

Zero Dynamics for Port-Hamiltonian Systems¹

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Abstract

The zero dynamics of infinite-dimensional systems can be difficult to characterize. The zero dynamics of boundary control systems are particularly problematic. In this paper the zero dynamics of port-Hamiltonian systems are studied. A complete characterization of the zero dynamics for a port-Hamiltonian systems with invertible feedthrough as another port-Hamiltonian system on the same state space is given. It is shown that the zero dynamics for any port-Hamiltonian system with commensurate wave speeds are well-defined, and are also a port-Hamiltonian system. Examples include wave equations with uniform wave speed on a network. A constructive procedure for calculation of the zero dynamics, that can be used for very large system order, is provided.

Keywords: Port-Hamiltonian system, distributed parameter systems, boundary control, zero dynamics, networks, coupled wave equations.

1. Introduction

The zeros of a system are well-known to be important to controller design; see for instance, the textbooks [6, 22]. For example, the poles of a system controlled with a constant feedback gain move to the zeros of the open-loop system as the gain increases. Furthermore, regulation is only possible if the zeros of the system do not coincide with the poles of the signal to be tracked. Another example is sensitivity reduction - arbitrary reduction of sensitivity is only possible if all the zeros are in the left half-plane. Right half-plane zeros restrict the achievable performance; see for example, [6]. The inverse of a system without right-hand-plane zeros can be approximated by a stable system, such systems are said to be minimum-phase and they are typically easier to control than non-minimum phase systems.

The zero dynamics are the dynamics of the system obtained by choosing the input u so that the output y is identically 0. This will only be possible for initial conditions in some subspace of the original subspace. For linear systems with ordinary differential equation models and a minimal realization, the eigenvalues of the zero dynamics correspond to the zeros of the transfer function. Zero dynamics are well understood for linear finite-dimensional systems, and have been extended to nonlinear finite-dimensional systems [10].

However, many systems are modeled by delay or partial differential equations. This leads to an infinite-dimensional state space, and also an irrational transfer function. As for finite-dimensional systems, the zero dynamics are important. The notion of minimum-phase can be extended to infinite-dimensional systems; see in particular [11] for a detailed study of conditions for second-order systems. Care needs to be taken since a system can have no right-hand-plane zeros and still

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fail to be minimum-phase. The simplest such example is a pure delay. Results on adaptive control and on high-gain feedback control of infinite-dimensional systems, see e.g. [18, 19, 20, 21, 25], require the system to be minimum-phase. Moreover, the sensitivity of an infinite-dimensional minimum-phase system can be reduced to an arbitrarily small level and stabilizing controllers exist that achieve arbitrarily high gain or phase margin [8].

Since the zeros are often not accurately calculated by numerical approximations [4, 5, 9, 17] it is useful to obtain an understanding of their behaviour in the original infinite-dimensional context. For infinite-dimensional systems with bounded control and observation, the zero dynamics have been calculated, although they are not always well-posed [23, 24, 30].

There are few results for zero dynamics for partial differential equations with boundary control and point observation. In [2, 3] the zero dynamics are found for a class of parabolic systems defined on an interval with collocated boundary control and observation. This was extended to the heat equation on an arbitrary region with collocated control and observation in [26]. In [15] the invariant zeros for a class of systems with analytic semigroup that includes boundary control/point sensing are defined and analysed.

The zero dynamics of an important class of boundary control systems, port-Hamiltonian systems [13, 16, 28] or systems of linear conservation laws [1], are established in this paper. Such models are derived using a variational approach and many situations of interest, in particular waves and vibrations, can be described in a port-Hamiltonian framework. A complete characterization of the zero dynamics for port-Hamiltonian systems with commensurate wave speeds is obtained. For any port-Hamiltonian system systems with invertible feedthrough, the zero dynamics are another port-Hamiltonian system on the same state space. Port-Hamiltonian systems with commensurate wave speeds can be written as a coupling of scalar systems with the same wave speed. For these systems the zero dynamics are shown to be well-defined, and are in fact a new port-Hamiltonian system. Preliminary versions of Theorem 2.5 (for constant coefficients) and of Theorem 3.3 (with an outline of the proof) appeared in [12].

A constructive procedure for calculation of the zero dynamics based on linear algebra is provided. This algorithm can be used on large networks, and does not use any approximation of the system of partial differential equations. The results are illustrated with several examples.

2. Port-Hamiltonian Systems

Consider systems of the form

$$\frac{\partial x}{\partial t}(\zeta, t) = P_1 \frac{\partial}{\partial \zeta} (\mathcal{H}(\zeta)x(\zeta, t)), \quad \zeta \in (0, 1), t \geq 0 \quad (1)$$

$$x(\zeta, 0) = x_0(\zeta), \quad \zeta \in (0, 1) \quad (2)$$

$$0 = W_{B,1} \begin{bmatrix} (\mathcal{H}x)(1, t) \\ (\mathcal{H}x)(0, t) \end{bmatrix}, \quad t \geq 0 \quad (3)$$

$$u(t) = W_{B,2} \begin{bmatrix} (\mathcal{H}x)(1, t) \\ (\mathcal{H}x)(0, t) \end{bmatrix}, \quad t \geq 0 \quad (4)$$

$$y(t) = W_C \begin{bmatrix} (\mathcal{H}x)(1, t) \\ (\mathcal{H}x)(0, t) \end{bmatrix}, \quad t \geq 0, \quad (5)$$

where P_1 is an Hermitian invertible $n \times n$ -matrix, $\mathcal{H}(\zeta)$ is a positive $n \times n$ -matrix for a.e. $\zeta \in (0, 1)$ satisfying $\mathcal{H}, \mathcal{H}^{-1} \in L^\infty(0, 1; \mathbb{C}^{n \times n})$, and $W_B := \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix}$ is a $n \times 2n$ -matrix of rank n . Such systems are said to be *port-Hamiltonian*, see [16, 28, 13], or systems of linear conservation laws [1]. Here, $x(\cdot, t)$ is the state of the system at time t , $u(t)$ represents the input of the system at time t and $y(t)$ the output of the system at time t .

A different representation of port-Hamiltonian systems, the diagonalized form, will be used. The matrices $P_1 \mathcal{H}(\zeta)$ possess the same eigenvalues counted according to their multiplicity as the matrix $\mathcal{H}^{1/2}(\zeta) P_1 \mathcal{H}^{1/2}(\zeta)$, and as $\mathcal{H}^{1/2}(\zeta) P_1 \mathcal{H}^{1/2}(\zeta)$ is diagonalizable the matrix $P_1 \mathcal{H}(\zeta)$ is diagonalizable

as well. Moreover, by our assumptions, zero is not an eigenvalue of $P_1\mathcal{H}(\zeta)$ and all eigenvalues are real, that is, there exists an invertible matrix $S(\zeta)$ such that

$$P_1\mathcal{H}(\zeta) = S^{-1}(\zeta) \underbrace{\text{diag}(p_1(\zeta), \dots, p_k(\zeta), n_1(\zeta), \dots, n_l(\zeta))}_{=:\Delta(\zeta)} S(\zeta).$$

Here $p_1(\zeta), \dots, p_k(\zeta) > 0$ and $n_1(\zeta), \dots, n_l(\zeta) < 0$. In the remainder of this article it is assumed that S and Δ are continuously differentiable on $(0, 1)$. Introducing the new state vector

$$z(\zeta, t) = \begin{bmatrix} z_+(\zeta, t) \\ z_-(\zeta, t) \end{bmatrix} = S(\zeta)x(\zeta, t), \quad \zeta \in [0, 1],$$

with $z_+(\zeta, t) \in \mathbb{C}^k$ and $z_-(\zeta, t) \in \mathbb{C}^l$, and writing

$$\Delta(\zeta) = \begin{bmatrix} \Lambda(\zeta) & \mathbf{0} \\ \mathbf{0} & \Theta(\zeta) \end{bmatrix},$$

where $\Lambda(\zeta)$ is a positive definite $k \times k$ -matrix and $\Theta(\zeta)$ is a negative definite $l \times l$ -matrix, the system (1)–(5) can be equivalently written as

$$\frac{\partial}{\partial t} z(\zeta, t) = \frac{\partial}{\partial \zeta} (\Delta(\zeta)z(\zeta, t)) + S(\zeta) \frac{S^{-1}(\zeta)}{d\zeta} \Delta(\zeta)z(\zeta, t), \quad (6)$$

