Extending Pappus' Theorem

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Let M_1 and M_2 be matroids such that M_2 arises from M_1 by relaxing a circuit-hyperplane. We will prove that if M_1 and M_2 are both representable over some finite field $GF(q)$, then M_1 and M_2 have highly structured representations. Roughly speaking, M_1 and M_2 have representations that can be partitioned into a bounded number of blocks each of which is "triangular", a property we call weakly block-triangular.

Geelen, Gerards and Whittle have announced that, under the hypotheses above, the matroids M_1 and M_2 both have pathwidth bounded by some constant depending only on q. That result plays a significant role in their announced proof of Rota's Conjecture. Bounding the pathwidth of M_1 and M_2 is currently the single most complicated part in the proof of Rota's Conjecture. Our result is intended as a step toward simplifying this part.

A matroid N is said to be a fragile minor of another matroid M if $M/C\backslash D = N$ for some $C, D \subseteq E(M)$, but $M/C'\backslash D' \neq N$ whenever $C \neq C'$ or $D \neq D'$. As a second result, we will prove that, given a $GF(q)$-representable matroid N, every $GF(q)$-representable matroid M having N as a fragile minor has a representation which is weakly block-triangular.

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Chapter 1

Introduction

1.1 Background and previous results

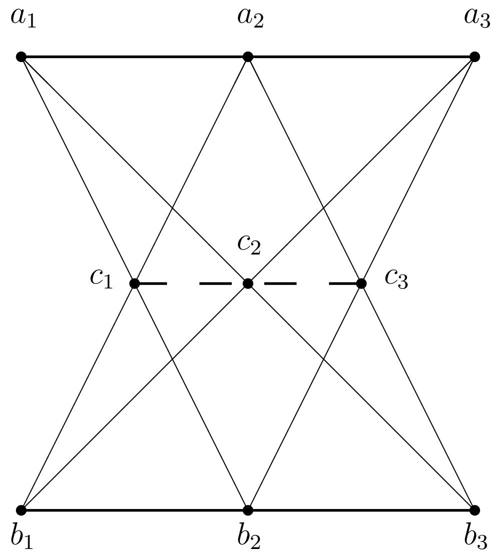
In circa 340 AD, long before matroids were formally introduced in the twentieth century, the very first theorem of matroid theory was proven by Pappus in the Byzantine city of Alexandria.

Let three points a_1, a_2, a_3 lie on a common line in the plane and let three other points b_1, b_2, b_3 lie on another common line in the plane. Construct the point c_1 as the intersection point of the line through a_1 and b_2 and the line through a_2 and b_1. Similarly, construct c_2 as the intersection point of the line through a_1 and b_3 and the line through a_3 and b_1 and c_3 as the intersection point of the line through a_2 and b_3 and the line through a_3 and b_2. Pappus' Theorem states that independently of how exactly the positions for a_1, a_2, a_3 and b_1, b_2, b_3 are chosen, the points c_1, c_2 and c_3 will lie on a common line; see Figure 1.1.

Nowadays, this picture can be interpreted as the drawing of a simple matroid M of rank three where the the points stand for the elements of the ground set of M and the lines indicate sets of rank two. Pappus' theorem shows that the rank-3-matroid M_1 on $\{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3\}$ whose set of lines is $\mathcal{L} = \{\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}, \{c_1, c_2, c_3\}, \{a_1, b_2, c_1\}, \{a_2, b_1, c_1\}, \{a_1, b_3, c_2\}, \{a_3, b_1, c_2\}, \{a_2, b_3, c_3\}, \{a_3, b_2, c_3\}\}$ is $\mathbb{R}$-representable, but the matroid M_2 whose set of lines is $\mathcal{L} - \{\{c_1, c_2, c_3\}\}$ is not. In fact, it turns out that M_2 is not representable over any field. Observe that the line $\{c_1, c_2, c_3\}$ is both a circuit and a hyperplane of M_1 and that M_2 arises from M_1 by relaxing that circuit-hyperplane.

This brings up the question whether the loss of representability is what should be expected in most cases when relaxing a circuit-hyperplane. For finite fields, this thesis

Figure 1.1: The Pappus matroid



yields a clear answer to this question. We will show that for any finite field $GF(q)$ the set of $GF(q)$-representable matroids with a circuit-hyperplane whose relaxation does not lose $GF(q)$-representability is very restricted. More exactly, we will prove that these matroids have highly structured representations. In order to provide some context for this result and its interpretations, we will now turn our attention to some related, more recent results in matroid theory. Any matroid notation in this thesis follows the conventions of the book "Matroid Theory" by James Oxley [6].

Matroids were introduced by Whitney in 1935 [13] generalizing the notion of independence in linear algebra as well as certain properties of graphs to more abstract objects. Since then, matroids have had a lot of influence in several fields of mathematics, in particular in combinatorics. Their importance spreads from geometry and algebra to algorithmic problems and combinatorial optimization.

Apart from that, a lot of research has been performed in order to get deeper insights into the structural properties of matroids. Similarly to graphs, there is a minor notion for matroids bringing up questions of forbidden minors for minor-closed classes. It is of particular interest to determine the minor-closed classes of matroids having only finitely many forbidden minors.[8]

For the case of graphs, an astonishingly pretty and complete answer to this question has been found. Paul Seymour and Neil Robertson proved in a series of papers that any minor-closed class of graphs only has a finite number of forbidden minors. The analogous statement is not true for matroids, as a family of matroids of rank 3 shows, see 14.1.2 in [6].

For any given field $\mathbb{F}$ it is easy to see that the class of $\mathbb{F}$-representable matroids is minor-closed. In 1971, Gian-Carlo Rota conjectured that there are only finitely many forbidden minors for the class of $GF(q)$-representable matroids for any finite field $GF(q)$.

Several partial results for fields of small cardinality have been proven providing some evidence for the truth of Rota's Conjecture. Already before the conjecture was stated, Tutte proved in 1958 that a matroid is binary if and only if it does not contain $U_{2,4}$ as a minor [11]. Seymour [10] and independently Bixby [1] proved in 1979 that a matroid is $GF(3)$-representable if and only if it does not contain $U_{2,5}, U_{3,5}$ the Fano plane or its dual as a minor. In 2000, Geelen, Gerards and Kapoor characterized the seven forbidden minors for $GF(4)$-representable matroids [3]. Despite these complete characterizations, there is no published proof of Rota's Conjecture for any field of size at least 5.

Nevertheless, in 2011, Geelen, Gerards and Whittle announced a proof for the general case of Rota's Conjecture that has not yet been published, see [4]. A significant part of this proof is concerned with the analysis of $GF(q)$-representable matroids having a circuit-hyperplane whose relaxation yields another $GF(q)$-representable matroid.

Let $M = (E, r)$ be a matroid and for any set $S \subseteq E$ let $\lambda(S) = r(S) + r(E - S) - r(M)$. The *pathwidth* of M is defined to be the minimum number t such that there is an ordering $(e_1, \ldots, e_n)$ of E such that $\lambda(\{e_1, \ldots, e_i\}) \leq t$ for all $i = 1, \ldots, n-1$. The proposed proof of Rota's Conjecture relies on the following conjecture:

Conjecture 1.1.1. *If M_1 and M_2 are $GF(q)$-representable matroids on a common ground set such that M_2 arises from M_1 by relaxing a circuit-hyperplane, then the pathwidth of M_1 and M_2 is bounded by a constant $K(q)$.*

This thesis provides some progress towards a simpler proof of that conjecture.

1.2 Block-triangular representations

An important tool in the analysis of representable matroids evidently is their representations. We will make strong use of a varied form of representations, so-called reduced representations, which are just representations with a suppressed identity matrix.

We will next explain what a reduced representation is. Let M be an $\mathbb{F}$-representable matroid for some field $\mathbb{F}$ and B be a basis of M. Let A be an $\mathbb{F}$-representation of M. As B is a basis of M, we can apply elementary row operations to make the columns of A indexed by B an identity matrix, so A is of the form $[I|A']$. We will now index the rows of A' by the elements of B in the order induced by the identity matrix. We will now call $A' \in \mathbb{F}^{B\times(E-B)}$ a *reduced representation* of M with respect to the basis B. Observe that deleting a column indexed by an element $e \in E - B$ yields a reduced representation of $M\backslash e$ with respect to B. Also observe that deleting the row indexed by an element $b_i \in B$ yields a reduced representation of M/b_i with respect to $B - \{b_i\}$ which is easily seen to be a basis of M/b_i.

In Chapter 2, more properties of reduced representations will be discussed. It will turn out that if M_1 and M_2 are two $GF(q)$-representable matroids such that M_2 arises from M_1 by relaxing a circuit-hyperplane, the reduced representations of M_1 and M_2 will display significant similarities. Given $\mathbb{F}$-representable matroids M and N for some field $\mathbb{F}$ such that N is a minor of M and a basis B_N of N we can find a basis B_M of M such that $B_M \cap E(N) = B_N$ and $M/(B_M - B_N)\backslash(E(M) - E(N) - B_M) = N$. We will also provide a lemma that makes a reduced representation of N with respect to a basis B_N visible in a reduced representation of M with respect to B_M.

It turns out that much of the work on some matroid properties can be discussed in the language of matrices rather than in the language of matroids. In order to make the statements about matrices more precise, we will need the following three definitions of structural properties of matrices.

Definition 1.2.1. A matrix A if *triangular* if it has at most two distinct entries and each non-empty submatrix has a constant row or column.

Algorithmically speaking, this means we can recursively delete constant rows and columns of A until the remaining matrix is empty.

For example,

$$A = \begin{bmatrix} \beta & \alpha & \beta \\ \beta & \alpha & \beta \\ \alpha & \alpha & \alpha \end{bmatrix}$$

is a triangular matrix. This can be seen by choosing the labelling a_2, a_3, a_4 for the rows and a_5, a_1, a_6 for the columns. On the other hand,

$$B = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$$

is not triangular as it does not contain any constant rows or column.

Given a matrix A with row index set I and column index set J as well as index sets $I' \subseteq I$ and $J' \subseteq J$, let $A[I', J']$ denote the submatrix of A which uses the row index set I' and the column index set J'. A *C-block decomposition* of an $I \times J$-matrix A is a tuple $((I_1, \ldots, I_C), (J_1, \ldots, J_C))$ where $(I_1, \ldots, I_C)$ is a partition of I and $(J_1, \ldots, J_C)$ is a partition of J.

Definition 1.2.2. For a non-negative integer C, an $I \times J$-matrix A is called *weakly C-block-triangular* if there is a C-block-decomposition $((I_1, \ldots, I_C), (J_1, \ldots, J_C))$ of A such that $A[I_k, J_l]$ is triangular for all $k, l \leq C$. The C-block-decomposition $((I_1, \ldots, I_C), (J_1, \ldots, J_C))$ is called a *certificate of weak C-block-triangularity.*

For example,

$$A = \left[\begin{array}{ccc|cc} \gamma & \delta & \delta & \epsilon & \epsilon \\ \gamma & \gamma & \delta & \zeta & \epsilon \\ \hline \beta & \alpha & \beta & \eta & \eta \\ \beta & \alpha & \beta & \eta & \eta \\ \alpha & \alpha & \alpha & \eta & \theta \end{array}\right]$$

is a weakly 2-block-triangular matrix where the certificate is indicated in the drawing.

Definition 1.2.3. For a non-negative integer C, an $I \times J$-matrix A for disjoint index sets I and J is called *C-block-triangular* if there is a C-block-decomposition $((I_1, \ldots, I_C), (J_1, \ldots, J_C))$ that is a certificate of weak C-block-triangularity and such that each non-empty submatrix of A has either a row that is constant in every partition class of $(J_1, \ldots, J_C)$ or a column that is constant in every partition class of $(I_1, \ldots, I_C)$. The C-block-decomposition $((I_1, \ldots, I_C), (J_1, \ldots, J_C))$ is then called a *certificate of C-block-triangularity.*

Algorithmically speaking, this means we can recursively delete rows and columns that are constant in each block until the remaining matrix is empty.

For example,

$$A = \left[\begin{array}{cc|cc} \beta & \alpha & \zeta & \zeta \\ \alpha & \alpha & \epsilon & \epsilon \\ \hline \gamma & \delta & \eta & \eta \\ \gamma & \gamma & \eta & \theta \end{array}\right]$$

is a 2-block-triangular matrix where the certificate is indicated in the drawing, labelling the rows by a_3, a_1, a_4, a_6 and labelling the columns by a_2, a_8, a_7, a_5.

Observe that it follows directly from the definition that it a matrix A that is block-triangular with certificate D is also weakly block-triangular with certificate D.

In Chapter 2, we will prove the following result that is the main motivation for the matrix analysis performed in this thesis.

Lemma 1.2.4. *Let M be a matroid with a reduced representation A over some field. If A is C-block-triangular for some integer C, then M has pathwidth at most $2C$.*

Therefore, the following conjecture implies Conjecture 1.1.1 :

Conjecture 1.2.5. *For each finite field $GF(q)$ there is an integer C such that, if M is a $GF(q)$-representable matroid with a circuit-hyperplane whose relaxation yields another $GF(q)$-representable matroid, then M has a C-block-triangular reduced representation.*

For small fields, the $GF(q)$-representable matroids having a circuit-hyperplane whose relaxation yields another $GF(q)$-representable matroid have been completely characterized proving the conjecture for these cases. Lucas [5] characterized the binary matroids of that kind. Truemper [12], and independently, Oxley and Whittle [7], proved a characterization of the ternary matroids of that kind. Only recently, Clark, Oxley and van Zwam did the same for quaternary matroids [2].

While we are not able to prove Conjecture 1.2.5, we can prove the following slightly weaker theorem:

Theorem 1.2.6. *For each finite field $GF(q)$ there is an integer C such that, if M is a $GF(q)$-representable matroid with a circuit-hyperplane whose relaxation yields another $GF(q)$-representable matroid, then M has a weakly C-block-triangular reduced representation.*

1.3 Obstructions to weak block-triangularity

We will now define families of matrices $\mathcal{A}_k, \mathcal{B}_k$ and $\mathcal{C}_k$ which are helpful to prove the theorems above. We say that a matrix A_1 is *isomorphic* to a matrix A_2 if A_2 can be obtained from A_1 by relabelling rows and columns.

The matrix $A(a, b, c, k)$ is the $k \times k$-matrix A over $GF(q)$ having entries a, b and c on, below and above the diagonal, respectively; for example:

$$A(a, b, c, 4) = \begin{bmatrix} a & c & c & c \\ b & a & c & c \\ b & b & a & c \\ b & b & b & a \end{bmatrix} .$$

Let $\mathcal{A}_k$ be the set of matrices isomorphic to a matrix of the form $A(a, b, c, k)$ where $a \neq b$ and $a \neq c$.

The matrix $B(a, b, c, d, k)$ is a $2k \times 2k$-matrix B over $GF(q)$ having the following block-structure:

$$B(a, b, c, d, k) = \begin{array}{|c|c|} \hline A(a, c, a, k) & A(b, b, a, k) \\ \hline A(c, c, d, k) & A(d, b, d, k) \\ \hline \end{array}.$$

For example,

$$B(a, b, c, d, 4) = \left(\begin{array}{cccc|cccc} a & a & a & a & b & a & a & a \\ c & a & a & a & b & b & a & a \\ c & c & a & a & b & b & b & a \\ c & c & c & a & b & b & b & b \\ \hline c & d & d & d & d & d & d & d \\ c & c & d & d & b & d & d & d \\ c & c & c & d & b & b & d & d \\ c & c & c & c & b & b & b & d \end{array} \right) .$$

Let $\mathcal{B}_k$ be the set of matrices isomorphic to a matrix of the form $B(a, b, c, d, k)$ where $\{a, d\} \cap \{b, c\} = \emptyset$.

The matrix $C(a, b, c, k)$ is a $3k \times 3k$-matrix C over $GF(q)$ having the following block-structure:

$$C(a, b, c, k) = \begin{array}{|c|c|c|} \hline A(a, a, a, k) & A(a, c, a, k) & A(b, b, a, k) \\ \hline A(a, a, c, k) & A(c, c, c, k) & A(c, b, c, k) \\ \hline A(b, a, b, k) & A(c, c, b, k) & A(b, b, b, k) \\ \hline \end{array}.$$

For example,

$$C(a,b,c,4) = \left(\begin{array}{cccc|cccc|cccc}
a & a & a & a & a & a & a & a & b & a & a & a \\
a & a & a & a & c & a & a & a & b & b & a & a \\
a & a & a & a & c & c & a & a & b & b & b & a \\
a & a & a & a & c & c & c & a & b & b & b & b \\
\hline
a & c & c & c & c & c & c & c & c & c & c & c \\
a & a & c & c & c & c & c & c & b & c & c & c \\
a & a & a & c & c & c & c & c & b & b & c & c \\
a & a & a & a & c & c & c & c & b & b & b & c \\
\hline
b & b & b & b & c & b & b & b & b & b & b & b \\
a & b & b & b & c & c & b & b & b & b & b & b \\
a & a & b & b & c & c & c & b & b & b & b & b \\
a & a & a & b & c & c & c & c & b & b & b & b
\end{array}\right).$$

Let $\mathcal{C}_k$ be the set of matrices isomorphic to a matrix of the form $C(a,b,c,k)$ where a,b,c are pairwise distinct.

Furthermore, let $\mathcal{A}$ denote $\bigcup\limits_{k=1}^{\infty} \mathcal{A}_k$, let $\mathcal{B}$ denote $\bigcup\limits_{k=1}^{\infty} \mathcal{B}_k$ and let $\mathcal{C}$ denote $\bigcup\limits_{k=1}^{\infty} \mathcal{C}_k$.

Chapter 3 will be mainly concerned with proving the following theorem that is a crucial step in the proof of Theorem 1.2.6.

