A Parallel Study of the Fock Space Approach to Classical and Free Brownian Motion

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The purpose of this thesis is to elaborate the similarities between classical and free probability by means of developing chaos decomposition of stochastic integrals driven by Brownian motion and its free counterparts in a parallel manner. The work focuses on constructing an apparatus that is general enough so that these similarities are apparent, yet not too general that their distinctions are completely obscured. In particular, we employ the notion of lattice paths to bring about the moment calculation of normally distributed random variables in the non-commutative probability environment; and we exploit the structure of the lattice of partition of n-elements, which underlies the relationships between stochastic integrations defined in the Itô and in the Wiener sense, to prove both classical and free chaos decomposition result.

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Introduction

Let us start this introduction with one of my all time favourite quotes by David Hilbert.

> The art of doing mathematics consists in finding that special case which contains all the gems of generality.

As the title of this thesis suggests, this special case is taken to be Brownian motion. However, as Brownian motion has so many properties, it is beyond the limitation of this thesis to draw the analogies on all aspects of Brownian motion. Therefore, we choose to further specialize in just one of its fundamental results, namely the chaos decomposition. With this in mind, the present thesis focuses on constructing an apparatus that is general enough so that the similarities between the terminologies/statements of classical and free probability are apparent, yet not too general that their distinctions are completely obscured.

In Chapter 1, we provide a brief account for the theory of non-commutative probability spaces. In particular, we see how the concept of moments in classical probability is captured by the framework of non-commutative probability space. However, one of the major differences is that given a collection of random variables $\{Y_i\}_{i\in I}$, for some index set I, free probability is interesting in the case when $\{Y_i\}_{i\in I}$ are related by free independence, whereas classical probability interest is in the case when $\{Y_i\}_{i\in I}$ are related by classical independence.

In Chapter 2, we concentrate on the common ground between classical and free independence: that is, they both depend on the results of moments of random variables. Here, we devise procedures in associating moment calculations of random variables of the form

$A + B$ to lattice paths. Most importantly, the lattice paths enables us to see clearly, cf. the proof of Lemma 2.11, that moments of $A + B$ depends only on the "dot product" of A, B, not so much on the exact values that the each of them takes, cf. Corollary 2.12. This fact is used to calculate the moments of normal distributions in the non-commutative environment. A similar but simpler calculation, which does not require weights of lattice paths, will lead to the moments of Wigner semicircle law, which is the counterpart of the Gaussian law in free probability.

At the beginning of Chapter 3 we briefly review the necessary rules in manipulating elements of tensor product space so as to define the full Fock space on which the creation and annihilation operators are defined. This chapter then finishes with the result that the semicircularly distributed random variables $\{X(h_i)\}_{i=1,2,\ldots,r}$, $r \in \mathbb{N}$, are freely independent if $\{h_1, h_2, \ldots, h_r\}$ is a collection of mutually orthogonal vectors. Chapter 4 mirrors the development of Chapter 3 and finishes with the similar result: normally distributed random variables $\{Q(h_i)\}_{i=1,2,\ldots,r}$, $r \in \mathbb{N}$, are (classically) independent if $\{h_1, h_2, \ldots, h_r\}$ is a collection of mutually orthogonal vectors.

Chapter 5 starts with the notion of dyadic step functions which are used to define stochastic integrals in both Itô and Wiener sense. It is worth pointing out that the main reason for involving the dyadic step functions in defining the stochastic integrals is that it makes the lattice-of-partition-relationship between the Itô and the Wiener integral apparent. This relationship plays a pivotal role in the proof of Lemma 5.18 which in turn plays the pivotal role in proving our chaos decomposition results, namely Theorem 5.19 and Theorem 6.11. Finally, in the same spirit as before, Chapter 6 mirrors the behaviour of Chapter 5 and develops its free counterparts.

Chapter 1

Background

1.1 Non-commutative probability space

Definition 1.1. Suppose $\mathcal{A}$ is a unital algebra (over $\mathbb{C}$). A pair $(\mathcal{A}, \varphi)$ is said to be a **non-commutative probability space** if φ is a unital linear functional on $\mathcal{A}$, i.e. if

[i] $\varphi : \mathcal{A} \to \mathbb{C}$ (Functional on $\mathcal{A}$: sending vectors in $\mathcal{A}$ into the underlying field),

[ii] $\forall a, b \in \mathcal{A}, \forall \lambda, \mu \in \mathbb{C}, \varphi(\lambda\, a + \mu\, b) = \lambda\, \varphi(a) + \mu\, \varphi(b)$ (Linearity), and

[iii] $\varphi(1_{\mathcal{A}}) = 1$ (Unital functional).

Note. Elements of non-commutative probability space are called **non-commutative random variables**.

Definition 1.2. An algebra $\mathcal{A}$ is said to be a $*$**-algebra** (over $\mathbb{C}$) if

[i] $* : \mathcal{A} \to \mathcal{A}$ given by $a \mapsto a^*$ satisfies $(a^*)^* = a$ (Involution),

[ii] $\forall a, b \in \mathcal{A}, (a\, b)^* = b^*\, a^*$ (Antihomomorphism), and

[iii] $\forall a, b \in \mathcal{A}, \forall \lambda, \mu \in \mathbb{C}, (\lambda\, a + \mu\, b)^* = \overline{\lambda}\, a^* + \overline{\mu}\, b^*$ (Antilinearity).

Remark 1.3. An element $a \in \mathcal{A}$ is said to be **self-adjoint** if $a^* = a$.

Remark 1.4. In the event that the $*$-algebra $\mathcal{A}$ is unital, one easily deduces that ${1_{\mathcal{A}}}^* = 1_{\mathcal{A}}$, since

$$1_{\mathcal{A}}\, {1_{\mathcal{A}}}^* = {1_{\mathcal{A}}}^* \implies (1_{\mathcal{A}}\, {1_{\mathcal{A}}}^*)^* = {1_{\mathcal{A}}}^{**} \implies 1_{\mathcal{A}}\, {1_{\mathcal{A}}}^* = 1_{\mathcal{A}} \implies {1_{\mathcal{A}}}^* = 1_{\mathcal{A}}.$$

Lemma 1.5. *Suppose $\mathcal{A}$ is a $*$-algebra. Then, for any arbitrary $x \in \mathcal{A}$, one has:*

$$x = a + i\, b,$$

for some self-adjoints $a, b \in \mathcal{A}$. Moreover, the pair (a, b) is uniquely determined for each $x \in \mathcal{A}$.

Proof. Take

$$a = \frac{x + x^*}{2}, \qquad b = \frac{x - x^*}{2\, i}. \tag{1.1}$$

To prove uniqueness, suppose $x = a' + i\, b'$ for some different self-adjoints $a', b' \in \mathcal{A}$. By taking the $*$-operation, we get: $x^* = a' - i\, b'$. Putting it together with Equation (1.1) yields the simultaneous equations:

$$\begin{cases} a + i\, b = a' + i\, b', \\ a - i\, b = a' - i\, b'. \end{cases}$$

Now by adding and subtracting these equations, we get: $(a, b) = (a', b')$ which completes the uniqueness. ☐

Lemma 1.6. *Suppose $\mathcal{A}$ is a unital $*$-algebra. Then, for any arbitrary self-adjoint $a \in \mathcal{A}$, one has:*

$$a = u^*\, u - v^*\, v,$$

for some $u, v \in \mathcal{A}$.

Proof. Take

$$u = \frac{a + 1_{\mathcal{A}}}{2}, \qquad v = \frac{a - 1_{\mathcal{A}}}{2}.$$

$\square$

Definition 1.7. A linear functional φ on $\mathcal{A}$ is said to be **positive** if

$$\varphi(a^* a) \geq 0, \qquad \forall a \in \mathcal{A}.$$

Note. Armed with such positive φ, the form $a^* a$ naturally signifies the notion of positive reals in the $*$-algebra setting.

Definition 1.8. A non-commutative probability space $(\mathcal{A}, \varphi)$ is said to be a **$*$-probability space** if

[i] the unital algebra $\mathcal{A}$ is a $*$-algebra, and

[ii] the unital linear functional φ on $\mathcal{A}$ is positive.

Lemma 1.9. *Let $(\mathcal{A}, \varphi)$ be a $*$-probability space. Then one has:*

$$\varphi(x^*) = \overline{\varphi(x)}, \qquad \forall x \in \mathcal{A}. \tag{1.2}$$

Note. A functional φ satisfying Equation (1.2) is said to be **self-adjoint**.

Proof. Pick an arbitrary $x \in \mathcal{A}$. By Lemma 1.5, we have: $x = a + i\,b$ for some self-adjoints $a, b \in \mathcal{A}$; hence

$$\overline{\varphi(x^*)} = \overline{\varphi((a+i\,b)^*)} = \overline{\varphi(a - i\,b)} = \overline{\varphi(a) - i\,\varphi(b)} = \varphi(a) + i\,\varphi(b),$$

where the last equality follows from

[i] applying Lemma 1.6 to the self-adjoints a, b, and

[ii] the positivity of the functional φ.

Finally, it is also clear from the linearity of φ that $\varphi(a) + i\,\varphi(b) = \varphi(x)$. $\square$

Lemma 1.10. *Let $(\mathcal{A}, \varphi)$ be a $*$-probability space. Then one has:*

$$|\varphi(b^*\,a)|^2 \leq \varphi(a^*\,a)\,\varphi(b^*\,b), \qquad \forall a, b \in \mathcal{A},$$

*which is commonly called the **Cauchy-Schwarz inequality** for the functional φ.*

Proof. Take arbitrary $u, v \in \mathcal{A}$. Let $t \in \mathbb{R}$ be a variable. By the positivity of φ, we obtain

$$\varphi\left((u + t\,v)^*\,(u + t\,v)\right) \geq 0.$$

By expanding the argument of φ and utilizing the linearity of φ, we get:

$$\varphi(u^*\,u) + (\varphi(u^*\,v) + \varphi(v^*\,u))\,t + \varphi(v^*\,v)\,t^2 \geq 0,$$

which, in view of Lemma 1.9, is a quadratic in $t \in \mathbb{R}$ with real coefficients. Thus it follows that the discriminant must be less than or equal to 0; hence

$$(\mathrm{Re}\{\varphi(u^*\,v)\})^2 \leq \varphi(u^*\,u)\,\varphi(v^*\,v), \qquad \forall u, v \in \mathcal{A}. \tag{1.3}$$

Returning to the Cauchy-Schwarz inequality, take arbitrary $a, b \in \mathcal{A}$. Since it is clear that the inequality holds trivially when $\varphi(b^*\,a) = 0$, we are left with the case when $\varphi(b^*\,a) \neq 0$. Putting $u = \frac{\varphi(b^*\,a)}{|\varphi(b^*\,a)|}\,b$ and $v = a$ into Equation (1.3) yields:

$$\left(\mathrm{Re}\left\{\frac{\overline{\varphi(b^*\,a)}}{|\varphi(b^*\,a)|}\,\varphi(b^*\,a)\right\}\right)^2 \leq \frac{\overline{\varphi(b^*\,a)}\,\varphi(b^*\,a)}{|\varphi(b^*\,a)|^2}\,\varphi(b^*\,b)\,\varphi(a^*\,a).$$

Finally, simplifying this last inequality indeed gives the desired result. $\square$

Note. It follows from Lemma 1.10 that if $a \in \mathcal{A}$ is non-zero such that $\varphi(a^*\,a) = 0$, then $\varphi(b\,a) = 0$ for all $b \in \mathcal{A}$. In such case, a is called a **degenerate element** for φ.

Definition 1.11. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space. The functional φ is said to be **faithful** if

$$\varphi(a^*\,a) = 0 \implies a = 0.$$

Having introduced the basic theory of the $*$-probability space, now we turn to look for mathematical objects that this structure applies to. As its name suggests, the most common candidates come from the probability theory. To make the connection, let us recall what we mean by a probability space. A triplet $(\Omega, \mathcal{F}, P)$ is called a **probability space** if

[i] Ω is a set/sample space, i.e. a collection consisting of elements/outcomes,

[ii] $\mathcal{F}$ is a σ-algebra of subsets of Ω; loosely speaking, a certain collection of Ω-subsets/events, and

[iii] $P : \mathcal{F} \to [0, 1]$ is a probability measure.

Furthermore, for each $p \geq 1$, we denote the space of functions

$$\left\{ X : \Omega \to \mathbb{C} \,\middle|\, X \text{ is } \mathcal{F}/\mathcal{B}_{\mathbb{C}}\text{-measurable and } \int_\Omega |X(\omega)|^p \, dP(\omega) < \infty \right\}$$

by $L^p(\Omega, \mathcal{F}, P)$, where $\mathcal{B}_{\mathbb{C}}$ stands for the σ-algebra generated by open sets in $\mathbb{C}$, i.e. the Borel σ-algebra. In particular, for $X \in L^p(\Omega, \mathcal{F}, P)$, we have $\int_\Omega X^p \, dP < \infty$, where $\int_\Omega X^p \, dP$ simply stands for $\int_\Omega (X(\omega))^p \, dP(\omega)$. Next, we try to get a feel of the relative "size" of each of these $L^p(\Omega, \mathcal{F}, P)$ spaces.

Proposition 1.12. *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Given $1 \leq p \leq q$, we have*

$$L^p(\Omega, \mathcal{F}, P) \supset L^q(\Omega, \mathcal{F}, P).$$

Proof. Pick an arbitrary $X \in L^q(\Omega, \mathcal{F}, P)$. Define a set

$$S := \{\omega \in \Omega \mid |X(\omega)| \leq 1\}.$$

Then, we estimate:

$$\begin{aligned}\int_\Omega |X|^p \, dP &= \int_S |X|^p \, dP + \int_{\Omega\setminus S} |X|^p \, dP \\ &\leq \int_S 1^p \, dP + \int_{\Omega\setminus S} |X|^q \, dP \leq 1 + \int_\Omega |X|^q \, dP < \infty,\end{aligned}$$

which says precisely that $X \in L^p(\Omega, \mathcal{F}, P)$. ☐

Definition 1.13. Let $(\Omega, \mathcal{F}, P)$ be a probability space. We define

$$L^{\infty-}(\Omega, \mathcal{F}, P) := \bigcap_{1\leq p<\infty} L^p(\Omega, \mathcal{F}, P).$$

In order to have $L^{\infty-}(\Omega, \mathcal{F}, P)$ serve as a component of a specific example of $*$-probability space, we must have at least the follow result.

Proposition 1.14. *$L^{\infty-}(\Omega, \mathcal{F}, P)$ is closed under multiplication.*

Proof. Recall the Cauchy-Schwarz inequality for integrals:

$$\int_\Omega |X\,Y| \, dP \leq \sqrt{\int_\Omega |X|^2 \, dP} \sqrt{\int_\Omega |Y|^2 \, dP} = \sqrt{\int_\Omega \overline{X}\, X \, dP} \sqrt{\int_\Omega \overline{Y}\, Y \, dP},$$

for all $\mathcal{F}/\mathcal{B}_{\mathbb{C}}$-measurable functions X, Y. Next, fix any $X, Y \in L^{\infty-}(\Omega, \mathcal{F}, P)$, and we shall show that $\int_\Omega |(X\,Y)^n| \, dP < \infty$ for all $n \in \mathbb{N}$. Now, by the Cauchy-Schwarz inequality for integrals, we estimate:

$$\begin{aligned}\int_\Omega |(X\,Y)^n| \, dP = \int_\Omega |X^n\, Y^n| \, dP &\leq \sqrt{\int_\Omega \overline{X^n}\, X^n \, dP} \sqrt{\int_\Omega \overline{Y^n}\, Y^n \, dP} \\ &= \sqrt{\int_\Omega |X|^{2\,n} \, dP} \sqrt{\int_\Omega |Y|^{2\,n} \, dP} < \infty,\end{aligned}$$

where the last inequality follows from $X, Y \in L^{\infty-}(\Omega, P) \subset L^{2\,n}(\Omega, P)$. Finally, since the product of measurable functions are again measurable, the proof is completed. ☐

Now that we have our possible candidate for the $\mathcal{A}$-component of a $*$-probability space $(\mathcal{A}, \varphi)$, what can serve for the φ-component? One possibility is the expectation. Suppose $X \in L^p(\Omega, \mathcal{F}, P)$. The **expectation** $\mathbb{E}(X)$ of the random variable X is given by

$$\mathbb{E}(X) := \int_\Omega X \, dP.$$

More generally, $\mathbb{E}(X^n)$ denotes the **n-th moment** of X, where $n \in \mathbb{N}$ is referred to as the **order** of the moment. This leads to the following convention

Definition 1.15. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and $a \in \mathcal{A}$. We call $\varphi(a^n)$ the **n-th moment** of a.

Remark 1.16. With these terminologies, $L^{\infty-}(\Omega, \mathcal{F}, P)$ can be understood as the algebra of complex random variables on Ω, which has finite moments of all orders.

Finally, we finish this section with two examples of $*$-probability space.

Example 1.17. The algebra $L^{\infty-}(\Omega, \mathcal{F}, P)$, where the vector multiplication is taken to be the usual pointwise product of two functions, together with the expectation $\mathbb{E}(\cdot)$ form a $*$-probability space. Notice that the positive unital linear functional $\mathbb{E}(\cdot)$ is faithful.

Example 1.18. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space and let d be a positive integer. Let $M_d(\mathcal{A})$ be the space of $d \times d$ matrices over $\mathcal{A}$

$$M_d(\mathcal{A}) := \{(a_{ij})_{i,j=1}^d \mid a_{ij} \in \mathcal{A}\},$$

with a $*$-operation $* : M_d(\mathcal{A}) \to M_d(\mathcal{A})$ given by

$$((a_{ij})_{i,j=1}^d)^* := ({a_{ji}}^*)_{i,j=1}^d,$$

and a linear functional $\varphi_d : M_d(\mathcal{A}) \to \mathbb{C}$ given by

$$\varphi_d((a_{ij})_{i,j=1}^d) = \frac{1}{d} \sum_{i=1}^d \varphi(a_{ii}).$$

Then, $(M_d(\mathcal{A}), \varphi_d)$ is a $*$-probability space.

1.2 Classical and free independence

Let us fix $(\mathcal{A}, \varphi)$ to be some non-commutative probability space and I to be some index set throughout this section. A collection $\{\mathcal{A}_i\}_{i\in I}$ of unital subalgebras of $\mathcal{A}$ is said to **commute** if for all $a \in \mathcal{A}_{i_1}, b \in \mathcal{A}_{i_2}$ with $i_1 \neq i_2$, we have $a\,b = b\,a$.

Definition 1.19. A collection $\{\mathcal{A}_i\}_{i\in I}$ of unital subalgebras of $\mathcal{A}$ is said to be **tensor independent** if $\{\mathcal{A}_i\}_{i\in I}$ commutes and if, for all $k \in \mathbb{N}$, we have the implication

$$\left.\begin{array}{l} a_j \in \mathcal{A}_{i(j)}\ (i(j) \in I) \text{ for all } j = 1, 2, \ldots, k \\ \substack{i(1)\neq i(2)\\ i(1)\neq i(3)\\ \vdots \\ i(1)\neq i(k)},\ \substack{i(2)\neq i(3)\\ \vdots \\ i(2)\neq i(k)},\ \ldots, i(k-1) \neq i(k) \end{array}\right\} \Longrightarrow \varphi(a_1\, a_2 \,\cdots\, a_k) = \varphi(a_1)\,\varphi(a_2)\,\cdots\,\varphi(a_k).$$

Definition 1.20. Random variables $a_i \in \mathcal{A}$, $i \in I$, are said to be **independent** or **classically independent** if the generated unital subalgebras $\mathrm{alg}(1, a_i)$, $i \in I$, are tensor independent.

Remark 1.21 (Classical independence)*.* Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Two random variables $a, b \in \mathcal{A}$ are independent if $a\,b = b\,a$ and $\varphi(a^j\, b^k) = \varphi(a^j)\,\varphi(b^k)$, for all $j, k = 1, 2, 3, \ldots$.

Definition 1.22. A collection $\{\mathcal{A}_i\}_{i\in I}$ of unital subalgebras of $\mathcal{A}$, is said to be **freely independent** if, for all $k \in \mathbb{N}$, we have the implication

$$\left.\begin{array}{l} a_j \in \mathcal{A}_{i(j)}\ (i(j) \in I) \text{ for all } j = 1, 2, \ldots, k \\ \varphi(a_j) = 0 \text{ for all } j = 1, 2, \ldots, k \\ i(1) \neq i(2), i(2) \neq i(3), \ldots, i(k-1) \neq i(k) \end{array}\right\} \Longrightarrow \varphi(a_1\, a_2 \,\cdots\, a_k) = 0.$$

Definition 1.23. Random variables $a_i \in \mathcal{A}$, $i \in I$, are said to be **free** (resp. $*$-free) or **freely independent** (resp. $*$-freely independent) if the generated unital subalgebras $\mathrm{alg}(1, a_i)$ (resp. $\mathrm{alg}(1, a_i, a_i{}^*)$), $i \in I$, are freely independent in $(\mathcal{A}, \varphi)$.

Let us illustrate how to use the definition of free independence with the a simple example.

Example 1.24. Suppose that the random variables $a, b \in \mathcal{A}$ in a non-commutative probability space $(\mathcal{A}, \varphi)$ are free. Express $\varphi(a\,b)$ in terms of $\varphi(a)$ and $\varphi(b)$.

