Ideal Clutters

by

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Author's Declaration

This thesis consists of material all of which I co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

This thesis is based on various collaborations with Gérard Cornuéjols, Ricardo Fukasawa, Bertrand Guenin, Natália Guričanová, Dabeen Lee, Kanstantsin Pashkovich and Laura Sanità. These include, but are not limited to, papers [1, 2, 3, 4, 5, 6, 7, 8, 9, 10].

Abstract

Let E be a finite set of *elements*, and let $\mathcal{C}$ be a family of subsets of E called *members*. We say that $\mathcal{C}$ is a *clutter* over *ground set* E if no member is contained in another. The clutter $\mathcal{C}$ is *ideal* if every extreme point of the polyhedron

$$\left\{x \in \mathbb{R}^E : \sum_{e \in C} x_e \geq 1 \ \forall C \in \mathcal{C}, x \geq \mathbf{0}\right\}$$

is integral.

Ideal clutters are central objects in Combinatorial Optimization, and they have deep connections to several other areas. To integer programmers, they are the underlying structure of set covering integer programs that are easily solvable. To graph theorists, they are manifest in the famous theorems of Edmonds and Johnson on T-joins, of Lucchesi and Younger on dijoins, and of Guenin on the characterization of weakly bipartite graphs; not to mention they are also the set covering analogue of perfect graphs. To matroid theorists, they are abstractions of Seymour's sums of circuits property as well as his f-flowing property. And finally, to combinatorial optimizers, ideal clutters host many minimax theorems and are extensions of totally unimodular and balanced matrices.

This thesis embarks on a mission to develop the theory of general ideal clutters.

In the first half of the thesis, we introduce and/or study tools for finding deltas, extended odd holes and their blockers as minors; identically self-blocking clutters; exclusive, coexclusive and opposite pairs; ideal minimally non-packing clutters and the $\tau = 2$ Conjecture; cuboids; cube-idealness; strict polarity; resistance; the sums of circuits property; and minimally non-ideal binary clutters and the f-Flowing Conjecture.

While the first half of the thesis includes many broad and high-level contributions that are accessible to a non-expert reader, the second half contains three deep and technical contributions, namely, a characterization of an infinite family of ideal minimally non-packing clutters, a structure theorem for ± 1-resistant sets, and a characterization of the minimally non-ideal binary clutters with a member of cardinality three.

In addition to developing the theory of ideal clutters, a main goal of the thesis is to trigger further research on ideal clutters. We hope to have achieved this by introducing a handful of new and exciting conjectures on ideal clutters.

Acknowledgements

I would like to thank Bertrand Guenin for nurturing me and Gérard Cornuéjols for believing in me.

I would also like to thank the examiners, whose constructive feedback improved the presentation of this thesis.

Dedication

I dedicate this thesis to my parents.

Table of Contents

List of Figures

Notes from the author

1. I have tried my best to write the thesis not as a thesis, but as a book on ideal clutters. This is achieved by taking the focus *away* from individual results and onto the connections between the various results and what they infer about ideal clutters. I have put a lot of care into keeping the flow of the thesis as smooth as possible.

2. Most of the results of the thesis come from 10 papers I have coauthored. In most cases, those papers contain more results and in much more detail, and they approach the results/problems from a different point of view. This helps preserve the value of the thesis as well as the value of the papers.

3. Chapter 1, the introduction, presents the underlying thread of the thesis, and provides representative results from the forthcoming chapters. Reading the introduction alone should give the reader a fairly good idea of what the entire thesis is about, and can be treated as an extended abstract. Chapters 2 through 6 contain broad and high-level contributions and should be accessible to non-experts. Chapters 7, 8 and 9 contain proofs of the three deep and technical contributions of the thesis. Chapter 10 concludes the thesis by reviewing the major conjectures on ideal clutters.

4. Seven chapters conclude with a section containing further notes. Each such section discusses conjectures and questions following the chapter, and possibly some relevant results that did not quite fit in the chapter.

5. At the request of an examiner, I have classified the results of the thesis stated in the introduction into three categories, from one star, two stars, to three stars, in increasing order of difficulty and depth. This classification is not completely objective.

Prologue

Let $G = (V, E)$ be a graph and take distinct vertices s, t. Edward Moore and Claude Shannon, in their 1956 paper titled "Reliable Circuits Using Less Reliable Relays" [57], proved the following:

> Let L be the minimum number of edges of an st-path, and let W be the minimum number of edges of an st-cut. Then $L \times W \leq |E|$.

A few years later in 1962, the physicist and mathematician Richard Duffin wrote a paper titled "The Extremal Length of a Network" [28]. There, seemingly unaware of the result above, Duffin generalized it to the following *width-length inequality*:

> To each edge e, assign a nonnegative length ℓ_e and a nonnegative width w_e. Then
> $$\min\left(\sum_{e\in C} \ell_e : C \subseteq E \text{ is an } st\text{-path}\right) \times \min\left(\sum_{e\in B} w_e : B \subseteq E \text{ is an } st\text{-cut}\right) \leq \sum_{e\in E} \ell_e w_e.$$

Just a year later, these two results on electrical circuits inspired Alfred Lehman, an eccentric yet brilliant mathematician. Lehman set out to investigate the width-length inequality in a more general setting. It is much easier to give an account of his results using current terminology.

Let E be a finite set of *elements*, and let $\mathcal{C}$ be a family of subsets of E called *members*. We say that $\mathcal{C}$ is a *clutter* over *ground set* E if no member is contained in another [30]. The *incidence matrix of* $\mathcal{C}$, denoted $M(\mathcal{C})$, is the $0, 1$ matrix whose columns are labeled by the elements and whose rows are the incidence vectors of the members.

A *cover* is a subset of E that intersects every member. A cover is *minimal* if it does not properly contain another cover. The *blocker of* $\mathcal{C}$, denoted $b(\mathcal{C})$, is another clutter over the same ground set E whose members are the minimal covers of $\mathcal{C}$. The blocking operator is an involution, that is, $b(b(\mathcal{C})) = \mathcal{C}$ [45, 30].

We say that $\mathcal{C}$ *satisfies the width-length inequality* if, for all $\ell, w \in \mathbb{R}^E_+$, the following inequality holds:

$$\min\left(\sum_{e\in C}\ell_e : C \in \mathcal{C}\right) \times \min\left(\sum_{e\in B} w_e : B \in b(\mathcal{C})\right) \le \sum_{e\in E}\ell_e w_e.$$

Observe that if a clutter satisfies the width-length inequality, then so does the blocker. For instance, the clutter of st-paths of G and its blocker, the clutter of minimal st-cuts, satisfy the width-length inequality.

In his 1979 paper titled "On the Width-Length Inequality" [49], Lehman gave the following characterization:

> $\mathcal{C}$ satisfies the width-length inequality if, and only if, every extreme point of the polyhedron $\left\{x \in \mathbb{R}^E_+ : M(\mathcal{C})x \ge \mathbf{1}\right\}$ is integral.

Notice that above, the integral extreme points of the polyhedron are precisely the incidence vectors of the minimal covers. Even though the paper was published in 1979, Lehman proved this result in 1963 and presented it in 1965 during a lecture at RAND to Ray Fulkerson.

A few years later in 1970-71, in a series of two papers titled "Blocking Polyhedra" and "Blocking and Anti-Blocking Pairs of Polyhedra" [38, 37], Fulkerson shed light on the polyhedral aspect of Lehman's characterization. Given an integer $n \ge 1$ and a nonnegative matrix A with n columns and without a row of all zeros, consider the polyhedron $P := \{x \in \mathbb{R}^n_+ : Ax \ge \mathbf{1}\}$. The *blocking polyhedron of* P is $Q := \{u \in \mathbb{R}^n_+ : u^\top x \ge 1 \ \forall x \in P\}$. Fulkerson proved the following:

> There is a nonnegative matrix B with n columns and without a row of all zeros, whose rows can be taken as the extreme points of P, such that
>
> $$Q = \{u \in \mathbb{R}^n_+ : Bu \ge \mathbf{1}\}.$$
>
> Moreover, P is the blocking polyhedron of Q.

In the context of Lehman's characterization, given that $\mathcal{C}, b(\mathcal{C})$ satisfy the width-length inequality, the sets

$$\left\{x \in \mathbb{R}^E_+ : M(\mathcal{C})x \ge \mathbf{1}\right\} \quad \text{and} \quad \left\{u \in \mathbb{R}^E_+ : M(b(\mathcal{C}))u \ge \mathbf{1}\right\}$$

give just one instance of blocking polyhedra.

Digging deeper Richard Duffin proved much more than the width-length inequality in his 1962 paper [28]. He obtained the width-length inequality as a consequence of the following result:

To each edge e, assign a positive number $r_e > 0$. Then the optimal values of the quadratic programs

$$\begin{array}{lll} \min & \sum_{e\in E} r_e^{-1} x_e^2 & \\ \text{s.t.} & x(C) \geq 1 & C \text{ is an } st\text{-path} \\ & x \geq \mathbf{0} & \end{array} \qquad \begin{array}{lll} \min & \sum_{e\in E} r_e u_e^2 & \\ \text{s.t.} & u(B) \geq 1 & B \text{ is an } st\text{-cut} \\ & u \geq \mathbf{0}, & \end{array}$$

are inverses of one another.

How did he prove this? Duffin viewed G as an unoriented electrical network, where each edge e is a wire with resistance r_e, and the vertices are junctions at which the wires are connected to one another. He then passed an electric current passing through G by connecting s, t to the poles of an external current source. He proved that the first quadratic program measures the joint conductance between s, t while the second program measures its inverse, the joint resistance between s, t.

Decades later, Seth Chaiken set out to extend this result to blocking polyhedra. In his 1987 paper titled "Extremal Length and Width of Blocking Polyhedra, Kirchhoff Spaces And Multiport Networks" [17], Chaiken proved the following beautiful generalization:

Take an integer $n \geq 1$ and n-dimensional blocking polyhedra P, Q, and let R be a positive definite $n \times n$ matrix. Then the optimal values of the quadratic programs

$$\min\left\{x^\top Rx : x \in P\right\} \quad \text{and} \quad \min\left\{u^\top R^{-1}u : u \in Q\right\}$$

are inverses of one another.

Later on in the same year, in a paper titled "Dual Gauge Programs, with Applications to Quadratic Programming and the Minimum-norm Problem" [36], Robert Freund extended the result above to the general framework of *gauge duality*. The idea behind this type of duality is best explained by Seth Chaiken himself in correspondence with the author:

> *"I recall feeling at the time that it was pretty cool for a kind of dual pair of optimization problems to have reciprocal optimal values rather than values that are negations of each other."*

Even though Freund's paper has received quite a bit of attention, Chaiken's paper has been cited only four times, as of writing this thesis: three times by himself and once by the host journal in a complete bibliography of their publications.

This rich and fascinating history, and the concepts and theories it has led to, only adds urgency to why ideal clutters need to be further studied. This thesis takes a tiny step in that direction.

Chapter 1

Introduction

Let $\mathcal{C}$ be a clutter over ground set E. Consider the polyhedron

$$Q(\mathcal{C}) := \left\{x \in \mathbb{R}^E : M(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}\right\} = \left\{x \in \mathbb{R}^E : x(C) \geq 1 \ \forall C \in \mathcal{C}, x \geq \mathbf{0}\right\},$$

where $x(C)$ is shorthand notation for $\sum_{e \in C} x_e$. $Q(\mathcal{C})$ is called the *set covering polyhedron associated with* $\mathcal{C}$ [11]. We say that $\mathcal{C}$ is *ideal* if every extreme point of $Q(\mathcal{C})$ is integral [25]. Rich examples of ideal clutters have been discovered time and again; let us name a few in chronological order:

1927 **Menger**: $\mathcal{C}$ is the clutter of st-paths of a graph [56],

1931 **Kőnig, Egreváry**: $\mathcal{C}$ is the clutter of edges of a simple bipartite graph [46, 33],

1938 **Gallai**: $\mathcal{C}$ is the clutter of directed st-paths of a directed graph [39],

1956 **Hoffman and Kruskal**: $M(\mathcal{C})$ is totally unimodular [44],

1967 **Edmonds**: $\mathcal{C}$ is the clutter of rooted arborescences of a directed graph [29, 37],

1972 **Berge**: $M(\mathcal{C})$ is balanced [13],

1973 **Edmonds and Johnson**: $\mathcal{C}$ is the clutter of minimal T-cuts of a graft with terminals T [32],

1979 **Lucchesi and Younger**: $\mathcal{C}$ is the clutter of minimal dicuts of a directed graph [53],

2001 **Guenin**: $\mathcal{C}$ is the clutter of odd circuits of a signed graph without an odd-K_5 minor [40].

The last three are fundamental classes of ideal clutters and the results are viewed as crowning achievements of the field of Combinatorial Optimization. Given these various classes, some of which are inherently different in nature, it should perhaps come as no surprise that testing idealness is difficult:

Theorem 1.1 (Ding et al. [27]). *Let $\mathcal{C}$ be a clutter over ground set E. Then the problem "Is $\mathcal{C}$ ideal?" is co-NP-complete.*

This result was proved by Ding, Feng and Zang in 2008. In fact, they proved that testing idealness is co-NP-complete for clutters where every element is used in exactly two members. Let us emphasize that for this theorem, the members of the clutter are explicitly provided as part of the input. Despite this setback, the goal of this thesis is to further the current knowledge of idealness and of ideal clutters.

1.1 Minors

The first notion needed for studying idealness is that of minors. Let $\mathcal{C}$ be a clutter over ground set E. Take disjoint subsets $I, J \subseteq E$. The *minor of* $\mathcal{C}$ obtained after *deleting* I and *contracting* J is the clutter over ground set $E - (I \cup J)$ whose members are[1]

$$\mathcal{C} \setminus I/J := \text{ the minimal sets of } \{C - J : C \in \mathcal{C}, C \cap I = \emptyset\}.$$

We say that $\mathcal{C} \setminus I/J$ is a *proper minor* if $I \cup J \neq \emptyset$. Minor operations reverse roles in the blocker, that is, $b(\mathcal{C} \setminus I/J) = b(\mathcal{C})/I \setminus J$ for disjoint subsets $I, J \subseteq E$ [69]. In terms of the set covering polyhedron, deleting I corresponds to projecting away the coordinates $(x_e : e \in I)$ while contracting J corresponds to restricting the coordinates $(x_e = 0 : e \in J)$. Since these operations preserve polyhedral integrality,

Remark 1.2 ([70]). *If a clutter is ideal, then so is every minor of it.*

In other words, if a clutter has a non-ideal minor, then it is non-ideal itself. As the reader should expect, a clutter is more often than not non-ideal. Can we pinpoint the minors that make a typical clutter non-ideal? We say that two clutters $\mathcal{C}_1, \mathcal{C}_2$ are *isomorphic*, and write $\mathcal{C}_1 \cong \mathcal{C}_2$, if one is obtained from the other after relabeling the ground set.

Take an integer $n \geq 3$. The *delta of dimension* n is the clutter over ground set $[n] := \{1, \ldots, n\}$ whose members are

$$\Delta_n := \{\{1,2\}, \{1,3\}, \ldots, \{1,n\}, \{2,3,\ldots,n\}\}.$$

Observe that the elements and members of Δ_n correspond to the points and lines of a degenerate projective plane, and that $b(\Delta_n) = \Delta_n$. Observed by Lehman as early as 1963, Δ_n is non-ideal [49]. Thus, if a clutter has a delta isomorphic minor, then it is non-ideal.[2] So, how can we look for such minors?

****Theorem 1.3** ([4], proved in Chapter 2). *Let $\mathcal{C}$ be a clutter over ground set E. If there is an element e and distinct members C_1, C_2, C such that $e \in C_1 \cap C_2$, $e \notin C$, and $C_1 \cup C_2 \subseteq C \cup \{e\}$, then there is a delta minor using element e that can be found in time $O\big(|E||\mathcal{C}|^2\big)$.*

As a consequence, we get the following:

***Theorem 1.4** ([4], proved in Chapter 2). *Let $\mathcal{C}$ be a clutter over ground set E. Then in time $O\big(|E|^4|\mathcal{C}|^5\big)$, one can find a delta minor or certify that none exists.*

Deltas and delta minors will be a recurring theme throughout the entire thesis; they will surface a few times when they are least expected.

[1] Given sets A and B, $A - B$ denotes the set of elements in A that are not in B.
[2] The prefix "isomorphic" from "isomorphic minor" will hereinafter be omitted.

Take an odd integer $n \geq 5$. An *odd hole of dimension* n is the clutter over ground set $[n]$ whose members are

$$\mathcal{C}_n^2 := \{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}\}.$$

Noticed once again by Lehman, odd holes and their blockers are also non-ideal. An *extended odd hole of dimension* n is any clutter over ground set $[n]$ whose minimum cardinality members are precisely

$$\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}.$$

We will see in Chapter 2 that extended odd holes and their blockers are also non-ideal.

****Lemma 1.5** ([9], proved in Chapter 2). *Let V be a finite set of cardinality at least 4, and let $\mathcal{C}$ be a clutter over ground set V where $\min\{|C| : C \in \mathcal{C}\} = 2$ and the minimum cardinality members correspond to the edges of a connected bipartite graph over vertex set V whose color classes are R, B. If R or B contains a member, then $\mathcal{C}$ has either a delta or an extended odd hole minor.*

This lemma leads to the following result:

***Theorem 1.6** ([9], proved in Chapter 2). *Let $\mathcal{C}$ be a clutter over ground set V, where no element belongs to every member, and there is a $w \in \mathbb{R}_+^V$ such that $w(C) > \frac{\mathbf{1}^\top w}{2}$ for all $C \in \mathcal{C}$. Then $\mathcal{C}$ has either a delta or the blocker of an extended odd hole minor.*

1.2 Blockers

The second notion needed for understanding idealness is that of blockers. As we mentioned in the thesis opening on the history of ideal clutters, idealness is equivalent to satisfying the width-length inequality:

Theorem 1.7 (Lehman [49]). *Let $\mathcal{C}$ be a clutter over ground set E. Then the following statements are equivalent:*

(i) $\mathcal{C}$ *is ideal,*

(ii) $\min\{w(C) : C \in \mathcal{C}\} \times \min\{\ell(B) : B \in b(\mathcal{C})\} \leq w^\top \ell$ *for all* $w, \ell \in \mathbb{R}_+^E$,

(iii) $b(\mathcal{C})$ *is ideal.*

So being ideal is closed under taking blockers. As a consequence of Chaiken's result [17], we prove the following:

****Theorem 1.8** (Abdi, Cornuéjols, Lee, proved in Chapter 3). *Take an ideal clutter that does not have a member or cover of cardinality at most one. Then there are either two disjoint members or two disjoint covers.*

A clutter $\mathcal{C}$ is *identically self-blocking* if $\mathcal{C} = b(\mathcal{C})$. The deltas, for instance, are identically self-blocking clutters. Quite early on, Claude Berge noticed the following characterization of such clutters:

Remark 1.9 ([14]). *A clutter is identically self-blocking if, and only if, it does not have disjoint members or disjoint covers.*

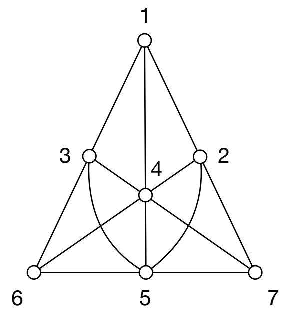


Figure 1.1: The Fano plane.

An identically self-blocking clutter is *trivial* if it is isomorphic to $\{\{1\}\}$. Theorem 1.8 and Remark 1.9 have the following consequence:

Corollary 1.10 (Abdi, Cornuéjols, Lee)**.** *A nontrivial identically self-blocking clutter is non-ideal.*

Under a certain assumption, we can strengthen this result by using Theorem 1.3:

***Theorem 1.11** (Abdi and Pashkovich, proved in Chapter 3)**.** *An identically self-blocking clutter with a member of cardinality two has a delta minor.*

The *Fano clutter* is the clutter over ground set $\{1,\ldots,7\}$ whose members are the lines of the Fano plane (see Figure 1.1):

$$\mathbb{L}_7 := \{\{1,3,6\},\{1,4,5\},\{1,2,7\},\{2,4,6\},\{2,3,5\},\{3,4,7\},\{5,6,7\}\}.$$

As the clutter of the lines of a projective plane, $\mathbb{L}_7$ is identically self-blocking. Since $\begin{pmatrix}\frac{1}{3} & \cdots & \frac{1}{3}\end{pmatrix}$ is an extreme point of $Q(\mathbb{L}_7)$, $\mathbb{L}_7$ is a non-ideal clutter, reaffirming Corollary 1.10. We conjecture the following:

Conjecture 1.12. *A nontrivial identically self-blocking clutter has one of* $\{\Delta_n : n \geq 3\} \cup \{\mathbb{L}_7, \mathcal{C}_5^2\}$ *as minor.*

We should point out that $\mathcal{C}_5^2$ is *not* identically self-blocking. We explain the rationale behind Conjecture 1.12 in Chapter 3.

1.3 Exclusive, coexclusive and opposite pairs

After minors and blockers, a third notion is needed for understanding idealness. Take a clutter $\mathcal{C}$ over ground set E, and take distinct elements e, f. We say that (e, f) is a *coexclusive pair* if every minimal cover contains at most one of e, f.

***Theorem 1.13** ([4], proved in Chapter 4)**.** *Let $\mathcal{C}$ be a clutter, and take distinct elements e, f. Then the following statements are equivalent:*

(i) (e, f) *is a coexclusive pair,*

(ii) *for all members* C_e, C_f *such that* $C_e \cap \{e, f\} = \{e\}$ *and* $C_f \cap \{e, f\} = \{f\}$*, there is another member contained in* $(C_e \cup C_f) - \{e, f\}$*,*

(iii) *for every extreme point* $x^\star$ *of* $Q(\mathcal{C})$*,* $x^\star_e + x^\star_f \leq 1$*.*

We say that (e, f) is an *exclusive pair* if every member contains at most one of e, f. We say that (e, f) is an *opposite pair* if it is both exclusive and coexclusive. There are several examples of ideal clutters with opposite pairs – let us present two of them here.

Let $D = (V, A)$ be a directed graph, where for every arc the opposite arc is also present. Let a, b be opposite arcs. Given distinct vertices s and t, (a, b) is an opposite pair in the clutter of directed st-paths, and given a vertex r, (a, b) is an opposite pair in the clutter of r-arborescences. Ignoring the directions of the arcs followed by identifying parallel edges, these two ideal clutters turn into clutters of st-paths and of spanning trees of an undirected graph labeled G, respectively, where the first clutter is ideal while the second one is not. Thinking of this operation in reverse, we start with two clutters associated with an undirected graph, one ideal and the other non-ideal, and by bidirecting the edges into opposite arcs, we turn them into ideal clutters. Let us extend this bidirecting trick to a general setting.

To *identify* elements e, f of clutter $\mathcal{C}$ is to replace it by the clutter over ground set $E - \{f\}$ whose members are

$$\mathcal{C}|_{e=f} := \text{ the minimal sets of } \big\{C \in \mathcal{C} : f \notin C\big\} \cup \big\{(C - \{f\}) \cup \{e\} : C \in \mathcal{C}, f \in C\big\}.$$

For instance, consider the ideal clutter $\mathbb{P}_4 := \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$. Then the clutter obtained from $\mathbb{P}_4$ after identifying $1, 4$ is $\Delta_3 = \{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$. Observe that identification is closed under taking blockers:

Remark 1.14. *Let* $\mathcal{C}$ *be a clutter, and take distinct elements* e, f*. Then* $b(\mathcal{C}|_{e=f}) = b(\mathcal{C})|_{e=f}$*.*

If the to-be-identified pair happens to be opposite, then the inverse of identification becomes an interesting operation. Let $\mathcal{C}$ be a clutter. A *single split of* $\mathcal{C}$ is another clutter which has an opposite pair whose identification gives back $\mathcal{C}$. We will provide a direct definition of single splits later. A clutter obtained from $\mathcal{C}$ after a series of single splits is called a *split of* $\mathcal{C}$.

Notice that as $(1, 4)$ is an opposite pair in $\mathbb{P}_4$, we may say that $\mathbb{P}_4$ is a single split of Δ_3. Moreover, it can be readily checked that the clutter of directed st-paths of D is a split of the clutter of st-paths of G, and the clutter of r-arborescences of D is a split of the clutter of spanning trees of G. By exploiting the geometry of idealness, we will see the following:

***Theorem 1.15** ([5], proved in Chapter 4)**.** *If a clutter is ideal, then so is any split of it.*

Thus, in terms of idealness, splitting does not make things worse, and as splits of Δ_3 and spanning tree clutters show, splitting can make a non-ideal clutter ideal. This made us wonder, does this phenomenon also hold for other minor-closed properties?

The *covering number of* $\mathcal{C}$, denoted $\tau(\mathcal{C})$, is the minimum cardinality of a cover, while the *packing number of* $\mathcal{C}$, denoted $\nu(\mathcal{C})$, is the maximum number of pairwise disjoint members. Clearly, $\tau(\mathcal{C}) \geq \nu(\mathcal{C})$.

If equality holds, we say that $\mathcal{C}$ *packs*. For instance, the clutter of st-paths of a graph packs because of Menger's theorem [56], while clutters such as the deltas do not.

We say that $\mathcal{C}$ has the *packing property* if every minor of $\mathcal{C}$, including the clutter itself, packs [24]. Lehman gave a powerful co-NP characterization of ideal clutters [50]; a fascinating consequence of his result is the following:

Theorem 1.16 (Lehman [24])**.** *If a clutter has the packing property, then it is ideal.*

Clutters with the packing property do not have a delta minor. Using this fact, we were able to prove the following more difficult analogue of Theorem 1.15:

****Theorem 1.17** ([5], proved in Chapter 4)**.** *If a clutter has the packing property, then so does any split of it.*

We say that $\mathcal{C}$ is *minimally non-packing* if it does not pack but every proper minor does. Notice that a clutter has the packing property if, and only if, it has no minimally non-packing minor. By Theorem 1.16, a minimally non-packing clutter is either ideal or *minimally non-ideal*, meaning that it is non-ideal but every proper minor is ideal. The second type, due to the powerful result of Lehman, is relatively well-understood. In contrast, the first type consisting of ideal minimally non-packing clutters is poorly understood, mainly because ideal clutters are still a mystery. Aware of this lack of understanding, we noticed the following:

Remark 1.18 ([4])**.** *Let $\mathcal{C}$ be a clutter and let (e,f) be a coexclusive pair. If $\mathcal{C}$ does not pack, then neither does $\mathcal{C}|_{e=f}$.*

Proof. Suppose that $\mathcal{C}$ does not pack. Clearly, $\nu(\mathcal{C}) \geq \nu(\mathcal{C}|_{e=f})$. Since (e,f) is coexclusive, every minimal cover contains at most one of e,f, so $\tau(\mathcal{C}|_{e=f}) = \tau(\mathcal{C})$. As a result, $\tau(\mathcal{C}|_{e=f}) = \tau(\mathcal{C}) > \nu(\mathcal{C}) \geq \nu(\mathcal{C}|_{e=f})$, so $\mathcal{C}|_{e=f}$ does not pack. $\square$

So, what if we made ideal minimally non-packing clutters even smaller by identifying coexclusive pairs that are present?

****Theorem 1.19** ([4], proved in Chapter 4)**.** *Let $\mathcal{C}$ be an ideal minimally non-packing clutter that has a coexclusive pair (e,f). Then,*

- *(i) $\mathcal{C}|_{e=f}$ is another ideal minimally non-packing clutter, or*
- *(ii) $\mathcal{C}|_{e=f}$ is not minimally non-packing, and every minimally non-packing minor has covering number two.*

The poor understanding of ideal minimally non-packing clutters stemmed, in part, from a lack of examples. In a paper by Lovász in 1972 [52], the first example appeared:

$$Q_6 := \{\{1,3,6\},\{1,4,5\},\{2,3,5\},\{2,4,6\}\}.$$

As the clutter of triangles of K_4, Q_6 established its importance in a seminal paper by Seymour in 1977, where he gave an excluded minor characterization of the matroids with the max-flow min-cut property [70].

The next example of an ideal minimally non-packing clutter was found by Schrijver in 1980 [63], and was used by him to refute a conjecture of Edmonds and Giles on the clutter of dijoins of a directed graph [31].

No more examples were known until 2000 when Cornuéjols, Guenin and Margot found a dozen sporadic instances, and an infinite family of such clutters which they labeled $\{Q_{r,t} : r, t \geq 1\}$ [24]. All of their sporadic instances, including Schrijver's example, had a coexclusive pair whose identification led to another sporadic instance, eventually collapsing to Q_6. However, none of the clutters in $\{Q_{r,t} : r, t \geq 1\}$, which included and extended $Q_6 = Q_{1,1}$, had a coexclusive pair (one might say that they sit at the bottom of the chains of ideal minimally non-packing clutters).

All of the *known* ideal minimally non-packing clutters, Cornuéjols, Guenin and Margot noticed, had one common feature: they all had covering number two. So they made the following conjecture:

The $\tau = 2$ Conjecture ([24])**.** *Every ideal minimally non-packing clutter has covering number two.*

This made outcome (ii) of Theorem 1.19 even more mysterious. It also prompted us to focus on the ideal minimally non-packing clutters not considered in Theorem 1.19 and with covering number two. So, using the fact that ideal clutters have no delta minor, we came up with the following characterization:

****Theorem 1.20** ([4], proved in Chapter 4)**.** *Let $\mathcal{C}$ be an ideal minimally non-packing clutter over ground set E. If $\mathcal{C}$ is without a coexclusive pair and has covering number two, then the following statements hold:*

(1) the minimum covers partition E,

(2) the minimum cardinality of a member is $\frac{|E|}{2}$,

(3) the members of minimum cardinality form an ideal non-packing cuboid.

Cuboids are defined shortly. In the meantime, we should point out that in all the known examples of clutters satisfying the hypotheses of Theorem 1.20, including $\{Q_{r,t} : r, t \geq 1\}$, the members are equicardinal.

1.4 Cuboids

Take an integer $n \geq 1$. A *cuboid* is a clutter whose ground set can be relabeled as $[2n]$ and every member C obeys

$$|C \cap \{2i-1, 2i\}| = 1 \qquad \forall i \in [n].$$

In particular, every member has cardinality n, and for each $i \in [n]$, $\{2i-1, 2i\}$ is a cover. For instance, the clutter Q_6 is a cuboid. Motivated by Theorem 1.20, when is a cuboid ideal? To answer this question, we need to compress the data defining a cuboid, and change frameworks as a result.

We will work over $\{0,1\}^n$, the vertices of the unit hypercube $[0,1]^n$. Take a set $S \subseteq \{0,1\}^n$. The *cuboid of S*, denoted cuboid(S), is the clutter over ground set $[2n]$ whose members have incidence vectors

$$(x_1, 1-x_1, \ldots, x_n, 1-x_n) \quad x \in S.$$

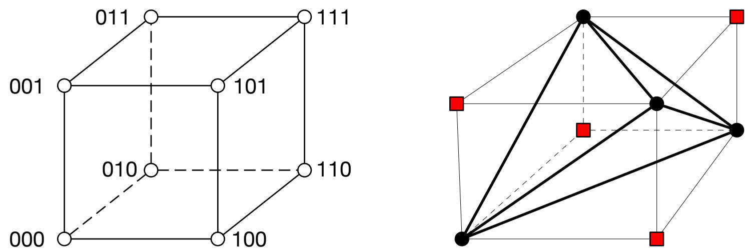


Figure 1.2: An illustration of the coordinate system, and the convex hull of $R_{1,1}$.

For instance, given $R_{1,1} := \{000, 110, 011, 101\} \subseteq \{0,1\}^3$, its cuboid is[3]

$$\mathrm{cuboid}(R_{1,1}) = \{\{2,4,6\},\{1,3,6\},\{2,3,5\},\{1,4,5\}\} = Q_6.$$

Notice that in fact, every cuboid over ground set $[2n]$ is the cuboid of an appropriate subset of $\{0,1\}^n$.

***Theorem 1.21** ([2], proved in Chapter 5)**.** *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then* $\mathrm{cuboid}(S)$ *is ideal if, and only if, S is cube-ideal.*

We say that S is *cube-ideal* if its convex hull, denoted $\mathrm{conv}(S)$, can be described by inequalities of the form

$$\begin{array}{rll} 0 \leq x_i \leq 1 & i \in [n] & (\textit{hypercube} \text{ inequalities}) \\ \displaystyle\sum_{i\in I} x_i + \sum_{j \in J}(1-x_j) \geq 1 & I, J \subseteq [n], I \cap J = \emptyset & (\textit{generalized set covering} \text{ inequalities}). \end{array}$$

Notice that generalized set covering inequalities[4] are precisely the inequalities that cut off (sub-)hypercubes of $\{0,1\}^n$. For instance, $R_{1,1}$ is cube-ideal because its convex hull is

$$\mathrm{conv}(R_{1,1}) = \left\{ x \in [0,1]^3 : \begin{array}{ll} (1-x_1) + x_2 + x_3 & \geq 1 \\ x_1 + (1-x_2) + x_3 & \geq 1 \\ x_1 + x_2 + (1-x_3) & \geq 1 \\ (1-x_1) + (1-x_2) + (1-x_3) & \geq 1 \end{array} \right\},$$

as illustrated in Figure 1.2. In particular, Theorem 1.21 reassures us that $Q_6 = \mathrm{cuboid}(R_{1,1})$ is indeed ideal.

Motivated by Theorem 1.21, when is a set cube-ideal? Given points $a, b \in \{0,1\}^n$, denote by $a \triangle b$ their coordinatewise sum modulo 2, and let

$$S \triangle a := \{y \triangle a : y \in S\}.$$

[3] For convenience, we will represent the points in $\{0,1\}^n$ as $0,1$ strings of length n.

[4] Interpreted as clause satisfaction inequalities for the Boolean satisfiability problem, it should come as no surprise that generalized set covering inequalities are fundamental and prevalent in the literature. Also referred to as *cropping* inequalities [47], these inequalities have surfaced as *cocircuit* inequalities valid for cycle polytopes of binary matroids [12], as *set covering* inequalities ($J = \emptyset$) for various set covering problems, and as *cover* inequalities ($I = \emptyset$) for the knapsack problem.

Take a coordinate $i \in [n]$. To *twist S at coordinate i* is to replace S by the set $S \triangle e_i$. This terminology is due to Bouchet [15]. As twists correspond to the change of variables $x_i \mapsto 1-x_i, i \in [n]$, and the hypercube and generalized set covering inequalities are closed under these transformations, if a set is cube-ideal then so is every twist of it.

Given a point $x \in \{0,1\}^n$, define the *induced clutter of S with respect to x* as the clutter over ground set $[n]$ whose members are

$$\mathrm{ind}(S \triangle x) := \text{the minimal sets of } \big\{C \subseteq [n] : \chi_C \in S \triangle x\big\}.$$

Here, χ_C denotes the characteristic vector of C. By using Lehman's powerful result on minimally non-ideal clutters, we will prove the following:

****Theorem 1.22** ([2], proved in Chapter 5)**.** *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then S is cube-ideal if, and only if, every induced clutter of S is ideal.*

For instance, the induced clutters of $R_{1,1}$ are equal to either $\{\emptyset\}$ or $\{\{1\},\{2\},\{3\}\}$, and as these clutters are clearly ideal, Theorem 1.22 reassures us that $R_{1,1}$ is indeed cube-ideal.

Conjecture 1.23 ([1])**.** *There exists an algorithm that given an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$ determines in time polynomial in $n, |S|$ whether or not S is cube-ideal.*

By Theorem 1.21, this conjecture equivalently predicts that idealness of cuboids can be tested in polynomial time, even though testing idealness of general clutters is co-NP-complete according to Theorem 1.1. This conjecture may suggest that cuboids form a small class of clutters, but this is actually not the case. To elaborate, take a clutter $\mathcal{C}$ over ground set E. The *set covering polytope of $\mathcal{C}$* is

$$P(\mathcal{C}) := \big\{x \in [0,1]^E : x(C) \geq 1 \ \forall\ C \in \mathcal{C}\big\}.$$

Proposition 1.24 (folklore; see [73], Proposition 3.2.1)**.** *Let $\mathcal{C}$ be a clutter. Then the integral extreme points of $P(\mathcal{C})$ are precisely the incidence vectors of the covers of $\mathcal{C}$. Moreover, $\mathcal{C}$ is ideal if, and only if, $P(\mathcal{C})$ is an integral polytope.*

We are now ready to show that studying cube-idealness is just as general as studying clutter idealness:

***Theorem 1.25** ([2])**.** *Let $\mathcal{C}$ be a clutter over ground set E, and let*

$$S := \{\chi_C : C \subseteq E \text{ contains a member}\} \subseteq \{0,1\}^E.$$

Then $\mathcal{C}$ is ideal if, and only if, S is cube-ideal.

(A word of caution to the reader: Theorem 1.25 does not imply that Conjecture 1.23 is at odds with Theorem 1.1; the set S above has a different size than the clutter $\mathcal{C}$.)

Proof. By Theorem 1.7, it suffices to show that $b(\mathcal{C})$ is ideal if, and only if, S is cube-ideal. ($\Rightarrow$) Assume that $b(\mathcal{C})$ is ideal. Then by Proposition 1.24, the set covering polytope $P(b(\mathcal{C}))$ is integral and its vertices are the incidence vectors of the covers of $b(\mathcal{C})$, i.e. the points in S. Hence, $P(b(\mathcal{C})) = \mathrm{conv}(S)$, implying in turn that S is cube-ideal. ($\Leftarrow$) Assume conversely that S is cube-ideal, that is, $\mathrm{conv}(S)$ is described by hypercube and generalized set covering inequalities. If

$$\sum_{i\in I} x_i + \sum_{j\in J}(1-x_j) \geq 1 \qquad I,J\subseteq [n], I\cap J=\emptyset$$

is valid for S, then so is the set covering inequality $\sum_{i\in I} x_i \geq 1$, as S is up-monotone. As a result, $\mathrm{conv}(S)$ is described by hypercube and set covering inequalities. Inevitably, $\mathrm{conv}(S) = P(b(\mathcal{C}))$, implying by Proposition 1.24 that $b(\mathcal{C})$ is an ideal clutter. $\square$

In addition, we will see in the next section and in Chapter 5 that some prominent conjectures and results on clutters can equivalently be formulated in terms of cuboids, demonstrating that cuboids, in fact, form an expansive class of clutters.

1.5 Binary spaces

Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. We say that S is a *vector space over* $GF(2)$, or simply a *binary space*, if $a\triangle b \in S$ for all (possibly equal) points $a, b \in S$. For instance, $R_{1,1}$ is a binary space. Notice that S is a binary space if, and only if, it is the cycle space of a binary matroid [60]. As for $R_{1,1}$, it is the cycle space of the graph on two vertices and three parallel edges between them. Relying on a result of Barahona and Grötschel [12], we prove the following characterization:

****Theorem 1.26** ([2], proved in Chapter 5). *Take an integer $n \geq 1$, a binary space $S \subseteq \{0,1\}^n$, and let M be the corresponding binary matroid. Then S is cube-ideal if, and only if, M has the sums of circuits property.*

A binary matroid M over ground set E has the *sums of circuits property* if for all $w \in \mathbb{R}^E_+$ satisfying

$$w(D - \{f\}) \geq w_f \quad \forall \text{ cocycles } D, \forall f \in D,$$

there exists for each circuit C an assignment $y_C \in \mathbb{R}_+$ such that

$$w = \sum \left(y_C \cdot \chi_C : C \text{ is a circuit}\right).$$

Seymour defined this rich property in 1979 where he proved that graphic matroids have the sums of circuits property [68] – let us prove this result using the tools developed so far.

Let $G = (V, E)$ be a graph and $T \subseteq V$ of even cardinality. A subset $J \subseteq E$ is called a *T-join* if the terminals are precisely the vertices incident with an odd number of non-loop edges in J.

Remark 1.27 ([2]). *Take a graph $G = (V, E)$ and let $S \subseteq \{0,1\}^E$ be its cycle space. Then every induced clutter of S is equal to, for some $T \subseteq V$ of even cardinality, the clutter of minimal T-joins of G.*

Proof. Take a point $\chi_A \in \{0,1\}^E$. Then

$$S\triangle\chi_A = \{\chi_C\triangle\chi_A : C \text{ is a cycle}\} = \{\chi_{C\triangle A} : C \text{ is a cycle}\} = \{\chi_J : J\triangle A \text{ is a cycle}\} = \{\chi_J : J \text{ is a } T\text{-join}\}$$

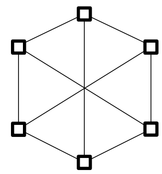


Figure 1.3: A graft representation of R_{10}.

where T is the set of odd-degree vertices of $A \subseteq E$. As a result, $\mathrm{ind}(S\triangle\chi_A)$ is the clutter of minimal T-joins of G, as required. $\square$

The pair (G,T) is referred to as a *graft*. The vertices in T are called *terminals*. A *T-cut* is a cut of the form $\delta(U)$ for some $U \subseteq V$ such that $|U \cap T|$ is odd. As mentioned already, the clutter of minimal T-cuts of (G,T) is ideal [32]. By Theorem 1.7, its blocker is also ideal:

Theorem 1.28 (Edmonds and Johnson [32], see [22], Theorems 1.21 and 2.1)**.** *Given a graft with terminals T, its clutter of minimal T-joins is ideal.*

Given the cycle space S of a graph, Remark 1.27 and Theorem 1.28 imply that the induced clutters of S are ideal. Thus, by Theorem 1.22, S is cube-ideal, so by Theorem 1.26,

Theorem 1.29 (Seymour [68])**.** *Graphic matroids have the sums of circuits property.*

In addition to graphic matroids, the Fano matroid and the cut matroid of the Wagner graph happen to have the sums of circuits property as well. After developing his so-called splitter theorems and a decomposition theorem for regular matroids [65], Seymour proved that these are *the* building blocks of binary matroids with the sums of circuits property, and obtained the following as a consequence:[5]

Theorem 1.30 (Seymour [66], (16.4))**.** *A binary matroid has the sums of circuits property if, and only if, it has none of $F_7^\star, R_{10}$, $M(K_5)^\star$ as a minor.*

$F_7^\star$ is the dual of the Fano matroid, R_{10} is the binary matroid whose graft representation is displayed in Figure 1.3, and $M(K_5)^\star$ is the cut matroid of K_5. Theorem 1.30, together with Theorem 1.26, completely characterizes cube-ideal binary spaces. After carefully studying the induced clutters of the cycle spaces corresponding to $F_7^\star, R_{10}, M(K_5)^\star$, we see that Theorems 1.22, 1.26 and 1.30 have the following consequence:

Corollary 1.31. *Take an integer $n \geq 1$ and a binary space $S \subseteq \{0,1\}^n$. Then S is cube-ideal if, and only if, every induced clutter has no $\mathbb{L}_7, \mathbb{O}_5, b(\mathbb{O}_5)$ minor.*

We will define $\mathbb{O}_5$ and $b(\mathbb{O}_5)$ shortly. Let us try to refine this consequence by asking the following question:

When is an induced clutter of a binary space ideal?

[5] To be accurate, he proved all of these results on the dual matroid, where the sums of circuits property corresponds to the ∞-*flowing property.*

Moving forward, we need another concept. Twists of binary spaces are referred to as *affine binary spaces*. Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Basic Linear Algebra implies that S is an affine binary space if, and only if, the symmetric difference of any odd number of points in S belongs to S. Notice that if S is an affine binary space, then for each $x \in S$, $S \triangle x$ is a binary space.

1.6 Binary clutters

A clutter is *binary* if the symmetric difference of any odd number of members contains a member [48].

Remark 1.32. *A clutter is binary if, and only if, it is an induced clutter of an affine binary space.*

Proof. ($\Rightarrow$) Let $\mathcal{C}$ be a binary clutter over ground set E, and let

$$S := \big\{\chi_C : C \subseteq E \text{ is the symmetric difference of an odd number of members}\big\}.$$

By definition, S is an affine binary space. As $\mathcal{C}$ is a binary clutter, it follows that $\mathrm{ind}(S) = \mathcal{C}$, as required. ($\Leftarrow$) Let $\mathcal{C}$ be an induced clutter of an affine binary space S. After a possible twisting, we may assume that $\mathcal{C} = \mathrm{ind}(S)$. The symmetric difference of any odd number of points in S is also a point of S, implying in turn that the symmetric difference of any odd number of members of $\mathcal{C}$ contains a member, so $\mathcal{C}$ is a binary clutter, as required. $\square$

So the question we ended the previous section with boils down to:

> When is a binary clutter ideal?

If a clutter is binary, then so is every minor of it [69]. So a co-NP approach presents itself:

> What are the minimally non-ideal binary clutters?

If a clutter is binary, then so is its blocker [48]. Moreover, if a clutter is minimally non-ideal, then so is its blocker by Theorem 1.7. Thus, whatever the family of minimally non-ideal binary clutters, it has to be closed under taking blockers.

Recall that $\mathbb{L}_7$ is the identically self-blocking clutter over ground set $\{1,\ldots,7\}$ of the lines of the Fano plane. We saw that $\mathbb{L}_7$ is a non-ideal clutter. In fact, $\mathbb{L}_7$ is a minimally non-ideal binary clutter [70].

To digress, neither the deltas nor the odd holes are binary clutters, so Conjecture 1.12 for binary clutters reduces to:

Conjecture 1.33. *A nontrivial identically self-blocking binary clutter has an $\mathbb{L}_7$ minor.*

There are two other known examples of minimally non-ideal binary clutters. Let $\mathbb{O}_5$ be the clutter of odd circuits of K_5 over ground set $E(K_5)$. As $\left(\frac{1}{3} \ \cdots \ \frac{1}{3}\right)$ is an extreme point of $Q(\mathbb{O}_5)$, $\mathbb{O}_5$ is non-ideal. In fact, $\mathbb{O}_5$ is another minimally non-ideal binary clutter [70]. As a result, $b(\mathbb{O}_5)$, which is the clutter of cut complements of K_5 over ground set $E(K_5)$, is also a minimally non-ideal binary clutter [70].