$$z(\zeta, 0) = z_0(\zeta), \quad \zeta \in (0, 1) \quad (7)$$

$$\begin{bmatrix} 0 \\ u(t) \end{bmatrix} = \underbrace{\begin{bmatrix} K_{0+} & K_{0-} \\ K_{u+} & K_{u-} \end{bmatrix}}_K \begin{bmatrix} \Lambda(1)z_+(1, t) \\ \Theta(0)z_-(0, t) \end{bmatrix} + \underbrace{\begin{bmatrix} L_{0+} & L_{0-} \\ L_{u+} & L_{u-} \end{bmatrix}}_L \begin{bmatrix} \Lambda(0)z_+(0, t) \\ \Theta(1)z_-(1, t) \end{bmatrix}, \quad (8)$$

$$y(t) = \underbrace{\begin{bmatrix} K_{y+} & K_{y-} \end{bmatrix}}_{K_y} \begin{bmatrix} \Lambda(1)z_+(1, t) \\ \Theta(0)z_-(0, t) \end{bmatrix} + \underbrace{\begin{bmatrix} L_{y+} & L_{y-} \end{bmatrix}}_{L_y} \begin{bmatrix} \Lambda(0)z_+(0, t) \\ \Theta(1)z_-(1, t) \end{bmatrix}, \quad (9)$$

where $t \geq 0$ and $\zeta \in (0, 1)$.

Next, consider well-posedness of the system (6)–(9), or equivalently of system (1)–(5). Well-posedness means that for every initial condition $z_0 \in L^2(0, 1; \mathbb{C}^n)$ and every input $u \in L^2_{\text{loc}}(0, \infty; \mathbb{C}^p)$ the mild solution z of the system (6)–(8) is well-defined in the state space $X := L^2(0, 1; \mathbb{C}^n)$ and the output (9) is well-defined in $L^2_{\text{loc}}(0, \infty; \mathbb{C}^m)$. See [13] for the precise definition and further results on well-posedness of port-Hamiltonian systems. To characterize well-posedness, define the matrices

$$K = \begin{bmatrix} K_0 \\ K_u \end{bmatrix} = \begin{bmatrix} K_{0+} & K_{0-} \\ K_{u+} & K_{u-} \end{bmatrix}, \quad L = \begin{bmatrix} L_0 \\ L_u \end{bmatrix} = \begin{bmatrix} L_{0+} & L_{0-} \\ L_{u+} & L_{u-} \end{bmatrix}.$$

Theorem 2.1. [31], [13, Thm. 13.2.2 and 13.3.1]. *The following are equivalent*

1. *The system (6)–(9) is well-posed on $L^2(0, 1; \mathbb{C}^n)$;*
2. *For every initial condition $z_0 \in L^2(0, 1; \mathbb{C}^n)$, the partial differential equation (6)–(8) with $u = 0$ possesses a unique mild solution on the state space $L^2(0, 1; \mathbb{C}^n)$. Furthermore, this solution depends continuously on the initial condition;*
3. *The matrix K is invertible.*

Thus, well-posedness of a port-Hamiltonian system is equivalent to well-posedness of a homogeneous partial differential equation; the boundedness of the input/state and state/output maps does not need to be checked separately. For the remainder of this paper it is assumed that K is invertible so that the control system is well-posed. The corresponding generator A of the C_0 -semigroup of the homogeneous system is given by [13]

$$Af = -(\Delta f)' + S(S^{-1})'\Delta f,$$

$$D(A) = \left\{ \Delta f \in H^1(0, 1; \mathbb{C}^n) \mid \begin{bmatrix} 0 \\ 0 \end{bmatrix} = K \begin{bmatrix} \Lambda(1)f_+(1) \\ \Theta(0)f_-(0) \end{bmatrix} + L \begin{bmatrix} \Lambda(0)f_+(0) \\ \Theta(1)f_-(1) \end{bmatrix} \right\}.$$

The resolvent operator of A is compact, and thus the spectrum of A contains only eigenvalues. For port-Hamiltonian systems, well-posedness implies that the system (6)–(9) is also regular, i.e, the transfer function $G(s)$ possesses a limit over the real line, see [31] or [13, Section 13.3]. Writing

$$K_y K^{-1} = \begin{bmatrix} * & E \end{bmatrix} \quad (10)$$

with $E \in \mathbb{C}^{m \times p}$, this limit of $G(s)$ over the real axis is E , see [13, Theorem 13.3.1].

Now consider zero dynamics for port-Hamiltonian systems.

Definition 2.2. *Consider the system (6)–(9) on the state space $X = L^2(0, 1; \mathbb{C}^n)$. The zero dynamics of (6)–(9) are the pairs $(z_0, u) \in X \times L^2_{\text{loc}}(0, \infty; \mathbb{C}^p)$ for which the mild solution of (6)–(9) satisfies $y = 0$. The largest output nulling subspace is*

$$V^* = \{z_0 \in X \mid \text{there exists a function } u \in L^2_{\text{loc}}(0, \infty; \mathbb{C}^p) : \\ \text{the mild solution of (6)–(9) satisfies } y = 0\}.$$

Thus, V^* is the space of initial conditions for which there exists a control u that “zeroes” the output.

Setting $y = 0$ in (9) reveals that the zero dynamics are described by

$$\frac{\partial}{\partial t} z(\zeta, t) = \frac{\partial}{\partial \zeta} (\Delta(\zeta) z(\zeta, t)) + S(\zeta) \frac{S^{-1}(\zeta)}{d\zeta} \Delta(\zeta) z(\zeta, t), \quad (11)$$

$$z(\zeta, 0) = z_0(\zeta), \quad \zeta \in (0, 1) \quad (12)$$

$$0 = \begin{bmatrix} K_0 \\ K_y \end{bmatrix} \begin{bmatrix} \Lambda(1)z_+(1, t) \\ \Theta(0)z_-(0, t) \end{bmatrix} + \begin{bmatrix} L_0 \\ L_y \end{bmatrix} \begin{bmatrix} \Lambda(0)z_+(0, t) \\ \Theta(1)z_-(1, t) \end{bmatrix}, \quad (13)$$

$$u(t) = K_u \begin{bmatrix} \Lambda(1)z_+(1, t) \\ \Theta(0)z_-(0, t) \end{bmatrix} + L_u \begin{bmatrix} \Lambda(0)z_+(0, t) \\ \Theta(1)z_-(1, t) \end{bmatrix}, \quad (14)$$

where $t \geq 0$ and $\zeta \in (0, 1)$. Note that system (11)–(14) is still in the format of a port-Hamiltonian system, but even regarding (14) as the (new) output, it needs not to be a well-posed port-Hamiltonian system since the new “ K -matrix”, $\begin{bmatrix} K_0 \\ K_y \end{bmatrix}$ can have rank less than n . The zero dynamics are a well-posed dynamical system if the system (11)–(14) with state-space V^* , no input and output u is well-posed.

The eigenvalues of the zero dynamics of the system are closely related to the invariant and transmission zeros of the system. For simplicity only the single-input single-output case is considered ($p = m = 1$).

Definition 2.3. [26, 4] *A complex number $\lambda \in \mathbb{C}$ is an invariant zero of the system (6)–(9) on the state space $X = L^2(0, 1; \mathbb{C}^n)$, if there exist $z \in H^1(0, 1; \mathbb{C}^n)$ and $u \in \mathbb{C}$ such that*

$$\lambda z(\zeta) = \frac{\partial}{\partial \zeta} (\Delta(\zeta) z(\zeta)) + S(\zeta) \frac{S^{-1}(\zeta)}{d\zeta} \Delta(\zeta) z(\zeta), \\ 0 = \begin{bmatrix} K_0 \\ K_y \end{bmatrix} \begin{bmatrix} \Lambda(1)z_+(1) \\ \Theta(0)z_-(0) \end{bmatrix} + \begin{bmatrix} L_0 \\ L_y \end{bmatrix} \begin{bmatrix} \Lambda(0)z_+(0) \\ \Theta(1)z_-(1) \end{bmatrix}, \\ u = K_u \begin{bmatrix} \Lambda(1)z_+(1) \\ \Theta(0)z_-(0, t) \end{bmatrix} + L_u \begin{bmatrix} \Lambda(0)z_+(0, t) \\ \Theta(1)z_-(1, t) \end{bmatrix},$$

Definition 2.4. *A complex number $s \in \mathbb{C}$ is a transmission zero of the system (6)–(9) if the transfer function satisfies $G(s) = 0$.*

If $\lambda \in \rho(A)$, where $\rho(A)$ denotes the resolvent set of A , then λ is an invariant zero if and only if λ is a transmission zero [13, Theorem 12.2.1]. Moreover, if the zero dynamics is well-posed, then the spectrum of the corresponding generator equals the set of invariant zeros of the system (6)–(9).

