Then, the heat trace is defined to be

$$H(t) = \int_M \text{Tr}(h(x, x, t)) dV_g(x) . \quad (2.32)$$

Essentially, this is the  $L_2(E)$  trace of  $h(x, x, t)$ . From Equation (2.27), one deduces

$$H(t) = \sum_{n=0}^{\infty} e^{-t\lambda_n} . \quad (2.33)$$

Once again, the convergence of the series on  $(0, \infty)$  is ensured by eigenvalue growth estimates [42, 74]. Notice that  $H(t)$  depends solely on the spectrum  $\{\lambda_n\}_{n=0}^{\infty}$ ; it is a spectral invariant. As will be discussed in Section 2.3, this quantity is our primary tool for the extraction of geometric information from the spectrum.

Before proceeding further, it is worthwhile to note that, for a nonnegative  $L$ , knowledge of  $H(t)$  is equivalent to knowledge of the spectrum. Manifestly, one can use the spectrum to compute  $H(t)$ . The other way is more involved. Consider the following expression:

$$\int_0^{\infty} \left( e^{-t\lambda} \sum_{n=0}^{\infty} \delta(\lambda - \lambda_n) \right) d\lambda = H(t) = \sum_{n=0}^{\infty} e^{-t\lambda_n} . \quad (2.34)$$

Thus,  $H(t)$  is the Laplace transform of a comb of Dirac deltas centered at the eigenvalues of  $L$ . When an eigenvalue is  $m$ -fold degenerate, the corresponding Dirac delta is multiplied by  $m$ . Consequently, one can obtain a representation of the spectrum as a Dirac comb by taking the inverse Laplace transform of  $H(t)$ . See standard texts, such as [54], for the theory of Laplace transforms.

### 2.2.3 Wave Kernel and Wave Trace

By analogy with the heat equation of  $L$ , one can study the wave equation of  $L$ .

$$\frac{\partial^2}{\partial t^2} \psi(x, t) + L\psi(x, t) = 0 . \quad (2.35)$$

As the wave equation is second order in time, the initial condition consists of a wave configuration  $\psi(x, 0)$  and a time derivative  $\dot{\psi}(x, 0)$ . Using a standard separation of variables approach [54], one assumes the solution has the form

$$\psi(x, t) = (a \cos(\omega t) + b \sin(\omega t)) \phi(x) . \quad (2.36)$$

for some constants  $a$  and  $b$ . Using this in the wave equation (Equation (2.35)) yields the following eigenvalue problem

$$L\phi(x) = \omega^2 \phi(x) . \quad (2.37)$$

This is, of course, the eigenvalue problem for  $L$  with  $\omega = \sqrt{\lambda_n}$  and  $\phi(x) = \psi_n(x)$ . Consequently, the general solution can be expressed as

$$\psi(x, t) = \sum_{n=0}^{\infty} (a_n \cos(\sqrt{\lambda_n} t) + b_n \sin(\sqrt{\lambda_n} t)) \psi_n(x) . \quad (2.38)$$

for coefficients  $\{a_n\}_{n=0}^{\infty}$  and  $\{b_n\}_{n=0}^{\infty}$  prescribed by the initial conditions. In terms of the initial conditions, the coefficients are determined by the expansions of  $\psi(x, 0)$  and  $\dot{\psi}(x, 0)$ :

$$\begin{aligned} \psi(x, 0) &= \sum_{n=0}^{\infty} a_n \psi_n(x) \\ \dot{\psi}(x, 0) &= \sum_{n=0}^{\infty} b_n \sqrt{\lambda_n} \psi_n(x) . \end{aligned} \quad (2.39)$$

Unlike the solution of the heat equation, the general solution of the wave equation cannot be written in terms of an operator  $L_2(E) \rightarrow L_2(E)$ . This is simply due to the fact that one needs two copies of  $L_2(E)$  to contain both the initial conditions. Consequently, one can build an operator  $L_2(E) \times L_2(E) \rightarrow L_2(E)$  sending the initial condition into the solution

at  $t > 0$ . This is not convenient. Instead, one assumes  $\dot{\psi}(x, 0) = 0$ . This reduces the initial condition and the solution to

$$\begin{aligned}\psi(x, 0) &= \sum_{n=0}^{\infty} a_n \psi_n(x) \\ \psi(x, t) &= \sum_{n=0}^{\infty} a_n \cos(\sqrt{\lambda_n t}) \psi_n(x) .\end{aligned}\tag{2.40}$$

This is analogous to the relationship between the initial condition and general solution for the heat equation. One then defines the wave kernel  $w(x, y, t)$  and wave operator  $\cos(\sqrt{L}t)$ :

$$\begin{aligned}w(x, y, t) &= \sum_{n=0}^{\infty} \cos(\sqrt{\lambda_n t}) \psi_n(x) \otimes \psi_n^*(y) \\ \cos(\sqrt{L}t)\phi &= \int_{\mathcal{M}} w(x, y, t) \phi(y) dV_g(y) .\end{aligned}\tag{2.41}$$

The  $L_2(E)$  trace of the wave operator yields the wave kernel:

$$W(t) = \sum_{n=0}^{\infty} \cos(\sqrt{\lambda_n t}) .\tag{2.42}$$

Just like the heat trace, this is a spectral invariant. For nonnegative  $L$ , it can be written as

$$W(t) = \int_{-\infty}^{\infty} e^{-i\omega t} \left( \sum_{n=0}^{\infty} \frac{1}{2} \left( \delta(\omega + \sqrt{\lambda_n}) + \delta(\omega - \sqrt{\lambda_n}) \right) \right) d\omega\tag{2.43}$$

This is a Fourier transform. Consequently, the spectrum of  $L$ , represented as a comb of Dirac deltas, can be recovered from  $W(t)$  via an inverse Fourier transform. Just like in the case of the heat trace, knowledge of the wave trace is equivalent to that of the spectrum.

Unlike the heat trace, the wave trace is not smooth. Indeed, it is a distribution rather than a function. This doesn't reduce its usefulness.

As a final note on the wave operator, it is useful to discuss the intuition behind the initial condition  $\dot{\psi}(x, 0) = 0$ . From a physical standpoint, this seems like a significant loss of generality. However, we are not interested in the physics of waves, but in the geometry

of the underlying manifold  $(\mathcal{M}, g)$ . The purpose of the wave equation in this context is to probe the geometry via the motion of waves. The velocity given by the initial condition interferes with this program, as one would need to disentangle its effects from those of the geometry. In sum, the initial velocity is of no interest for our purposes.

## 2.3 Geometric Information in Spectral Invariants

The heat and wave traces introduced above are the main tools used to extract geometric information from the spectrum of Laplace-type operators. This is achieved through their relationship to the corresponding kernels, which, in turn, are related to the operator under study. Since Laplace-type operators have the Riemannian metric as the coefficient in their leading term (see Equation (2.1)), it is not surprising that their traces contain geometric information. We focus our attention on the heat trace, as the wave trace is a much more complicated object. We briefly discuss it at the end of the section.

Consider the heat trace. While  $H(t)$  is smooth on  $(0, \infty)$ ,  $\lim_{t \rightarrow 0} H(t) = \infty$ . One can obtain an asymptotic expansion as  $t \rightarrow 0$ .

$$H(t) \sim \frac{1}{(4\pi t)^{N/2}} \sum_{n=0}^{\infty} a_n t^{n/2} \quad (2.44)$$

The coefficients  $a_n$  are computed in terms of the coefficients  $u_n(x, y)$  of the local expansion of  $h(x, y, t)$  near the diagonal  $x = y$ . Recall that these coefficients are used in the proof of the existence of  $h(x, y, t)$ . Since they depend solely on the spectrum, the coefficients  $a_n$  are often termed the heat invariants of  $L$ .

The coefficients  $a_n$  are influenced by two distinct sources: the interior of the manifold and the boundary. The interior geometry only contributes to  $a_n$  with  $n$  even. That is, in the case of a manifold without boundary, only integer powers of  $t$  will appear in the asymptotic expansion. Terms due to the boundary conditions, however, generically influence all of the  $a_n$ . See [43] for the details.

Since asymptotic expansions are unique, one can obtain the  $\{a_n\}_{n=0}^{\infty}$  from knowledge of the spectrum alone. This allows one to extract the geometric information contained in these coefficients. Depending on the precise operator under study, the information one can extract is slightly different. We concentrate our attention on the Laplace-Beltrami operator. This is done for simplicity's sake. The asymptotics of the heat trace of general Laplace-type operators are treated in [41]. As it turns out, the formulas for the coefficients

$a_0$ ,  $a_1$  and  $a_2$  for other Laplace-type operators are similar to those for the Laplace-Beltrami operator.

For the Laplace-Beltrami operator on a manifold without boundary the first two non-vanishing coefficients in the asymptotic expansion of the heat trace are

$$\begin{aligned} a_0 &= \int_{\mathcal{M}} dV_g = \text{Vol}(\mathcal{M}) \\ a_2 &= \int_{\mathcal{M}} \frac{1}{6} R dV_g . \end{aligned} \tag{2.45}$$

where  $\text{Vol}(\mathcal{M})$  is the volume of  $(\mathcal{M}, g)$  and  $R$  is the scalar curvature. Proofs of these expressions can be found in [25, 96]. A relatively simple heuristic computation can also be found in [78].

When  $\mathcal{M}$  has a boundary equipped with Dirichlet or Neumann boundary conditions, it turns out that the coefficients  $a_0$  and  $a_2$  remain unchanged. In terms of  $\text{Vol}(\partial\mathcal{M})$ , the volume of the boundary, the coefficient  $a_1$  is given by

$$a_1 = \begin{cases} -\frac{1}{4}\text{Vol}(\partial\mathcal{M}) & , \text{ Dirichlet boundary conditions} \\ \frac{1}{4}\text{Vol}(\partial\mathcal{M}) & , \text{ Neumann boundary conditions.} \end{cases} \tag{2.46}$$

A heuristic computation of  $a_1$  for the case of a domain in  $\mathbb{R}^3$  can be found in [5]. Starting from  $a_3$  onwards, the expressions for the coefficients become increasingly more complicated and harder to interpret. The situation becomes even worse with general Laplace-type operators and more complicated boundary conditions. This situation is best illustrated by the formula for  $a_5$  (Theorem 3.6.5 of [43]). It is worth the look.

The complexity of the expressions for the heat invariants leads one to ask whether there exists a more compact way to write them. This is indeed the case. In [90], an alternative expression for the coefficients  $u_n(x)$  of the asymptotic expansion of the diagonal  $h(x, x, t)$  of the heat kernel is obtained. To compute the heat invariants it remains to integrate those coefficients over the volume form. The expressions in [90] take a more compact form by expressing the coefficients  $u_n(x)$  in terms of the Laplace-Beltrami operator and the distance function of the manifold, rather than the usual expression in terms of the metric tensor and its derivatives. However, this shorter form of the heat invariants is of little help for the extraction of geometric information from the spectrum.

From the above formulas, one sees that the spectrum of the Laplace-Beltrami operator determines the volume of the manifold and the volume of its boundary, if any. Moreover,

note the factor  $\frac{1}{(4\pi t)^{N/2}}$  in the asymptotic expansion. Since the heat trace can be computed from the spectrum, the following theorem holds.

**Theorem 2.3.** *Let  $(\mathcal{M}, g)$  and  $(\mathcal{N}, h)$  be two compact Riemannian manifolds with isospectral Laplace-Beltrami operators. If  $\mathcal{M}$  and  $\mathcal{N}$  have boundaries, let them be equipped with the same boundary conditions. Then,  $\mathcal{M}$  and  $\mathcal{N}$  have the same dimension, volume and their boundaries have the same volume.*

The asymptotic result regarding the volume of a manifold can also be recast in a different form, originally due to Weyl for the special case of planar domains. Historically, the next theorem can be seen as the first result in spectral geometry. It was known to hold for a number of special cases where the spectrum is computable and there was significant interest in establishing its general validity.

**Theorem 2.4** (Weyl Estimate). *Let  $(\mathcal{M}, g)$  be a compact Riemannian manifold of dimension  $N$  and let  $\{\lambda_n\}_{n=0}^\infty$  be the spectrum of its Laplace-Beltrami operator. Let  $\beta(N)$  denote the volume of a unit ball in  $\mathbb{R}^N$ . Then, for large  $n$ , the eigenvalues obey the following asymptotic expansion:*

$$\lambda_n \sim \left( \frac{(2\pi)^N}{\text{Vol}(\mathcal{M})\beta(N)} \right)^{2/N} n^{2/N} . \quad (2.47)$$

Figure 2.1 illustrates the Weyl estimate for a flat 2-torus.

We do not prove the above theorem here. An interested reader can find a generalization of Weyl's proof to domains in  $\mathbb{R}^N$  in [8]. The proof uses a clever construction that doesn't depend on the theory discussed in the present thesis. The main idea is to cover the domain by a lattice of squares (or cubes in higher dimensions). Then the eigenvalues of the domain can be estimated using the eigenvalues of the cubes in the lattice. Since the eigenvalues of a cube can be readily computed, this yields the desired growth estimate.

A more modern proof relies on the Hardy-Littlewood-Karamata theorem (see [39] for a precise statement). Informally speaking, this theorem links the asymptotics of a function with the asymptotics of its Laplace transform. Recall that, by Equation (2.34), the heat trace is a Laplace transform and that Equation (2.45) (specifically the expression for  $a_0$ ) expresses its asymptotic behavior. Applying the Hardy-Littlewood-Karamata theorem directly yields the desired result. A full proof following this strategy can be found in [33]. Notice that in this approach the asymptotic expansion of the heat kernel is known in advance through different means. Weyl's approach, on the other hand, obtains the growth estimate from scratch.

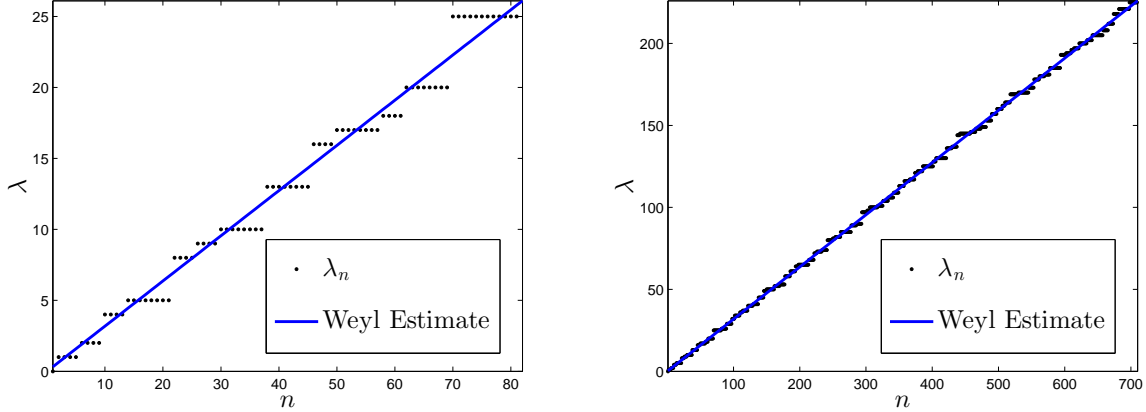


Figure 2.1: Weyl estimate for a flat 2-torus constructed by imposing periodic boundary conditions on the square  $[0, 2\pi) \times [0, 2\pi)$  in  $\mathbb{R}^2$ . The eigenvalues are given by  $k^2 + l^2$  for  $k, l \in \mathbb{Z}$ . The Weyl estimate reads  $\lambda_n \sim \frac{1}{\pi}n$ .

As mentioned previously, the coefficients in the asymptotic expansion of the heat kernel rapidly become unwieldy and hard to interpret. For operators more complicated than the Laplace-Beltrami operator, the situation is even worse. Nonetheless, there are results that rely upon the heat trace coefficients of such operators. For instance, the proof of the next theorem uses the asymptotic expansions of the Hodge Laplacian on 0, 1 and 2-forms [88].

We begin by setting up some notation. Let  $a_n^p(\mathcal{M})$  denote the  $n^{\text{th}}$  coefficient of the heat trace asymptotics of the Hodge Laplacian on  $p$ -forms on  $(\mathcal{M}, g)$ . Recall that  $(\mathcal{M}, g)$  is said to be Einstein if its Ricci tensor is given by  $R_{ij} = Cg_{ij}$  for some constant  $C$  [10]. In particular, it means that  $(\mathcal{M}, g)$  has constant scalar curvature  $R = NC$ . Also recall that the norm of a tensor is given by the square root of its contraction with its metric dual. For instance, for the Riemann curvature tensor,  $\|R_{ijkl}\| = \sqrt{R_{ijkl}R^{ijkl}}$ . This is a slight abuse of notation, as the left-hand side should not have any indices, while the right-hand side respects Einstein's summation convention. The purpose of the indices on the left-hand is to distinguish the Riemann tensor  $R_{ijkl}$  from the Ricci tensor  $R_{ij}$  and the scalar curvature  $R$ .

**Theorem 2.5.** *Let  $(\mathcal{M}, g)$  and  $(\mathcal{N}, h)$  be boundaryless compact Riemannian manifolds of dimension  $N \geq 2$ . Then, the following statements hold.*

- (i) *If  $a_4^p(\mathcal{M}) = a_4^p(\mathcal{N})$  for  $p = 0, 1$ , then  $(\mathcal{M}, g)$  is flat if and only if  $(\mathcal{N}, h)$  is flat.*



(ii) If  $a_4^p(\mathcal{M}) = a_4^p(\mathcal{N})$  for  $p = 0, 1$  and  $a_n^0(\mathcal{M}) = a_n^0(\mathcal{N})$  for  $n = 0, 2$ , then

- $(\mathcal{M}, g)$  is of constant sectional curvature  $K$  if and only if  $(\mathcal{N}, h)$  is of constant sectional curvature  $K$ .
- $(\mathcal{M}, g)$  is an Einstein manifold with scalar curvature  $NC$  if and only if  $(\mathcal{N}, h)$  is an Einstein manifold with scalar curvature  $NC$ .

(iii) If  $a_4^p(\mathcal{M}) = a_4^p(\mathcal{N})$  for  $p = 0, 1, 2$ , then  $a_4^p(\mathcal{M}) = a_4^p(\mathcal{N})$  for all  $0 \leq p \leq N$  and

$$\begin{aligned} \int_{\mathcal{M}} R^2 dV_g &= \int_{\mathcal{N}} R^2 dV_h \\ \int_{\mathcal{M}} \|R_{ij}\|^2 dV_g &= \int_{\mathcal{N}} \|R_{ij}\|^2 dV_h \\ \int_{\mathcal{M}} \|R_{ijkl}\|^2 dV_g &= \int_{\mathcal{N}} \|R_{ijkl}\|^2 dV_h . \end{aligned} \tag{2.48}$$

(iv) If  $a_4^p(\mathcal{M}) = a_4^p(\mathcal{N})$  for  $p = 0, 1, 2$  and  $a_n^0(\mathcal{M}) = a_n^0(\mathcal{N})$  for  $n = 0, 2$ , then  $(\mathcal{M}, g)$  has constant scalar curvature  $R$  if and only if  $(\mathcal{N}, h)$  has constant scalar curvature  $R$ .

*Proof.* See [88]. □

Recall that isospectrality of the Hodge Laplacian on  $p$ -forms implies the equality of the coefficients  $a_n^p$ . Thus, all of the conditions on the coefficients  $a_n^p$  in the above theorem are satisfied in the case of isospectral manifolds.

This concludes the present review of the geometric information one can extract from the asymptotics of the heat trace. Notice that the results reviewed here are insufficient to completely reconstruct manifolds from their spectra. Instead, only a few global properties, such as flatness, are accessible. Even then, in order to use those results to establish the properties of a manifold, one must have a reference manifold with the same spectra, or at least the same first few coefficients  $a_n^p$ . Moreover, the geometric properties of this reference manifold must be known. This is not an easy task in general. This observation was one of our original motivations for seeking new approaches to spectral geometry.

We close this section by briefly discussing the wave trace. It is very hard to do the wave trace justice due to the amount of material one needs to introduce in order to paint a full picture. See [112] for a review.

Similarly to the heat trace, the wave trace can be asymptotically expanded to yield spectral invariants with geometric content. The difference is the point about which one expands. Similarly to the heat trace,

$$\lim_{t \rightarrow 0} W(t) = \lim_{t \rightarrow 0} \sum_{n=0}^{\infty} \cos(\sqrt{\lambda_n t}) = \infty . \quad (2.49)$$

An expansion about  $t = 0$  is possible and yields invariants similar to those of the heat equation. The key feature of the wave trace is that other expansions are of interest. The values of  $t$  around which such expansions are possible are the lengths of periodic geodesics. In the case of a manifold with boundary, the class of curves under study is enlarged by billiard ball orbits, that is periodic paths that are geodesics in the bulk of the manifold and that reflect off the boundary according to the laws of geometric optics. The set of lengths of such geodesics is known as the length spectrum and is denoted  $\text{Lsp}(\mathcal{M}, g)$ .

One then considers the singular support  $\text{SingSupp}(W(t))$  of  $W(t)$ , that is the complement of the set of all  $t$  on which  $W(t)$  is smooth. It is known that

$$\text{SingSupp}(W(t)) \subset \{0\} \cup \text{Lsp}(\mathcal{M}, g) . \quad (2.50)$$

See [26, 29, 30, 34] for the boundaryless case. The wave trace can then be asymptotically expanded around the lengths of periodic geodesics. Note that there is no difference between length and duration, as the speed of the waves is set to one. The invariants thus obtained are of a different nature than the heat invariants. Given a periodic geodesic  $\gamma$ , the wave invariants will only depend upon the metric in a neighborhood of  $\gamma$ . This marks a significant difference with the heat invariants. Indeed heat invariants are computed as integrals whose integrands can be determined locally in the neighborhood of a point (the integral is still taken globally over  $\mathcal{M}$ ). In that sense, the wave invariants are less local than the heat invariants. However, they are also more local, as a change in the metric outside the neighborhood of  $\gamma$  will not alter the nature of  $\gamma$  as a periodic geodesic and its wave invariants will remain the same. The heat invariants, in contrast, are generically sensitive to changes in metric.

## 2.4 Spectral Geometry

In this section, we review results regarding the main problem of spectral geometry: the identification of shape from spectrum. The results discussed here mostly use the Laplace-Beltrami spectrum. Broadly speaking, the results can be divided into positive and negative ones. The positive results are cases where identification of shape from spectrum is possible.

Correspondingly, negative results are counterexamples to this program: isospectral non-isometric manifolds. We begin by covering positive results, after which we review the counterexamples. In particular, we sketch the Sunada construction, a famous technique for constructing counterexamples.

### 2.4.1 Positive Results

There is a limited number of positive results in spectral geometry. Unlike the counterexamples discussed later, there is no general technique that can be used to find manifolds that can be determined from their spectra. One thus often has to settle for quite weak results.

This review follows [32]. In particular, we adopt the terminology of [32] regarding the classification of positive results. Manifolds that can be determined by their spectra from a wide class of manifolds are said to be *spectrally unique* in that class. A weaker notion is that of *local uniqueness*, which is just spectral uniqueness in a neighborhood of the studied shape, given an appropriate topology. This can be further weakened by studying continuous deformations of a manifold. A manifold that admits no continuous isospectral deformations is said to be *spectrally rigid*. Clearly, the following hierarchy of properties holds.

$$\text{Spectral uniqueness} \implies \text{Local uniqueness} \implies \text{Spectral rigidity} \quad (2.51)$$

We begin by discussing results of the spectral uniqueness variety. The most powerful results of this type can be found in the work of Steve Zelditch and his collaborators [110, 111, 113, 61]. There, special classes of manifolds are studied and spectral uniqueness results for the Laplace-Beltrami operator are proven for generic members of those classes. A key assumption is that the manifolds are in some sense analytic. In the case of the surfaces of revolution studied in [110], the requirement is that the surface is generated by the rotation of an analytic curve. In the other cases, the manifolds under study are reflection symmetric domains in  $\mathbb{R}^N$  with analytic boundaries. Then, some non-degeneracy conditions are imposed on the periodic geodesics of the studied manifolds. Those conditions are generic, in the sense that, in a suitable topology, they are satisfied by a residual set within the class of manifolds (see Appendix A for the definition of residual sets). Those non-degeneracy conditions enable one to study the wave trace asymptotics around a convenient periodic geodesic. Then, one shows that the coefficients in the expansion determine all of the Taylor coefficients of the boundary near an endpoint of the geodesic (or of the surface

in the case of the surface of revolution). Then, the assumption of analyticity allow one to recover the manifold.

Another powerful result is valid on generic surfaces of revolution of dimension  $N$  [57]. For the sake of notational simplicity, we discuss the more familiar two-dimensional case. The general case is analogous. Let  $\phi$  denote the azimuthal angle on the surface and let  $x$  denote the other coordinate. The surface is defined as the rotation of a radial function  $R(x)$ . The generator of azimuthal rotations  $\frac{\partial}{\partial\phi}$  has spectrum  $\{m\}_{m\in\mathbb{Z}}$ . To each  $m \in \mathbb{Z}$  corresponds a discrete family  $\{\lambda_{mk}\}_k$  of eigenvalues of the Laplace-Beltrami operator. Then, under suitable generic conditions on the surface, the knowledge of the joint spectrum  $\{m, \lambda_{mk}\}_{m,k}$  of  $\frac{\partial}{\partial\phi}$  and  $\Delta$  allows one to explicitly reconstruct the surface. The proof of this result is rather interesting, as it relies on an asymptotic analysis of the motion of a quantum mechanical particle on the surface. This is also known as a semiclassical analysis. See [99] for a general treatment of this technique. Once the angular part of the Laplacian is taken care of, one is left with solving a one-dimensional Schrödinger eigenvalue problem. The potential  $V(x)$  in that operator depends solely on  $R(x)$ . In this picture, every Laplacian eigenvalue corresponds to an energy level of the system. For every energy level  $E$ , one can consider the classically allowed region  $\{x : V(x) \leq E\}$ . In general, this region will be disconnected into multiple potential wells. Then, the semiclassical treatment allows one to recover the number and width of those wells. Since the energy  $E$  can be varied, the shape of the individual wells can be recovered. It remains to position those potential wells in the correct order. This is done by finding the extrema of the potential through a variation of  $E$ . As  $E$  is increased, one can find signatures of new classically allowed wells appearing and of old wells fusing into larger ones. The former indicates that  $E$  has reached a local minimum of the potential, while the latter indicates that  $E$  has reached a local maximum. Since the shape of the individual wells is known, this provides the distance between consecutive minima and maxima. The potential  $V(x)$  is then recovered through a combinatorial procedure.  $R(x)$  can then be calculated from  $V(x)$ , solving the inverse spectral problem.

It is also known that generic flat  $N$ -dimensional tori can be distinguished from other such tori by their Laplace-Beltrami spectrum [108].

The other spectral uniqueness results are much more specialized. Indeed, they deal with the uniqueness of some standard shapes, such as balls and spheres in  $\mathbb{R}^N$ .

Possibly the best known result of this type is that balls in  $\mathbb{R}^N$  possess a Laplace-Beltrami spectrum distinct from all other compact domains in  $\mathbb{R}^n$  with smooth boundary. This result follows from the heat coefficients  $a_0$  and  $a_1$  (Equations (2.45), (2.46) and Theorem 2.3), which give the volume of the domain  $\Omega$  and its boundary  $\partial\Omega$ . Since balls uniquely minimize

the ratio  $\text{Vol}(\partial\Omega)/\text{Vol}(\Omega)$  among all domains in  $\mathbb{R}^N$  [38], the result is established.

Analogous results are widely reported to hold for the standard  $N$ -dimensional sphere  $S^N = \{x \in \mathbb{R}^{N+1} : \|x\| = 1\}$ , see the reviews in [10, 32], for example. While it is likely that standard spheres have a unique spectrum, we find the proofs somewhat dubious. To the best of our knowledge, and according to the citations in the reviews, the claim originates in the works of Tanno [104, 105]. The proof strategy is roughly the following. First, one uses the first few heat coefficients to show that a manifold isospectral to a manifold of constant sectional curvature  $K$  is also of constant sectional curvature  $K$ . Then, without explicitly presenting an argument, Tanno concludes that manifolds isospectral to the standard round sphere  $S^N$  are isometric to it. The best candidate for this missing step is the uniqueness theorem for simply connected space forms. Recall that complete manifolds of constant sectional curvature are known as space forms. Simply connected space forms of a given sectional curvature  $K$  are unique up to isometry [10]. The issue with Tanno's argument is that simple connectedness is never established. With that condition as an additional hypothesis, the results hold<sup>2</sup>.

Assuming the simple connectedness hypothesis to be fulfilled, the following results can be stated. For  $N \leq 6$ ,  $S^N$  is uniquely determined by its Laplace-Beltrami spectrum [104]. For arbitrary  $N$ , a spectral uniqueness result can be obtained using the spectra of the Hodge Laplacian on 0 and 1-forms. It is an application of Theorem 2.5, originally proven in [88] and used by Tanno in [104]. The best known result in arbitrary dimension using solely the Laplace-Beltrami spectrum is one of local uniqueness [105].

Local uniqueness and spectral rigidity results have been obtained for wider classes of manifolds. For instance, it is known that flat manifolds without boundary are locally unique [72].

Similarly, boundaryless manifolds of constant negative sectional curvature are known to be locally unique [98]. If one relaxes the assumption that the sectional curvature is constant, spectral rigidity results can be obtained. In dimension 2, the spectral rigidity of negatively curved manifolds has been established in [55]. Later, the result was extended to negatively curved manifolds of arbitrary dimension obeying a curvature estimate [56]. The latter result has since been extended to arbitrary negatively curved manifolds [31].

Finally, it is known that ellipses are spectrally rigid among domains with the same reflection symmetries as the ellipse [60].

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<sup>2</sup>See Note 9.7.2.1 in [10]. There, it is observed that the heat coefficients of a space form are determined by its sectional curvature and volume. This is not sufficient to reconstruct a general space form.

## 2.4.2 Counterexamples

An important direction in spectral geometry is the construction of counterexamples, that is isospectral non-isometric manifolds. Some counterexamples are constructed in an *ad hoc* manner, while others are made using systematic approaches. The ultimate goal of this endeavor is to achieve a complete understanding of isospectrality by obtaining a construction procedure (or a number of distinct construction procedures) that can be used to obtain all possible counterexamples. This would provide a complete classification of all counterexamples. This objective has not yet been achieved. Nonetheless, multiple systematic constructions are available. Moreover, some of the *ad hoc* examples have since been recast as special cases of the systematic techniques.

