Linearly-dense classes of matroids with bounded branch-width

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Let M be a non-empty minor-closed class of matroids with bounded branch-width that does not contain arbitrarily large simple rank-2 matroids. For each non-negative integer n we denote by $ex(n)$ the size of the largest simple matroid in M that has rank at most n. We prove that there exist a rational number Δ and a periodic sequence $(a_0, a_1, \ldots)$ of rational numbers such that $ex(n) = \Delta n + a_n$ for each sufficiently large integer n.

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Chapter 1

Introduction

This thesis analyzes the extremal members of linearly-dense minor-closed classes of matroids. We say that a class of matroids is *minor-closed* if it is closed under taking minors and under isomorphism. In addition, we say that M has an N*-minor* when M has a minor isomorphic to N. For a class $\mathcal{M}$ of matroids, we let $ex_{\mathcal{M}}(n)$, denote the maximum number of elements of a simple rank-n matroid in $\mathcal{M}$. The following theorem of Geelen, Kabell, Kung, and Whittle [3] describes the possible behaviors of $ex_{\mathcal{M}}$ for minor-closed classes of matroids.

Theorem 1.1 (Growth Rate Theorem)**.** *Let $\mathcal{M}$ be a minor-closed class of matroids. Then exactly one of the following is true:*

(i) There exists a real number c such that for all $n \in \mathbb{N}$, $ex_{\mathcal{M}}(n) \leq cn$.

(ii) There exists a real number c such that for all $n \in \mathbb{N}$, $ex_{\mathcal{M}}(n) \leq cn^2$ and $\mathcal{M}$ contains all graphic matroids.

(iii) There exists a prime-power q and a real number c such that for all $n \in \mathbb{N}$, $ex_{\mathcal{M}}(n) \leq cq^n$ and $\mathcal{M}$ contains all matroids representable over $GF(q)$.

(iv) $\mathcal{M}$ contains U_n^2 for all $n \in \mathbb{N}$, and thus has unbounded density.

We say that a minor-closed class $\mathcal{M}$ of matroids is *linearly-dense* if there exists a constant c such that $|M| \leq cr(M)$ for any simple matroid $M \in \mathcal{M}$, that is, if $\mathcal{M}$ is of type (i) in the Growth Rate Theorem. Geelen, Kung, and Whittle [3] showed that $\mathcal{M}$ is linearly-dense if and only if there exists a graph G and an integer n such that $\mathcal{M}$ contains neither the graphic matroid of G, $M(G)$, nor the n-point line $U_{2,n}$. As the class of graphic matroids has arbitrarily large branch-width, if $\mathcal{M}$ has bounded branch-width, then $\mathcal{M}$ is linearly-dense if and only if there exists an integer n such that $\mathcal{M}$ does not contain $U_{2,n}$. This is useful as it allows us to bound the size of boundaries of small separations on matroids in $\mathcal{M}$.

Given a matroid M, we let $\epsilon(M)$ denote the number of rank-one flats in M and we define its *density*, $d(M)$, as the ratio of the size of its simplification to its rank, that is, $d(M) = \frac{\epsilon(M)}{r(M)}$. Given a linearly-dense minor-closed class $\mathcal{M}$ of matroids, we say that $\mathcal{M}$ has *limiting density* Δ where

$$\Delta = \limsup_{n\to\infty}(\frac{ex_{\mathcal{M}}(n)}{n}).$$

For a class $\mathcal{M}$ of matroids, we let $ex_{\mathcal{M}}(n)$, denote the maximum number of elements of a simple rank-n matroid in $\mathcal{M}$. We say a sequence $(a_0, a_1, a_2, \cdots)$ of rational numbers is *periodic with period* p if $a_i = a_{i+p}$ for every non-negative integer i. We will first show that linearly-dense minor-closed classes of matroids with bounded branch-width have rational limiting density, that is, given such a class, there exists a rational constant Δ for which large extremal members have density roughly equal to Δ. Next, we show that the function describing the density of large extremal matroids in such a class is eventually periodic; we do this by providing structural characterizations of some large extremal matroids of the class.

We can observe the above properties in many well-known matroid theory results. For example, let $\mathcal{M}$ be the class of graphic matroids with no $M(K_{3,3})$-minor. It is easy to show that for $n \geq 3$,

$$ex_{\mathcal{M}}(n) = 3n - \begin{cases} 2, & \text{when } n \equiv 2 \mod 3, \\ 3, & \text{otherwise.} \end{cases}$$

and further, that equality is attained by cycle matroids of planar triangulations when $n \equiv 0, 1 \pmod 3$ and equality is attained by 2-sums of copies of K_5 when $n \equiv 2 \pmod 3$. From this we can express $ex_{\mathcal{M}}(n)$ as $ex_{\mathcal{M}}(n) = 3n - a_n$ where $(a_3, a_4, a_5, \cdots) = (3, 3, 2, 3, 3, 2, \cdots)$ is a periodic sequence with period 3. Our main theorem proves this behavior for all linearly-dense minor-closed classes of matroids with bounded branch-width as the following.

Theorem 1.2. *Let $\mathcal{M}$ be a linearly-dense minor-closed class of matroids with branch-width at most k. Then there exists a rational number Δ, and a periodic sequence $(a_0, a_1, a_2, \cdots)$ of rational numbers such that $ex_{\mathcal{M}}(n) = \Delta n + a_n$ for sufficiently large n.*

Theorem 1 generalizes the following result of Kapadia [6]:

Theorem 1.3. *Let $\mathbb{F}$ be a finite field and let $\mathcal{M}$ be a minor-closed class of $\mathbb{F}$-representable matroids with branch-width at most k. Then there exists a rational number Δ, and a periodic sequence $(a_0, a_1, a_2, \cdots)$ of rational numbers such that $ex_{\mathcal{M}}(n) = \Delta n + a_n$ for sufficiently large n.*

To prove Theorem 1.2, we many of the techniques presented in Kapadia's proof. However, Kapadia's proof uses the fact that minor-closed classes of $\mathbb{F}$-representable matroids with bounded branch-width are well-quasi-ordered, proved by Geelen, Gerards, and Whittle [2].

Theorem 1.4. *Let $\mathbb{F}$ be a finite field and k be an integer. Then each collection of $\mathbb{F}$-representable matroids with branch-width at most k has two members such that one is isomorphic to a minor of the other.*

Unfortunately, Theorem 1.4 does not extend to all linearly-dense classes of matroids with bounded branch-width. For example, the class of spikes is linearly dense and has bounded branch-width, but is not well-quasi-ordered. So we relax the idea of a well-quasi-order under the minor relation to a well-quasi-order with respect to possible densities attained by taking minors.

Theorem 1.5. *If $\mathcal{M}$ is an infinite set of simple matroids with bounded branch-width, then there exist distinct matroids $N, M \in \mathcal{M}$ and a simple minor N' of N such that $r(N') = r(M)$ and $|N'| = |M|$.*

We conjecture that Theorem 1.5 does not require bounded branch-width.

Conjecture 1.6. *If $\mathcal{M}$ is an infinite set of simple matroids, then there exist distinct matroids $N, M \in \mathcal{M}$ and a simple minor N' of N such that $r(N') = r(M)$ and $|N'| = |M|$.*

Geelen, Gerards, and Whittle [5] conjectured that Theorem 1.2 does not require bounded branch-width.

Conjecture 1.7. *Let $\mathcal{M}$ be a linearly-dense minor-closed class of matroids. Then there exists a rational number Δ, and a periodic sequence $(a_0, a_1, a_2, \cdots)$ of rational numbers such that $ex_{\mathcal{M}}(n) = \Delta n + a_n$ for sufficiently large n.*

They also conjecture that the limiting density of such classes is attained by a subfamily of bounded branch-width.

Conjecture 1.8. *If $\mathcal{M}$ is a linearly-dense minor-closed class of matroids, then there is a minor-closed subclass $\mathcal{M}'$ of $\mathcal{M}$ with bounded branch-width such that $ex_{\mathcal{M}'}(n) = ex_{\mathcal{M}}(n)$ for each non-negative integer n.*

Note that as a corollary of Theorem 1.2, a proof of Conjecture 1.8 implies a proof of Conjecture 1.7. For the remainer of the text, we follow the notation and terminology of Oxley [8].

1.1 Connectivity systems and branch-width

Given a ground set S and a function $\lambda : 2^S \to \mathbb{R}$, we say that λ is *submodular* if for all $A, B \subseteq S$, $\lambda(A) + \lambda(B) \geq \lambda(A \cup B) + \lambda(A \cap B)$ and we say that λ is *symmetric* if for all $A \subseteq S$, $\lambda(A) = \lambda(S \setminus A)$. If λ is both symmetric and submodular, we call the pair $(S,)$ a *connectivity system*. Given a matroid M and a subset A of its ground set S, the *connectivity function* of M, denoted $_M : 2^S \to \mathbb{Z}$, is defined by $\lambda_M(A) = r_M(A) + r_M(S \setminus A) - r_M(S) + 1$, so note that (S, λ_M) is a connectivity system. For a matroid M on ground set S, we let $K_M = (S, \lambda_M)$.

Given a tree T, we let $L(T)$ denote the set of leaves of T. We call T a *cubic tree* if every vertex in T has degree either one or three. Given a connectivity system (S, λ), a *branch-decomposition* of (S, λ) is a pair (T, μ) where T is a cubic tree and $\mu : S \to L(T)$ is an injective function. Given a subtree T' of T, we define the *set displayed by* T' as the set $\{s \in S : \mu(s) \in L(T')\}$. The *width* $\lambda(e)$ of an edge e of T is given by $\lambda(e) = \lambda(S')$ where S' is the set displayed by some component of $T - e$. Note that $\lambda(S')$ is unique as λ is symmetric. The *width* of a branch-decomposition (T, μ) is given by the maximum width of edges in T. The *branch-width* of M is the minimum width among all branch-decompositions of (S, λ).

Given disjoint subsets $A, B \subseteq S$, we define $\kappa(A, B)$ to be the minimum value $\lambda(C)$ where $A \subseteq C \subseteq S \setminus B$. Let (T, μ) be a branch-decomposition of M and let e and f be edges in T. We let $T(e, f)$ be the set displayed by the component of $T - e$ not containing f and we say edges e and f are linked if $\kappa(T(e, f), T(f, e))$ is equal to the minimum edge-width among edges in the unique minimal path in T containing both e and f. We say that a branch-decomposition (T, μ) is *linked* if every pair of edges in T is linked. The following theorem, proved by Geelen, Gerards, and Whittle in [2], will be helpful in bounding the size of parts in a linked dissection.

Theorem 1.9. *A connectivity system with branch-width n has a linked branch decomposition of width n.*

Let (S, λ) be a connectivity system and let X be a subset of S. We let $S \circ X = (S - X) \cup \{X\}$ and let $\lambda \circ X : 2^{S \circ X} \to \mathbb{R}$ be the function where for any $A \subseteq S - X$ we let $(\lambda \circ X)(A) = \lambda(A)$ and $(\lambda \circ X)(A \cup \{X\}) = \lambda(A \cup X)$. It is simple exercise to show that the pair $(S \circ X, \lambda \circ X)$ is a connectivity system.

Lemma 1.10. *If (S, λ) is a connectivity system with branch-width k and $X \subseteq S$ has $\lambda(X) = t$, then $(S \circ X, \lambda \circ X)$ has branch-width at most $k + t$.*

Proof. Let (T, μ) be a branch-decomposition of (S, λ) and let s be some element in X. Let (T, μ_s) be the branch-decomposition of $(S \circ X, \lambda \circ X)$ obtained by letting μ_s be the restriction of μ to $S \setminus X$ along with the definition $\mu_s(\{X\}) = \mu(s)$. To find the width of (T, μ_s), we look at an arbitrary edge e of T and let A be the set displayed by the subtree of $T - e$ containing $\{X\}$. Submodularity of λ gives us an upper bound for the weight of e as

$$(\lambda \circ X)(A) = \lambda((A - \{X\}) \cup X) \leq \lambda(e) + \lambda(X) \leq k + t.$$

$\square$

Lemma 1.10 and Theorem 1.9 will be helpful in proving Theorem 2.1 as together they allow us to take a minor-closed class $\mathcal{M}$ of matroids with bounded branch-width, and find linked branch-decompositions of bounded branch-width corresponding to low order separations of matroids in $\mathcal{M}$.

In [10] Robertson and Seymour proved that graphs with bounded branch-width are well-quasi-ordered under the minor relation. We use the same techniques to prove Theorem 2.1. Both proofs heavily rely on their Lemma on Cubic Trees that we will see soon.

1.2 Lemma on Cubic Trees

A *directed tree* is a triple (T, ϕ, ψ), where $T = (V, E)$ is a tree and $\phi, \psi : E \to V$ such that for each edge $e \in E$, $e = \{\phi(e), \psi(e)\}$. For an edge $e \in E$ with $\phi(e) = u$ and $\psi(e) = v$, we call u the *tail* of e and v the *head* of e. Conceptually, we can imagine the edge e is "pointing" from u to v. For a vertex $v \in V$, an edge $e \in E$ is an *out-edge* of v if $\phi(e) = v$. The *outdegree* of v is the number of out-edges incident to v, that is, $|\{e \in E : \phi(e) = v\}|$. Given a directed tree (T, ϕ, ψ) and edges $e, f \in E$, there exists a *directed path* from e to f in (T, ϕ, ψ), if there exists a sequence of edges $e_1 = e, e_2, e_3, \cdots, e_k = f$ such that for $i \in [k-1]$, $\psi(e_i) = \phi(e_{i+1})$.