If the feedthrough operator of the original system is invertible, then the zero dynamics system is well-posed on the entire state space, and is also a port-Hamiltonian system.

Theorem 2.5. *Assume that the system has the same number of inputs as outputs. Then the zero dynamics are well-posed on the entire state space if and only if the feedthrough operator E of the original system is invertible.*

Proof: This was proven in [12] in the case of a constant coefficient matrix \mathcal{H} . The proof presented here is more complete, and includes the generalization to variable coefficients. The feedthrough operator E of the original system is given by $[* E] = K_y K^{-1}$ (see (10)). It will first be shown that invertibility of E is equivalent to invertibility of the “ K -matrix” of equation (13):

$$\tilde{K} := \begin{bmatrix} K_0 \\ K_y \end{bmatrix}.$$

If E is singular, then there is $u \neq 0$ in the kernel of E , and

$$K_y K^{-1} \begin{bmatrix} 0 \\ u \end{bmatrix} = 0.$$

Combining this with the fact that $K_0 K^{-1} = [I \ 0]$,

$$\tilde{K} K^{-1} \begin{bmatrix} 0 \\ u \end{bmatrix} = \begin{bmatrix} K_0 \\ K_y \end{bmatrix} K^{-1} \begin{bmatrix} 0 \\ u \end{bmatrix} = 0.$$

Thus \tilde{K} is singular. Assume next that \tilde{K} is singular. Thus there exists non-zero $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that

$$\begin{bmatrix} K_0 \\ K_y \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (15)$$

This implies that

$$K \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} K_0 \\ K_u \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ z \end{bmatrix},$$

where $z \neq 0$, since K is invertible. Thus

$$Ez = K_y K^{-1} \begin{bmatrix} 0 \\ z \end{bmatrix} = K_y \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

and thus E is not invertible.

Assume now that E is invertible, then by the above equivalence with the invertibility of \tilde{K} and Theorem 2.1 for every initial condition there exists a solution of (11)–(13). Since z is now determined, u is determined by (14). Now it is straightforward to see that the functions z and u satisfy (6)–(8) and the corresponding output y satisfies $y = 0$.

If for every $z_0 \in L^2(0, 1; \mathbb{C}^n)$ there exists a solution of (11)–(14), then the functions z and u satisfy (6)–(8). Since K is invertible, the solution depends continuously on the initial condition. By construction, z is the solution of the homogeneous equation (11)–(13), and Theorem 2.5 implies the invertibility of \tilde{K} . \square

Example 1 in [12] illustrates calculation of the zero dynamics in the case where E is invertible.

It is very common though for the feedthrough to be non-invertible. This more challenging situation is considered in the next two sections.

3. Commensurate constant wave speed

In this section, the following class of port-Hamiltonian systems is considered:

$$\frac{\partial}{\partial t} z(\zeta, t) = -\lambda_0 \frac{\partial}{\partial \zeta} z(\zeta, t), \quad (16)$$

$$z(\zeta, 0) = z_0(\zeta), \quad \zeta \in (0, 1) \quad (17)$$

$$\begin{bmatrix} 0 \\ u(t) \end{bmatrix} = -\lambda_0 K z(0, t) - \lambda_0 L z(1, t), \quad (18)$$

$$y(t) = -\lambda_0 K_y z(0, t) - \lambda_0 L_y z(1, t). \quad (19)$$

If \mathcal{H} is constant, then (6)–(9) is of the form (16)–(19) with $-\lambda_0$ replaced by a diagonal (constant) and invertible matrix Δ . On the diagonal of the matrix Δ are the possible different wave speeds of the system. If the ratio of any pair of diagonal entries of Δ is rational, then the system (6)–(9) can be equivalently written in form (16)–(19) by dividing the intervals to adjust the propagation periods. This is a standard procedure and is illustrated in Example 3.1. The following simple reflection makes positive wave speeds into negative wave speed, while keeping the same absolute speed

$$\tilde{z}_k(\zeta, t) := z_k(1 - \zeta, t).$$

It is good to remark that the system (16)–(19) will in general have larger matrices than the original system (6)–(9). However, for simplicity, still denote the size by n .

Example 3.1. *Consider the following system with commensurable wave speeds*

$$\begin{aligned}\frac{\partial z_1}{\partial t} &= -\frac{\partial z_1}{\partial \zeta}, \\ \frac{\partial z_2}{\partial t} &= -\frac{1}{2} \frac{\partial z_2}{\partial \zeta},\end{aligned}$$

with $\zeta \in [0, 1]$, $t \geq 0$ and

$$\begin{aligned}\begin{bmatrix} 0 \\ u(t) \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} z(0, t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} z(1, t) \\ y(t) &= \begin{bmatrix} 0 & 0 \end{bmatrix} z(0, t) + \begin{bmatrix} 1 & 1 \end{bmatrix} z(1, t).\end{aligned}$$

This system has not a uniform wave speed, but can be written equivalently as a system with one wave speed. To reach this goal, split the second equation in two and obtain the following equivalent system

$$\begin{aligned}\frac{\partial z_1}{\partial t} &= -\frac{\partial z_1}{\partial \zeta}, \\ \frac{\partial z_{2a}}{\partial t} &= -\frac{\partial z_{2a}}{\partial \zeta}, \\ \frac{\partial z_{2b}}{\partial t} &= -\frac{\partial z_{2b}}{\partial \zeta},\end{aligned}$$

with $\zeta \in [0, 1]$, $t \geq 0$, $z_{2b}(\zeta, t) = z_2(\zeta/2, t)$ and $z_{2a}(\zeta, t) = z_2((1 + \zeta)/2, t)$ and

$$\begin{aligned}\begin{bmatrix} 0 \\ 0 \\ u(t) \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} z(0, t) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} z(1, t) \\ y(t) &= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} z(0, t) + \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} z(1, t).\end{aligned}$$

This transformation also works if $\mathcal{H}(\zeta)$ is diagonal a.e. $\zeta \in (0, 1)$ and the ratio of the numbers $\tau_i := \int_0^1 \frac{1}{\mathcal{H}(\zeta)_{ii}} d\zeta$ are pairwise rational [27].

It is now shown that the zero dynamics can be well-defined through the input and output equations.

It is well-known that the solution of (16) is given by $z(\zeta, t) = f(1 - \zeta + \lambda_0 t)$ for $t \geq 0$ and some function f . Using this fact, we write the system (16)–(19) equivalently as

$$f(t) = z_0(1 - t), \quad t \in [0, 1], \quad (20)$$

$$\begin{bmatrix} 0 \\ u(t) \end{bmatrix} = -\lambda_0 K f(1 + \lambda_0 t) - \lambda_0 L f(\lambda_0 t), \quad t \geq 0, \quad (21)$$

$$y(t) = -\lambda_0 K_y f(1 + \lambda_0 t) - \lambda_0 L_y f(\lambda_0 t), \quad t \geq 0. \quad (22)$$

Since the system is well-posed, the matrix K is invertible (Theorem 2.1). Thus, equivalently

$$f(t) = z_0(1-t), \quad t \in [0, 1], \quad (23)$$

$$f(1 + \lambda_0 t) = -K^{-1}Lf(\lambda_0 t) - \lambda_0^{-1}K^{-1} \begin{bmatrix} 0 \\ u(t) \end{bmatrix}, \quad t \geq 0, \quad (24)$$

$$y(t) = (\lambda_0 K_y K^{-1} L - \lambda_0 L_y) f(\lambda_0 t) + K_y K^{-1} \begin{bmatrix} 0 \\ u(t) \end{bmatrix}, \quad t \geq 0. \quad (25)$$