Solution. In order to be able to use the freeness condition, we must have an expression involving centered random variables. This leads to the consideration of the expression $(a - \varphi(a)\,1)\,(b - \varphi(b)\,1)$. By definition, $(a - \varphi(a)\,1)$ belongs to the generated unital subalgbra $\mathrm{alg}(1, a)$; similarly, $(b - \varphi(b)\,1) \in \mathrm{alg}(1, b)$. So, by the freeness, we get $\varphi\left((a - \varphi(a)\,1)\,(b - \varphi(b)\,1)\right) = 0$. On the other hand, by the linearity of φ, we deduce that $\varphi\left((a - \varphi(a)\,1)\,(b - \varphi(b)\,1)\right) = \varphi\,(a\,b) - \varphi\,(a)\;\varphi\,(b)$. Rearranging give: $\varphi\,(a\,b) = \varphi\,(a)\;\varphi\,(b)$.

Based on Example 1.24, it is tempting to speculate that there might be a remark similar to Remark 1.21, which holds for freely independent random variables. This is, however, not true. For instance, if random variables $a, b \in \mathcal{A}$ are free, then it can be shown that

$$\varphi(a\,b\,a\,b) = \varphi(a^2)\,\varphi(b)^2 + \varphi(a)^2\,\varphi(b^2) - \varphi(a)^2\,\varphi(b)^2,$$

where its calculation can be found in [7, Remarks 5.16.].

Next, we recall that a random variable is classically understood as a measurable function. Furthermore, measure theoretic framework captures the intuitive meaning of a random variable Y in ways that makes sense of the integral $\int_\Omega f(Y)\,dP$, for a function $f : \mathbb{R} \to \mathbb{R}$, while avoiding making reference to what specific value the function $f(Y(\omega))$ takes for each outcome $\omega \in \Omega$. This leads to the consequence that given a random variable Y, we are primarily concerned with its distribution (known as the push-forward of Y in the measure theoretic framework), whose behaviour is captured by the linear functional

$$\int_\Omega f(Y)\,dP$$

defined on some set of which f is an element. Following this trend, we give the next definition.

Definition 1.25 (Distribution of a non-commutative random variable)**.** Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and $a \in \mathcal{A}$ be self-adjoint. A probability measure μ on $\mathbb{R}$ is said to be a distribution of a if

$$\varphi(f(a)) = \int_{\mathbb{R}} f(z)\, d\mu(z), \tag{1.4}$$

for any polynomial f. If there is just one such measure μ satisfying all the equalities (there are many polynomials) of Equation (1.4), we call such μ the distribution of a.

The present thesis deals with only two types of distributions, namely the normal distribution and the Wigner semicircle law. In the event that we wish to say that a non-commutative random variable b has the Wigner semicircle law, it suffices to verify that all moments of b agree with the moments of the Wigner semicircle law. This is because the associated (in the above sense) linear functional of Wigner semicircle law is completely determined by what it does on the polynomials, which is a result of the Stone–Weierstrass theorem. On the other hand, if we want to show that a non-commutative random variable c is normally distributed, then it is again sufficient to verify that all moments of c agree with the moments of the normal distribution. The reason for this is that the moments of normal distribution satisfies the so-called Carleman's condition. The Carleman's condition can be found in [1] and [10, pp. 294-296].

Chapter 2

Fundamentals of moment calculation

In this chapter, we shall give a simplified version of what is to come. Throughout this chapter, take $\mathcal{H}$ to be a separable complex Hilbert space with a countable orthonormal basis $\{\, \delta_n \mid n = 0, 1, 2, \dots \}$. Denote $\mathcal{L}_0 := \{\, X : \mathcal{H}_0 \to \mathcal{H}_0 \mid X \text{ is linear} \}$, where $\mathcal{H}_0 := \left\{ \sum_{j=1}^{n} d_j\, \delta_j \;\middle|\; d_j \in \mathbb{C}, n \in \mathbb{N} \right\}$ is the linear span of the set $\{\, \delta_n \mid n = 0, 1, 2, \dots \}$. Moreover, we define a unital linear functional φ_0 on the set of operators $\mathcal{L}_0$ by the formula

$$\varphi_0(X) := \langle\, X\, \delta_0 \;,\; \delta_0 \,\rangle_{\mathcal{H}} \,,$$

where $\langle \,\cdot\, ,\, \cdot\, \rangle_{\mathcal{H}}$ is the inner product on $\mathcal{H}$. Then $(\mathcal{L}_0, \varphi_0)$ is a non-commutative probability space, and it is the toy model on which we shall see a glimpse of pieces of classical/free Brownian motion, namely random variables with normal distributions and with semicircle distributions. Without further ado, let us proceed to finding/constructing/defining these random variables (linear operators on $\mathcal{H}_0$).

2.1 Creation and annihilation operators

Since a linear map on $\mathcal{H}_0$ is completely determined by where it sends its basis to, we shall define the following random variables by specifying what they do on the orthonormal basis $\{\, \delta_n \mid n = 0, 1, 2, \dots \}$.

Definition 2.1 (Creation and Annihilation operator, free case)**.**

$$S(\delta_n) = \delta_{n+1}, \qquad \forall n = 0, 1, 2, \ldots,$$

$$S^*(\delta_n) = \delta_{n-1}, \qquad \forall n = 1, 2, \ldots,$$

and $S^*(\delta_0) = 0 \in \mathcal{H}$.

Remark 2.2. The annihilation operator S^* is indeed the adjoint operator for the random variable S. This can be seen from verifying the relationship

$$\langle\, S\,\delta_p\ ,\ \delta_q\,\rangle_{\mathcal{H}} = \langle\, \delta_p\ ,\ S^*\delta_q\,\rangle_{\mathcal{H}}\,, \qquad \forall p, q = 0, 1, 2, \ldots,$$

which follows from the fact that $\{\, \delta_n \mid n = 0, 1, 2, \ldots \}$ is an orthonormal basis and their defining formulas. Indeed, both sides of the above displayed equation are equal to 1 whenever $p = q - 1$, and are equal to 0 otherwise.

Definition 2.3 (Creation and Annihilation operator, classical case)**.**

$$T(\delta_n) = \sqrt{n+1}\,\delta_{n+1}, \qquad \forall n = 0, 1, 2, \ldots,$$

$$T^*(\delta_n) = \sqrt{n}\,\delta_{n-1}, \qquad \forall n = 1, 2, \ldots,$$

and $T^*(\delta_0) = 0 \in \mathcal{H}$.

Remark 2.4. The annihilation operator T^* is indeed the adjoint operator for the random variable T. This is verified exactly the same way as in the proof for Remark 2.2.

The main purpose of this chapter is to calculate moments of the random variables $S + S^*$ and $T + T^*$. To simplify our writing, we shall use the term "raising" operator to refer to the creation operators in both the free and classical case; similarly, we use the term "lowering " operator to refer to the annihilation operators.

Definition 2.5. Let $\mathcal{H}_0$ and $\mathcal{L}_0$ be given as above. Then $A \in \mathcal{L}$ is called a **raising** operator if

$$A(\delta_p) = a(p+1)\,\delta_{p+1},$$

where $a(p)$, $p = 0, 1, 2, \ldots$, are some scalars; and $B \in \mathcal{L}$ is called a **lowering** operator if

$$B(\delta_q) = b(q)\,\delta_{q-1}; \quad B(\delta_0) = 0 \in \mathcal{H},$$

where $b(q)$, $q = 1, 2, \ldots$, are some scalars.

2.2 Combinatorial tools

For the sake of convenience, we denote $\mathcal{P}_n$, $n = 1, 2, 3, \ldots$, to be the set of lattice paths that start from $(0, 0)$ in the plane integer lattice $\mathbb{Z} \times \mathbb{Z}$ with exactly n steps such that each step can only be either ascending (along the vector $(1, 1)$) or descending (along the vector $(1, -1)$). In computing moments $\varphi_0((A + B)^p)$, $p = 1, 2, \ldots$, of a raising operator A and a lowering operator B from the first principle, it turns out that it is useful to be able to identify each term of the expansion of $(A + B)^p$ with a lattice path $l \in \mathcal{P}_p$ in the following manner. While reading a summand of an expansion of $(A + B)^p$ from right to left,

[i] each time when one encounters a lowering operator B, the lattice path goes down by one step (along the vector $(1, -1)$), and

[ii] each time when one encounters a raising operator A, the lattice path goes up by one step (along the vector $(1, 1)$).

This procedure (map l) of assigning a product Y of exactly p-many creation or annihilation operators to a lattice path $l_Y \in \mathcal{P}_p$ is clearly well defined.

An obvious benefit of the above procedure—thanks to the linearity of φ_0—is that now the computation of $\varphi_0(X^p)$ is broken down into finitely many smaller pieces each of which is an answer to the question: what is the weight (function) of the corresponding lattice path.

Definition 2.6. Let $\mathcal{H}$ be a complex Hilbert space over $\mathbb{C}$ with an orthonormal basis $\{\,\delta_n \mid n = 0, 1, 2, \ldots\}$; and let $\mathcal{L}_0$, $\mathcal{H}_0$ and φ_0 be given as at the start of this chapter.

Suppose $A, B \in \mathcal{L}_0$ satisfies the relations

$$A(\delta_p) = a(p+1)\,\delta_{p+1}; \quad B(\delta_q) = b(q)\,\delta_{q-1}; \quad B(\delta_0) = 0 \in \mathcal{H}, \tag{2.1}$$

where $a(p)$, $b(q)$ are some scalars for all $p = 0, 1, 2, \ldots$, $q = 1, 2, \ldots$. Define the sequence-valued map $\gamma_{A,B} : \{-1, 1\} \to \{a, b\}$ to be given by $\gamma(1) = a(\cdot)$ and $\gamma(-1) = b(\cdot)$. Then the scalar-valued map $W_{A,B}$ defined on the set of all plane integer lattice paths given by

$$W_{A,B}(l) := \prod_{s=0}^{\text{maximum number of steps of } l} \gamma_{A,B}(l(s+1) - l(s))\,(\max\{l(s), l(s+1)\})$$

is called the **weight of the lattice path** l with respect to the linear operators A and B.

With these terminologies and the way they are being introduced, we see easily that, for any product Y of finitely many linear operators A, B that satisfies Equation (2.1), we have

$$\varphi_0(Y) = W_{A,B}(l_Y); \tag{2.2}$$

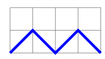
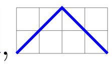
hence

$$\varphi_0((A+B)^p) = \sum_{l \in \mathcal{P}_p} W_{A,B}(l). \tag{2.3}$$

In fact, we may use the fact that $\{\, \delta_n \mid n = 0, 1, 2, \ldots \}$ is an orthonormal basis and the way the functional φ_0 is defined to further simplify the summation on the right hand side of Equation (2.3). To this end, let us introduce the so-called Dyck paths.

Definition 2.7. A **Dyck path** of length $2n$ is a lattice path from $(0, 0)$ to $(2n, 0)$ in the plane integer lattice $\mathbb{Z} \times \mathbb{Z}$ consisting of only two kinds of step: up-steps (along the vector $(1, 1)$) and down-steps (along the vector $(1, -1)$), such that the path never goes below the x-axis. We denote $\mathcal{D}_{2\,n}$ to be the collection of all Dyck paths of length $2\,n$.

As an example, we may write:

,  $\in \mathcal{D}_4.$

These are the only two Dyck paths of length 2. In general, it can be shown that $|\mathcal{D}_{2\,n}|$ is equal to the nth Catalan number $\mathrm{Cat}_n := \frac{(2\,n)!}{n!\,(n+1)!}$.

Lemma 2.8. *Let A, B be two linear operators satisfying Equation (2.1), p_1 be any odd number, and p_2 be any even number. Then we have following two statements. If Y_1 is a product of p_1-many operators A or B, then $W_{A,B}(l_{Y_1}) = 0$. If Y_2 is a product of p_2-many operators A or B, then*

$$W_{A,B}(l_{Y_2}) = \begin{cases} \varphi_0(Y_2), & \textit{if } l_{Y_2} \textit{ is a Dyck path,} \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Observe that $Y_1\,\delta_0$ is always equal to either $0 \in \mathcal{H}$ or some scalar c times δ_q for some odd number q, as a result of p_1 being odd and Equation (2.1). The result holds trivially when $Y_1\,\delta_0 = 0$. For the other cases, we use Equation (2.2), and get

$$W_{A,B}(l_{Y_1}) = \varphi_0(Y_1) = \langle\, c\,\delta_q \,,\ \delta_0 \,\rangle_{\mathcal{H}} = c\ \langle\, \delta_q \,,\ \delta_0 \,\rangle_{\mathcal{H}} = 0,$$

where the last equality follows from $\{\, \delta_n \mid n = 0, 1, 2, \dots \}$ being an orthonormal basis.

Now, we move on to the proof for the second statement. Suppose the lattice path l_{Y_2} ends at some coordinate $(x, r) \in \mathbb{Z} \times \mathbb{Z}$ with $r \neq 0$. Similar as before, we get

$$W_{A,B}(l_{Y_2}) = \begin{cases} c_1\ \langle\, \delta_r \,,\ \delta_0 \,\rangle_{\mathcal{H}}\,, & \text{if } r \geq 1, \\ c_2\ \langle\, 0 \,,\ \delta_0 \,\rangle_{\mathcal{H}}\,, & \text{otherwise,} \end{cases}$$

for some scalars c_1 and c_2. Next, suppose that l_{Y_2} is some lattice path such that it drops below 0 somewhere. Let c be the smallest possible integer such that $(c, -1) \in \mathbb{Z} \times \mathbb{Z}$ is a point on the path l_{Y_2}. Clearly, c exists and is finite. We observe that after applying $(c-1)$-many A or B of the product Y_2 to the vector δ_0, due to Equation (2.1), we obtain a vector $c_4\,\delta_0$ for some scalar c_4. The minimality of c implies that the cth operation in the expression Y_2 is an operator B. Finally, use $B(\delta_0) = 0$ and we get $W_{A,B}(l_{Y_2}) = 0$. Thus, if l_{Y_2} is not a Dyck path, then $W_{A,B}(l_{Y_2})$ must be zero. $\square$

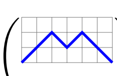
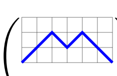
In words, Lemma 2.8 enables us to say that given any pair of linear operators A, B satisfying Equation (2.1), the task of computing the pth moment (in particular, when p is even) of the random variable $A+B$ comes down to summing over all weights $W_{A,B}(l)$ of l, where l runs through all Dyck paths of length p. Now, if one returns to have another look at Equation (2.1), it should not be surprising to find out that it is possible to have two pairs of random variables (A_1, B_2) and (A_2, B_2) such that $W_{A_1,B_1} = W_{A_2,B_2}$ on all Dyck paths. As we shall see below, one such example is to take one pair as (T, T^*) (cf. Definition 2.3) and the other pair as (S, U) (cf. Definition 2.1 for the definition of S), where

$$U(\delta_n) := n\,\delta_{n-1}, \qquad \forall n = 1, 2, \ldots, \tag{2.4}$$

and $U(\delta_0) = 0 \in \mathcal{H}$. Let us now devise a mechanism to prove in general whether two weights W_{A_1,B_1}, W_{A_2,B_2} agrees on all Dyck paths or not.

Definition 2.9. Take any $l \in \cup_{n=1}^{\infty} \mathcal{D}_{2\,n}$ and w.l.o.g. suppose that $l \in \mathcal{D}_{2\,k}$ for some $k \in \mathbb{N}$. We define the **content of maximum** $\mathrm{Cnt}(l)$ of l to be the partition $(1^{r_1}\ 2^{r_2}\ \cdots\ k^{r_k})$ of the integer $r_1 \cdot 1 + r_2 \cdot 2 + \cdots + r_k \cdot k$, where r_i records the number of local maximum of the Dyck path l that is of height i for all $i = 1, 2, 3, \ldots, k$. Similarly, we define the **content of minimum** $\mathrm{cnt}(l)$ of l to be the partition $(1^{p_1}\ 2^{p_2}\ \ldots\ k^{p_k})$ of the integer $p_1 \cdot 1 + p_2 \cdot 2 + \cdots + p_k \cdot k$, where p_i records the number of local minimum of the Dyck path l that is of height i for all $i = 1, 2, 3, \ldots, k$.

For example, we have:

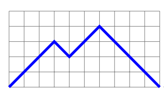
Cnt () $= (1^0\ 2^2\ 3^0)$; cnt () $= (1^1\ 2^0\ 3^0)$.

Definition 2.10. Let $(1^{r_1}\ 2^{r_2}\ \cdots\ k^{r_k})$ be a partition of a number $k \in \mathbb{N}$, and A, B be two random variables satisfying Equation (2.1). We define the (A,B)-factorial function $\mathrm{fcl}_{A,B}\left((1^{r_1}\, 2^{r_2}\, \cdots\, k^{r_k})\right)$ to be equal to the value $\left(\prod_{i=1}^{1} a(i)\, b(i)\right)^{r_1} \left(\prod_{i=1}^{2} a(i)\, b(i)\right)^{r_2} \cdots \left(\prod_{i=1}^{k} a(i)\, b(i)\right)^{r_k}$.

Lemma 2.11. *Let A, B be two random variables satisfying Equation (2.1) and $l \in \mathcal{D}_{2\,k}$ for some $k \in \mathbb{N}$. Then*

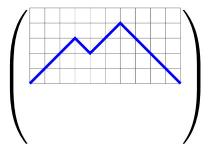
$$W_{A,B}(l) = \frac{fcl_{A,B}(Cnt(l))}{fcl_{A,B}(cnt(l))}. \tag{2.5}$$

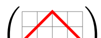
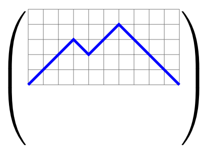
Proof. First of all, we notice that between any two consecutive local maxima of the path l, there must exist one and only one local minimum and the path l can be partitioned into all intervals of consecutive local maxima. Therefore if there is any method that ensures the validity of Equation (2.5) over any arbitrary interval of two consecutive local maxima, then we are done. Indeed, there is such a method and it suffices to outline it for, say when

$l =$ .

Then we have

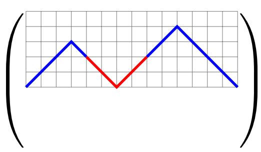
$$\mathrm{fcl}_{A,B}(\mathrm{cnt}(l))\, W_{A,B}(l)$$

$$= \mathrm{fcl}_{A,B}\left(1^0\ 2^1\ 3^0\ 4^0\ 5^0\right)\ W_{A,B}$$ 

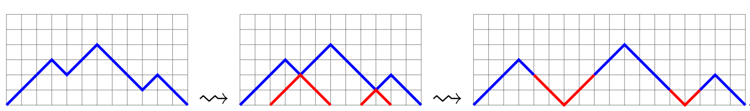
$$= W_{A,B}$$  $$W_{A,B}$$ 

$$= \underbrace{a(1)b(1)\,a(2)b(2)}_{\text{red}}\ \underbrace{a(1)a(2)a(3)b(3)a(3)a(4)b(4)b(3)b(2)b(1)}_{\text{blue}}$$

$$= \underbrace{a(1)a(2)a(3)b(3)}_{\text{blue}}\ \underbrace{b(2)b(1)a(1)a(2)}_{\text{red}}\ \underbrace{a(3)a(4)b(4)b(3)b(2)b(1)}_{\text{blue}}$$

$$= W_{A,B}$$  $$= \mathrm{fcl}_{A,B}\left(1^0\ 2^0\ 3^1\ 4^1\ 5^0\right) = \mathrm{fcl}_{A,B}(\mathrm{Cnt}(l)),$$

and rearranging yields the desired result. It is important to note that the above method always works because for a general Dyck path, we can always have the following flow of augmented diagrams:



$\square$

Thus, we have our result on how to test whether two weights W_{A_1,B_1}, W_{A_2,B_2} agree on all Dyck paths or not.

Corollary 2.12. *Let $\mathcal{H}$ be a complex Hilbert space with an orthonormal basis $\{\,\delta_n \mid n = 0, 1, 2, \dots\}$ and A_1, A_2, B_1, B_2 be linear operators on $\mathcal{H}_0$:=*

$\left\{ \sum_{j=1}^{n} d_j\, \delta_j \;\middle|\; d_j \in \mathbb{C}, n \in \mathbb{N} \right\}$ *satisfying*

$$\begin{cases} A_1(\delta_p) = a_1(p+1)\, \delta_{p+1}; \quad B_1(\delta_q) = b_1(q)\, \delta_{q-1}; \quad B_1(\delta_0) = 0 \in \mathcal{H}; \\ A_2(\delta_p) = a_2(p+1)\, \delta_{p+1}; \quad B_2(\delta_q) = b_2(q)\, \delta_{q-1}; \quad B_2(\delta_0) = 0 \in \mathcal{H}, \end{cases}$$

where $a_1(p)$, $b_1(q)$, $a_2(p)$, $b_2(q)$ *are some scalars for all* $p = 0, 1, 2, \ldots,$ $q = 1, 2, \ldots.$ *Then* $W_{A_1,B_1} = W_{A_2,B_2}$ *on all Dyck paths if and only if*

$$a_1(q)\, b_1(q) = a_2(q)\, b_2(q),$$

for all $q = 1, 2, 3, \ldots.$

Proof. This is a direct consequence of Definition 2.10 and Lemma 2.11. □

Now we arrive at a theorem that uses all the lemmas proven so far in this section.

