5-Choosability of Planar-plus-two-edge Graphs

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, 2017

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

We prove that graphs that can be made planar by deleting two edges are 5-choosable. To arrive at this, first we prove an extension of a theorem of Thomassen. Second, we prove an extension of a theorem Postle and Thomas. The difference between our extensions and the theorems of Thomassen and of Postle and Thomas is that we allow the graph to contain an inner 4-list vertex. We also use a colouring technique from two papers by Dvořák, Lidický and Škrekovski, and independently by Compos and Havet.

Acknowledgements

First of all I thank my supervisor Bruce Richter for his support and for his generosity. Secondly, I thank my younger brother Ali for making my life easier since he came to Waterloo and for his help in the Algebraic Enumeration course.

Special thanks to Edward Lee. I wouldn't have survived in C&O without his help and his care.

I also thank my other office-mates, Christos Stratopoulos for his compassion, and Cedric (a.k.a. Zhuan Khye Koh) and Sharat Ibrahimpur for always being there for help.

Ghislain Mckay taught me to draw wheels and spirals in Latex, and before that Thomas Kelly helped me to make my first Latex drawings. I thank them both for that and thank them again with Jamie de Jong, Owen Hill, and Alan Arroyo Guevara for always helping me whenever I asked for help.

I cannot forget to mention Nishad Kothari's generosity when he agreed easily to give me a lecture on matchings, bricks and braces, and Pfaffian Orientations. It was an entertaining three-hour lecture by a really good teacher. Thank you Nishad.

Thanks to my professors David Wagner and Stephen Vavasis for their understanding and encouragement when I thought I am going to fail. I also thank Penny Haxell for her feedback on the thesis.

Finally, I thank Melissa Cambridge for her help, moral support, and for always doing her best to make everything easy for everyone.

Table of Contents

Chapter 1

Introduction

In Graph Theory, one of the most fundamental theorems is the Four Colour Theorem. Many colouring theorems and conjectures are either extensions of or inspired by the Four Colour Theorem. One direction of extension is colourability of graphs that are close to planar, another is list-colourings of planar graphs.

A major generalization in the first direction is Albertson's Conjecture, which states that if a graph has chromatic number r, then its crossing number is at least that of K_r. In case $r = 5$, this conjecture is equivalent to the Four Colour Theorem. The conjecture is proved for $r \leq 16$ (cf. [9], [2] and [3]).

In the second direction we see Thomassen's famous 5-choosability theorem for planar graphs [12] and Voigt's examples of planar graphs that are not 4-choosable [14].

The study of list-colourability of graphs that are not far from planar is also a natural growing line of research. Compos and Havet [4], and independently Dvořák, Lidický, and Škrekovski [5] proved that graphs with two crossings are 5-choosable.

Here we prove in Theorem 2.1.1 that graphs that can be made planar by deleting two edges, no matter how many crossings there are (there can be arbitrarily many), are 5-choosable. This should give the maximum number of edges that can be added to a planar graph without losing 5-choosability, since for example K_6 is a graph that can be made planar by deleting three edges but it has chromatic number 6.

As mentioned above, 5-choosability of planar graphs was proved by Thomassen, that was in 1994. In 2011, Compos and Havet proved (in a minor theorem, Theorem 3, in the paper [4] where they prove 5-choosability of graphs with two crossings) that graphs that can be made planar by deleting one edge are also 5-choosable.

The bigger ambition behind our work was to prove a list-colouring analogue of Theorem 4.1 in [7], not the main theorem there, by Erman, Havet, Lidický and Pangrác, 2011. In that theorem they prove that if a graph can be made planar by deleting a set of at most $2k$ edges, then it is $(4+k)$-colourable. The proof of that colouring theorem is a simple induction on k, but this seems not to go that simply with list-colouring.

In a plane graph G, the *outer walk* is the boundary of the infinite face, and an *inner vertex* is a vertex not in the outer walk. If L is a list-assignment of G, then for $v \in V(G)$, $L(v)$, or just v for short, is a *k-list* if $|L(v)| \geq k$. Our proof is in three stages.

(1) An extension of a theorem of Thomassen from 2007 [13]. We prove that a plane graph with a precoloured path of length at most two on the outer walk and an inner vertex with a list of size at least four is colourable unless it contains a wheel-like structure attached to the outer walk and the attachment vertices have few colours in their lists. This is Theorem 4.3.1.

(2) An extension of a theorem of Postle and Thomas from 2015 [11]. This is concerned with colouring plane graphs with two 2-lists on the outer walk and one inner 4-list that do not contain a wheel attached to the outer walk with centre the 4-list. This is Theorem 2.1.3, proved in Section 4.4. In the proof of this theorem, the proofs of Case 1 of Claim 4.4.11, and Case 2 of Claim 4.4.15, are proved and written by Bruce Richter.

(3) We colour a part of a shortest path between the two edges carefully so that after deleting its coloured vertices we obtain a graph with a list assignment similar to that in (2). This is shown in Chapter

3. This technique of colouring carefully a shortest path between two bad configurations was done twice before in 2011 to prove that graphs with two crossings are 5-choosable, by Dvořák, Lidický, and Škrekovski [5], and independently by Compos and Havet [4].

In this work, we measure how far from planar the graph is by the number of edges to delete to obtain a planar graph. There are other ways to measure this. These include the crossing number, the distance between crossings, and the number of vertices to delete to remove all the crossings. Also whether the crossings are independent (that is the edges involved in them do not have end-vertices in common) affects the chromatic number and the choice number.

Dvořák, Lidický and Mohar proved that every graph drawn in the plane so that the distance between every pair of crossings is at least 15 is 5-choosable [6]. In the same paper they also allowed some vertices to have lists of size four only, as long as they are far apart and far from the crossings.

Inspired by this, one possible way of extending our work is to answer the following question.

Question 1.0.1. *What is the choice number of a graph that can be made planar by deleting edges $\{e_1, \cdots, e_k\}$ such that for every distinct i and j, the distance between any crossing with e_i and any crossing with e_j is at least d ?*

In 2009 [9] Oporowski and Zhao asked whether graphs of crossing number at most 5 and clique number at most 5 are 5-colourable. In 2011 [7], Erman, Havet, Lidický and Pangrác answered this question in the negative (Theorem 1.3) but they showed that graphs with crossing number at most 4 and clique number at most 5 are 5-colourable (Theorem 1.4).

They also showed in the same paper [7] that if a graph with clique number at most 5 has three edges whose removal leaves the graph planar, then it is 5-colourable (Theorem 1.6). Furthermore, they proved that if a graph G has clique number at most 6 and there is a set of at most seven edges whose deletion from G results in a planar graph, then G is 6-colourable (Theorem 6.2). The last theorem in that paper, Theorem 6.12, states that if a triangle-free graph contains a set of at most four edges whose deletion results in a planar graph, then it is 4-choosable.

Given this it is natural to ask the following question.

Question 1.0.2. *If a graph does not contain K_6 as a subgraph and can be made planar by deleting three edges, is it* 5*-choosable ?*

In the same paper [7], Erman, Havet, Lidický and Pangrác also proved that if a K_4-free graph has a drawing in the plane in which no two crossings are dependent, then it is 4-colourable (Theorem 6.11). There has been more research in the relationship between the independence of crossings and chromatic number. In this respect Albertson conjectured that if a graph can be drawn in the plane such that all its crossings are independent, then its chromatic number is at most 5. He proved in 2008, [1], that this is true for graphs of crossing number at most 3. Wenger [15] extended Albertson's result to graphs with four crossings. Later in 2010, Král' and Stacho proved the conjecture for any number of independent crossings [8].

It is also natural to try to extend or prove analogues of those results for list-colouring.

Chapter 2

From 5-Choosability to Inner 4-Lists

2.1 The Problem

The goal of this thesis is to prove the following theorem.

Theorem 2.1.1. *Let G be a graph. If there are edges e_1 and e_2 such that $G - \{e_1, e_2\}$ is planar, then G is 5-choosable.*

In this chapter, we reduce the problem to that of list-colouring a plane graph G' containing either

(1) two inner 4-lists, each of which is the centre of a wheel attached to the outer walk of G' or

(2) two outer (that is on the outer walk) 2-lists and one inner 4-list that is not the centre of a wheel attached to the outer walk of G' (but still may be the centre of a wheel).

In case G' is as in (1), we colour it by Proposition 2.1.2 stated below, and in case it is as in (2), we colour it by Theorem 2.1.3 stated below.
Notation: For a plane graph G, let ∂G denote the subgraph of G consisting of those vertices and edges incident with the infinite face.

Proposition 2.1.2. *Let G be a plane graph and let x and y be two inner vertices of G that are the centres of wheels W_1 and W_2, respectively, in G. Suppose that, for $i \in \{1, 2\}$, $V(\partial W_i) \subseteq V(\partial G)$. Let L be a list assignment such that:*

(a) for every $v \in \partial G$, $|L(v)| \geq 3$;

(b) $|L(x)| = |L(y)| = 4$; and

(c) otherwise, $|L(v)| \geq 5$.

Then G is L-colourable.

Theorem 2.1.3. *Let G be a plane graph and let u and w be two vertices in ∂G. Suppose that x is an inner vertex of G such that, if x is the centre of a wheel W in G, then $V(\partial W) \not\subseteq V(\partial G)$. Let L be a list assignment of G such that:*

(a) $|L(x)| \geq 4$;

(b) $|L(u)| \geq 2$ and $|L(w)| \geq 2$;

(c) for every $v \in V(\partial G) \setminus \{u, w\}$, $|L(v)| \geq 3$; and

(d) otherwise, $|L(v)| \geq 5$.

Then G is L-colourable.

Proposition 2.1.2 is proved in Section 4.2, and Theorem 2.1.3 is proved in Section 4.4.

2.2 Reducing the problem to plane graphs

In this section, we explain how to find an appropriate plane subgraph G' of G with list-assignment L' for G' satisfying either Proposition 2.1.2 or Theorem 2.1.3. An L'-colouring of G' will yield the desired colouring for G.

We start with a minimum counterexample G to Theorem 2.1.1 and choose two edges e_1 and e_2 of G so that $G - \{e_1, e_2\}$ is planar and L is a

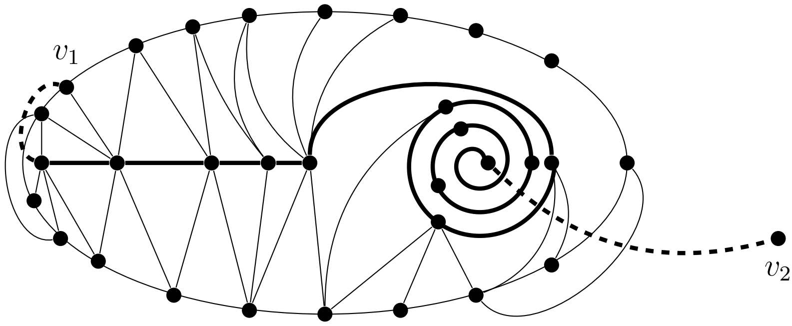


Figure 2.1: v_1 is in V_F and is adjacent to one more vertex other than u_1 in Q.

5-list-assignment of G for which G has no L-colouring. The main effort is to find a suitable shortest path Q in $G - \{e_1, e_2\}$ from a vertex incident with e_1 to a vertex incident with e_2. We obtain G' by deleting all or all but one end of Q from G.

Definitions of $\mathbf{G'}$, $\mathbf{Q}$, $\mathbf{u_1}$, $\mathbf{u_2}$, $\mathbf{v_1}$, and $\mathbf{v_2}$:

For $i = 1, 2$, let $e_i = u_i v_i$. Fix an embedding of $G - \{e_1, e_2\}$ in the plane. Let Q be a shortest $\{u_1, v_1\}\{u_2, v_2\}$-path in $G - \{e_1, e_2\}$, and set $G' = G - V(Q)$. Clearly Q is contained in one face F of G'. Let V_F denote the vertices of the boundary of F. See Figure 2.1. We may assume without loss of generality that Q is a path between u_1 and u_2.

For a vertex v, let $N(v)$ denote the set of neighbours of v in G. We show the following.

Proposition 2.2.1. *There is an L-colouring φ of Q such that for every vertex v in V_F, $|L(v) \backslash \{\varphi(z) \mid z \in V(Q) \cap N(v)\}| \geq 3$. In particular, for $i = \{1, 2\}$, if the end v_i of e_i that is not in Q is also not on the boundary of F, then it has at least four available colours.*

Note that if v_1 (or v_2) is not in V_F, then its only neighbour in $V(Q)$ is u_1 (or respectively u_2), and so it has a list of size at least 4 after deleting the colours of its neighbours in Q from its list.

To prove that such a colouring of Q exists, first note that we may assume that $|V(Q)| \geq 3$ since otherwise every vertex in V_F has at most two neighbours in Q and so still has a list of size at least 3. More generally, a vertex v in V_F cannot have two neighbours at distance 3 or more in Q, as otherwise there is a shorter path in G from u_1 to u_2. We summarize this as follows.

Observation 2.2.2. *Any vertex v in V_F has at most three neighbours in Q and the distance in Q between any two of its neighbours is at most 2. In particular, if v has three neighbours in Q, then those three neighbours are consecutive in Q.*

Let $Q = u_1 z_1 \cdots z_n u_2$, and rename u_1 as z_0 and u_2 as z_{n+1}. Then,

Observation 2.2.3. *For every $k \in \{0, 1, \cdots, n\}$, the only vertex in $\{z_0, \cdots, z_k\}$ adjacent to z_{k+1} is z_k.*

We need the following two lemmas frequently in the thesis, Lemmas 2.2.5 and 2.2.6. They are about extending the colouring of a cycle of length at most four to the interior of the cycle when the interior contains one 4-list. Such a colouring is extendable unless the cycle has length four and the 4-list is adjacent to all the vertices of the cycle.

The proof of those two lemmas needs Theorem 4.3.1. However, the proof of Theorem 4.3.1 itself requires the proof of the claim that, for minimum counterexamples, there are no triangles with nonempty interior and no 4-cycles that contain vertices other than the 4-list in their interior.

The proof of that claim is literally the same as the proofs of the two lemmas combined except for the reference to Theorem 4.3.1. In the proofs of the lemmas we refer to the theorem generally while in the proof of the claim we refer to the theorem as an induction hypothesis valid for the interiors of the cycles, which are smaller subgraphs than a minimum counterexample.

Thus, to avoid writing the same proof twice, and to avoid vicious circles, we write the statements of the lemmas below with the premise *" If Theorem 4.3.1 is true"*. Actually we use a special case of Theorem

4.3.1, which we state below as Proposition 2.2.4, and so you will find *"If Proposition 2.2.4 is true"* in the statements of Lemmas 2.2.5 and 2.2.6. In this way we can refer to those lemmas in the proof of Theorem 4.3.1 as well as outside it.

Proposition 2.2.4. *Let G be a plane graph, z a vertex in ∂G, and x a vertex in $G - V(\partial G)$. Let L be a list assignment such that:*

(a) $L(z)$ is a singleton;

(b) for every $v \in V(\partial G) \setminus \{z\}$, $|L(v)| \geq 3$;

(c) $|L(x)| \geq 4$; and

(d) otherwise, $|L(v)| \geq 5$.

Then G has an L-colouring.

Lemma 2.2.5. *Let H be a plane graph such that ∂H is a triangle and let x be a vertex of $H - V(\partial H)$. Let φ be a colouring of ∂H and let L be a list assignment on H such that:*

(a) for every vertex v of ∂H, $L(v) = \{\varphi(v)\}$;

(b) for every vertex v of $H - (V(\partial H) \cup \{x\})$, $|L(v)| \geq 5$; and

(c) $|L(x)| \geq 4$.

If Proposition 2.2.4 is true for $H - V(\partial H)$, then H has an L-colouring.

Proof. Let H be a minimum counterexample. We may assume that there is no vertex in the interior of ∂H adjacent to all the vertices of ∂H. If there is such a vertex, we can colour that vertex then by minimality extend the colouring to the interiors of each one of the three triangles it creates with its adjacencies.

Delete from the lists of the vertices in the interior of ∂H the colours of their neighbours in ∂H. We colour the interior of ∂H with this new list assignment as described below.

Since every vertex in the interior of ∂H is adjacent to at most two vertices in ∂H, every vertex in the outer walk of a block in $H - V(\partial H)$ has at least three colours in its list, except for x, which may have a list of size two. Thus, any such block is colourable either by Thomassen's Theorem 3.2.6 or by Proposition 2.2.4 as a start.

Now we describe how the colouring proceeds. Start by colouring a block containing x by Proposition 2.2.4 or Theorem 3.2.6 of Thomassen, depending on whether x is an inner vertex of the block or on its outer walk. Then move to colour an uncoloured block containing an already coloured vertex by Theorem 3.2.6 of Thomassen. This shows how to colour a component containing x.

To colour a component not containing x, we can start by colouring any block in the component and then move to an uncoloured block containing an already coloured vertex.

Note that we should not move from a coloured block to one that has no coloured vertices in the same component since then when we return to colour an adjacent block to the first block, it has two coloured vertices.

□

Lemma 2.2.6. *Let H be a plane graph such that ∂H is a 4-cycle and let x be a vertex of $H - V(\partial H)$. Let φ be a colouring of ∂H and let L be a list assignment on H such that:*

(a) for every vertex v of ∂H, $L(v) = \{\varphi(v)\}$;

(b) for every vertex v of $H - (V(\partial H) \cup \{x\})$, $|L(v)| \geq 5$; and

(c) $|L(x)| \geq 4$.

If Proposition 2.2.4 is true for $H - V(\partial H)$, and x is not adjacent to all the vertices of ∂H, then H has an L-colouring.

Proof. If $H - V(\partial H)$ does not contain a block containing x as an inner vertex that contains two vertices each adjacent to three vertices of C, we colour $H - V(\partial H)$ as follows. Start by colouring a block containing x and then move to an uncoloured block containing an already coloured vertex.

In colouring the different blocks we use Theorem 3.2.6 of Thomassen, Theorem 3.2.7 of Compos and Havet, or Proposition 2.2.4.

Now we show how to colour a block with two vertices each adjacent to three vertices of C if it contains x as an inner vertex. This is the same as colouring a plane graph with two 2-lists on the outer cycle, all the other lists on the outer cycle are 4-lists, one inner 4-list, and all the other inner lists are 5-lists.

There are two possibilities. If one of the 2-lists is not in any chord of the block, we can colour it, delete it, then colour the smaller block, which has only one vertex with less than three colours on its outer cycle (and so is colourable by Proposition 2.2.4). If both 2-lists lie on chords of the block, colour the vertex that has a 2-list on that chord (or one of them if the chord has the two 2-lists as its end-vertices), delete it, then colour the smaller blocks, moving from a block to an adjacent one, using Theorems 3.2.6, 3.2.7. □

We also need the following lemma for the proof of Proposition 2.2.1.

Lemma 2.2.7. *If T is a separating triangle in $G - \{e_1, e_2\}$, then each of e_1 and e_2 has one end-vertex in the interior of T and the other in its exterior.*

Proof. Suppose for a contradiction that there is a separating triangle in $G - \{e_1, e_2\}$, and note that a separating triangle in $G - \{e_1, e_2\}$ may, in G, have one of its edges crossed by either e_1 or e_2.

Let T be a separating triangle in $G - \{e_1, e_2\}$, and let G_1 and G_2 be the subgraphs of G induced by $V(T)$ and the vertices in the exterior and the interior of T. Choose the labeling so that G_1 contains at least as many of e_1,e_2 as G_2 does. Furthermore, we may assume that, if either e_1 or e_2 is contained in either of G_1 or G_2, that e_1 is contained in G_1.

Recall that G is a minimum counterexample to Theorem 2.1.1. Therefore, we can colour G_1 by minimality. We have the following cases:

(a) *e_1 and e_2 are both contained in G_1.*

The subgraph $G_2 - V(T)$ is planar and there is at most one vertex in it that is adjacent to all the three vertices of T. After deleting from the list of every vertex v of $G_2 - V(T)$ the colours of the vertices in $N(v) \cap V(T)$, Thomassen's Theorem 3.2.6 shows $G_2 - V(T)$ has an L-colouring extending that of G_1 to all of G.

(b) *e_1 is contained in G_1 but e_2 is not contained in any of G_1 or G_2.*

Since e_2 is not contained in any of G_1 or G_2, it does not have an end-vertex in T. Assume without loss of generality that the end-vertex of e_2 in $V(G_1) - V(T)$ is u_2 and the end-vertex in $V(G_2) - V(T)$ is v_2. Delete from $L(v_2)$ the colour of u_2. Then now we have a coloured triangle with interior (or exterior) consisting of vertices that have lists of size at least 5 except for one vertex that has a list of size at least 4. The interior of such a triangle is colourable by Lemma 2.2.5.

(c) *e_1 is contained in G_1 and e_2 is contained in G_2.*

In this case there are at most two vertices of $\{u_1, u_2, v_1, v_2\}$ in the interior of T. Let z and w be two vertices of $\{u_1, u_2, v_1, v_2\}$ in the interior of T. Note that each of z and w still has a list of size at least 4 after deleting the colour of its neighbour in the subgraph induced by the two edges e_1 and e_2.

Since in this case there is symmetry between G_1 and G_2, we may assume without loss of generality that G_1 is the subgraph induced by the vertices in the exterior of T and $V(T)$.

We may also assume that T does not contain any other separating triangles. Therefore, if there is a vertex in the interior of T that is adjacent to all the three vertices of T, then it is the only vertex in the interior of T. In this case the interior of T is colourable as this vertex has in its list a colour different from the colours of the three vertices of $V(T)$ and the colour of its possible unique neighbour in the exterior of T.

Therefore, we may assume that every vertex in the interior of T is adjacent to at most two vertices in $V(T)$. Delete from the lists of the vertices in the interior of T the colours of their neighbours in G_1. Then every vertex in the interior of T, including z and w, has a list of size at least three. This is true for z and w because they have no neighbours in the exterior of T. If any of z and w has a neighbour in the exterior of T, then one of the edges of T is e_1 or e_2, but this returns us to part (a) where the two edges are contained in G_1.

Now we can extend the colouring to the interior of T by Theorem 3.2.6 of Thomassen.

We have shown that we can colour G in all the cases, and so we have a contradiction. □

2.2.1 Colouring vertices in Q

Here we prove Proposition 2.2.1.

Proof. Recall that $Q = z_0 z_1 \cdots z_n z_{n+1}$, where $z_0 = u_1$ and $z_{n+1} = u_2$. Colour z_0 by any colour $\varphi(z_0)$ in $L(z_0)$ and then colour z_1 by any colour $\varphi(z_1)$ in $L(z_1) \setminus \{\varphi(z_0)\}$. Suppose that, for some $k \in \{2, 3, \cdots, n+1\}$, $\varphi(z_0), \varphi(z_1), \cdots, \varphi(z_{k-1})$ are defined. For $k < n+1$, let $R_k = V_F \cup \{z_{k+1}, \cdots, z_{n+1}\}$, and let $R_{n+1} = V_F$. For a vertex v, let $B_k(v)$ be the list obtained from $L(v)$ by deleting the colours of the neighbours of v in $\{z_0, z_1, \cdots, z_{k-1}\}$. We show below by induction on k that we can choose the colour $\varphi(z_k) \in B_k(z_k)$ $(=L(z_k) \setminus \{\varphi(z_{k-1})\})$ in such a way that $|B_{k+1}(v)| \geq 3$ for every $v \in R_k$.

We have the following two cases.

Case 1. *No vertex of R_k has three neighbours in $\{z_0, \cdots, z_k\}$.*

Then $|B_{k+1}(v)| \geq 3$ for all $v \in R_k$ regardless of how we define $\varphi(z_k)$. In particular, if a vertex $v \in R_k$ is adjacent to three vertices in $\{z_0, \cdots, z_{k-1}\}$ or to at most two vertices in $\{z_0, \cdots, z_{k-1}, z_k\}$, then, regardless of how we define $\varphi(z_k)$, $|B_{k+1}(v)| \geq 3$.

Case 2. *There is a vertex y in R_k that has three neighbours in $\{z_0, \cdots, z_k\}$.*

If y is adjacent to three vertices in $\{z_0, \cdots, z_{k-1}\}$, then Observation 2.2.2 shows that those are all the vertices it is adjacent to in $\{z_0, \cdots, z_{k-1}, z_k\}$. Then, $B_{k+1}(y) = B_k(y)$ and this has at least three colours by the induction hypothesis.

Therefore we may assume that y is adjacent to z_k. Again by Observation 2.2.2, this means that the neighbours of y in Q are z_{k-2}, z_{k-1} and z_k. By planarity of $G - \{e_1, e_2\}$, and since no end vertex of e_1 or e_2 is adjacent to three vertices in Q, there is at most one other vertex w such that w is adjacent to z_{k-2}, z_{k-1} and z_k. We show we can have one of the following:

(1) a recolouring of z_{k-1} and a colour for z_k such that every vertex still has at least three colours, or

(2) a rerouting of Q so that there is at most one vertex adjacent to z_{k-2}, z_{k-1}, and z_k.

For (1): We go back to the step where we were to colour z_{k-1}. Each of y and w is adjacent to only z_{k-2}, z_{k-1} and z_k in Q, therefore, the only coloured neighbour of y and w at this step is z_{k-2}. Thus, each of y and w still has four available colours.

If $L(y) \setminus \{\varphi(z_{k-2})\} = L(z_{k-1}) \setminus \{\varphi(z_{k-2})\} = L(w) \setminus \{\varphi(z_{k-2})\} = S$, then colour z_k by a colour from $L(z_k) \setminus S$. With this colouring, each of y, z_{k-1} and w still has four available colours. Thus, regardless of how we colour z_{k-1}, each of y and w will have three available colours.

If $L(z_{k-1}) \setminus \{\varphi(z_{k-2})\} \neq L(y) \setminus \{\varphi(z_{k-2})\}$ or $L(z_{k-1}) \setminus \{\varphi(z_{k-2})\} \neq L(v) \setminus \{\varphi(z_{k-2})\}$, then there is a colour c in $L(z_{k-1}) \setminus \{\varphi(z_{k-2})\}$ such that either $|L(y) \setminus \{\varphi(z_{k-2}), c\}| \geq 4$ or $|L(v) \setminus \{\varphi(z_{k-2}), c\}| \geq 4$.

Suppose without loss of generality that $|L(y) \setminus \{\varphi(z_{k-2}), c\}| \geq 4$. Then colour z_{k-1} with c. If $|L(v) \setminus \{\varphi(z_{k-2}), c\}| = 3$, then there is a colour d in $L(z_k) \setminus (L(v) \setminus \{\varphi(z_{k-2}), c\})$. Colour z_k with d.

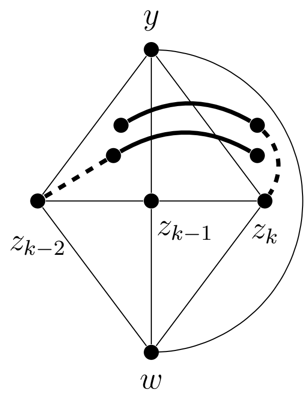


Figure 2.2: The vertices $w_m = w$ and y are adjacent. The dashed lines are parts of Q, and the thick edges are e_1 and e_2.

Now we need to show that every vertex still has a list of size at least three after this recolouring of z_{k-1}. We have already shown this for y and w, and clearly this holds for any vertex adjacent to at most two vertices in $z_0, \cdots, z_{k-1}, z_k$, and for any vertex adjacent to three vertices in $z_0, \cdots, z_{k-2}$. *The only possible obstruction for this is a vertex adjacent to* $z_{k-3}, z_{k-2}, z_{k-1}$.
For (2): Consider the longest sequence $w_1 \cdots w_m$ of vertices such that:

(i) $w_1 = w$;

(ii) for $i \in \{1, \cdots, m\}$, $w_i \neq y$; and

(iii) for every i, w_i is adjacent to z_{k-2}, w_{i-1} and z_k,

If v_m is not adjacent to y, then replace $z_{k-2}z_{k-1}z_k$ by $z_{k-2}w_m z_k$ in Q.

Now that there is only one vertex adjacent to the three vertices in this part of (the new) Q, namely v_{m-1} is adjacent to z_{k-2}, v_m, and z_k, we can choose the colour of z_k to be the unique colour in $L(z_k) \setminus L(w_{m-1})$ Recall that after colouring v_m (the new z_{k-1}), $|L(z_k)| = 4$ while $|L(w_{m-1})| = 3$.

Thus v_m and y are adjacent, then there is no clear way for rerouting Q that will make (2) satisfied. See Figure 2.2. However, we can reroute Q such that (1) is satisfied.

Note that the subgraph induced by $y, z_{k-2}, z_{k-1}, z_k, w, w_1, \cdots w_m$ is a plane graph with every face bounded by a triangle. By Lemma 2.2.7, the end-vertices of e_1 and e_2 are in exactly two of those triangles, T_1 and T_2. Since there are no separating 4-cycles that have all the end-vertices of e_1 and e_2 on one side, T_1 and T_2 intersect in at most one vertex. Also for the same reason m is at most 3, and if T_1 and T_2 are disjoint, then they are distance one apart.

There are a few cases for which triangles are T_1 and T_2, with the three possible values for m. In each of those cases it is not hard to show there is a rerouting of Q such that: if there is a vertex adjacent to z_{k-3}, z_{k-2}, and z_{k-1} (the new one), then either there is a crossing avoiding e_1 and e_2, or there is a shorter path than Q. Then, since $G - \{e_1, e_2\}$ is embedded in the plane, there is no vertex adjacent to z_{k-3}, z_{k-2}, and z_{k-1}. This was the only problematic situation for (1).

For example, in Figure 2.2, if we replace $z_{k-2}z_{k-1}z_k$ by $z_{k-2}wz_k$ in Q, then by planarity, the only vertex that can be adjacent to all of z_{k-3}, z_{k-2}, and w is z_{k-1}. This gives a shorter path than Q as we can replace $z_{k-3}z_{k-2}z_{k-1}$ by $z_{k-3}z_{k-1}$.

□

2.2.2 Colouring G'

To know whether we can colour G', defined in Page 7, after colouring Q as described above and deleting the colours of $V(Q)$ from the lists of their neighbours, we need to know the answer to the following question.

Question 2.2.8. *Let H be a plane graph and let x and y be two distinct vertices in $V(H) \setminus V(\partial H)$. Let L be a list assignment such that:*

(a) $|L(x)| = |L(y)| = 4$;

(b) for every vertex $v \in \partial H$, $|L(v)| \geq 3$; *and*

(c) otherwise, $|L(v)| \geq 5$.