Defining

$$\begin{aligned} A_d &= -K^{-1}L, & B_d &= -\lambda_0^{-1}K^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}, \\ C_d &= -\lambda_0 K_y A_d - \lambda_0 L_y, & D_d &= -\lambda_0 K_y B_d, \end{aligned} \quad (26)$$

equation (24)–(25) can be written as

$$\begin{aligned} f(1 + \lambda_0 t) &= A_d f(\lambda_0 t) + B_d u(t), \\ y(t) &= C_d f(\lambda_0 t) + D_d u(t). \end{aligned}$$

Define for $n \in \mathbb{N}$ the functions $z_d(n) \in L^2(0, 1; \mathbb{C}^n)$, $u_d(n) \in L^2(0, 1; \mathbb{C}^p)$, and $y_d(n) \in L^2(0, 1; \mathbb{C}^m)$ by $z_d(0)(\zeta) := z_0(1 - \zeta)$, $z_d(n)(\zeta) = f(n + \zeta)$ for $n \geq 0$ and

$$u_d(n)(\zeta) = u\left(\frac{n + \zeta}{\lambda_0}\right), \quad y_d(n)(\zeta) = y\left(\frac{n + \zeta}{\lambda_0}\right), \quad n \in \mathbb{N}.$$

Thus equations (16)–(19) can be equivalently rewritten as

$$z_d(n+1)(\zeta) = A_d z_d(n)(\zeta) + B_d u_d(n)(\zeta) \quad (27)$$

$$(z_d(0))(\zeta) = z_0(1 - \zeta) \quad (28)$$

$$y_d(n)(\zeta) = C_d z_d(n)(\zeta) + D_d u_d(n)(\zeta) \quad (29)$$

This representation is very useful, not only for the zero dynamic, but also for other properties like stability.

Theorem 3.2. [14, Corollary 3.7] *The system (16)–(19) is exponentially stable if and only if the spectral radius of A_d satisfies $r(A_d) < 1$ or equivalently if $\sigma_{\max}(A_d) < 1$.*

Further sufficient conditions for exponential stability can be found in [1, 7, 13]. In particular, exponential stability is implied by the condition $KK^* - LL^* > 0$, [1, Thm. 3.2] and [13, Lemma 9.1.4]. However, the condition $KK^* - LL^* > 0$ is in general not necessary, see [13, Example 9.2.1].

It will now be shown that the zero dynamics of systems of the form (16)–(19) are again a port-Hamiltonian system, but with possibly a smaller state, that is, instead of $L^2(0, 1; \mathbb{C}^n)$ the state space will be $L^2(0, 1; \mathbb{C}^k)$ with $0 \leq k \leq n$. First, it is shown that the problem of determining the zero dynamics for (16)–(19) can be transformed into determining the zero dynamics for the finite-dimensional discrete-time system described by the matrices A_d , B_d , C_d and D_d .

Theorem 3.3. *Let $z_0 \in L^2(0, 1; \mathbb{C}^n)$. Then the following are equivalent.*

1. *There exists an input $u \in L^2_{loc}(0, \infty; \mathbb{C}^p)$ such that the output y of (16)–(19) with initial condition $z(\cdot, 0) = z_0$ is identically zero;*
2. *$z_0 \in L^2(0, 1; V_d^*)$, where $V_d^* \subset \mathbb{C}^n$ is the largest output nulling subspace of the discrete-time system $\Sigma(A_d, B_d, C_d, D_d)$ with state space \mathbb{C}^n given by*

$$w(n+1) = A_d w(n) + B_d u(n), \quad y(n) = C_d w(n) + D_d u(n). \quad (30)$$

In particular, the largest output nulling subspace V^ of (16)–(19) is given by $V^* = L^2(0, 1; V_d^*)$.*

Proof: The system (16)–(19) can be equivalently written as as (27)–(29). In these equations the input, state and output were still spatially dependent. However, the time axis has been split as $[0, \infty) = \cup_{n \in \mathbb{N}} [n, (n+1))$. Thus condition 1. is equivalent to

1' There exists a sequence $(u_d(n))_{n \in \mathbb{N}} \subseteq L^2(0, 1; \mathbb{C}^m)$ and a set $\Omega \subset (0, 1)$ whose complement has measure zero such that for every $\zeta \in \Omega$,

$$\begin{aligned} z_d(n+1)(\zeta) &= A_d z_d(n)(\zeta) + B_d u_d(n)(\zeta), \\ (z_d(0))(\zeta) &= z_0(1 - \zeta). \\ 0 &= C_d z_d(n)(\zeta) + D_d u_d(n)(\zeta), \end{aligned} \quad (31)$$

Clearly, condition 1' implies that $z_0(\zeta) \in V_d^*$ a.e., where V_d^* denotes the largest output nulling subspace of the finite-dimensional system (30). Since trivially $z_0 \in L^2(0, 1; V_d^*)$, condition 2 follows.

The system (A_d, B_d, C_d, D_d) is a finite-dimensional discrete-time system. Let $V_d^* \subset \mathbb{C}^n$ indicate the largest output nulling subspace. Then there exists a matrix \mathcal{K} such that the output-nulling control is given by $u_d(n) = \mathcal{K}z_d(n)$, see [29]. Referring now to (31), if $z_0 \in L^2(0, 1; V_d^*)$ then the output-nulling control $(u_d(n))_{n \in \mathbb{N}}$ for system (31) satisfies $u_d(n) \in L^2(0, 1; \mathbb{C}^p)$. Condition 2 thus implies condition 1'. \square

For many partial differential equation systems, the largest output nulling subspace is not closed and the zero dynamics are not well-posed, [24, 30]. However, for systems of the form (16)–(19) the largest output nulling subspace is closed, and the zero dynamics are well-posed. The following theorem provides a characterization of the largest output nulling subspace of $\Sigma(A_d, B_d, C_d, D_d)$ and hence of the zero dynamics for the original partial differential equation. The proof can be found in [12].

Theorem 3.4. Define $E = -\begin{bmatrix} K_o \\ K_y \end{bmatrix}$, $F = \begin{bmatrix} L_o \\ L_y \end{bmatrix}$. The initial condition v_0 lies in the largest output nulling subspace V_d of $\Sigma(A_d, B_d, C_d, D_d)$ if and only if there exists a sequence $\{v_k\}_{k \geq 1} \subset \mathbb{C}^n$ such that

$$E v_{k+1} = F v_k, \quad k \geq 0. \quad (32)$$

Furthermore, the largest output nulling subspace V_d^* satisfies $V_d^* = \cap_{k \geq 0} V^k$, where $V^0 = \mathbb{C}^n$, $V^{k+1} = V^k \cap F^{-1} E V^k$.

Thus in addition to the well-known V^* -algorithm for finite-dimensional systems, see [1, p. 91], Theorem 3.4 provides an alternative algorithm. It remains to show that the system restricted to the output nulling subspace is again port-Hamiltonian.

Theorem 3.5. For the port-Hamiltonian system (16)–(19) the zero dynamics is well-posed, and the dynamics restricted to the largest output nulling subspace is a port-Hamiltonian system without inputs.

Proof: By Theorem 3.3, the largest output nulling subspace V^* of (16)–(19) is given by $V^* = L^2(0, 1; V_d^*)$. If $V_d^* = \{0\}$, then there is nothing to prove, and so assume that V_d^* is a non-trivial subspace of \mathbb{C}^n . It is well-known that there exists a matrix F_d such that [29]

$$(A_d + B_d F_d) V_d^* \subset V_d^*.$$

Therefore, using Theorem 3.3 and (27)–(29), it is easy to see that for the choice $u_d(n)(\zeta) := F_d z_d(n)(\zeta)$ the output $y_d(n)(\zeta)$ is zero provided the initial condition z_0 lies in $L^2(0, 1; V_d^*)$. Using the definition of the A_d, B_d, C_d, D_d, u_d and z_d , it follows that for $z_0 \in L^2(0, 1; V_d^*)$ there exists a function f satisfying

$$f(t) = z_0(1 - t), \quad t \in [0, 1], \quad (33)$$

$$f(1 + \lambda_0 t) = -K^{-1} L f(\lambda_0 t) - \lambda_0^{-1} K^{-1} \begin{bmatrix} 0 \\ F_d f(\lambda_0 t) \end{bmatrix}, \quad t \geq 0, \quad (34)$$

$$0 = (\lambda_0 K_y K^{-1} L - \lambda_0 L_y) f(\lambda_0 t) + K_y K^{-1} \begin{bmatrix} 0 \\ F_d f(\lambda_0 t) \end{bmatrix}, \quad t \geq 0. \quad (35)$$