Theorem 2.13. *Let* $(\mathcal{H}, \langle \cdot\, , \,\cdot \rangle_{\mathcal{H}})$ *be a complex Hilbert space with an orthonormal basis* $\{ \delta_n \mid n = 0, 1, 2, \ldots \}$, *and* $(\mathcal{L}_0, \varphi_0)$ *be the non-commutative probability space, where* $\mathcal{L}_0 := \{ X : \mathcal{H}_0 \to \mathcal{H}_0 \mid X \text{ is linear} \}$, $\mathcal{H}_0 := \left\{ \sum_{j=1}^{n} d_j\, \delta_j \;\middle|\; d_j \in \mathbb{C}, n \in \mathbb{N} \right\}$, *and* $\varphi_0(X) := \langle X\, \delta_0 \,,\, \delta_0 \rangle_{\mathcal{H}}$. *Suppose* $A, B \in \mathcal{L}_0$ *satisfies the relations*

$$A(\delta_p) = a(p+1)\, \delta_{p+1}; \quad B(\delta_q) = b(q)\, \delta_{q-1}; \quad B(\delta_0) = 0 \in \mathcal{H},$$

where $a(p)$, $b(q)$ *are some scalars for all* $p = 0, 1, 2, \ldots,$ $q = 1, 2, \ldots.$ *Then*

$$\varphi_0((A+B)^p) = \begin{cases} 0, & \text{if } p \text{ is odd,} \\ \sum_{l \in \mathcal{D}_p} \dfrac{fcl_{A,B}(Cnt(l))}{fcl_{A,B}(cnt(l))}, & \text{if } p \text{ is even.} \end{cases}$$

Proof. By Equation (2.3), we have

$$\varphi_0((A+B)^p) = \sum_{l \in \mathcal{P}_p} W_{A,B}(l).$$

By Lemma 2.8, we deduce that

$$\varphi_0((A+B)^p) = \begin{cases} 0, & \text{if } p \text{ is odd,} \\ \sum_{l\in\mathcal{D}_p} W_{A,B}(l), & \text{if } p \text{ is even.} \end{cases}.$$

Finally, applying Lemma 2.11 yields the result. □

In view of Definition 2.10 and the above theorem, we realize that the moments $\varphi_0((A+B)^p)$ are determined by what values the products $a_1 b_1, a_2 b_2, \dots$ take, not so much by the specific values $a_1, a_2, \dots, b_1, b_2, \dots$. We shall use an example to demonstrate what this entails.

Example 2.14.

$$\varphi_0((T+T^*)^p) = \varphi_0((S+U)^p),$$

for all $p = 0, 1, 2, \dots$. (Refer to Equation (2.4), Definition 2.3, and Definition 2.1 for the defining properties of these linear operators S, U, T, T^*.)

Solution. By Theorem 2.13, we have

$$\varphi_0((S+U)^p) = \begin{cases} 0, & \text{if } p \text{ is odd,} \\ \sum_{l\in\mathcal{D}_p} \dfrac{\mathrm{fcl}_{S,U}(\mathrm{Cnt}(l))}{\mathrm{fcl}_{S,U}(\mathrm{cnt}(l))}, & \text{if } p \text{ is even.} \end{cases}.$$

However, by Definition 2.10, we easily see that $\mathrm{fcl}_{S,U} = \mathrm{fcl}_{T,T^*}$ on any partition of any number. Finally, applying Theorem 2.13 again to $\varphi_0((T+T^*)^p)$ yields the desired result.

In order to further simplify our answer to $\varphi_0((T+T^*)^p)$, we need a stand alone result about the pair partitions.

Definition 2.15. We denote $\Pi(n)$ to be the lattice of partition of n-elements $[n] := \{1, 2, \dots, n\}$. $\eta \in \Pi(n)$ simply means that η is a collection of sets such that each number k, $k = 1, 2, 3, \dots, n$, belongs to one and only one of the sets of the collection η. We also call each set of the collection η a block.

Definition 2.16. Let $\Pi(n)$ be a lattice of partition of $[n]$. We define

$$\Pi^{(j)}(n) := \{ \{V_1, V_2, \ldots, V_s\} \in \Pi(n) \mid |V_i| = j, i = 1, 2, , \ldots, s \} ,$$

for each $j = 1, 2, \ldots, n$.

In this manner, $\Pi^{(2)}(n)$ becomes the notation for the so-called collection of pair partition of $[n]$, if $n = 2, 4, 6, \ldots$. Below is the result on pair partitions.

Lemma 2.17. *For any positive integer n, we have*

$$|\Pi^{(2)}(2\,n)| = (2\,n-1)!! := (2\,n-1)\,(2\,n-3)\,\cdots\,(3)(1).$$

Proof. Consider any arrangement ω of the numbers $1, 2, \ldots, 2n$ in a row, there are $(2\,n)!$ such arrangements. For each ω, we may swap the first number with the second number and this swapped arrangement is considered to be a double counting for the purpose of pair partition. There are n such double counting; hence a discount factor 2^n. Also, for the purpose of pair partition, the ordering of these blocks (of size 2) does not matter; hence a discount factor of $n!$. Therefore,

$$|\Pi^{(2)}(2\,n)| = \frac{(2\,n)!}{2^n\,n!}.$$

On the other hand, algebraically we have

$$2^n\,n! = (2\,n)\,(2\,(n-1))\,(2\,(n-2))\,\cdots\,(2\cdot 2)\,(2\cdot 1).$$

Substituting in the above algebraic formula gives the desired result. □

Next, we want make a connection between the number $|\Pi^{(2)}(2\,n)|$ of pair partitions and the weight $W_{S,U}$. To do so, we proceed similarly as it is done at the beginning of the present section. We shall assign each pair partition of $[p]$, $p = 2, 4, 6, \ldots$ to a Dyck path $l \in \mathcal{D}_p$.

Definition 2.18. Given a pair partition $\eta \in \Pi^{(2)}(p)$, $p = 2, 4, 6, \ldots$, we define a function $\gamma_\eta : [p] \to \{+1, -1\}$ by

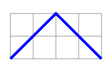
$$\gamma_\eta(k) = \begin{cases} 1, & \text{if } k \text{ is the smaller number of the block (of which } k \text{ is an element to) of } \eta, \\ -1, & \text{otherwise.} \end{cases}$$

Define the associated Dyck path $l_\eta \in \mathcal{D}_p$ of the pair partition $\eta \in \Pi^{(2)}(p)$ inductively by

$$l_\eta(0) = 0; \qquad l_\eta(k) = l_\eta(k-1) + \gamma_\eta(k),$$

for all $k = 1, 2, 3, \ldots, p$.

Clearly, this assignment (map) $l_{(\cdot)}$ from $\Pi^{(2)}(p)$ to $\mathcal{D}_p$ is well-defined, but it is not injective. Here is a simple example.

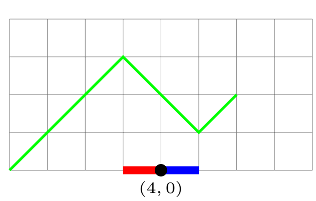
$l_{\{\{1,3\},\{2,4\}\}} =$  $= l_{\{\{1,4\},\{2,3\}\}}$.

Lemma 2.19. *For any arbitrary* $\lambda \in \mathcal{D}_p$, $p = 2, 4, 6, \ldots$, *Define the set* $K_\lambda := \{\, \eta \in \Pi^{(2)}(p) \mid l_\eta = \lambda \,\}$. *Then*

$$|K_\lambda| = W_{S,U}(\lambda),$$

for all $\lambda \in \mathcal{D}_p$.

Proof. Take an arbitrary $\lambda \in \mathcal{D}_p$. Due to the way $l_{(\cdot)}$ is defined, it is clear that if $\lambda(k) - \lambda(k-1) = 1$, then the block of which k is an element must have k as the smaller element. Next, we shall use an example to illustrate what information the statement $\lambda(k) - \lambda(k-1) = -1$ entails. Let us consider an initial segment of a Dyck path



.

The lattice step over the red interval $[(3,0),(4,0)]$ being descending implies that the number 4 could have been the larger number of the blocks $\{1,*\},\{2,*\},\{3,*\}$ of a pair partition η, under the map $l_{(\cdot)} : \Pi^{(2)}(p) \to \mathcal{D}_p$, where $*$ are the places to which the number 4 can go; similarly, the lattice step over the blue interval $[(4,0),(5,0)]$ being descending implies that the number 5 could have been the larger number of the blocks $\{1,*\},\{2,*\},\{3,*\}$ which is not filled by the number 4. This pattern yields the result that

$$|K_\lambda| = \prod_{\substack{j=1,2,\ldots,p \text{ such that} \\ \lambda(j)-\lambda(j-1)=-1}} \lambda(j-1),$$

which together with the definition of $W_{S,U}(\lambda)$ (cf. Definition 2.6), Equation (2.4), and Definition 2.1 yield the desired result. $\square$

Corollary 2.20. *If p is an even number, then*

$$\sum_{l \in \mathcal{D}_p} W_{S,U}(l) = (p-1)!!.$$

Proof. This follows from Lemma 2.17 and Lemma 2.19. $\square$

2.3 Moments of random variables

Denote $Q_c := T + T^*$ (cf. Definition 2.3) and $Q_f := S + S^*$ (cf. Definition 2.1). In this section, we shall prove that the moments of Q_c agree with that of a standard normal random variable and that the moments of Q_f agree with that of a random variable with standard Wigner semicircle distribution (when the radius $R = 2$), as claimed at the beginning of this chapter.

Lemma 2.21.

$$\varphi_0({Q_c}^p) = \begin{cases} 0, & \textit{if } p \textit{ is odd,} \\ (p-1)!!, & \textit{if } p \textit{ is even.} \end{cases}$$

Proof. Since T is a raising operator and T^* is a lowering operator, by Equation (2.3), we get $\varphi_0({Q_c}^p) = \sum_{l\in\mathcal{P}_p} W_{T,T^*}(l)$. By Lemma 2.8, we obtain

$$\varphi_0({Q_c}^p) = \begin{cases} 0, & \text{if } p \text{ is odd,} \\ \sum_{l\in\mathcal{D}_p} W_{T,T^*}(l), & \text{if } p \text{ is even.} \end{cases}$$

Finally, when p is even, we apply Corollary 2.12 and Corollary 2.20 to finish the proof. □

Lemma 2.22.

$$\varphi_0({Q_f}^p) = \begin{cases} 0, & \textit{if } p \textit{ is odd,} \\ |\mathcal{D}_p|, & \textit{if } p \textit{ is even.} \end{cases}$$

Proof. Since S is a raising operator and S^* is a lowering operator, by Equation (2.3), we get $\varphi_0({Q_f}^p) = \sum_{l\in\mathcal{P}_p} W_{S,S^*}(l)$. By Lemma 2.8, we obtain

$$\varphi_0({Q_f}^p) = \begin{cases} 0, & \text{if } p \text{ is odd,} \\ \sum_{l\in\mathcal{D}_p} W_{S,S^*}(l), & \text{if } p \text{ is even.} \end{cases}$$

Next, it is obvious that $W_{S,S^*}(l) = 1$ for all $l \in \mathcal{P}_p \supset \mathcal{D}_p$; hence the desired result. □

Proposition 2.23. *Q_c is a standard Gaussian random variable and Q_f is a Wigner semi-circle law.*

Proof. The first statement follows from Lemma 2.21. The second statement follows from Lemma 2.22 and the well-known fact that $|\mathcal{D}_p|$ is equal to the pth Catalan number. □

Chapter 3

Full Fock space

3.1 Definition of the full Fock space

Given vector spaces $V_1, V_2, \ldots, V_n$ over $\mathbb{C}$, we use the standard notation $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ for their tensor product. This is a vector space over $\mathbb{C}$, spanned by elements of the form $a_1 \otimes a_2 \otimes \cdots \otimes a_n$ with $a_1 \in V_1, a_2 \in V_2, \ldots, a_n \in V_n$; moreover it has the property: whenever $F : V_1 \times V_2 \times \cdots \times V_n \to V$ is a multilinear map into a vector space V, there exists a linear map $f : V_1 \otimes V_2 \otimes \cdots \otimes V_n \to V$, uniquely determined in such a way that

$$f(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = F(a_1, a_2, \ldots, a_n),$$

for all $a_1 \in V_1, a_2 \in V_2, \ldots, a_n \in V_n$.

In particular, we denote $A^{\otimes n} := \underbrace{A \otimes A \otimes \cdots \otimes A}_{n \text{ copies}}$ and call a vector of the form $a_1 \otimes a_2 \otimes \cdots \otimes a_n$ **a simple tensor of length** $n \in \mathbb{N}$. By convention, we take $A^{\otimes 0}$ to be the scalar field, i.e. $A^{\otimes 0} = \mathbb{C}\,\Omega$, where Ω is the vacuum vector. For any $a \in A$, we have $a^{\otimes 0} = \Omega$, i.e. the vacuum vector Ω is a simple tensor of length 0.

On the other hand, we use the standard notation $\bigoplus_{i=1}^{\infty} \mathcal{H}_i$ for the direct sum of Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \ldots$. It can be shown that the completion $\overline{\bigoplus_{i=1}^{\infty} \mathcal{H}_i}$ of the direct sum $\bigoplus_{i=1}^{\infty} \mathcal{H}_i$

is a Hilbert space with the inner product given by

$$\langle (a_1, a_2, \dots) , (b_1, b_2, \dots) \rangle_{\bigoplus_{i=1}^{\infty} \mathcal{H}_i} := \langle a_1 , b_1 \rangle_{\mathcal{H}_1} + \langle a_2 , b_2 \rangle_{\mathcal{H}_2} + \cdots . \tag{3.1}$$

Furthermore, if we impose on the vector space $\bigotimes_{i=1}^{k} \mathcal{H}_i$, $k \in \mathbb{N}$, the inner product given by

$$\langle a_1 \otimes a_2 \otimes \cdots \otimes a_k , b_1 \otimes b_2 \otimes \cdots \otimes b_k \rangle_{\bigotimes_{i=1}^{k} \mathcal{H}_i} := \langle a_1 , b_1 \rangle_{\mathcal{H}_1} \langle a_2 , b_2 \rangle_{\mathcal{H}_2} \cdots \langle a_k , b_k \rangle_{\mathcal{H}_2} , \tag{3.2}$$

then the completion $\overline{\bigotimes_{i=1}^{k} \mathcal{H}_i}$ of the tensor product $\bigotimes_{i=1}^{k} \mathcal{H}_i$ becomes a Hilbert space.

Definition 3.1 (Complexification)**.** Let $(\mathcal{H}, \langle \cdot , \cdot \rangle_{\mathcal{H}})$ be a real Hilbert space and $i = \sqrt{-1}$. Its complexification $\mathcal{H}_{\mathbb{C}}$ is a complex Hilbert space $(V, \langle \cdot , \cdot \rangle_V)$ with the vector space V being specified by the set $\{ a + i\, b \mid a, b \in \mathcal{H} \}$ and the inner product given by

$$\langle a + i\, b , c + i\, d \rangle_V := \langle a , c \rangle_{\mathcal{H}} + i \langle a , d \rangle_{\mathcal{H}} + i \langle b , c \rangle_{\mathcal{H}} - \langle b , d \rangle_{\mathcal{H}} .$$

Definition 3.2 (Full Fock space)**.** Let $\mathcal{H}$ be a Hilbert space over $\mathbb{R}$. We define the **full Fock space** $\mathcal{F}(\mathcal{H}_{\mathbb{C}})$ over $\mathcal{H}_{\mathbb{C}}$ to be the completion $\overline{\bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathbb{C}}^{\otimes n}}$ of the set $\bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathbb{C}}^{\otimes n}$, where the completion is taken with respect to the formula prescribed by Equation (3.1) and Equation (3.2).

Remark 3.3. If the Hilbert space $\mathcal{H}$ has an orthonormal basis $\{ \delta_i \in \mathcal{H} \mid i \in I \}$, for some index set I. Then the set $\{\Omega\} \bigcup \{ \delta_{i_1} \otimes \delta_{i_2} \otimes \cdots \otimes \delta_{i_n} \in \mathcal{F}(\mathcal{H}) \mid i_1, i_2, i_3, \dots, i_n \in I, n \in \mathbb{N} \}$ is an orthonormal basis for the full Fock space $\mathcal{F}(\mathcal{H})$.

3.2 Creation and annihilation operators

Definition 3.4. Fix any $h \in \mathcal{H}_{\mathbb{C}}$. For each non-negative n, we define the n-th left creation operator $L_n^+(h) : \mathcal{H}_{\mathbb{C}}^{\otimes n} \to \mathcal{H}_{\mathbb{C}}^{\otimes (n+1)}$ to be given by

$$L_n^+(h)(v_1 \otimes \cdots \otimes v_n) := h \otimes v_1 \otimes \cdots \otimes v_n.$$

Definition 3.5 (Left creation operator)**.** Fix any $h \in \mathcal{H}_{\mathbb{C}}$. We define the left creation operator $L^+(h) : \bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathbb{C}}{}^{\otimes n} \to \bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathbb{C}}{}^{\otimes n}$ to be given by

$$L^+(h) := \bigoplus_{n=0}^{\infty} L_n^+(h).$$

By the universal property of tensor product, the left creation operator $L^+(h)$ can be extended so that it is an operator on the full Fock space $\mathcal{F}(\mathcal{H}_{\mathbb{C}})$.

Definition 3.6. Fix any $h \in \mathcal{H}_{\mathbb{C}}$. For each non-negative n, we define the n-th left annihilation operator $L_n^-(h) : \mathcal{H}_{\mathbb{C}}{}^{\otimes n} \to \mathcal{H}_{\mathbb{C}}{}^{\otimes(n-1)}$ to be given by

$$L_n^-(h)(v_1 \otimes \cdots \otimes v_n) := \langle\, h \,,\, v_1 \,\rangle_{\mathcal{H}_{\mathbb{C}}} \, v_2 \otimes v_3 \otimes \cdots \otimes v_n.$$

Definition 3.7 (Left annihilation operator)**.** Fix any $h \in \mathcal{H}_{\mathbb{C}}$. We define the left annihilation operator $L^-(h) : \bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathbb{C}}{}^{\otimes n} \to \bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathbb{C}}{}^{\otimes n}$ to be given by

$$L^-(h) := \bigoplus_{n=0}^{\infty} L_n^-(h).$$

By the universal property of tensor product, the left creation operator $L^+(h)$ can be extended so that it is an operator on the full Fock space $\mathcal{F}(\mathcal{H}_{\mathbb{C}})$.

Lemma 3.8. *For any $h \in \mathcal{H}_{\mathbb{C}}$, we have*

$$L^-(h)L^+(h) = \langle\, h \,,\, h \,\rangle_{\mathcal{H}_{\mathbb{C}}} \, I,$$

where I is the identity operator on the full Fock space $\mathcal{F}(\mathcal{H}_{\mathbb{C}})$.

Proof. It follows from Remark 3.3, Definition 3.5, and Definition 3.7. □

Define a positive, unital, linear functional $\mathbb{E}(\cdot)$ on the set of linear operators that acts on the full Fock space $\mathcal{F}(\mathcal{H}_{\mathbb{C}})$. given by

$$\mathbb{E}[\cdot] := \langle \cdot\, \Omega \,,\, \Omega \,\rangle_{\mathcal{F}(\mathcal{H}_{\mathbb{C}})} \,.$$

Theorem 3.9. *Let r be a positive integer and $\{h_1, h_2, \ldots, h_r\} \subset \mathcal{H}_{\mathbb{C}}$ be a collection of mutually orthogonal vectors, and $\sigma : \{1, 2, \ldots, r\} \to \{1, 2, \ldots, r\}$ be a map. Then*

(3.3)

$$\mathbb{E}\left[L^+(h_{\sigma(r)})^{p_r} L^-(h_{\sigma(r)})^{q_r} L^+(h_{\sigma(r-1)})^{p_{r-1}} L^-(h_{\sigma(r-1)})^{q_{r-1}} \cdots L^+(h_{\sigma(1)})^{p_1} L^-(h_{\sigma(1)})^{q_1}\right] = 0,$$

for any non-negative integers $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_r$, provided that

[i] *at least one of $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_r$ is non-zero, and*

[ii] *for all pair of integers $j > s$, we have the implication:*

$$q_j \neq 0, q_{j-1} = 0, \ldots, q_{s+1} = 0, p_s \neq 0, p_{s+1} = 0, \ldots, p_{j-1} = 0 \implies \sigma(j) \neq \sigma(s).$$

Proof. Denote $L := L^+(h_{\sigma(r)})^{p_r} L^-(h_{\sigma(r)})^{q_r} \cdots L^+(h_{\sigma(1)})^{p_1} L^-(h_{\sigma(1)})^{q_1}$. Firstly, suppose that all $q_1, q_2, \ldots, q_r$ are zero. Then, by the definitions of L^+, we see that the vector $L\,\Omega \in \mathcal{F}(\mathcal{H}_{\mathbb{C}})$ must be a simple tensor of some positive length. With Remark 3.3, this gives Equation (3.3).