Does H have an L-colouring ?

We do not know the answer to this question. However, in Section 4.1, we prove Proposition 2.1.2, which states that it is true in the special case that each of x and y is the centre of a wheel attached to the outer walk of H.

So we may suppose that in G' at least one of v_1 and v_2 is not the centre of a wheel whose outer cycle is attached to the boundary of F. We may assume without loss of generality that v_1 satisfies this. This includes also the case when v_1 is in V_F.

In this case, with a slight modification described below to the colouring of Q described above, we come to a list assignment L' of G' such that, for some two vertices w_1 and w_2 in V_F:

(a) if $v_1 \notin V_F$, then $|L'(v_1)| \geq 4$;

(b) for every $v \in V_F \setminus \{w_1, w_2\}$, $|L'(v)| \geq 3$; and

(c) $|L'(w_1)| \geq 2$ and $|L'(w_2)| \geq 2$,

Theorem 2.1.3 states that G' is L'-colourable, and is proved in Section 4.4.

Let us for the moment call the situation when, in a plane graph, the vertices on its outer boundary have 3-lists and the other vertices have 5-lists, the *primary situation*. A plane graph in the primary situation is known to be colourable by Thomassen's Theorem 3.2.6.

Note that in the list assignment of Question 2.2.8, the total number of colours lost from the primary situation is 2, one lost at x and one lost at y. In the list assignment L' (above), the total number of colours lost is 3 (in case $v_1 \notin V_F$). However, the question of L'-colourability of G' is less difficult than that since we added the condition that the unique 4-list vertex (if exists) is not contained in a certain structure (not the centre of a wheel attached to the boundary of F).

The conclusion of this short comparison between those two list colouring problems, Question 2.2.8 and Theorem 2.1.3, is that those two problems almost have the same rank of difficulty. One reason why we found the latter easier is that there is a ready proof to try to make an adaptation of, that is the proof of Theorem 3.2.8 of Postle and Thomas for which Theorem 2.1.3 is an extension.

Now we show how to come to the list assignment L' of G' with the properties mentioned above.

Colour Q as described above, and then uncolour u_2. Now u_2 has four available colours, since by Observation 2.2.3, the only neighbour of u_2 in Q is z_n. Also v_2 still has five colours if it is not in V_F.

Note also that at most two of the neighbours of u_2 on the boundary of F have neighbours in Q other than u_2 (because $G - \{e_1, e_2\}$ is embedded in the plane).

This means that for all neighbours y of u_2 on the boundary of F, except possibly two, $|B_{n+1}(y)| \geq 5$. If $v \in V_F$ is adjacent to u_2 and other vertices in Q, we know from the construction in the proof of Proposition 2.2.1 that $|B_{n+1}(v)| \geq 3$.

Let w_1 and w_2 be two vertices in V_F such that $|B_{n+1}(w_1)| \geq 3$ and $|B_{n+1}(w_2)| \geq 3$. For every $i \in \{1, 2\}$, let a_i be a colour in $B_{n+1}(u_2) \setminus B_{n+1}(w_i)$ if $|B_{n+1}(w_i)| = 3$, and let it be any colour in $B_{n+1}(u_2)$ otherwise. If $a_1 \neq a_2$, let $S = \{a_1, a_2\}$, and if $a_1 = a_2$, let b be any colour different from a_1 in $B_{n+1}(u_2)$, and let $S = \{a_1, b\}$. In any case, for every $i \in \{1, 2\}$, either $|B_{n+1}(w_i)| \geq 4$ or there is at most one colour in $S \cap B_{n+1}(w_i)$.

Therefore, if we delete the colours in S from the lists of the neighbours of u_2 different from v_2, we have at most two 2-lists on the boundary of F. All the other vertices on the boundary of F have 3-lists and $L(v_2)$ is still a 5-list.

Now since v_1 is not the centre of a wheel whose outer cycle is attached to the boundary of F, there is a colouring φ of G' by Theorem 2.1.3 if it is true. Then colour u_2 with a colour in $S \setminus \{\varphi(v_2)\}$.

Chapter 3

Preliminaries

3.1 Introduction

Our extension Theorem 4.3.1 of Thomassen's Theorem 3.2.4 asserts that as long as there is no exceptional configuration, G has an L-colouring. Although our result has more exceptions, Thomassen already had to deal with some. Fortunately, ours are also 'wheel-like' structures that attach to the outer boundary.

The purpose of this chapter is to thoroughly analyze the exceptional configurations that occur in Theorem 4.3.1.

We begin by recalling the main results of Thomassen, Compos and Havet, and Postle and Thomas. Then we introduce Thomassen's exceptions, the 'generalized wheels'. We will need a complete understanding of the list assignments L of these exceptions that do not yield L-colourings. The most basic and important example is a 'broken wheel', which is fully analysed in Section 3.2.

In Section 3.3, we discuss material from Postle [10] that gives us as a direct consequence the ability to extend a single pre-coloured vertex on the outer walk of a plane graph to a complete colouring of the graph. Here it is important to show that we can do so to avoid a particular colouring of some other path of length one that is also on the outer walk. The avoided colouring is one that does not extend to a colouring of some generalized wheel in the original graph. This combines with the analysis

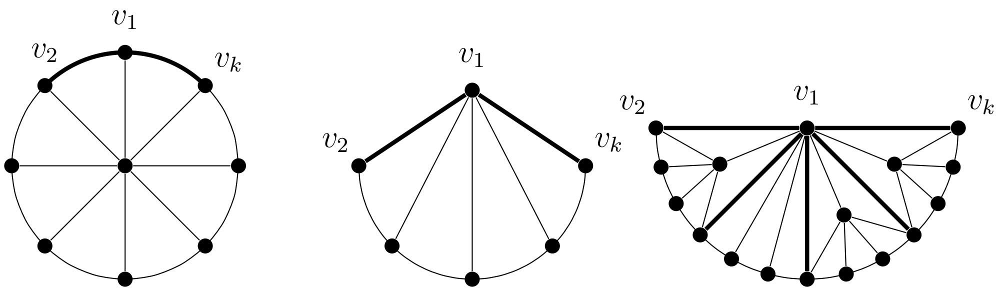


Figure 3.1: generalized wheels with principal path $v_2 v_1 v_k$.

of the generalized wheels to show that there is always an extension to the whole graph.

3.2 Wheel-Like Structures

In this section we introduce a number of wheel-like structures that appear as exceptions to colouring. We recall several previous results concerning list colourings of plane graphs, culminating in our Lemma 3.2.11. This result completely determines the list assignments L of a broken wheel W for which there is no L-colouring of W.

Thomassen [13] provided the first example of a theorem of the form 'either there is an L-colouring or there is an exception'. This is the model for our Theorem 4.3.1 and is used repeatedly in our proofs. We state his theorem below after the following relevant definitions.

Definition 3.2.1. [13] (Broken Wheel) A *broken wheel* is a graph that consists of a cycle $C = v_1 v_2 \cdots v_k v_1$ and, for all $i = 3, 4, \cdots, k-1$, the edge $v_1 v_i$. The vertex v_1 is called the *major vertex* and the path $v_2 v_1 v_k$ is called the *principal path* of the wheel.

See Figures 3.3, 3.4, and 3.5 for examples of broken wheels.

Definition 3.2.2. The broken wheel is *even* or *odd* if the length of its outer walk is even or odd, respectively.

Definition 3.2.3. [13] (Generalized Wheel) A graph G is a *generalized wheel* with *principal path uvw* if G is either a wheel, a broken wheel, or the union of two generalized wheels G_1 and G_2 with principal paths uvz and zvw, respectively, such that $G_1 \cap G_2$ is just the path vz.

See Figure 3.1 for examples of generalized wheels.

Theorem 3.2.4. *(Thomassen [13]) Let G be a plane graph such that ∂G is a cycle $v_1 v_2 \cdots v_k v_1$. Let φ be a colouring of $P := v_2 v_1 v_k$, and let L be a list assignment such that:*

(a) for $i \in \{1, 2, k\}$, $L(v_i) = \{\varphi(v_i)\}$;

(b) for $i \in \{3, 4, \cdots, k-1\}$, $|L(v_i)| \geq 3$; and

(c) otherwise, $|L(v)| \geq 5$.

Then either G has an L-coloring or G contains a subgraph G' such that:

(1) G' is a generalized wheel with principal path P;

(2) $V(\partial G') \subseteq V(\partial G)$; and

(3) for all $v \in V(\partial G') \setminus V(P)$, $L(v)$ is of size exactly 3.

From generalized wheels we define another wheel-like structure that we call a *wheel of wheels*. The notion of *double bellows* introduced in [10, P. 51] includes some, but not all, of our wheels of wheels.

Definition 3.2.5. (Wheel of Wheels) A graph G is a *wheel of wheels* if it is obtained from two generalized wheels by identifying their principal paths. The special case when one of the two generalized wheels is a broken wheel and the other one is a wheel is called a *double-centred wheel*.

See Figures 3.2, 4.3, and 4.6 for examples of wheels of wheels.

In his breakthrough 1994 paper [12] proving 5-choosability of planar graphs, Thomassen introduced what is now the standard approach to proving other 5-list-colouring theorems for planar graphs. He proved it by proving the following stronger theorem.

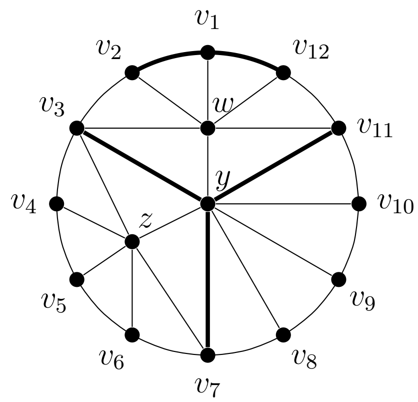


Figure 3.2: A wheel of wheels with centre y and three sections, a broken wheel and two wheels with centres w and z

Theorem 3.2.6. *(Thomassen [12]). Let G be a plane graph and $P = v_1v_2$ a path of length one contained in ∂G. Let L be a list assignment for G such that:*

(a) for all $v \in V(G) \setminus V(\partial G)$, $|L(v)| \geq 5$;

(b) for all $v \in V(\partial G) \setminus V(P)$, $|L(v)| \geq 3$; and

(c) $L(v_1)$ and $L(v_2)$ are unequal singletons.

Then G is L-colourable.

In 2011, Compos and Havet [4] proved a variation of Thomassen's result in which the vertices with singleton lists are not adjacent.

Theorem 3.2.7. *(Compos and Havet [4]) Suppose G is a plane graph and x, y and z are three distinct vertices in ∂G. Let L be a list assignment such that:*

(a) for all $v \in V(G) \setminus V(\partial G)$, $|L(v)| \geq 5$;

(b) for all $v \in V(\partial G) \setminus \{x, y, z\}$, $|L(v)| \geq 4$;

(c) $L(x) \neq L(y)$, $|L(z)| \geq 3$; and

(d) $L(x)$ and $L(y)$ are singletons that are unequal in case x and y are adjacent.

Then G is L-colourable.

In 2015, Postle and Thomas published the following theorem which solves the situation when there are two lists of size 2. This theorem implies Theorem 3.2.6 of Thomassen. It is also one of our main tools. We introduced an extension of this theorem, Theorem 2.1.3, that we need for the proof of the main theorem of this thesis.

Theorem 3.2.8. *(Postle and Thomas [11]) Let G be a plane graph, and let v_1 and v_2 be distinct vertices in ∂G. Let L be a list assignment for G such that:*

(a) for all $v \in V(G) \setminus V(\partial G)$, $|L(v)| \geq 5$;

(b) for all $v \in V(\partial G) \setminus \{v_1, v_2\}$, $|L(v)| \geq 3$; and

(c) $|L(v_1)| = |L(v_2)| = 2$.

Then G is L-colourable.

We will prove in Lemma 4.2.1 an analogue of the following lemma of Thomassen [13]. In our case we need one inner 4-list and non-extendable colourings of a path of length one.

Definition 3.2.9. Let H be a subgraph of a graph G, L a list assignment of G, and φ an L-colouring of H. The colouring φ is *good* if it is extendable to an L-colouring of G and *bad* otherwise.

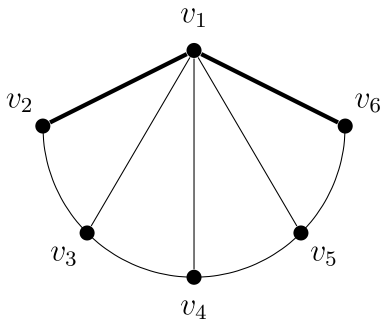
The following lemma is a rephrasing of [13, Lemma 1].

Lemma 3.2.10. *Assume G is a generalized wheel that is not a broken wheel, with outer cycle $C : v_1v_2 \cdots v_kv_1$. Let L be a list assignment of G such that:*

(a) for all $v \in V(G) \setminus V(C)$, $|L(v)| \geq 5$; and

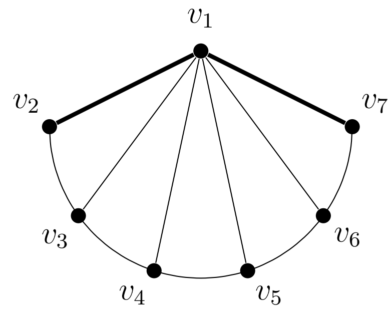
(b) for all $v \in V(C)$, $|L(v)| \geq 3$.

Then there is at most one coloring of the path $v_kv_1v_2$ that cannot be extended to an L-coloring of G.



$L(v_1) = \{1\}$,
$L(v_2) = \{3\}$,
$L(v_3) = \{1, 2, 3\}$,
$L(v_4) = \{1, 2, 3\}$,
$L(v_5) = \{1, 2, 3\}$ and
$L(v_6) = \{2\}$.

Figure 3.3: All colourings of the principal path that are permutations of $\{1, 2, 3\}$ are unextendable while all the colourings $1, 2, 1$, $1, 3, 1$, $2, 1, 2$, $2, 3, 2$, $3, 1, 3$ and $3, 2, 3$ are extendable. Note that the outer cycle is even.



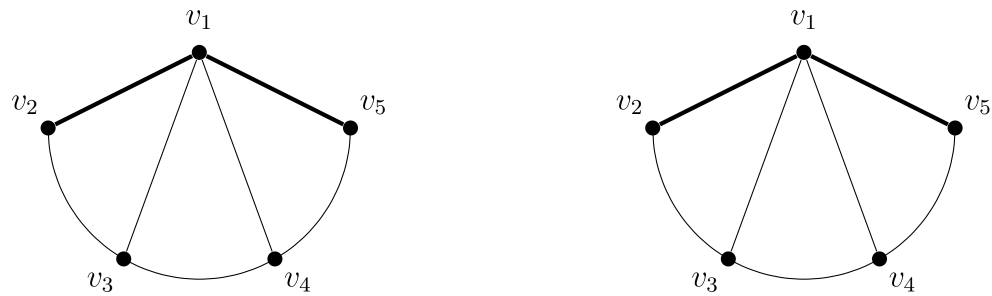
$L(v_1) = \{1\}$,
$L(v_2) = \{3\}$,
$L(v_3) = \{1, 2, 3\}$,
$L(v_4) = \{1, 2, 3\}$,
$L(v_5) = \{1, 2, 3\}$,
$L(v_6) = \{1, 2, 3\}$ and
$L(v_7) = \{2\}$.

Figure 3.4: All the colourings $1, 2, 1$, $1, 3, 1$, $2, 1, 2$, $2, 3, 2$, $3, 1, 3$ and $3, 2, 3$, of the principal path, are unextendable while all the colourings that are permutations of $\{1, 2, 3\}$ are extendable. Note that the outer cycle is odd.

Lemma 3.2.10 is about generalized wheels that are not broken wheels. Here we prove the following lemma about the unextendable colourings of the principal path of a broken wheel.

When we say that abc is a colouring of the path $u_1u_2u_3$ or that the path $u_1u_2u_3$ is coloured abc we mean that u_1, u_2, and u_3 are given the colours a, b, and c, respectively. Now, when a, b, and c are different, we can regard the colouring abc as a permutation of $\{a, b, c\}$.

Lemma 3.2.11. *Let W be a broken wheel with outer cycle $v_1v_2 \cdots v_kv_1$ and principal path $P := v_2v_1v_k$, and let L be a list assignment of W such*



$L(v_1) = \{2\}$,
$L(v_2) = \{1\}$,
$L(v_3) = \{1, 2, 4\}$,
$L(v_4) = \{2, 3, 4\}$ and
$L(v_5) = \{3\}$.

$L(v_1) = \{4\}$,
$L(v_2) = \{1\}$,
$L(v_3) = \{1, 2, 4\}$,
$L(v_4) = \{2, 3, 4\}$ and
$L(v_5) = \{3\}$.

Figure 3.5: The only difference between the two list assignments is the colour of the middle vertex of the principal path. Both colourings of the principal path are unextendable.

that, for every $i \notin \{1, 2, k\}$, $L(v_i) \geq 3$. *If there is more than one bad colouring for* P*, then all the bad colourings are from one of the following five cases.*

(1) *They are all the permutations of a fixed* 3*-set* S*. In this case,* W *is even, all the lists are equal to* S*, and all the colourings of* P *of the form aba with* a *and* b *having values in* S *are good.*

(2) *They are all the colourings of* P *of the form aba taken from a fixed* 3*-set* S*. In this case,* W *is odd, all the lists are equal to* S*, and all the colourings of* P *that are permutations of* S *are good.*

(3) *They are two colourings cae and cbe that agree on* v_2 *and* v_k *but give* v_1 *different colours.*

(4) *They are two colourings abe and bae that give* v_k *the same colour and alternate the colours of* v_2 *and* v_1*, or they are two colourings cab and cba that give* v_2 *the same colour and alternate the colours of* v_1 *and* v_k*.*

(5) *They are two colourings aba and bab.*

Proof. We start with a helpful claim.

Claim 3.2.12. *If there is a bad colouring for P, then all the lists of $v_3, \cdots, v_{k-1}$ are of size exactly three and every colour of v_1 involved in a bad colouring of P is contained in all those lists. Consequently, at most three colours of v_1 are involved in bad colourings of P. Moreover, if φ and φ' are distinct bad colourings of P such that $\varphi(v_1) = \varphi'(v_1)$, then $\varphi(v_2) \neq \varphi'(v_2)$ and $\varphi(v_k) \neq \varphi'(v_k)$.*

Proof. Let φ be a bad colouring of P and suppose that P is coloured with φ. Colour the vertices from v_3 in ascending order of indices. If some vertex v_i has two colours in its list that are both different from the colours of its coloured neighbours we can stop colouring at this point then start colouring from v_{k-1} in descending order of indices. Then v_i is colourable.

Therefore when P is coloured with φ, in colouring from v_3 in ascending order of indices, each vertex v_i is forced to be coloured by the unique colour in its list different from the colours of v_1 and v_{i-1}. Similarly, in colouring from v_{k-1} in descending order of indices, each vertex v_i is forced to be coloured by the unique colour in its list different from the colours of v_1 and v_{i+1}.

Thus, if there is a bad colouring for P, then all the lists of $v_3, \cdots, v_{k-1}$ are of size exactly three, and every colour of v_1 involved in a bad colouring of P is contained in all those lists. Therefore, at most three colours of v_1 are involved in bad colourings of P.

Suppose φ_1 and φ_2 are colourings of P that agree on v_1, v_k, but differ on v_2. Starting with v_{k-1}, all the vertices will have their colours forced in both colourings. Thus, at most one of φ_1 and φ_2 can be bad.

Similarly, if two colourings of P agree on the colours of v_1, v_2 but differ on v_k, then at most one of them is bad. □

Now, we have the following cases.

Case 1: *There are three colours of v_1 involved in bad colourings.*

Let a, b, c be the three colours. Then all the vertices $v_3, \cdots, v_{k-1}$ have the same list $S := \{a, b, c\}$. Any colouring that gives v_2 a colour not in S

is good since we can colour the vertices from v_{k-1} in descending order of indices and then v_3 is colourable since the colour of one of its neighbours, namely v_2, is not in $L(v_3)$. Similarly any colouring that gives v_k a colour not in S is good. Therefore, all the bad colourings give v_2, v_1 and v_k colours from S.

Now consider any colouring of P with colours from S. We may suppose without loss of generality that v_1 is coloured a and v_2 is coloured b. If we colour the vertices from v_3 in ascending order of indices, the vertices with an odd index are coloured c and the vertices with an even index are coloured b. Therefore, if W is even, then the colouring bab is good and the colouring bac is bad; this is (1). However, if W is odd, then the colouring bab of P is bad and the colouring bac is good; this is (2).

Case 2: *P has more than one bad colouring and at most two colours of v_1 are involved in bad colourings of P.*

Let φ and φ' be two bad colourings of P. We show that $\varphi(v_1) \neq \varphi'(v_1)$. Suppose for a contradiction that $\varphi(v_1) = \varphi'(v_1)$. Then by Claim 3.2.12, $\varphi(v_2) \neq \varphi'(v_2)$ and $\varphi(v_k) \neq \varphi'(v_k)$. Suppose that φ and φ' are b_1ac_1 and b_2ac_2 respectively. Since both b_1ac_1 and b_2ac_2 are bad colourings, $L(v_3) = \{a, b_1, b_2\}$ and $L(v_{k-1}) = \{a, c_1, c_2\}$.

By considering colouring from v_3 in ascending order of indices in both cases, when P is coloured b_1ac_1 and b_2ac_2, we find, since both colourings are bad and the colour of each v_i is forced by the colour of v_{i-1}, that all the lists are equal to $\{a, b_1, b_2\}$. Therefore, $\{b_1, b_2\} = \{c_1, c_2\} = \{c, b\}$ for some b and c, φ and φ' have values in the 3-set $S := \{a, b, c\}$ and all the lists are equal to S.

Now it is not hard to see that, depending on the parity of W, either all the permutations of S are bad colourings of P or all the colourings of the form $\theta\lambda\theta$ taken from S are bad colourings of P. In either case this means that there are 3 colours of v_1 involved in bad colourings of P. This contradicts our assumption that there are at most two colours of v_1 involved in bad colourings of P.

We conclude there are exactly two bad colourings φ and φ' of P and that $\varphi(v_1) \neq \varphi'(v_2)$.

Let a and b be the two colours of v_1 involved in bad colourings of P and suppose that c_1ae_1 and c_2be_2 are the two bad colourings. Since $L(v_3)$ contains both a and c_1 as well as b and c_2, $L(v_3) = \{a, b, c\}$ for some $c \notin \{a, b\}$, ($c_2 = c$ or $c_2 = a$) and ($c_1 = b$ or $c_1 = c$). Similarly, $L(v_{k-1}) = \{a, b, e\}$ for some $e \notin \{a, b\}$, ($e_1 = b$ or $e_1 = e$) and ($e_2 = a$ or $e_2 = e$). Note that c and e may be equal.

Thus the different possibilities of the two bad colourings can be viewed as the elements of $\{cae, bae, cab, bab\} \times \{cbe, cba, abe, aba\}$ (that is, the two bad colourings may be one of the 16 pairs in this Cartesian product). The ones that belong to one of the cases in the statement of the theorem are (cae, cbe), (bae, abe), (cab, cba), and (bab, aba). Those are respectively cases (3), (4), (4), and (5) in the statement of the Lemma.

We can partition the remaining possibilities of the two bad colourings into groups as follows (in all four cases $\{\theta, \lambda\} = \{a, b\}$):

(i) either v_2 or v_k has the same colour in both colourings, that is, $\{(cae, cba), (cae, abe), (bae, cbe), (cab, cbe)\}$;

(ii) one of the two colourings is of the form $\theta\lambda\theta$ and the other is either $c\theta\lambda$ or $\lambda\theta e$, that is, $\{(bae, aba), (cab, aba), (bab, abe), (bab, cba)\}$;

(iii) one of the two colourings is of the form $\theta\lambda\theta$ and the other is $c\theta e$, that is, $\{(cae, aba), (bab, cbe)\}$;

(iv) one of the two colourings is of the form $c\theta\lambda$ and the other is $\theta\lambda e$, that is, $\{(bae, cba), (cab, abe)\}$.

We can take (cae, cba), (bab, abe), (cae, aba) and (bae, cba) to be representatives of each of these four groups, respectively.

Note that all the lists of the vertices $v_3, \cdots, v_{k-1}$ contain both a and b. Then, in case P is given a bad colouring that gives v_2 a colour outside $\{a, b\}$, in colouring from v_3 in ascending order of indices, the vertices v_i with i odd are forced to be coloured from $\{a, b\}$ while the vertices v_i with i even are forced to be coloured from outside $\{a, b\}$.

Similarly, in colouring from v_{k-1} in descending order of indices, the vertices v_i with i of a parity different from that of k are forced to be coloured from $\{a, b\}$ while the vertices v_i with i of the same parity as k are forced to be coloured from outside $\{a, b\}$.

In case P is given a bad colouring that gives v_2 a colour in $\{a, b\}$, in colouring from v_3 in ascending order of indices, the vertices v_i with i odd are forced to be coloured from outside $\{a, b\}$ while the vertices v_i with i even are forced to be coloured from $\{a, b\}$.

Similarly, in colouring from v_{k-1} in descending order of indices, the vertices v_i with i of a parity different from that of k are forced to be coloured from outside $\{a, b\}$ while the vertices v_i with i of the same parity as k are forced to be coloured from $\{a, b\}$.

Now note that there are two consecutive non-equal lists $L(v_r) \neq L(v_{r+1})$ (since otherwise all the lists are equal and there are three colours of v_1 involved in bad colourings). Then $L(v_r) = \{a, b, f_r\}$ and $L(v_{r+1}) = \{a, b, f_{r+1}\}$ where $f_r \neq f_{r+1}$. For each group, there are four cases, depending on the parities of k and r. This requires a total of sixteen easy checks that at most one of the two colourings is bad, left to the reader.

□

3.3 Avoiding a Colouring

The main result in this section is Corollary 3.3.9. This result shows we may precolour a vertex and forbid a particular colouring of a path of length one, both in the outer walk, and still have an extension to a colouring of the plane graph. This corollary is a simple consequence of Theorem 3.3.7, below, proved by Postle [10].

Most of this section consists of providing the definitions from [10] that are needed to state Theorem 3.3.7. The concepts introduced here are used later in the thesis, in particular to state Theorem 4.3.1, which is our extension of Thomassen's Theorem 3.2.4 to allow an inner 4-list.

Definition 3.3.1. (Canvas [10]) A triple (G, S, L) is a *canvas* if G is a plane graph, S is a subgraph of ∂G, and L is a list assignment of the vertices of G such that:

(a) for all $v \in V(G) \setminus V(\partial G)$, $|L(v)| \geq 5$;

(b) for all $v \in V(\partial G) \setminus V(S)$, $|L(v)| \geq 3$; and

(c) there exists a proper L-colouring of S.

In this definition, it is possible that two vertices of S are adjacent in G, but not in S. Thus, even if S has a proper L-colouring, it need not be the case that the subgraph of G induced by $V(S)$ has a proper L-colouring.

In this work, we allow one vertex in $V(G) \setminus V(\partial G)$ to have a 4-list. This necessitates one more entry in the definition of canvas. We also say that (G, S, L, x) is a canvas if x is an inner vertex of G (that is not on its outer boundary) such that $|L(x)| = 4$ and $(G - x, S, L)$ is a canvas.

We also need a few slightly different notions of 'subcanvas', given in the following definition.

Definition 3.3.2. Let (G, S, L, x) be a canvas.

(a) A canvas (G', S', L', x) is a *subcanvas* of (G, S, L, x), and a canvas (G', S', L') is a *subcanvas* of (G, S, L) or (G, S, L, x) if:

   i. G' is a subgraph of G such that $V(\partial G') \subseteq V(\partial G)$;

   ii. L' is the restriction of L to the vertices of G'; and

   iii. S' is any subgraph of $\partial G'$ that has a proper L-colouring.

(b) A canvas (G', S', L', x) is a *semi-subcanvas* of (G, S, L, x), and a canvas (G', S', L') is a *semi-subcanvas* of (G, S, L) or (G, S, L, x) if there is a vertex $s \in V(\partial G')$ not in $V(\partial G)$ such that:

   i. G' is a subgraph of G such that $V(\partial G') \setminus \{s\} \subseteq V(\partial G)$;

   ii. L' is the restriction of L to the vertices of G'; and

   iii. S' is any subgraph of $\partial G'$ that has a proper L-colouring.

The right drawing of Figure 4.1 shows a broken wheel semi-subcanvas, namely the graph bounded by $xv_2v_3v_4v_5x$.

(c) For $k \in \{3, 4\}$, a subcanvas or semi-subcanvas (G', S', L') of (G, S, L, x) is *k-restricted* if, for every vertex v in the the intersection of the outer boundaries of G and G', but not in S', $|L(v)| \leq k$.

Theorem 3.3.7 below is concerned with interactions of sets of colourings of two paths P and P' of length one in ∂G. The set $\Phi(P', \mathcal{C})$, of colourings of P' that extend to all of G such that the restrictions to P are in a particular set $\mathcal{C}$, is required to contain a *government* if $\mathcal{C}$ contains a government. Theorem 3.3.7 asserts that, in case $\mathcal{C}$ consists of one colouring, there is only one obstruction - an *accordion* - to the existence of such a government for P'. There are no obstructions in case $\mathcal{C}$ contains a government.

Definition 3.3.3. (Government [10]) Let $\mathcal{C} = \{\varphi_1, \varphi_2, \cdots, \varphi_k\}$, $k \geq 2$, be a collection of distinct colourings of a path $P = p_1 p_2$ of length one. For $p \in P$, let $\mathcal{C}(p)$ denote the set $\{\varphi(p) \mid \varphi \in \mathcal{C}\}$. The collection $\mathcal{C}$ is:

(a) a *dictatorship* if there exists $i \in \{1, 2\}$ such that $\varphi_j(p_i)$ is the same for all $1 \leq j \leq k$, in which case, p_i is the *dictator* of $\mathcal{C}$;

(b) a *democracy* if $k = 2$ and $\varphi_1(p_1) = \varphi_2(p_2)$ and $\varphi_2(p_1) = \varphi_1(p_2)$; and

(c) a *government* if it is either a dictatorship or a democracy.