Equations (34)-(35) can be equivalent written as

$$0 = -\lambda_0 K_{ext} f(1 + \lambda_0 t) - \lambda_0 L_{ext} f(\lambda_0 t), \quad (36)$$

with

$$K_{ext} = \begin{bmatrix} K \\ K_y \end{bmatrix} \quad (37)$$

and some matrix L_{ext} . Since $z_0 \in L^2(0, 1; V_d^*)$, for all t and almost all $\zeta \in [0, 1]$, $f(\zeta + \lambda_0 t) \in V_d^*$. Thus, K_{ext} and L_{ext} can be restricted to V_d^* and equation (36) can equivalently be written with matrices $K_{ext}|_{V_d^*}$ and $L_{ext}|_{V_d^*}$. Since K is part of the the matrix K_{ext} , the matrix $K_{ext}|_{V_d^*}$ has rank equal to the dimension of V_d^* . Let P be the projection onto the range of $K_{ext}|_{V_d^*}$. This leads to

$$0 = -\lambda_0 P K_{ext}|_{V_d^*} f(1 + \lambda_0 t) - \lambda_0 P L_{ext}|_{V_d^*} f(\lambda_0 t). \quad (38)$$

Define $K_{V_d^*} := P K_{ext}|_{V_d^*}$ and $L_{V_d^*} := P L_{ext}|_{V_d^*}$. The above equation is the solution of the partial differential equation

$$\frac{\partial}{\partial t} z(\zeta, t) = -\lambda_0 \frac{\partial}{\partial \zeta} z(\zeta, t), \quad (39)$$

$$0 = -\lambda_0 K_{V_d^*} z(0, t) - \lambda_0 L_{V_d^*} z(1, t) \quad (40)$$

on the state space $L^2(0, 1; V_d^*)$. Since $K_{V_d^*}$ is invertible, Theorem 2.1 implies that this system is a well-posed port-Hamiltonian system. \square

In the following section a present a second method to obtain the zero dynamics for systems with one dimensional input and output spaces is developed. The advantage of this method is that a transformation to a discrete system is not needed and non-constant wave speed is possible.

4. Zero dynamics of port-Hamiltonian systems with commensurate wave speed

In this section the zero dynamics of systems of the form (16)-(19) with one dimensional input and output spaces and (possibly) non-constant wave speed are defined. The class of systems considered has the form

$$\frac{\partial}{\partial t} z(\zeta, t) = -\frac{\partial}{\partial \zeta} (\lambda_0(\zeta) z(\zeta, t)) \quad (41)$$

$$0 = K_0(\lambda_0(0)z(0, t)) + L_0(\lambda_0(1)z(1, t)) \quad (42)$$

$$u(t) = K_u(\lambda_0(0)z(0, t)) + L_u(\lambda_0(1)z(1, t)) \quad (43)$$

$$y(t) = K_y z(0, t) + L_y z(1, t). \quad (44)$$

Here $K_0, L_0 \in \mathbb{C}^{(n-1) \times n}$, $K_u, K_y, L_u, L_y \in \mathbb{C}^{1 \times n}$ and $\lambda_0 \in L^\infty(0, 1)$ satisfying $0 < m \leq \lambda_0(\zeta) \leq M$ for almost every $\zeta \in (0, 1)$ and constants $m, M > 0$. It will be assumed throughout this section that the port-Hamiltonian system (41)–(44) is a well posed linear system with state space $L^2(0, 1; \mathbb{C}^n)$ or equivalently that the matrix $\begin{bmatrix} K_0 \\ K_u \end{bmatrix}$ is an invertible $n \times n$ -matrix, see Theorem 2.1. The corresponding generator A of the C_0 -semigroup of the homogeneous system is given by [13]

$$A f = -(\lambda_0 f)', \quad D(A) = \left\{ \lambda_0 f \in H^1(0, 1; \mathbb{C}^n) \mid \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} K_0 \\ K_u \end{bmatrix} (\lambda_0 f)(0) + \begin{bmatrix} L_0 \\ L_u \end{bmatrix} (\lambda_0 f)(1) \right\}.$$

Denote by $G(s)$ the transfer function of the port-Hamiltonian system (41)–(44). Since the port-Hamiltonian system is assumed to be well-posed, there exists a right half plane

$$\mathbb{C}_\alpha := \{s \in \mathbb{C} \mid \operatorname{Re} s > \alpha\}$$

such that $G : \mathbb{C}_\alpha \rightarrow \mathbb{C}$ is an analytic and bounded function. Define

$$p := \int_0^1 \lambda_0^{-1}(s) ds.$$

Moreover, using [13, Theorem 12.2.1] for $s \in \rho(A)$, where $\rho(A)$ denotes the resolvent set of A , and $u \in \mathbb{C}$ the number $G(s)u$ is (uniquely) determined by

$$0 = (K_0 + L_0 e^{-sp})v, \quad (45)$$

$$u = (K_u + L_u e^{-sp})v, \quad (46)$$

$$G(s)u = (K_y + L_y e^{-sp})v \quad (47)$$

for some $v \in \mathbb{C}^n$.

Lemma 4.1. *There exists $\mu \in \mathbb{R}$ such that, for $s \in \mathbb{C}_\mu$, $G(s) = 0$ if and only if the matrix $\begin{bmatrix} K_0 + L_0 e^{-sp} \\ K_y + L_y e^{-sp} \end{bmatrix}$ is not invertible.*

Proof: Since the matrix $\begin{bmatrix} K_0 \\ K_u \end{bmatrix}$ is invertible and A generates a C_0 -semigroup there is a $\mu \in \mathbb{R}$ such that $\rho(A) \subseteq \mathbb{C}_\mu$ and

$$\begin{bmatrix} K_0 + L_0 e^{-sp} \\ K_u + L_u e^{-sp} \end{bmatrix} = \begin{bmatrix} K_0 \\ K_u \end{bmatrix} + \begin{bmatrix} L_0 \\ L_u \end{bmatrix} e^{-sp}$$

is invertible for $s \in \mathbb{C}_\mu$.

Assume now $G(s) = 0$ for some $s \in \mathbb{C}_\mu$. Then (45)–(47) imply that there exists $v \in \mathbb{C}^n$ such that

$$\begin{aligned} 0 &= (K_0 + L_0 e^{-sp})v, \\ 1 &= (K_u + L_u e^{-sp})v, \\ 0 &= (K_y + L_y e^{-sp})v. \end{aligned}$$

Because $\begin{bmatrix} K_0 + L_0 e^{-sp} \\ K_u + L_u e^{-sp} \end{bmatrix}$ is invertible, it yields $v \neq 0$. Thus $\begin{bmatrix} K_0 + L_0 e^{-sp} \\ K_y + L_y e^{-sp} \end{bmatrix}$ is not invertible.

Conversely, assume that for some $s \in \mathbb{C}_\mu$, $\begin{bmatrix} K_0 + L_0 e^{-sp} \\ K_y + L_y e^{-sp} \end{bmatrix}$ is not invertible. Then there exists a non-zero vector $v \in \mathbb{C}^n \setminus \{0\}$ such that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} K_0 + L_0 e^{-sp} \\ K_y + L_y e^{-sp} \end{bmatrix} v.$$

Set $u := (K_u + L_u e^{-sp})v$. Since $\begin{bmatrix} K_0 + L_0 e^{-sp} \\ K_u + L_u e^{-sp} \end{bmatrix}$ is invertible, it follows that $u \neq 0$. However, $G(s)u = 0$ by (45)–(47), which implies $G(s) = 0$. \square

Theorem 4.2. *Suppose that $G(s) \not\equiv 0$. Then the zero dynamics of the port-Hamiltonian system (41)–(44) are again a well-posed port-Hamiltonian system with wave speed $-\lambda_0$ and possibly a smaller state space. More precisely, there exists $k \in \{0, \dots, n\}$ such that the zero dynamics is described by the port-Hamiltonian system*

$$\begin{aligned} \frac{\partial}{\partial t} w(\zeta, t) &= -\frac{\partial}{\partial \zeta} (\lambda_0(\zeta) w(\zeta, t)) \\ 0 &= K_w(\lambda_0(0)w(0, t)) + L_w(\lambda_0(1)w(1, t)). \end{aligned}$$

with state space $L^2(0, 1; \mathbb{C}^k)$ and the $k \times k$ -matrix K_w is invertible.