To date, $\mathbb{L}_7, \mathbb{O}_5, b(\mathbb{O}_5)$ are the only known minimally non-ideal binary clutters. In 1977, Seymour made the following conjecture:

The f-Flowing Conjecture ([70, 66]). *Up to isomorphism, $\mathbb{L}_7, \mathbb{O}_5, b(\mathbb{O}_5)$ are the only minimally non-ideal binary clutters.*

We will prove that,

*****Theorem 1.34** ([6], proved in Chapter 9). *Up to isomorphism, $\mathbb{L}_7, \mathbb{O}_5$ are the only minimally non-ideal binary clutters with a member of cardinality three.*

1.7 Strict polarity

Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Recall from Theorems 1.21 and 1.22 that cuboid(S) is ideal if, and only if, every induced clutter of S is ideal. Since every induced clutter picks up only local information about S, we may view idealness as a "local" property. To formalize this, let $\mathcal{P}$ be a minor-closed property defined on clutters. We say that $\mathcal{P}$ is 2-*local* if, for every integer $n \geq 1$ and set $S \subseteq \{0,1\}^n$, the following statements are equivalent:

- cuboid(S) has property $\mathcal{P}$,
- every induced clutter of S has property $\mathcal{P}$.

(The prefix 2- in 2-local is added to accommodate for a more general locality feature idealness satisfies, which we will not discuss in this thesis.) For instance, idealness is a 2-local property. What about the packing property? Well, cuboid($R_{1,1}$) $= Q_6$ does not pack, while its induced clutters have the packing property as they are equal to either $\{\emptyset\}$ or $\{\{1\},\{2\},\{3\}\}$. Thus, the packing property is not 2-local, which at a high level, is a rift from idealness. Let us extract the non-2-local essence of the packing property.

Two points a, b in $\{0,1\}^n$ are *antipodal* if $a + b = \mathbf{1}$. Take a set $S \subseteq \{0,1\}^n$. We will refer to the points in S as *feasible* and to the points in $\overline{S}$ as *infeasible*. (Here, $\overline{S} := \{0,1\}^n - S$.) We say that S is *polar* if either there are antipodal feasible points, or the feasible points all agree on a coordinate:

$$\{x, \mathbf{1} - x\} \subseteq S \text{ for some } x \in \{0,1\}^n \qquad \text{or} \qquad S \subseteq \{x : x_i = a\} \text{ for some } i \in [n] \text{ and } a \in \{0,1\}.$$

Observe that S is polar if, and only if, cuboid(S) packs. Notice further that if S is polar, then so is any twist of it.

Take a coordinate $i \in [n]$. The set obtained from $S \cap \{x : x_i = 0\}$ after dropping coordinate i is called the 0-*restriction of S over coordinate i*, and the set obtained from $S \cap \{x : x_i = 1\}$ after dropping coordinate i is called the 1-*restriction of S over coordinate i*. If S' is obtained from S after 0- and 1-restricting some coordinates, then we say that S' is a *restriction of S*. A restriction of S is *proper* if it is not equal to S.

We say that S is *strictly polar* if every restriction of it, including S itself, is polar. We will see:

****Theorem 1.35** ([2], proved in Chapter 5). *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then* cuboid(S) *has the packing property if, and only if, S is strictly polar and every induced clutter of S has the packing property.*

As a consequence, once strict polarity is enforced, the packing property becomes 2-local, behaving just like idealness. We conjecture that strict polarity does far more:

The Polarity Conjecture ([2]). *Take an integer $n \geq 1$ and a strictly polar set $S \subseteq \{0,1\}^n$. Then* $\mathrm{cuboid}(S)$ *has the packing property if, and only if,* $\mathrm{cuboid}(S)$ *is ideal.*

We will see that,

****Theorem 1.36** ([2], proved in Chapter 5). *The Polarity Conjecture is equivalent to the $\tau = 2$ Conjecture.*

The Polarity Conjecture prompts the following question: when is a set strictly polar?

****Theorem 1.37** ([2], proved in Chapter 5). *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then the following statements are equivalent:*

(i) S is not strictly polar,

(ii) there are distinct points $a, b, c \in S$ such that the restriction of S containing them of smallest dimension is not polar.

As a result, in time $O(n|S|^4)$ one can certify whether or not S is strictly polar.

A set is *strictly non-polar* if it is not polar but every proper restriction is polar. Observe that a set is strictly polar if, and only if, it has no strictly non-polar restriction. Theorem 1.37 equivalently states that every strictly non-polar set has three distinct feasible points that do not all agree on a coordinate. For instance, $R_{1,1}$ is strictly non-polar, and no three points of it agree on a coordinate.

1.8 Strict non-polarity

Recall that a minimally non-packing clutter is either ideal or minimally non-ideal. We will see that,

***Theorem 1.38** ([2], proved in Chapter 5). *A minimally non-packing cuboid is ideal.*

This theorem tells us where to look for ideal minimally non-packing clutters – among cuboids. In fact, because of the following remark, we will look for them among strictly non-polar sets:

Remark 1.39 ([2]). *Take an integer $n \geq 3$ and a set $S \subseteq \{0,1\}^n$. If* $\mathrm{cuboid}(S)$ *is minimally non-packing, then S is strictly non-polar.*

Take an integer $n \geq 1$. We say that two sets $S_1, S_2 \subseteq \{0,1\}^n$ are *isomorphic*, and write $S_1 \cong S_2$, if one is obtained from the other after relabeling and twisting the coordinates. We set out to generate non-isomorphic strictly non-polar sets. How? The *skeleton graph G_n of $\{0,1\}^n$* is the graph whose vertices are the points in $\{0,1\}^n$ where two points are adjacent if they differ in exactly one coordinate. Take a set $S \subseteq \{0,1\}^n$ and an integer $k \geq 0$. We say that S has *degree at most k* if every infeasible point has at most k infeasible neighbors (in G_n). We say that S has *degree k* if it has degree at most k and not $k-1$.

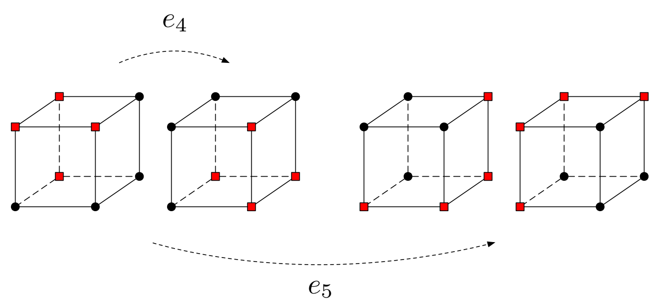


Figure 1.4: An illustration of R_5, a strictly non-polar set of degree 2. Round points are feasible while square points are infeasible.

****Theorem 1.40** ([2], explained in Chapter 5). *Up to isomorphism, there are precisely* 745 *strictly non-polar sets of dimension at most* 7 *and degree at most* 4*,* 716 *of which have ideal minimally non-packing cuboids.*

The 745 sets above are provided explicitly in [2]. Theorem 1.40 is proved by a computer code we have written for generating strictly non-polar sets of bounded degree. Let us emphasize that these 716 ideal minimally non-packing cuboids are pairwise non-isomorphic and have up to $14 = 2 \times 7$ elements.

We noticed a pattern among the generated strictly non-polar sets; let us elaborate. Notice that if $S \subseteq \{0,1\}^n$ is non-polar, then $|S| \leq 2^{n-1}$. We say that $S \subseteq \{0,1\}^n$ is *half-dense* if $|S| = 2^{n-1}$. Out of the 745 strictly non-polar sets above, 75 are half-dense, while most of the other ones are *nearly* half-dense. For instance, every strictly non-polar set of dimension 6 and degree 4, of which there are 682, has cardinality between 27 and 32.

So what can we prove in general about strictly non-polar sets of bounded degree? It is not difficult to see that $R_{1,1}$ is, up to isomorphism, the only strictly non-polar set of degree at most one [4]. Using Mantel's Theorem [55], as well as Theorem 1.3 on finding delta minors, we prove the following:

****Theorem 1.41** ([2], proved in Chapter 5). *Take an integer $k \geq 2$ and a strictly non-polar set S of degree k, whose dimension is n. Then $k \leq n \leq 2k+1$. Moreover, if $n = 2k+1$, then S is half-dense, every infeasible point has exactly k infeasible neighbors, and* cuboid(S) *is an ideal minimally non-packing clutter.*

Thus, Theorem 1.40 yields a complete list of all strictly non-polar sets of degree 2 and 3: up to isomorphism, there are precisely two strictly non-polar sets of degree 2 [4], while there are 11 strictly non-polar sets of degree 3 [2]. For instance,

$$\begin{aligned} R_5 :=& \{00000, 10000, 11000, 11100, 11110, 01110, 00110, 00010\} \\ & \cup \{01001, 01101, 00101, 10101, 10111, 10011, 11011, 01011\} \subseteq \{0,1\}^5, \end{aligned}$$

as displayed in Figure 1.4, is strictly non-polar of degree 2 and dimension $5 = (2 \times 2) + 1$. Notice that R_5 is half-dense, every infeasible point has exactly 2 infeasible neighbors, and according to Theorem 1.41, cuboid(R_5) is ideal minimally non-packing. In fact, cuboid(R_5) is equal to Q_{10}, an ideal minimally non-packing clutter found in [4].

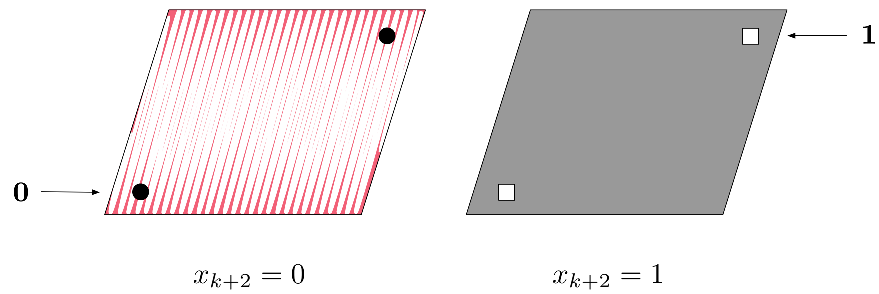


Figure 1.5: An illustration of $R_{k,1}$. The shaded plane on the left is infeasible, while the filled-in plane on the right is feasible. Round points are feasible while square points are infeasible.

There is in fact an infinite class of half-dense strictly non-polar sets, extending $R_{1,1}$. For each integer $k \geq 1$, let

$$R_{k,1} := \left\{(\mathbf{0}^{k+1}, 0), (\mathbf{1}^{k+1}, 0)\right\} \cup \left\{(x, 1) : x \in \{0,1\}^{k+1} - \left\{\mathbf{0}^{k+1}, \mathbf{1}^{k+1}\right\}\right\} \subseteq \{0,1\}^{k+2},$$

as displayed in Figure 1.5. ($\mathbf{0}^{k+1}, \mathbf{1}^{k+1}$ denote the all-zeros and all-ones vectors of dimension $k+1$, respectively.) Then $\{R_{k,1} : k \geq 1\}$ are strictly non-polar sets. In fact, $\{\text{cuboid}(R_{k,1}) : k \geq 1\} = \{Q_{k,1} : k \geq 1\}$, part of the infinite class of ideal minimally non-packing clutters found in [24].

Other than being half-dense and strictly non-polar, $\{R_{k,1} : k \geq 1\}$ and R_5 have another common feature: they are 1-*resistant*, a concept defined shortly. We will prove the following theorem:

*****Theorem 1.42** ([3], proved in Chapter 7)**.** *Up to isomorphism, $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ are the only half-dense strictly non-polar sets that are 1-resistant.*

So what are 1-resistant sets? And what are they resistant to?

1.9 Resistance

Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. For a coordinate $i \in [n]$, the *projection of S over i* is the subset of $\{0,1\}^{n-1}$ obtained from S after dropping coordinate i. A set obtained from S after a series of projections and restrictions is called a *minor of S*. The minor is *proper* if at least one projection or restriction is applied.

Remark 1.43 ([2])**.** *If a set is cube-ideal, then so is any minor of it.*

Consider the 3-dimensional sets

$$P_3 := \{110, 011, 101\} \qquad \text{and} \qquad S_3 := \{110, 011, 101, 111\},$$

as displayed in Figure 1.6. Their induced clutters with respect to the origin is the non-ideal clutter $\Delta_3 = \{\{1,2\},\{2,3\},\{3,1\}\}$, so P_3, S_3 are not cube-ideal by Theorem 1.22. In fact, up to isomorphism, P_3

Figure 1.6: An illustration of P_3 and S_3, the smallest non-cube-ideal sets.

and S_3 are the only non-cube-ideal sets of dimension at most 3. Thus, by Remark 1.43, cube-ideal sets have no P_3, S_3 minor. So, in pursuit of better understanding cube-ideal sets, we set out to investigate sets without a P_3, S_3 minor, and as a first step, in this thesis, we study only sets that do not have a P_3, S_3 minor even after we locally change the set.

We say that S is 2-*resistant* if, for every subset $X \subseteq \{0,1\}^n$ of cardinality at most two, $S \cup X$ has no P_3, S_3 minor. We will see that,

****Theorem 1.44** ([1], proved in Chapter 6)**.** *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then S is 2-resistant if, and only if, every infeasible component is a hypercube or has maximum degree at most two.*

Here, an *infeasible component of S* is a connected component of $G_n[\overline{S}]$.[6]

We say that S is ± 1-*resistant* if, for every subset $X \subseteq \{0,1\}^n$ of cardinality at most one, $S \triangle X$ has no P_3, S_3 minor. We will prove the following structure theorem for ± 1-resistant sets:

*****Theorem 1.45** ([1], proved in Chapter 8)**.** *Take an integer $n \geq 1$ and a ± 1-resistant set $S \subseteq \{0,1\}^n$. Then one of the following statements holds:*

(i) $S \cong A_k \times \{0,1\}^{n-k}$ *for some* $k \in \{2, \ldots, n\}$, *where* $A_k = \{\mathbf{0}, \mathbf{1}\} \subseteq \{0,1\}^k$,

(ii) $S \cong B_k \times \{0,1\}^{n-k}$ *for some* $k \in \{3, \ldots, n\}$, *where* $B_k = \{\mathbf{0}, e_1, \mathbf{1}\} \subseteq \{0,1\}^k$,

(iii) $S \cong C_8 \times \{0,1\}^{n-4}$, *where* $C_8 = \{0000, 1000, 1010, 1011, 1111, 0111, 0101, 0100\} \subseteq \{0,1\}^4$,

(iv) $S \cong D_k \times \{0,1\}^{n-k}$ *for some* $k \in \{3, \ldots, n\}$, *where* $D_k = \{\mathbf{0}, e_2, \mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\} \subseteq \{0,1\}^k$,

(v) *S is a hypercube, or*

(vi) *every infeasible component of S is a hypercube.*

Here, given integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq \{0,1\}^{n_1}, S_2 \subseteq \{0,1\}^{n_2}$, the *product of S_1 and S_2* is

$$S_1 \times S_2 = \{(x, y) : x \in S_1, y \in S_2\} \subseteq \{0,1\}^{n_1+n_2}.$$

More generally, we say that S is 1-*resistant* if, for every subset $X \subseteq \{0,1\}^n$ of cardinality at most one, $S \cup X$ has no P_3, S_3 minor. Observe that 2- and ± 1-resistant sets are 1-resistant.

***Theorem 1.46** ([3], proved in Chapter 6)**.** *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then in time $O(n^2|S|^2)$, one can test whether or not S is 1-resistant.*

[6]Given a graph $G = (V, E)$ and $U \subseteq V$, $G[U]$ denotes the subgraph of G induced on vertex set U.

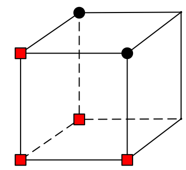


Figure 1.7: An illustration of a fragile set.

Take a set $F \subseteq \{0,1\}^3$ such that

$$F \cap \{000, 100, 010, 001, 101, 011\} = \{101, 011\};$$

see Figure 1.7 for an illustration. We refer to F, and any set isomorphic to it, as *fragile*.

Remark 1.47. *A 3-dimensional set is fragile if, and only if, it is not 1-resistant.*

Proof. ($\Rightarrow$) Take a fragile set $F \subseteq \{0,1\}^3$, where $F \cap \{000, 100, 010, 001, 101, 011\} = \{101, 011\}$. Then $F \cup \{110\}$ is either P_3 or S_3, so F is not 1-resistant. ($\Leftarrow$) is immediate also. $\square$

We will prove the following characterization of 1-resistant sets:

****Theorem 1.48** ([3], proved in Chapter 6)**. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then the following statements are equivalent:*

(i) S is 1-resistant,

(ii) S has no fragile minor,

(iii) for each $x \in \{0,1\}^n$, the members of $\mathrm{ind}(S \triangle x)$ are pairwise disjoint,

(iv) S has no fragile restriction and no $\{\mathbf{0}^k, \mathbf{1}^k - e_1\}, k \geq 4$ isomorphic restriction.[7]

Theorem 1.46 is in fact a consequence of this theorem, and as another consequence, we will show that,

***Theorem 1.49** ([3], proved in Chapter 6)**.** *Every 1-resistant set is cube-ideal.*

[7] The prefix "isomorphic" from "isomorphic restriction" will hereinafter be omitted.

Chapter 2

Minors

Take an integer $n \geq 3$. Recall that Δ_n is the clutter over ground set $[n]$ whose members are $\{1,2\}$, $\{1,3\},\ldots,\{1,n\},\{2,3,\ldots,n\}$, and that $b(\Delta_n) = \Delta_n$.

Remark 2.1. *The deltas $\{\Delta_n : n \geq 3\}$ are non-ideal.*

Proof. It suffices by Theorem 1.7 to prove that the deltas violate the width-length inequality for suitable widths and lengths. Take an integer $n \geq 3$. Let $w := (n-2, 1, \ldots, 1)$ and $\ell := (1, 1, \ldots, 1)$. Then

$$\min\{w(C) : C \in \Delta_n\} \times \min\{\ell(B) : B \in b(\Delta_n) = \Delta_n\} = (n-1) \times 2 > (n-2) + (n-1) = w^\top \ell,$$

so the width-length inequality is violated, as required. □

Take an odd integer $n \geq 5$. Recall that an extended odd hole of dimension n is any clutter over ground set $[n]$ whose minimum cardinality members are $\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}$, the edges of an odd circuit of length n.

Remark 2.2. *Extended odd holes and their blockers are non-ideal.*

Proof. Take an odd integer $n \geq 5$ and let $\mathcal{C}$ be an extended odd hole of dimension n. Since it intersects each one of the members $\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}$, every cover has cardinality at least $\lceil \frac{n}{2} \rceil = \frac{n+1}{2}$. Thus, for $w := \ell := (1,1,\ldots,1)$,

$$\min\{w(C) : C \in \mathcal{C}\} \times \min\{\ell(B) : B \in b(\mathcal{C})\} \geq 2 \times \frac{n+1}{2} = n+1 > n = w^\top \ell,$$

so the width-length inequality is violated. By Theorem 1.7, both $\mathcal{C}$ and $b(\mathcal{C})$ are non-ideal, as required. □

The careful reader will notice that the deltas and blockers of extended odd holes are non-ideal for a common reason:

Remark 2.3. *Let $\mathcal{C}$ be a clutter over ground set V such that $\tau(\mathcal{C}) \geq 2$. If there exists $w \in \mathbb{R}_+^V$ such that $w(C) > \frac{\mathbf{1}^\top w}{2}$ for every member C, then $\mathcal{C}$ is non-ideal.*

Proof. Let $\ell := (1, 1, \ldots, 1) \in \mathbb{R}_+^V$. Then

$$\min\{w(C) : C \in \mathcal{C}\} \times \min\{\ell(B) : B \in b(\mathcal{C})\} > \frac{\mathbf{1}^\top w}{2} \times \tau(\mathcal{C}) \geq \mathbf{1}^\top w = \ell^\top w,$$

so the width-length inequality is violated, and thereby, the result follows from Theorem 1.7. $\square$

Deltas, extended odd holes and their blockers are among the simplest examples of non-ideal clutters, and in this chapter, we explore different ways to find them as minors. In §2.1, we provide two tools for finding delta minors, and provide as a consequence a polynomial time algorithm for finding delta minors. In §2.2, we prove a lemma for finding delta or extended odd hole minors, and in §2.3, we show that in the context of the preceding remark, we can in fact find a delta or the blocker of an extended odd hole minor.

2.1 Delta minors

Take a clutter $\mathcal{C}$ over ground set V. Take an element u and a member C *not* containing u. We say that C is *u-redundant* if there is another member C' such that $u \in C'$ and $C' - \{u\} \subsetneq C$. The member C' is called a *cause* of the redundancy. Equivalently, C is u-redundant if it is not a member of $\mathcal{C}/u$. We say that C is *doubly u-redundant* if it is u-redundant with at least two distinct causes.

Theorem 1.3 **([4]).** *Let $\mathcal{C}$ be a clutter over ground set V, and take an element $u \in V$. If there is a doubly u-redundant member, then there is a delta minor using element u that can be found in time $O\big(|V||\mathcal{C}|^2\big)$.*

Proof. We may assume that $\mathcal{C}$ is minor-minimal with respect to having a doubly u-redundant member. We will show that $\mathcal{C}$ is in fact a delta. To this end, let C be a doubly u-redundant member, that is, $u \notin C$ and for two distinct members C_1, C_2 containing u, we have that $C_1 - \{u\} \subsetneq C$ and $C_2 - \{u\} \subsetneq C$.

Claim 1. $C_1 \cap C_2 = \{u\}$.

Proof of Claim. The follows from the minimality assumption, because for $I := (C_1 \cap C_2) - \{u\}$, the minor $\mathcal{C}/I$ has $C - I$ as a doubly u-redundant member with causes $C_1 - I$ and $C_2 - I$. $\diamond$

Claim 2. $\{u\} \cup C = V$.

Proof of Claim. This also follows from the minimality assumption, because for $J := V - (\{u\} \cup C)$, $\mathcal{C} \setminus J$ has C as a doubly u-redundant member with the same causes. $\diamond$

Claim 3. $|C_1| = |C_2| = 2$.

Proof of Claim. Suppose for a contradiction that one of C_1, C_2, say C_1, has cardinality at least 3. Pick an element $w \in C_1 - \{u\}$, and note that by Claim 1, $w \notin C_2$. Consider the minor $\mathcal{C}' := \mathcal{C}/w$, which has $C_1' := C_1 - \{w\}$ and $C' := C - \{w\}$ as members. Notice that C_2 contains a member C_2' of $\mathcal{C}'$, for which it is easy to see that $u \in C_2'$ and $C_2' \neq \{u\}$. But now, $\mathcal{C}'$ has C' as a doubly u-redundant member with causes C_1', C_2', a contradiction to our minimality assumption. $\diamond$

Let $X := \big\{v \in V : \{u, v\} \text{ is a member}\big\}$. By Claim 3, $|X| \geq 2$, and by Claim 2, $X \subseteq C$.

Claim 4. *$X = C$.*

Proof of Claim. For if not, pick an element $w \in C - X$, and note that $C - \{w\}$ is doubly u-redundant for $\mathcal{C}/w$ with causes C_1, C_2, contradicting the minimality assumption. $\diamond$

As a result, $\mathcal{C}$ has the following members:

$$\{\{u, v\} : v \in C\} \cup \{C\}.$$

Since $\mathcal{C}$ is a clutter, it cannot have any other members, implying in turn that $\mathcal{C}$ is a delta, as required. It can be readily checked that our constructive proof leads to an algorithm that runs in time $O\big(|V||\mathcal{C}|^2\big)$. $\square$

Take distinct elements f, g of $\mathcal{C}$. Recall that (f, g) is an exclusive pair if every member contains at most one of f, g. As a consequence of Theorem 1.3, we get the following:

Corollary 2.4 ([4])**.** *Let $\mathcal{C}$ be a clutter without a delta minor, and take distinct elements u, v, w. If $\{u, v\}, \{u, w\}$ are members, then (v, w) is an exclusive pair.*

Proof. Assume that $\{u, v\}, \{u, w\}$ are members. If there is a member C containing v and w, then C is doubly u-redundant with causes $\{u, w\}$ and $\{u, w\}$, so by Theorem 1.3, $\mathcal{C}$ has a delta minor, which is not the case. Thus, every member contains at most one of v and w, i.e. (v, w) is exclusive. $\square$

Observe that an exclusive pair of elements remains exclusive in every minor where the elements are present. This observation, together with Corollary 2.4, leads to the following:

Theorem 1.4 ([4])**.** *Let $\mathcal{C}$ be a clutter over ground set V. Then in time $O(|V|^4|\mathcal{C}|^5)$, one can find a delta minor or certify that none exists.*

Proof. We claim that the following statements are equivalent:

(i) $\mathcal{C}$ does not have a delta minor,

(ii) for all distinct members C_1, C_2 with $C_1 \cap C_2 \neq \emptyset$ and for all elements u, v, w with $u \in C_1 \cap C_2, v \in C_1 - C_2, w \in C_2 - C_1$, the following holds: for $X := (C_1 \cup C_2) - \{u, v, w\}$ and $\mathcal{C}' := \mathcal{C}/X$, either $\{u, v\} \notin \mathcal{C}'$ or $\{u, w\} \notin \mathcal{C}'$ or (v, w) is an exclusive pair in $\mathcal{C}'$.

(ii) ⇒ (i): Assume that (i) does not hold. Suppose that $\mathcal{C}$ has a delta minor, obtained after deleting $I \subseteq V$ and contracting $J \subseteq V$. Pick elements $u, v, w \in V - (I \cup J)$ such that $\{u, w\}, \{u, w\}$ are members of the delta minor. Notice that (v, w) is not exclusive in the delta minor, and are therefore not exclusive in $\mathcal{C}$. Let C_1, C_2 be members of $\mathcal{C}$ such that $\{u, v\} \subseteq C_1 \subseteq \{u, v\} \cup J$ and $\{u, w\} \subseteq C_2 \subseteq \{u, w\} \cup J$. It can be readily checked that C_1, C_2 and u, v, w do not satisfy (ii). Thus, (ii) does not hold. **(i) ⇒ (ii)**: Assume that (i) holds. Take $C_1, C_2, u, v, w, X, \mathcal{C}'$ as in (ii) where $\{u, v\} \in \mathcal{C}'$ and $\{u, w\} \in \mathcal{C}'$. Since $\mathcal{C}$ has no delta minor, neither does $\mathcal{C}'$, so by Corollary 2.4, (v, w) is exclusive in $\mathcal{C}'$, so (ii) holds. Hence, (i) and (ii) are equivalent. Since (ii) may be verified in time $O\big(|V|^3|\mathcal{C}|^2 \cdot |V||\mathcal{C}|^2 \cdot |\mathcal{C}|\big)$, and if (ii) does not hold, a delta minor can be found in time $O\big(|V||\mathcal{C}|^2\big)$ using Theorem 1.3, we can find a delta minor or certify that none exists in time $O(|V|^4|\mathcal{C}|^5)$. □

Theorem 1.3 has another consequence, namely a second tool for finding delta minors:

Theorem 2.5. *Take a clutter $\mathcal{C}$ and distinct elements u, v such that $\{u, v\}$ is both a member and a minimal cover. Then the following statements are equivalent:*

(i) there are members C_u, C_v where $C_u \cap \{u, v\} = \{u\}$, $C_v \cap \{u, v\} = \{v\}$ and $C_u \cap C_v \neq \emptyset$,

(ii) there exist a member C_u and a minimal cover B_u where $C_u \cap \{u, v\} = B_u \cap \{u, v\} = \{u\}$ and $C_u \cap B_u \neq \{u\}$,

(iii) there is a delta minor using both u and v.

Proof. **(i) ⇒ (ii)**: Suppose that (i) holds. Pick an element $w \in C_u \cap C_v$, and choose a minimal cover B such that $B \cap C_v = \{w\}$. In particular, $v \notin B$ and since $B \cap \{u, v\} \neq \emptyset$, it follows that $u \in B$. Since $C_u \cap B \supseteq \{u, w\}$, (ii) holds for $B_u := B$.

(ii) ⇒ (iii): Suppose next that (ii) holds. Notice that there is no delta minor using exactly one of u, v; for if one is deleted (resp. contracted), the other turns into a minimal cover (resp. member) of cardinality one. It therefore suffices to show the existence of a delta minor using at least one of u, v. We proceed recursively as follows: we either find a delta minor using one of u, v, or we find a proper minor where $\{u, v\}$ is both a member and a minimal cover and (ii) is satisfied – this process will eventually terminate with a delta minor. We may assume that $C_u = B_u$, because for $I := C_u - B_u$ and $J := B_u - C_u$, the minor $\mathcal{C}/I \setminus J$ still has $\{u, v\}$ as a member and a minimal cover and satisfies (ii). Let C_v be a member such that $C_v \cap \{u, v\} = \{v\}$. Note that $C_v \cap C_u = C_v \cap B_u \neq \emptyset$. If $C_v - \{v\} \subseteq C_u$, then C_u is doubly v-redundant with causes $\{u, v\}, C_v$, so $\mathcal{C}$ has a delta minor using v by Theorem 1.3. We may therefore assume that $X := (C_v - \{v\}) - C_u \neq \emptyset$. If there is a member C contained in $\{u\} \cup X$, then C_v is doubly u-redundant with causes $\{u, v\}, C$, so $\mathcal{C}$ has a delta minor using u by Theorem 1.3. Thus, we may assume that no member is contained in $\{u\} \cup X$. Subsequently, $\{u, v\}$ is a member and a minimal cover for the minor $\mathcal{C}' := \mathcal{C}/X$. Let C'_u be a member of $\mathcal{C}'$ contained in C_u; as $C'_u \cap \{u, v\} \neq \emptyset$ we have $u \in C'_u$. Since B_u is a minimal cover of $\mathcal{C}'$ and $C'_u \cap B_u \neq \{u\}$, $\mathcal{C}'$ satisfies (ii), so we can recurse. This shows that (iii) holds.

(iii) ⇒ (i): Suppose finally that (iii) holds, that is, there are disjoint element subsets I, J such that $(I \cup J) \cap \{u, v\} = \emptyset$ and $\mathcal{C}/I \setminus J$ is a delta. Clearly, $\{u, v\}$ is both a member and a minimal cover of this minor, so there are two other members C'_u, C'_v of this minor such that $C'_u \cap \{u, v\} = \{u\}$, $C'_v \cap \{u, v\} = \{v\}$, and since this minor is a delta, we have that $C'_u \cap C'_v \neq \emptyset$. Now let C_u, C_v be members of $\mathcal{C}$ such that $C'_u \subseteq C_u \subseteq C'_u \cup I$ and $C'_v \subseteq C_v \subseteq C'_v \cup I$. Then $C_u \cap \{u, v\} = \{u\}, C_v \cap \{u, v\} = \{v\}$ and $C_u \cap C_v \supseteq C'_u \cap C'_v \neq \emptyset$, so (i) holds. □

As a consequence,

Corollary 2.6. *Let $\mathcal{C}$ be a clutter that has members of the form $\{u,v\}, C_u, C_v$ such that $C_u \cap \{u,v\} = \{u\}, C_v \cap \{u,v\} = \{v\}$. Then at least one of the following statements holds:*

(i) $C_u \cap C_v = \emptyset$,

(ii) $(C_u \cup C_v) - \{u,v\}$ *contains a member, or*

(iii) $\mathcal{C}$ *has a delta minor using both u and v.*

Proof. Suppose that neither (i) nor (ii) hold. Since (ii) does not hold, the complement of $(C_u \cup C_v) - \{u,v\}$ is a cover so it contains a minimal cover B. Since B intersects both C_u and C_v, it follows that $\{u,v\} \subseteq B$. Let $\mathcal{C}' := \mathcal{C} \setminus (B - \{u,v\})$. Then $\{u,v\}$ is both a member and a minimal cover of $\mathcal{C}'$, and C_u, C_v are still members of $\mathcal{C}'$. As (i) does not hold, $C_u \cap C_v \neq \emptyset$, so by Theorem 2.5, $\mathcal{C}'$ has a delta minor using both u, v, implying in turn that (iii) holds. □

2.2 Extended odd hole minors

Given a simple graph, we treat each edge as a vertex subset of cardinality two, and in this chapter only, we treat each circuit as a vertex subset.

Remark 2.7. *Let V be a finite set of cardinality at least 3, and let $\mathcal{C}$ be a clutter over ground set V where $\min\{|C| : C \in \mathcal{C}\} = 2$ and the minimum cardinality members correspond to edges of a graph $G = (V, E)$. If G is not bipartite, then $\mathcal{C}$ has a Δ_3 or an extended odd hole minor.*

Proof. Suppose that G is not bipartite. Let $C \subseteq V$ be an odd circuit of G with the smallest number of vertices. Our minimality assumption implies that C has no chords. If $|C| = 3$, then $\mathcal{C} \setminus (V - C) \cong \Delta_3$. Otherwise, $|C| \geq 5$ and $\mathcal{C} \setminus (V - C)$ is an extended odd hole, as required. □

Let $G = (V, E)$ be a connected graph. A *block* is a maximal vertex-induced subgraph of G that is 2-(vertex)-connected. (To clarify, a block has at least two vertices and may form an edge.) The *block graph of G* is the graph X constructed as follows:

- X is a bipartite graph,
- the cut-vertices of G form one color class of X,
- the blocks of G form the other color class of X, and
- cut-vertex u and block B are adjacent in X if $u \in V(B)$.

It is well-known that X is a tree (see [26], Lemma 3.1.4). Observe that every cut-vertex of G has degree at least 2 in X, so the leaves of X correspond to blocks of G, and we will refer to those blocks as *leaf blocks*. For each block of G, we will refer to its cut-vertices as *boundary* vertices and to its other vertices as *interior* vertices. We are now ready for the main result of this section:

Lemma 1.5 ([9]). *Let V be a finite set of cardinality at least 4, and let $\mathcal{C}$ be a clutter over ground set V where $\min\{|C| : C \in \mathcal{C}\} = 2$ and the minimum cardinality members correspond to the edges of a connected bipartite graph G over vertex set V whose color classes are R, B. If R contains a member, then $\mathcal{C}$ has either a delta or an extended odd hole minor.*

Proof. Let us proceed by induction on $|V| \geq 4$. For $|V| = 4$, it can be readily checked that $\mathcal{C} \cong \Delta_4$, so we are done. For the induction step, assume that $|V| \geq 5$. Let $C \subseteq V$ be a member of $\mathcal{C}$ contained in the color class R. Notice that $|C| \geq 3$. Let us refer to the vertices of G in C as *terminals*, and to the other vertices as *non-terminals*. Observe that the terminals form a stable set of G.

Claim 1. *Let u be a non-terminal vertex of G. If the terminals of G belong to the same component of $G \setminus u$, then $\mathcal{C} \setminus u$ has a delta or an extended odd hole minor.*

Proof of Claim. Let W be the set of vertices of $G \setminus u$ that do not belong to the component containing the terminals. Consider the clutter $\mathcal{C} \setminus u \setminus W$; its minimum cardinality members correspond to the edges of the connected bipartite graph $G \setminus u \setminus W$. As $G \setminus u \setminus W$ has at least 4 vertices, and the member $C \in \mathcal{C} \setminus u \setminus W$ is still a subset of one of the color classes of $G \setminus u \setminus W$, the induction hypothesis implies that $\mathcal{C} \setminus u \setminus W$ has a delta or an extended odd hole minor, as claimed. $\diamond$

We may therefore assume that every non-terminal vertex of G is a cut-vertex separating the terminals. In particular, G is not 2-connected. Let us then consider the blocks of G. Since every non-terminal is a cut-vertex, the interior vertices of each block are terminals. Let X be the block graph of G; recall that X is a tree.

Claim 2. *Every leaf block of G consists of precisely two vertices and an edge between them, where one vertex belongs to the boundary and is a non-terminal, and the other vertex belongs to the interior, is a terminal and has degree one in G.*

Proof of Claim. By definition, the leaf blocks are precisely those blocks with exactly one boundary vertex. Consider a leaf block B, viewed as a vertex-induced subgraph of G. Then B has a unique boundary vertex, and as each block has at least two vertices, B has at least one interior vertex. As the interior vertices are terminals and therefore form a stable set, B must have exactly one interior vertex and therefore exactly one edge, implying in turn that its boundary vertex is a non-terminal vertex. This also implies that the interior vertex has degree one in G. $\diamond$

Claim 3. *If a non-terminal vertex is adjacent to at least two terminals, then $\mathcal{C}$ has a delta minor.*

Proof of Claim. Suppose that a non-terminal vertex u is adjacent to two terminals v and w. Then both $\{u, v\}$ and $\{u, w\}$ are members of $\mathcal{C}$, and $\{v, w\} \subseteq C$. Since u is not a terminal, u is not contained in C. Thus, C is a doubly u-redundant member of $\mathcal{C}$ with causes $\{u, v\}, \{u, w\}$, so by Theorem 1.3, $\mathcal{C}$ has a delta minor. $\diamond$

We may therefore assume that every non-terminal vertex is adjacent to at most one terminal. In particular, by Claim 2,

Claim 4. *Every non-terminal vertex belongs to at most one leaf block. That is, X does not have leaf edges that share a vertex.*

Using this, we prove the following:

Claim 5. *There exists a leaf block with boundary vertex u such that $G \setminus u$ has exactly two components. That is, X has a vertex u of degree two that is incident with a leaf edge.*

Proof of Claim. Pick a leaf vertex B of X, and among all the other leaf vertices, pick one B' that is farthest from B. Let u be the vertex of X adjacent to B'. Clearly u has degree at least two, as it lies the path joining B, B'. Notice that by Claim 4, the edge uB' is the unique leaf edge incident with u. This fact, together with our maximal choice of B', implies that u has degree exactly two, as required. $\diamond$

Let B be the leaf block from Claim 5, let u be the boundary vertex, and let v be the interior vertex. Then $G \setminus u \setminus v$ is a connected bipartite graph. Recall from Claim 2 that u is a non-terminal, v is a terminal, and that v has degree one in G. Since there are at least two terminals in $G \setminus u \setminus v$, and the terminals form a stable set, it follows that $G \setminus u \setminus v$ has at least 3 vertices.

Claim 6. *$\mathcal{C} \setminus u/v$ has a delta or extended odd hole minor.*

Proof of Claim. Let $\mathcal{C}' := \mathcal{C} \setminus u/v$. Observe that $C' := C - \{v\} \in \mathcal{C}'$. Since v has degree one in G, and u is its neighbor, it follows that $\min\{|D| : D \in \mathcal{C}'\} \geq 2$. Therefore, each edge of $G \setminus u \setminus v$ corresponds to a minimum cardinality member of $\mathcal{C}'$ (and not necessarily vice versa). In particular,

$$\min\{|D| : D \in \mathcal{C}'\} = 2.$$

If $|V - \{u, v\}| = 3$, then $\mathcal{C} \setminus u/v \cong \Delta_3$, so we are done. We may therefore assume that $|V - \{u, v\}| \geq 4$. Let G' be the graph over vertex set $V - \{u, v\}$ corresponding to the minimum cardinality members of $\mathcal{C}'$. Notice that $E(G \setminus u \setminus v) \subseteq E(G')$. In particular, G' is connected. If G' is not bipartite, then by Remark 2.7, $\mathcal{C}'$ has a Δ_3 or an extended odd hole minor, so we are done. Otherwise, G' is bipartite. Since C' is contained in one of the color classes of $G \setminus u \setminus v$, it must also be contained in one of the color classes of G'. It therefore follows from the induction hypothesis that $\mathcal{C}'$ has a delta or an extended odd hole minor, as required. $\diamond$

This claim completes the induction step, thereby finishing the proof. $\square$

2.3 Blockers of extended odd hole minors

We have the following consequence of Lemma 1.5:

Theorem 2.8 ([9])**.** *Let V be a finite set of cardinality at least 3, and let $\mathcal{C}$ be a clutter over ground set V such that $\tau(\mathcal{C}) \geq 2$. If every member has cardinality at least $\frac{|V|+1}{2}$, then $\mathcal{C}$ has either a delta or the blocker of an extended odd hole minor.*

Proof. Observe that $|V| \geq 3$. We will proceed by induction on $|V|$. For the base case $|V| = 3$, observe that $\mathcal{C} \cong \Delta_3$, so we are done. For the induction step, assume that $|V| \geq 4$.

Claim 1. *If an element v does not appear in a cover of cardinality two, then $\mathcal{C} \setminus v$ has a delta or the blocker of an extended odd hole minor.*

Proof of Claim. Assume that v does not appear in a cover of cardinality two. Then $\tau(\mathcal{C} \setminus v) \geq 2$. As every member of $\mathcal{C} \setminus v$ has cardinality at least $\frac{|V|+1}{2} \geq \frac{|V-\{v\}|+1}{2}$, the induction hypothesis implies that $\mathcal{C} \setminus v$ has a delta or the blocker of an extended odd hole minor, as required. ◇

We may therefore assume that $\tau(\mathcal{C}) = 2$ and that every element appears in a minimum cover. Let $G = (V, E)$ be the graph whose edges correspond to the minimum covers of $\mathcal{C}$.

Claim 2. *If G is not bipartite, then $\mathcal{C}$ has a Δ_3 or the blocker of an extended odd hole minor.*

Proof of Claim. It follows from Remark 2.7 that $b(\mathcal{C})$ has a Δ_3 or an extended odd hole minor, as required. ◇

We may therefore assume that G is a bipartite graph. Take a component H of G where $V(H)$ is partitioned into color classes U_1, U_2 such that $|U_2| \geq |U_1|$.

Claim 3. *If $\tau(\mathcal{C} \setminus U_2/U_1) \geq 2$, then $\mathcal{C} \setminus U_2/U_1$ has a delta or the blocker of an extended odd hole minor.*

Proof of Claim. Suppose that $\tau(\mathcal{C} \setminus U_2/U_1) \geq 2$. Observe that every member of $\mathcal{C} \setminus U_2/U_1$ has at least

$$\frac{|V|+1}{2} - |U_1| \geq \frac{1}{2}(|V| - |U_1| - |U_2| + 1)$$

elements. It therefore follows from the induction hypothesis that $\mathcal{C} \setminus U_2/U_1$ has a delta or the blocker of an extended odd hole minor, as required. ◇

We may therefore assume that $\tau(\mathcal{C} \setminus U_2/U_1) \leq 1$.

- If $H = G$, then $U_1 \cup U_2 = V$. Then $|U_2| \geq \frac{|V|}{2}$ because $|U_2| \geq |U_1|$. Since every member has cardinality at least $\frac{|V|+1}{2}$, U_2 is a cover of $\mathcal{C}$.
- Otherwise, H is a proper component of G, so $b(\mathcal{C}) \setminus U_1/U_2$ has a member of cardinality at most 1.

Either way, there is a member B of $b(\mathcal{C})$ such that $B \cap U_1 = \emptyset$ and $|B - U_2| \leq 1$. Observe that as H is a component of G, the graph G has no edge with exactly one end in $U_1 \cup U_2$. In particular, $|B| \geq 3$, $|B \cap U_2| \geq 2$ and $|U_1 \cup U_2| \geq 3$.

Claim 4. *$b(\mathcal{C})$ has a delta or an extended odd hole minor.*

Proof of Claim. Let $b(\mathcal{C}')$ be the minor obtained from $b(\mathcal{C})$ after deleting $V - (U_1 \cup U_2 \cup B)$ and contracting $B - U_2$; so $b(\mathcal{C}')$ has ground set $U_1 \cup U_2$. Notice that $B' := B \cap U_2$ is a member of $b(\mathcal{C}')$. Since G has no edge with exactly one end in $U_1 \cup U_2$, each edge of H is a minimum cardinality member of $b(\mathcal{C}')$. Let G' be the graph over vertex set $U_1 \cup U_2$ whose edges are the minimum cardinality members of $b(\mathcal{C}')$. As $E(H) \subseteq E(G')$, G' is a connected graph. If G' is not bipartite, it then follows from Remark 2.7 that $b(\mathcal{C}')$ has a Δ_3 or an extended odd hole minor, so we are done. Otherwise, G' is bipartite. Observe that the color classes of G' are inevitably U_1 and U_2, so $|B'| \geq 3$ and $|U_1 \cup U_2| \geq 4$. Since $B' \subseteq U_2$, it follows from Lemma 1.5 that $b(\mathcal{C}')$ has a delta or an extended odd hole minor, as claimed. ◇

Thus, $\mathcal{C}$ has a delta or the blocker of an extended odd hole minor, thereby completing the induction step. $\square$

Let $\mathcal{C}$ be a clutter over ground set V, and take distinct elements $u, v \in V$. We say that u, v are *duplicates* if every member contains u if and only if it contains v. We say that u, v are *replicates* if no member contains both u and v, and for each $C \subseteq V - \{u, v\}$, $C \cup \{u\}$ is a member if and only if $C \cup \{v\}$ is a member. We leave the following as an easy exercise for the reader:

Remark 2.9 (see [9])**.** *Take a clutter $\mathcal{C}$ and distinct elements u, v. Then u, v are duplicates in $\mathcal{C}$ if, and only if, u, v are replicates in $b(\mathcal{C})$.*

Using this remark, we prove the following:

Remark 2.10 ([9])**.** *The deltas and the blockers of extended odd holes do not have duplicates.*

Proof. It follows from the definition that the deltas do not have duplicates. To prove that the blockers of extended odd holes do not have duplicates, it suffices by Remark 2.9 to show that extended odd holes do not have replicates. Pick an odd integer $n \geq 5$ and let $\mathcal{C}$ be an extended odd hole over ground set $[n]$ whose minimum cardinality members are $\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}, \{n, 1\}$. Suppose for a contradiction that $\mathcal{C}$ has replicates $i, j \in [n]$. After a possible relabeling, we may assume that $i = 1$. As $1, j$ do not appear in a member together, $j \in \{3, 4, \ldots, n-1\}$. However, as $1, j$ are replicates and $\{1,2\}, \{1,n\}$ are members, it follows that $\{j, 2\}, \{j, n\}$ are members also, a contradiction as $\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}, \{n, 1\}$ are the only members of cardinality two. Thus, extended odd holes do not have replicates, as required. $\square$

Let $\mathcal{C}$ be a clutter over ground set V, take an element v, and let $\bar{v}$ be a new element. To *duplicate* v is to replace $\mathcal{C}$ by the clutter over ground set $V \cup \{\bar{v}\}$ whose members are

$$\{C : v \notin C \in \mathcal{C}\} \cup \{C \cup \{\bar{v}\} : v \in C \in \mathcal{C}\}.$$

To *replicate* v is to replace $\mathcal{C}$ by the clutter over ground set $V \cup \{\bar{v}\}$ whose members are[1]

$$\{C : v \notin C \in \mathcal{C}\} \cup \{C \triangle \{v, \bar{v}\} : v \in C \in \mathcal{C}\}.$$

A *duplication of* $\mathcal{C}$ is any clutter obtained after duplicating a series of elements, while a *replication of* $\mathcal{C}$ is any clutter obtained after replicating a series of elements.