A *rooted tree* is a finite directed tree with exactly one leaf vertex having outdegree one. We call this unique leaf vertex the *root vertex* and we call the unique edge incident to the root vertex the *root edge*. We call all other edges incident to leaves *leaf edges*. An *infinite rooted forest* is a countable collection of rooted trees. Given an infinite rooted forest F, we denote the edge set of F by $E(F)$. We say that (F, l, r) is an *infinite binary forest* if F is an infinite rooted forest of cubic trees and $l, r : E(F) \to E(F)$ are functions on the non-leaf edges of F such that each non-leaf edge e in $E(F)$ has $\{l(e), r(e)\} = \{e_1, e_2\}$ where $\phi(e_1) = \phi(e_2) = \psi(e)$.

Given an infinite rooted forest F, we call a function $\lambda : E(F) \to [n]$ an *n-edge-labelling* of F. Let λ be an n-edge-labelling on an infinite rooted forest F, and let e and f be edges in F. We say that e is *λ-linked* to f if F contains a directed path P from e to f and $\lambda(e) = \lambda(f)$ is equal to the minimum λ-value among all edges of P. The following lemma proved by Robertson and Seymour [10] is key in showing that we can construct a well-quasi-order on matroid separations with respect to density, but first we need some definitions. A *quasi-order* is a pair $(X, \preceq)$ where X is a set and $\preceq$ is a binary relation which is both reflexive and transitive. An *antichain* of $(X, \preceq)$ is a collection of pairwise incomparable elements of X.

Lemma 1.11 (Lemma on Cubic Trees)**.** *Let (F, l, r) be an infinite binary forest with an n-edge-labelling λ. Moreover, let $\preceq$ be a quasi-order on the edges of F with no infinite strictly descending sequences, such that $e \preceq f$ whenever f is λ-linked to e. If the leaf edges of F are well-quasi-ordered by $\preceq$ but the root edges of F are not, then F contains an infinite sequence $(e_0, e_1, \cdots)$ of nonleaf edges such that:*

(i) $\{e_0, e_1, \cdots\}$ *is an antichain with respect to* $\preceq$ *;*

(ii) $l(e_0) \preceq \cdots \preceq l(e_{i-1}) \preceq l(e_i) \preceq \cdots$ *;*

(iii) $r(e_0) \preceq \cdots \preceq r(e_{i-1}) \preceq r(e_i) \preceq \cdots$ *.*

The following theorem of Tutte's [11] helps to show that linked branch-decompositions will have the λ-linked property necessary to apply the Lemma on Cubic Trees in our proof of Theorem 2.1.

Theorem 1.12 (Tutte's Linking Theorem)**.** *Let X and Y be disjoint subsets of the ground set of a matroid M. Then we have $\kappa_M(X, Y) \geq n$ if and only if there exists a minor M' of M with ground set $X \cup Y$ such that $\lambda_{M'}(X) \geq n$.*

1.3 Linked Dissections

Let M be a matroid. Given a partition (A, B) of the ground set of M, we say that (M, A, B) is a *k-separation*, or a separation of *order* k, if both $|A|, |B| \geq k$ and $\lambda_M(A) < k$. We call the intersection of the closures of A and B the *boundary* of (M, A, B) and denote it $\partial_M(A, B)$.

A *k-dissection* of a matroid M is a partition of the ground set of M into non-empty subsets $\mathcal{D} = (S_1, S_2, \cdots, S_n)$ such that for each $i \in [n]$, $(S_1 \cup S_2 \cup \cdots \cup S_i, S_{i+1} \cup S_{i+2} \cup \cdots \cup S_n)$ is a k-separation. For a k-dissection $\mathcal{D}$ of length n, we let $\partial_{\mathcal{D}}(1) = \emptyset$ and for $i \in \{2, \cdots, n\}$, we let $\partial_{\mathcal{D}}(i) = \partial_M(S_1 \cup \cdots \cup S_{i-1}, S_i \cup \cdots \cup S_n)$. For k-dissections $(S_1, S_2, \cdots, S_n)$ and $(T_1, T_2, \cdots T_m)$ of a matroid, we say that $(S_1, S_2, \cdots, S_n)$ *contains* $(T_1, T_2, \cdots, T_m)$ if there is some subsequence $i_1, i_2, \cdots, i_{m-1}$ of $1, 2, \cdots, n$ such that

(i) $T_1 = S_1 \cup S_2 \cup \cdots \cup S_{i_1}$,

(ii) for $2 \leq j \leq m-1$, $T_j = S_{i_{j-1}+1} \cup \cdots \cup S_{i_j}$ and

(iii) $T_m = S_{i_{m-1}+1} \cup \cdots \cup S_n$.

Containment will be a key to proving the theorem on periodicity as we can highlight useful parts of dissections. Another useful tool will be removing unwanted parts of dissections by taking minors. To do this we use an extension of Tutte's Linking Theorem to matroid dissections.

Theorem 1.13 (Tutte's Linking Theorem for dissections)**.** *If $\mathcal{D} = (S_1, S_2, \cdots, S_n)$ is a k-dissection of a matroid M where for some $i \in \{2, 3, \cdots, n-1\}$,*

$$r(\partial_{\mathcal{D}}(i)) = r(\partial_{\mathcal{D}}(i+1)) \leq \kappa_M(S_1 \cup \cdots \cup S_{i-1}, S_{i+1} \cup \cdots \cup S_n),$$

then there exists a partition (C_i, D_i) of S_i such that $(S_1, S_2, \cdots, S_{i-1}, S_{i+1}, S_{i+2}, \cdots, S_n)$ is a k-dissection of $M/C_i \backslash D_i$.

In the above theorem, we say that we *collapse* the part S_i of the dissection $(S_1, \cdots, S_n)$ of M to obtain the dissection $(S_1, \cdots, S_{i-1}, S_{i+1}, \cdots, S_n)$ of $M/C_i \backslash D_i$. We say that a linked dissection $\mathcal{D}$ *collapses* to the linked dissection $\mathcal{D}'$ if $\mathcal{D}$ can be obtained from $\mathcal{D}$ by collapsing some parts of $\mathcal{D}$. Later we aim to use dissections for which each part satisfies the conditions to apply Tutte's Linking Theorem. This will give a convenient relation between a quasi-order on dissections and collapsability. For a matroid M, and a partition (C, D) of the ground set of M, we say $\mathcal{D} = (C, D; S_1, \cdots, S_n)$ is a *linked k-dissection* of M if $(S_1, \cdots, S_n)$ is a k-dissection where

(i) for $i \in \{2, \cdots, n\}$, $r(\partial_{\mathcal{D}}(i)) = k$,

(ii) $\kappa_M(S_1, S_n) = k$,

(iii) and for $i \in \{2, \cdots, n-1\}$, $(C|_{S_i}, D|_{S_i})$ is a partition of S_i for which $\lambda_{M/C_i \setminus D_i}(S_1 \cup \cdots \cup S_{i-1}) = k$.

We may use the pair $(M, \mathcal{D})$ to denote a linked dissection $\mathcal{D}$ of M. Note that Tutte's Linking Theorem implies the existence of such a partition (C, D) of the ground set of M if M has a k-dissection satisfying (i) and (ii). Given a subset $I \subseteq [n]$, we say that the dissection $(M, \mathcal{D})$ *collapses to* $(M[I], \mathcal{D}[I])$ if we can collapse $\mathcal{D}$ to a linked dissection where the parts are given by parts of $\mathcal{D}$ indexed by I. As a note, if we have a linked dissection $(M, \mathcal{D}) = (C, D; S_1, S_2, \cdots, S_n)$ and $I = \{i, i+1, \cdots, j-1, j\}$ for some $1 \leq i < j \leq n$, then $(M[I], \mathcal{D}[I])$ is the restriction of $(M, \mathcal{D})$ to the parts indexed by I. Similarly, we say that $(M, \mathcal{D})$ collapses to $(M \circ I, \mathcal{D} \circ I)$ if we can collapse $\mathcal{D}$ to a linked dissection by collapsing the parts indexed by I.

In addition, we note that paths in a branch-decomposition of a matroid M induce dissections of M. The following lemma proved by Ben-David and Geelen in [1] implies that large matroids with low branch-width have long linked dissections.

Lemma 1.14. *If a matroid M has a k-dissection of length at least n^{k+1}, then it has a linked l-dissection of length at least n for some $l \leq k$.*

Corollary 1.15. *Let M be a matroid with branch-width at most k. If $\epsilon(M) > 2^{n^{k+1}}$, then M has a linked l-dissection of length at least n for some $l \leq k$.*

Proof. Let S be the ground set of M and let T be a branch-decomposition of M of width k. Since T is cubic and $\epsilon(M) > 2^{n^{k+1}}$, there exists a path $P = e_1 e_2 \cdots e_{n^{k+1}}$ in T where $\{e_1, \cdots, e_{n^{k+1}}\} \subseteq E(T)$. Recall that given edges e and f of T, $T(e, f)$ is the component of $T - e$ not containing f. Let $S_1 = T(e_1, e_2)$, and for $i \in \{2, \cdots, n^{k+1}\}$, let $S_i = T(e_i, e_{i+1}) \setminus T(e_{i-1}, e_i)$. Then $(S_1, \cdots, S_{n^{k+1}})$ is a k-dissection of M and the result follows from Lemma 1.14. □

In a similar proof, we can see that if a matroid is large, then we can obtain a long dissection with parts bounded in size.

Lemma 1.16. *Let $\mathcal{M}$ be a linearly-dense minor-closed class of matroids with branch-width at most k. Then for any simple matroid $M \in \mathcal{M}$ with $|M| > 2^n$, M has a k-dissection $(M, \mathcal{D}) = (S_1, \cdots, S_n)$ of length n where* *(i) for* $i \in \{1, 2, \cdots, n-1\}$*,* $|S_i| \le 2^i$*,*

(ii) and for $1 \le i < j \le n$*,* $\kappa_M(T_1 \cup T_2 \cup \cdots \cup T_i, T_j \cup T_{j+1} \cup \cdots \cup T_n) = min_{i \le t < j}\{r(\partial_{\mathcal{D}}(t))\}$*.*

Proof. By Theorem 1.9, we can let T_M be a linked branch decomposition of width at most k. Let $P = v_1 v_2 \cdots v_t$ be a maximal path in T_M. Since T_M is cubic, we have $t \ge n$. For $i \in [t]$, let S_i be the set displayed by the component of $T_M - \{v_i\}$ not containing any other vertices of P. See that $(S_1, S_2, \cdots, S_n)$ is a k-dissection of M. In addition, since P is maximal and T_M is cubic, for $i \in [t-1]$, we have $|S_i| \le 2^i$. In addition, since T_M is a linked branch decomposition, (ii) is satisfied.

□

Let M be a matroid with ground set S. Given subsets X and Y of S, the *local connectivity* between X and Y is $\sqcap(X, Y) = r(X) + r(Y) - r(X \cup Y)$. Note that given a dissection $(S_1, S_2, \cdots, S_n)$ of a matroid M, we have

$$r(M) = r(S_1) + \sum_{i=2}^{n} r(S_i) - \sqcap(S_1 \cup \cdots \cup S_{i-1}, S_i).$$

Given a dissection $\mathcal{D} = (S_1, S_2, \cdots, S_n)$ of a matroid M, let $\sqcap_{\mathcal{D}}(i) = \sqcap_M(S_1 \cup \cdots \cup S_{i-1}, S_i)$. Given a matroid M, and a q-dissection $\mathcal{D}$ of M, we call $\mathcal{D}$ an (l, q)-*dissection* if for $i \in \{2, 3, \cdots, n-1\}$, $\sqcap_{\mathcal{D}}(i) = l$. We may use (l, q)-dissections to easily calculate the density of a matroid.

Observation 1.17. *Let* M *be a simple matroid with an* (l, q)*-dissection* $\mathcal{D} = (S_1, S_2, \cdots, S_n)$*. The density of* M *is*

$$\begin{aligned} d(M) = \frac{\epsilon(M)}{r(M)} &= \frac{\sum_{i=1}^{n} |S_i|}{r(S_1) + \sum_{i=2}^{n} r(S_i) - \sqcap_{\mathcal{D}}(i)} \\ &= \frac{\sum_{i=1}^{n} |S_i|}{(r(S_1) + r(S_n) - q) + \sum_{i=2}^{n-1} (r(S_i) - l)}. \end{aligned}$$

It is important to note that given a linked (l, q)-dissection $(M, \mathcal{D})$, the linked k-dissection obtained by collapsing a part of $\mathcal{D}$ by using Tutte's Linking Theorem is also a linked (l, q)-dissection.

1.4 Pruned Matroids

One of the key ideas for proving that linearly-dense minor-closed classes have rational limiting density is the notion of pruned matroids and pruned sequences of matroids. These were introduced by Kapadia in [6]. We may use pruned matroids to find the specific density of individual parts of a matroid dissection. For any $\delta > 0$ and any positive integer k, we say that a matroid M is *(δ, k)-pruned* if any minor N of M with rank $r(N) \geq r(M) - k$ satisfies the following inequality:

$$\epsilon(M) - \epsilon(N) \geq (d(M) - \delta)(r(M) - r(N))$$

We say that a sequence of matroids $\{M_i\}_{i \in \mathbb{N}}$ is a *pruned sequence* if for any $\delta > 0$ and positive integer k, there exists some positive integer M such that for any integer $m \geq M$, M_m is (δ, k)-pruned. We use the equivalent, one-parameter definition, that a sequence of matroids is a pruned sequence if for any positive integer n, there exists a positive integer M such that for any integer $m \geq M$, M_m is $(\frac{1}{n}, n)$-pruned.