Proof: The zero dynamics are defined by the equations

$$\frac{\partial}{\partial t} z(\zeta, t) = -\frac{\partial}{\partial \zeta} (\lambda_0(\zeta) z(\zeta, t)) \quad (48)$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} K_0 \\ K_y \end{bmatrix} (\lambda_0(0)z(0, t)) + \begin{bmatrix} L_0 \\ L_y \end{bmatrix} (\lambda_0(1)z(1, t)). \quad (49)$$

Since there is one input and one output, and $\text{rank} \begin{bmatrix} K_0 \\ K_u \end{bmatrix} = n$, the rank of the matrix $\begin{bmatrix} K_0 \\ K_y \end{bmatrix}$ equals $n - 1$ or n .

If $\text{rank} \begin{bmatrix} K_0 \\ K_y \end{bmatrix} = n$, that is, this matrix is invertible, then the zero dynamics is well-defined on the whole state space $L^2(0, 1; \mathbb{C}^n)$, see Theorem 2.5. Theorem 2.1 implies that the zero dynamics are well-posed on the state space $L^2(0, 1; \mathbb{C}^n)$. Thus $k = n$ and the theorem is proved.

Suppose next that $\text{rank} \begin{bmatrix} K_0 \\ K_y \end{bmatrix} = n - 1$. Then K_y is a linear combination of the rows of K_0 and there is an invertible transformation, a row reduction, so that (49) is equivalent to

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ 0 & 0 \end{bmatrix} (\lambda_0(0)z(0, t)) + \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} (\lambda_0(1)z(1, t)). \quad (50)$$

Here $K_{11}, L_{11} \in \mathbb{C}^{(n-1) \times (n-1)}$ and $L_{22} \in \mathbb{C}$. Since $\text{rank}[K_{11} \ K_{12}] = n - 1$, column transformations lead to a representation where the matrix K_{11} is invertible. Assume now that this has been done.

Since K_{11} is invertible, and G is not equivalently zero, Lemma 4.1, implies that there exists $s_0 \in \mathbb{C}$ such that both $T_1 := K_{11} + L_{11}e^{-s_0 p}$ and

$$T := \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} := \begin{bmatrix} K_{11} + L_{11}e^{-s_0 p} & K_{12} + L_{12}e^{-s_0 p} \\ L_{21}e^{-s_0 p} & L_{22}e^{-s_0 p} \end{bmatrix} \quad (51)$$

are invertible. Defining the Schur complement of T with respect to T_1 ,

$$S = T_4 - T_3 T_1^{-1} T_2,$$

$$\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} = \begin{bmatrix} I & 0 \\ T_3 T_1^{-1} & I \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & T_1^{-1} T_2 \\ 0 & I \end{bmatrix}.$$

Since T_1 and T are invertible, S is invertible and

$$T^{-1} := \begin{bmatrix} T_1^{-1} + T_1^{-1} T_2 S^{-1} T_3 T_1^{-1} & -T_1^{-1} T_2 S^{-1} \\ -S^{-1} T_3 T_1^{-1} & S^{-1} \end{bmatrix}.$$

Now apply the state transformation

$$\tilde{z} = Tz.$$

The equations (50) are equivalent to

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} K_{11} & K_{12} \\ 0 & 0 \end{bmatrix} T^{-1} (\lambda_0(0) \tilde{z}(0, t)) + \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} T^{-1} (\lambda_0(1) \tilde{z}(1, t)) \\ &= \begin{bmatrix} K_{11}(T_1^{-1} + T_1^{-1} T_2 S^{-1} T_3 T_1^{-1}) - K_{12} S^{-1} T_3 T_1^{-1} & K_{12} S^{-1} - K_{11} T_1^{-1} T_2 S^{-1} \\ 0 & 0 \end{bmatrix} (\lambda_0(0) \tilde{z}(0, t)) \\ &\quad + \begin{bmatrix} L_{11}(T_1^{-1} + T_1^{-1} T_2 S^{-1} T_3 T_1^{-1}) - L_{12} S^{-1} T_3 T_1^{-1} & L_{12} S^{-1} - L_{11} T_1^{-1} T_2 S^{-1} \\ 0 & e^{s_0 p} \end{bmatrix} (\lambda_0(1) \tilde{z}(1, t)). \end{aligned}$$

Also the system of partial differential equations (48) are equivalent to

$$\frac{\partial}{\partial t} \tilde{z}(\zeta, t) = -\frac{\partial}{\partial \zeta} (\lambda_0(\zeta) \tilde{z}(\zeta, t)). \quad (52)$$

Thus, the transformed partial differential equation is identical to the original. The general solution

$$\tilde{z}_n(\zeta, t) = \frac{c}{\lambda_0(\zeta)} e^{\int_0^\zeta \lambda_0^{-1}(s) ds - t}$$

and the boundary condition $\tilde{z}_n(1, t) = 0$ imply that $\tilde{z}_n \equiv 0$.

Define

$$w := \begin{bmatrix} \tilde{z}_1 \\ \vdots \\ \tilde{z}_{n-1} \end{bmatrix}, \quad (53)$$

and the matrices

$$\begin{aligned} K_w &:= K_{11}(T_1^{-1} + T_1^{-1}T_2S^{-1}T_3T_1^{-1}) - K_{12}S^{-1}T_3T_1^{-1} \\ L_w &:= L_{11}(T_1^{-1} + T_1^{-1}T_2S^{-1}T_3T_1^{-1}) - L_{12}S^{-1}T_3T_1^{-1} \\ K_{w12} &:= K_{12}S^{-1} - K_{11}T_1^{-1}T_2S^{-1}. \end{aligned}$$

Thus it yields

$$\begin{bmatrix} K_{11} & K_{12} \\ 0 & 0 \end{bmatrix} T^{-1} = \begin{bmatrix} K_w & K_{w12} \\ 0 & 0 \end{bmatrix}.$$

Here K_{w12} is a $(n-1) \times 1$ -matrix and $\text{rank } K_w \geq n-2$. The zero dynamics is described by the reduced port-Hamiltonian system

$$\begin{aligned} \frac{\partial w}{\partial t} &= -\frac{\partial}{\partial \zeta}(\lambda_0 w), \\ 0 &= K_w(\lambda_0(0)w(0, t)) + L_w(\lambda_0(1)w(1, t)). \end{aligned}$$

The reduced system is well-posed on $L^2(0, 1; \mathbb{C}^{n-1})$ if and only K_w is invertible; that is, K_w has rank $n-1$. If K_w is invertible, then the theorem is proved.

Now suppose that $\text{rank } K_w = n-2$. As in the first part, elementary row and column transformations can be used to put the boundary conditions for the reduced system into the form, again indicating the state variables by w ,

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{K}_{11} & \tilde{K}_{12} \\ 0 & 0 \end{bmatrix} (\lambda_0(0)w(0, t)) + \begin{bmatrix} \tilde{L}_{11} & \tilde{L}_{12} \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix} (\lambda_0(1)w(1, t)).$$

where \tilde{K}_{11} is invertible. Define

$$\tilde{T}(s) = K_w + L_w e^{-sp}.$$

In order to repeat the above procedure, a complex number s such that \tilde{T} and $\tilde{K}_{11} + \tilde{L}_{11}e^{-sp}$ are both invertible is needed. Set $s = s_0$. Define

$$X = T_1^{-1} + T_1^{-1}T_2S^{-1}T_3T_1^{-1}.$$

Recalling that $T_1 = K_{11} + e^{-s_0p}L_{11}$, $T_2 = K_{12} + e^{-s_0p}L_{12}$,

$$\begin{aligned} K_w + L_w e^{-s_0p} &= K_{11}X - K_{12}S^{-1}T_3T_1^{-1} + e^{-s_0p}L_{11}X - e^{-s_0p}L_{12}S^{-1}T_3T_1^{-1} \\ &= T_1X - T_2S^{-1}T_3T_1^{-1} \\ &= I + T_2S^{-1}T_3T_1^{-1} - T_2S^{-1}T_3T_1^{-1} \\ &= I. \end{aligned}$$

Thus, with $s = s_0$, $\tilde{T}(s)$ is invertible. Define

$$f_w : \mathbb{C}_\alpha \rightarrow \mathbb{C}, \quad f_w(s) = \det[\tilde{T}(s)].$$

and so $f_w(s_0) = 1$. Since f_w is analytic, there is a sequence s_n , $\text{Res}_n \rightarrow \infty$ with $f(s_n) \neq 0$. Choose then s_w so that $\tilde{K}_{11} + \tilde{L}_{11}e^{-s_w p}$ is invertible. Repeating the previous procedure leads to a port-Hamiltonian system with state-space $L^2(0, 1; \mathbb{C}^{n-2})$. Since each iteration leads to a state-space with fewer number of state variables, this procedure is guaranteed to converge within n steps. \square

Since the zero dynamics are a well-posed dynamical system, the following result is immediate.