We begin by sketching the Sunada construction, a prototypical representation-theoretic technique used to obtain isospectral non-isometric manifolds. Then, we review counterexample construction procedures based on Riemannian submersions. Finally, we review different types of known counterexamples. For a more complete survey, see [45].

### Sunada Construction

The Sunada construction, first described in [100], is a very important method for constructing non-isometric isospectral manifolds. We use it to illustrate the general ideas involved in the construction of counterexamples. For a review of various proofs and applications of Sunada's theorem, see [16]. Our treatment is inspired by the one found in [45].

The general idea of the Sunada method is to obtain two manifolds whose spectra are subsets of the spectrum of a reference manifold. The core of the construction lies in ensuring that those subsets are equal. The procedure outlined below works for Laplace-type operators that commute with isometries. For the sake of notational simplicity, we only consider the Laplace-Beltrami case. None of the key steps of the construction are modified in the general case.

Let  $(\mathcal{M}, g)$  be a compact oriented Riemannian manifold with or without boundary and let  $\Delta$  be the Laplace-Beltrami operator on it. Notice that  $\Delta$  commutes with isometries. That is, if  $j : \mathcal{M} \rightarrow \mathcal{M}$  is an isometry,  $j^*$  is the induced pullback and  $\psi \in C^\infty(\mathcal{M})$ , the following holds

$$\Delta(j^*\psi) = j^*(\Delta\psi) . \tag{2.52}$$

In particular, if  $\psi_n$  is an eigenfunction of  $\Delta$  with eigenvalue  $\lambda_n$ ,

$$\Delta(j^*\psi_n) = \lambda_n(j^*\psi_n) . \tag{2.53}$$

Consequently,  $j^*$  maps eigenspaces of  $\Delta$  into themselves. Note that it doesn't necessarily map eigenfunctions into themselves. In general,  $j^*\psi_n \neq \psi_n$ . Those eigenfunctions that are mapped into themselves will soon become important.

Let  $H_1$  be a finite group of isometries of  $(\mathcal{M}, g)$ . Suppose that  $H_1$  acts freely, that is, if  $h(x) = x$  for some  $h \in H_1$  and  $x \in \mathcal{M}$ , then  $h$  is the identity isometry of  $\mathcal{M}$ . In other words, nontrivial isometries in  $H_1$  have no fixed points in  $\mathcal{M}$ . The orbit of a point  $x \in \mathcal{M}$  under the action of  $H_1$  is defined to be  $O(x) = \{y \in \mathcal{M} : y = h(x) \text{ for some } h \in H_1\}$ . The orbits of the points of  $\mathcal{M}$  form a manifold denoted  $H_1 \backslash \mathcal{M}$  [40]. It is known as the quotient of  $\mathcal{M}$  by  $H_1$ . Define  $\pi : \mathcal{M} \rightarrow H_1 \backslash \mathcal{M}$  by  $x \mapsto O(x)$ .  $H_1 \backslash \mathcal{M}$  can be equipped with a Riemannian metric by requiring  $\pi$  to be a local isometry.

**Definition 2.6** (Local Isometry). *Let  $(\mathcal{M}, g)$  and  $(\mathcal{N}, h)$  be Riemannian manifolds. A smooth map  $v : \mathcal{M} \rightarrow \mathcal{N}$  is said to be a local isometry if for every  $x \in \mathcal{M}$  there is an open neighborhood  $U \subset \mathcal{M}$  such that  $x \in U$ ,  $v(U) \subset \mathcal{N}$  is open and  $v|_U : U \rightarrow v(U)$  is an isometry [63].*

The metric on  $H_1 \backslash \mathcal{M}$  defined by this procedure is known as the quotient metric.

Correspondingly, the Laplace-Beltrami operator on  $H_1 \backslash \mathcal{M}$  is closely related to that on  $\mathcal{M}$ . Begin by noting that functions on  $H_1 \backslash \mathcal{M}$  can be identified with those functions on  $\mathcal{M}$  that are invariant under  $H_1$ . Indeed, let  $f \in C^\infty(\mathcal{M})$  be such that  $h^*f = f$  for all  $h \in H_1$ . Then,  $f(O(x))$  is a well-defined smooth function on  $H_1 \backslash \mathcal{M}$ . Similarly, a function  $F \in C^\infty(H_1 \backslash \mathcal{M})$  defines an  $H_1$ -invariant smooth function on  $\mathcal{M}$  by  $\tilde{F}(x) = F(O(x))$ .

Consequently, the eigenfunctions of  $\Delta$  on  $H_1 \backslash \mathcal{M}$  induce  $H_1$ -invariant eigenfunctions of  $\Delta$  on  $\mathcal{M}$  with the same eigenvalues. Moreover, all  $H_1$ -invariant eigenfunctions on  $\mathcal{M}$  are induced in that way. Indeed, the contrary would lead to a contradiction, as an  $H_1$ -invariant eigenfunction on  $\mathcal{M}$  induces an eigenfunction on  $H_1 \backslash \mathcal{M}$  that, in turn, induces the original eigenfunction on  $\mathcal{M}$ .

Thus, the spectrum of the Laplacian on  $H_1 \backslash \mathcal{M}$  can be obtained from that on  $\mathcal{M}$  by the following procedure. Let  $V_n$  be the eigenspace of  $\Delta$  on  $\mathcal{M}$  with eigenvalue  $\lambda_n$ . Determine all  $H_1$ -invariant eigenfunctions in  $V_n$ . Let  $V_n^{H_1}$  denote the subspace of  $V_n$  spanned by those functions and let  $D_{1,n}$  denote its dimension. If  $D_{1,n} \neq 0$ , then  $\Delta$  on  $H_1 \backslash \mathcal{M}$  has an eigenvalue  $\lambda_n$  with multiplicity  $D_{1,n}$ . Repeating the procedure for all  $n$  yields the spectrum of  $\Delta$  on  $H_1 \backslash \mathcal{M}$ .

Now, let  $H_2$  be another freely acting finite group of isometries of  $(\mathcal{M}, g)$ . We want the quotient  $H_2 \backslash \mathcal{M}$  to be isospectral to  $H_1 \backslash \mathcal{M}$ . Let  $D_{2,n}$  denote the dimension of  $V_n^{H_2}$ , the  $H_2$ -invariant subspace of  $V_n$ . In order to ensure isospectrality, we need a condition on  $H_1$  and  $H_2$  that ensures that  $D_{1,n} = D_{2,n}$  for all  $n$ .

This condition can be obtained through representation theoretic considerations. Indeed, note that the action of  $H_1$  and  $H_2$  on  $V_n$  can be seen as the representation of a finite group on a finite dimensional vector space. It is convenient to consider  $H_1$  and  $H_2$  as subgroups of a larger finite group of isometries  $G$ . We need the following technical lemma.

**Lemma 2.7** (Projector onto  $V^H$ ). *Let  $H$  be a finite group represented on a finite dimensional vector space  $V$ . Let  $V^H$  denote the subspace of  $V$  spanned by vectors  $v$  that obey  $hv = v$  for all  $h \in H$ . Then, the projection on  $V^H$  is given by*

$$P_H = \frac{1}{|H|} \sum_{h \in H} h, \quad (2.54)$$

where  $|H|$  is the cardinality of  $H$ .

*Proof.* Let  $v \in V^H$ . Then, since  $hv = v$  for all  $h \in H$ ,

$$P_H v = \frac{1}{|H|} \sum_{h \in H} hv = \frac{1}{|H|} |H| v = v. \quad (2.55)$$

Thus,  $P_H$  acts as desired on  $V^H$ . We now turn our attention to the remainder of the vector space.

By standard representation theory, it is known that finite dimensional representations of finite groups are completely reducible [102]. This means that, after an appropriate choice of inner product and basis for  $V$ , the orthogonal complement  $V^{H\perp}$  of  $V^H$  is an invariant subspace of the representation. Let  $u \in V^{H\perp}$ . Then  $P_H u \in V^{H\perp}$ . Let  $a \in H$ .

$$\begin{aligned} aP_H u &= a \frac{1}{|H|} \sum_{h \in H} hu = \frac{1}{|H|} \sum_{h \in H} (ah)u \\ &= \frac{1}{|H|} \sum_{h \in H} hu \\ &= P_H u \end{aligned} \quad (2.56)$$



Since  $a$  is arbitrary,  $P_H u \in V^H$ . This is a contradiction unless  $P_H u = 0$ . Thus,  $P_H$  maps the orthogonal complement of  $V^H$  to zero. Since  $P_H$  is linear, this completes the proof.  $\square$

As a consequence of this lemma, the dimension of  $V_n^{H_i}$  can be computed as the trace of  $P_{H_i}$ :

$$D_{i,n} = \text{Tr}(P_{H_i}) = \frac{1}{|H_i|} \sum_{h \in H_i} \text{Tr}(h) \quad \text{for } i = 1, 2. \quad (2.57)$$

Our goal is to obtain a condition that ensures that  $D_{n,1} = D_{n,2}$ . By the cyclic property of the trace,  $\text{Tr}(aha^{-1}) = \text{Tr}(h)$  for all  $a, h \in G$ . Let  $[h] = \{k \in G : \exists a \in G \text{ such that } k = aha^{-1}\}$  be the conjugacy class of  $h \in G$ . Notice that, for all  $k \in [h]$ ,  $\text{Tr}(k) = \text{Tr}(h)$ . It is thus natural to use the conjugacy classes of  $G$  to compute  $\text{Tr}(P_{H_i})$ . Since, the conjugacy classes are equivalence classes, the formula for  $D_{i,n}$  becomes,

$$D_{i,n} = \left( \sum_{h \in G} |H_i \cap [h]| \right)^{-1} \left( \sum_{h \in G} |H_i \cap [h]| \text{Tr}(h) \right). \quad (2.58)$$

Consider the following definition.

**Definition 2.8** (Almost Conjugate Subgroup). *Let  $H_1$  and  $H_2$  be subgroups of a finite group  $G$ .  $H_1$  and  $H_2$  are said to be almost conjugate if every conjugacy class of  $G$  intersects them in the same number of elements. Symbolically,*

$$|H_1 \cap [h]| = |H_2 \cap [h]|, \quad \forall h \in G. \quad (2.59)$$

Thus, for almost conjugate subgroups  $H_1$  and  $H_2$ ,  $D_{1,n} = D_{2,n}$ . By the above discussion, the following theorem is proven.

**Theorem 2.9** (Sunada). *Let  $(\mathcal{M}, g)$  be a compact Riemannian manifold. Let  $G$  be a finite group of isometries of  $(\mathcal{M}, g)$  with two almost conjugate subgroups  $H_1$  and  $H_2$ . Suppose that  $H_1$  and  $H_2$  act freely. Then, the quotient manifolds  $H_1 \backslash \mathcal{M}$  and  $H_2 \backslash \mathcal{M}$  equipped with the quotient metrics are isospectral for all Laplace-type operators that commute with isometries.*

Notice that the above theorem does not ensure that  $H_1 \backslash \mathcal{M}$  and  $H_2 \backslash \mathcal{M}$  are non-isometric. Indeed, consider the case of conjugate  $H_1$  and  $H_2$ , that is, the case when

there exists  $h \in G$  such that  $H_2 = hH_1h^{-1}$ . As the terminology suggests,  $H_1$  and  $H_2$  are then almost conjugate and Sunada's theorem applies. Let  $O_1(x)$  be the orbit of  $x \in \mathcal{M}$  under the action of  $H_1$  and let  $O_2(y)$  be the orbit of  $y \in \mathcal{M}$  under the action of  $H_2$ . Set  $y = hx$ . Then,  $O_2(y) = hO_1(x)$ . This is an isometry from  $H_1 \backslash \mathcal{M}$  to  $H_2 \backslash \mathcal{M}$ .

This can be an obstacle in the construction of counterexamples. Consequently, one must make sure that  $H_1$  and  $H_2$  are not conjugate. (Even then, one can show that there always exists a local isometry (see Definition 2.6) between the manifolds constructed by the Sunada method.) This has to hold not only within  $G$ , but also within the whole isometry group of  $(\mathcal{M}, g)$ . This is very hard in general, as one rarely has access to the whole isometry group. The practical solution to this issue is to use *ad hoc* means to verify that the quotient manifolds  $H_1 \backslash \mathcal{M}$  and  $H_2 \backslash \mathcal{M}$  are not isometric. Up until that last verification step, one can only make sure that  $H_1$  and  $H_2$  are almost conjugate, but not conjugate, within  $G$ .

The Sunada construction is part of a wider class of representation-theoretic techniques used to construct isospectral non-isometric manifolds. The broad strokes of those methods are the same as in the Sunada approach. For example, see [52] for an analogous construction involving factoring Lie groups with left-invariant metrics by discrete subgroups. The result is a continuous family of isospectral non-isometric manifolds.

A weaker variant of Sunada's theorem has been obtained in [89]. There, the almost conjugacy condition is replaced by a weaker one. The resulting theorem ensures isospectrality for the Laplace-Beltrami operator, but not necessarily any other Laplace-type operator.

## Riemannian Submersions

Riemannian submersions can be used to construct isospectral non-isometric manifolds. In a sense, the overall direction of this approach is opposite to the one used in the Sunada technique. Instead of factoring a reference manifold into isospectral factors, the goal here is to prove the isospectrality of two manifolds from the isospectrality of carefully chosen submanifolds. Our discussion is based on the one found in [45].

Begin by recalling that a submersion  $\pi : \mathcal{M} \rightarrow \mathcal{N}$  is a smooth map between manifolds that induces a surjective push-forward  $\pi_* : T\mathcal{M} \rightarrow T\mathcal{N}$ . Assuming that  $\mathcal{M}$  and  $\mathcal{N}$  are equipped with Riemannian metrics, the submersion  $\pi$  is said to be Riemannian if  $\pi_* : T\mathcal{M} \supset \ker(\pi_*)^\perp \rightarrow T\mathcal{N}$  is an isometry. A fiber of  $\pi$  (the pre-image  $\pi^{-1}x$  for  $x \in \mathcal{N}$ ) is said to be totally geodesic if all geodesics of  $\mathcal{M}$  that start tangent to the fiber stay tangent to the fiber. The following lemma then holds [106].

**Lemma 2.10.** *Let  $\pi : \mathcal{M} \rightarrow \mathcal{N}$  be a Riemannian submersion with totally geodesic fibers. Let  $\Delta_{\mathcal{M}}$  and  $\Delta_{\mathcal{N}}$  be the Laplace-Beltrami operators on  $\mathcal{M}$  and  $\mathcal{N}$ , respectively. Then, for all  $f \in C^\infty(\mathcal{N})$ ,*

$$\Delta_{\mathcal{N}}f = \Delta_{\mathcal{M}}\pi^*f \tag{2.60}$$

*In particular, the spectrum of  $\Delta_{\mathcal{N}}$  is the same as the spectrum of  $\Delta_{\mathcal{M}}$  restricted to  $\pi^*C^\infty(\mathcal{N})$ .*

Under some additional hypotheses, an analogous lemma holds for Hodge Laplacians [41]. The following theorem enables one to construct isospectral non-isometric manifolds. Some familiarity with the fundamentals of principal bundles is required (see [81, 62]).

**Theorem 2.11.** *Let  $T$  be a torus of dimension at least 2. Let  $\mathcal{M}$  and  $\mathcal{N}$  be principal  $T$ -bundles equipped with Riemannian metrics. Let the fibers of  $\mathcal{M}$  and  $\mathcal{N}$  be totally geodesic flat tori. For each subtorus  $S \subset T$  of codimension at most 1, suppose that the quotient manifolds  $S \backslash \mathcal{M}$  and  $S \backslash \mathcal{N}$  have isospectral Laplace-Beltrami operators. Then  $\mathcal{M}$  and  $\mathcal{N}$  are isospectral.*

*Sketch of Proof.* The idea of this proof is to decompose the spaces of square integrable functions  $L_2(\mathcal{M})$  and  $L_2(\mathcal{N})$  into spaces of functions constant on various subtori  $S$  of  $T$ . The maps  $\mathcal{M} \rightarrow S \backslash \mathcal{M}$  and  $\mathcal{N} \rightarrow S \backslash \mathcal{N}$  are Riemannian submersions with totally geodesic fibers. By hypothesis, the Laplace-Beltrami operators on  $S \backslash \mathcal{M}$  and  $S \backslash \mathcal{N}$  are isospectral. By Lemma 2.10, the Laplace-Beltrami operators on  $\mathcal{M}$  and  $\mathcal{N}$  are then isospectral on the space of functions constant on  $S$ . By varying  $S$ , the space on which the Laplace-Beltrami operators are isospectral is eventually shown to be all of  $L_2$ .

See [45] for a more detailed sketch of a proof. A full proof can be found in [46]. □

This approach is used in [46] to construct isospectral deformations of metrics on spheres in dimension  $N \geq 8$  and balls in dimension  $N \geq 9$ . Those metrics can be chosen arbitrarily close to the standard ones.

## Overview of Counterexamples

In this section we review a number of qualitatively different pairs or families of isospectral non-isometric manifolds. The purpose of this review is to showcase various restrictions one can impose on the studied manifolds without ruling out the existence of non-isometric isospectrality.

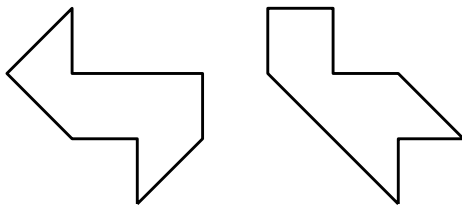


Figure 2.2: Famous isospectral non-isometric domains in the plane. Dirichlet boundary conditions are imposed.

A number of counterexamples are known for planar domains. The most famous pair of non-isometric isospectral domains is illustrated on Figure 2.2. Its construction can be cast in terms of the Sunada method. See [48] for an informal discussion of this example and [49] for a sketch of a proof. An explicit proof can be found in [24]. See also the review in [44].

To the best of our knowledge, all known counterexamples in the plane are concave with non-smooth boundaries. In dimension 4 and higher, convex isometric non-isospectral manifolds are known [47]. In the hyperbolic plane, convex isospectral non-isometric polygons have been constructed [50].

As mentioned previously, the Sunada construction always yields pairs of locally isometric manifolds, in the sense that one can always construct at least one local isometry (see Definition 2.6) between them. It is thus of interest to determine whether all pairs of isospectral manifolds admit local isometries between them. This question has been solved in [101]. There, a pair of isospectral manifolds that does not admit local isometries between them is constructed. The spectra are computed explicitly. This construction can be recast in terms of Riemannian submersions [45].

Recall that two metrics  $g$  and  $h$  over  $\mathcal{M}$  are said to be conformally equivalent if there exists a function  $f \in C^\infty(\mathcal{M})$  such that  $g = fh$ . In [19], isospectral sets of non-isometric conformally equivalent metrics are constructed.

Finally, counterexamples can form continuous families rather than mere pairs [52]. Continuous families isospectral for the Laplace-Beltrami operator, but not the Hodge Laplacian on 1-forms have been constructed in [53]. Moreover, continuous families of conformally equivalent counterexamples are known to exist [17].

## 2.5 Compactness of Isospectral Sets

Another approach to understanding isospectrality is to study the properties of isospectral sets. Specifically, the goal is to show that the set of manifolds isospectral to a given reference manifold is not too big. This is achieved by establishing that such sets are compact in a natural topology. In this section, we review such compactness results.

The topology in which the isospectral sets are to be shown to be compact is defined as follows. First, one constructs all Sobolev spaces of positive integer order on covariant 2-tensors on  $\mathcal{M}$ . See [74, 42, 81] for the definition of Sobolev spaces on arbitrary vector bundles. Then, a sequence of metrics  $\{g_i\}_i$  on  $\mathcal{M}$  that converges for all Sobolev norms is said to converge in the  $C^\infty$  sense. This can be extended to the convergence of manifolds. Consider a sequence of Riemannian manifolds  $\{(\mathcal{M}_i, g_i)\}_i$ . This sequence is said to converge to a Riemannian manifold  $(\mathcal{M}, g)$  in the  $C^\infty$  sense if there exists a sequence of diffeomorphisms  $\chi_i : \mathcal{M} \rightarrow \mathcal{M}_i$  such that the sequence of pullback metrics  $\{\chi_i^* g_i\}_i$  on  $\mathcal{M}$  converges to  $g$  in the  $C^\infty$  sense. This topology is thus valid for equivalence sets of isometric manifolds (Riemannian structures) and not merely Riemannian metrics.

The first compactness results were obtained for boundaryless manifolds in dimension 2, as well as for simply connected planar domains in [82]. The proof involves the heat invariants as well as the determinant of the Laplace-Beltrami operator. This object is a spectral invariant that has to be carefully defined since the Laplace-Beltrami operator is unbounded. See [78] for a non-rigorous introduction to the determinant of the Laplacian.

In higher dimensions, analogous results are harder to obtain. In dimension 3, it is known that sets of conformally equivalent isospectral metrics are compact [23]. The conformal equivalence assumption can be removed at the cost of a different one. There is some choice in the matter. For instance, one can assume a lower bound on the length of the shortest periodic geodesic [3]. Alternatively, one can assume either that the isospectral manifolds have negative sectional curvature or Ricci curvature uniformly bounded below [18].

Finally, in arbitrary dimension, it is known that isospectral families of metrics over a fixed differentiable manifold are compact if their sectional curvatures are uniformly bounded above and below [114].

Outside of the above compactness results, not much is known about the structure of isospectral sets. This observation was part of our motivation to study the geometry of isospectral sets. Our results regarding the non-convexity of sets of isospectral conformally equivalent metrics in dimension 2 can be found in [Chapter 5](#).

## Chapter 3

# Numerical Explorations of Infinitesimal Inverse Spectral Geometry

One of the key questions of spectral geometry is whether one generically can recover small changes in shape from small changes in spectrum. This is what we term infinitesimal inverse spectral geometry. In one way or another, the remainder of this thesis is concerned with this problem. As one might expect, from the title of this thesis if nothing else, infinitesimal inverse spectral geometry is related to the perturbation theory of Laplace-type operators. From now on, we restrict our attention to the Laplace-Beltrami operator. In the coming chapters we will present a rigorous treatment of the perturbations of said operator. For now, however, we pursue a simpler endeavor. Our goal here is to numerically assess whether the program of inverse infinitesimal spectral geometry is possible.

The results presented in this chapter were obtained by the author of this thesis and were originally published in [86]. An early version of this can also be found in the author's Master's thesis [85]. A similar numerical treatment for a Laplacian on graphs can be found in [1]. Finally, on the more practical side, techniques similar to those used here were applied to the design of musical instruments, specifically metallophones [11]. There, the usual bars are replaced with whimsical shapes that nonetheless produce the correct notes when struck.

In Section 3.1, we begin by discussing the general ingredients needed for our numerical approach. Then, in Section 3.2, we apply said approach to the Laplace-Beltrami operator on a set of planar domains. In Section 3.3 we conclude this chapter by discussing possible

applications of our approach to more general manifolds.

### 3.1 General Numerical Setup

To reiterate, we want to obtain a numerical algorithm that would reconstruct small changes of shape from small changes in spectrum. A number of ingredients are needed in order to accomplish this. We begin by introducing a space of shapes  $\mathcal{G}$ . The nature of the elements of  $\mathcal{G}$  can remain unspecified for now. The approach that we discuss here is quite general and can be applied to a variety of objects other than Laplacians on manifolds, such as graph Laplacians or Schrödinger potentials. We keep these options open for now. Of course, we will later restrict our attention to a particular class of manifolds.

The space of shapes  $\mathcal{G}$  can contain shapes equivalent by isometry, or any other appropriate notion of equivalence if  $\mathcal{G}$  is not a set of manifolds. Furthermore, suppose that  $\mathcal{G}$  is parametrized in a well behaved way by  $\mathbb{R}^M$ . That is, we can abuse terminology and call every point in  $\mathbb{R}^M$  a shape. The coordinates  $\{x_n\}_{n=1}^M$  in  $\mathbb{R}^M$  will be called the *shape degrees of freedom*.

Suppose that to each shape in  $\mathcal{G}$  one can associate an eigenvalue spectrum. The precise nature of that spectrum also does not need to be specified for now. In fact, even when the space  $\mathcal{G}$  is composed of compact Riemannian manifolds, there are multiple choices of operators whose spectra one may consider. One can even use multiple spectra at once.

Since numerical methods are to be employed to find said spectra, only a finite number  $N$  of eigenvalues will be accessible. They will be called the *spectral degrees of freedom*. For instance, one might consider the ten lowest eigenvalues of the Laplace-Beltrami operator. Then,  $N = 10$ . Alternatively, one might use, say, the smallest five Laplace-Beltrami eigenvalues together with the smallest five eigenvalues of the Hodge Laplacian on 1-forms. Then,  $N = 10$  as well. The key idea is that one can construct a *spectral map*  $\sigma : \mathbb{R}^M \rightarrow \mathbb{R}^N$  that maps the  $M$  shape degrees of freedom into the  $N$  spectral degrees of freedom  $\{\lambda_n\}_{n=1}^N$ . If desired, one can further alter this map. For instance, one can consider the reciprocals  $1/\lambda_n$  of the eigenvalues rather than the eigenvalues  $\lambda_n$  themselves, as we do in Section 3.2. In any case, we suppose that all such choices have been made and denote the resulting spectral map by  $\sigma$ . The study of the spectral geometry of  $\mathcal{G}$  can then be replaced by the study of the spectral map  $\sigma$ .

As our goal is to compare different shapes and their spectra, we need to introduce metrics (in the sense of metric space, not necessarily Riemannian metrics) on the spaces of shapes and spectra. This is a necessary step since numerical methods have finite precision



and will never yield exact isospectrality or isometry. We will have to content ourselves with isospectrality and isometry up to some thresholds  $\varepsilon_\sigma$  and  $\varepsilon_G$ , respectively. Since the spectra are points in  $\mathbb{R}^N$ , we employ the standard Euclidean distance.

The situation with the space of shapes is more subtle. Suppose that  $\mathcal{G}$  is equipped with a complete metric  $d_G(\cdot, \cdot)$ . Once again, this is a metric in the sense of metric space, not necessarily a Riemannian metric. Recall that  $\mathcal{G}$  is allowed to contain equivalent shapes, by isometry in our case. Let  $[\mathcal{G}]$  denote the set of such equivalence classes. Assuming that each class  $g \in [\mathcal{G}]$  is a compact subset of  $\mathcal{G}$ , one can define a complete metric on  $[\mathcal{G}]$ . This is a fairly strong, but very convenient assumption. See Section 3.3 for a possible way of avoiding this hypothesis. Recall the definition of the Hausdorff distance.

**Definition 3.1** (Hausdorff Distance). *Let  $(X, d)$  be a metric space. The Hausdorff distance  $d_H(\cdot, \cdot)$  on the family of all bounded, non-empty closed subsets of  $(X, d)$  is defined by letting*

$$d_H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}, \quad (3.1)$$

where  $d(a, B) = \inf_{b \in B} d(a, b)$ .

If  $(X, d)$  is complete,  $d_H(\cdot, \cdot)$  is a complete metric on the set of compact subsets of  $(X, d)$  [36]. Consequently, we can define a complete metric  $d_{[\mathcal{G}]}(\cdot, \cdot)$  on  $[\mathcal{G}]$  as the Hausdorff distance on equivalence classes of  $\mathcal{G}$ . This is the metric that we will be using to verify whether two elements of  $\mathcal{G}$  are equivalent. In practice, this requires the knowledge of the equivalence classes of  $\mathcal{G}$ . In the case of manifolds, one must be able to generate all of the isometries. This is a hard task in general. In Section 3.2, we overcome this by studying an especially well behaved set of manifolds. In Section 3.3, we propose a possible way to entirely sidestep this issue.

We are now ready to discuss our numerical algorithm. Pick a starting shape  $A$  and a target shape  $B$ . Our goal is to construct a path  $P(t)$  starting at  $A$  and ending at a shape equivalent to  $B$ . For now view  $P(t)$  as a differentiable path. In truth, it will be a sequence of points approximating a differentiable path.

The only tools we allow ourselves in the construction of  $P(t)$  are local behavior of  $\sigma$  near  $P(t)$  and the target spectrum  $\sigma(B)$ . We will achieve that by expressing  $P(t)$  as the solution of a differential equation.

We cannot *a priori* guarantee that  $P(t)$  will reach its target point. Instead, we require that the distance between the current and target spectrum is nonincreasing along the path:

$$\frac{d}{dt}\|\sigma(B) - \sigma(P(t))\| \leq 0 . \quad (3.2)$$

This immediately gives us a first approach to constructing  $P(t)$ . Indeed, consider the following gradient descent equation:

$$\frac{d}{dt}P(t) = -\frac{1}{2}\mathbf{grad}\left(\|\sigma(B) - \sigma(P(t))\|^2\right) . \quad (3.3)$$

The reason for the presence of the factor  $1/2$  and the squaring of the Euclidean distance will become apparent shortly. At this point, we have replaced the inverse spectral problem by a local optimization algorithm.

If  $P(t)$  satisfies this equation, it satisfies all of the requirements that we imposed above. However, the gradient descent equation is not the best approach here. Define the Jacobian matrix of  $\sigma$ :

$$\mathcal{J} = \begin{pmatrix} \frac{\partial\lambda_1}{\partial x_1} & \frac{\partial\lambda_1}{\partial x_2} & \frac{\partial\lambda_1}{\partial x_3} & \cdots & \frac{\partial\lambda_1}{\partial x_M} \\ \frac{\partial\lambda_2}{\partial x_1} & \frac{\partial\lambda_2}{\partial x_2} & \frac{\partial\lambda_2}{\partial x_3} & \cdots & \frac{\partial\lambda_2}{\partial x_M} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\lambda_N}{\partial x_1} & \frac{\partial\lambda_N}{\partial x_2} & \frac{\partial\lambda_N}{\partial x_3} & \cdots & \frac{\partial\lambda_N}{\partial x_M} \end{pmatrix} . \quad (3.4)$$

Let  $v_\sigma = \sigma(B) - \sigma(P)$  and denote the transpose of  $\mathcal{J}$  by  $\mathcal{J}^T$ . A straightforward component-wise computation can be used to express Equation (3.3) as

$$\frac{d}{dt}P(t) = -\frac{1}{2}\mathbf{grad}\left(\|\sigma(B) - \sigma(P(t))\|^2\right) = \mathcal{J}^T v_\sigma . \quad (3.5)$$

This expression is the reason why we use the square of the Euclidean distance as well as the factor  $1/2$  in Equation (3.3). This new form suggests an improvement to the gradient descent method. Let  $\mathcal{J}^+$  denote the pseudoinverse of  $\mathcal{J}$ . See [95] for a definition. In brief, the pseudoinverse inverts all that can be inverted and acts like 0 elsewhere. More importantly, given a linear equation  $Ax = v$ ,  $x = A^+v$  is a least-squares solution. This means that  $x$  is a vector such that  $\|Ax - v\|$  is minimal. Consider the following, alternative, equation for  $P(t)$ .

$$\frac{d}{dt}P(t) = \mathcal{J}^+v_\sigma \quad (3.6)$$

By the least squares property of  $\mathcal{J}^+$ , this equation makes  $P(t)$  vary in a way that is as close as possible to a straight line towards the target spectrum. This is conceptually similar to the idea of inverting the spectral map. Indeed, at each  $t$ , we locally pseudo-invert the tangent to the spectral map. Note that this is a canonical generalization of the notion of the inverse. An extremely important benefit to our purposes is that the pseudoinverse is defined for maps between spaces of different dimension. In a sense, this will allow us to study how well the  $M$  shape degrees of freedom are encoded in the  $N$  spectral degrees of freedom.