Theorem 1.3.1. *Let $GF(q)$ be a finite field. For every integer k there is an integer n such that every matrix over $GF(q)$ that is not weakly n-block-triangular has a submatrix in $\mathcal{A}_k, \mathcal{B}_k$ or $\mathcal{C}_k$.*

This proof will first characterize all matrices A such that A does not have a constant row or column but all of its proper submatrices do. It turns out that these matrices are exactly the matrices in $\mathcal{B}_1$ and $\mathcal{C}_1$.

After that, we will apply some Ramsey theory to show that a matrix consisting of a bounded number of blocks that are weakly C-block-triangular with respect to some bounded number C are themselves weakly block-triangular with respect to some bounded number. The analogous statement is not true for block-triangularity and that is the reason why the proofs of Theorem 1.2.6 cannot be easily generalized to prove Conjecture 1.2.5.

We will then finish the proof by a decomposition argument. The characterization will be relatively good in the sense that the sets $\mathcal{A}_k, \mathcal{B}_k$ and $\mathcal{C}_k$ are fairly small and allow to prove the desired matroid results in Chapter 4.

It would be desirable though to find a family $\mathcal{F}$ of matrices satisfying the properties of Theorem 1.3.1 such that for all k there is an n such that no matrix in $\mathcal{F}$ of size at least n is weakly k-block-triangular. Such a family could be considered a complete characterization of weakly block-triangular matrices. While this is easily seen to fail for $\mathcal{B}$ and $\mathcal{C}$, we will be able to show that for all k there is an n such that no matrix in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ of size at least n is k-block-triangular. This can be seen as an indication that the matrices in $\mathcal{A}_k, \mathcal{B}_k$ and $\mathcal{C}_k$ could play a crucial role in proving Conjecture 1.1.1.

The fourth chapter is concerned with showing that the results of Chapter 3 are applicable to matroid theory. In particular, we will show the following theorem. Observe that if M_1, M_2 are matroids such that M_2 arises from M_1 by relaxing a circuit-hyperplane H, then $H - \{e\} \cup \{f\}$ is a basis of both M_1 and M_2 for all $e \in H$ and all $f \in E(M_1) - H$.

Theorem 1.3.2. *For some finite field $GF(q)$, let M_1 and M_2 be $GF(q)$-representable matroids such that M_2 arises from M_1 by relaxing a circuit-hyperplane H and let A be a reduced representation of M_1 or M_2 with respect to a basis B of the form $H - \{e\} \cup \{f\}$. Then, after scaling its rows and columns, A does not contain any submatrix in $\mathcal{A}_{q^4}, \mathcal{B}_2, \mathcal{C}_3$.*

We will derive Theorem 1.2.6 from Theorem 1.3.2 and Theorem 1.3.1.

1.4 Fragile minors

The fifth chapter deals with fragile minors of representable matroids.

A matroid N is a *fragile* minor of another matroid M if $M/C\backslash D = N$ for some $C, D \subseteq E(M)$, but $M/C'\backslash D' \neq N$ whenever $C \neq C'$ or $D \neq D'$. The following conjecture is interesting as, together with Lemma 1.2.4, it implies that every matroid that has a certain fragile minor has bounded pathwidth:

Conjecture 1.4.1. *For each finite field $GF(q)$ and $GF(q)$-representable matroid N there is an integer C such that if M is a $GF(q)$-representable matroid having N as a fragile minor, then M has a C-block-triangular reduced representation.*

While we cannot prove this conjecture, we can prove the following weaker version:

Theorem 1.4.2. *For each finite field $GF(q)$ and $GF(q)$-representable matroid N there is an integer C such that if M is a $GF(q)$-representable matroid having N as a fragile minor, then M has a weakly C-block-triangular reduced representation.*

As a crucial step in the analysis of fragile minors, we will prove the following theorem:

Theorem 1.4.3. *Let $\mathbb{F}$ be a finite field of order q, let N be an $\mathbb{F}$-representable matroid and M an $\mathbb{F}$-representable matroid having N as a fragile minor. Then there is a field $\mathbb{F}_1$ of order $q^{2^{|N|+1}}$ and $\mathbb{F}_1$-representable matroids M_1, M_2 such that:*

(i) *M_2 is obtained from M_1 by relaxing a circuit-hyperplane,*

(ii) *$M/C\backslash D = M_1/c\backslash d$ for some partition (C, D) of $E(N)$ and some $c, d \in E(M_1)$.*

This theorem connects fragile minors and circuit-hyperplane relaxations and has several impacts on the questions discussed in this thesis. It gives us the possibility to apply results for circuit-hyperplane relaxations to fragile minors. We will explain how Theorem 1.4.3 and Theorem 1.2.6 imply Theorem 1.4.2. Furthermore, we will show that Theorem 1.4.3 and Conjecture 1.2.5 imply Conjecture 1.4.1.

Most of the theory in this thesis, in particular Theorem 1.2.6, as well as the rough ideas to prove it have been developed by Geelen, Gerards and Whittle during the process of proving Rota's Conjecture. Our work can be viewed as an effort to work out the details of this proof. During this process, we came to a surprising result that made the proof easier than expected, namely that for Theorem 1.3.2, the matrices in $\mathcal{B}$ and $\mathcal{C}$ can be chosen of constant size, meaning independently of q and their concrete entries.

Chapter 2

Matroids and matrices

This chapter is meant to clarify some of the relationships between several technical objects and provide the necessary preliminaries for the main proofs in the following chapters. Recall that for all integers C, a C-block-triangular matrix is also weakly C-block-triangular. In the first part we will show that the converse is false.

The second part collects some basic properties of reduced representations considering the settings of circuit-hyperplane relaxations and minors. It also gives a theorem that connects block-triangularity of reduced representations with the pathwidth of the underlying matroid.

2.1 Block-triangularity and weak block-triangularity

If A is a matrix that is k-block-triangular with a certificate D, then A is also weakly k-block-triangular with certificate D. Unfortunately the converse is false. For example, the matrices in $\mathcal{B}$ are weakly 2-block-triangular by definition, but Lemma 2.1.2 below shows that the matrices in $\mathcal{B}$ have unbounded block-triangularity. We also prove similar results for $\mathcal{A}$ and $\mathcal{C}$, although while the matrices in $\mathcal{C}$ are weakly 3-block-triangular, the matrices in $\mathcal{A}$ have unbounded weak block-triangularity. This highlights the importance of the matrix classes $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ for the purpose of finding a good obstruction for block-triangularity.

We will first need to introduce a bit of notation that will facilitate the proofs of the above statements. Recall that a k-block-decomposition of an $I \times J$-matrix A is a tuple $((I_1, \dots, I_C), (J_1, \dots, J_C))$ where $(I_1, \dots, I_C)$ is a partition of I and $(J_1, \dots, J_C)$ is a partition of J. One block-decomposition of A *refines* another if each part of the first is contained in a part of the second. Note that, given a C_1-block-decomposition and a C_2-block-decomposition, there is a n-block-decomposition that refines them both, with $n \leq C_1 C_2$.

A block-decomposition $((I_1, \ldots, I_C), (J_1, \ldots, J_C))$ of an $I \times I$-matrix is *symmetric* if $I_i = J_i$ for each $i \in \{1, \ldots, C\}$. Now let A be an $I \times J$-matrix in $\mathcal{A}_k$ and consider an isomorphism from A to $A(a, b, c, k)$. Observe that the isomorphism is unique whenever $k \geq 3$. A block-decomposition $((I_1, \ldots, I_C), (J_1, \ldots, J_C))$ of A is *symmetric* with respect to the isomorphism if, for each $i \in \{1, \ldots, C\}$, the sets I_i and J_i map to the same set under the given isomorphism.

We will use the fact that if A is a C-block-triangular matrix with certificate of block-triangularity $((I_1, \ldots, I_k), (J_1, \ldots, J_k))$, then any submatrix A' of A on $I' \subseteq I$ and $J' \subseteq J$ is k-block-triangular with certificate $((I_1 \cap I', \ldots, I_k \cap I'), (J_1 \cap J', \ldots, J_k \cap J'))$.

We are now ready give the proofs of the facts mentioned above. Recall that for example a matrix $A(a, b, c, 4) \in \mathcal{A}_4$ is of the form

$$\begin{bmatrix} a & c & c & c \\ b & a & c & c \\ b & b & a & c \\ b & b & b & a \end{bmatrix}$$

for some a, b, c such that $a \neq b$ and $a \neq c$.

Lemma 2.1.1. *Matrices in $\mathcal{A}_{2k^2}$ are not weakly k-block-triangular, so in particular not k-block-triangular.*

Proof. The statement is trivial for $k = 1$. Let A be a matrix in $\mathcal{A}_{2k^2}$ for some $k \geq 2$ where the rows as well as the columns of A are labelled by $I = \{1, \ldots, 2k^2\}$ in the canonical order. For the sake of a contradiction, we will assume that A is weakly k-block-triangular. So there is a certificate $(\mathcal{I}_0, \mathcal{J}_0)$ of k-block-triangularity.

Observe that because of the symmetry of A, $(\mathcal{J}_0, \mathcal{I}_0)$ also is a certificate of k-block-triangularity. Therefore, there is a symmetric refinement $\mathcal{I}$ of $(\mathcal{I}_0, \mathcal{J}_0)$ and $(\mathcal{J}_0, \mathcal{I}_0)$ that is a certificate of k^2-block-triangularity.

By a majority argument there must be some partition class I^* of $\mathcal{I}$ that is of size at least 2, without loss of generality exactly 2. By construction,

$$A[I^*, I^*] = \begin{bmatrix} a & c \\ b & a \end{bmatrix} .$$

We can see that $A[I^*, I^*]$ does not contain a constant row or column. This is a contradiction to $\mathcal{I}$ being a certificate for k^2-block-triangularity. $\square$

Recall that a matrix $B(a, b, c, d, k) \in \mathbb{B}_k$ is of the form

$$B(a,b,c,d,k) = \begin{array}{|c|c|}\hline A(a,c,a,k) & A(b,b,a,k) \\ \hline A(c,c,d,k) & A(d,b,d,k) \\ \hline \end{array},$$

where $\{a, d\} \cap \{b, c\} = \emptyset$.

Lemma 2.1.2. *Matrices in $\mathcal{B}_{4k^4}$ are not k-block-triangular.*

Proof. Let $A = B(a, b, c, d, 4k^4)$ be a matrix in $\mathcal{B}_{4k^4}$ and let the rows and columns be indexed in the following way:

$$\begin{array}{c c} & \begin{array}{cc} \quad J_\gamma \quad & \quad J_\delta \quad \end{array} \\ \begin{array}{c} I_\alpha \\ I_\beta \end{array} & \begin{bmatrix} A(a,c,a,k) & A(b,b,a,k) \\ A(c,c,d,k) & A(d,b,d,k) \end{bmatrix} \end{array}.$$

For the sake of a contradiction, we will asumme that A is k-block-triangular. So there is a certificate $(\mathcal{I}, \mathcal{J})$ of k-block-triangularity. By refining, we can now obtain a certificate $(\mathcal{I}', \mathcal{J}')$ of $2k^4$-block-triangularity whose restriction is symmetric in all of $A[I_\alpha, J_\gamma], A[I_\alpha, J_\delta], A[I_\beta, J_\gamma]$ and $A[I_\beta, J_\delta]$ and such that $(\mathcal{I}', \mathcal{J}')$ refines $((I_\alpha, I_\beta), (J_\gamma, J_\delta))$.

By a majority argument, there must be some partition class I_1 of $\mathcal{I}'$ of size at least 2, without loss of generality exactly 2, all of whose elements are in I_α. Using the symmetry conditions, we can find I_2, J_1, J_2 such that

$$A[I_1 \cup I_2, J_1 \cup J_2] = \begin{array}{|cc|cc|}\hline a & a & b & a \\ c & a & b & b \\ \hline c & d & d & d \\ c & c & b & d \\ \hline \end{array},$$

where the partition classes of $\mathcal{I}'$ are indicated in the drawing. $A[I_1 \cup I_2, J_1 \cup J_2]$ does not have a row a row or column that is constant in every row of $\mathcal{J}'$ or $\mathcal{I}'$, respectively. This is a contradiction to $(\mathcal{I}', \mathcal{J}')$ being a certificate of $2k^4$-block-triangularity. $\square$

Recall that a matrix $C(a, b, c, d, 3) \in \mathcal{C}_k$ is of the form

$$C(a,b,c,k) = \begin{array}{|c|c|c|}\hline A(a,a,a,k) & A(a,c,a,k) & A(b,b,a,k) \\ \hline A(a,a,c,k) & A(c,c,c,k) & A(c,b,c,k) \\ \hline A(b,a,b,k) & A(c,c,b,k) & A(b,b,b,k) \\ \hline \end{array}.$$

for distinct a, b and c.

Lemma 2.1.3. *Matrices in $\mathcal{C}_{6k^6}$ are not k-block-triangular.*

Proof. Let $A = C(a, b, c, d, 6k^6)$ be a matrix in $\mathcal{B}_{6k^6}$ and let the rows and columns be indexed in the following way:

$$\begin{array}{cc} & \begin{array}{ccc} J_\delta & J_\epsilon & J_\zeta \end{array} \\ \begin{array}{c} I_\alpha \\ I_\beta \\ I_\gamma \end{array} & \begin{bmatrix} A(a,a,a,k) & A(a,c,a,k) & A(b,b,a,k) \\ A(a,a,c,k) & A(c,c,c,k) & A(c,b,c,k) \\ A(b,a,b,k) & A(c,c,b,k) & A(b,b,b,k) \end{bmatrix} \end{array}.$$

For the sake of a contradiction, we will assume that A is k-block-triangular. So there is a certificate $(\mathcal{I}, \mathcal{J})$ of k-block-triangularity. By refining, we can now obtain a certificate $(\mathcal{I}', \mathcal{J}')$ of $3k^6$-block-triangularity whose restriction is symmetric in $A[I_k, J_l]$ for all $k \in \{\alpha, \beta, \gamma\}$ and $l \in \{\delta, \epsilon, \zeta\}$ and such that $(\mathcal{I}', \mathcal{J}')$ is a refinement of $((I_\alpha, I_\beta, I_\gamma), (J_\delta, J_\epsilon, J_\zeta))$.

By a majority argument, there must be some partition class I_1 of $\mathcal{I}'$ of size at least 2, without loss of generality exactly 2, all of whose elements are in I_α. Using the symmetry conditions, we can find I_2, I_3, J_1, J_2, J_3 such that

$$A[I_1 \cup I_2 \cup I_3, J_1 \cup J_2 \cup J_3] = \begin{array}{|cc|cc|cc|} \hline a & a & a & a & b & a \\ a & a & c & a & b & b \\ \hline a & c & c & c & c & c \\ a & a & c & c & b & c \\ \hline b & b & c & b & b & b \\ a & b & c & c & b & b \\ \hline \end{array}.$$

where the partition classes of $\mathcal{I}'$ are indicated in the drawing. A' does not have a row or column that is constant in every row of $\mathcal{J}'$ or $\mathcal{I}'$, respectively. This is a contradiction to $(\mathcal{I}', \mathcal{J}')$ being a certificate of $3k^6$-block-triangularity. □

2.2 Reduced representations

We will now discuss some important properties of reduced representations that will play crucial roles in this thesis. Recall that a reduced representation of a matroid M with respect to a basis B is a matrix $A \in \mathbb{F}^{B \times E(M) - B}$ over some field $\mathbb{F}$ such that the matrix $[I|A]$ is a representation of M where I is a $B \times B$ identity matrix. Clearly a matroid is completely determined by its reduced representation. Another important property is that if A is a reduced representation of a matroid M with respect to some basis B, and A' is obtained from A by row and column scalings, then A' is also reduced representation of M with respect to B.

Lemma 2.2.1. *Let M be a matroid with a reduced representation of the form*

$$A = \begin{array}{c|cc} & X_2 & Y_2 \\ \hline X_1 & A_1 & A_2 \\ Y_1 & A_3 & A_4 \end{array}.$$

Then $\lambda_M(X_1 \cup X_2) = r(A_2) + r(A_3)$

Proof. Observe that A_2 is a reduced representation of $M/Y_1 \backslash X_2$ and therefore $r(A_2) = r_{M/Y1}(Y_2)$. Similarly, $r(A_3) = r_{M/X1}(X_2)$. Also, as $X_1 \cup Y_1$ is a basis of M, we have $r(M) = r_M(X_1) + r_M(Y_1)$. Therefore, we can calculate:

$$\begin{aligned} &\lambda_M(X_1 \cup X_2) = r_M(X_1 \cup X_2) + r_M(Y_1 \cup Y_2) - r(M) \\ =&r_{M/X_1}(X_2) + r_M(X_1) + r_{M/Y1}(Y_2) + r_M(Y_1) - r(M) \\ =&r_{M/X_1}(X_2) + r_{M/Y1}(Y_2) = r(A_3) + r(A_2). \end{aligned}$$

□

We are now ready to prove Lemma 1.2.4, which is the following statement.