Corollary 4.3. *The invariant zeroes are contained in a left-hand-plane.*

One consequence of calculating the zero dynamics using the original port-Hamiltonian form is that it is easy to obtain the input u that zeroes the output. Suppose only one state space reduction

in Theorem 4.2 is needed. The state space of the zero dynamics is $L^2(0, 1; \mathbb{C}^{n-1})$. From (43) and (51)

$$\begin{aligned} u(t) &= K_u \lambda_0(0) z(0, t) + L_u \lambda_0(1) z(1, t) \\ &= K_u \lambda_0(0) T^{-1} \tilde{z}(0, t) + L_u \lambda_0(1) T^{-1} \tilde{z}(1, t). \end{aligned}$$

In the zero dynamics, $\tilde{z}_n \equiv 0$. Defining \tilde{K}_u to be the first $n - 1$ columns of $K_u \lambda_0(0) T^{-1}$ and defining \tilde{L}_u similarly, the zeroing input is

$$u(t) = \tilde{K}_u w(0, t) + \tilde{L}_u w(1, t)$$

where w is defined in (53). For the situation where more than one state space reduction is needed, the calculation is similar, except that a transformation matrix T is needed for each reduction.

5. Computation

Theorem 4.2 leads to a characterization of the zero dynamics as a port-Hamiltonian system of smaller dimension. Moreover, the proof is constructive and can be used in an algorithm to calculate the zero dynamics using standard linear algebra algorithms, see the box on the following page. Zero dynamics can be calculated exactly for large system order; that is those with a large number of nodes. Furthermore, Theorem 3.2 can be used to check stability.

Several examples are now presented to illustrate the calculation of zero dynamics.

Example 5.1. Consider the system from Example 3.1, written in the equal wave speed form. For zero dynamics,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_K z(0, t) + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}}_L z(1, t). \quad (54)$$

The rank of $K = 2$ and so the zero dynamics are defined on a smaller state space than the original. Applying one iteration of the algorithm yields (with $s_0 = 0$)

$$TP = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \quad K_w = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}, \quad L_w = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

The last row of the transformation matrix TP indicates that for zero dynamics

$$z_1 + z_{2a} \equiv 0$$

and the first two rows define the remaining state variables:

$$\tilde{z}_1 = 2z_1 + z_{2b}, \quad \tilde{z}_2 = z_{2a} - z_{2b}.$$

(Because $z_1 + z_{2a} \equiv 0$, \tilde{z}_1 can also be written $-2z_{2a} + z_{2b}$.) The rank $K_w = 2$. The zero dynamics are well-defined on $L^2(0, 1; \mathbb{C}^2)$ with governing differential equation (52) and boundary conditions

$$\begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix} \tilde{z}(0, t) + \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \tilde{z}(1, t).$$

From the definition of the control system in Example 3.1,

$$u(t) = z_{2b}(0, t)$$

is the zeroing input.

This example could be done by hand. The definition of the zero dynamics (54) implies that $z_1 + z_{2a} = 0$. Substitution into the equations yields the zero dynamics.

Algorithm: Calculation of Zero Dynamics

The data are: wave speed $p = \int_0^1 \frac{1}{\lambda_0(\xi)} d\xi$, boundary condition matrices K_0, L_0 , and output matrices K_y, L_y . The dimension of the system is n , the number of columns in K_0 . Define

$$K = \begin{bmatrix} K_0 \\ K_y \end{bmatrix}.$$

If K is invertible the zero dynamics are well-defined with n state variables. Otherwise do the following calculations.

1. Perform LU-decomposition of K : $P_{\ell u}K = M_{\ell}M_u$ where M_{ℓ} is lower triangular, M_u is upper triangular and $P_{\ell u}$ is a permutation matrix.
2. If necessary permute last column of M_u with earlier column so that rank of top left $n - 1$ block is $n - 1$; call the permutation matrix P . Partition M_uP and $M_{\ell}^{-1}P_{\ell u}LP$ similarly as

$$M_u = \begin{bmatrix} K_{11} & K_{12} \\ [0 \dots 0] & 0 \end{bmatrix}, \quad M_{\ell}^{-1}P_{\ell u}L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}.$$

3. Define the matrices

$$T_1 = K_{11} + L_{11}e^{-ps_0}, \quad T = \begin{bmatrix} T_1 & K_{12} + L_{12}e^{-ps_0} \\ L_{21}e^{-ps_0} & L_{22}e^{-ps_0} \end{bmatrix}$$

for s_0 so that both matrices are invertible. (The existence of such an s_0 is guaranteed if the transfer function is not identically zero. A simple way find a suitable s_0 is to start with $s_0 = 0$ and then increase by an arbitrary amount until both matrices are invertible.)

4. Decompose T^{-1} using the same decomposition as for K_u and construct the inverse of T using the Schur complement. Letting X be the solution of $T_3 = XT_1$, define

$$S_i = (T_4 - XT_2)^{-1}.$$

(Note S is a scalar.) Only the 2 left blocks of T^{-1} are needed:

$$(T^{-1})_{11} = T_1^{-1}(I + T_2S_kX), \quad (T^{-1})_{21} = -S_iX.$$

5. The boundary matrices for the reduced system are

$$K_w = K_{11}(T^{-1})_{11} + K_{12}(T^{-1})_{21}, \quad L_w = L_{11}(T^{-1})_{11} + L_{12}(T^{-1})_{21}.$$

6. The new variables are $\tilde{z}_1 \dots \tilde{z}_{n-1}$ where $\tilde{z} = TPz$, the differential equation is

$$\frac{\partial}{\partial t} \tilde{z}(\zeta, t) = -\frac{\partial}{\partial \zeta} (\lambda_0(\xi) \tilde{z}(\zeta, t))$$

and the boundary conditions are

$$K_w \lambda_0(0) \tilde{z}(0, t) + L_w \lambda_0(1) \tilde{z}(1, t).$$

If rank $K_w = n - 1$, the algorithm is complete. If not, return to the first step with $K = K_w$, $L = L_w$ and repeat the process.

Example 5.2.

$$\frac{\partial x_i}{\partial t} = -\frac{\partial x_i}{\partial \zeta}, \quad i = 1, 2, 3.$$

with

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \\ u(t) \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}}_K x(0, t) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(1, t) \\ y(t) &= [0 \ 0 \ 0] x(0, t) + [1 \ 0 \ 0] x(1, t). \end{aligned} \quad (55)$$

The rank of K in (55) is 3 and so the system is well-posed. The transfer function is not identically zero.

Zero dynamics require

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(0, t) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} x(1, t). \quad (56)$$

Applying the algorithm yields (with $s_0 = 0$)

$$TP = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \quad K_w = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad L_w = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The third row of TP implies that $z_1 \equiv 0$. The reduced states are

$$\tilde{z}_2 = -z_1 + z_2 = z_2, \quad \tilde{z}_3 = z_1 - z_3 = -z_3.$$

Since K_w does not have full rank, the algorithm needs to be repeated; but with K_w, L_w as the boundary matrices. This yields

$$(TP)_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (K_w)_2 = [1], \quad (L_w)_2 = [0].$$

Thus $\tilde{z}_2 = z_2 \equiv 0$ and $\tilde{z}_3(0) = -z_3(0) = 0$.

This example is also simple enough to do by hand. The original equations (56) are already row-reduced, and imply $x_1 \equiv 0$. The reduced system must have $x_2 \equiv 0$.

Either calculation leads to one non-zero equation, for x_3 with the boundary condition

$$x_3(0, t) = 0.$$

The system equations (55) imply that in order to achieve this, $u(t) = x_3(1, t)$.