Definition 3.3.4. (Accordion [10]) A graph G is an *accordion* with ends distinct paths P_1 and P_2 of length one if either:

(a) G is a generalized wheel with principal path $P_1 \cup P_2$; or

(b) G is the union $G_1 \cup G_2$ of two accordions G_1 and G_2 with ends P_1, U and U, P_2, respectively, such that $G_1 \cap G_2 = U$.

Definition 3.3.5. (1-Accordion [10]) Let $T = (G, P, L)$ be a canvas where P is a path of length one and, for all $v \in V(P)$, $|L(v)| = 1$. Let P' be a path of length one in ∂G. Then T is a *1-accordion* from P to P' if G is an accordion whose ends are P and P' and there exists exactly one L-colouring of G.

Definition 3.3.6. [10] Suppose that $T = (G, P, L)$ is a canvas such that P is a path of length one in ∂G, and $\mathcal{C}$ is a collection of L-colourings of P. If P' is another path of length one in ∂G, then $\Phi_G(P', \mathcal{C})$ denotes the collection of colourings of P' that can be extended to a colouring φ of G such that φ restricted to P is a colouring in $\mathcal{C}$. The subscript G is dropped when the graph is clear from context.

Theorem 3.3.7. *[10] Let $T = (G, P, L)$ be a canvas, where P is a path of length one, and let P' be a path of length one distinct from P. Let $\mathcal{C}$ be a non-empty set of L-colourings of P such that, if $|\mathcal{C}| \geq 2$, then $\mathcal{C}$ contains a government. Then $\Phi(P', \mathcal{C})$ does not contain a government if and only if T contains a subcanvas T' such that T' is a 1-accordion from P to P' and $\mathcal{C} = \{\varphi\}$, where φ is the restriction to P of the unique colouring of T'.*

We have the following two corollaries of this theorem. We use the first in the proofs of Theorem 4.4.2 and Lemma 2.1.2, while we use the second in the proof of Theorem 4.3.1.

Corollary 3.3.8. *Let G be a plane graph, and let P and P' be two paths of length one in ∂G. Let L be a list assignment such that:*

(a) for every $v \in V(\partial G)$, $|L(v)| \geq 3$; and

(b) otherwise, $|L(v)| \geq 5$.

If there is a government $\mathcal{C}$ of L-colourings of P, then there exists a government $\mathcal{C}'$ of L-colourings of P' such that every colouring in $\mathcal{C}'$ is extendable to a colouring of G whose restriction to P is in $\mathcal{C}$.

Proof. This is a special case of Theorem 3.3.7. □

Corollary 3.3.9. *Let G be a plane graph, $P = v_1v_2$ a path of length one in ∂G, and z a vertex in $V(\partial G) \setminus V(P)$. Let L be a list assignment for G such that:*

(a) $L(z)$ is a singleton;

(b) for every $v \in V(\partial G) \setminus \{z\}$, $|L(v)| \geq 3$; and

(c) otherwise, $|L(v)| \geq 5$.

If f is an L-colouring of P, then there is an L-colouring of G such that its restriction to P is different from f.

Proof. Let a be the colour of z and let y be a neighbour of z in ∂G. Let $\mathcal{C}$ be the dictatorship consisting of the two colourings of zy having z coloured a and y coloured with different colours from $L(y)\setminus\{a\}$. Theorem 3.3.7 implies $\Phi(P, \mathcal{C})$ contains a government. A government contains at least two colourings, therefore, there is a colouring in $\Phi(P, \mathcal{C})$ different from f. □

Postle also proved a similar theorem to 3.3.7 for unions of two governments; a *confederacy.*

Definition 3.3.10. (Confederacy [10]) Let $\mathcal{C}$ be a collection of colourings of a path $P = p_1p_2$ of length one. Then $\mathcal{C}$ is a *confederacy* if $\mathcal{C}$ is the union of two governments but is not a government.

The *harmonicas* referred to in the following theorem are complicated-to-describe graphs. We will only use this theorem in the form of Corollary 3.3.12, in which case it is clear that the harmonica exception does not arise. Thus, it is not necessary for us to know what a harmonica is here. However, we define harmonicas in the proof of Case 1 of Claim 4.4.11 where we need them.

Theorem 3.3.11. *[10] Let (G, P, L) be a canvas and P, P' be paths of length one in ∂G. Given a collection $\mathcal{C}$ of colourings of P such that $\mathcal{C}$ is either a government or a confederacy, then $\Phi(P', \mathcal{C})$ contains a confederacy unless $\mathcal{C}$ is a government and there exists a subgraph G' of G such that $(G', P \cup P', L)$ is a harmonica from P to P' with government $\mathcal{C}$.*

Corollary 3.3.12. *Let G be a plane graph, and let P and P' be two paths of length one in ∂G. Let L be a list assignment such that:*

(a) for every $v \in V(\partial G)$, $|L(v)| \geq 3$; and

(b) otherwise, $|L(v)| \geq 5$.

If there is a confederacy $\mathcal{C}$ *of* L*-colourings of* P*, then there exists a confederacy* $\mathcal{C}'$ *of* L*-colourings of* P' *such that every colouring in* $\mathcal{C}'$ *is extendable to a colouring of* G *whose restriction to* P *is in* $\mathcal{C}$*.*

Chapter 4

Inner 4-Lists

4.1 Introduction

The main results in this chapter are Theorems 4.3.1 and 4.4.2. Theorem 4.3.1 is an extension of Theorem 3.2.4 of Thomassen, and Theorem 4.4.2 is an extension of Theorem 4.4.1 of Postle and Thomas.

In Theorem 4.3.1 we prove that, if we change the statement of Theorem 3.2.4 to allow one inner 4-list, then more wheel-like structures need to be excluded than the generalized wheels so that the colouring of P extends to G.

In Theorem 4.4.2 we prove that we can change the statement of Theorem 4.4.1 to allow one inner 4-list if we add a few conditions on x. Those conditions are concerned with the adjacencies between x and P and with the situation when x is the centre of a wheel.

In Section 4.2 we prove the lemmas we need for the proofs of the theorems. We prove analogues of Lemma 3.2.10 for wheels, double-centred wheels, and wheels of wheels with centre a 4-list vertex. We also prove Proposition 2.1.2.

In Lemma 3.2.10, Thomassen proved that there is at most one bad colouring for the principal path of a generalized wheel that is not a broken wheel. Here we prove that there is at most one bad colouring of a path of length one on the outer walk of a wheel with centre a 4-list. We also prove that there is at most one bad colouring of a path of length two on the outer walk of a wheel of wheels under certain conditions.

4.2 Lemmas

In this section we prove analogues of Lemma 3.2.10. First we prove in Lemma 4.2.1 that there is at most one bad colouring of a path of length one on the outer walk of a wheel with centre a 4-list vertex. Second we prove in Lemmas 4.2.4 and 4.2.5 that there is at most one bad colouring of a path of length two on the outer walk of a wheel of wheels containing exactly one inner 4-list under certain conditions. Fortunately, the conditions are exceptions that do not occur in a minimum counterexample of Theorem 4.3.1; there should be no separating 4-cycles with interiors consisting of 5-list only, and no separating triangles. We also prove Proposition 2.1.2.

The last lemma in this section, Lemma 4.2.6, is concerned with choosing an appropriate colouring for a path P of length three on the outer walk of a wheel with centre a 4-list. The colouring is chosen so that it extends to the wheel and is chosen from two confederacies for the first and last length-one subpaths of P.

We start with the Lemma about extending a colouring of a path of length one to a wheel with centre a 4-list. The proof is almost the same as the proof of Lemma 3.2.10 of Thomassen. See Figure 4.1.

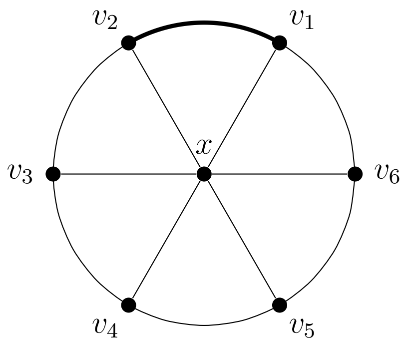
Lemma 4.2.1. *Let G be a wheel with centre x and outer cycle C, and let P be a path of length one in C. Let L be a list assignment such that:*

(a) for all $v \in V(C)$, $|L(v)| \geq 3$; and

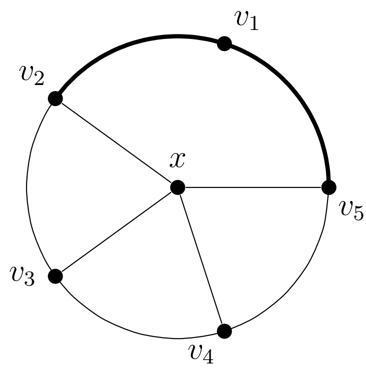
(b) $|L(x)| \geq 4$.

Then at most one colouring of P is unextendable to G.

Proof. Let $C : v_1v_2 \cdots v_kv_1$ and $P = v_1v_2$. Suppose that v_1 and v_2 are coloured $f(v_1)$ and $f(v_2)$ respectively, and that this colouring is unextendable to G. We will show that $f(v_1)$ and $f(v_2)$ are uniquely defined in terms of the other lists.



$L(v_1) = \{1\}$,
$L(v_2) = \{2\}$,
$L(v_3) = \{2, 3, 4\}$,
$L(v_4) = \{3, 4, 5\}$,
$L(v_5) = \{3, 4, 5\}$,
$L(v_6) = \{1, 3, 4\}$ and
$L(x) = \{1, 2, 3, 4\}$.



$L(v_1) = \{1\}$,
$L(v_2) = \{2\}$,
$L(v_3) = \{1, 2, 4\}$,
$L(v_4) = \{1, 3, 4\}$,
$L(v_5) = \{3\}$ and
$L(x) = \{1, 2, 3, 4\}$.

Figure 4.1: Wheels with centre a 4-list. The colouring of the thick path is bad.

First, there are at most two colours in $L(v_3) \setminus \{f(v_2)\}$. Suppose there are more, and let L' be the list assignment of $G - v_1 - v_2$ obtained by deleting from $L(v)$, for every $v \in G$, the colours of its neighbours in P.

Then $|L'(x)| \geq 2$, $|L'(v_k)| \geq 2$ (we may assume that $k > 3$), $|L'(v_3)| \geq 3$ (by assumption) and $L(v) \geq 3$ otherwise. This is colourable by Thomassen (the two 2-lists are adjacent and we can colour them first to have a pre-coloured edge). Similarly, $L(v_k) \setminus \{f(v_1)\}$ consists of exactly two colours. Let $L(v_3) \setminus \{f(v_2)\} = \{\alpha, \beta\}$, and $L(v_k) \setminus \{f(v_1)\} = \{\gamma, \delta\}$.

Now we show that $L(x) \setminus \{f(v_1), f(v_2)\} = \{\alpha, \beta\} = \{\gamma, \delta\}$. Suppose for contradiction that $L(x) \setminus \{f(v_1), f(v_2)\}$ has a colour ϵ distinct from α and β. We can then colour x by ϵ, give v_3 the list $\{\alpha, \beta, \epsilon\}$ and extend the colouring to $G - v_1 - v_2$ by Thomassen, a contradiction. That $\{\gamma, \delta\} = L(x) \setminus \{f(v_1), f(v_2)\}$ too follows by symmetry.

Thus $L(v_3)$ and $L(v_k)$ have precisely two colours in common and $f(v_1)$ is the unique colour in $L(v_k) \setminus L(v_3)$ and $f(v_2)$ is the unique colour in $L(v_3) \setminus L(v_k)$. $\square$

It is now convenient to restate and prove Proposition 2.1.2 since we use Lemma 4.2.1 in the proof. We use Theorem 4.3.1 in the proof as well. Theorem 4.3.1 is stated and proved in Section 4.3 which is dedicated for it. Proposition 2.1.2 is not used in the proof of Theorem 4.3.1, and so there is no vicious circle.

Proposition 2.1.2. *Let G be a plane graph and let x and y be two inner vertices of G that are the centres of wheels W_1 and W_2, respectively, in G. Suppose that, for $i \in \{1, 2\}$, $V(\partial W_i) \subseteq V(\partial G)$. Let L be a list assignment such that:*

(a) *for every $v \in \partial G$, $|L(v)| \geq 3$;*

(b) *$|L(x)| = |L(y)| = 4$; and*

(c) *otherwise, $|L(v)| \geq 5$.*

Then G is L-colourable.

Proof.

Claim 4.2.2. *G is 2-connected.*

Proof. Suppose for a contradiction that G has a cut vertex. If one of the blocks contain both x and y, we colour this block by induction then colour the rest of the graph by Theorem 3.2.6. If x and y are contained in different blocks, we colour the block containing x first by Theorem 4.3.1 (the theorem has no conditions in case the precoloured path is empty), then colour the rest of the graph also by Theorem 4.3.1 (the theorem has no conditions in case the precoloured path consists of one vertex). □

Hence C is a cycle and we may suppose that $C = v_1 \cdots v_k v_1$.

Claim 4.2.3. *There are chords $v_l v_m$ and $v_r v_s$, $l < r < s < m$ such that the subgraph H bounded by $v_l \cdots v_r v_s \cdots v_m v_l$ has all its inner vertices having lists of size at least five, and such that $v_l v_m$ is an edge in the outer cycle of the wheel W_1 with centre x and $v_r v_s$ is an edge in the outer cycle of the wheel W_2 with centre y.*

Proof. Follows from planarity and symmetry of x and y. □

Now by Lemma 4.2.1 there is at most one colouring of v_rv_s unextendable to W_2. Let a be a colour in $L(v_s)$ different from the colour involved in the unique colouring of v_rv_s unextendable to W_2, and let b and c be two different colours in $L(v_r) \setminus \{a\}$. Let $\mathcal{C} := \{\varphi_1, \varphi_2\}$ where φ_1 and φ_2 are two colourings of v_rv_s defined by $\varphi_1(v_s) = \varphi_2(v_s) = a$, $\varphi_1(v_r) = b$ and $\varphi_2(v_r) = c$. Then $\mathcal{C}$ is a dictatorship, that is it contains a government, and so by Corollary 3.3.8, $\Phi_H(v_lv_m, \mathcal{C})$ contains a government.

Again by Lemma 4.2.1, there is at most one colouring of v_lv_m unextendable to W_1. Colour v_lv_m by a colouring from $\Phi_H(v_lv_m, \mathcal{C})$ (recall that a government contains at least two different colourings) different from the unique colouring unextendable to W_1. Then extend that colouring to H such that the colouring of v_rv_s is in $\mathcal{C}$ (we can do this by the definition of $\Phi_H(v_lv_m, \mathcal{C})$). Now colour W_2 and W_1 then colour each of the remaining uncoloured parts of G by Theorem 3.2.6. □

Now we prove the lemmas concerning double-centred wheels and wheels of wheels.

Lemma 4.2.4. *Let W be a double-centred wheel with centres x and y and outer cycle $C := v_1v_2 \cdots v_kv_1$, and let L be a list assignment of W such that:*

(a) for every $v \in V(C)$, $|L(v)| \geq 3$;

(b) $|L(x)| \geq 4$; and

(c) otherwise, $|L(v)| \geq 5$.

Suppose also that:

(i) x is not the centre of a wheel whose outer cycle is a triangle; and

(ii) y is not the centre of a wheel whose outer cycle is a triangle or a 4-cycle.

Then there is at most one bad colouring of $P := v_2v_1v_k$.

Proof. We consider three cases, depending on which vertices of P are adjacent to x and y. See Figures 4.2 and 4.3 for examples of the three cases.

Case 1. *x is adjacent to all vertices of P.*

In this case, there are indices r and s such that $2 \leq r \leq s \leq k$; for $i \in \{1, \cdots, r\} \cup \{s, \cdots, k\}$, v_i is adjacent to x ; and for $i \in \{r, \cdots, s\}$, v_i is adjacent to y.

Note that v_r and v_s are not adjacent since y is not the centre of a wheel whose outer cycle is a triangle. Let φ be a bad colouring of P. By Lemma 3.2.10, there is at most one colouring of $v_r x v_s$ that is unextendable to the wheel with centre y and outer cycle $v_r v_{r+1} \cdots v_s x v_r$.

Therefore, there is exactly one colour in $L(x) \setminus \{\varphi(v_2), \varphi(v_1), \varphi(v_k)\}$, let a denote this colour. Then the lists of the vertices v_3 to v_r are

$$\{\varphi(v_2), a, a_3\}, \{a, a_3, a_4\}, \cdots, \{a, a_{r-1}, a_r\}$$

and the lists of v_{k-1} to v_s are

$$\{\varphi(v_k), a, a_{k-1}\}, \{a, a_{k-1}, a_{k-2}\}, \cdots, \{a, a_{s+1}, a_s\}.$$

Any colouring that gives P a set of colours different from $\{\varphi(v_2), \varphi(v_1), \varphi(v_k)\}$ allows x to be coloured by a colour different from its colour in the unique bad colouring of $v_r x v_s$, and so we consider only colourings of P that permute the colours of φ.

Let φ' be a colouring of P different from φ such that $\varphi(V(P)) = \varphi'(V(P))$. Then either $\varphi'(v_2) \neq \varphi(v_2)$ or $\varphi'(v_k) \neq \varphi(v_k)$. Suppose without loss of generality that $\varphi'(v_2) \neq \varphi(v_2)$. In case P is coloured by φ', x is still forced to be coloured by a but the vertices from v_3 to v_r can now be coloured $\varphi(v_2), a_3, \cdots, a_{r-1}$ instead of $a_3, a_4 \cdots, a_r$ (they are forced to be coloured so when P is coloured by φ). Now the bad colouring of $v_r x v_s$ is avoided (since v_r is coloured differently).

Case 2. *Neither x nor y is adjacent to all vertices of P.*

In this case, there is an index s, $2 < s < k$, such that: for $i \in \{1, \cdots, s\}$, v_i is adjacent to x; and for $i \in \{1\} \cup \{s, \cdots, k\}$, v_i is adjacent to y.

We have two broken wheels W_1 and W_2 with principal paths $P_1 := v_2 x v_s$ and $P_2 := v_k y v_s$ respectively (W_1 is bounded by the cycle $v_2 v_3 \cdots v_s x v_s v_2$ and W_2 is bounded by the cycle $v_k v_{k-1} \cdots v_s y v_k$). Let φ be a bad colouring of P.

When P is coloured by φ, the vertex v_2 in P_1 is coloured $\varphi(v_2)$. Then according to the five possibilities of Lemma 3.2.11, the bad colourings of $x v_s$ for W_1 are a subset of a set of two colourings that:

(a) either alternate the colours of x and v_s, for example $\{ab, ba\}$ (call this the *first type*); or

(b) both give v_s the same colour but change the colour of x, for example $\{ac, bc\}$ (call this the *second type*).

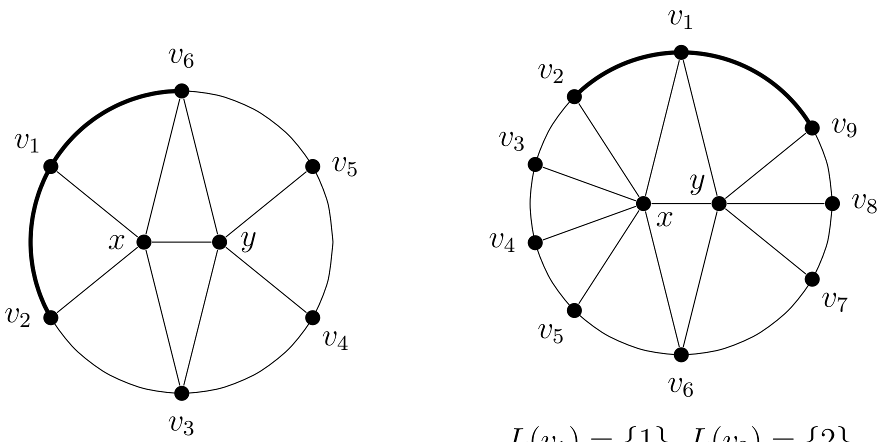
Similarly for $y v_s$ and W_2.

Subcase 2.1. *Both $x v_s$ and $y v_s$ have their bad colourings of the second type, or $s = 3$ (so that v_2 and v_s are adjacent) and $y v_s$ has its bad colourings of the second type.*

In this case, it is two colours of v_s that we want to avoid in order to avoid the bad colourings of $x v_s$ and $y v_s$. Since v_s has a list of size at least three, we can avoid those two colours, colour x, then colour y, and then extend the colouring to W_1 and W_2. This means that φ is not a bad colouring, a contradiction.

Note that we do not have to consider the case when $s = k - 1$ as we considered $s = 3$ since y is adjacent to at least four vertices on C.

Subcase 2.2. *At least one of $x v_s$ and $y v_s$ has its bad colourings of the first type.*



$L(v_1) = \{2\}$, $L(v_2) = \{3\}$, $L(v_3) = \{2, 3, 4\}$, $L(v_4) = \{2, 3, 5\}$, $L(v_5) = \{1, 3, 5\}$, $L(v_6) = \{1\}$, $L(x) = \{1, 2, 3, 4\}$ and $L(y) = \{1, 2, 3, 4, 5\}$.

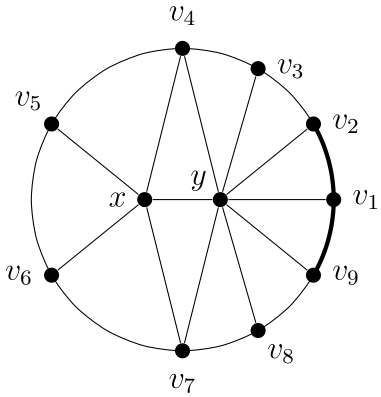
$L(v_1) = \{1\}$, $L(v_2) = \{2\}$, $L(v_3) = \{2, 3, 4\}$, $L(v_4) = \{2, 3, 4\}$, $L(v_5) = \{2, 3, 4\}$, $L(v_6) = \{2, 3, 4\}$, $L(v_7) = \{2, 3, 4\}$, $L(v_8) = \{3, 4, 5\}$, $L(v_9) = \{5\}$, $L(x) = \{1, 2, 3, 4\}$ and $L(y) = \{1, 2, 3, 4, 5\}$.

Figure 4.2: Double-Centred Wheels.

It is easy to see that $L(v_s)$ cannot contain a colour not involved in a bad colouring of yv_s or xv_s (not equal to $\varphi(v_2)$ in case $s = 3$). In case $s \neq 3$, we may assume without loss of generality that each of xv_s and yv_s has two bad colourings (not only one) since otherwise the problem is easier.

Since there are at most two colours of v_s involved in bad colourings of yv_s, there is a colour in $L(v_s)$ that is not involved in a bad colouring of yv_s. This colour either equals $\varphi(v_2)$ in case $s = 3$ or is involved in a bad colouring of xv_s otherwise). Similarly, there is a colour in $L(v_s)$ that is not involved in a bad colouring of xv_s (or is different from $\varphi(v_2)$) but is involved in a bad colouring of yv_s.

Note that in case $s = 3$, any colour of x when v_s is coloured $\varphi(v_2)$ can be counted as involved in a bad colouring. Then whether the bad colourings of xv_s are of the first or the second type or whether $s = 3$, there are at least two colours of x involved in bad colourings. Denote those two colours by a and b.



$L(v_1) = \{2\}$, $L(v_2) = \{1\}$, $L(v_3) = \{1, 4, 5\}$, $L(v_4) = \{1, 4, 5\}$,
$L(v_5) = \{1, 4, 5\}$, $L(v_6) = \{2, 4, 5\}$, $L(v_7) = \{2, 4, 5\}$, $L(v_8) = \{3, 4, 5\}$,
$L(v_9) = \{3\}$, $L(x) = \{1, 2, 4, 5\}$ and $L(y) = \{1, 2, 3, 4, 5\}$.

Figure 4.3: Double-Centred Wheels.

We have the following four cases.

(i) $s = 3$ *and the bad colourings of* yv_s *are of the first type.*

In this case, $L(v_3) = \{\varphi(v_2), c, d\}$ where the bad colourings of yv_s are $\{cd, dc\}$. If $L(x)$ contains c we colour x with c, colour v_3 with d, then colour y with a colour different from c and d. Then the colouring of $v_k y v_s$ is extendable to W_2. Thus we may assume that $L(x)$ does not contain c and similarly does not contain d, i.e., each of a and b is different from c and d.

(ii) *Both* xv_s *and* yv_s *have their bad colourings of the first type.*

Suppose that the bad colourings of yv_s are $\{cd, dc\}$. Since we assumed that a and b are two colours of x involved in bad colourings of xv_s and the bad colourings of xv_s are of the first type, those bad colourings are ab and ba.

Recall that $L(v_s)$ contains a colour that is not involved in a bad colouring of yv_s but is involved in a bad colouring of xv_s. If $L(x)$ contains a colour different from a and b we can colour x with this colour, colour v_s by a or b depending on which of them is in $L(v_s)$ and is different from c and d. Colour y, then extend the colouring to W_1 and W_2. Thus $L(x) \setminus \{\varphi(v_2), \varphi(v_1)\} = \{a, b\}$.

Consider the two cases when $s = 3$ or xv_s has its bad colourings of the first type, while yv_s has its bad colourings of the first type. In case $s = 3$ colour v_s by a colour different from $\varphi(v_2)$. In the second case, colour x by a or by b, and then colour v_s by a colour different from b or respectively a. Then yv_s is forced to be coloured cd or dc. This means that $\{a, b\} \cap \{\varphi(v_1), \varphi(v_k), c, d\} = \emptyset$ and $L(y)$ contains $\varphi(v_1)$, $\varphi(v_k)$, a, b, c and d, a contradiction. To see this recall that c and d are different from $\varphi(v_1)$ and $\varphi(v_k)$ by definition, that is since cd and dc are bad colourings of yv_s when P is coloured by φ.

Thus for at least one of a and b, yv_s can be coloured by a good colouring for W_2 (such that the colour given to v_s together with the colour given to x make a good colouring of xv_s). This means that φ is not a bad colouring, a contradiction.

(iii) *The bad colourings of xv_s are of the second type and of yv_s are of the first type.*

Then there is a colour c such that the bad colourings of xv_s are $\{ca, cb\}$ and there are colours e and f such that the bad colourings of yv_s are $\{ef, fe\}$.

Since $L(v_s)$ does not contain a colour that avoids the bad colourings of both xv_s and yv_s, $c \notin \{e, f\}$ and $L(v_s) = \{c, e, f\}$. As $|L(y)| \geq 5$, there is a colour $d \in L(y) \setminus \{\varphi(v_1), \varphi(v_k), e, f\}$ (note also that $\{e, f\} \cap \{\varphi(v_1), \varphi(v_k)\} = \emptyset$ by the definition of e and f). If $d \neq a$, we can colour y by d, colour x by a then colour v_s by either e or f depending on which of them is different from a. This colouring is extendable to W_1 and W_2, a contradiction. Thus $d = a$ and also by symmetry $d = b$. Thus $a = b$, a contradiction.

(iv) *The bad colourings of xv_s are of the first type and of yv_s are of the second type.*

The bad colourings of xv_s are $\{ab, ba\}$ and there are colours c, e and d such that the bad colourings of yv_s are $\{cd, ce\}$.

It is not hard to see that the assumption that φ is a bad colouring for P implies that $L(v_s) = \{a, b, c\}$, $L(x) = \{\varphi(v_1), \varphi(v_2), a, b\}$ and $\{\varphi(v_1), \varphi(v_k), c, d, e\} \subseteq L(y)$. If c is not in $L(y)$ then by colouring v_s with c we avoid the bad colourings of xv_s and still have three colours in $L(y)$ enough to avoid d and e. Now we consider any colouring φ' of P different from φ and show it is a good colouring.

Suppose for a contradiction that φ' is a bad colouring for P, and let P be coloured by φ'. If $\varphi(v_2) \neq \varphi'(v_2)$, then the set of bad colourings of xv_s is different from what it was in the case when P is coloured φ. Similarly, if $\varphi(v_k) \neq \varphi'(v_k)$, then the set of bad colourings of yv_s is different from what it was in the case when P is coloured φ. Since $L(v_s) = \{a, b, c\}$, in case P is coloured φ', the set of bad colourings of xv_s is either $\{bc, cb\}$ or $\{ca, ac\}$.

We may assume without loss of generality it is $\{ca, ac\}$. Thus a and c are colours of x involved in bad colourings of v_2xv_s when P is coloured φ'. Since a and b are colours of x involved in a bad colouring of v_2xv_s when P is coloured φ, all the lists of the vertices v_i with $3 \leq i \leq s$ are equal to $\{a, b, c\}$. In particular, since $L(v_3) = \{a, b, \varphi(v_2)\}$, $\varphi(v_2) = c$.

Since we assumed that the bad colourings of xv_s are $\{ac, ca\}$ in case P is coloured φ', the colour of v_s involved in the bad colourings of yv_s is b. Since $L(v_{s+1}) = \{c, d, e\}$, $b \in \{d, e\}$. We may assume without loss of generality that $b = d$, and so the bad colourings of yv_s in case P is coloured φ', are bc and be.

Thus e and c are colours of y involved in bad colourings of v_kyv_s when P is coloured φ', and, $d (= b)$ and e are colours of y involved in bad colourings of v_kyv_s when P is coloured φ. Thus all the lists of the vertices v_i with $s \leq i \leq k-1$ are equal to $\{b, c, e\}$. In particular, since $L(v_{k-1}) = \{\varphi(v_k), d, e\}$ (and $d = b$), $\varphi(v_k) = c$.

Now that we know that $\varphi(v_2) = \varphi(v_k) = c$, we see that in the case when P was coloured φ, we could have coloured v_s with c, x with b, and y by a colour different from $\varphi(v_1)$, $c = \varphi(v_k)$ (the colours of v_s and v_k) and $b(= d)$ (the colour of x). This gives v_2xv_s and v_kyv_s colourings extendable to W_1 and W_2, a contradiction to the assumption that φ is a bad colouring of P.

Case 3. *y is adjacent to all vertices of P.*

In this case, there are indices r and s, $2 \leq r \leq s \leq k$, such that: for $i \in \{r, \cdots, s\}$, v_i is adjacent to x; and for $i \in \{1, \cdots, r\} \cup \{s, \cdots, k\}$, v_i is adjacent to y.

Note that v_r and v_s are not consecutive on C since x is not the centre of a triangle, and either $r \neq 2$ or $s \neq k$ since y is not the centre of a 4-cycle. We have a broken wheel W_1 with major vertex x and outer cycle $xv_rv_{r+1} \cdots v_sx$ and a wheel W_2 with centre y and outer cycle $v_1 \cdots v_rxv_sv_{s+1} \cdots v_kv_1$.