Similarly to the gradient approach of Equation (3.3), the pseudoinverse approach of Equation (3.6) is a local optimization algorithm. Indeed, the distance to the target spectrum is also non-decreasing along the path. Consider

$$\begin{aligned} \frac{d}{dt}\|v_\sigma\|^2 &= 2v_\sigma \cdot \frac{d}{dt}v_\sigma \\ &= 2v_\sigma \cdot \left( \cancel{\frac{d}{dt}\sigma(B)} - \frac{d}{dt}\sigma(P) \right) \\ &= -2v_\sigma \cdot \left( \mathcal{J} \frac{d}{dt}P(t) \right) \\ &= -2v_\sigma \cdot (\mathcal{J}\mathcal{J}^+v_\sigma) . \end{aligned} \quad (3.7)$$

The last line follows from Equation (3.6). Since  $\mathcal{J}\mathcal{J}^+$  is the orthogonal projection onto the image of  $\mathcal{J}$ ,  $v_\sigma \cdot (\mathcal{J}\mathcal{J}^+v_\sigma) \geq 0$ , which establishes that solutions to the pseudoinverse method also satisfy Equation (3.2), as required.

While it is more discerning than the gradient method, the pseudoinverse method can also get stuck in local minima of  $\|v_\sigma\|^2$ . In fact,  $\ker(\mathcal{J}^T) = \ker(\mathcal{J}^+)$  [95]. Thus, for a given  $v_\sigma$ , the right-hand side of Equation (3.5) vanishes if and only if the right-hand side of Equation (3.6) vanishes. The two methods thus get stuck at the same shapes.

An advantage of the pseudoinverse approach, albeit one that we do not use in the present thesis, is the possibility of setting secondary objectives for the optimization process [22]. Notice that  $\mathbb{1} - \mathcal{J}^+\mathcal{J}$  is the projector onto  $\ker(\mathcal{J}) = \text{coker}(\mathcal{J}^+)$ , the subspace that does not influence our objective function to first order. For example, given a function  $\rho : \mathbb{R}^M \rightarrow \mathbb{R}$ , we can require for it to be increased (possibly maximized) alongside our main optimization by setting

$$\frac{d}{dt}P(t) = \mathcal{J}^+v_\sigma + (\mathbb{1} - \mathcal{J}^+\mathcal{J})\mathbf{grad}(\rho(P)) . \quad (3.8)$$

We use the pseudoinverse approach in what follows. In order to numerically solve Equation (3.6), we begin by discretizing the time variable  $t$ . From now on,  $t \in \mathbb{N}$  represents the number of the step in the solution. Then, we numerically integrate Equation (3.6) by Euler's method:

$$P(t+1) = P(t) + S\mathcal{J}^+(t)v_\sigma(t) , \quad (3.9)$$

where  $S$  is a variable step size.  $\mathcal{J}$  is computed using symmetric finite differences.

The value of  $S$  is varied according to the following rules. Let  $u > 1$  and  $d < 1$  be two positive constants. If  $\sigma(P(t+1))$  is closer to the target spectrum  $\sigma(B)$  than  $\sigma(P(t))$ , the new step size is increased to  $Su$ . If  $\sigma(P(t+1))$  is further away from  $\sigma(B)$  than  $\sigma(P(t))$ , the step is cancelled. Then, the step size is reduced to  $Sd$  and the step is attempted again under the same rules. In our investigations, we used  $u = 1.1$  and  $d = 0.7$ . These values work well as they provide noticeable changes in step size without being too extreme. Ultimately, however, the choice of  $u$  and  $d$  is arbitrary.

Were we to exactly solve equation (3.6), the situation where  $d_\sigma(\sigma(P(t+1)), \sigma(B)) > d_\sigma(\sigma(P(t)), \sigma(B))$  would never arise. In the discrete setting, however, such things are possible. This is why we compute  $P(t+1)$  again with a smaller step size.

The increase in step size serves to speed up the otherwise quite lengthy integration. When the initial shape  $A$  and the target shape  $B$  start close together, which will be the case in most of our study, this is of little consequence. However, when  $A$  and  $B$  are taken to be very different from each other and, more to the point, have very different spectra, this proves to be a major advantage. Of course, the exponential increase in step size will eventually produce a step size so large we have to drop all pretense of approximating a path satisfying a differential equation. Nonetheless, this approach remains a sound optimization algorithm, as the shapes it produces will have a spectrum that is closer and closer to the target spectrum.

The integration terminates if either of two conditions are fulfilled. The first one is the convergence to the desired spectrum, up to tolerance. That is, if  $d_\sigma(\sigma(P(t)), \sigma_B) \leq \varepsilon_\sigma$ . The second one is the reduction of the step size below some very small threshold  $\varepsilon_S > 0$ . This occurs when the algorithm is repeatedly unable to reduce  $S$  in order to compute a  $P(t+1)$  that satisfies  $d_\sigma(\sigma(P(t+1)), \sigma(B)) \leq d_\sigma(\sigma(P(t)), \sigma(B))$ . In other words, this occurs when the algorithm is stuck in a local optimum.

The outcome of the algorithm is a final shape  $P_f$ . One can then compare  $P_f$  and  $\sigma(P_f)$  to the target shape  $B$  and target spectrum  $\sigma(B)$  using the metrics and tolerances introduced above. Recall that  $d_{[\mathcal{G}]}(\cdot, \cdot)$  takes into account the equivalence of shapes. Isometry is the relevant equivalence relation in the case of manifolds. Then, four outcomes are possible.

- (1) Success: both the shapes and the spectra match.
- (2) Local optimum: neither the shapes nor the spectra match.
- (3) Potential counterexample: the spectra match, but the shapes do not.
- (4) Numerical artifact: the shapes match, but the spectra do not.

The first two outcomes have already been discussed at length and require no further explanation. Regarding the third outcome, it is important to remember that the algorithm only uses  $N$  eigenvalues. Consequently all that was found is a pair of shapes isospectral on  $N$  eigenvalues, up to tolerance. It is not necessarily a pair of isospectral non-isometric manifolds. The fourth outcome is an unavoidable consequence of the way we compare the final shape and spectrum to the target ones. Indeed, we tacitly assumed that balls of radius  $\varepsilon_{\mathcal{G}}$  in the space of shapes map to balls of radius  $\varepsilon_{\sigma}$  in the space of spectra, which is not the case in general. Consequently, it is impossible to completely eliminate possibility (4) by choosing compatible thresholds  $\varepsilon_{\mathcal{G}}$  and  $\varepsilon_{\sigma}$ .

We are interested in the success rates of our algorithm. In order to do that, we run the algorithm for randomly generated pairs of initial and target shapes  $(A, B)$ . We then can compute the success rates as functions of the initial shape distance  $d_{[\mathcal{G}]}(A, B)$  between  $A$  and  $B$  and the number  $N$  of considered eigenvalues. For the purposes of such statistics, the dubious cases of outcome (4) are considered as failures. Consequently, we underestimate our success rate.

Moreover, we study the prevalence of counterexamples. We compute the proportion of potential counterexamples (outcome (3)), among all runs that reach the target spectrum (outcomes (1) and (3)). By studying this quantity's dependence upon the number of eigenvalues  $N$ , it is possible to see whether the counterexamples are common or not.

This concludes the general setup of our method of numerical investigation. In the next section, we apply it to a particular class of planar domains.

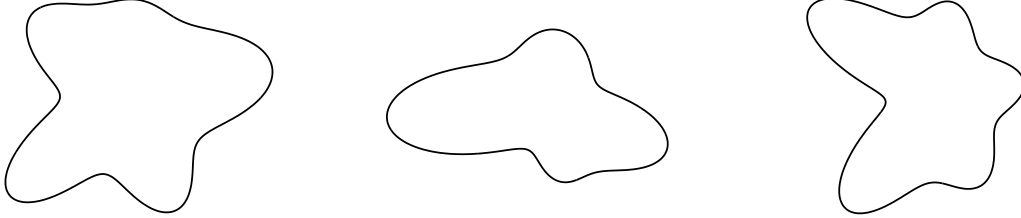


Figure 3.1: Three shapes generated using Equation (3.10). In all the cases, the number of shape degrees of freedom is  $M = 11$ .

### 3.2 Star-Shaped Planar Domains

In this section, we apply our numerical approach to a family of compact star-shaped domains in  $\mathbb{R}^2$ . Those domains are defined by their boundaries. In standard polar coordinates  $(r, \phi)$ , we set

$$r(\phi) = a + b \exp \left( C_0 + \sum_{k=1}^{\frac{M-1}{2}} [C_k \cos(k\phi) + S_k \sin(k\phi)] \right). \quad (3.10)$$

In the above, the Fourier coefficients  $\{C_k\}_{k=0}^{\frac{M-1}{2}}$  and  $\{S_k\}_{k=1}^{\frac{M-1}{2}}$  form the coordinates for  $\mathbb{R}^M$ . We choose  $a = 0.1$  and  $b = 0.9$  so that the vanishing of all Fourier coefficients produces a disk of radius 1. Notice that  $r(\phi) > 0$  for all  $\phi \in [0, 2\pi)$ .

The spectral map  $\sigma : \mathbb{R}^M \rightarrow \mathbb{R}^N$  is chosen to output the lowest  $N$  eigenvalues of the standard Laplacian with Dirichlet boundary conditions, in ascending order. The spectra are computed using Freefem++, a freely available finite element solver [59]. The technical limitations of said solver require the presence of some  $a \gg 10^{-7}$  in equation (3.10). Otherwise, the choice  $a = 0$  and  $b = 1$  would have provided a sufficiently rich space of shapes. For an introduction to finite element methods, see [4], among many others.

As mentioned previously, we then take the reciprocals  $1/\lambda_n$  of the eigenvalues. This seems to increase the success rates, likely due to the fact that smaller eigenvalues are typically less sensitive to small changes in shape than the large ones. In effect, using  $1/\lambda_n$  rather than  $\lambda_n$  seems to encourage the algorithm to first find a roughly correct shape and then make smaller adjustments on it.

The only missing ingredient is a metric on the space of shapes. Notice that the boundaries of the domains are compact subsets of  $\mathbb{R}^2$ . Indeed, they are continuous images of the circle  $S^1$ , which is compact. Thus, the Euclidean distance on  $\mathbb{R}^2$  can be used to induce a Hausdorff distance on the boundaries (see Definition 3.1). We use that as the metric  $d_{\mathcal{G}}(\cdot, \cdot)$ .

We still need to define a metric  $d_{[\mathcal{G}]}(\cdot, \cdot)$  on the isometry equivalent classes of  $\mathcal{G}$ . Recall that all isometries of the plane can be produced as a composition of a translation, as reflexion and a rotation. For simplicity, we slightly alter the space of shapes under study. Instead of leaving the domains as produced by Equation (3.10), we translate them so that their barycenters lie at the origin. This takes care of all translation isometries. The remaining isometries can be expressed as either a rotation or a reflexion with respect to the  $x$  axis followed by a rotation. Since  $S^1$  is compact and rotations are continuous, the isometry equivalence class of a given domain is the union of two compact sets: those shape equivalent by rotation alone and those equivalent by a reflexion followed by a rotation. Since finite unions of compacts are compact, the isometry equivalence class of a domain is compact in the  $d_{\mathcal{G}}(\cdot, \cdot)$  metric. We can thus define  $d_{[\mathcal{G}]}(\cdot, \cdot)$  as a Hausdorff distance on those equivalence classes. To reiterate, our metric on  $[\mathcal{G}]$  is a Hausdorff distance of a Hausdorff distance.

The range of the shape degrees of freedom under study gives rise to domains with a diameter roughly between 1 and 10. We set the threshold for isometry to be much smaller than this scale:  $\varepsilon_{\mathcal{G}} = 0.005$ . The choice of a spectral tolerance is more subtle. Given a large sample of shapes isometric up to tolerance, it is tempting to set the spectral tolerance to the smallest number such that all isometric shapes are isospectral. This, however, results in more near-isospectral non-isometric shapes counting as isospectral. Many false counterexamples to the determination of shape from spectrum are then detected. In the end, we chose to be stricter on the notion of isospectrality and set  $\varepsilon_{\sigma} = \sqrt{10^{-9}} \approx 3.16 \cdot 10^{-5}$ . This is roughly 10 times lower than what compatibility with isometry would suggest.

We applied our algorithm to domains generated with  $\frac{M-1}{2} = 1 \dots 5$ . In other words, we considered  $M = 3, 5, 7, 9, 11$ . The case  $M = 1$  results in discs which are determined by their spectrum (see Chapter 2). The omitted even values of  $M$  produce the same shapes, due to the following trigonometric identity.

$$a \cos(\phi) + b \sin(\phi) = \sqrt{a^2 + b^2} \sin(\phi + \delta) \quad , \quad \text{for some } \delta \in [0, 2\pi) \quad (3.11)$$

Depending on  $M$ , between 1250 and 1750 pairs of initial and target shapes  $(A, B)$  were randomly generated. Pairs that started with  $d_{[\mathcal{G}]}(A, B) \leq \varepsilon_{\mathcal{G}}$  were automatically rejected.

Then, for each pair, the algorithm was ran for different numbers of eigenvalues  $N = 1 \dots 40$ . An example of a successful run of the algorithm is illustrated in Figure 3.2.

We will only be discussing the results in the case  $M = 11$ , as it illustrates the general behavior of all the studied cases. Moreover, it is the most complex case considered.

The first piece of information we are interested in is the rate of success of our algorithm. Recall that we identify the initial shape distance  $d_{[\mathcal{G}]}(A, B)$  and the number of eigenvalues  $N$  as the variables that can influence the success rate. In order to probe their influence, we represent the success rate in two ways.

First, we compute the proportion of successes among runs such that  $d_{[\mathcal{G}]}(A, B) \in [a, b]$  for some  $a, b$ . For  $N = 40$ , this is illustrated on Figure 3.3. Note the high rate of success for nearby  $A$  and  $B$  as well as the rapid decay of the success rate as  $d_{[\mathcal{G}]}(A, B)$  increases.

We are mostly interested in the success rate at small initial shape distances, as this is what probes the local invertibility of the spectral map. Moreover, we want to see the dependence of the short range success rate on the number of eigenvalues used. In order to do that, we introduce the following quantity:

$$A(d, N) = \frac{\text{Number of successful runs with } N \text{ eigenvalues and } d_{[\mathcal{G}]}(A, B) \leq d}{\text{Number of runs with } N \text{ eigenvalues and } d_{[\mathcal{G}]}(A, B) \leq d}. \quad (3.12)$$

It essentially is a success rate for  $d_{[\mathcal{G}]}(A, B) \in (\varepsilon_{\mathcal{G}}, d]$  using  $N$  eigenvalues. For  $M = 11$ ,  $A(d, N)$  is illustrated on Figure 3.4. Consider the top edge of the graph. This corresponds to runs with small  $d_{[\mathcal{G}]}(A, B)$ . Notice that the success rate is very high there. Indeed, it seems to go to 1 in the limit of small  $d_{[\mathcal{G}]}(A, B)$ , assuming that  $N$  is sufficiently large. This indicates that it is possible to reconstruct small changes in shape from small changes in spectrum, assuming enough eigenvalues are considered.

The question of how many eigenvalues are needed remains. One could expect that a match between the number of shape and spectral degrees of freedom is required. Note that the space of shapes studied here has one redundant degree of freedom, due to isometry by rotation. Thus, one would expect  $A(d, N)$  to change behavior near  $N = M - 1 = 10$ . This is indeed the case.  $A(d, N)$  increases dramatically as  $N$  goes from 1 to roughly 10, after which it stays near 1 and only slightly improves with  $N$ . This is very encouraging, as it indicates that this strategy could be applied in the case of infinitely many shape degrees of freedom. If the number of eigenvalues required for local reconstruction grows at the same rate as the number of shape degrees of freedom, the joint limit  $M, N \rightarrow \infty$  could plausibly be taken.



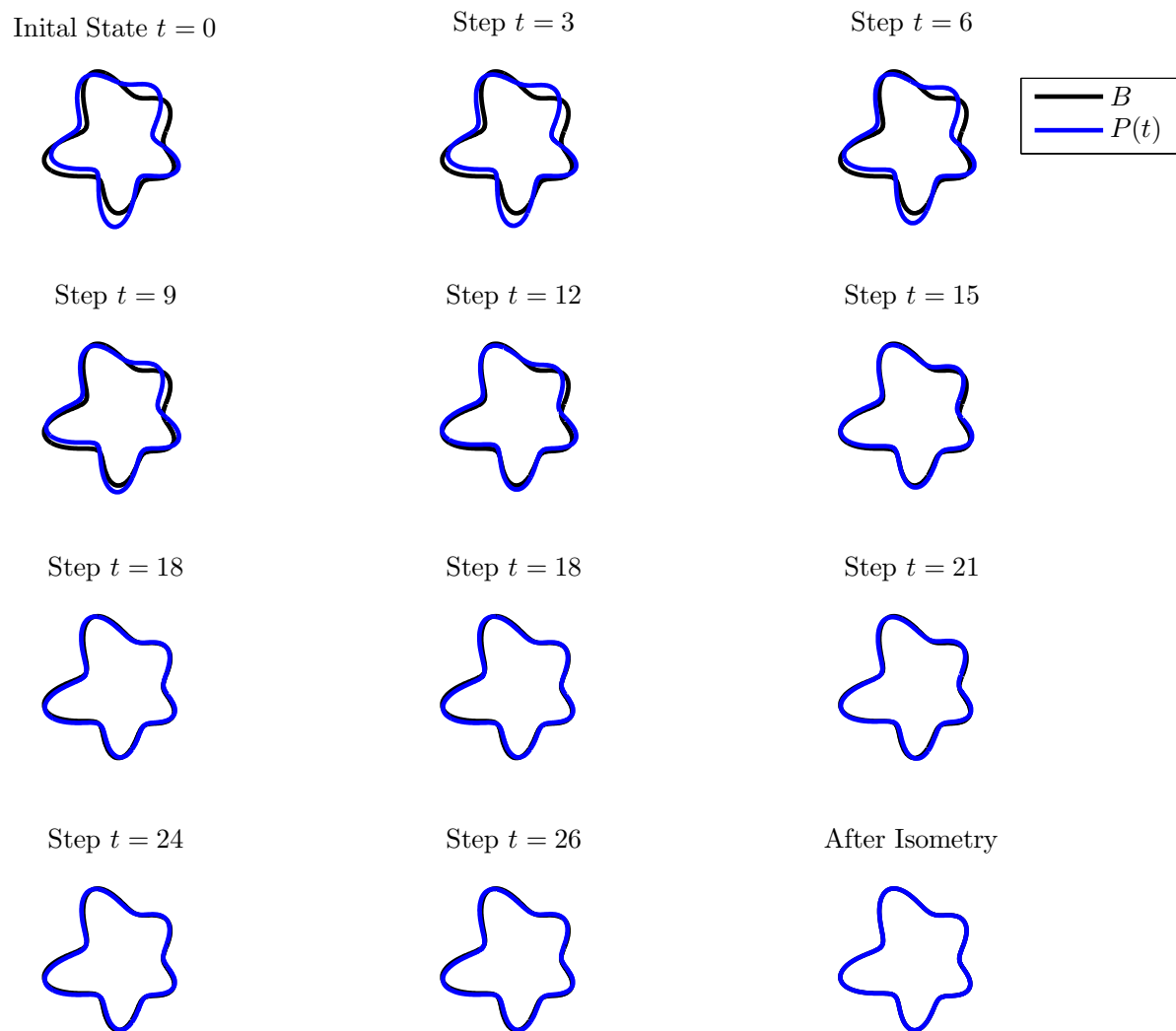


Figure 3.2: Successful run of the algorithm for  $M = 11$  and  $N = 20$ .

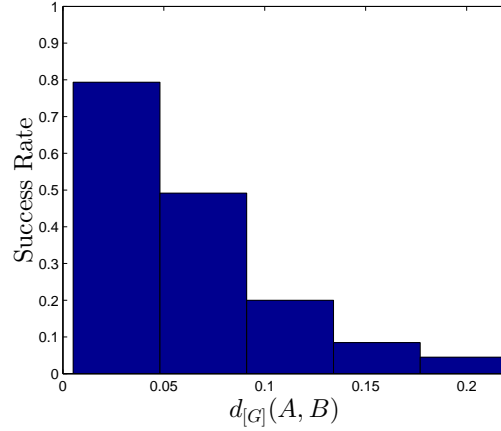


Figure 3.3: Dependence of the success rate on the initial shape distance  $d_{[G]}(A, B)$  for  $M = 11$  and  $N = 40$ . Lower values of  $N$  produce analogous graphs, but with lower success rates. Originally published in [86].

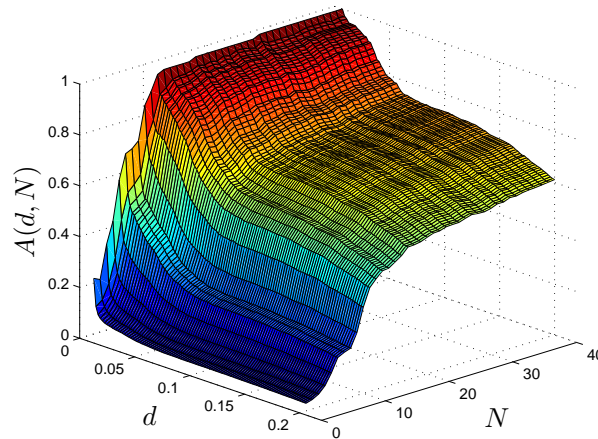


Figure 3.4: Dependence of the success rate on the initial shape distance  $d_{[G]}(A, B)$  and the number of eigenvalues  $N$  for  $M = 11$ . Lower values of  $M$  produce similar results. Originally published in [86].

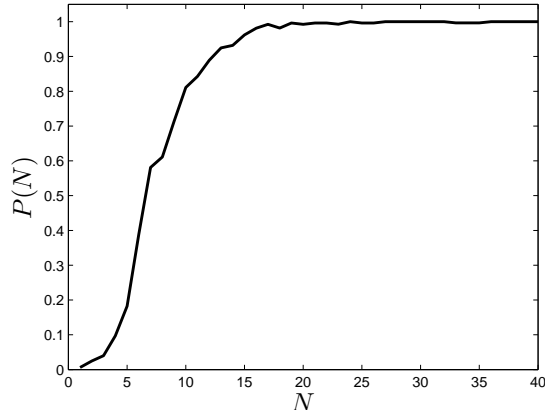


Figure 3.5: Proportion of isometric runs among isospectral ones as a function of  $N$  for  $M = 11$ . Similar results are obtained for other values of  $M$ . Originally published in [86].

It remains to discuss whether our algorithm has found any isospectral non-isometric manifolds. Due to the nature of our tools we can unfortunately not answer this question. In fact, were we to find a pair of non-isometric shapes isospectral for the first  $N$  eigenvalues, there is no guarantee that this isospectrality would remain when  $N + 1$  eigenvalues are considered. The best we can do is to estimate the proportion of runs resulting in an isometric pair of shapes among all runs that produce shapes isospectral on the first  $N$  eigenvalues. Call this proportion  $P(N)$ . Since isospectral non-isometric manifolds are expected to be rare, this proportion is expected to go to 1 as  $N$  increases and it becomes increasingly harder to find shapes isospectral on the first  $N$  eigenvalues. This is indeed what can be observed on Figure 3.5, where the behavior of  $P(N)$  is illustrated. This is an indication of the rarity of isospectral non-isometric manifolds.

### 3.3 Possible Extensions

The results that we obtained for the star-shaped planar domains defined in Equation (3.10) are very encouraging. Indeed, they indicate that isospectral non-isometric manifolds are rare and that it is possible to locally reconstruct small changes in shape from small changes in spectrum.

It would be of interest to extend such studies to other types of manifolds. Most ingredi-

ents for this endeavor are readily available. Common finite element solvers, say FreeFem++ [59], can be used to compute Laplacian eigenvalues in dimensions 2 and 3 and it is theoretically possible to construct solvers for higher dimensions. The pseudoinverse approach of Equation (3.6) applies as long as one has a way to parametrize the shapes under study by  $\mathbb{R}^M$ .

Still, one key piece is missing. That piece is a way to check for isometries. For domains in  $\mathbb{R}^3$ , a strategy similar to the one used here can be adopted. It is not so for curved manifolds.

Fortunately, there is a result that can be used to numerically find isometries between generic Riemannian manifolds. The appropriate notion of genericity is the non-degeneracy of the Laplace-Beltrami spectrum. Indeed, metrics that give rise to such Laplacians form a residual set in the  $C^\infty$  topology [6]. Given such a manifold  $(\mathcal{M}, g)$ , a point  $x \in \mathcal{M}$  is said to be generic if none of the eigenfunctions of the Laplace-Beltrami operator vanish at  $x$ . The set of such points has full measure in  $\mathcal{M}$  as the nodal sets of the eigenfunctions have a dimension smaller than that of the manifold [27]. Since a countable union of sets of measure zero is also of measure zero, its complement has full measure. The following theorem holds.

**Theorem 3.2.** *Let  $(\mathcal{M}, g)$  and  $(\mathcal{N}, h)$  be two compact Riemannian manifolds with non-degenerate Laplace-Beltrami spectrum. This is a generic condition. Let  $h_{\mathcal{M}}(x, y, t)$  and  $h_{\mathcal{N}}(\tilde{x}, \tilde{y}, t)$  denote the heat kernels of the Laplace-Beltrami operators of  $(\mathcal{M}, g)$  and  $(\mathcal{N}, h)$ , respectively. Let  $p \in \mathcal{M}$  be generic in the sense given above. Let  $f : \mathcal{M} \rightarrow \mathcal{N}$  be a map such that  $f(p)$  is generic and*

$$h_{\mathcal{M}}(p, y, t) = h_{\mathcal{N}}(f(p), f(y), t) \quad , \quad \text{for all } t > 0 \text{ and all } y \in \mathcal{M}. \quad (3.13)$$

*Then  $f$  is an isometry.*

A proof of the above theorem, as well as a numerical algorithm for shape matching can be found in [83]. Moreover, one can use this algorithm to construct approximate isometries between manifolds if an exact isometry is unavailable.

We suggest that one could use this method to compare shapes in a generalization of our algorithm. Once a best possible map  $f$  is found and a convenient norm is chosen, a notion of distance between shapes could be provided by

$$d(\mathcal{M}, \mathcal{N}) = \|h_{\mathcal{M}}(p, y, t) - h_{\mathcal{N}}(f(p), f(y), t)\| . \quad (3.14)$$

In the above, we purposefully leave the norm unspecified, as multiple choices are valid. Realistically, due to the nature of numerical methods, one would only have access to a discretized version of the heat kernel. Heat kernels would then be represented as finite arrays of real numbers. This would allow for the usage of the usual Euclidean distance, for example. Note that this notion of distance between shapes does not require the assumption that the isometry equivalence sets in  $\mathcal{G}$  are compact, unlike the notion of distance used in Section 3.1.

We conclude this section with a discussion of a rigorous notion of distance between compact Riemannian manifolds based on the heat kernel. See [67] for the details. For  $\varepsilon > 0$  and using the notation of Theorem 3.2, not necessarily continuous maps  $p : \mathcal{M} \rightarrow \mathcal{N}$  and  $q : \mathcal{N} \rightarrow \mathcal{M}$  are said to be  $\varepsilon$ -spectral approximations if

$$\begin{aligned} e^{-(t+1/t)} |h_{\mathcal{M}}(t, x, y) - h_{\mathcal{N}}(t, p(x), p(y))| < \varepsilon \quad , \quad \forall t > 0, \quad x, y \in \mathcal{M} \\ e^{-(t+1/t)} |h_{\mathcal{M}}(t, q(a), q(b)) - h_{\mathcal{N}}(t, a, b)| < \varepsilon \quad , \quad \forall t > 0, \quad a, b \in \mathcal{N} . \end{aligned} \tag{3.15}$$

The infimum of such  $\varepsilon$  is said to be the *spectral distance* between the Riemannian manifolds  $(\mathcal{M}, g)$  and  $(\mathcal{N}, h)$ . If  $(\mathcal{M}, g)$  and  $(\mathcal{N}, h)$  admit no  $\varepsilon$ -spectral approximations, it is understood that the spectral distance between them is  $\infty$ . This yields a metric (in the sense of metric space) on the set of isometry equivalence classes of compact Riemannian manifolds.

The spectral distance is of theoretical importance in the study of sequences of Riemannian manifolds. More specifically, it arises in the study of the convergence of the associated Laplacians. The sequences of manifolds under study can admit limit manifolds with topologies and dimensions different from those of the elements of the sequences, making questions of convergence nontrivial. See [65, 66, 103] for applications. We also refer the reader to [76] for a discussion of an alternative notion of spectral distance, as well as discussion of various notions of distance between Riemannian manifolds.

For the purposes of numerical experiments, the spectral distance may prove too expensive to compute in a reasonable time. Thus, an approximation to it in the sense of Theorem 3.2 and Equation (3.14) is likely preferable.

# Chapter 4

## Analytic Perturbation Theory

Having numerically demonstrated the viability of infinitesimal inverse spectral geometry, we turn our attention to the theoretical tools required to perturbatively study the Laplace-Beltrami spectrum. This endeavor will occupy all of this chapter, and most of the next one. The aim of this chapter is to introduce the fundamentals of the analytic perturbation theory of eigenvalues of linear operators.