Lemma 1.18 (Kapadia [6])**.** *Let $\mathcal{M}$ be a linearly-dense minor-closed class of matroids with limiting density $\Delta > 0$. Then there exists a pruned sequence $\{M_i\}_{i \in \mathbb{N}}$ in $\mathcal{M}$ such that $d(M_i) \to \Delta$ and $r(M_i) \to \infty$.*

Proof. Let n be a positive integer and let M be a matroid in $\mathcal{M}$ that is not $(\frac{1}{n}, n)$-pruned. Then there exists a minor N of M such that $r(N) \geq r(M) - n$ and

$$\begin{aligned} \epsilon(M) - \epsilon(N) &< (d(M) - \frac{1}{n})(r(M) - r(N)) \\ &< d(M)(r(M) - r(N)) \\ &= \epsilon(M) - d(M)r(N) \end{aligned}$$

so we have $d(N) > d(M)$ and $\epsilon(M) - \epsilon(N) < (d(N) - \frac{1}{n})(r(M) - r(N))$. Note that these properties are transitive. If N is not $(\frac{1}{n}, n)$-pruned so there exists a matroid H which is a minor of N such that $r(H) \geq r(M) - n$ and $\epsilon(N) - \epsilon(H) < (d(N) - \frac{1}{n})(r(N) - r(H))$. Therefore, $d(M) > d(N) > d(H)$ and as a result

$$\begin{aligned} \epsilon(M) - \epsilon(H) &= (\epsilon(M) - \epsilon(N)) + (\epsilon(N) - \epsilon(H)) \\ &< (d(M) - \frac{1}{n})(r(M) - r(N)) + (d(N) - \frac{1}{n})(r(N) - r(H)) \\ &< (d(H) - \frac{1}{n})(r(M) - r(N)) + (d(H) - \frac{1}{n})(r(N) - r(H)) \\ &= (d(H) - \frac{1}{n})(r(M) - r(H)) \end{aligned}$$

Now we define r_n to be some positive integer where $r_n \geq n$ and $d(M) < \Delta + \frac{1}{3n}$ for any matroid M in $\mathcal{M}$ with $r(M) \geq r_n$. We know such an integer exists, as otherwise would contradict the limiting density of $\mathcal{M}$. Assume that $\mathcal{M}$ has no $(\frac{1}{n}, n)$-pruned matroid of rank at least $r_n + n$. Furthermore, we let ϵ_n be the maximum size of any matroid M in $\mathcal{M}$ with $r(M) \leq r_n + n$.

We let M be a matroid in $\mathcal{M}$ with density $d(M) > \Delta - \frac{1}{3n}$ and rank $r(M) > 3n\epsilon_n$. and let $(M_0 = M, M_1, \cdots, M_t = N)$ be a maximal sequence of matroids such that

(i) M_i is a minor of M_{i-1} with $r(M_{i-1}) \geq r(M_i) - n$ for each $i \in \{1, 2, \cdots, t\}$,

(ii) $\epsilon(M_{i-1}) - \epsilon(M_i) < (d(M_{i-1}) - \frac{1}{n})(r(M_{i-1}) - r(M_i))$ for $i \in \{1, 2, \cdots, t\}$,

(iii) and $r(M_{t-1}) \geq r_n + n$.

Assume that $r(N) < r_n + n$, so by definition $\epsilon(N) \leq \epsilon_n$. Thus $\epsilon(M) - \epsilon_n \leq (d(N) - \frac{1}{n})r(M)$ as otherwise we would have

$$\begin{aligned}
\epsilon(M) - \epsilon(N) &\geq \epsilon(M) - \epsilon_n \\
&> (d(N) - \frac{1}{n})r(M) \\
&\geq (d(N) - \frac{1}{n})(r(M) - r(N)),
\end{aligned}$$

a contradiction. But we know that $r(N) \geq r_n$, so $d(N) < \Delta + \frac{1}{3n} < d(M) + \frac{2}{3n}$. However, this means that

$$\begin{aligned}
\epsilon(M) - \epsilon_n &\leq (d(N) - \frac{1}{n})r(M) \\
&< (d(M) - \frac{1}{3n})r(M),
\end{aligned}$$

a contradiction. Hence N is $(\frac{1}{n}, n)$-pruned by maximality. Furthermore, N has density $d(N) \geq d(M) > \Delta - \frac{1}{3n}$ and rank $r(N) \geq r_n + n$. $\square$

Chapter 2

Well-quasi-order

2.1 A well-quasi-order on matroid separations

Recall that a quasi-order is a pair $(X, \preceq)$ where X is a set and $\preceq$ is a binary relation which is both reflexive and transitive, and that an antichain of a quasi-order is a collection of pairwise incomparable elements. We will often say that $\preceq$ is a quasi-order on X. A *chain* of $(X, \preceq)$ is a strictly increasing sequence. A *well-quasi-order* is a quasi-order with no infinite anti-chain and no infinite strictly decreasing sequence. The quasi-orders that we consider have no infinite strictly decreasing sequences.

Given a minor-closed class $\mathcal{M}$ of matroids, we let $\mathcal{S}(\mathcal{M}) = \{(M, A, B) : M \in \mathcal{M}, A \cup B = E(M), A \cap B = \emptyset\}$ denote the set of all matroid separations on matroids in the class $\mathcal{M}$. In addition, let $\mathcal{S}_t(\mathcal{M})$ denote the restriction of $\mathcal{S}(\mathcal{M})$ to separations of order at most t. If for any non-negative integer n, $\mathcal{M}$ excludes the n-point line, $\mathcal{U}_{2,n}$, then we can bound the size of the boundaries of separations in $\mathcal{S}_t(\mathcal{M})$. We will denote the maximum size of the boundary of a t-separation of a matroid in $\mathcal{M}$ by $\mathcal{K}_t(\mathcal{M})$.

Let M be a matroid on ground set S. Recall that $\epsilon(M)$ is the number of rano-one flats in M. Given a subset S' of S, we define $\epsilon_M(S')$ to be the number of rank-one flats in the restriction of M to S', that is, $\epsilon_M(S') = \epsilon(M|_{S'})$. Given matroid separations (M, A, B) and (N, A', B'), we say that (N, A', B') is a *minor* of (M, A, B) if there exist subsets C and D of $A \setminus \partial_M(A, B)$ such that $N = M_{/C \setminus D}$, $B' = B$, and $\lambda_N(A') = \lambda_M(A)$. We define the quasi-order $(\mathcal{S}(\mathcal{M}), \preceq)$ by letting $(M, A, B) \preceq (M', A', B')$ if (M', A', B') has a minor (N, A_N, B_N) for which

(i) $r_N(A') = r_M(A)$,

(ii) $\lambda_N(A') = \lambda_M(A)$,

(iii) and $\epsilon_N(A' \setminus \partial_N(A', B')) = \epsilon_M(A \setminus \partial_M(A, B))$.

We prove that for any integer t, $\preceq$ is a well-quasi-order when restricted to $\mathcal{S}_t(\mathcal{M})$.

Theorem 2.1. *Let $\mathcal{M}$ be a linearly-dense minor-closed class of matroids with branch-witdth at most k and let t be a non-negative integer. Then $(\mathcal{S}_t(\mathcal{M}), \preceq)$ is a well-quasi-order.*

Proof. Assume the opposite, that there exist matroid separations $\{(M_i, A_i, B_i)\}_{i\in\mathbb{N}}$ of $\mathcal{S}_t(\mathcal{M})$ that form an antichain under the order $\preceq$. We may assume that each M_i is simple. By taking a subsequence we can assume that each separation is of the same order. For $i \in \mathbb{N}$, since the connectivity system K_{M_i} has branch-width at most k, by Lemma 1.10, $K_{M_i} \circ B_i$ has branch-width at most $k + t$. For $i \in \mathbb{N}$, let (T_i, μ_i) denote a linked branch-decomposition of $K_{M_i} \circ B_i$ with width at most $k + t$. Furthermore, let $T'_i = (T_i, \phi_i, \psi_i)$ be the rooted tree of T_i with root vertex $\mu_i(\{B_i\})$. Now let (F, l, r) be some infinite binary forest given by the collection of rooted trees $\{T_i\}_{i\in\mathbb{N}}$ along with an orientation defined by l and r.

Then for any $i \in \mathbb{N}$ and $e \in T_i$, e describes a separation of M_i call it (M_i, A_e, B_e) where we let A_e be the set displayed by $T_i - e$ not containing $\{B_i\}$. So for rooted trees $T_i, T_j \in (F, l, r)$ and for edges $e \in T_i$,$f \in T_j$ we say that $e \preceq_F f$ when $(M_i, A_e, B_e) \preceq (M_j, A_f, B_f)$. By our assumption, the root edges of (F, l, r) form an antichain. In addition, for a leaf edge e of F, A_e is of bounded size since branch-decompositions are injective, so the leaf edges of F are well-quasi-ordered. Let $\Lambda : E(F) \to \mathbb{Z}$ be the $(k + t)$-edge-labelling given by letting $\Lambda(e) = (\lambda \circ B_i)(e) = \lambda_{M_i}(A_e)$ where $e \in T_i$.

Let e and f be edges of T_i in F for which f is Λ-linked to e. Since (T_i, μ_i) is linked, we have that

$$\kappa_{M_i}(A_e, B_f) = \min_{B_f\subseteq X\subseteq A_e}\{\lambda_{M_i}(X) = (\lambda \circ B_i)(X)\} = \lambda_{M_i}(A_e) = \lambda_{M_i}(B_f),$$

so by Tutte's Linking Theorem there exist subsets C and D of $A_f \setminus A_e$ such that

$$\lambda_{(M_i)/C\setminus D}(A_e) = \lambda_{M_i}(A_e).$$

Let C' and D' be the restrictions of C and D to $A_f \setminus \partial_{M_i}(A_f, B_f)$, respectively, and let (M'_i, A'_f, B'_f) be the separation where $M'_i = (M_i)_{/C'\setminus D'}$, $A'_f = A_f \setminus (C' \cup D')$, and $B'_f = B_f$. By construction, (M'_i, A'_f, B'_f) is a minor of (M_i, A_f, B_f) for which $\lambda_{M'_i}(A'_f) = \lambda_{M_i}(A_e)$, $r_{M'_i}(A'_f) = r_{M_i}(A_e)$, and $A'_f \subseteq (A_e \cup \partial_{M_i}(A_f, B_f))$. So $(M_i, A_e, B_e) \preceq (M_i, A_f, B_f)$, and thus $e \preceq_F f$. Therefore, we may apply the Lemma on Cubic Trees to find edges $\{e_1, e_2, \cdots\}$ in (F, l, r) which form an antichain, and have $l(e_1) \preceq_F l(e_2) \preceq_F \cdots$ and $r(e_1) \preceq_F r(e_2) \preceq_F \cdots$. Let $T_{i_1}, T_{i_2}, \cdots$ be the rooted branch decompositions of (F, l, r) containing $e_1, e_2, \cdots$, respectively. Without loss of generality, we may assume that $i_j = j$ for $j \in \mathbb{N}$. Thus we have chains $(M_1, A_{l_1}, B_{l_1}) \preceq (M_2, A_{l_2}, B_{l_2}) \preceq \cdots$ and $(M_1, A_{r_1}, B_{r_1}) \preceq (M_2, A_{r_2}, B_{r_2}) \preceq \cdots$.

For $i \in \mathbb{N}$, let

$$\partial_i = (\partial_{M_i}(A_{l_i}, B_{l_i}) \cup \partial_{M_i}(A_{r_i}, B_{r_i})) \setminus \partial_{M_i}(A_i, B_i).$$

Since (T_i, μ_i) has branch-width at most $k+t$, $r_{M_i}(\partial_i) \leq 2(k+t)$. Thus $|\partial_i| \leq \mathcal{K}_{2(k+t)}(\mathcal{M})$, and without loss of generality, we may assume that

$$M|_{\partial_1} \cong M|_{\partial_2} \cong \cdots .$$

For $i \in \mathbb{N}$ let $(M'_i, A'_{l_i}, B'_{l_i})$ be a minor of (M_i, A_{l_i}, B_{l_i}) for which $\epsilon_{M'_i}(A'_{l_i} \setminus \partial_{M'_i}(A'_{l_i}, B'_{l_i})) = \epsilon_{M_i}(A_{l_1})$, $r_{M'_i}(A'_{l_i}) = r_{M_1}(A_{l_1})$, and $\lambda_{M'_i}(A'_{l_i}) = \lambda_{M_i}(A_{l_i}) = \lambda_{M_1}(A_{l_1})$. Assume that for $i \in \mathbb{N}$, we obtain M'_i as a minor of M_i by contracting C_{l_i} and deleting D_{l_i} where C_{l_i} and D_{l_i} are subsets of $A_i \setminus \partial_{M_i}(A_i, B_i)$. In addition, we may assume that we obtained similar minors for each $i \in \mathbb{N}$ and (M_i, A_{r_i}, B_{r_i}) by contracting C_{r_i} and deleting D_{r_i} where C_{r_i} and D_{r_i} are subsets of $A_i \setminus \partial_{M_i}(A_i, B_i)$.

Now for $i \in \mathbb{N}$, let $C_i = C_{l_i} \cup C_{r_i}$,$D_i = D_{l_i} \cup D_{r_i}$, and $M'_i = (M_i)_{/C_i \setminus D_i}$. Since C_i and D_i are subsets of A_i, M'_i has a separation (M'_i, A'_i, B_i) where $A'_i = A_i \setminus (C_i \cup D_i)$. Thus for $i \in \mathbb{N}$, we have $\lambda_{M'_i}(A'_i) = \lambda_{M_i}(A_i) = \lambda_{M_1}(A_1)$ and since $\partial_{M_i}(A_{l_i}, A_{r_i}) \subseteq \partial_i$, we have $r_{M'_i}(A'_i) = r_{M_1}(A_1)$. In addition, since M_i is simple, a point e in $\partial_{M'_i}(A'_i, B_i) \cap A'_i$ must be in ∂_i, we have

$$\begin{aligned}\epsilon_{M'_i}(A'_i \setminus \partial_{M'_i}(A'_i, B_i)) &= \epsilon_{M'_i}(A'_i \setminus \partial_i) + \epsilon_{M'_i}(A'_i \cap (\partial_i \setminus \partial_{M'_i}(A'_i, B_i))) \\ &= \epsilon_{M_1}(A_1 \setminus \partial_1) + \epsilon_{M_1}(A_1 \cap (\partial_1 \setminus \partial_{M_1}(A_1, B_1))).\end{aligned}$$

Therefore for $i \in \mathbb{N}$, we have that $(M_1, A_1, B_1) \preceq (M_i, A_i, B_i)$, a contradiction.