Let φ be a bad colouring of P. In every case in Lemma 3.2.11, there are at most four colours that appear in bad colourings of v_rxv_s for W_1. Suppose that the number is at most three, that is there is a set S of size three such that for every bad colouring ψ of v_rxv_s, $\{\psi(v_r), \psi(x), \psi(v_s)\} \subseteq S$. Since $|L(x)| = 4$, there is a colour θ in $L(x) \setminus S$. Since $|L(y)| \geq 5$, there is a colour λ in $L(y)$ different from θ, $\varphi(v_2)$, $\varphi(v_1)$ and $\varphi(v_k)$.

When P is coloured with φ, we can colour y with λ, colour the vertices from v_3 to v_r in ascending order of indices, then colour the vertices from v_{k-1} to v_s in descending order of indices. If v_r or v_s receives a colour not in S, we are done. If both v_r and v_s receive colours in S then we can colour x with θ. This colouring is extendable to W_1, a contradiction.

Therefore there are four distinct colours a, b, c and d such that the bad colourings of v_rxv_s are acb and adb. Recall that either $r \neq 2$ or $s \neq k$. We may assume without loss of generality that $r \neq 2$. Let e and f be two colours in $L(y) \setminus \{\varphi(v_2), \varphi(v_1), \varphi(v_k)\}$.

If when P is coloured φ, v_r is forced to be coloured a whether we colour y with e or with f, then e and f are in the lists of all the vertices v_i with $3 \leq i \leq r$. In particular $L(v_3) = \{\varphi(v_2), e, f\}$.

Thus in this case, there is no third colour in $L(y) \setminus \{\varphi(v_2), \varphi(v_1), \varphi(v_k)\}$ that forces v_r to be coloured a. If there is such a colour, then $L(v_3)$ contains that colour besides $\varphi(v_2)$, e and f, i.e. it has size four, but v_3 has degree three and so we could have coloured v_r differently from a then colour v_3 at the end.

Now we may assume without loss of generality that $L(y) \setminus \{\varphi(v_2), \varphi(v_1), \varphi(v_k)\} = \{e, f\}$ and that both colours force v_r to be coloured a. Now consider any colouring φ' different from φ.

If the set $L(y) \setminus \{\varphi'(v_2), \varphi'(v_1), \varphi'(v_k)\}$ contains a colour g different from e and f that forces v_r to be coloured a then the lists of all the vertices v_i with $3 \leq i \leq r$ are equal to $\{e, f, g\}$, a contradiction. Note that $a \notin \{e, f, g\}$ since v_r receives the colour a when its neighbour y is coloured e, f and g. Therefore we may assume that $L(y) \setminus \{\varphi'(v_2), \varphi'(v_1), \varphi'(v_k)\} = L(y) \setminus \{\varphi(v_2), \varphi(v_1), \varphi(v_k)\} = \{e, f\}$.

Thus φ and φ' are just different permutations of the same three colours among the vertices of P. Then either $\varphi(v_2) \neq \varphi'(v_2)$ or $\varphi(v_k) \neq \varphi'(v_k)$. If $s = k$ and $\varphi(v_k) \neq \varphi'(v_k)$ then the bad colouring of $v_r x v_s$ is avoided. If $s \neq k$ then the argument above with v_r could by symmetry have been done with v_s. Therefore we may assume without loss of generality that $\varphi(v_2) \neq \varphi'(v_2)$.

Thus at least one of e and f does not force v_r to be coloured a when P is coloured φ' since otherwise $L(v_3)$ contains $\varphi'(v_2)$ besides $\varphi(v_2)$, e and f, i.e. has size four, a contradiction. Hence φ' is a good colouring and φ is the only bad colouring for P. □

Note that if a wheel of wheels is not a double-centred wheel then it has a well-defined centre.

Lemma 4.2.5. *Let W be a wheel of wheels that is neither a wheel nor a double-centred wheel. Suppose that W has outer cycle $C := v_1 v_2 \cdots v_k v_1$ and an inner vertex x. Let L be a list assignment of W such that:*

(a) for every $v \in V(C)$, $|L(v)| \geq 3$;

(b) $|L(x)| \geq 4$*; and*

(c) otherwise, $|L(v)| \geq 5$*.*

Suppose also that:

(i) there are no separating triangles; and

(ii) there are no separating 4*-cycles with interior consisting of* 5*-lists only.*

Then there is at most one bad colouring of $P := v_2 v_1 v_k$*.*

Proof. Let φ be a bad colouring of P.

Case 1. *The centre of* W *is* x*.*

Let r and s be the smallest and largest indices respectively such that $r, s \geq 2$ and x is adjacent to v_r and v_s (as 5 and 9 in Figure 4.4). If the subgraph G_1 bounded by $xv_rv_{r+1}\cdots v_sx$ is not a broken wheel with principal path v_rxv_s, then by Lemma 3.2.10 there is at most one colouring of v_rxv_s that is unextendable to G_1.

In that case delete from $L(x)$ the colour of x in that colouring. Again by Lemma 3.2.10 the subgraph G_2 bounded by $v_1v_2\cdots v_rxv_s\cdots v_kv_1$ is colourable, with this new list of $L(x)$, when P is coloured by any colouring different from φ.

If G_1 is a broken wheel with principal path v_rxv_s, then W consists of three sections, since W is neither a wheel nor a double centred wheel. Two of those sections are wheels and together they form G_2, the third section is the broken wheel G_1, and x is adjacent to v_1, v_r and v_s.

Let z_1 be the centre of the wheel bounded by $v_1v_2\cdots v_rxv_1$, and z_2 be the centre of the wheel bounded by $v_1xv_s\cdots v_kv_1$ (as in Figure 4.4). Let W_1 be the broken wheel bounded by $v_rz_1v_2\cdots v_r$ and W_2 the broken wheel bounded by $v_kz_2v_s\cdots v_k$. Note that $v_2z_1v_r$ is the principal path of W_1 and $v_kz_2v_s$ is the principal path of W_2.

Now when P is coloured with φ, there are at most two colourings of z_1v_r that are bad for W_1 and at most two colourings of z_2v_s that are bad

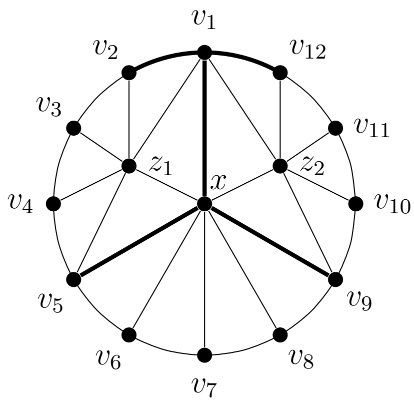


Figure 4.4: A wheel of wheels with centre x and three sections, a broken wheel and two wheels with centres z_1 and z_2.

for W_2. Therefore, there are at least four colourings of z_1v_r that are good for W_1 and at least four colourings of z_2v_s that are good for W_2 (this is true whether v_r is adjacent to v_2 or not and whether v_s is adjacent to v_k or not).

Note that, two colourings of z_2v_s that give v_s different colours have extensions to x such that the resulting colourings of xv_s are still different at v_s. Similarly, two colourings of z_2v_s that give v_s the same colour but give z_2 different colours can be extended by giving different colours to x so that the resulting colourings of xv_s are different.

Note also that, by Lemma 3.2.11, we have two cases for the colours of v_s involved in good colourings of z_2v_s. One case is, there are at least three colours of v_s involved in good colourings of z_2v_s and at least one of those colours is involved in two good colourings of z_2v_s. The other case is, there are only two colours of v_s involved in good colourings of z_2v_s and each of them is involved in at least two good colourings of z_2v_s.

Thus, there is a set of four pairwise distinct colourings of xv_s such that each of those colourings is compatible with at least one of the good colourings of z_2v_s.

There is also a set of four pairwise distinct colourings of G_1 such that each of them is compatible with at least one of the good colourings of xv_s and such that their restrictions to xv_r are pairwise distinct. That the

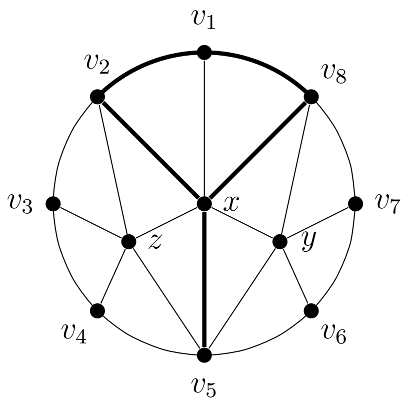


Figure 4.5: $L(v_1) = \{4\}$, $L(v_2) = \{1\}$, $L(v_3) = \{1, 4, 5\}$, $L(v_4) = \{3, 4, 5\}$, $L(v_5) = \{2, 3, 5\}$, $L(v_6) = \{1, 4, 5\}$, $L(v_7) = \{1, 3, 4\}$, $L(v_8) = \{3\}$, $L(x) = \{1, 2, 3, 4\}$ and $L(y) = L(z) = \{1, 2, 3, 4, 5\}$.

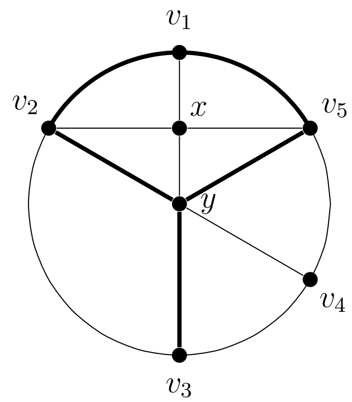
restrictions to xv_r are pairwise distinct can be seen by extending each colouring of xv_s by colouring from v_s to v_r in descending order of indices as long as we are forced (that is have only one colour choice).

If the two colourings of xv_s have different colours of x, then this is still the case in the resulting two colourings of xv_r. On the other hand, if they give x the same colour but give v_s different colours, then, as long as we are forced, we have different colours with both colourings of xv_s at every step along the descending indices. If we have choice at some index, then we can jump to v_r and give it different colours with each colouring of xv_s then go in ascending order of indices until we come back to the vertex were we have choice.

Now from the set of four pairwise distinct colourings of xv_r we have a set of four pairwise distinct colourings of z_1v_r such that each of them is compatible with φ and with at least one of the four colourings of xv_s. Since there are at most two bad colourings of z_1v_r for W_1, at least two of the four colourings we obtained for z_1v_r are good for W_1. Thus we have a colouring of the whole graph.

Case 2. *The centre of W is not x.*

Let y be the centre of W.



$L(v_1) = \{2\}$, $L(v_2) = \{1\}$, $L(v_3) = \{1, 2, 5\}$, $L(v_4) = \{2, 3, 5\}$, $L(v_5) = \{3\}$, $L(x) = \{1, 2, 3, 4\}$ and $L(y) = \{1, 2, 3, 4, 5\}$.

Figure 4.6: Wheels of Wheels

Subcase 2.1. *x is adjacent to all the vertices of P.*

The subgraph $G - v_1$ is not a broken wheel with principal path v_2xv_k since W is not a wheel. Therefore by Lemma 3.2.10 there is at most one colouring of v_2xv_k unextendable to $G - v_1$. Since φ is a bad colouring, there is only one colour c in $L(x) \setminus \{\varphi(v_2), \varphi(v_1), \varphi(v_k)\}$ and $\varphi(v_2)c\varphi(v_k)$ is the unique colouring of v_2xv_k unextendable to $G - v_1$.

Any colouring of P that is different from φ at v_2 or at v_k gives v_2xv_k a colouring different from its unique bad colouring. So consider a colouring ψ of P such that $\psi(v_2) = \varphi(v_2)$, $\psi(v_k) = \varphi(v_k)$ but $\psi(v_1) \neq \varphi(v_1)$. Then $\varphi(v_1) \in L(x) \setminus \{\psi(v_2), \psi(v_1), \psi(v_k)\}$, and so we can colour x with $\varphi(v_1)$, which is a colour different from c, and avoid the unique bad colouring of v_2xv_k.

Subcase 2.2. *x is not adjacent to v_1.*

See Figures 4.7 and 4.8 for examples. Note that x is the centre of a wheel section of W. Then there are indices l and m, $l, m \geq 2$ such that the vertices of C in the wheel section centred at x are v_i such that

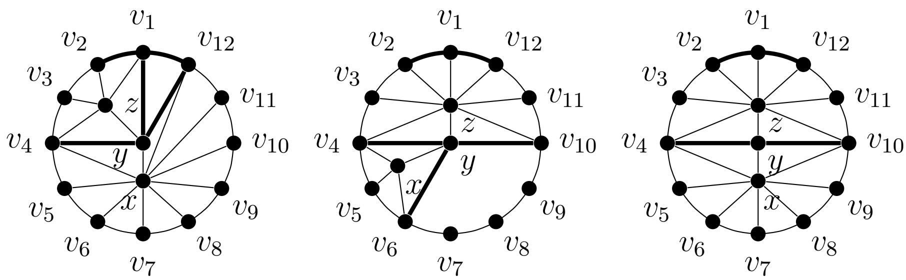


Figure 4.7: The case when x is not adjacent to v_1.

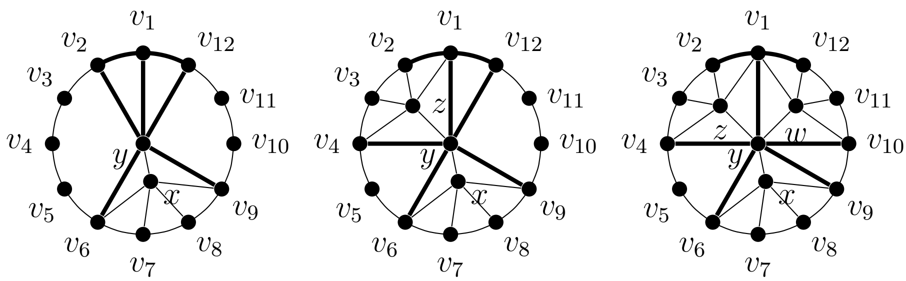


Figure 4.8: The case when x is not adjacent to v_1.

$l \leq i \leq m$. Thus y is adjacent to v_l and v_m.

By Lemma 3.2.11, the colourings of $v_l x v_m$ unextendable to the broken wheel bounded by $xv_l \cdots v_m x$:

(1) either have values in a fixed 3-set; or

(2) are two colourings that give v_l the same colour and give v_m the same colour but give x two different colours.

In case (1), since $|L(x)| = 4$, $L(x)$ contains a colour c that does not appear in any of the bad colourings of $v_l x v_m$ (not as a colour of x nor v_l nor v_m). Now consider the 2-chord $v_l y v_m$, it has P on one side and x on the other side. The side of $v_l y v_m$ containing P is colourable by Theorem 3.2.4 if y is given the list $L(y) \setminus \{c\}$ and P is coloured φ since $|L(y) \setminus \{c\}| \geq 4$.

Then, if this colouring gives v_l or v_m the colour c, the bad colourings of $v_l x v_m$ are avoided, and if it does not, then we still can colour x the colour

c since neither y nor v_l nor v_m is coloured c. Thus the bad colourings of $v_l x v_m$ are avoided and the graph is colourable.

In case (2), suppose that the two bad colourings of $v_l x v_m$ are acb and adb. We need to show that when P is coloured by any colouring different from φ, we can colour the side of $v_l y v_m$ containing P such that either v_l is not coloured a or v_m is not coloured b.

Let r and s be the smallest and largest indices respectively such that $r, s \geq 2$ and y is adjacent to v_r and v_s, and let G' be the subgraph bounded by $v_1 \cdots v_r y v_s \cdots v_k v_1$. The subgraph G' can be:

- a union of two wheels that intersect only in yv_1;
- the union of two triangles that intersect only in yv_1 (as in the leftmost drawing of Figure 4.8);
- the union of a wheel and a triangle that intersect only in yv_1 (as in the leftmost drawing in Figure 4.7); or
- a wheel.

Note that x is not adjacent to v_2 and v_k together since then we will have a separating 4-cycle with interior consisting of 5-lists only.

• *Suppose G' is the union of two wheels.*

Let w be the centre of the wheel with principal path $v_k v_1 y$ (as in the rightmost drawing of Figure 4.8). Let f be the colour of y in the unique colouring of yv_1v_2 unextendable to the subgraph H_1 bounded by $v_1 v_2 \cdots v_l y v_1$.

Consider the subgraph H_2 bounded by $v_{k-1} w y v_m \cdots v_{k-1}$ with the list assignment L' defined as, $L'(v_{k-1}) = L(v_{k-1}) \setminus \{\varphi(v_k)\}$, $L'(w) = L(w) \setminus \{\varphi(v_1), \varphi(v_k)\}$, $L'(y) = L(y) \setminus \{\varphi(v_1), f\}$, $L'(v_m) = L(v_m) \setminus \{b\}$. Note that $m \neq k-1$ since there are no separating 4-cycles whose interior consists of 5-lists only, and $L'(v) = L(v)$ otherwise.

This subgraph, H_2, is L'-colourable by Theorem 3.2.8. Fix an L' colouring of H_2 and note that now the bad colourings of $v_l x v_m$ are avoided since v_m is coloured by a colour different from b. Now colour H_1 (note that v_l and v_m are not adjacent since there are no separating triangles, and so the colourings of H_1 and H_2 are compatible), colour x, then colour the broken wheel bounded by $xv_l \cdots v_m x$.

- *Suppose G' is the union of two triangles.*

Then since W is not a double centred wheel, at least one of the two subgraphs bounded by $yv_2 \cdots v_l y$ and $yv_m \cdots v_k v_k y$ is not a broken wheel. Suppose without loss of generality that the subgraph H bounded by $yv_2 \cdots v_l y$ is not a broken wheel (then in particular it is not a triangle). Then by Lemma 3.2.10 there is at most one colouring of $v_2 y v_l$ that is unextendable to H.

Therefore there is at least one colour in $L(v_l)$ that is different from a and from the colour of v_l in that unique bad colouring of $v_l y v_2$. Colour v_l with that colour (this is safe since v_l and v_2 are not adjacent since H is not a triangle and since there are no separating triangles), colour y, then colour H.

Now colour the subgraph bounded by $yv_m \cdots v_k v_k y$ by Theorem 3.2.6 (this colouring is compatible with the colouring of H since v_m and v_l are not adjacent). Finally colour x then colour the broken wheel bounded by $xv_l \cdots v_m x$.

- *Suppose G' is the union of a wheel and a triangle.*

Suppose without loss of generality that the triangle is $v_k v_1 y$ and the wheel is the subgraph bounded by $v_1 v_2 \cdots v_r y v_1$ and that it has centre z (as in the middle drawing of Figure 4.8). Suppose first that $l \neq r$ and $m \neq s$ and that the subgraph H bounded by $v_k y v_m \cdots v_k$ is not a broken wheel.

Colour v_m by a colour different from b and from the colour of v_m in the unique colouring of $v_k y v_m$ unextendable to H. Delete that colour from the lists of y and x, then the subgraph bounded by $yzv_3 \cdots v_l y$ is colourable by Theorem 3.2.8 (the 2-lists are at y and v_3, after deleting from their lists the colours of their neighbours in P). Now colour H then colour the broken wheel bounded by $xv_l \cdots v_m x$.

Consider now the case when H is a broken wheel. The list $L(y) \setminus \{\varphi(v_1), \varphi(v_k)\}$ contains two colours different from the colour of y in the unique colouring of yv_1v_2 unextendable to the subgraph bounded by $yv_1v_2 \cdots v_l y$. Colour y with one of those two colours then colour the vertices from v_{k-1} to v_m in descending order of indices.

If the colour v_m receives is b, recolour y by the other colour then recolour the vertices from v_{k-1} to v_m again in descending order of indices. Since the first colour given to y forced v_m to be coloured b, and since v_k is still coloured the same, now v_m is either forced to be coloured a colour different from b or has the choice to be coloured a colour different from b. Finally colour the subgraph bounded by $yv_1v_2 \cdots v_l y$, then colour the broken wheel bounded by $xv_l \cdots v_m x$.

Now suppose that $m = s = k$ but $l \neq r$. Suppose that the colour of v_k is b, so we have to avoid colouring v_l with a. Delete a from the list of v_l, colour the subgraph bounded by $yzv_3 \cdots v_l y$ by Theorem 3.2.8 (the 2-lists are at v_l and v_3), then colour the broken wheel bounded by $xv_l \cdots v_k x$.

If $m = s = k$ and $l = r$ (as in the leftmost drawing of Figure 4.7), then colour z and v_r such that the colour of v_r is different from a and the colouring of $v_2 z v_r$ is extendable to the broken wheel bounded by $zv_2 \cdots v_r z$.

Such a colouring exists since with v_2 coloured $\varphi(v_2)$, there are still three colours in the list of z and three colours in the list of v_r as v_r and v_2 are not adjacent, and by Lemma 3.2.11, there are at most two bad colourings of $v_r z$, either of the form ef and fe, or gf and ge. Now colour y, then colour the broken wheel bounded by $xv_r \cdots v_k x$.

- *Suppose G' is a wheel.*

Let z be the centre of G'. We may suppose that at most one of $r = l$ and $s = m$ holds since otherwise we have a separating 4-cycle with interior consisting of 5-lists only as in the rightmost drawing of Figure 4.7 (the graph is colourable in this case though but we will not write the proof).

By symmetry there are only two cases, $l = r$ and $m \neq s$, or both $l \neq r$ and $m \neq s$. In the first case, if $r \neq 2$, colour z (which has two available colours) a colour such that when colouring the vertices from v_3 to v_r in ascending order of indices, v_r receives a colour different from a. Now colour the vertice from v_{k-1} to v_s in descending order of indices, colour y, colour the graph bounded by $yv_m \cdots v_s y$ by Theorem 3.2.6, then colour the broken wheel bounded by $xv_r \cdots v_m x$.

If $r = 2$, colour the vertices from v_{k-1} to v_s in descending order of indices. Let H be the subgraph bounded by $yv_m \cdots v_s y$. In case H is not a broken wheel, colour y (which now has two available colours) a colour that is different from the colour of y in the unique colouring of $v_m y v_s$ unextendable to H. In case H is a broken wheel, colour y a colour that does not force v_m to be coloured b. Now colour H then colour the broken wheel bounded by $xv_2 \cdots v_m x$.

In the second case, $l \neq r$ and $m \neq s$, we again colour the subgraph $G' - y$. Colour y a colour with a colour that is different from the colour of y in the unique colouring of $v_m y v_s$ unextendable to H in case H is not a broken wheel. In case H is a broken wheel, colour y a colour that does not force v_m to be coloured b. Then we colour H, then by Theorem 3.2.6 we colour the subgraph bounded by $yv_r \cdots v_l y$, and finally we colour the broken wheel bounded by $xv_l \cdots v_m x$.

Now we consider the case when x is adjacent to v_1 and only one other vertex of P (as in Figure 4.9). We assume without loss of generality that that vertex is v_2. Suppose first that y is adjacent to v_k. Let r be the smallest index such that $r \geq 2$ and y is adjacent to v_r.

Thus W consists of three parts, a wheel with centre x and outer cycle $v_1 v_2 \cdots v_r y v_1$, a triangle $v_1 y v_k v_1$, and a generalized wheel W' that is not a broken wheel (since W is not a double-centred wheel) bounded by $yv_r \cdots v_k y$ (see the leftmost drawing in Figure 4.9).

There are at least two colours in $L(x) \setminus \{\varphi(v_1), \varphi(v_2)\}$. Colour x with one of those two colours then colour the vertices from v_3 to v_r in ascending order of indices. If this colouring gives v_r the colour of the unique colouring of $v_r y v_k$ unextendable to W', we recolour x with the

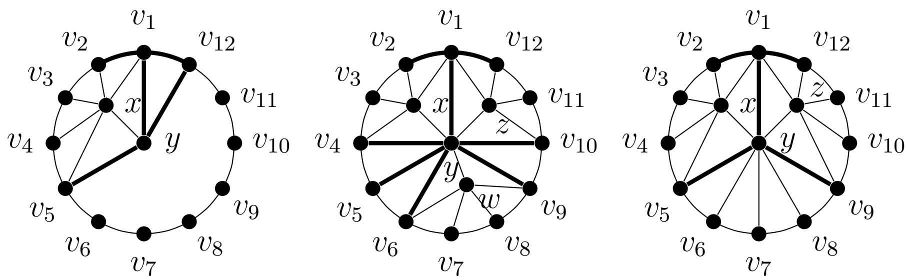


Figure 4.9: The case when x is adjacent to v_1 and only one other vertex of P.

other colour then again colour the vertices from v_3 to v_r in ascending order of indices. Then v_r receives a colour different from the one of the previous colouring and so the bad colouring of $v_r y v_k$ is avoided. Now colour y (still has a colour in its list since only four of its neighbours are coloured, v_k, v_1, x and v_r) then colour W'.

Suppose now that y is not adjacent to v_k and let v_r and v_s be as before. Then the graph W_1 bounded by $v_1 v_2 \cdots v_r y v_1$ is a wheel with centre x, the graph W_2 bounded by $v_1 y v_s \cdots v_k v_1$ is a wheel with centre a 5-list vertex z, and the graph W_3 bounded by $y v_r \cdots v_s y$ is a generalized wheel (can be a broken wheel), as in the middle and rightmost drawings in Figure 4.9.

If W_3 is not a broken wheel then by Lemma 3.2.10 there is at most one colouring of $v_r y v_s$ that is bad for W_3. We can choose a colour for x such that when we colour the vertices from v_3 to v_r in ascending order of indices we have v_r coloured by a colour different from that of the unique bad colouring of $v_r y v_s$ for W_3. Then of the two colours remaining for y (the only coloured neighbours are v_1, x and v_r, since v_k is not a neighbour of y by assumption), choose the one such that the bad colouring of $y v_1 v_k$ for W_2 is avoided. Colour W_2 then colour W_3.

Now suppose that W_3 is a broken wheel. Let W_1' be the broken wheel bounded by $x v_2 \cdots v_r x$.

• If the bad colourings of $v_r y v_s$ for W_3 have their values in some fixed 3-set we say that they are of *type 1*; and

• If they are two colourings that give v_r the same colour and give v_s the same colour but give y different colours, we say that they are of *type 2*.

We have the following cases.

(1) (i) *$r = 3$ and W_3 is a triangle, or*

(ii) *W_3 is not a triangle and the bad colourings of v_3yv_s for W_3 are of type 1.*

Consider first a colouring ψ of P whose restriction to v_1v_k is not a part of a bad colouring of yv_1v_k for W_2. Let a and b be the two colours in $L(x)$ not in $\{\psi(v_2), \psi(v_1)\}$ and $\{c, d\}$ the two colours in $L(y) \setminus \{a, b, \psi(v_1)\}$.

In case (i), if when colouring y with one of c or d then colouring W_2, the colour v_4 receives allows v_3 to be coloured, we are done. In case (ii), if $|L(v_3) \setminus \{\psi(v_2), c\}| \geq 2$ or $|L(v_3) \setminus \{\psi(v_2), d\}| \geq 2$, we are done.

Therefore we may assume that $L(v_3) = \{\psi(v_2), c, d\}$. Now colour y with one of a and b (note that none of them is in $L(v_3)$), colour W_2. Then at most two of the coloured neighbours of v_3 have colours in $L(v_3)$.

At least one of c and d is not used in colouring v_4 in case (i), or at least one of them makes the colouring of v_3yv_s good for W_3 with the given colours for y and v_s, in case (ii). Colour v_3 with the appropriate one of c or d. Finally colour x (it is colourable since v_3 has a colour not in $L(x) \setminus \{\psi(v_2), \psi(v_1)\}$).

Now we consider colourings of P whose restriction to v_1v_k is a part of a bad colouring of yv_1v_k for W_2. Since there is only one colouring of yv_1v_k bad for W_2, any two such colourings of P differ only at v_2. By what we have just proved, and since φ is a bad colouring of P by assumption, the restriction of φ to v_1v_k is a part of a bad colouring of yv_1v_k for W_2.

Let ψ be a colouring such that $\varphi(v_1) = \psi(v_1)$ and $\varphi(v_k) = \psi(v_k)$ but $\varphi(v_2) \neq \psi(v_2)$. We show that ψ is a good colouring. Again let

$L(x) \setminus \{\psi(v_2), \psi(v_1)\} = \{a, b\}$ and $L(y) \setminus \{a, b, \psi(v_1)\} = \{c, d\}$, and let $L(x) \setminus \{\varphi(v_2), \varphi(v_1)\} = \{e, f\}$ and $L(y) \setminus \{e, f, \varphi(v_1)\} = \{g, h\}$.

If both $c\psi(v_1)\psi(v_k)$ and $d\psi(v_1)\psi(v_k)$ are good colourings of yv_1v_k for W_2, then ψ is a good colouring of P (and similarly for φ, g and h). From this and the uniqueness of the bad colouring of yv_1v_k for W_2, we may assume without loss of generality that $c = g$ and $c\psi(v_1)\psi(v_k)$ (which is the same as $c\varphi(v_1)\varphi(v_k)$) is the unique bad colouring of yv_1v_k for W_2.

Then $d\psi(v_1)\psi(v_k)$ and $h\varphi(v_1)\varphi(v_k)$ are good colourings of yv_1v_k for W_2. When P is coloured φ (ψ), colour y with h (d) then colour W_2. If in case (i) the colour v_4 receives does not allow v_3 to be coloured, or if in case (ii), $|L(v_3) \setminus \{\psi(v_2), c\}| = 1$ or $|L(v_3) \setminus \{\psi(v_2), d\}| = 1$, then $\{h, \varphi(v_2)\} \subseteq L(v_3)$ ($\{d, \psi(v_2)\} \subseteq L(v_3)$).

Since $\psi(v_2) \neq \varphi(v_2)$ and $|L(v_3)| = 3$, $h = d$ or $h = \psi(v_2)$ or $d = \varphi(v_2)$. Suppose first that $h = d$. Since $c = g$, $\{c, d\} = \{g, h\}$, i.e., $L(y) \setminus \{a, b, \psi(v_1)\} = L(y) \setminus \{e, f, \varphi(v_1)\}$. Since $\psi(v_1) = \varphi(v_1)$, $\{a, b\} = \{e, f\}$, i.e., $L(x) \setminus \{\psi(v_2), \psi(v_1)\} = L(x) \setminus \{\varphi(v_2), \varphi(v_1)\}$. This gives a contradiction since $\psi(v_2)$, $\psi(v_1)$, $\varphi(v_2)$ and $\varphi(v_1)$ are all in $L(x)$ and $\psi(v_1) = \varphi(v_1)$ but $\psi(v_2) \neq \varphi(v_2)$. If any of $\psi(v_2)$ or $\psi(v_1)$ is not in $L(x)$, then with P coloured ψ we can colour y by a or b, colour W_2, colour W_3 then colour x. Similarly for $\varphi(v_2)$ and $\varphi(v_1)$ with P coloured φ.