Lemma (Restatement of Lemma 1.2.4)**.** *Let M be a matroid with a reduced representation A over some field. If A is C-block-triangular for some integer C, then M has pathwidth at most $2C$.*

Proof. Let M be a representable matroid having a C-block-triangular reduced representation A with respect to some basis B. Let I be the row index set and J be the column index set of A. There is a certificate of C-block-triangularity $((I_1, \dots, I_C), (J_1, \dots, J_C))$. Observe that this induces an ordering $(e_1, \dots, e_n)$ of $I \cup J$ such that the row or column of $A[I \cap \{e_i, \dots, e_n\}, J \cap \{e_i, \dots, e_n\}]$ which is indexed by e_i is constant in every partition class of $\mathcal{J}$ or $\mathcal{I}$, respectively. Recall that $E(M) = I \cup J$ by definition and so it suffices to prove that for every $i = 1, \dots, n-1$,we have $\lambda(\{e_1, \dots, e_i\}) \leq 2C$. For that purpose, fix some $i \in \{1, \dots, n-1\}$. Let I_a be defined as $B \cap \{e_1, \dots, e_i\}$ and J_a be $(E(M) - B) \cap \{e_1, \dots, e_i\}$. Similarly, let I_b be defined as $B \cap \{e_{i+1}, \dots, e_n\}$ and J_b be $(E(M) - B) \cap \{e_{i+1}, \dots, e_n\}$. Let $A_1 = A[I_a, J_a], A_2 = A[I_a, J_b], A_3 = A[I_b, J_a]$ and $A_4 = A[I_b, J_b]$, so

$$A = \begin{array}{cc} & \begin{array}{cc} J_a & J_b \end{array} \\ \begin{array}{c} I_a \\ I_b \end{array} & \left[\begin{array}{cc} A_1 & A_2 \\ A_3 & A_4 \end{array}\right] \end{array}.$$

Observe that $\mathcal{I}_a = (I_1 \cap I_a, \dots, I_C \cap I_a)$ is a partition of I_a, $\mathcal{I}_b = (I_1 \cap I_b, \dots I_C \cap I_b)$ is a partition of I_b and, analogously, $\mathcal{J}_a = (J_1 \cap J_a, \dots, J_C \cap J_a)$ is a partition of J_a and $\mathcal{J}_b = (J_1 \cap J_b, \dots, J_C \cap J_b)$ is a partition of J_b. Observe that every partition class of $\mathcal{I}_b$ is a subset of a partition class of $\mathcal{I}$.

As every row of I_a is of lower index then every column of J_b, every row of I_a is constant in every partition class of $\mathcal{J}_b$, so the rows of A_2 are constant in every partition class of $\mathcal{J}_b$. It follows that any two columns of A_2 in the same partition class of $\mathcal{J}_b$ are identical, so A_2 has at most C distinct columns, so $r(A_2) \leq C$.

Similarly, as every column of J_a is of lower index then every row of I_b, every column of J_a is constant in every partition class of $\mathcal{I}_b$, so the columns of A_3 are constant in every partition class of $\mathcal{I}_b$. It follows that any two rows of A_3 in the same partition class of $\mathcal{I}_b$ are identical, so A_3 has at most C distinct rows, so $r(A_3) \leq C$.

It so follows by Lemma 2.2.1 that $\lambda(I_a \cup J_a) = r(A_2) + r(A_3) \leq 2C$. This finishes the proof. □

We will now prove an important lemma that describes circuit-hyperplane relaxations in the context of reduced representations. Observe that if M_1 and M_2 are matroids such that M_2 is obtained from M_1 by relaxing a circuit-hyperplane H, for any $e \in H$ and $f \in E(M_1) - H$, $H - \{e\} \cup \{f\}$ is a basis of both M_1 and M_2.

Lemma 2.2.2. *Let M_1 and M_2 be rank-r matroids on the same ground set E such that M_2 is obtained from M_1 by relaxing a circuit-hyperplane H, and let $B = (H - \{e\}) \cup \{f\}$ for some $e \in B$ and $f \in E - B$. If A_1 and A_2 are reduced representations of M_1 and M_2, respectively, with respect to B, then*

1. *For each $I \subseteq B$ and $J \subseteq E - B$, the matrices $A_1[I, J]$ and $A_2[I, J]$ have the same rank unless $I = \{f\}$ and $J = \{e\}$,*

2. *Up to scaling, A_1 and A_2 have the form*

$$\begin{array}{cc} & \begin{array}{cccc} & & & e \end{array} \\ \begin{array}{c} \\ \\ \\ f \end{array} & \left[\begin{array}{ccc|c} & & & 1 \\ & A_1' & & \vdots \\ & & & 1 \\ \hline 1 & \dots & 1 & 0 \end{array}\right] \end{array} \text{ and } \begin{array}{cc} & \begin{array}{cccc} & & & e \end{array} \\ \begin{array}{c} \\ \\ \\ f \end{array} & \left[\begin{array}{ccc|c} & & & 1 \\ & A_2' & & \vdots \\ & & & 1 \\ \hline 1 & \dots & 1 & 1 \end{array}\right] \end{array} \text{,respectively.}$$

Proof. As H is a circuit in M_1, we have $A_1(f,e) = 0$ and $A_1(g,e) \neq 0$ for any element $g \in H - \{f\}$. Similarly, since $E - H$ is a circuit in M_1^*, we have $A_1(f,g) \neq 0$ for each $g \in E - (H \cup \{e\})$. Since $M_2 \backslash e = M_1 \backslash e$ and $M_2/f = M_1/f$, we have $A_2(a,e) \neq 0$ for all $a \in E - (H \cup \{f\})$ and and $A_2(f,b) \neq 0$ for all $b \in H - \{e\}$. Moreover, since H is a basis in M_2, we have $A_2(f,e) \neq 0$. It follows that A_1 and A_2 can be scaled to have the required form. □

The next lemma roughly speaking states the converse of the previous one, as it shows that matrices with similar properties as above are reduced representations of matroids M_1, M_2 such that M_2 arises from M_1 by relaxing a circuit-hyperplane.

Lemma 2.2.3. *Let M_1 and M_2 be matroids on a common ground set E with $r(M_1) = r(M_2)$ and let $H \subseteq E$. If M_1 and M_2 agree in rank on all subsets of E except H and $r_{M_1}(H) < r_{M_2}(H)$, then H is a circuit-hyperplane in M_1 and M_2 is obtained from M_1 by relaxing H.*

Proof. Observe first that $r_{M_1}(H) < r_{M_2}(H) \leq |H|$, so H is dependent in M_1. Let $B \subseteq E$ be a basis of H in M_2. If $B \neq H$, then $r_{M_1}(B) = r_{M_2}(B) = r_{M_2}(H) > r_{M_1}(H)$. So H is independent in M_2 and so are all its proper subsets. As M_1 agrees on them with M_2, they are independent in M_1. It follows that H is a circuit of M_1.

Observe that $r^*_{M_1}(E - H) < r^*_{M_2}(E - H)$ and $r^*_{M_1}(X) = r^*_{M_2}(X)$ for all $X \subseteq E$ that are different from $E - H$. It follows by the argumentation above that $E - H$ is a circuit of M_1^* and therefore that H is a hyperplane of M_1.

It follows immediately from the rank condition that M_2 arises from M_1 by relaxing this circuit-hyperplane. □

We will finally prove a small lemma that allows us to make a reduced representation of a matroid N visible in the reduced representation of a matroid M if N is a minor of M. This will be applied in Chapter 5. For a minor N of M, we will say that B *displays* N if $(M/B - E(N))|E(N) = N$. Observe that if A is a reduced representation of M with respect to a basis B displaying N, then deleting the rows of A in $B - E(N)$ and deleting the columns of A in $E(M) - (X \cup B)$ yields a reduced representation of N with respect to $B - E(N)$.

Lemma 2.2.4. *Let M be a matroid and N be a minor of M. Then there is a basis of M that displays N.*

Proof. By the scum theorem (see 3.3.2 in [6]), there is an independent set C and a coindependent set D such that $M/C\backslash D = N$. As D is coindependent, $E(M) - D$ is spanning and so there is some basis B that is disjoint from D. As C is independent, we can repeatedly apply the third independence axiom in order to find a basis B' such that $C \subseteq B' \subseteq B \cup C$. It follows that $M[E(N), B'] = M/(B' - E(N))\backslash(E(M) - (E(N) \cup B')) = M/C\backslash D = N$, so B' displays N. $\square$

Chapter 3

Obstructions for weak block-triangularity

This chapter will highlight the significant role played by the matrix classes $\mathcal{A}_k, \mathcal{B}_k$ and $\mathcal{C}_k$, which were defined in the introduction. In particular, our main goal is to prove Theorem 1.3.1. We will start with three lemmas which we will later apply in the proof of the main theorem, shortening the latter significantly. The first one characterizes all matrices that do not have a constant row or column but all of whose proper submatrices do.

3.1 Characterizing minimal matrices without constant row or column

Let $\mathcal{D}_2$ denote the set of 2×2 matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with $\{a, d\} \cap \{b, c\} = \emptyset$, and let $\mathcal{D}_3$ denote the set of 3×3 matrices that are isomorphic to

$$\begin{bmatrix} a & b & b \\ a & a & c \\ c & b & c \end{bmatrix}$$

where a, b and c are distinct. Observe that $\mathcal{D}_2 = \mathcal{B}_1$ and $\mathcal{D}_3 = \mathcal{C}_1$. Now let $\mathcal{D} = \mathcal{D}_2 \cup \mathcal{D}_3$.

Lemma 3.1.1. *If A is a matrix that does not contain a constant row or column, then A contains a submatrix in $\mathcal{D}$.*

Proof. We may assume that A is submatrix-minimal having no constant row or column.

So, for some integers m and n, let A be a matrix on $\{1, \dots, m\} \times \{1, \dots, n\}$ such that A does not contain a constant row or column but all of its proper submatrices do. Observe that any 2×2-matrix without a constant row or column is in $\mathcal{B}_1$, so we may assume that A has at least 3 rows or columns and will then prove that A is in $\mathcal{C}_1$. We will start by excluding small counterexamples.

Claim 1. *If A is not a 2×2-matrix, then A contains at least 3 rows and 3 columns.*

Proof. A matrix with only one row (column) has obviously constant columns (rows), so without loss of generality we may assume that A has two rows and n columns for some $n \geq 3$ aiming for a contradiction. As the first column is not constant, we know that $A(1, 1) = a$ and $A(2, 1) = b$ for some $a \neq b$.

As the first row of A is not constant, there is some $j \in \{2, \dots, n\}$ such that $A(1, j) = c$ for some $c \neq a$. As the j-th column of A is not constant, it follows that $A(2, j) \neq c$. As $A[\{1, 2\}, \{1, j\}]$ must contain a constant row or column, it follows that $A(2, j) = b$. Observe that $b \neq c$.

Similarly, as the second row of A is not constant, there is some $k \in \{2, \dots, n\}$ such that $A(2, k) = d$ for some $d \neq b$. As the k-th column of A is not constant, it follows that $A(1, k) \neq d$. As $A[\{1, 2\}, \{1, k\}]$ must contain a constant row or column, it follows that $A(1, k) = a$. Observe that $a \neq d$.

Therefore, we get that $c, d \notin \{a, b\}$ and therefore $A[\{1, 2\}, \{j, k\}]$ does not contain a constant row or column, a contradiction. ☐

For the rest of the proof, we will assume that A has $m \geq 3$ rows and $n \geq 3$ columns. We will now continue with proving some facts about the structure of A.

Claim 2. *There is an injective function $f : \{1, \dots, m\} \to \{1, \dots, n\}$ such that after the deletion of the i-th row, the remainder of the $f(i)$-th column is constant.*

Proof. After deleting row i, A must contain at least one constant column by definition. For all i, let $f(i) = j$ where j is the index of such a column. We still need to prove that f is injective. Assume otherwise, so $f(i_1) = f(i_2)$ for some $i_1, i_2 \in \{1, \dots, m\}$. As A contains at least three rows $\{1, \dots, m\} - \{i_1, i_2\}$ is nonempty. As $f(i_1) = j$, it follows that $A_{i,j} = A_{i_2,j}$ for all $i \in \{1, \dots, m\} - \{i_1, i_2\}$. As $f(i_1) = j$, it follows that $A_{i,j} = A_{i_1,j}$ for all $i \in \{1, \dots, m\} - \{i_1, i_2\}$. Therefore, it follows that the j-th column is constant, a contradiction. ☐

Applying the claim to A^T, we get that there also is an injective function $g : \{1, \dots, n\} \to \{1, \dots, m\}$ such that after the deletion of the j-th column, the remainder of the $g(j)$-th row is constant.

It therefore follows that $m = n$ and that f and g are both bijections. As f and g are in particular surjective, every row and column has exactly one special entry that is different from all others.

Claim 3. *The bijections f and g are uniquely defined.*

Proof. It suffices to prove that f is uniquely defined. Assume otherwise, so there are distinct bijections f_1 and f_2 such that the remainders of the $f_1(i)$-th column and the $f_2(i)$-th column are constant after deleting the row i for all $i \in \{1, \dots, m\}$. By assumption, there are some $i_1, j_1, j_2 \in \{1, \dots, m\}$ with $j_1 \neq j_2$ such that $f_1(i_1) = j_1$ and $f_2(i_1) = j_2$. It follows that the remainder of the column j_1 is constant after deleting the row i_1. As f_2 is a bijection, there is some $i_2 \neq i_1$ such that $f_2(i_2) = j_1$. It follows that the remainder of the column j_1 is constant after deleting the row i_1. As the column j_1 has at least three elements, it follows that it is constant in A, a contradiction. ☐

This yields that whenever we delete a row there is exactly one constant column and whenever we delete a column there is exactly one constant row.

Without loss of generality, we will assume that the special element of the first row is its first entry.

Claim 4. *There are no $i, j \in \{1, \dots, m\}$ such that after deleting the i-th row the remainder of the j-th column is constant and after deleting the j-th column the remainder of the i-th row is constant.*

Proof. Assume that this is not the case, and without loss of generality $i = j = 1$. So A is of the form

$$\begin{bmatrix} a & b & \dots & b \\ c & & & \\ \vdots & & & \\ c & & & \end{bmatrix}.$$

Now consider $A' = A[\{2,\ldots,m\},\{2,\ldots,m\}]$. As the first row is the unique one whose remainder is constant after deleting the first column of A, A' does not contain a constant row. Similarly, A' does not contain a constant column. This is a contradiction to the choice of A as being minimal with the property of not having a constant row or column. ☐

By symmetry, we may now assume that the i-th element in row i is special for row i for all $i = 1,\ldots,m$ and, by Claim 4 that the second element in column 1 is special for column 1. Let $A(1,1) = a$, $A(1,2) = b$ and $A(2,1) = c$. Observe that $a \notin \{b,c\}$. As the first row has $A(1,1)$ for its special element, $A(1,m) = \ldots = A(1,3) = A(1,2) = b$. As the second row has $A(2,2)$ for its special element, $A(2,m) = \ldots = A(2,3) = A(2,1) = c$. As the first column has $A(2,1)$ for its special element, $A(m,1) = \ldots = A(3,1) = A(1,1) = a$. As the third row has $A(3,3)$ for its special element, $A(3,m) = \ldots = A(3,4) = A(3,2) = A(3,1) = a$. So A is of the form

$$\begin{bmatrix} a & b & b & b & \ldots & b \\ c & & c & c & \ldots & c \\ a & a & & a & \ldots & a \\ a & & & & & \\ \vdots & & & & & \\ a & & & & & \end{bmatrix}.$$

As $A[\{1,2\},\{1,2\}]$ must contain a constant row or column, it follows that $A(2,2)$ is either b or c. As $A(2,2)$ is the special element of the second row, we know that $A(2,2) \neq c$, so $A(2,2) = b$. Observe that this also implies that $b \neq c$. As $A[\{2,3\},\{2,3\}]$ must contain a constant row or column, it follows that $A(3,3)$ is either a or c. As $A(3,3)$ is the special element of the second row, we know that $A(3,3) \neq a$, so $A(2,2) = c$.

This yields

$$A[\{1,2,3\},\{1,2,3\}] = \begin{bmatrix} a & b & b \\ c & b & c \\ a & a & c \end{bmatrix}.$$

Observe that $A[\{1,2,3\},\{1,2,3\}]$ does not have a constant row or column, which means that $A[\{1,2,3\},\{1,2,3\}]$ is the whole matrix, $A = A[\{1,2,3\},\{1,2,3\}]$. As A is in $\mathcal{D}_3$, this finishes the proof. ☐

3.2 Some properties of weak block-triangularity

The next lemma will be helpful to take care of matrices that do not contain matrices in $\mathcal{B}_1$ or $\mathcal{C}_1$.

Lemma 3.2.1. *A matrix over some finite field $GF(q)$ all of whose submatrices have a constant row or column is q-block-triangular.*

Proof. Let A be an $I \times J$-matrix over $GF(q)$ where, without loss of generality $I \cap J = \emptyset$. It suffices to prove that there is a q-block-decomposition $(\mathcal{I}, \mathcal{J})$ such that $A[I_0, J_0]$ has only two distinct entries for all $I_0 \in \mathcal{I}$ and $J_0 \in \mathcal{J}$. As all submatrices of A have a constant row or column, we can recursively delete constant rows and columns. Therefore, there is an ordering $(a_1, \dots, a_n)$ of $I \cup J$ and a function $\Gamma : I \cup J \to GF(q)$ such that , for each $i \in \{1, \dots, n\}$, the entries in the row or column of $A[I \cap \{a_i, \dots, a_n\}, J \cap \{a_i, \dots, a_n\}]$ which is indexed by a_i are all equal to $\Gamma(a_i)$. Let $I_e = \{a_i \in I | \Gamma(a_i) = e\}$ and $J_e = \{a_i \in J | \Gamma(a_i) = e\}$ for all $e \in GF(q)$.

Let $A_{e,f}$ denote $A[I_e, J_f]$ for $e, f \in GF(q)$. We will prove that $A_{e,f}$ is a triangular matrix for all $e, f \in GF(q)$ finishing the proof of the claim.

We will prove that the only entries of $A_{e,f}$ are e and f. Consider $A(a_i, a_j)$ with $a_i \in I_e$ and $a_j \in J_f$. Assume first that a_i is of lower index in the ordering above than a_j. This implies that the row indexed by a_i of $A[I \cap \{a_i, \dots, a_n\}, J \cap \{a_i, \dots, a_n\}]$ is constant of e. As $A(a_i, a_j)$ is a part of that row, it follows that $A(a_i, a_j) = e$. Similarly, if a_j is of lower index in the ordering above than a_i, it follows that $A(a_i, a_j) = f$. ☐

We need one more lemma that makes a matrix highly structured if all of its blocks are.