Analytic perturbation theory concerns itself with one-parameter families of operators of the form

$$H(t) = H^{(0)} + tH^{(1)} + t^2H^{(2)} + \dots , \quad (4.1)$$

where the  $\{H^{(i)}\}_{i=0}^{\infty}$  are linear operators on some Hilbert space  $\mathcal{H}$  and  $t$  is a real or complex parameter in the neighborhood of 0. Consider the following eigenvalue problem:

$$H(t)\psi_n(t) = \lambda_n(t)\psi_n(t) , \quad (4.2)$$

where  $\psi_n(t) \in \mathcal{H}$  is a family of eigenvectors and  $\lambda_n(t)$  a family of eigenvalues. The index  $n$  serves to enumerate the distinct families of eigenvalues and eigenvectors of  $H(t)$ . The two goals of analytic perturbation theory are, first, to establish the conditions necessary for those families to be convergent power series in  $t$  and, second, to compute the coefficients of those series:

$$\begin{aligned} \lambda_n(t) &= \lambda_n^{(0)} + t\lambda_n^{(1)} + t^2\lambda_n^{(2)} + \dots \\ \psi_n(t) &= \psi_n^{(0)} + t\psi_n^{(1)} + t^2\psi_n^{(2)} + \dots \end{aligned} \quad (4.3)$$

Notice that in all of the above the superscript in parenthesis matches the power of the parameter  $t$ . That superscript is known as the order of the perturbation. Thus,  $H^{(1)}$  is the first order perturbation to  $H^{(0)}$ ,  $H^{(2)}$  the second order perturbation etc. The objects with the superscript (0) are termed unperturbed. Sometimes, the term correction is used instead of perturbation, especially for the eigenvalues and eigenvectors.

We are mostly concerned with the practical side of perturbation theory, that is the computation of eigenvalue corrections, as the perturbation series that we study in the later chapters have been shown to converge.

The first two eigenvalue corrections are sufficient for our purposes. We obtain the expressions for said corrections in two distinct ways. In Section 4.1 we present a heuristic computation of these corrections using an approach typical of quantum mechanical textbooks. Then, in Section 4.2, we sketch the rigorous approach to the computation of the eigenvalue corrections.

## 4.1 Heuristic Perturbation Theory

In this section, we pursue a heuristic approach to the determination of the first and second order eigenvalue corrections. Unlike the treatments of perturbation theory found in textbooks on quantum mechanics, we do not assume that the perturbations  $H^{(i)}$  for  $i \geq 1$  are self-adjoint or even symmetric (see Appendix B for a discussion of the difference between symmetric and self-adjoint operators). In fact, the perturbations of the Laplace-Beltrami operator studied in Chapter 5 are not symmetric in general.

We begin by computing the eigenvalue corrections in the case of a nondegenerate unperturbed spectrum. Despite the title of this section, this first computation is rigorous. Then, we proceed to treat the case of a finitely degenerate spectrum. There, the failings of the heuristic approach will become apparent. The expressions obtained here remain, nonetheless, valid.

### 4.1.1 Nondegenerate Case

Consider a self-adjoint operator  $H^{(0)}$  on a Hilbert space  $\mathcal{H}$  with inner product  $\langle \cdot, \cdot \rangle$ . Let the eigenvectors  $\{\psi_n^{(0)}\}_n$  of  $H^{(0)}$  form a countable orthonormal basis for  $\mathcal{H}$  and let the corresponding eigenvalues  $\{\lambda_n^{(0)}\}_n$  be nondegenerate. If one further assumes that  $H^{(i)} = 0$  for  $i \geq 2$  and that  $H^{(1)}$  is self-adjoint, the following formulas for the eigenvalue corrections can be found in quantum mechanics textbooks, such as [77, 28].

$$\begin{aligned}\lambda_n^{(1)} &= \langle \psi_n^{(0)}, H^{(1)} \psi_n^{(0)} \rangle \\ \lambda_n^{(2)} &= \sum_{k \neq n} \frac{|\langle \psi_k^{(0)}, H^{(1)} \psi_n^{(0)} \rangle|^2}{\lambda_n^{(0)} - \lambda_k^{(0)}}\end{aligned}\tag{4.4}$$

For our purposes, we require a slight generalization of these results. Namely, we must consider the case of non-symmetric perturbations  $H^{(i)}$  for  $i \geq 1$  and of a perturbation with changing inner product. As will be seen in Chapter 5, this situation arises when dealing with perturbations of Laplacians induced by perturbations of the Riemannian metric. The procedure followed here is entirely analogous to the one found in the standard textbooks on quantum mechanics [77, 28].

Let  $\{G^{(i)}\}_{i \geq 1}$  be bounded self-adjoint operators that define the following family of inner products in some neighborhood of  $t = 0$ .

$$\langle \cdot, \cdot \rangle_t = \langle \cdot, \cdot \rangle + t \langle \cdot, G^{(1)} \cdot \rangle + t^2 \langle \cdot, G^{(2)} \cdot \rangle + \dots\tag{4.5}$$

Consider the eigenvalue problem for  $H(t)$  (Equation (4.2)). Using the expansions for  $H(t)$ ,  $\psi_n(t)$  and  $\lambda_n(t)$  (Equations (4.1) and (4.3)) both the right-hand and left-hand side of Equation (4.2) become polynomials in  $t$ . Equating the coefficients order by order, up to second order, yields

$$\begin{aligned}H^{(0)} \psi_n^{(0)} &= \lambda_n^{(0)} \psi_n^{(0)} \\ H^{(0)} \psi_n^{(1)} + H^{(1)} \psi_n^{(0)} &= \lambda_n^{(0)} \psi_n^{(1)} + \lambda_n^{(1)} \psi_n^{(0)} \\ H^{(0)} \psi_n^{(2)} + H^{(1)} \psi_n^{(1)} + H^{(2)} \psi_n^{(0)} &= \lambda_n^{(0)} \psi_n^{(2)} + \lambda_n^{(1)} \psi_n^{(1)} + \lambda_n^{(2)} \psi_n^{(0)}.\end{aligned}\tag{4.6}$$

The first line of the above is the unperturbed eigenvalue problem for  $H^{(0)}$ . The next two lines encode the first and second order contributions to the eigenvalue problem of  $H(t)$ , respectively.

Before proceeding further with the computation of the eigenvalue corrections, it is necessary to choose a normalization for the eigenvectors. The intuitive choice would be to set  $\langle \psi_n(t), \psi_n(t) \rangle_t = 1$ . However, this is not the most convenient approach. Instead, following [97], we require

$$\langle \psi_n^{(0)}, \psi_n(t) \rangle_t = 1.\tag{4.7}$$



Order by order in  $t$ , up to second order, this condition becomes

$$\begin{aligned}
\langle \psi_n^{(0)}, \psi_n^{(0)} \rangle &= 1 \\
\langle \psi_n^{(0)}, \psi_n^{(1)} \rangle &= -\langle \psi_n^{(0)}, G^{(1)} \psi_n^{(0)} \rangle \\
\langle \psi_n^{(0)}, \psi_n^{(2)} \rangle &= -\langle \psi_n^{(0)}, G^{(1)} \psi_n^{(1)} \rangle - \langle \psi_n^{(0)}, G^{(2)} \psi_n^{(0)} \rangle .
\end{aligned} \tag{4.8}$$

Now, following the standard procedure, we take the unperturbed inner product of the first order part of equation (4.6) with  $\psi_n^{(0)}$ .

$$\begin{aligned}
\langle \psi_n^{(0)}, H^{(0)} \psi_n^{(1)} \rangle + \langle \psi_n^{(0)}, H^{(1)} \psi_n^{(0)} \rangle &= \langle \psi_n^{(0)}, \lambda_n^{(0)} \psi_n^{(1)} \rangle + \langle \psi_n^{(0)}, \lambda_n^{(1)} \psi_n^{(0)} \rangle \\
\lambda_n^{(0)} \langle \psi_n^{(0)}, \psi_n^{(1)} \rangle + \langle \psi_n^{(0)}, H^{(1)} \psi_n^{(0)} \rangle &= \lambda_n^{(0)} \langle \psi_n^{(0)}, \psi_n^{(1)} \rangle + \lambda_n^{(1)}
\end{aligned} \tag{4.9}$$

The second line of the above is obtained using the self-adjointness of  $H^{(0)}$  and the fact that  $\psi_n^{(0)}$  is normalized. Isolating  $\lambda_n^{(1)}$  gives:

$$\boxed{\lambda_n^{(1)} = \langle \psi_n^{(0)}, H^{(1)} \psi_n^{(0)} \rangle} . \tag{4.10}$$

Similarly, taking the inner product of the first order part of the eigenvalue problem with  $\psi_k^{(0)}$  for  $k \neq n$  and using the self-adjointness of  $H^{(0)}$  together with the orthogonality of  $\psi_n^{(0)}$  and  $\psi_k^{(0)}$  yields

$$\begin{aligned}
\langle \psi_k^{(0)}, H^{(0)} \psi_n^{(1)} \rangle + \langle \psi_k^{(0)}, H^{(1)} \psi_n^{(0)} \rangle &= \langle \psi_k^{(0)}, \lambda_n^{(0)} \psi_n^{(1)} \rangle + \langle \psi_k^{(0)}, \lambda_n^{(1)} \psi_n^{(0)} \rangle \\
\implies \lambda_k^{(0)} \langle \psi_k^{(0)}, \psi_n^{(1)} \rangle + \langle \psi_k^{(0)}, H^{(1)} \psi_n^{(0)} \rangle &= \lambda_n^{(0)} \langle \psi_k^{(0)}, \psi_n^{(1)} \rangle \\
\implies \langle \psi_k^{(0)}, \psi_n^{(1)} \rangle &= \frac{\langle \psi_k^{(0)}, H^{(1)} \psi_n^{(0)} \rangle}{\lambda_n^{(0)} - \lambda_k^{(0)}} .
\end{aligned} \tag{4.11}$$

Together with the normalization condition of Equation (4.8), this gives the following expansion of  $\psi_n^{(1)}$  in the eigenbasis of  $H^{(0)}$ :

$$\psi_n^{(1)} = \sum_{k \neq n} \frac{\langle \psi_k^{(0)}, H^{(1)} \psi_n^{(0)} \rangle}{\lambda_n^{(0)} - \lambda_k^{(0)}} \psi_k^{(0)} - \langle \psi_n^{(0)}, G^{(1)} \psi_n^{(0)} \rangle \psi_n^{(0)} . \tag{4.12}$$

This expression is necessary in order to obtain the expression for the second order eigenvalue correction  $\lambda_n^{(2)}$ . Begin by taking the inner product of the second order part of the eigenvalue problem (third line of Equation (4.6)) and  $\psi_n^{(0)}$ :

$$\begin{aligned} \langle \psi_n^{(0)}, H^{(0)} \psi_n^{(2)} \rangle + \langle \psi_n^{(0)}, H^{(1)} \psi_n^{(1)} \rangle + \langle \psi_n^{(0)}, H^{(2)} \psi_n^{(0)} \rangle = \\ \langle \psi_n^{(0)}, \lambda_n^{(0)} \psi_n^{(2)} \rangle + \langle \psi_n^{(0)}, \lambda_n^{(1)} \psi_n^{(1)} \rangle + \langle \psi_n^{(0)}, \lambda_n^{(2)} \psi_n^{(0)} \rangle . \end{aligned} \quad (4.13)$$

By the self-adjointness of  $H^{(0)}$  and the normalization of  $\psi_n^{(0)}$ ,

$$\lambda_n^{(2)} = \langle \psi_n^{(0)}, H^{(1)} \psi_n^{(1)} \rangle + \langle \psi_n^{(0)}, H^{(2)} \psi_n^{(0)} \rangle - \langle \psi_n^{(0)}, \lambda_n^{(1)} \psi_n^{(1)} \rangle . \quad (4.14)$$

Then, using the expression for  $\psi_n^{(1)}$  (Equation (4.12)) and the first order normalization condition (Equation (4.8), line two) the following expression for  $\lambda_n^{(2)}$  is obtained:

$$\begin{aligned} \lambda_n^{(2)} = \sum_{k \neq n} \frac{\langle \psi_k^{(0)}, H^{(1)} \psi_n^{(0)} \rangle}{\lambda_n^{(0)} - \lambda_k^{(0)}} \langle \psi_n^{(0)}, H^{(1)} \psi_k^{(0)} \rangle - \langle \psi_n^{(0)}, G^{(1)} \psi_n^{(0)} \rangle \langle \psi_n^{(0)}, H^{(1)} \psi_n^{(0)} \rangle \\ + \langle \psi_n^{(0)}, H^{(2)} \psi_n^{(0)} \rangle + \lambda_n^{(1)} \langle \psi_n^{(0)}, G^{(1)} \psi_n^{(0)} \rangle . \end{aligned} \quad (4.15)$$

Finally, it remains to recall that  $\lambda_n^{(1)} = \langle \psi_n^{(0)}, H^{(1)} \psi_n^{(0)} \rangle$  in order to get

$$\boxed{\lambda_n^{(2)} = \sum_{k \neq n} \frac{\langle \psi_n^{(0)}, H^{(1)} \psi_k^{(0)} \rangle \langle \psi_k^{(0)}, H^{(1)} \psi_n^{(0)} \rangle}{\lambda_n^{(0)} - \lambda_k^{(0)}} + \langle \psi_n^{(0)}, H^{(2)} \psi_n^{(0)} \rangle} . \quad (4.16)$$

Compare this expression to the case of a symmetric  $H^{(1)}$  (Equation (4.4)). The only change is in the numerator of the first term. This has important consequences for our endeavors. Specifically, the numerator is no longer guaranteed to be positive. As explained at the end of Section 5.3.2, this is an obstacle to the generalization of the main results of this thesis.

Also note neither  $\lambda_n^{(1)}$  (Equation (4.10)) nor  $\lambda_n^{(2)}$  (Equation (4.16)) depend on the perturbation of the inner product. This is to be expected, as the eigenvalues of an operator do not depend upon the inner product. Thus, strictly speaking, it was unnecessary to set  $\langle \psi_n^{(0)}, \psi_n(t) \rangle_t = 1$ . The usual normalization with respect to  $\langle \cdot, \cdot \rangle$  would have yielded the same results. We will return to this point in the next section while discussing perturbations of an operator with a finitely degenerate spectrum. An important feature of the rigorous theory sketched in Section 4.2 is that no choice of normalization is ever necessary.

## 4.1.2 Finitely Degenerate Spectrum

Here, we heuristically obtain the procedure for the calculation of the first two eigenvalue corrections for the case of a finitely degenerate eigenvalue of a self-adjoint operator. Compared to the nondegenerate case presented above, the material in this section contains significantly less standard and less rigorous material. Outside of questions of convergence, which are nontrivial, all of the results in the nondegenerate case were obtained through legitimate means. Here, however, many of the expressions obtained do not, strictly speaking, make sense. That will require us to reinterpret the results of our formal manipulations as the discussion goes on. Nonetheless, we believe that those formal manipulations are highly instructive, as they shed light on the otherwise quite obtuse rigorous theory developed in the subsequent sections.

Let  $\lambda_n^{(0)}$  be the eigenvalue of  $H^{(0)}$  corresponding to an  $N$ -dimensional eigenspace. If  $N > 1$ , the choice of the basis of eigenvectors in that eigenspace is no longer unique up to a scalar factor. This turns out to be a very important feature of degenerate perturbation theory, as making the correct choice of eigenbasis will be necessary to obtain the eigenvalue corrections. Consequently, we cannot make a choice of eigenbasis yet. This makes it impossible to use the strategy employed in the nondegenerate case, as it relies on a certain eigenbasis being chosen. Instead, we treat the eigenspaces as separate objects using projection operators.

Let  $P_n(t)$  be a family of projection operators onto the subspace spanned by the eigenvectors of  $H(t)$  belonging to the eigenvalue  $\lambda_n(t)$ . Such operators are termed eigenprojections. Then, the following equation holds.

$$H(t)P_n(t) = \lambda_n(t)P_n(t) \quad (4.17)$$

We assume, without proof for now, that  $P_n(t)$  can be written as a power series in  $t$ :

$$P_n(t) = P_n^{(0)} + tP_n^{(1)} + t^2P_n^{(2)} + \dots \quad (4.18)$$

For now, let  $P_n^{(0)}$  be the projection onto the  $\lambda_n^{(0)}$  eigenspace of  $H^{(0)}$ . This assumption will not hold in general, but is useful in the meantime. Combining Equations (4.17) and (4.18) yields, order by order in  $t$ ,

$$\begin{aligned} H^{(0)}P_n^{(0)} &= \lambda_n^{(0)}P_n^{(0)} \\ H^{(0)}P_n^{(1)} + H^{(1)}P_n^{(0)} &= \lambda_n^{(0)}P_n^{(1)} + \lambda_n^{(1)}P_n^{(0)} \\ H^{(0)}P_n^{(2)} + H^{(1)}P_n^{(1)} + H^{(2)}P_n^{(0)} &= \lambda_n^{(0)}P_n^{(2)} + \lambda_n^{(1)}P_n^{(1)} + \lambda_n^{(2)}P_n^{(0)}. \end{aligned} \quad (4.19)$$

Rewrite the first order equation as follows

$$\lambda_n^{(1)} P_n^{(0)} = (H^{(0)} - \lambda_n^{(0)}) P_n^{(1)} + H^{(1)} P_n^{(0)}. \quad (4.20)$$

Our goal is to isolate  $\lambda_n^{(1)}$ . Consider the above equation on the vector space  $P_n^{(0)}\mathcal{H}$ , that is on the  $\lambda_n^{(0)}$  eigenspace of  $H^{(0)}$ . Let the identity on  $P_n^{(0)}\mathcal{H}$  be denoted by  $\mathbb{1}_{P_n^{(0)}\mathcal{H}}$ . Then, multiplying by  $P_n^{(0)}$  on both sides, the equation becomes

$$\begin{aligned} \lambda_n^{(1)} \mathbb{1}_{P_n^{(0)}\mathcal{H}} &= [P_n^{(0)} (H^{(0)} - \lambda_n^{(0)})] P_n^{(1)} P_n^{(0)} + P_n^{(0)} H^{(1)} P_n^{(0)} P_n^{(0)} \\ &= P_n^{(0)} H^{(1)} P_n^{(0)}. \end{aligned} \quad (4.21)$$

Now, the left-hand side of the equation is a multiple of the identity, while the right-hand side is merely a diagonalizable matrix. Thus, the equality need not hold and the above expression need not make sense. We thus need to reinterpret this relation in a way that brings meaning back into it. This is precisely the situation referred to in the introduction of this section. Begin by diagonalizing  $P_n^{(0)} H^{(1)} P_n^{(0)}$  on  $P_n^{(0)}\mathcal{H}$ . Let  $\{\lambda_{ni}^{(1)}\}_{i=1}^I$  denote the set of the  $I$  distinct eigenvalues of  $P_n^{(0)} H^{(1)} P_n^{(0)}$  and let  $\{Q_{ni}^{(1)}\}_{i=1}^I$  denote the corresponding eigenprojections. It is very important not to confuse  $Q_{ni}^{(1)}$  with  $P_n^{(1)}$ , as those are very different objects, a fact which we emphasize by denoting them with different letters. Note that  $P_n^{(0)} Q_{ni}^{(1)} = Q_{ni}^{(1)}$ . Then, on  $Q_{ni}^{(1)}\mathcal{H}$ , Equation (4.21) can be recast in a meaningful form:

$$\lambda_n^{(1)} \mathbb{1}_{Q_{ni}^{(1)}\mathcal{H}} = Q_{ni}^{(1)} H^{(1)} Q_{ni}^{(1)} = \lambda_{ni}^{(1)} \mathbb{1}_{Q_{ni}^{(1)}\mathcal{H}}. \quad (4.22)$$

The values of  $\lambda_n^{(1)}$  that satisfy this equation for some  $i$  correspond to the eigenvalues of  $P_n^{(0)} H^{(1)} P_n^{(0)}$  on  $P_n^{(0)}\mathcal{H}$ . Thus, at first order, the  $N$ -fold degenerate eigenvalue  $\lambda_n^{(0)}$  splits into  $I$  eigenvalues of the form  $\lambda_n^{(0)} + t\lambda_{ni}^{(1)}$  for  $i = 1, \dots, I$ . The multiplicity of  $\lambda_n^{(0)} + t\lambda_{ni}^{(1)}$  is equal to the dimension of the subspace  $Q_{ni}^{(1)}\mathcal{H}$ . This gives us a procedure to calculate all the possible values of  $\lambda_n^{(1)}$ .

The second order corrections can further split the degenerate eigenvalues. Begin by considering the second order part of Equation (4.19) (third line) rewritten as follows:

$$\lambda_n^{(2)} P_n^{(0)} = (H^{(0)} - \lambda_n^{(0)}) P_n^{(2)} + (H^{(1)} - \lambda_n^{(1)}) P_n^{(1)} + H^{(2)} P_n^{(0)}. \quad (4.23)$$

Similarly to the first order case, consider this equation on the subspace  $Q_{ni}^{(1)}\mathcal{H}$ .

$$\begin{aligned}
\lambda_n^{(2)} \mathbb{1}_{Q_{ni}^{(1)} \mathcal{H}} &= \left[ \cancel{Q_{ni}^{(1)} (H^{(0)} - \lambda_n^{(0)})} \right] P_n^{(2)} Q_{ni}^{(1)} + Q_{ni}^{(1)} (H^{(1)} - \lambda_n^{(1)}) P_n^{(1)} Q_{ni}^{(1)} \\
&\quad + Q_{ni}^{(1)} H^{(2)} P_n^{(0)} Q_{ni}^{(1)} \\
&= Q_{ni}^{(1)} (H^{(1)} - \lambda_n^{(1)}) P_n^{(1)} Q_{ni}^{(1)} + Q_{ni}^{(1)} H^{(2)} Q_{ni}^{(1)}
\end{aligned} \tag{4.24}$$

Here, a situation similar to the one encountered during the computation of first order corrections arises. Immediately, one recognizes that the possible values of  $\lambda_n^{(2)}$  are given by the eigenvalues of the right-hand side on the subspace  $Q_{ni}^{(1)} \mathcal{H}$ . In order to proceed further, we need  $P_n^{(1)}$ . We will compute it using a method analogous to the one used to compute  $\psi_n^{(1)}$  in the nondegenerate case. Some modifications are necessary since the object in question is an operator rather than a vector. Consider the following expression for  $P_n^{(1)}$ .

$$\begin{aligned}
P_n^{(1)} &= \mathbb{1} P_n^{(1)} \mathbb{1} \\
&= \left( \sum_{k \neq n} P_k^{(0)} + P_n^{(0)} \right) P_n^{(1)} \left( \sum_{k \neq n} P_k^{(0)} + P_n^{(0)} \right) \\
&= \sum_{k \neq n} \sum_{l \neq n} P_k^{(0)} P_n^{(1)} P_l^{(0)} + \sum_{k \neq n} P_k^{(0)} P_n^{(1)} P_n^{(0)} + \sum_{k \neq n} P_n^{(0)} P_n^{(1)} P_k^{(0)} + P_n^{(0)} P_n^{(1)} P_n^{(0)}
\end{aligned} \tag{4.25}$$

We thus have to compute three type of terms of the form  $P_k^{(0)} P_n^{(1)} P_l^{(0)}$  with various admissible values of  $k$  and  $l$ .

We begin by arguing that  $P_n^{(0)} P_n^{(1)} P_n^{(0)} = 0$ . This is plausible, as it means that the first order eigenvector corrections will lie outside the unperturbed eigenspace. Note that we do not prove this. Recall the situation in the nondegenerate case. There, computing the form of the first order correction to the eigenvector required the additional assumption of a choice of normalization. In the usual quantum mechanical treatments of perturbation theory, say [28], this yields that  $\psi_n^{(1)}$  is orthogonal to  $\psi_n^{(0)}$ . In the more general case discussed in Section 4.1.1, an additional component lying in the original eigenspace must be introduced in order to compensate for the changing inner product. This is quite peculiar, as eigenvectors remain eigenvectors, no matter the choice of normalization. There should thus be a normalization-independent argument for the vanishing of  $P_n^{(0)} P_n^{(1)} P_n^{(0)}$ . This will have to wait for the rigorous theory sketched later in this chapter.

We can compute the rest of the terms through legitimate means. Multiplying the second line of Equation (4.19) by  $P_k^{(0)}$  on the left and  $P_l^{(0)}$  on the right results in the following.

$$\begin{aligned}
P_k^{(0)} H^{(0)} P_n^{(1)} P_l^{(0)} + P_k^{(0)} H^{(1)} P_n^{(0)} P_l^{(0)} &= \lambda_n^{(0)} P_k^{(0)} P_n^{(1)} P_l^{(0)} + \lambda_n^{(1)} P_k^{(0)} P_n^{(0)} P_l^{(0)} \\
\lambda_k^{(0)} P_k^{(0)} P_n^{(1)} P_l^{(0)} + \delta_{nl} P_k^{(0)} H^{(1)} P_n^{(0)} &= \lambda_n^{(0)} P_k^{(0)} P_n^{(1)} P_l^{(0)} + \lambda_n^{(1)} \delta_{kn} \delta_{nl} P_n^{(0)}
\end{aligned}$$

When  $k \neq n$  and  $l \neq n$ , this implies that  $P_k^{(0)} P_n^{(1)} P_l^{(0)} = 0$ . Thus, the first term in Equation (4.25) vanishes. When  $k \neq n$  and  $l = n$ , we get

$$P_k^{(0)} P_n^{(1)} P_n^{(0)} = \frac{P_k^{(0)} H^{(1)} P_n^{(0)}}{\lambda_n^{(0)} - \lambda_k^{(0)}}. \quad (4.26)$$

Using this in Equation (4.25) gives

$$P_n^{(1)} = \sum_{m \neq n} P_n^{(0)} P_n^{(1)} P_m^{(0)} + \sum_{k \neq n} \frac{P_k^{(0)} H^{(1)} P_n^{(0)}}{\lambda_n^{(0)} - \lambda_k^{(0)}}. \quad (4.27)$$

Since  $Q_{ni}^{(1)} \mathcal{H} \subset P_n^{(0)} \mathcal{H}$  and  $P_k^{(0)} P_n^{(0)} = \delta_{kn} P_k^{(0)}$ ,  $P_k^{(0)} Q_{ni}^{(1)} = \delta_{kn} Q_{ni}^{(1)}$ . Thus, the following holds

$$P_n^{(1)} Q_{ni}^{(1)} = \sum_{k \neq n} \frac{P_k^{(0)} H^{(1)} Q_{ni}^{(1)}}{\lambda_n^{(0)} - \lambda_k^{(0)}}. \quad (4.28)$$

We can now return to the calculation of the second order eigenvalue corrections. Using the above in Equation (4.24), together with the fact that  $Q_{ni}^{(1)} P_m^{(0)} = \delta_{mn} Q_{ni}^{(1)}$ , results in the final expression for the operator that needs to be diagonalized on  $Q_{ni}^{(1)} \mathcal{H}$  in order to obtain all the possible values of second order corrections to the  $\lambda_n^{(0)} + t\lambda_{ni}^{(1)}$  family of eigenvalues.

$$\lambda_n^{(2)} \mathbb{1}_{Q_{ni}^{(1)} \mathcal{H}} = \sum_{k \neq n} \frac{Q_{ni}^{(1)} H^{(1)} P_k^{(0)} H^{(1)} Q_{ni}^{(1)}}{\lambda_n^{(0)} - \lambda_k^{(0)}} + Q_{ni}^{(1)} H^{(2)} Q_{ni}^{(1)} \quad (4.29)$$

Let's now summarize the above discussion. For each distinct eigenvalue  $\lambda_n^{(0)}$  of  $H^{(0)}$  with eigenspace  $P_n^{(0)} \mathcal{H}$ , define the following two operators:

$$\boxed{
\begin{aligned}
\Lambda_n^{(1)} &= P_n^{(0)} H^{(1)} P_n^{(0)} \\
\Lambda_n^{(2)} &= \sum_{k \neq n} \frac{P_n^{(0)} H^{(1)} P_k^{(0)} H^{(1)} P_n^{(0)}}{\lambda_n^{(0)} - \lambda_k^{(0)}} + P_n^{(0)} H^{(2)} P_n^{(0)}.
\end{aligned}
} \quad (4.30)$$

The first order eigenvalue corrections are given by the eigenvalues  $\{\lambda_{ni}^{(1)}\}_i$  of  $\Lambda_n^{(1)}$  on the eigenspace  $P_n^{(0)}\mathcal{H}$  of  $\lambda_n^{(0)}$ . Let  $\{Q_{ni}^{(1)}\}_i$  denote the corresponding eigenprojections. Then, the second order eigenvalue corrections  $\{\lambda_{nij}^{(2)}\}_j$  to the family of eigenvalues  $\lambda_n^{(0)} + t\lambda_{ni}^{(1)}$  are given by the eigenvalues of  $\Lambda_n^{(2)}$  on the space  $Q_{ni}^{(1)}\mathcal{H}$ . In sum, the unperturbed eigenvalue  $\lambda_n^{(0)}$  splits into families indexed by  $i$ , which in turn split into families indexed by  $j$ . This process can be iterated further, if necessary. Since, the unperturbed eigenvalue is finitely degenerate, the last split must occur at some finite order. Note that this doesn't imply that the degeneracy will necessarily be lifted.

This diagonalization procedure can be expressed in a different way. Let  $\{\psi_n^{(0)}\}_n$  be an orthonormal eigenbasis of  $H^{(0)}$  such that  $\Lambda_n^{(1)}$  is diagonal on the eigenspaces of  $H^{(0)}$  and  $\Lambda_n^{(2)}$  is diagonal on the eigenspaces of  $\Lambda_n^{(1)}$ . Then, the eigenvalue corrections can be expressed in a form similar to the nondegenerate ones

$$\begin{aligned} \lambda_n^{(1)} &= \langle \psi_n^{(0)}, H^{(1)}\psi_n^{(0)} \rangle \\ \lambda_n^{(2)} &= \sum_{\substack{k \\ \lambda_k^{(0)} \neq \lambda_n^{(0)}}} \frac{\langle \psi_n^{(0)}, H^{(1)}\psi_k^{(0)} \rangle \langle \psi_k^{(0)}, H^{(1)}\psi_n^{(0)} \rangle}{\lambda_n^{(0)} - \lambda_k^{(0)}} + \langle \psi_n^{(0)}, H^{(2)}\psi_n^{(0)} \rangle. \end{aligned} \quad (4.31)$$

For our purposes, this last form is the most convenient one. Note that in the above we dropped the indices  $i$  and  $j$ , as the index  $n$  now goes through all basis vectors, rather than eigenspaces. Also, higher order perturbations will not spoil these formulas, as they will simply involve further choices of basis within the eigenspaces of  $\Lambda_n^{(2)}$ . No matter the basis thus chosen, it will yield the same results at the level of the first two corrections. This is enough for our purposes, as only the existence of a basis in which Equation (4.31) holds is needed.