Secondly, suppose at least one element of the set $\{q_1, q_2, \ldots, q_r\}$ is non-zero. W.l.o.g. we may take k to be the smallest amongst $\{1, 2, \ldots, r\}$ such that q_k is non-zero. If $p_{k-1}, \ldots, p_1$ are all zero, then Equation (3.3) holds trivially as $L^-(h_{\sigma(k)})\,\Omega = 0 \in \mathcal{F}(\mathcal{H}_{\mathbb{C}})$. If $p_{k-1}, \ldots, p_1$ are not all zero, then there exist the largest integer $t \in \{1, 2, \ldots k-1\}$ such that p_t is non-zero. By the maximality of t and the minimality of k, we deduce that L now becomes

$$L^+(h_{\sigma(r)})^{p_r} L^-(h_{\sigma(r)})^{q_r} \cdots L^+(h_{\sigma(k)})^{p_k} L^-(h_{\sigma(k)})^{q_k} L^+(h_{\sigma(t)})^{p_t} \cdots L^+(h_{\sigma(1)})^{p_1}.$$

Also, by the assumption of the theorem, we have $\sigma(k) \neq \sigma(t)$. Next, when applying $L^-(h_{\sigma(k)})^{q_k}$ to the vector $L^+(h_{\sigma(t)})^{p_t} \cdots L^+(h_{\sigma(1)})^{p_1}\,\Omega$, we see that the result must be a simple tensor with a scalar coefficient that is a multiple of $\langle\, h_{\sigma(k)}\,,\; h_{\sigma(t)} \,\rangle_{\mathcal{H}_{\mathbb{C}}}$. Since the set $\{h_1, h_2, \ldots, h_r\} \subset \mathcal{H}_{\mathbb{C}}$ is a collection of mutually orthogonal vectors, this means that Equation (3.3) holds again, finishing the proof. $\square$

3.3 Freely independent random variables $X(h)$

Fix some $h \in \mathcal{H}_{\mathbb{C}}$. Define $X(h) := L^+(h) + L^-(h)$ on the full Fock space $\mathcal{F}(\mathcal{H}_{\mathbb{C}})$, cf. Definition 3.5 and Definition 3.7.

Lemma 3.10.

$$\langle\, X(h)^p \, \Omega \,,\, \Omega \,\rangle_{\mathcal{F}(\mathcal{H}_{\mathbb{C}})} = \begin{cases} 0, & \textit{if } p \textit{ is odd,} \\ \left(\langle\, h \,,\, h \,\rangle_{\mathcal{H}_{\mathbb{C}}}\right)^{\frac{p}{2}} \, |\mathcal{D}_p|, & \textit{if } p \textit{ is even.} \end{cases}$$

Proof. By Definition 3.5, we notice that the left creation operator $L^+(h)$ is a raising operator on the linear subspace $\left\{ \sum_{j=0}^{n} d_j \, h^{\otimes j} \;\middle|\; d_j \in \mathbb{C}, n \in \mathbb{N} \right\} \subset \mathcal{F}(\mathcal{H}_{\mathbb{C}})$; similarly, by Definition 3.7, we also observe that the left annihilation operator $L^-(h)$ is a lowering operator on the linear subspace $\left\{ \sum_{j=0}^{n} d_j \, h^{\otimes j} \;\middle|\; d_j \in \mathbb{C}, n \in \mathbb{N} \right\} \subset \mathcal{F}(\mathcal{H}_{\mathbb{C}})$. Thus, by the proof of Lemma 2.22, the result follows. ☐

Lemma 3.11. *The linear span of the set* $\{\, L^+(h)^p \, L^-(h)^q \mid p, q = 0, 1, 2, \ldots. \}$ *is an algebra.*

Proof. The closure property of the vector addition and the scalar multiplication follows from the the definition of a linear span. It remains to check the closure property of the vector multiplication. Pick any non-negative integers p_1, p_2, q_1, q_2 such that $(p_1, q_1) \neq (0, 0)$ and $(p_2, q_2) \neq (0, 0)$. Then, by Lemma 3.8,

$$\begin{aligned}
& L^+(h)^{p_1} L^-(h)^{q_1} L^+(h)^{p_2} L^-(h)^{q_2} \\
&= \begin{cases} L^+(h)^{p_1} L^-(h)^{q_1 - p_2} \langle\, h \,,\, h \,\rangle_{\mathcal{H}_{\mathbb{C}}}{}^{p_2} \, L^-(h)^{q_2}, & \text{if } q_1 \geq p_2, \\ L^+(h)^{p_1} \langle\, h \,,\, h \,\rangle_{\mathcal{H}_{\mathbb{C}}}{}^{q_1} \, L^+(h)^{p_2 - q_1} L^-(h)^{q_2}, & \text{otherwise,} \end{cases} \\
&= \begin{cases} \langle\, h \,,\, h \,\rangle_{\mathcal{H}_{\mathbb{C}}}{}^{p_2} \, L^+(h)^{p_1} L^-(h)^{q_1 - p_2} L^-(h)^{q_2}, & \text{if } q_1 \geq p_2, \\ \langle\, h \,,\, h \,\rangle_{\mathcal{H}_{\mathbb{C}}}{}^{q_1} \, L^+(h)^{p_1} L^+(h)^{p_2 - q_1} L^-(h)^{q_2}, & \text{otherwise,} \end{cases} \\
&= \begin{cases} \langle\, h \,,\, h \,\rangle_{\mathcal{H}_{\mathbb{C}}}{}^{p_2} \, L^+(h)^{p_1} L^-(h)^{q_1 - p_2 + q_2}, & \text{if } q_1 \geq p_2, \\ \langle\, h \,,\, h \,\rangle_{\mathcal{H}_{\mathbb{C}}}{}^{q_1} \, L^+(h)^{p_1 + p_2 - q_1} L^-(h)^{q_2}, & \text{otherwise.} \end{cases}
\end{aligned}$$

In either of the form above, we see that they are always of the desired form; hence the closure property of the vector multiplication. $\square$

Theorem 3.12. *Let r be a positive integer and $\{h_1, h_2, \ldots, h_r\} \subset \mathcal{H}_{\mathbb{C}}$ be a collection of mutually orthogonal vectors. Then the random variables $X(h_1), X(h_2), \ldots, X(h_r)$ are freely independent.*

Proof. According to Definition 1.23, we have to consider the unital algebras $\mathcal{A}_1 := \text{alg}(1, X(h_1)), \mathcal{A}_2 := \text{alg}(1, X(h_2)), \ldots, \mathcal{A}_r := \text{alg}(1, X(h_r))$. We notice that, for each $i = 1, 2, \ldots, r$, the unital algebra $\mathcal{A}_i$ is a subset of the linear span of the set $\{ L^+(h_i)^p L^-(h_i)^q \mid p, q = 0, 1, 2, \ldots . \}$. This is because of Lemma 3.11 and the fact that $\mathcal{A}_i$ is the smallest algebra containing the operator $L^+(h_i) + L^-(h_i)$ and the operator $1 = L^+(h_i)^0 + L^-(h_i)^0$. Therefore, we are left with the verification of Definition 1.22 for the linear span of the set $\{ L^+(h_i)^p L^-(h_i)^q \mid p, q = 0, 1, 2, \ldots . \}$. This, however, follows from Theorem 3.9. $\square$

Chapter 4

Bosonic Fock space

Throughout this chapter, we shall fix some Hilbert space $\mathcal{H}$ over $\mathbb{R}$. Recall that $\mathcal{F}(\mathcal{H}_{\mathbb{C}}) := \bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathbb{C}}{}^{\otimes n}$ is called the full Fock space over the complexification $\mathcal{H}_{\mathbb{C}}$ (or the associated free Fock space). The inner product endowed on $\mathcal{F}(\mathcal{H}_{\mathbb{C}})$ is outlined in Section 3.1.

4.1 Preliminaries

Definition 4.1 (Symmetric product)**.** Let $u_1, u_2, \ldots, u_n \in \mathcal{H}_{\mathbb{C}}$. We define

$$u_1 \circ u_2 \circ \cdots \circ u_n := \frac{1}{n!} \sum_{\sigma \in S_n} u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \cdots \otimes u_{\sigma(n)},$$

where S_n stands for the group of permutations of the set $\{1, 2, \ldots, n\}$, $n \in \mathbb{N}$. In particular, we denote $u^{\circ n} := \underbrace{u \circ u \circ \cdots \circ u}_{n \text{ copies}}$.

Corollary 4.2. *For any $u_1, u_2, \ldots, u_n \in \mathcal{H}_{\mathbb{C}}$, we have*

$$u_1 \circ u_2 \circ \cdots \circ u_n = u_{\sigma(1)} \circ u_{\sigma(2)} \circ \cdots \circ u_{\sigma(n)},$$

for any permutation σ of the set $\{1, 2, \ldots, n\}$, $n \in \mathbb{N}$.

Corollary 4.3. *For any $u \in \mathcal{H}_{\mathbb{C}}$, we have*

$$u^{\circ n} = u^{\otimes n}.$$

Proof.

$$u^{\circ n} := \frac{1}{n!} \sum_{\sigma \in S_n} u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \cdots \otimes u_{\sigma(n)} = \frac{1}{n!} n!\, u \otimes u \otimes \cdots \otimes u = u^{\otimes n},$$

where $u_1, u_2, \ldots, u_n = u$. □

Definition 4.4. [6, p.55]. Let ${\mathcal{H}_{\mathbb{C}}}^{\circ n}$ be the linear subspace of ${\mathcal{H}_{\mathbb{C}}}^{\otimes n}$ generated by all vectors of the form $u_1 \circ u_2 \circ \cdots \circ u_n$, where $u_1, u_2, \ldots u_n \in \mathcal{H}_{\mathbb{C}}$. We call the completion $\overline{{\mathcal{H}_{\mathbb{C}}}^{\circ n}}$ (with respect to the inner product on $\mathcal{F}(\mathcal{H})$) the n-th symmetric power of $\mathcal{H}_{\mathbb{C}}$.

Note. Similar to the convention on the tensor product, we denote $A^{\circ n} := \underbrace{A \circ A \circ \cdots \circ A}_{n \text{ copies}}$ and take ${\mathcal{H}_{\mathbb{C}}}^{\circ 0} = \mathbb{C}\,\Omega$, where Ω is the vacuum vector.

Definition 4.5 (Bosonic Fock space over $\mathcal{H}_{\mathbb{C}}$)**.**

$$\Gamma(\mathcal{H}_{\mathbb{C}}) := \overline{\bigoplus_{i=0}^{\infty} {\mathcal{H}_{\mathbb{C}}}^{\circ n}},$$

where the completion is taken with respect to the inner product on $\mathcal{F}(\mathcal{H})$.

In order to get a better feel for the Bosonic Fock space $\Gamma(\mathcal{H}_{\mathbb{C}})$, we look deeper into the n-th symmetric power of $\mathcal{H}_{\mathbb{C}}$ for some arbitrary natural number n. Our first step is to have a polarization formula for the symmetric products

Proposition 4.6. *[4, p. 89]. Let $u_1, u_2, \ldots, u_n \in \mathcal{H}_{\mathbb{C}}$ and denote $\{\pm 1\} := \{1, -1\}$. Then*

$$u_1 \circ u_2 \circ \cdots \circ u_n = \frac{1}{n!\, 2^n} \sum_{\epsilon \in \{\pm 1\}^n} \left(\prod_{k=1}^{n} \epsilon_k \right) (\epsilon_1\, u_1 + \cdots + \epsilon_n\, u_n)^{\otimes n}.$$

Proof. Expanding the right hand side of the above equality gives:

$$\begin{aligned}
&\frac{1}{n!\,2^n}\sum_{\epsilon\in\{\pm1\}^n}\left(\prod_{k=1}^{n}\epsilon_k\right)(\epsilon_1\,u_1+\cdots+\epsilon_n\,u_n)^{\otimes n}\\
&=\frac{1}{n!\,2^n}\sum_{\epsilon\in\{\pm1\}^n}\left(\prod_{k=1}^{n}\epsilon_k\right)\sum_{j_1,\ldots,j_m=1}^{n}\epsilon_{j_1}\,u_{j_1}\otimes\cdots\otimes\epsilon_{j_n}\,u_{j_n}\\
&=\frac{1}{n!\,2^n}\sum_{\epsilon\in\{\pm1\}^n}\sum_{\sigma\in S_n}u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes\cdots\otimes u_{\sigma(n)}\\
&=\frac{1}{n!\,2^n}\,2^n\left(\sum_{\sigma\in S_n}u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes\cdots\otimes u_{\sigma(n)}\right)=u_1\circ u_2\circ\cdots\circ u_n.
\end{aligned}$$

□

It follows from Proposition 4.6 that we have the next corollary.

Corollary 4.7. *The linear span of the set* $\{\, v\otimes v\otimes\cdots\otimes v \mid v\in\mathcal{H}_{\mathbb{C}}\,\}$ *is equal to the linear span of the set* $\{\, u_1\circ u_2\circ\cdots\circ u_n \mid u_1,u_2,\ldots,u_n\in\mathcal{H}_{\mathbb{C}}\,\}$*.*

We remark an important consequence of the above corollary below.

Remark 4.8. A linear map on ${\mathcal{H}_{\mathbb{C}}}^{\circ n}$, $n\in\mathbb{N}$, is uniquely determined by how it acts on the set $\{\, v^{\otimes n} \mid v\in\mathcal{H}_{\mathbb{C}}\,\}$. As a consequence, a linear map on $\Gamma(\mathcal{H}_{\mathbb{C}})$ is completely determined by how it acts on the set $\cup_{n=0}^{\infty}\{\, v^{\otimes n} \mid v\in\mathcal{H}_{\mathbb{C}}\,\}$.

4.2 Creation and annihilation operators

Definition 4.9 ((n,k)-creation operators)**.** Fix any $h\in\mathcal{H}_{\mathbb{C}}$. For each pair (n,k) of integers such that n is non-negative and $1\le k\le n+1$, we define the (n,k)-creation operator $a^+_{n,k}(h):{\mathcal{H}_{\mathbb{C}}}^{\otimes n}\to{\mathcal{H}_{\mathbb{C}}}^{\otimes(n+1)}$ to be given by

$$a^+_{n,k}(h)(v_1\otimes\cdots\otimes v_n):=v_1\otimes\cdots\otimes v_{k-1}\otimes h\otimes v_k\otimes\cdots\otimes v_n.$$

Definition 4.10 (Creation operators)**.** Fix any $h \in \mathcal{H}_{\mathbb{C}}$ and non-negative integer n. We define the creation operator $a_n^+(h) : {\mathcal{H}_{\mathbb{C}}}^{\otimes n} \to {\mathcal{H}_{\mathbb{C}}}^{\otimes (n+1)}$ to be given by

$$a_n^+(h) := \frac{1}{\sqrt{n+1}} \sum_{k=0}^{n} a_{n,k}^+(h).$$

Definition 4.11 (Creation operators on $\mathcal{F}(\mathcal{H}_{\mathbb{C}})$)**.** Fix any $h \in \mathcal{H}_{\mathbb{C}}$. We define the creation operator $a^+(h) : \bigoplus_{n=0}^{\infty} {\mathcal{H}_{\mathbb{C}}}^{\otimes n} \to \bigoplus_{n=0}^{\infty} {\mathcal{H}_{\mathbb{C}}}^{\otimes n}$ to be given by

$$a^+(h) := \bigoplus_{n=0}^{\infty} a_n^+(h).$$

By the universal property of tensor product, the creation operator $a^+(h)$ can be extended so that it is an operator on the full Fock space $\mathcal{F}(\mathcal{H}_{\mathbb{C}})$.

Lemma 4.12. *[4, p. 81]. For any $h, v \in \mathcal{H}_{\mathbb{C}}$ and any non-negative integer n, we have*

$$\begin{aligned} a^+(h)(v^{\otimes n}) &= \frac{1}{\sqrt{n+1}} \sum_{k=0}^{n} v^{\otimes k} \otimes h \otimes v^{\otimes (n-k)} \\ &= \sqrt{n+1} \left(h \circ \underbrace{v \circ \cdots \circ v}_{n \text{ copies}} \right) \end{aligned}$$

Proof. The first equality follows directly from Definition 4.9 and Definition 4.10. For the second equality, we compute

$$
\begin{aligned}
&\sum_{k=0}^{n} v^{\otimes k} \otimes h \otimes v^{\otimes(n-k)} \\
&= \frac{1}{n!}\left(n!\,(h \otimes v \otimes \cdots \otimes v) + n!\,(v \otimes h \otimes \cdots \otimes v) + \cdots + n!\,(v \otimes v \otimes \cdots \otimes h)\right) \\
&= \frac{1}{n!}\Bigg(\sum_{\sigma \in S_n} (h \otimes v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}) + \sum_{\sigma \in S_n} (v_{\sigma(1)} \otimes h \otimes \cdots \otimes v_{\sigma(n)}) + \cdots \\
&\qquad\qquad \cdots + \sum_{\sigma \in S_n} (v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes h)\Bigg) \\
&= \frac{1}{n!}(n+1)\left(\sum_{\tau \in S_{n+1}} u_{\tau(1)} \otimes u_{\tau(2)} \otimes \cdots \otimes u_{\tau(n+1)}\right) = (n+1)\left(h \circ \underbrace{v \circ \cdots \circ v}_{n \text{ copies}}\right),
\end{aligned}
$$

where $v_1, v_2, \ldots, v_n = v$, $u_1, u_2, \ldots, u_n = v$ and $u_{n+1} = h$. $\square$

Definition 4.13 ((n,k)-annihilation operators)**.** Fix any $h \in \mathcal{H}_{\mathbb{C}}$. Let n be any non-negative integer and k be a positive integer such that $k \leq n$. We define the (n,k)-annihilation operator $a^-_{n,k}(h) : {\mathcal{H}_{\mathbb{C}}}^{\otimes n} \to {\mathcal{H}_{\mathbb{C}}}^{\otimes(n-1)}$ to be given by

$$a^-_{n,k}(h)(v_1 \otimes \cdots \otimes v_n) := \langle\, h\ ,\ v_k \,\rangle_{\mathcal{H}_{\mathbb{C}}}\ v_1 \otimes \cdots \otimes v_{k-1} \otimes v_{k+1} \otimes \cdots \otimes v_n.$$

Definition 4.14 (Annihilation operators)**.** Fix any $h \in \mathcal{H}_{\mathbb{C}}$ and non-negative integer n. We define the annihilation operator $a^-_n(h) : {\mathcal{H}_{\mathbb{C}}}^{\otimes n} \to {\mathcal{H}_{\mathbb{C}}}^{\otimes(n-1)}$ to be given by

$$a^-_n(h) := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} a^-_{n,k}(h).$$

Definition 4.15 (Annihilation operators on $\mathcal{F}(\mathcal{H}_{\mathbb{C}})$)**.** Fix any $h \in \mathcal{H}_{\mathbb{C}}$. We define the annihilation operator $a^+(h) : \bigoplus_{n=0}^{\infty} {\mathcal{H}_{\mathbb{C}}}^{\otimes n} \to \bigoplus_{n=0}^{\infty} {\mathcal{H}_{\mathbb{C}}}^{\otimes n}$ to be given by

$$a^-(h) := \bigoplus_{n=0}^{\infty} a^-_n(h).$$

By the universal property of tensor product, the annihilation operator a^- can be extended so that it is an operator on the full Fock space $\mathcal{F}(\mathcal{H}_\mathbb{C})$.

Lemma 4.16. *[4, p. 81]. For any $h, v \in \mathcal{H}_\mathbb{C}$ and any non-negative integer n, we have*

$$a^-(h)(v^{\otimes n}) = \sqrt{n}\ \langle\, h\ ,\ v\,\rangle_{\mathcal{H}_\mathbb{C}}\ v^{\otimes(n-1)} = \sqrt{n}\ \langle\, h\ ,\ v\,\rangle_{\mathcal{H}_\mathbb{C}}\ v^{\circ(n-1)}.$$

Proof. The first equality follows from Definition 4.13 and Definition 4.14. The second equality follows from Corollary 4.3. □

4.3 Joint moments of the position operator $Q(h)$

Fix some $h \in \mathcal{H}_\mathbb{C}$. Define the position operator $Q(h) := a^+(h) + a^-(h)$ on the bosonic Fock space $\Gamma(\mathcal{H}_\mathbb{C}) \subset \mathcal{F}(\mathcal{H}_\mathbb{C})$, cf. Definition 4.11 and Definition 4.15.