□

2.2 Well-quasi-order on linked dissections

Given a minor-closed class $\mathcal{M}$ of matroids with branch-width at most n, let $\mathcal{D}_k(\mathcal{M})$ be the set of all closed, linked k-dissections of matroids in $\mathcal{M}$. As shown in Section 2.2, there exists a well-quasi-order $\preceq$ on $\mathcal{S}_{2k}(\mathcal{M})$. We define a quasi-order $\preceq_{\mathcal{D}}$ on $\mathcal{D}_k(\mathcal{M})$ after some definitions.

Since we only care about the density of matroids in our class, and are less worried about the local structure, we introduce the following definition. If $(M, \mathcal{D}) = (C, D; S_1, \cdots, S_n)$ and $(M', \mathcal{D}') = (C', D'; T_1, \cdots, T_m)$ are linked dissections, then we say $(M, \mathcal{D})$ and $(M', \mathcal{D}')$ are *density equivalent* if $n = m$ and for $i \in [n]$, $\epsilon_M(S_i) = \epsilon_{M'}(T_i)$ and $r_M(S_i) - \sqcap_{\mathcal{D}}(i) = r_{M'}(T_i) - \sqcap_{\mathcal{D}'}(i)$. We see then that if simple matroids M and M' have density equivalent linked dissections, then we have that $\epsilon(M) = \epsilon(M')$ and $r(M) = r(M')$, and as a result $d(M) = d(M')$. In addition, if $n = m$ and for all $i \in [n]$, we have the following relation between separations $(M', T_i, T \setminus T_i) \preceq (M, S_i, S \setminus S_i)$, then we say that the dissection D *conforms to* D'. We introduce a quasi-order on matroid dissections using this terminology.

Let $(M, \mathcal{D}) = (C, D; S_1, S_2, \cdots, S_n)$ and $(M', \mathcal{D}') = (C', D'; T_1, T_2, \cdots, T_m)$ be linked dissections with M and M' in $\mathcal{M}$. We say $(M, \mathcal{D}) \preceq_{\mathcal{D}} (M', \mathcal{D}')$ if there exist indices $1 \leq i_1 < i_2 < \cdots < i_n \leq m$ such that $(T_1, \cdots, T_m)$ collapses to $(T_{i_1}, \cdots, T_{i_n})$, $(T_{i_1}, \cdots, T_{i_n})$ conforms to $(S_1, \cdots, S_n)$, and for $i \in [n]$, $\sqcap_{\mathcal{D}}(i) = \sqcap_{\mathcal{D}'}(i_n)$. We see that this says that there exists a minor M'' of M' with a linked dissection $(M'', \mathcal{D}'')$ which is density equivalent to $(M, \mathcal{D})$ up to some parallel elements. We will see how to deal with this in Section 2.3.

Higman [4] proved that given a finite set A which is well-quasi-ordered, the set of finite sequences over A is well-quasi-ordered by subsequence majorization. We can see that the quasi-order $\preceq_{\mathcal{D}}$ is equivalent to a quasi-order on finite sequences of tuples, $((M, A, B), t)$ with $(M, A, B) \in \mathcal{S}_{2k}(\mathcal{M})$ and $0 \leq t \leq 2k$, defined by subsequence majorization. So by Higman's Lemma, since $\preceq$ is a well-quasi-order and the finite product of well-quasi-orders is a well-quasi-order, we have that $\preceq_{\mathcal{D}}$ is a well-quasi-order on $\mathcal{D}_k(\mathcal{M})$. We detail Higman's Lemma more precisely.

Let $(A, \preceq)$ be a quasi-order and let $V(A)$ be the set of finite sequences of elements in A. Then define the quasi-order $(V(A), \preceq_V)$ where $(a_0, a_1, \cdots, a_n) \preceq_V (b_0, b_1, \cdots b_m)$ if there exist integers $0 \leq i_1 < i_2 < \cdots < i_n \leq m$ such that for $j \in [n]$, $a_j \preceq b_{i_j}$. Higman [4], proved the following theorem.

Theorem 2.2. *If $(A, \preceq)$ is a well-quasi-order, then $(V(A), \preceq_V)$ is a well-quasi-order.*

Therefore, we see that by definition of $\preceq_{\mathcal{D}}$, since $(\mathcal{S}_{2k}(\mathcal{M}), \preceq)$ is a well-quasi-order, so is $(\mathcal{D}_k(\mathcal{M}), \preceq_{\mathcal{D}})$.

2.3 Notes on parallel elements

Now is a good time to note that the definition of density equivalent dissections hide parallel elements lying in separation boundaries. Since applying Tutte's Linking Theorem can introduce parallel elements, we must be careful to clean these up in our proofs. In addition, we consider how to properly keep track of elements in our dissections through applying Tutte's Linking Theorem and our constructed well-quasi-order. We introduce some observations and theorems which allow us to do so.

Given a dissection $\mathcal{D} = (C, D; S_1, \cdots, S_n)$ of a matroid M with ground set S, for $i \in \{2, 3, \cdots, n-1\}$ we consider the separations $(M, S_i, S \setminus S_i)$. For ease of notation we denote the boundaries of such separations $\partial_{\mathcal{D}}(S_i) = \partial_M(S_i, S \setminus S_i)$. In applying our well-quasi-order on separations and obtaining a minor, we delete and contract subsets of $S_i \setminus \partial_{\mathcal{D}}(S_i)$. See how the following adaptation of Observation 1.17 will give us information on the density of a dissection after applying our well-quasi-order.

Observation 2.3. *The density of an (l, q)-dissection $\mathcal{D} = (S_1, S_2, \cdots, S_n)$ of a simple matroid M is*

$$d(M) = \frac{\epsilon(M)}{r(M)} = \frac{\sum_{i=1}^{n} |S_i|}{r(S_1) + \sum_{i=2}^{n}(r_M(S_i) - \sqcap_{\mathcal{D}}(i))}$$
$$= \frac{\sum_{i=1}^{n}(|S_i \setminus \partial_{\mathcal{D}}(S_i)| + |S_i \cap \partial_{\mathcal{D}}(S_i)|)}{r(S_1) + r(S_n) - k + \sum_{i=1}^{n}(r_M(S_i) - l)}.$$

Our next observation pertains to parallel elements arising from minor operations. We call rank one flats in a matroid *parallel classes*. If a minor operation increases the size of a parallel class, we say that it *extends* the parallel class. When applying Tutte's Linking Theorem we may extend parallel classes, however, the number of parallel classes extended is bounded as we see in the following observation:

Observation 2.4. *Let $\mathcal{M}$ be a linearly-dense minor-closed class of matroids with branch-width at most k. Then if $(M, \mathcal{D}) \in \mathcal{D}_k(\mathcal{M})$ is a linked k-dissection of length n, then for $i \in \{2, \cdots, n-1\}$, the dissection $(M \circ \{i\}, \mathcal{D} \circ \{i\})$ extends at most $\mathcal{K}_k(\mathcal{M})$ parallel classes by introducing parallel elements. In addition, each parallel class is extended by at most one element.*

This is true as the only parallel classes that will occur from collapsing a dissection will lie in the identified boundaries of the dissection. Furthermore, we may note the following which helps us to keep track of where elements which extend parallel classes lie:

Observation 2.5. *Let $(M, \mathcal{D}) = (C, D; S_1, S_2, \cdots, S_n)$ be a linked k-dissection and let e be an element which extends a parallel class when collapsing S_i for some $i \in \{2, 3, \cdots, n-1\}$. We may assume that $e \in S_{i+1}$.*

Our next three theorems come as corollaries to a well-known Ramsey theory result. For the following we only consider simple graphs. A graph $G = (V, E)$ is a *clique* if for any distinct vertices $u, v \in V$, $\{u, v\} \in E$. Given a subset V' of V, the *induced subgraph* $G[V']$ of G is the graph (V', E') where $E' = \{\{u, v\} : \{u, v\} \in E; u, v \in V'\}$. Given a graph G, an *n-edge-coloring* of G is a function $\phi : E \to [n]$. Given an n-edge-coloring of G, we say that a subgraph H of G is *monochromatic* if there exists an integer $c \in \{1, 2, \cdots, n\}$ such that $\phi(e) = c$ for every edge e in H. Let c and r be non-negative integers with $c \geq 3$. We define $R(c; r)$ as the minimum integer such that given a clique $G = K_{R(c;r)}$ and any r-edge-coloring $\phi : E(G) \to [r]$, there exists some vertex set $V' \subseteq V(G)$ such that $|V'| \geq c$ and $G[V']$ is monochromatic. Ramsey [9] proved that $R(c; r)$ exists for any such values of c and r.

Theorem 2.6. *Let k be a non-negative integer. Then there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that if $\mathcal{D} = (S_1, S_2, \cdots, S_m)$ is a k-dissection and $m > f(n)$, then for some $l \in \{0, 1, 2, \cdots, k\}$, $\mathcal{D}$ contains an (l, q)-dissection $(T_1, T_2, \cdots, T_n)$.*

Proof. Let $\mathcal{D} = (S_1, S_2, \cdots, S_m)$ be a k-dissection of a matroid M and assume that $m > R(n; k)$. We define an auxiliary graph G by letting G be a clique on the vertex set $V(G) = [m]$. Let ϕ be the k-edge-coloring of $V(G)$ defined by $\phi(\{i, j\}) = \sqcap_M(S_1 \cup \cdots \cup S_i, S_{i+1} \cup \cdots \cup S_j)$ where $i < j$. Since $m > R(n; k)$, there exists a vertex set $V' \subseteq V$ such that $|V'| = n$ and the clique induced by V' is monochromatic. So for some $l \in \{0, 1, 2, \cdots, k\}$, $\phi(e) = l$ for every $e \in E(G[V'])$. We let $i_1 < i_2 < \cdots < i_n$ be integers such that $\{i_1, i_2, \cdots, i_n\} = V'$.

Let $(T_1, T_2, \cdots, T_n)$ be the dissection of M given by

(i) $T_1 = S_1 \cup S_2 \cup \cdots S_{i_1}$,

(ii) $T_n = S_{i_{n-1}+1} \cup S_{i_{n-1}+2} \cup \cdots \cup S_n$,

(iii) and for $j \in \{2, 3, \cdots, n-1\}$, $T_j = S_{i_j+1} \cup S_{i_j+2} \cup \cdots \cup S_{i_{j+1}}$.

By definition of ϕ, $(T_1, T_2, \cdots, T_n)$ is an (l, q)-dissection. $\square$

For our next theorem we need to note that for a linked dissection $(M, \mathcal{D})$ and any part S_i of $\mathcal{D}$, the size of $\partial_{\mathcal{D}}(S_i)$ is bounded. Since $\partial_{\mathcal{D}}(S_i) \subseteq cl(\partial_{\mathcal{D}}(i) \cup \partial_{\mathcal{D}}(i+1))$, the rank of $\partial_{\mathcal{D}}(S_i)$ is bounded as $r(\partial_{\mathcal{D}}(S_i)) \leq r(\partial_{\mathcal{D}}(i)) + r(\partial_{\mathcal{D}}(i+1)) = 2k$ where k is the maximum branch-width of matroids in our class. As a result, we know that $\epsilon_M(\partial_{\mathcal{D}}(S_i)) \leq \mathcal{K}_{2k}(\mathcal{M})$.

Theorem 2.7. *Let $\mathcal{M}$ be a linearly-dense minor-closed class of matroids with branch-width at most k and let $(M, \mathcal{D})$ be a linked dissection in $\mathcal{D}_k(\mathcal{M})$. There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that if $I \subseteq \mathbb{N}$ is a set indexing parts of $\mathcal{D}$ with $|I| \geq f(n)$, then there exists a subset $I' \subseteq I$ of size n such that:*

(i) M has a simple minor M' with a linked k-dissection $(M', \mathcal{D}') = (C', D'; S'_1, S'_2, \cdots, S'_{n+1})$ which is a restriction of $(M[I'], \mathcal{D}[I'])$, and

(ii) there exists an integer $c \in \{0, 1, 2, \cdots, \mathcal{K}_{2k}(\mathcal{M})\}$ such that for $i \in \{1, 2, 3, \cdots, n+1\}$, $|\partial_{\mathcal{D}'}(S'_i) \cap S'_i| = c$.

Proof. Let $I = \{i_1, i_2, \cdots, i_t\}$ where $i_1 < i_2 < \cdots < i_t$ be an indexing set for a simple linked dissection $(M, \mathcal{D}) = (C, D; S_1, S_2, \cdots, S_m)$ in $\mathcal{D}_k(\mathcal{M})$ and assume that $t \geq R(n; \mathcal{K}_{2k}(\mathcal{M}))$. We define an auxiliary graph G by letting G be a clique on the vertex set $V(G) = I$. Then we define a coloring $\phi : E(G) \to [\mathcal{K}_{2k}(\mathcal{M})]$ as follows. For $j, l \in \{1, 2, \cdots, t\}$ where $j < l$, let $I(j, l) = [m] \setminus \{i_j + 1, i_j + 2, \cdots, i_l - 1\}$, that is, the set indexing all parts in $\mathcal{D}$ between S_{i_j} and S_{i_l}. Then let $\phi(\{i_j, i_l\}) = |\partial_{\mathcal{D}}(S_{i_l}) \cap S_{i_l}| - c(j, l)$ where $c(j, l)$ is the number of parallel elements created by collapsing $(M, \mathcal{D})$ to $(M[I(j, l)], \mathcal{D}[I(j, l)])$.