We are now ready to prove the following theorem:

Theorem 1.6 ([9])**.** *Let $\mathcal{C}$ be a clutter over ground set V such that $\tau(\mathcal{C}) \geq 2$. If there is a $w \in \mathbb{R}^V_+$ such that $w(C) > \frac{\mathbf{1}^\top w}{2}$ for every member C, then $\mathcal{C}$ has either a delta or the blocker of an extended odd hole minor.*

[1]Given sets A and B, $A \triangle B$ denotes the symmetric difference $(A \cup B) - (A \cap B)$.

Proof. After tweaking the widths, if necessary, we may assume that w has rational entries. Pick an appropriate integer $N \geq 1$ such that Nw_v is an integer for each $v \in V$. Let $\mathcal{C}'$ be the clutter over ground set V' obtained from $\mathcal{C}$ as follows: for each $v \in V$,

- if $Nw_v = 0$ then contract v,
- otherwise, duplicate $Nw_v - 1$ times the element v.

Observe that $\tau(\mathcal{C}') \geq 2$, and as $\min\{N \cdot w(C) : C \in \mathcal{C}\} > \frac{N(\mathbf{1}^\top w)}{2}$, every member of $\mathcal{C}'$ has cardinality more than $\frac{|V'|}{2}$. It therefore follows from Theorem 2.8 that $\mathcal{C}'$ has a delta or the blocker of an extended odd hole minor. Since the deltas and the blockers of extended odd holes do not have duplicates by Remark 2.10, this minor gives rise to a delta or the blocker of an extended odd hole minor in $\mathcal{C}$, as required. $\square$

Let us wrap up this section with the following conjecture:

Conjecture 2.11. *There exists a polynomial $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ such that the following statement holds:*

> *Let $\mathcal{C}$ be a clutter over ground set V such that $\tau(\mathcal{C}) \geq 2$. If there is a $w \in \mathbb{R}^V_+$ such that $w(C) > \frac{\mathbf{1}^\top w}{2}$ for every member C, then in time $f(|V|, |\mathcal{C}|)$, one can find a delta or the blocker of an extended odd hole minor.*

2.4 Further notes

Recall Theorem 1.1, stating that testing clutter idealness is a co-NP-complete problem. The culprit is the following hardness result:

Theorem 2.12 (Ding et al. [27], follows from Theorems 2.4 and 3.1)**.** *Let $\mathcal{C}$ be a clutter over ground set V where every element is used in exactly two members. Then the problem "Does $\mathcal{C}$ have an odd hole minor?" is NP-complete.*

This result implies in particular that, given a generic clutter, the problem "Does $\mathcal{C}$ have a delta or an extended odd hole minor?" is NP-complete. We believe however that Theorem 1.6 suggests the following:

Conjecture 2.13. *There exists a finite family $\mathcal{F}$ of non-ideal clutters and a polynomial $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ such that the following statement holds:*

> *Let $\mathcal{C}$ be a clutter over ground set V without a minor in $\mathcal{F}$. Then in time $f(|V|, |\mathcal{C}|)$, one can find a delta or the blocker of an extended odd hole minor, or certify that none exists.*

A related conjecture is the following:

Conjecture 2.14. *There exists an algorithm that given a clutter $\mathcal{C}$ over ground set V finds in time polynomial in $|V|, |\mathcal{C}|$ a minor $\mathcal{C}'$ such that*

$$\tau(\mathcal{C}') \geq 2 \quad \text{and} \quad \nu(\mathcal{C}') = 1,$$

or certify that none exists.

Chapter 3

Blockers

Take an integer $n \geq 1$ and a nonnegative matrix A with n columns and without a row of all zeros. Consider the polyhedron

$$P := \left\{x \in \mathbb{R}^n_+ : Ax \geq \mathbf{1}\right\}.$$

Recall that the blocking polyhedron of P is

$$Q := \{u \in \mathbb{R}^n_+ : u^\top x \geq 1 \ \forall x \in P\}.$$

Fulkerson showed that there exists a nonnegative matrix B with n columns and without a row of all zeros such that

$$Q = \left\{u \in \mathbb{R}^n_+ : Bu \geq \mathbf{1}\right\}$$

and that the blocking polyhedron of Q is P [38, 37].

Ideal clutters, as discussed in the prologue, give rise to pairs of blocking polyhedra. To elaborate, let $\mathcal{C}$ be an ideal clutter over ground set $[n]$. Recall from Theorem 1.7 that $b(\mathcal{C})$ is also an ideal clutter. Then the sets

$$Q(\mathcal{C}) = \left\{x \in \mathbb{R}^n_+ : M(\mathcal{C})x \geq \mathbf{1}\right\} \quad \text{and} \quad Q(b(\mathcal{C})) = \left\{u \in \mathbb{R}^n_+ : M(b(\mathcal{C}))u \geq \mathbf{1}\right\}$$

are blocking polyhedra [38, 37].

Let R be an $n \times n$ positive definite matrix. In 1987, Seth Chaiken studied the following pair of convex quadratic programs

$$(P) \quad \begin{array}{ll} \min & x^\top R x \\ \text{s.t.} & Ax \geq \mathbf{1} \\ & x \geq \mathbf{0} \end{array} \qquad\qquad (P') \quad \begin{array}{ll} \min & u^\top R^{-1} u \\ \text{s.t.} & Bu \geq \mathbf{1} \\ & u \geq \mathbf{0} \end{array}$$

and noticed a duality phenomenon between the two [17]. We will study these convex quadratic programs and the duality between the two in this chapter.

In §3.1, we study the program (P) and its (Lagrangian) dual, and show how (P) can be used to find disjoint members in a clutter. In §3.2, we reprove Chaiken's result in regards to the relationship between (P) and (P'), and use it to deduce a basic fact about blocking pairs of ideal clutters.[1] In §3.3, we study

[1] Chaiken's result is reproved because we find the proof, which is the same as the original one, interesting.

identically self-blocking clutters. In §3.4, we will briefly discuss *gauge duality*, a general framework founded by Robert Freund [36] the same year, which encompasses the duality phenomenon observed by Chaiken.

3.1 Convex quadratic programming over set covering polyhedra

Chaiken's result relies on solving a convex quadratic program over the set covering polyhedron; let us first apply strong duality to this program:

Proposition 3.1 (see Chaiken [17])**.** *Take an integer $n \geq 1$, let A be a nonnegative matrix with n columns and without a row of all zeros, and let R be a positive definite $n \times n$ matrix. Consider the dual pair of quadratic programs:*

$$(P)\quad \begin{array}{ll} \min & x^\top R x \\ \text{s.t.} & Ax \geq \mathbf{1} \\ & x \geq \mathbf{0} \end{array} \qquad\qquad (D)\quad \begin{array}{ll} \max & \mathbf{1}^\top y - x^\top R x \\ \text{s.t.} & A^\top y \leq 2Rx \\ & y \geq \mathbf{0}. \end{array}$$

Then there exist $x^\star, y^\star$ such that

- *(i) $x^\star$ is the unique optimal solution of (P),*
- *(ii) $(y^\star, x^\star)$ is an optimal solution of (D),*
- *(iii) given that α is the optimal value of (P) and (D), we have that ${y^\star}^\top A x^\star = \mathbf{1}^\top y^\star = 2\alpha$.*

Proof. Notice that (P) is a convex quadratic program with a strictly convex objective function that is feasible and bounded from below by 0. As a result, (P) has a unique optimal solution $x^\star$, so (i) holds. Since strong duality and the complementary slackness conditions (i.e. the Karush-Kuhn-Tucker conditions) hold for convex quadratic programs, an optimal solution $(y^\star, z^\star)$ for (D) satisfies the following equations:

$$\begin{aligned} {x^\star}^\top R x^\star &= \mathbf{1}^\top y^\star - {z^\star}^\top R z^\star \\ \langle y^\star, Ax^\star - \mathbf{1} \rangle &= 0 \\ \langle x^\star, 2Rz^\star - A^\top y^\star \rangle &= 0. \end{aligned}$$

Thus, to prove (ii) and (iii), it suffices to show that $z^\star = x^\star$. The equations above imply the following:

$${x^\star}^\top R x^\star + {z^\star}^\top R z^\star = \mathbf{1}^\top y^\star = {y^\star}^\top A x^\star = 2{z^\star}^\top R x^\star.$$

Since R is positive definite, there is a nonsingular $n \times n$ matrix D such that $R = D^\top D$. Combining this with the equation above yields the following:

$$0 = {x^\star}^\top R x^\star + {z^\star}^\top R z^\star - 2{z^\star}^\top R x^\star = \langle Dx^\star - Dz^\star, Dx^\star - Dz^\star \rangle.$$

Thus, $Dx^\star - Dz^\star = \mathbf{0}$, implying in turn that $x^\star = z^\star$, as required. □

We prove the following as a consequence:

Theorem 3.2 (Abdi, Cornuéjols, Lee)**.** *Let $\mathcal{C}$ be a clutter, and let α be the optimal value of*

$$(P) \quad \begin{array}{lll} \min & x^\top x & \\ s.t. & x(C) \geq 1 & \forall C \in \mathcal{C} \\ & x \geq \mathbf{0}. & \end{array}$$

Then $\mathcal{C}$ has at least α disjoint members. Moreover, if every member has cardinality at least two, then $\mathcal{C}$ has more than α disjoint members.

Proof. Let $A := M(\mathcal{C})$, the incidence matrix of $\mathcal{C}$. Then (P) and its dual can be written as:

$$(P) \quad \begin{array}{ll} \min & x^\top x \\ \text{s.t.} & Ax \geq \mathbf{1} \\ & x \geq \mathbf{0} \end{array} \qquad (D) \quad \begin{array}{ll} \max & \mathbf{1}^\top y - x^\top x \\ \text{s.t.} & A^\top y \leq 2x \\ & y \geq \mathbf{0}. \end{array}$$

By Proposition 3.1, there exist $x^\star, 2y^\star$ such that $x^\star$ is the unique optimal solution to (P), $(2y^\star, x^\star)$ is an optimal solution to (D) and

$$\mathbf{1}^\top y^\star = \alpha.$$

After dropping rows of A, if necessary, we may assume that $y^\star > \mathbf{0}$. It therefore follows from the complementary slackness conditions that $Ax^\star = \mathbf{1}$. Combining this equation with the inequality $A^\top y^\star \leq x^\star$ gives us

$$AA^\top y^\star \leq \mathbf{1}.$$

Write $\mathcal{C} = \{C_1, \ldots, C_m\}$, and for ease of notation, let us reindex $y^\star \in \mathbb{R}^{\mathcal{C}}_+$ as $y^\star \in \mathbb{R}^m_+$. Let $\nu := \nu(\mathcal{C})$, the maximum number of pairwise disjoint members of $\mathcal{C}$. After a possible relabeling, we may assume that $C_1, \ldots, C_\nu$ are pairwise disjoint members. The matrix inequality $AA^\top y^\star \leq \mathbf{1}$ for the first ν rows gives:

$$\sum_{i=1}^{m} |C_i \cap C_j| \cdot y_i^\star \leq 1 \quad \forall j \in [\nu].$$

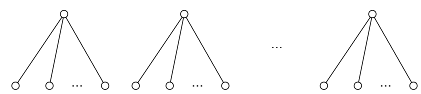
Adding these up yields:

$$\begin{aligned}
\nu &\geq \sum_{j=1}^{\nu} \sum_{i=1}^{m} |C_i \cap C_j| \cdot y_i^\star \\
&= \sum_{j=1}^{\nu} \sum_{i=1}^{\nu} |C_i \cap C_j| \cdot y_i^\star + \sum_{j=1}^{\nu} \sum_{i=\nu+1}^{m} |C_i \cap C_j| \cdot y_i^\star \\
&= \sum_{i=1}^{\nu} |C_i| \cdot y_i^\star + \sum_{i=\nu+1}^{m} \left(\sum_{j=1}^{\nu} |C_i \cap C_j| \right) y_i^\star \\
&\geq \sum_{i=1}^{\nu} |C_i| \cdot y_i^\star + \sum_{i=\nu+1}^{m} y_i^\star \\
&\geq \sum_{i=1}^{m} y_i^\star + (|C_1| - 1) y_1^\star
\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{1}^\top y^\star + (|C_1| - 1)y_1^\star \\
&= \alpha + (|C_1| - 1)y_1^\star
\end{aligned}
$$

where the second inequality follows from the maximal choice of $C_1, \dots, C_\nu$. As a result, $\nu \geq \alpha$, and if $|C_1| \geq 2$ then $\nu > \alpha$, as required. $\square$

Let us illustrate this via an example. Take integers $\nu, k \geq 1$, and let G be a graph that is the vertex-disjoint union of ν stars, each of which has $k+1$ vertices:



Let $\mathcal{C}$ be the clutter over ground set $V(G)$ of the edges of G. Then the optimal solution of (P) assigns $\frac{k}{k+1}$ to the center of each star and $\frac{1}{k+1}$ to all the other vertices. As a result,

$$\alpha = \left(\left(\frac{k}{k+1}\right)^2 + k \cdot \left(\frac{1}{k+1}\right)^2\right) \cdot \nu = \frac{k}{k+1} \cdot \nu,$$

which is at most $\nu = \nu(\mathcal{C})$.

3.2 Chaiken's hidden gem

Having analyzed the involved convex quadratic programs and their duals, we are now ready to state and reprove Chaiken's main result on blocking polyhedra:

Theorem 3.3 (Chaiken [17]). *Take an integer $n \geq 1$, let P, Q be blocking n-dimensional polyhedra, and let R be an $n \times n$ positive definite matrix. Let $x^\star$ be the optimal solution of*

$$(P) \quad \min\left\{x^\top R x : x \in P\right\}$$

with objective value α. Then $\frac{1}{\alpha} R x^\star$ is the optimal solution of

$$(P') \quad \min\left\{u^\top R^{-1} u : u \in Q\right\}$$

with objective value $\frac{1}{\alpha}$.

Proof. Let us first prove that,

Claim 1. *(P') has optimal value at least $\frac{1}{\alpha}$.*

Proof of Claim. Take a point $u \in Q$, a feasible solution to (P'). Its objective value is

$$u^\top R^{-1} u = \frac{1}{\alpha}(x^{\star\top} R x^\star)(u^\top R^{-1} u) = \frac{1}{\alpha}(x^{\star\top} R x^\star)(u^\top R^{-1} R R^{-1} u) \geq \frac{1}{\alpha} x^{\star\top} R R^{-1} u = \frac{1}{\alpha} x^{\star\top} u \geq \frac{1}{\alpha},$$

where the first inequality is the Cauchy-Schwarz inequality and the second inequality follows from the fact that $x^\star, u$ belong to the blocking polyhedra P, Q, respectively. $\diamond$

Let A be a nonnegative matrix with n columns and without a row of all zeros such that $P = \{x \geq \mathbf{0} : Ax \geq \mathbf{1}\}$. Consider (P) and its dual program:

$$
(P) \quad \begin{array}{ll} \min & x^\top Rx \\ \text{s.t.} & Ax \geq \mathbf{1} \\ & x \geq \mathbf{0} \end{array} \qquad\qquad (D) \quad \begin{array}{ll} \max & \mathbf{1}^\top y - x^\top Rx \\ \text{s.t.} & A^\top y \leq 2Rx \\ & y \geq \mathbf{0}. \end{array}
$$

By Proposition 3.1, there is a $y^\star$ such that $(y^\star, x^\star)$ is optimal for (D) and satisfies $\mathbf{1}^\top y^\star = 2\alpha$.

Claim 2. $\frac{1}{\alpha} Rx^\star \in Q$.

Proof of Claim. Pick a point $x \in P$. Then

$$\left\langle x, \frac{1}{\alpha} Rx^\star \right\rangle = \frac{1}{\alpha} x^\top Rx^\star \geq \frac{1}{2\alpha} x^\top A^\top y^\star \geq \frac{1}{2\alpha} \mathbf{1}^\top y^\star = 1$$

where the first inequality follows from $x \geq \mathbf{0}$ and $2Rx^\star \geq A^\top y^\star$, while the second inequality follows from $Ax \geq \mathbf{1}$ and $y^\star \geq \mathbf{0}$. Since the inequality above holds for all $x \in P$, it follows that $\frac{1}{\alpha} Rx^\star \in Q$. $\diamond$

Thus, $\frac{1}{\alpha} Rx^\star$ is feasible for (P'). Its objective value is

$$\left(\frac{1}{\alpha} Rx^\star\right)^\top R^{-1} \left(\frac{1}{\alpha} Rx^\star\right) = \frac{1}{\alpha^2} x^{\star\top} RR^{-1}Rx^\star = \frac{1}{\alpha^2} x^\star Rx^\star = \frac{1}{\alpha}.$$

It therefore follows from Claim 1 that $\frac{1}{\alpha} Rx^\star$ is the optimal solution for (P') with objective value $\frac{1}{\alpha}$. $\square$

This result was proved independently by Lovász in 2001 for the special case when R is a diagonal matrix [51]. As an immediate consequence of Theorem 3.3,

Corollary 3.4. *Let $\mathcal{C}$ be an ideal clutter over ground set $[n]$, and let R be an $n \times n$ positive definite matrix. Let α, β be the optimal values of*

$$\min\left\{x^\top Rx : x \in Q(\mathcal{C})\right\} \quad \textit{and} \quad \min\left\{u^\top R^{-1}u : u \in Q(b(\mathcal{C}))\right\},$$

respectively. Then $\alpha\beta = 1$.

We are now ready to prove the following theorem:

Theorem 1.8 (Abdi, Cornuéjols, Lee)**.** *Take an ideal clutter that does not have a member or cover of cardinality at most one. Then there are either two disjoint members or two disjoint covers.*

Proof. Let $\mathcal{C}$ be an ideal clutter that does not have a member or cover of cardinality at most one. By Theorem 1.7, the blocker $b(\mathcal{C})$ is also ideal. Let α, β be the optimal values of

$$\min\left\{x^\top x : x \in Q(\mathcal{C})\right\} \quad \text{and} \quad \min\left\{u^\top u : u \in Q(b(\mathcal{C}))\right\},$$

respectively. Since the members of $\mathcal{C}, b(\mathcal{C})$ have cardinality at least two, it follows from Theorem 3.2 that $\nu(\mathcal{C}) > \alpha$ and $\nu(b(\mathcal{C})) > \beta$. However, $\alpha\beta = 1$ by Corollary 3.4, so either $\alpha \geq 1$ or $\beta \geq 1$. As a result, either $\nu(\mathcal{C}) \geq 2$ or $\nu(b(\mathcal{C})) \geq 2$, implying in turn that $\mathcal{C}$ has either two disjoint members or two disjoint covers. □

3.3 Identically self-blocking clutters

Recall that a clutter is identically self-blocking if it is equal to its blocker. As noted in the introduction, Theorem 1.8 has the following rather immediate corollary:

Corollary 1.10 (Abdi, Cornuéjols, Lee)**.** *A nontrivial identically self-blocking clutter is non-ideal.*

We conjecture the following strengthening of this result:

Conjecture 1.12. *A nontrivial identically self-blocking clutter has one of* $\{\Delta_n : n \geq 3\} \cup \{\mathbb{L}_7, \mathcal{C}_5^2\}$ *as minor.*

So far, the only examples of nontrivial identically self-blocking clutters that we have seen are $\{\Delta_n : n \geq 3\} \cup \{\mathbb{L}_7\}$. So how does $\mathcal{C}_5^2$ come into play? Let $\mathcal{A}, \mathcal{B}$ be clutters over the same ground set E, and take a new element e. The *join of* $\mathcal{A}, \mathcal{B}$ is the clutter over ground set $E \cup \{e\}$ whose members are

$$\mathcal{A} \vee_e \mathcal{B} := \text{ the minimal sets of } \left\{\{e\} \cup A : A \in \mathcal{A}\right\} \cup \left\{\{e\} \cup B : B \in \mathcal{B}\right\} \cup \left\{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\right\}.$$

Clearly, $\mathcal{A} \vee_e \mathcal{B} = \mathcal{B} \vee_e \mathcal{A}$. (We will drop the subscript e whenever there is no ambiguity.)

Proposition 3.5 ([10])**.** *Let $\mathcal{A}, \mathcal{B}$ be clutters over the same ground set, and take a new element e. Then $b(\mathcal{A} \vee_e \mathcal{B}) = b(\mathcal{A}) \vee_e b(\mathcal{B})$.*

Proof. It suffices to prove that

(1) every minimal cover of $\mathcal{A} \vee \mathcal{B}$ contains a member of $b(\mathcal{A}) \vee b(\mathcal{B})$,

(2) every member of $b(\mathcal{A}) \vee b(\mathcal{B})$ contains a minimal cover of $\mathcal{A} \vee \mathcal{B}$.

(1) Let C' be a minimal cover of $\mathcal{A} \vee \mathcal{B}$. Assume in the first case that $e \notin C'$. Then as C' intersects every set of $\big\{\{e\} \cup A : A \in \mathcal{A}\big\} \cup \big\{\{e\} \cup B : B \in \mathcal{B}\big\}$, C' is a cover of both $\mathcal{A}$ and $\mathcal{B}$. As a result, there exist $A' \in b(\mathcal{A})$ and $B' \in b(\mathcal{B})$ such that $A' \cup B' \subseteq C'$, so C' contains a member of $b(\mathcal{A}) \vee b(\mathcal{B})$. Assume in the second case that $e \in C'$ and C' is a cover of $\mathcal{A}$. Then for some $A' \in b(\mathcal{A})$, $\{e\} \cup A' \subseteq C'$, implying in turn that C' contains a member of $b(\mathcal{A}) \vee b(\mathcal{B})$. Assume in the remaining case that $e \in C'$ and C' is not a cover of $\mathcal{A}$. Then there is a member A of $\mathcal{A}$ disjoint from C'. Since C' intersects every set of $\{A \cup B : B \in \mathcal{B}\}$, C' must be a cover of $\mathcal{B}$. Thus, for some $B' \subseteq b(\mathcal{B})$, $\{e\} \cup B' \subseteq C'$, so C' contains a member of $b(\mathcal{A}) \vee b(\mathcal{B})$.
(2) Let $A' \in b(\mathcal{A})$ and $B' \in b(\mathcal{B})$. It is easy to see that $\{e\} \cup A', \{e\} \cup B', A' \cup B'$ are covers of $\mathcal{A} \vee \mathcal{B}$, implying in turn that they contain a minimal cover of $\mathcal{A} \vee \mathcal{B}$. $\square$

As an immediate consequence,

Theorem 3.6 ([10])**.** *For every clutter $\mathcal{C}$, the join $\mathcal{C} \vee b(\mathcal{C})$ is an identically self-blocking clutter.*

For instance, the clutter

$$\mathcal{C}_5^2 \vee_6 b(\mathcal{C}_5^2) = \big\{\{6,1,2\},\{6,2,3\},\{6,3,4\},\{6,4,5\},\{6,5,1\},\{1,2,4\},\{2,3,5\},\{3,4,1\},\{4,5,2\},\{5,1,3\}\big\}$$

is another example of an identically self-blocking clutter. This clutter has none of $\{\Delta_n : n \geq 6\} \cup \{\mathbb{L}_7\}$ as a minor, and in fact, its only minimally non-ideal minors are $\mathcal{C}_5^2, b(\mathcal{C}_5^2)$. This clutter is the reason we have added $\mathcal{C}_5^2$ to the list of minors in Conjecture 1.12. (For each odd integer $n \geq 7$, the join $\mathcal{C}_n^2 \vee b(\mathcal{C}_n^2)$ also has $\mathcal{C}_5^2$ as a minor.)

Theorem 3.6 tells us how to easily generate identically self-blocking clutters with the use of the join operator. This operator, in fact, generates every such clutter:

Remark 3.7. *Let $\mathcal{C}$ be identically self-blocking. Then for every element e, $\mathcal{C} \setminus e$ and $\mathcal{C}/e$ are blockers and $\mathcal{C} = (\mathcal{C} \setminus e) \vee_e (\mathcal{C}/e)$.*

For example, $\Delta_n = \{\{2,3,\ldots,n\}\} \vee_1 b(\{\{2,3,\ldots,n\}\})$ for each $n \geq 3$, while $\mathbb{L}_7 = Q_6 \vee_7 b(Q_6)$.

In closing, let us prove Conjecture 1.12 when there is a member of cardinality two; we will need Theorem 2.5 from the previous chapter.

Theorem 1.11 (Abdi and Pashkovich)**.** *An identically self-blocking clutter with a member of cardinality two has a delta minor.*

Proof. Let $\mathcal{C}$ be an identically self-blocking clutter with a member $\{e,f\}$ of cardinality two. By definition, $\{e,f\}$ is also a minimal cover. So there are members C_e, C_f such that $C_e \cap \{e,f\} = \{e\}$ and $C_f \cap \{e,f\} = \{f\}$. By definition, C_f is also a minimal cover, so $C_e \cap C_f \neq \emptyset$. It therefore follows from Theorem 2.5 that $\mathcal{C}$ has a delta minor, as required. $\square$

3.4 Further notes

Theorem 3.3 is an instance of a general duality phenomenon discovered in 1987 by Robert Freund, called *gauge duality*. For instance, Theorem 3.3 can be extended as follows:

Theorem 3.8 (Freund [36])**.** *Take an integer $n \geq 1$, let P, Q be blocking n-dimensional polyhedra, and let $p, q \geq 1$ be real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then the optimal values of* $\min\{\|x\|_p : x \in P\}$ *and* $\min\{\|u\|_q : u \in Q\}$ *are inverses of one another.*

Recall that $\|\cdot\|_p : \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ denotes the ℓ_p-norm mapping

$$x \mapsto \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}.$$

Question 3.9. *Is there an analogue of Theorem 3.2 for other norms?*

Chapter 4

Exclusive, coexclusive and opposite elements

Let $\mathcal{C}$ be a clutter and take distinct elements e, f. Recall that (e, f) is a coexclusive pair if every minimal cover contains at most one of e, f.

Theorem 1.13 ([4]). *Let $\mathcal{C}$ be a clutter and take distinct elements e, f. Then the following statements are equivalent:*

(i) (e, f) is a coexclusive pair,

(ii) for all members C_e, C_f such that $C_e \cap \{e, f\} = \{e\}$ and $C_f \cap \{e, f\} = \{f\}$, there is another member contained in $(C_e \cup C_f) - \{e, f\}$,

(iii) for every extreme point $x^\star$ of the set covering polyhedron $Q(\mathcal{C})$, $x^\star_e + x^\star_f \le 1$.

Proof. **(i) $\Rightarrow$ (ii)**: Suppose that (e, f) is a coexclusive pair. Take members C_e, C_f where $C_e \cap \{e, f\} = \{e\}$ and $C_f \cap \{e, f\} = \{f\}$. Suppose for a contradiction that $(C_e \cup C_f) - \{e, f\}$ does not contain a member. Then the complement of $(C_e \cup C_f) - \{e, f\}$ is a cover, so it contains a minimal cover B. As B intersects both C_e, C_f, it follows that $\{e, f\} \subseteq B$, a contradiction as (e, f) is coexclusive. Thus, $(C_e \cup C_f) - \{e, f\}$ contains a member, so (ii) holds. **(ii) $\Rightarrow$ (iii)**: Take an extreme point $x^\star$ of $Q(\mathcal{C}) = \{x \ge \mathbf{0} : M(\mathcal{C})x \ge \mathbf{1}\}$. If $x^\star_e = 0$ or $x^\star_f = 0$, then clearly $x^\star_e + x^\star_f \le 1$. Otherwise, there is a member C_e with $e \in C_e$ and a member C_f with $f \in C_f$ such that $x^\star(C_e) = x^\star(C_f) = 1$. If $\{e, f\} \subseteq C_e$, then $x^\star_e + x^\star_f \le x^\star(C_e) = 1$. We may therefore assume that $C_e \cap \{e, f\} = \{e\}$, and similarly, $C_f \cap \{e, f\} = \{f\}$. It now follows from (ii) that there is a member $C \subseteq (C_e \cup C_f) - \{e, f\}$. Then

$$x^\star_e + x^\star_f + 1 \le x^\star_e + x^\star_f + x^\star(C) \le x^\star(C_e) + x^\star(C_f) = 2,$$

proving (iii). **(iii) ⇒ (i)**: Let B be a minimal cover, and let $x^\star$ be the corresponding characteristic vector. Then $x^\star$ is an extreme point of $Q(\mathcal{C})$, so by (iii), $x^\star_e + x^\star_f \leq 1$, implying in turn that $|B \cap \{e,f\}| \leq 1$. Thus, (e,f) is coexclusive, so (i) holds. □

Recall that (e,f) is an exclusive pair if every member contains at most one of e,f, and that (e,f) is an opposite pair if it is both exclusive and coexclusive. As a consequence of Theorem 1.13,

Theorem 4.1 ([5]). *Let $\mathcal{L}$ be a clutter and take distinct elements e,f. Then (e,f) is an opposite pair if, and only if, for all members L_e, L_f such that $e \in L_e$ and $f \in L_f$, there is a member contained in $(L_e \cup L_f) - \{e,f\}$.*

Proof. (⇒) Take members L_e, L_f such that $e \in L_e$ and $f \in L_f$. Since (e,f) is exclusive, it follows that $L_e \cap \{e,f\} = \{e\}$ and $L_f \cap \{e,f\} = \{f\}$. Since (e,f) is coexclusive, it follows from Theorem 1.13 that $(L_e \cup L_f) - \{e,f\}$ contains a member. (⇐) By Theorem 1.13, (e,f) is coexclusive. It remains to show that (e,f) is exclusive. Suppose for a contradiction that there is a member L containing both e,f. Then for $L_e := L_f := L$, the hypothesis tells us that $(L_e \cup L_f) - \{e,f\} = L - \{e,f\}$ contains a member, a contradiction as $\mathcal{L}$ is a clutter. □

In this chapter, we will have an in-depth study of exclusive, coexclusive and opposite pairs. Along the way, we will see how we can take advantage of such pairs, directly and indirectly, to help us study idealness and the packing property. We will show that splitting preserves idealness (§4.1) as well as the packing property (§4.2). Moreover, our study reveals chains of ideal minimally non-packing clutters (§4.4), and gives rise to cuboids (§4.5), objects central to the rest of the thesis.

4.1 Splitting preserves idealness.

Let $\mathcal{C}$ be a clutter. Recall that a single split of $\mathcal{C}$ is another clutter $\mathcal{L}$ which has an opposite pair (e,f) such that $\mathcal{L}|_{e=f} = \mathcal{C}$. Let us first provide a direct and explicit definition of single splits, for which we need the following remark:

Remark 4.2 ([5]). *Let $\mathcal{L}$ be a clutter with an opposite pair (e,f). Take members L_e, L_f such that $e \in L_e$ and $f \in L_f$. Then neither of $L_e - \{e\}, L_f - \{f\}$ is contained in the other. In particular,*

$$\mathcal{L}|_{e=f} = \big\{L \in \mathcal{L} : f \notin L\big\} \cup \big\{L \triangle \{e,f\} : f \in L \in \mathcal{L}\big\};$$

so there is a one-to-one correspondence between the members of $\mathcal{L}$ and the members of its identification $\mathcal{L}|_{e=f}$.

Proof. Suppose for a contradiction that one of $L_e - \{e\}, L_f - \{f\}$ is contained in the other. After relabeling e,f, if necessary, we may assume that $L_e - \{e\} \subseteq L_f - \{f\}$. By Theorem 4.1, $(L_e \cup L_f) - \{e,f\}$ contains a member L. However, as $L_e - \{e\} \subseteq L_f - \{f\}$, it follows that $L \subseteq L_f - \{f\}$, a contradiction as $\mathcal{L}$ is a clutter. □

This remark, together with Theorem 4.1, tells us that every single split is obtained as follows:

Remark 4.3 ([5]). *Let $\mathcal{C}$ be a clutter over ground set E. For an element e, let $\mathrm{port}_1, \mathrm{port}_2$ be a partition of $\{C \in \mathcal{C} : e \in C\}$ such that for all $C_1 \in \mathrm{port}_1$ and $C_2 \in \mathrm{port}_2$, $(C_1 \cup C_2) - \{e\}$ contains another member. Take a new element f, and let $\mathcal{L}$ be the clutter over ground set $E \cup \{f\}$ whose members are*

$$\big\{C : C \in \mathcal{C}, C \notin \mathrm{port}_2\big\} \cup \big\{C \triangle \{e, f\} : C \in \mathrm{port}_2\big\}.$$

Then (e, f) is an opposite pair in $\mathcal{L}$. Moreover, $\mathcal{L}|_{e=f} = \mathcal{C}$, so $\mathcal{L}$ is a single split of $\mathcal{C}$.

Since (e, f) is an opposite pair for $\mathcal{L}$, it is also an opposite pair for the blocker $b(\mathcal{L})$. Thus, there is also a one-to-one correspondence between the members of $b(\mathcal{L})$ and the members its identification $b(\mathcal{L})|_{e=f}$. Recall however from Remark 1.14 that $b(\mathcal{L})|_{e=f} = b(\mathcal{L}|_{e=f})$. Thus, there is also a one-to-one correspondence between the minimal covers of $\mathcal{L}$ and the minimal covers of its identification $\mathcal{L}|_{e=f}$.

Recall that a split is what is obtained after a series of single splits. We are now ready to prove the following theorem:

Theorem 1.15 ([5]). *If a clutter is ideal, then so is any split of it.*

Proof. It suffices to prove this for single splits. Let $\mathcal{C}$ be a clutter over ground set E, and let $\mathcal{L}$ be a single split of it over ground set $E \cup \{f\}$. That is, $\mathcal{L}$ has an opposite pair (e, f) such that $\mathcal{L}|_{e=f} = \mathcal{C}$. Assume that $\mathcal{C}$ is ideal. Suppose for a contradiction that $\mathcal{L}$ is non-ideal, and let $x^\star \in \mathbb{R}_+^{E \cup \{f\}}$ be a fractional extreme point of $Q(\mathcal{L})$. After relabeling e and f, if necessary, we may assume that $x_e^\star \geq x_f^\star$. Define $y \in \mathbb{R}_+^E$ as follows: $y_e := \max\{x_e^\star, x_f^\star\} = x_e^\star$ and for each $g \in E - \{e\}$, $y_g := x_g^\star$. For each $C \in \mathcal{C}$ and its corresponding member L in $\mathcal{L}$,

$$y(C) = \begin{cases} x^\star(L) & \text{if } f \notin L \\ x^\star(L) - x_f^\star + x_e^\star & \text{if } f \in L, \end{cases}$$

so because $x^\star(L) \geq 1$, we have that $y(C) \geq 1$. As a result, $y \in Q(\mathcal{C})$ and since $\mathcal{C}$ is ideal, there exist an integer $n \geq 1$ and members $B_1, \ldots, B_n$ of $b(\mathcal{C})$ so that y is at least as large as a convex combination of $\chi_{B_1}, \ldots, \chi_{B_n}$:

$$y \geq \sum_{i=1}^{n} \lambda_i \chi_{B_i}$$

for some $\lambda \in \mathbb{R}_+^n$ such that $\sum_{i=1}^n \lambda_i = 1$. For each $i \in [n]$, let K_i be the corresponding member of B_i in $b(\mathcal{L})$, and let

$$x := \sum_{i=1}^{n} \lambda_i \chi_{K_i} \in Q(\mathcal{L}).$$

Since $x^\star$ is a fractional extreme point, the equation above implies that $x^\star \not\geq x$. However, for each $g \in E - \{e\}$, $x_g^\star = y_g \geq x_g$ and

$$x_e^\star = y_e \geq x_e + x_f.$$

Hence, $x_f > x_f^\star$ and in particular, $x_e^\star > 0$. Therefore, since $x^\star$ is an extreme point of $Q(\mathcal{L})$, there exists

$L_e \in \mathcal{L}$ with $e \in L_e$ such that $x^*(L_e) = 1$. But then, since $f \notin L_e$ and $x \in Q(\mathcal{L})$, it follows that

$$1 = x^\star(L_e) \geq x(L_e) \geq 1,$$

so equality holds throughout. In particular, $x^\star_e = x_e$, implying that $x_f = 0$, a contradiction with $x_f > x^\star_f$. Hence, $\mathcal{L}$ is ideal. $\square$

4.2 Splitting preserves the packing property.

Let $\mathcal{C}$ be a clutter. A *packing in* $\mathcal{C}$ is a collection of pairwise disjoint members. Recall that $\mathcal{C}$ packs if $\tau(\mathcal{C}) = \nu(\mathcal{C})$, that is, if the minimum cardinality of a cover is equal to the maximum cardinality of a packing.

Remark 4.4 ([5])**.** *Let $\mathcal{C}$ be a clutter, and let $\mathcal{L}$ be a split of it. Then $\tau(\mathcal{C}) = \tau(\mathcal{L})$. Moreover, if $\mathcal{C}$ packs, then so does $\mathcal{L}$.*

Proof. We may assume that $\mathcal{L}$ is a single split of $\mathcal{C}$. The one-to-one correspondence between the minimal covers of $\mathcal{C}$ and those of $\mathcal{L}$ tells us that $\tau(\mathcal{L}) = \tau(\mathcal{C})$. If $\mathcal{C}$ packs, then it has a packing of cardinality $\tau(\mathcal{C})$ in $\mathcal{C}$, naturally leading to a packing of cardinality $\tau(\mathcal{L})$ in $\mathcal{L}$, so $\mathcal{L}$ packs. $\square$

Recall that $\mathcal{C}$ has the packing property if every minor of it, including $\mathcal{C}$ itself, packs. In this section, we set out to prove the analogue of Theorem 1.15 for the packing property. We will need the following tool for finding delta minors, a more general version of which was proved by the authors:

Theorem 4.5 ([5])**.** *Take a clutter $\mathcal{L}$ with an opposite pair (e, f). If there exist $L_e \in \mathcal{L}$ and $K_f \in b(\mathcal{L})$ such that $L_e \cap \{e, f\} = \{e\}, K_f \cap \{e, f\} = \{f\}$ and $|L_e \cap K_f| = 1$. Then $\mathcal{L}|_{e=f}$ has a delta minor.*

Proof. Pick the element g of $\mathcal{L}$ such that $L_e \cap K_f = \{g\}$. Let $I := L_e - \{e, g\}$, $J := K_f - \{f, g\}$ and $\mathcal{L}' := \mathcal{L}/I \setminus J$. Notice that for $\mathcal{L}'$, (e, f) is an opposite pair, $\{e, g\}$ is a member, and $\{f, g\}$ is a minimal cover. So $\mathcal{L}'$ has a member L_1 such that $L_1 \cap \{f, g\} = \{f\}$. Since (e, f) is opposite, it follows from Theorem 4.1 that $(\{e, g\} \cup L_1) - \{e, f\} = (L_1 - \{f\}) \cup \{g\}$ contains a member L_2 of $\mathcal{L}'$. Clearly, $g \in L_2$.

Let $\mathcal{C}' := \mathcal{L}'|_{e=f}$. Let $C := (L_1 - \{f\}) \cup \{e\}$. Then $\{e, g\}, L_2, C$ are members of $\mathcal{C}'$ by Remark 4.2. In fact, C is a doubly g-redundant member of $\mathcal{C}'$ with causes $\{e, g\}, L_2$, so by Theorem 1.3, $\mathcal{C}'$ has a delta minor. Since $\mathcal{C}' = \mathcal{L}'|_{e=f} = (\mathcal{L}/I \setminus J)|_{e=f} = (\mathcal{L}|_{e=f})/I \setminus J$, it follows that $\mathcal{L}|_{e=f}$ has a delta minor, as required. $\square$

Recall that if a clutter has the packing property, then it does not have a delta minor. We will make use of this below:

Lemma 4.6 ([5])**.** *Take a clutter $\mathcal{L}$ with an opposite pair (e, f). If $\mathcal{L}|_{e=f}$ has the packing property, then*

(1) $\mathcal{L} \setminus f$ packs,

(2) for all $L \in \mathcal{L}$ and $K \in b(\mathcal{L})$ such that $e \in L, f \in K$ or $f \in L, e \in K$, we have $|L \cap K| \geq 2$,

(3) if a minimum cover of $\mathcal{L}$ does not contain e, then $\mathcal{L}/e \setminus f$ packs,

(4) in fact, $\mathcal{L}/e \setminus f$ always packs, and

(5) $\mathcal{L}/e$ packs.

Proof. Let $\mathcal{C} := \mathcal{L}|_{e=f}$. Suppose that $\mathcal{C}$ has the packing property. By Remark 4.4, $\tau(\mathcal{L}) = \tau(\mathcal{C})$ and $\mathcal{L}$ packs.

(1) Observe that $\tau(\mathcal{L}) - 1 \leq \tau(\mathcal{L} \setminus f) \leq \tau(\mathcal{L})$. If $\tau(\mathcal{L}) - 1 = \tau(\mathcal{L} \setminus f)$, then any maximum packing in $\mathcal{L}$ yields a packing in $\mathcal{L} \setminus f$ of cardinality $\tau(\mathcal{L} \setminus f)$, implying that $\mathcal{L} \setminus f$ packs. Otherwise, we have that $\tau(\mathcal{L} \setminus f) = \tau(\mathcal{L})$. So no minimum cover of $\mathcal{L}$ contains f.

If there is no minimum cover of $\mathcal{L}$ using e either, then $\tau(\mathcal{L} \setminus \{e, f\}) = \tau(\mathcal{L})$ and since $\mathcal{L} \setminus \{e, f\} = \mathcal{C} \setminus e$ packs, $\mathcal{L} \setminus f$ packs also.

Otherwise, there is a minimum cover K_e of $\mathcal{L}$ using e. Note that K_e is also a minimum cover of $\mathcal{C}$. Let $C_1, \ldots, C_\tau$ be a packing of $\mathcal{C}$, where $\tau = \tau(\mathcal{C}) = \tau(\mathcal{L}) = \tau(\mathcal{L} \setminus f)$. Let $L_1, \ldots, L_\tau$ be their corresponding members in $\mathcal{L}$. If for some $j \in [\tau]$, $f \in L_j$, then $e \in C_j$ so $K_e \cap C_j = \{e\}$, implying that $K_e \cap L_j = \emptyset$, which cannot be. Hence, $f \notin L_1 \cup \cdots \cup L_\tau$, so $L_1, \ldots, L_\tau$ yields a packing in $\mathcal{L} \setminus f$, so $\mathcal{L} \setminus f$ packs.

(2) Suppose for a contradiction that $|L \cap K| = 1$. By symmetry, we may assume that $e \in L, f \in K$. Then by Theorem 4.5, $\mathcal{C}$ has a delta minor, a contradiction as $\mathcal{C}$ has the packing property.

(3) Suppose that there is a minimum cover of $\mathcal{L}$ avoiding element e. Then $\tau(\mathcal{L}/e) = \tau(\mathcal{L})$. If there is a minimum cover of $\mathcal{L}$ using f, then $\tau(\mathcal{L}/e \setminus f) = \tau(\mathcal{L}) - 1$, so a packing for $\mathcal{L}$ also yields a packing for $\mathcal{L}/e \setminus f$. We may therefore assume that no minimum cover of $\mathcal{L}$ uses f, so $\tau := \tau(\mathcal{L}/e \setminus f) = \tau(\mathcal{L})$.

If no minimum cover of $\mathcal{L}$ uses e either, then $\tau = \tau(\mathcal{L} \setminus \{e, f\})$, so a packing for $\mathcal{C} \setminus e = \mathcal{L} \setminus \{e, f\}$ yields one for $\mathcal{L}/e \setminus f$. Otherwise, there is a minimum cover K_e of $\mathcal{L}$ that uses e. Let $L_1, \ldots, L_\tau$ be a packing for $\mathcal{L}$. Note that $|L_i \cap K_e| = 1$ for each $i \in [\tau]$. So from (2) it follows that $f \notin L_1 \cup \ldots \cup L_\tau$, so $L_1, \ldots, L_\tau$ also yields a packing for $\mathcal{L}/e \setminus f$.

(4) Denote by E the ground set of $\mathcal{L}$. We will need the following claim:

Claim. *Suppose that L_e, L_f are disjoint members of $\mathcal{L}$ where $e \in L_e$ and $f \in L_f$. Then there exist disjoint $L, L' \in \mathcal{L}$ contained in $(L_e \cup L_f) - \{e, f\}$.*

Proof of Claim. By (2) every minimal cover using e (resp. f) intersects L_f (resp. L_e) at least twice. As a result, given that $F = \big(E - (L_e \cup L_f)\big) \cup \{e, f\}$, we have $\tau(\mathcal{L} \setminus F) \geq 2$. Since $\mathcal{L} \setminus F = \mathcal{C} \setminus F$ packs, the result follows. $\diamond$

Let $L_1, \ldots, L_k, \ldots, L_{k+\ell}, \ldots, L_{k+\ell+r}$ be a packing for $\mathcal{L}/\{e, f\}$ where

$$\begin{aligned} k + \ell + r = \tau(\mathcal{L}/\{e, f\}) &\geq \tau(\mathcal{L}/e) \geq \tau(\mathcal{L}) \\ \{e\} \cup L_j \in \mathcal{L} &\quad j = 1, \ldots, k \\ \{f\} \cup L_j \in \mathcal{L} &\quad j = k+1, \ldots, k+\ell \\ L_j \in \mathcal{L} &\quad j = k+\ell+1, \ldots, k+\ell+r. \end{aligned}$$

By (3) we may assume that there is a minimum cover K_e of $\mathcal{L}$ that contains e. Then $e \in K_e$ and by (2), K_e intersects each one of $L_{k+1}, \ldots, L_{k+\ell}$ at least twice. As a result, $\tau(\mathcal{L}) = |K_e| \geq 1 + 2\ell + r$ which, together with $k + \ell + r \geq \tau(\mathcal{L})$, implies that $k > \ell$. By the claim above, for each $j \in [\ell]$, we can find disjoint $L_j^1, L_j^2 \in \mathcal{L}$ such that $L_j^1 \cup L_j^2 \subseteq L_j \cup L_{k+j}$. Observe now that

$$\begin{array}{rl} L_j^1, L_j^2 & j = 1, \ldots, \ell \\ L_j & j = \ell+1, \ldots, k \\ L_j & k+\ell+1, \ldots, k+\ell+r \end{array}$$

is a packing of size $k + \ell + r$ in $\mathcal{L}/e \setminus f$. However, $\tau(\mathcal{L}/e \setminus f) \leq \tau(\mathcal{L}/\{e, f\}) = k + \ell + r$, implying that $\mathcal{L}/e \setminus f$ packs.