Lemma 3.2.2. *If A is a matrix of the form*

$$A = \begin{array}{|c|c|c|} \hline A_{1,1} & \dots & A_{1,n} \\ \hline \vdots & \ddots & \vdots \\ \hline A_{n,1} & \dots & A_{n,n} \\ \hline \end{array} \quad ,$$

such that each of the blocks $A_{k,l}$ is weakly t-block-triangular for some $t \in \mathbb{N}$, then A is weakly nt^n-block-triangular.

Proof. Let I_k be the index set of the rows and J_l be the index set of the columns of $A_{k,l}$. Without loss of generality all these sets are pairwise disjoint. By assumption $A_{k,l}$ is weakly

t-block-triangular for all k and l and so by definition, for all k, l there is a certificate $((I_{k,l,1}, \ldots, I_{k,l,t}), (J_{l,k,1}, \ldots, J_{l,k,t}))$ of weak t-block-triangularity. We will now define new partitions for every I_k and every J_l in the following way:

For every k and for every sequence $s \in \{1, \ldots, t\}^n$ define $I_{k,s} = \bigcap_{l=1}^{n} I_{k,l,s_l}$. Similarly, for given l and for every sequence $s \in \{1, \ldots, t\}^n$ define $J_{l,s} = \bigcap_{k=1}^{n} J_{l,k,s_k}$. Observe that this partitions I and J each into at most nt^n partition classes. Also observe that for every k, l, and for all sequences r, s, $I_{k,r}$ is a subset of I_{k,l,r_l} and $J_{l,s}$ is a subset of J_{l,k,s_k}. It follows that $A[I_{k,r}, J_{l,s}]$ is a submatrix of $A[I_{k,l,r_l}, J_{l,k,s_k}]$ which is triangular by assumption. As the property of being triangular is hereditary under taking submatrices, it follows that $A[I_{k,r}, J_{l,s}]$ is triangular. So, $((I_{k,s} : k = 1, \ldots, n, s \in \{1, \ldots, t\}^n), (J_{l,s} : l = 1, \ldots, n, s \in \{1, \ldots, t\}^n))$ is a certificate of weak nt^n-block-triangularity. This finishes the proof. ☐

3.3 The main proof of Theorem 1.3.1

We are now ready to give the main proof of Theorem 1.3.1, which, for convenience, will be reformulated in the following way :

Theorem (Restatement of Theorem 1.3.1)**.** *There is a function $f : \mathbb{N}^2 \to \mathbb{N}$, such that for each q and $k \in \mathbb{N}$, if A is a matrix over some finite field $GF(q)$, then either*

(i) A is weakly $f(q, k)$-block-triangular, or

(ii) A has a submatrix in $\mathcal{A}_k$, $\mathcal{B}_k$ or $\mathcal{C}_k$.

for all $k \in \mathbb{N}$.

Proof. For the proof of this theorem, we will need to introduce the notion of a special matrix. It will prove useful as it helps constructing an intermediate step in the proof of the theorem.

Definition 3.3.1. We will recursively define a special matrix of depth t. A special matrix of depth 1 is a matrix in $\mathcal{A}_1$, $\mathcal{B}_1$ or $\mathcal{C}_1$. Given a special matrix A_0 of depth $t - 1$, we construct a special matrix of depth t by one of the following three operations:

(i)

$$A = \begin{array}{|c|ccc|}\hline a & b & \dots & b \\ \hline c & & & \\ \vdots & & A_0 & \\ c & & & \\ \hline \end{array} \, ,$$

where $a, b, c \in GF(q)$ and $a \notin \{b, c\}$

(ii)

$$A = \begin{array}{|cc|ccc|}\hline a & b & a & \dots & a \\ c & d & d & \dots & d \\ \hline c & b & & & \\ \vdots & \vdots & & A_0 & \\ c & b & & & \\ \hline \end{array} \, ,$$

where $a, b, c, d \in GF(q)$ and $\{a, d\} \cap \{b, c\} = \emptyset$,

(iii)

$$A = \begin{array}{|ccc|ccc|}\hline a & a & b & a & \dots & a \\ a & c & c & c & \dots & c \\ b & c & b & b & \dots & b \\ \hline a & c & b & & & \\ \vdots & \vdots & \vdots & & A_0 & \\ a & c & b & & & \\ \hline \end{array} \, ,$$

where $a, b, c \in GF(q)$ are distinct.

According to which of the above conditions is satisfied, we will say that the l-th block of a special matrix is of type $(i), (ii), (iii)$, respectively. For example,

$$A = \left(\begin{array}{c|ccc|cc|c|c} a & c & c & c & c & c & c & c \\ \hline e & f & f & g & f & f & f & f \\ e & f & h & h & h & h & h & h \\ e & g & h & g & g & g & g & g \\ \hline e & f & h & g & i & j & i & i \\ e & f & h & g & k & l & l & l \\ \hline e & f & h & g & k & j & m & n \\ \hline e & f & h & g & k & j & o & p \end{array} \right) \, ,$$

is a special matrix of depth 4, where the blocks are of type $(i), (i), (ii), (iii), (i)$, respectively. Observe that the entries in different blocks do not need to be different.

Observe that a matrix of the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

with $\{a, d\} \cap \{b, c\} = \emptyset$ is a special matrix of depth 2 and a special matrix of depth 1.

The main part of the proof of Theorem 3.3 is divided into two lemmas whose combination then yields the result easily.

The most difficult part of the proof is subsumed in the following result that finds big special submatrices in all matrices that are not weakly block-triangular with respect to some bounded number. Its proof makes use of the three preceding lemmas.

Lemma 3.3.2. *There is a function $g : \mathbb{N}^2 \to \mathbb{N}$ such that, if A is a matrix over $GF(q)$ and $t \in \mathbb{N}$, then for all $t \in \mathbb{N}$ either*

(i) A contains a special matrix of depth t, or

(ii) A is weakly $g(q, t)$-block-triangular.

Proof. We will show that the statement holds for the function q that is recursively defined by $g(q, 1) = 1$ and $q(q, t + 1) = g(q, t)(q^3 + 1)^{g(q,t)}$

By Lemma 3.2.1 we can assume that A contains a submatrix without constant row or column. As every 1×1-matrix is a special matrix of depth 1, the statement is trivial for t=1, so we can assume it holds for all integers up to some t.

By Lemma 3.1.1 , it follows that A contains a submatrix A_1 in $\mathcal{B}_1$ or $\mathcal{C}_1$, so there are matrices A_2, A_3 and A_4 such that

$$A = \begin{array}{|c|c|} \hline A_1 & A_2 \\ \hline A_3 & A_4 \\ \hline \end{array}\ .$$

Let l be the number of rows of A_1; thus $l \leq 3$. Observe that A_2 has l rows and A_3 has l columns. We will now partition the row index set of A_4 in sets $(I_v : v \in GF(q)^l)$ such that the index i belongs to the index set I_v if the i-th row of A_3 equals v. Observe that there are at most q^3 partition classes as A_3 has l columns.

Similarly, we will partition the column index set of A_4 in sets $(J_v : v \in GF(q)^l)$ such that the index j belongs to the index set J_v if the j-th column of A_2 equals v. Again, there are at most q^3 partition classes as A_2 has l rows. For $v, w \in GF(q)^l$, let $A_{v,w} = A_4[I_v, I_w]$. We can now apply the inductive hypothesis to each of the $A_{v,w}$, yielding that each of them either is weakly $g(q,t)$-block-triangular or contains a special matrix of depth t. We will need to distinguish two cases.

Case 1. *$A_{v,w}$ is $g(q,t)$-block-triangular for all $v, w \in GF(q)^l$.*

We need to prove that A is weakly C-block-triangular for some constant number C depending only on q. By assumption $A_{v,w}$ is weakly $g(q,t)$-block-triangular for all v and w. Observe that A_1 is trivially weakly 3-block-triangular and also $A_2[\{1,\dots,l\}, J_w]$ for all $w \in GF(q)^l$ as well as $A_3[I_v, \{1,\dots,l\}]$ for all $v \in GF(q)^l$ are weakly 3-block-triangular. So we can apply Lemma 3.2.2 to see that A is weakly $g(q,t)(q^3+1)^{g(q,t)}$-block-triangular. This finishes the case.

Case 2. *There are some v, w such that $A_{v,w}$ contains a special matrix of depth t.*

Consider a submatrix A'_4 of $A_{v,w}$ that is isomorphic to a special matrix of depth t. By construction the rows in A_3 and the columns of A_2 that extend to A'_4 are all the same. So adding the rows and columns of A_1 yields a matrix of the following form where the columns of A'_3 and the rows of A'_2 are constant:

$$A' = \begin{array}{|c|c|} \hline A_1 & A'_2 \\ \hline A'_3 & A'_4 \\ \hline \end{array} \,.$$

We will prove that A' has a special submatrix of depth $t+1$. If there are some $i, j \leq l$ such that $A'_{i,j}$ is different from the entry in the i-th row of A'_2 and the entry in the j-th column of A'_3, we can get a special matrix of depth $t+1$ with the first block of type (i).

Theefore, we may assume this is not the case and can finish the proof by a small case analysis.

Subcase 1. *A_1 is in $\mathcal{B}_1$*

We know that A' is of the following form:

$$\left(\begin{array}{cc|ccc} a & b & e & \dots & e \\ c & d & f & \dots & f \\ \hline g & h & & & \\ \vdots & \vdots & & A_4' & \\ g & h & & & \end{array}\right).$$

As the first block is not of type (i), it follows that either $e = a$ or $g = a$. Assume first that $e = a$. As $b \neq a$, it follows that $h = b$. Similar arguments yield $f = d$ and $g = c$. This makes A' a special matrix of depth $t + 1$ where the first block is of type (ii).

If on the other hand $g = a$, it follows that $f = c$. We can reduce this to the other case by switching the first two rows and relabelling a, b, c, d.

Subcase 2. *A_1 is in $\mathcal{C}_1$*

We know that A' is of the following form:

$$\left(\begin{array}{ccc|ccc} a & a & b & d & \dots & d \\ a & c & c & e & \dots & e \\ b & c & b & f & \dots & f \\ \hline g & h & i & & & \\ \vdots & \vdots & \vdots & & A_4' & \\ g & h & i & & & \end{array}\right).$$

As the first block is not of type (i), it follows that either $d = a$ or $g = a$. Assume first that $d = a$. By similar arguments as before, it follows that $i = b$, $e = c$, $g = a$, $f = b$ and $h = c$. This makes A' a special matrix of depth $t + 1$ where the first block is of type (iii).

Assuming that $g = a$, the analogous argumentation yields the same matrix, so again A' is a special matrix of depth $t + 1$ where the first block is of type (iii). ☐

We will now show how to find a large matrix in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ in a special matrix of sufficient depth.

Lemma 3.3.3. *Every special matrix over $GF(q)$ of depth $3kq^4$ has a submatrix in $\mathcal{A}_k, \mathcal{B}_k$ or $\mathcal{C}_k$.*

Proof. Let A be a special matrix of depth $3kq^4$ for some k. It follows that A either contains kq^4 blocks of type (i), kq^4 blocks of type (ii) or kq^4 blocks of type (iii).

Case 1. *A contains kq^4 blocks of type (i).*

We will prove that A contains a matrix in $\mathcal{A}_k$. Recall that for example a matrix in $\mathcal{A}_4$ is of the form

$$A(a,b,c,4) = \begin{bmatrix} a & c & c & c \\ b & a & c & c \\ b & b & a & c \\ b & b & b & a \end{bmatrix},$$

where $a \notin \{b,c\}$. We will first delete the rows and columns corresponding to all other blocks of A. Now observe that every block i is characterized by the element $A_{i,i}$ and by the elements in the constant part of the i-th row and column. As there are only q possibilities for each of these parameters and there are kq^4 blocks, there must be some a,b,c that characterize at least k of the blocks. Keeping only these rows and columns yields the matrix $A(a,b,c,k)$.

Case 2. *A contains kq^4 blocks of type (ii).*

We will prove that A contains a matrix in $\mathcal{B}_k$. Recall that for example a matrix in $\mathcal{B}_4$ is of the form

$$B(a,b,c,d,4) = \left(\begin{array}{cccc|cccc} a & a & a & a & b & a & a & a \\ c & a & a & a & b & b & a & a \\ c & c & a & a & b & b & b & a \\ c & c & c & a & b & b & b & b \\ \hline c & d & d & d & d & d & d & d \\ c & c & d & d & b & d & d & d \\ c & c & c & d & b & b & d & d \\ c & c & c & c & b & b & b & d \end{array}\right),$$

where $\{a,d\} \cap \{b,c\} = \emptyset$. We will first delete the rows and columns corresponding to all other blocks of A. Now observe that every block is characterized by its four parameters. As there are only q possibilities for each of these parameters and there are kq^4 blocks, there must be some a,b,c,d that characterize at least k of the blocks. Keeping only the rows and columns contained by these blocks yields a matrix A' of the form

$$\left(\begin{array}{cc|ccc} a & b & a & \dots & a \\ c & d & d & \dots & d \\ \hline c & b & \ddots & & \\ \vdots & \vdots & & a & b \\ c & b & & c & d \end{array}\right).$$

So A' contains k blocks. Denote rows in each block i by $(i,1)$ and $(i,2)$ and the columns in each block by $(j,1)$ and $(j,2)$. Reordering the rows and columns in lexicographical order starting with the second coordinate yields the matrix $B(a,b,c,d,k)$.

Case 3. *A contains kq^4 blocks of type (iii).*

We will prove that A contains a matrix in $\mathcal{C}_k$. Recall that for example a matrix in $\mathcal{C}_4$ is of the form

$$C(a,b,c,4) = \left(\begin{array}{cccc|cccc|cccc}
a & a & a & a & a & a & a & a & b & a & a & a \\
a & a & a & a & c & a & a & a & b & b & a & a \\
a & a & a & a & c & c & a & a & b & b & b & a \\
a & a & a & a & c & c & c & a & b & b & b & b \\
\hline
a & c & c & c & c & c & c & c & c & c & c & c \\
a & a & c & c & c & c & c & c & b & c & c & c \\
a & a & a & c & c & c & c & c & b & b & c & c \\
a & a & a & a & c & c & c & c & b & b & b & c \\
\hline
b & b & b & b & c & b & b & b & b & b & b & b \\
a & b & b & b & c & c & b & b & b & b & b & b \\
a & a & b & b & c & c & c & b & b & b & b & b \\
a & a & a & b & c & c & c & c & b & b & b & b
\end{array}\right),$$

where a, b and c are pairwise distinct. We will first delete the rows and columns corresponding to all other blocks of A. Now observe that every block is characterized by its three parameters. As there are only q possibilities for each of these parameters and there are kq^4 blocks, there must be some a, b, c, d that characterize at least k of the blocks. Keeping only the rows and columns of these blocks yields a matrix A' of the form

$$\left(\begin{array}{ccc|cccc}
a & a & b & a & \dots & \dots & a \\
a & c & c & c & \dots & \dots & c \\
b & c & b & b & \dots & \dots & b \\
\hline
a & c & b & \ddots & & & \\
\vdots & \vdots & \vdots & & a & a & b \\
\vdots & \vdots & \vdots & & a & c & c \\
a & c & b & & b & c & b
\end{array}\right).$$

So A' contains k blocks. Denote rows in each block i by $(i,1)(i,2)$ and $(i,3)$ and the columns in each block by $(j,1)$, $(j,2)$ and $(j,3)$. Reordering the rows and columns in lexicographical order starting with the second coordinate yields the matrix $C(a,b,c,k)$. □

Applying the two lemmas, we can now finish the proof of the main theorem. Let $f : \mathbb{N}^2 \to \mathbb{N}$ be defined as $f(q, k) = g(q, 3kq^4)$. Let A be a matrix over some finite field $GF(q)$ and let k be an integer. We may assume that A is not $f(q, k)$-block-triangular. So by Lemma 3.3.2, A contains a special matrix of depth $3kq^4$. Now applying Lemma 3.3.3, we get that A contains a matrix in $\mathcal{A}_k, \mathcal{B}_k$ or $\mathcal{C}_k$. This finishes the proof. $\square$

Chapter 4

Representations of circuit-hyperplane relaxations

This chapter will mainly be concerned with proving a strengthening of Theorem 1.3.2: In order to do this, we will first examine a set of matrices possibly appearing as reduced representations and determine whether they are singular or non-singular. We will then apply these results to the reduced representations of two matroids M_1 and M_2 such that M_2 arises from M_1 by relaxing a circuit-hyperplane, yielding a contradiction to Lemma 2.2.2 if they contain matrices in $\mathcal{A}_{q^2}, \mathcal{B}_2$ or $\mathcal{C}_3$ in a certain way. The first part is concerned with determining whether certain matrices in $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are singular, results that will be needed to get the contradiction in the end.

4.1 Singular and non-singular matrices

Lemma 4.1.1. *Let A be a $(k+1) \times (k+1)$-matrix of the form*

$$\begin{bmatrix} a & b & \dots & b & 1 \\ b & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & b & \vdots \\ b & \dots & b & a & 1 \\ 1 & \dots & \dots & 1 & \epsilon \end{bmatrix}.$$

over some finite field $GF(q)$, where $a \neq b$, and $\epsilon \in \{0, 1\}$. If q divides k, then A is singular if and only if $\epsilon = 1$.