Lemma 4.17.

$$\langle\, Q(h)^p\,\Omega\ ,\ \Omega\,\rangle_{\mathcal{F}(\mathcal{H}_\mathbb{C})} = \begin{cases} 0, & \textit{if } p \textit{ is odd} \\ \left(\langle\, h\ ,\ h\,\rangle_{\mathcal{H}_\mathbb{C}}\right)^{\frac{p}{2}}\ (p-1)!!, & \textit{if } p \textit{ is even} \end{cases}$$

Proof. By Corollary 4.3 and Lemma 4.12, we notice that the creation operator $a^+(h)$ is a raising operator on a linear subspace $\left\{ \sum_{j=0}^n d_j\, h^{\otimes j} \,\middle|\, d_j \in \mathbb{C}, n \in \mathbb{N} \right\} \subset \Gamma(\mathcal{H}_\mathbb{C}) \subset \mathcal{F}(\mathcal{H}_\mathbb{C})$; similarly, by Corollary 4.3 and Lemma 4.16, we also observe that the annihilation operator $a^-(h)$ is a lowering operator on a linear subspace $\left\{ \sum_{j=0}^n d_j\, h^{\otimes j} \,\middle|\, d_j \in \mathbb{C}, n \in \mathbb{N} \right\} \subset \Gamma(\mathcal{H}_\mathbb{C}) \subset \mathcal{F}(\mathcal{H}_\mathbb{C})$. Thus, by the proof of Lemma 2.21, the result follows. □

Lemma 4.18. *For any $h_1, h_2 \in \mathcal{H}_\mathbb{C}$, we have*

$$a^+(h_1)\, a^+(h_2) = a^+(h_2)\, a^+(h_1).$$

Proof. In view of Remark 4.8, it suffices to show that for any arbitrary choice $v \in \mathcal{H}_\mathbb{C}$, we have $a^+(h_1)\, a^+(h_2)\, (v^{\otimes n}) = a^+(h_2)\, a^+(h_1)\, (v^{\otimes n})$, for any $n = 1, 2, \ldots$. By Lemma 4.12, we have

$$a^+(h_1)\, a^+(h_2)\left(v^{\otimes n}\right) = \sqrt{n+2}\,\sqrt{n+1}\,\left(h_1 \circ h_2 \circ \underbrace{v \circ \cdots \circ v}_{n \text{ copies}}\right);$$

and similarly,

$$a^+(h_2)\, a^+(h_1)\left(v^{\otimes n}\right) = \sqrt{n+2}\,\sqrt{n+1}\,\left(h_2 \circ h_1 \circ \underbrace{v \circ \cdots \circ v}_{n \text{ copies}}\right).$$

Finally, Corollary 4.2 completes the proof. $\square$

Lemma 4.19. *For any $h_1, h_2 \in \mathcal{H}_{\mathbb{C}}$, we have*

$$a^-(h_1)\, a^-(h_2) = a^-(h_2)\, a^-(h_1).$$

Proof. In view of Remark 4.8, it suffices to show that for any arbitrary choice $v \in \mathcal{H}_{\mathbb{C}}$, we have $a^-(h_1)\, a^-(h_2)\,(v^{\otimes n}) = a^-(h_2)\, a^-(h_1)\,(v^{\otimes n})$, for any $n = 1, 2, \ldots$. By Lemma 4.16, we have

$$a^-(h_1)\, a^-(h_2)\left(v^{\otimes n}\right) = \sqrt{n-1}\,\langle\, h_1\,,\, v\,\rangle_{\mathcal{H}_{\mathbb{C}}}\,\sqrt{n}\,\langle\, h_2\,,\, v\,\rangle_{\mathcal{H}_{\mathbb{C}}}\, v^{\otimes(n-2)} = a^-(h_2)\, a^-(h_1)\left(v^{\otimes n}\right).$$

$\square$

Lemma 4.20. *For any $h_1, h_2 \in \mathcal{H}_{\mathbb{C}}$ such that $\langle\, h_2\,,\, h_1\,\rangle_{\mathcal{H}_{\mathbb{C}}} = 0$, we have*

$$a^+(h_1)\, a^-(h_2) = a^-(h_2)\, a^+(h_1).$$

Proof. In view of Remark 4.8, it suffices to show that for any arbitrary choice $v \in \mathcal{H}_{\mathbb{C}}$, we have $a^+(h_1)\, a^-(h_2)\,(v^{\otimes n}) = a^-(h_2)\, a^+(h_1)\,(v^{\otimes n})$, for any $n = 1, 2, \ldots$. By Lemma 4.12 and

Lemma 4.16, we have

$$
\begin{aligned}
a^{+}(h_1)\, a^{-}(h_2) \left(v^{\otimes n}\right) &= \sqrt{n}\, \langle\, h_2 \,,\, v \,\rangle_{\mathcal{H}_{\mathbb{C}}}\, a^{+}(h_1) \left(v^{\otimes (n-1)}\right) \\
&= \sqrt{n}\, \langle\, h_2 \,,\, v \,\rangle_{\mathcal{H}_{\mathbb{C}}}\, \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} v^{\otimes k} \otimes h_1 \otimes v^{\otimes (n-1-k)} \\
&= \langle\, h_2 \,,\, v \,\rangle_{\mathcal{H}_{\mathbb{C}}} \sum_{k=0}^{n-1} v^{\otimes k} \otimes h_1 \otimes v^{\otimes (n-1-k)}.
\end{aligned}
$$

On the other hand, we also compute, by Lemma 4.12 and Definition 4.14,

$$
\begin{aligned}
&a^{-}(h_2)\, a^{+}(h_1) \left(v^{\otimes n}\right) = a^{-}(h_2) \left(\frac{1}{\sqrt{n+1}} \sum_{s=0}^{n} v^{\otimes s} \otimes h_1 \otimes v^{\otimes (n-s)} \right) \\
&= \frac{1}{\sqrt{n+1}} \sum_{s=0}^{n} a^{-}(h_2) \left(v^{\otimes s} \otimes h_1 \otimes v^{\otimes (n-s)}\right) \\
&= \frac{1}{\sqrt{n+1}} \sum_{s=0}^{n} \frac{1}{\sqrt{n+1}} \left(\sum_{t=0}^{n-1} \langle\, h_2 \,,\, v \,\rangle_{\mathcal{H}_{\mathbb{C}}} \left(v^{\otimes t} \otimes h_1 \otimes v^{\otimes (n-t-1)}\right) + \langle\, h_2 \,,\, h_1 \,\rangle_{\mathcal{H}_{\mathbb{C}}} \left(v^{\otimes n}\right) \right) \\
&= \sum_{t=0}^{n-1} \langle\, h_2 \,,\, v \,\rangle_{\mathcal{H}_{\mathbb{C}}} \left(v^{\otimes t} \otimes h_1 \otimes v^{\otimes (n-t-1)}\right),
\end{aligned}
$$

where the last equality follows from the hypothesis $\langle\, h_2 \,,\, h_1 \,\rangle_{\mathcal{H}_{\mathbb{C}}} = 0$. $\square$

Lemma 4.21. *For any $h_1, h_2 \in \mathcal{H}_{\mathbb{C}}$ such that $\langle\, h_2 \,,\, h_1 \,\rangle_{\mathcal{H}_{\mathbb{C}}} = 0$, we have*

$$
Q(h_1)\, Q(h_2) = Q(h_2)\, Q(h_1).
$$

Proof. Simply expand the expression $(a^{+}(h_1) + a^{-}(h_1))\, (a^{+}(h_2) + a^{-}(h_2))$ and then apply Lemma 4.18, Lemma 4.19, and Lemma 4.20. $\square$

Fix some $h \in \mathcal{H}_{\mathbb{C}}$. Define a positive, unital, linear functional $\mathbb{E}(\cdot)$ on the set of linear operators that acts on the bosonic Fock space $\Gamma(\mathcal{H}_{\mathbb{C}})$ given by

$$
\mathbb{E}[\cdot] = \langle \cdot\, \Omega \,,\, \Omega \,\rangle_{\mathcal{F}(\mathcal{H}_{\mathbb{C}})}\,.
$$

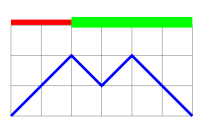
Now we shall mimic our approach in Section 2.2 so as to calculate $\mathbb{E}\left[Q(h_1)^{p_1}\, Q(h_2)^{p_2}\right]$ for any non-negative integers p_1, p_2 and $h_1, h_2 \in \mathcal{H}_{\mathbb{C}}$ such that $\langle\, h_2\ ,\ h_1\,\rangle_{\mathcal{H}_{\mathbb{C}}} = 0$. By expanding the expression $Q(h_1)^{p_1}\, Q(h_2)^{p_2}$ in terms of operators $a^+(h_1)$, $a^-(h_1)$, $a^+(h_2)$, and $a^-(h_2)$, we see that $Q(h_1)^{p_1}\, Q(h_2)^{p_2}$ can always be written as a finite sum of terms(summands) of the form

$$Y = \underbrace{a^{\pm}(h_1)\, a^{\pm}(h_1)\ \cdots\ a^{\pm}(h_1)}_{p_1 \text{ copies}}\ \underbrace{a^{\pm}(h_2)\, a^{\pm}(h_2)\ \cdots\ a^{\pm}(h_2)}_{p_2 \text{ copies}},$$

where the symbol $a^{\pm}(\cdot)$ is a placeholder for the operators $a^{+}(\cdot)$ or $a^{-}(\cdot)$. For each specific summand Y, we identify its associated lattice path $l_Y \in \mathcal{P}_{p_1+p_2}$ in the following manner. While reading the summand Y from right to left,

- [i] each time when one encounters an annihilation operator $a^-(h_2)$, the lattice path goes down by one step (along the vector $(1,-1)$),
- [ii] each time when one encounters a creation operator $a^+(h_2)$, the lattice path goes up by one step (along the vector $(1,1)$),
- [iii] after exactly p_1 many encounters of operators $a^-(h_2)$ or $a^+(h_2)$, one shall never run into operators $a^-(h_2)$ and $a^+(h_2)$ ever again and one shall start encountering operators $a^-(h_1)$ and $a^+(h_1)$,
- [iv] each time when one encounters an annihilation operator $a^-(h_1)$, the lattice path goes down by one step (along the vector $(1,-1)$), and
- [v] each time when one encounters a creation operator $a^+(h_1)$, the lattice path goes up by one step (along the vector $(1,1)$).

As an example, if $Y = a^-(h_1)\, a^-(h_1)\, a^+(h_1)\, a^-(h_1)\, a^+(h_2)\, a^+(h_2)$, then

$l_Y =$  $\in \mathcal{D}_{2+4}$,

where the red line indicates that the region is due to the vector $h_2 \in \mathcal{H}_{\mathbb{C}}$ and the green line indicates that the region is due to the vector $h_1 \in \mathcal{H}_{\mathbb{C}}$. Next, we mimic Definition 2.6

and define the weight $W(\cdot)$ of $l \in \mathcal{P}_{p_1+p_2}$ by

$$W(l) := \left(\prod_{s_2=0}^{p_2} \gamma_{a^+(h_2),a^-(h_2)}(l(s_2+1) - l(s_2)) \left(\max\{l(s_2), l(s_2+1)\}\right) \right) \times \\ \left(\prod_{s_1=p_2+1}^{p_1+p_2} \gamma_{a^+(h_1),a^-(h_1)}(l(s_1+1) - l(s_1)) \left(\max\{l(s_1), l(s_1+1)\}\right) \right).$$

Then, similar to Equation (2.3), we have

$$\mathbb{E}\left[Q(h_1)^{p_1}\, Q(h_2)^{p_2}\right] = \sum_{l \in \mathcal{P}_{p_1+p_2}} W(l).$$

And, by the proof of Lemma 2.8, we deduce that

$$\mathbb{E}\left[Q(h_1)^{p_1}\, Q(h_2)^{p_2}\right] = \begin{cases} \sum_{l \in \mathcal{D}_{p_1+p_2}} W(l), & \text{if } p_1 + p_2 \text{ is even,} \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

Lemma 4.22. *For any non-negative integers p_1, p_2 and $h_1, h_2 \in \mathcal{H}_{\mathbb{C}}$ such that $\langle\, h_2\ ,\ h_1 \,\rangle_{\mathcal{H}_{\mathbb{C}}} = 0$, we have*

$$\mathbb{E}\left[Q(h_1)^{p_1}\, Q(h_2)^{p_2}\right] = \mathbb{E}\left[Q(h_1)^{p_1}\right] \mathbb{E}\left[Q(h_2)^{p_2}\right].$$

Proof. By Equation (4.1) and Lemma 4.17, it remains to show the implication

$$l(p_2) \neq 0 \implies W(l) = 0,$$

for any $l \in \mathcal{D}_{p_1+p_2}$. Let Y be the product of $a^+(h_1)$, $a^-(h_1)$, $a^+(h_2)$, or $a^-(h_2)$ such that $l_Y = l$. (Y is unique as the map $l_{(\cdot)}$ is bijective.) Notice that $l(p_2) \neq 0$ means that after applying p_2 many operators $a^+(h_2)$ or $a^-(h_2)$ of Y to the vacuum vector Ω, we have a scalar multiple of a simple tensor of h_2 of positive length $l(p_2)$. In order to bring this simple tensor down to being of length 0, at some point in applying the operators $a^+(h_1)$ or $a^-(h_1)$ in the expression Y, the scalar $\langle\, h_2\ ,\ h_1 \,\rangle_{\mathcal{H}_{\mathbb{C}}}$ must pop up when the operator $a^-(h_1)$ is annihilating the vector h_2 in the simple tensor, leading to $W(l_Y) = 0$. This finishes the proof. $\square$

Corollary 4.23. *the random variables $Q(h_1)$ and $Q(h_2)$ are independent for any $h_1, h_2 \in \mathcal{H}_{\mathbb{C}}$ such that $\langle\, h_2\ ,\ h_1 \,\rangle_{\mathcal{H}_{\mathbb{C}}} = 0$.*

Proof. This follows from Remark 1.21, Lemma 4.21, and Lemma 4.22. $\square$

Corollary 4.24. *Let r be a positive integer and $\{h_1, h_2, \ldots, h_r\} \subset \mathcal{H}_{\mathbb{C}}$ be a collection of mutually orthogonal vectors. Then the random variables $Q(h_1), Q(h_2), \ldots, Q(h_r)$ are (classically) independent.*

Proof. If one checks the proofs of Lemma 4.21 and Lemma 4.22, then it is easily seen that there is nothing stopping us from proving the equivalent results when r-many vectors are involved. $\square$

Chapter 5

Classical chaos decomposition

Throughout this entire chapter, let us denote $\mathcal{H}$ be the collection of real-valued square integrable functions on the half interval $[0, \infty)$, equipped with the usual L^2-norm. Furthermore, let $\mathcal{H}_{\mathbb{C}}$ be its complexification.

5.1 Stochastic integrals

Definition 5.1. For each $m = 1, 2, \ldots$, we define a map $t_{(\cdot)}^{(m)} : \{0, 1, 2, \ldots, m\,2^m\} \to [0, \infty)$ to obey the following condition: $\left\{0 = t_0^{(m)} < t_1^{(m)} < \cdots < t_{m\,2^m}^{(m)} = m\right\} \subset \mathbb{R}$ is a partition of the interval $[0, m]$ into $(m\,2^m)$-many subintervals of equal length.

Definition 5.2. For all $m = 1, 2, \ldots$ and $i = 1, 2, \ldots, m\,2^m$, we denote

$$\left[t_i^{(m)}\right) := \left[t_{i-1}^{(m)}, t_i^{(m)}\right) \subset \mathbb{R}.$$

Definition 5.3. For any positive integer n, $f \in {\mathcal{H}_{\mathbb{C}}}^{\otimes n}$ is said to be a **dyadic step function** if there exists a positive integer m such that

$$f = \sum_{i_n=1}^{m\,2^m} \cdots \sum_{i_2=1}^{m\,2^m} \sum_{i_1=1}^{m\,2^m} a_{i_1, i_2, \ldots, i_n} \, \mathbb{1}_{\left[t_{i_1}^{(m)}\right) \times \left[t_{i_2}^{(m)}\right) \times \cdots \times \left[t_{i_n}^{(m)}\right)},$$

where $a_{i_1,i_2,\ldots,i_n}$ are some complex numbers and $\mathbb{1}_{(\cdot)}$ is the usual indicator function on $\left[t^{(m)}_{i_1}\right) \times \left[t^{(m)}_{i_2}\right) \times \cdots \times \left[t^{(m)}_{i_n}\right) \subset [0,\infty)^n$.

Theorem 5.4. *The collection of dyadic step functions is dense in ${\mathcal{H}_{\mathbb{C}}}^{\otimes n}$.*

Proof. By the property of tensor product and complexification, it suffices to show that the collection of dyadic step functions on the half interval $[0,\infty) \subset \mathbb{R}$ is dense (w.r.t. $\langle \cdot\,,\,\cdot \rangle$, i.e. the L^2-norm) in the Hilbert space $(L^2_{\mathbb{R}}([0,\infty)), \langle \cdot\,,\,\cdot \rangle)$. To this end, we recall the following results from real analysis and measure theory.

[i] Continuous functions with compact support can be uniformly approximated by step functions.

[ii] The set of continuous functions with compact support is dense in $L^2_{\mathbb{R}}(\mathbb{R})$.

[iii] Uniform convergence within a bounded interval implies convergence in the L^2-norm.

Finally, since every step function can be written as a limit of dyadic step functions, we can complete the proof. $\square$

Definition 5.5 (Stochastic integral $I^{(Q)}_n(\cdot)$ defined in the Itô sense)**.** Let $f \in {\mathcal{H}_{\mathbb{C}}}^{\otimes n}$. By Theorem 5.4, we have dyadic step functions

$$f^{(m)} = \sum_{i_n=1}^{m\,2^m} \cdots \sum_{i_2=1}^{m\,2^m} \sum_{i_1=1}^{m\,2^m} a^{(m),(f)}_{i_1,i_2,\ldots,i_n}\, \mathbb{1}_{\left[t^{(m)}_{i_1}\right) \times \left[t^{(m)}_{i_2}\right) \times \cdots \times \left[t^{(m)}_{i_n}\right)}, \qquad m = 1, 2, \ldots,$$

where $a^{(m),(f)}_{i_1,i_2,\ldots,i_n}$ are some complex numbers, such that $f^{(m)} \xrightarrow[m\to\infty]{{\mathcal{H}_{\mathbb{C}}}^{\otimes n}} f$. Then we define the n-th **stochastic integral $I^{(Q)}_n(\cdot)$ with respective to the linear operator** Q in the Itô sense to be given by

$$I^{(Q)}_n(f) = \lim_{m\to\infty} \sum_{\substack{i_n=1\\ i_n \neq i_{n-1}\\ \vdots \\ i_n \neq i_2 \\ i_n \neq i_1}}^{m\,2^m} \cdots \sum_{\substack{i_2=1\\ i_2\neq i_1}}^{m\,2^m} \sum_{i_1=1}^{m\,2^m} a^{(m),(f)}_{i_1,i_2,\ldots,i_n}\, Q\left(\mathbb{1}_{\left[t^{(m)}_{i_1}\right)}\right) Q\left(\mathbb{1}_{\left[t^{(m)}_{i_2}\right)}\right) \cdots Q\left(\mathbb{1}_{\left[t^{(m)}_{i_n}\right)}\right).$$

Remark 5.6. It is conceptually helpful to point out that stochastic integrals $I_n^{(Q)}(\cdot)$ is a linear operator acting on the full Fock space $\mathcal{F}(\mathcal{H}_{\mathbb{C}})$. We write $I_n(f) \in \mathcal{L}(\mathcal{F}(\mathcal{H}_{\mathbb{C}}))$ which is defined to be the set of linear operators acting on $\mathcal{F}(\mathcal{H}_{\mathbb{C}})$.

In order to illustrate the benefit of involving dyadic step functions in Definition 5.5, let us try to apply Definition 5.5 to a dyadic step function

$$g = \sum_{i_n=1}^{k\,2^k} \cdots \sum_{i_2=1}^{k\,2^k} \sum_{i_1=1}^{k\,2^k} a_{i_1,i_2,\ldots,i_n}^{(k),(g)} \, \mathbb{1}_{\left[t_{i_1}^{(k)}\right) \times \left[t_{i_2}^{(k)}\right) \times \cdots \times \left[t_{i_n}^{(k)}\right)}.$$

If r is any integer that is greater than k, then we can always algebraically express

$$g = \sum_{i_n=1}^{r\,2^r} \cdots \sum_{i_2=1}^{r\,2^r} \sum_{i_1=1}^{r\,2^r} a_{i_1,i_2,\ldots,i_n}^{(r),(g)} \, \mathbb{1}_{\left[t_{i_1}^{(r)}\right) \times \left[t_{i_2}^{(r)}\right) \times \cdots \times \left[t_{i_n}^{(r)}\right)},$$

where the scalars $a_{i_1,i_2,\ldots,i_n}^{(r),(g)}$ take the trivial choices of the scalars $a_{i_1,i_2,\ldots,i_n}^{(k),(g)}$. Let us consider their respective stochastic integrals

$$\sum_{\substack{i_n=1 \\ i_n \neq i_{n-1} \\ \vdots \\ i_n \neq i_2 \\ i_n \neq i_1}}^{k\,2^k} \cdots \sum_{\substack{i_2=1 \\ i_2 \neq i_1}}^{k\,2^k} \sum_{i_1=1}^{k\,2^k} a_{i_1,i_2,\ldots,i_n}^{(k),(g)} \, Q\left(\mathbb{1}_{\left[t_{i_1}^{(k)}\right)}\right) Q\left(\mathbb{1}_{\left[t_{i_2}^{(k)}\right)}\right) \cdots Q\left(\mathbb{1}_{\left[t_{i_n}^{(k)}\right)}\right), \tag{5.1}$$

and

$$\sum_{\substack{i_n=1 \\ i_n \neq i_{n-1} \\ \vdots \\ i_n \neq i_2 \\ i_n \neq i_1}}^{r\,2^r} \cdots \sum_{\substack{i_2=1 \\ i_2 \neq i_1}}^{r\,2^r} \sum_{i_1=1}^{r\,2^r} a_{i_1,i_2,\ldots,i_n}^{(r),(g)} \, Q\left(\mathbb{1}_{\left[t_{i_1}^{(r)}\right)}\right) Q\left(\mathbb{1}_{\left[t_{i_2}^{(r)}\right)}\right) \cdots Q\left(\mathbb{1}_{\left[t_{i_n}^{(r)}\right)}\right). \tag{5.2}$$

Due to the linearity of Q and the fact that the scalars $a_{i_1,i_2,\ldots,i_n}^{(r),(g)}$ take the trivial choices of the scalars $a_{i_1,i_2,\ldots,i_n}^{(k),(g)}$, we conclude that Expression 5.1 and Expression 5.2 are algebraically identical. This shows that our stochastic integrals are well defined on dyadic step functions.