## 4.2 Sketch of the Rigorous Treatment

In this section we sketch a rigorous treatment of analytic perturbation theory. More accurately, we sketch the process through which the eigenvalue corrections are computed. We do not discuss the existence of the perturbation expansions.

For an introduction to this subject in a quantum mechanical context, see [77]. A very thorough treatment of the finite dimensional case can be found in [7]. A reader interested in

the existence of perturbation expansions, but not in explicit computations of the eigenvalue corrections can consult [92, 93].

In Section 4.2.1, we begin by introducing holomorphic functional calculus, an important tool in perturbation theory. In particular, it allows to straightforwardly compute projectors onto the eigenspaces of an operator given that operator's resolvent. It can also be used to establish the convergence of the perturbation expansion [68], although we do not pursue this here. Then, in Section 4.2.2, we obtain power series for the resolvent of a perturbed operator. Together with the methods of holomorphic functional calculus, this yields power series for the eigenprojections of  $H(t)$ . Those series do not, however, distinguish between the different branches taken by a degenerate eigenvalue whose degeneracy has been lifted by the perturbation. Section 4.2.3 remedies that fact by a method known as reduction theory. There, the formulas for the first two eigenvalue corrections are obtained. Our discussion follows the one in [7].

## 4.2.1 Holomorphic Functional Calculus

Holomorphic functional calculus is a powerful formalism that allows one to define holomorphic functions of operators. It is necessary for our purposes, as it provides an invaluable tool for degenerate perturbation theory. Specifically, it is used to compute the projections onto the eigenspaces of perturbed operators.

First, an operator-valued integral is introduced. Then it is used to justify the existence of the Cauchy-Riesz integral, the main tool of the functional calculus. Finally, that integral is used to define holomorphic<sup>1</sup> functions of operators, including eigenprojections.

One can define the integral of a continuous family of operators on a Banach space as the limit of a Riemann sum.

**Definition 4.1** (Operator Integral). *Let  $B(t)$ , for  $t \in [a, b]$  be a family of bounded operators on some Banach space. Suppose this family to be continuous in the operator norm topology. Let  $P = \{[t_0, t_1], [t_1, t_2], \dots, [t_{n-1}, t_n]\}$  be a partition of the interval  $[a, b]$  into  $n$  parts such that  $a = t_0 < t_1 < \dots < t_n = b$ . For all  $i = 1 \dots n$ , pick  $t_i^* \in [t_{i-1}, t_i]$ . Under the condition that  $\lim_{n \rightarrow \infty} \sup_i (t_{i+1} - t_i) = 0$ , the integral of  $B(t)$  is defined as the  $n \rightarrow \infty$  limit of Riemann sums:*

$$\int_a^b B(t) dt = \lim_{n \rightarrow \infty} \sum_{i=1}^n B(t_i^*) (t_i - t_{i-1}) . \quad (4.32)$$

---

<sup>1</sup>In fact, one can use this to define piecewise-holomorphic functions of operators, that is functions that are defined on a domain of disjoint open sets on each of which they are holomorphic.



The above limit converges in the operator norm topology.

In the case of an operator on a finite dimensional space, this corresponds to an integration of the matrix elements. Of course, a Lebesgue analogue of the above integral can be defined. The Riemann variant is sufficient for our purposes.

Now, recall the Cauchy integral formula:

$$f(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\lambda)}{\lambda - z} d\lambda . \quad (4.33)$$

where  $f$  is a holomorphic function on some open set  $D \subset \mathbb{C}$ ,  $\Gamma$  is a simple closed curve in  $D$  that separates  $D$  into an interior  $I$  and an exterior  $E$  and  $z \in I$ . The goal is to obtain an analogous formula for functions of operators. Let  $T$  be an operator. Tentatively, write

$$f(T) = \frac{1}{2\pi i} \oint_{\Gamma} f(\lambda)(\lambda - T)^{-1} d\lambda . \quad (4.34)$$

It remains to make sense of this expression. Begin by noting that  $(\lambda - T)^{-1}$  is the resolvent of  $T$ :

$$R(\lambda, T) = (\lambda - T)^{-1} . \quad (4.35)$$

Recall that the resolvent set of  $T$  is the set of all  $\lambda \in \mathbb{C}$  such that  $(\lambda - T)$  has a dense range, is invertible on its range and the inverse  $(\lambda - T)^{-1}$  is bounded [79]. By definition, the spectrum of  $T$  is the complement of the resolvent set [79]. Thus, the right-hand side of the expression converges as an operator integral as long as the curve  $\Gamma$  lies in the resolvent set of  $T$  or, equivalently, outside of the spectrum of  $T$ . Consequently, the above integral exists.

It remains to ascertain that this definition of a function of an operator is meaningful. In order to do that, it is sufficient to verify that monomials in  $T$  are reproduced correctly. By linearity, so will be polynomials. We will only verify this in the case of bounded operators. First, we need a technical lemma.

**Lemma 4.2** (Resolvent for large  $\lambda$ ). *Let  $R(\lambda, T)$  be the resolvent of a bounded operator  $T$ . Then, for  $|\lambda| > \|T\|$ ,*

$$R(\lambda, T) = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}} . \quad (4.36)$$

*Proof.* Recall the Neumann series [79]:

$$(\mathbb{1} - B)^{-1} = \sum_{n=0}^{\infty} B^n . \quad (4.37)$$

This series converges for  $\|B\| < 1$ . Then, for  $|\lambda| > \|T\|$ ,

$$R(\lambda, T) = \frac{1}{\lambda} \left( \mathbb{1} - \frac{1}{\lambda} T \right)^{-1} = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}} . \quad (4.38)$$

□

We now can verify that the definition of a holomorphic function of an operator given above is meaningful.

**Lemma 4.3.** *Let  $T$  be bounded and let  $R(\lambda, T)$  be its resolvent. Let  $\Gamma$  be a simple closed curve lying outside the region  $|\lambda| > \|T\|$ . Then,*

$$\frac{1}{2\pi i} \oint_{\Gamma} \lambda^m R(\lambda, T) d\lambda = T^m . \quad (4.39)$$

*Proof.* From Lemma 4.2, the following series expansion is valid for  $|\lambda| > \|T\|$ :

$$\lambda^m R(\lambda, T) = \sum_{n=0}^{\infty} \lambda^{m-n-1} T^n . \quad (4.40)$$

By Cauchy's residue theorem,

$$\frac{1}{2\pi i} \oint_{\Gamma} \lambda^{m-n-1} d\lambda = \delta_{mn} . \quad (4.41)$$

Then, the desired result is obtained:

$$\frac{1}{2\pi i} \oint_{\Gamma} \lambda^m R(\lambda, T) d\lambda = \sum_{n=0}^{\infty} T^n \frac{1}{2\pi i} \oint_{\Gamma} \lambda^{m-n-1} d\lambda = T^m . \quad (4.42)$$

□

In the case of unbounded operators, additional complications arise. For our purposes, however, only operators that can be expressed as weighted sums of projections are of interest. This significantly simplifies matters.

**Definition 4.4.** (*Holomorphic function of a weighted sum of projections*) Let  $T$  be an operator on a Hilbert space  $\mathcal{H}$  that can be represented as a weighted sum of mutually orthogonal projections  $\{P_n\}_n$ :

$$T = \sum_n \lambda_n P_n , \quad (4.43)$$

where the  $\{\lambda_n\}_n$  are distinct complex numbers such that, for each  $n$ , there exists  $\varepsilon = \varepsilon(n) > 0$  such that  $|\lambda_n - \lambda_m| > \varepsilon$  for  $m \neq n$ . Let  $f(\lambda)$  be a function that is holomorphic in an open neighborhood of each  $\lambda_n$ . For each  $n$ , let  $\Gamma_n$  be a simple closed curve enclosing  $\lambda_n$  and no other  $\lambda_m$ . Suppose that each  $\Gamma_n$  is entirely contained in a domain of holomorphicity of  $f(\lambda)$ . Then  $f(T)$  is defined as

$$f(T) = \frac{1}{2\pi i} \oint_{\bigcup_n \Gamma_n} f(\lambda) R(\lambda, T) d\lambda . \quad (4.44)$$

For our purposes, the case  $f(\lambda) = 1$  is of special interest, as it can be used to construct projections onto eigenspaces. For  $T$  as above,

$$R(\lambda, T) = \sum_m \frac{P_m}{\lambda - \lambda_m} . \quad (4.45)$$

Applying Cauchy's integral formula (Equation (4.33)) to a single curve  $\Gamma_n$  (rather than  $\bigcup_n \Gamma_n$ ) yields

$$\frac{1}{2\pi i} \oint_{\Gamma_n} R(\lambda, T) d\lambda = \frac{1}{2\pi i} \oint_{\Gamma_n} \sum_m \frac{P_m}{\lambda - \lambda_m} d\lambda = P_n . \quad (4.46)$$

Similarly, by choosing an appropriate integration curve  $\Gamma$ , the same approach can be used to build projectors onto multiple eigenspaces at once. This is done by picking  $\Gamma$  such that it encloses the eigenvalues corresponding to the eigenspaces of interest and no other eigenvalue. This crucial fact will be used later.

The key feature of this way of obtaining projection operators is that it turns an ordinarily quite involved process into a linear operation, provided that the resolvent  $(\lambda - T)^{-1}$

is known. In the next section, we obtain series expansions for the resolvent for the purpose of using them in such integrals.

## 4.2.2 Power Series of the Resolvent and Eigenprojections

The goal of this section is to compute the first few terms of the power series of the resolvent of a perturbed operator. Thanks to the theory developed in the previous section, this will allow us to compute the power series for the eigenprojections of said operator.

Let  $T$  and  $T_0$  be two neighboring operators, in a sense to be specified below. Our goal is to obtain a series representation of  $R(\lambda, T)$  in terms of  $R(\lambda, T_0)$ . Assume that  $\lambda$  is in the resolvent sets of both operators. We begin by a formal computation. The relevant convergence criteria are discussed below.

Consider the following formal manipulations.

$$\begin{aligned}
 R(\lambda, T) &= (\lambda - T)^{-1} = ((\lambda - T_0) - (T - T_0))^{-1} \\
 &= \left( (\mathbb{1} - (T - T_0)(\lambda - T_0)^{-1})(\lambda - T_0) \right)^{-1} \\
 &= \left( (\mathbb{1} - (T - T_0)R(\lambda, T_0))(\lambda - T_0) \right)^{-1} \\
 &= (\lambda - T_0)^{-1} (\mathbb{1} - (T - T_0)R(\lambda, T_0))^{-1}
 \end{aligned} \tag{4.47}$$

Applying the Neumann series (Equation (4.37)) to the above gives

$$R(\lambda, T) = R(\lambda, T_0) \sum_{n=0}^{\infty} ((T - T_0)R(\lambda, T_0))^n. \tag{4.48}$$

In the case of bounded  $T$  and  $T_0$ , this series is guaranteed to converge for  $\|T - T_0\| < \|R(\lambda, T_0)\|^{-1}$ . In the unbounded case, the situation is more complicated and will not be treated here. The interested reader can consult [68].

We can now return to our explicitly perturbative goals. Set

$$\begin{aligned}
 T_0 &= H^{(0)} \\
 T = H(t) &= H^{(0)} + tH^{(1)} + t^2H^{(2)} + \dots
 \end{aligned} \tag{4.49}$$

The resolvent of  $H(t)$  can then be expressed as

$$R(\lambda, H(t)) = R(\lambda, H^{(0)}) \sum_{n=0}^{\infty} \left( \sum_{i=1}^{\infty} t^i H^{(i)} R(\lambda, H^{(0)}) \right)^n . \quad (4.50)$$

We are primarily interested in  $R(\lambda, H(t))$  as a power series in  $t$ .

$$R(\lambda, H(t)) = \sum_{i=1}^{\infty} t^i R_i(\lambda) \quad (4.51)$$

The coefficients  $\{R_i(\lambda)\}_{i=1}^{\infty}$  can be read off from Equation (4.50). The first three coefficients are

$$\begin{aligned} R_0(\lambda) &= R(\lambda, H^{(0)}) \\ R_1(\lambda) &= R_0(\lambda) H^{(1)} R_0(\lambda) \\ R_2(\lambda) &= R_0(\lambda) H^{(2)} R_0(\lambda) + R_0(\lambda) H^{(1)} R_0(\lambda) H^{(1)} R_0(\lambda) . \end{aligned} \quad (4.52)$$

Let  $\lambda_n^{(0)}$  be an eigenvalue of  $H^{(0)}$  isolated from the rest of the spectrum. Let  $[\lambda_n^{(0)}](t)$  denote the family of eigenvalues of  $H(t)$  that go to  $\lambda_n^{(0)}$  in the limit  $t \rightarrow 0$ . The theorems establishing the existence of perturbation expansions guarantee that, for sufficiently small  $t$ , there exists  $\delta > 0$  such that the only eigenvalues of  $H(t)$  in the ball  $\|\lambda - \lambda_n^{(0)}\| < \delta$  are the eigenvalues in  $[\lambda_n^{(0)}](t)$  [93].

Let  $\Gamma_n$  be a simple closed curve enclosing  $\lambda_n^{(0)}$ , but no other unperturbed eigenvalue. Consequently, for sufficiently small  $t$ , the same curve  $\Gamma_n$  will enclose all of the  $[\lambda_n^{(0)}](t)$  and no other perturbed eigenvalue. Thus, the projection onto the eigenspaces of  $H(t)$  corresponding to the eigenvalues  $[\lambda_n^{(0)}](t)$  can be computed using the curve  $\Gamma_n$  and Equation (4.46). This yields

$$P_n(t) = \oint_{\Gamma_n} R(\lambda, H(t)) d\lambda . \quad (4.53)$$

Note that this is not an eigenprojection, as it does not correspond to a single eigenvalue, except for the unperturbed case  $t = 0$ .  $P_n(t)$  can be expanded in power series in  $t$ .

$$P_n(t) = \sum_{i=1}^{\infty} t^i P_n^{(i)} \quad (4.54)$$

The coefficients  $P_n^{(i)}$  can be determined from the series for the resolvent. From Equation (4.52), one immediately deduces that  $P_n^{(0)}$  is the projector onto the unperturbed eigenspace, as expected. The expression for  $P_n^{(1)}$  will require a bit more work.

$$P_n^{(1)} = \frac{1}{2\pi i} \oint_{\Gamma_n} R_0(\lambda) H^{(1)} R_0(\lambda) d\lambda \quad (4.55)$$

Assuming that  $H^{(0)}$  is a weighted sum of projections, one has

$$\begin{aligned} H^{(0)} &= \sum_n \lambda_n^{(0)} P_n^{(0)} \\ R_0(\lambda) &= \sum_n \frac{P_n^{(0)}}{\lambda - \lambda_n^{(0)}} . \end{aligned} \quad (4.56)$$

The expression for  $P_n^{(1)}$  then becomes

$$\begin{aligned} P_n^{(1)} &= \frac{1}{2\pi i} \oint_{\Gamma_n} \sum_k \frac{P_k^{(0)}}{\lambda - \lambda_k^{(0)}} H^{(1)} \sum_m \frac{P_m^{(0)}}{\lambda - \lambda_m^{(0)}} d\lambda \\ &= \frac{1}{2\pi i} \oint_{\Gamma_n} \sum_k \sum_m \frac{P_k^{(0)}}{\lambda - \lambda_k^{(0)}} H^{(1)} \frac{P_m^{(0)}}{\lambda - \lambda_m^{(0)}} d\lambda \\ &= \sum_k \sum_m P_k^{(0)} H^{(1)} P_m^{(0)} \frac{1}{2\pi i} \oint_{\Gamma_n} \frac{1}{\lambda - \lambda_k^{(0)}} \frac{1}{\lambda - \lambda_m^{(0)}} d\lambda . \end{aligned} \quad (4.57)$$

A straightforward application of Cauchy's integral formula yields

$$P_n^{(1)} = \sum_{m \neq n} \frac{P_n^{(0)} H^{(1)} P_m^{(0)}}{\lambda_n^{(0)} - \lambda_m^{(0)}} + \sum_{k \neq n} \frac{P_k^{(0)} H^{(1)} P_n^{(0)}}{\lambda_n^{(0)} - \lambda_k^{(0)}} . \quad (4.58)$$

Compare this to Equations (4.25) and (4.27) from Section 4.1.2. The heuristic approach correctly determined the second term. The first term will turn out not to matter, as in the heuristic case. Most importantly, we never had to specifically require that  $P_n^{(0)} P_n^{(1)} P_n^{(0)} = 0$ . Recall that this was the analogue of fixing a particular normalization of  $\psi_n(t)$  in the heuristic approach for the degenerate case. The current approach makes such choices unnecessary.

### 4.2.3 Reduction Theory

In this section, we sketch what is known as the reduction process, or reduction theory. This is the rigorous version of the heuristic approach to degenerate perturbation theory adopted in Section 4.1.2. We present an abridged version of the very thorough discussion found in [7].

In the previous section, we obtained a power series for  $P_n(t)$ , the projector onto the eigenspaces of  $H(t)$  associated to the eigenvalues  $[\lambda_n^{(0)}](t)$ . We haven't however yet separated  $P_n(t)$  into subprojections onto the individual eigenspaces of  $H(t)$ . Neither have we computed the associated eigenvalues. These two tasks turn out to be related.

The main idea of reduction theory is to iterate the process outlined in the previous section. As was shown there, it is straightforward to obtain the first order term of the projection onto the subspace associated to the eigenvalues  $[\lambda_n^{(0)}](t)$ . This is the operation that will be iterated.

Consider the following family of operators.

$$B_n(t) = \frac{1}{t} (H(t) - \lambda_n^{(0)}) P_n(t) \quad (4.59)$$

This can be written as a power series

$$B_n(t) = \sum_{k=0} t^k B_n^{(k)} . \quad (4.60)$$

Since  $H(t)P_n(t) = P_n(t)H(t)$ ,  $B_n(t)P_n(t) = P_n(t)B_n(t)$ . This can be used to express the lowest order term of the series for  $B_n(t)$  in a particularly convenient form.

$$\begin{aligned} B_n^{(0)} &= B_n(0) = P_n(0)B_n(0) \\ &= P_n^{(0)} (H^{(1)}P_n^{(0)} + (H^{(0)} - \lambda_n^{(0)}) P_n^{(1)}) \\ &= P_n^{(0)} H^{(1)} P_n^{(0)} + \cancel{[P_n^{(0)} (H^{(0)} - \lambda_n^{(0)})] P_n^{(1)}} \\ &= P_n^{(0)} H^{(1)} P_n^{(0)} \end{aligned} \quad (4.61)$$

Thus,  $B_n(t)$  can be viewed as a perturbation of  $B_n^{(0)} = P_n^{(0)} H^{(1)} P_n^{(0)}$ . The process of perturbation theory developed so far can begin anew on this operator. Notice that  $B_n^{(0)}$  acts like 0 outside of the  $\lambda_n^{(0)}$  eigenspace of  $H^{(0)}$ . It will thus typically have a large eigenspace with eigenvalue 0. We are not interested in that eigenspace. Consequently, we start by

determining its eigenvalues  $\{\lambda_{n\rho}^{(0)}\}_\rho$  and its eigenprojections  $\{Q_{n\rho}^{(0)}\}_\rho$  on the subspace  $P_n^{(0)}\mathcal{H}$ . For technical reasons that will become clear shortly, we assume all of the  $\{\lambda_{n\rho}^{(0)}\}_\rho$  to be nonzero. This can be done without loss of generality as one can always substitute the study of  $B(t)$  for the study of  $B(t) + \alpha P_n^{(0)}$  with a suitable constant  $\alpha$ . This will merely shift the  $\{\lambda_{n\rho}^{(0)}\}_\rho$  by  $\alpha$ , a change that can be undone later. The goal of this procedure is to separate the eigenvalues of  $B_n^{(0)}$  on  $P_n^{(0)}\mathcal{H}$  from its zero eigenvalue on  $(\mathbb{1} - P_n^{(0)})\mathcal{H}$ .

Let  $\Gamma_{n\rho}$  be a simple closed curve separating  $\lambda_{n\rho}^{(0)}$  from the rest of the spectrum of  $B_n^{(0)}$ , including 0. Let  $[\lambda_{n\rho}](t)$  denote the eigenvalues of  $B_n(t)$  that go to  $\lambda_{n\rho}$  as  $t \rightarrow 0$ . The projection onto all the eigenspaces of  $B_n(t)$  associated with  $[\lambda_{n\rho}](t)$  can then be defined as follows.

$$Q_{n\rho}(t) = \oint_{\Gamma_{n\rho}} R(\lambda, B(t)) d\lambda \quad (4.62)$$

This expression is the reason why we required that the  $\{\lambda_{n\rho}^{(0)}\}_\rho$  be nonzero.  $Q_{n\rho}(t)$  can be expanded in power series in  $t$ .

$$Q_{n\rho}(t) = \sum_{k=0} t^k Q_{n\rho}^{(k)} \quad (4.63)$$

We are now at the same point in the analysis of  $B(t)$  as we were in the analysis of  $H(t)$  at the start of this section. Correspondingly, we retrace our steps and define a new family of operators:

$$C_{n\rho}(t) = \frac{1}{t} (B_n(t) - \lambda_{n\rho}^{(0)}) Q_{n\rho}(t) . \quad (4.64)$$

By an argument similar to the one for  $B_n(t)$ , this is a perturbation of  $C_{n\rho}^{(0)} = Q_{n\rho}^{(0)} B_n^{(1)} Q_{n\rho}^{(0)}$ . Denote the eigenvalues of  $C_{n\rho}^{(0)}$  on the subspace  $Q_{n\rho}^{(0)}\mathcal{H}$  by  $\{\lambda_{n\rho\mu}^{(0)}\}_\mu$ . Let  $\{Q_{n\rho\mu}^{(0)}\}_\mu$  be the associated eigenprojections. One can then proceed as with  $B(t)$ . Without loss of generality, assume that none of the  $\{\lambda_{n\rho\mu}^{(0)}\}_\mu$  vanish. Then, the projections on the  $[\lambda_{n\rho\mu}](t)$  eigenvalues of  $C_{n\rho}^{(0)}$  are given by

$$Q_{n\rho\mu}(t) = \oint_{\Gamma_{n\rho\mu}} R(\lambda, C(t)) d\lambda . \quad (4.65)$$



The pattern of the reduction process should be apparent by now. It can be continued indefinitely. We will not pursue this here. Instead, we will now show that the eigenvalue corrections obtained through the reduction process match the ones derived heuristically in Section 4.1.2.

The corrections to the eigenvalue  $\lambda_n^{(0)}$  of  $H^{(0)}$  are given by

$$\begin{aligned}\lambda_n(t) &= \lambda_n^{(0)} + t\lambda_n^{(1)} + t^2\lambda_n^{(2)} + \dots \\ \lambda_n^{(1)} &= \lambda_{n\rho}^{(0)} \\ \lambda_n^{(2)} &= \lambda_{n\rho\mu}^{(0)} .\end{aligned}\tag{4.66}$$

Thus, the possible values of  $\lambda_n^{(1)}$  are given by the eigenvalues of  $P_n^{(0)}H^{(1)}P_n^{(0)}$  on  $P_n^{(0)}\mathcal{H}$ . This is in agreement with the heuristic results. Similarly, the possible values of  $\lambda_n^{(2)}$  are given by the eigenvalues of  $Q_{n\rho}^{(0)}B_n^{(1)}Q_{n\rho}^{(0)}$  on  $Q_{n\rho}^{(0)}\mathcal{H}$ . It remains to compute  $Q_{n\rho}^{(0)}B_n^{(1)}Q_{n\rho}^{(0)}$  in order to confirm the results of our heuristic derivation.

From Equation (4.59), one deduces

$$B_n^{(1)} = (H^{(0)} - \lambda_n^{(0)})P_n^{(2)} + H^{(1)}P_n^{(1)} + H^{(2)}P_n^{(0)} .\tag{4.67}$$

Using the expression for  $P_n^{(1)}$  obtained in Equation (4.58) yields

$$Q_{n\rho}^{(0)}B_n^{(1)}Q_{n\rho}^{(0)} = \sum_{k \neq n} \frac{Q_{n\rho}^{(0)}H^{(1)}P_k^{(0)}H^{(1)}Q_{n\rho}^{(0)}}{\lambda_n^{(0)} - \lambda_k^{(0)}} + Q_{n\rho}^{(0)}H^{(2)}Q_{n\rho}^{(0)} .\tag{4.68}$$

This is the same expression as the one obtained heuristically in Section 4.1.2. This concludes our treatment of perturbation theory.

# Chapter 5

## Perturbations of the Laplace-Beltrami Spectrum

In this chapter, we finally get to the meat of our investigations. Indeed, by the end of Section 5.3, we will have obtained the main results of this thesis. Namely, we will have shown that isospectral sets of conformally equivalent metrics on surfaces contain no convex subsets.

Before achieving that, however, we need to obtain the expressions for the corrections of the Laplace-Beltrami eigenvalues subject to a perturbation of the Riemannian metric. We do this in two steps. First, in Section 5.1, we obtain the expressions for the corrections of the Laplace-Beltrami operator itself. Then, in Section 5.2, we use the formulas obtained in Chapter 4 to compute the eigenvalue corrections. While the proof of our main results only requires the expressions for the case of conformal perturbations in dimension 2, we also discuss the general case. This is done for the sake of completeness.

Note that all the perturbations series studied here have been shown to converge in some neighborhood of  $t = 0$  [6].

### 5.1 Perturbations of the Laplace-Beltrami Operator

In this section, we carry out a calculation of the perturbations of the Laplace-Beltrami operator induced by a perturbation of the metric. We begin by considering the simplest case: a conformal perturbation of the metric on a manifold of dimension 2. Then, we compute the general case.

While we are by no means the first to compute the Laplace-Beltrami corrections (see [9, 12, 13]), such results are not commonly found in the literature. Moreover, to the best of our knowledge, the approach used here is original, except for the case of conformal perturbations on surfaces.

Before proceeding further, we establish some notation. Given a Riemannian metric  $g$ , the induced metric on 1-forms is denoted  $\bar{g}$ . Thus, in local coordinates,  $g$  is the lower indices metric  $g_{ij}$ , while  $\bar{g}$  is the upper-indices metric  $g^{ij}$ . We sometimes refer to  $\bar{g}$  as the inverse metric. This terminology is motivated by the fact that, in the local coordinate basis, the matrices  $g_{ij}$  and  $g^{ij}$  are inverses of each other. Moreover, we continue to use the notation for orders of perturbation established in Chapter 4:

$$\begin{aligned} g &= g^{(0)} + tg^{(1)} + t^2g^{(2)} + \dots \\ \bar{g} &= \bar{g}^{(0)} + t\bar{g}^{(1)} + t^2\bar{g}^{(2)} + \dots \\ \Delta &= \Delta^{(0)} + t\Delta^{(1)} + t^2\Delta^{(2)} + \dots \end{aligned} \tag{5.1}$$

$\Delta^{(0)}$  is always the Laplace-Beltrami operator induced by  $g^{(0)}$ .

### 5.1.1 Conformal Perturbation on Surfaces

Let  $(\mathcal{M}, g^{(0)})$  be a compact oriented Riemannian manifold without boundary. Suppose that  $\mathcal{M}$  is of dimension  $N = 2$ . Let  $\psi \in C^\infty(\mathcal{M})$ . Recall the local expression for the Laplace-Beltrami operator for a metric  $g$ :

$$\Delta\psi = -\frac{1}{\sqrt{|\det(g)|}}\partial_i\left(\sqrt{|\det(g)|}g^{ij}\partial_j\psi\right). \tag{5.2}$$

Let  $f \in C^\infty(\mathcal{M})$ ,  $f > 0$ . Then,  $fg$  is a valid metric on  $\mathcal{M}$ . The following formulas can easily be shown to hold:

$$\begin{aligned} |\det(fg)| &= f^2|\det(g)| \\ [fg]^{ij} &= \frac{1}{f}g^{ij}. \end{aligned} \tag{5.3}$$

Consequently, the Laplace-Beltrami operator induced by  $fg$  is

$$\begin{aligned}\Delta\psi &= -\frac{1}{\sqrt{f^2|\det(g)|}}\partial_i\left(\sqrt{f^2|\det(g)|}\frac{1}{f}g^{ij}\partial_j\psi\right) \\ &= \frac{1}{f}\Delta^{(0)}\psi.\end{aligned}\tag{5.4}$$

This is inconvenient. It would be significantly simpler to have a formula linear in  $f$ . This can be achieved by considering  $f\bar{g}$ , rather than  $fg$ . Under such a transformation,

$$\Delta\psi = f\Delta^{(0)}\psi.\tag{5.5}$$

The usage of  $\bar{g}$ , rather than  $g$ , will be more convenient in the general case as well. Now, let  $\{f^{(i)}\}_{i=1}^\infty$  be a sequence of smooth functions. For sufficiently small  $|t|$ , consider the following perturbation of  $\bar{g}^{(0)}$

$$\bar{g} = \left(1 + \sum_{i=1}^\infty t^i f^{(i)}\right)\bar{g}^{(0)}.\tag{5.6}$$

This induces the following perturbation of the Laplacian:

$$\Delta = \left(1 + \sum_{i=1}^\infty t^i f^{(i)}\right)\Delta^{(0)}.\tag{5.7}$$

Order by order in  $t$ ,

$$\boxed{\Delta^{(i)} = f^{(i)}\Delta^{(0)}, \text{ for all } i \geq 1.}\tag{5.8}$$

This is an exceptionally simple expression. As will be seen below, this is special to the case of conformal transformations in dimension  $N = 2$ . This simplicity is part of what enables us to prove the results of Section 5.3.

### 5.1.2 General Perturbation of the Metric

In this section, we compute the first two corrections to the Laplace-Beltrami operator induced by a perturbation of the metric.