(5) Suppose for a contradiction that $\mathcal{L}/e$ does not pack. If there is a minimum cover of $\mathcal{L}$ that does not use e, then $\tau(\mathcal{L}/e) = \tau(\mathcal{L})$, so the packing in $\mathcal{L}$ gives a packing in $\mathcal{L}/e$, which is not the case. Hence, every minimum cover of $\mathcal{L}$ uses e, so $\tau(\mathcal{L}/e) \geq \tau(\mathcal{L}) + 1$.

If $\tau(\mathcal{L}/e \setminus f) = \tau(\mathcal{L}/e)$, then the packing in $\mathcal{L}/e \setminus f$ from (4) yields one in $\mathcal{L}/e$, which again cannot be the case. Hence, we have $s := \tau(\mathcal{L}/e \setminus f) = \tau(\mathcal{L}/e) - 1$. Together with the inequality above, we have $s \geq \tau(\mathcal{L})$.

Let $L_1, \ldots, L_s$ be a packing and let K be a cover of cardinality s, in $\mathcal{L}/e \setminus f$. Note $K \cup \{f\}$ is a minimum cover of $\mathcal{L}/e$, and in particular, $K \cup \{f\} \in b(\mathcal{L})$. Since every minimum cover of $\mathcal{L}$ uses e and has size at most s, there exists $j \in [s]$ such that $L_j \cup \{e\} \in \mathcal{L}$. But then

$$\left|\left(L_j \cup \{e\}\right) \cap \left(K \cup \{f\}\right)\right| = |L_j \cap K| = 1,$$

contradicting (2). $\square$

We are now ready to prove the following theorem:

Theorem 1.17 ([5]). *If a clutter has the packing property, then so does any split of it.*

Proof. Once again, it suffices to prove this for single splits. Let $\mathcal{C}$ be a clutter over ground set E, and let $\mathcal{L}$ be a single split of it over ground set $E \cup \{f\}$. That is, $\mathcal{L}$ has an opposite pair (e, f) such that $\mathcal{L}|_{e=f} = \mathcal{C}$. Assume that $\mathcal{C}$ has the packing property. We need to show that every minor of $\mathcal{L}$ packs. Take a minor $\mathcal{L}' := \mathcal{L}/I \setminus J$ of $\mathcal{L}$. If $\{e, f\} \subseteq I$ or $\{e, f\} \subseteq J$, then $\mathcal{L}'$ is also a minor of $\mathcal{C}$, so it packs. If $\{e, f\} \cap (I \cup J) = \emptyset$, then $\mathcal{L}'$ is a single split of a minor of $\mathcal{C}$, so by Remark 4.4, $\mathcal{L}'$ packs. Otherwise, up to relabeling e and f, one of the following holds:

(i) $e \in I$ and $f \in J$: by applying Lemma 4.6 (4) to $\mathcal{C}/(I - \{e\}) \setminus (J - \{f\})$ and $\mathcal{L}/(I - \{e\}) \setminus (J - \{f\})$, we get that $\mathcal{L}'$ packs,

(ii) $e \notin I \cup J$ and $f \in J$: by applying Lemma 4.6 (1) to $\mathcal{C}/I \setminus (J - \{f\})$ and $\mathcal{L}/I \setminus (J - \{f\})$, we get that $\mathcal{L}'$ packs,

(iii) $e \in I$ and $f \notin I \cup J$: by applying Lemma 4.6 (5) to $\mathcal{C}/(I-\{e\}) \setminus J$ and $\mathcal{L}/(I-\{e\}) \setminus J$, we get that $\mathcal{L}'$ packs.

Thus, every minor of $\mathcal{L}$ packs, so $\mathcal{L}$ has the packing property. □

4.3 When does coexclusive identification preserve properties?

In the previous two sections, we studied the operation of splitting and showed that it preserves idealness and the packing property. In this section, we will study the inverse operation, more specifically, the operation of identifying coexclusive elements. Such identifications, in contrast to splitting, do not preserve either idealness or the packing property. For instance, $\mathbb{P}_4 = \{\{1,2\},\{2,3\},\{3,4\}\}$ has $(1,4)$ as a coexclusive pair and its identification leads to Δ_3, proof that an ideal clutter with the packing property can identify to a non-ideal clutter that does not pack.

In this section, we characterize when coexclusive identifications preserve idealness and the packing property. The tools developed in this section will be used in the next section.

Theorem 4.7 ([4]). *Take a clutter $\mathcal{C}$ with a coexclusive pair (e,f). If $\mathcal{C}$ has the packing property, then the following statements are equivalent:*

(i) $\mathcal{C}|_{e=f}$ has the packing property,

(ii) every minor of $\mathcal{C}|_{e=f}$ with covering number at least two has two disjoint members,

(iii) for all members C_e, C_f of $\mathcal{C}$ where $C_e \cap \{e,f\} = \{e\}$ and $C_f \cap \{e,f\} = \{f\}$, there are members C, C' such that

$$C \cap \{e,f\} = \emptyset, \qquad C \cap C' \subseteq C_e \cap C_f \quad \text{and} \quad C \cup C' \subseteq C_e \cup C_f.$$

Proof. Let E be the ground set of $\mathcal{C}$. **(i) ⇒ (ii)**: This follows from the definition of the packing property. **(ii) ⇒ (iii)**: Let $I := C_e \cap C_f$ and $J := E - (C_e \cup C_f)$. Consider the minor $\mathcal{C}' := \mathcal{C}/I \setminus J$. Since $C_e - I$ and $C_f - I$ are disjoint members of $\mathcal{C}'$, we get that $\tau(\mathcal{C}') \geq 2$. As (e,f) is coexclusive in $\mathcal{C}$, it is also coexclusive in the minor $\mathcal{C}'$, so $\tau\big(\mathcal{C}'|_{e=f}\big) = \tau(\mathcal{C}') \geq 2$. Since $\mathcal{C}'|_{e=f} = \mathcal{C}|_{e=f}/I \setminus J$, it follows that $\tau\big(\mathcal{C}|_{e=f}/I \setminus J\big) \geq 2$. So (ii) implies the existence of two disjoint members in $\mathcal{C}|_{e=f}/I \setminus J$ – these disjoint members will correspond to members C, C' of $\mathcal{C}$ satisfying (iii). **(iii) ⇒ (i)**: Since (iii) is a minor-closed property, it suffices to show that $\mathcal{C}|_{e=f}$ packs. To this end, let $\tau := \tau(\mathcal{C}) = \tau\big(\mathcal{C}|_{e=f}\big)$. Since $\mathcal{C}$ packs, it has τ pairwise disjoint members $C_1, \ldots, C_\tau$, where $C_i \cap \{e,f\} = \emptyset$ for each $i \in \{3, \ldots, \tau\}$. If one of C_1, C_2 is also disjoint from $\{e,f\}$, then these members yield the desired packing of cardinality τ in $\mathcal{C}|_{e=f}$. We may therefore assume that $C_1 \cap \{e,f\} = \{e\}$ and $C_2 \cap \{e,f\} = \{f\}$. It then follows from (iii) that there are disjoint members C, C' contained in $C_1 \cup C_2$ such that $C \cap \{e,f\} = \emptyset$. It is easy to see that $C, C', C_3, \ldots, C_\tau$ gives rise to a packing of cardinality τ in $\mathcal{C}|_{e=f}$, as required. □

Next, we set out to prove an analogue of Theorem 4.7 for idealness. Let $\mathcal{C}$ be a clutter over ground set E. Consider the dual pair of linear programs

$$(P) \quad \begin{array}{ll} \min & \sum (x_g : g \in E) \\ \text{s.t.} & x(C) \geq 1 \quad \forall C \in \mathcal{C} \\ & x \geq \mathbf{0} \end{array} \qquad\qquad (D) \quad \begin{array}{ll} \max & \sum (y_C : C \in \mathcal{C}) \\ \text{s.t.} & \sum (y_C : g \in C \in \mathcal{C}) \leq 1 \quad \forall g \in E \\ & y \geq \mathbf{0}. \end{array}$$

A feasible solution to the dual program (D) is called a *fractional packing* and its *value* is the objective value of the solution. When $\mathcal{C}$ is ideal, basic polyhedral theory dictates that a minimum cover yields an optimal solution to (P), and thus by strong duality for linear programs, there exists a fractional packing of value $\tau(\mathcal{C})$ ([20], Theorems 3.7 and 4.1).

Let (e,f) be a coexclusive pair, and take members C_e, C_f such that

$$C_e \cap \{e,f\} = \{e\} \quad \text{and} \quad C_f \cap \{e,f\} = \{f\}. \tag{$\diamond$}$$

A *fractional disentangling of C_e and C_f* is a vector $y \in \mathbb{R}^{\mathcal{C}}_+$ where

- $\sum \big(y_C : C \in \mathcal{C}\big) = 2$,
- $\sum \big(y_C : C \cap \{e,f\} \neq \emptyset\big) \leq 1$, and
- for each element g,

$$\sum \big(y_C : g \in C\big) \leq \begin{cases} 0 & \text{if } g \notin C_e \cup C_f \\ 1 & \text{if } g \in C_e \triangle C_f \\ 2 & \text{if } g \in C_e \cap C_f. \end{cases}$$

Notice that a fractional disentangling is the fractional analogue of the two members in part (iii) of Theorem 4.7. We are now ready to state the fractional analogue of that theorem:

Theorem 4.8 ([4]). *Take a clutter $\mathcal{C}$ with a coexclusive pair (e,f). If $\mathcal{C}$ is ideal, then the following statements are equivalent:*

(i) $\mathcal{C}|_{e=f}$ is ideal,

(ii) every minor of $\mathcal{C}|_{e=f}$ with covering number at least two has a fractional packing of value two,

(iii) all members C_e, C_f of $\mathcal{C}$ satisfying $(\diamond)$ have a fractional disentangling,

(iv) for all members C_e, C_f of $\mathcal{C}$ satisfying $(\diamond)$, the inequality

$$x\big(C_e - \{e\}\big) + x\big(C_f - \{f\}\big) + x_e \geq 2$$

is valid for $Q\big(\mathcal{C}|_{e=f}\big)$.

Proof. Denote by E the ground set of $\mathcal{C}$. **(i) $\Rightarrow$ (ii)**: Notice that every minor of $\mathcal{C}|_{e=f}$ is ideal, so (ii) follows immediately from our discussion above on fractional packings. **(ii) $\Rightarrow$ (iii)**: As in Theorem 4.7, let $I := C_e \cap C_f$ and $J := E - (C_e \cup C_f)$. Consider the minor $\mathcal{C}' := \mathcal{C}/I \setminus J$. Since $C_e - I$ and $C_f - I$ are disjoint members of $\mathcal{C}'$, we get that $\tau(\mathcal{C}') \geq 2$. As (e,f) is coexclusive in $\mathcal{C}$, it is also coexclusive in the minor $\mathcal{C}'$, and the corresponding identification is $\mathcal{C}'|_{e=f} = \mathcal{C}|_{e=f}/I \setminus J$. Moreover, $\tau\big(\mathcal{C}'|_{e=f}\big) = \tau(\mathcal{C}') \geq 2$. Now (ii) implies the existence of a fractional packing $y \in \mathbb{R}^{\mathcal{C}'|_{e=f}}_+$ of value 2. Consider the natural extension of y to $\mathbb{R}^{\mathcal{C}}_+$ where members of $\mathcal{C}$ present in the identified minor $\mathcal{C}'|_{e=f}$ are assigned the same value as before, and all the other members are assigned 0. It can be readily checked that this natural extension is a fractional disentangling of C_e and C_f. **(iii) $\Rightarrow$ (iv)**: Let $y \in \mathbb{R}^{\mathcal{C}}_+$ be a fractional disentangling of C_e and C_f. Since the members of $\mathcal{C}$ are in correspondence with the sets in the family

$$\widehat{\mathcal{C}|_{e=f}} := \big\{C : f \notin C \in \mathcal{C}\big\} \cup \big\{(C \cup \{e\}) - \{f\} : f \in C \in \mathcal{C}\big\},$$

we may regard y as a vector in $\mathbb{R}^{\widehat{\mathcal{C}|_{e=f}}}_+$. Observe further that $\mathcal{C}|_{e=f}$ consists of the minimal sets in $\widehat{\mathcal{C}|_{e=f}}$. As a result, for every $x \in Q\big(\mathcal{C}|_{e=f}\big)$, we have

$$\begin{aligned}
0 &\le \sum_{C \in \widehat{\mathcal{C}|_{e=f}}} y_C\big(x(C) - 1\big)\\
&= \sum_{C \in \widehat{\mathcal{C}|_{e=f}}} y_C \cdot x(C) - \sum_{C \in \widehat{\mathcal{C}|_{e=f}}} y_C\\
&= \sum_{g \in E - \{f\}} x_g \left[\sum \big(y_C : g \in C \in \widehat{\mathcal{C}|_{e=f}}\big)\right] - 2\\
&= x_e \left[\sum \big(y_C : e \in C \in \widehat{\mathcal{C}|_{e=f}}\big)\right] + \sum_{g \notin C_e \cup C_f} x_g \left[\sum \big(y_C : g \in C \in \widehat{\mathcal{C}|_{e=f}}\big)\right]\\
&\quad + \sum_{g \in (C_e \triangle C_f) - \{e,f\}} x_g \left[\sum \big(y_C : g \in C \in \widehat{\mathcal{C}|_{e=f}}\big)\right] + \sum_{g \in C_e \cap C_f} x_g \left[\sum \big(y_C : g \in C \in \widehat{\mathcal{C}|_{e=f}}\big)\right] - 2\\
&\le x_e \cdot 1 + \sum_{g \notin C_e \cup C_f} x_g \cdot 0 + \sum_{g \in C_e \triangle C_f - \{e,f\}} x_g \cdot 1 + \sum_{g \in C_e \cap C_f} x_g \cdot 2 - 2\\
&= x\big(C_e - \{e\}\big) + x\big(C_f - \{f\}\big) + x_e - 2,
\end{aligned}$$

where the second equation and the last inequality follow from the fact that y is a fractional disentangling. The last equation proves (iv). **(iv) $\Rightarrow$ (i)**: Since $\mathcal{C}$ is ideal, the linear system

$$\begin{aligned}
x(C - \{e,f\}) + z(C \cap \{e,f\}) &\ge 1 \quad \forall C \in \mathcal{C}\\
x_g &\ge 0 \quad \forall g \in E - \{e,f\}\\
z_e, z_f &\ge 0
\end{aligned}$$

describes the dominant of $\mathrm{conv}\big\{\chi_B : B \in b(\mathcal{C})\big\}$. We now add a new variable x_e with the additional linear constraint $x_e = z_e + z_f$. It can be readily checked that the dominant of $\mathrm{conv}\big\{\chi_{B'} : B' \in b(\mathcal{C}|_{e=f})\big\}$ can be described by this new linear system after eliminating the variables z_e, z_f. After applying the Fourier-Motzkin Elimination method to do so, we see that the dominant of $\mathrm{conv}\big\{\chi_{B'} : B' \in b(\mathcal{C}|_{e=f})\big\}$ is described by

$$\begin{aligned}
x(C) &\ge 1 \quad \forall\ C \in \mathcal{C}|_{e=f}\\
x_g &\ge 0 \quad \forall\ g \in E - \{f\}\\
x\big(C_e - \{e\}\big) + x\big(C_f - \{f\}\big) + x_e &\ge 2 \quad \text{for all } C_e, C_f \text{ satisfying } (\diamond).
\end{aligned}$$

However, it follows from (iv) that the last line of inequalities are all redundant, implying in turn that as the dominant of $\mathrm{conv}\big\{\chi_{B'} : B' \in b(\mathcal{C}|_{e=f})\big\}$, $Q\big(\mathcal{C}|_{e=f}\big)$ is an integral polyhedron, thereby proving (i). $\square$

4.4 Chains of ideal minimally non-packing clutters

Recall that a clutter is minimally non-packing if it does not pack but every proper minor does. The following is a consequence of Theorem 4.7:

Proposition 4.9 ([4]). *Take a clutter $\mathcal{C}$ with a coexclusive pair (e,f). If $\mathcal{C}$ is minimally non-packing, then either*

(i) $\mathcal{C}|_{e=f}$ is minimally non-packing, or

(ii) every minimally non-packing minor of $\mathcal{C}|_{e=f}$ has covering number two.

Proof. Denote by E the ground set of $\mathcal{C}$. As $\mathcal{C}$ does not pack, $\mathcal{C}|_{e=f}$ does not pack either by Remark 1.18. Thus, there exist disjoint subsets $I, J \subseteq E - \{f\}$ such that $\mathcal{C}|_{e=f} \setminus I/J$ is minimally non-packing minor. If $I = J = \emptyset$ then (i) holds. Otherwise, $I \cup J \neq \emptyset$. In particular, $\mathcal{C}|_{e=f} \setminus I/J$ cannot be a minor of $\mathcal{C}$, implying in turn that $e \notin I \cup J$. Consider the clutter $\mathcal{C}' := \mathcal{C} \setminus I/J$. As a proper minor of $\mathcal{C}$, $\mathcal{C}'$ has the packing property. Its identification $\mathcal{C}'|_{e=f} = \mathcal{C}|_{e=f} \setminus I/J$ however, is minimally non-packing. Thus, by Theorem 4.7, a minor of $\mathcal{C}'|_{e=f}$ with covering number at least two does not have two disjoint members. Since every proper minor of $\mathcal{C}'|_{e=f}$ packs, the clutter $\mathcal{C}'|_{e=f}$ itself does not have two disjoint members. This implies that

$$\tau\big(\mathcal{C}|_{e=f} \setminus I/J\big) = \tau\big(\mathcal{C}'|_{e=f}\big) = 2;$$

for if not, then deleting any element of $\mathcal{C}'|_{e=f}$ yields a proper non-packing minor, which cannot be. Thus, (ii) holds. $\square$

Blockers of odd holes form an infinite class of examples satisfying Proposition 4.9 (ii). To elaborate, take an odd integer $n \geq 5$. Consider the odd hole of dimension n:

$$\mathcal{C}_n^2 = \big\{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}\big\}.$$

Let $\mathcal{C}_3^2 := \Delta_3$. It is well-known that $\mathcal{C}_3^2, \mathcal{C}_5^2, \mathcal{C}_7^2, \ldots$ and their blockers $b(\mathcal{C}_3^2), b(\mathcal{C}_5^2), b(\mathcal{C}_7^2), \ldots$ are (non-ideal) minimally non-packing clutters. Notice that $(n-1,1)$ is a coexclusive pair for $b(\mathcal{C}_n^2)$, and that $b\big(\mathcal{C}_n^2\big)|_{n-1=1}$ has $b(\mathcal{C}_{n-2}^2)$ as its only minimally non-packing minor, whose covering number is two, thereby verifying Proposition 4.9 (ii).

$$M := \begin{pmatrix} & & 1 & 1 & 1 & & 1 & \\ & & 1 & 1 & & 1 & & 1 \\ 1 & 1 & & & & 1 & 1 & \\ 1 & 1 & & & 1 & & & 1 \\ 1 & & 1 & 1 & & 1 & 1 & \\ & 1 & 1 & & 1 & & & 1 \\ & 1 & 1 & & 1 & & 1 & \\ 1 & 1 & & 1 & & 1 & & 1 \end{pmatrix} \quad \text{identifies to} \quad M(Q_6 \otimes 1) := \begin{pmatrix} & 1 & 1 & 1 & & 1 & \\ & 1 & 1 & & 1 & & 1 \\ 1 & & & & 1 & 1 & \\ 1 & & & 1 & & & 1 \\ 1 & 1 & & 1 & & 1 & \\ 1 & & 1 & & 1 & & 1 \end{pmatrix}.$$

As mentioned in the introduction, the dozen sporadic examples of minimally non-packing clutters introduced by Cornuéjols, Guenin and Margot [24] obey Proposition 4.9 (i). For instance, consider the (ideal) minimally non-packing clutter whose incidence matrix is M as displayed above (for readability's sake, the zeros are removed). Using Theorem 1.13 (ii), it is easily seen that the first and second columns correspond to a coexclusive pair, and that identifying them leads to another minimally non-packing clutter $Q_6 \otimes 1$; the second and third columns here form a coexclusive pair and identifying them gives the minimally non-packing clutter Q_6, verifying Proposition 4.9 (i). What is more, in these examples, not only is $\mathcal{C}|_{e=f}$ minimally non-packing, but it is also ideal – this is not a coincidence, and to see this, we need two ingredients. The first ingredient is a consequence of Theorem 4.8:

Proposition 4.10 ([4])**.** *Take an ideal clutter $\mathcal{C}$ with a coexclusive pair (e, f). If $\mathcal{C}|_{e=f}$ is minimally non-ideal, then $\tau(\mathcal{C}) = \tau\big(\mathcal{C}|_{e=f}\big) = 2$.*

Proof. Suppose that $\mathcal{C}|_{e=f}$ is minimally non-ideal. It then follows from Theorem 4.8 that a minor of $\mathcal{C}|_{e=f}$ with covering number at least two has no fractional packing of value two. As every proper minor of it is ideal, $\mathcal{C}|_{e=f}$ itself has no fractional packing of value two. As a consequence, $\tau\big(\mathcal{C}|_{e=f}\big) = 2$; for if not, deleting any element of $\mathcal{C}|_{e=f}$ keeps the covering number at least two, so because the minor is ideal, it has a fractional packing of value two, corresponding to a fractional packing of the same value in $\mathcal{C}|_{e=f}$, which cannot be. $\square$

The second ingredient is a straightforward consequence of Lehman's result on minimally non-ideal clutters:

Theorem 4.11 (Lehman [50], see [67, 61])**.** *Let $\mathcal{C}$ be a minimally non-ideal clutter whose covering number is two. Then $\mathcal{C}$ is either a delta or the blocker of an extended odd hole. Moreover, each minimum cover is contained in a member.*

We are now ready to prove the following theorem:

Theorem 1.19 ([4]). *Let $\mathcal{C}$ be an ideal minimally non-packing clutter that has a coexclusive pair (e, f). Then,*

(i) $\mathcal{C}|_{e=f}$ is another ideal minimally non-packing clutter, or

(ii) $\mathcal{C}|_{e=f}$ is not minimally non-packing, and every minimally non-packing minor has covering number two.

Proof. If $\mathcal{C}|_{e=f}$ is not minimally non-packing, then Proposition 4.9 implies that every minimally non-packing minor of it has covering number two, so (ii) holds. We may therefore assume that $\mathcal{C}|_{e=f}$ is minimally non-packing. To prove (i), we need to show that $\mathcal{C}|_{e=f}$ is ideal. Suppose for a contradiction that $\mathcal{C}|_{e=f}$ is non-ideal. Then $\mathcal{C}|_{e=f}$ must be minimally non-ideal, by Theorem 1.16. Since $\mathcal{C}$ is ideal, Proposition 4.10 implies that $\mathcal{C}|_{e=f}$ has covering number two. Thus, by Theorem 4.11, $\mathcal{C}|_{e=f}$ is either a delta or the blocker of an extended odd hole.

Claim. *If $\mathcal{C}|_{e=f} = \Delta_n$ for some integer $n \geq 4$, then $\mathcal{C}$ has a delta minor and is therefore non-ideal.*

Proof of Claim. Recall that $\Delta_n = \big\{\{1,2\},\{1,3\},\ldots,\{1,n\},\{2,3,\ldots,n\}\big\}$. Let us assume that $\mathcal{C}$ has ground set $[n+1]$ and the identified coexclusive pair is either $(1,n+1)$ or $(n,n+1)$. In the first case, $b(\mathcal{C})$ has $\{2,3,\ldots,n\}$ as a member, and since $n\geq 4$, it also has two members among $\{1,2\},\{1,3\},\ldots,\{1,n\}$ or two members among $\{n+1,2\},\{n+1,3\},\ldots,\{n+1,n\}$. Either way, $\{2,3,\ldots,n\}$ is a doubly redundant member of $b(\mathcal{C})$, so $b(\mathcal{C})$, and therefore $\mathcal{C}$, has a delta minor by Theorem 1.3. In the second case, $b(\mathcal{C})$ has $\{1,2\},\{1,3\}$ as members as well as

$$\{2,3,\ldots,n-1,n\} \quad \text{or} \quad \{2,3,\ldots,n-1,n+1\}$$

as a member. But again, both the potential members above are doubly 1-redundant, so $b(\mathcal{C})$, and therefore $\mathcal{C}$, has a delta minor by Theorem 1.3, finishing the proof of the claim. $\diamond$

Since $\mathcal{C}$ is ideal, it therefore follows that $\mathcal{C}|_{e=f}$ is either Δ_3 or the blocker of an extended odd hole. We will prove that $\mathcal{C}$ packs, thereby achieving the desired contradiction. To this end, let us assume that $\mathcal{C}$ has ground set $[2n]$ for some integer $n\geq 2$, that $(e,f)=(1,2n)$ and that $\mathcal{C}|_{1=2n}$ has $\{1,2\},\{2,3\},\ldots,\{2n-2,2n-1\},\{2n-1,1\}$ as its minimum covers.

As $\mathcal{C}$ is ideal, $b(\mathcal{C})$ is also ideal by Theorem 1.7. Notice that $b(\mathcal{C})$ has $\{2,3\},\{3,4\},\ldots,\{2n-2,2n-1\}$ as members. If $b(\mathcal{C})$ has $\{1,2\},\{2n-1,1\}$ as members also, then $b(\mathcal{C})\setminus 2n$ is Δ_3 or an extended odd hole, so by Remark 2.2, $b(\mathcal{C})$ is non-ideal, which is not the case. Thus, $b(\mathcal{C})$ does not simultaneously have $\{1,2\},\{2n-1,1\}$ as members, and by the symmetry between $1,2n$, it does not simultaneously have $\{2n,2\},\{2n-1,2n\}$ as members. After possibly relabeling 1 and $2n$, we may therefore assume that $b(\mathcal{C})$ has $\{1,2\},\{2n-1,2n\}$ as members. To summarize, $\mathcal{C}$ has $\{1,2\},\{2,3\},\ldots,\{2n-1,2n\}$ as minimum covers.

Since $\mathcal{C}$ is ideal, it has a fractional packing $y\in\mathbb{R}^{\mathcal{C}}_+$ of value two. By the complementary slackness conditions for linear programs, whenever $y_C>0$, C must intersect every minimum cover exactly once. However, the only such subsets of $[2n]$ are $\{1,3,\ldots,2n-1\}$ and $\{2,4,\ldots,2n\}$, and since the fractional packing has value two, these two subsets must be members of $\mathcal{C}$. As a result, $\mathcal{C}$ has disjoint members, and as $\tau(\mathcal{C})=\tau\big(\mathcal{C}|_{e=f}\big)=2$, it follows that $\mathcal{C}$ packs, a contradiction. $\square$

4.5 The birth of cuboids

Theorem 1.19 considers ideal minimally non-packing clutters that have a coexclusive pair. In this section, we deal with ideal minimally non-packing clutters without a coexclusive pair and with covering number two. We will need the following consequence of Theorem 1.3:

Proposition 4.12 ([4])**.** *Let $\mathcal{C}$ be a clutter without a delta minor, and take distinct elements e,f,g. If $\{e,f\},\{e,g\}$ are minimal covers, then (f,g) is a coexclusive pair.*

Proof. Assume that $\{e,f\},\{e,g\}$ are minimal covers. If there is a minimal cover B containing f and g, then B is a doubly e-redundant member of $b(\mathcal{C})$ with causes $\{e,f\},\{e,g\}$, so $b(\mathcal{C})$, and therefore $\mathcal{C}$, has a delta minor by Theorem 1.3, which is not the case. Thus, every minimal cover of $\mathcal{C}$ contains at most one of f,g, i.e. (f,g) is coexclusive. $\square$

We will also need the following technical lemma:

Lemma 4.13 ([4])**.** *Take an integer $n \geq 1$ and a clutter $\mathcal{C}$ over ground set $[2n]$, where for each $i \in [n]$, $\{2i-1, 2i\}$ intersects every member exactly once. Then the following statements are equivalent:*

(i) $b(\mathcal{C})$ *is ideal,*

(ii) $\operatorname{conv}\{\chi_C : C \in \mathcal{C}\} = Q(b(\mathcal{C})) \cap \{x : x_{2i-1} + x_{2i} = 1 \ \forall i \in [n]\}$.

Proof. **(i) $\Rightarrow$ (ii)**: Since $\chi_C \in \{x : x_{2i-1} + x_{2i} = 1 \ \forall i \in [n]\}$ for every member C, the inclusion $\subseteq$ holds. Let us prove the reverse inclusion $\supseteq$. Since $b(\mathcal{C})$ is ideal,

$$Q(b(\mathcal{C})) = \operatorname{conv}\{\chi_C : C \in \mathcal{C}\} + \mathbb{R}^{2n}_{+}.$$

It is easy to see that this equation implies the reverse inclusion. **(ii) $\Rightarrow$ (i)**: Let $x^\star$ be an extreme point of $Q(b(\mathcal{C}))$. It suffices to show that $x^\star$ is integral. Since $\{2i-1, 2i\}$ is a cover of $\mathcal{C}$, it contains a member of $b(\mathcal{C})$, so $x^\star_{2i-1} + x^\star_{2i} \geq 1$. Moreover, since $(2i-1, 2i)$ is exclusive in $\mathcal{C}$, it is coexclusive in $b(\mathcal{C})$, so by Theorem 1.13 (iii), $x^\star_{2i-1} + x^\star_{2i} \leq 1$. So for each $i \in [n]$, $x^\star_{2i-1} + x^\star_{2i} = 1$, implying in turn by (ii) that $x^\star \in \operatorname{conv}\{\chi_C : C \in \mathcal{C}\}$. Since $x^\star$ is an extreme point, it must be one of the characteristic vectors and hence integral, as required. $\square$

Take an integer $n \geq 1$. Recall that a cuboid is a clutter whose ground set can be relabeled as $[2n]$ and every member C obeys

$$|C \cap \{2i-1, 2i\}| = 1 \qquad \forall i \in [n].$$

We are ready to prove the following theorem:

Theorem 1.20 ([4])**.** *Let $\mathcal{C}$ be an ideal minimally non-packing clutter over ground set E. If $\mathcal{C}$ is without a coexclusive pair and has covering number two, then the following statements hold:*

(1) the minimum covers partition E,

(2) the minimum cardinality of a member is $\frac{|E|}{2}$,

(3) the members of minimum cardinality form an ideal non-packing cuboid.

Proof. Suppose that $\mathcal{C}$ is without a coexclusive pair and $\tau(\mathcal{C}) = 2$. Since $\mathcal{C}$ is minimally non-packing,

every element e is contained in a minimum cover.

For if not, then $\tau(\mathcal{C} \setminus e) = \tau(\mathcal{C})$, and because $\mathcal{C}$ does not pack, $\mathcal{C} \setminus e$ does not pack either, contradicting the minimality of $\mathcal{C}$. In fact,

Claim 1. *Every element is contained in exactly one minimum cover, i.e. the minimum covers partition E, so (1) holds.*

Proof of Claim. Suppose for a contradiction that an element, say e, is contained in minimum covers $\{e,f\},\{e,g\}$. Since $\mathcal{C}$ is ideal, it does not have a delta minor, so by Proposition 4.12, the pair (f,g) is coexclusive, a contradiction to our hypothesis. $\diamond$

Let's relabel the elements so that $E=[2n]$ and the minimum covers are $\{1,2\},\{3,4\},\ldots,\{2n-1,2n\}$, for some integer $n\geq 1$. In particular, each member has cardinality at least n, as it contains an element from each minimum cover. In fact,

Claim 2. *The minimum cardinality of a member is $n=\frac{|E|}{2}$, so (2) holds.*

Proof of Claim. It suffices to present a member of cardinality n. Well, since $\mathcal{C}$ is ideal, it has a fractional packing $y\in\mathbb{R}^{\mathcal{C}}_{+}$ of value two. Pick a member C for which $y_C>0$. By the complementary slackness conditions for linear programs, C must intersect every minimum cover exactly once, i.e. $|C|=n$. $\diamond$

Now let $\mathcal{C}_0:=\big\{C\in\mathcal{C}:|C|=n\big\}$. Clearly, $\mathcal{C}_0$ is a cuboid. As argued above, every fractional packing of value two picks only members from $\mathcal{C}_0$, so $\tau(\mathcal{C}_0)\geq 2$, and since $2=\tau(\mathcal{C})\geq\tau(\mathcal{C}_0)$, we get that $\tau(\mathcal{C}_0)=2$. (In fact, by strict complementarity, $\mathcal{C}_0$ consists precisely of the members that are used in at least one fractional packing of value two.) Moreover, as $\mathcal{C}$ does not have two disjoint members, neither does $\mathcal{C}_0$, so $\mathcal{C}_0$ does not pack. In fact,

Claim 3. *$\mathcal{C}_0$ is an ideal non-packing cuboid, so (3) holds.*

Proof of Claim. We just showed above that $\mathcal{C}_0$ is a non-packing cuboid. It remains to show that $\mathcal{C}_0$ is ideal. By Theorem 1.7 and Lemma 4.13, it suffices to show that

$$\mathrm{conv}\big\{\chi_C: C\in\mathcal{C}_0\big\}=Q\big(b(\mathcal{C}_0)\big)\cap\big\{x: x_{2i-1}+x_{2i}=1\ \forall i\in[n]\big\}.$$

The inclusion $\subseteq$ holds clearly. To prove the reverse inclusion $\supseteq$, pick a point $x^\star$ in the set on the right-hand side. As $Q\big(b(\mathcal{C}_0)\big)\subseteq Q\big(b(\mathcal{C})\big)$, we have that $x^\star\in Q\big(b(\mathcal{C})\big)$. Since $\mathcal{C}$ is ideal, so is $b(\mathcal{C})$ by Theorem 1.7, implying that for some $\lambda\in\mathbb{R}^{\mathcal{C}}_{+}$ with $\sum_{C\in\mathcal{C}}\lambda_C=1$,

$$x^\star\geq\sum_{C\in\mathcal{C}}\lambda_C\chi_C.$$

Since for each $i\in[n]$, $x^\star_{2i-1}+x^\star_{2i}=1$ and $\{2i-1,2i\}$ is a cover of $\mathcal{C}$, equality must hold above, and for all $C\in\mathcal{C}-\mathcal{C}_0$, $\lambda_C=0$. Hence, $x^\star\in\mathrm{conv}\big\{\chi_C: C\in\mathcal{C}_0\big\}$, thereby proving the claim. $\diamond$

Claims 1, 2 and 3 finish the proof of Theorem 1.20. $\square$

4.6 Further notes

We showed in Theorem 1.15 that splitting preserves idealness and saw examples of non-ideal clutters with an ideal split. Can we always hope for an ideal split?

Theorem 4.14 ([5])**.** *If a clutter has an ideal split, then every minimally non-ideal minor is a delta.*

Thus, we cannot always hope for an ideal split. Nonetheless, there is another interesting example of a non-ideal clutter with an ideal split. Let $P = (V, \leq)$ be a partially ordered set. The associated *comparability graph* is an (undirected) graph over vertex set V, where distinct vertices u, v are neighbors if $u \leq v$ or $v \leq u$.

Theorem 4.15 ([5]). *The clutter of edges of a comparability graph splits to the clutter of edges of a bipartite graph.*

How? Let $G = (V, E)$ be a comparability graph, associated with the partially ordered set $P = (V, \leq)$. If G is triangle-free, then it is already bipartite. Otherwise, G has a triangle, so there are vertices u, v, w such that $u < v < w$. Let G' be the graph obtained from $G \setminus v$ by introducing two new vertices v_1 and v_2, where v_1 has neighbors $\{x \in V : x < v\}$ and v_2 has neighbors $\{y \in V : v < y\}$. Then the clutter of edges of G splits to the clutter of edges of G', another comparability graph with fewer triangles.[1] Repeating this procedure eventually yields the bipartite graph promised above.

A common extension of splitting, duplication and replication

Let $\mathcal{C}$ be a clutter over ground set E, and take an element e. Denote by

$$\mathrm{port}(\mathcal{C}, e) := \{C \in \mathcal{C} : e \in C\}.$$

An *e-fragmentation* of $\mathcal{C}$ consists of a partition of $\mathrm{port}(\mathcal{C}, e)$ into nonempty parts so that for all members C, C' from different parts, $(C \cup C') - \{e\}$ contains another member. Thus, the partition of $\mathrm{port}(\mathcal{C}, e)$ into just one part is a valid e-fragmentation.

Remark 4.16. *Take a clutter $\mathcal{C}$ and an element e. Let $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_k$ be an e-fragmentation, and let $\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_\ell$ be another one. Then $\bigcup (\mathcal{F}_i \cap \mathcal{G}_j : i \in [k], j \in [\ell])$ is another e-fragmentation.*

As a result, there is a unique e-fragmentation $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_k$ with the maximum number of parts. We will refer to each $\mathcal{F}_i$ as an *e-fragment of* $\mathcal{C}$.

Let $\mathcal{F}, \mathcal{F}'$ be possibly empty subsets of $\mathrm{port}(\mathcal{C}, e)$, each of which is the union of some e-fragments. Introduce a new element f. Let $\mathcal{L}$ be the clutter over ground set $E \cup \{f\}$ whose members are the minimal sets of

$$\big\{C \cup \{f\} : C \in \mathrm{port}(\mathcal{C}, e)\big\} \cup \big\{C : C \in \mathcal{F}\big\} \cup \big\{C \triangle \{e, f\} : C \in \mathcal{F}'\big\} \cup \big\{C : e \notin C \in \mathcal{C}\big\}.$$

We say that $\mathcal{L}$ is obtained from $\mathcal{C}$ after *cleaving* e. Notice that,

- if $\mathcal{F}, \mathcal{F}'$ partition $\mathrm{port}(\mathcal{C}, e)$, then cleaving is equivalent to splitting, by Remark 4.3,
- if $\mathcal{F} = \mathcal{F}' = \emptyset$, then cleaving is equivalent to duplication, and
- if $\mathcal{F} = \mathcal{F}' = \mathrm{port}(\mathcal{C}, e)$, then cleaving is equivalent to replication.

As a result, cleaving may be regarded as a common generalization of splitting, duplication and replication. Generalizing Theorem 1.15, we conjecture that,

[1] In effect, our split destroyed a star cutset in the complement of G.

Conjecture 4.17. *Cleaving preserves idealness.*

We also conjecture that,

Conjecture 4.18. *Cleaving preserves the packing property.*

This conjecture holds for splitting by Theorem 1.17, and can be easily shown to hold for duplication. However, a major open question since 1993 has been to settle it for replication:

The Replication Conjecture (Conforti and Cornuéjols [19])**.** *If a clutter has the packing property, then so does any replication of it.*

In fact, we conjecture that

Conjecture 4.19. *Conjecture 4.18 is equivalent to the Replication Conjecture.*

Bertrand Guenin and I believe to have shown Conjectures 4.17 and 4.19 to be true. However, since our proofs are not publicly available, we refrain from calling these conjectures theorems.

Chapter 5

Cuboids

Take an integer $n \geq 1$. Recall that a cuboid is a clutter whose ground set can be relabeled as $[2n]$ where every member C satisfies

$$|C \cap \{2i-1, 2i\}| = 1 \quad \forall i \in [n].$$

Lemma 5.1 ([2]). *The incidence matrix of a minimally non-ideal clutter does not have complementary columns. In particular, cuboids cannot be minimally non-ideal.*

Proof. Suppose for a contradiction that the incidence matrix of $\mathcal{C}$, a minimally non-ideal clutter, has complementary columns. That is, there are distinct elements e, f such that

$$|C \cap \{e, f\}| = 1 \quad \forall C \in \mathcal{C}.$$

In particular, $\mathcal{C}$ has covering number two and $\{e, f\}$ is a minimum cover. It follows from Theorem 4.11 that $\{e, f\}$ is contained in a member, contradicting the equation above. □

As an immediate consequence,

Theorem 1.38 ([2])**.** *A minimally non-packing cuboid is ideal.*

Proof. A minimally non-packing clutter is either ideal or minimally non-ideal, by Theorem 1.16. As cuboids cannot be minimally non-ideal by Lemma 5.1, minimally non-packing cuboids must be ideal. □

In this chapter, we study ideal cuboids as well as cuboids with the packing property. Our study reveals

- a geometric rift between idealness and the packing property,

- the equivalence of cube-ideal binary spaces and binary matroids with the sums of circuits property,
- that the geometry supports the $\tau = 2$ Conjecture,
- that there are at least 716 ideal minimally non-packing cuboids over at most 14 elements,
- a local and global structure theorem for ideal minimally non-packing cuboids of bounded degree and maximum dimension.

In particular, two characterizations of ideal cuboids are provided in §5.1, and a characterization of cube-ideal binary spaces is given in §5.2. In §5.3, we characterize cuboids with the packing property, and show that the $\tau = 2$ Conjecture is equivalent to the Polarity Conjecture. In §5.4, we characterize minimally non-packing cuboids, and in §5.5, we study minimally non-packing cuboids of bounded degree and describe a pseudocode for generating hundreds of such cuboids.

5.1 Cube-idealness

Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Recall that S is cube-ideal if $\mathrm{conv}(S)$, its convex hull, is described by hypercube inequalities $0 \leq x_i \leq 1, i \in [n]$ as well as generalized set covering inequalities:

$$\sum_{i \in I} x_i + \sum_{j \in J} (1 - x_j) \geq 1 \quad I, J \subseteq [n], I \cap J = \emptyset.$$

Recall further that $\mathrm{cuboid}(S)$ is the clutter over ground set $[2n]$ whose members have incidence vectors $(x_1, 1-x_1, \ldots, x_n, 1-x_n), x \in S$; notice that every cuboid is obtained in this manner.

Theorem 1.21 **([2]).** *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then* $\mathrm{cuboid}(S)$ *is ideal if, and only if, S is cube-ideal.*

Proof. Let $\mathcal{C} := \mathrm{cuboid}(S)$. By Theorem 1.7, it suffices to show that $b(\mathcal{C})$ is ideal if, and only if, S is cube-ideal. For each $i \in [n]$, $\{2i-1, 2i\}$ intersects every member exactly once, so we may apply Lemma 4.13 from the previous chapter partially characterizing when $b(\mathcal{C})$ ideal. ($\Leftarrow$) Assume that $b(\mathcal{C})$ is ideal. Then by Lemma 4.13,

$$\mathrm{conv}\big\{\chi_C : C \in \mathcal{C}\big\} = Q\big(b(\mathcal{C})\big) \cap \big\{x : x_{2i-1} + x_{2i} = 1 \ \forall i \in [n]\big\}.$$

By projecting away the even coordinates, we get that

$$\mathrm{conv}(S) = \left\{y \in [0,1]^n : \sum \big(y_i : 2i-1 \in B\big) + \sum \big(1 - y_j : 2j \in B\big) \geq 1 \quad \forall B \in b(\mathcal{C})\right\}.$$

As a result, S is cube-ideal. ($\Rightarrow$) Assume conversely that S is cube-ideal. Then

$$\mathrm{conv}(S) = \left\{y \in [0,1]^n : \sum \big(y_i : i \in I\big) + \sum \big(1 - y_j : j \in J\big) \geq 1 \quad \forall \ (I,J) \in \mathcal{V}\right\},$$

for some appropriate set $\mathcal{V}$. We may assume that for each $(I,J)\in\mathcal{V}$, $I\cap J=\emptyset$. After the change of variables $y_i\mapsto x_{2i-1}$ and $1-y_i\mapsto x_{2i}$ to the equation above, we get that

$$\operatorname{conv}\{\chi_C : C\in\mathcal{C}\}=\left\{x\in\mathbb{R}^{2n}_+ : \begin{array}{l}\sum\left(x_{2i-1}:i\in I\right)+\sum\left(x_{2j}:j\in J\right)\geq 1\quad\forall\,(I,J)\in\mathcal{V}\\ x_{2i-1}+x_{2i}=1\quad\forall\, i\in[n]\end{array}\right\}.$$

Together with Lemma 4.13, this equation implies that $b(\mathcal{C})$ is an ideal clutter, as required. □

Recall that the induced clutter of S with respect to a point $x\in\{0,1\}^n$ has ground set $[n]$ and members

$$\operatorname{ind}(S\triangle x)=\text{the minimal sets of }\{C\subseteq[n]:\chi_C\in S\triangle x\}.$$

Remark 5.2 ([2]). *Take an integer $n\geq 1$ and a set $S\subseteq\{0,1\}^n$. The induced clutters $\operatorname{ind}(S\triangle x), x\in\{0,1\}^n$ are in bijection with the minors of $\operatorname{cuboid}(S)$ obtained after contracting, for each $i\in[n]$, exactly one of $2i-1,2i$.*

This easy remark, together with Lemma 5.1, has the following consequence:

Theorem 1.22 ([2]). *Take an integer $n\geq 1$ and a set $S\subseteq\{0,1\}^n$. Then S is cube-ideal if, and only if, every induced clutter of S is ideal.*

Proof. Let $\mathcal{C}:=\operatorname{cuboid}(S)$. By Theorem 1.21, it suffices to show that $\mathcal{C}$ is ideal if, and only if, every induced clutter of S is ideal. ($\Rightarrow$) Assume that $\mathcal{C}$ is ideal. Then all of its minors are ideal, so by Remark 5.2, the induced clutters of S are ideal. ($\Leftarrow$) Assume that $\mathcal{C}$ is non-ideal. Pick disjoint $I,J\subseteq[2n]$ such that the minor $\mathcal{C}':=\mathcal{C}\setminus I/J$ is minimally non-ideal. Clearly, $\tau(\mathcal{C}')\geq 2$ and by Lemma 5.1, $M(\mathcal{C}')$ does not have complementary columns; these facts imply that for each $i\in[n]$,

- if $I\cap\{2i-1,2i\}\neq\emptyset$ then $J\cap\{2i-1,2i\}\neq\emptyset$, and so
- $J\cap\{2i-1,2i\}\neq\emptyset$.