Although there might exist several sequence $\{f^{(m)}\}_{m=1}^{\infty}$ of dyadic step functions approaching a general $f \in \mathcal{H}_{\mathbb{C}}{}^{\otimes n}$, the stochastic $I_n(f)$ always exists and is unique in the sense that it is the L^2-limit of the Cauchy sequence $\{I_n\left(f^{(m)}\right)\}_{m=1}^{\infty}$. In other words, take any positive integers $p > q$. We want to show that

$$\mathbb{E}\left[\left(I_n^{(Q)}(f^{(p)}) - I_n^{(Q)}(f^{(q)})\right)^2\right] \xrightarrow[p,q\to\infty]{} 0.$$

From Definition 5.3, we have

$$I_n^{(Q)}(f^{(p)}) = \sum_{\substack{i_n=1\\ i_n\neq i_{n-1}\\ \vdots \\ i_n\neq i_2\\ i_n\neq i_1}}^{p\,2^p} \cdots \sum_{\substack{i_2=1\\ i_2\neq i_1}}^{p\,2^p} \sum_{i_1=1}^{p\,2^p} a_{i_1,i_2,\ldots,i_n}^{(p),(f^{(p)})}\, Q\left(\mathbb{1}_{\left[t_{i_1}^{(p)}\right)}\right) Q\left(\mathbb{1}_{\left[t_{i_2}^{(p)}\right)}\right) \cdots Q\left(\mathbb{1}_{\left[t_{i_n}^{(p)}\right)}\right),$$

and

$$I_n^{(Q)}(f^{(q)}) = \sum_{\substack{j_n=1\\ j_n\neq j_{n-1}\\ \vdots \\ j_n\neq j_2\\ j_n\neq j_1}}^{p\,2^p} \cdots \sum_{\substack{j_2=1\\ j_2\neq j_1}}^{p\,2^p} \sum_{j_1=1}^{p\,2^p} a_{j_1,j_2,\ldots,j_n}^{(p),(f^{(q)})}\, Q\left(\mathbb{1}_{\left[t_{j_1}^{(p)}\right)}\right) Q\left(\mathbb{1}_{\left[t_{j_2}^{(p)}\right)}\right) \cdots Q\left(\mathbb{1}_{\left[t_{j_n}^{(p)}\right)}\right).$$

Use them in the expression $\left(I_n^{(Q)}(f^{(p)}) - I_n^{(Q)}(f^{(q)})\right)^2$. Then, as the set $\left\{\mathbb{1}_{\left[t_1^{(p)}\right)}, \mathbb{1}_{\left[t_2^{(p)}\right)}, \ldots, \mathbb{1}_{\left[t_{p\,2^p}^{(p)}\right)}\right\}$ is a collection of mutually orthogonal vectors in $\mathcal{H}_{\mathbb{C}}$, by Lemma 4.21, we deduce that $\left(I_n^{(Q)}(f^{(p)}) - I_n^{(Q)}(f^{(q)})\right)^2$ is a finite sum of terms of the form

$$c_1\, Q\left(\mathbb{1}_{\left[t_{i_1}^{(p)}\right)}\right)^{s_1} \cdots Q\left(\mathbb{1}_{\left[t_{i_n}^{(p)}\right)}\right)^{s_n} Q\left(\mathbb{1}_{\left[t_{j_1}^{(p)}\right)}\right)^{s_{n+1}} \cdots Q\left(\mathbb{1}_{\left[t_{j_n}^{(p)}\right)}\right)^{s_{n+n}},$$

for some scalar c_1, which is determined by $a_{i_1,i_2,\ldots,i_n}^{(p),(f^{(p)})}$ and $a_{j_1,j_2,\ldots,j_n}^{(p),(f^{(q)})}$, and $s_1, s_2, \ldots, s_{2\,n} = 1, 2$. From the linearity of $\mathbb{E}\left[\cdot\right]$, we see that if we have the following two results:

[i] $\mathbb{E}\left[Q\left(\mathbb{1}_{\left[t^{(p)}_{i_1}\right)}\right)^{s_1}\cdots Q\left(\mathbb{1}_{\left[t^{(p)}_{i_n}\right)}\right)^{s_n} Q\left(\mathbb{1}_{\left[t^{(p)}_{j_1}\right)}\right)^{s_{n+1}}\cdots Q\left(\mathbb{1}_{\left[t^{(p)}_{j_n}\right)}\right)^{s_{2n}}\right] = 0$, if one of the elements of $\{s_1, s_2, \ldots, s_{2n}\}$ takes the value 1, and

[ii] $\mathbb{E}\left[Q\left(\mathbb{1}_{\left[t^{(p)}_{i_1}\right)}\right)^{2} Q\left(\mathbb{1}_{\left[t^{(p)}_{i_2}\right)}\right)^{2}\cdots Q\left(\mathbb{1}_{\left[t^{(p)}_{i_n}\right)}\right)^{2}\right] = \left(\frac{1}{2^p}\right)^n$,

then

$$\mathbb{E}\left[\left(I_n^{(Q)}(f^{(p)}) - I_n^{(Q)}(f^{(q)})\right)^2\right] = \lim_{p\to\infty}\left[\sum_{\substack{i_n=1\\ i_n\neq i_{n-1}\\ \vdots\\ i_n\neq i_2\\ i_n\neq i_1}}^{p\,2^p}\cdots\sum_{\substack{i_2=1\\ i_2\neq i_1}}^{p\,2^p}\sum_{i_1=1}^{p\,2^p}\left(a^{(p),(f^{(p)})}_{i_1,i_2,\ldots,i_n} - a^{(p),(f^{(q)})}_{i_1,i_2,\ldots,i_n}\right)^2\left(\frac{1}{2^p}\right)^n\right]$$

$$= \lim_{p\to\infty}\left[\sum_{i_n=1}^{p\,2^p}\cdots\sum_{i_1=1}^{p\,2^p}\left(a^{(p),(f^{(p)})}_{i_1,i_2,\ldots,i_n} - a^{(p),(f^{(q)})}_{i_1,i_2,\ldots,i_n}\right)^2\left(\frac{1}{2^p}\right)^n\right] = \left\langle\, f^{(p)} - f^{(q)}\ ,\ f^{(p)} - f^{(q)}\,\right\rangle \xrightarrow[p,q\to\infty]{} 0.$$

In fact, the above two used results follow from Corollary 4.24 and Lemma 4.17.

Remark 5.7. Alternatively, we may show that $I_n(f)$ exists and is unique by showing that

$$\mathbb{E}\left[\left(I_n^{(Q)}(f)\right)^p\right] < \infty,$$

for any $p \in \mathbb{N}$, i.e. $I_n^{(Q)}(f)$ has finite moments of all order.

Thus, we obtain our classical Brownian motion, namely $I_1^{(Q)}\left(\mathbb{1}_{[0,t)}\right)$, $t \in [0,\infty)$. It is readily verifiable that this linear operators on the full Fock spaces $\mathcal{F}(\mathcal{H}_{\mathbb{C}})$ satisfies every single requirement of the following definition of classical Brownian motion, written in terms of classical random variables (meaning measurable functions).

Definition 5.8. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A map $X : [0,\infty)\times\Omega \to \mathbb{R}$ is called a **stochastic process** if

[i] for each $t \in [0,\infty)$, $X(t,\cdot)$ is a random variable on $(\Omega, \mathcal{F}, P)$, and

[ii] for each $\omega \in \Omega$, $X(\cdot,\omega)$ is a measurable function (called a sample path).

Usually, we write the random variable $X(t,\cdot)$ as X_t. This way, a stochastic process $X(t,\omega)$ may also be thought of as a collection $\{X_t\}_{t\in[0,\infty)}$ of random variables X_t, partially ordered by the time variable $t \in [0,\infty)$.

Definition 5.9. A stochastic process $B(t,\omega)$ is called a **Brownian motion** if

[i] the random variable $B(0,\cdot) = 0$ almost surely, i.e. $P\left(\{\,\omega \in \Omega \mid B(0,\omega) = 0\,\}\right) = 1$,

[ii] for any $0 \le s < t$, the random variable $B_t - B_s$ is normally distributed with mean 0 and variance $t - s$, i.e. for any $a \le b$, we have

$$P\left(\{a \le B_t - B_s \le b\}\right) = \frac{1}{\sqrt{2\,\pi\,(t-s)}} \int_a^b e^{-\frac{x^2}{2\,(t-s)}}\,dx,$$

[iii] $B(t,\omega)$ has independent increment, i.e., for any $0 \le t_1 < t_2 < \cdots < t_n$, the random variables

$$B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}},$$

are independent, and

[iv] the sample paths of $B(t,\omega)$ are almost surely continuous, i.e.

$$P\left(\{\,\omega \in \Omega \mid B(\cdot,\omega) \text{ is continuous}\,\}\right) = 1.$$

5.2 Solution spaces

In practice, we wish to use mathematical objects to model physical phenomenon. More specifically, our stochastic integrals now enable us to describe a random physical event driven by the classical Brownian motion. For example, the linear operator

$$
\begin{aligned}
&I_1^{(Q)}\left(\mathbb{1}_{[0,1)}\right) + 3\, I_1^{(Q)}\left(\mathbb{1}_{[1,2)}\right) + I_1^{(Q)}\left(\mathbb{1}_{[2,3)}\right) I_1^{(Q)}\left(\mathbb{1}_{[2,5)}\right) \\
&\qquad = I_1^{(Q)}\left(\mathbb{1}_{[0,1)}\right) + 3\, I_1^{(Q)}\left(\mathbb{1}_{[1,2)}\right) + I_1^{(Q)}\left(\mathbb{1}_{[2,3)}\right) I_1^{(Q)}\left(\mathbb{1}_{[2,3)}\right) + I_1^{(Q)}\left(\mathbb{1}_{[2,3)}\right) I_1^{(Q)}\left(\mathbb{1}_{[3,5)}\right)
\end{aligned}
$$

may be interpreted as a random variable modelling the vertical displacement $x(t)$ of a point particle, under the principle that it (regardless its current/past heights) has an equal chance of travelling upwards and downwards by the same amount, say 1 unit length per 1 unit time, with the following specification:

[i] During the time period between $t = 0$ to $t = 1$, the difference $x(1) - x(0)$ is normally distributed with variance 1,

[ii] During the time period between $t = 1$ to $t = 2$, the difference $x(2) - x(1)$ is normally distributed with variance $3^2\,(2-1)$,

[iii] During the time period between $t = 2$ to $t = 3$, the difference $x(3) - x(2)$ is a random variable whose distribution is the result of multiplying a normally distributed random variable of variance 1 by itself (notice that this is a multiplication of two dependent random variables), and

[iv] During the time period between $t = 3$ to $t = 5$, the difference $x(5) - x(3)$ is a random variable whose distribution is the result of multiplying two independent normally distributed random variables, one with variance 1 and the other with variance 2.

Definition 5.10. Denote I to be the identity operator on the full Fock space $\mathcal{F}(\mathcal{H}_\mathbb{C})$. We define $\mathcal{L}_Q$ to be the closure (L^2-norm) of the linear span of the set $\left\{ I_1^{(Q)}(f_1)\, I_1^{(Q)}(f_2)\, \cdots I_1^{(Q)}(f_p) \;\middle|\; f_1, f_2, \ldots, f_p \in \mathcal{H}_\mathbb{C}, p \in \mathbb{N} \right\} \cup \{I\}$.

In view of the above discussion, $\mathcal{L}_Q$ can be understood as the collection of all possible ways that we ever wish to be able to write down as our solution to a random physical phenomenon, which is driven by the classical Brownian motion.

Definition 5.11 (Stochastic integral $(W)I_n^{(Q)}(\cdot)$ defined in the Wiener sense)**.** Let $f \in \mathcal{H}_{\mathbb{C}}{}^{\otimes n}$. By Theorem 5.4, we have dyadic step functions

$$f^{(m)} = \sum_{i_n=1}^{m\,2^m} \cdots \sum_{i_2=1}^{m\,2^m} \sum_{i_1=1}^{m\,2^m} a_{i_1,i_2,\ldots,i_n}^{(m),(f)} \mathbb{1}_{\left[t_{i_1}^{(m)}\right) \times \left[t_{i_2}^{(m)}\right) \times \cdots \times \left[t_{i_n}^{(m)}\right)}, \qquad m = 1, 2, \ldots,$$

where $a_{i_1,i_2,\ldots,i_n}^{(m),(f)}$ are some complex numbers, such that $f^{(m)} \xrightarrow[m\to\infty]{\mathcal{H}_{\mathbb{C}}{}^{\otimes n}} f$. Then we define the n-th **stochastic integral** $(W)I_n^{(Q)}(\cdot)$ **with respective to the linear operator** Q in the Wiener sense to be given by

$$(W)I_n^{(Q)}(f) = \lim_{m\to\infty} \sum_{i_n=1}^{m\,2^m} \cdots \sum_{i_2=1}^{m\,2^m} \sum_{i_1=1}^{m\,2^m} a_{i_1,i_2,\ldots,i_n}^{(m),(f)} Q\left(\mathbb{1}_{\left[t_{i_1}^{(m)}\right)}\right) Q\left(\mathbb{1}_{\left[t_{i_2}^{(m)}\right)}\right) \cdots Q\left(\mathbb{1}_{\left[t_{i_n}^{(m)}\right)}\right).$$

Remark 5.12. We observe that $(W)I_1^{(Q)}(f) = I_1^{(Q)}(f)$ for any $f \in \mathcal{H}_{\mathbb{C}}$. Also, the existence/well-definiteness of $I_1^{(Q)}(f)$ is dealt with when we introduced the stochastic integral defined in the Itô sense previously.

Lemma 5.13. *Let $n \in \mathbb{N}$, and $f \in \mathcal{H}_{\mathbb{C}}{}^{\otimes n}$. Write $f = f_1 \otimes f_2 \otimes \cdots \otimes f_n$ for some $f_1, \ldots, f_n \in \mathcal{H}_{\mathbb{C}}$. Then*

$$(W)I_n^{(Q)}(f) = I_1^{(Q)}(f_1)\, I_1^{(Q)}(f_2) \,\cdots\, I_1^{(Q)}(f_n).$$

Proof. Let us recall the classical limit result on functions:

$$\lim_{x\to\infty} h(x), \lim_{x\to\infty} g(x) \text{ both exists} \quad\Longrightarrow\quad \left(\lim_{x\to\infty} h(x)\right) \left(\lim_{x\to\infty} g(x)\right) = \lim_{x\to\infty} h(x)\, g(x).$$

Simply mimic the proof of the above implication and we obtain an equivalent implication. Its equivalent hypothesis is true as the existence/well-definiteness of $I_1^{(Q)}(f)$ is dealt with when we introduced the stochastic integral defined in the Itô sense previously. ☐

Lemma 5.14.

$$\mathcal{L}_Q = \mathit{Im}((W)I_1^{(Q)}) + \mathit{Im}((W)I_2^{(Q)}) + \cdots + \mathit{Im}((W)I_n^{(Q)}) + \cdots,$$

where $Im(\cdot)$ denotes the image set. For example, $Im((W)I_3^{(Q)}) := \left\{ (W)I_3^{(Q)}(f) \;\middle|\; f \in \mathcal{H}_{\mathbb{C}}{}^{\otimes 3} \right\}$.

Proof. This follows from the definition of $\mathcal{L}_Q$ and Lemma 5.13. □

Let us recall that $L_B^2(\Omega)$ classically denotes the complex Hilbert space of square integrable functions on the probability space $(\Omega, \mathcal{F}^B, P)$, where $\mathcal{F}^B$ stands for the natural filtration generated by the Brownian motion $B(t)$, $t \geq 0$. It is clear from [5, Theorem 9.4.5 on p.161] that we have the next result.

Theorem 5.15.

$$L_B^2(\Omega) = \mathcal{L}_Q.$$

5.3 Orthogonality of stochastic integrals

In this section, we prove one of the most important result that demonstrates the strength of stochastic integrals defined in the Itô sense.

Theorem 5.16. *Let $r, n_1, n_2, \ldots, n_r \in \mathbb{N}$ and $f_1 \in \mathcal{H}_{\mathbb{C}}{}^{\otimes n_1}, f_2 \in \mathcal{H}_{\mathbb{C}}{}^{\otimes n_2}, \ldots, f_r \in \mathcal{H}_{\mathbb{C}}{}^{\otimes n_r}$ and denote $m = \max\{n_1, n_2, \ldots, n_r\}$. If $|\{ x \in \{n_1, n_2, \ldots, n_r\} \mid x = m \}| = 1$ and $2\,m > \sum_{j=1}^{r} n_j$, then $\mathbb{E}\left[I_{n_1}^{(Q)}(f_1)\, I_{n_2}^{(Q)}(f_2)\, \cdots\, I_{n_r}^{(Q)}(f_r) \right] = 0$.*

Proof. W.l.o.g. take $n_r = m$. It suffices to prove for the case when $f_1, f_2, \ldots, f_r$ are dyadic step functions. Let s be the largest number of steps amongst these step functions. Then, we have

$$I_{n_1}^{(Q)}(f_1) = \sum_{\substack{i_{n_1}=1 \\ i_{n_1}\neq i_{n_1-1} \\ \vdots \\ i_{n_1}\neq i_2 \\ i_{n_1}\neq i_1}}^{s\,2^s} \cdots \sum_{\substack{i_2=1 \\ i_2\neq i_1}}^{s\,2^s} \sum_{i_1=1}^{s\,2^s} a_{i_1,i_2,\ldots,i_{n_1}}^{(s),(f_1)} \; Q\left(\mathbb{1}_{\left[t_{i_1}^{(s)}\right)}\right) Q\left(\mathbb{1}_{\left[t_{i_2}^{(s)}\right)}\right) \cdots Q\left(\mathbb{1}_{\left[t_{i_{n_1}}^{(s)}\right)}\right),$$

$$\vdots$$

$$I_{n_r}^{(Q)}(f_r) = \sum_{\substack{i_{n_r}=1 \\ i_{n_r}\neq i_{n_r-1} \\ \vdots \\ i_{n_r}\neq i_2 \\ i_{n_r}\neq i_1}}^{s\,2^s} \cdots \sum_{\substack{i_2=1 \\ i_2\neq i_1}}^{s\,2^s} \sum_{i_1=1}^{s\,2^s} a_{i_1,i_2,\ldots,i_{n_r}}^{(s),(f_r)} \; Q\left(\mathbb{1}_{\left[t_{i_1}^{(s)}\right)}\right) Q\left(\mathbb{1}_{\left[t_{i_2}^{(s)}\right)}\right) \cdots Q\left(\mathbb{1}_{\left[t_{i_{n_r}}^{(s)}\right)}\right).$$

Therefore, $I_{n_1}^{(Q)}(f_1)\, I_{n_2}^{(Q)}(f_2) \cdots I_{n_r}^{(Q)}(f_r)$ must be a finite sum of terms which are some scalar (determined by $a_{i_1,i_2,\ldots,i_{n_1}}^{(s),(f_1)}, \ldots, a_{i_1,i_2,\ldots,i_{n_r}}^{(s),(f_r)}$) multiple of the terms of the form

$$\underbrace{Q\left(\mathbb{1}_{[\cdot)}\right) Q\left(\mathbb{1}_{[\cdot)}\right) \cdots Q\left(\mathbb{1}_{[\cdot)}\right)}_{\text{denote this as } \clubsuit} Q\left(\mathbb{1}_{\left[t_{i_1}^{(s)}\right)}\right)^{k_1} Q\left(\mathbb{1}_{\left[t_{i_2}^{(s)}\right)}\right)^{k_2} \cdots Q\left(\mathbb{1}_{\left[t_{i_{n_r}}^{(s)}\right)}\right)^{k_{n_r}},$$

where $k_1, k_2, \ldots, k_{n_r}$ are some positive integers. Notice that $\clubsuit$ is a product of $\sum_{j=1}^{r-1} n_j$-many $Q\left(\mathbb{1}_{[\cdot)}\right)$'s. Now, by the hypothesis of this theorem and the restrictions imposed on the region of summation of $I_{n_r}^{(Q)}(f_r)$, we see that, for each and every summand of $I_{n_1}^{(Q)}(f_1)\, I_{n_2}^{(Q)}(f_2) \cdots I_{n_r}^{(Q)}(f_r)$, there must exist at least one $k \in \{k_1, k_2, \ldots, k_{n_r}\}$ such that, even after the simplification (using Lemma 4.21), $k = 1$. W.l.o.g. take $k_1 = 1$. Then, by the linearity of $\mathbb{E}[\cdot]$, Corollary 4.24, and the fact that $\mathbb{E}\left[Q\left(\mathbb{1}_{[\cdot)}\right)\right] = 0$, we deduce the desired conclusion. $\square$

Corollary 5.17. *For any $p, q \in \mathbb{N}$ such that $p > q$ and $f \in \mathcal{H}_{\mathbb{C}}^{\otimes p}, g \in \mathcal{H}_{\mathbb{C}}^{\otimes q}$, we have* $\mathbb{E}\left[I_p^{(Q)}(f)\, I_q^{(Q)}(g)\right] = 0$

5.4 Classical chaos decomposition

Let $\Pi(n)$ be a lattice of partition of n-elements, cf. Definition 2.15. We define

$$\Pi^{(\leq 2)}(n) := \left\{ \{V_1, V_2, \ldots, V_s\} \in \Pi(n) \;\middle|\; \max_{1\leq i\leq s} |V_i| \leq 2 \right\},$$

i.e. the collection of partitions whose maximum block size is less than or equal to 2. Moreover, for each $\sigma \in \Pi(n), 1 \leq i \leq n$ define

$$\sigma^{(i)} := \{\, V \in \sigma \mid |V| = i \,\}.$$

Now we can make an important observation: For any positive integer n, the lattice $\Pi(n)$ can be partitioned into the following four classes

$$\begin{aligned}
&C_1 := \{\{\{1\},\{2\},\{3\},\ldots,\{n\}\}\},\\
&C_2 := \Pi^{(\leq 2)}(n)\backslash C_1,\\
&C_3 := \{\, \{V_1, V_2, \ldots, V_s\} \in \Pi(n) \mid \exists i \in \{1,2,\ldots,s\} \text{ s.t. } |V_i| \text{ is odd} \,\} \setminus \left(C_1 \bigcup C_2 \right),\\
&C_4 := \Pi(n)\backslash \left(C_1 \bigcup C_2 \bigcup C_3 \right).
\end{aligned}$$

Notice that every elements of class C_4 has at least one block of even size $q \geq 4$.