If $h = \psi(v_2)$ (or if $d = \varphi(v_2)$), then with P coloured ψ (or φ), colour y by h (d) so that two of the neighbours of v_3 and x are coloured the same.

(2) $r = 3$ *and* W_3 *is not a triangle and the bad colourings of* v_3yv_s *for* W_3 *are of type 2.*

In this case, colour v_3 by a colour different from its unique colour involved in the two bad colourings of v_3yv_s for W_3, colour x, then of the remaining two colours for y choose one that avoids the bad colouring of yv_1v_k for W_2, colour W_2 and finally colour W_3.

(3) $r \neq 3$.

Let a and b be the two colours in $L(x) \setminus \{\varphi(v_2), \varphi(v_1)\}$. Let P be coloured with φ. We show first that the set of good colourings of v_rx for W_1' (the broken wheel bounded by $xv_2 \cdots v_rx$) either contains ab and ba, or there is $c \in L(v_r)$ such that the set of good colourings contains ca and cb. Note that we know from Lemma 3.2.11 that with v_2 coloured $\varphi(v_2)$, the bad colourings of v_rx for W_1' have one of those two sets of forms.

If the bad colourings of v_rx for W_1' are ab and ba (or they are two colourings of this form with two other colours), let c be any colour in $L(v_r) \setminus \{a, b\}$ (or in $L(v_r)$ delete the two colours). Then ca and cb are good colourings of v_rx for W_1'. If the bad colourings of v_rx for W_1' are da and db then either $L(v_r) \setminus \{d\} = \{a, b\}$ or there is a colour c in $L(v_r) \setminus \{d, a, b\}$. In the first case ab and ba are good colourings, and in the second case ca and cb are good colourings.

Now we consider each of those two cases of the good colourings of v_rx for W_1'. If ab and ba are good colourings, let e and f be the two colours in $L(y) \setminus \{a, b, \varphi(v_1)\}$. Colour y by the colour of e and f that avoids the bad colourings of yv_1v_k for W_2 then colour W_2.

Depending on the colour v_s (v_{r+1} in case W_3 is a triangle) receives, colour v_rx either ab or ba. By Lemma 3.2.11, with fixed colours of y and v_s it is at most one colour of v_r that makes the colouring of v_ryv_s bad for W_3. If W_3 is a triangle, one of a or b will be different from the colour of v_{r+1} - the colour of y, which is either e or f is already different from both a and b. Now colour W_1'.

Suppose now that ca and cb are good colourings of v_rx for W_1'. Then $(L(y) \setminus \{c, a, \varphi(v_1)\}) \cup (L(y) \setminus \{c, b, \varphi(v_1)\})$ contains at least three colours (since each set in the union contains at least two elements and the two sets are different).

If v_r is coloured c, colouring y with each of those three colours then colouring the vertices from v_{r+1} to v_s in ascending order of indices gives three different colours at v_s (corresponding to the three colours at y). One of the three colours v_s can receive from this procedure avoids all the bad colourings of $v_s z$ (the colourings that with v_k coloured $\varphi(v_k)$ make the colouring of $v_s z v_k$ unextendable to the broken wheel W_2' bounded by $zv_s \cdots v_k z$).

Now colour v_r with c then y with the colour that makes v_s receive the right colour to colour W_2' after colouring the broken wheel W_3 in ascending order of indices. Also colour W_3 using this procedure, colour W_2', colour x with a or b (whichever of them is available, since the colour y is coloured with may be a or b), and finally colour W_1'.

□

We also have the following lemma about choosing an appropriate colouring from two confederacies for two paths of length one, at distance one, in the outer walk of a wheel with centre a 4-list vertex. We want the colouring of the two paths (which is a colouring of a path of length three) to be extendable to the wheel.

Lemma 4.2.6. *Let G be a wheel with centre x, and let $uu'v'v$ be a path of length three in ∂G. Let L be a list assignment of G such that:*

- *for $v \in V(\partial G)$, $|L(v)| \geq 3$; and*
- *$|L(x)| \geq 4$.*

Let $\mathcal{C}_u$ and $\mathcal{C}_v$ be confederacies for uu' and vv' respectively. Then, there are colourings $\varphi_u \in \mathcal{C}_u$ and $\varphi_v \in \mathcal{C}_v$ such that:

- *$\varphi_u \cup \varphi_v$ is a proper colouring of the subgraph induced by $\{u, u', v', v\}$; and*
- *$\varphi_u \cup \varphi_v$ is extendable to G.*

Proof. For a proper colouring $\varphi_u \cup \varphi_v$ of the subgraph induced by $\{u, u', v', v\}$ to be extendable to G, the following two conditions should be satisfied:

- $L(x)$ contains at least one colour not in $\{\varphi_v(v), \varphi_v(v'), \varphi_u(u), \varphi_u(u')\}$; and

- there is at least one such colour c such that the colouring $\varphi_u(u)c\varphi_v(v)$ of uxv is extendable to the broken wheel $W := G - \{u', v'\}$.

Suppose that there is a colouring $\varphi_v \in \mathcal{C}_v$ such that, for every colouring $\varphi_u \in \mathcal{C}_u$, either $L(x) = \{\varphi_u(u), \varphi_u(u'), \varphi_v(v), \varphi_v(v')\}$, or $\varphi_u(u') = \varphi_v(v')$.

When we write $cd \in \mathcal{C}_v$, we mean that c is the colour of v' and d is the colour of v, similarly for $\mathcal{C}_u$.

Suppose that $L(x) = \{a, b, c, d\}$, $cd \in \mathcal{C}_v$, and $\mathcal{C}_u$ consists of a subset of $\{ab, ba\}$ and colourings that give u' the colour c (or we say *start with* c). This means that there are at least two colourings starting with c in $\mathcal{C}_u$.

Claim 4.2.7. *$\mathcal{C}_v$ does not contain dc.*

Proof. Suppose for a contradiction that $\mathcal{C}_v$ contains dc. If u and v are adjacent, then the colouring dc for $v'v$ and one of the two colourings starting with c in $\mathcal{C}_u$ make a proper colouring of the cycle $uu'v'vu$. This colouring is extendable to x since u' and v are both coloured c.

Thus, u and v are not adjacent. If the colouring dc for $v'v$ with the two colourings starting with c in $\mathcal{C}_u$ force uxv to be coloured by a colouring not extendable to the broken wheel W, then the two colourings starting with c in $\mathcal{C}_u$ are ca and cb, and the bad colourings of uxv for W are the permutations of $\{a, b, c\}$.

Now we prove that any colouring in $\mathcal{C}_v$ either starts with c or d. Suppose there is a colouring φ in $\mathcal{C}_v$ that starts with e, where $e \notin \{c, d\}$. If $\varphi(v)$ is not in $\{a, b, c\}$, then we colour $v'v$ with φ. This guarantees that all the bad colourings of uxv are avoided. Then, we colour $u'u$ by any of the colourings in $\mathcal{C}_u$ that start with c, then extend the colouring of $uu'v'v$ to x and then to W.

Thus, $\varphi(v)$ is in $\{a, b, c\}$. If $\varphi = ec$, then when we colour $v'v$ with ec and $u'u$ with any of ca or cb, we can colour x with d and hence have the bad colourings of uxv avoided. If $\varphi = ea$ (eb), then we colour $v'v$ with

φ and colour $u'u$ with ca (repectively cb). This guarantees that the bad colourings of uxv are avoided.

Thus $\mathcal{C}_v$ is the union of $\{cd, dc\}$ and:

(*i*) *a dictatorship with dictator* v' *in which the colour of* v' *is* c*,*

(*ii*) *a dictatorship with dictator* v' *in which the colour of* v' *is* d*, or*

(*iii*) *a colouring* cf *with* $f \neq d$*, and a colouring* dg *with* $g \neq c$*.*

In the cases (i) and (iii), there is a colouring cf in $\mathcal{C}_v$ such that $f \neq d$. We colour $v'v$ with cf and $u'u$ with ab, then we colour x with d. This avoids all the bad colourings of uxv. In case (ii), there is a colouring dg in $\mathcal{C}_v$ such that $g \neq c$. If $g \notin \{a, b\}$, then colouring $v'v$ with dg avoids all the bad colourings of uxv. If $g = a$ ($g = b$), colour $u'u$ with ca (repectively cb). □

Now consider a colouring $\varphi \in \mathcal{C}_v$ that starts with a colour $e \neq c$. We have the following three cases.

Case 1. $\varphi(v) = c$.

In this case, by Claim 4.2.7, $e \neq d$. Since $\mathcal{C}_v$ contains cd but not dc, and $\mathcal{C}_v$ is a confederacy (a union of two governments), $\mathcal{C}_v$ either contains a colouring fd with $f \neq c$ (this is Case 2) or a colouring cf with $f \neq d$. Thus, we assume $\mathcal{C}_v$ contains a colouring cf with $f \neq d$.

Colour $v'v$ with φ and $u'u$ with ab. If the union of those two colourings is not extendable to G, then we have one of the follwing two cases.

Subcase 1.1. $e = a$.

If each of the two colourings starting with c in $\mathcal{C}_u$ with the colouring $\varphi = ac$ for $v'v$ does not extend to G, then those two colourings are cb and cd and the bad colourings of uxv are the permutations of $\{b, d, c\}$. In this case we colour $v'v$ with cf, and $u'u$ with ab.

Subcase 1.2. *bdc is a colouring of uxv unextendable to W.*

In this case we also colour $v'v$ with cf, and $u'u$ with ab.

Case 2. $\varphi(v) = d$.

Recall that $\varphi(v') = e \neq c$. If colouring $v'v$ with ed and $u'u$ with ab does not extend to G, then the bad colourings of uxv are either the permutations of $\{b, c, d\}$ or two colourings, including bcd, that give u and v the colours b and d respectively.

Suppose that $e \neq b$. At least one of the two colourings in $\mathcal{C}_u$ that start with c ends with a colour not in $\{c, b\}$. Let ψ be such a colouring. Colour $u'u$ with ψ and $v'v$ with φ, and then colour x with b. This avoids the bad colourings of uxv.

Thus, $e = b$. That is, the two colourings we know in $\mathcal{C}_v$ are cd and bd. If one of the colourings in $\mathcal{C}_u$ that start with c ends with a colour not in $\{a, b\}$, then giving this colouring to $u'u$ and $\varphi = bd$ to $v'v$ avoids the bad colourings of uxv and allows x to be coloured a.

Thus, the two colourings in $\mathcal{C}_u$ that start with c are ca and cb. If colouring $u'u$ with cb and $v'v$ with bd does not avoid the bad colourings of uxv, then the bad colourings of uxv are bcd and bad.

Since $\mathcal{C}_v$ is a confederacy, it is not a government, and so not a dictatorship with dictator v. Thus, there is a colouring in $\mathcal{C}_v$ that ends with a colour different from d. Let φ' be such a colouring. If $\varphi'(v') \neq a$, then colour $u'u$ with ab and $v'v$ with φ'. Since $\varphi'(v) \neq d$, the bad colourings of uxv are avoided, and we just need to show that there is a colour available for x. If $\varphi'(v') \neq d$ as well, then x can be coloured d. If $\varphi'(v') = d$, then $\varphi'(v) \neq c$ by Claim 4.2.7, and so we can colour x by c.

Thus, $\varphi'(v') = a$. Colour $u'u$ with ca and $v'v$ with φ'. Since $\varphi'(v) \neq d$, x can be coloured with d.

Case 3. $\varphi(v) \notin \{c, d\}$.

Let f denote $\varphi(v)$. Thus, $\varphi = ef$. If colouring $v'v$ with φ and $u'u$ with ab is not extendable, then bcf is a bad colouring of uxv. Thus, with v coloured f, the bad colourings of ux are either $\{bc, cb\}$ or $\{bc, bg\}$ for $g \neq c$.

At least one of the two colourings in $\mathcal{C}_u$ that start with c ends with a colour different from b. Let ψ be such a colouring. Colour $u'u$ with ψ and $v'v$ with φ. This avoids the bad colourings of uxv but may not be extendable to x. If $\varphi \cup \psi$ is not extendable to x, then either $\psi = ca$ and $\varphi = db$ ($\varphi \neq bd$ since $f \neq d$), or $\psi = cd$ and $\varphi = ab$ or ba. We consider only one of those three possibilities, the other ones can be proved using similar arguments.

If there is in $\mathcal{C}_u$ a colouring that starts with c and ends with a colour not in $\{a, b\}$, then the union of this colouring for $u'u$ and db for $v'v$ extends to G. Thus, the two colourings in $\mathcal{C}_u$ that start with c are ca and cb.

If the colouring cb for $u'u$, as ab, union the colouring db for $v'v$ does not extend to G, then bab as bcb are bad colourings for uxv. We may assume the harder case without loss of generality, that is bab and bcb are not the only bad colourings of uxv, but also aba, aca, cbc, and cac.

Note that the colourings in $\mathcal{C}_v$ that we know are cd and db, and they are in two different governments. If $\mathcal{C}_v$ contains a colouring that starts with c and ends with a colour not in $\{d, b\}$, then colouring $v'v$ with that colouring and $u'u$ with ab extends to G.

Thus, the second colouring in the government containing cd in $\mathcal{C}_v$ is cb. If $\mathcal{C}_v$ contains a colouring that starts with d different from db, then colouring $v'v$ by that colouring and $u'u$ with ab extends to G since dc is not in $\mathcal{C}_v$ by Claim 4.2.7.

If $\mathcal{C}_v$ contains a colouring that ends with b different from db and cb, then colouring $v'v$ with that colouring and $u'u$ with ca extends to G. Also from Case 2 we know that $\mathcal{C}_v$ does not contain a colouring that ends with d different from cd, in particular it does not contain bd. Thus, cb is the only colouring in the government containing db in $\mathcal{C}_v$.

Now we know that $\mathcal{C}_u$ contains $\{ab, cb, ca\}$, and $\mathcal{C}_v = \{cd, cb, db\}$. We colour $u'u$ with ab and $v'v$ with cb. Then, x can be coloured d, and so the bad colourings of uxv are avoided. □

4.3 An Extension of a Theorem of Thomassen

In this section we state and prove our extension of Theorem 3.2.4 of Thomassen.

Theorem 4.3.1. *Let (G, P, L, x) be a canvas, where P is a path of length at most two. Given a fixed L-colouring φ of P, then G has an L-colouring extending φ unless:*

(a) P has length one and G contains a 3-restricted subcanvas that is a wheel with centre x; or

(b) P has length two and G contains:

(i) a 4-restricted subcanvas that is a wheel with centre x;

(ii) a 3-restricted subcanvas that is a wheel of wheels containing x (either as its centre or the centre of one of the smaller wheel sections);

(iii) a 3-restricted semi-subcanvas that is a broken wheel with major vertex x and principal path whose end-vertices are the end-vertices of P; or

(iv) a 3-restricted subcanvas that is a generalized wheel that does not contain x as an inner vertex.

Proof. This is an adaptation of Thomassen's proof of 3.2.4. Let G be a minimum conterexample. We assume without loss of generality that G is a near-triangulation.

Claim 4.3.2. *G is 2-connected.*

Proof. If G has a cut vertex, we colour the block containing x by minimality then colour the rest of the graph either by Theorem 3.2.6 or Theorem 3.2.4. □

Now let $C := v_1v_2\cdots v_kv_1$ be the outer cycle of G and suppose that $P = v_1$, $P = v_2v_1$ or $P = v_2v_1v_k$ in case P is of length zero, one or two respectively.

Claim 4.3.3. $|C| \geq 6$, *there are no separating triangles, and if there is a separating* 4*-cycle then its interior consists of* x *only (in particular there is no separating* 4*-cycle with all its interior vertices having* 5*-lists).*

Proof. (1) $|C| \geq 4$ *and there are no separating triangles.*

Assume for a contradiction that $|C| = 3$ or that there is a separating 3-cycle. Let C' be a 3-cycle with nonempty interior such that the subgraph induced by C' and its interior does not contain a separating 3-cycle. Note that C' is C in case G contains no separating 3-cycles. Colour C' and its exterior by minimality, and let G' be the subgraph of G induced by the vertices in the interior of C'.

Let L' be the list assignment of G' such that, for every $v \in V(G')$, $L'(v)$ is obtained from $L(v)$ by deleting the colours of the neighbours of v in C'. By minimality, this theorem, and so also Proposition 2.2.4, is true for G'. Thus, G' is L' colourable by Lemma 2.2.5.

(2) $|C| \geq 5$ *and if there is a separating* 4*-cycle then its interior consists of* x *only, and* x *is adjacent to all the four vertices of the cycle.*

Assume that $|C| = 4$ or that there is a separating 4-cycle. Let C' be a 4-cycle with nonempty interior such that the subgraph induced by C' and its interior does not contain a separating 4-cycle. Note that C' is C in case G contains no separating 4-cycles. Colour C' and its exterior by minimality, and let G' be the subgraph of G induced by the vertices in the interior of C'.

Let L' be the list assignment of G' such that, for every $v \in V(G')$, $L'(v)$ is obtained from $L(v)$ by deleting the colours of the neighbours of v in C'. By minimality, this theorem, and so also Proposition 2.2.4, is true for G'. Thus, if x is not adjacent to all the vertices of C', G' is L' colourable by Lemma 2.2.6.

(3) $|C| \geq 6$.

Suppose that $|C| = 5$. There is no 5-list vertex that is adjacent to all the vertices of C since there are no separating triangles and we assumed that x is in the interior of C.

The 4-list vertex x is not adjacent to all the vertices of C since this is one of the obstructions in case P has length at least 1 and in case P has length is a vertex or is empty this wheel with centre x is colourable.

We saw in the preceding paragraph that x is not adjacent to all 5-vertices of C. Since G is a near-triangulation and there are no separating 4-cycles with only interior 5-lists, x is not adjacent to four vertices of C. Such a wheel with centre x is an obstruction in case P has length at least one, but in case P is empty or is a vertex, G is colourable (G consists of a vertex of degree two joined to a wheel with centre x whose outer cycle is of length four). Therefore, if there is a vertex adjacent to four vertices of C, it is not x.

Suppose there is a vertex v in the interior of C adjacent to four vertices of C; there is at most one such vertex. This divides the interior of C into three triangles and one 4-cycle. Let C' be the 4-cycle. Because there are no separating triangles, all the vertices other than v in the interior of C are in the interior of C'; in particular, x is in the interior of C'.

By (2), x is the only vertex in the interior of C'. By (1), and since G is a near triangulation, x is adjacent to all the vertices of C'. Thus, G is a double-centred wheel with centres v and x. This yields the contradiction that G is either L-colourable or an obstruction, depending on whether P has length less than two or equal to two.

Thus, we may assume that every interior vertex of C is adjacent to at most three vertices of C. If there are three vertices in the interior of C such that each one of them has three neighbours in C, then there is a separating 4-cycle with interior consisiting of 5-lists only.

Thus, there can be at most two vertices in the interior of C that are adjacent to three vertices of C. If there is exactly one vertex in the interior of C that is adjacent to three vertices of C we can colour the interior of C by colouring the block containing that vertex first.

If there are two vertices in the interior of C that are adjacent to three vertices of C, we colour C first. Then we may need to colour a block in which all the interior vertices have 5-lists, there are three 2-lists on its outer walk and all the other vertices on the outer walk have 4-lists, or a block that contains x in its interior, has two outer 2-lists, one outer 3-list and all the other lists on the outer walk are 4-lists.

Both types of block are colourable by deleting (the appropriate) one of the outer 2-lists or 3-list. Then, whether the deleted vertex is on a chord of the block or not, the resulting smaller blocks are colourable. We may need to delete one vertex from one of the smaller blocks to colour it by minimality or Theorems 3.2.6, 3.2.7, or 3.2.8.

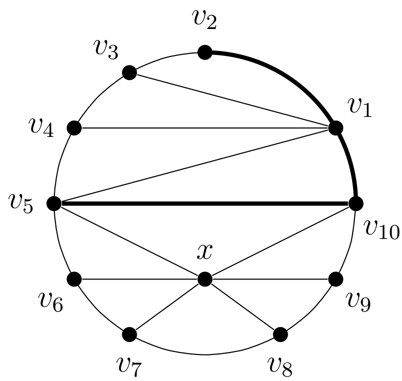
Therefore every vertex in the interior of C is adjacent to at most two vertices of C and so the interior of C is colourable by colouring the block containing x first.

□

Claim 4.3.4. *C has no chords.*

Proof. (1) If there is a chord that has x and P on one side, colour that side first (and if P is empty, colour the side containing x first), then colour the other side by Theorem 3.2.6.

(2) If P consists of a vertex and there is a chord that has P and x on different sides, choose the closest such chord to x. Colour the side containing P first by Theorem 3.2.6. Since G is a counterexample, this colouring is unextendable to the side containing x, and so by minimality, this side contains a 3-restricted wheel subcanvas with centre x. This subcanvas contains the chord since there are no closer chords to x that separate it from P, and then, in fact, this



$L(v_1) = \{2\}$, $L(v_2) = \{3\}$, $L(v_3) = \{1, 2, 3\}$, $L(v_4) = \{1, 2, 3\}$, $L(v_5) = \{1, 2, 3, 4\}$, $L(v_6) = \{2, 3, 4\}$, $L(v_7) = \{2, 3, 5\}$, $L(v_8) = \{2, 3, 5\}$, $L(v_9) = \{1, 2, 3\}$, $L(v_{10}) = \{1\}$ and $L(x) = \{1, 2, 3, 4\}$.

Figure 4.10: uncolourable even though one of the vertices on the outer cycle has a list of size greater than three; $|L(v_4)| = 4$.

subcanvas is all of the side containing x since by (1) there is no chord that has x and P on the same side. By Lemma 4.2.1, there is at most one colouring of the chord that is unextendable to the side containing x. By Lemma 3.3.9 there is a colouring of the side containing P that avoids the unique unextendable colouring to the side containing x.

(3) If P has length one and there is a chord that has P and x on different sides, also choose such a chord that is closest to x. As in (2) if we colour the side containing P by Theorem 3.2.6, then, since this is a counterexample, this colouring is not extendable to the other side. Thus, the side containing x is a wheel with centre x. If we show that the end-vertices of the chord both have lists of size at most three then this wheel is a 3-restricted subcanvas and so we have a contradiction (since now, as P has length one, this is an obstruction).

We know by Lemma 4.2.1 that it is at most one colouring of the chord that is unextendable to the wheel with centre x. If one end-vertex of the chord has more than three colours, we delete the colour involved in the unique unextendable colouring from its list. After deleting that colour, the side containing P is still colourable by Theorem 3.2.6 and that colouring is extendable to the side containing x since we deleted the colour involved in its unique unextendable colouring.

(4) If P has length two and there is a chord that has P and x on different sides, we can colour the side containing P by Theorem 3.2.4 (it does not contain an obstruction since the obstructions of Theorem 3.2.4 are a subset of the obstructions of this theorem). The rest of the argument is the same as that in (3) except that now we need to argue that the end-vertices of the chord both have lists of size at most four.

If one end-vertex of the chord has a list of size greater than four, we delete from its list the unique colour involved in the colouring of the chord unextendable to the side containing x. This deletion does not introduce any new 3-lists, and since the all the possible obstructions to colouring the side containing P are 3-restricted by Theorem 3.2.4, it is still colourable. This colouring is extendable to the side containing x, a contradiction. Thus the wheel with centre x is a 4-restricted subcanvas, a contradiction.

(5) If P has length two and there is a chord that has v_1, the middle vertex of P, as one end-vertex, there are two cases.

Case 1. *There are such chords on both sides of x.*

Choose the closest such chords to x on both sides. Now the graph is divided into three parts. The outer parts are colourable by Theorem 3.2.6, but this colouring is not extendable to the middle part since this is a counterexample. From this, and by our choice of the chords, by (1) and (4) of this claim, and by (2) of Claim 4.3.3, the middle part is either a wheel with centre x or a wheel of wheels containing x.

Let v_i and v_j be the end-vertices of the two chords different from v_1. Note that v_i and v_j are not adjacent in C since there are no separating triangles.

In case the middle subgraph is a wheel with centre x, in order to have a contradiction, we need to show that neither of the two corner vertices, v_i and v_j, between it and the other two parts has a list of size bigger than four. In case the middle part is a wheel of wheels that is not a wheel, we need to show that each corner vertex has a list of size at most three.

Let A_j denote the part adjacent to v_1v_j and A_i denote the part adjacent to v_1v_i. Assume without loss of generality that $i < j$ and that v_j has a list of size greater than four if the middle part is a wheel, and greater than three if it is a wheel of wheels that is not a wheel. Then, if $j = k-1$, delete from $L(v_j)$ the colour of v_k. If $j \neq k-1$ and A_j is a broken wheel, then $A_j - v_j$ is also a broken wheel. Since G does not contain a 3-restricted broken wheel subcanvas, at least one of the vertices v_l with $j < l < k$ has a list of size greater than three, and so all the colours of v_j are good for A_j in this case. If A_j is not a broken wheel delete from $L(v_j)$ the colour involved in the unique unextendable colouring of $v_jv_1v_k$ to A_j.

Colour v_i (the other corner vertex) by a colour different from that involved in the unique unextendable colouring of A_i if A_i is not a broken wheel. If $i = 3$, colour v_i by a colour different from that of v_2. If $i \neq 3$ and A_i is a broken wheel, then at least one of the vertices v_l with $2 < l < i$ has a list of size greater than three. Thus, we can colour v_i by any colour in this case and have the colouring of $v_iv_1v_2$ extendable to A_i.

Now we can colour the middle part. If it is a wheel with centre x we colour it by colouring x then colouring the vertices from v_i toward v_j (v_j is colourable since it still has at least four colours and it has degree 3 in the middle part).

If it is a wheel of wheels, since we assumed that $|L(v_j)| > 3$ and have deleted at most one colour from it, it still contains a colour different from that of v_1 and the colour involved in the unique colouring, given by Lemmas 4.2.4 and 4.2.5, of $v_j v_1 v_i$ unextendable to the wheel of wheels. We colour v_j with this colour then colour the wheel of wheels. Note that there are no separating 4-cycles with interiors consisting of 5-lists only and no separating triangles, and so the conditions of the lemmas are satisfied.

Case 2. *There are chords only on one side of x.*

Also choose the closest such chord to x. The side not containing x is colourable but the colouring is not extendable to the side containing x, and so the side containing x is either a wheel with centre x or a wheel of wheels. In case it is a wheel, then, as before, the corner vertex cannot have more than four colours. In case it is a wheel of wheels, then, as before, the corner vertex cannot have more than three colours. Thus we have a contradiction in both cases.

□

Claim 4.3.5. *Let u be an inner vertex that is joined to two vertices v and w on C such that P is contained in one of the two vw-paths in C. Let H be the subgraph bounded by vuw and the vw-path in C not containing P. Then H is either a broken wheel or a wheel with centre x.*

Proof. **Case 1.** $u = x$.

Assume H is not a broken wheel. Suppose we have chosen the closest two such vertices (v and w) to P. The side of vxw containing P is colourable since it has a list of size greater than three on its outer boundary; $L(x)$.

Since this is a counter-example, this colouring is not extendable to H. Then, by Theorem 3.2.4 and Claim 4.3.4, H is a generalized wheel with principal path vxw and with all outer boundary vertices other than v and w having lists of size exactly three. Delete from the list of x the colour of the unique unextendable colouring of vxw to H. If the side of vxw containing P is colourable after deleting that colour, we colour it then colour H, a contradiction.

If the side of vxw containing P is not colourable after deleting that colour, this means it is a generalized wheel with principal path P and all its outer lists other than those of P are of size three. Since this subgraph (the side of vxw containing P) cannot contain any chords other than v_1x, it is either a wheel or a union of two wheels that intersect only in v_1x.

Therefore, the union of the two sides of vxw, that is G, is a wheel of wheels with all outer lists other than those of P of size three, contradicting the hypothesis of the theorem.

Case 2. $|L(u)| = 5$ *and x is on the same side of vux as P.*

Assume that H is not a broken wheel and again suppose that v and w are chosen closest to P. The side containing P and x is colourable since it has a list of size greater than three on its outer boundary; $|L(u)|$ (the obstruction to colouring this side cannot be a wheel with centre x because of Case 1). We can delete from the list of u the colour of the unique unextendable colouring to H. After deleting that colour, $L(u)$ still has at least four colours and so the side of vuw containing P and x is still colourable.

Case 3. *u has a 5-list and x and P are on different sides of vuw. In this case assume that v and w, contrary to the previous two cases, are chosen furthest from P.*

We can colour the side containing P, but this colouring is not extendable to the other side. Thus the side containing x is (not just contains, because of the choice "furthest") either a wheel with centre x (in that case we are done) or is a wheel of wheels. If this side contains a broken wheel semi-subcanvas with principal path uxw then, since this is a near-triangulation and there are no chords (that is v and w are not adjacent on a chord), x is adjacent to u and this side is again a wheel with centre

x.

If it is a wheel of wheels, it has only one unextendable colouring and we can delete its colour from the list of u and still have the side containing P colourable since u has then a list of size at least four. All the obstructions to colouring that side should have lists of size exactly three on their outer boundary and they contain u since there are no chords in G.

Finally, after colouring the side of vuw containing P, we colour the side containing x (which is colourable now since the colour of u involved in the unique bad colouring of vuw was deleted from $L(u)$ before colouring the side containing P). □

Claim 4.3.6. *If P has length at least one, then no interior neighbour is adjacent to two vertices that are the ends of a path Q in C of length at most two such that $P \subseteq Q$.*

Proof. Suppose that there is such an inner vertex u. If $u = x$, then by Claim 4.3.5, part (2) of Claim 4.3.3 (no separating 4-cycles with interior all 5-lists) and Claim 4.3.4 (no chords), we have that G is a wheel with centre x. Each outer vertex cannot have a list of size greater than 3, since otherwise we can colour the wheel. This is one of the obstructions, a contradiction.

Thus $|L(u)| = 5$. Suppose first that P and x are on different sides of v_2uv_k (or v_2uv_1). Then by Claim 4.3.5, x is adjacent to v_2 and v_k, a contradiction since there are no separating 4-cycles with interiors consisting of 5-lists only and no separating triangles at all.