By Remark 5.2, the latter implies that $\mathcal{C}'$ is a minor of an induced clutter of S, implying in turn that an induced clutter of S is non-ideal, as required. □

5.2 The sums of circuits property

Take an integer $n\geq 1$ and a set $S\subseteq\{0,1\}^n$. Recall that S is a binary space if the symmetric difference of any two feasible points, that are possibly equal, is also feasible. In this section, we characterize when a binary space is cube-ideal.

Recall that S is an affine binary space if the symmetric difference of any odd number of feasible points is also feasible. Notice that affine binary spaces are nothing but twists of binary spaces. Basic Linear Algebra implies that S is an affine binary space if, and only if,

$$S = \{x \in \{0,1\}^n : Ax \equiv b \pmod 2\}$$

for a $0,1$ matrix A and a $0,1$ vector b of appropriate dimensions. Notice further that S is a binary space if, and only if, $b = \mathbf{0}$. We will need the following routine application of the Gaussian elimination method:

Lemma 5.3 (folklore)**.** *Take integers $m, n \geq 1$, an $m \times n$ matrix A, and an m-dimensional vector b with $0,1$ entries. If the system $Ax \equiv b \pmod 2$ does not have a solution in $\{0,1\}^n$, then there exists a vector $c \in \{0,1\}^m$ such that $c^\top A \equiv \mathbf{0}$ and $c^\top b \equiv 1 \pmod 2$.*[1]

Moving forward, assume that S is a binary space represented as

$$S = \{x \in \{0,1\}^n : Ax \equiv \mathbf{0} \pmod 2\}.$$

By definition, S is the cycle space of a binary matroid M (see [60]). We refer to M as the *associated* binary matroid. The cocycle space of M is precisely the binary space generated by the rows of A ([60], Proposition 9.2.2).

Theorem 5.4 ([2])**.** *Take an integer $n \geq 1$ and a binary space $S \subseteq \{0,1\}^n$, and let M be the associated binary matroid. Then S is cube-ideal if, and only if,*

$$\mathrm{conv}(S) = \{x \in [0,1]^n : x(F) - x(D - F) \leq |F| - 1 \quad \forall \textit{ cocycles } D \textit{ and odd subsets } F \subseteq D\}.$$

Proof. ($\Leftarrow$) Notice that each inequality $x(F) - x(D-F) \leq |F| - 1$ can be rewritten as

$$\sum_{i \in D-F} x_i + \sum_{j \in F} (1 - x_j) \geq 1,$$

which is a generalized set covering inequality. Thus, S is cube-ideal. ($\Rightarrow$) Suppose conversely that S is cube-ideal. We first prove that

$$\mathrm{conv}(S) \subseteq \{x \in [0,1]^n : x(F) - x(D - F) \leq |F| - 1 \quad \forall \text{ cocycles } D \text{ and odd subsets } F \subseteq D\}.$$

Denote by P the polytope on the right-hand side. To prove this inclusion, it suffices to show that for every cycle C, χ_C belongs to P. Well, for every cocycle D and odd subset $F \subseteq D$, we have $C \cap D \neq F$ because $|C \cap D|$ is even, so if $F \subseteq C$ then $C \cap (D - F) \neq \emptyset$, implying in turn that

$$\chi_C(F) - \chi_C(D - F) \leq |F| - 1.$$

Thus, $\chi_C \in P$. To prove the reverse inclusion, it suffices to show that every inequality defining $\mathrm{conv}(S)$ is valid for P. Since S is cube-ideal, $\mathrm{conv}(S)$ is described by hypercube inequalities – which are valid for P – and by generalized set covering inequalities. Take disjoint subsets $I, J \subseteq [n]$ such that $\sum_{i \in I} x_i + \sum_{j \in J}(1 - x_j) \geq 1$ is a defining inequality of $\mathrm{conv}(S)$.

[1] This lemma may be viewed as the Farkas lemma for binary spaces. It is essentially equivalent to the following result proved by Lehman ([48], (44)): given a binary matroid M and element e, the two clutters $\{C - \{e\} : C \text{ is a circuit}, e \in C\}$ and $\{D - \{e\} : C \text{ is a cocircuit}, e \in D\}$ are blockers of one another.

Claim. *There is a cocycle D such that $D \subseteq I \cup J$ and $|D \cap J|$ is odd.*

Proof of Claim. To see this, write

$$S = \{x \in \{0,1\}^n : Ax \equiv \mathbf{0} \pmod 2\}$$

for some $0,1$ matrix A. Let d be the sum of the columns in J of A, and let B be the submatrix of A obtained after dropping columns $I \cup J$. Since $\sum_{i\in I} x_i + \sum_{j\in J}(1 - x_j) \geq 1$ is valid for every point of S, the system

$$By \equiv d \pmod 2$$

has no $0,1$ solution. (For if y is a solution, then by setting the coordinates in I to 0 and the coordinates in J to 1, we can extend y to a feasible point x of S for which $\sum_{i\in I} x_i + \sum_{j\in J}(1 - x_j) = 0$, which is not the case.) By Lemma 5.3, there is a $0,1$ vector c such that

$$c^\top B \equiv \mathbf{0} \quad \text{and} \quad c^\top d \equiv 1 \pmod 2.$$

Consider the cocycle $D \subseteq [n]$ for which $\chi_D = c^\top A$. Then the first equation implies that $D \subseteq I \cup J$, while the second equation implies that $|D \cap J|$ is odd, as required. ◇

Let $F := D \cap J$. Then F is an odd subset of the cocycle D. Observe that the inequality

$$\sum_{i\in I} x_i + \sum_{j\in J}(1 - x_j) \geq 1$$

is dominated by the inequality

$$\sum_{i\in D-F} x_i + \sum_{j\in F}(1 - x_j) \geq 1$$

because $D - F \subseteq I$ and $F \subseteq J$. However, the inequality above is equivalent to $x(F) - x(D - F) \leq |F| - 1$, so it is valid for P. As a result, every inequality defining $\mathrm{conv}(S)$ is valid for P, so $\mathrm{conv}(S) \supseteq P$. Hence, $\mathrm{conv}(S) = P$, thereby finishing the proof. □

Since S is a binary space, $S \triangle x = S$ for every feasible point x. Taking advantage of this transitive property of binary spaces, Barahona and Grötschel proved that to describe the facets of $\mathrm{conv}(S)$, it suffices to have a facet description of the polyhedral cone generated by S:

$$\mathrm{cone}(S) := \left\{ \sum_{x\in S} \alpha_x x : \alpha \in \mathbb{R}^S_+ \right\} \subseteq \{0,1\}^n.$$

Theorem 5.5 (Barahona and Grötschel [12])**.** *Take an integer $n \geq 1$ and a binary space $S \subseteq \{0,1\}^n$. Then*

$$\mathrm{conv}(S) = \{x \in [0,1]^n : x(F) - x(D - F) \leq |F| - 1 \quad \forall \textit{ cocycles } D \textit{ and odd subsets } F \subseteq D\} \tag{1}$$

if, and only if,

$$\mathrm{cone}(S) = \{x \in \mathbb{R}^n_+ : x_f - x(D - \{f\}) \leq 0 \quad \forall \textit{ cocycles } D \textit{ and } f \in D\}. \tag{2}$$

Proof. ($\Rightarrow$) Suppose that (1) holds. As $\mathbf{0} \in S$, the facets of $\mathrm{conv}(S)$ tight at $\mathbf{0}$ describe the conic hull of S. Since the cocycle inequality

$$x(F) - x(D - F) \leq |F| - 1 \quad \text{cocycle } D \text{ and odd subset } F \subseteq D$$

is tight at $\mathbf{0}$ if and only if $|F| = 1$, (2) holds. ($\Leftarrow$) Conversely, suppose that (2) holds. To prove that (1) holds, let

$$\sum_{i \in I} a_i x_i + \sum_{j \in [n]-I} a_j(1 - x_j) \geq b \qquad a \in \mathbb{R}^n_+, b \in \mathbb{R} \tag{$\diamond$}$$

be a facet-defining inequality for $\mathrm{conv}(S)$. It suffices to show that ($\diamond$) is equivalent to a cocycle inequality. To this end, take a point $u \in S$ that lies on this facet. Consider the change of variables $x_i \mapsto 1 - x_i$ for the indices in $\{i \in [n] : u_i = 1\}$; this mapping sends the above inequality to the inequality

$$\sum_{i \in I: u_i = 0} a_i x_i + \sum_{i \in I: u_i = 1} a_i(1 - x_i) + \sum_{j \in [n]-I: u_j = 0} a_j(1 - x_j) + \sum_{j \in [n]-I: u_j = 1} a_j x_j \geq b, \tag{$\star$}$$

and the set S to the set $S \triangle u = \{x \triangle u : x \in S\}$. Then ($\star$) is a facet-defining inequality for $S \triangle u$ and the facet contains the point $\mathbf{0} = u \triangle u \in S \triangle u$. Hence, ($\star$) also defines a facet for $\mathrm{cone}(S \triangle u)$. However, since S is a binary space, $S \triangle u$ is just the original set S, so ($\star$) defines a facet of $\mathrm{cone}(S)$. By (2), there is a cocycle $D \subseteq [n]$ and an element $f \in D$ such that ($\star$) is equivalent to the inequality

$$x_f - x(D - \{f\}) \leq 0.$$

Take the cycle $C \subseteq [n]$ such that $u = \chi_C$. Reverting back the change of variables, we see that ($\diamond$) is equivalent to

$$x(F) - x(D - F) \leq |F| - 1$$

where $F = (C \cap D) \triangle \{f\}$. Since $|C \cap D|$ is even, it follows that $|F|$ is odd, so ($\diamond$) is equivalent to a cocycle inequality, as required. ☐

Recall that the binary matroid M associated with S has the sums of circuits property if for all $w \in \mathbb{R}^n_+$ satisfying

$$w(D - \{f\}) \geq w_f \quad \forall \text{ cocycles } D \text{ and } f \in D,$$

there exists for each circuit C an assignment $y_C \in \mathbb{R}_+$ such that

$$w = \sum \left(y_C \cdot \chi_C : C \text{ is a circuit} \right).$$

We are now ready to prove the following:

Theorem 1.26 ([2])**.** *Take an integer $n \geq 1$, a binary space $S \subseteq \{0,1\}^n$, and let M be the corresponding binary matroid. Then S is cube-ideal if, and only if, M has the sums of circuits property.*

Proof. (⇒) Suppose that S is cube-ideal. Then by Theorems 5.4 and 5.5,

$$\operatorname{cone}(S) = \big\{x \in \mathbb{R}^n_+ : x_f - x(D - \{f\}) \le 0 \quad \forall \text{ cocycles } D \text{ and } f \in D\big\}.$$

Now pick weights $w \in \mathbb{R}^n_+$ satisfying

$$w(D - \{f\}) \ge w_f \quad \forall \text{ cocycles } D \text{ and } f \in D,$$

that is, $w \in \operatorname{cone}(S)$. Thus, there exists $\alpha \in \mathbb{R}^S_+$ such that $w = \sum_{x \in S} \alpha_x x$. As S is the cycle space of M, we may view α as a vector assigning nonnegative values to each cycle, so

$$w = \sum \left(\alpha_C \cdot \chi_C : C \text{ is a cycle}\right).$$

Since every nonempty cycle is the disjoint union of some circuits, the assignment α naturally leads to an assignment y of nonnegative values to the circuits such that

$$w = \sum \left(y_C \cdot \chi_C : C \text{ is a circuit}\right).$$

Thus, M has the sums of circuits property. (⇐) Suppose conversely that M has the sums of circuits property. By Theorems 5.4 and 5.5, it suffices to show that

$$\operatorname{cone}(S) = \big\{x \in \mathbb{R}^n_+ : x_f - x(D - \{f\}) \le 0 \quad \forall \text{ cocycles } D \text{ and } f \in D\big\}. \tag{3}$$

(⊆) Observe that for every cycle C,

$$C \cap D \neq \{f\} \quad \forall \text{ cocycles } D \text{ and } f \in D,$$

so χ_C belongs to the set on the right-hand side of (3). Thus, S belongs to the set on the right-hand side of (3), implying the inclusion ⊆. (⊇) For the reverse inclusion, pick a point $x \in \mathbb{R}^n_+$ such that

$$x_f - x(D - \{f\}) \le 0 \quad \forall \text{ cocycles } D \text{ and } f \in D.$$

As M has the sums of circuits property, there exists for each circuit C an assignment $y_C \in \mathbb{R}_+$ such that

$$x = \sum \left(y_C \cdot \chi_C : C \text{ is a circuit}\right),$$

so $x \in \operatorname{cone}(S)$, as required. □

5.3 Strict polarity

Let $\mathcal{P}$ be a minor-closed property defined on clutters. Recall that $\mathcal{P}$ is a 2-local property if for all integers $n \ge 1$ and sets $S \subseteq \{0,1\}^n$, cuboid(S) has property $\mathcal{P}$ if, and only if, every induced clutter of S has property $\mathcal{P}$. As a consequence of Theorems 1.21 and 1.22, idealness is a 2-local property.

What about the packing property?

Consider the set $R_{1,1} = \{000, 110, 101, 011\}$. Its cuboid is Q_6, which does not pack, while its induced clutters are equal to $\{\emptyset\}$ or $\big\{\{1\},\{2\},\{3\}\big\}$, both of which have the packing property. Thus, in contrast to idealness, the packing property is non-2-local. Let us extract the non-2-local essence of the packing property.

Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Recall that S is polar if either there are antipodal feasible points or the feasible points all agree on a coordinate. Recall further that S is strictly polar if every restriction of it, including S itself, is polar.

Remark 5.6 ([2]). *Take an integer $n \geq 1$ and a strictly polar set $S \subseteq \{0,1\}^n$. Then every minor of S is polar.*

Proof. Let S' be a minor of S. As twisting and relabeling clearly preserves polarity, we may assume that only restrictions and projections are applied to obtain S'. Pick disjoint sets $I, J, K \subseteq [n]$ such that S' is obtained after 0-restricting I, 1-restricting J and projecting away K; among all possible I, J, K we may assume that K is minimal, so that no single projection can be replaced by a single restriction. Let R be the restriction of S obtained after 0-restricting I and 1-restricting J; notice that S' is obtained from R after projecting away K. Since S is strictly polar, it follows from the definition that R is polar. If R contains antipodal points, then the same points give antipodal points in the projection S'. Otherwise, the points in R agree on a coordinate, so by the minimality of K, the points in the projection S' also agree on the same coordinate. Either way, we see that S' is polar, as required. □

Set minor operations can be defined directly in terms of cuboids:

Remark 5.7 ([4]). *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then, for each $i \in [n]$, the following statements hold:*

- *If S' is the 0-restriction of S over i, then* $\mathrm{cuboid}(S') = \mathrm{cuboid}(S) \setminus (2i-1)/2i$.
- *If S' is the 1-restriction of S over i, then* $\mathrm{cuboid}(S') = \mathrm{cuboid}(S)/(2i-1) \setminus 2i$.
- *If S' is the projection of S over i, then* $\mathrm{cuboid}(S') = \mathrm{cuboid}(S)/\{2i-1, 2i\}$.

If S' is a minor of S, then we say that $\mathrm{cuboid}(S')$ is a *cuboid minor* of $\mathrm{cuboid}(S)$. We are now ready for the following characterization of strict polarity:

Proposition 5.8 ([2]). *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. The following statements are equivalent: (i) S is strictly polar, (ii) every cuboid minor of* $\mathrm{cuboid}(S)$ *packs, (iii) every minor of* $\mathrm{cuboid}(S)$ *with covering number at least two has two disjoint members.*

Proof. **(i) ⇒ (ii)**: Since S is strictly polar, Remark 5.6 implies that every minor of S is polar, so every cuboid minor of $\mathrm{cuboid}(S)$ packs. **(ii) ⇒ (iii)**: Let $\mathcal{C}$ be a minor of $\mathrm{cuboid}(S)$ such that $\tau(\mathcal{C}) \geq 2$ and every element of $\mathcal{C}$ is contained in a member. We need to show that $\mathcal{C}$ has two disjoint members. To this end, pick disjoint $I, J \subseteq [2n]$ such that $\mathrm{cuboid}(S) \setminus I/J = \mathcal{C}$. As $\tau(\mathcal{C}) \geq 2$, for each $i \in [n]$ such that $I \cap \{2i-1, 2i\} \neq \emptyset$, we must have that $J \cap \{2i-1, 2i\} \neq \emptyset$. Let $\mathcal{C}'$ be the cuboid minor of $\mathrm{cuboid}(S)$ obtained after deleting I and contracting $\{2j-1 : j \in [n], 2j \in I\} \cup \{2j : j \in [n], 2j-1 \in I\} \subseteq J$. By (ii), the cuboid $\mathcal{C}'$ packs. Since $\tau(\mathcal{C}) \geq 2$ and every element of $\mathcal{C}$ is contained in a member, we see that $\tau(\mathcal{C}') = 2$, implying in turn that $\mathcal{C}'$ has two disjoint members. Since $\mathcal{C}$ is a contraction minor of $\mathcal{C}'$, we get that $\mathcal{C}$ has two disjoint members, too. **(iii) ⇒ (i)**: In particular, every cuboid minor of $\mathrm{cuboid}(S)$ packs, so every minor of S is polar, implying in turn that S is strictly polar. □

We are now ready to prove the following theorem:

Theorem 1.35 **([2]).** *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then* $\text{cuboid}(S)$ *has the packing property if, and only if, S is strictly polar and every induced clutter of S has the packing property.*

Proof. ($\Rightarrow$) Suppose that $\text{cuboid}(S)$ has the packing property. Then every minor of $\text{cuboid}(S)$ has the packing property. Thus, by Remark 5.2, every induced clutter of S has the packing property. Moreover, every cuboid minor of $\text{cuboid}(S)$ packs, so every restriction of S is polar, implying in turn that S is strictly polar.

($\Leftarrow$) Suppose that S is strictly polar and every induced clutter of S has the packing property. Assume for a contradiction that $\text{cuboid}(S)$ does not have the packing property. Let $\mathcal{C}$ be a non-packing minor of $\text{cuboid}(S)$. As S is strictly polar, it follows from Proposition 5.8 that $\tau(\mathcal{C}) \geq 3$. Pick disjoint subsets $I, J \subseteq [2n]$ such that $\mathcal{C} = \text{cuboid}(S) \setminus I/J$. Since $\tau(\mathcal{C}) \geq 3$, it follows that for each $i \in [n]$, $J \cap \{2i-1, 2i\} \neq \emptyset$. Hence, $\mathcal{C}$ is a minor of an induced clutter of S by Remark 5.2, implying in turn that an induced clutter of S does not have the packing property, a contradiction. $\square$

As a consequence, once strict polarity is enforced, the packing property becomes 2-local, just like idealness. We conjecture that strict polarity does far more than that:

The Polarity Conjecture ([2])**.** *Take an integer $n \geq 1$ and a strictly polar set $S \subseteq \{0,1\}^n$. Then* $\text{cuboid}(S)$ *has the packing property if, and only if,* $\text{cuboid}(S)$ *is ideal.*

Since the packing property implies idealness by Theorem 1.16, and since cuboid idealness is equivalent to cube-idealness by Theorem 1.21, we may rephrase the Polarity Conjecture as follows:

The Polarity Conjecture (rephrased)**.** *If a set is cube-ideal and strictly polar, then its cuboid has the packing property.*

We will prove that that this conjecture is equivalent to the following conjecture:

The $\tau = 2$ Conjecture ([24])**.** *Every ideal minimally non-packing clutter has covering number two.*

To this end, we need the following proposition:

Proposition 5.9 ([2]). *Let $\mathcal{C}$ be a clutter over ground set E, where every minor with covering number at least two has two disjoint members. Let*

$$S := \{\chi_C : C \subseteq E \text{ contains a member}\} \subseteq \{0,1\}^E.$$

Then S is strictly polar.

Proof. Take disjoint $I, J \subseteq E$. Let $S' \subseteq \{0,1\}^{E-(I\cup J)}$ be obtained from S after 0-restricting the coordinates I and 1-restricting the coordinates J. It suffices to show that S' is polar. Notice that

$$S' = \left\{\chi_{C'} : C' \subseteq E - (I \cup J) \text{ contains a member of } \mathcal{C} \setminus I/J\right\} \subseteq \{0,1\}^{E-(I\cup J)}.$$

By assumption, either $\mathcal{C} \setminus I/J$ has a cover of cardinality one, or two disjoint members. This implies that either the points in S' all agree on a coordinate, or S' contains antipodal points. Hence, S' is polar, as required. $\square$

We are now ready to prove the promised equivalence:

Theorem 1.36 ([2]). *The Polarity Conjecture is equivalent to the $\tau = 2$ Conjecture.*

Proof. Assume first that the $\tau = 2$ Conjecture is true, that is, every ideal minimally non-packing clutter has covering number two. Take an integer $n \geq 1$ and a cube-ideal, strictly polar set $S \subseteq \{0,1\}^n$. Take an induced clutter $\mathcal{C}$ of S. Since S is cube-ideal, it follows from Theorem 1.22 that $\mathcal{C}$ is ideal. Since S is strictly polar, it follows from Proposition 5.8 that every minor of $\mathcal{C}$ with covering number at least two has two disjoint members. Thus, since the $\tau = 2$ Conjecture is true, it follows that $\mathcal{C}$ does not have a minimally non-packing minor, so $\mathcal{C}$ has the packing property. As a result, every induced clutter of S has the packing property, and as S is strictly polar, it follows from Theorem 1.35 that cuboid(S) has the packing property. Hence, the Polarity Conjecture is true.

Assume conversely that the $\tau = 2$ Conjecture is false, that is, there is an ideal minimally non-packing clutter $\mathcal{C}$ such that $\tau(\mathcal{C}) \geq 3$. Then every proper minor of $\mathcal{C}$ packs. Moreover, for an arbitrary element e, $\tau(\mathcal{C} \setminus e) \geq 2$, so $\mathcal{C} \setminus e$, and therefore $\mathcal{C}$, has two disjoint members. Thus, every minor of $\mathcal{C}$ with covering number at least two has two disjoint members. Given that E is the ground set of $\mathcal{C}$, let

$$S := \{\chi_C : C \subseteq E \text{ contains a member}\} \subseteq \{0,1\}^E.$$

It then follows from Theorem 1.25 and Proposition 5.9 that S is cube-ideal and strictly polar. Since $\mathcal{C} = \text{ind}(S)$, $\mathcal{C}$ is a minor of cuboid(S) by Remark 5.2, so cuboid(S) does not have the packing property. Hence, the Polarity Conjecture is false. $\square$

5.4 Strict non-polarity

Recall that a set is strictly non-polar if it is non-polar but every proper restriction is polar. Clearly, a set is strictly polar if and only if it has no strictly non-polar restriction.

Lemma 5.10 ([2]). *Take an integer $n \geq 3$ and a strictly non-polar set $S \subseteq \{0,1\}^n$. Then there exist distinct points $a, b \in S$ such that for $I := \{i \in [n] : a_i = b_i\}$ the following statement holds: for every $x \in S$, either $x_i = a_i$ for all $i \in I$, or $x_i = 1 - a_i$ for all $i \in I$.*

Proof. Consider the incidence matrix $M(\text{cuboid}(S))$, whose column labels are $[2n]$. After possibly relabeling and twisting the elements of S, we may assume that

(1) among all the columns in $M(\text{cuboid}(S))$, column 1 has the maximum number of zeros, and

(2) for each $j \in \{2, \ldots, n\}$, there is a point $x \in S$ such that $x_1 = 0$ and $x_j = 0$.

Let $I \subseteq [n]$ be the set of coordinates i such that $S \subseteq \{x \in \{0,1\}^n : x_i = x_1\}$. Notice that $1 \in I$, and since S is not polar, $I \neq [n]$. Let $S' \subseteq \{0,1\}^{[n]-I}$ be obtained from S after 0-restricting the coordinates in I. As S is strictly non-polar, and $I \neq \emptyset$, it follows that S' is polar.

Claim. *S' has antipodal points.*

Proof of Claim. Suppose otherwise. Since S' is polar, there exist an $a \in \{0,1\}$ and a coordinate $j \in [n] - I$ such that
$$S' \subseteq \left\{ y \in \{0,1\}^{[n]-I} : y_j = a \right\}.$$
Together with our choice of I, this implies that for each $x \in S$: if $x_1 = 0$ then $x_j = a$. Thus by (2) we must have that $a = 0$. Hence, in the incidence matrix $M(\text{cuboid}(S))$, column $2j - 1$ has just as many zeros as column 1, so by (1), for each $x \in S$: if $x_1 = 1$ then $x_j = 1$. But then j must have belonged to I, a contradiction. ◇

Let a', b' be antipodal points of S', and let a, b be the corresponding points in S – these are the desired points. □

We are now ready to prove the following theorem:

Theorem 1.37 ([2]). *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then the following statements are equivalent:*

(i) S is not strictly polar,

(ii) there are distinct points $a, b, c \in S$ such that the restriction of S containing them of smallest dimension is not polar.

As a result, in time $O(n|S|^4)$ one can certify whether or not S is strictly polar.

Proof. **(ii) ⇒ (i)** holds trivially. **(i) ⇒ (ii)**: Let $S' \subseteq \{0,1\}^J$ be a strictly non-polar restriction of S. It suffices to show that S' has three points that do not all agree on a coordinate of J. By Lemma 5.10, there are distinct points $a', b' \in S'$ such that for $I := \{i \in J : a'_i = b'_i\}$ the following statement holds: for every $x \in S'$, either $x_i = a'_i$ for all $i \in I$, or $x_i = 1 - a'_i$ for all $i \in I$. As S' is non-polar, $I \neq \emptyset$. Since the points in S' do not all agree on a coordinate, there exists a point $c' \in S' - \{a', b'\}$ such that $c'_i = 1 - a'_i$ for all $i \in I$. Then the points a', b', c' do not all agree on a coordinate of J. The points a', b', c' of S' correspond naturally to some points a, b, c of S, respectively, and as a', b', c' do not all agree on a coordinate in J, it follows that S' is the smallest restriction of S containing a, b, c. Thus, since S' is not polar, (ii) holds.

For any three points a, b, c in S, it takes time $O(n|S|)$ to determine whether or not the smallest restriction of S containing a, b, c is polar. Thus, testing (ii) takes time $O(n|S|^4)$, and because (i) and (ii) are equivalent, testing strict polarity takes time $O(n|S|^4)$. □

Consider the strictly non-polar sets $P_3 = \{110, 101, 011\}$, $S_3 = \{110, 101, 011, 111\}$ and $R_{1,1} = \{000, 110, 101, 011\}$. The cuboid of $R_{1,1}$ is Q_6, a minimally non-packing clutter, while the cuboids of P_3, S_3 have a Δ_3 minor and are therefore not minimally non-packing. So what makes the cuboid of a strictly non-polar set minimally non-packing? We will need the following immediate remark:

Remark 5.11 ([2])**.** *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then an induced clutter of a minor of S is a minor of an induced clutter of S.*

Take an integer $n \geq 3$ and a strictly non-polar set $S \subseteq \{0,1\}^n$. We say that S is *critically non-polar* if for each $i \in [n]$, both the 0- and 1-restrictions of S over coordinate i have antipodal feasible points.

Theorem 5.12 ([2])**.** *Take an integer $n \geq 3$ and a strictly non-polar set $S \subseteq \{0,1\}^n$. Then the following statements are equivalent:*

(i) cuboid(S) *is minimally non-packing,*

(ii) S is critically non-polar, and the induced clutters of S have the packing property.

Proof. **(i) ⇒ (ii)**: Since cuboid(S) is minimally non-packing, its proper minors – including all of the induced clutters by Remark 5.2 – have the packing property. Take a coordinate $i \in [n]$. As cuboid$(S)/(2i-1)$ has covering number two, it has two disjoint members, which correspond to antipodal points in the 1-restriction of S over coordinate i. Similarly, as cuboid$(S)/2i$ has covering number two, it has two disjoint members, which correspond to antipodal points in the 0-restriction of S over coordinate i. Thus, S is critically non-polar.

(ii) ⇒ (i): By Remark 5.11, the induced clutters of the minors of S also have the packing property. Hence, since proper restrictions of S are strictly polar, it follows from Theorem 1.35 that for each $i \in [n]$, cuboid$(S) \setminus (2i-1)/2i$ and cuboid$(S) \setminus 2i/(2i-1)$ have the packing property, implying in turn that all proper deletion minors of cuboid(S) have the packing property. It remains to show that for each nonempty $J \subseteq [2n]$, cuboid$(S)/J$ packs. If $J \cap \{2i-1, 2i\} \neq \emptyset$ for each $i \in [n]$, then cuboid$(S)/J$ is a minor of an induced clutter of S by Remark 5.2, so cuboid(S) packs. Otherwise, cuboid$(S)/J$ has covering number two. Take a coordinate $j \in [n]$ such that $J \cap \{2j-1, 2j\} \neq \emptyset$. Since both the 0- and 1-restrictions of S over coordinate j have antipodal feasible points, both cuboid$(S)/(2j-1)$ and cuboid$(S)/2j$ have disjoint members; one of these two pairs of disjoint members corresponds to a pair of disjoint members in cuboid$(S)/J$, so cuboid$(S)/J$ packs, as required. □

5.5 Strictly non-polar sets of bounded degree

Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Take another integer $k \geq 0$. Recall that S has degree at most k if every infeasible point has at most k infeasible neighbors, and if in addition there is an infeasible point with exactly k infeasible neighbors, then S has degree k. We will describe a pseudocode for generating strictly non-polar sets of constant degree and dimension, for which we need a few definitions.

A *partial set* is a triple $P = (F, I, U)$ where F, I, U partitions $\{0,1\}^n$. We refer to F, I and U as the *feasible* points, *infeasible* points and *undecided* points of P, respectively. If $U = \emptyset$, then F is the *corresponding set* of P. Take an integer $k \in \{0, 1, \ldots, n\}$ and a set $S \subseteq \{0,1\}^k$. The *n-dimensional partial set originating from S* is the partial set whose feasible and infeasible points are $S \times \{\mathbf{0}^{n-k}\}$ and $\overline{S} \times \{\mathbf{0}^{n-k}\}$, respectively. We are now ready to describe an algorithm for finding the strictly non-polar sets of constant degree and dimension:

Input: degree $k \in \{0, 1, 2, \ldots\}$ and dimension $n \in \{k, k+1, k+2, \ldots\}$

Output: all non-isomorphic strictly non-polar sets of degree k and dimension n

Algorithm

(1) Enumerate all non-isomorphic subsets of $\{0,1\}^k$ all of whose proper restrictions are polar.

Call these sets *configurations*. Observe that each configuration is either strictly polar or strictly non-polar. Let $\mathcal{P}$ be the family of all n-dimensional partial sets originating from a configuration.

(2) While $\mathcal{P}$ has a partial set P with an undecided point:

- (a) If P has an undecided point whose antipodal is feasible, update P by making the undecided point infeasible.
- (b) If P has an infeasible point with k infeasible neighbors, update P by making the undecided neighbors feasible.
- (c) Otherwise, take an undecided point q. Let P_1 and P_2 be the partial sets obtained from P after making q feasible and infeasible, respectively. Set $\mathcal{P} := \mathcal{P} \triangle \{P, P_1, P_2\}$.

At this point, the partial sets in $\mathcal{P}$ have no undecided point. Let $\mathcal{S}$ be the family of sets corresponding to the partial sets in $\mathcal{P}$.

(3) From every isomorphic class in $\mathcal{S}$, keep only one set and filter out the other ones.

(4) Output the sets in $\mathcal{S}$ that are strictly non-polar.

End of Algorithm

After running the code on Racket, Version 6.11 [34] for $k \in \{0, 1, 2, 3, 4\}$ and $n \in \{3, 4, 5, 6, 7\}$, we get the following result:

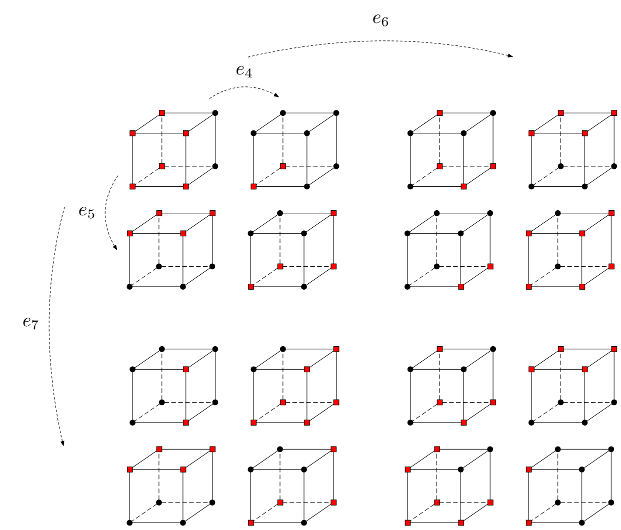


Figure 5.1: An illustration of the strictly non-polar sets of degree 4 and dimension 7. Round points are feasible and square points are infeasible.

Theorem 1.40 ([2]). *Up to isomorphism, there are precisely* 745 *strictly non-polar sets of dimension at most* 7 *and degree at most* 4*,* 716 *of which have ideal minimally non-packing cuboids.*

An explicit description of the 745 strictly non-polar sets is provided in [2]. See Figure 5.1 for the largest strictly non-polar set that we have generated: this set is half-dense, every infeasible point has 4 infeasible neighbors, and its cuboid is an ideal minimally non-packing clutter.

Let us now study strictly non-polar sets of bounded degree. A graph is *triangle-free* if it has no circuit with three edges, and it is *simple* if it has no loops or parallel edges. We will need the following classic result known as Mantel's Theorem:

Theorem 5.13 (Mantel [55])**.** *For an integer $n \geq 3$, every triangle-free simple graph on n vertices has at most $\lfloor n^2/4 \rfloor$ edges, and this bound is achieved only by the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.*

Take an integer $n \geq 1$ and points $a, b \in \{0,1\}^n$. Denote by $\mathrm{dist}(a,b)$ the *Hamming distance* between a and b, that is, $\mathrm{dist}(a,b)$ is the number of coordinates a, b disagree on. We are now ready to prove the following lemma:

Lemma 5.14 ([2])**.** *Take integers $n \geq 3$, $k \in \{0,1,\ldots,n\}$ and a set $S \subseteq \{0,1\}^n$ that is not polar, has degree k, and has none of $P_3, S_3, R_{1,1}$ as a restriction. Take an infeasible point x whose set of feasible neighbors is F and whose set of infeasible neighbors is I, where $|I| = k$. Then the following statements hold:*

(1) We have that

$$\frac{(n-k-1)^2-1}{4} \leq \left|\{x\triangle y\triangle z : y,z \in F, y \neq z\} \cap S\right| \leq \left|\{x\triangle y\triangle z : y,z \in I, y \neq z\} \cap S\right| \leq \frac{k^2}{4}.$$

(2) We have that $n \leq 2k+1$.

(3) If $n = 2k+1$, *then* $k \geq 2$, *every point in* F *has exactly* k *feasible neighbors, every point in* I *has exactly* k *infeasible neighbors, and there is a partition of* I *into parts* I_1, I_2 *such that* $\left||I_1| - |I_2|\right| \leq 1$ *and for distinct* $y, z \in I$,

$$x\triangle y\triangle z \in S \Leftrightarrow |I_1 \cap \{y,z\}| = 1.$$

Proof. Let us start with the following claim:

Claim 1. *Every feasible point has at most* k *feasible neighbors.*

Proof of Claim. S is not polar, so it does not have antipodal points, implying in turn that $G_n[S]$ is isomorphic to a subgraph of $G_n[\overline{S}]$.[2] Thus, as $G_n[\overline{S}]$ has maximum degree at most k, so does $G_n[S]$, so every feasible point has at most k feasible neighbors. $\diamond$

In particular, since $n \geq 3$ and S has no $R_{1,1}$ restriction, it follows that $k \geq 1$.

Claim 2. *There exist no* $x \in \overline{S}$ *and coordinates* $1 \leq i < j < k \leq n$ *such that*

- *Type I:* $x\triangle e_i, x\triangle e_j, x\triangle e_k \in S$ *and* $x\triangle e_i\triangle e_j, x\triangle e_i\triangle e_k, x\triangle e_j\triangle e_k \in \overline{S}$, *or*
- *Type II:* $x\triangle e_i, x\triangle e_j, x\triangle e_k \in \overline{S}$ *and* $x\triangle e_i\triangle e_j, x\triangle e_i\triangle e_k, x\triangle e_j\triangle e_k \in S$.

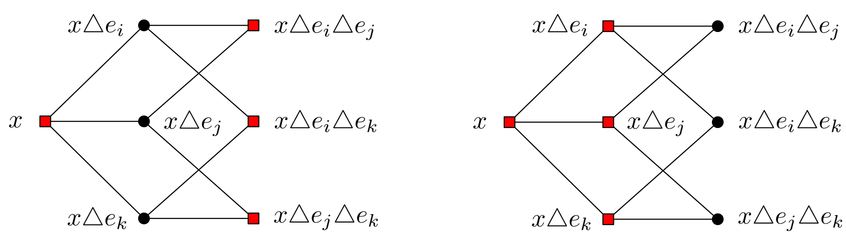


Figure 5.2: The forbidden restrictions of Claim 2. Round points are in S and square points are in $\overline{S}$.

Proof of Claim. Depending on whether or not the point $x\triangle e_i\triangle e_j\triangle e_k$ is feasible, Type I gives an $R_{1,1}$ or a P_3 restriction, while Type II gives an S_3 or a P_3 restriction; as S has none of these restrictions, both restrictions are forbidden. $\diamond$

[2] Two graphs are *isomorphic* if one can be obtained from the other after relabeling the vertices.

Since S has degree k, there is an infeasible point with precisely k infeasible points adjacent to it. After a possible twisting and relabeling, if necessary, we may assume that $\mathbf{0}$ is infeasible, its neighbors $e_1, \dots, e_{n-k}$ are feasible and its other neighbors $e_{n-k+1}, \dots, e_n$ are infeasible. We will partition the points of $\{0,1\}^n$ at Hamming distance 2 from $\mathbf{0}$ into three sets as follows:

$$\begin{aligned}
X &:= \{e_i + e_j \ : \ 1 \le i < j \le n-k\} \\
Y &:= \{e_i + e_j \ : \ 1 \le i \le n-k < j \le n\} \\
Z &:= \{e_i + e_j \ : \ n-k+1 \le i < j \le n\}.
\end{aligned}$$

That is, the set X consists of all the points $e_i + e_j$ such that $e_i, e_j \in S$, Z of all the points $e_i + e_j$ such that $e_i, e_j \in \overline{S}$, and Y of all the remaining points at Hamming distance 2 from $\mathbf{0}$. (If $k = 1$ then $Z = \emptyset$.) We now use Claim 2 to deduce some bounds on the number of infeasible and feasible points in X and Z.

Claim 3. *The following statements hold:*

- $|X \cap \overline{S}| \le \left(\frac{n-k}{2}\right)^2$.
- $|Z \cap S| \le \left(\frac{k}{2}\right)^2$. *If* $|Z \cap S| \ge \left(\frac{k}{2}\right)^2 - \frac{1}{4}$, *then there is a partition of* $\{e_{n-k+1}, \dots, e_n\}$ *into parts* I_1, I_2 *such that* $\big||I_1| - |I_2|\big| \le 1$ *and for distinct* $e_i, e_j \in I_1 \cup I_2$,

$$e_i + e_j \in S \Leftrightarrow |\{e_i, e_j\} \cap I_1| = 1.$$

Proof of Claim. Consider the simple graph G on vertices $\{e_1, \dots, e_{n-k}\}$ and edges $\{e_i e_j : e_i + e_j \in \overline{S}\}$, which is in bijection with $X \cap \overline{S}$. By Claim 2, S has no restriction of Type I, implying in turn that G is triangle-free. Thus, by Theorem 5.13, $|X \cap \overline{S}| \le \left(\frac{n-k}{2}\right)^2$. This proves the first part. For the next part, consider the simple graph G' on vertices $\{e_{n-k+1}, \dots, e_n\}$ and edges $\{e_i e_j : e_i + e_j \in S\}$, which is in bijection with $Z \cap S$. By Claim 2, there is no restriction of Type II, implying in turn that G' is triangle-free. Thus, by Theorem 5.13, $|Z \cap S| \le \left(\frac{k}{2}\right)^2$ and if $|Z \cap S| \ge \left(\frac{k}{2}\right)^2 - \frac{1}{4}$, then G' is a complete bipartite graph with bipartition I_1, I_2 such that $\big||I_1| - |I_2|\big| \le 1$, as required. $\diamond$

Define $A := \{(i,j) : e_i \in S, e_i + e_j \in S\}$ and $B := \left\{(i,j) : e_i \in \overline{S}, e_i + e_j \in S\right\}$.

Claim 4. *The following inequalities hold:*

$$2|X \cap S| + |Y \cap S| = |A| \le (n-k)k \le |B| = 2|Z \cap S| + |Y \cap S|.$$

In particular, $|Z \cap S| \ge |X \cap S|$ *and if equality holds, then every point in* $\{e_1, \dots, e_{n-k}\}$ *has precisely* k *feasible neighbors and every point in* $\{e_{n-k+1}, \dots, e_n\}$ *has precisely* k *infeasible neighbors.*

Proof of Claim. For all distinct i, j with $e_i + e_j \in X \cap S$, the two pairs $(i,j), (j,i)$ belong to A; for all distinct i, j with $e_i + e_j \in Y \cap S$, exactly one of $(i,j), (j,i)$ belongs to A; and for all distinct i, j with $e_i + e_j \in Z \cap S$, neither of $(i,j), (j,i)$ belongs to A. Hence, $|A| = 2|X \cap S| + |Y \cap S|$. Similarly, $|B| = 2|Z \cap S| + |Y \cap S|$. On the one hand, each point in $S \cap \{e_i : i \in [n]\} = \{e_1, \dots, e_{n-k}\}$ has at most k feasible neighbors by Claim 1, so

$$|A| \le |S \cap \{e_i : i \in [n]\}| \times k = (n-k)k.$$

On the other hand, each point in $\overline{S} \cap \{e_i : i \in [n]\} = \{e_{n-k+1}, \ldots, e_n\}$ has at least $n-k$ feasible neighbors by assumption, so because $\mathbf{0} \in \overline{S}$,

$$|B| \geq (n-k) \times |\overline{S} \cap \{e_i : i \in [n]\}| = (n-k)k.$$

All of these (in)equalities put together prove the claim. $\diamond$

Hence, by Claims 3 and 4,

$$\frac{(n-k-1)^2 - 1}{4} = \binom{n-k}{2} - \left(\frac{n-k}{2}\right)^2 \leq |X| - |X \cap \overline{S}| = |X \cap S| \leq |Z \cap S| \leq \frac{k^2}{4}.$$

This proves (1). Since $k, n-k-1$ are both integers and $k \geq 1$, we must have that $n-k-1 \leq k$, implying in turn that $n \leq 2k+1$, so (2) holds. To prove (3), assume that $n = 2k+1$. Since S is not polar and is not one of $P_3, S_3, R_{1,1}$, it follows that $2k+1 = n \geq 4$, so $k \geq 2$. Since $n = 2k+1$, the inequalities above imply that $|Z \cap S| \geq \frac{k^2-1}{4}$ and $|Z \cap S| = |X \cap S|$, so Claims 3 and 4 prove (3), as required. $\square$

We are now ready to prove the following theorem, for which we rely on Theorem 1.3 on finding delta minors:

Theorem 1.41 ([2]). *Take an integer $k \geq 2$ and a strictly non-polar set S of degree k, whose dimension is n. Then $k \leq n \leq 2k+1$. Moreover, if $n = 2k+1$, then S is half-dense, every infeasible point has exactly k infeasible neighbors, and* $\mathrm{cuboid}(S)$ *is an ideal minimally non-packing clutter.*

Proof. Clearly, $n \geq k$. If $S \in \{P_3, S_3\}$, then $n = 3 \leq 7 = 2k+1$, so we are done. We may therefore assume that S has no P_3, S_3 restriction. Moreover, $S \neq R_{1,1}$ as $k > 0$, so S has no $R_{1,1}$ restriction. As a result, we may apply Lemma 5.14. Choosing x to be any infeasible point with exactly k infeasible neighbors, Lemma 5.14 (2) implies that $n \leq 2k+1$. Suppose now that $n = 2k+1$.

Claim 1. *For each $x \in \{0,1\}^{2k+1}$, we have that $|S \cap \{x, \mathbf{1}-x\}| = 1$. In particular, $|S| = 2^{2k}$.*

Proof of Claim. There are no antipodal feasible points, so $|S \cap \{x, \mathbf{1}-x\}| \leq 1$. Suppose for contradiction that both $x, \mathbf{1}-x$ are infeasible. The infeasible point x has at most k infeasible neighbors, so it has at least $k+1$ feasible neighbors. Similarly, the infeasible point $\mathbf{1}-x$ has at most k infeasible neighbors, so it has at least $k+1$ feasible neighbors. By the Pigeonhole Principle, there are antipodal feasible points, one in the neighborhood of x and the other in the neighborhood of $\mathbf{1}-x$, a contradiction. $\diamond$

For a point in $\overline{S}$ define its *degree* to be the number of infeasible points adjacent to it, and for a point in S define its *degree* to be the number of feasible points adjacent to it.

Claim 2. *If a point in $\{0,1\}^{2k+1}$ has degree k, then so do all the points of $\{0,1\}^{2k+1}$ adjacent to it.*

Proof of Claim. Lemma 5.14 (3) proves the claim for infeasible points. To conclude that the same holds for all feasible points, notice that $\overline{S} = S\triangle\mathbf{1}$ by Claim 1. Thus, if a feasible point x has degree k, the infeasible point $\mathbf{1}-x$ also has degree k and so do all the points adjacent to it, implying in turn that all the points adjacent to x have degree k as well. This finishes the proof of the claim. $\diamond$

Since there is at least one point whose degree is k, Claim 2 implies that every point of $\{0,1\}^n$ has degree k. Thus, every infeasible point has exactly k infeasible neighbors.