Let  $(\mathcal{M}, g^{(0)})$  be the unperturbed manifold, assumed to be of dimension  $N$ . It is convenient to rewrite the coordinate expression of the Laplace-Beltrami operator as follows:

$$\begin{aligned}
\Delta\psi &= -\frac{1}{\sqrt{|\det(g)|}}\partial_i\left(\sqrt{|\det(g)|}g^{ij}\partial_j\psi\right) \\
&= -\left(\frac{\partial_i\sqrt{|\det(g)|}}{\sqrt{|\det(g)|}}\right)g^{ij}\partial_j\psi - \partial_i(g^{ij}\partial_j\psi) \\
&= -\left(\partial_i\log\left(\sqrt{|\det(g)|}\right)\right)g^{ij}\partial_j\psi - \partial_i(g^{ij}\partial_j\psi).
\end{aligned} \tag{5.9}$$

This expression contains two distinct types of quantities that change under perturbation of the metric, the inverse metric  $g^{ij}$  and the logarithm of the square root of the determinant of the metric  $\log\left(\sqrt{|\det(g)|}\right)$ . Neither are linear in perturbations of the metric  $g$ , which is quite inconvenient. This reproduces the issue that we already faced in the previous section. The solution to this problem is the same: considering perturbations of the inverse metric  $\bar{g}$ . Let  $\{\bar{g}^{(i)}\}_{i=1}^{\infty}$  be a sequence of symmetric contravariant two-tensors. They will be the corrections to  $\bar{g}^{(0)}$  for sufficiently small  $|t|$ :

$$\bar{g} = \sum_{i=0}^{\infty} \bar{g}^{(i)} t^i. \tag{5.10}$$

Only  $\log\left(\sqrt{|\det(g)|}\right)$  remains to be expanded in power series in  $t$ . For that purpose, it is convenient to write the perturbation of  $\bar{g}$  in multiplicative form. Let  $\{h^{(i)}\}_{i=1}^{\infty}$  be the  $(1, 1)$  tensors defined by

$$\bar{g}^{(i)kl} = -\bar{g}^{(0)kp}h_p^{(i)l}. \tag{5.11}$$

For simplicity of notation, we will drop the indices and denote  $\bar{g}^{(0)}h^{(i)} = \bar{g}^{(0)kp}h_p^{(i)l}$ . This allows us to write

$$\bar{g} = \bar{g}^{(0)}\left(\mathbb{1} - t\sum_{j=1}^{\infty} h^{(j)}t^{j-1}\right). \tag{5.12}$$

We are now in position to compute the determinant of  $g$ .

$$\begin{aligned}
\det(g) &= \det(\bar{g})^{-1} \\
&= \left( \det(\bar{g}^{(0)}) \det \left( \mathbb{1} - t \sum_{j=1}^{\infty} h^{(j)} t^{j-1} \right) \right)^{-1}
\end{aligned} \tag{5.13}$$

Recall [58] that for any square matrix  $A$ ,  $\det(A) = \exp(\text{Tr}(\log(A)))$ , as long as the appropriate series converge, which is the case here as  $t$  is assumed near 0. The series expression for the matrix logarithm [58] is

$$\log(\mathbb{1} - A) = - \sum_{m=1}^{\infty} \frac{1}{m} A^m . \tag{5.14}$$

Notice that our choice of sign for  $h^i$  is adapted for this formula. Also, note that the trace of  $h^{(i)}$  is defined:  $\text{Tr}(h^{(i)}) = h_k^{(i)k}$ . Then, using this series, we obtain

$$\det \left( \mathbb{1} - t \sum_{j=1}^{\infty} h^{(j)} t^{j-1} \right) = \exp \left( - \sum_{m=1}^{\infty} \frac{t^m}{m} \text{Tr} \left( \left( \sum_{j=1}^{\infty} h^{(j)} t^{j-1} \right)^m \right) \right) . \tag{5.15}$$

Consequently,  $\log \left( \sqrt{|\det(g)|} \right)$  becomes, up to second order in the perturbation parameter  $t$ ,

$$\begin{aligned}
\log \left( \sqrt{|\det(g)|} \right) &= \log \left( \sqrt{|\det(g^{(0)})|} \right) + \frac{1}{2} \sum_{m=1}^{\infty} \frac{t^m}{m} \text{Tr} \left( \left( \sum_{j=1}^{\infty} h^{(j)} t^{j-1} \right)^m \right) \\
&\approx \log \left( \sqrt{|\det(g^{(0)})|} \right) + t \left( \frac{1}{2} \text{Tr}(h^{(1)}) \right) \\
&\quad + t^2 \left( \frac{1}{2} \text{Tr}(h^{(2)}) + \frac{1}{4} \text{Tr}(h^{(1)} h^{(1)}) \right) + \dots
\end{aligned} \tag{5.16}$$

We are now ready to compute the first two corrections to the Laplace-Beltrami operator. Together with the original expansion for the inverse metric (Equation (5.10)), the above expression for  $\log \left( \sqrt{|\det(g)|} \right)$  can be inserted into Equation (5.9) to obtain the induced perturbation of the Laplacian.

$$\Delta^{(1)}\psi = -\frac{1}{\sqrt{|\det(g^{(0)})|}}\partial_i \left( \sqrt{|\det(g^{(0)})|}\bar{g}^{(1)ij}\partial_j\psi \right) - \frac{1}{2} (\partial_i \text{Tr}(h^{(1)})) \bar{g}^{(0)ij}\partial_j\psi \quad (5.17)$$

$$\begin{aligned} \Delta^{(2)}\psi = & -\frac{1}{\sqrt{|\det(g^{(0)})|}}\partial_i \left( \sqrt{|\det(g^{(0)})|}\bar{g}^{(2)ij}\partial_j\psi \right) - \frac{1}{2} (\partial_i \text{Tr}(h^{(2)})) \bar{g}^{(0)ij}\partial_j\psi \\ & - \frac{1}{2} (\partial_i \text{Tr}(h^{(1)})) \bar{g}^{(1)ij}\partial_j\psi - \frac{1}{4} (\partial_i \text{Tr}(h^{(1)}h^{(1)})) \bar{g}^{(0)ij}\partial_j\psi \end{aligned} \quad (5.18)$$

The above formulas are valid for arbitrary perturbations of the metric. They become significantly simpler when specialized to conformal perturbations. As before, let  $\{f^{(i)}\}_{i=1}^{\infty}$  be smooth functions and set  $\bar{g}^{(i)} = f^{(i)}\bar{g}^{(0)}$  and, consequently,  $h^{(i)} = -f^{(i)}\mathbb{1}$ . In other words,

$$\bar{g} = \bar{g}^{(0)} \left( 1 + \sum_{i=1}^{\infty} t^i f^{(i)} \right). \quad (5.19)$$

The traces in Equations (5.17) and (5.18) can then be easily computed:

$$\begin{aligned} \text{Tr}(h^{(1)}) &= -f^{(1)}N \\ \text{Tr}(h^{(2)}) &= -f^{(2)}N \\ \text{Tr}(h^{(1)}h^{(1)}) &= \text{Tr}\left((f^{(1)})^2\mathbb{1}\right) = (f^{(1)})^2 N. \end{aligned} \quad (5.20)$$

Substituting these expressions into Equations (5.17) and (5.18) yields, after some simplifications,

$$\begin{aligned} \Delta_c^{(1)}\psi &= f^{(1)} (\Delta^{(0)}\psi) + \frac{N-2}{2} (\partial_i f^{(1)}) \bar{g}^{(0)ij} (\partial_j\psi) \\ \Delta_c^{(2)}\psi &= f^{(2)} (\Delta^{(0)}\psi) + \frac{N-2}{2} (\partial_i f^{(2)}) \bar{g}^{(0)ij} (\partial_j\psi). \end{aligned} \quad (5.21)$$

The subscript  $c$  in  $\Delta_c^{(1)}$  and  $\Delta_c^{(2)}$  indicates that the perturbation is strictly conformal. This notation will become useful shortly. Notice that, unlike in Equations (5.17) and (5.18),

there is no second order contribution to the Laplacian arising from the first order correction to the metric. When  $N = 2$ , the second terms in the expressions for  $\Delta^{(1)}$  and  $\Delta^{(2)}$  vanish. The expressions derived in Section 5.1.1 are thus recovered.

The relative simplicity of the conformally perturbed expressions raises the following question: can one more easily express general perturbations of the Laplace-Beltrami operator by decomposing  $\bar{g}^{(i)}$  into a conformal component and a remainder? Indeed one can. In order to do so, one must obtain the general properties of said remainder.

Set  $h_p^{(i)l} = -f^{(i)}\delta_p^l + r_p^{(i)l}$  with  $f^{(i)}$  defined as above. Consequently,  $\bar{g}^{(i)kl} = f^{(i)}\bar{g}^{(0)kl} - \bar{g}^{(0)kp}r_p^{(i)l}$ . Of course, we require  $\tilde{g}^{(i)kl} = \bar{g}^{(0)kp}r_p^{(i)l}$  to be symmetric. Otherwise, the perturbation would fail to be symmetric. Moreover, we require  $r_p^{(i)l}$  to be orthogonal to  $\delta_p^l$  in the inner product induced by  $g^{(0)}$ :

$$0 = r_p^{(i)l}\delta_l^p = r_p^{(i)p} = \text{Tr}(r) . \quad (5.22)$$

Thus,  $r^{(i)}$  is traceless. This is quite encouraging, considering the prevalence of traces in Equations (5.17) and (5.18).

Before proceeding further, a short digression is in order. Some metric perturbations can be seen as arising from the pullback of infinitesimal diffeomorphisms, that is the pullback induced by the flow of a vector field. These perturbations can be ignored, as they lead to isometric and thus isospectral manifolds. However, these perturbations are not orthogonal to the space of conformal perturbations. One thus cannot split the set of possible metric perturbations (symmetric covariant two-tensors) into those conformal to the unperturbed metric, those induced by diffeomorphisms and the rest. Instead, a different  $L_2(\mathcal{M})$ -orthogonal decomposition holds. The space of two-tensors splits into tensors conformal to the unperturbed metric, tensors representing a change in the conformal equivalence class due to a diffeomorphism and so-called transverse traceless tensors, which we do not define here. See [109] for the details. We do not use this decomposition in the present thesis. Obtaining alternative formulas taking into account this decomposition of the space of symmetric covariant two-tensors is a direction for future work.

We now return to the computation of the Laplace-Beltrami corrections. Directly from the tracelessness of  $r$ ,

$$\text{Tr}(h^{(i)}) = -f^{(i)}N \quad , \quad \text{for all } i. \quad (5.23)$$

It remains to compute  $\text{Tr}(h^{(1)}h^{(1)})$ .



$$\begin{aligned}
\mathrm{Tr}(h^{(1)}h^{(1)}) &= \left(-f^{(1)}\delta_l^p + r_l^{(1)p}\right) \left(-f^{(1)}\delta_p^l + r_p^{(1)l}\right) \\
&= (f^{(1)})^2 N - f^{(1)}\delta_l^p r_p^{(1)l} - f^{(1)}r_l^{(1)p}\delta_p^l + r_l^{(1)p}r_p^{(1)l} \\
&= (f^{(1)})^2 N - f^{(1)}\mathrm{Tr}(r^{(1)}) - f^{(1)}\mathrm{Tr}(r^{(1)}) + \mathrm{Tr}(r^{(1)}r^{(1)}) \\
&= (f^{(1)})^2 N + \mathrm{Tr}(r^{(1)}r^{(1)})
\end{aligned} \tag{5.24}$$

We recover the conformal term  $(f^{(1)})^2 N$ , as well as a new contribution  $\mathrm{Tr}(r^{(1)}r^{(1)})$ . Note that this last contribution will only vanish if  $r^{(1)} = 0$ . Indeed, in the inner product induced by  $g^{(0)}$ ,  $\|r^{(1)}\|^2 = \mathrm{Tr}(r^{(1)}r^{(1)})$ .

We are now ready to produce the expressions for the corrections of the Laplace-Beltrami operator. We can summarize our results in the following lemma.

**Lemma 5.1.** *Let  $\bar{g}(t) = \sum_{i=1}^{\infty} t^i \bar{g}^{(i)}$  be a perturbation of the metric on 1-forms on a manifold  $\mathcal{M}$  of dimension  $N$ . For all  $i$ , write  $\bar{g}^{(i)kl} = f^{(i)}\bar{g}^{(0)kl} - \bar{g}^{(0)kp}r_p^{(i)l}$ , where  $f^{(i)} \in C^\infty(\mathcal{M})$ ,  $r_p^{(i)l}$  is traceless and  $\tilde{g}^{(i)kl} = \bar{g}^{(0)kp}r_p^{(i)l}$  is symmetric. Let  $\psi \in C^\infty(\mathcal{M})$ . Then, the first two corrections to the Laplace-Beltrami operator take the following form:*

(I) *If  $r^{(i)} = 0$  for all  $i$ , the perturbation is said to be conformal and the first two corrections to the Laplace-Beltrami operator are*

$$\begin{aligned}
\Delta_c^{(1)}\psi &= f^{(1)}(\Delta^{(0)}\psi) + \frac{N-2}{2}(\partial_i f^{(1)})\bar{g}^{(0)ij}(\partial_j\psi) \\
\Delta_c^{(2)}\psi &= f^{(2)}(\Delta^{(0)}\psi) + \frac{N-2}{2}(\partial_i f^{(2)})\bar{g}^{(0)ij}(\partial_j\psi) .
\end{aligned} \tag{5.25}$$

(II) *In the general case,*

$$\begin{aligned}
\Delta^{(1)}\psi &= \Delta_c^{(1)}\psi + \frac{1}{\sqrt{|\det(g^{(0)})|}}\partial_i \left( \sqrt{|\det(g^{(0)})|}\tilde{g}^{(1)ij}\partial_j\psi \right) \\
\Delta^{(2)}\psi &= \Delta_c^{(2)}\psi + \frac{1}{\sqrt{|\det(g^{(0)})|}}\partial_i \left( \sqrt{|\det(g^{(0)})|}\tilde{g}^{(2)ij}\partial_j\psi \right) \\
&\quad + \frac{N}{2}(\partial_i f^{(1)})\tilde{g}^{(1)ij}\partial_j\psi - \frac{1}{4}(\partial_i \mathrm{Tr}(r^{(1)}r^{(1)}))\bar{g}^{(0)ij}\partial_j\psi .
\end{aligned} \tag{5.26}$$

*Proof.* Part (I) was already proven above. Part (II) can be obtained by comparing the general expressions in Equations (5.17) and (5.18) to the conformal ones. The additional terms due to nonzero  $r^{(i)}$  can then be directly read off. We note for future reference the negative sign in  $\bar{g}^{(i)kl} = f^{(i)}\bar{g}^{(0)kl} - \tilde{g}^{(0)kl}$ .  $\square$

## 5.2 Laplace-Beltrami Eigenvalue Corrections

In this section, we use the expressions obtained previously to compute the first two eigenvalue corrections due to a perturbation of the metric. As in the previous section, we begin by the case of conformal perturbation on surfaces. We then obtain the first two eigenvalue corrections in the case of a conformal perturbation on a manifold of arbitrary dimension. Finally, we compute the first order correction in the case of a general perturbation of the metric.

### 5.2.1 Conformal Perturbation on Surfaces

In order to apply the perturbation formulas (Equation (4.31)), one needs to compute  $\langle \psi_m, \Delta^{(1)}\psi_n \rangle$  for arbitrary  $m$  and  $n$ , as well as  $\langle \psi_n, \Delta^{(2)}\psi_n \rangle$ . This is particularly simple in the case of conformal perturbations on surfaces. Indeed, by Lemma 5.1, for any  $m, n$  and  $i = 1, 2$ ,

$$\begin{aligned} \langle \psi_n, \Delta^{(i)}\psi_m \rangle &= \langle \psi_n, f^{(i)}\Delta^{(i)}\psi_m \rangle \\ &= \lambda_m^{(0)} \langle \psi_n, f^{(i)}\psi_m \rangle . \end{aligned} \tag{5.27}$$

Since multiplication by  $f^{(1)}$  is a self-adjoint operator, the  $\Lambda_n^{(1)}$  and  $\Lambda_n^{(2)}$  operators defined in Equation (4.30) are diagonalizable on the relevant spaces. Then, applying Equation (4.31) yields:

**Lemma 5.2.** *Let  $\mathcal{M}$  be a two-dimensional compact Riemannian manifold without boundary. Let  $\bar{g}(t) = \sum_{i=0}^{\infty} t^i f^{(i)}\bar{g}^{(0)}$  with  $f^{(i)} \in C^\infty(\mathcal{M})$  for all  $i$ . After an appropriate choice of orthonormal eigenbasis of  $\Delta^{(0)}$ , the first two corrections to the unperturbed Laplace-Beltrami eigenvalue  $\lambda_n^{(0)}$  are:*

$$\begin{aligned}
\lambda_n^{(1)} &= \lambda_n^{(0)} \langle \psi_n^{(0)}, f^{(1)} \psi_n^{(0)} \rangle \\
\lambda_n^{(2)} &= \sum_{\substack{k \\ \lambda_k^{(0)} \neq \lambda_n^{(0)}}} \frac{\lambda_n^{(0)} \lambda_k^{(0)} \left| \langle \psi_n^{(0)}, f^{(1)} \psi_k^{(0)} \rangle \right|^2}{\lambda_n^{(0)} - \lambda_k^{(0)}} + \lambda_n^{(0)} \langle \psi_n^{(0)}, f^{(2)} \psi_n^{(0)} \rangle .
\end{aligned} \tag{5.28}$$

*Note: If the spectrum is nondegenerate, any orthonormal eigenbasis suffices. See Chapter 4.*

In the above, we used the self-adjointness of multiplication by  $f^{(1)}$  to obtain the numerator in the series.

## 5.2.2 General Perturbation of the Metric

The general case is significantly more complicated. Since expressions of the form  $\langle \psi_m, \Delta^{(i)} \psi_n \rangle$  are linear in  $\Delta^{(i)}$ , it is possible to use Lemma 5.1 to separately compute the contribution due to the conformal and non-conformal parts of the metric perturbation.

Before computing the conformal contributions to  $\langle \psi_m, \Delta^{(i)} \psi_n \rangle$ , we need some technical machinery.

**Definition 5.3** (Structure Constants). *Let  $\{\psi_n\}_{n=0}^\infty$  be an orthonormal basis of smooth functions for  $L_2(\mathcal{M})$ .  $\mathcal{M}$  is assumed compact. The inner product on  $L_2(\mathcal{M})$  is taken to be the one induced by the Riemannian metric  $g$  via the volume form  $dV_g$ . The structure constants  $C_{nmi}$  are defined as*

$$\begin{aligned}
\psi_n \psi_m &= \sum_{i=0}^{\infty} C_{nmi} \psi_i \\
C_{nmi} &= \int_{\mathcal{M}} \psi_n \psi_m \psi_i dV_g .
\end{aligned} \tag{5.29}$$

The notion of structure constants is more general [95]. The definition used here is specialized for the case of the algebra of smooth functions of a compact manifold.

**Lemma 5.4.** *Let  $(\mathcal{M}, g)$  be a compact Riemannian manifold and let  $\{\psi_n\}_{n=0}^\infty$  be an orthonormal eigenbasis of the Laplace-Beltrami operator on  $\mathcal{M}$ . Let  $g(d\psi_n, d\psi_m) \in C^\infty(\mathcal{M})$  denote the pointwise inner product of  $d\psi_n$  and  $d\psi_m$ . Then,*

$$g(d\psi_n, d\psi_m) = \frac{1}{2} \sum_{i=0}^{\infty} D_{nmi} \psi_i , \quad (5.30)$$

where  $D_{nmi} = (\lambda_n + \lambda_m - \lambda_i)C_{nmi}$ .

*Proof.* Begin by recalling the product rule for the Laplace-Beltrami operator [96]. For  $f, h \in C^\infty(\mathcal{M})$ ,

$$\Delta(fh) = (\Delta f)h + f(\Delta h) - 2g(df, dh) . \quad (5.31)$$

This formula can be straightforwardly obtained in the center of a Riemann normal coordinate system. In our case, it can be recast as

$$\begin{aligned} g(d\psi_n, d\psi_m) &= \frac{1}{2} \left( (\Delta\psi_n)\psi_m + \psi_n(\Delta\psi_m) - \Delta(\psi_n\psi_m) \right) \\ &= \frac{1}{2} \left( (\lambda_n + \lambda_m)\psi_n\psi_m - \Delta(\psi_n\psi_m) \right) \\ &= \frac{1}{2} \left( (\lambda_n + \lambda_m) \left( \sum_{i=0}^{\infty} C_{nmi} \psi_i \right) - \Delta \left( \sum_{i=0}^{\infty} C_{nmi} \psi_i \right) \right) \\ &= \frac{1}{2} \sum_{i=0}^{\infty} (\lambda_n + \lambda_m - \lambda_i) C_{nmi} \psi_i . \end{aligned} \quad (5.32)$$

□

In local coordinates,

$$g(df, dh) = (\partial_i f) g^{ij} (\partial_j h) . \quad (5.33)$$

Recall that similar terms occur in the expressions for the Laplacian corrections (Lemma 5.1). The purpose of Lemma 5.4 is to compute such terms. Also, this lemma, as well as other analogous results, will be discussed in Chapter 6.

Let's begin by computing the corrections in the conformal case. By Lemma 5.1, the first two conformal corrections ( $i = 1, 2$ ) to the Laplacian have the same form.

$$\langle \psi_n^{(0)}, \Delta_c^{(i)} \psi_m^{(0)} \rangle = \lambda_n^{(0)} \langle \psi_n^{(0)}, f^{(i)} \psi_m^{(0)} \rangle + \frac{N-2}{2} \langle \psi_n^{(0)}, g^{(0)}(df^{(i)}, d\psi_m^{(0)}) \rangle \quad (5.34)$$

Expand  $f^{(i)}$  as

$$f^{(i)} = \sum_{k=0}^{\infty} a_k^{(i)} \psi_k^{(0)}. \quad (5.35)$$

Then,

$$\langle \psi_n^{(0)}, \Delta_c^{(i)} \psi_m^{(0)} \rangle = \lambda_m^{(0)} \langle \psi_n^{(0)}, f^{(i)} \psi_m^{(0)} \rangle + \frac{N-2}{2} \left\langle \psi_n^{(0)}, \sum_{k=0}^{\infty} a_k^{(i)} g^{(0)}(d\psi_k^{(0)}, d\psi_m^{(0)}) \right\rangle. \quad (5.36)$$

Using Lemma 5.4,

$$\begin{aligned} \langle \psi_n^{(0)}, \Delta_c^{(i)} \psi_m^{(0)} \rangle &= \lambda_m^{(0)} \langle \psi_n^{(0)}, f^{(i)} \psi_m^{(0)} \rangle + \frac{N-2}{4} \sum_{k=0}^{\infty} a_k^{(i)} \sum_{l=0}^{\infty} D_{kml} \langle \psi_n^{(0)}, d\psi_l^{(0)} \rangle \\ &= \lambda_m^{(0)} \langle \psi_n^{(0)}, f^{(i)} \psi_m^{(0)} \rangle + \frac{N-2}{4} \sum_{k=0}^{\infty} a_k^{(i)} D_{kmn}. \end{aligned} \quad (5.37)$$

Consider the summation in the last term.

$$\begin{aligned} \sum_{k=0}^{\infty} a_k^{(i)} D_{kmn} &= \sum_{k=0}^{\infty} a_k^{(i)} (\lambda_k^{(0)} + \lambda_m^{(0)} - \lambda_n^{(0)}) \int_{\mathcal{M}} \psi_k^{(0)} \psi_m^{(0)} \psi_n^{(0)} dV_{g^{(0)}} \\ &= \int_{\mathcal{M}} \left( \sum_{k=0}^{\infty} a_k^{(i)} \lambda_k^{(0)} \psi_k^{(0)} \right) \psi_m^{(0)} \psi_n^{(0)} dV_{g^{(0)}} \\ &\quad + (\lambda_m^{(0)} - \lambda_n^{(0)}) \int_{\mathcal{M}} \left( \sum_{k=0}^{\infty} a_k^{(i)} \psi_k^{(0)} \right) \psi_m^{(0)} \psi_n^{(0)} dV_{g^{(0)}} \\ &= \int_{\mathcal{M}} (\Delta^{(0)} f^{(i)}) \psi_m^{(0)} \psi_n^{(0)} dV_{g^{(0)}} + (\lambda_m^{(0)} - \lambda_n^{(0)}) \int_{\mathcal{M}} f^{(i)} \psi_m^{(0)} \psi_n^{(0)} dV_{g^{(0)}} \\ &= \langle \psi_n^{(0)}, (\Delta^{(0)} f^{(i)}) \psi_m^{(0)} \rangle + (\lambda_m^{(0)} - \lambda_n^{(0)}) \langle \psi_n^{(0)}, f^{(i)} \psi_m^{(0)} \rangle \end{aligned} \quad (5.38)$$

All together, the above proves the technical lemma below.

**Lemma 5.5.** For  $i = 1, 2$ , let  $\Delta_c^{(i)}$  be as in Lemma 5.1. Then,

$$\begin{aligned} \langle \psi_n^{(0)}, \Delta_c^{(i)} \psi_m^{(0)} \rangle &= \left( \lambda_m^{(0)} + \frac{N-2}{4} (\lambda_m^{(0)} - \lambda_n^{(0)}) \right) \langle \psi_n^{(0)}, f^{(i)} \psi_m^{(0)} \rangle \\ &+ \frac{N-2}{4} \langle \psi_n^{(0)}, (\Delta^{(0)} f^{(i)}) \psi_m^{(0)} \rangle . \end{aligned} \quad (5.39)$$

We are now ready to obtain the first two eigenvalue corrections in the case of a conformal perturbation of the metric. The expressions obtained here are rather complicated, so we begin by defining a few shorthands.

**Definition 5.6** (Shorthands). Define the following expressions:

$$\begin{aligned} A_{kn} &= \left( \lambda_n^{(0)} + \frac{N-2}{4} (\lambda_n^{(0)} - \lambda_k^{(0)}) \right) \\ \Psi_{kn} &= \langle \psi_k^{(0)}, f^{(1)} \psi_n^{(0)} \rangle \\ \Phi_{kn} &= \langle \psi_k^{(0)}, (\Delta^{(0)} f^{(1)}) \psi_n^{(0)} \rangle . \end{aligned} \quad (5.40)$$

**Lemma 5.7** (Eigenvalue Corrections: Conformal Case). Let  $\lambda_{nc}^{(1)}$  and  $\lambda_{nc}^{(2)}$  denote the first two corrections to  $\lambda_n^{(0)}$  in the case of a strictly conformal perturbation. Upon an appropriate choice of an orthonormal eigenbasis of  $\Delta^{(0)}$ , the eigenvalue corrections can be expressed as follows.

$$\lambda_{nc}^{(1)} = \left\langle \psi_n^{(0)}, \left( \lambda_n^{(0)} f^{(1)} + \frac{N-2}{4} (\Delta^{(0)} f^{(1)}) \right) \psi_n^{(0)} \right\rangle \quad (5.41)$$

$$\begin{aligned} \lambda_{nc}^{(2)} &= \sum_{\substack{k \\ \lambda_k^{(0)} \neq \lambda_n^{(0)}}} \frac{A_{nk} A_{kn} |\Psi_{kn}|^2 + \left( \lambda_n^{(0)} + \lambda_k^{(0)} \right) \frac{N-2}{4} \Psi_{kn} \Phi_{kn} + \left( \frac{N-2}{4} \right)^2 |\Phi_{kn}|^2}{\lambda_n^{(0)} - \lambda_k^{(0)}} \\ &+ \left\langle \psi_n^{(0)}, \left( \lambda_n^{(0)} f^{(2)} + \frac{N-2}{4} (\Delta^{(0)} f^{(2)}) \right) \psi_n^{(0)} \right\rangle . \end{aligned} \quad (5.42)$$

*Note:* If the spectrum is nondegenerate, any choice of orthonormal eigenbasis yields the result. See Chapter 4.

*Proof.* Notice that, by Lemma 5.5, the  $\Lambda_n^{(1)}$  operator of this problem (see Equation (4.30)) is diagonalizable on the eigenspaces of  $\Delta^{(0)}$ . This is due to the fact that multiplication by functions is self-adjoint. Thus,  $\lambda_{nc}^{(1)}$  can be written in the form in Equation (4.31).

The expression for  $\lambda_{nc}^{(2)}$  requires some additional work. We begin by computing the numerator arising in the series in Equation (4.31). Using the fact that multiplication by  $f^{(1)}$  and  $(\Delta^{(0)} f^{(1)})$  are self-adjoint with respect to the  $L_2(\mathcal{M})$  inner product, as well as the fact that  $A_{nk} + A_{kn} = \lambda_n^{(0)} + \lambda_k^{(0)}$ ,

$$\begin{aligned} \langle \psi_n^{(0)}, \Delta_c^{(1)} \psi_k^{(0)} \rangle \langle \psi_k^{(0)}, \Delta_c^{(1)} \psi_n^{(0)} \rangle = & \\ & A_{nk} A_{kn} \left| \langle \psi_k^{(0)}, f^{(1)} \psi_n^{(0)} \rangle \right|^2 \\ & + \left( \lambda_n^{(0)} + \lambda_k^{(0)} \right) \frac{N-2}{4} \langle \psi_k^{(0)}, f^{(1)} \psi_n^{(0)} \rangle \langle \psi_k^{(0)}, (\Delta^{(0)} f^{(1)}) \psi_n^{(0)} \rangle \\ & + \left( \frac{N-2}{4} \right)^2 \left| \langle \psi_k^{(0)}, (\Delta^{(0)} f^{(1)}) \psi_n^{(0)} \rangle \right|^2. \end{aligned} \quad (5.43)$$

The term due to  $\Delta_c^{(2)}$  (second term in Equation (4.30)) is obtained in a manner entirely analogous to the first order correction. It remains to show that the  $\Lambda_n^{(2)}$  operator of this problem (see Equation (4.30)) is diagonalizable on the eigenspaces of  $\Lambda_n^{(1)}$ . This follows from the self-adjointness of the multiplication by functions. The form in Equation (4.31) can thus be used. Applying the shorthands of Definition 5.6 yields the desired result.  $\square$

It remains to compute the eigenvalue corrections in the case of a general perturbation. Due to the complexity of the second order expressions, we limit ourselves to the first order corrections. Indeed, even at the level of conformal perturbations, it was necessary to resort to a modified notation to compactly write the formulas. The interested reader can carry out the computations using the expressions in Lemma 5.1. The last technique useful for this computation, namely integration by parts, is illustrated in the proof of the following lemma. Depending on one's purposes, however, integration by parts may be unnecessary.