Now we shall demonstrate the relationships between the stochastic integrals defined in the Itô sense and in the Wiener sense. Take any $f, g, h \in \mathcal{H}_{\mathbb{C}}$. By Lemma 5.13, we have

$$\begin{aligned}
&I_1^{(Q)}(f)\, I_1^{(Q)}(g)\, I_1^{(Q)}(h)\\
&\quad = \lim_{m\to\infty} \left[\sum_{k=1}^{m\,2^m} \sum_{j=1}^{m\,2^m} \sum_{i=1}^{m\,2^m} \underbrace{a_i^{(m),(f)}\, a_j^{(m),(g)}\, a_k^{(m),(h)}\, Q\left(\mathbb{1}_{\left[t_i^{(m)}\right)}\right) Q\left(\mathbb{1}_{\left[t_j^{(m)}\right)}\right) Q\left(\mathbb{1}_{\left[t_k^{(m)}\right)}\right)}_{\text{denote this as } \mathcal{Q}_{i,j,k}} \right]
\end{aligned}$$

Next, we exploit the lattice structure of the region of summation, i.e. we have

$$\sum_{k=1}^{m\,2^m}\sum_{j=1}^{m\,2^m}\sum_{i=1}^{m\,2^m}\mathcal{Q}_{i,j,k} = \sum_{\{\{i\},\{j\},\{k\}\}}^{m\,2^m}\mathcal{Q}_{i,j,k} + \sum_{\{\{i,j\},\{k\}\}}^{m\,2^m}\mathcal{Q}_{i,j,k} + \sum_{\{\{i,k\},\{j\}\}}^{m\,2^m}\mathcal{Q}_{i,j,k} + \sum_{\{\{j,k\},\{i\}\}}^{m\,2^m}\mathcal{Q}_{i,j,k} + \sum_{\{\{i,j,k\}\}}^{m\,2^m}\mathcal{Q}_{i,j,k}, \tag{5.3}$$

where $\sum_{\{\{i\},\{j\},\{k\}\}}^{m\,2^m}$ denotes $\sum_{\substack{k=1\\k\neq j\\k\neq i}}^{m\,2^m}\sum_{\substack{j=1\\j\neq i}}^{m\,2^m}\sum_{i=1}^{m\,2^m}$; $\sum_{\{\{i,j\},\{k\}\}}^{m\,2^m}$ denotes $\sum_{\substack{k=1\\k\neq i}}^{m\,2^m}\sum_{j=i}^{m\,2^m}\sum_{i=1}^{m\,2^m}$; $\sum_{\{\{i,k\},\{j\}\}}^{m\,2^m}$ denotes $\sum_{k=i}^{m\,2^m}\sum_{\substack{j=1\\j\neq i}}^{m\,2^m}\sum_{i=1}^{m\,2^m}$; $\sum_{\{\{j,k\},\{i\}\}}^{m\,2^m}$ denotes $\sum_{k=j}^{m\,2^m}\sum_{j=1}^{m\,2^m}\sum_{\substack{i=1\\i\neq j}}^{m\,2^m}$; and $\sum_{\{\{i,j,k\}\}}^{m\,2^m}$ denotes $\sum_{k=i}^{m\,2^m}\sum_{j=i}^{m\,2^m}\sum_{i=1}^{m\,2^m}$. Now, we shall deal with each of the summations separately.

First of all, $\{\{i\},\{j\},\{k\}\}\in C_1$. By Definition 5.5, we get

$$\begin{aligned}
&\lim_{m\to\infty}\left[\sum_{\{\{i\},\{j\},\{k\}\}}^{m\,2^m}\mathcal{Q}_{i,j,k}\right]\\
&=\lim_{m\to\infty}\left[\sum_{\{\{i\},\{j\},\{k\}\}}^{m\,2^m} a_{i,j,k}^{(m),(f\otimes g\otimes h)}\, Q\left(\mathbb{1}_{\left[t_i^{(m)}\right)}\right)\, Q\left(\mathbb{1}_{\left[t_j^{(m)}\right)}\right)\, Q\left(\mathbb{1}_{\left[t_k^{(m)}\right)}\right)\right] = I_3^{(Q)}(f\otimes g\otimes h).
\end{aligned}$$

Secondly, we see that $\{\{i,j\},\{k\}\},\{\{i,k\},\{j\}\},\{\{j,k\},\{i\}\}\in C_2$. It suffices for us to know how to deal with the summation corresponding to, say $\{\{i,j\},\{k\}\}$.

$$\begin{aligned}
\lim_{m\to\infty}\left[\sum_{\{\{i,j\},\{k\}\}}^{m\,2^m}\mathcal{Q}_{i,j,k}\right] &= \lim_{m\to\infty}\left[\sum_{\{\{i,j\},\{k\}\}}^{m\,2^m} a_i^{(m),(f\,g)}\, a_k^{(m),(h)}\, \left(Q\left(\mathbb{1}_{\left[t_i^{(m)}\right)}\right)\right)^2 Q\left(\mathbb{1}_{\left[t_k^{(m)}\right)}\right)\right] \\
&= \left(\lim_{m\to\infty}\left[\sum_{\substack{i=1\\ i\neq k}}^{m\,2^m} a_i^{(m),(f\,g)}\, \left(Q\left(\mathbb{1}_{\left[t_i^{(m)}\right)}\right)\right)^2\right]\right)\left(\lim_{m\to\infty}\left[\sum_{k=1}^{m\,2^m} a_k^{(m),(h)}\, Q\left(\mathbb{1}_{\left[t_k^{(m)}\right)}\right)\right]\right) \\
&= \langle\, f\ ,\ g\,\rangle_{\mathcal{H}_{\mathbb{C}}}\ I_1^{(Q)}(h),
\end{aligned}$$

where the last equality holds in probability, cf. [5, Lemma 7.2.3.]. Similarly,

$$\begin{aligned}
\lim_{m\to\infty}\left[\sum_{\{\{i,k\},\{j\}\}}^{m\,2^m}\mathcal{Q}_{i,j,k}\right] &= \langle\, f\ ,\ h\,\rangle_{\mathcal{H}_{\mathbb{C}}}\ I_1^{(Q)}(g), \\
\lim_{m\to\infty}\left[\sum_{\{\{j,k\},\{i\}\}}^{m\,2^m}\mathcal{Q}_{i,j,k}\right] &= \langle\, g\ ,\ h\,\rangle_{\mathcal{H}_{\mathbb{C}}}\ I_1^{(Q)}(f).
\end{aligned}$$

Thirdly, we see that $\{\{i,j,k\}\}\in C_3$.

$$\begin{aligned}
\sum_{\{\{i,j,k\}\}}^{m\,2^m}\mathcal{Q}_{i,j,k} = \sum_{\{\{i,j,k\}\}}^{m\,2^m} a_i^{(m),(f\,g\,h)}\, \left(Q\left(\mathbb{1}_{\left[t_i^{(m)}\right)}\right)\right)^3 \\
= \frac{1}{m\,2^m}\sum_{i=1}^{m\,2^m}\left(\left(m\,2^m\, a_i^{(m),(f\,g\,h)}\right)^{\frac{1}{3}} Q\left(\mathbb{1}_{\left[t_i^{(m)}\right)}\right)\right)^3.
\end{aligned}$$

Observe that the random variables $\left\{\left(\left(m\,2^m\, a_i^{(m),(f\,g\,h)}\right)^{\frac{1}{3}} Q\left(\mathbb{1}_{\left[t_i^{(m)}\right)}\right)\right)^3\right\}_{i=1}^{m\,2^m}$ are i.i.d. with zero expectation. This is because the random variables $\left\{\left(m\,2^m\, a_i^{(m),(f\,g\,h)}\right)^{\frac{1}{3}} Q\left(\mathbb{1}_{\left[t_i^{(m)}\right)}\right)\right\}_{i=1}^{m\,2^m}$ are independent, normally distributed with mean 0 and the odd moments of a Gaussian random variable are zero. Therefore, by the law of large numbers, we get

$$\lim_{m\to\infty}\left[\sum_{\{\{i,j,k\}\}}^{m\,2^m} \mathcal{Q}_{i,j,k}\right] = 0,$$

with the equality holding almost surely. At this point, since all partitions in $\Pi(3)$ have been dealt with, we have obtained an explicit formula for expressing $(W)I_3^{(Q)}(f\otimes g\otimes h)$ in terms of $I_3^{(Q)}(\cdot)$, $I_2^{(Q)}(\cdot)$, and $I_1^{(Q)}(\cdot)$. In order to complete the general case when the dimension n can take any positive integer, let us take $e \in \mathcal{H}_{\mathbb{C}}$ and consider the partition $\{\{i,j,k,l\}\} \in C_4$. Rather than going at the numbers $\left\{ a_i^{(m),(e\,f\,g\,h)} \;\middle|\; m = 1,2,\dots, i = 1,2,\dots, m\,2^m \right\}$ straight away, we suppose that $e,f,g,h \in \mathcal{C}_c([0,\infty),\mathbb{C})$ (continuous complex-valued function with compact support defined on the half interval $[0,\infty)$). So,

$$\sum_{\{\{i,j,k,l\}\}}^{m\,2^m} \mathcal{Q}_{i,j,k,l} = \sum_{\{\{i,j,k,l\}\}}^{m\,2^m} a_i^{(m),(e\,f\,g\,h)} \left(Q\left(\mathbb{1}_{\left[t_i^{(m)}\right)}\right)\right)^4$$

Now, as $m \to \infty$, we get

$$\begin{aligned}\sum_{\{\{i,j,k,l\}\}}^{m\,2^m} \mathcal{Q}_{i,j,k,l} \approx \sum_{\{\{i,j,k,l\}\}}^{m\,2^m} a_i^{(m),(e\,f\,g\,h)} &\left(3\left(t_i^{(m)} - t_{i-1}^{(m)}\right)\left(t_i^{(m)} - t_{i-1}^{(m)}\right)\right)\\ &= \left(\frac{1}{m\,2^m}\right) \sum_{\{\{i,j,k,l\}\}}^{m\,2^m} 3\,a_i^{(m),(e\,f\,g\,h)} \left(t_i^{(m)} - t_{i-1}^{(m)}\right),\end{aligned}$$

where the factor 3 comes from the 4th moment of normal distribution, cf. Lemma 4.17; hence

$$\lim_{m\to\infty}\left[\sum_{\{\{i,j,k,l\}\}}^{m\,2^m} \mathcal{Q}_{i,j,k,l}\right] = \lim_{m\to\infty}\left[\left(\frac{1}{m\,2^m}\right)\int_0^\infty 3\,e(x)\,f(x)\,g(x)\,h(x)\,dx\right] = 0, \tag{5.4}$$

as $\int_0^\infty e(x)\,f(x)\,g(x)\,h(x)\,dx$ is finite following from $e,f,g,h \in \mathcal{C}_c([0,\infty),\mathbb{C})$. Since $\mathcal{C}_c([0,\infty),\mathbb{C})$ is dense in $\mathcal{H}_{\mathbb{C}}$, it follows that Equation (5.4) holds for general $e,f,g,h \in \mathcal{H}_{\mathbb{C}}$. Finally collecting the techniques for dealing with terms in C_1, C_2, C_3, and C_4 gives the following lemma

Lemma 5.18. *Let n be a positive integer, and $f_1, f_2, \dots, f_n \in \mathcal{H}_{\mathbb{C}}$. Then*

$$\prod_{l=1}^{n} I_1^{(Q)}(f_l) = \sum_{\sigma \in \Pi^{(\leq 2)}(n)} \left[I_{|\sigma^{(1)}|}^{(Q)} \left(\bigotimes_{\{k\} \in \sigma^{(1)}} f_k \right) \prod_{\{i,j\} \in \sigma^{(2)}} \langle f_i \, , \, f_j \rangle_{\mathcal{H}_{\mathbb{C}}} \right].$$

Theorem 5.19.

$$\mathcal{L}_Q = Im(I_1^{(Q)}) \oplus Im(I_2^{(Q)}) \oplus \cdots \oplus Im(I_n^{(Q)}) \oplus \cdots .$$

Proof. It follows from Lemma 5.18, Lemma 5.14, and Corollary 5.17. □

It is worth noting that our product formula is different from the usual product formula known in classical probability, cf. [8, Proposition 1.1.3].

Proposition 5.20. *Let f be a symmetric function of p variables, and g be a symmetric function of q variables. Then*

$$I_p^{(Q)}(f)\, I_q^{(Q)}(g) = \sum_{r=0}^{\min\{p,q\}} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}^{(Q)}(f \otimes_r g),$$

where

$$f \otimes_r g := \int_0^\infty \cdots \int_0^\infty f(t_1, \dots, t_{p-r}, s_1, \dots, s_r)\, g(t_{p+1}, \dots, t_{p+q-r}, s_1, \dots, s_r)\, ds_1 \cdots ds_r.$$

We shall not give a proof to the above lemma. Instead, we shall give a proof to the following specific case in our notation.

Proposition 5.21. *Let $p \in \mathbb{N}$, $f \in \mathcal{H}_{\mathbb{C}}^{\otimes p}$, and $g \in \mathcal{H}_{\mathbb{C}}$. Then*

$$\begin{aligned} I_p^{(Q)}(f)\, I_1^{(Q)}(g) &= I_{p+1}^{(Q)}(f \otimes g) + I_{p-1}^{(Q)}\left(\int_0^\infty f(s, x_2, \dots, x_{p-1}, x_p)\, g(s) ds \right) + \cdots \\ &\cdots + I_{p-1}^{(Q)}\left(\int_0^\infty f(x_1, x_2, \dots, s, x_p)\, g(s) ds \right) + I_{p-1}^{(Q)}\left(\int_0^\infty f(x_1, x_2, \dots, x_{p-1}, s)\, g(s) ds \right). \end{aligned} \tag{5.5}$$

Proof. By Definition 5.5, we may write

$$I_p^{(Q)}(f) = \lim_{m\to\infty} \sum_{\substack{i_p=1\\ i_p\neq i_{p-1}\\ \vdots\\ i_p\neq i_2\\ i_p\neq i_1}}^{m\,2^m} \cdots \sum_{\substack{i_2=1\\ i_2\neq i_1}}^{m\,2^m} \sum_{i_1=1}^{m\,2^m} a^{(m),(f)}_{i_1,i_2,\ldots,i_p}\, Q\left(\mathbb{1}_{\left[t^{(m)}_{i_1}\right)}\right)\, Q\left(\mathbb{1}_{\left[t^{(m)}_{i_2}\right)}\right) \cdots Q\left(\mathbb{1}_{\left[t^{(m)}_{i_n}\right)}\right);$$

$$I_1^{(Q)}(g) = \lim_{m\to\infty} \sum_{i_{p+1}=1}^{m\,2^m} a^{(m),(g)}_{i_{p+1}}\, Q\left(\mathbb{1}_{\left[t^{(m)}_{i_{p+1}}\right)}\right). \tag{5.6}$$

Now, we focus on the region of summation of Equation (5.6), and we see that

$$\sum_{i_{p+1}=1}^{m\,2^m} = \sum_{\substack{i_{p+1}=1\\ i_{p+1}\neq i_p\\ \vdots\\ i_{p+1}\neq i_2\\ i_{p+1}\neq i_1}}^{m\,2^m} + \sum_{\substack{i_{p+1}=1\\ i_{p+1}=i_p}}^{m\,2^m} + \sum_{\substack{i_{p+1}=1\\ i_{p+1}=i_{p-1}}}^{m\,2^m} + \cdots + \sum_{\substack{i_{p+1}=1\\ i_{p+1}=i_1}}^{m\,2^m}.$$

It is now obvious how we obtain the first term on the right hand side of Equation (5.5).

As for the other terms, it suffices to show how to obtain one of them.

$$
\begin{aligned}
& I_p^{(Q)}(f)\left(\lim_{m\to\infty}\sum_{\substack{i_{p+1}=1\\ i_{p+1}=i_1}}^{m\,2^m} a_{i_{p+1}}^{(m),(g)}\, Q\left(\mathbb{1}_{\left[t_{i_{p+1}}^{(m)}\right)}\right)\right)\\
&= \lim_{m\to\infty}\sum_{\substack{i_p=1\\ i_p\neq i_{p-1}\\ \vdots\\ i_p\neq i_2\\ i_p\neq i_1}}^{m\,2^m}\cdots\sum_{\substack{i_2=1\\ i_2\neq i_1}}^{m\,2^m}\sum_{i_1=1}^{m\,2^m} a_{i_1,i_2,\ldots,i_p}^{(m),(f)}\, a_{i_1}^{(m),(g)}\left(Q\left(\mathbb{1}_{\left[t_{i_1}^{(m)}\right)}\right)\right)^2 Q\left(\mathbb{1}_{\left[t_{i_2}^{(m)}\right)}\right)\cdots Q\left(\mathbb{1}_{\left[t_{i_n}^{(m)}\right)}\right)\\
&\approx \lim_{m\to\infty}\sum_{\substack{i_p=1\\ i_p\neq i_{p-1}\\ \vdots\\ i_p\neq i_2\\ i_p\neq i_1}}^{m\,2^m}\cdots\sum_{\substack{i_2=1\\ i_2\neq i_1}}^{m\,2^m}\sum_{i_1=1}^{m\,2^m} a_{i_1,i_2,\ldots,i_p}^{(m),(f)}\, a_{i_1}^{(m),(g)}\left(\frac{1}{2^m}\right) Q\left(\mathbb{1}_{\left[t_{i_2}^{(m)}\right)}\right)\cdots Q\left(\mathbb{1}_{\left[t_{i_n}^{(m)}\right)}\right)\\
&= \lim_{m\to\infty}\sum_{\substack{i_p=1\\ i_p\neq i_{p-1}\\ \vdots\\ i_p\neq i_2}}^{m\,2^m}\cdots\sum_{i_2=1}^{m\,2^m} a_{i_2,\ldots,i_p}^{(m),\left(\int_0^\infty f(s,x_2,\ldots,x_{p-1},x_p)\,g(s)ds\right)}\, Q\left(\mathbb{1}_{\left[t_{i_2}^{(m)}\right)}\right)\cdots Q\left(\mathbb{1}_{\left[t_{i_n}^{(m)}\right)}\right),
\end{aligned}
$$

where the last equality follows from

$$
\lim_{m\to\infty}\left[\sum_{i_1=1}^{m\,2^m} a_{i_1,i_2,\ldots,i_p}^{(m),(f)}\, a_{i_1}^{(m),(g)}\left(\frac{1}{2^m}\right)\right] = \int_0^\infty f(s,x_2,\ldots,x_{p-1},x_p)\,g(s)ds,
$$

and $\approx$ follows again from [5, Lemma 7.2.3.]. $\square$

Corollary 5.22. *Let $p\in\mathbb{N}$, $f\in{\mathcal{H}_{\mathbb{C}}}^{\otimes p}$ be symmetric function, and $g\in\mathcal{H}_{\mathbb{C}}$. Then*

$$
I_p^{(Q)}(f)\, I_1^{(Q)}(g) = I_{p+1}^{(Q)}(f\otimes g) + p\, I_{p-1}^{(Q)}\left(f\otimes_1 g\right).
$$

In fact the proof for Proposition 5.20 in [8, Proposition 1.1.3] is an inductive proof on q with Corollary 5.22 being the inductive base.

We finish this section with a discussion on what Theorem 5.19 entails in practice. Suppose, through some methods, we find a solution $Z \in \mathcal{L}_Q$ to an physical phenomenon. By Theorem 5.19, we may write

$$Z = c_1 + I_1^{(Q)}(f_1) + I_2^{(Q)}(f_2) + \cdots ,$$

for some scalar c_1, and $f_1 \in {\mathcal{H}_{\mathbb{C}}}^{\otimes 1}, f_2 \in {\mathcal{H}_{\mathbb{C}}}^{\otimes 2}, \ldots$. Then, by Corollary 5.17,

$$\mathbb{E}\left[Z^2\right] = \mathbb{E}\left[{I_1^{(Q)}(f_1)}^2\right] + \mathbb{E}\left[{I_2^{(Q)}(f_2)}^2\right] + \mathbb{E}\left[{I_3^{(Q)}(f_3)}^2\right] + \cdots ,$$

for any $r \in \mathbb{N}$. This type of formula does not hold for stochastic integrals $(W)I_n^{(Q)}$ defined in the Wiener sense. However, if one wishes to have some feasible interpretations of Z, like the one sketched at the beginning of Section 5.2, the stochastic integrals $I_n^{(Q)}$ actually does not allow it. But, this is not too much of a trouble—Due to the lattice structure of the proof of Lemma 5.18, we can use the Möbius inversion formula to obtain the following.