Now suppose that P and x are on the same side of v_2, u, v_k. The subgraph W bounded by $uv_2 \cdots v_k u$ is a broken wheel by Claim 4.3.5. Colour W; the cycle $v_1v_2uv_kv_1$ or $v_1v_2uv_1$ is now coloured. By (1) and (2) in Claim 4.3.3, the interior of $v_1v_2uv_kv_1$ or $v_1v_2uv_1$ is colourable unless x is adjacent to all the vertices of this cycle.

Now if the interior of this cycle is colourable, we have a colouring of G and so have a contradiction. If it is not colourable, G is a double-centred wheel (a union of a wheel with centre x and a broken wheel with major vertex u). This is also a contradiction because every uncoloured vertex on C has a list of size exactly three since otherwise we can colour the wheel with centre x first and extend any colouring that v_2uv_k receives to the broken wheel W. □

Claim 4.3.7. *We may assume that P has length at least one.*

Proof. To prove this we need to show that the graph does not contain a wheel subcanvas with centre x even if P has length less than one (i.e. empty or consisting of exactly one vertex). Then we can colour an edge or a neighbour of the precoloured vertex to turn our counterexample into one with a precoloured path of length one.
If the graph contains a wheel subcanvas with centre x, it must be all of the graph since we proved before that there are no chords. By Lemma 4.2.1, a wheel with centre a 4-list without precoloured vertices or with only one precoloured vertex is colourable. □

Claim 4.3.8. *x is not adjacent to both v_3 and v_{k-1}.*

Proof. If both v_3 and v_{k-1} are adjacent to x, then let G' be the graph bounded by the cycle $C' := v_1v_2v_3xv_{k-1}v_kv_1$, and let W be the broken wheel bounded by $xv_3 \cdots v_{k-1}x$ (we know that W is a broken wheel by Claim 4.3.5). Note that the interior of C' consists of 5-lists only.

Colour W first. Then colour the interior of the now-coloured C' as follows. If every vertex in the interior of C' is adjacent to at most two vertices in C', then the result follows from Theorem 3.2.6.

If every vertex in the interior of C' is adjacent to at most three vertices of C', then there are at most three vertices each adjacent to three vertices of C'. If there are three such vertices, then they have 2-lists after deleting the colours of their neighbours in C', and every other vertex in the interior of C' has a list of size at least four. In this case, any block in the interior of C' can be coloured as follows. Delete one of the 2-lists, then the resulting graph is colourable either by Theorem 3.2.7 or Theorem 3.2.8.

If there are two such vertices, then all the other interior vertices of C', except possibly one, are adjacent to at most one vertex of C'. Thus, the interior is colourable, block after block, by Theorem 3.2.7.

Therefore, we may assume that there are vertices in the interior of C' adjacent to more than three vertices of C'. There can be at most two vertices adjacent to four vertices of C'. Suppose there are two such vertices y and z. Then all the other interior vertices of C' are adjacent to at most one vertex of C'.

The vertices y and z may be the only vertices in the interior of C' and they may be adjacent. Otherwise, the interior of C' is colourable, block after block, by Theorem 3.2.7.

In case the interior of C' consists of y and z, we colour G as follows. Note first that the interior of G consists of the three vertices x, y and z. The vertices x, y and z make a triangle, y is adjacent to v_1, v_2, v_3, z is adjacent to v_1, v_k, v_{k-1}, and x is adjacent to $v_3, \cdots, v_{k-1}$.

By Lemma 3.2.11, there is a set S of size 3 such that the colours of v_3, x, v_{k-1}, that appear in colourings of v_3xv_{k-1} unextendable to the broken wheel bounded by $xv_3 \cdots v_{k-1}x$, are all contained in S. Since $|L(x)| \geq 4$, there is a colour c in $L(x) \setminus S$.

If $L(v_3) \setminus \{\varphi(v_2)\} = \{c, d\}$, $L(v_{k-1}) \setminus \{\varphi(v_k)\} = \{c, d'\}$ for some colours d and d', $L(y) = \{\varphi(v_1), \varphi(v_2), d, c, e\}$, and $L(z) = \{\varphi(v_1), \varphi(v_{k-1}), d', c, e\}$ for some colour e, then colour v_3 by c, colour v_{k-1} by d', and colour x by any other colour f (f may equal e). This colouring of v_3xv_{k-1} is extendable to the broken wheel bounded by $xv_3 \cdots v_{k-1}x$ and $L(y) \setminus \{\varphi(v_1), \varphi(v_2), c, f\} \neq L(z) \setminus \{\varphi(v_1), \varphi(v_{k-1}), d', f\}$. Thus, the colouring is also extendable to y and z.

Otherwise, colour x by c, then colour v_3 and v_{k-1} by colours such that the remaining available colours in $L(y)$ and $L(z)$ are different.

If there is a vertex adjacent to four vertices of C' and a vertex adjacent to three vertices of C', then there can be at most one vertex of the remaining vertices that is adjacent to two vertices of C' and so the interior is colourable, block after block, by Theorem 3.2.7 in this case also.

Therefore, we may assume there are vertices adjacent to more than four vertices of C'. Now since there are no sparating 4-cycles with interior 5-lists only, the interior of the cycle consists of only one vertex that is either adjacent to all the vertices of the cycle or to only five of them.

Actually it is adjacent to all of them since this is a triangulation and any chord of this cycle is either a chord of the outer cycle of G as well or is a chord connecting x to a vertex of P and so creates a bigger broken wheel containing W that we can consider to be W.

Note that the fact that any cycle, whose interior is not empty, inside C' is a separating cycle follows from the fact that there are vertices outside C' since $|C| \geq 6$ by Claim 4.3.3 and only five vertices of C are in C'.

Now we have a contradiction since one side of v_3xv_{k-1} is a wheel (the interior of C') and the other is a broken wheel, namely W, and so G is a wheel of wheels. This wheel of wheels has all its outer boundary vertices, except for those in P, having lists of size exactly three since first if a vertex in $V(C) - V(C')$ has a list of size greater than three. We can colour G' first then colour W.

Second, if one of v_3 and v_{k-1} has a list of size greater than three, say $|L(v_3)| > 3$, then there is a colour in $L(v_3)$ that avoids all the colourings of v_3xv_{k-1} unextendable to W. Colour v_3 with that colour, colour the centre of G', colour v_{k-1}, colour x, then colour W. □

We may assume without loss of generality that v_3 is not adjacent to x. Consider the subgraph $G - v_3$, choose two colours from $L(v_3) \setminus \{\varphi(v_2)\}$ and delete them from the lists of the neighbours of v_3 other than v_4, let L' be the resulting list assignment. By induction, if $G - v_3$ does not contain any of the obstructions, then it is L'-colourable, then its colouring is extendable to G, a contradiction. Therefore $G - v_3$ contains one of the obstructions B.

Since B is not an obstruction of G, ∂B contains at least one vertex in the interior of C that is a neighbour of v_3. Since there are no separating 4-cycles with interior consisting of neighbours of v_3 only, and since C has no chords, $G - v_3$ is not a wheel. Therefore, $G - v_3$ is either a double-centred wheel or a wheel of wheels, or B is a proper subgraph of $G - v_3$.

Let $w_1, \cdots, w_n$ be the neighbours of v_3 in the interior of C from v_2 to v_4. If $G - v_3$ has a proper subgraph that is one of the obstructions, then there are i and j, $1 \leq i \leq n$ and $j \in \{1, \cdots, k\} \setminus \{2, 3, 4\}$, such that w_i is adjacent to v_j. Let s be the maximum such j different from 1, if exists, and let s be 4 otherwise. Let r be the minimum i such that w_i is adjacent to v_s. By Claim 4.3.5, the subgraph bounded by $w_r v_3 v_4 \cdots v_s w_r$ is either a broken wheel or a wheel with centre x. However, v_3 is not adjacent to x, so it is a broken wheel.

If the subgraph H bounded by $v_1 v_2 w_1 \cdots w_r v_s \cdots v_k v_1$ does not contain one of the obstructions, then it is colourable by minimality. Then the colouring of $w_r v_s$ is extendable to the broken wheel bounded by $w_r v_4 \cdots v_s$ w_r by Theorem 3.2.6. This gives a colouring of $G - v_3$, extendable to G, a contradiction.

Therefore, H contains one of the obstructions. By the choice of r and s, and since C has no chords, this obstruction is either H itself, or one of $w_1, \cdots, w_r$ is adjacent to v_1, say w_i is, and the subgraph bounded by $v_1 w_i \cdots w_r v_s \cdots v_k v_1$ is one of the obstructions. But since there is no separating 4-cycle with interior containing vertices other than x, $i = 1$, and there are at most three neighbours of v_3 in the interior of C. We summarize this in the following claim.

Claim 4.3.9. *There are at most three neighbours of v_3 in the interior of C, and if H is not an obstruction, then w_1 is adjacent to v_1 and the subgraph B_H of H bounded by $v_1 w_1 \cdots w_r v_s \cdots v_k v_1$ is an obstruction.*

We consider every possible case for the obstruction B_H contained in H, a wheel (as a proper subgraph of $G - v_3$), a generalized wheel, a double-centred wheel, and a wheel of wheels, and we consider with every case the two cases of whether or not w_1 is adjacent to v_1.

By the choice of r and s, by Claim 4.3.6 and Claim 4.3.5, if B_H is a generalized wheel, then it is a wheel.

In case B_H is a wheel, since G is not a double-centred wheel, B_H can only be as shown in Figure 4.11, that is v_3 has exactly two neighbours in the interior of C, those two neighbours are w_1 and w_2, w_1 is adjacent to v_1, w_2 is adjacent to v_i for some $i > 4$, and if t is the maximum such i, then the subgraph bounded by $v_1 w_1 w_2 v_t \cdots v_k v_1$ is a wheel with centre x.

Case 1. *B_H is a wheel.*

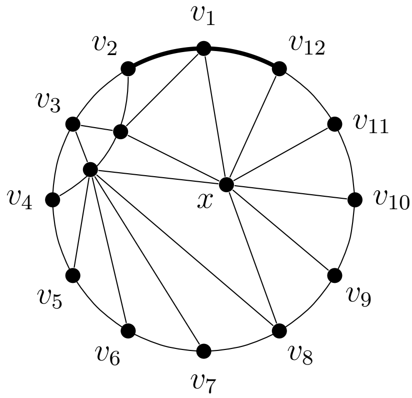


Figure 4.11: The obstruction is a wheel.

In this case we colour G as follows (See Figure 4.11). We colour xv_t and v_3w_2 so as to avoid the colourings of v_kxv_t and $v_3w_2v_t$ unextendable to the respective broken wheels. Colour v_t by a colour that avoids the bad colourings of xv_t and then colour x.

If with the colour v_t has now, there is only one colour of v_3 that can make the colouring of $v_3w_2v_t$ bad, colour v_3 by a different colour, colour w_1 then colour w_2. If the bad colourings of v_3w_2 with the now fixed colour of v_t are ab and ba, let c be a colour in $L(w_2)$ different from a, b, the colour of x and the colour of w_t. If c is not in $L(w_1) \setminus \{\varphi(v_1), \varphi(v_2)\}$, colour w_2 by c, colour v_3, then colour w_1.

If $c \in L(w_1) \setminus \{\varphi(v_1), \varphi(v_2)\}$, then colour v_3 by the same colour of x in case the colour of x is one of a or b, and in case the colour of x is neither a nor b, colour v_3 by the colour of a and b that is not in $L(w_1) \setminus \{\varphi(v_1), \varphi(v_2)\}$. In case both a and b with the colour of x and c are in $L(w_1) \setminus \{\varphi(v_1), \varphi(v_2)\}$, this means that $|L(w_1)| \geq 6$ and there is no problem in the first place, also in case v_3 contains a colour different from a and b there is no problem. Finally colour w_2 by c.

Case 2. *$G - v_3$ is a double-centred wheel.*

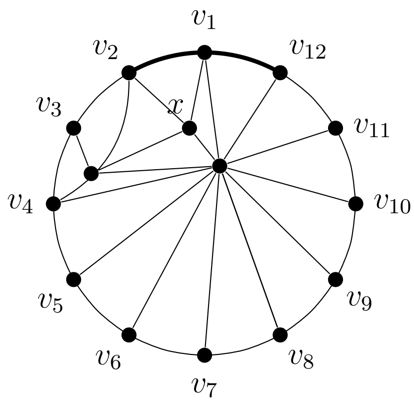


Figure 4.12: $G - v_3$ is a double-centred wheel.

In this case, G can only be as shown in Figure 4.12, that is v_3 has exactly one neighbour, w_1, in the interior of C and x is adjacent to only w_1, v_1 and v_2 and the second centre, y. We colour G as follows. After choosing two colours in $L(v_3)$ and deleting them from the lists of the neighbours of v_3 in the interior of C, we colour yw_1 such that the colouring of $v_k y w_1$ is extendable to the broken wheel bounded by $v_k y w_1 v_4 \cdots v_k$.

If only one colour of w_1 is involved in bad colourings of $w_1 y v_k$, colour w_1 by a different colour, colour x then colour y. So suppose that the bad colourings of yw_1 are ab and ba. If there is a colour in $L(w_1)$ different from a, b, $\varphi(v_2)$ and the two colours kept for v_3, colour w_1 with that colour, colour x then colour y. Therefore, we may assume that the only colours in $L(w_1)$ different from $\varphi(v_2)$ and the two colours kept for v_3 are a and b.

Since $|L(y)| \geq 5$, there is a colour c different from a and b in $L(y) \setminus \{\varphi(v_1), \varphi(v_k)\}$. Since $|L(x)| = 4$, either one of a or b is not in $L(x)$, or one of $\varphi(v_1)$ and $\varphi(v_2)$ is a or b. In either case, there is a colour of a and b that if we colour w_1 with, x still has two available colours. Colour w_1 by the appropriate colour of a and b, colour y by c then colour x.

Case 3. *B_H is a double-centred wheel.*

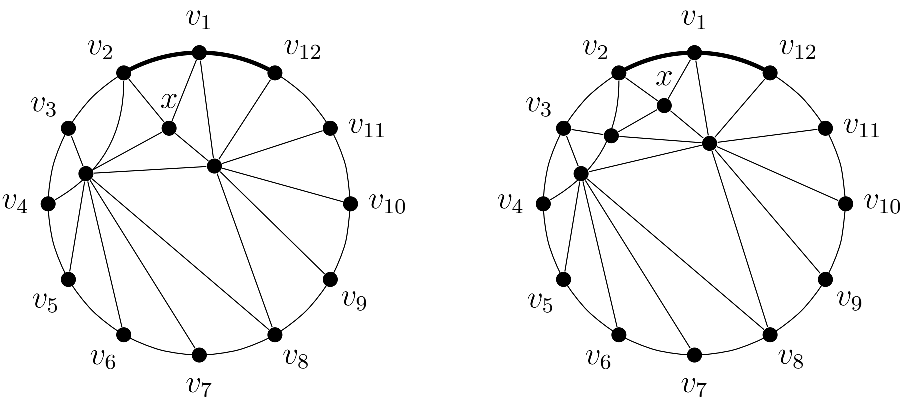


Figure 4.13: The obstruction contained in H is a double-centred wheel.

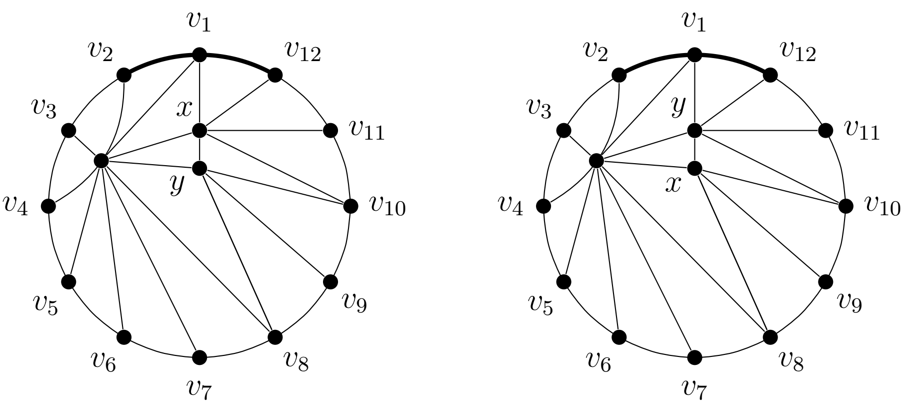


Figure 4.14: The obstruction contained in H is a double-centred wheel.

Let y be the centre different from x. We consider first the case when v_1 and w_1 are adjacent to y, and x is the centre of a wheel bounded by the cycle $v_1v_2w_1yv_1$ (See Figure 4.13). As in Case 2, we can colour w_1yv_k such that its colouring is extendable to the broken wheel bounded by $v_kyw_1v_s \cdots v_ky$ or $v_kyw_1w_2v_s \cdots v_ky$. After the double-centred wheel is coloured, we can by Theorem 3.2.6 extend the colouring of w_1v_s to the broken wheel bounded by $w_1v_4 \cdots v_sw_1$, or extend the colouring of w_2v_s to the broken wheel bounded by $w_2v_4 \cdots v_sw_2$. Finally colour v_3.

The remaining possibilities when the obstruction is a double-centred wheel are shown in Figures 4.14, 4.15 and 4.16. In each of those possibilities we consider the cases of whether or not the wheel centred at x has

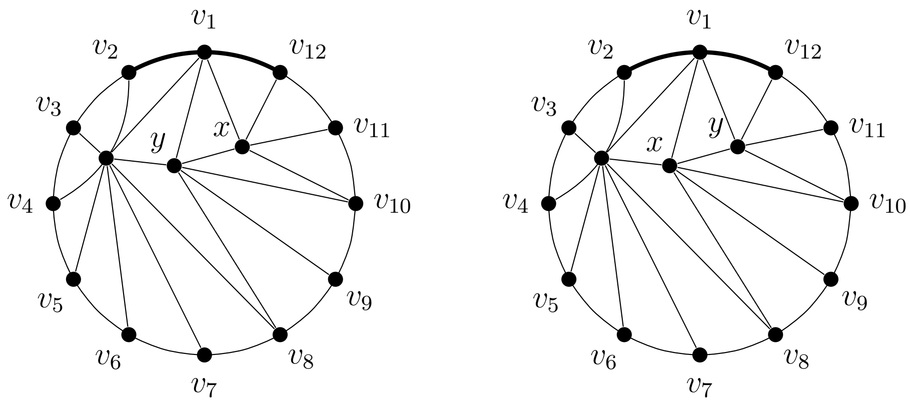


Figure 4.15: The obstruction contained in H is a double-centred wheel.

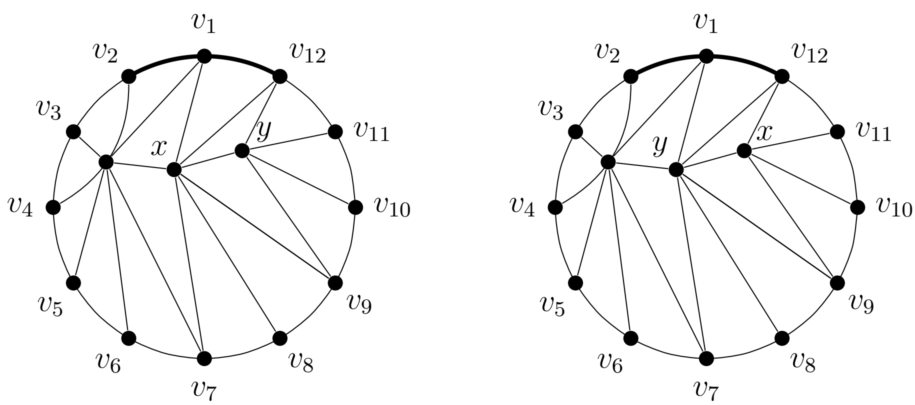


Figure 4.16: The obstruction contained in H is a double-centred wheel.

its outer cycle of length four.

Let t be the unique index different from 1 such that both x and y are adjacent to v_t ($t = 10$ in Figures 4.14 and 4.15). The cases shown in Figure 4.14 are when the other vertex to which both x and y are adjacent is w_1, and in Figure 4.15 this vertex is v_1.

In the case shown in the left drawing of Figure 4.14, we colour G as follows. If one of the colours in $L(v_t)$ that avoid the bad colourings of $v_k x v_t$ (for the broken wheel bounded by $v_k x v_t \cdots v_k$) is not in $L(x) \setminus \{\varphi(v_1), \varphi(v_2)\}$, we colour v_t by such a colour, say c. We now colour v_s by a colour that avoids the bad colourings of $v_t y v_s$ (with the now fixed colour of v_t), and if possible we choose it so as to also avoid the bad colourings of $v_s w_1 v_2$.

If it is not possible to choose the colour of v_s so as to avoid the bad colourings of $v_s w_1 v_2$, then the bad colourings of $v_s w_1$ are of the form ab and ba. In any case by fixing the colour of v_s, at least one of the two colours left in the list of $L(w_1)$ makes the colouring of $v_s w_1 v_2$ good.

Colour w_1 with such a colour, colour x (which is colourable even though four of its neighbours are already coloured because v_t is coloured c which was chosen to be outside $L(x) \setminus \{\varphi(v_1), \varphi(v_2)\}$), then colour y and extend the colourings to the respective broken wheels. If there is no colour as c, then the good colourings of $v_t x$ are of the form de and ed.

If one of e and d, say d, is such that colouring v_t with it avoids the bad colourings of $v_t y v_s$, then colour v_t with d, colour x with e, colour v_s by a colour that avoids the bad colourings of $v_s w_1 v_2$, colour w_1, then colour y.

Therefore, we can assume that both d and e are colours of v_t involved in bad colourings of $v_t y v_s$. By Lemma 3.2.11, there is at most one more colour (different from e and d) that appears in a bad colouring of $v_t y v_s$ (either as a colour of v_t, y or v_s), and so, since $|L(y)| \geq 5$, there are at least two colours in $L(y) \setminus \{d, e\}$ that avoid the bad colourings of $v_t y v_s$.

One of those two colours, say g, is different from the colour, say f, in $L(w_1) \setminus \{\varphi(v_1), \varphi(v_2)\}$ that avoids the bad colourings of $v_2 w_1 v_s$. Colour y with g, colour w_1 with f, then colour x with e or d depending on whether

$f = d$ or $f = e$, then colour v_t and v_s.

In the case shown in the right drawing of Figure 4.14, we use a new argument than in the previous cases. Since, $|L(v_3) \setminus \{\varphi(v_2)\}| \geq 2$ and $|L(w_1) \setminus \{\varphi(v_1), \varphi(v_2)\}| \geq 3$, there are two dictatorships for v_3w_1 with dictator v_3. Let $\mathcal{C}_{v_3w_1}$ denote the union of those two dictatorships. Similarly, there is a confederacy $\mathcal{C}_{v_{k-1}y}$ for $v_{k-1}y$.

By Corollary 3.3.12, there is a confederacy $\mathcal{C}_{w_1v_s}$ for w_1v_s ($s = 8$ in this drawing) such that every colouring of w_1v_s in $\mathcal{C}_{w_1v_s}$ extends to a colouring of the broken wheel bounded by $w_1v_3 \cdots v_sw_1$ whose restriction to v_3w_1 is in $\mathcal{C}_{v_3w_1}$. Similarly, there is a confederacy $\mathcal{C}_{yv_t}$ for yv_t that corresponds to $\mathcal{C}_{v_{k-1}y}$ in the subgraph bounded by $v_{k-1}yv_t \cdots v_{k-1}$.

By Lemma 4.2.6, there is a colouring $\varphi \in \mathcal{C}_{w_1v_s}$ and a colouring $\psi \in \mathcal{C}_{yv_t}$ such that $\varphi \cup \psi$ extends to a colouring of the wheel with centre x (bounded by $yw_1v_s \cdots v_ty$).

The remaining subcases, shown in Figures 4.15 and 4.16, can be coloured using similar techniques.

Case 4. *$G - v_3$ is a wheel of wheels that is neither a wheel nor a double-centred wheel.*

Again since there are no separating 4-cycles with interior consisting only of neighbours of v_3, there is exactly one neighbour w_1 of v_3 in the interior of C and w_1 is adjacent to the centre of the wheel of wheels $G - v_3$ (See Figure 4.17).

Let y be the centre of $G - v_3$. By Claim 4.3.6, and since every section of a wheel of wheels is either a wheel or a broken wheel, v_1 is also adjacent to y. Now we have the 4-cycle $v_1v_2w_1yv_1$. Then either y is adjacent to v_2 or the interior of this cycle consists of x alone and together with the cycle they make a wheel.

By Claim 4.3.5 any wheel section of $G - v_3$ that is not centred at x either contains v_1 and v_2 or w_1 and v_4, and since there is no separating 4-cycle with interior that contains any vertex other than x, y is adjacent to at most one of v_2 and v_4 and any 5-list vertex different from y is adjacent

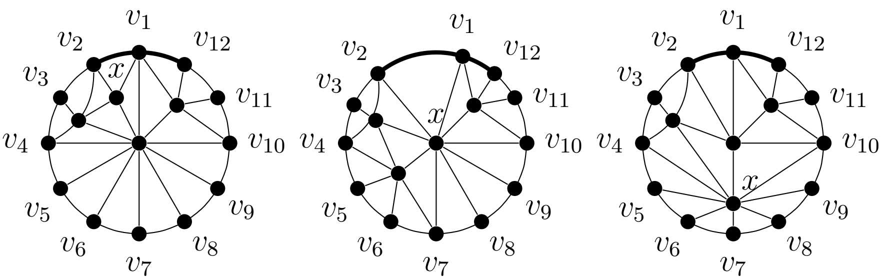


Figure 4.17: $G - v_3$ is a wheel of wheels.

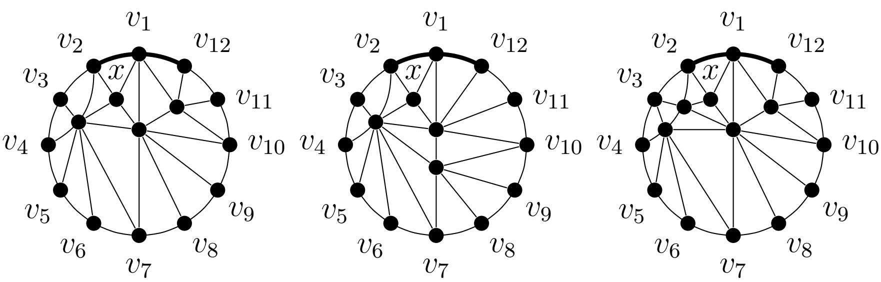


Figure 4.18: The obstruction contained in H is a wheel of wheels.

to at most one of v_1 and w_1 if $y \neq x$.

Now we show how to colour G. If x is the centre of $G - v_3$ then x is adjacent to v_2 since there are no vertices in the interior of the 4-cycle $v_1v_2w_1xv_1$ (See the middle drawing of Figure 4.17). Since x is adjacent to v_2 and there is no separating 4-cycle with interior containing vertices other than x, x is not adjacent to v_4.

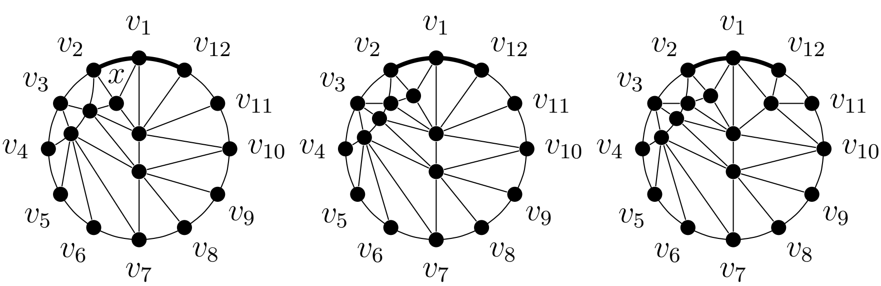


Figure 4.19: The obstruction contained in H is a wheel of wheels.

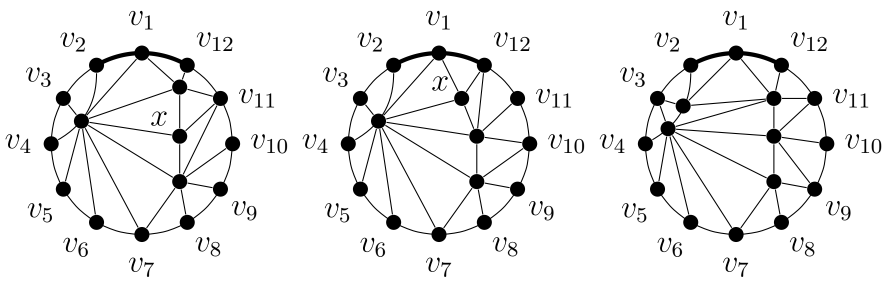


Figure 4.20: The obstruction contained in H is a wheel of wheels.

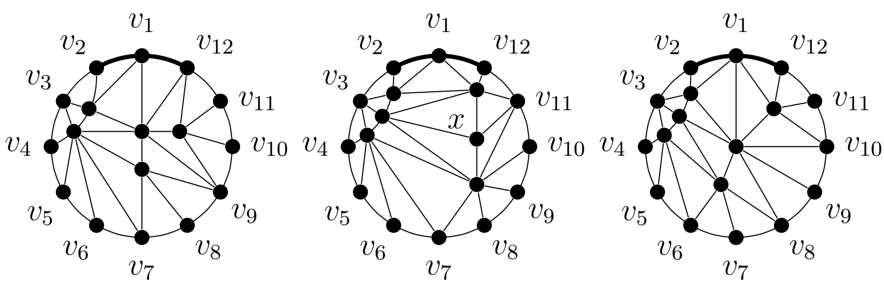


Figure 4.21: The obstruction contained in H is a wheel of wheels.

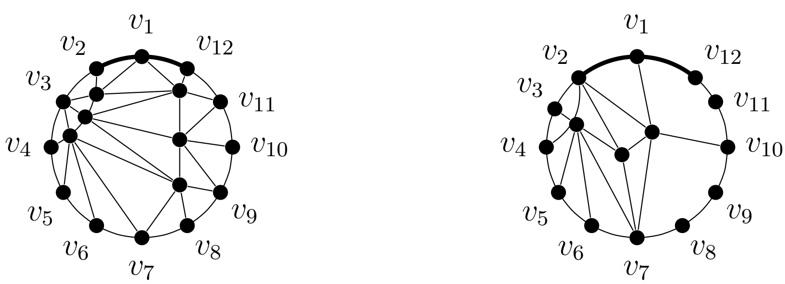


Figure 4.22: The obstruction contained in H is a wheel of wheels.