Now since we know that for any $i \in [m]$, $|\partial_{\mathcal{D}}(S_i)| \leq \mathcal{K}_{2k}(\mathcal{M})$, ϕ is well-defined. Since $t > R(n; \mathcal{K}_{2k}(\mathcal{M}))$, there exists a subset I' of I such that $|I'| = n$ and the induced subgraph $G[I']$ is monochromatic. Let $i'_1 < i'_2 < \cdots < i'_n$ be the integers of I' and define the set

$$J = \bigcap_{j=1}^{n-1} I(i'_j, i'_{j+1}).$$

We can see that J is the set indexing all parts of $\mathcal{D}$ which lie between parts indexed by I', excluding those indexed by I' themselves. That is to say,

$$(\mathcal{M}[J], \mathcal{D}[J]) = (C|_{M[J]}, D|_{M[J]}; S_1, S_2, \cdots, S_{i'_1 - 1}, S_{i'_1}, S_{i'_2}, \cdots, S_{i'_n}, S_{i'_n + 1}, \cdots, S_m).$$

Now let $(M', \mathcal{D}') = (C', D'; T_1, T_2, \cdots, T_{n+1})$ be the dissection obtained from $(M[J], \mathcal{D}[J])$ as follows. Let $T_1 = S_1 \cup S_2 \cup \cdots \cup S_{i'_1 - 1}$. Then for $j \in [n]$, let T_{j+1} be the restriction of $S_{i'_j}$ to elements which are not parallel to any elements in $T_1 \cup T_2 \cup \cdots \cup T_j$ in $M[J]$. We note that (M', D') is simple and for $j \in \{2, 3, \cdots, n+1\}$, we have $|\partial_{\mathcal{D}'}(T_j) \cap T_j| = \phi(\{i'_j, i'_{j+1}\})$. Since $G[I']$ is monochromatic, we have obtained a linked k-dissection with the desired properties, up to a technicality.

One may note that in constructing the T_i by deleting parallel elements, maybe we are left with some T_i which are empty. However, if we start with a linked dissection in which each part is of size at least $\mathcal{K}_{2k}(\mathcal{M}) + 1$ then by construction each T_i will be non-empty. $\square$

Chapter 3

Main Results

3.1 Proving that the limiting density is rational

We can prove the lower bound on the limiting density is rational as follows:

Theorem 3.1. *Let $\mathcal{M}$ be a linearly-dense minor-closed class of matroids with limiting density $\Delta > 0$ and branch-width at most k. Then there exist integers l and q where $l \leq q \leq k$, and there exists a sequence of matroids $H_1, H_2, \cdots$ in $\mathcal{M}$ such that for each $n \in \mathbb{N}$, there exists a $(\frac{1}{n}, n)$-pruned matroid $M_n \in \mathcal{M}$ where:*

(i) M_n has density $d(M_n) \geq \Delta - \frac{1}{n}$ and

(ii) M_n has a linked (l, q)-dissection $\mathcal{D}_n = (C_n, D_n; S_1^n, S_2^n, \cdots, S_n^n)$ where $M_n[\{i\}] \cong H_i$ for $i \in [n]$.

Proof. By Lemma 1.18, there exists a pruned sequence $M_1, M_2, \cdots$ of matroids in $\mathcal{M}$ such that $r(M_i) \to \infty$ and $d(M_i) \to \Delta$. By definition of matroid density, we may assume that the M_i are simple. By choosing a subsequence of $M_1, M_2, \cdots$, we may assume that for $i \in \mathbb{N}$, M_i is $(\frac{1}{i}, i)$-pruned and has density at least $d(M_i) \geq \Delta - \frac{1}{i}$. Similarly, since the M_i are simple, we may assume that for $i \in \mathbb{N}$, $\epsilon(M_i) > 2^i$, that is to say, by Lemma 1.16, M_i has a k-dissection $\mathcal{D}_i = (S_1^i, S_2^i, \cdots, S_i^i)$ of length i for which each $j \in [i-1]$ satisfies $|S_j^i| \leq 2^{j-1}$, and for each $1 \leq j < l \leq i$, $\kappa_{M_i}(S_1^i \cup \cdots \cup S_j^i, S_l^i \cup \cdots \cup S_i^i) = \min_{j<t\leq l}\{r(\partial_{\mathcal{D}_i}(t))\}$.

Fix an integer $n \in \mathbb{N}$. Then since for each $i > n$, we have $|S_1^i \cup S_2^i \cup \cdots \cup S_n^i| \leq 2^n$, there are a finite number of isomorphism classes on the restrictions $M_i[\{1, 2, \cdots, n\}]$. Thus by choosing a subsequence of $M_1, M_2, \cdots$, we may assume that for integers $i, j \geq n$,

$$M_i[\{1, 2, \cdots, n\}] \cong M_j[\{1, 2, \cdots, n\}].$$

By doing this in order for integers $n = 1, 2, \cdots$ we may assume that for $n \in \mathbb{N}$,

$$M_n \cong M_{n+1}[\{1, 2, \cdots, n\}].$$

Now since the natural numbers are well-ordered and the sequence $r(\partial_{\mathcal{D}_1}(1)), r(\partial_{\mathcal{D}_2}(2)), \cdots$ is bounded above by k, $\liminf_{i\in\mathbb{N}}\{r(\partial_{\mathcal{D}_i}(i))\} = q$ for some $q \leq k$. Thus there exists a subsequence $i_1, i_2, \cdots$ of $1, 2, \cdots$ where $q = r(\partial_{\mathcal{D}_{i_1}}(i_1)) = r(\partial_{\mathcal{D}_{i_2}}(i_2)) = \cdots$ and for $j \geq i_1$, $r(\partial_{\mathcal{D}_j}(j)) \geq q$. Then we define $T_1 = S_1 \cup \cdots \cup S_{i_1}$ and for $j \in \mathbb{N}$, we let $T_{j+1} = S^{i_{j+1}}_{i_j+1} \cup S^{i_{j+1}}_{i_j+2} \cup \cdots \cup S^{i_{j+1}}_{i_{j+1}}$, and we can see that for $j \in \mathbb{N}$, M_{i_j} has a q-dissection $(T_1, T_2, \cdots, T_j)$. Furthermore, since $\kappa_{M_{i_j}}(T_1, T_j) = \kappa_{M_{i_j}}(S^i_1 \cup \cdots \cup S^i_{i_1}, S^i_{i_{j-1}+1} \cup \cdots \cup S^i_{i_j}) = \min_{i_1 \leq t < i_j}\{r(\partial_{\mathcal{D}_{i_t}})\} = q$, we have that $(T_1, T_2, \cdots, T_j)$ is linked.

For $i \in \mathbb{N}$, let $(M'_i, \mathcal{D}'_i)$ be the linked q-dissection $(T_1, T_2, \cdots, T_j)$. Since $\sqcap_{\mathcal{D}'_i}(i)$ is bounded above by q for every $i \in \mathbb{N}$, there exists an integer $l \in \{0, 1, 2, \cdots, q\}$ such that $\sqcap_{\mathcal{D}'_i}(i) = l$ for infinitely many $i \in \mathbb{N}$. Fix such an l, then by applying Tutte's Linking Theorem, without loss of generality, we may assume that $(M'_i, \mathcal{D}'_i)$ is a linked (l, q)-dissection for each $i \in \mathbb{N}$.

□

Theorem 3.2. *Let $\mathcal{M}$ be a linearly-dense minor-closed class of matroids with limiting density $\Delta > 0$ and branch-width at most k. Then the limiting density of $\mathcal{M}$ is rational.*

Proof. Let $H_1, H_2, \cdots$ be matroids in $\mathcal{M}$ as described in Theorem 3.1. Fix an integer $i \geq 2$ and let $r_i = r(H_i)$. Then by Theorem 3.1, for $n > \max\{r_i, i\}$, there exists a $(\frac{1}{n}, n)$-pruned matroid $M_n \in \mathcal{M}$ with a linked (l, q)-dissection $\mathcal{D}_n = (C_n, D_n; S^n_1, S^n_2, \cdots, S^n_n)$ such that $M_n[\{i\}] \cong H_i$. Since $\mathcal{D}_n$ is a linked (l, q)-dissection, by Tutte's linking theorem, $(M_n, \mathcal{D}_n)$ collapses to $(M_n \circ \{i\}, \mathcal{D}_n \circ \{i\})$ where $r(M_n \circ \{i\}) = r(M_n) - (r(H_i) - \sqcap_{\mathcal{D}_n}(i))$. Since M_n is $(\frac{1}{n}, n)$-pruned, and $|r(M_n) - r(M_n \circ \{i\})| \leq r_i$ we get the following inequality:

$$\begin{aligned}
\epsilon(H_i) = |S_i| &= \epsilon(M_n) - \epsilon(M_n \circ \{i\}) \\
&\geq (d(M_n) - \frac{1}{n})(r(M_n) - r(M_n \circ \{i\})) \\
&= (\Delta - \frac{2}{n})(r(H_i) - \sqcap_{\mathcal{D}_n}(i))
\end{aligned}$$

Given that n can be chosen to be arbitrarily large, we have that $\epsilon(H_i) \geq \Delta(r(H_i) - \sqcap_{\mathcal{D}_n}(i))$.

The following paragraph allows us to observe the "asymptotic" behavior of separation boundaries in our linked dissections with respect to a single part. For example, see how possibly $\partial_{M_5}(S_3^5, S^5 \setminus S_3^5) \neq \partial_{M_7}(S_3^7, S^7 \setminus S_3^7)$. This is necessary to consider since application of our well-quasi-order is with regards to a restriction outside of separation boundaries.

Now let $M_1, M_2, \cdots$ be matroids of $\mathcal{M}$ as described in the last paragraph of Theorem 3.1 with linked (l,q)-dissections $(M_i, \mathcal{D}_i) = (C_i, D_i; S_1^i, S_2^i, \cdots, S_i^i)$ on ground set S^i. For any positive integers i and j where $j \geq i$, let A_i^j be $\partial_{\mathcal{D}_j}(S_i^j)$. Note that for a fixed i we have $\epsilon_{M_i}(A_i^i) \leq \epsilon_{M_{i+1}}(A_i^{i+1}) \leq \cdots$ where all are bounded above by $\mathcal{K}_{2k}(\mathcal{M})$. Therefore, for any positive integer i, there exists a positive integer $c_i \leq \mathcal{K}_{2k}(\mathcal{M})$ where for some n_i we have $c_i = \epsilon_{M_{n_i}}(A_i^{n_i}) = \epsilon_{M_{n_i+1}}(A_i^{n_i+1}) = \cdots$. Note that we may choose the n_i such that $n_1 < n_2 < \cdots$.

Now consider the sequence of separations given by

$$J_i = (M_{n_i}, S_i^{n_i}, S^{n_i} \setminus S_i^{n_i}).$$

By Theorem 2.1 there exists a subsequence $J_{i_1} \preceq J_{i_2} \preceq \cdots$ where $\preceq$ is our well-quasi-order on matroid separations. To tidy subscripts, for $j \in \mathbb{N}$ let $n(j) = n_{i_j}$. By Theorem 2.7, for any positive integer t, there exists

(i) an integer $c_t \in \{0, 1, \cdots, \mathcal{K}_{2k}(\mathcal{M})\}$,

(ii) a subset I_t of $\{n(1), n(2), \cdots\}$ with $|I_t| = t$,

(iii) and a restriction $(M'_t, \mathcal{D}'_t) = (C'_t, D'_t; T_1^t, T_2^t, \cdots, T_t^t)$ of $(M[I_t], \mathcal{D}[I_t])$ such that $|\partial_{D'}(T_i^t) \cap T_i^t| = c_t$ for each $i \in [t]$.

Since there are a bounded number of possible values for c_t, there exists some integer $c \in \{0, 1, \cdots, \mathcal{K}_{2k}\}$ and sequence $I_{t_1}, I_{t_2}, \cdots$ for which $t_1 < t_2 < \cdots$ and the dissection given by (iii) is realized by c. By the process of construction of the J_i, the auxiliary graphs given by applying Theorem 2.7 in each step will be extensions of colorings given by previous steps. Therefore we may assume that $I_{t_1} \subseteq I_{t_2} \subseteq \cdots$.

Since $J_{i_1} \preceq J_{i_2} \preceq \cdots$, for $j \in \mathbb{N}$ there exist subsets C_j^j and D_j^j of T_j^j such that $\epsilon_{M'_j}(T_j^j \setminus (C_j^j \cup D_j^j \cup \partial_{\mathcal{D}_j}(T_j^j)) = \epsilon_{M'_1}(T_1^1 \setminus \partial_{\mathcal{D}_1}(T_1^1))$ and $r(T_j^j \setminus (C_j^j \cup D_j^j)) = r(T_1^1)$. Recall we may obtain these dissections in such a way that for positive integers i and j with $i < j$, the restriction of $\mathcal{D}'_j$ to $\bigcup_{t=1}^{i} T_i^j$ is isomorphic to $\mathcal{D}'_i$, so we may identify the C_j^j and D_j^j over the dissections. We denote $(T_i^j)' = (T_i^j)_{/C_i^i \setminus D_i^i}$.