**Lemma 5.8** (Eigenvalue Corrections: General Case). *Let the metric  $g^{(0)}$  be subject to a general perturbation. Upon an appropriate choice of an orthonormal eigenbasis of  $\Delta^{(0)}$ , the first order eigenvalue correction can be expressed as follows.*

$$\lambda_n^{(1)} = \lambda_{nc}^{(1)} - \langle d\psi_n^{(0)}, r^{(1)} d\psi_n^{(0)} \rangle \quad (5.44)$$

Where  $\lambda_{nc}^{(1)}$  is as in Lemma 5.7,  $r^{(1)}d\psi_n^{(0)} = r_p^{(1)i}\partial_i\psi_n^{(0)}$  and the inner product is the  $L_2$  inner product on 1-forms induced by  $g^{(0)}$ . Note: If the spectrum is nondegenerate, any orthonormal eigenbasis of  $\Delta^{(0)}$  suffices.

*Proof.* First, we compute  $\langle \psi_n^{(0)}, \Delta^{(1)}\psi_n^{(0)} \rangle$ . By Lemmas 5.1 and 5.7,

$$\begin{aligned} \langle \psi_n^{(0)}, \Delta^{(1)}\psi_n^{(0)} \rangle &= \left\langle \psi_n^{(0)}, \left( \lambda_n^{(0)} f^{(1)} + \frac{N-2}{4} (\Delta^{(0)} f^{(1)}) \right) \psi_n^{(0)} \right\rangle \\ &\quad + \left\langle \psi_n^{(0)}, \frac{1}{\sqrt{|\det(g^{(0)})|}} \partial_i \left( \sqrt{|\det(g^{(0)})|} \tilde{g}^{(1)ij} \partial_j \psi_n^{(0)} \right) \right\rangle. \end{aligned} \quad (5.45)$$

The second term can be rewritten in a more convenient form.

$$\begin{aligned} &\left\langle \psi_n^{(0)}, \frac{1}{\sqrt{|\det(g^{(0)})|}} \partial_i \left( \sqrt{|\det(g^{(0)})|} \tilde{g}^{(1)ij} \partial_j \psi_n^{(0)} \right) \right\rangle = \\ &= \int_{\mathcal{M}} \psi_n^{(0)} \frac{1}{\sqrt{|\det(g^{(0)})|}} \partial_i \left( \sqrt{|\det(g^{(0)})|} \tilde{g}^{(1)ij} \partial_j \psi_n^{(0)} \right) dV_{g^{(0)}} \\ &= - \int_{\mathcal{M}} (\partial_i \psi_n^{(0)}) \tilde{g}^{(1)ij} (\partial_j \psi_n^{(0)}) dV_{g^{(0)}} = - \langle \partial_i \psi_n^{(0)}, \tilde{g}^{(1)ij} \partial_j \psi_n^{(0)} \rangle \end{aligned} \quad (5.46)$$

The last line was obtained by integrating the second line by parts and using the fact that, locally,  $dV_{g^{(0)}} = \sqrt{|\det(g^{(0)})|} dx^1 \dots dx^N$ . By the symmetry of  $\tilde{g}^{(1)ij}$ , this brings  $\langle \psi_n^{(0)}, \Delta^{(1)}\psi_n^{(0)} \rangle$  into a symmetric form. One can thus choose an eigenbasis of  $\Delta^{(0)}$  such that Equation (4.31) holds.

It remains to express this in a theoretically more appealing form. Indeed, as the expression is right now, it contains what looks like expressions in local coordinates and a global inner product. This can be remedied as follows. Recall that  $\tilde{g}^{(1)kl} = \bar{g}^{(0)kp} r_p^{(1)l}$ . One can see  $r_p^{(1)l}$  as acting on 1-forms. For our purposes, the relevant local form of this action is  $r^{(1)}d\psi_n^{(0)} = r_p^{(1)i}\partial_i\psi_n^{(0)}$ . We can then give our expressions an invariant meaning:

$$\langle \partial_i \psi_n^{(0)}, \tilde{g}^{(1)ij} \partial_j \psi_n^{(0)} \rangle = \langle d\psi_n^{(0)}, r^{(1)}d\psi_n^{(0)} \rangle. \quad (5.47)$$

The inner product on the right-hand side is the  $L_2$  inner product on 1-forms induced by  $g^{(0)}$ . This completes the proof.  $\square$

An expression for  $\lambda_n^{(2)}$  can be obtained through similar means. We leave this exercise to the reader.



## 5.3 Geometry of Conformal Isospectral Sets on Surfaces

In this section we study the geometric properties of isospectral sets of conformally equivalent metrics on surfaces. We begin by presenting a general argument for the triviality of certain isospectral sets of operators. This argument does not involve any Riemannian geometry and is solely based on perturbation theory in Hilbert spaces developed in Chapter 4. Then, we specialize it to perturbations of Laplace-Beltrami operators on surfaces induced by conformal perturbations of the metric. We then show that sets of isospectral conformally equivalent metrics contain no convex subsets. The results presented here were originally published in [87] with the author of this thesis as the principal contributor.

### 5.3.1 General Argument

In this section, we consider perturbations of an operator  $H^{(0)}$  on a Hilbert space  $\mathcal{H}$ . We assume that all the relevant power series converge in a neighborhood of  $t = 0$ . Furthermore, the eigenvectors of  $H^{(0)}$  are assumed to span the whole space. The notation used in what follows is the same as in Chapter 4. For convenience, we repeat the key points here.

$$\begin{aligned}
 H(t)\psi_n(t) &= \lambda_n(t)\psi_n(t) \\
 H(t) &= H^{(0)} + tH^{(1)} + t^2H^{(2)} + \dots \\
 \lambda_n(t) &= \lambda_n^{(0)} + t\lambda_n^{(1)} + t^2\lambda_n^{(2)} + \dots \\
 \psi_n(t) &= \psi_n^{(0)} + t\psi_n^{(1)} + t^2\psi_n^{(2)} + \dots
 \end{aligned}
 \tag{5.48}$$

Furthermore, for simplicity, assume the  $H^{(i)}$  to be self-adjoint. Also, for now, let the perturbation be strictly of first order. In other words,  $H^{(i)} = 0$  for  $i \geq 2$ .

We now require  $H(t)$  to form an isospectral family. In particular, the first two eigenvalue corrections must vanish for all  $n$ . By Equation (4.31),

$$\begin{aligned}
 0 &= \langle \psi_n^{(0)}, H^{(1)}\psi_n^{(0)} \rangle \\
 0 &= \sum_{\substack{k \\ \lambda_k^{(0)} \neq \lambda_n^{(0)}}} \frac{\left| \langle \psi_k^{(0)}, H^{(1)}\psi_n^{(0)} \rangle \right|^2}{\lambda_n^{(0)} - \lambda_k^{(0)}}.
 \end{aligned}
 \tag{5.49}$$

If the spectrum of  $H^{(0)}$  is degenerate, the above expression only holds after an appropriate choice of orthonormal eigenbasis, as explained in Chapter 4. We will work in such a basis.

Consider Equation (5.49). Our goal is to use it to show that  $H^{(1)}$  vanishes under a reasonable set of assumptions. That would establish that the isospectral family  $H(t)$  is trivial. The first order part of Equation (5.49) expresses that the diagonal of  $H^{(1)}$  vanishes in the  $\{\psi_n^{(0)}\}_n$  basis. It remains to show that all off-diagonal elements  $\langle \psi_k^{(0)}, H^{(1)}\psi_n^{(0)} \rangle$  for  $k \neq n$  also vanish.

This is where the second order perturbations come in. Indeed, the numerator in the series in Equation (5.49) contains the off-diagonal elements  $\langle \psi_k^{(0)}, H^{(1)}\psi_n^{(0)} \rangle$  for  $\lambda_k^{(0)} \neq \lambda_n^{(0)}$ . The remaining elements  $\langle \psi_k^{(0)}, H^{(1)}\psi_n^{(0)} \rangle$  for  $\lambda_k^{(0)} = \lambda_n^{(0)}$  and  $k \neq n$  vanish due to the choice of eigenbasis of  $H^{(0)}$ . See Chapter 4 for details.

An additional hypothesis on the spectrum of  $H^{(0)}$  is required to proceed further. Suppose that the eigenvalues of  $H^{(0)}$  are finitely degenerate and can be numbered such that

$$\lambda_0^{(0)} \leq \lambda_1^{(0)} \leq \lambda_2^{(0)} \leq \lambda_3^{(0)} \dots \quad (5.50)$$

Thus, we require  $H^{(0)}$  to mimic the spectral properties of Laplace-type operators on compact manifolds (see Theorem 2.2). Only property (iv) of Theorem 2.2 is not fully needed here. Indeed, it can be relaxed to have  $\sup_n \{\lambda_n^{(0)}\}$  as the only accumulation point in the spectrum. We need to arrange the spectrum in this way in order to construct an inductive argument. General accumulation points can get in the way of this endeavor.

Consider the second order eigenvalue correction to  $\lambda_0^{(0)}$ .

$$\begin{aligned} 0 &= \sum_{\substack{k \\ \lambda_k^{(0)} \neq \lambda_0^{(0)}}} \frac{|\langle \psi_k^{(0)}, H^{(1)}\psi_0^{(0)} \rangle|^2}{\lambda_k^{(0)} - \lambda_0^{(0)}} \\ &= - \sum_{\substack{k \\ \lambda_k^{(0)} > \lambda_0^{(0)}}} \frac{|\langle \psi_k^{(0)}, H^{(1)}\psi_0^{(0)} \rangle|^2}{|\lambda_0^{(0)} - \lambda_k^{(0)}|} \end{aligned} \quad (5.51)$$

We obtain that a sum of terms of the same sign vanishes. Thus, every term must vanish individually.

$$\left| \langle \psi_k^{(0)}, H^{(1)} \psi_0^{(0)} \rangle \right|^2 = 0 \quad , \quad \text{for all } k \text{ such that } \lambda_k^{(0)} > \lambda_0^{(0)}. \quad (5.52)$$

If  $\lambda_1^{(0)} = \lambda_0^{(0)}$ , the choice of eigenbasis ensures that  $\left| \langle \psi_0^{(0)}, H^{(1)} \psi_1^{(0)} \rangle \right|^2 = 0$ . Otherwise, the above result, combined with the fact that  $H^{(1)}$  is self-adjoint also yields  $\left| \langle \psi_0^{(0)}, H^{(1)} \psi_1^{(0)} \rangle \right|^2 = 0$ . Consequently, the correction to  $\lambda_1^{(0)}$  can always be expressed as

$$0 = - \sum_{\substack{k \\ \lambda_k^{(0)} > \lambda_1^{(0)}}} \frac{\left| \langle \psi_k^{(0)}, H^{(1)} \psi_1^{(0)} \rangle \right|^2}{\left| \lambda_1^{(0)} - \lambda_k^{(0)} \right|}. \quad (5.53)$$

Similarly to the  $\lambda_0^{(0)}$  case, we deduce that

$$\left| \langle \psi_k^{(0)}, H^{(1)} \psi_1^{(0)} \rangle \right|^2 = 0 \quad , \quad \text{for all } k \text{ such that } \lambda_k^{(0)} > \lambda_1^{(0)}. \quad (5.54)$$

This process can be inductively repeated to yield

$$\langle \psi_i^{(0)}, H^{(1)} \psi_j^{(0)} \rangle = 0 \quad , \quad \text{for all } i, j. \quad (5.55)$$

This is precisely what we sought to show, as it ensures that  $H^{(1)} = 0$  and thus that the perturbation is trivial. For the sake of simplicity, we assumed that  $H^{(2)} = 0$ . This assumption can be significantly weakened. Recall that

$$\lambda_n^{(2)} = \sum_{\substack{k \\ \lambda_k^{(0)} \neq \lambda_n^{(0)}}} \frac{\left| \langle \psi_k^{(0)}, H^{(1)} \psi_n^{(0)} \rangle \right|^2}{\lambda_n^{(0)} - \lambda_k^{(0)}} + \langle \psi_n^{(0)}, H^{(2)} \psi_n^{(0)} \rangle. \quad (5.56)$$

The above inductive argument remains valid if  $\langle \psi_n^{(0)}, H^{(2)} \psi_n^{(0)} \rangle \leq 0$  for all  $n$ . In addition to  $H^{(1)} = 0$ , the proof then produces  $\langle \psi_n^{(0)}, H^{(2)} \psi_n^{(0)} \rangle = 0$  for all  $n$ . Moreover, notice that it is unnecessary to require all of  $H(t)$  to form an isospectral family. Indeed, it is sufficient to ask that  $\lambda_n^{(1)} = \lambda_n^{(2)} = 0$  for all  $n$ .

We are now ready to apply this strategy to conformal perturbations of metrics on surfaces. Up to a few technicalities, the argument remains unchanged.

### 5.3.2 Non-Convexity of Isospectral Sets on Surfaces

We now return to geometric considerations. Let  $(\mathcal{M}, g^{(0)})$  be a compact, boundaryless, oriented Riemannian manifold of dimension 2. We consider conformal perturbations of the inverse metric  $\bar{g}^{(0)}$ :

$$\bar{g}(t) = \bar{g}^{(0)} (1 + t f^{(1)} + t^2 f^{(2)} + \dots) . \quad (5.57)$$

By Lemma 5.2,

$$\begin{aligned} \lambda_n^{(1)} &= \lambda_n^{(0)} \langle \psi_n^{(0)}, f^{(1)} \psi_n^{(0)} \rangle \\ \lambda_n^{(2)} &= \sum_{\substack{k \\ \lambda_k^{(0)} \neq \lambda_n^{(0)}}} \frac{\lambda_n^{(0)} \lambda_k^{(0)} \left| \langle \psi_n^{(0)}, f^{(1)} \psi_k^{(0)} \rangle \right|^2}{\lambda_n^{(0)} - \lambda_k^{(0)}} + \lambda_n^{(0)} \langle \psi_n^{(0)}, f^{(2)} \psi_n^{(0)} \rangle . \end{aligned} \quad (5.58)$$

Since the eigenvalues of the Laplace-Beltrami operator are non-negative, these expressions have a form suitable for the inductive argument of the previous section. We immediately obtain the following lemma.

**Lemma 5.9.** *Let  $(\mathcal{M}, g)$  be a compact, connected, boundaryless, oriented Riemannian manifold of dimension 2. Let  $\bar{g} = \bar{g}^{(0)} (1 + \sum_{i=1}^{\infty} t^i f^{(i)})$  be a conformal perturbation of the metric on 1-forms with  $f^{(2)} \leq 0$ . If the first two eigenvalue corrections vanish, then  $f^{(1)} = 0$  and  $\langle \psi_n^{(0)}, f^{(2)} \psi_n^{(0)} \rangle = 0$  for all  $n$ .*

*Proof.* Up to a minor modification, the lemma follows from the inductive strategy of Section 5.3.1 with the operators  $H^{(1)}$  and  $H^{(2)}$  replaced with  $f^{(1)}$  and  $f^{(2)}$ , respectively. The issue is with the eigenspace with eigenvalue 0. This eigenspace consists of functions constant on the connected components of  $\mathcal{M}$ . Since  $\mathcal{M}$  is assumed connected, this eigenspace has dimension 1. The corresponding eigenvalue and eigenfunction are  $\lambda_0^{(0)} = 0$  and  $\psi_0^{(0)}$  (a constant function), respectively.

The inductive argument of Section 5.3.1 can be used to show that

$$\langle \psi_i^{(0)}, f^{(1)} \psi_j^{(0)} \rangle = 0 \quad , \quad \text{for all } i, j \geq 1. \quad (5.59)$$

However, the case where either  $i$  or  $j$  is zero remains. Pick  $j \geq 1$ . Then, since the  $\{\psi_n^{(0)}\}_{n=0}^{\infty}$  form an orthonormal basis, there exists a constant  $c$  such that

$$f^{(1)}\psi_j^{(0)} = c\psi_0^{(0)} . \tag{5.60}$$

In other words,  $f^{(1)}\psi_j^{(0)}$  is a constant function. Let  $x \in \mathcal{M}$  be such that  $\psi_j^{(0)}(x) = 0$ . Such points are known to exist, as  $\langle \psi_j^{(0)}, \psi_0^{(0)} \rangle = 0$  and  $\psi_0^{(0)}$  is constant. Consequently,  $\psi_j^{(0)}$  has to change sign in order for the integral defining the  $L_2(\mathcal{M})$  inner product to vanish. The smoothness of  $\psi_j^{(0)}$  (see Theorem 2.2) guarantees that  $x$  exists. Thus,  $f^{(1)}(x)\psi_j^{(0)}(x) = 0$ . Consequently,  $f^{(1)}(p) = 0$  on the complements of the nodal sets of  $\psi_j^{(0)}$ , that is on all points on which  $\psi_j^{(0)}$  is nonzero. In [27], it is shown that nodal sets of Laplace-Beltrami eigenfunctions are submanifolds of  $\mathcal{M}$  of lower dimension. By continuity of  $f^{(1)}$ , we then get  $f^{(1)} = 0$ , as desired.  $\square$

This result can be used to study the geometric structure of sets of isospectral conformally equivalent metrics on surfaces. Recall that such metrics have been shown to exist [19]. Before proceeding further, consider two metrics  $g_a$  and  $g_b$ . The path  $(1 - \tau)g_a + \tau g_b$  for  $\tau \in [0, 1]$  is composed of metrics. Indeed,  $(1 - \tau)g_a + \tau g_b$  is symmetric for any  $\tau \in [0, 1]$ . Moreover, it is the sum of two positive-definite tensors and is thus positive definite. Consequently, the set of metrics over  $\mathcal{M}$  is equipped with a notion of straight line. The convexity of subsets of that set can thus be studied. The same holds for sets of metrics on 1-forms. In fact, we begin by studying the convexity properties of isospectral sets of such “inverse” metrics.

**Theorem 5.10.** *Let  $\bar{\mathcal{G}}$  be an isospectral set of conformally equivalent metrics on 1-forms over a surface  $\mathcal{M}$  assumed connected, oriented, compact and without boundary. Then,  $\bar{\mathcal{G}}$  contains no convex subset composed of more than one element.*

*Proof.* Suppose the contrary. Let  $\bar{g}_1$  and  $\bar{g}_2$  be distinct elements of a nonempty convex subset of  $\bar{\mathcal{G}}$ . In other words, for  $\tau \in [0, 1]$ ,  $\tau\bar{g}_1 + (1 - \tau)\bar{g}_2 \subset \bar{\mathcal{G}}$ .

We can obtain a perturbation expansion of the eigenvalues in a neighborhood of any  $\tau_0 \in [0, 1]$ . Since  $\bar{g}_1$  and  $\bar{g}_2$  are conformally equivalent, this situation can be seen as a conformal perturbation with  $f^{(i)} = 0$  for  $i \geq 2$ . Lemma 5.9 then applies, yielding that  $f^{(1)} = 0$ . Thus,  $\bar{g}_1 = \bar{g}_2$ , a contradiction.  $\square$

We now turn our attention to the space of metrics, rather than that of inverse metrics. An analogous result holds there as well. Note that straight lines in the space of metrics do not map to straight lines in the space of inverse metrics. The two results are thus distinct. Moreover, the proof technique in the case of the space of metrics is slightly different. Indeed, it requires a specialized expression for the eigenvalue corrections.

**Lemma 5.11.** *Let  $g(t) = g^{(0)} (1 + tf^{(1)})$  with  $f^{(1)} \in C^\infty(\mathcal{M})$  be a strictly first order conformal perturbation of the metric on a manifold of dimension 2. Then, after an appropriate choice of eigenbasis of  $\Delta^{(0)}$ , the Laplace-Beltrami eigenvalue corrections are given by:*

$$\begin{aligned}\lambda_n^{(1)} &= -\lambda_n^{(0)} \langle \psi_n^{(0)}, f^{(1)} \psi_n^{(0)} \rangle \\ \lambda_n^{(2)} &= \sum_{\substack{k \\ \lambda_k^{(0)} \neq \lambda_n^{(0)}}} \frac{\left(\lambda_n^{(0)}\right)^2 \left| \langle \psi_k^{(0)}, f^{(1)} \psi_n^{(0)} \rangle \right|^2}{\lambda_n^{(0)} - \lambda_k^{(0)}} + \lambda_n^{(0)} \left| \langle \psi_n^{(0)}, f^{(1)} \psi_n^{(0)} \rangle \right|^2 .\end{aligned}\quad (5.61)$$

*Note: If the spectrum of  $\Delta^{(0)}$  is nondegenerate, any orthonormal eigenbasis suffices. See Chapter 4.*

*Proof.* All of our results apply to perturbations of inverse metric. Thus, we need to compute the induced perturbation of the inverse metric  $\bar{g}$  before proceeding further. For small  $|t|$ , we can use the Taylor expansion:

$$\bar{g}(t) = \bar{g}^{(0)} (1 + tf^{(1)})^{-1} = \bar{g}^{(0)} \left( 1 - tf^{(1)} + t^2 (f^{(1)})^2 + \dots \right) .\quad (5.62)$$

The formula for  $\lambda_n^{(1)}$  then immediately follows from Lemma 5.2. The expression for  $\lambda_n^{(2)}$  requires some additional work. Expand  $f^{(1)} \psi_n^{(0)}$  in the eigenbasis of the Laplacian.

$$f^{(1)} \psi_n^{(0)} = \sum_{k=0}^{\infty} \langle \psi_k^{(0)}, f^{(1)} \psi_n^{(0)} \rangle \psi_k^{(0)}\quad (5.63)$$

This allows us to write  $\langle \psi_n^{(0)}, (f^{(1)})^2 \psi_n^{(0)} \rangle$  in a more convenient way.

$$\begin{aligned}\langle \psi_n^{(0)}, (f^{(1)})^2 \psi_n^{(0)} \rangle &= \sum_{k=0}^{\infty} \left| \langle \psi_k^{(0)}, f^{(1)} \psi_n^{(0)} \rangle \right|^2 \\ &= \left| \langle \psi_n^{(0)}, f^{(1)} \psi_n^{(0)} \rangle \right|^2 + \sum_{\substack{k \\ \lambda_k^{(0)} \neq \lambda_n^{(0)}}} \left| \langle \psi_k^{(0)}, f^{(1)} \psi_n^{(0)} \rangle \right|^2\end{aligned}\quad (5.64)$$

The terms omitted from the summation above are guaranteed to vanish due to the choice of eigenbasis described in Chapter 4. One can then make the following simplification:

$$\begin{aligned}
& \sum_{\substack{k \\ \lambda_k^{(0)} \neq \lambda_n^{(0)}}} \frac{\lambda_k^{(0)} \lambda_n^{(0)} |\langle \psi_k^{(0)}, f^{(1)} \psi_n^{(0)} \rangle|^2}{\lambda_n^{(0)} - \lambda_k^{(0)}} + \lambda_n^{(0)} \sum_{\substack{k \\ \lambda_k^{(0)} \neq \lambda_n^{(0)}}} |\langle \psi_k^{(0)}, f^{(1)} \psi_n^{(0)} \rangle|^2 \\
&= \sum_{\substack{k \\ \lambda_k^{(0)} \neq \lambda_n^{(0)}}} \frac{(\lambda_n^{(0)})^2 |\langle \psi_k^{(0)}, f^{(1)} \psi_n^{(0)} \rangle|^2}{\lambda_n^{(0)} - \lambda_k^{(0)}}
\end{aligned} \tag{5.65}$$

Applying Lemma 5.2 gives the desired result.  $\square$

Everything is now in place to prove the metric analogue of Theorem 5.10.

**Theorem 5.12.** *Let  $\mathcal{G}$  be an isospectral set of conformally equivalent metrics over a surface  $\mathcal{M}$  assumed connected, oriented, compact and without boundary. Then,  $\mathcal{G}$  contains no convex subset composed of more than one element.*

*Proof.* Begin by considering a strictly first order conformal perturbation of a metric. Suppose this perturbation isospectral. By Lemma 5.11 above,

$$\begin{aligned}
0 &= -\lambda_n^{(0)} \langle \psi_n^{(0)}, f^{(1)} \psi_n^{(0)} \rangle \\
0 &= \sum_{\substack{k \\ \lambda_k^{(0)} \neq \lambda_n^{(0)}}} \frac{(\lambda_n^{(0)})^2 \left| \langle \psi_k^{(0)}, f^{(1)} \psi_n^{(0)} \rangle \right|^2}{\lambda_n^{(0)} - \lambda_k^{(0)}}.
\end{aligned} \tag{5.66}$$

The  $\lambda_n^{(0)} \left| \langle \psi_n^{(0)}, f^{(1)} \psi_n^{(0)} \rangle \right|^2$  term in  $\lambda_n^{(2)}$  vanishes due to the first order isospectrality prescribing that  $\lambda_n^{(0)} \langle \psi_n^{(0)}, f^{(1)} \psi_n^{(0)} \rangle = 0$ . Once again, we obtain a form suitable for the inductive argument of Section 5.3.1. We thus have that  $f^{(1)} = 0$ . This rules out the existence of straight paths of isospectral metrics, completing the proof.  $\square$

The significance of Theorems 5.10 and 5.12 will be discussed at length in Chapter 7. For now, note the difference between their proof strategies. Indeed, the proof of Theorem 5.10 relied on Lemma 5.9, which requires that  $f^{(2)} \leq 0$ . In Theorem 5.12, on the other hand,  $f^{(2)} = (f^{(1)})^2 \geq 0$ . Nonetheless, Lemma 5.11 shows that the expressions for the

eigenvalue corrections conspire to once again produce the desired result. The condition  $f^{(2)} \leq 0$  of Lemma 5.9 is thus merely sufficient for our purposes, rather than necessary.

Before proceeding further, we would also like to draw attention to the fact that Theorems 5.10 and 5.12 deal with metrics and not with isometry equivalence classes of Riemannian manifolds. Consequently, our results can be applied to isospectral sets that contain isometric metrics. In particular, our results hold on sets of isometric metrics, which are of course isospectral.

Finally, it would be of interest to generalize our results to manifolds of higher dimension. Assuming that analogous theorems indeed hold, the proof technique would have to be adjusted. Specifically, the form of the eigenvalue corrections of Lemma 5.7 does not guarantee that the numerator arising in the formula for  $\lambda_n^{(2)}$  is positive. This would not be a problem if  $\Delta^{(1)}$  were symmetric. Then, the strategy of Section 5.3.1 would apply verbatim.



# Chapter 6

## Some Observations on the Integrals of Products of Eigenfunctions

Integrals of products of eigenfunctions arise in the perturbation theory of the Laplace-Beltrami operator studied in the previous chapter. In this chapter, we make a few observations and conjectures regarding such operations. The content of this chapter is the most speculative of the present thesis. In Section 6.1, we discuss the squares of the eigenfunctions. More specifically, we are interested in the subspace of  $L_2(\mathcal{M})$  spanned by them. In Section 6.2, we turn our attention to products of three and more eigenfunctions. There, we derive a class of formulas similar to the Rayleigh quotient and speculate on their significance.

The integrals of products of eigenfunctions have attracted some attention in the past. We refer the reader to [20, 21] for estimates of the  $L_2$  norm of certain products of eigenfunctions.

### 6.1 On the Squares of Eigenfunctions

Let  $(\mathcal{M}, g^{(0)})$  be a compact, oriented Riemannian manifold without boundary of dimension  $N = 2$ . Recall the first two perturbative corrections to the eigenvalues of the Laplace-Beltrami operator subject to a conformal perturbation of the inverse metric (Lemma 5.2).

$$\begin{aligned}
\lambda_n^{(1)} &= \lambda_n^{(0)} \langle \psi_n^{(0)}, f^{(1)} \psi_n^{(0)} \rangle \\
\lambda_n^{(2)} &= \sum_{\substack{k \\ \lambda_k^{(0)} \neq \lambda_n^{(0)}}} \frac{\lambda_n^{(0)} \lambda_k^{(0)} \left| \langle \psi_n^{(0)}, f^{(1)} \psi_k^{(0)} \rangle \right|^2}{\lambda_n^{(0)} - \lambda_k^{(0)}} + \lambda_n^{(0)} \langle \psi_n^{(0)}, f^{(2)} \psi_n^{(0)} \rangle
\end{aligned} \tag{6.1}$$

The first order term can be rewritten as

$$\lambda_n^{(1)} = \lambda_n^{(0)} \left\langle (\psi_n^{(0)})^2, f^{(1)} \right\rangle. \tag{6.2}$$

Requiring the perturbation to be isospectral, implies that  $\lambda_n^{(1)} = 0$  for all  $n$ . This produces an orthogonality relation between  $(\psi_n^{(0)})^2$  and  $f^{(1)}$ . Thus, if  $f^{(1)} \in \text{span}_n((\psi_n^{(0)})^2)$ , this implies that  $f^{(1)} = 0$ . The perturbation of the metric is thus trivial to first order. This implies that the second order correction is reduced to

$$\lambda_n^{(2)} = \lambda_n^{(0)} \langle \psi_n^{(0)}, f^{(2)} \psi_n^{(0)} \rangle \tag{6.3}$$

Similarly, if  $f^{(2)} \in \text{span}_n((\psi_n^{(0)})^2)$ , isospectrality implies that  $f^{(2)} = 0$ . This is very similar to the results obtained in Lemma 5.9. However, as long as one can justify that  $f^{(1)}, f^{(2)} \in \text{span}_n((\psi_n^{(0)})^2)$ , the proof is significantly simpler.

One then has to ask: what space do the squares of the eigenfunctions span? It is tempting to conjecture that they span all of  $L_2(\mathcal{M})$ . This, however, cannot be true in full generality. Indeed, consider the usual flat 2-torus  $T^2$  with coordinates  $(x, y) \in [0, 2\pi) \times [0, 2\pi)$ . The Laplace-Beltrami operator then has eigenvalues of the form  $(m^2 + n^2)$  for  $m, n \in \mathbb{N}$ . The corresponding eigenfunctions can take the following forms.

$$\begin{aligned}
\psi_{mn} &= \cos(mx) \cos(ny) & \psi_{mn} &= \cos(mx) \sin(ny) \\
\psi_{mn} &= \sin(mx) \sin(ny) & \psi_{mn} &= \sin(mx) \cos(ny)
\end{aligned} \tag{6.4}$$

Recall the following trigonometric identities:

$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2} \quad \cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}. \tag{6.5}$$

Thus, the squares of the eigenfunctions on the torus become

$$\begin{aligned}
\psi_{mn}^2 &= \frac{(1 + \cos(2mx))(1 + \cos(2ny))}{4} & \psi_{mn}^2 &= \frac{(1 + \cos(2mx))(1 - \cos(2ny))}{4} \\
\psi_{mn}^2 &= \frac{(1 - \cos(2mx))(1 - \cos(2ny))}{4} & \psi_{mn}^2 &= \frac{(1 - \cos(2mx))(1 + \cos(2ny))}{4}
\end{aligned} \tag{6.6}$$

All that remains is the constant degree of freedom and cosines with even frequency. This is not enough to span all of  $L_2(T^2)$ . The conjecture thus fails to hold in this case. An analogous discussion can be carried out in the case of the standard sphere, once again resulting in the failure of the conjecture.