Therefore, if l is the least number ≥ 5 such that x is adjacent to v_l, the section bounded by $xw_1v_4 \cdots v_lx$ is a wheel not a broken wheel. Consequently, since this wheel section contains w_1, which is an interior vertex of C different from x, if m is maximum such that x is adjacent to v_m, the subgraph bounded by $xv_2 \cdots v_mx$ does not contain a generalized wheel subcanvas with principal path v_2xv_m. By Theorem 3.2.4, any colouring of v_2xv_m is extendable to the subgraph bounded by $xv_2 \cdots v_mx$.

By Claim 4.3.6, and since x is adjacent to v_2, it is not adjacent to v_k. Therefore, the section bounded by $v_1xv_m \cdots v_kv_1$ is a wheel not a broken wheel, and so by Lemma 3.2.10 there is at most one colouring of xv_1v_k unextendable to that section. Colour x by a colour different from its colour in that colouring of xv_1v_k, extend the colouring to the section bounded by $v_1xv_m \cdots v_kv_1$, then extend it to the subgraph bounded by $xv_2 \cdots v_mx$.

We can suppose now that the centre y is different from x. Let l and m, different from 1, 2 and 3, be minimum and maximum respectively such that y is adjacent to v_l and v_m.

Choose two colours in $L(v_3) \setminus \{\varphi(v_2)\}$ and delete them from $L(w_1)$. Consider first the case when x is the centre of the 4-cycle $v_1v_2w_1yv_1$. Suppose that y is adjacent to v_k. If $L(x) \setminus \{\varphi(v_1), \varphi(v_2)\}$ is equal to the set of two colours in $L(w_1)$ that remain after deleting the two chosen colours for v_3 and deleting $\varphi(v_2)$, let this set be $\{a, b\}$.

Colour y by a colour different from $\varphi(v_k)$ (if it is coloured, that is if $|V(P)| = 3$, and if it is not coloured, colour it first), $\varphi(v_1)$, a and b. Then colour xw_1 either ab or ba depending on which of them makes the colouring of w_1yv_k (with the now two fixed colours of y and $\varphi(v_k)$) extendable to the subgraph bounded by $v_kyw_1v_4 \cdots v_k$. Finally, colour this subgraph then colour v_3 by a colour different from the colour of v_4.

If $L(x) \setminus \{\varphi(v_1), \varphi(v_2)\} = \{a, b\}$ and the two colours remaining in $L(w_1)$ after deleting the two chosen colours for v_3 and deleting $\varphi(v_2)$ include a colour $c \notin \{a, b\}$, colour y by a colour different from $\varphi(v_1)$ and $\varphi(v_k)$ and the colour of the unique colouring of w_1yv_k unextendable to the subgraph bounded by $v_kyw_1v_4 \cdots v_k$.

If the colour given to y is c, colour w_1 first (still has one colour in its list) then colour x (a and b are different from c). If it is different from c, colour x (by a or b) then colour w_1 by c. Extend the colouring to the subgraph bounded by $v_k y w_1 v_4 \cdots v_k$ then to v_3.

Therefore, in case x is the centre of the 4-cycle $v_1 v_2 w_1 y v_1$, we may assume that y is not adjacent to v_k, and so the section bounded by $v_1 y v_m \cdots v_k v_1$ is a wheel not a broken wheel. But note that now the subgraph bounded by $v_m y w_1 v_4 \cdots v_m$ may be a broken wheel as in the leftmost drawing of Figure 4.17.

Again if $L(x) \setminus \{\varphi(v_1), \varphi(v_2)\} = \{a, b\}$ and the two available colours for w_1 are also a and b, colour y by a colour different from $\varphi(v_1)$, a, b and the colour of the unique colouring of $y v_1 v_k$ unextendable to the wheel section bounded by $v_1 y v_m \cdots v_k v_1$.

Extend the colouring to the wheel section bounded by $v_1 y v_m \cdots v_k v_1$, then colour $x w_1$ either ab or ba depending on which of them makes the colouring of $w_1 y v_m$ extendable to the subgraph bounded by $v_m y w_1 v_4 \cdots v_m$ (since now y and v_m are coloured, it is only one colour of w_1 that can make the colouring of $w_1 y v_m$ bad).

If w_1 has an available colour c different from a and b, then if the colour y receives from the colouring described above is c, we have only one colour left to colour w_1 with and which may make the colouring of $w_1 y v_m$ unextendable to the subgraph bounded by $v_m y w_1 v_4 \cdots v_m$.

Let z be the centre of the wheel bounded by $v_1 y v_m \cdots v_k v_1$. We colour w_1 by c then try to colour y, v_m and z such that the colourings of $w_1 y v_m$ and $v_k z v_m$ are extendable to the respective broken wheels. With the now fixed colours of w_1 and $\varphi(v_k)$ there are two possibilities for the type of the bad colourings of $y v_m$ and two possibilities for the types of the bad colourings of $z v_m$ (cf. Lemma 3.2.11).

If the bad colourings of $y v_m$ are de and ed and the bad colourings of $z v_m$ are fg and gf, colour v_m by a colour different from f and g, by this we have avoided the bad colourings of $z v_m$. If this colour is different from d and e, then we have also avoided the bad colourings of $y v_m$, and if it is one of d and e then we still can colour y by a colour different from d, e,

c and $\varphi(v_1)$ and so avoid the bad colourings of yv_m. Then colour z and extend the colourings to the respective broken wheels, and finally colour v_3 and x (recall that w_1 is coloured $c \notin L(x) \setminus \{\varphi(v_1), \varphi(v_2)\}$).

If the two bad colourings of yv_m have v_m coloured the same and the two bad colourings of zv_m have v_m coloured the same, we can avoid all bad colourings by colouring v_m by the third colour (different from the one involved in the bad colourings of $v_m y$ and the one involved in the bad colourings of $v_m z$).

If the bad colourings of yv_m are de and ed while the bad colourings of $v_m z$ involve only one colour f of v_m, colour v_m by the colour of d and e different from f, colour y by a colour different from d, e, c and $\varphi(v_1)$, then colour z. The case when the bad colourings of $v_m z$ are fg and gf and the bad colourings of $v_m y$ have one colour for v_m is similar.

Now we may assume that y is adjacent to v_2 (See the rightmost drawing of Figure 4.17). Then by Claim 4.3.6, y is not adjacent to v_k, and so the section bounded by $v_1 y v_m \cdots v_k v_1$ is a wheel not a broken wheel. If this section is centred at x, then there are at most two colours of y that with the given colours of v_1 and v_k make the colouring of $v_k v_1 y$ unextendable to this section.

Colour y by a colour different from those two colours and from $\varphi(v_1)$ and $\varphi(v_2)$, and extend the colouring to the wheel section bounded by $v_1 y v_m \cdots v_k v_1$. Now consider the subgraph bounded by $yv_2 \cdots v_m y$, which now has the path $v_2 y v_m$ coloured. Since y is adjacent to v_2 and there is no separating 4-cycle with interior consisting of neighbours of v_3, y is not adjacent to v_4, and so the section bounded by $yw_1 v_4 \cdots v_l y$ is a wheel not a broken wheel. This wheel contains an interior vertex of C different from y, therefore, the subgraph bounded by $yv_2 \cdots v_m y$ does not contain a generalized wheel subcanvas with principal path $v_2 y v_m$, and so the colouring of $v_2 y v_m$ is extendable to it by Theorem 3.2.4.

Suppose now that the wheel section bounded by $v_1 y v_m \cdots v_k v_1$ is centred at a 5-list vertex, and so the wheel section with centre x lies somewhere else, let t be maximum such that v_t is in the wheel section centred at x. As we showed above the section bounded by $yw_1 v_4 \cdots v_l y$ is a wheel, and so the subgraph bounded by $yv_2 \cdots v_l y$ is colourable whatever colouring is given to $v_2 y v_l$ by Theorem 3.2.4.

Therefore, colour y by a colour different from $\varphi(v_1)$, $\varphi(v_2)$, the colour of the unique colouring of yv_1v_k unextendable to the subgraph bounded by $v_k v_1 y v_t \cdots v_k$, and the colour of the unique colouring of yv_t unextendable to the wheel section centred at x. Extend the colouring to the subgraph bounded by $v_k v_1 y v_t \cdots v_k$ then to the wheel with centre x then to the rest the subgraph bounded by $v_k v_1 y v_l \cdots v_k$ so that $v_l y v_2$ is coloured, then extend the colouring to the subgraph bounded by $yv_2 \cdots v_l y$.

Case 5. *B_H is a wheel of wheels that is neither a wheel nor a double-centred wheel.*

See Figures 4.18, 4.19, 4.20, 4.21, and 4.22. This case can be proved by arguments similar to those of the preceding cases. □

4.4 An Extension of a Theorem of Postle and Thomas

In this section we prove Theorem 2.1.3, which states that a plane graph with two 2-lists on the outer walk and one inner 4-list is colourable if the 4-list vertex is not the centre of a wheel attached to the outer walk of the graph. This is an extension of Theorem 3.2.8 of Postle and Thomas. To prove their theorem, they proved a stronger theorem, Theorem 4.4.1, below. We also prove Theorem 2.1.3 through a stronger theorem which is an extension of Theorem 4.4.1. This is Theorem 4.4.2.

Excluding wheels with centre the 4-list vertex is not a necessary condition, but it enables us to go on in the proof of Theorem 4.4.2 following the method of Postle and Thomas in the proof of Theorem 4.4.1, [11, Theorem 3.1].

The proofs of Claims 4.3.6 and 4.3.9 are different from the proofs of the corresponding claims in Theorem 4.4.1. We use Corollaries 3.3.8 and

3.3.12. Those corollaries were not used in the proof of Theorem 4.4.1. However, in many parts, the proof of 4.4.2 is almost the same as the proof of 4.4.1.

Theorem 4.4.1. *[11] Let (G, S, L) be a canvas, where S has two components: a path P and an isolated vertex u with $|L(u)| \geq 2$. Assume that if $|V(P)| \geq 2$, then G is 2-connected, u is not adjacent to an internal vertex of P and there does not exist a chord of the outer walk of G with an end in P which separates a vertex of P from u. Let L_0 be a set of size two. If $L(v) = L_0$ for all $v \in V(P)$, then G has an L-colouring, unless $L(u) = L_0$ and $V(S)$ induces an odd cycle in G.*

Theorem 4.4.2. *Let (G, S, L, x) be a canvas such that S consists of a path P and an isolated vertex u. Assume:*

(a) all vertices of P have the same list L_0 of size 2;

(b) if $|V(P)|$ is 1 or 2, then x is not the centre of a wheel subcanvas of G;

(c) if $|V(P)| > 2$, then x is not the centre of a wheel in G; and

(d) if $|V(P)| \geq 2$, then:

i. G is 2-connected;

ii. x is not adjacent to two vertices at an odd distance in P;

iii. u is not adjacent to an internal vertex of P; and

iv. there is no chord of the outer walk of G having an end in P that separates a vertex of P from u.

Then either G is L-colourable or $L(u) = L_0$ and $V(S)$ induces an odd cycle in G.

Proof. Let (G, S, L, x) be a counterexample with $|V(G)|$ minimum and, subject to that, with $|V(P)|$ maximum. Let C be the outer walk of G.

Claim 4.4.3. *G is 2-connected.*

Proof. Suppose for a contradiction that G is not 2-connected. Then by assumption $(d) - i$, $|V(P)| = 1$. Let z be a cut vertex of G. Then G can be expressed as $G = G_1 \cup G_2$, where $V(G_1) \cap V(G_2) = \{z\}$ and $V(G_1) \setminus V(G_2)$ and $V(G_2) \setminus V(G_1)$ are both non-empty.

If u and P are in the same one of G_1 and G_2, then we colour the one containing them by minimality or by Theorem 3.2.8, then colour the other side by Theorem 3.2.6 or Theorem 4.3.1. Therefore we may assume without loss of generality that $u \in V(G_2) \setminus V(G_1)$ and the unique vertex of P is in $V(G_1) \setminus V(G_2)$.

Now consider the canvas (G_1, S_1, L), where $S_1 = P + z$, the graph obtained from P by adding z as an isolated vertex. There exists an L-colouring φ_1 of G_1 either by Theorem 4.3.1 or Theorem 3.2.6. Let L_1 be the list assignment of G_1 such that $L_1(v) = L(v)$ for every $v \in V(G_1)\setminus\{z\}$ and $L_1(z) = L(z) \setminus \{\varphi_1(z)\}$. Since $|V(G_1)| < |V(G)|$, there exists an L_1-colouring φ_2 of G_1. Note that $\varphi_1(z) \neq \varphi_2(z)$.

Let L_2 be the list assignment of G_2 such that $L_2(z) = \{\varphi_1(z), \varphi_2(z)\}$ and $L_2(v) = L(v)$ for all $v \in V(G_2)\setminus\{z\}$. Consider the canvas (G_2, S_2, L_2), where S_2 consists of the isolated vertices z and u. Since $|V(G_2)| < |V(G)|$, there exists an L_2-colouring φ of G_2. Letting i be such that $\varphi_i(z) = \varphi(z)$, $\varphi \cup \varphi_i$ is an L-colouring of G, a contradiction. □

Claim 4.3.4 shows G is 2-connected and, therefore, every face of G is bounded by a cycle. In particular, C is a cycle.. Let v_1 and v_2 be the two neighbours of the end-vertices of P in $V(C) \setminus V(P)$.

Claim 4.4.4. *There is no chord of C with an end in P.*

Proof. Suppose for a contradiction that there is a chord with an end in P. By assumption $(d) - iv$. in the statement of the theorem, both P and u are on the same side of the chord. Colour the side of the chord containing P and u first by minimality or by Theorem 3.2.8 depending on whether or not x is on the same side, then colour the other side by Theorem 4.3.1 (which is colourable since it contains no wheel subcanvas with centre x) or by Theorem 3.2.6. □

Claim 4.4.5. *There is no chord of C that has P and u on the same side.*

Proof. If such a chord exists, colour the side containing P and u by minimality, then extend the colouring to the other side by Theorem 3.2.6. □

Claim 4.4.6. $v_1 \neq v_2$.

Proof. Suppose for a contradiction that $v_1 = v_2$. Then $v_1 = v_2 = u$. If C is an odd cycle, then either $L(u) = L_0$ and we are done or $L(u) \setminus L_0 \neq \emptyset$. Thus, we may assume $L(u) \setminus L_0 \neq \emptyset$ and C has an L-colouring. Thus, whether C is odd or even, either we are done or C has an L-colouring φ.

Let $G' := G \setminus V(P)$ and let L' be the list assignment of G' such that, $L(u) = \{\varphi(u)\}$, and for every v in $V(G') \setminus \{u\}$, $L'(v)$ is obtained from $L(v)$ by deleting the colours φ gives to its neighbours in P.

Since x is not adjacent to two vertices at an odd distance in P, and since at most two colours are used in colouring P, $|L'(v)| \geq 3$ for every vertex v different from u on the outer walk of G'. Therefore, G' has an L'-colouring φ' by Theorem 4.3.1 or Theorem 3.2.6 such that $\varphi'(u) = \varphi(u)$, and thus G has an L-colouring, a contradiction. □

Claim 4.4.7. *For $i \in \{1, 2\}$, $L_0 \subseteq L(v_i)$ and $|L(v_i)| = 3$.*

Proof. By symmetry, it suffices to prove the claim for $i = 1$. Suppose for a contradiction that $|L(v_1) \setminus L_0| \geq 2$.

Case 1. $|V(P)| \geq 3$ *or x is adjacent to a vertex in P.*

In this case, there is a colour c in L_0 such that $|L(v_1) \setminus \{c\}| \geq 3$. Colour the neighbour of v_1 in P by this colour, and then extend the colouring to P. Let L' be the list assignment of $G - P$ such that, for every $v \in V(G - P)$, $L'(v)$ is obtained from $L(v)$ by deleting the colours of the neighbours of v in P.

If x is adjacent to a vertex in P, then by hypothesis (d)-ii and Theorem 3.2.8, $G - P$ is L'-colourable. If x is not adjacent to a vertex in P, and if $|V(P)| \geq 3$, then x is not the centre of any wheel (even if not a subcanvas of G) by hypothesis (c). Thus, x is not the centre of a wheel

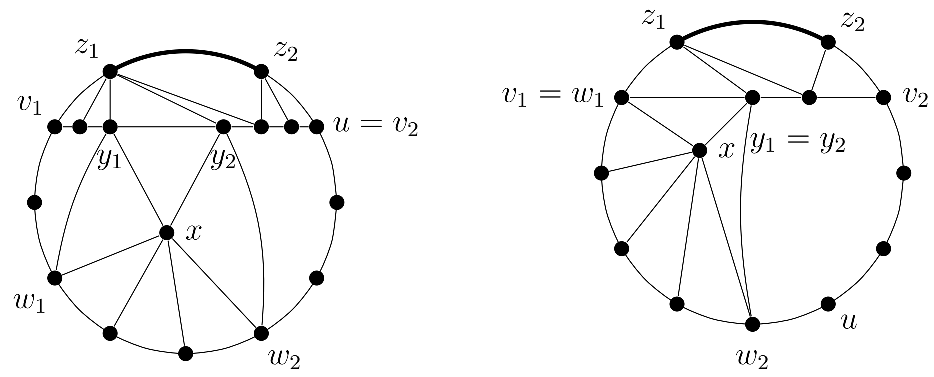


Figure 4.23: x is the centre of a wheel subcanvas of $G - P$ that is not a subcanvas of G.

subcanvas in $G - P$. Therefore, $G - P$ is L'-colourable by minimality.

Case 2. $|V(P)| \leq 2$ *and x is not adjacent to P.*

In this case, if x is not the centre of a wheel subcanvas of $G - P$, then G is colourable. Thus, we assume that x is the centre of a wheel W that is a subcanvas of $G - P$ (but not a subcanvas of G). See Figure 4.23 for examples.

Note that even though, due to hypothesis (d)-ii, we cannot assume that the interior of G is triangulated, we may assume that the neighbours of the vertices of P form a path Q. Let y_1 and y_2 be the two vertices in $W \cap Q$ closest to v_1 and v_2 respectively, with distance measured in Q.

Since this is a minimum counterexample, there are no separating 4-cycles with interior consisting of 5-lists only. Therefore, there are at most three vertices in $\partial W \cap Q$. Again since this is a minimum counterexample, there is no triangle with centre x. Thus, ∂W contains vertices from C.

Let z_1 and z_2 be the vertices of P adjacent to v_1 and v_2 respectively, and let w_1 and w_2 be respectively the neighbours of y_1 and y_2 in $V(C) \cap V(\partial W)$. For $i \in \{1, 2\}$, let H_i be the subgraph bounded by the $v_i w_i$-path in ∂G not containing P, $w_i y_i$, and $y_i Q v_i$.

Let $N(P)$ denote the set of vertices that have a neighbour in P. Let L' be the list assignment of $G - P$ such that, for every $v \in (V(G - P) \setminus$

$\{v_1, v_2\}) \cap N(P)$, $L'(v) = L(v) \setminus L_0$, and otherwise $L'(v) = L(v)$. Note that, whether $u = v_i$ for some $i \in \{1, 2\}$ or not, there is an L'-confederacy $\mathcal{C}_{uw}$ for uw, where w is any neighbour of u in $V(\partial G) \setminus V(P)$.

For $i \in \{1, 2\}$, if u is in H_i, then by Corollary 3.3.12, $\mathcal{C}_{y_i w_i} := \Phi_{H_i}(y_i w_i, \mathcal{C}_{uw})$ contains an L'-confederacy. For $i \in \{1, 2\}$, let t_i be the neighbour of v_i in $V(\partial G) \setminus V(P)$, and let $\mathcal{C}_{v_i t_i}$ be an L'-confederacy for $v_i t_i$. For $i \in \{1, 2\}$, if u is not in H_i, then by Corollary 3.3.12, $\mathcal{C}'_{y_i w_i} := \Phi_{H_i}(y_i w_i, \mathcal{C}_{v_i t_i})$ contains an L'-confederacy.

Since we assumed $|L(v_1) \setminus L_0| \geq 2$, we can choose $\mathcal{C}_{v_1 t_1}$ such that the colours of v_1 are not in L_0. Since a confederacy contains at least three colourings, for $i \in \{1, 2\}$, we can choose colourings for $y_i w_i$ from $\mathcal{C}_{y_i w_i}$ or $\mathcal{C}'_{y_i w_i}$, depending on whether u is in H_i or not, such that together they are extendable to W.

In case y_1 and y_2 are adjacent, we can find such colourings for $y_1 w_1$ and $y_2 w_2$ by Lemma 4.2.6. The other two possibilities for y_1 and y_2 are, the case when $y_1 = y_2$, and the case when there is exactly one vertex between y_1 and y_2 on $\partial W \cap Q$ (recall that $\partial W \cap Q$ contains at most three vertices). In those two cases, we can prove lemmas similar to Lemma 4.2.6 about the existence of appropriate colourings for $y_1 w_1$ and $y_2 w_2$.

There are two broken wheels in W with principal paths $y_1 x y_2$ and $w_1 x w_2$. The one with principal path $y_1 x y_2$ is bounded by a 4-cycle or a triangle since there are at most three vertices in $V(\partial W) \cap V(Q)$. We extend the colourings of $y_1 w_1$ and $y_2 w_2$ to x first and then to the broken wheels; so the colourings should also be chosen such that x still has an available colour.

Now for $i \in \{1, 2\}$, we extend the colouring of $y_i w_i$ to H_i, and then colour P starting with z_2. □

Claim 4.4.8. *For $i \in \{1, 2\}$, if $v_i \neq u$, then either v_i is the end of a chord of C that separates P from u, or x is adjacent to v_i and a vertex in P.*

Proof. By symmetry it suffices to prove the claim for v_1. Suppose that $v_1 \neq u$ and that it is not the end of a chord that separates u from P.

Let P' be the path obtained from P by adding v_1, let $S' = P' + u$, and let L' be the list assignment of G defined by $L'(v_1) = L_0$ and $L'(v) = L(v)$ for all $v \in V(G) \setminus \{v_1\}$. Consider the canvas (G, S', L'). This canvas is not colourable since (G, S, L) is not. As (G, S, L) was chosen so that $|V(P)|$ is maximized, if (G, S', L') satisfies the hypotheses of the theorem, then $G[V(S')]$ is an odd cycle and $L(u) = L_0$.

Since by Claim 4.3.3 there is no chord of C with an end in P, u is not adjacent to an internal vertex of P'. From this, and since we assumed v_1 is not the end of a chord that separates a vertex of P' from u, then either P' and x do not satisfy (d)-ii, or $G[V(S')]$ is an odd cycle and $L(u) = L_0$.

If P' and x do not satisfy (d)-ii, then, since P and x satisfy (d)-ii, x is adjacent to v_1 and a vertex in P. Thus, suppose that $G[V(S')]$ is an odd cycle and $L(u) = L_0$. Then, since by Claim 4.4.4 there is no chord with an end in P, $u = v_2$.

By Claim 4.4.5, there is no chord that has P and u on the same side. Thus, u is adjacent to v_1 in C. That is, $V(C) = V(P) \cup \{v_1, v_2\}$.

Colour v_1 by the unique colour in $L(v_1) \setminus L_0$, then extend the colouring to $C - v_1$ using L_0. Now we have the following two cases.

Case 1. $|V(P)|$ *is either* 1 *or* 2.

In this case, C is a triangle or a 4-cycle. Then G colourable unless C is a 4-cycle and $C + x$ is a wheel. Since G contains no wheel subcanvas with centre x, G is colourable.

Case 2. $|V(P)| > 2$.

In this case, delete from the lists of the vertices in the interior of C the colours of their neighbours in P. Then, the subgraph G' consisting of the union of the interior of C and v_1v_2 now has v_1v_2 coloured and has its other outer boundary vertices having lists of size at least three. Since $|V(P)| > 2$, G does not contain any wheel with centre x. Thus, G' does not contain a wheel subcanvas with centre x, and so it is colourable by

Theorem 4.3.1 or Theorem 3.2.6. □

Let Q be the path in C obtained by adding v_1 and v_2 to P.

Claim 4.4.9. *If w_1 and w_2 are two consecutive neighbours of x in Q such that $\{w_1, w_2\} \neq \{v_1, v_2\}$, then the interior of the cycle xw_1Qw_2x is empty.*

Proof. Let w_1 and w_2 be two vertices as in the statement of the claim. Suppose for a contradiction that the interior of xw_1Qw_2x is not empty. We may assume without loss of generality that $w_2 \notin \{v_1, v_2\}$.

Colour xw_1Qw_2x with its exterior by induction. Let G' be the subgraph x, w_1, and the vertices in the interior of xw_1Qw_2x. Let L' be the list assignment of G' such that for every $v \in V(G) \setminus V(G')$, $L'(v)$ is obtained from $L(v)$ by deleting the colours of the neighbours of v in $G - V(G')$.

There is a precoloured path of length one in $\partial G'$, namely xw_1, and since $w_2 \notin \{v_1, v_2\}$, every vertex in $\partial G'$ not in this path has a list of size at least three. Thus, G' is colourable by Thomassen's Theorem 3.2.6. □

Claim 4.4.10. *If $\{v_1, v_2\} \cap \{u\} = \emptyset$, then at least one of v_1 and v_2 is the end of a chord separating P from u.*

Proof. Suppose for a contradiction that v_1 and v_2 are both different from u and none of them is the end of a chord separating P from u. By Claim 4.4.8, for $i \in \{1, 2\}$, x is adjacent to v_i and a vertex in P at an odd distance from v_i in Q. From this, and hypothesis $(d) - ii$, we have that Q is of even length.

By Claim 4.4.9, and since x is adjacent to a vertex in P, the interior of xv_1Qv_2x is empty of vertices. Thus, we try to colour $G - P$ a colouring φ such that:

- for $i \in \{1, 2\}$, $\{\varphi(x), \varphi(v_i)\} \neq L_0$, and
- if $\varphi(v_1)$ and $\varphi(v_2)$ are both in L_0, then $\varphi(v_1) = \varphi(v_2)$.

Then we extend φ to P.

Since xv_1Qv_2x is empty, this is equivalent to finding such a colouring in case P consists of one vertex. Denote this vertex by z. Now we may assume that the graph contains only two 2-lists, u and z.

Let u_1 (u_2) be the neighbour of u that belongs to the path not containing z between v_1 (v_2) and u in C. We prove that at least one of v_1, v_2, u_1, and u_2 is the end of a chord that separates u and z.

Since $\{v_1, v_2\} \cap \{u\} = \emptyset$ by hypothesis, $\{u_1, u_2\} \cap \{z\} = \emptyset$. Suppose for a contradiction that no one of v_1, v_2, u_1, and u_2 is the end of a chord that separates u and z. Then, by symmetry, x is adjacent to u_1, u, and u_2.

For $i \in \{1, 2\}$, let Q_i be the path between u_i and v_i in $C - \{u, z\}$. Since G does not contain a wheel subcanvas with centre x, the subgraphs bounded by $xv_1Q_1u_1x$ and $xv_2Q_2u_2x$ are not both broken wheels.

We may assume without loss of generality that the subgraph bounded by $xv_1Q_1u_1x$ is not a broken wheel. Then, there is at most one colouring of v_1xu_1 unextendable to it. Delete from $L(x)$ the colour involved in that colouring. Now, the subgraph bounded by $xv_1zv_2Q_2u_2uu_1x$ is colourable by induction and its colouring is extendable to G, a contradiction.

Thus, at least one of v_1, v_2, u_1, and u_2 is the end of a chord that separates u and z. If u_1 or u_2 is the end of such a chord, and v_1 and v_2 are not, then we exchange the names of u_1 and v_1, u_2 and v_2, and u and z in case $P = z$.

If $P \neq z$, then we can pick one neighbour z of x in P, delete the rest of P and add the edges zv_1, zv_2. This is a smaller instance and so is colourable. This colouring extends to a colouring of G. □

Claim 4.4.11. *For $i \in \{1, 2\}$, if $v_i \neq u$, then v_i is the end of a chord of C that separates P from u.*

Proof. By symmetry, it suffices to prove the claim for $i = 1$. Suppose for a contradiction that $v_1 \neq u$, and v_1 is not the end of a chord that separates P from u. By Claim 4.4.8, x is adjacent to v_1 and a vertex in P at an odd distance from v_1 in Q.

By Claim 4.4.10, we have the following two cases.

Case 1. *v_2 is the end of a chord separating P from u.*

This text, the proof of Case 1, was prepared by Bruce Richter, following our discussions on how to resolve this case. We thank Luke Postle for his suggestions.

We apply Theorem 3.3.11. In this instance, our application requires knowledge of an harmonica.

Definition. *Let $T = (G, P \cup P', L)$ be a canvas such that P and P' are distinct paths of length one and let $\mathcal{C}$ be a government for P. Then T is an* harmonica *from P to P' with government $\mathcal{C}$ if one of:*

1. *$\mathcal{C}$ is a dictatorship, $G = P \cup P'$, and the dictator of $\mathcal{C}$ is the vertex of $P \cap P'$;*

2. *$\mathcal{C}$ is a dictatorship with dictator z having colour c, there is another path P'' of length one and T contains an harmonica H from P'' to P', $T = H \cup P$, z is adjacent to both ends u, v of P'', $c \in L(u) = L(v)$, $|L(u)| = 3$, and the government $\mathcal{C}''$ is the democracy using $L(u) \setminus \{c\}$;*

3. *$\mathcal{C}$ is a democracy $\{(a, b), (b, a)\}$, there is a vertex z adjacent to both ends u, v of P, $L(z) = \{a, b, c\}$ and, $G - u$ is an harmonica from the path zv to P' with the dictatorship $\{(c, a), (c, b)\}$ having z as dictator.*

It is routine to see that if T is an harmonica, then, from P to P', there is a sequence $\mathcal{C}_0 = \mathcal{C}$, $\mathcal{C}_1$, $\dots$, $\mathcal{C}_k$ of governments that alternate between dictatorships and democracies. These correspond precisely to the alternation between adding the two vertices of a democracy to a dictator in (2) and the dictator to the democracy in (3).