Finally, for $j \in \mathbb{N}$, let $C(j) = \bigcup_{t=1}^{j} C_t^t$, $D(j) = \bigcup_{t=1}^{j} D_t^t$ and let $M_j' = (M_j)_{/C(j) \setminus D(j)}$. We can now calculate the density of M_j' as

$$
\begin{aligned}
d(M_j') = \frac{\epsilon(M_j')}{r(M_j')} &= \frac{\sum_{i=1}^{j} \epsilon_{M_j'}((T_i^j)')}{r(T_1^1) + \sum_{l=2}^{j}(r((T_i^j)') - \sqcap_{\mathcal{D}_j'}(i))} \\
&= \frac{\sum_{i=1}^{j}[\epsilon_{M_j'}(T_i^j \setminus \partial_{\mathcal{D}_j'}((T_i^j)')) + |\partial_{\mathcal{D}_j'}((T_i^j)')|]}{r(T_1^1) + r(T_j^j) - q + \sum_{l=2}^{j-1}(r((T_i^j)') - l)} \\
&= \frac{j(\epsilon_{M_1'}(T_1^1 \setminus \partial_{\mathcal{D}_j'}(T_1^1)) + |\partial_{\mathcal{D}_j'}((T_1^j)')|)}{(2l-q) + j(r(T_1^1) - l)} \\
&= \frac{\epsilon_{M_1'}(T_1^1)}{(r(T_1^1) - l) + \frac{(2l-q)}{j}}
\end{aligned}
$$

But we also know that as M_j' is a minor of M_j, we have that

$$
\limsup_{j \to \infty} d(M_j') = \frac{\epsilon_{M_1'}(T_1^1)}{r(T_1^1) - l} \leq \Delta.
$$

This inequality gives us that $\epsilon(H_{i_1}) = \epsilon_{M_1'}(T_1^1) \leq \Delta(r(T_1^1) - l) = \Delta(r(H_{i_1}) - l)$, which finalizes the proof that Δ is rational. $\square$

3.2 Periodicity

Let $\mathcal{M}$ be a linearly-dense minor-closed class of matroids with limiting density Δ and let i be a non-negative integer. Let $f_i : \mathcal{M} \to \mathbb{Z}$ be the function defined by $f_i(M) = \epsilon(M) - \Delta(r(M) - i)$. We further define functions $g_i : \mathcal{D}_{2k}(\mathcal{M}) \to \mathbb{Z}$ by letting $g_i(M, \mathcal{D}) = f_{\sqcap_{\mathcal{D}}(i)}(M|_{S_i})$ when S_i is defined in $\mathcal{D}$. The functions f_i and g_i will tell us how close the density of matroids and specific parts of a dissection are to the limiting density of the class we are analyzing.

We can make a few observations about the functions f_i and g_i with respect to dissections of matroids in $\mathcal{M}$. Let $(M, \mathcal{D})$ be a linked dissection of a simple matroid $M \in \mathcal{M}$. Then we can see that

$$\begin{aligned}
f_0(M) &= \epsilon(M) - \Delta(r(M)) \\
&= \sum_{i=1}^{n} \epsilon_M(S_i) - \Delta[(r(M|_{S_1})) + \sum_{i=2}^{n}(r(M|_{S_i}) - \sqcap_{\mathcal{D}}(i))] \\
&= \epsilon_M(S_1) - \Delta(r(M|_{S_1}))) + \sum_{i=2}^{n}[\epsilon_M(S_i) - \Delta(r(M|_{S_i}) - \sqcap_{\mathcal{D}}(i))] \\
&= f_0(M|_{S_1}) + \sum_{i=2}^{n} f_{\sqcap_{\mathcal{D}}(i)}(M|_{S_i}) \\
&= \sum_{i=1}^{n} g_i(M, \mathcal{D}).
\end{aligned}$$

So it is natural to think of the density of a matroid by the taking a dissection of that matroid and looking at the densities of each part. Given a part S_i of a dissection $(M, \mathcal{D})$, we call $g_i(M, \mathcal{D})$ the *contribution* of S_i to $\mathcal{D}$. We define positive, negative, zero and non-zero contribution as one naturally would. We define the *total positive contribution* to a dissection $\mathcal{D}$ as the sum of the contributions over all parts of the dissection with positive contribution. We define total negative contribution similarly.

Next, for $n \in \mathbb{N}$, we let $ex_{\mathcal{M}}(f, n) = \max\{f_0(M) : M \in \mathcal{M}, r(M) = n\}$. We note that $ex_{\mathcal{M}}(n, f) = ex_{\mathcal{M}}(n) - \Delta n$ and claim that $ex_{\mathcal{M}}(f, n)$ is bounded. So we are saying that within our given class, extremal matroids must have density which accurately represent the limiting density of the class. To do so, we start with the following observation that says the contribution of parts of dissections is bounded. This is key in the application of Theorem 3.7 as boundedness allows us to apply the general Ramsey result discussed in Section 2.3.

First, we introduce some functions. For a linked dissection $(M, \mathcal{D})$ of length n and for $i, j \in \{1, 2, \cdots, n\}$ with $i < j$, we let $p_{i,j}(M, \mathcal{D})$ be the number of parallel classes extended by collapsing $(M, \mathcal{D})$ to $(M', \mathcal{D}') = (M[I(i,j)], \mathcal{D}[I(i,j)])$ where

$$I(i,j) = [n] \setminus \{i, i+1, i+2, \cdots, j-2, j-1\}.$$

Furthermore, we let

$$g_{i,j}(M, \mathcal{D}) = \sum_{k=i}^{j-1} g_k(M, \mathcal{D}).$$

Lemma 3.3. *Let $\mathcal{M}$ be a linearly-dense minor-closed class of matroids with branch-width at most k. There exists an integer c such that if $(M,\mathcal{D}) = (C, D; S_1, S_2, \cdots, S_n)$ is a linked (l,q)-dissection in $\mathcal{D}_k(\mathcal{M})$, then for all $i \in \{1, 2, \cdots, n\}$, $g_i(M,\mathcal{D}) < c$.*

Proof. It suffices to prove this for a fixed $l, q \in \{0, 1, 2, \cdots, k\}$. Assume the opposite, that there exists a sequence of linked (l,q)-dissections $(M_1,\mathcal{D}_1)$, $(M_2,\mathcal{D}_2), \cdots$ and integers $i_1, i_2, \cdots$ such that

$$g_{i_1}(M_1,\mathcal{D}_1) < g_{i_2}(M_2,\mathcal{D}_2) < \cdots$$

and for $j \in \mathbb{N}$, let N_j be the restriction $M_j[\{i_j\}]$. For simplicity of writing, in this proof we write $g_1(M,\mathcal{D})$ when we mean $g_1(M,\mathcal{D}) - \Delta l$. We may assume that for $j \in \mathbb{N}$, any minor N'_j of N_j has $f_l(N'_j) < f_l(N_j)$. It is important to note that $r(N_j) \to \infty$ as otherwise there would exist an integer r such that $r(N_j) \leq r$ for all $j \in \mathbb{N}$ and as a result we would have

$$\lim_{j\to\infty} d(N_j) = \lim_{j\to\infty} \frac{f_l(N_j) + \Delta(r(N_j) - l)}{r(N_j)} = \infty,$$

a contradiction. Therefore we have that $f_l(N_j) \to \infty$ and $\epsilon(N_j) \to \infty$. By taking a subsequence we may assume that for $j \in \mathbb{N}$, $\epsilon(N_j) > 2^{j^{l+1}}$. So by Corollary 1.15, we may assume that each N_j has a linked (l,q)-dissection $(N_j,\mathcal{D}_j) = (C^j, D^j; S^j_1, \cdots, S^j_j)$. Furthermore, we may assume that $f_l(N_j) > j$.

Consider the linked dissection $(M_i,\mathcal{D}_i)$ of N_i and let $j, m \in \{1, 2, \cdots, i\}$ where $j < m$. Furthermore, let $I = \{j, j+1, \cdots, m-1, m\}$, $(N'_i,\mathcal{D}'_i) = (M_i \circ [I], \mathcal{D}_i \circ [I])$, and $S' = S^i_j \cup S^i_{j+1} \cup \cdots \cup S^i_{m-1} \cup S^i_m$. We can see that

$$\begin{aligned} f_l(N_i) &= \epsilon(N_i) - \Delta(r(N_i) - l) \\ &> \epsilon(N'_i) - \Delta(r(N'_i) - l) \\ &= (\epsilon(N_i) - \epsilon_{N'_i}(S') - p_{i,j}(N_i,\mathcal{D}_i)) - \Delta((r(N_i) - r(S') + l) - l)\,, \end{aligned}$$

which shows us that $g_{j,k}(N_i,\mathcal{D}_i) + p_{j,k}(N_i,\mathcal{D}_i) > 0$.

Claim 3.4. *There exists a linked (l,q)-dissection $(N_i,\mathcal{D}'_i) = (C', D'; T_1, T_2, \cdots, T_n)$ such that for $j \in \{1, 2, \cdots, n-1\}$, $g_j(N_i,\mathcal{D}'_i) > 0$.*

Proof. Let j be minimal such that $g_j(N_i,\mathcal{D}_i) \leq 0$. Then let $k \geq j$ be minimal such that $g_{k+1}(N_i,\mathcal{D}'_i) > 0$. For $t \in \{j, j+1, \cdots k\}$, let $\mathcal{P}_t$ be the set of elements in S_{t+1} which extend parallel classes when collapsing $S_j \cup S_{j+1} \cup \cdots \cup S_t$. Furthermore, for $t \in \{j, j+1, \cdots, k\}$, move $|g_{j,t}(N_i,\mathcal{D}_i)|+1$ elements from $\mathcal{P}_t$ to S_t. We note that this is possible as $|\mathcal{P}_t| + g_{j,t}(N_i,\mathcal{D}_i) > 0$. □

So for $j \in \mathbb{N}$, and $m \in [j-1]$, we have $\epsilon(N_j[\{m\}]) \geq \Delta(r(N_j|[\{m\}]) - l) + 1$. Thus by Observation 1.17 we have that

$$\begin{aligned} d(N_j) &= \frac{\sum_{m=1}^{j-1} \epsilon(N_j[\{m\}])}{\sum_{m=1}^{j-1}(r(N_j[\{m\}]) - l) + l} \\ &\geq \frac{\sum_{m=1}^{j-1}(\Delta(r(N_j[\{m\}]) - l) + 1)}{\sum_{m=1}^{j-1}(r(N_j[\{m\}]) - l) + l} \\ &\geq \Delta \frac{\sum_{m=1}^{j-1}(r(N_j[\{m\}]) - l)}{\sum_{m=1}^{j-1}(r(N_j[\{m\}]) - l) + l} + \frac{j-1}{(j-1)R + l} \end{aligned}$$

where $R = \max\{(r(H) - l) : f_l(H) > 0, H \in \mathcal{M}\}$, which exists as separations in $\mathcal{M}$ are well-quasi-ordered and f_l is entirely determined by the rank and size of a matroid. Thus we have $\lim_{j\to\infty} d(M_j) = \Delta + \frac{1}{R} > \Delta$, a contradiction. $\square$

Lemma 3.5. *Let $\mathcal{M}$ be a linearly-dense minor-closed class of matroids with branch-width at most k and let M be an extremal member of $\mathcal{M}$. There exists an integer c such that if $(M, \mathcal{D}) = (C, D; S_1, S_2, \cdots, S_n)$ is a linked (l, q)-dissection in $\mathcal{D}_k(\mathcal{M})$, then for all $i \in \{1, 2, \cdots, n\}$, $|g_i(M, \mathcal{D})| < c$.*

Proof. We see that Lemma 3.3 gives an upper bound. To show a lower bound, we look back to proof of Theorem 3.2 where we construct a sequence of matroids $M_1, M_2, \cdots$ in $\mathcal{M}$ with linked (l, q)-dissections $\mathcal{D}_1 = (C_1, D_1; S_1^1), \mathcal{D}_2 = (C_2, D_2; S_1^2, S_2^2), \cdots$ such that for each $i, j \in \mathbb{N}$, $\epsilon(M_i[\{j\}]) = \epsilon(H)$ and $r(M_i[\{j\}]) = r(H)$ for some $H \in \mathcal{M}$ satisfying that $\epsilon(H) = \Delta(r(H) - l)$. Now for $n \in \mathbb{N}$, we see that either $n \leq r(H)$ and we let $t = 1$, or there exists some integer $t \geq 2$ such that $(t-2)(r(H)-l) + r(H) < n \leq (t-1)(r(H)-l) + r(H)$. Then we may take a minor M_t' of M_t by deleting elements of S_1^t such that $r(M_t') = n$. But since we only affected elements of S_1^t, we have that $f_0(M_t') \geq f_0(H) - \epsilon(H) + (t-1)f_l(H) = f_0(H) - \epsilon(H)$.

Therefore if M is an extremal member of $\mathcal{M}$, $f_0(M)$ is bounded below. Furthermore, given a linked dissection $(M, \mathcal{D}) = (C, D; S_1, S_2, \cdots, S_n)$, since $g_i(M, \mathcal{D})$ is bounded above for each $i \in \{1, 2, \cdots, n\}$, and since $g_{i,j}(M, \mathcal{D}) = g_i(M, \mathcal{D}')$ where

$$(M, \mathcal{D}') = (C, D; S_1, S_2, \cdots S_{i-1}, T, S_j, S_{j+1}, \cdots, S_n)$$

and $T = S_i \cup S_{i+1} \cup \cdots \cup S_{j-1}$, the lemma follows from containment.

$\square$

While these lemmas are necessary for the proof of Theorem 3.7, they are also important for describing classes of dissections as we will see later. Our classification involves how close the densities of end parts of a dissection are to the limiting density of our minor-closed class. Thus, a necessary but trivial corollary is that given a dissection of an extremal matroid of our class, the contributed density of the end parts is bounded. This will allow us to apply our well-quasi-order on separations to the end parts of our dissections as well by restricting the well-quasi-order to separations with a fixed contributed density.