However, the flat torus and standard sphere are highly symmetric manifolds with very degenerate spectra. They cannot represent a generic situation. Indeed, in the degenerate case, the choice of orthonormal eigenbasis is not unique. The squared eigenfunctions are correspondingly not unique either.

In order to remedy this, we restrict our attention to the case of nondegenerate spectrum. This is a generic situation. Precisely, given a manifold  $\mathcal{M}$ , metrics inducing Laplace-Beltrami operators with nondegenerate spectrum form a residual set in the  $C^\infty$  topology [6] (see Appendix A for a definition of residual sets). The eigenfunctions can be chosen to be normalized. They are then unique up to a sign, which will disappear upon squaring.

The hypothesis of a nondegenerate spectrum could still prove insufficient to ensure that the squares of the eigenfunctions are sufficiently varied to span all of  $L_2(\mathcal{M})$ . One can however expect that a sufficiently bumpy metric would have little relationship between the squares of the eigenfunctions. This is why we introduce an additional, unspecified, genericity condition in the formulation of the conjecture below.

**Conjecture 6.1** (Squares of Eigenfunctions). *Let  $(\mathcal{M}, g)$  be an oriented compact Riemannian manifold without boundary. Suppose that  $(\mathcal{M}, g)$  is in some sense generic among manifolds with nondegenerate Laplace-Beltrami spectrum. Then  $L_2(\mathcal{M}) = \text{span}_n((\psi_n)^2)$ , where the  $\{\psi_n\}_{n=0}^\infty$  are the eigenfunctions of the Laplace-Beltrami operator.*

We will now numerically explore this conjecture. We consider a 2-torus with a metric conformal to the flat one. The conformal factor is a randomly generated positive function. Specifically, we first generate a linear combination of eigenfunctions of the flat torus (Equation (6.1),  $m, n = 1, 2$ ) with uniformly distributed coefficients between  $-1$  and  $1$ . Then, the resulting function is rescaled to take values in  $[-1/2, 1/2]$ . Adding  $1$  to it results in a function with values in  $[1/2, 3/2]$ . This is appropriate as a conformal factor. Generically, the resulting manifold will not have nontrivial symmetries and the eigenvalues will be nondegenerate.

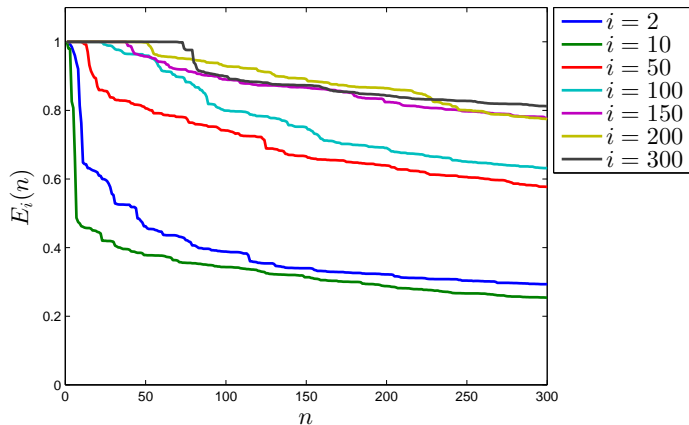


Figure 6.1: Decay of the  $E_i(n)$  with  $n$  for various values of  $i$ .

We use a finite difference method to compute the eigenfunctions of the resulting Laplacian. Then, we compute their squares and construct an orthogonal projection onto their span. This is done by applying the Gram-Schmidt process to  $\{\psi_i^2\}_i$ . One then obtains an orthonormal basis onto which one can readily project.

Let  $P_n$  denote the orthogonal projection onto  $\text{span}_{i=1}^n \{\psi_i^2\}$ . We can then study how well do the squares of the eigenfunctions span the eigenfunctions themselves. Specifically, we study the behavior of the following error:

$$E_i(n) = \|\psi_i - P_n \psi_i\|_{L_2}. \quad (6.7)$$

If, for all  $i$ , we have the limit  $\lim_{n \rightarrow \infty} E_i(n) = 0$ , the squares of the eigenfunctions can be linearly combined to form any element of a basis for  $L_2(\mathcal{M})$ . Thus, they would then span the whole space. If, for some  $i$ ,  $\lim_{n \rightarrow \infty} E_i(n) = c > 0$ , then  $\psi_i$  has a component with norm  $c$  in the orthogonal complement of  $\text{span}_{i=1}^{\infty} \{\psi_i^2\}$ . This would rule out the conjecture.

In Figure 6.1, we illustrate the errors  $E_i(n)$  for various values of  $i \geq 1$  and  $n = 1 \dots 300$  on a generically chosen manifold. Note that the constant eigenfunction  $\psi_0$  is always perfectly reproduced, which is why we do not study  $E_0(n)$ . Note that the decay of the  $E_i(n)$  with  $n$  is very slow, especially for high  $i$ .

It is difficult to draw conclusions from this. The  $E_i(n)$  have not settled into obvious limiting behaviors. It is unclear whether  $E_i(n)$  decays to 0, or to some constant  $c > 0$ .

Assuming that our conjecture holds, this would mean that the squares of the eigenfunctions span the eigenfunctions in an inefficient manner. Indeed, one would need all of the  $\{\psi_i^2\}_{i=0}^\infty$  to reproduce any given  $\psi_i$ . Contrast this with the eigenbasis of the flat torus of Equation (6.1). There,  $\psi_i$  is either perfectly reproduced by  $P_n$  for a finite  $n$  or not reproduced at all.

In sum, it is unclear whether our conjecture holds. However, from perturbation theoretic considerations, it is of great interest to know the properties of the  $\{\psi_i^2\}_{i=0}^\infty$ . To the best of our knowledge, this subject has only been touched once in the past, also in a perturbative context [35]. Introduce the following set of functions:

$$\mathcal{A}_0(\mathcal{M}, g) = \left\{ \phi \in C^\infty(\mathcal{M}) : \int_{\mathcal{M}} \phi \, dV_g = 0 \right\} . \quad (6.8)$$

Recall the expression for the first order eigenvalue correction due to a conformal perturbation (Lemma 5.7). Set  $f^{(1)} = \phi \in \mathcal{A}_0(\mathcal{M}, g)$ . The expression for the first eigenvalue correction can be seen as a quadratic form:

$$q_{k\phi}(\psi) = \left\langle \psi^2, \left( \lambda_k^{(0)} \phi + \frac{N-2}{4} (\Delta^{(0)} \phi) \right) \right\rangle . \quad (6.9)$$

In [35], the following result is proven.

**Lemma 6.2.** *Let  $(\mathcal{M}, g)$  be a compact Riemannian manifold and let  $E_k$  be the eigenspace corresponding to  $\lambda_k$ . Then, the following two conditions are equivalent.*

- (i) *For all  $\phi \in \mathcal{A}_0(\mathcal{M}, g)$ , the quadratic form  $q_{k\phi}(\psi)$  is indefinite on  $E_k$ .*
- (ii) *There exists a finite family of eigenfunctions  $\{u_i\}_{i=1}^d \subset E_k$  such that*

$$\sum_{i=1}^d u_i^2 = 1 . \quad (6.10)$$

This lemma arises in the study of metrics critical for the eigenvalue  $\lambda_k$ . Intuitively, metrics critical for  $\lambda_k$  are those metrics that have  $\lambda_k^{(1)} = 0$  for all perturbations. This is the appropriate analogue of a critical point of a function, that is a point where the gradient of the function vanishes. See [35] for details.

## 6.2 Rayleigh-Like Quotients

In the previous section, we have discussed the squares of the Laplace-Beltrami eigenfunctions. We will now study general products of eigenfunctions. In Lemma 5.4 we have shown the following formula

$$\frac{1}{2}(\lambda_n + \lambda_m - \lambda_k) \int_{\mathcal{M}} \psi_n \psi_m \psi_k dV_g = \int_{\mathcal{M}} g(d\psi_n, d\psi_m) \psi_k dV_g . \quad (6.11)$$

In a sense, this formula generalizes the Rayleigh quotient. Indeed, set  $n = m$  and  $k = 0$ . Then,

$$\begin{aligned} \lambda_n \int_{\mathcal{M}} |\psi_n|^2 dV_g &= \int_{\mathcal{M}} g(d\psi_n, d\psi_n) dV_g \\ \implies \lambda_n &= \frac{\int_{\mathcal{M}} g(d\psi_n, d\psi_n) dV_g}{\int_{\mathcal{M}} |\psi_n|^2 dV_g} . \end{aligned} \quad (6.12)$$

If we allow ourselves to ignore the possibility of division by zero, our result can be rewritten in a similar form.

$$\frac{1}{2}(\lambda_n + \lambda_m - \lambda_k) = \frac{\int_{\mathcal{M}} g(d\psi_n, d\psi_m) \psi_k dV_g}{\int_{\mathcal{M}} \psi_n \psi_m \psi_k dV_g} \quad (6.13)$$

Notice that this quantity is spectrally determined. That is, assuming that the right-hand side is defined, we have obtained a spectral invariant. In fact, we can do better. We can construct an infinite family of expressions of this form.

**Lemma 6.3** (Rayleigh-like Formulas). *Let  $\{\psi_{n_i}\}_{i=1}^I$ ,  $I \geq 2$  be a sequence eigenfunctions of the Laplace-Beltrami operator with eigenvalues  $\{\lambda_{n_i}\}_{i=1}^I$ . Then, for any  $k$ ,*

$$\begin{aligned} &\left( \sum_{i=1}^I \lambda_{n_i} - \lambda_k \right) \left( \int_{\mathcal{M}} \psi_k \prod_{i=1}^I \psi_{n_i} dV_g \right) \\ &= \int_{\mathcal{M}} \psi_k \sum_{i=1}^I \sum_{j \neq i} g(d\psi_{n_i}, d\psi_{n_j}) \left( \prod_{l \neq i, j} \psi_{n_l} \right) dV_g . \end{aligned} \quad (6.14)$$

*Proof.* First, we need to obtain a generalized product rule for the Laplacian. This is easiest at the center of a Riemann normal coordinate system. There [96],

$$\Delta = - \sum_{k=1}^N \partial_k^2 \quad g(d\psi, d\phi) = \sum_{k=1}^N (\partial_k \psi)(\partial_k \phi) . \quad (6.15)$$

Applying this to a product of  $I$  eigenfunctions yields,

$$\begin{aligned} \Delta \prod_{i=1}^I \psi_{n_i} &= - \sum_{k=1}^N \partial_k^2 \prod_{i=1}^I \psi_{n_i} = - \sum_{k=1}^N \partial_k \sum_{i=1}^I (\partial_k \psi_{n_i}) \prod_{j \neq i} \psi_{n_j} \\ &= - \sum_{k=1}^N \sum_{i=1}^I \left( (\partial_k^2 \psi_{n_i}) \prod_{j \neq i} \psi_{n_j} + (\partial_k \psi_{n_i}) \left( \sum_{j \neq i} \partial_k \psi_{n_j} \prod_{l \neq i, j} \psi_{n_l} \right) \right) \\ &= \sum_{i=1}^I \lambda_{n_i} \prod_{j=1}^I \psi_{n_j} - \sum_{i=1}^I \sum_{j \neq i} g(d\psi_{n_i}, d\psi_{n_j}) \prod_{l \neq i, j} \psi_{n_l} \end{aligned} \quad (6.16)$$

The proof is completed by taking the  $L_2(\mathcal{M})$  inner product product of both sides with  $\psi_k$  and using the self-adjointness of  $\Delta$  on the left-hand side.  $\square$

If the denominator does not vanish, we can write the above lemma as a generalized Rayleigh quotient.

$$\sum_{i=1}^I \lambda_{n_i} - \lambda_k = \frac{\int_{\mathcal{M}} \psi_k \sum_{i=1}^I \sum_{j \neq i} g(d\psi_{n_i}, d\psi_{n_j}) \left( \prod_{l \neq i, j} \psi_{n_l} \right) dV_g}{\int_{\mathcal{M}} \psi_k \prod_{i=1}^I \psi_{n_i} dV_g} \quad (6.17)$$

Once again, this is a spectrally determined quantity. It is thus of interest to know under which conditions this is a legitimate way of expressing the above lemma.

Even in the simplest case of Equation (6.13), this is an important issue. Indeed, set  $k = 0$ . Then  $\psi_k$  is the constant eigenfunction and

$$C_{nm0} = \int_{\mathcal{M}} \psi_n \psi_m \psi_0 dV_g = \psi_0 \delta_{nm} . \quad (6.18)$$

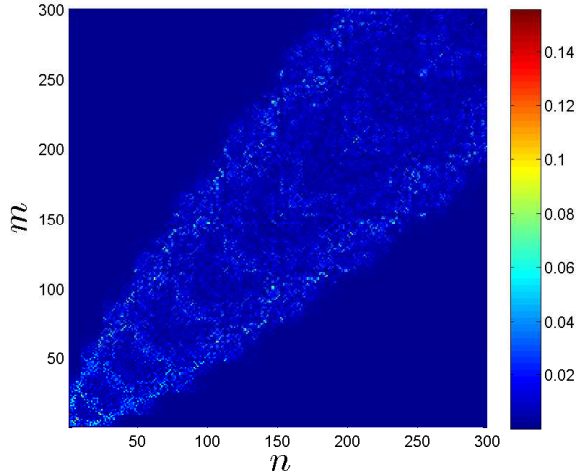


Figure 6.2: Absolute value of the integral of the product of three eigenfunctions ( $|C_{nmk}|$ ). The indices  $n$  and  $m$  vary, while  $k = 10$ . Notice the apparent vanishing of the integrals far from the diagonal  $n = m$ .

Besides this trivial example, we have to resort to numerics to determine when the denominators vanish. In Figure 6.2, we plot the absolute values of  $C_{nmk}$  for  $k = 10$  and  $n, m = 0 \dots 300$ . The manifold used is the same as the one used in the previous section. While the finite precision of the numerical methods does not allow us to conclusively find pairs  $(m, n)$  such that  $C_{nmk}$  vanishes, it seems that nonzero values of  $C_{nmk}$  can be found near the diagonal  $n = m$  (but not necessarily on the diagonal itself). Far away from the diagonal, the  $C_{nmk}$  seem to vanish.

For integrals of products of more functions, the relationship will likely be different. Let

$$C_{nmkl} = \int_{\mathcal{M}} \psi_n \psi_m \psi_k \psi_l dV_g . \quad (6.19)$$

In Figure 6.3, we plot  $C_{nmkl}$  for  $k = 10$  and  $l = 300$ . Except for low values of  $n$  and  $m$ , the integral seems to be nonzero. This leads us to make the following conjecture.

**Conjecture 6.4** (Rayleigh-Like Formulas). *Let  $(\mathcal{M}, g)$  be a generic compact Riemannian manifold. Then, for sufficiently large  $I$  and generic  $\{n_i\}_{i=1}^I$ , Equation (6.17) is valid.*

At the moment, we cannot provide even a tentative definition of what it would mean for  $\{n_i\}_{i=1}^I$  to be generic. Our conjecture is motivated by the apparent loss of structure



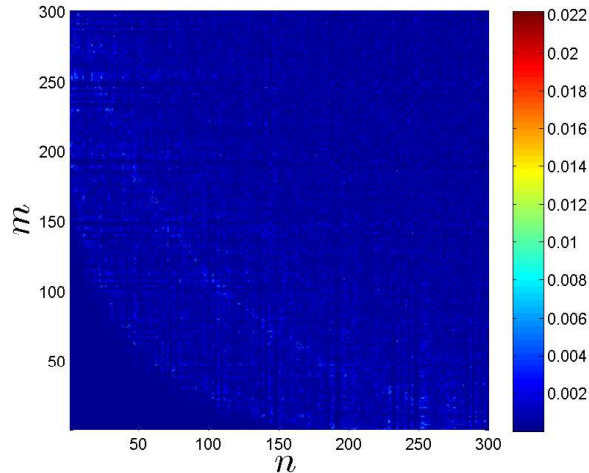


Figure 6.3: Absolute value of the integral of the product of four eigenfunctions ( $|C_{nmkl}|$ ). The indices  $n$  and  $m$  vary, while  $k = 10$  and  $l = 300$ . Notice the apparent vanishing of the integrals for small  $n$  and  $m$ .

between Figure 6.2 and Figure 6.3. Indeed, pairs  $(n, m)$  with apparently nonzero integrals are much more uniformly distributed in Figure 6.3. We expect that, for a large enough  $I$  (number of multiplied eigenfunctions), the loss of structure would be even more pronounced such that most choices  $(n_1, \dots, n_I)$  would yield a nonzero denominator for Equation (6.17).

Supposing that our conjecture holds, it would be of interest to know if one can extract any geometric information from Equation (6.17). Considering that the numerator contains the metric, it is plausible that this is the case. Since the left-hand side of Equation (6.17) depends solely on the spectrum, this would provide a new way to extract geometric information from the spectrum.

To close this chapter, we would like to speculate on the relationship between the generalized Rayleigh quotients of Equation (6.17) and the structure constants  $C_{mnk}$  of  $C^\infty(\mathcal{M})$  (see Definition 5.3). This is important as knowledge of the structure constants jointly with the Laplace-Beltrami spectrum is equivalent to the knowledge of the Riemannian manifold.

Indeed, the structure constants determine  $C^\infty(\mathcal{M})$  as an abstract algebra. This is enough to reconstruct  $\mathcal{M}$  as a differentiable manifold (see [80]) together with the eigenfunctions as functions on  $\mathcal{M}$ , rather than abstract elements of an algebra. Together with the knowledge of the spectrum of the Laplace-Beltrami operator this allows one to recover

the Riemannian metric. There are multiple ways of doing this. One of them is to deduce the leading order coefficients of the Laplace-Beltrami operator  $(-g^{ij})$ .

Consequently, the extraction of geometric information from the generalized Rayleigh quotients can be approached from a different direction. One could attempt to use them to obtain some information about the  $C_{mnk}$ , which would be of use for the reconstruction of manifolds from spectra.

# Chapter 7

## Conclusion and Outlook

In [69], it was observed that the key difficulty with understanding the relationship between shape and spectrum is the high nonlinearity of the spectral map. Following this line of thought, we studied the problem of inverse spectral geometry in an infinitesimal or perturbative regime.

In Chapter 3, we began our investigations by a numerical exploration of the possibility of reconstructing small changes in shape from small changes in spectrum. The numerical experiment was carried out on a set of star-shaped planar domains. It has shown the reconstruction to be possible on the condition that the number of considered Laplace-Beltrami eigenvalues matches the number of shape degrees of freedom. This indicates that, at least in the studied space of shapes, it is possible to preserve the local reconstruction of changes of shape from changes in spectrum in the limit of infinitely many shape degrees of freedom. It would be of interest to check whether this relationship holds in other spaces of shapes. Especially interesting is the case of curved manifolds. This case is more difficult than that of the planar domains due to the fact that the space of isometries of general manifolds is much richer than that of  $\mathbb{R}^2$ . In that case, we propose to compare shapes using the heat kernel matching methods developed in [83]. Moreover, the perturbation formulas obtained in Chapter 5 can be used to speed up the computation of the Jacobian matrix used by our numerical algorithm.

Motivated by the success of our numerical explorations, we pursued a perturbative approach to spectral geometry. Specifically, we turned our attention to isospectral perturbations. This choice warrants a few comments. In our work, perturbation theory is used in a way opposite to the usual one. This is a consequence of the fact that inverse spectral geometry is indeed an inverse problem. A typical application of the perturbative

expansions seeks to approximate the spectral decomposition of an operator using the more easily accessible spectral decomposition of a neighboring operator. In practice, this means starting with a well-understood operator and adding small corrections to it. Physically, this corresponds to taking small, previously neglected, interactions into consideration. Our usage of perturbation theory is very different. Indeed, by assuming the isospectrality of perturbations, we begin with the knowledge of the eigenvalue corrections. In the case of a self-adjoint perturbation (Section 5.3.1), we can use that to deduce the vanishing of the corrections to the operator. Interestingly, very little knowledge of the unperturbed operator is necessary. One merely has to require that the eigenvectors span the whole Hilbert space and that the eigenvalues are finitely degenerate and can be numbered such that  $\lambda_0^{(0)} \leq \lambda_1^{(0)} \leq \lambda_2^{(0)} \leq \dots$ . This is the type of information provided by spectral theorems, such as the one for Laplace-type operators (Theorem 2.2).

The fact that the perturbations of the Laplace-Beltrami operator induced by perturbations of the metric are not self-adjoint complicates matters. Nonetheless, a situation similar to the self-adjoint one was recovered in the case of conformal perturbations of metrics on surfaces. This led us to rule out the existence of certain types of isospectral deformations of surfaces. In particular, strictly linear conformal perturbations of either metrics  $g_{ij}$  or inverse metrics  $g^{ij}$  were found to never be isospectral. This was then used to disprove the existence of straight paths in the spaces of metrics and inverse metrics. A fundamentally local statement about perturbations was thus converted in a global result about the non-existence of convex sets of isospectral, conformally equivalent metrics on surfaces (Theorems 5.10 and 5.12). To the best of our knowledge, this is the first result regarding the geometry (or the linear structure), rather than the topology, of isospectral sets.

The generalization of such non-convexity results to manifolds of higher dimension is a natural direction of future research. As evidenced by the expressions for the eigenvalue correction of Lemmas 5.7 and 5.8, the inductive argument of Section 5.3.1 will likely have to be modified or, perhaps, replaced entirely. The main issue is that one can no longer guarantee that the numerator in the expression for the second order eigenvalue corrections is positive. Nonetheless, we expect that analogous results can be shown to hold in higher dimensions. It is possible that using a different decomposition of the degrees of freedom of the metric, such as the one in [109], would shed some light on this issue.

As a byproduct of our investigations into perturbations of Laplace-Beltrami operators, we arrived at two conjectures (see Chapter 6). The first one is that, generically, the squares of the eigenfunctions span all of  $L_2(\mathcal{M})$ . We found some numerical evidence for this. Proving this conjecture would provide one with powerful tools for further perturbative investigations, especially into conformal perturbations. In fact, if one assumes that this

conjecture holds, the proof of our main results (Theorems 5.10 and 5.12) becomes significantly simpler. The proof would no longer require considering the second order eigenvalue corrections, as the first order ones would already be sufficient.

The second conjecture stated in Chapter 6 has to do with the generalized Rayleigh quotients of Lemma 6.3 and Equation (6.17). These formulas are spectral invariants that could plausibly be used to extract geometric information from the Laplace-Beltrami eigenvalues. The issue with those invariants is an uncertain domain of definition. We conjecture that, generically, they are defined. We obtained some numerical evidence in support of this conjecture. The question whether those expressions can be used to extract geometric information from the spectrum remains open.

To conclude this thesis, we would like to speculate regarding the implications of our non-convexity results for the rarity of isospectrality. Recall that the program of identifying shape from spectrum would be put on solid mathematical footing if the set of all families of isospectral non-isometric metrics formed a meager set (see Appendix A for a definition) in the space of all metrics on a manifold.

Were there countably many isospectral families, this problem would be reduced to showing that each set of isospectral non-isometric metrics is nowhere dense in the set of all metrics. Recall that a set is nowhere dense if its closure has empty interior [71]. Also recall that pre-images of closed sets under continuous maps are closed. Since eigenvalues are continuous functions of the metric (see [6]) and countable intersections of closed sets are closed, the isospectral sets are closed. It remains to show that they have empty interior. This is very easy to prove, as it is sufficient to show that any member of such a family can be deformed in a way that changes its spectrum. Note that a perturbation  $(1+t)g_{ij}$  of  $g_{ij}$  changes the volume of the manifold, no matter how small  $|t|$  is. Since the volume is spectrally determined (see Equation (2.45)), this perturbation exits the isospectral set. Thus, any given isospectral set is nowhere dense.

There are however uncountably many isospectral non-isometric families. Indeed, given one such family, applying the deformation  $(1+t)g_{ij}$  to all members of the family yields another, different, isospectral family <sup>1</sup>. This results from the fact that this changes the Laplace-Beltrami operator from  $\Delta$  to  $(1+t)^{-1}\Delta$ . This merely rescales the spectrum, which preserves the isospectrality of the family. Since there are uncountably many real numbers, this can be used to construct uncountably many isospectral non-isometric families.

Consequently, it is likely that one has to establish a result stronger than nowhere density for the isospectral sets. Note that our Lemmas 5.9 and 5.11 are such results.

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<sup>1</sup>We find it mildly amusing that the same deformation can be used to show both the nowhere density of an isospectral set and the fact that there exist uncountably many such sets.

Indeed, ruling out isospectral perturbations of the form  $(1 + tf^{(1)})g_{ij}$ , with  $f^{(1)} \in C^\infty(\mathcal{M})$ , is more than what is required to establish nowhere density. Whether this is enough to show that counterexamples to the program of inverse spectral geometry form a negligible set is the subject of further research.

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# Appendix A

## Meager and Residual Sets

The notions of meager and residual sets arise when discussing whether a given property is satisfied by generic points of a topological space. In other words, those notions formalize what it means for a subset of a topological space to be non-generic and thus negligible. For a complete discussion of meager and residual sets we refer the reader to textbooks on general topology, say [75, 107].

Begin by recalling the definition of a nowhere dense set.

**Definition A.1** (Nowhere Dense Set). *Let  $X$  be a topological space and let  $A \subset X$ .  $A$  is said to be nowhere dense in  $X$  if its closure  $\bar{A}$  has empty interior.*

We can now proceed to define meager and residual sets.

**Definition A.2** (Meager Set). *Let  $X$  be a topological space and let  $A \subset X$ .  $A$  is said to be meager if it is a countable union of nowhere dense sets.*

**Definition A.3** (Residual Set). *Let  $X$  be a topological space and let  $A \subset X$ .  $A$  is said to be residual if its complement is meager.*

As the name suggests, meager sets are negligible. Correspondingly, their complements (the residual sets) are considered non-negligible. The long term goal of spectral geometry then becomes to show that, given an appropriate topology, a residual set of Riemannian manifolds can be determined (up to isometry) from the spectra of some Laplace-type operators. This has been shown to hold in some special classes of manifolds, see [110, 111, 113].



Neglecting meager subsets is analogous to neglecting subsets of measure zero in a measure space. By probabilistic arguments, points in a set of measure zero occur with vanishing probability and will thus *almost never* be of relevance. Note that this is an analogy, not an equivalence. This analogy is explored in [84].

We close this section with a minor comment on the terminology used here. Meager sets are also known as sets of first category. Sets that are not meager are known as sets of second category. This terminology is non-descriptive, which is why we prefer calling sets of first category meager. Nonetheless, this terminology is used by many texts as it historically arose in the formulation of the Baire category theorem, which we include here for the sake of completeness.

**Theorem A.4** (Baire Category Theorem). *Let  $X$  be a complete metric space. Then,  $X$  is of second category when viewed as a subset of itself. In other words,  $X$  is not a countable union of nowhere dense subsets.*

# Appendix B

## Symmetric and Self-Adjoint Operators

In this appendix, we briefly discuss the difference between symmetric and self-adjoint operators on Hilbert spaces. We refer the reader to [2, 91] for the details.

Let  $\mathcal{H}$  be a Hilbert space with inner product  $\langle \cdot, \cdot \rangle$ . Let  $A$  be a linear operator on  $\mathcal{H}$ . If  $A$  is unbounded, it is not possible to define  $A$  on all of  $\mathcal{H}$ . Let  $D(A) \subset \mathcal{H}$  be the domain of definition of  $A$ .  $A$  is said to be densely defined if  $D(A)$  is dense in  $\mathcal{H}$ .

**Definition B.1** (Symmetric Operator). *Let  $A$  be a densely defined linear operator on  $\mathcal{H}$ .  $A$  is said to be symmetric if  $\langle \psi, A\phi \rangle = \langle A\psi, \phi \rangle$  for all  $\psi, \phi \in D(A)$ .*

The symmetry of an operator is insufficient to guarantee that it obeys a spectral theorem. Self-adjointness, on the other hand, provides such a condition [91]. Begin by recalling the definition of the adjoint of  $A$ .

**Definition B.2.** *Let  $A$  be a densely defined linear operator on  $\mathcal{H}$ . Let  $D(A^\dagger) \subset \mathcal{H}$  be the set of  $\phi \in \mathcal{H}$  for which there exists  $\eta \in \mathcal{H}$  such that*

$$\langle A\psi, \phi \rangle = \langle \psi, \eta \rangle \quad , \quad \forall \psi \in D(A) \quad . \quad (\text{B.1})$$

*With  $\phi$  and  $\eta$  as above, the adjoint  $A^\dagger$  of  $A$  is defined as*

$$\begin{aligned} A^\dagger : D(A^\dagger) &\rightarrow \mathcal{H} \\ \phi &\mapsto \eta \quad . \end{aligned} \quad (\text{B.2})$$

Notice that, in general,  $D(A^\dagger) \neq D(A)$ . This difference in domains is what distinguishes symmetric operators from self-adjoint ones. Indeed, consider the definition of self-adjointness.

**Definition B.3** (Self-Adjoint Operator). *A densely defined operator  $A$  is said to be self-adjoint if  $A = A^\dagger$ . In other words,  $A$  is self-adjoint if and only if it is symmetric and  $D(A^\dagger) = D(A)$ .*

Under certain conditions, the domain of a symmetric operator can be extended to make the operator self-adjoint. This procedure is known as a self-adjoint extension. In the case of differential operators, the extension takes the form of imposing boundary conditions. We refer the reader to [2] for the details.

Self-adjoint extensions of differential operators are frequently done implicitly. Indeed, this is how we approach the spectral theorem for Laplace-type operators (Theorem 2.2).