Proof. We will subtract the last row b times from all other rows. This yields the following matrix where a' denotes $a-b$ which is non-zero as $a \neq b$:

$$\begin{bmatrix} a' & 0 & \dots & 0 & 1-b\epsilon \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & a' & 1-b\epsilon \\ 1 & \dots & \dots & 1 & \epsilon \end{bmatrix}.$$

Observe that if $b=1$ and $\epsilon=1$, all the entries above the diagonal of A are zero and all the entries on the diagonal are non-zero. Therefore, A is non-singular and we are done in that case. Otherwise, we can now multiply all rows but the last one with $(1-b\epsilon)^{-1}$ giving the following matrix where a'' denotes $a'(1-b\epsilon)^{-1}$, which is obviously non-zero:

$$\begin{bmatrix} a'' & 0 & \dots & 0 & 1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & a'' & 1 \\ 1 & \dots & \dots & 1 & 1 \end{bmatrix}.$$

Finally, we will add all other rows $-(a'')^{-1}$times onto the last row. Observing that $q=0$ in $GF(q)$, we get the following matrix:

$$\begin{bmatrix} a'' & 0 & \dots & 0 & 1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & a'' & 1 \\ 0 & \dots & \dots & 0 & \epsilon \end{bmatrix}.$$

If $\epsilon=0$, then A has a row of zeros and therefore is singular. If, on the other hand, $\epsilon=1$ then all the entries below the diagonal are zero and all the diagonal entries of A are non-zero, so A is non-singular. □

Lemma 4.1.2. *Let A be a $(k+1)\times(k+1)$-matrix of the form*

$$\begin{bmatrix} a & c & \dots & c & 1 \\ b & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & c & \vdots \\ b & \dots & b & a & 1 \\ 1 & \dots & \dots & 1 & \epsilon \end{bmatrix}.$$

over some finite field $GF(q)$, where a, b and c are pairwise distinct, $\epsilon \in \{0, 1\}$ and $b\epsilon \neq 1$. If $q - 1$ divides k, then A is singular if and only if $\epsilon = 1$.

Proof. Firstly, we will subtract the last row b times from all other rows. This yields the following matrix where a' denotes $a - b$ and c' denotes $c - b$, which are both non-zero:

$$\begin{bmatrix} a' & c' & \dots & c' & 1-b\epsilon \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & c' & \vdots \\ 0 & \dots & 0 & a' & 1-b\epsilon \\ 1 & \dots & \dots & 1 & \epsilon \end{bmatrix}.$$

As $b\epsilon \neq 1$, we can now multiply all rows but the last one with $(1 - b\epsilon)^{-1}$ yielding the following matrix where a'' denotes $a'(1 - b\epsilon)^{-1}$ and c'' denotes $c'(1 - b\epsilon)^{-1}$:

$$\begin{bmatrix} a'' & c'' & \dots & c'' & 1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & c'' & \vdots \\ 0 & \dots & 0 & a'' & 1 \\ 1 & \dots & \dots & 1 & 1 \end{bmatrix}.$$

Observe that a'' and c'' are both non-zero and also $a'' \neq c''$. Now consider the vector

$$x = (-\frac{1}{a''}, -\frac{1}{a''}(1 - \frac{c''}{a''}), -\frac{1}{a''}(1 - \frac{c''}{a''})^2, \dots, -\frac{1}{a''}(1 - \frac{c''}{a''})^{k-1}, 1) \in GF(q)^k$$

and let $v = xA$. We can calculate the entries of v. For all $1 \leq j \leq k$, we get:

$$\begin{aligned} v_j &= \sum_{i=1}^{j-1} -\frac{1}{a''}\left(1 - \frac{c''}{a''}\right)^{i-1} c'' - \frac{1}{a''}\left(1 - \frac{c''}{a''}\right)^{j-1} a'' + 1 \\ &= -\frac{c''}{a''}\sum_{i=0}^{j-2}\left(1 - \frac{c''}{a''}\right)^{i} - \left(1 - \frac{c''}{a''}\right)^{j-1} + 1 \\ &= -\frac{c''}{a''}\frac{1 - \left(1 - \frac{c''}{a''}\right)^{j-1}}{1 - \left(1 - \frac{c''}{a''}\right)} - \left(1 - \frac{c''}{a''}\right)^{j-1} + 1 \\ &= -\left(1 - \left(1 - \frac{c''}{a''}\right)^{j-1}\right) - \left(1 - \frac{c''}{a''}\right)^{j-1} + 1 \end{aligned}$$

$$= 0.$$

Furthermore, we can calculate:

$$\begin{aligned} v_k &= -\frac{1}{a''}\sum_{i=1}^{k}\left(1-\frac{c''}{a''}\right)^{i-1}+\epsilon \\ &= -\frac{1}{a''}\sum_{i=0}^{k-1}\left(1-\frac{c''}{a''}\right)^{i}+\epsilon \\ &= -\frac{1}{a''}\frac{1-\left(1-\frac{c''}{a''}\right)^{k}}{1-\left(1-\frac{c''}{a''}\right)}+\epsilon \\ &= \epsilon, \end{aligned}$$

as the $(q-1)$-th power of any number in $GF(q)$ is 1. So, in total, this yields $v = (0, \dots, 0, \epsilon)$. Observe that v is in the row space of A. If $\epsilon = 0$, v is the zero vector and so A is singular. If $\epsilon = 1$, v and the first k rows of A and v span $GF(q)^{k+1}$, so A is non-singular. □

Lemma 4.1.3. *Let A be a 5×5-matrix of the form*

$$\begin{bmatrix} a & a & b & a & 1 \\ c & a & b & b & 1 \\ c & d & d & d & 1 \\ c & c & b & d & 1 \\ 1 & 1 & 1 & 1 & \epsilon \end{bmatrix}.$$

over some finite field $GF(q)$, where $\{a,d\} \cap \{b,c\} = \emptyset$, $\epsilon \in \{0,1\}$, $b\epsilon \neq 1$ and $c\epsilon \neq 1$. Then A is non-singular if and only if $\epsilon = 1$.

Proof. We will now add the last row $-c$ times to all other rows yielding the following matrix where $a' = a - c$, $b' = b - c$ and $d' = d - c$:

$$\begin{bmatrix} a' & a' & b' & a' & 1-c\epsilon \\ 0 & a' & b' & b' & 1-c\epsilon \\ 0 & d' & d' & d' & 1-c\epsilon \\ 0 & 0 & b' & d' & 1-c\epsilon \\ 1 & 1 & 1 & 1 & \epsilon \end{bmatrix}.$$

As $c\epsilon \neq 1$, we can now multiply every row but the last one with $(1 - c\epsilon)^{-1}$ yielding the following matrix where $a'' = a'(1 - c\epsilon)^{-1}$, $b'' = b'(1 - c\epsilon)^{-1}$ and $d'' = d'(1 - c\epsilon)^{-1}$:

$$\begin{bmatrix} a'' & a'' & b'' & a'' & 1 \\ 0 & a'' & b'' & b'' & 1 \\ 0 & d'' & d'' & d'' & 1 \\ 0 & 0 & b'' & d'' & 1 \\ 1 & 1 & 1 & 1 & \epsilon \end{bmatrix}.$$

We will now add the last row $-a''$ times to the first row and $-d''$ times to the third row yielding the following matrix:

$$\begin{bmatrix} 0 & 0 & b'' - a'' & 0 & 1 - a''\epsilon \\ 0 & a'' & b'' & b'' & 1 \\ -d'' & 0 & 0 & 0 & 1 - d''\epsilon \\ 0 & 0 & b'' & d'' & 1 \\ 1 & 1 & 1 & 1 & \epsilon \end{bmatrix}.$$

We will next add the last column $-b''$ times to the third column yielding the following matrix:

$$\begin{bmatrix} 0 & 0 & a''(b''\epsilon - 1) & 0 & 1 - a''\epsilon \\ 0 & a'' & 0 & b'' & 1 \\ -d'' & 0 & -b''(1 - d''\epsilon) & 0 & 1 - d''\epsilon \\ 0 & 0 & 0 & d'' & 1 \\ 1 & 1 & 1 - b''\epsilon & 1 & \epsilon \end{bmatrix}.$$

Now consider the vector

$$x = (\frac{d'' - b''}{a''(1 - b''\epsilon)d''}, -\frac{1}{a''}, \frac{1}{d''}, \frac{b'' - a''}{a''d''}, 1) \in GF(q)^5$$

and let $v = xA$.

We can calculate the entries of v.

$$\begin{aligned} v_1 &= \frac{1}{d''}\left(-d''\right) + 1 \\ &= 0, \\ v_2 &= -\frac{1}{a''}a'' + 1 \\ &= 0, \\ v_3 &= \frac{d'' - b''}{a''\left(1 - b''\epsilon\right)d''}\left(a''\left(b''\epsilon - 1\right)\right) + \frac{1}{d''}\left(-b''\left(1 - d''\epsilon\right)\right) + 1 - b''\epsilon \end{aligned}$$

$$
\begin{aligned}
&= -1 + \frac{b''}{d''} - \frac{b''}{d''} + b''\epsilon + 1 - b''\epsilon \\
&= 0, \\
v_4 &= -\frac{1}{a''}b'' + \frac{b'' - a''}{a''d''}d'' + 1 \\
&= -\frac{b''}{a''} + \frac{b''}{a''} - 1 + 1 \\
&= 0, \\
v_5 &= \frac{d'' - b''}{a''\left(1 - b''\epsilon\right)d''}\left(1 - a''\epsilon\right) - \frac{1}{a''} \\
&+ \frac{1}{d''}\left(1 - d''\epsilon\right) + \frac{b'' - a''}{a''d''} + \epsilon \\
&= \frac{\left(d'' - b''\right)\left(1 - a\epsilon\right)}{a''\left(1 - b''\epsilon\right)d''} - \frac{\left(1 - b''\epsilon\right)d''}{a''\left(1 - b''\epsilon\right)d''} \\
&+ \frac{\left(1 - d''\epsilon\right)\left(1 - b''\epsilon\right)a''}{a''\left(1 - b''\epsilon\right)d''} + \frac{\left(b'' - a''\right)\left(1 - b''\epsilon\right)}{a''\left(1 - b''\epsilon\right)d''} + \frac{a''\left(1 - b''\epsilon\right)d''\epsilon}{a''\left(1 - b''\epsilon\right)d''} \\
&= \frac{1}{a''\left(1 - b''\epsilon\right)d''}\Bigg(d'' - a''d''\epsilon - b'' + a''b''\epsilon - d'' + b''d''\epsilon + a'' - a''b''\epsilon - a''d''\epsilon + a''b''d''\epsilon^2 \\
&+ b'' - \left(b''\right)^2\epsilon - a'' + a''b''\epsilon + a''d''\epsilon - a''b''d''\epsilon^2\Bigg) \\
&= \frac{b''d''\epsilon - a''d''\epsilon - \left(b''\right)^2\epsilon + a''b''\epsilon}{a''\left(1 - b''\epsilon\right)d''} \\
&= \frac{-\epsilon\left(b'' - a''\right)\left(b'' - d''\right)}{a''\left(1 - b''\epsilon\right)d''}.
\end{aligned}
$$

So, in total, this yields

$$
v = (0, 0, 0, 0, \frac{-\epsilon(b'' - a'')(b'' - d'')}{a''(1 - b''\epsilon)d''})
$$

. Observe that v is in the row space of A. If $\epsilon = 0$, v is the zero vector and so A is singular. If $\epsilon = 1$, the last entry of v is non-zero as $a'' \neq b''$ and $b'' \neq d''$. We will prove that the first four rows of A and v span $GF(q)^5$. As $A(5,5)$ and $A(4,4)$ are nonzero, it suffices to prove that $A[\{1,2,3\},\{1,2,3\}]$=

$$
\begin{bmatrix}
0 & 0 & a''(b''\epsilon - 1)'' \\
0 & a'' & 0 \\
-d'' & 0 & -b''(1 - d''\epsilon)
\end{bmatrix}
$$

is non-singular. All entries above its anti-diagonal are zero, while all the entries on the anti-diagonal are non-zero. It follows that $A[\{1,2,3\},\{1,2,3\}]$ is non-singular.

Therefore, A is non-singular. $\square$

Lemma 4.1.4. *Let A be a 10×10-matrix of the form*

$$\begin{bmatrix} a & a & a & a & a & a & b & a & a & 1 \\ a & a & a & c & a & a & b & b & a & 1 \\ a & a & a & c & c & a & b & b & b & 1 \\ a & c & c & c & c & c & c & c & c & 1 \\ a & a & c & c & c & c & b & c & c & 1 \\ a & a & a & c & c & c & b & b & c & 1 \\ b & b & b & c & b & b & b & b & b & 1 \\ a & b & b & c & c & b & b & b & b & 1 \\ a & a & b & c & c & c & b & b & b & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \epsilon \end{bmatrix}.$$

over some finite field $GF(q)$, where a, b, c are pairwise distinct, $\epsilon \in \{0,1\}$, $a\epsilon \neq 1$ and $b\epsilon \neq 1$. Then A is non-singular if and only if $\epsilon = 1$.

Proof. We will add the last row $-a$ times to all other rows yielding the following matrix where $b' = b - a$ and $c' = c - a$:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & b' & 0 & 0 & 1-a\epsilon \\ 0 & 0 & 0 & c' & 0 & 0 & b' & b' & 0 & 1-a\epsilon \\ 0 & 0 & 0 & c' & c' & 0 & b' & b' & b' & 1-a\epsilon \\ 0 & c' & c' & c' & c' & c' & c' & c' & c' & 1-a\epsilon \\ 0 & 0 & c' & c' & c' & c' & b' & c' & c' & 1-a\epsilon \\ 0 & 0 & 0 & c' & c' & c' & b' & b' & c' & 1-a\epsilon \\ b' & b' & b' & c' & b' & b' & b' & b' & b' & 1-a\epsilon \\ 0 & b' & b' & c' & c' & b' & b' & b' & b' & 1-a\epsilon \\ 0 & 0 & b' & c' & c' & c' & b' & b' & b' & 1-a\epsilon \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \epsilon \end{bmatrix}.$$

As $a \neq 1$, we can multiply all rows but the last one by $(1 - a\epsilon)^{-1}$ yielding the following matrix where $b'' = b'(1-a\epsilon)^{-1}$ and $c'' = c'(1-a\epsilon)^{-1}$:

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & b'' & 0 & 0 & 1 \\
0 & 0 & 0 & c'' & 0 & 0 & b'' & b'' & 0 & 1 \\
0 & 0 & 0 & c'' & c'' & 0 & b'' & b'' & b'' & 1 \\
0 & c'' & c'' & c'' & c'' & c'' & c'' & c'' & c'' & 1 \\
0 & 0 & c'' & c'' & c'' & c'' & b'' & c'' & c'' & 1 \\
0 & 0 & 0 & c'' & c'' & c'' & b'' & b'' & c'' & 1 \\
b'' & b'' & b'' & c'' & b'' & b'' & b'' & b'' & b'' & 1 \\
0 & b'' & b'' & c'' & c'' & b'' & b'' & b'' & b'' & 1 \\
0 & 0 & b'' & c'' & c'' & c'' & b'' & b'' & b'' & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \epsilon
\end{bmatrix}.$$

We will now add the last row $-c''$ times to the fourth row and $-b''$ times to the seventh row. This yields

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & b'' & 0 & 0 & 1 \\
0 & 0 & 0 & c'' & 0 & 0 & b'' & b'' & 0 & 1 \\
0 & 0 & 0 & c'' & c'' & 0 & b'' & b'' & b'' & 1 \\
-c'' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-c''\epsilon \\
0 & 0 & c'' & c'' & c'' & c'' & b'' & c'' & c'' & 1 \\
0 & 0 & 0 & c'' & c'' & c'' & b'' & b'' & c'' & 1 \\
0 & 0 & 0 & c''-b'' & 0 & 0 & 0 & 0 & 0 & 1-b''\epsilon \\
0 & b'' & b'' & c'' & c'' & b'' & b'' & b'' & b'' & 1 \\
0 & 0 & b'' & c'' & c'' & c'' & b'' & b'' & b'' & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \epsilon
\end{bmatrix}.$$

We will now add the last column $-c''$ times to the fourth column and $-b''$ times to the seventh column. This yields

$$\begin{bmatrix}
0 & 0 & 0 & -c'' & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b'' & 0 & 1 \\
0 & 0 & 0 & 0 & c'' & 0 & 0 & b'' & b'' & 1 \\
-c'' & 0 & 0 & c''(c''\epsilon-1) & 0 & 0 & b''(c''\epsilon-1) & 0 & 0 & 1-c''\epsilon \\
0 & 0 & c'' & 0 & c'' & c'' & 0 & c'' & c'' & 1 \\
0 & 0 & 0 & 0 & c'' & c'' & 0 & b'' & c'' & 1 \\
0 & 0 & 0 & b''(c''\epsilon-1) & 0 & 0 & b''(b''\epsilon-1) & 0 & 0 & 1-b''\epsilon \\
0 & b'' & b'' & 0 & c'' & b'' & 0 & b'' & b'' & 1 \\
0 & 0 & b'' & 0 & c'' & c'' & 0 & b'' & b'' & 1 \\
1 & 1 & 1 & 1-c''\epsilon & 1 & 1 & 1-b''\epsilon & 1 & 1 & \epsilon
\end{bmatrix}$$

Now consider the vector $x =$

$$\left(\frac{(b''-c'')(c''\epsilon-1)}{(c'')^2(b''\epsilon-1)}, -\frac{(c''-b'')^2}{b''(c'')^2}, \frac{c''-b''}{b''c''}, \frac{1}{c''}, -\frac{b''}{(c'')^2}, \frac{b''-c''}{(c'')^2}, \frac{b''-c''}{b''c''(b''\epsilon-1)}, -\frac{1}{b''}, \frac{1}{c''}, 1\right)$$

$\in GF(q)^{10}$ and let $v = xA$.