Turning now to the proof of Case 1, let u' be any boundary neighbour of u and $\mathcal{C}_{-1}$ be any confederacy of possible colourings of u and u'; P_{-1} is the path (u, u'). Applying Theorem 3.3.11, there is a confederacy $\mathcal{C}_0$ at the path $P_0 = (v_2, w)$ such that any colouring of P_0 with a colouring in $\mathcal{C}_0$ L-colours the portion of G on the side of P_0 that contains P_{-1} so that P_{-1} is coloured with a colouring from $\mathcal{C}_{-1}$.

If the confederacy $\mathcal{C}_0$ can be chosen so that, for some colour $c \in L_0$, no colouring in $\mathcal{C}_0$ colours v_2 with c, then we proceed as follows. Colour P starting with c on the neighbour of v_2. Delete P and remove the colours of their P neighbours from all the lists of the neighbours of P, other than v_1. Notice that we have deleted at most one colour from $L(x)$, so x has at least three colours. We retain the original $L(v_1)$ (even though one of its colours appears on its P-neighbour).

Theorem 3.3.11 again implies (v_1, x) has a confederacy $\mathcal{C}_x$ such that each of its colourings extends to an L-colouring of the other portion of G created by cleaving on v_2w, with the colouring of v_2w coming from $\mathcal{C}_0$. Since at least one of the colourings in $\mathcal{C}_x$ colours v_1 so that its colour is different from the colour of its P-neighbour, we are done in this case.

In the remaining case, some colouring in $\mathcal{C}_0$ uses the colour of the P-neighbour of v_1. By a small case-checking, it is easy to see that there is a government G_0 contained in the set of colourings involved in $\mathcal{C}_0$ and a colour $c \in L_0$ such that no colouring in G_0 colours v_2 with c. Again colour P by starting with c on the P-neighbour of v_2, delete P, and, except from $L(v_1)$ delete the colours of the P-neighbours of all the vertices.

Applying Theorem 3.3.11 again, either we get a confederacy at (v_1, x) or we get an harmonica. In the case of the confederacy, we finish as in the case there was a colour $c \in L_0$ such that no colouring in $\mathcal{C}_0$ coloured v_2 with c. Thus, we may assume that there is an harmonica H from $P_0 = (v_2, w)$ to (x, v_1) with government G_0 (that does not colour v_2 with c). See Figure 4.24.

Notice that $L_0 \subseteq L(v_1)$ but the colour of the P-neighbours of x is now not in $L(x)$. Therefore, $L(v_1) \neq L(x)$, so the government at the (x, v_1) end of H is not a democracy. It follows that this government is a dictatorship.

Let G_0, G_1, $\dots$, G_k be the sequence of governments obtained in the

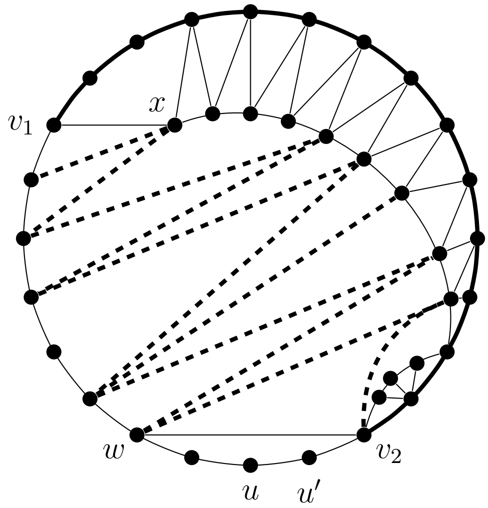


Figure 4.24: The dashed edges are in a harmonica from wv_2 to v_1x.

harmonica H. For each G_i that is a dictatorship, let z_i be the dictator and let c_i be the colour of z_i in G_i. For each G_i that is a democracy, let $\{w_i, y_i\}$ be the vertices of G_i and let L_i be the set of two colours used in G_i. The following claim will be helpful for the remainder of the proof.

Let J be the subgraph of G obtained from the portion of G cleaved by v_2w that contains xv_1 by deleting $V(P)$.

Claim. *For each i such that G_i is a democracy, one of w_i, y_i is in the boundary walk of J from w to v_1 that does not include either v_2 or x. The other of w_i, y_i is in the boundary walk of J from v_2 to x that does not include either v_1 or w.*

Proof. If $i = 0$, then the result is trivial. Otherwise, the government G_k at x, v_1 is a dictatorship, so $i < k$. Thus, there is a dictator z_{i+1} joined to w_i and y_i. Planarity shows that w_i, y_i are in the different boundary walks. □

We choose the labelling so that w_i is on the wv_1-subpath of the boundary of J and y_i is on the v_2x-subpath. Note that each y_i has a list of size 3, so every y_i is adjacent to vertices of P having different colours. In particular, $L(y_i) \cap L_0 = \varnothing$.

Note that being an harmonica implies, for all i such that G_i is a democracy, $|L(z_{i+1}) \cap L(y_i)| \geq 2$. Since $L(y_i) \cap L_0 = \varnothing$, we see that $|L(z_{i+1}) \cap L_0| \leq 1$. Since $|L(v_1) \cap L_0| = 2$, we conclude that v_i is not a dictator. It follows that x is the dictator z_k of G_k.

We now show that we can finish the L-colouring of G. We know that x is joined to the democracy w_{k-1}, y_{k-1}. The colour of x is c_k. We can colour v_1 with a colour in $L(v_1)$ that is neither c_k nor the colour of the P-neighbour of v_1.

Letting P_k denote the boundary walk in J from w_{k-1} to v_1, we consider the problem of colouring the portion J_k of J bounded by $P_k \cup (v_1, x, w_{k-1})$. Suppose first that J_k does not contain a broken wheel centred at x. Colour w_{k-1} with a colour different from c_1 and, if it is adjacent to v_1, the colour of v_1. Now apply Theorem 3.2.4 to colour J_k.

In the other case, x is the centre of a broken wheel, so x is adjacent to all the vertices of P_k (there are no chords of P_k). Since there are no separating 3-cycles, J_k is this broken wheel. We colour P_k starting from the v_1 end. Since w_{k-1} has the two colours in L_{k-1} different from c_k, it can be coloured from L_{k-1}. This forces the colour of y_{k-1} to the other colour in L_{k-1}.

For the next iteration, there is a dictator z_{k-2} adjacent to both w_{k-1} and y_{k-1}. We colour z_{k-2} with c_{k-2}. Let P_{k-2} be the boundary walk in $J - z_{k-2}$ joining either w_{i-3} to w_{i-1} or y_{i-3} to y_{i-1}. (This is the general situation; we will discuss the possibility that $z_{k-2} \in \{w, v_2\}$ at the end.)

Then P_{k-2} together with the edges from its ends to z_{k-2} bounds a region J_{k-2}, which can be coloured exactly as we did J_k above. Continuing in this way (there is an obvious induction that is left to the reader), we come to the remaining possibility that $z_0 \in \{w, v_2\}$. But exactly the same argument applies. The vertex in $\{w, v_2\} \setminus \{z_0\}$ has two different colours in its dictatorship, so that the corresponding region J_0 can be coloured in the same way as the earlier J_i's.

Case 2. $v_2 = u$.

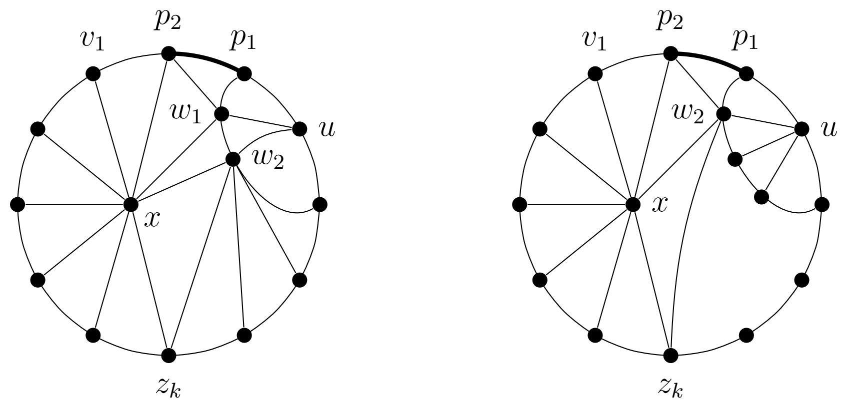


Figure 4.25: Case 2 in Claim 4.4.11.

Subcase 2.1. $L(u) = L_0$.

In this case, extend the path P to include the vertex u. Since P was chosen of maximum length, x is adjacent to u and a vertex in P at an odd distance from u. Since x is adjacent to v_1 and a vertex at an odd distance from v_1, again the path Q (the extension of P to include v_1 and u) is of even length.

If P consists of one vertex, then the side G' of v_1xu not containing P is not a broken wheel since G is not a wheel with centre x. In case P has more than one vertex, G' may be a broken wheel.

If G' is not a broken wheel, then we colour u and v_1 the same colour from L_0, colour x by a colour not in L_0 and different from the unique colour involved in the colouring of v_1xu unextendable to G'. Such a colouring is extendable to P and G', that is to G.

Let a and b be the colours of L_0. Then, the bad colourings of v_1xu for G' are either $\{aca, ada\}$ for some two colours c and d, or $\{aba, aca, bab, bcb, cac, cbc\}$ for some colour c. In both cases, there is a colouring of v_1xu that gives v_1 and u the same colour from L_0 yet avoids all the bad colourings. Such a colouring is extendable also to P, and so G is colourable.

Subcase 2.2. $L(u) \neq L_0$.

Colour u by a colour not in L_0 then delete u and delete this colour from the lists of the neighbours of u. The neighbour of u in $C - P$ may now have a list of size 2. The subgraph $G - u$ is colourable unless it contains a wheel subcanvas with centre x.

Let W be a wheel subcanvas with centre x. By hypothesis (c) in the statement of this Theorem, P contains at most two vertices. Let $P = p_1p_2$, where p_1 is adjacent to u. Since W is a subcanvas of $G-u$ but not of G, ∂W contains vertices that are neighbours of u in the interior of C. It contains at most two such vertices since there are no separating triangles. See Figure 4.25.

In case ∂W contains one neighbour of u in the interior of C we denote it by w_2, and in case they are two we let w_1 denote the one that has neighbours in P, and let w_2 denote the other one. Suppose that $C = up_1p_2v_1z_1 \cdots z_nu$, and that the largest i such that z_i is in ∂W is k.

If the subgraph bounded by $uw_2z_k \cdots z_nu$ is a broken wheel, then the subgraph bounded by $uw_1p_2v_1z_1 \cdots z_nu$ in case u has two neighbours in the interior of C, and G in case u has one such neighbour, are double-centred wheels.

In case the subgraph $uw_1p_2v_1z_1 \cdots z_nu$ is a double-centred wheel, we give u a colour that avoids the unique colouring of uw_2p_2, given by Lemma 4.2.4, unextendable to this double-centred wheel. Then, p_1, p_2, w_2 in this order, and then extend the colouring to G. See the left drawing of Figure 4.25.

In case G is a double-centred wheels, colour u a colour that avoids the unique colouring of up_1p_2, given by Lemma 4.2.4, unextendable to G.

Thus, the subgraph bounded by $uw_2z_k \cdots z_nu$ is not a broken wheel. In case u has one neighbour in the interior of C, we colour G as follows. Colour p_2 by a colour that avoids the unique colouring of p_2w_2 unextendable to W given by Lemma 4.2.1. Then, colour p_1, u, and then of the two colours remaining in $L(w_2)$, choose the one that avoids the unique colouring of uw_2z_k unextendable to the subgraph bounded by $uw_2z_k \cdots z_nu$. Now, extend this colouring to W, and then to the rest of G.

In case u has two neighbours in the interior of C, and the subgraph bounded by $uw_2z_k \cdots z_nu$ is not a broken wheel, the subgraph bounded by $uw_1xz_k \cdots z_nu$ is not a generalized wheel with principal path uw_1z. Thus, any colouring of uw_1x is extendable to subgraph bounded by $uw_1xz_k \cdots z_nu$.

Since $|L(x)| \geq 4$, there is a colour in $L(x)$ that avoids all the bad colourings of v_1xz_k for the broken wheel bounded by $xp_2v_1z_1 \cdots z_kx$. Colour x with that colour, colour p_2, p_1, u, w_1 in this order, then extend the colouring to the subgraph bounded by $uw_1xz_k \cdots z_nu$, and then to the broken wheel bounded by $xp_2v_1z_1 \cdots z_kx$. □

Claim 4.4.12. *v_1v_2 is a chord of C.*

Proof. By Claim 4.4.6, $v_1 \neq v_2$. Thus, we may assume without loss of generality that $v_1 \neq u$. By the previous claim, v_1 is the end of a chord of C that separates u from P. This and Claim 4.4.4 imply that $v_2 \neq u$ as well. Again by the previous claim, v_2 is the end of a chord of C which separates u from P. By planarity and 2-connectedness of G, it follows that v_1v_2 is a chord of C. □

Claim 4.4.13. $|V(P)| = 1$.

Proof. Suppose for a contradiction that $|V(P)| \geq 2$ and let G_1 and G_2 be the subgraphs such that $G = G_1 \cup G_2$, $V(G_1) \cap V(G_2) = \{v_1, v_2\}$, $V(P) \subseteq V(G_1)$ and $u \in V(G_2)$. Let $y \notin V(G)$ be a new vertex and construct a new graph G' with $V(G') = V(G_2) \cup \{y\}$ and $E(G') = E(G_2) \cup \{yv_1, yv_2\}$. Let $L(y) = L_0$. Consider the canvas (G', S', L), where S' consists of the isolated vertices y and u. Since $|V(P)| \geq 2$, $|V(G')| < |V(G)|$. By minimality of (G, S, L), there exists an L-colouring φ of G'. Hence there exists L-colouring φ of G_2, where $\{\varphi(v_1), \varphi(v_2)\} \neq L_0$.

We extend φ to an L-colouring of $P \cup G_2$. First colour P. If $|V(P)| = 2$, then $G[V(P) \cup \{v_1, v_2\}]$ is a 4-cycle, and its interior is colourable since there is no wheel subcanvas with centre x in G. If $|V(P)| > 2$, let $L'(v_1) = \{\varphi(v_1)\}$ and $L'(v_2) = \{\varphi(v_2)\}$, and for $v \in V(G_1) \setminus (V(P) \cup \{v_1, v_2\})$ let $L'(v)$ be obtained from $L(v)$ by deleting the colours of the neighbours of v in P. Then $G_1 \setminus V(P)$ is L'-colourable either by Theorem 4.3.1 or

Theorem 3.2.6 (in case x is adjacent to P) since G does not contain any wheel with centre x (as $|V(P)| > 2$). The union of the colourings of G_1 and G_2 is a colouring of G, a contradiction. □

Let z be such that $V(P) = \{z\}$.

Since v_1 and v_2 are adjacent, both u_1 and u_2 are different from z (Recall the definition of u_1 and u_2 from the proof of Claim 4.4.10). Planarity and 2-connectedness of G imply that at least one of u_1 and u_2 is not an end of a chord that separates z from u different from u_1u_2. Therefore, we have by symmetry from Claims 4.4.11 and 4.4.12 that u_1 and u_2 are adjacent. We also have by symmetry and Claim 4.4.7 that $|L(u_1)| = |L(u_2)| = 3$ and $L(u)$ is contained in both $L(u_1)$ and $L(u_2)$.

Claim 4.4.14. *$L(v_1) = L(v_2)$ or $L(u_1) = L(u_2)$.*

Proof. In the latter case we exchange the names of v_1 and u_1, v_2 and u_2, z and u, and L_0 and $L(u)$.

Suppose that $L(v_1) \neq L(v_2)$ and $L(u_1) \neq L(u_2)$. Since G is planar, either v_1 is not an end of a chord of C separating v_2 from u, or v_2 is not an end of a chord separating v_1 from u. Assume without loss of generality that v_1 is not in a chord of C separating v_2 from u. This implies that v_1 is not an end of a chord in C other than v_1v_2. Let v' be the vertex in C distinct from v_2 and z that is adjacent to v_1.

Let $c \in L(v_1) \setminus L_0$. Let $G' = G - \{z, v_1\}$, and $L'(v)$ be either $L(v) \setminus \{c\}$, if v is adjacent to v_1, or $L(v)$, otherwise. Note that $|L'(v_2)| \geq 3$ as $L(v_1) \neq L(v_2)$ and $L_0 \subseteq L(v_1) \cap L(v_2)$. Let S' consist of the isolated vertices v' and u.

Case 1. *G' does not contain a wheel subcanvas with centre x.*

In this case, G' has an L' colouring. If $u \neq v'$, this follows from the minimality of G. If $u = v'$, this follows from Theorem 4.3.1 or Theorem 3.2.6. Since this L'-colouring of G' can be extended to an L-colouring of G, we have a contradiction.

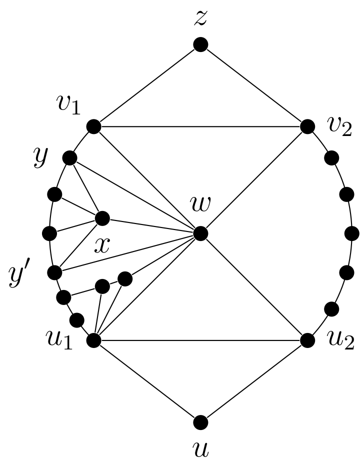


Figure 4.26: W is a wheel subcanvas of $G-\{z,v_1\}$, $G-\{z,v_2\}$, $G-\{u,u_1\}$, and $G-\{u,u_2\}$.

Case 2. *G' contains a wheel subcanvas W with centre x.*

Since this subcanvas is not a subcanvas of G, it contains in its outer boundary a vertex from the interior of C that is a neighbour of v_1. By the definition of subcanvas, any vertex of ∂W not in C is a neighbour of v_1. Now, since there are no separating 4-cycles with interior consisting of 5-lists only, there are at most two vertices of ∂W in the interior of C.

Again by planarity, either u_1 is not the end of a chord of C separating z from u_2, or u_2 is not the end of a chord separating z from u_1. Thus, by symmetry with v_1 and v_2, W is a wheel subcanvas of either $G-\{u,u_1\}$ or $G-\{u,u_2\}$. Now, if there are two vertices of ∂W in the interior of C, then both are adjacent to v_1 as well as to u_1 or u_2. Therefore, it is exactly one vertex of ∂W in the interior of C; call it w.

In case W is a subcanvas of $G-\{u,u_1\}$, G contains the path v_1wu_1, and in case W is a subcanvas of $G-\{u,u_2\}$, G contains the path v_1wu_2 (those are not symmetric). By planarity, in both cases, none of the vertices v_1, v_2, u_1, and u_2 is the end of a chord that separates z from u other than v_1v_2 and u_1u_2. Therefore, W is a wheel subcanvas of $G-\{z,v_2\}$, $G-\{u,u_1\}$, and $G-\{u,u_2\}$ (See Figure 4.26).

For $i\in\{1,2\}$, let H_i be the subgraph bounded by v_iwu_i and the

v_iu_i-path in $C - z$. Then W is either contained in H_1 or in H_2. Assume without loss of generality that W is contained in H_1 and let y (y') be the vertex in $V(W) \cap V(C)$ that is closest to v_1 (u_1) with distance measured in $C - z$. Let H_3 (H_4) be the subgraph of H_1 bounded by v_1wy (u_1wy') and the path in $C - z$ between v_1 (u_1) and y (y').

We colour G as follows. Let a be the unique colour in $L(v_1) \setminus L_0$, and b the unique colour in $L(u_2) \setminus L(u)$. There is a dictatorship $\mathcal{C}_1$ for v_1w such that v_1 is the dictator and its colour in every colouring in $\mathcal{C}_1$ is a, and such that the colours given to w by the colourings in $\mathcal{C}_1$ are all different from b.

By Corollary 3.3.8, there is a government $\mathcal{C}_2$ for yw such that evey colouring of yw in $\mathcal{C}_2$ is extendable to a colouring of H_3 whose restriction to v_1w is in $\mathcal{C}_1$. Choose from $\mathcal{C}_2$ a colouring for yw different from the unique bad colouring for W given by Lemma 4.2.1, and then extend that colouring to both W and H_3.

Now $y'w$ is coloured (since W is), extend its colouring to a colouring of H_4 by Theorem 3.2.6, and then colour u_2 by b. The colour b of u_2 is different from the colour of u_1 since it is not in $L(u_1)$ by the assumption $L(u_1) \neq L(u_2)$ and the fact that $L(u)$ is contained in both $L(u_1)$ an $L(u_2)$. It is also different from the colour of w by our choice of the government $\mathcal{C}_1$.

Now extend the colouring of wu_2 to a colouring of H_2 by Theorem 3.2.6 (recall that $L(v_2)$ has three colours different from the colour of v_1). Finally, colour z and u (they are colourable since v_1 is coloured by a colour not in L_0 and u_2 is coloured by a colour not in $L(u)$). □

By symmetry between z, v_1, v_2 and u, u_1, u_2, assume that $L(v_1) = L(v_2)$.

Claim 4.4.15. *One of v_1, v_2 is the end of a chord of C distinct from v_1v_2 that separates u from z.*

Proof. Suppose for a contradiction that there is no such chord. Let $c \in L(v_1) \setminus L_0 = L(v_2) \setminus L_0$, and let L_1 be a set of size two such that $c \in L_1 \subseteq L(v_1)$. Let $L_1(v_1) = L_1(v_2) = L_1$ and $L_1(v) = L(v)$ for all $v \in$

$V(G) \setminus \{z, v_1, v_2\}$. Let P' denote the path with vertex-set $\{v_1, v_2\}$ and consider the canvas $(G - z, P' + u, L_1)$. Note that $G - z$ is 2-connected, since G is 2-connected and since there are no vertices in the interior of the triangle zv_1v_2z.

Since P' has no internal vertex, and there is no chord with an end in P' which separates a vertex of P' from u, and since $G - z$ contains no wheels subcanvas with centre x, and G is a counterexample, hypothesis (d)-ii is not satisfied in $G - z$. That is, x is adjacent to v_1 and v_2.

Case 1. $L(u_1) = L(u_2)$.

In this case, by symmetry, x is adjacent to u_1 and u_2. At most one of the subgraphs H_i bounded by v_ixu_i and the v_iu_i-path in $C - z$, $i \in \{1, 2\}$, is a broken wheel since G contains no wheel subcanvas with centre x. Thus, for at least one of H_i, $i \in \{1, 2\}$, it is at most one colouring of v_ixu_i that is not extendable to it.

We choose the colours of v_1, v_2, u_1, u_2, and x such that at least one of v_1 and v_2 is not coloured from L_0, at least one of u_1 and u_2 is not coloured from $L(u)$, and for $i \in \{1, 2\}$, the colouring of v_ixu_i is extendable to H_i.

Case 2. $L(u_1) \neq L(u_2)$.

In this case, let d denote the unique colour in $L(u_1) \setminus L(u)$. Note that $d \notin L(u_2)$ by assumption. Colour u_1 with d, and delete d from the lists of the neighbours of u_1. Then, $G - \{u_1, u\}$ contains two 2-lists, namely, z and the neighbour of u_1 on C different from u and u_2. If $G - \{u_1, u\}$ is not colourable by induction, then it contains a wheel subcanvas W with centre x.

For W not to be a subcanvas of G, it has to contain in its outer walk vertices that are neighbours of u_1 in the interior of C. It cannot contain more than two such vertices since there are no separating 4-cycles with interior consisting of 5-lists. Similarly, $G - \{u_2, u\}$ contains a wheel subcanvas with centre x.

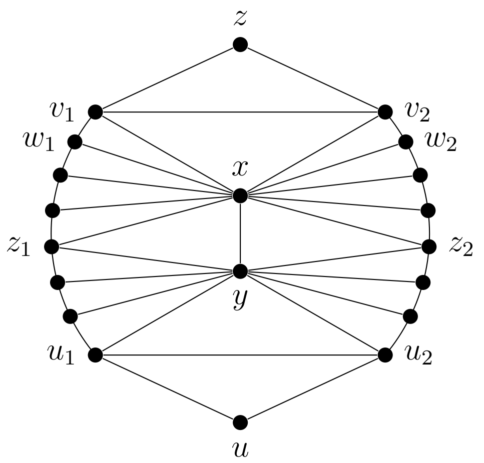


Figure 4.27: $G - \{u_1, u_2\}$ contains a wheel subcanvas with centre x that is not a subcanvas of G.

Thus, the unique vertex in the interior of C that is in ∂W is the common neighbour of u_1 and u_2 in the interior of C. See Figure 4.27.

Let y be the common neighbour of u_1 and u_2 in the interior of C. For $i \in \{1,2\}$, let z_i be the common neighbour of x and y in the v_iu_i-path in $C - z$.

The following text was prepared by Bruce Richter.

Let w_1 and w_2 be the boundary neighbours of v_1 and v_2, respectively that are adjacent to x. Recall that $L(z) = \{a, b\}$ and $L(v_1) = L(v_2) = \{a, b, c\}$.

Lemma 1. *Suppose there is an L-colouring ϕ of either:*

1. *for some $i \in \{1,2\}$, v_i and x such that $\phi(v_i) = c$ and both $|L(w_i) \setminus \{\phi(x), \phi(v_i)\}| \geq 2$ and $|L(v_{3-i}) \setminus \{\phi(x), \phi(v_i)\}| \geq 2$; or*
2. *v_1, v_2, and x such that $c \in \{\phi(v_1), \phi(v_2)\}$ and, for both $i = 1, 2$, $|L(w_i) \setminus \{\phi(x), \phi(v_i)\}| \geq 2$.*

Then there is an L-colouring of G.

Proof. Extend ϕ by colouring y to avoid $\phi(x)$, $L(u_2) \setminus L(u_1)$, and any two colours in $L(z_2) \setminus \phi(x)$. Notice that there are still two colours available at u_1, but possibly only one at z_1. There are two colours available at all the vertices from u_2 to either w_2 or v_2. Starting by colouring u_2 with $L(u_2) \setminus L(u_1)$, we colour up to z_2 and on to either w_2 or v_2. On the other side, colour up and down from z_1 to either w_1 or v_1 going up, and down to u_1. □

It remains to show that there is a colouring ϕ satisfying one of the hypotheses of the lemma.

Claim 1. *If there is an $i \in \{1,2\}$ such that $c \notin L(w_i)$, then there is an L-colouring ϕ of x, v_i such that $\phi(v_i) = c$ and both $|L(w_i) \setminus \{\phi(x), \phi(v_i)|\}| \geq 2$ and $|L(v_{3-i}) \setminus \{\phi(x), \phi(v_i)|\}| \geq 2$.*

Proof. Colour v_i with c and x with a colour in $L(x) \setminus \{a, b, c\}$. Since $c \notin L(w_i)$, $|L(w_i) \setminus \{\phi(x), \phi(v_i)|\}| \geq 2$. That $\phi(x) \notin \{a, b, c\}$, $\phi(x) \notin L(v_{3-i})$, so $|L(v_{3-i}) \setminus \{\phi(x), \phi(v_i)|\}| \geq 2$. □

Claim 2. *If $L(x) \setminus (L(w_1) \cup L(w_2)) \neq \varnothing$, then there is an L-colouring ϕ of x, v_1, v_2 such that $c \in \{\phi(v_1), \phi(v_2)\}$ and, for both $i = 1, 2$, $|L(w_i) \setminus \{\phi(x), \phi(v_i)\}| \geq 2$.*

Proof. By Claim 1, we may assume $c \in L(w_1) \cap L(w_2)$. Set $\phi(x)$ to be in $L(x) \setminus (L(w_1) \cup L(w_2))$ (so $\phi(x) \neq c$), $\phi(v_1) = c$, and $\phi(v_2) \in \{a, b\} \setminus \{\phi(x)\}$. □

Claim 3. *If, for some $i \in \{1,2\}$, $L(x) \setminus (L(w_i) \cup \{a, b\}) \neq \varnothing$, then there is an L-colouring ϕ of x and v_i such that $\phi(v_i) = c$ and that both $|L(w_i) \setminus \{\phi(x), \phi(v_i)|\}| \geq 2$ and $|L(v_{3-i}) \setminus \{\phi(x), \phi(v_i)|\}| \geq 2$.*

Proof. Choose $\phi(x) \in L(x) \setminus (L(w_i) \cup \{a, b\})$. Colour v_i with c. □

At this point:

1. Claim 2 shows we may assume $L(x) \subseteq L(w_1) \cup L(w_2)$; and

2. Claim 3 shows we may assume, for $i = 1, 2$, $L(x) \subseteq L(w_i) \cup \{a, b\}$.

Since $|L(x)| > |L(w_1)|$, Item 2 shows either a or b is in $L(x) \setminus L(w_1)$; we choose the labelling of a, b so that $a \in L(x) \setminus L(w_1)$. Now Item 1 and the fact that $|L(x)| > |L(w_2)|$ shows that $b \in L(x) \setminus L(w_2)$.

Set $\phi(x) = a$, $\phi(v_1) = c$, and $\phi(v_2) = b$ to leave two choices for each of w_1 and w_2.

Here ends the text prepared by Bruce Richter. $\square$

Now, suppose without loss of generality that v_2 is the end of a chord of C distinct from v_1v_2 that separates u from z. Choose such a chord v_2y such that y is closest to v_1 measured by the distance in $C - v_2$. Let G_1 and G_2 be the connected subgraphs of G such that $V(G_1) \cap V(G_2) = \{v_2, y\}$, $G_1 \cap G_2 = G$, $z \in V(G_1)$ and $u \in V(G_2)$.

Select a colour c as follows. If v_1 is adjacent to y, let $c \in L(v_1) \setminus L_0 = L(v_2) \setminus L_0$. Note that in this case $V(G_1) = \{z, v_1, v_2, y\}$ (since the interiors of the triangles zv_1v_2z and yv_1v_2y are colourable as in Claim 4.3.3). If v_1 is not adjacent to y, consider the canvas (G_1, P'', L), where $P'' = zv_2y$.

Since G_1 does not contain a wheel subcanvas with centre x, and since it is not a broken wheel (as y was chosen to be the closest neighbour of v_2 to v_1 and we are assuming here it is not adjacent to v_1), then by Lemmas 3.2.10, 4.2.4 and 4.2.5, there is at most one colouring of P'' that does not extend to G_1. If such a bad colouring of P'' exists, let c be the colour of y in that colouring, otherwise let c be arbitrary.

Consider the canvas (G_2, S', L'), where S' consists of the isolated vertices y and u, $L'(y) = L(y) \setminus \{c\}$ and $L'(v) = L(v)$ otherwise. As $|V(G_2)| < |V(G)|$, there exists an L'-colouring of G_2. This colouring is extendable to an L-colouring of G by the choice of c, a contradiction.

$\square$

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