Corollary 3.6. *Let $\mathcal{M}$ be a linearly-dense minor-closed class of matroids with branch-width at most k. Let $(M, \mathcal{D}) = (C, D; S_1, S_2, \cdots, S_n)$ be a linked (l, q)-dissection of a matroid $M \in \mathcal{M}$. Then there exists a constant c_0 such that if M satisfies $\epsilon(M) = ex_{\mathcal{M}}(r(M))$, then $|g_1(M, \mathcal{D})| + |g_n(M, \mathcal{D})| < c_0$.*

To account for parallel elements which may arise from taking minors, we must adapt Theorem 2.7. We introduce a function which relates the contributed density of a part of a dissection to the number of parallel classes extended when applying Tutte's Linking Theorem to some parts of a dissection. For a linked dissection $(M, \mathcal{D})$ of length n and for $i, j \in \{1, 2, \cdots, n\}$, we let

$$h_{i,j}(M, \mathcal{D}) = g_j(M, \mathcal{D}) - p_{i,j}(M, \mathcal{D})$$

and note that $h_{i,j}(M, \mathcal{D}) = g_j(M', \mathcal{D}')$ where M' is the matroid obtained from M by removing elements from S_j which extended parallel classes by collapsing $S_{i+1} \cup S_{i+2} \cup \cdots \cup S_{j-1}$.

Theorem 3.7. *Let $\mathcal{M}$ be a linearly-dense minor-closed class of matroids with branch-width at most k and let $(M, \mathcal{D})$ be a linked dissection in $\mathcal{D}_k(\mathcal{M})$ that satisfies $\epsilon(M) = ex_{\mathcal{M}}(r(M))$. There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that if $I \subseteq \mathbb{N}$ is a set indexing parts of $\mathcal{D}$ with $|I| > f(n)$, then there exists a subset $I' \subseteq I$ of size $n + 1$ and an integer c such that $h_{i,j}(M, \mathcal{D}) = c$ for all $i, j \in I'$ with $i < j$.*

Proof. The proof follows the same argument of Theorem 2.7 by constructing an auxiliary graph with an edge coloring defined by h. Bounding both g and p bounds h giving a finite number of color classes, thus allowing us to apply the general ramsey theorem. □

Since the function h gives us insight as to the contributed density of parts of our dissection after taking minors, we may rephrase this theorem. It tells us that if we begin with a long enough linked dissection $(M, \mathcal{D})$ of an extremal matroid M of our class, then we can obtain a minor M' of M with a linked dissection $(M', \mathcal{D}')$ of a certain length. In addition, the contributed density of each interior part is fixed. Our next theorem shows that we may shift elements in our linked dissection to ensure the contributed density of interior parts is zero. Thus we can obtain arbitrarily long dissections for which interior parts have contributed density equal to the limiting density of our class. Then since the contributions of the two end parts of a dissection are bounded, we have some control over the density of a dissection through taking minors. However, to prove this theorem we need a lemma of Kapadia [6].

Lemma 3.8. *[Kapadia]*

Let k, c, P, and N be integers where $0 \leq c \leq P-1$. If $N \geq kP$ and $a_1, \cdots, a_N$ is a sequence of N integers, then there are integers m and l so that $l \geq k$ and $\sum_{i=m+1}^{m+l} a_i \equiv c \pmod{P}$.

Since we know that the contributed density of any part of a dissection of an extremal matroid is bounded, we can use this lemma to extract parts of our dissection with zero contributed density. This is simple by letting P be larger than the constant bounding contributed densities of parts and letting the a_i be the contributed densities of parts.

Corollary 3.9. *Let $\mathcal{M}$ be a linearly-dense minor-closed class of matroids with branch-width at most k. There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that if $(M, \mathcal{D}) = (C, D; S_1, S_2, \cdots, S_m)$ is a linked dissection in $\mathcal{D}_k(\mathcal{M})$ where $m > f(n)$, then*

(i) $(M, \mathcal{D})$ contains a linked dissection $(M, \mathcal{D}') = (C, D; T_1, T_2, \cdots, T_{m'})$,

(ii) and there exists a set $I \subseteq [m']$ of size at least n such that $g_i(M, \mathcal{D}') = 0$ for each $i \in I$.

Theorem 3.10. *Let $\mathcal{M}$ be a linearly-dense minor-closed class of matroids with branch-width at most k. There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that if $(M, \mathcal{D}) = (C, D; S_1, S_2, \cdots, S_m)$ is a linked dissection in $\mathcal{D}_k(\mathcal{M})$ where $m > f(n)$ and $\epsilon(M) = ex_{\mathcal{M}}(r(M))$, then*

(i) $(M, \mathcal{D})$ contains a linked dissection $(M, \mathcal{D}') = (C', D'; T_1, T_2, \cdots, T_{m'})$,

(ii) and there exists a set $I \subseteq [m']$ of size $n+1$ such that for $i, j \in I$ where $i < j$, $h_{i,j}(M, \mathcal{D}') = 0$.

Proof. First, we must acknowledge that there exists some integer N such that applying Theorem 3.7 to dissections of length at least N yields a non-positive c value as otherwise would suggest a contradiction to the limiting density of $\mathcal{M}$, following the same argument of Lemma 3.3. Therefore, we may assume that in all cases c is non-positive. Since $h_{i,j} = g_j - p_{i,j}$, if we can ensure that $g_j = 0$ for our indexing set, we have that $h_{i,j} = -p_{i,j}$ and both g and p are fixed over our indexing set and a shift of elements can ensure h values of 0.

Let M be an extremal matroid of $\mathcal{M}$, let α be the function described in Theorem 3.7, and let β be the function described in Corollary 3.9. Assume that $\epsilon(M) > 2^{\beta(\alpha(2n)+1)+1}$, so by Corollary 1.15, M has a linked dissection $(M, \mathcal{D}) = (C, D; S_1, S_2, \cdots, S_m)$ where $m > \beta(\alpha(2n) + 1)$. By Corollary 3.9, $(M, \mathcal{D})$ contains a linked dissection $(M, \mathcal{D}') = (C, D; T_1, T_2, \cdots, T_{m'})$ for which there exists an indexing set $I \subseteq [m']$ such that $|I| > \alpha(2n)$ and $g_i(M, \mathcal{D}) = 0$ for each $i \in I$. Thus by Theorem 3.7, there exists a subset $I' \subseteq I$ of size $2n + 2$ such that for $i, j \in I'$ with $i < j$, $h_{i,j}(M, \mathcal{D}') = c$ for some fixed integer c. We assume that $2n + 2 > N$, so c is non-positive. If $c = 0$, then we consider any subset J of I' of size n, and the linked dissection $(M, \mathcal{D}')$ and set J satisfy the desired properties, so we assume that c is negative.

Let $\{i_1, i_2, \cdots, i_{2n+2}\} = I'$ where $i_1 < i_2 < \cdots < i_{2n+2}$. Then for $t \in [n + 1]$, let $\mathcal{P}_t$ be the set of elements in $S_{i_{2t}}$ which extend parallel classes when considering the application of Tutte's Linking Theorem in obtaining $(M \circ [J], \mathcal{D}' \circ [J])$ where $J = \{i_{2t-1} + 1, i_{2t-1} + 2, \cdots, i_{2t} - 1\}$. Let $(M, \mathcal{D}'')$ be the dissection obtained from $(M, \mathcal{D}')$ by shifting the elements of $\mathcal{P}_t$ from $S_{i_{2t}}$ to $S_{i_{2t-1}}$ for each $t \in [n + 1]$. Then we have that $g_{i_{2t-1}}(M, \mathcal{D}'') = -c$ for each $t \in [n + 1]$. Finally, we let $I_{odd} = \{i_1, i_3, \cdots, i_{2n+1}\}$. Note that for $i, j \in I_{odd}$ where $i < j$,

$$p_{i,j}(M, \mathcal{D}'') = p_{i,j}(M, \mathcal{D}') = -h_{i,j}(M, \mathcal{D}') = -c$$

by construction of $(M, \mathcal{D}'')$ and Observation 2.5. Therefore, we have that

$$h_{i,j}(M, \mathcal{D}'') = g_j(M, \mathcal{D}'') - p_{i,j}(M, \mathcal{D}'') = -c - c = 0.$$

Thus the linked dissection $(M, \mathcal{D}'')$ and the set I_{odd} satisfy the desired properties, completing the proof.

□

We now look at the idea of restricting our well-quasi-order on matroid separations to separations with a fixed contributed density. For $l \in \{0, 1, 2, \cdots, 2k\}$, let $\mathcal{C}^l \subseteq \mathcal{S}_l(\mathcal{M})$ be the set of l-separations of matroids in $\mathcal{M}$ with zero contribution in an (l, q)-dissection, that is,

$$\mathcal{C}^l = \{(M, A, B) : M \in \mathcal{M}, \epsilon_M(A) = \Delta_{\mathcal{M}}(r(A) - l)\}.$$

Recall that the product of well-quasi-orders is a well-quasi-order, and define the well-quasi-order $(\mathcal{S}_l(\mathcal{M})\times\mathbb{Z}, \preceq')$ where $(M, A, B) \preceq' (M', A', B')$ if both $(M, A, B) \preceq (M', A', B')$ and $|\partial_M(A, B) \cap A| = |\partial_{M'}(A', B') \cap A'|$. Note that then if $(M, A, B) \preceq (M', A', B')$, then the separation obtained from (M', A', B') satisfying $\preceq'$ has the same contribution as (M, A, B). We will see this well-quasi-order come into play in Theorem 3.13.

So for $l \in \{0, 1, 2, \cdots, k\}$, we let $\mathcal{C}^l_{min}$ be the set of minimal elements of $\mathcal{C}^l$ with respect to the well-quasi-order $\preceq'$ and note that $\mathcal{C}^l_{min}$ is finite. We now let P be the integer defined by

$$P = \prod_{l=0}^{k} \prod_{(M,A,B)\in\mathcal{C}^l_{min}} (r(A) - l).$$

To prove Theorem 1.2, we describe classes of dissections which satisfy the desired periodic behavior and show that finitely many of these classes cover all but finitely many extremal values of $\mathcal{M}$. Therefore we see that $\mathcal{M}$ also satisfies the desired periodic behavior. Let A, B, and C be separations in $\mathcal{S}_{2k}(\mathcal{M})$. Then we let $\langle A, B, C\rangle$ denote the class of linked (l, q)-dissections $(M, \mathcal{D}) = (C, D; S_1, S_2, \cdots, S_n)$ in $\mathcal{D}_k(\mathcal{M})$ such that $(M, S_1, S \setminus S_1) \cong A$, $(M, S_n, S \setminus S_n) \cong C$, and for $i \in \{2, 3, \cdots, n-1\}$, $(M, S_i, S \setminus S_i) \cong B$ with respect to $\preceq'$.

Further, we let $\mathcal{T}_k(\mathcal{M})$ be the set of all such $\langle A, B, C\rangle$ for which $B \in \mathcal{C}_{min}$. Since $\preceq_{\mathcal{D}}$ is a well-quasi-order, $\mathcal{T}_k(\mathcal{M})$ is a finite set up to congruence in $\preceq_{\mathcal{D}}$. The final observations we need for our next theorem are with regards to the contributed density of parts of dissections after taking minors.

Observation 3.11. *Let $(M, \mathcal{D}) = (C, D; S_1, S_2, \cdots, S_n)$, let $i, j \in \{2, 3, \cdots, n-1\}$ where $i < j$, and let $(M', \mathcal{D}') = (M \circ [\{i, i+1, \cdots, j-1\}], \mathcal{D} \circ [\{i, i+1, \cdots, j-1\}])$. Then $f_0(M') = f_0(M) - g_{i,j}(M, \mathcal{D}) - p_{i,j}(M, \mathcal{D})$.*

We can use this observation to see what happens when we collapse a dissection to an indexed set of parts.

Observation 3.12. *Let $(M, \mathcal{D}) = (C, D; S_1, S_2, \cdots, S_n)$ be a linked (l, q)-dissection, and let $I \subseteq [n]$ where $I = \{i_1, i_2, \cdots, i_m\}$ and $i_1 < i_2 < \cdots < i_m$. Furthermore, let $(M', \mathcal{D}') = (M[I'], \mathcal{D}[I'])$ where*

$$I' = I \cup \{1, 2, \cdots, i_1 - 1\} \cup \{i_m + 1, i_m + 2, \cdots, n\}.$$

Then

$$f_0(M') = f_0(M) - g_{i_1,i_m}(M, \mathcal{D}) + \sum_{j=1}^{m-1} h_{i_j,i_{j+1}}(M, \mathcal{D}).$$

Proof. We define the parts of $(M', \mathcal{D}')$ as

$$(C', D'; T_1, T_2, \cdots, T_{i_1}, T_{i_1}+1, \cdots, T_{i_1}+m-1, T_{i_1}+m, \cdots, T_{n'}).$$

Then for $j \in \{0, 1, \cdots, m-1\}$,

$$g_{i_1+j}(M', \mathcal{D}') = h_{i_1+j, i_1+(j+1)}(M, \mathcal{D}).$$

So we have that

$$\begin{aligned}
f_0(M') &= \sum_{s=1}^{n'} g_s(M', \mathcal{D}') \\
&= \sum_{s=1}^{i_1-1} g_s(M', \mathcal{D}') + \sum_{s=i_1+m}^{n'} g_s(M', \mathcal{D}') + \sum_{j=0}^{m-2} g_{i_1+j}(M', \mathcal{D}') \\
&= \sum_{s=1}^{i_1-1} g_s(M, \mathcal{D}) + \sum_{s=i_m}^{n} g_s(M, \mathcal{D}) + \sum_{j=1}^{m-1} h_{i_j, i_{j+1}}(M, \mathcal{D}) \\
&= \sum_{s=1}^{n} g_s(M, \mathcal{D}) - g_{i_1, i_m}(M, \mathcal{D}) + \sum_{j=1}^{m-1} h_{i_j, i_{j+1}}(M, \mathcal{D}) \\
&= f_0(M) - g_{i_1, i_m}(M, \mathcal{D}) + \sum_{j=1}^{m-1} h_{i_j, i_{j+1}}(M, \mathcal{D})
\end{aligned}$$

□

Theorem 3.13. *Let $\mathcal{M}$ be a linearly-dense minor-closed class of matroids with branch-width at most k. There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that if M satisfies that $\epsilon(M) = ex_{\mathcal{M}}(r(M)) > f(n)$, then for some $\langle A, B, C\rangle$ in $\mathcal{T}_k(\mathcal{M})$, for any non-negative integer $n_0 \leq n+2$, M has a minor M' with a simple linked (l, q)-dissection $(M', \mathcal{D}')$ such that:*

(i) $(M', \mathcal{D}')$ *has length* n_0,

(ii) $f_0(M) = f_0(M')$,

(iii) $(M', \mathcal{D}') \in \langle A, B, C\rangle$,

(iv) *and the linked dissection* $(M', \mathcal{D}')$ *of length* $n+2$ *satisfies* $r(M) \equiv r(M') \pmod{P}$.