Example 5.3. Consider a larger system with $n = 10$. Suppose the wave speed λ_0 is such that $-\int_0^1 \lambda_0(\xi) d\xi = -1$. The entries in the boundary matrices are zero, except that

$$\begin{aligned} K_0(1, 2) &= 1, & K_0(1, 9) &= -3, & K_0(2, 3) &= 1, & K_0(2, 2) &= -1, & K_0(3, 6) &= 1, & K_0(3, 10) &= 2, \\ K_0(4, 1) &= -5, & K_0(4, 6) &= 2, & K_0(5, 10) &= 6, & K_0(5, 9) &= -4, & K_0(6, 8) &= 4, & K_0(6, 1) &= -2, \\ K_0(7, 6) &= 1, & K_0(7, 7) &= 3, & K_0(8, 3) &= -2, & K_0(8, 8) &= 1, & K_0(8, 5) &= -5, \\ K_0(9, 1) &= 1, & K_0(9, 6) &= 5, & K_0(9, 9) &= -1; \\ K_u(1, 4) &= 1; \\ L_y(1, 2) &= 1, & L_y(1, 4) &= -2. \end{aligned}$$

Since

$$\text{rank} \begin{bmatrix} K_0 \\ K_u \end{bmatrix} = 10$$

this system is well-posed. Also, the transfer function G is not identically 0; in particular $G(0) \neq 0$. Applying the algorithm with $s_0 = 0$ yields

$$TP = \begin{bmatrix} -5 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & -5 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2.5 & 0 & 0 & 0.5 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 5.4 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0.1852 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -0.1481 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1852 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_w = I_9, \quad L_w = 0_{9 \times 9}.$$

For zero dynamics, $z_2 - 2z_4 \equiv 0$ and the zeroing input is

$$u(t) = K_u TPz(0, t) = -2.5z_5(0, t) + 0.5z_8(0, t) - 3z_9(0, t).$$

6. Conclusions

In this paper, zero dynamics were formally defined for port-Hamiltonian systems. If the feedthrough operator is invertible, then the zero dynamics are again a port-Hamiltonian system of the same order. In general, however, the feedthrough operator is not invertible. For many infinite-dimensional systems, where the feedthrough is not invertible, the zero dynamics are not well-defined. It has been shown in this paper that provided the system can be rewritten as a network of waves with the same speed, the zero dynamics are always well-defined, and are a port-Hamiltonian system. Furthermore, a numerical method to construct the zero dynamics using the original partial differential equation has been described. Finite-dimensional approximations, which can be inaccurate in calculation of zeros, are not needed. The approach applies to systems with commensurate but non-equal wave speeds, and this generalization will be explored in future work. The extension to multi-input multi-output systems also needs to be established.

References

- [1] G. Bastin and J.-M. Coron. *Stability and Boundary Stabilization of 1-D Hyperbolic Systems*, volume 88 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser/Springer, [Cham], 2016. Subseries in Control.
- [2] C. I. Byrnes, D. S. Gilliam, A. Isidori, and V. I. Shubov. Zero dynamics modeling and boundary feedback design for parabolic systems. *Math. Comput. Modelling*, 44(9-10):857–869, 2006.
- [3] C. I. Byrnes, D.S. Gilliam, and J. He. Root-locus and boundary feedback design for a class of distributed parameter systems. *SIAM Jour. on Control and Optimization*, 32(5):1364–1427, 1994.
- [4] A. Cheng and K.A. Morris. Accurate zeros approximation for infinite-dimensional systems. In *42nd IEEE Conference on Decision and Control*, Honolulu, Hawaii, 2003.
- [5] R.L. Clark. Accounting for out-of-bandwidth modes in the assumed modes approach: Implications on colocated output feedback control. *ASME J. Dyn. Sys. Meas. and Cont.*, 119:390–395, 1997.
- [6] J.C. Doyle, B.A. Francis, and A.R. Tannenbaum. *Feedback Control Theory*. MacMillan Publishing Co., 1992.
- [7] K.-J. Engel. Generator property and stability for generalized difference operators. *J. Evol. Equ.*, 13(2):311–334, 2013.
- [8] C. Foias, H. Özbay, and A. Tannenbaum. *Robust Control of Infinite Dimensional Systems. Frequency Domain Methods*. Lecture Notes in Control and Information Sciences 209. Springer-Verlag, Berlin, 1996.
- [9] J.R. Grad and K.A. Morris. Calculation of achievable broadband noise reduction using approximations. *Engineering applications and computational algorithms, Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms, suppl.*, pages 438–443, 2003.
- [10] A. Isidori. *Nonlinear Control Systems*. Springer, 1999.
- [11] B. Jacob, K.A. Morris, and C. Trunk. Minimum-phase infinite-dimensional second-order systems. *IEEE Trans. Auto. Cont.*, 52(9):1654–1665, 2007.
- [12] B. Jacob, K.A. Morris, and H.J. Zwart. Zero dynamics for waves on networks' In *5th IFAC Workshop on Control of Lagrangian and Hamiltonian Methods for Nonlinear Control*. IFAC, 2015.
- [13] B. Jacob and H.J. Zwart. *Linear Port-Hamiltonian Systems on Infinite-Dimensional Spaces*. Basel: Birkhäuser, 2012.

- [14] B. Klöss. Difference operators as semigroup generators. *Semigroup Forum*, 81(3):461–482, 2010.
- [15] T. Kobayashi. Low-gain adaptive stabilization of infinite-dimensional second-order systems. *J. Math. Anal. Appl.*, 275(2):835–849, 2002.
- [16] Y. Le Gorrec, H. Zwart, and B. Maschke. Dirac structures and boundary control systems associated with skew-symmetric differential operators. *SIAM Journal on Control and Optimization*, 44(5):1864–1892, 2005.
- [17] D.K. Lindner, K.M. Reichard, and L.M. Tarkenton. Zeros of modal models of flexible structures. *IEEE Trans. Autom. Control*, 38(9):1384–1388, 1993.
- [18] H. Logemann and D.H. Owens. Robust high-gain feedback control of infinite-dimensional minimum-phase systems. *IMA J. Math. Control Inf.*, 4:195–220, 1987.
- [19] H. Logemann and S. Townley. Adaptive control of infinite-dimensional systems without parameter estimation: An overview. *IMA J. Math. Control Inf.*, 14(2):175–206, 1997.
- [20] H. Logemann and S. Townley. Adaptive low-gain integral control of multivariable well-posed linear systems. *SIAM J. Control Optimization*, 41(6):1722–1732, 2003.
- [21] H. Logemann and H. Zwart. On robust PI-control of infinite-dimensional systems. *SIAM J. Control Optimization*, 30(3):573–593, 1992.
- [22] K.A. Morris. *Introduction to Feedback Control*. Harcourt/ Academic Press, Burlington, MA, 2001.
- [23] K.A. Morris and R. Rebarber. Feedback invariance for SISO infinite-dimensional systems. *Math. of Control, Signals and Systems*, 19:313–335, 2007.
- [24] K.A. Morris and R.E. Rebarber. Zeros of SISO infinite-dimensional systems. *International Journal of Control*, 83(12):2573–2579, 2010.
- [25] S. Nikitin and M. Nikitina. High gain output feedbacks for systems with distributed parameters. *Math. Models Methods Appl. Sci.*, 9(6):933–940, 1999.
- [26] T. Reis and T. Selig. Zero dynamics and root locus for a boundary controlled heat equation. *Math. Control Signals Systems*, 27(3):347–373, 2015.
- [27] M. Suzuki, J. Imura, and K. Aihara. Analysis and stabilization for networked linear hyperbolic systems of rational dependent conversation laws. *Automatica*, 49:3210–3221, 2013.
- [28] J.A. Villegas. *A port-Hamiltonian Approach to Distributed Parameter Systems*. PhD thesis, Universiteit Twente, 2007.
- [29] W.M. Wonham. *Linear Multivariable Control*, volume 10 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, third edition, 1985. A geometric approach.
- [30] H. Zwart. *Geometric Theory for Infinite Dimensional Systems*. Springer-Verlag, 1989.
- [31] H. Zwart, Y. Le Gorrec, B Maschke, and J. Villegas. Well-posedness and regularity of hyperbolic boundary control systems on a one-dimensional spatial domain. *ESAIM: Control, Optimisation and Calculus of Variations*, 16(4):1077–1093, 2010.