Next we show that cuboid(S) is an ideal minimally non-packing clutter. By Theorem 1.38, it suffices to show that cuboid(S) is minimally non-packing.

Claim 3. *S is critically non-polar.*

Proof of Claim. By our hypothesis, S is strictly non-polar. Let $S' \subseteq \{0,1\}^{2k}$ be a single restriction of S. It suffices to show that S' has antipodal feasible points. Suppose otherwise. Since S is strictly non-polar, S' is polar, so the points in S' must all agree on a coordinate. In particular, some infeasible point of S' has at least $2k-1$ infeasible neighbors, implying in turn that an infeasible point of S has at least $2k-1$ infeasible neighbors. However, every infeasible point of S has exactly k infeasible neighbors, so $k \geq 2k-1$, a contradiction as $k \geq 2$. $\diamond$

Thus, by Theorem 5.12, it suffices to show that the induced clutters of S have the packing property. We need the following:

Claim 4. *The induced clutters of proper restrictions of S do not have a delta minor.*

Proof of Claim. Let S' be a proper restriction of S. As S is strictly non-polar, S' is strictly polar. Thus by Proposition 5.8, every minor of cuboid(S') with covering number at least two has two disjoint members. In particular, cuboid(S') does not have a delta minor, implying in turn that the induced clutters of S' do not have a delta minor, as required. $\diamond$

Take a point $x \in \overline{S}$. By symmetry, it suffices to show that $\mathrm{ind}(S\triangle x)$ has the packing property. After a possible twisting, we may assume that $x = \mathbf{0}$. The infeasible point $\mathbf{0}$ has exactly k infeasible neighbors; after a possible relabeling, we may assume that $\{e_1,\ldots,e_{k+1}\} \subseteq S$ and $\{e_{k+2},\ldots,e_{2k+1}\} \subseteq \overline{S}$. By Lemma 5.14 (3), there is a partition $I_1 \cup I_2$ of $\{e_{k+2},\ldots,e_{2k+1}\}$ such that $\big||I_1|-|I_2|\big| \leq 1$ and for all distinct $e_i, e_j \in \{e_{k+2},\ldots,e_{2k+1}\}$,

$$e_i + e_j \in S \Leftrightarrow \big|I_1 \cap \{e_i,e_j\}\big| = 1.$$

Notice that since $k \geq 2$, $|I_1| + |I_2| \geq 2$.

Claim 5. *Let $S' \subseteq \{0,1\}^{I_1 \cup I_2}$ be obtained from S after 0-restricting coordinates $[k+1]$. Then* $\mathrm{ind}(S') = \big\{\{i,j\} : e_i \in I_1, e_j \in I_2\big\}$.

Proof of Claim. We know that $\big\{\{i,j\} : e_i \in I_1, e_j \in I_2\big\}$ are the only members of $\mathrm{ind}(S')$ of cardinality at most two. Suppose for a contradiction that $\mathrm{ind}(S')$ has another member $C \subseteq I_1 \cup I_2$. Then $|C| \geq 3$. After possibly relabeling I_1 and I_2, we may assume that $|C \cap I_2| \geq 2$. Pick distinct coordinates $j, j' \in C \cap I_2$ and pick an arbitrary $i \in I_1$. Notice that $\{i,j\}, \{i,j'\}, C$ are members of $\mathrm{ind}(S')$, implying in particular that $i \notin C$, and so by Theorem 1.3, $\mathrm{ind}(S')$ has a delta minor, thereby contradicting Claim 4. $\diamond$

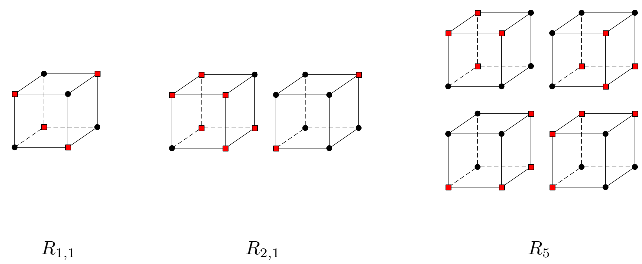


Figure 5.3: An illustration of the strictly non-polar sets of degree at most 2.

As a result,

$$\mathrm{ind}(S) = \big\{\{1\},\{2\},\ldots,\{k+1\}\big\} \cup \big\{\{i,j\} : e_i \in I_1, e_j \in I_2\big\}.$$

It can be readily checked that this clutter has the packing property, as required. $\square$

As a consequence, Theorem 1.40 has already generated all the strictly non-polar sets of degree at most 3:

Corollary 5.15 ([4]). *Up to isomorphism, $R_{1,1}, R_{2,1}, R_5$ are the only strictly non-polar sets of degree at most two.*

See Figure 5.3 for an illustration of $R_{1,1}, R_{2,1}, R_5$.

Corollary 5.16 ([2]). *Up to isomorphism, there are exactly 11 strictly non-polar sets of degree three.*

See Figure 5.4 at the end of the chapter for an illustration of these strictly non-polar sets; the ones that have an ideal minimally non-packing cuboid are those with dimension at least 5.

5.6 Further notes

Consider the following question:

> When is a polytope defined by the hypercube inequalities as well as some generalized set covering inequalities integral?

In 1998, Guenin [42] and, independently, Nobili and Sassano [58] studied this question (and reduced it to clutter idealness, in the same vein as in Theorem 1.21). This question was also studied by Coppersmith and Lee [21]. The question above is dual to the following question studied in this chapter:

> When is a set cube-ideal?

Lee [47] also studied this question.

Minimally non-packing cuboids

We saw in Theorem 5.12 that if a set has a minimally non-packing cuboid, then the set is critically non-polar; we also know from Theorems 1.21 and 1.38 that the set is also cube-ideal. We conjecture the converse:

Conjecture 5.17 ([2]). *Take an integer $n \geq 3$ and a set $S \subseteq \{0,1\}^n$. Then* $\text{cuboid}(S)$ *is minimally non-packing if, and only if, S is cube-ideal and critically non-polar.*

In fact,

Theorem 5.18 ([2]). *If the Polarity Conjecture is true, then so is Conjecture 5.17.*

The Replication Conjecture

We are not the first ones to notice the special role cuboids play among clutters. In 2008, Flores, Gitler and Reyes [35] came across cuboids, too – and called them 2-*partitionable* clutters – while they were studying the Replication Conjecture. In particular, they studied the conjecture for cuboids:

Conjecture 5.19. *If a cuboid has the packing property, then so does any replication of it.*

However, this conjecture is just as strong as the Replication Conjecture:

Theorem 5.20 ([2]). *The Replication Conjecture is equivalent to Conjecture 5.19.*

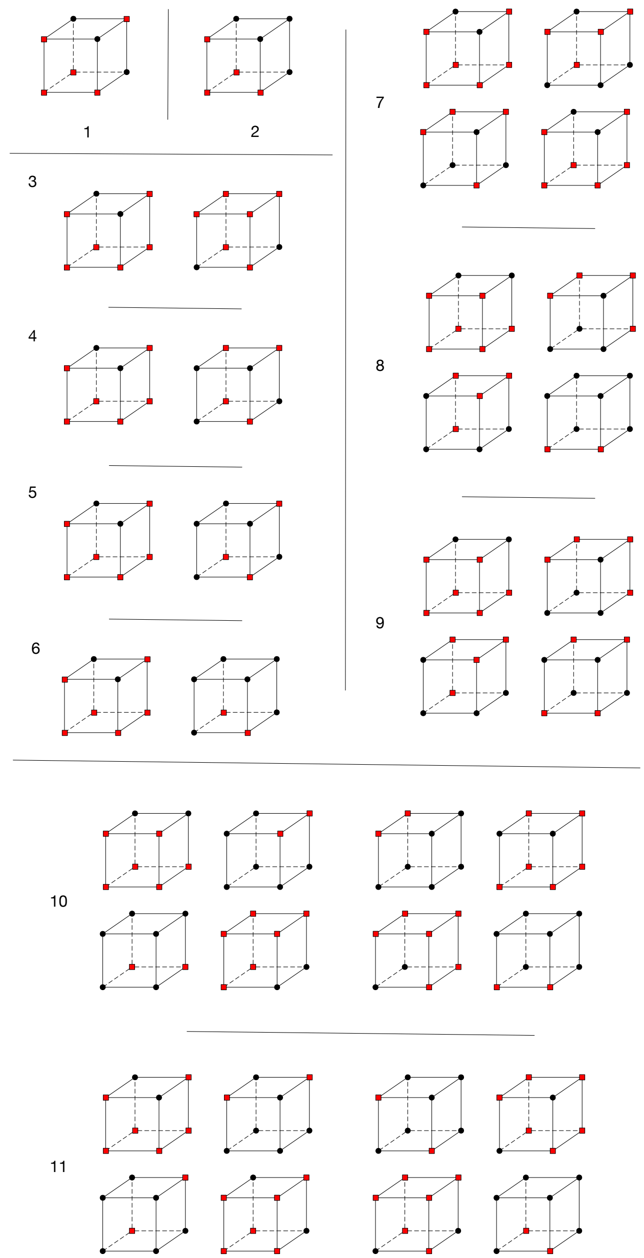


Figure 5.4: An illustration of the strictly non-polar sets of degree 3.

Chapter 6

Resistant sets

Consider the sets $P_3 = \{110, 101, 011\} \subseteq \{0,1\}^3$ and $S_3 = \{110, 101, 011, 111\} \subseteq \{0,1\}^3$, as displayed in Figure 6.1. Recall that P_3, S_3 are, up to isomorphism, the only non-cube-ideal sets of dimension at most 3. In particular, cube-ideal sets do not have a P_3, S_3 minor by Remark 1.43. What can we say about sets that do not have a P_3, S_3 minor even after we locally change the set?

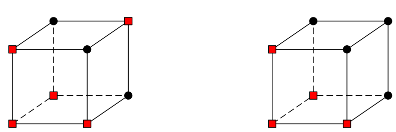
Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Recall that

- S is 1-resistant if for every subset $X \subseteq \{0,1\}^n$ of cardinality at most one, $S \cup X$ has no P_3, S_3 minor,
- S is 2-resistant if for every subset $X \subseteq \{0,1\}^n$ of cardinality at most two, $S \cup X$ has no P_3, S_3 minor,
- S is ± 1-resistant if for every subset $X \subseteq \{0,1\}^n$ of cardinality at most one, $S \triangle X$ has no P_3, S_3 minor.

Observe that 2- and ± 1-resistant sets are 1-resistant also. For an explanation of the origin of resistance, and why it is defined this way, see §6.4.

Remark 6.1. *If a set is* 1*-resistant, then so is every minor of it.*

Proof. Being 1-resistant is clearly closed under restrictions; it remains to show that it is also closed under projections. To this end, take an integer $n \geq 1$ and a 1-resistant set $S \subseteq \{0,1\}^n$. Let $S' \subseteq \{0,1\}^{n-1}$ be



P_3 S_3

Figure 6.1: An illustration of P_3 and S_3, the smallest non-cube-ideal sets. Round points are feasible while square points are infeasible.

the projection of S over coordinate n. Suppose for a contradiction that S' is not 1-resistant. Then for some $X' \subseteq \{0,1\}^{n-1}$ of cardinality at most one, $S' \cup X'$ has a P_3, S_3 minor. Since S has no P_3, S_3 minor, $X' \neq \emptyset$, so $X' = \{x'\}$ for some $x' \in \{0,1\}^{n-1}$. Then $S \cup \{(x', 0)\}$ has $S' \cup X'$ as a projection, and has a P_3, S_3 minor as a consequence, implying in turn that S is not 1-resistant, a contradiction. $\square$

In §6.1 we describe the excluded minors defining 1-resistance; we also find the defining excluded restrictions, thereby leading to a polynomial time algorithm for testing 1-resistance. There we also characterize 1-resistance in terms of induced clutters, which in turn leads to a proof that 1-resistant sets are cube-ideal and to a characterization of when the cuboid of a 1-resistant set has the packing property. In §6.2 we prove a structure theorem for 2-resistant sets and list, as a consequence, the strictly non-polar sets that are 2-resistant. In §6.3 we find the excluded minors defining ± 1-resistance.

6.1 Testing 1-resistance, idealness and the packing property

Take a set $F \subseteq \{0,1\}^3$ such that

$$F \cap \{000, 100, 010, 001, 101, 011\} = \{101, 011\}.$$

Recall that F, and any set isomorphic to it, is called fragile. Recall from Remark 1.47 that fragile sets are precisely the 3-dimensional sets that are not 1-resistant.

Theorem 1.48 ([3]). *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then the following statements are equivalent:*

(i) S is 1-resistant,

(ii) S has no fragile minor,

(iii) for each $x \in \{0,1\}^n$, the members of $\mathrm{ind}(S \triangle x)$ are pairwise disjoint,

(iv) S has no fragile restriction and no $\{\mathbf{0}^k, \mathbf{1}^k - e_1\}, k \geq 4$ restriction.

Proof. **(i) ⇒ (ii)**: Assume that S is 1-resistant. Then every minor of S is also 1-resistant by Remark 6.1. As fragile sets are not 1-resistant by Remark 1.47, it follows that S has no fragile minor.

(ii) ⇒ (iii): Suppose that an induced clutter of S has intersecting members. It suffices to prove that S has a fragile minor. After possibly twisting S, we may assume that $\mathcal{C} := \mathrm{ind}(S)$ has intersecting members.

Claim 1. *There exist disjoint $I, J \subseteq [n]$ such that $\mathcal{C} \setminus I/J$ has ground set $\{x, y, z\}$ and has $\{x, z\}, \{y, z\}$ among its members.*

Proof of Claim. Among all pairs of intersecting members in $\mathcal{C}$, pick an intersecting pair C_1, C_2 whose union is minimal. Our minimal choice of C_1, C_2 implies that every member of $\mathcal{C}$ contained in $C_1 \cup C_2$ is either C_1 or C_2 or it contains $C_1 \triangle C_2$. Take elements $x \in C_1 - C_2$, $y \in C_2 - C_1$ and $z \in C_1 \cap C_2$. Let

$I := [n] - (C_1 \cup C_2)$ and $J := [n] - (I \cup \{x, y, z\})$. It is easy to see that $\mathcal{C} \setminus I/J$ has ground set $\{x, y, z\}$ and has $\{x, z\}, \{y, z\}$ among its members. $\diamond$

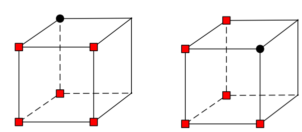
Consider now the minor $S' \subseteq \{0,1\}^{\{x,y,z\}}$ of S obtained after 0-restricting coordinates I and projecting away coordinates J. Since $\text{ind}(S) = \mathcal{C}$, it follows that $\text{ind}(S') = \mathcal{C} \setminus I/J$ has $\{x, z\}, \{y, z\}$ as members, implying in turn that S' is fragile. Thus, S has a fragile minor.

(iii) $\Rightarrow$ (iv): Suppose that the induced clutters of S do not have intersecting members. Remark 5.11 from Chapter 5 tells us that the induced clutters of minors of S are minors of the induced clutters of S. Thus, just like the induced clutters of S, the induced clutters of minors of S do not have intersecting members. Now, given a fragile set, its induced clutter with respect to the origin has intersecting members $\{1, 2\}, \{2, 3\}$. Also, given the set $\{\mathbf{0}^k, \mathbf{1}^k - e_1\}$ for some integer $k \geq 4$, the induced clutter with respect to $e_1 + e_2$ has intersecting members $\{1, 2\}$ and $\{1, 3, 4, \ldots, k\}$. As a result, S has no fragile restriction and no $\{\mathbf{0}^k, \mathbf{1}^k - e_1\}, k \geq 4$ restriction.

(iv) $\Rightarrow$ (i): Assume that S is not 1-resistant. We will show that S has either a fragile restriction or a $\{\mathbf{0}^k, \mathbf{1}^k - e_1\}$ restriction, for some $k \in \{4, \ldots, n\}$. As S is not 1-resistant, $S \cup X$ has one of P_3, S_3 as minor for some set $X \subseteq \{0,1\}^n$ of cardinality at most one. So there is a set $Y \subseteq \{0,1\}^3$ of cardinality at most one such that S has one of $P_3 - Y, S_3 - Y$ as a minor. As $P_3 - Y, S_3 - Y$ are not 1-resistant, they are fragile by Remark 1.47, so S has a fragile minor. We will need the following two claims:

Claim 2. *Suppose that $R \subseteq \{0,1\}^4$ has no fragile restriction and its projection over coordinate 4 is fragile. Then R is a twist of $\{\mathbf{0}^4, \mathbf{1}^4 - e_1\}$.*

Proof of Claim. For $i \in \{0, 1\}$, let $R_i \subseteq \{0,1\}^3$ be the i-restriction of R over coordinate 4. Since the projection of R over 4 is fragile, it follows that $\{000, 100, 010, 001\} \subseteq \overline{R_0}$ and $\{000, 100, 010, 001\} \subseteq \overline{R_1}$. Moreover, as R_0 and R_1 are not fragile, we may assume that $R_0 \cap \{101, 011\} = \{011\}$ and $R_1 \cap \{101, 011\} = \{101\}$:



Once again, as R_0 and R_1 are not fragile, it follows that $110 \notin R_0 \cup R_1$. Since the 1-restriction of R over coordinate 1 is not fragile, we get that $111 \notin R_0$, and since the 1-restriction of R over coordinate 2 is not fragile, $111 \notin R_1$. Thus, R is a twist of $\{\mathbf{0}^4, \mathbf{1}^4 - e_1\}$, as claimed. $\diamond$

Claim 3. *Take an integer $k \geq 4$ and a set $R \subseteq \{0,1\}^{k+1}$ without a $\{\mathbf{0}^k, \mathbf{1}^k - e_1\}$ restriction. If the projection of R over coordinate $k+1$ is $\{\mathbf{0}^k, \mathbf{1}^k - e_1\}$, then R is a twist of $\{\mathbf{0}^{k+1}, \mathbf{1}^{k+1} - e_1\}$.*

Proof of Claim. For $i \in \{0, 1\}$, let $R_i \subseteq \{0,1\}^k$ be the i-restriction of R over coordinate $k+1$. Clearly, $R_i \subseteq \{\mathbf{0}^k, \mathbf{1}^k - e_1\}$ for each $i \in \{0, 1\}$. As equality cannot hold, we may assume that $R_0 \cap \{\mathbf{0}^k, \mathbf{1}^k - e_1\} = \{\mathbf{0}^k\}$ and $R_1 \cap \{\mathbf{0}^k, \mathbf{1}^k - e_1\} = \{\mathbf{1}^k - e_1\}$, implying in turn that R is a twist of $\{\mathbf{0}^{k+1}, \mathbf{1}^{k+1} - e_1\}$. $\diamond$

Suppose that a fragile minor of S is obtained after applying k single projections and $n - k - 3$ single restrictions, for some $k \in \{0, \ldots, n-3\}$. If $k = 0$, then S has a fragile restriction, so we are done. We may therefore assume that $k \geq 1$ and S has no fragile restriction. It follows from Claim 2 that S has a

$\{\mathbf{0}^4, \mathbf{1}^4 - e_1\}$ minor obtained after applying $k-1$ single projections and $n-k-3$ single restrictions. If $k=1$, then S has a $\{\mathbf{0}^4, \mathbf{1}^4 - e_1\}$ restriction. We may therefore assume that $k \geq 2$ and S has no $\{\mathbf{0}^4, \mathbf{1}^4 - e_1\}$ restriction. Now by repeatedly applying Claim 3, we see that S has one of $\{\mathbf{0}^\ell, \mathbf{1}^\ell - e_1\}, \ell \in \{5, \ldots, k+3\}$ as a restriction. $\square$

Take an integer $n \geq 1$. Recall that for points $a, b \in \{0,1\}^n$, the Hamming distance between a, b, denoted $\mathrm{dist}(a,b)$, is the number of coordinates a, b disagree on. We are ready to prove the following:

Theorem 1.46 **([3]).** *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then in time $O(n^2|S|^2)$, one can test whether or not S is 1-resistant.*

Proof. We will take advantage of Theorem 1.48, which states that $S \subseteq \{0,1\}^n$ is 1-resistant if, and only if, it has no fragile restriction and no $\{\mathbf{0}^k, \mathbf{1}^k - e_1\}, k \in \{4, \ldots, n\}$ restriction. For $k \in \{3, 4, \ldots, n\}$, consider the following algorithm:

1. for every pair of points x, y of S at Hamming distance $k-1$,
   (a) let $I := \{i \in [n] : x_i = y_i\}$,
   (b) for every coordinate $i \in I$,
      i. let $S' \subseteq \{0,1\}^k$ be the restriction of S over coordinates $I - \{i\}$ containing (images of) x and y,
      ii. if $k = 3$ and S' is fragile, then output "S has a fragile restriction",
      iii. if $k \geq 4$ and S' is isomorphic to $\{\mathbf{0}^k, \mathbf{1}^k - e_1\}$, then output "$S$ has a $\{\mathbf{0}, \mathbf{1} - e_1\} \subseteq \{0,1\}^k$ restriction",
   (c) if $k = 3$ and (ii) fails for every $i \in I$, then change the pair x, y,
   (d) if $k \geq 4$ and (iii) fails for every $i \in I$, then change the pair x, y,
2. if $k = 3$ and (ii) fails for every pair x, y, then output "S has no fragile restriction",
3. if $k \geq 4$ and (iii) fails for every pair x, y, then output "S has no $\{\mathbf{0}^k, \mathbf{1}^k - e_1\}$ restriction".

The correctness of this algorithm is clear; its running time is $\binom{|S|}{2} \times (n-k+1)$. Thus, by Theorem 1.48, one can test whether or not S is 1-resistant in time $\sum_{k=3}^{n} \binom{|S|}{2} \times (n-k+1) = O(n^2|S|^2)$, as required. $\square$

Another important consequence of Theorem 1.48 is that 1-resistant sets are cube-ideal. To explain this, we need the following obvious fact:

Remark 6.2. *A clutter whose members are pairwise disjoint is ideal.*

We are now ready to prove the following theorem:

Theorem 1.49 ([3]). *Every* 1*-resistant set is cube-ideal.*

Proof. Take an integer $n \geq 1$ and a 1-resistant set $S \subseteq \{0,1\}^n$. By Theorem 1.48, every induced clutter of S has only pairwise disjoint members, so by Remark 6.2, every induced clutter of S is ideal. It now follows from Theorem 1.22 that S is cube-ideal, as required. □

Hence, the cuboids of 1-resistant sets are ideal by Theorem 1.21. Do they have the packing property? No. For instance, $R_{1,1}$ is 1-resistant, yet its cuboid does not pack. So, when does the cuboid of a 1-resistant set have the packing property? We will need the following immediate remark:

Remark 6.3. *A clutter whose members are pairwise disjoint has the packing property.*

Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Recall that S is polar if either there are antipodal feasible points, or the feasible points all agree on a coordinate. Recall further that S is strictly polar if every restriction, including S itself, is polar. Combining Remark 6.3 with Theorem 1.35 yields the following:

Theorem 6.4 ([3])**.** *Take an integer $n \geq 1$ and a* 1*-resistant set $S \subseteq \{0,1\}^n$. Then* cuboid(S) *has the packing property if, and only if, S is strictly polar.*

Proof. ($\Rightarrow$) follows from Theorem 1.35. ($\Leftarrow$) Suppose that S is strictly polar. By Theorem 1.48, every induced clutter of S has only pairwise disjoint members and therefore has the packing property, by Remark 6.3. It therefore follows from Theorem 1.35 that cuboid(S) has the packing property. □

So when is a 1-resistant set strictly polar? Even though we are not able to answer this question, in Chapter 7, we will find all the 1-resistant strictly non-polar sets that are half-dense.

6.2 The structure of 2-resistant sets and strict polarity

Let us start with the following remark, whose proof is almost identical to that of Remark 6.1:

Remark 6.5 ([1])**.** *If a set is* 2*-resistant, then so is every minor of it.*

Proof. Being 2-resistant is clearly closed under restrictions; it remains to show that it is also closed under projections. To this end, take an integer $n \geq 1$ and a 2-resistant set $S \subseteq \{0,1\}^n$. Let $S' \subseteq \{0,1\}^{n-1}$ be the projection of S over coordinate n. Suppose for a contradiction that S' is not 2-resistant. Then for some $X' \subseteq \{0,1\}^{n-1}$ of cardinality at most two, $S' \cup X'$ has a P_3, S_3 minor. Let $X := \{(x', 0) : x' \in X'\}$. Then $S \cup X$ has $S' \cup X'$ as a projection, and has a P_3, S_3 minor as a consequence, implying in turn that S is not 2-resistant, a contradiction. □

As a consequence,

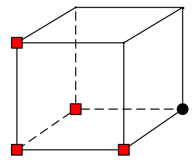


Figure 6.2: The excluded minor for 2-resistant sets.

Proposition 6.6 ([1])**.** *A set is* 2*-resistant if, and only if, it has no minor* $F \subseteq \{0,1\}^3$ *such that* $F \cap \{000, 100, 010, 001, 110\} = \{110\}$*. (See Figure 6.2 for an illustration of* F*.)*

Proof. ($\Rightarrow$) $F \cup \{101, 011\}$ is either P_3 or S_3, so F is not 2-resistant, so a 2-resistant set has no F minor by Remark 6.5. ($\Leftarrow$) Take an integer $n \geq 3$ and a set $S \subseteq \{0,1\}^n$ that is not 2-resistant. We need to show that S has an F minor. As S is not 2-resistant, there is a subset $X \subseteq \{0,1\}^n$ of cardinality at most two such that $S \cup X$ has a P_3, S_3 minor. Thus, there is a subset $Y \subseteq \{0,1\}^3$ of cardinality at most two such that S has a $P_3 - Y, S_3 - Y$ minor. After possibly relabeling the coordinates, we see that $P_3 - Y$ and $S_3 - Y$ are the desired minor. $\square$

We are now ready to prove the following theorem:

Theorem 1.44 ([1]). *Take an integer* $n \geq 1$ *and a set* $S \subseteq \{0,1\}^n$*. Then* S *is* 2*-resistant if, and only if, every infeasible component is a hypercube or has maximum degree at most two.*

Proof. ($\Rightarrow$) Let $S \subseteq \{0,1\}^n$ be a 2-resistant set.

Claim 1. *Let* x *be an infeasible point with at least three infeasible neighbors. If* $x \triangle e_i, x \triangle e_j$ *are infeasible for distinct* $i, j \in [n]$*, then* $x \triangle e_i \triangle e_j$ *is also infeasible.*

Proof of Claim. Suppose for a contradiction that $x \triangle e_i \triangle e_j$ is feasible. Since x has at least three infeasible neighbors, there is a coordinate $k \in [n] - \{i, j\}$ such that $x \triangle e_k$ is infeasible. Then the 3-dimensional restriction of S containing $x \triangle e_i, x \triangle e_j, x \triangle e_k$ is a set $F \subseteq \{0,1\}^3$ such that $F \cap \{000, 100, 010, 001, 110\} = \{110\}$, a contradiction to Proposition 6.6. $\diamond$

Claim 2. *Let* x *be an infeasible point with at least three infeasible neighbors. Let* $k \geq 3$ *be the number of infeasible neighbors of* x*. Then the* k*-dimensional hypercube containing* x *and its infeasible neighbors is infeasible.*

Proof of Claim. After a possible twisting and relabeling, if necessary, we may assume that $x = \mathbf{0}$ and its infeasible neighbors are $e_1, \ldots, e_k$. We need to show that for all subsets $I \subseteq [k]$, $\sum_{i \in I} e_i \in \overline{S}$. We will proceed by induction on $|I| \geq 0$. The base cases $|I| \in \{0, 1\}$ hold by assumption, and the case $|I| = 2$ follows from Claim 1. For the induction step, assume that $|I| \geq 3$. After a possible relabeling, if necessary, we may assume that $I = [\ell]$. Let $y := \sum_{i=1}^{\ell-2} e_i$. By the induction hypothesis, y and its three neighbors

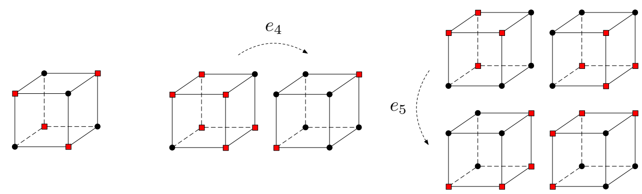


Figure 6.3: An illustration of the strictly non-polar sets that are 2-resistant.

$y\triangle e_{\ell-2}, y\triangle e_{\ell-1}, y\triangle e_\ell$ are all infeasible. It therefore follows from Claim 1 that $y\triangle e_{\ell-1}\triangle e_\ell = \sum_{i=1}^{\ell} e_i$ is infeasible, thereby completing the induction step. $\diamond$

Let $K \subseteq \overline{S}$ be an infeasible component, and let k be the maximum number of infeasible neighbors of a point in K. If $k \leq 2$, then K has maximum degree at most two. Otherwise, $k \geq 3$. It then follows from Claim 2 that K contains a k-dimensional hypercube. Our maximal choice of k in turn implies that K is in fact the k-dimensional hypercube. Thus, every infeasible component is a hypercube or has maximum degree at most two.

($\Leftarrow$) Suppose conversely that every infeasible component of S is a hypercube or has maximum degree at most two. It can be readily checked that,

Claim 3. *If S' is a minor of S, then every infeasible component of S' is a hypercube or has maximum degree at most two.*

As a consequence, S does not have a minor $F \subseteq \{0,1\}^3$ such that $F \cap \{000, 100, 010, 001, 110\} = \{110\}$. Thus, by Proposition 6.6, S is 2-resistant. $\square$

Recall that a set is strictly non-polar if it is non-polar but every proper restriction is polar. Consider the sets $R_{1,1}, R_{2,1}, R_5$ displayed in Figure 6.3. These sets are strictly non-polar sets, and as every each infeasible component has maximum degree at most two, they are 2-resistant by Theorem 1.44. In fact, as was shown in Corollary 5.15 of Chapter 5, these sets are, up to isomorphism, the only strictly non-polar sets whose infeasible components have maximum degree at most two. Leveraging this result, we will prove that these sets are also the only strictly non-polar sets that are 2-resistant. To do so, we need the following lemma:

Lemma 6.7 ([1]). *Take an integer $n \geq 5$ and a set $S \subseteq \{0,1\}^n$ of maximum degree at most two. Then $|S| \geq 2^{n-1}$.*

Proof. It suffices to prove this for $n = 5$, as the general case follows from a simple inductive argument. For $i, j \in \{0,1\}$, let $S_{ij} \subseteq \{0,1\}^3$ be the restriction of S obtained after i-restricting coordinate 4 and j-restricting coordinate 5. We may assume that $|S_{00}| + |S_{10}| \leq 7$ and $|S_{00}| \leq 3$. After a possible twisting of coordinates $1,2,3$, we may assume that $\{000, 111\} \subseteq S_{00} \subseteq \{000, 111, 110\}$. This implies that $\{001, 101, 011\} \subseteq S_{10}$. Since $|S_{00}| + |S_{10}| \leq 7$, we get that $S_{00} = \{000, 111, 110\}$ and therefore $S_{10} = \{001, 101, 011, 110\}$. Since S has maximum degree at most two, it follows that

$\{100, 010, 001, 101, 011\} \subseteq S_{01}$ and $\{000, 100, 010\} \subseteq S_{11}$, implying in turn that $|S_{01}| + |S_{11}| \geq 8$. In fact, as S has maximum degree at most two, $|S_{01}| + |S_{11}| > 8$, so $|S| \geq 7 + 9 = 16$, as required. $\square$

Using this lemma, we prove the following:

Lemma 6.8 ([1]). *Take an integer $n \geq 5$ and a nonempty set $S \subseteq \{0,1\}^n$, where every infeasible component is a hypercube or has maximum degree at most two. If S has no $R_{1,1}$ restriction and one of its infeasible components is a hypercube of dimension at least 3, then*

- *$|S| \geq 2^{n-1}$, and*
- *if $|S| = 2^{n-1}$, then S is either a hypercube of dimension $n-1$ or the union of antipodal hypercubes of dimension $n-2$.*

Proof. We will prove this by induction on $n \geq 5$. The base case $n = 5$ is clear. For the induction step, assume that $n \geq 6$. For $i \in \{0,1\}$, let $S_i \subseteq \{0,1\}^{n-1}$ be the i-restriction of S over coordinate n. If one of S_0, S_1 is empty, then the other one must be $\{0,1\}^{n-1}$, so S is a hypercube of dimension $n-1$ and the induction step is complete. We may therefore assume that S_0, S_1 are nonempty.

Assume in the first case that S has an infeasible hypercube of dimension ≥ 4 active in, say, direction e_n. Then both S_0, S_1 have infeasible hypercubes of dimension ≥ 3. Thus by the induction hypothesis, $|S_0| \geq 2^{n-2}$ and $|S_1| \geq 2^{n-2}$, implying in turn that $|S| = |S_0| + |S_1| \geq 2^{n-1}$. Assume next that $|S| = 2^{n-1}$. Then $|S_0| = |S_1| = 2^{n-2}$, so by the induction hypothesis, each S_i is either a hypercube of dimension $n-2$ or the union of antipodal hypercubes of dimension $n-3$. If one of S_0, S_1 is a hypercube, then as every infeasible component of S is a hypercube or has maximum degree at most two, S is either a hypercube of dimension $n-1$ or the union of antipodal hypercubes of dimension $n-2$. Otherwise, each one of S_0, S_1 is the union of two antipodal hypercubes of dimension $n-3 \geq 3$. As S has no $R_{1,1}$ restriction, it must be that $S_0 = S_1$, implying in turn that S is the union of antipodal hypercubes of dimension $n-2$, thereby completing the induction step.

Assume in the remaining case that every infeasible component of S has maximum degree at most two or is a (3-dimensional) cube. By assumption, one of the infeasible components is a cube, which we may assume is contained in S_0. By the induction hypothesis, $|S_0| \geq 2^{n-2}$ and if equality holds, then S_0 is either a hypercube of dimension $n-2$ or the union of antipodal hypercubes of dimension $n-3$. If S_1 has an infeasible component that is a cube, then the induction hypothesis implies that $|S_1| \geq 2^{n-2}$, and if not, S_1 has maximum degree at most two, so by Lemma 6.7, $|S_1| \geq 2^{n-2}$. Either way, $|S_1| \geq 2^{n-2}$, so $|S| = |S_0| + |S_1| \geq 2^{n-1}$. We claim that equality cannot hold. Suppose for a contradiction that $|S| = 2^{n-1}$. Then $|S_0| = |S_1| = 2^{n-2}$. Then S_0 is either a hypercube of dimension $n-2 \geq 4$ or the union of antipodal hypercubes of dimension $n-3 \geq 3$. As S has no infeasible hypercube of dimension ≥ 4, it follows that $n = 6$ and S_0 is the union of antipodal cubes, and as $|S_1| = 2^{n-2}$, it follows that S has an $R_{1,1}$ restriction, a contradiction to our assumption. This completes the induction step. $\square$

We are now ready to prove the following:

Theorem 6.9 ([1]). *Up to isomorphism, $R_{1,1}, R_{2,1}, R_5$ are the only strictly non-polar sets that are 2-resistant.*

Proof. Take an integer $n \geq 1$ and a 2-resistant set $S \subseteq \{0,1\}^n$ without an $R_{1,1}, R_{2,1}, R_5$ restriction. We need to show that S is polar. By Theorem 1.44, every infeasible component is a hypercube or has maximum degree at most two. If S has maximum degree at most two, then by Corollary 5.15, S is polar. Otherwise, S has an infeasible hypercube of dimension ≥ 3. If $n = 4$ or $S = \emptyset$, then S is clearly polar. Otherwise, $n \geq 5$ and $S \neq \emptyset$. By Lemma 6.8, $|S| \geq 2^{n-1}$; if equality holds, then S is either a hypercube or the union of antipodal hypercubes, so S is clearly polar. Otherwise, $|S| > 2^{n-1}$, implying in particular that there are antipodal feasible points, so S is polar, as required. $\square$

This result, together with Theorem 6.4, has the following immediate consequence:

Corollary 6.10 ([1])**.** *Take an integer $n \geq 1$ and a 2-resistant set $S \subseteq \{0,1\}^n$. Then the following statements are equivalent:*

(i) $\text{cuboid}(S)$ *has the packing property,*

(ii) S is strictly polar,

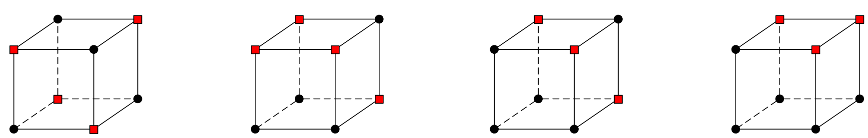
(iii) S has no $R_{1,1}, R_{2,1}, R_5$ restriction.

6.3 An excluded minor characterization of ± 1-resistant sets

Let us start with the following obvious remark:

Remark 6.11. *If a set is ± 1-resistant, then so is every restriction of it.*

Is ± 1-resistance a minor-closed property? The answer turns out to be yes, but the reason is not as straightforward as it was for 1- and 2-resistance. To see why, consider the four 3-dimensional sets displayed below:



$R_{1,1}$ F_1 F_2 F_3

and written as

$$\begin{aligned} R_{1,1} &= \{000, 110, 101, 011\} \\ F_1 &= \{000, 100, 010, 111\} \\ F_2 &= \{000, 100, 010, 001, 111\} \\ F_3 &= \{000, 100, 010, 001, 110\}. \end{aligned}$$

Observe that these four sets are not ± 1-resistant, as for each set, the removal of some feasible point results in a set isomorphic to either P_3 or S_3. Below we will prove that a ± 1-resistant set has none of these four sets as a minor. We will frequently appeal to Remark 6.1, implying that every minor of a ± 1-resistant set is 1-resistant.

Lemma 6.12 ([1]). *A ± 1-resistant set has no $R_{1,1}, F_1, F_2, F_3$ minor.*

Proof. Let $N \in \{R_{1,1}, F_1, F_2, F_3\}$. Take an integer $n \geq 1$ and a ± 1-resistant set $S \subseteq \{0,1\}^n$. By Remark 6.11, every restriction of S is also ± 1-resistant. We may therefore assume that no proper restriction of S has an N minor. Suppose for a contradiction that S has an N minor. Then N must be a projection of S. We may assume that N is obtained from S after projecting away coordinates $4, \ldots, n$. Pick $x^\star \in N$ such that $N - \{x^\star\}$ is isomorphic to either P_3 or S_3. Since S is ± 1-resistant however, it must have at least two feasible points projecting onto $x^\star$; we may assume that one point is contained in $\{x \in \{0,1\}^n : x_4 = 0\}$ and another is contained in $\{x \in \{0,1\}^n : x_4 = 1\}$. Let $R \subseteq \{0,1\}^4$ be obtained from S after projecting away coordinates $5, \ldots, n$. Then N is obtained from R after projecting away coordinate 4. For $i \in \{0,1\}$, let R_i be the i-restriction of R over coordinate 4. Observe that $R_0 \cup R_1 = N$ and $x^\star \in R_0 \cap R_1$. Since no proper restriction of S has an N minor, $R_0 \subsetneq N$ and $R_1 \subsetneq N$. There are four cases:

(1) $N = R_{1,1}$: As $R_0 \cap R_1 \neq \emptyset$, it follows that $|R_0| + |R_1| > |N| = 4$, so either $|R_0| \geq 3$ or $|R_1| \geq 3$. As $R_0 \subsetneq N$ and $R_1 \subsetneq N$, it follows that either $|R_0| = 3$ or $|R_1| = 3$, implying in turn that one of R_0, R_1 is isomorphic to P_3. As a result, S has a P_3 minor, a contradiction.

(2) $N = F_1$: As 000 is the only point whose removal from F_1 yields a set isomorphic to one of P_3, S_3, it follows that $x^\star = 000$. So $000 \in R_0 \cap R_1$. Since the 0-restriction of R over coordinate 2 (resp. 1) is 1-resistant, it follows that $100 \in R_0 \cap R_1$ (resp. $010 \in R_0 \cap R_1$). But then as $R_0 \cup R_1 = N$, either $R_0 = N$ or $R_1 = N$, a contradiction.

(3) $N = F_2$: Note that $x^\star = 111$, so $111 \in R_0 \cap R_1$. As S has no P_3 minor, no $R_{1,1}$ minor by (1) and no F_1 minor by (2), we have that $R_0 \ncong P_3, R_{1,1}, F_1$ and $R_1 \ncong P_3, R_{1,1}, F_1$. If $000 \notin R_0$, then as $R_0 \ncong R_{1,1}, P_3$ and as R_0 is 1-resistant, it follows that $R_0 = \{111\}$, implying in turn that $R_1 = N$, which is not the case. Thus, $000 \in R_0$ and similarly, $000 \in R_1$. But now, since $R_0 \subsetneq N$ and $R_1 \subsetneq N$, it follows that either $R_0 \cong F_1$ or $R_1 \cong F_1$, a contradiction.

(4) $N = F_3$: Note that $x^\star = 110$, so $110 \in R_0 \cap R_1$. Since the 1-restriction of R over coordinate 1 (resp. 2) is 1-resistant, it follows that $100 \in R_0 \cap R_1$ (resp. $010 \in R_0 \cap R_1$). We may assume, after possibly twisting coordinate 4, that $001 \in R_0$. Since $R_0 \neq N$, it follows that $R_0 \cong F_1$. Thus, S has an F_1 minor, a contradiction to (2).

In each one of the four cases, we reached a contradiction. This finishes the proof of the lemma. □

We are ready to prove an excluded minor characterization of ± 1-resistance:

Theorem 6.13 ([1]). *A set is ± 1-resistant if, and only if, it is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3$ minor.*

Proof. ($\Rightarrow$) A ± 1-resistant set is clearly 1-resistant, and has no $R_{1,1}, F_1, F_2, F_3$ minor by Lemma 6.12. ($\Leftarrow$) Take an integer $n \geq 3$ and a set $S \subseteq \{0,1\}^n$ that is not ± 1-resistant. If S is not 1-resistant, then we are done. Otherwise, there is a feasible point x such that $S - \{x\}$ has a P_3, S_3 minor. Thus, there is a set $Y \subseteq \{0,1\}^3$ of cardinality at most one such that S has one of $P_3 \cup Y, S_3 \cup Y$ as a minor. However, the sets $P_3 \cup Y, S_3 \cup Y$ are isomorphic to either $P_3, S_3, R_{1,1}, F_1, F_2, F_3$, implying in turn that S is either not 1-resistant or has one of $R_{1,1}, F_1, F_2, F_3$ as a minor. □

This theorem, together with Theorem 1.48, has the following immediate consequence:

Corollary 6.14 ([1])**.** *Being ± 1-resistant is a minor-closed property, and the excluded minors are fragile sets and $R_{1,1}, F_1, F_2, F_3$.*

Just like 2-resistance, in Chapter 8, we will find a structure theorem for ± 1-resistant sets, and show as a consequence that ± 1-resistance implies strict polarity.

6.4 Further notes

The origin of resistance

The Polarity Conjecture predicts that a cube-ideal and strictly polar set will always have a cuboid with the packing property. Finding a counterexample to this conjecture was our original motivation for studying resistance. To elaborate, our plan was to write a computer code to generate a cube-ideal and strictly polar set whose cuboid did not have the packing property.

We began by generating strictly polar sets. How? We started from an arbitrary set. At each iteration, we made a suitably chosen infeasible point feasible so as to destroy non-polar restrictions. Eventually, we destroyed all the non-polar restrictions, and ended up with a strictly polar set.

Not surprisingly, our strictly polar set was quite dense most of the time. As a result, due to the 2-locality of idealness, the strictly polar set was usually cube-ideal.

To increase our chances of finding a counterexample, we then looked for feasible points whose removal from the set would keep the set strictly polar. This procedure gave us a sequence of strictly polar sets starting with the one generated above, and each next set obtained from the preceding one by making a feasible point infeasible.

We then studied the cuboids of these strictly polar sets. Except for the ones at the very end of the sequence, the sets were most often cube-ideal and their cuboids had the packing property. The sets at the very end of the sequence were most often non-cube-ideal and their cuboids had an odd hole of dimension 5 (and rarely, dimension 7).

Theorem 1.49 explains why the sets in the sequence were most often cube-ideal, and Theorem 6.4 explains why their cuboids had the packing property. Meanwhile, Proposition 5.8 explains why at the very end of the sequence, the cuboids had odd holes of dimension 5 and dimension 7 as minors, because they are the smallest minimally non-packing clutters whose covering number is at least three.

Needless to say, we have not found a counterexample to the Polarity Conjecture.

Testing 2-resistance and ± 1-resistance

We saw in Theorem 1.46 that 1-resistance can be tested in polynomial time. By finding the excluded restrictions for the class of 2-resistant sets and for the class of ± 1-resistant sets, we can also show the following:

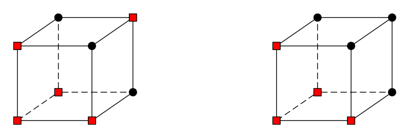
Theorem 6.15 ([1]). *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then in time $O(n^3|S|)$, one can test whether or not S is 2-resistant.*

Theorem 6.16 ([1]). *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then in time $O(n^2|S|^2)$, one can test whether or not S is ± 1-resistant.*

Chapter 7

The 1-resistant strictly non-polar sets that are half-dense

Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Recall that S is 1-resistant if, for all subsets $X \subseteq \{0,1\}^n$ of cardinality at most one, $S \cup X$ has no P_3, S_3 minor.