Proof. Let $N' = nP^2 + 1$, $N'' = |\mathcal{C}_{min}|N' + 2$, and let $N''' = 2cN''$ where c is the constant given by Lemma 3.5. Furthermore, let α be the function described in Theorem 3.10, let β be the function described in Theorem 2.6, and let M be an extremal matroid of $\mathcal{M}$ with $\epsilon(M) > 2^{\beta(\alpha(N''')+1)}$. By Corollary 1.15, for some $q \leq k$, M has a linked q-dissection of length at least $\beta(\alpha(N''') + 1)$. Then by Theorem 2.6, for some $l \leq q$, M has a linked (l, q)-dissection of length at least $\alpha(N''') + 1$.

So by Theorem 3.10, M has a linked (l, q)-dissection $(M, \mathcal{D}) = (C, D; S_1, \cdots, S_m)$ such that there exists a set $I \subseteq [m]$ of size $N''' + 1$ satisfying (ii) of Theorem 3.10. For $j \in [N''']$, let $a_j = g_{i_j, i_{j+1}}(M, \mathcal{D})$. By Lemma 3.8, there exists an integer $t \in [N''']$ and an integer $k \geq N''$ such that $\sum_{i=t}^{t+k} a_i \equiv 0 \pmod{2c}$. Using Observation 3.12, we see that it must be that $\sum_{i=t}^{t+k} a_i = 0$ as otherwise would imply that either $|g_{i_t, i_{t+k}}(M, \mathcal{D})| > 2c$, or $|g_{1, i_t}(M, \mathcal{D})| + |g_{i_{t+k}, m}(M, \mathcal{D})| > 2c$, both of which contradict Lemma 3.5.

So we consider the minor $(M', \mathcal{D}') = (M[I'], \mathcal{D}[I'])$ where

$$\begin{aligned} I' = \{1, 2, \cdots, i_t\} &\cup \{i_{t+1}, i_{t+2}, \cdots, i_{t+k}\} \\ &\cup \{i_{t+k} + 1, i_{t+k} + 2, \cdots, m\}. \end{aligned}$$

So $(M', \mathcal{D}') = (C', D'; T_1, T_2, \cdots, T_{N''+2})$ where $T_1 = S_1 \cup S_2 \cup \cdots \cup S_{i_t - 1}$, $T_{N''+2} = S_{i_{t+k}+1} \cup S_{i_{t+k}+2} \cup \cdots \cup S_n$ and for $j \in \{2, 3, \cdots, N'' + 1\}$, $T_j = S_{i_j}$. We know from the previous paragraph and Observation 3.12 that $f_0(M') = f_0(M)$. Furthermore, from the properties of h, we know that for $i \in \{2, 3, \cdots, N'' + 1\}$, $g_i(M', \mathcal{D}') = 0$.

For $i \in \{2, \cdots, N''+1\}$, there exists an element (M_i, A_i, B_i) in $\mathcal{C}_{min}$ such that $(M', T_i, T \setminus T_i)$ conforms to (M_i, A_i, B_i) under the order $\preceq'$. Then by definition of N'', there exist integers $2 \leq i_1 < i_2 < \cdots < i_{N'} \leq N'' + 1$ such that for each $j \in [N']$, $(M', T_{i_j}, T \setminus T_{i_j})$ conforms to the same element in $\mathcal{C}^l_{min}$. For the remainder of this proof, we call that element (N, A, B), and we let $I'' = \{i_1, i_2, \cdots, i_{N'}\}$. Furthermore, we let $\tilde{C} = \bigcup_{i \in I''} \tilde{C}_i$ and $\tilde{D} = \bigcup_{i \in I''} \tilde{D}_i$ where $\tilde{C}_i$ and $\tilde{D}_i$ are the deletion and contraction sets given by the relation $(N, A, B) \preceq' (M', T_i, T \setminus T_i)$.

Now for $j \in [N' - 1]$, let $a_j = r_M(S_{i_j+1} \cup S_{i_j+2} \cup \cdots \cup S_{i_{j+1}-1}) + l$. Then by Lemma 3.8, there exists an integer m such that $\sum_{j=m}^{m+(nP-1)} a_i \equiv r(M') - r(M) \pmod{P}$. We then collapse the dissection $(M', \mathcal{D}')$ to $(M'', D'') = (M[J], D[J])$ where

$$\begin{aligned} J = \{1, 2, \cdots, i_m\} &\cup \{i_{m+1}, i_{m+2}, \cdots, i_{m+nP-1}\} \\ &\cup \{i_{m+nP}, i_{m+nP} + 1, \cdots, N'\}. \end{aligned}$$

Note that by (ii) of Theorem 3.7, we have $f_0(M'') = f_0(M') = f_0(M)$, and since $\sum_{j=m}^{m+(nP-1)} a_j \equiv$

$r(M') - r(M) \pmod P$, we have

$$\begin{aligned} r(M'') &\equiv r(M') - (r(M') - r(M)) \\ &\equiv r(M) \qquad\qquad \pmod P. \end{aligned}$$

For ease of notation, we relabel $(M'', \mathcal{D}'')$ by $(M', \mathcal{D}')$ and we relabel the parts of $\mathcal{D}'$ to $(S_1, S_2, \cdots, S_t)$ and let $i = i_m$. In addition, we note that $(M', \mathcal{D}')$ contains the dissection $(M', \mathcal{D}'') = (T_1, T_2, \cdots, T_{nP+2})$ where $T_1 = S_1 \cup S_2 \cup \cdots S_{i-1}$, $T_{nP+2} = S_{i+nP} \cup S_{i+nP+1} \cup \cdots \cup S_{N'}$, and for $j \in \{2, 3, \cdots, nP+1\}$, $T_j = S_{i+j-2}$. So $(M', \mathcal{D}'')$ is a linked (l, q)-dissection of length $nP + 2$ satisfying (ii) of Theorem 3.7 and for $i \in \{2, 3, \cdots, nP+1\}$, $(M', T_i, T \setminus T_i)$ conforms to (N, A, B). Again for ease of notation, we relabel the dissection $(M', \mathcal{D}'')$ by $(M, \mathcal{D})$.

Next, for $j \in \{2, 3, \cdots nP+1\}$, we let $b_j = r(T_j) - r_0$ where $r_0 = r_N(A)$ of the element (N, A, B). This definition of b_j along with Lemma 3.8 will allow us to ensure that our dissection conforms to a dissection of the same rank modulo P. By Lemma 3.8, there exists some integer $s \in \{0, 1, \cdots, n(P-1)\}$ such that $\sum_{j=s}^{s+(n-1)} b_j \equiv 0 \pmod P$. We then see that $(M, \mathcal{D})$ contains the dissection $(M, \mathcal{D}') = (T', T_{s+1}, T_{s+2}, \cdots, T_{s+n-1}, T'')$ where $T' = T_1 \cup T_2 \cup \cdots T_s$ and $T'' = T_{s+n} \cup T_{s+n+1} \cup \cdots \cup T_{nP+2}$. Finally, we relabel $\tilde{C}$ and $\tilde{D}$ to be the restrictions of $\tilde{C}$ and $\tilde{D}$ to $T_{s+1} \cup T_{s+2} \cup \cdots \cup T_{s+n-1}$, and let $(M', \mathcal{D}'')$ be the linked (l, q)-dissection obtained by contracting $\tilde{C}$ and deleting $\tilde{D}$ in the linked (l, q)-dissection $(M, \mathcal{D}')$.

We can see that $(M', \mathcal{D}'')$ has length n, satisfies (ii) of Theorem 3.7, and for $i \in \{1, 2, \cdots, n-1\}$, $(M', T_{s+i}, T \setminus T_{s+i}) \cong (N, A, B)$ with respect to our well-quasi-order on separations. So (i) and (iii) are satisfied. Furthermore, by construction (ii) and (iv) are satisfied, completing the proof.

$\square$

We now explain how we get from obtaining these structured minors to obtaining extremal matroids of our class. Let n be a natural number. We define $EX_{\mathcal{M}}(n)$ be the set of all extremal members of $\mathcal{M}$ with rank n. Then for any triple $\langle A, B, C \rangle$ in $\mathcal{T}_k(\mathcal{M})$, we say that $\langle A, B, C \rangle$ *covers* n if we may find maximal length dissections of the form described in Theorem 3.13 from a member of $EX_{\mathcal{M}}(n)$. More precisely, there exists an integer m such that for the function f described in Theorem 3.13, $f(m) < n \leq f(m+1)$, and a matroid $M \in EX_{\mathcal{M}}(n)$ satisfies that for any $m_0 \leq m$, M has a minor with a linked dissection satisfying (i),(ii),(iii), and (iv) of Theorem 3.13. So we may rephrase Theorem 3.13. Given an extremal matroid of our class which is large enough, it is covered by some triple in $\mathcal{T}_k(\mathcal{M})$ which preserves its f_0 value.

We say that $\langle A, B, C\rangle$ realizes n if there exists a linked dissection $(M, \mathcal{D})$ in $\mathcal{D}_k(\mathcal{M})$ such that M is extremal in $\mathcal{M}$, $r(M) = n$, and $(M, \mathcal{D})$ is in $\langle A, B, C\rangle$. We can see that if $(M, \mathcal{D})$ is a linked (l, q)-dissection in $\langle A, B, C\rangle$ is simple, then

$$\begin{aligned} f_0(M) &= f_0(A) + (n-2)f_l(B) + f_q(C) \\ &= f_0(A) + f_q(C) \\ &= \epsilon_M(A) + \epsilon_M(C) - \Delta(r(A) + r(C) - q). \end{aligned}$$

Thus we see that for a triple $\langle A, B, C\rangle$ of $\mathcal{T}_k(\mathcal{M})$, $f_0(M)$ is fixed for any linked dissection $(M, \mathcal{D})$ in $\langle A, B, C\rangle$. The following lemmas are the last components to the proof of Theorem 1.2.

Lemma 3.14. *Let $\mathcal{M}$ be a linearly-dense minor-closed class of matroids with branch-width at most k and let I be an infinite subset of $\mathbb{N}$. If $\langle A, B, C\rangle$ covers I, then $\langle A, B, C\rangle$ realizes I.*

Proof. We can see that if we wish to realize some $i \in I$, we may take some $i' \in I$ where $i' > f(i)$ as described in Theorem 3.13 and look at the set of covers of the form $\langle A, B, C\rangle$. We then take the set of covers given by a dissection of an element of $EX_{\mathcal{M}}(i)$ of the form $\langle A, B, C\rangle$. Then by a simple comparison of the relations $(i) - (iv)$ with the fact that f_0 is fixed for these matroids since they came from covers, we see that one of our covers given by i' realizes i. □

Lemma 3.15. *Let $\mathcal{M}$ be a linearly-dense minor-closed class of matroids with branch-width at most k and let I be an infinite subset of $\mathbb{N}$. Then there exists an infinite subset I' of I and a triple $\langle A, B, C\rangle$ of $\mathcal{T}_k(\mathcal{M})$ such that $\langle A, B, C\rangle$ realizes I'.*

Proof. Let I be an infinite subset of $\mathbb{N}$. Since we know that $\mathcal{T}_k(\mathcal{M})$ has finitely many elements, there exists some triple $\langle A, B, C\rangle$ in $\mathcal{T}_k(\mathcal{M})$ such that for some $I' \subseteq I$, $\langle A, B, C\rangle$ covers I'. Now let $\mathcal{T}_k(\mathcal{M}, I')$ be the set of all such triples. Note that by Corollary 3.6, if $(M, \mathcal{D})$ is a length n dissection of some triple of $\mathcal{T}_k(\mathcal{M})$, then $g_1(M, \mathcal{D}) + g_n(M, \mathcal{D})$ is bounded. Therefore, we may let $\langle A, B, C\rangle$ be a triple in $\mathcal{T}_k(\mathcal{M}, I')$ which maximizes this value. By Lemma 3.14, we have that $\langle A, B, C\rangle$ realizes an infinite subset of I. □

Therefore we have finitely many triples $\langle A, B, C\rangle$ in $\mathcal{T}_k(\mathcal{M})$ which cover all but finitely many of the natural numbers. We have seen each triple $\langle A, B, C\rangle$ satisfies that a linked dissection $(M, \mathcal{D})$ in $\langle A, B, C\rangle$ has $\epsilon(M) = \Delta r(M) + a$ for some fixed rational a. The union of finitely many such functions yields a function as described in 1.2, completing the proof.

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