We can calculate the entries of v.

$$\begin{aligned}
v_1 &= \frac{1}{c''}(-c'') + 1 \\
&= 0, \\
v_2 &= -\frac{1}{b''}b'' + 1 \\
&= 0, \\
v_3 &= -\frac{b''}{(c'')^2}c'' - \frac{1}{b''}b'' + \frac{1}{c''}b'' + 1 \\
&= 0, \\
v_4 &= \frac{(b''-c'')(c''\epsilon-1)}{(c'')^2(b''\epsilon-1)}(-c'') \\
&+ \frac{1}{c''}c''(c''\epsilon-1) + \frac{b''-c''}{b''c''(b''\epsilon-1)}b''(c''\epsilon-1) + 1 - c''\epsilon \\
&= -\frac{(b''-c'')(c''\epsilon-1)}{c''(b''\epsilon-1)} + c''\epsilon - 1 + \frac{(b''-c'')(c''\epsilon-1)}{c''(b''\epsilon-1)} + 1 - c''\epsilon \\
&= 0, \\
v_5 &= \frac{c''-b''}{b''c''}c'' - \frac{b''}{(c'')^2}(c'') + \frac{b''-c''}{(c'')^2}c'' - \frac{1}{b''}c'' + \frac{1}{c''}c'' + 1 \\
&= \frac{c''}{b''} - 1 - \frac{b''}{c''} + \frac{b''}{c''} - 1 - \frac{c''}{b''} + 1 + 1 \\
&= 0, \\
v_6 &= -\frac{b''}{(c'')^2}c'' + \frac{b''-c''}{(c'')^2}c'' - \frac{1}{b''}b'' + \frac{1}{c''}c'' + 1 \\
&= -\frac{b''}{c''} + \frac{b''}{c''} - 1 - 1 + 1 + 1 \\
&= 0,
\end{aligned}$$

$$
\begin{aligned}
v_7 &= \frac{1}{c''} b'' \left(c''\epsilon - 1\right) + \frac{b'' - c''}{b''c'' \left(b''\epsilon - 1\right)} b'' \left(b''\epsilon - 1\right) + 1 - b''\epsilon \\
&= b''\epsilon - \frac{b''}{c''} + \frac{b''}{c''} - 1 + 1 - b''\epsilon \\
&= 0, \\
v_8 &= -\frac{\left(c'' - b''\right)^2}{b'' \left(c''\right)^2} b'' + \frac{c'' - b''}{b''c''} b'' - \frac{b''}{\left(c''\right)^2} c'' + \frac{b'' - c''}{\left(c''\right)^2} b'' - \frac{1}{b''} b'' + \frac{1}{c''} b'' + 1 \\
&= -1 + 2\frac{b''}{c''} - \frac{\left(b''\right)^2}{\left(c''\right)^2} + 1 - \frac{b''}{c''} - \frac{b''}{c''} + \frac{\left(b''\right)^2}{\left(c''\right)^2} - \frac{b''}{c''} - 1 + \frac{b''}{c''} + 1 \\
&= 0, \\
v_9 &= \frac{c'' - b''}{b''c''} b'' - \frac{b''}{\left(c''\right)^2} c'' + \frac{b'' - c''}{\left(c''\right)^2} c'' - \frac{1}{b''} b'' + \frac{1}{c''} b + 1 \\
&= \frac{c'' - b''}{c''} - \frac{b''}{c''} + \frac{b'' - c''}{c''} - 1 + \frac{b''}{c''} + 1 \\
&= 0, \\
v_{10} &= \frac{\left(b'' - c''\right)\left(c''\epsilon - 1\right)}{\left(c''\right)^2 \left(b''\epsilon - 1\right)} 1 - \frac{\left(c'' - b''\right)^2}{b'' \left(c''\right)^2} 1 + \frac{c'' - b''}{b''c''} 1 \\
&+ \frac{1}{c''} \left(1 - c''\epsilon\right) - \frac{b''}{\left(c''\right)^2} 1 + \frac{b'' - c''}{\left(c''\right)^2} 1 + \frac{b'' - c''}{b''c'' \left(b''\epsilon - 1\right)} \left(1 - b''\epsilon\right) \\
&- \frac{1}{b''} 1 + \frac{1}{c''} 1 + \epsilon \\
&= \frac{\left(b'' - c''\right)\left(c''\epsilon - 1\right)}{\left(c''\right)^2 \left(b''\epsilon - 1\right)} - \frac{\left(c'' - b''\right)^2}{b'' \left(c''\right)^2} + \frac{1}{b''} - \frac{1}{c''} + \frac{1}{c''} - \epsilon - \frac{b''}{\left(c''\right)^2} \\
&+ \frac{b''}{\left(c''\right)^2} - \frac{1}{c''} + \frac{b'' - c''}{b''c'' \left(b''\epsilon - 1\right)} \left(1 - b''\epsilon\right) - \frac{1}{b''} + \frac{1}{c''} + \epsilon \\
&= \frac{\left(b'' - c''\right)\left(c''\epsilon - 1\right)}{\left(c''\right)^2 \left(b''\epsilon - 1\right)} - \frac{\left(c'' - b''\right)^2}{b'' \left(c''\right)^2} + \frac{\left(b'' - c''\right)\left(1 - b''\epsilon\right)}{b''c'' \left(b''\epsilon - 1\right)} \\
&= \frac{1}{b'' \left(b''\epsilon - 1\right) \left(c''\right)^2} \Bigg(b'' \left(b'' - c''\right) \left(c''\epsilon - 1\right) \\
&- \left(b''\epsilon - 1\right) \left(c'' - b''\right)^2 + c'' \left(b'' - c''\right) \left(1 - b''\epsilon\right) \Bigg) \\
&= \frac{1}{b'' \left(b''\epsilon - 1\right) \left(c''\right)^2} \Bigg(\left(b''\right)^2 c''\epsilon - \left(b''\right)^2 - b'' \left(c''\right)^2 \epsilon + b''c''
\end{aligned}
$$

$$
\begin{aligned}
&-\left(b''\right)^{3}\epsilon+2\left(b''\right)^{2}c''\epsilon-b''\left(c''\right)^{2}\epsilon+\left(b''\right)^{2}-2b''c''+\left(c''\right)^{2}\\
&+b''c''-\left(b''\right)^{2}c''\epsilon-\left(c''\right)^{2}+\left(c''\right)^{2}b''\epsilon\Big)\\
&=\frac{-\epsilon\left(\left(b''\right)^{3}+2\left(b''\right)^{2}c''+b''\left(c''\right)^{2}\right)}{b''\left(b''\epsilon-1\right)\left(c''\right)^{2}}\\
&=\frac{-\epsilon\left(b''\right)\left(b''-c''\right)^{2}}{b''\left(b''\epsilon-1\right)\left(c''\right)^{2}}.
\end{aligned}
$$

So, in total, this yields

$$v=(0,0,0,0,0,0,0,0,0,\frac{-\epsilon\left(b''\right)\left(b''-c''\right)^{2}}{b''\left(b''\epsilon-1\right)\left(c''\right)^{2}})$$

. Observe that v is in the row space of A. If $\epsilon=0$, v is the zero vector and so A is singular. If $\epsilon=1$, the last entry of v is non-zero as $b''\neq 0$ and $b''\neq c''$. We will prove that the first nine rows of A and v span $GF(q)^{10}$. As all entries not in these blocks are zero, it suffices to prove that $A[\{1,4,7\},\{1,4,7\}]$ and $A[\{2,3,5,6,8,9\},\{2,3,5,6,8,9\}]$ are non-singular for $\epsilon=1$.

We know that $A[\{1,4,7\},\{1,4,7\}]$=

$$\begin{bmatrix} 0 & -c'' & 0\\ -c'' & c''(c''-1) & b''(c''-1)\\ 0 & b''(c''-1) & b''(b''-1)\end{bmatrix}.$$

After exchanging the last two columns and the last two rows, we get the following matrix:

$$\begin{bmatrix} 0 & 0 & -c''\\ 0 & b''(b''-1) & b''(c''-1)\\ -c'' & b''(c''-1) & c''(c''-1)\end{bmatrix}.$$

All entries above its anti-diagonal are zero, while all the entries on the anti-diagonal are non-zero. It follows that $A[\{1,4,7\},\{1,4,7\}]$ is non-singular.

We know that $A[\{2,3,5,6,8,9\},\{2,3,5,6,8,9\}]=$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & b'' & 0 \\ 0 & 0 & c'' & 0 & b'' & b'' \\ 0 & c'' & c'' & c'' & c'' & c'' \\ 0 & 0 & c'' & c'' & b'' & c'' \\ b'' & b'' & c'' & b'' & b'' & b'' \\ 0 & b'' & c'' & c'' & b'' & b'' \end{bmatrix}.$$

As there is only one nonzero entry in the first row, we can expand to get the following matrix:

$$\begin{bmatrix} 0 & 0 & c'' & 0 & b'' \\ 0 & c'' & c'' & c'' & c'' \\ 0 & 0 & c'' & c'' & c'' \\ b'' & b'' & c'' & b'' & b'' \\ 0 & b'' & c'' & c'' & b'' \end{bmatrix}.$$

As there is only one nonzero entry in the first column we can expand to get the following matrix:

$$\begin{bmatrix} 0 & c'' & 0 & b'' \\ c'' & c'' & c'' & c'' \\ 0 & c'' & c'' & c'' \\ b'' & c'' & c'' & b'' \end{bmatrix}.$$

We will now add the second column $-\frac{b''}{c''}$ times to the last column yielding the following matrix:

$$\begin{bmatrix} 0 & c'' & 0 & 0 \\ c'' & c'' & c'' & c''-b'' \\ 0 & c'' & c'' & c''-b'' \\ b'' & c'' & c'' & 0 \end{bmatrix}.$$

As there is only one nonzero entry in the first row, we expand to get the following matrix:

$$\begin{bmatrix} c'' & c'' & c''-b'' \\ 0 & c'' & c''-b'' \\ b'' & c'' & 0 \end{bmatrix}.$$

We will now add the second row -1 times to the first row yielding

$$\begin{bmatrix} c'' & 0 & 0 \\ 0 & c'' & c''-b'' \\ b'' & c'' & 0 \end{bmatrix}.$$

Now exchanging the second and the third row yields

$$\begin{bmatrix} c'' & 0 & 0 \\ b'' & c'' & 0 \\ 0 & c'' & c''-b'' \end{bmatrix}.$$

All entries above its diagonal are zero, while all the entries on the diagonal are non-zero. It follows that the matrix is non-singular. ☐

4.2 The main proof of Theorems 1.2.6 and 1.3.2

We will prove Theorems 1.2.6 and 1.3.2 in this chapter. We will first prove some lemmas to shorten the proof of Theorem 1.3.2 which will follow them. Afterwards, the proof of Theorem 1.2.6 will be an easy combination of Theorem 1.3.2, Theorem 1.3.1 and Lemma 3.2.2.

In order to do this we will first need to introduce the notion of an auxiliary graph which is helpful in proving some lemmas providing strong similarities between the representations of two matroids M_1 and M_2 such that M_2 arises from M_1 by relaxing a circuit-hyperplane. Given a matrix A on disjoint row and column sets X and Y, define the simple graph G_A on the vertex set $V(G_A) = X \times Y$ and with the edge set $E(G_A) = \{(i_1, j_1)(i_2, j_2) : A(i_1, j_1) = A(i_2, j_2) \text{ and } (i_1 = i_2 \text{ or } j_1 = j_2)\}$. Observe that $G_{A[I,J]} = G_A[I \times J]$ for all $I \subseteq X$ and $J \subseteq Y$.

Lemma 4.2.1. *Let A_1 and A_2 be $X \times Y$-matrices with $X \cap Y = \emptyset$ such that A_1 is of the form*

$$\left[\begin{array}{ccc|c} & & & 1 \\ & A_1' & & \vdots \\ & & & 1 \\ \hline 1 & \dots & 1 & 0 \end{array}\right],$$

A_2 is of the form

$$\left[\begin{array}{ccc|c} & & & 1 \\ & A_2' & & \vdots \\ & & & 1 \\ \hline 1 & \dots & 1 & 1 \end{array}\right]$$

and $A_1[I, J]$ and $A_2[I, J]$ are either both singular or both non-singular for all non-empty $I \subseteq X$ and $J \subseteq Y$ such that $|I| = |J|$ and $(I, J) \neq (\{s\}, \{t\})$, where s and t index the last row and column, respectively. Then $G_{A_1'} = G_{A_2'}$.

Proof. Assume otherwise. By definition $G_{A_1'}$ and $G_{A_2'}$ have the same vertex set, so there is an edge in $E(G_{A_1'}) \triangle E(G_{A_2'})$. Assume first there is some edge $(i_1, j_1)(i_2, j_2) \in E(G_{A_1'}) - E(G_{A_2'})$. By definition, either $i_1 = i_2$ or $j_1 = j_2$. By symmetry, we may assume that $i_1 = i_2$. It follows by definition that $A_1(i_1, j_1) = A_1(i_1, j_2)$, but $A_2(i_1, j_1) \neq A_2(i_1, j_2)$, therefore

$$A_1[\{i_1, s\}, \{j_1, j_2\}] = \begin{bmatrix} A_1(i_1, j_1) & A_1(i_1, j_1) \\ 1 & 1 \end{bmatrix},$$

which is singular whereas

$$A_2[\{i_1, s\}, \{j_1, j_2\}] = \begin{bmatrix} A_2(i_1, j_1) & A_2(i_1, j_2) \\ 1 & 1 \end{bmatrix},$$

which is non-singular as $A_2(i_1, j_1) \neq A_2(i_1, j_2)$. This is a contradiction to the singularity assumption.

The proof that $E(G_{A_2'}) - E(G_{A_1'}) = \emptyset$ is completely analogous. □

In order to facilitate some of the following arguments, we need to introduce some notation. For every natural number k, let $\mathcal{A}_k^+$ be the set of matrices in $\mathcal{A}_k$ that are isomorphic to a matrix $A(a, b, c, k)$ such that $b = c$ and let $\mathcal{A}_k^-$ be the set of matrices in $\mathcal{A}_k$ that are isomorphic to a matrix $A(a, b, c, k)$ such that $b \neq c$. Observe that $(\mathcal{A}_k^+, \mathcal{A}_k^-)$ is a partition of $\mathcal{A}_k$ for all $k \geq 3$.

Lemma 4.2.2. *Let k be a positive integer and let q_1 and q_2 be prime powers. Now let A_1 and A_2 be matrices in $GF(q_1)^{X \times Y}$ and $GF(q_2)^{X \times Y}$, respectively, where $G_{A_1} = G_{A_2}$. If either A_1 or A_2 contains a matrix in $\mathcal{A}_{q_1 q_2 k}^+$, then there exist k-element subsets $I \subseteq X$ and $J \subseteq Y$ such that $A_1[I, J]$ and $A_2[I, J]$ are in $\mathcal{A}_k^+$.*

Proof. By symmetry, we may assume that A_1 contains a matrix in $\mathcal{A}^{+}_{q_1q_2k}$. By deleting all other rows and columns we may assume that $A_1 = A(a, b, b, q_1q_2k)$ for some distinct $a, b \in GF(q_1)$. For $e \in \{a, b\}$, let $Z_e = \{(i, j)|A_1(i, j) = e\}$. Observe that Z_b induces a connected component of G_{A_1} as $q_1q_2k \geq 3$ and therefore also of G_{A_2}. This implies that there is some $b' \in GF(q_2)$ such that $A_2(i, j) = b'$ for all $(i, j) \in Z_b$.

Observe that there are q_1q_2k elements in Z_a, so there is a subset S of Z_a of size k as well as some $a' \in GF(q_2)$ such that $A_2(i, j) = a'$ for all $(i, j) \in S$. There are no edges between S and Z_b but as $q_1q_2k \geq 2$ there are some $i \in X$ and $j_1, j_2 \in Y$ such that $(i, j_1) \in S$ and $(i, j_2) \in Z_b$. It follows that $a' \neq b'$. Now let I denote the set of all $i \in X$ such that there is some $j \in Y$ such that $(i, j) \in S$ and, similarly, let J denote the set of all $j \in Y$ such that there is some $i \in X$ such that $(i, j) \in S$.

We can now see that $A_1[I, J] = A(a, b, b, k)$ and $A_2[I, J] = A(a', b', b', k)$, so $A_1[I, J]$ and $A_2[I, J]$ are both in $\mathcal{A}^{+}_{k}$ as $b \neq c$ and $b' \neq c'$. ☐

Lemma 4.2.3. *Let k be a positive integer, let q_1 and q_2 be prime powers. Now let A_1 and A_2 be matrices in $GF(q_1)^{X\times Y}$ and $GF(q_2)^{X\times Y}$, respectively, where $G_{A_1} = G_{A_2}$. If either A_1 or A_2 contains a matrix in $\mathcal{A}^{-}_{q_1q_2k}$, then there exist k-element subsets $I \subseteq X$ and $J \subseteq Y$ such that $A_1[I, J]$ and $A_2[I, J]$ are in $\mathcal{A}^{-}_{k}$.*

Proof. By symmetry, we may assume that A_1 contains a matrix in $\mathcal{A}^{-}_{q_1q_2k}$. By deleting all other rows and columns we may assume that $A_1 = A(a, b, c, q_1q_2k)$ for some pairwise distinct a, b and c. For all $e \in \{a, b, c\}$, let $Z_e = \{(i, j)|A_1(i, j) = e\}$. Observe that Z_b and Z_c induce connected components of G_{A_1} and therefore also of G_{A_2}. This implies that there is some $b' \in GF(q_2)$ such that $A_2(i, j) = b'$ for all $(i, j) \in Z_b$ and similarly, there is some c' such that $A_2(i, j) = c'$ for all $(i, j) \in Z_c$. There are no edges between Z_b and Z_c but as $q_1q_2k \geq 3$ there are some $i \in X$ and $j_1, j_2 \in Y$ such that $(i, j_1) \in Z_a$ and $(i, j_2) \in Z_b$. It follows that $b' \neq c'$.