Lemma 5.23. *Let n be a positive integer, and $f_1, f_2, \ldots, f_n \in L^2_{\mathbb{R}}([0,\infty))$. Then*

$$I_n\left(\bigotimes_{k=1}^{n} f_k\right) = \sum_{\sigma \in \Pi^{(\leq 2)}(n)} \left[(-1)^{|\sigma^{(2)}|} \prod_{\{i,j\} \in \sigma^{(2)}} \left(\int_0^\infty f_i(x) f_j(x)\, dx\right) \prod_{\{k\} \in \sigma^{(1)}} I(f_k)\right].$$

Proof. Let us just demonstrate where the Möbius inversion formula is being used in trying to obtain the above formula, as the reverse of Lemma 5.18. If we use the same example for proving Lemma 5.18, namely take $n = 3$, then, in the same spirit as Equation (5.3), we write

$$
\begin{aligned}
\sum_{\{\{i\},\{j\},\{k\}\}}^{m\,2^m} = (1)\left(\sum_{\geq\{\{i\},\{j\},\{k\}\}}^{m\,2^m}\right) + \\
(-1)\left(\sum_{\geq\{\{i,j\},\{k\}\}}^{m\,2^m}\right) + (-1)\left(\sum_{\geq\{\{i,k\},\{j\}\}}^{m\,2^m}\right) + (-1)\left(\sum_{\geq\{\{j,k\},\{i\}\}}^{m\,2^m}\right) + \\
(2)\left(\sum_{\geq\{\{i,j,k\}\}}^{m\,2^m}\right),
\end{aligned}
\tag{5.7}
$$

where

[i] the coefficients in front of the summations are determined by the Möbius function, which are known explicitly for every dimension $n = 1, 2, 3, \ldots,$ and

[ii] the partial order $\geq$ used in each of the summation is given by the refinement on the lattice of partitions $\Pi(n)$ of n-elements, e.g.

$$
\sum_{\geq\{\{i,j\},\{k\}\}}^{m\,2^m} = \sum_{\{\{i,j\},\{k\}\}}^{m\,2^m} + \sum_{\{\{i,j,k\}\}}^{m\,2^m} \left(= \sum_{\substack{k=1\\k\neq i}}^{m\,2^m} \sum_{j=i}^{m\,2^m} \sum_{i=1}^{m\,2^m} + \sum_{k=i}^{m\,2^m} \sum_{j=i}^{m\,2^m} \sum_{i=1}^{m\,2^m} \right).
$$

Finally, we proceed in the same manner as the proof of Lemma 5.18 and notice that the summations on the right hand of Equation (5.7) now corresponds to the stochastic integrals defined in the Wiener sense, cf. Definition 5.11. $\square$

It is worth noting that the types of formulas in Lemma 5.18 and Lemma 5.23 and their proofs are inspired by the work [9].

Chapter 6

Free chaos decomposition

Throughout this entire chapter, let us denote $\mathcal{H}$ be the collection of real-valued square integrable functions on the half interval $[0, \infty)$, equipped with the usual L^2-norm. Furthermore, let $\mathcal{H}_{\mathbb{C}}$ be its complexification.

6.1 Free stochastic integrals

The reader is advised to compare the next definition to Definition 5.5.

Definition 6.1 (Free stochastic integral $I_n^{(X)}(\cdot)$ defined in the Itô sense)**.** Let $f \in {\mathcal{H}_{\mathbb{C}}}^{\otimes n}$. By Theorem 5.4, we have dyadic step functions

$$f^{(m)} = \sum_{i_n=1}^{m\,2^m} \cdots \sum_{i_2=1}^{m\,2^m} \sum_{i_1=1}^{m\,2^m} a^{(m),(f)}_{i_1,i_2,\ldots,i_n}\, \mathbb{1}_{\left[t^{(m)}_{i_1}\right) \times \left[t^{(m)}_{i_2}\right) \times \cdots \times \left[t^{(m)}_{i_n}\right)}, \qquad m = 1, 2, \ldots,$$

where $a^{(m),(f)}_{i_1,i_2,\ldots,i_n}$ are some complex numbers, such that $f^{(m)} \xrightarrow[m\to\infty]{{\mathcal{H}_{\mathbb{C}}}^{\otimes n}} f$. Then we define the n-th **free stochastic integral** $I_n^{(X)}(\cdot)$ **with respective to the linear operator** X in the Itô sense to be given by

$$I_n^{(X)}(f) = \lim_{m\to\infty} \sum_{\substack{i_n=1\\ i_n\neq i_{n-1}\\ \vdots\\ i_n\neq i_2\\ i_n\neq i_1}}^{m\,2^m} \cdots \sum_{\substack{i_2=1\\ i_2\neq i_1}}^{m\,2^m} \sum_{i_1=1}^{m\,2^m} a_{i_1,i_2,\ldots,i_n}^{(m),(f)}\, X\left(\mathbb{1}_{\left[t_{i_1}^{(m)}\right)}\right) X\left(\mathbb{1}_{\left[t_{i_2}^{(m)}\right)}\right) \cdots X\left(\mathbb{1}_{\left[t_{i_n}^{(m)}\right)}\right).$$

Although there might exist several sequence $\{f^{(m)}\}_{m=1}^{\infty}$ of dyadic step functions approaching a general $f \in \mathcal{H}_{\mathbb{C}}{}^{\otimes n}$, the stochastic $I_n(f)$ always exists and is unique in the sense that it is the L^2-limit of the Cauchy sequence $\{I_n\left(f^{(m)}\right)\}_{m=1}^{\infty}$. In other words, take any positive integers $p > q$. We want to show that

$$\mathbb{E}\left[\left(I_n^{(X)}(f^{(p)}) - I_n^{(X)}(f^{(q)})\right)^2\right] \xrightarrow[p,q\to\infty]{} 0.$$

Thanks to Definition 5.3, we have

$$I_n^{(X)}(f^{(p)}) = \sum_{\substack{i_n=1\\ i_n\neq i_{n-1}\\ \vdots\\ i_n\neq i_2\\ i_n\neq i_1}}^{p\,2^p} \cdots \sum_{\substack{i_2=1\\ i_2\neq i_1}}^{p\,2^p} \sum_{i_1=1}^{p\,2^p} a_{i_1,i_2,\ldots,i_n}^{(p),(f^{(p)})}\, X\left(\mathbb{1}_{\left[t_{i_1}^{(p)}\right)}\right) X\left(\mathbb{1}_{\left[t_{i_2}^{(p)}\right)}\right) \cdots X\left(\mathbb{1}_{\left[t_{i_n}^{(p)}\right)}\right),$$

and

$$I_n^{(X)}(f^{(q)}) = \sum_{\substack{j_n=1\\ j_n\neq j_{n-1}\\ \vdots\\ j_n\neq j_2\\ j_n\neq j_1}}^{p\,2^p} \cdots \sum_{\substack{j_2=1\\ j_2\neq j_1}}^{p\,2^p} \sum_{j_1=1}^{p\,2^p} a_{j_1,j_2,\ldots,j_n}^{(p),(f^{(q)})}\, X\left(\mathbb{1}_{\left[t_{j_1}^{(p)}\right)}\right) X\left(\mathbb{1}_{\left[t_{j_2}^{(p)}\right)}\right) \cdots X\left(\mathbb{1}_{\left[t_{j_n}^{(p)}\right)}\right).$$

Plugging them into the expression $\left(I_n^{(X)}(f^{(p)}) - I_n^{(X)}(f^{(q)})\right)^2$, we see that

$\left(I_n^{(X)}(f^{(p)}) - I_n^{(X)}(f^{(q)})\right)^2$ is a finite sum of terms of the form

$$c_1\, X\left(\mathbb{1}_{\left[t_{i_1}^{(p)}\right)}\right) \cdots X\left(\mathbb{1}_{\left[t_{i_n}^{(p)}\right)}\right) X\left(\mathbb{1}_{\left[t_{j_1}^{(p)}\right)}\right) \cdots X\left(\mathbb{1}_{\left[t_{j_n}^{(p)}\right)}\right),$$

for some scalar c_1, which is determined by $a_{i_1,i_2,\ldots,i_n}^{(p),(f^{(p)})}$ and $a_{j_1,j_2,\ldots,j_n}^{(p),(f^{(q)})}$. Here, we observe that the set $\left\{\mathbb{1}_{\left[t_1^{(p)}\right)}, \mathbb{1}_{\left[t_2^{(p)}\right)}, \ldots, \mathbb{1}_{\left[t_{p\,2p}^{(p)}\right)}\right\}$ is a collection of mutually orthogonal vectors in $\mathcal{H}_{\mathbb{C}}$. Thanks to the linearity of $\mathbb{E}\left[\cdot\right]$, it is now sufficient to verity that

$$\mathbb{E}\left[\left[X\left(\mathbb{1}_{\left[t_{i_1}^{(p)}\right)}\right) \cdots X\left(\mathbb{1}_{\left[t_{i_n}^{(p)}\right)}\right) X\left(\mathbb{1}_{\left[t_{j_1}^{(p)}\right)}\right) \cdots X\left(\mathbb{1}_{\left[t_{j_n}^{(p)}\right)}\right)\right]\right] = 0. \tag{6.1}$$

In the event $\left[t_{i_n}^{(p)}\right) \neq \left[t_{j_1}^{(p)}\right)$, Equation (6.1) follows from Theorem 3.12 and Lemma 3.10. In the case $\left[t_{i_n}^{(p)}\right) = \left[t_{j_1}^{(p)}\right)$, we shall use the fact that $X(h) := L^+(h) + L^-(h)$ and expand the expression $X\left(\mathbb{1}_{\left[t_{i_1}^{(p)}\right)}\right) \cdots X\left(\mathbb{1}_{\left[t_{i_n}^{(p)}\right)}\right) X\left(\mathbb{1}_{\left[t_{j_1}^{(p)}\right)}\right) \cdots X\left(\mathbb{1}_{\left[t_{j_n}^{(p)}\right)}\right)$ as a finite sum of terms L^+ and L^-. Then by the proof of Lemma 3.11 we see that the term $X\left(\mathbb{1}_{\left[t_{i_n}^{(p)}\right)}\right) X\left(\mathbb{1}_{\left[t_{j_1}^{(p)}\right)}\right)$ can be written as a finite sum of terms that is a scalar multiple of $L^+\left(\mathbb{1}_{\left[t_{j_1}^{(p)}\right)}\right)^{r_1} L^-\left(\mathbb{1}_{\left[t_{j_1}^{(p)}\right)^{r_2}}\right)^{r_2}$, where $r_1 = 1, 2$ and $r_2 = 1, 2$. Finally, by Theorem 3.9, we deduce that Equation (6.1) holds again.

Thus, we have obtained our free Brownian motion, explicitly written as $I_1^{(X)}\left(\mathbb{1}_{[0,t)}\right)$, $t \in [0, \infty)$. It is readily verifiable that this linear operators on the full Fock spaces $\mathcal{F}(\mathcal{H}_{\mathbb{C}})$ satisfies every single requirement of the following definitions of free Brownian motion, written in terms of random variables.

Definition 6.2. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. A collection $\{W_t\}_{t\geq 0}$ of self-adjoint random variables $W_t \in \mathcal{A}$, $t \in [0, \infty)]$ is called a **free Brownian motion** if

- [i] the random variable $W_0 = 0$,
- [ii] for any $0 \leq s < t$, the random variable $W_t - W_s$ has the Wigner semicircle distribution

of mean 0 and variance $t-s$, i.e. for any $a \leq b$, we have

$$P\left(\{a \leq W_t - W_s \leq b\}\right) = \frac{1}{2\,\pi\,(t-s)} \int_a^b \sqrt{4\,(t-s) - x^2}\, dx,$$

and

[iii] $\{W_t\}_{t\geq 0}$ has independent increment, i.e., for any $0 \leq t_1 < t_2 < \cdots < t_n$, the random variables

$$W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}},$$

are freely independent.

Remark 6.3. The free central limit theorem being the reason for replacing the normally distributed increment by the semicircular distributed increment.

6.2 Free solution spaces

Here we follow the same development as we dealt with in the classical case; hence the next definition is very similar to Definition 5.10

Definition 6.4. Denote I to be the identity operator on the full Fock space $\mathcal{F}(\mathcal{H}_{\mathbb{C}})$. We define $\mathcal{L}_X$ to be the closure (L^2-norm) of the linear span of the set $\left\{ I_1^{(X)}(f_1)\, I_1^{(X)}(f_2) \cdots I_1^{(X)}(f_p) \,\middle|\, f_1, f_2, \ldots, f_p \in \mathcal{H}_{\mathbb{C}}, p \in \mathbb{N} \right\} \cup \{I\}$.

Definition 6.5 (Free stochastic integral $(W)I_n^{(X)}(\cdot)$ defined in the Wiener sense)**.** Let $f \in {\mathcal{H}_{\mathbb{C}}}^{\otimes n}$. By Theorem 5.4, we have dyadic step functions

$$f^{(m)} = \sum_{i_n=1}^{m\,2^m} \cdots \sum_{i_2=1}^{m\,2^m} \sum_{i_1=1}^{m\,2^m} a_{i_1,i_2,\ldots,i_n}^{(m),(f)}\, \mathbb{1}_{\left[t_{i_1}^{(m)}\right) \times \left[t_{i_2}^{(m)}\right) \times \cdots \times \left[t_{i_n}^{(m)}\right)}, \qquad m = 1, 2, \ldots,$$

where $a_{i_1,i_2,\ldots,i_n}^{(m),(f)}$ are some complex numbers, such that $f^{(m)} \xrightarrow[m\to\infty]{{\mathcal{H}_{\mathbb{C}}}^{\otimes n}} f$. Then we define the n-th **free stochastic integral** $(W)I_n^{(X)}(\cdot)$ **with respective to the linear operator** X

in the Wiener sense to be given by

$$(W)I_n^{(X)}(f) = \lim_{m\to\infty} \sum_{i_n=1}^{m\,2^m} \cdots \sum_{i_2=1}^{m\,2^m} \sum_{i_1=1}^{m\,2^m} a_{i_1,i_2,\ldots,i_n}^{(m),(f)} \, X\left(\mathbb{1}_{\left[t_{i_1}^{(m)}\right)}\right) X\left(\mathbb{1}_{\left[t_{i_2}^{(m)}\right)}\right) \cdots X\left(\mathbb{1}_{\left[t_{i_n}^{(m)}\right)}\right).$$

Remark 6.6. We observe that $(W)I_1^{(X)}(f) = I_1^{(X)}(f)$ for any $f \in \mathcal{H}_{\mathbb{C}}$. Also, the existence/well-definiteness of $I_1^{(X)}(f)$ is dealt with when we introduced the free stochastic integral defined in the Itô sense previously.

Lemma 6.7. *Let $n \in \mathbb{N}$, and $f \in \mathcal{H}_{\mathbb{C}}{}^{\otimes n}$. Write $f = f_1 \otimes f_2 \otimes \cdots \otimes f_n$ for some $f_1, \ldots, f_n \in \mathcal{H}_{\mathbb{C}}$. Then*

$$(W)I_n^{(X)}(f) = I_1^{(X)}(f_1)\, I_1^{(X)}(f_2) \, \cdots \, I_1^{(X)}(f_n).$$

Lemma 6.8.

$$\mathcal{L}_X = Im((W)I_1^{(X)}) + Im((W)I_2^{(X)}) + \cdots + Im((W)I_n^{(X)}) + \cdots,$$

where $Im(\cdot)$ denotes the image set. For example, $Im((W)I_2^{(X)}) := \left\{ (W)I_2^{(X)}(f) \;\middle|\; f \in \mathcal{H}_{\mathbb{C}}{}^{\otimes 2} \right\}$.

6.3 Orthogonality of free stochastic integrals

Theorem 6.9. *Let $r, n_1, n_2, \ldots, n_r \in \mathbb{N}$ and $f_1 \in \mathcal{H}_{\mathbb{C}}{}^{\otimes n_1}, f_2 \in \mathcal{H}_{\mathbb{C}}{}^{\otimes n_2}, \ldots, f_r \in \mathcal{H}_{\mathbb{C}}{}^{\otimes n_r}$ and denote $m = \max\{n_1, n_2, \ldots, n_r\}$. If $r \geq 2$ and $m \geq 2$, then $\mathbb{E}\left[I_{n_1}^{(X)}(f_1)\, I_{n_2}^{(X)}(f_2) \, \cdots \, I_{n_r}^{(X)}(f_r)\right] = 0$.*

Proof. The proof starts in the exact same way as the proof for Theorem 5.16, and then we are led to consider terms of the form

$$Q\left(\mathbb{1}_{[\cdot)}\right)^{k_1} Q\left(\mathbb{1}_{[\cdot)}\right)^{k_2} \cdots Q\left(\mathbb{1}_{[\cdot)}\right)^{k_{n_1+n_2+\cdots+n_r}},$$

where $k_1, k_2, \ldots, k_{n_1+n_2+\cdots+n_r}$ are some positive integers. Then use the linearity of $\mathbb{E}[\cdot]$, Theorem 3.12, Lemma 3.10, the proof of Lemma 3.11, and Theorem 3.9 to finish the proof. $\square$

6.4 Free chaos decomposition

Lemma 6.10. *Let n be a positive integer, and $f_1, f_2, \ldots, f_n \in \mathcal{H}_{\mathbb{C}}$. Then*

$$\prod_{l=1}^{n} I_1^{(X)}(f_l) = \sum_{\sigma \in \Pi^{(\leq 2)}(n)} \left[I_{|\sigma^{(1)}|}^{(X)} \left(\bigotimes_{\{k\} \in \sigma^{(1)}} f_k \right) \prod_{\{i,j\} \in \sigma^{(2)}} \langle\, f_i \,,\, f_j \,\rangle_{\mathcal{H}_{\mathbb{C}}} \right].$$

Proof. The combinatorial aspect of the proof is exactly the same as the proof for Lemma 5.18. The only differences are:

[i] For partitions of type C_2, we use Lemma 3.10 to obtain

$$\lim_{m\to\infty} \left(X \left(\mathbb{1}_{\left[t_i^{(m)}\right)} \right) \right)^2 = \lim_{m\to\infty} \frac{1}{2^m}.$$

[ii] For partitions of type C_3, we use the free law of large numbers.

□

Theorem 6.11.

$$\mathcal{L}_X = Im(I_1^{(X)}) \oplus Im(I_2^{(X)}) \oplus \cdots \oplus Im(I_n^{(X)}) \oplus \cdots .$$

Proof. It follows from Lemma 6.10,Theorem 6.9, and Lemma 6.8. □

It is worth noting that the above theorem is consistent with the free chaos decomposition appearing in [3, p. 399].

Future outlook

In light of our parallel treatment of the classical and free chaos decomposition of the respective solution spaces, one of the most natural directions in which this project can continue proceeding in the future is to ask how we can recover the crucial information

$$\left\{a^{(m),(f)}_{i_1,i_2,\ldots,i_n}\right\}^{m\,2^m}_{i_1,i_2,\ldots,i_n=1}, \qquad m \in \mathbb{N},$$

in Definitions 5.5 and Definitions 6.1 from a stochastic process, cf. Definition 5.8. Let us clarify what this direction is about in the following way. Suppose we are given a collection $\{Y_t\}_{t\in[0,\infty)}$ of random variables Y_t, ordered by a time parameter t. Moreover, we further impose that these random variables Y_t, $t \in [0,\infty)$, are related in such a way that they are the results of integrating (in the sense of Definitions 5.5 or Definitions 6.1) $f\,\mathbb{1}_{[0,t)}$, $t \in [0,\infty)$, for some scalar-valued square integrable functions f defined on the half interval $[0,\infty)$, respectively. (Incidentally, this means that this collection $\{Y_t\}_{t\in[0,\infty)}$ is a subset of a solution space, cf. Definition 5.10 or Definition 6.4.) Now our direction of investigation can be stated as: how we can recover this f, given that we have such stochastic process $\{Y_t\}_{t\in[0,\infty)}$.

Hopefully, with the help of this thesis, the following awkward-looking formula now suggests itself as one of the very first few ansatzes.

$$f(t) \overset{?}{=} \sqrt{\lim_{h\to 0} \frac{\mathbb{E}\left[(Y_{t+h} - Y_t)^2\right]}{h^2}}, \tag{6.2}$$

for each $t \in [0,\infty)$. Indeed, this inverse formula works for any step function. As a demonstration, take f to be some step function, say

$$f = 2\,\mathbb{1}_{[0,1)} + 3\,\mathbb{1}_{[1,3)},$$

and take $Y_t := I_1^{(Q)}\left(f\,\mathbb{1}_{[0,t)}\right)$. Then Equation (6.2) is readily verifiable from Definitions 5.5, Corollary 4.24, Remark 1.21, and Lemma 4.17.

This idea of looking for the inverse of an integration, or more precisely the time derivative of a stochastic process driven by Brownian motion, is not a new one. In fact, in classical probability, it falls into the scope of white noise analysis. It would be interesting to be able to devise a definition that represents faithfully the notion of white noise and to explore its connection to Equation (6.2), cf. [11].

Also, as mentioned at the end of Chapter 1, the present thesis deals with only two types of distributions, namely the normal distribution and the Wigner semicircle law. Another well-known pair of distributions that exhibits the similarity between classical and free probability is the Poisson and free Poisson distributions. Therefore, another direction of investigation could be to mimic the approach of this thesis and ask for a chaos decomposition result for the Poisson-related stochastic integrals, cf. [2].

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