P_3 $\qquad$ S_3

Here and throughout this chapter, round points are feasible and square points are infeasible. Suppose that S is 1-resistant. We saw in Theorem 1.49 that S is cube-ideal, so cuboid(S) is ideal by Theorem 1.21. When does cuboid(S) have the packing property? Theorem 6.4 partially answered this question: cuboid(S) has the packing property if, and only if, S is strictly polar. So,

Question 7.1. *What are the* 1*-resistant strictly non-polar sets?*

Even though we have not been able to answer this question, in this chapter, we will find all the 1-resistant strictly non-polar sets that are half-dense. What are they? Consider the set

$$\begin{aligned} R_5 = &\{00000, 10000, 11000, 11100, 11110, 01110, 00110, 00010\} \\ &\cup \{01001, 01101, 00101, 10101, 10111, 10011, 11011, 01011\} \subseteq \{0,1\}^5, \end{aligned}$$

as displayed in Figure 7.1. Also, consider for each integer $k \geq 1$ the set

$$R_{k,1} = \left\{(\mathbf{0}^{k+1}, 0), (\mathbf{1}^{k+1}, 0)\right\} \cup \left\{(x, 1) : x \in \{0,1\}^{k+1} - \left\{\mathbf{0}^{k+1}, \mathbf{1}^{k+1}\right\}\right\} \subseteq \{0,1\}^{k+2},$$

as displayed in Figure 7.2. As discussed in the introduction, $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ are half-dense strictly non-polar sets. These sets are also 1-resistant:

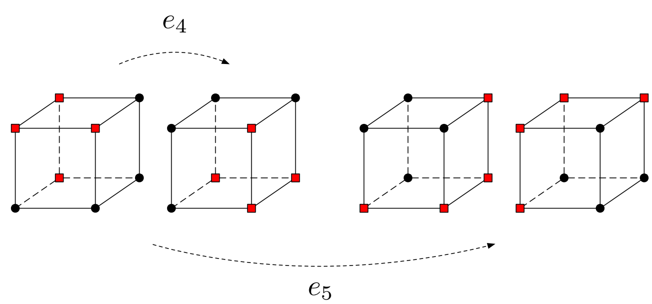


Figure 7.1: An illustration of R_5.

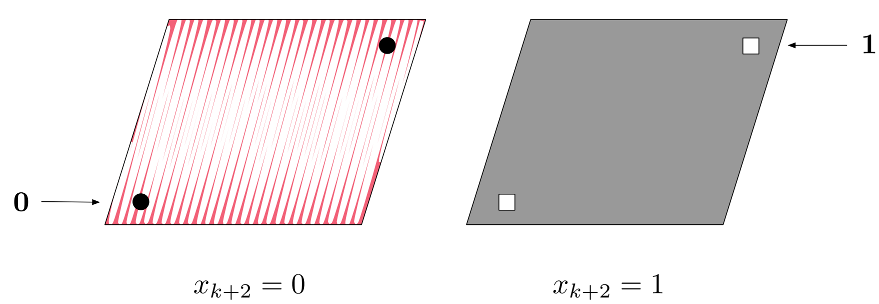


Figure 7.2: An illustration of $R_{k,1}$. The shaded plane on the left is infeasible, while the filled-in plane on the right is feasible. The two round points are feasible while the two square points are infeasible.

Remark 7.2. *$\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ are 1-resistant.*

Proof. Since its infeasible components are circuits and therefore have maximum degree two, R_5 is 2-resistant by Theorem 1.44. That is, for every subset $X \subseteq \{0,1\}^5$ of cardinality at most two, $R_5 \cup X$ has no P_3, S_3 minor. In particular, R_5 is 1-resistant.

Next take an integer $k \geq 1$. Then every induced clutter of $R_{k,1}$ is either $\{\emptyset\}, \{\{1\},\{2\},\ldots,\{k+2\}\}$, or $\{A, B, \{k+2\}\}$ for some nonempty subsets $A, B \subseteq [k+1]$ such that $A \cap B = \emptyset$ and $A \cup B = [k+1]$. In particular, the induced clutters of $R_{k,1}$ do not have intersecting members. Thus, $R_{k,1}$ is 1-resistant by Theorem 1.48. $\square$

Moving forward, we need a few tools. Recall that for points $a, b \in \{0,1\}^n$, $\mathrm{dist}(a,b)$ denotes the number of coordinates a, b disagree on. Recall further that G_n is the skeleton graph of $\{0,1\}^n$.

Remark 7.3. *For an integer $n \geq 1$, the following statements hold:*

- *For points $a, b, c \in \{0,1\}^n$, $\mathrm{dist}(a,b) + \mathrm{dist}(b,c) \geq \mathrm{dist}(a,c)$.*
- *For points $a, b \in \{0,1\}^n$, every (a,b)-path in G_n has at least $\mathrm{dist}(a,b)$ many edges.*

An (a,b)-path whose vertices, as traversed from a to b, are $a = v_0, v_1, \ldots, v_k = b$ will be represented as the sequence $(v_0, v_1, \ldots, v_k)$. The *length of a path* is the number of edges it has. An (a,b)-path of G_n is *straight* if it has length exactly $\mathrm{dist}(a,b)$.

Remark 7.4. *Take an integer $n \geq 1$. Then the following statements hold:*

- *Take distinct points a, b at Hamming distance $\ell \geq 1$, and let P be an (a,b)-path of G_n. Then P is straight if, and only if, there are ℓ distinct coordinates $i_1, \ldots, i_\ell$ such that*

$$P = (a, a\triangle e_{i_1}, a\triangle e_{i_1}\triangle e_{i_2}, \ldots, a\triangle e_{i_1}\triangle e_{i_2}\triangle\cdots\triangle e_{i_\ell} = b).$$

- *Pick distinct points a, b, c such that $\mathrm{dist}(a,b) + \mathrm{dist}(b,c) = \mathrm{dist}(a,c)$. If P is a straight (a,b)-path and Q a straight (b,c)-path of G_n, then $P \cup Q$ is a straight (a,c)-path of G_n.*[1]

Given a set $S \subseteq \{0,1\}^n$, a path in $G_n[S]$ is called a *feasible path* and a path in $G_n[\overline{S}]$ is called an *infeasible path*. We will prove the following theorem:

Theorem 7.5. *Take an integer $n \geq 3$ and a 1-resistant set $S \subseteq \{0,1\}^n$ that is non-polar. If every straight infeasible path has length at most $n-1$, then S has one of $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ as a restriction.*

The main result of this chapter is a consequence of Theorem 7.5:

> **Theorem 1.42 ([3]).** *Up to isomorphism, $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ are the only half-dense strictly non-polar sets that are 1-resistant.*

[1] If P is an (a,b)-path and Q is a (b,c)-path, then $P \cup Q$ denotes the (a,c)-walk that first traverses the vertices of P from a to b, and then traverses the vertices of Q from b to c.

Proof. Take an integer $n \geq 3$ and a half-dense strictly non-polar set $S \subseteq \{0,1\}^n$ that is 1-resistant. Since S is non-polar and half-dense, for each $x \in \{0,1\}^n$, one of $x, \mathbf{1}-x$ is feasible and the other one is infeasible. In particular, there is no antipodal pair of infeasible points. Since a straight path of length n has antipodal points as its ends by Remark 7.4, every straight infeasible path must have length at most $n-1$. Hence, as S is 1-resistant and non-polar, Theorem 7.5 implies that S has one of $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ as a restriction. As $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ are non-polar, and S is strictly non-polar, S must be isomorphic to one of $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$, as required. □

The rest of this chapter is dedicated to proving Theorem 7.5. In §7.1, we prove three key lemmas studying the different *propagation* features that 1-resistant sets satisfy. There, as an application, we prove Theorem 7.5 for the case when every straight infeasible path has length at most $n-2$. In §7.2, we introduce *straight circuits* and see how they can be used to obtain the desired restrictions. Finally, we prove Theorem 7.5 in §7.3.

7.1 Propagations in 1-resistant sets

Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. A *feasible component of* S is a component of $G_n[S]$. We say that S is *connected* if $G_n[S]$ is a connected graph. Let us start with the following proposition which best illustrates the title of this section:

Proposition 7.6. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$, where for all $x \in \{0,1\}^n$ and distinct $i, j \in [n]$, the following statement holds:*

if $x, x\triangle e_i, x\triangle e_j \in S$ then $x\triangle e_i \triangle e_j \in S$.

Then every feasible component of S is a hypercube.

Proof. We claim that the following statement holds for all subsets $I \subseteq [n]$ of cardinality at least two:

($\star$) If $x \in S$ and $x\triangle e_i \in S$ for each $i \in I$, then $x\triangle \left(\sum_{i\in I} e_i\right) \in S$.

We prove ($\star$) by induction on $|I| \geq 2$. The base case $|I| = 2$ is our hypothesis. For the induction step, assume that $k := |I| \geq 3$. After a possible twisting and relabeling, we may assume that $x = \mathbf{0}$ and $I = \{e_1, \ldots, e_k\}$. To prove ($\star$), we need to show that $\sum_{i=1}^k e_i \in S$. Let $y := \sum_{i=1}^{k-2} e_i$. If $k = 3$ then $y \in S$ by assumption, and if $k \geq 4$ then $y \in S$ by the induction hypothesis. Moreover, the induction hypothesis implies that $y\triangle e_{k-1}, y\triangle e_k \in S$. As a result, $\sum_{i=1}^k e_i = y\triangle e_{k-1}\triangle e_k \in S$ by our hypothesis, thereby completing the induction step. Thus, ($\star$) holds for every subset $I \subseteq [n]$ of cardinality at least two.

Take a feasible component S' of S. Let d be the maximum number of feasible neighbors of a point in S'. If $d \leq 1$, then $|S'| \in \{1,2\}$, so S' is clearly a hypercube. Otherwise, $d \geq 2$. After a possible twisting and relabeling, we may assume that $\mathbf{0}, e_1, \ldots, e_d \in S'$. Then for all subsets $I \subseteq [d]$ of cardinality at least two, $\sum_{i\in I} e_i \in S$ by ($\star$). As a result,

$$\left\{x \in \{0,1\}^n : x_j = 0, j \in [n] - [d]\right\} \subseteq S'.$$

Since every feasible point in S' has at most d feasible neighbors, equality holds above, so S' is a hypercube, as required. □

So the condition "if $x, x\triangle e_i, x\triangle e_j \in S$ then $x\triangle e_i\triangle e_j \in S$" has a *propagating effect*, ensuring that every feasible component is a hypercube. As a consequence,

Corollary 7.7. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then every feasible component of S is a hypercube if, and only if, S has no $\{00, 10, 01\}$ restriction.*

Proof. ($\Rightarrow$) If every feasible component is a hypercube, then there is no 2-dimensional restriction with exactly three feasible points, so there is no $\{00, 10, 01\}$ restriction. ($\Leftarrow$) Assume that S has no $\{00, 10, 01\}$ restriction. Then for all $x \in \{0,1\}^n$ and distinct $i, j \in [n]$: if $x, x\triangle e_i, x\triangle e_j \in S$ then $x\triangle e_i\triangle e_j \in S$. Thus, by Proposition 7.6, every feasible component of S is a hypercube. $\square$

Take a set $F \subseteq \{0,1\}^3$ such that

$$F \cap \{000, 100, 010, 001, 101, 011\} = \{101, 011\}.$$

Recall that F, and any set isomorphic to it, is called fragile. Recall from Theorem 1.48 that 1-resistance is equivalent to excluding fragile restrictions and $\{\mathbf{0}^k, \mathbf{1}^k - e_1\}, k \geq 4$.

Corollary 7.7 says that excluding $\{00, 10, 01\}$ restrictions has a propagating effect. In the same vein, we will see that excluding fragile restrictions and $\{\mathbf{0}^k, \mathbf{1}^k - e_1\}, k \geq 4$ restrictions, which is equivalent to 1-resistance, entails propagations.

7.1.1 The Plane, Sight and Path Propagation Lemmas

Lemma 7.8 (Plane Propagation). *Take an integer $n \geq 1$ and a 1-resistant set $S \subseteq \{0,1\}^n$. If $S \cap \{x : x_n = 0\} = \emptyset$, then S is a hypercube.*

Proof. Let $S_1 \subseteq \{0,1\}^n$ be the 1-restriction of S over coordinate n.

Claim 1. *Every feasible component of S_1 is a hypercube.*

Proof of Claim. Suppose otherwise. It follows from Corollary 7.7 that S_1 has a $\{00, 10, 01\}$ restriction. As $S \cap \{x : x_n = 0\} = \emptyset$, it follows that S has a $\{000, 100, 010\}$ restriction. However, $\{000, 100, 010\}$ is fragile, a contradiction to Theorem 1.48. $\diamond$

Claim 2. *S_1 is connected.*

Proof of Claim. Suppose otherwise. Let a, b be distinct feasible points at minimum Hamming distance and from different feasible components. Since a, b belong to different feasible components, it follows that $k := \mathrm{dist}(a, b) \geq 2$. Our minimality assumption implies that the restriction of S_1 containing a, b as antipodal points does not contain another feasible point, so it is isomorphic to $\{\mathbf{0}^k, \mathbf{1}^k\}$. Since $S \cap \{x : x_n = 0\} = \emptyset$, it follows that S has a $\{\mathbf{0}^{k+1}, \mathbf{1}^{k+1} - e_1\}$ restriction. As S is 1-resistant, Theorem 1.48 implies that $k = 2$. However, $\{\mathbf{0}^3, \mathbf{1}^3 - e_1\}$ is fragile, so S has a fragile restriction, a contradiction to Theorem 1.48. $\diamond$

Claims 1 and 2 together imply that S_1 is a hypercube, so S is a hypercube because $S \cap \{x : x_n = 0\} = \emptyset$, as required. $\square$

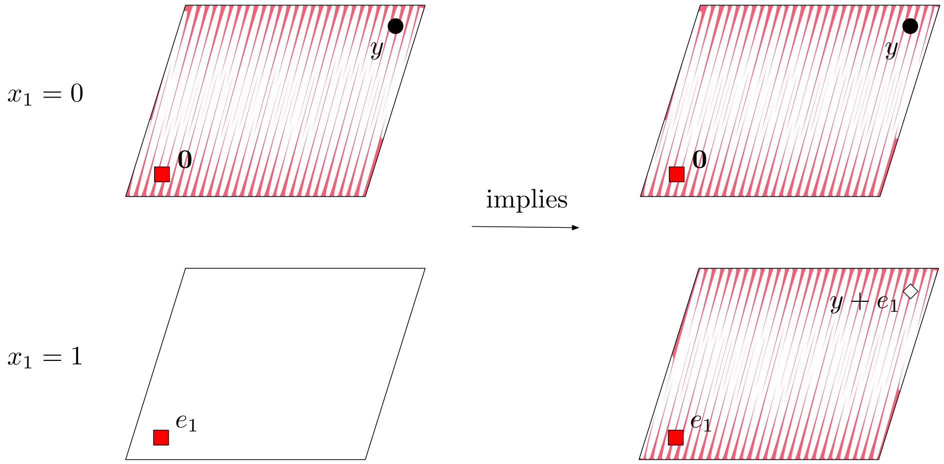


Figure 7.3: An illustration of Remark 7.9. Round points are feasible, the square points and the shaded region are infeasible, while the diamond point can be either.

For the next lemma, let us start with the following remark illustrated in Figure 7.3:

Remark 7.9. *Take an integer $n \geq 1$ and a 1-resistant set $S \subseteq \{0,1\}^n$, where $\mathbf{0}, e_1$ are infeasible. Assume that y is a minimal feasible point such that $y_1 = 0$. Then $\{z \in S : z \leq y + e_1, z_1 = 1\} \subseteq \{y + e_1\}$.*

Proof. Suppose otherwise. Pick a minimal point z of $\{z \in S : z \leq y+e_1, z_1 = 1\}$. Our contrary assumption implies that $z \neq y + e_1$, and therefore, z is also a minimal point of S. Moreover, as e_1 is infeasible, $z \neq e_1$. Pick members $C, C' \in \mathrm{ind}(S)$ such that $y = \chi_C$ and $z = \chi_{C'}$. Then $C \cap C' \neq \emptyset$, a contradiction as S is 1-resistant. $\square$

Take an integer $n \geq 1$ and a 1-resistant set $S \subseteq \{0,1\}^n$. A *valid pair* is a pair $[x, y]$ where x is infeasible, y is feasible, and $y \triangle x$ is a minimal feasible point of $S \triangle x$. If $[x, y]$ is a valid pair, we will say that x *sees* y. Remark 7.9 has the following immediate consequence:

Lemma 7.10 (Sight Propagation). *Take an integer $n \geq 1$, a 1-resistant set $S \subseteq \{0,1\}^n$ and a valid pair $[x, y]$. For a coordinate $i \in [n]$ such that $x \triangle e_i$ is infeasible, exactly one of the following statements holds:*

(i) $y \triangle e_i$ is feasible and $[x \triangle e_i, y \triangle e_i]$ is valid,

(ii) $y \triangle e_i$ is infeasible and $[x \triangle e_i, y]$ is valid.

Proof. After a possible twisting and relabeling, if necessary, we may assume that $x = \mathbf{0}$ and $i = 1$. As $[\mathbf{0}, y]$ is valid, y is a minimal feasible point. If $y_1 = 1$, then clearly (ii) holds and (i) does not. Otherwise, $y_1 = 0$. Then by Remark 7.9, $\{z \in S : z \leq y + e_1, z_1 = 1\}$ is either $\emptyset$ or $\{y + e_1\}$. In the first case, (ii) holds and (i) does not, while in the second case, (i) holds and (ii) does not. $\square$

The Sight Propagation Lemma has a subtle implication, which leads to the third propagation lemma. Take an integer $n \geq 1$, a 1-resistant set $S \subseteq \{0,1\}^n$ and an infeasible point x. A *valid sequence for* x is a nonempty sequence $(i_1, i_2, \ldots, i_k)$ of (not necessarily distinct) coordinates in $[n]$ such that the points

$$x \triangle e_{i_1}, x \triangle e_{i_1} \triangle e_{i_2}, \ldots, x \triangle e_{i_1} \triangle e_{i_2} \triangle \cdots \triangle e_{i_k}$$

are infeasible. Take a valid pair $[x, y]$ and a valid sequence $(i_1, \ldots, i_k)$ for x. In what follows, we will define the *trajectory of* $[x, y]$ *along* $(i_1, \ldots, i_k)$ as a sequence $(t_1, \ldots, t_k)$ whose entries are in $\{0, 1\}$, and given the sequence, we define the *image of* $[x, y]$ *along* $(i_1, \ldots, i_k)$ as

$$\mathrm{im}[x, y](i_1, \ldots, i_k) := y + \sum_{j=1}^{k} t_j e_{i_j} \mod 2.$$

The sequence $(t_1, \ldots, t_k)$ is defined as follows:

- for a valid pair $[x, y]$ and a valid sequence (i) of length 1, the trajectory of $[x, y]$ along (i) is

$$T[x, y](i) := \begin{cases} (1) & \text{if } y \triangle e_i \in S \\ (0) & \text{if } y \triangle e_i \notin S, \end{cases}$$

- for a valid pair $[x, y]$ and a valid sequence $(i_1, \ldots, i_k)$ of length at least 2, the trajectory of $[x, y]$ along $(i_1, \ldots, i_k)$ is defined recursively as follows: let $y' := \mathrm{im}[x, y](i_1, \ldots, i_{k-1})$ and

$$T[x, y](i_1, \ldots, i_k) := \begin{cases} T[x, y](i_1, \ldots, i_{k-1}) \cup (1) & \text{if } y' \triangle e_{i_k} \in S \\ T[x, y](i_1, \ldots, i_{k-1}) \cup (0) & \text{if } y' \triangle e_{i_k} \notin S. \end{cases}$$

The following is an immediate consequence of the Sight Propagation Lemma:

Remark 7.11. *Take an integer $n \geq 1$, a 1-resistant set $S \subseteq \{0,1\}^n$, a valid pair $[x, y]$ and a valid sequence $(i_1, \ldots, i_k)$ for x. Then $\mathrm{im}[x, y](e_{i_1}, \ldots, e_{i_k})$ is feasible and is seen by $x \triangle e_{i_1} \triangle \cdots \triangle e_{i_k}$.*

We are now equipped for the next lemma:

Lemma 7.12 (Path Propagation). *Take an integer $n \geq 1$, a 1-resistant set $S \subseteq \{0,1\}^n$ and a straight infeasible (a, b)-path P contained in $\{x : x_n = 0\}$. If $a \triangle e_n, b \triangle e_n$ are feasible, then for every vertex v of P, $v \triangle e_n$ is feasible.*

Proof. If a, b are the only vertices of P, then we are clearly done. Otherwise, as P is straight and contained in $\{x : x_n = 0\}$, we may assume by Remark 7.4 that after a possible relabeling,

$$P = \left(a = \mathbf{0}, e_1, e_1 + e_2, \ldots, \sum_{i=1}^{k} e_i = b \right),$$

where $k \in \{2, \ldots, n-1\}$. Assuming that $a \triangle e_n = e_n$ and $b \triangle e_n = e_n + \sum_{i=1}^{k} e_i$ are feasible, we need to show that the points $e_n + \sum_{i=1}^{j} e_i, j \in [k-1]$ are feasible. To this end, as P is infeasible, the sequence

$(1,\dots,k)$ is valid for $\mathbf{0}$. Consider the valid pair $[\mathbf{0}, e_n]$ and the valid sequence $(1,\dots,k)$. Let

$$(t_1,\dots,t_k) := T[\mathbf{0}, e_n](1,\dots,k)$$
$$y := \mathrm{im}[\mathbf{0}, e_n](1,\dots,k) = e_n + \sum_{i=1}^{k} t_i e_i.$$

By Remark 7.11, y is a feasible point seen by $\mathbf{0} + \sum_{i=1}^{k} e_i = b$.

Claim. $y = e_n + \sum_{i=1}^{k} e_i$.

Proof of Claim. We know that b sees y, and clearly, b sees $b \triangle e_n = e_n + \sum_{i=1}^{k} e_i$ too. Suppose for a contradiction that $y \neq b \triangle e_n$. As S is 1-resistant, the members of $\mathrm{ind}(S \triangle b)$ are pairwise disjoint, so the supposedly distinct points $y \triangle b$ and $b \triangle e_n \triangle b = e_n$ must have disjoint supports, a contradiction as $(y \triangle b)_n = 1$. $\diamond$

As an immediate consequence, $t_1 = t_2 = \cdots = t_k = 1$. Take a coordinate $j \in [k-1]$. Then the image of the valid pair $[\mathbf{0}, e_n]$ along the valid sequence $(1,\dots,j)$ for $\mathbf{0}$ is

$$\mathrm{im}[\mathbf{0}, e_n](1,\dots,j) = e_n + \sum_{i=1}^{j} t_i e_i = e_n + \sum_{i=1}^{j} e_i.$$

Thus, by Remark 7.11, $e_n + \sum_{i=1}^{j} e_i$ is feasible, as required. $\square$

7.1.2 A special case of Theorem 7.5

The Path Propagation Lemma has the following consequence:

Theorem 7.13. *Take an integer $n \geq 3$ and a 1-resistant set $S \subseteq \{0,1\}^n$ without antipodal points. If every straight infeasible path has length at most $n-2$, then S has an $R_{1,1}$ restriction.*

Proof. Let $m \leq n-2$ be the maximum length of a straight infeasible path. Since S does not have antipodal points, it follows that

($\star$) every straight feasible path has length at most m.

Let $P := (v_0, v_1, \dots, v_m)$ be a maximum length straight infeasible path. After a possible twisting and relabeling, we may assume that $v_0 = \mathbf{0}$ and $v_j = v_{j-1} \triangle e_j$ for $j \in [m]$, by Remark 7.4. Our maximal choice of P implies that for each $j \in \{m+1,\dots,n\}$, the points $v_0 \triangle e_j, v_m \triangle e_j$ are feasible. Thus, by the Path Propagation Lemma,

($\diamond$) for each $j \in \{m+1,\dots,n\}$, $P \triangle e_j$ is a feasible path.

If $m = n-2$, then $v_0 \triangle e_{m+1} = e_{n-1}$ and $v_m \triangle e_{m+2} = \mathbf{1} - e_{n-1}$ are feasible points by ($\diamond$), which cannot be the case as there are no antipodal feasible points. Thus, $m \leq n-3$. Let

$$R := S \cap \big\{x : x_i = 0, i \notin \{m+1, m+2, m+3\}\big\}.$$

By assumption, $\mathbf{0} \notin R$, and by ($\diamond$), $e_{m+1}, e_{m+2}, e_{m+3} \in R$. Moreover, by ($\star$), $P \triangle e_{m+1}, P \triangle e_{m+2}, P \triangle e_{m+3}$ are maximal straight feasible paths, so $e_{m+1} \triangle e_{m+2}, e_{m+2} \triangle e_{m+3}, e_{m+3} \triangle e_{m+1} \notin R$. As S is 1-resistant, it does not have a fragile restriction by Theorem 1.48, so $e_{m+1} \triangle e_{m+2} \triangle e_{m+3} \in R$. As a result, after dropping coordinates $[n] - \{m+1, m+2, m+3\}$ from R we obtain an $R_{1,1}$, so S has an $R_{1,1}$ restriction, as required. $\square$

Observe that Theorem 7.13 proves Theorem 7.5 for when every straight infeasible path has length at most $n-2$. We will next analyze the remaining case.

7.2 Straight circuits

Take an integer $n \geq 2$. Let C be a circuit of G_n whose vertices, denoted $V(C)$, are $v_0, v_1, \ldots, v_k$ in clockwise order. We will represent C as the sequence $(v_0, v_1, \ldots, v_k, v_0)$. Take a point $v \in \{0,1\}^n$, an integer $\ell \in \{2, \ldots, n\}$, and distinct coordinates $i_1, \ldots, i_\ell \in [n]$. Denote by $(v : i_1, i_2, \ldots, i_\ell)$ the circuit

$$(v_0, v_1, \ldots, v_\ell, \ldots, v_{2\ell-1}, v_{2\ell} = v_0)$$

where $v_0 = v$ and $v_j = v_{j-1} \triangle e_{i_j}$ and $v_{\ell+j} = v_{\ell+j-1} \triangle e_{i_j}$ for each $j \in [\ell]$. We will refer to $(v : i_1, i_2, \ldots, i_\ell)$ as a *straight circuit*. (Notice that any point of the straight circuit can be a starting point.) The *length of a circuit* is the number of edges it has. Given a set $S \subseteq \{0,1\}^n$, we refer to every circuit of $G_n[S]$ as a *feasible circuit* and to every circuit of $G_n[\overline{S}]$ as an *infeasible circuit*.

The purpose of this section is to prove the following statement:

> Take an integer $n \geq 4$ and a 1-resistant set $S \subseteq \{0,1\}^n$ that is non-polar. Assume that there is a straight infeasible circuit K of length $2(n-1)$ contained in $\{x : x_n = 0\}$ such that $V(K \triangle e_n) \subseteq S$. Then S has one of $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ as a restriction.[2]

This tool is crucial for proving the remaining case of Theorem 7.5. To prove this statement, let us start with the following lemma that is widely referenced throughout this section:

Lemma 7.14 (**Straight Circuit).** *Take an integer $n \geq 4$ and a 1-resistant set $S \subseteq \{0,1\}^n$ without antipodal points. Let K be a straight infeasible circuit of length $2(n-1)$ contained in $\{x : x_n = 0\}$ such that $V(K \triangle e_n) \subseteq S$. Then for a vertex $v \in \{x : x_n = 0\} - V(K)$ that is adjacent to a vertex of K, either*

$$\{v, \mathbf{1} - v \triangle e_n\} \subseteq \overline{S} \quad \textit{and} \quad \{v \triangle e_n, \mathbf{1} - v\} \subseteq S$$

or

$$\{v, \mathbf{1} - v \triangle e_n\} \subseteq S \quad \textit{and} \quad \{v \triangle e_n, \mathbf{1} - v\} \subseteq \overline{S}.$$

[2]If C is a circuit, then $C \triangle x$ denotes the circuit whose vertices are the vertices of C twisted by x.

Proof. After a possible relabeling and twisting, we may assume that

$$K = (\mathbf{0} : 1, 2, \ldots, n-1) = (v_0, v_1, \ldots, v_{n-1}, \ldots, v_{2n-3}, v_{2n-2} = v_0),$$

where $v_0 = \mathbf{0}$ and $v_j = v_{j-1} \triangle e_j$ and $v_{n-1+j} = v_{n-1+j-1} \triangle e_j$ for each $j \in [n-1]$.

Claim. *Take a vertex $w \in \{x : x_n = 0\} - V(K)$ that is adjacent to a vertex of K. If $w \in \overline{S}$ then $w \triangle e_n \in S$.*

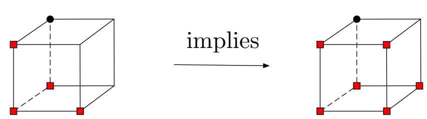
Proof of Claim. By the symmetry between the vertices of K, we may assume that w is adjacent to v_0, that is, $w = v_0 \triangle e_i$ for some $i \in [n-1] - \{1, n-1\}$. Let

$$P := (v_{n-1+i}, v_{n-1+i+1}, \ldots, v_{2n-2} = v_0) \quad \text{and} \quad Q := (v_0, v_1, \ldots, v_{i-1}) \quad \text{and} \quad R := P \cup Q.$$

Notice that R is a straight subpath of the infeasible circuit K. The Path Propagation Lemma implies that the feasible points in $R \triangle e_i$ should form a path. Thus, since $v_0 \triangle e_i = w \in \overline{S}$, it follows that either $V(P \triangle e_i) \subseteq \overline{S}$ or $V(Q \triangle e_i) \subseteq \overline{S}$. By symmetry, we may assume that $V(P \triangle e_i) \subseteq \overline{S}$. Consider now the straight infeasible path $P' := (v_0) \cup [P \triangle e_i]$ whose ends are v_0 and $v_{n-1+i} \triangle e_i = v_{n-1+i-1}$. Since $\{v_0 \triangle e_n, v_{n-1+i-1} \triangle e_n\} \subseteq V(K \triangle e_n) \subseteq S$, it follows from the Path Propagation Lemma that $P' \triangle e_n$ is a feasible path. In particular, we have that $v_0 \triangle e_i \triangle e_n = w \triangle e_n \in S$. $\diamond$

Now take a vertex $v \in \{x : x_n = 0\} - V(K)$ that is adjacent to a vertex of K. Assume in the first case that $v \in \overline{S}$. By the claim, $v \triangle e_n \in S$. Since S does not contain antipodal points, we get that $\mathbf{1} - v \triangle e_n \in \overline{S}$. Since $\mathbf{1} - v \triangle e_n$ is also adjacent to a vertex of K, and is not a vertex of K, it follows from the claim that $\mathbf{1} - v \in S$. Assume in the remaining case that $v \in S$. As S does not contain antipodal points, $\mathbf{1} - v \in \overline{S}$, so by the claim, we have $\mathbf{1} - v \triangle e_n \in S$, so its antipodal point $v \triangle e_n$ belongs to $\overline{S}$. This finishes the proof of the lemma. $\square$

Before moving on, we should point out that the results in this section will make heavy use of the Sight Propagation Lemma, most often applied as illustrated in the following figure,



and in most cases, we will leave it to the reader to identify the 3-dimensional cube where the Sight Propagation Lemma is being applied.

7.2.1 When there are no $R_{1,1}, R_5$ restrictions

We will need the following technical lemma:

Lemma 7.15. *Take an integer $n \geq 6$ and a 1-resistant set $S \subseteq \{0,1\}^n$ without antipodal points and without an $R_{1,1}, R_5$ restriction. Suppose $K := (\mathbf{0} : 1, 2, 3, 4, 5, \ldots, n-1)$ is a straight infeasible circuit, $V(K \triangle e_n) \subseteq S$, and $\{e_2, e_2 + e_3, e_3, e_1 + e_3\} \subseteq S$. Then, for each $i \in [n-4]$,*

$$K_i := (\mathbf{0} : 4, \ldots, 3+i, 1, 2, 3, 3+i+1, \ldots, n-1)$$

is a straight infeasible circuit, $V(K_i \triangle e_n) \subseteq S$ and $\{e_2, e_2 + e_3, e_3, e_1 + e_3\} \triangle e_4 \triangle \cdots \triangle e_{3+i} \subseteq S$.

Proof. We proceed by induction on $i \geq 1$. Let us first prove the base case $i = 1$, which we restate as follows:

> ($\star$) Take an integer $n \geq 6$ and a 1-resistant set $S \subseteq \{0,1\}^n$ without antipodal points and without an $R_{1,1}, R_5$ restriction. Suppose $K := (\mathbf{0} : 1,2,3,4,5,\ldots,n-1)$ is a straight infeasible circuit, $V(K \triangle e_n) \subseteq S$, and $\{e_2, e_2+e_3, e_3, e_1+e_3\} \subseteq S$. Then, $K' := (\mathbf{0} : 4,1,2,3,5,\ldots,n-1)$ is a straight infeasible circuit,$V(K' \triangle e_n) \subseteq S$ and $\{e_2, e_2+e_3, e_3, e_1+e_3\} \triangle e_4 \subseteq S$.

Define $P_0, P_1, Q_0, Q_1 \subseteq \{0,1\}^4$ as follows: P_0 (resp. P_1) is obtained after 0-restricting coordinates $5,\ldots,n-1$ and 0-restricting coordinate n (resp. 1-restricting coordinate n), and Q_0 (resp. Q_1) is obtained after 1-restricting coordinates $5,\ldots,n-1$ and 0-restricting coordinate n (resp. 1-restricting coordinate n). Since $V(K) \subseteq \overline{S}$ and $V(K \triangle e_n) \subseteq S$, it follows that

$$\{0000, 1000, 1100, 1110, 1111\} \subseteq \overline{P_0} \qquad \{0000, 1000, 1100, 1110, 1111\} \subseteq P_1$$
$$\{1111, 0111, 0011, 0001, 0000\} \subseteq \overline{Q_0} \qquad \{1111, 0111, 0011, 0001, 0000\} \subseteq Q_1.$$

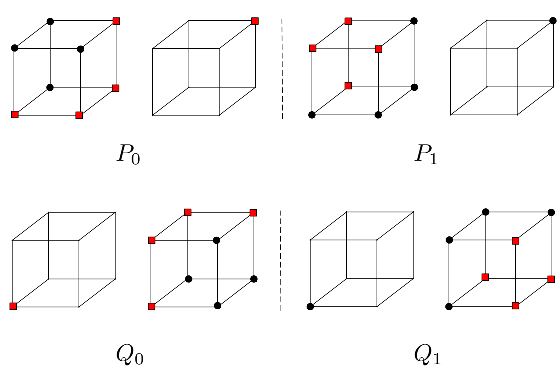
By assumption, we also know that

$$\{0100, 0110, 0010, 1010\} \subseteq P_0.$$

In $\{0,1\}^n$, each one of these points belongs to $\overline{K}$ and is adjacent to a vertex of K, so by the Straight Circuit Lemma,

$$\{0100, 0110, 0010, 1010\} \subseteq \overline{P_1}$$
$$\{1011, 1001, 1101, 0101\} \subseteq Q_0$$
$$\{1011, 1001, 1101, 0101\} \subseteq \overline{Q_1}.$$

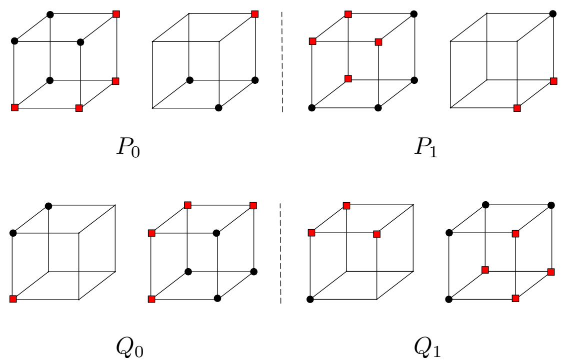
See the following figure illustrating the inclusions listed so far:



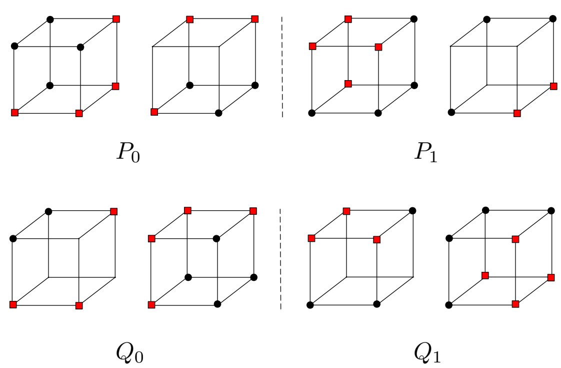
In the following claim, we will take advantage of the assumption that S has no $R_{1,1}, R_5$ restriction.

Claim 1. $\{0001, 1001, 1101\} \subseteq \overline{P_0} \cap P_1$ *and* $\{1110, 0110, 0010\} \subseteq \overline{Q_0} \cap Q_1$.

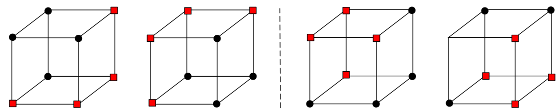
Proof of Claim. We will first show that $1101 \in \overline{P_0}$. Suppose for a contradiction that $1101 \in P_0$. Since S is 1-resistant, it follows from the Sight Propagation Lemma that $1001 \in P_0$. As the vertices in $\{0,1\}^n$ corresponding to $1101, 1001$ are in $\overline{K}$ and adjacent to vertices of K, the Straight Circuit Lemma implies that $\{1101, 1001\} \subseteq \overline{P_1}$ and $\{0010, 0110\} \subseteq Q_0 \cap \overline{Q_1}$. Since the restriction of $(P_0 \times \{0\}) \cup (P_1 \times \{1\})$ obtained after 1-restricting coordinate 2 and 0-restricting coordinate 3 is neither P_3 nor $R_{1,1}$, it follows that $0101 \in P_0$. As S does not contain antipodal points, we get that $1010 \in \overline{Q_1}$:



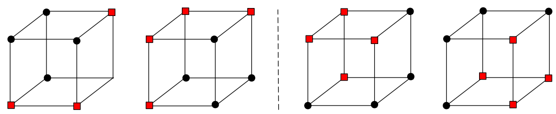
Since the 0-restriction of Q_1 over coordinate 2 is 1-resistant, it follows that $1000 \in Q_1$, so by the Straight Circuit Lemma, $1000 \in \overline{Q_0}$ and $0111 \in \overline{P_0} \cap P_1$. As the 1-restriction of Q_1 over coordinate 3 is 1-resistant, we have $1110 \in Q_1$, and so by the Straight Circuit Lemma, $1110 \in \overline{Q_0}$ and $0001 \in \overline{P_0} \cap P_1$:



Since the 1-restriction of P_0 over coordinate 3 is 1-resistant, we get that $1011 \in P_0$, and by the Straight Circuit Lemma, $1011 \in \overline{P_1}$ and $0100 \in Q_0 \cap \overline{Q_1}$. As the restriction of $(Q_0 \times \{0\}) \cup (Q_1 \times \{1\})$ obtained after 1-restricting coordinate 1 and 0-restricting coordinate 4 is 1-resistant and has no $R_{1,1}$ restriction, it follows that $1100 \in Q_1$ and $1010 \in Q_0$. Since S does not have antipodal points, we get that $0011 \in \overline{P_0}$ and $0101 \in \overline{P_1}$:



P_0 $\qquad$ P_1



Q_0 $\qquad$ Q_1

Since the 0-restriction of P_1 over coordinate 2 is 1-resistant, it follows that $0011 \in P_1$, implying in turn that $(P_0 \times \{0\}) \cup (P_1 \times \{1\}) \cong R_5$, so S has an R_5 restriction, a contradiction. Thus, $1101 \in \overline{P_0}$.

It follows from the Sight Propagation Lemma that $\{1001, 0001\} \subseteq \overline{P_0}$. By the Straight Circuit Lemma, $\{0001, 1001, 1101\} \subseteq P_1$ and $\{1110, 0110, 0010\} \subseteq \overline{Q_0} \cap Q_1$, as claimed. $\diamond$

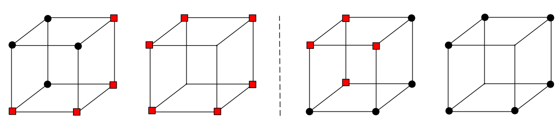
Recall that $K' = (\mathbf{0} : 4, 1, 2, 3, 5, \ldots, n-1)$. Notice that by Claim 1, K' is a straight infeasible circuit such that $V(K' \triangle e_n) \subseteq S$. In the following claim, we will apply the Straight Circuit Lemma to the straight infeasible circuit K'.

Claim 2. $\{0101, 0111, 0011, 1011\} \subseteq P_0$.

Proof of Claim. Since P_1 is 1-resistant, it follows from the Sight Propagation Lemma that either

$$\{0011, 0111\} \subseteq P_1 \quad \text{or} \quad \{0011, 0111\} \subseteq \overline{P_1}.$$

We claim that $\{0011, 0111\} \subseteq \overline{P_1}$. Suppose for a contradiction that $\{0011, 0111\} \subseteq P_1$. After applying the Straight Circuit Lemma to K', we get that $\{0011, 0111\} \subseteq \overline{P_0}$:



P_0 $\qquad$ P_1

Since P_0 is 1-resistant, it follows from Theorem 1.48 that P_0 has no fragile restriction. However, either its 0-restriction over coordinate 2 or its 1-restriction over coordinate 3 is fragile, a contradiction. Thus, $\{0011, 0111\} \subseteq \overline{P_1}$. After applying the Straight Circuit Lemma to K', we get that $\{0011, 0111\} \subseteq P_0$. As P_0 is 1-resistant, it follows that $\{0101, 1011\} \subseteq P_0$, as required. $\diamond$

By Claim 2, $\{e_2, e_2 + e_3, e_3, e_1 + e_3\} \triangle e_4 \subseteq S$. This proves $(\star)$ and in turn the base case $i = 1$. For the induction step, assume that $i \geq 2$. Then an application of $(\star)$ to K_{i-1}, instead of K, implies that $K_i := (\mathbf{0} : 4, \ldots, 3+i, 1, 2, 3, 3+i+1, \ldots, n-1)$ is a straight infeasible circuit, $V(K_i \triangle e_n) \subseteq S$, and $\{e_2, e_2 + e_3, e_3, e_1 + e_3\} \triangle e_4 \triangle \cdots \triangle e_{3+i} \subseteq S$, thereby completing the induction step. $\square$

Let $D_3 := \{000, 100, 110, 111\} \subseteq \{0,1\}^3$. Using the preceding lemma, we prove the following:

Proposition 7.16. *Take an integer $n \geq 5$ and a 1-resistant set $S \subseteq \{0,1\}^n$ without antipodal points and without an $R_{1,1}, R_5$ restriction. Let K be a straight infeasible circuit of length $2(n-1)$ contained in $\{x : x_n = 0\}$ such that $V(K \triangle e_n) \subseteq S$. Then S does not have a D_3 restriction whose infeasible points all belong to K.*

Proof. After a possible relabeling and twisting, we may assume that $K = (\mathbf{0} : 1, 2, \ldots, n-1)$. Suppose for a contradiction that S has a D_3 restriction whose infeasible points all belong to K. By symmetry, we may assume that the D_3 restriction is obtained after 0-restricting coordinates $4, \ldots, n$, that is, $\{e_2, e_2 + e_3, e_3, e_1 + e_3\} \subseteq S$. Assume in the first case that $n \geq 6$. It then follows from Lemma 7.15 that for each $i \in [n-4]$, $K_i := (\mathbf{0} : 4, \ldots, 3+i, 1, 2, 3, 3+i+1, \ldots, n-1)$ is a straight infeasible circuit,$V(K_i \triangle e_n) \subseteq S$, and $\{e_2, e_2 + e_3, e_3, e_1 + e_3\} \triangle e_4 \triangle \cdots \triangle e_{3+i} \subseteq S$. In particular, setting $i = n-4$, we get that

$$e_3 \triangle e_4 \triangle e_5 \triangle \cdots \triangle e_{n-1} \in S.$$

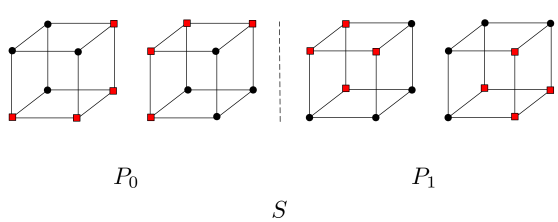
However, $e_3 \triangle e_4 \triangle e_5 \triangle \cdots \triangle e_{n-1} \in K \subseteq \overline{S}$, a contradiction. Assume in the remaining case that $n = 5$. Let $P_0 \subseteq \{0,1\}^4$ (resp. $P_1 \subseteq \{0,1\}^4$) be the 0-restriction (resp. 1-restriction) of S over coordinate 5. Since $V(K) \subseteq \overline{S}$ and $V(K \triangle e_n) \subseteq S$, it follows that

$$\begin{aligned} \{0000, 1000, 1100, 1110, 1111, 0111, 0011, 0001\} &\subseteq \overline{P_0} \\ \{0000, 1000, 1100, 1110, 1111, 0111, 0011, 0001\} &\subseteq P_1. \end{aligned}$$

As the 0-restriction of P_0 over coordinate 4 yields D_3, we also know that $\{0100, 0110, 0010, 1010\} \subseteq P_0$. In $\{0,1\}^5$, each one of these points belongs to $\overline{K}$ and is adjacent to a vertex of K, so by the Straight Circuit Lemma,

$$\begin{aligned} \{0100, 0110, 0010, 1010\} &\subseteq \overline{P_1} \\ \{1011, 1001, 1101, 0101\} &\subseteq P_0 \\ \{1011, 1001, 1101, 0101\} &\subseteq \overline{P_1}. \end{aligned}$$

The points given above determine that $S \cong R_5$,



a contradiction. $\square$

7.2.2 Finding $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ restrictions

For an integer $n \geq 3$, we will need the following property defined on the points x in $\{0,1\}^n$:

$$x \text{ is feasible} \Leftrightarrow \mathbf{1} - x\triangle e_n \text{ is feasible} \Leftrightarrow x\triangle e_n \text{ is infeasible} \Leftrightarrow \mathbf{1} - x \qquad (\diamond)$$

Lemma 7.17. *Take an integer $n \geq 5$ and a 1-resistant set $S \subseteq \{0,1\}^n$ that does not have antipodal points, and let $K := (\mathbf{0} : 1, 2, \ldots, n-1)$ be a straight infeasible circuit such that $V(K\triangle e_n) \subseteq S$. Suppose that $\ell \in \{2, \ldots, n-3\}$ is an integer such that the points in $\{x : x_{\ell+1} = \cdots = x_n = 0\}$ satisfy $(\diamond)$ and the feasible points in there form a hypercube. Then one of the following statements hold:*

- *S has one of $\{R_{k,1} : 1 \leq k \leq \ell\} \cup \{R_5\}$ as a restriction, or*
- *the points in $\{x : x_{\ell+2} = \cdots = x_n = 0\}$ satisfy $(\diamond)$ and the feasible points in there form a hypercube.*

Proof. Let us proceed by induction on $\ell \geq 2$.