Observe that there are q_1q_2k elements in Z_a, so there is a subset S of Z_a of size k as well as some $a' \in GF(q_2)$ such that $A_2(i, j) = a'$ for all $(i, j) \in S$. There are no edges between S and Z_b but as $q_1q_2k \geq 2$ there are some $i \in X$ and $j_1, j_2 \in Y$ such that $(i, j_1) \in S$ and $(i, j_2) \in Z_b$. It follows that $a' \neq b'$. Similarly, $a' \neq c'$. Now let I denote the set of all i such that there is some $j \in Y$ such that $(i, j) \in S$ and, similarly, let J denote the set of all j such that there is some $i \in X$ such that $(i, j) \in S$.

We can now see that $A_1[I, J] = A(a, b, c, k)$ and $A_2[I, J] = A(a', b', c', k)$, so $A_1[I, J]$ and $A_2[I, J]$ are both in $\mathcal{A}^{-}_{k}$ as $b \neq c$ and $b' \neq c'$. ☐

Lemma 4.2.4. *Let q_1 and q_2 be prime powers. Now let A_1 and A_2 be matrices in $GF(q_1)^{X \times Y}$ and $GF(q_2)^{X \times Y}$, respectively, where $G_{A_1} = G_{A_2}$. If either of $A_1[I, J]$ or $A_2[I, J]$ is in $\mathcal{B}_2$ for some $I \subseteq X$ and $J \subseteq Y$, then both $A_1[I, J]$ or $A_2[I, J]$ are in $\mathcal{B}_2$.*

Proof. Without loss of generality we will assume that $A_1[I, J] = B(a, b, c, d, 2)$ for some $a, b, c, d \in GF(q_1)$ such that $\{a, d\} \cap \{b, c\} = \emptyset$ and that $I = X$ and $J = Y$. For all $e \in \{a, b, c, d\}$, let $Z_e = \{(i, j)|A_1(i, j) = e\}$. Observe that Z_e induces a connected component of G_{A_1} and therefore also of G_{A_2} for all $e \in \{a, b, c, d\}$. This implies that there are some $a', b', c', d' \in GF(q_2)$ such that $A_2(i, j) = a'$ for all $(i, j) \in Z_a$, $A_2(i, j) = b'$ for all $(i, j) \in Z_b$, $A_2(i, j) = c'$ for all $(i, j) \in Z_c$ and $A_2(i, j) = d'$ for all $(i, j) \in Z_d$. There are no edges between Z_a and Z_b but there is some $i \in X$ and $j_1, j_2 \in Y$ such that $(i, j_1) \in Z_a$ and $(i, j_2) \in Z_b$. It follows that $a' \neq b'$. Similarly, $a' \neq c'$,$d' \neq b'$ and $d' \neq c'$, so $\{a', d'\} \cap \{b', c'\} = \emptyset$ and therefore $A_2[I, J] \in \mathcal{B}_2$. ☐

Lemma 4.2.5. *Let q_1 and q_2 be prime powers. Now let A_1 and A_2 be matrices in $GF(q_1)^{X \times Y}$ and $GF(q_2)^{X \times Y}$, respectively, where $G_{A_1} = G_{A_2}$. If either of $A_1[I, J]$ or $A_2[I, J]$ is in $\mathcal{C}_3$ for some $I \subseteq X$ and $J \subseteq Y$, then both $A_1[I, J]$ or $A_2[I, J]$ are in $\mathcal{C}_3$.*

Proof. Without loss of generality we will assume that $A_1[I, J] = C(a, b, c, 3)$ for some pairwise distinct $a, b, c \in GF(q_1)$ and that $I = X$ and $J = Y$. For all $e \in \{a, b, c\}$, let $Z_e = \{(i, j)|A_1(i, j) = e\}$. Observe that Z_e induces a connected component of G_{A_1} and therefore also of G_{A_2} for all $e \in \{a, b, c\}$. This implies that there are some $a', b', c' \in GF(q_2)$ such that $A_2(i, j) = a'$ for all $(i, j) \in Z_a$, $A_2(i, j) = b'$ for all $(i, j) \in Z_b$ and $A_2(i, j) = c'$ for all $(i, j) \in Z_c$. There are no edges between Z_a and Z_b but there is some $i \in X$ and $j_1, j_2 \in Y$ such that $(i, j_1) \in Z_a$ and $(i, j_2) \in Z_b$. It follows that $a' \neq b'$. Similarly, $a' \neq c'$ and $b' \neq c'$, so $A_2[I, J] \in \mathcal{C}_3$. ☐

We are now ready to start with the main proof of Theorem 1.3.2. We will prove the following, slightly stronger restatement. For facility, let $\mathcal{S}$ denote $\mathcal{A}^+_{q_1^2 q_2^2} \cup \mathcal{A}^-_{q_1 q_2 (q_1-1)(q_2-1)} \cup \mathcal{B}_2 \cup \mathcal{C}_3$.

Theorem (Restatement of Theorem 1.3.2)**.** *For any pair of finite fields $GF(q_1), GF(q_2)$, let M_1 be a $GF(q_1)$-representable and M_2 be a $GF(q_2)$-representable matroid on a common ground set E such that M_2 arises from M_1 by relaxing a circuit-hyperplane H and let B be a basis of M_1 and M_2 of the form $H - \{e\} \cup \{f\}$. Let A_1 and A_2 be reduced $GF(q_1)$ and $GF(q_2)$-representations of M_1 and M_2, respectively, with respect to B. Then, after rescaling rows and columns neither A_1 nor A_2 contain a matrix in $\mathcal{S}$ whose row set is disjoint from $\{f\}$ and whose column set is disjoint from $\{e\}$.*

Proof. Let M_1 be a $GF(q_1)$-representable and M_2 be a $GF(q_2)$-representable matroid on a common ground set E such that M_2 arises from M_1 by relaxing a circuit-hyperplane H, and let B be a basis of the form $H - \{e\} \cup \{f\}$ for M_1 and M_2, let A_1 be a reduced $GF(q_1)$-representation of M_1 with respect to B and let A_2 be a reduced $GF(q_2)$-representation of M_2 with respect to B. We have that $A_1(e, f) = 0$, $A_2(e, f) \neq 0$ and all other elements in the row e and the column f are nonzero for both A_1 and A_2. By rescaling, we can achieve that $A_1 = \begin{array}{cc} & \begin{array}{cccc} & & & e \end{array} \\ \begin{array}{c} \\ \\ \\ f \end{array} & \left[\begin{array}{ccc|c} & & & 1 \\ & A_1' & & \vdots \\ & & & 1 \\ \hline 1 & \dots & 1 & 0 \end{array}\right] \end{array}$ and $A_2 = \begin{array}{cc} & \begin{array}{cccc} & & & e \end{array} \\ \begin{array}{c} \\ \\ \\ f \end{array} & \left[\begin{array}{ccc|c} & & & 1 \\ & A_2' & & \vdots \\ & & & 1 \\ \hline 1 & \dots & 1 & 1 \end{array}\right] \end{array}$,respectively.

If A_1' or A_2' contain a matrix $\tilde{A}$ in $\mathcal{S}$, there are index sets I, J such that $A_1'[I, J]$ and $A_2'[I, J]$ are both in $\mathcal{S}_0$ and are both of the same kind where $\mathcal{S}_0 = \mathcal{A}^+_{q_1 q_2} \cup \mathcal{A}^-_{(q_1-1)(q_2-1)} \cup \mathcal{B}_2 \cup \mathcal{C}_3$. This follows from Lemma 4.2.2, Lemma 4.2.3, Lemma 4.2.4 or Lemma 4.2.5, whichever kind $\tilde{A}$ is of. Observe that $A'2(i, j) \neq 1$ for all $i \in B - \{f\}$ and $j \in E(M) - B - \{e\}$ as otherwise $A_1[\{i, f\}, \{j, e\}]$ is non-singular and $A_2[\{i, f\}, \{j, e\}]$ is singular, contradicting Lemma 2.2.2. We can therefore apply the lemmas 4.1.1, 4.1.2, 4.1.3 and 4.1.4, yielding that $A_1[I \cup \{f\}, J \cup \{e\}]$ is singular and $A_1[I \cup \{f\}, J \cup \{e\}]$ is nonsingular. This is a contradiction to the singularity condition stated in Lemma 2.2.2. This finishes the proof.

□

We will now conclude the following slight strengthening of Theorem 1.2.6. This will be an easy application of some previous results.

Theorem (Restatement of Theorem 1.2.6)**.** *For every pair of finite fields $GF(q_1), GF(q_2)$ there is some constant $K(q_1, q_2)$ such that if M_1 and M_2 are $GF(q)$-representable matroids on a common ground set E such that M_2 arises from M_1 by relaxing a circuit-hyperplane H and A is a reduced representation of M_1 or M_2 with respect to a basis of the form $H - \{e\} \cup \{f\}$ for some $e \in H$ and $f \in E - H$, then A is weakly $K(q_1, q_2)$-block-triangular after scaling.*

Proof. Let $GF(q_1), GF(q_2)$ be a pair of finite fields and let M_1 and M_2 be $GF(q)$-representable matroids such that M_2 arises from M_1 by relaxing a circuit-hyperplane H. Let B be a basis of the form $H - \{e\} \cup \{f\}$ for some $e \in H$ and $f \in E - H$ and let A_1 be a reduced representation of M_1 with respect to B on disjoint index sets I and J. By Theorem 1.3.2 after scaling $A_1[H - \{e\}, E - H - \{f\}]$ does not contain a submatrix in $\mathcal{A}_{q_1^2 q_2^2}, \mathcal{B}_2$ or $\mathcal{C}_3$. It follows now from Theorem 1.3.1 that there is some number $n_1(q_1, q_2)$ such that

$A_1[H-\{e\}, E-H-\{f\}]$ is weakly $n_1(q_1,q_2)$-block-triangular. As we only append a single row and column, this means that A_1 is weakly $q_1^2 n_1(q_1,q_2)$-block-triangular.

Similarly, if A_2 is a reduced representation of M_2 with respect to B, there is some constant with respect to which A_2 is weakly block-triangular. Choosing $K(q_1,q_2)$ to be the maximum of the two constants finishes the proof. □

Chapter 5

Fragile minors of matroids

This chapter is concerned with finding a connection between two different objects of matroid theory; circuit-hyperplane relaxations and fragile minors. Recall that a matroid N is a fragile minor of another matroid M if $M/C\backslash D = N$ for some $C, D \subseteq E(M)$ but $M/C'\backslash D' \neq N$ whenever $C \neq C'$ or $D \neq D'$. We will start with the main proof of Theorem 1.4.3. Afterwards, we will have Theorem 1.4.2 as a first corollary of Theorem 1.4.3 and, as a second corollary of Theorem 1.4.3, we will prove that Conjecture 1.2.5 implies Conjecture 1.4.1.

5.1 The proof of Theorem 1.4.3

We will first introduce the notion of matrix fragility which will allow us to formulate several lemmas as matrix results that can be easily transformed to matroid results. The definition is motivated by Lemmas 5.1.2 and 5.1.3.

Definition 5.1.1. Let A be an $X \times Y$-matrix with $X \cap Y = \emptyset$ and let $I \subseteq X$ and $J \subseteq Y$ such that $A[I, J] = 0$. Then A is called (I, J)-*fragile* if for all $I' \subseteq X - I$ and $J' \subseteq Y - J$ such that $I' \cup J' \neq \emptyset$, it holds that $\text{rank}(A[I', J']) < \text{rank}(A[I' \cup I, J' \cup J])$.

If I, J are disjoint finite sets, then we denote by $R(I, J)$ the matroid with ground set $I \cup J$ such that I is a set of coloops and J is a set of loops. We call matroids of the form $R(I, J)$ *isolated*; thus a matroid is isolated if each of its components has size one.

We say that for a matroid M on ground set E, two subsets $X \subseteq E$ and $Y \subseteq E$ are *skew* if $r_M(X) + r_M(Y) = r_M(X \cup Y)$.

Lemma 5.1.2. *Let M be a representable matroid having a fragile minor N and let B be a basis of M displaying N. If A is a reduced representation of M with respect to B of the form*

$$A = \begin{array}{c|cc} & E(M)-E(N)-B & E(N)-B \\ \hline B-E(N) & A_1 & A_2 \\ E(N)\cap B & A_3 & A_4 \end{array},$$

then the matrix

$$A' = \begin{array}{c|cc} & E(M)-E(N)-B & E(N)-B \\ \hline B-E(N) & A_1 & A_2 \\ E(N)\cap B & A_3 & 0 \end{array}.$$

is $(E(N)\cap B, E(N)-B)$-fragile.

Proof. Assume otherwise. Then there are some $C \subseteq E(M)-E(N)-B$ and some $D \subseteq B-E(N)$ such that $\operatorname{rank}(A'[D\cup(E(N)\cap B), C\cup(B-E(N))]) = \operatorname{rank}(A'[D,C])$. Without loss of generality, we may assume that $C = E(M)-E(N)-B$ and $D = B-E(N)$. Consider the matrix

$$A'' = \begin{array}{ccccc} D & X & C & Y & Y' \\ \left[\begin{matrix} I \\ 0 \end{matrix}\right. & \begin{matrix} 0 \\ I \end{matrix} & \begin{matrix} A_1 \\ A_3 \end{matrix} & \begin{matrix} A_2 \\ A_4 \end{matrix} & \left.\begin{matrix} A_2 \\ 0 \end{matrix}\right] \end{array},$$

where $X = B\cap E(N)$ and $Y = E(N)-B$ and let $M'' = M(A'')$. We will first need some claims about M''.

Claim 1. $N = (M''/Y')|E(N)$

Proof. In order to contract Y', we apply a number of row operations to the rows in the top part of A'' until the columns of Y' are either all zeros or contain a single one in a row that does not contain any other nonzero entry in a column indexed by Y'. As we only applied row operations, the bottom part of the matrix representing $M''|E(N)$ is still identical. So by removing these rows, we get that $(M''/Y')|E(N)$ is represented by $A'' = \begin{array}{cc} X & Y \\ \left[\begin{matrix} 0 \\ I \end{matrix}\right. & \left.\begin{matrix} 0 \\ A_4 \end{matrix}\right] \end{array}$, so $(M''/Y')|E(N) = N$ □

Claim 2. $r''_M(C \cup Y') = r''_M(C)$

Proof. By assumption $\mathrm{rank}(A_1) = \mathrm{rank}(A')$. As A_1 is a submatrix of the representation of $M''|C$ and A' is the representation of $C \cup Y'$, the statement follows. ☐

Claim 3. *C is skew to $E(N)$ in M''/Y'.*

Proof. By the block structure of the representation of $X \cup C$, we have that $r''_M(X \cup C) = r''_M(X) + \mathrm{rank}(A_1)$. By assumption, $\mathrm{rank}(A_1) = \mathrm{rank}(A')$ and therefore $r''_M(C) = \mathrm{rank}(A_1)$. This yields $r_M(X \cup C) = r''_M(X) + r''_M(C)$.

We will prove that $r_{M''/Y'}(C) + r_{M''/Y'}(E(N)) = r_{M''/Y'}(E(N) \cup C)$.

The rank formula yields $r_{M''/Y'}(C) + r_{M''/Y'}(X) = r_{M''}(Y' \cup C) - r_{M''}(Y') + r_{M''}(Y' \cup X) - r_{M''}(Y')$.

As C spans Y' in M'' by Claim 2 and $r_{M''}(X) + r_{M''}(Y') = r_{M''}(X \cup Y')$ by construction, the above equals $r_{M''}(X \cup C) - r_{M''}(Y') = r_{M''/Y'}(X \cup C)$, so $r_{M''/Y'}(C) + r_{M''/Y'}(X) = r_{M''/Y'}(X \cup C)$.

As X spans $E(N)$ in M'', it follows that $r_{M''/Y'}(C) + r_{M''/Y'}(E(N)) = r_{M''/Y'}(E(N) \cup C)$. ☐

We will now give the main proof. By Claim 1, $N = (M''/Y')|E(N)$. By Claim 3, C and $E(N)$ are skew and so $E(N)$ is not influenced by whether elements in C are deleted or contracted and so $N = ((M''/Y')|(E(N) \cup C))/C = ((M''/C)/Y')|E(N)$. By Claim 2, this equals $(M''/C)|E(N)$ which is the same as $(M/C)|E(N)$.

This is a contradiction to N being a fragile minor of M. ☐

Lemma 5.1.3. *Let A be an $X \times Y$-matrix that is (I, J)-fragile and let $M = M[I|A]$. Then M is $R(I, J)$-fragile.*

Proof. Assume otherwise, then there are some $I_1 \subseteq X - I$ and $J_1 \subseteq Y - J$ such that $M/(X - I_1 \cup J_1)\backslash(Y - J_1 \cup I_1) = N$. Without loss of generality, we may assume that $I_1 = X - I$ and $J_1 = Y - J$, so $M/J_1\backslash I_1 = N$. This means in particular that $r_{M/J_1\backslash I_1}(I) = r_N(I) = |I|$ and $r_{M/J_1\backslash I_1}(J) = r_N(J) = 0$.

Using the rank formula, this means that

$$\begin{aligned} |I| &= r_{M/J_1\backslash I_1}(I) = r_{M/J_1}(I) = r_M(I \cup J_1) - r_M(J_1) \\ &= r_{M/I}(J_1) + r_M(I) - r_M(J_1) = |I| + \mathrm{rank}(A[I_1, J_1]) - r_M(J_1) \end{aligned}$$

so $\operatorname{rank}(A[I_1, J_1]) = r_M(J_1)$.

Furthermore, $0 = r_{M/J_1 \setminus I_1}(I) = r_M(I_1 \cup J_1) - r_M(J_1) = \operatorname{rank}(A) - r_M(J_1)$.

Therefore, $\operatorname{rank}(A) = \operatorname{rank}(A[I_1, J_1])$, which is a contradiction to A being (I, J)-fragile.

□

Lemma 5.1.4. *Let M be an $\mathbb{F}$-representable matroid with a fragile minor N, and let B be a basis of N. Then there is an $\mathbb{F}$-representable matroid M' such that $E(M) = E(M')$, M' has a fragile minor $R(B, E(N) - B)$, and $M'/B \setminus (E(N) - B) = M/B \setminus (E(N) - B)$.*

Proof. Let M, N, B as in the statement and consider a reduced representation A of M with respect to a basis containing B. Let A' be the matrix which arises from A by replacing $A[B \cap E(N), E(N) - B)]$ by an all-zero matrix. By Lemma 5.1.2, A' is $(B \cap E(N), E(N) - B)$-fragile. Let $M' = M(A')$. By Lemma 5.1.3, $R(B, E(N) - B)$ is a fragile minor of M'. Also, by construction, $M'/B \setminus (E(N) - B) = M/B \setminus (E(N) - B)$. □

For some prime power q and some natural number n, consider the nested set of fields $\mathbb{F}_0 \subsetneq \mathbb{F}_1 \subsetneq \ldots \subsetneq \mathbb{F}_{n+1}$ where $\mathbb{F}_i = GF(q^{2^i})$. A vector $x \in \mathbb{F}_{n+1}^n$ is called *hyper-extending for q* if $x_i \in \mathbb{F}_{i+1} - \mathbb{F}_i$ for all $i = 1, \ldots, n$.

Now consider an $X \times Y$-matrix A over some finite field $GF(q)$ of the form

$$A = \begin{array}{c|cc} & Y-J & J \\ \hline X-I & A_1 & A_2 \\ I & A_3 & 0 \end{array}$$

for some $I \subseteq X$ and $J \subseteq Y$. Let v_i denote the row of A_3 which is indexed by $i \in I$ and let w_j denote the column of A_2 which is indexed by $j \in J$. Now consider an ordering $a = (a_1, \ldots, a_n)$ of $I \cup J$ such that $\{a_1, \ldots, a_{|I|}\} = I$ and $\{a_{|I|+1}, \ldots, a_n\} = J$. For some hyper-extending vector x of length n, now consider the $(X - I \cup \{s\}) \times (Y - J \cup \{t\})$-matrix A' with

$$A' = \begin{array}{c|cc} & Y-J & t \\ \hline X-I & A_1 & \sum\limits_{j=|I|+1}^{|J|} x_j w_{a_j} \\ s & \sum\limits_{i=1}^{|I|} x_i v_{a_i} & 0 \end{array}$$

Such a matrix is called a *free homomorphism of A with respect to (I, J) and a*. We will also refer to x_i as $x_a i$.

Lemma 5.1.5. *Let A be an $X \times Y$-matrix over $GF(q)$ and let $I \subseteq X$ and $J \subseteq Y$ such that $A[I,J]$ is an all-zero matrix and A is (I,J)-fragile. Now let A' be an $(X - I \cup \{s\}) \times (Y - J \cup \{t\})$-free homomorphism of A with respect to (I,J) and some ordering a. Then A' is $(\{s\},\{t\})$-fragile.*

Proof. Assume otherwise, then there are some sets $I' \subseteq X - I$ and $J' \subseteq Y - J$ such that $\operatorname{rank}(A'[I' \cup \{s\}, J' \cup \{t\}]) = \operatorname{rank}(A'[I', J'])$. Without loss of generality, we may assume that $I' = X - I$ and $J' = Y - J$. As A is (I,J)-fragile, it follows that either $\operatorname{rank}(A[X, Y - J]) > \operatorname{rank}(A[X - I, Y - J])$ or $\operatorname{rank}(A[X,Y]) > \operatorname{rank}(A[X, Y - J])$.

Assume first that $\operatorname{rank}(A[X, Y - J]) > \operatorname{rank}(A[X - I, Y - J])$. Recall that the row s of A' is defined by $v_s = \sum\limits_{i \in I} x_i v_i$, where v_i is the row indexed by i of A. As v_s is freely placed among $\{v_i : i \in I\}$ and the rows of $A[X - I, Y - J]$ do not span all the rows $\{v_i : i \in I\}$ by assumption, they also do not span v_s. It follows that $\operatorname{rank}(A'[I' \cup \{s\}, J']) > \operatorname{rank}(A'[I', J'])$ and therefore $\operatorname{rank}(A'[I' \cup \{s\}, J' \cup \{t\}]) > \operatorname{rank}(A'[I', J'])$, a contradiction to the assumption.

Assume now that $\operatorname{rank}(A[X,Y]) > \operatorname{rank}(A[X, Y - J])$ and recall that the column t of A' is defined by $w_t = \sum\limits_{j \in J} x_j w_j$, where w_j is the row of A which is indexed by j. As w_t is freely placed among $\{w_j : j \in J\}$ and the columns of $A[X, Y - J]$ do not span all the rows $\{w_j : j \in J\}$ by assumption, they also do not span w_t. It follows that $\operatorname{rank}(A'[I' \cup \{s\}, J' \cup \{t\}]) > \operatorname{rank}(A'[I' \cup \{s\}, J'])$ and therefore $\operatorname{rank}(A'[I' \cup \{s\}, J' \cup \{t\}]) > \operatorname{rank}(A'[I', J'])$, a contradiction to the assumption.

This finishes the proof.

□

The following result gives a matroidal interpretation to Lemma 5.1.5, it shows that in order to prove Conjecture 1.4.1, it suffices to consider the case that N has only two elements.

Lemma 5.1.6. *Let $R = R(I,J)$ be an isolated matroid and let $R' = R(\{e\},\{f\})$ where $e \in I$ and $f \in J$. If M is a $GF(q)$-representable matroid having R as a fragile minor, then there is a $GF(q^{2^{|R|}})$-representable matroid M' such that R' is a fragile minor of M' and $M'/e\backslash f = M/I\backslash J$.*

Proof. Let A be a reduced representation on $X \times Y$ of M with respect to a basis displaying R. Obviously $A[I,J] = 0$. By Lemma 5.1.2, A is (I,J)-fragile. Now consider a free homomorphism A' of A on $X - I \cup \{e\} \times Y - J \cup \{f\}$. It follows from Lemma 5.1.5 that A' is $(\{e\}, \{f\})$-fragile. It therefore follows by Lemma 5.1.3 that M' has R' as a fragile minor. □

Lemma 5.1.7. *Let A_1 be an $X \times Y$-matrix over some finite field $GF(q)$ and let there be $s \in X$ and $t \in Y$ such that $A_1(s,t) = 0$ and A_1 is $(\{s\}, \{t\})$-fragile. Now let $\mathbb{F}$ be an extension field of $GF(q)$ and let $z \in \mathbb{F} - GF(q)$. Now let A_2 be an $X \times Y$-matrix over $\mathbb{F}$ which is defined by*

$$A_2(i,j) = \begin{cases} z & \textit{if } i = s \textit{ and } j = t \\ A(i,j) & \textit{else} \end{cases}.$$

Then for any $I \subseteq X$ and $J \subseteq Y$ such that $|I| = |J|$ and either $I \neq \{s\}$ or $J \neq \{t\}$, it holds that $A_1[I,J]$ and $A_2[I,J]$ are either both singular or non-singular.

Proof. If $s \notin I$ or $t \notin J$, $A_1[I,J] = A_2[I,J]$ and so the statement trivially holds. So we may assume for the rest of the proof that $s \in I$ and $t \in J$. Let $C = A_1[I - \{s\}, J - \{t\}] = A_2[I - \{s\}, J - \{t\}]$ Observe that by Laplace expansion, we can get that $det(A_2[I,J]) = zdet(C) + det(A_1[I,J])$. Therefore, if C is singular, we get that $det(A_2[I,J]) = det(A_1[I,J])$ and so the statement holds. So we can now assume that C is non-singular and will show that under these conditions both $A_1[I,J]$ and $A_2[I,J]$ are non-singular.

As C is non-singular and $A_1[I,J]$ has one more row and column and higher rank than C, $A_1[I,J]$ is non-singular. We still need to prove that $A_2[I,J]$ is non-singular. Recall that $det(A_2[I,J]) = zdet(C) + det(A_1[I,J])$. As z is in $\mathbb{F} - GF(q)$ and $det(C)$ and $det(A_1[I,J])$ are in $GF(q)$, $det(A_2[I,J])$ is in $\mathbb{F} - GF(q)$ and so in particular non-zero. This finishes the proof. □

Lemma 5.1.8. *Let $N = R(\{e\}, \{f\})$, let M be a $GF(q)$-representable matroid having N as a fragile minor and let $C, D \subseteq E(M)$ such that $M/C \backslash D = N$. Then $C \cup \{e\}$ is a circuit-hyperplane of M and the matroid obtained by relaxing $C \cup \{e\}$ is $GF(q^2)$-representable.*

Proof. Let A_1 be a reduced $GF(q)$-representation of M on $X \times Y$ with respect to $C \cup \{e\}$. As $A_1(e,f) = 0$, A_1 is $\{e\}, \{f\}$-fragile by Lemma 5.1.2. Now let A_2 be defined as

$$A_2(i,j) = \begin{cases} z & \text{if } i = e \text{ and } j = f \\ A(i,j) & \text{else} \end{cases},$$

where $z \in GF(q^2) - GF(q)$. By Lemma 5.1.7, $A_1[I, J]$ and $A_2[I, J]$ are either both singular or both nonsingular for all $I \subseteq X$ and $J \subseteq Y$ such that $|I| = |J|$ and either $I \neq \{s\}$ or $J \neq \{t\}$. Hence the statement follows by Lemma 2.2.3 and the fact that the relaxation of M is equal to $M(A_n)$ and therefore $GF(q^2)$-representable by construction. ☐

We will now prove the following slightly more technical, but stronger version of Theorem 1.4.3:

Theorem 5.1.9 (Restatement of Theorem 1.4.3)**.** *Let $\mathbb{F}$ be a finite field of order q, let N be an $\mathbb{F}$-representable matroid and M an $\mathbb{F}$-representable matroid having N as a fragile minor. Furthermore, let B be a basis of M displaying N. Then there is a field $\mathbb{F}_1$ of order $q^{2^{|N|+1}}$ and $\mathbb{F}_1$-representable matroids M_1, M_2 on $E(M) - E(N) \cup \{c, d\}$ such that:*

(i) $B - B_N \cup \{d\}$ is a circuit-hyperplane of M_1,

(ii) M_2 is obtained from M_1 by relaxing a circuit-hyperplane $B - B_N \cup \{d\}$, and

(iii) $M/C\backslash D = M_1/c\backslash d$ for some partition (C, D) of $E(N)$.

Furthermore, there is a $B \times (E(M) - B)$-matrix A_1 over $\mathbb{F}$ and a $(B - E(N) \cup \{c\}) \times (E(M) - E(N) - B \cup \{d\})$-matrix A_2 such that:

(i) A_1 is a reduced representation of M with respect to B,

(ii) A_2 is a reduced representation of M_1 with respect to $B - E(N) \cup \{c\}$, and

(iii) $A_1[B - E(N), E(M) - B - E(N)] = A_2[B - E(N), E(M) - B - E(N)]$.

Proof. Let A be a reduced representation of M with respect to B, so A is of the form

$$A = \begin{array}{c|cc} & E(M) - E(N) - B & E(N) - B \\ \hline B - E(N) & A_1 & A_2 \\ B \cap E(N) & A_3 & A_4 \end{array}.$$

Now consider the matrix A' where

$$A' = \begin{array}{c|cc} & E(M) - E(N) - B & E(N) - B \\ \hline B - E(N) & A_1 & A_2 \\ B \cap E(N) & A_3 & 0 \end{array}.$$

Let M' be the matroid which has A' as a reduced representation and let N' be the matroid that has

$$\begin{array}{cc} & E(N)-B \\ B \cap E(N) & \left[\begin{array}{c} \quad 0 \quad \end{array}\right] \end{array}$$

as its reduced representation. By Lemma 5.1.4, N' is a fragile minor of M'.

Now let A'' be a free homomorphism of A' with respect to $(E(N) \cap B, E(N) - B)$. Observe that the index sets of A'' are $B - E(N) \cup \{c\}$ and $E(M) - B - E(N) \cup \{d\}$ and that A'' is a matrix over $\mathbb{F}_{|N|}$. Let M'' be the matroid that has A'' as a reduced representation and let N'' be the matroid that has

$$\begin{array}{cc} & \{t\} \\ \{s\} & \left[\begin{array}{c} 0 \end{array}\right] \end{array}$$

as a reduced representation. By Lemma 5.1.6, N'' is a fragile minor of M''. Hence, $E(M'') - E(N'')$ cannot contain a loop or coloop and so no row of A'' in $B - E(N)$ and no column of A'' in $E(M) - B - E(N)$ is constant of zeros. Now let $z \in \mathbb{F}_{|N|+1} - \mathbb{F}_|N|$ and define C by

$$C(i,j) = \begin{cases} z & \text{if } i = c \text{ and } j = d \\ A''(i,j) & \text{else} \end{cases}.$$

Let $M_1 = M''$ and let M_2 be the matroid that has C as a reduced representation. By Lemma 5.1.8, $B - E(N) \cup \{d\}$ is a circuit-hyperplane of M_1 and M_2 arises from M_1 by relaxing $B - E(N) \cup \{d\}$. By construction, M_1 and M_2 are representable over $\mathbb{F}_{n+1} = GF(q^{2^{|N|+1}})$. Also, by construction, $A''[B - E(N), E(M) - B - E(N)] = C[B - E(N), E(M) - B - E(N)]$ and therefore, $M/B \cap E(N) \backslash E(N) - B = M_1/\{c\}\backslash\{d\}$. This finishes the proof. □

5.2 Applications of Theorem 1.4.3

In this part we will prove Theorem 1.4.2. We will prove the following, slightly stronger version.

Theorem (Restatement of Theorem 1.4.2)**.** *For every $GF(q)$-representable matroid N there is a constant C such that, if M is a matroid having N as a fragile minor and A is a reduced $GF(q)$-representation of M with respect to a basis B displaying N, then A is weakly C-block-triangular.*

Proof. Let A be a matrix as in the statement, so A is of the form

$$A = \begin{array}{cc} & \begin{array}{cc} E(M) - E(N) - B & E(N) - B \end{array} \\ \begin{array}{c} B - E(N) \\ B \cap E(N) \end{array} & \left[\begin{array}{cc} A_1 & A_2 \\ A_3 & A_4 \end{array}\right] \end{array}.$$

By Theorem 5.1.9, A_1 is a submatrix of a $(B - E(N) \cup \{c\}) \times (E(M) - E(N) - B \cup \{d\})$-matrix A' over $GF(q^{2^{|N|+1}})$ which is a reduced representation of a matroid M_1 which has a circuit-hyperplane H with $|H - (B - E(N) \cup \{c\})| = 1$ whose relaxation yields another $GF(q^{2^{|N|+1}})$-representable matroid. It follows by Theorem 1.2.6 that there is some constant C_1 such that A' is weakly C_1-block-triangular after row and column scalings. This yields that A' is weakly qC_1-block-triangular. As A_1 is a submatrix of A', A_1 is also weakly qC_1-block-triangular.

Observe that for any weakly C-block-triangular matrix A, the matrix that arises from A by appending a row or column is weakly qC-block-triangular. This can be seen by choosing the index classes in which the new row/column is constant and refining them with a certificate of weak C-block-triangularity for A.

Applying the above fact repeatedly yields that A is weakly $q^{|N|+1}C_1$-block-triangular, a number depending only on N and q. □

Theorem (Conjecture 1.2.5 implies Conjecture 1.4.1)**.** *Assuming the truth of Conjecture 1.2.5, for every $GF(q)$-representable matroid N there is a constant C such that, if M is a matroid having N as a fragile minor and A is a reduced $GF(q)$-representation of M with respect to a basis B displaying N, then A is C-block-triangular.*

Proof. Let A be a matrix as in the statement, so A is of the form

$$A = \begin{array}{cc} & \begin{array}{cc} E(M) - E(N) - B & E(N) - B \end{array} \\ \begin{array}{c} B - E(N) \\ B \cap E(N) \end{array} & \left[\begin{array}{cc} A_1 & A_2 \\ A_3 & A_4 \end{array}\right] \end{array}.$$

By Theorem 5.1.9, A_1 is a submatrix of a $(B - E(N) \cup \{c\}) \times (E(M) - E(N) - B \cup \{d\})$-matrix A' over $GF(q^{2^{|N|+1}})$ which is a reduced representation of a matroid M_1 which has a circuit-hyperplane H with $|H - (B - E(N) \cup \{c\})| = 1$ whose relaxation yields another $GF(q^{2^{|N|+1}})$-representable matroid. It follows now by Conjecture 1.2.5 that there is some constant C_1 such that A' is C_1-block-triangular. As A_1 is a submatrix of A', A_1 is also C_1-block-triangular.

Observe that for any C-block-triangular matrix A, the matrix that arises from A by appending a row or column is qC-block-triangular. This can be seen by choosing the index classes in which the new row/column is constant, refining them with a certificate of weak C-block-triangularity for A and observing that the newly appended row/column is trivially constant in all partition classes.

Applying the above fact repeatedly yields that A is $q^{|N|}C_1$-block-triangular, a number depending only on N and q. $\square$

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