For the base case, assume that $\ell = 2$. Notice that every point in $\{x : x_4 = \cdots = x_n = 0\}$ either belongs to K or is adjacent to a vertex of K. It therefore follows from the Straight Circuit Lemma that every point in $\{x : x_4 = \cdots = x_n = 0\}$ satisfies $(\diamond)$. Suppose that the feasible points in $\{x : x_4 = \cdots = x_n = 0\}$ do not form a hypercube. Let $H \subseteq \{0,1\}^4$ be the 0-restriction of S over coordinates $4, \ldots, n-1$. Since $V(K) \subseteq \overline{S}$ and $V(K\triangle e_n) \subseteq S$, we see that $\{0000, 1000, 1100, 1110\} \subseteq \overline{H}$ and $\{0001, 1001, 1101, 1111\} \subseteq H$. As the 0-restriction of H over the last coordinate is not a hypercube, one of the following inclusions must hold:

- $\{0100, 1010\} \subseteq H$: By $(\diamond)$, $\{0101, 1011\} \subseteq \overline{H}$. If $|\{0010, 0110\} \cap H| = 0$, then by $(\diamond)$, $H \cong R_{2,1}$, so S has an $R_{2,1}$ restriction. If $|\{0010, 0110\} \cap H| = 1$, then by $(\diamond)$, H, and therefore S, has an $R_{1,1}$ restriction. Otherwise, when $|\{0010, 0110\} \cap H| = 2$, then $H \cong D_3$ and so S has a D_3 restriction whose infeasible points all belong to K, so by Proposition 7.16, S has one of $R_{1,1}, R_5$ as a restriction.
- $\{0110, 1010\} \subseteq H$: Since H is 1-resistant, it follows that $0100 \in H$, so by the preceding case, S has one of $R_{1,1}, R_{2,1}, R_5$ as a restriction.
- $\{0100, 0010\} \subseteq H$: Since H is 1-resistant, it follows that $1010 \in H$, so by the first case, S has one of $R_{1,1}, R_{2,1}, R_5$ as a restriction.

In each case, we see that S has one of $R_{1,1}, R_{2,1}, R_5$ as a restriction, thereby proving the base case $\ell = 2$.

For the induction step, assume that $\ell \geq 3$. Then $n \geq 6$. Let $S' := S \cap \{x : x_{\ell+1} = \cdots = x_n = 0\}$. By assumption, S' is a (possibly empty) hypercube, which excludes the points $\mathbf{0}, \sum_{i=1}^{\ell} e_i$ as these two points belong to the infeasible circuit K. Since S' is a hypercube and the points in $\{x : x_{\ell+1} = \cdots = x_n = 0\}$ satisfy $(\diamond)$,

> (1) every infeasible point of $\{x : x_{\ell+1} = \cdots = x_n = 0\}$ appears on a straight infeasible circuit $K' := (\mathbf{0} : i_1, \ldots, i_\ell, \ell+1, \ldots, n-1)$ such that $V(K'\triangle e_n) \subseteq S$, where $i_1, \ldots, i_\ell$ is some permutation of $1, \ldots, \ell$.

We will use (1) throughout the proof to reroute the circuit K. Notice that together with the Straight Circuit Lemma, (1) implies that

(2) every point of $\{x : x_{\ell+2} = \cdots = x_n = 0\}$ adjacent to an infeasible point of $\{x : x_{\ell+1} = \cdots = x_n = 0\}$ satisfies ($\diamond$).

As a result, if the feasible points in $\{x : x_{\ell+2} = \cdots = x_n = 0\}$ form a hypercube, then every infeasible point in $\{x : x_{\ell+2} = \cdots = x_n = 0\}$ satisfies ($\diamond$), and so every infeasible point of $\{x : x_{\ell+2} = \cdots = x_n = 0\}$ appears on a straight infeasible circuit $K' := (\mathbf{0} : i_1, \ldots, i_{\ell+1}, \ell+2, \ldots, n-1)$ such that $V(K' \triangle e_n) \subseteq S$, where $i_1, \ldots, i_{\ell+1}$ is some permutation of $1, \ldots, \ell+1$. Thus, by the Straight Circuit Lemma,

(3) if the feasible points in $\{x : x_{\ell+2} = \cdots = x_n = 0\}$ form a hypercube, then every infeasible point in $\{x : x_{\ell+2} = \cdots = x_n = 0\}$ satisfies ($\diamond$).

Suppose that S has none of $\{R_{k,1} : 1 \leq k \leq \ell\} \cup \{R_5\}$ as a restriction. By (3), it suffices to show that the feasible points in $\{x : x_{\ell+2} = \cdots = x_n = 0\}$ form a hypercube. As $\mathbf{0}, \sum_{i=1}^{\ell} e_i \notin S'$, it follows that S' is a hypercube of dimension at most $\ell - 2$. There are five cases:

(i) $S' = \emptyset$,

(ii) S' is nonempty, of dimension at most $\ell - 3$, and has no vertex adjacent to $\sum_{i=1}^{\ell} e_i$,

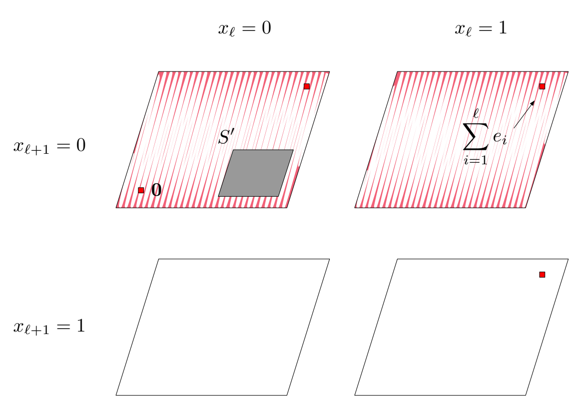
(iii) S' is nonempty, of dimension at most $\ell - 3$, and has a vertex adjacent to $\sum_{i=1}^{\ell} e_i$,

(iv) S' is of dimension $\ell - 2$ and $\ell = 3$,

(v) S' is of dimension $\ell - 2$ and $\ell \geq 4$.

(i) In this case, it follows from the Plane Propagation Lemma that the feasible points in $\{x : x_{\ell+2} = \cdots = x_n = 0\}$ form a hypercube, thereby completing the induction step.

(ii) In this case, after possibly relabeling coordinates $1, \ldots, \ell$ and rerouting K according to (1), we may assume that $S' \subseteq \{x \in \{0,1\}^n : x_\ell = 0\}$ while K remains as $(\mathbf{0} : 1, \ldots, n-1)$. Consider the following illustration of $\{x : x_{\ell+2} = \cdots = x_n = 0\}$:



The filled-in parallelogram shows the feasible points of S', while the shaded area and the square vertices indicate infeasible points. As $S' \neq \emptyset$, the infeasible point $\sum_{i=1}^{\ell} e_i$ sees a feasible point in S', so by the Sight Propagation Lemma, $\sum_{i=1}^{\ell+1} e_i$ sees a feasible point in $S' \cup (S' \triangle e_{\ell+1})$. In particular,

(4) $S'\triangle e_{\ell+1}\triangle e_\ell$ contains an infeasible point,

and $\sum_{i=1}^{\ell+1} e_i - e_\ell$ is infeasible, and by the Straight Circuit Lemma, $\sum_{i=1}^{\ell+1} e_i - e_\ell$ satisfies ($\diamond$). Consider now the straight infeasible circuit

$$K' := (\mathbf{0} : 1, \ldots, \ell-1, \ell+1, \ell, \ldots, n-1)$$

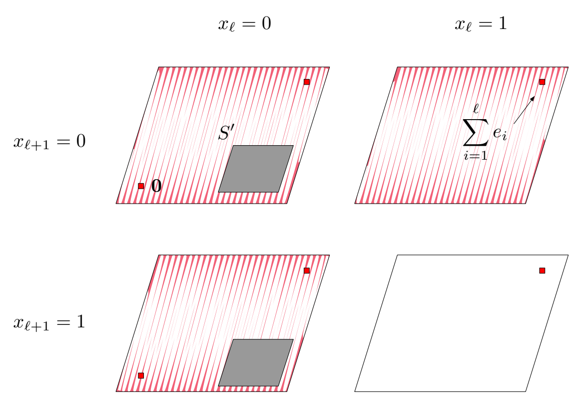
such that $V(K'\triangle e_n) \subseteq S$. Let us apply the induction hypothesis to K' given that the points in $\{x : x_\ell = x_{\ell+1} = x_{\ell+2} = \cdots = x_n = 0\}$ satisfy ($\diamond$) and its feasible points form a hypercube. The induction hypothesis implies that the points in $\{x : x_\ell = x_{\ell+2} = \cdots = x_n = 0\}$ also satisfy ($\diamond$) and its feasible points form a hypercube. In particular, by (2), the points in $\{x : x_{\ell+2} = \cdots = x_n = 0\}$ all satisfy ($\diamond$). Moreover, as $S' \neq \emptyset$, $S \cap \{x : x_\ell = x_{\ell+2} = \cdots = x_n = 0\}$ is either S' or $S' \cup (S'\triangle e_{\ell+1})$.

Assume in the first case that $S \cap \{x : x_\ell = x_{\ell+2} = \cdots = x_n = 0\} = S'$. We then must have that

$$S \cap \{x : x_{\ell+2} = \cdots = x_n = 0\} = S'.$$

Suppose not. Pick the closest pair of feasible vertices a, b such that $a \in S'$ and $b \in \{x : x_{\ell+2} = \cdots = x_n = 0\} - S'$. Since the points in $\{x : x_{\ell+2} = \cdots = x_n = 0\}$ satisfy ($\diamond$), it follows that the restriction of S containing $a, b\triangle e_n$ as antipodal points is one of $\{R_{k,1} : 1 \leq k \leq \ell\}$ as a restriction, a contradiction. Thus, the equation above holds, implying in turn that the feasible points in $\{x : x_{\ell+2} = \cdots = x_n = 0\}$ form a hypercube, thereby completing the induction step.

Assume in the remaining case that $S \cap \{x : x_\ell = x_{\ell+2} = \cdots = x_n = 0\} = S' \cup (S'\triangle e_{\ell+1})$:



Consider the straight infeasible circuit

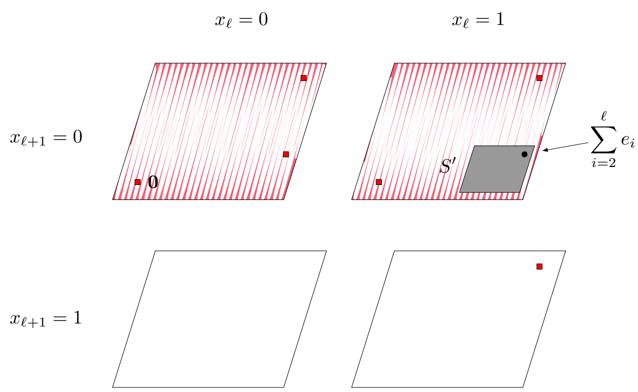
$$K'' := (e_{\ell+1} : 1, \ldots, \ell, \ell+2, \ldots, n-1, \ell+1)$$

such that $V(K''\triangle e_n) \subseteq S$. Let us apply the induction hypothesis to K'' given that the points in $\{x : x_\ell = x_{\ell+2} = \cdots = x_n = 0, x_{\ell+1} = 1\}$ satisfy ($\diamond$) and its feasible points form a hypercube. The induction hypothesis implies that the feasible points in $\{x : x_{\ell+2} = \cdots = x_n = 0, x_{\ell+1} = 1\}$ form a hypercube. That is, $S \cap \{x : x_{\ell+2} = \cdots = x_n = 0, x_{\ell+1} = 1\}$ is either $S'\triangle e_{\ell+1}$ or $(S'\triangle e_{\ell+1}) \cup (S'\triangle e_{\ell+1}\triangle e_\ell)$. However, the latter is not possible by (4), so $S \cap \{x : x_{\ell+2} = \cdots = x_n = 0, x_{\ell+1} = 1\} = S' \triangle e_{\ell+1}$ and

$$S \cap \{x : x_{\ell+2} = \cdots = x_n = 0\} = S' \cup (S' \triangle e_{\ell+1}).$$

Thus, the feasible points in $\{x : x_{\ell+2} = \cdots = x_n = 0\}$ form a hypercube, thereby completing the induction step.

(iii) In this case, as S' has dimension at most $\ell - 3$, it cannot have a vertex adjacent to $\mathbf{0}$. So, after possibly relabeling coordinates $1, \ldots, \ell$ and rerouting K according to (1), we may assume that $S' \subseteq \{x : x_\ell = 1, x_1 = 0, x_2 = 1\}$ and $\sum_{i=2}^{\ell} e_i \in S'$ while K remains as $(\mathbf{0} : 1, \ldots, n-1)$:



Consider the straight circuit

$$K' := (e_\ell : 1, \ldots, \ell - 1, \ell + 1, \ldots, n - 1, \ell).$$

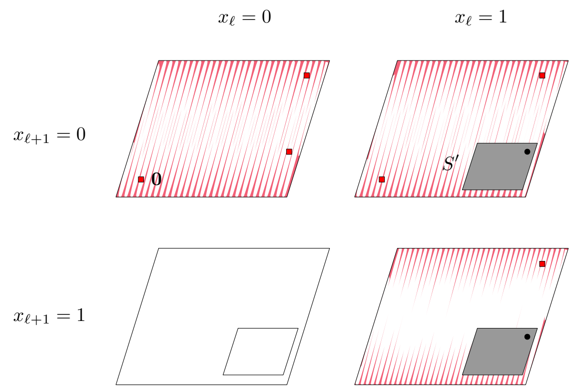
Since e_ℓ is infeasible and satisfies ($\diamond$) by (2), it follows that K' is infeasible and $K' \triangle e_n$ is feasible. Let us apply the induction hypothesis to K', given that the points in $\{x : x_{\ell+1} = x_{\ell+2} = \cdots = x_n = 0, x_\ell = 1\}$ satisfy ($\diamond$) and its feasible points form a hypercube. The induction hypothesis implies that the points in $\{x : x_{\ell+2} = \cdots = x_n = 0, x_\ell = 1\}$ satisfy ($\diamond$) and its feasible points form a hypercube. Together with (2), this implies that all the points in $\{x : x_{\ell+2} = \cdots = x_n = 0\}$ satisfy ($\diamond$).

Assume in the first case that $S \cap \{x : x_{\ell+2} = \cdots = x_n = 0, x_\ell = 1\} = S'$. Then we must have that

$$S \cap \{x : x_{\ell+2} = \cdots = x_n = 0\} = S'.$$

Suppose otherwise. Pick a closest pair of feasible points a, b such that $a \in S'$ and $b \in \{x : x_{\ell+2} = \cdots = x_n = 0\} - S'$. As the points in $\{x : x_{\ell+2} = \cdots = x_n = 0\}$ satisfy ($\diamond$), it follows that the restriction of S containing $a, b \triangle e_n$ as antipodal points is one of $\{R_{k,1} : 1 \leq k \leq \ell\}$, a contradiction. Thus, $S \cap \{x : x_{\ell+2} = \cdots = x_n = 0\} = S'$. So the feasible points in $\{x : x_{\ell+2} = \cdots = x_n = 0\}$ form a hypercube, thereby completing the induction step.

Assume in the remaining case that $S \cap \{x : x_{\ell+2} = \cdots = x_n = 0, x_\ell = 1\} = S' \cup (S' \triangle e_{\ell+1})$:



We claim that all the points in $S'\triangle e_\ell \triangle e_{\ell+1}$ are infeasible. Suppose for a contradiction that, for some $x \in S'$, $x\triangle e_\ell \triangle e_{\ell+1} \in S$. Recall that $S' \subseteq \{x : x_1 = 0, x_2 = 1\}$. For $i \in \{1, 2\}$, consider the 3-dimensional cube $H_i \subseteq \{0,1\}^3$ containing $x\triangle e_\ell, x\triangle e_{\ell+1}, x\triangle e_i$. Notice that for $i \in \{1, 2\}$,

$$\{x, x\triangle e_{\ell+1}, x\triangle e_\ell \triangle e_{\ell+1}\} \subseteq S \quad \text{and} \quad \{x\triangle e_i, x\triangle e_\ell, x\triangle e_i \triangle e_\ell, x\triangle e_i \triangle e_{\ell+1}\} \subseteq \overline{S}.$$

Thus, since H_1, H_2 are not fragile by Theorem 1.48, it follows that $x\triangle e_1 \triangle e_\ell \triangle e_{\ell+1}, x\triangle e_2 \triangle e_\ell \triangle e_{\ell+1} \in S$. To summarize, setting $y := x\triangle e_{\ell+1}$, $\{y\triangle e_\ell, y\triangle e_1 \triangle e_\ell, y\triangle e_2 \triangle e_\ell, y\} \subseteq S$. Moreover, $\{y\triangle e_1, y\triangle e_2, y\triangle e_1 \triangle e_2\} \subseteq \overline{S}$. As a result, since S does not contain antipodal points and the points in $\{x : x_{\ell+2} = \cdots = x_n = 0\}$ satisfy ($\diamond$), it follows that the 3-dimensional restriction of S containing $\{y\triangle e_1, y\triangle e_2, y\triangle e_\ell\}\triangle \mathbf{1}$ is fragile, a contradiction to Theorem 1.48. Thus, all the points in $S'\triangle e_\ell \triangle e_{\ell+1}$ are infeasible.

We next claim that

$$S \cap \{x : x_{\ell+2} = \cdots = x_n = 0\} = S' \cup (S'\triangle e_{\ell+1}).$$

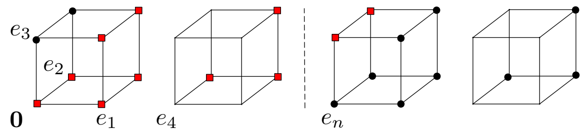
Suppose otherwise. Pick the closest pair of feasible points a, b such that $a \in S'\triangle e_{\ell+1}$ and $b \in \{x : x_{\ell+2} = \cdots = x_n = 0\} - [S' \cup (S'\triangle e_{\ell+1})]$. Since all the points in $S'\triangle e_\ell \triangle e_{\ell+1}$ are infeasible, it follows that $\mathrm{dist}(a, b) \geq 2$. Consider now the restriction of S containing $a, b\triangle e_n$ as antipodal points; because all the points in $\{x : x_{\ell+2} = \cdots = x_n = 0\}$ satisfy ($\diamond$), this restriction is one of $\{R_{k,1} : 1 \leq k \leq \ell - 1\}$, a contradiction. Thus, $S \cap \{x : x_{\ell+2} = \cdots = x_n = 0\} = S' \cup (S'\triangle e_{\ell+1})$, implying in particular that the feasible points in $\{x : x_{\ell+2} = \cdots = x_n = 0\}$ form a hypercube, thereby completing the induction step.

(iv) After possibly relabeling coordinates $1, 2, 3$ and rerouting K according to (1), we may assume that $S' = \{e_3, e_2 + e_3\}$ while K remains as $(\mathbf{0} : 1, \ldots, n-1)$. By the Straight Circuit Lemma, $\overline{S} \cap \{x : x_4 = \cdots = x_{n-1} = 0, x_n = 1\} = \{e_3 + e_n, e_2 + e_3 + e_n\}$.

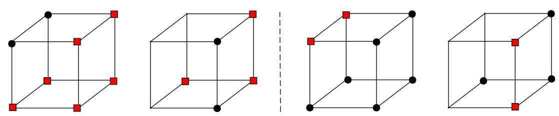
Consider the straight circuit

$$K_1 := (e_2 : 1, 3, 4, 5, \ldots, n-1, 2).$$

By (2), K_1 is infeasible and $K_1 \triangle e_n$ is feasible. The induction hypothesis applied to K_1 implies that the feasible points in $\{x : x_5 = \cdots = x_n = 0, x_2 = 1\}$ form a hypercube, implying in turn that $\{e_2 + e_4, e_1 + e_2 + e_4\} \subseteq \overline{S}$, and so by (2), $\{e_2 + e_4 + e_n, e_1 + e_2 + e_4 + e_n\} \subseteq S$:



We claim that $e_1+e_3+e_4 \in \overline{S}$. Suppose for a contradiction that $e_1+e_3+e_4 \in S$. By (2), $e_1+e_3+e_4+e_n \in \overline{S}$. By the Sight Propagation Lemma, $e_1 + e_4 \in S$, and so by (2), $e_1 + e_4 + e_n \in \overline{S}$:



Consider the 3-dimensional restriction of S containing $e_3 + e_1, e_3 + e_4, e_3 + e_n$; as this restriction is neither P_3 nor $R_{1,1}$, it follows that $e_3 + e_4 \in S$. If $e_4 \in S$, then as S does not have antipodal points and $\mathbf{0}, e_1, e_1+e_3$ satisfy ($\diamond$) by (2), the 3-dimensional restriction of S containing $\{e_1, e_3, e_4\}\triangle\mathbf{1}$ is fragile, thereby contradicting Theorem 1.48. Otherwise, $e_4 \notin \overline{S}$. By the Sight Propagation Lemma, $e_2 + e_3 + e_4 \in S$. Consider the straight circuit

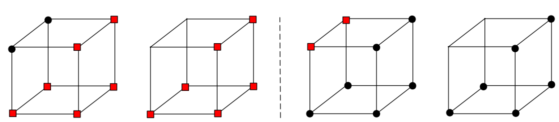
$$K_2 := (\mathbf{0} : 4, 2, 1, 3, 5, \ldots, n-1).$$

By (2), K_2 is infeasible and $K_2\triangle e_n$ is feasible. However, the 3-dimensional restriction of S containing $e_4 + e_1, e_4 + e_2, e_4 + e_3$ is a D_3 whose infeasible points all belong to K_3, so by Proposition 7.16, S has an $R_{1,1}, R_5$ restriction, a contradiction. Thus, $e_1 + e_3 + e_4 \in \overline{S}$, and so by (2), $e_1 + e_3 + e_4 + e_n \in S$.

Consider the straight circuit

$$K_3 := (\mathbf{0} : 1, 3, 4, 2, 5, \ldots, n-1).$$

By (2), K_3 is infeasible and $K_3\triangle e_n$ is feasible. The induction hypothesis applied to K_3 tells us that the feasible points in $\{x : x_5 = \cdots = x_n = 0, x_2 = 0\}$ form a hypercube, implying in turn that $\{e_4, e_1+e_4\} \subseteq \overline{S}$. By (2), $\{e_4 + e_n, e_1 + e_4 + e_n\} \subseteq S$:



Resistance now implies that the feasible points in $\{x : x_5 = \cdots = x_n = 0\}$ form a hypercube, thereby completing the induction step.

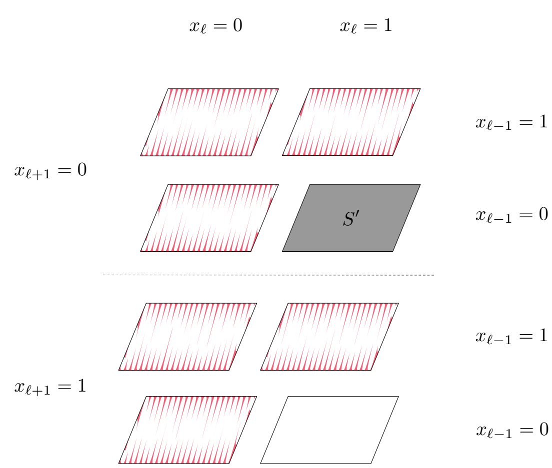
(v) After possibly relabeling coordinates $1, \ldots, \ell$ and rerouting K according to (1), we may assume that $S' = \{x : x_{\ell-1} = 0, x_\ell = 1, x_{\ell+2} = \cdots = x_n = 0\}$ while K remains as $(\mathbf{0} : 1, \ldots, n-1)$. As $\ell - 2 \geq 2$, the points in S' are active in directions $1, 2$. Let us apply the induction hypothesis to the straight infeasible circuit K but with a different starting point $(e_1 : 2, \ldots, n-1, 1)$, given that the points in $\{x : x_{\ell+1} = \cdots = x_n = 0, x_1 = 1\}$ satisfy ($\diamond$) and its feasible points $S' \cap \{x : x_1 = 1\}$ form a hypercube; and to the straight infeasible circuit $K_4 := (e_2 : 1, 3, \ldots, n-1, 2)$ satisfying $V(K_4\triangle e_n) \subseteq S$, given that the points in $\{x : x_{\ell+1} = \cdots = x_n = 0, x_2 = 1\}$ satisfy ($\diamond$) and its feasible points $S' \cap \{x : x_2 = 1\}$ form a hypercube. The induction hypothesis implies that

(5) the points in $\{x : x_{\ell+2} = \cdots = x_n = 0, x_1 = 1\}$ satisfy ($\diamond$) and its feasible points form a hypercube,

and that the points in $\{x : x_{\ell+2} = \cdots = x_n = 0, x_2 = 1\}$ satisfy ($\diamond$) and its feasible points form a hypercube. The latter implies in particular that $\sum_{i=1}^{\ell+1} e_i - e_1 \in \overline{S}$. We will next apply the induction hypothesis to the straight infeasible circuit $K_5 := (\mathbf{0} : 2, \ldots, \ell+1, 1, \ell+1, \ldots, n-1)$ satisfying $V(K_5 \triangle e_n) \subseteq S$, given that the points in $\{x : x_{\ell+1} = \cdots = x_n = 0, x_1 = 0\}$ satisfy ($\diamond$) and its feasible points $S' \cap \{x : x_1 = 0\}$ form a hypercube. The induction hypothesis tells us that

(6) the points in $\{x : x_{\ell+2} = \cdots = x_n = 0, x_1 = 0\}$ satisfy ($\diamond$) and its feasible points form a hypercube.

By (5) and (6), the points in $\{x : x_{\ell+2} = \cdots = x_n = 0\}$ satisfy ($\diamond$) and the feasible points of $\{x : x_{\ell+2} = \cdots = x_n = 0\}$ are contained in $S' \cup (S' \triangle e_{\ell+1})$:



After applying the Plane Propagation Lemma to the 0-restriction of S over coordinates $\ell+2, \ldots, n$, we see that the feasible points in $\{x : x_{\ell+2} = \cdots = x_n = 0\}$ must in fact form a hypercube, thereby completing the induction step. This finishes the proof of the lemma. $\square$

We are now ready to prove the main result of this section:

Theorem 7.18. *Take an integer $n \geq 4$ and a 1-resistant set $S \subseteq \{0,1\}^n$ that is non-polar. Assume that there is a straight infeasible circuit K of length $2(n-1)$ contained in $\{x : x_n = 0\}$ such that $V(K \triangle e_n) \subseteq S$. Then S has one of $\{R_{k,1} : 1 \leq k \leq n-2\} \cup \{R_5\}$ as a restriction.*

Proof. After a possible twisting and relabeling, we may assume that $K = (\mathbf{0} : 1, 2, \ldots, n-1)$.

Claim. *If $n = 4$, then $S \cong R_{2,1}$.*

Proof of Claim. Suppose that $n = 4$. As $V(K) \subseteq \overline{S}$ and $V(K \triangle e_4) \subseteq S$, it follows that

$$\{0000, 1000, 1100, 1110, 0110, 0010\} \subseteq \overline{S} \quad \text{and} \quad \{0001, 1001, 1101, 1111, 0111, 0011\} \subseteq S.$$

Since S is non-polar, $|\{1010, 0100\} \cap S| \geq 1$. Since $1010, 0100$ are both adjacent to a vertex of K, it follows from the Straight Circuit Lemma that $\{1010, 0100\} \subseteq S$ and $\{0101, 1011\} \subseteq \overline{S}$, implying in turn that $S \cong R_{2,1}$, as required. $\diamond$

We may therefore assume that $n \geq 5$. By the Straight Circuit Lemma, the points of $\{x : x_3 = \cdots = x_n = 0\}$ satisfy ($\diamond$). Also, as $\{x : x_3 = \cdots = x_n = 0\}$ contains at most one feasible point, the hypotheses of Lemma 7.17 hold for $\ell = 2$. If S has one of $\{R_{k,1} : 1 \leq k \leq n-3\} \cup \{R_5\}$ as a restriction, then we are done. Otherwise, after applying Lemma 7.17 for $\ell = 2, \ldots, n-3$ in this order, we see that the points in $\{x : x_{n-1} = x_n = 0\}$ satisfy ($\diamond$), implying in turn that all the points in $\{0,1\}^n$ satisfy ($\diamond$), and that $S' := S \cap \{x : x_{n-1} = x_n = 0\}$ is a hypercube. Since S is non-polar and ($\diamond$) holds, it follows that $S' \neq \emptyset$. Pick a closest pair of feasible points a, b such that $a \in S'$ and $b \in (S \cap \{x : x_n = 0\}) - S' = S' \triangle \mathbf{1} \triangle e_n$. Notice that $\operatorname{dist}(a, b) \geq 2$. It follows from ($\diamond$) that the restriction of S containing $a, b \triangle e_n$ as antipodal points is one of $\{R_{k,1} : 1 \leq k \leq n-2\}$. In either one of the two cases, S has one of $\{R_{k,1} : 1 \leq k \leq n-2\} \cup \{R_5\}$ as a restriction, as required. $\square$

7.3 Proof of Theorem 7.5

Using Theorems 7.13 and 7.18, we are ready to prove Theorem 7.5 stating the following:

> Take an integer $n \geq 3$ and a 1-resistant set $S \subseteq \{0,1\}^n$ that is non-polar. If every straight infeasible path has length at most $n-1$, then S has one of $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ as a restriction.

Proof. If there is no straight infeasible path of length $n-1$, then S has an $R_{1,1}$ restriction by Theorem 7.13, so we are done. Otherwise, there is a straight infeasible path $P := (v_0, v_1, \ldots, v_{n-1})$ of length $n-1$, which by assumption is maximal. After a possible relabeling and twisting, if necessary, we may assume that $V(P) \subseteq \{x : x_n = 0\}$. Maximality of P implies that $v_0 \triangle e_n, v_{n-1} \triangle e_n$ are feasible, so by the Path Propagation Lemma, $P \triangle e_n$ is a feasible path. As S does not contain antipodal points, it follows that the path $Q := P \triangle e_n \triangle \mathbf{1}$ is infeasible. Since Q is a straight infeasible (v_{n-1}, v_0)-path, and $v_{n-1} \triangle e_n, v_0 \triangle e_n \in S$, we get from the Path Propagation Lemma that $Q \triangle e_n$ is a feasible path. Consider the straight infeasible circuit $K := P \cup Q$ of length $2(n-1)$ contained in $\{x : x_n = 0\}$. We just showed that $V(K \triangle e_n) \subseteq S$. Thus, by Theorem 7.18, S has one of $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ as a restriction, as required. $\square$

We saw already that, as a consequence of this theorem, $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ are, up to isomorphism, the only 1-resistant strictly non-polar sets that are half-dense. However, Question 7.1, asking for all of the 1-resistant strictly non-polar sets, remains open. In fact, we cannot even answer the following question:

Question 7.19. *Is there a 1-resistant non-polar set $S \subseteq \{0,1\}^n$ such that $|S| < 2^{n-1}$?*

To be able to fully answer these questions, it seems that structure theorems for 1-resistant sets are needed. For instance, we saw a structure theorem for 2-resistant sets in Theorem 1.44, and used it to find the 2-resistant strictly non-polar sets in Theorem 6.9. Furthermore, in Chapter 8, we will provide a structure theorem for ± 1-resistant sets, and conclude as a consequence that ± 1-resistance implies strict polarity.

7.4 Further notes

Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq \{0,1\}^{n_1}, S_2 \subseteq \{0,1\}^{n_2}$. The *reflective product of* S_1, S_2 is

$$S_1 * S_2 := (S_1 \times S_2) \cup (\overline{S_1} \times \overline{S_2}) \subseteq \{0,1\}^{n_1+n_2}.$$

In words, the reflective product $S_1 * S_2$ is obtained from S_1 after replacing each feasible point by a copy of S_2 and each infeasible point by a copy of $\overline{S_2}$. Observe that $\overline{S_1 * S_2} = \overline{S_1} * S_2 = S_1 * \overline{S_2}$. The strictly non-polar sets studied in this chapter are proper reflective products, namely,

$$R_{k,1} = \{\mathbf{0}^{k+1}, \mathbf{1}^{k+1}\} \star \{0\}$$

while

$$R_5 = \{0000, 1000, 1100, 1110, 1111, 0111, 0011, 0001\} \star \{0\}.$$

Theorem 7.20 ([3])**.** *Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq \{0,1\}^{n_1}$ and $S_2 \subseteq \{0,1\}^{n_2}$. If $S_1, \overline{S_1}, S_2, \overline{S_2}$ are 1-resistant, then so are $S_1 * S_2, \overline{S_1 * S_2}$.*

An application of Theorem 1.42, the main result of this chapter, is the following characterization:

Theorem 7.21 ([3])**.** *Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq \{0,1\}^{n_1}, S_2 \subseteq \{0,1\}^{n_2}$, where $S_1, \overline{S_1}, S_2, \overline{S_2}$ are nonempty and 1-resistant. Then $S_1 * S_2$ is strictly polar if, and only if, $S_1 * S_2$ has none of $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ as a restriction.*

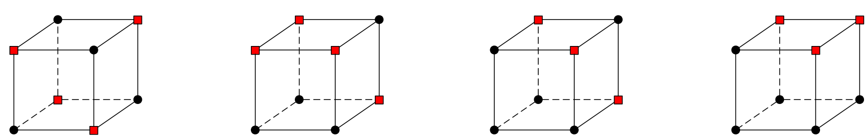
Chapter 8

The structure of ±1-resistant sets

Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Recall that S is ± 1-resistant if for each subset $X \subseteq \{0,1\}^n$ of cardinality at most one, $S \triangle X$ has no P_3, S_3 minor.

P_3 S_3

Here and throughout this chapter, round points are feasible and square points are infeasible. Recall that S is 1-resistant if for each subset $X \subseteq \{0,1\}^n$ of cardinality at most one, $S \cup X$ has no P_3, S_3 minor. By Theorem 6.13, S is ± 1-resistant if, and only if, S is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3$ minor.



$R_{1,1}$ F_1 F_2 F_3

In this chapter, we will take this characterization further and provide a structure theorem for ± 1-resistant sets. As a consequence, we will see that ± 1-resistance implies strict polarity.

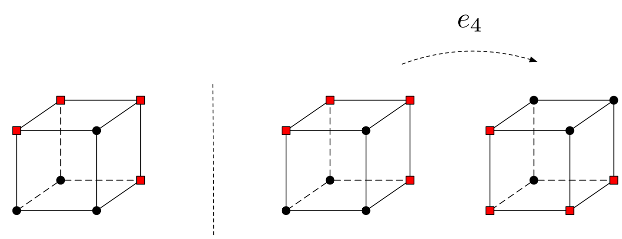
Our structure theorem is a consequence of three results. For an integer $k \geq 2$ let $A_k := \{\mathbf{0}, \mathbf{1}\} \subseteq \{0,1\}^k$, and for an integer $k \geq 3$ let $B_k := \{\mathbf{0}, e_1, \mathbf{1}\} \subseteq \{0,1\}^k$.

Theorem 8.1. *Take an integer $n \geq 2$ and a 1-resistant set $S \subseteq \{0,1\}^n$ without an $R_{1,1}, F_1, F_2, F_3$ minor. If S is not connected, then either*

- *$S \cong A_k \times \{0,1\}^{n-k}$ for some $k \in \{2, \ldots, n\}$,*

- $S \cong B_k \times \{0,1\}^{n-k}$ *for some* $k \in \{3, \dots, n\}$*, or*
- S *has a* D_3 *minor.*

Here, $D_3 = \{000, 100, 010, 101\} \subseteq \{0,1\}^3$. Let $C_8 := \{0000, 1000, 0100, 1010,\ 0101, 0111, 1111, 1011\} \subseteq \{0,1\}^4$.

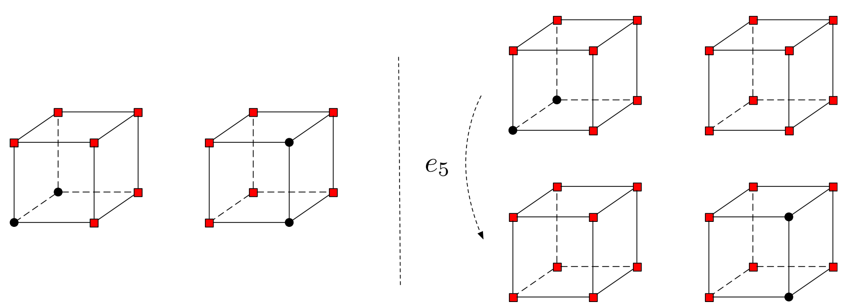


D_3 C_8

Theorem 8.2. *Take an integer* $n \geq 3$ *and a* 1*-resistant set* $S \subseteq \{0,1\}^n$ *without an* $R_{1,1}, F_1, F_2, F_3$ *minor. If* S *has a* D_3 *minor, then either*

- $S \cong C_8 \times \{0,1\}^{n-4}$*, or*
- $S \cong D_k \times \{0,1\}^{n-k}$ *for some* $k \in \{3, \dots, n\}$.

Here, for each integer $k \geq 4$, $D_k = \{\mathbf{0}, e_2, \mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\} \subseteq \{0,1\}^k$.



D_4 D_5

Theorem 8.3. *Take an integer* $n \geq 1$ *and a* 1*-resistant set* $S \subseteq \{0,1\}^n$ *without an* $R_{1,1}, F_1, F_2, F_3$ *minor. If* S *is connected and has no* D_3 *minor, then either*

- S *is a hypercube, or*
- *every infeasible component of* S *is a hypercube.*

The main result of this chapter is a consequence of these three results:

Theorem 1.45 ([1]). *Take an integer $n \geq 1$ and a ±1-resistant set $S \subseteq \{0,1\}^n$. Then one of the following statements holds:*

(i) $S \cong A_k \times \{0,1\}^{n-k}$ *for some* $k \in \{2, \dots, n\}$,

(ii) $S \cong B_k \times \{0,1\}^{n-k}$ *for some* $k \in \{3, \dots, n\}$,

(iii) $S \cong C_8 \times \{0,1\}^{n-4}$,

(iv) $S \cong D_k \times \{0,1\}^{n-k}$ *for some* $k \in \{3, \dots, n\}$,

(v) S *is a hypercube, or*

(vi) every infeasible component of S *is a hypercube.*

Proof. By Theorem 6.13, S is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3$ minor. If S is not connected and has no D_3 minor, then (i) or (ii) holds by Theorem 8.1. If S has a D_3 minor, then (iii) or (iv) holds by Theorem 8.2. Otherwise, S is connected and has no D_3 minor. Then by Theorem 8.3, either (v) or (vi) holds. □

We prove Theorems 8.1, 8.2 and 8.3 in §8.2, §8.3 and §8.5, respectively. In §8.6, we show that ±1-resistance implies strict polarity. A result we will appeal to throughout this chapter is the Plane Propagation Lemma (7.8) proved in Chapter 7:

> Take an integer $n \geq 1$ and a 1-resistant set $S \subseteq \{0,1\}^n$. If $S \cap \{x : x_n = 0\} = \emptyset$, then S is a hypercube.

8.1 Bridges

Take an integer $n \geq 2$. For a point $x \in \{0,1\}^n$ and distinct coordinates $i, j \in [n]$ such that $x_i = x_j = 0$, we refer to $\{x, x+e_i, x+e_j, x+e_i+e_j\}$ as a *square* that *initiates at* x and is *active in directions* e_i, e_j. Two squares are *parallel* if they are active in the same pair of directions. Two parallel squares are *neighbors* if the points they initiate from are neighbors.

Take a set $S \subseteq \{0,1\}^n$. A *bridge* is a square that contains feasible points from different feasible components. Notice that a bridge contains exactly two feasible points, which are non-adjacent and belong to different feasible components. In this section, we will prove the following statement:

> Take an integer $n \geq 3$ and let $S \subseteq \{0,1\}^n$ be a set that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3$ minor. Then every pair of bridges are parallel.

We will need three lemmas to prove this statement.

Lemma 8.4. *Take an integer $n \geq 3$ and a set $S \subseteq \{0,1\}^n$, where direction e_n is not active in any bridge. If S' is obtained from S after projecting away coordinate n, then the feasible components of S project onto different feasible components of S'.*

Proof. For a point $x \in \{0,1\}^n$, denote by $x' \subseteq \{0,1\}^{n-1}$ the point obtained from x after dropping the n^{th} coordinate. To prove the lemma, it suffices to show that if K is a feasible component of S and $x \in S - K$, then $\operatorname{dist}(x', y') \geq 2$ for all $y \in K$. Well, since x does not belong to the component K, $\operatorname{dist}(x, y) \geq 2$ for all $y \in K$, implying in turn that

$$\operatorname{dist}(x', y') \geq \operatorname{dist}(x, y) - 1 \geq 1 \quad \forall\, y \in K.$$

In particular, $x' \notin \{y' : y \in K\}$. Suppose for a contradiction that $\operatorname{dist}(x', y') = 1$ for some $y \in K$. As the inequalities above are held at equality, there must be a coordinate $i \in [n-1]$ such that $y = x \triangle e_i \triangle e_n$. But then $\{x, x\triangle e_i, x\triangle e_n, x\triangle e_i\triangle e_n\}$ would be a bridge that is active in direction e_n, contrary to our assumption. Hence,

$$\operatorname{dist}(x', y') \geq 2 \quad \forall y \in K,$$

as required. $\square$

Lemma 8.5. *Take an integer $n \geq 3$ and a set $S \subseteq \{0,1\}^n$ that is 1-resistant and has no $R_{1,1}, F_1, F_2$ restriction. Take a point $x \in \{0,1\}^n$ and distinct coordinates $i, j, k \in [n]$. Then the following statements hold:*

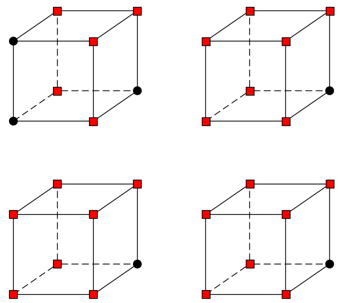
(i) If $x\triangle e_i, x\triangle e_j, x\triangle e_k \in \overline{S}$, then $\left|\{x\triangle e_i\triangle e_j, x\triangle e_j\triangle e_k, x\triangle e_k\triangle e_i\} \cap S\right| \leq 1$.

(ii) If $x \in S$ and $\{x, x\triangle e_i, x\triangle e_j, x\triangle e_i\triangle e_j\}$ is a bridge, then $\{x\triangle e_i\triangle e_k, x\triangle e_j\triangle e_k\} \cap S = \emptyset$.

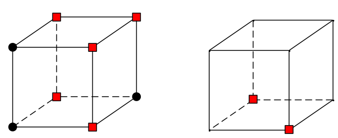
(iii) If $x \in S$ and $\{x, x\triangle e_i, x\triangle e_j, x\triangle e_i\triangle e_j\}$ is a bridge, then $\left|\{x\triangle e_k, x\triangle e_i\triangle e_j\triangle e_k\} \cap S\right| \geq 1$.

Proof. After a possible twisting and relabeling, if necessary, we may assume that $x = \mathbf{0}$ and $i = 1, j = 2, k = 3$. Let $S' \subseteq \{0,1\}^3$ be the restriction of S obtained after 0-restricting coordinates $4, \ldots, n$. **(i)**: Suppose that $e_1, e_2, e_3 \in \overline{S}$. Assume for a contradiction that two of $e_1+e_2, e_2+e_3, e_3+e_1$, say e_1+e_2, e_2+e_3 belong to S. If $e_1 + e_3 \in S$, then S' is isomorphic to one of $P_3, S_3, R_{1,1}, F_2$, which cannot occur as S is 1-resistant and has no $R_{1,1}, F_2$ restriction. Otherwise, $e_1 + e_3 \in \overline{S}$. Since $S' \ncong P_3$ and S is 1-resistant, it follows that $\mathbf{0}, e_1 + e_2 + e_3 \in S$, implying in turn that $S' \cong F_1$, a contradiction as S has no F_1 restriction. **(ii)**, **(iii)**: Suppose that $\mathbf{0} \in S$ and $\{\mathbf{0}, e_1, e_2, e_1 + e_2\}$ is a bridge. Then $e_1 + e_2 \in S$ and $e_1, e_2 \in \overline{S}$. Let us first prove (ii), that $\{e_1 + e_3, e_2 + e_3\} \cap S = \emptyset$. Suppose otherwise. After possibly relabeling coordinates $1, 2$, we may assume that $e_1 + e_3 \in S$. Since $\mathbf{0}, e_1 + e_2$ are in different feasible components, it follows that $|\{e_3, e_1 + e_2 + e_3\}| \cap S| \leq 1$. After possibly twisting coordinates $1, 2$, we may assume that $e_3 \in \overline{S}$. Since $e_1, e_2, e_3 \in \overline{S}$, we get from (i) that $|\{e_1 + e_2, e_2 + e_3, e_3 + e_1\} \cap S| \leq 1$, a contradiction. Thus, $\{e_1 + e_3, e_2 + e_3\} \cap S = \emptyset$, so (ii) holds. Since S is 1-resistant, it follows immediately that $\{e_3, e_1 + e_2 + e_3\} \cap S \neq \emptyset$, so (iii) holds. $\square$

Lemma 8.6. *Take a set $S \subseteq \{0,1\}^5$ that is 1-resistant, has no $R_{1,1}, F_1, F_2, F_3$ minor, and in every minor, including S itself, every pair of bridges are parallel. If $\mathbf{0} \in S$ and $\{\mathbf{0}, e_1, e_2, e_1 + e_2\}$ is a bridge without neighboring bridges, then after possibly twisting coordinates 1 and 2, we have that $S = \{\mathbf{0}, e_3, e_1 + e_2, e_1 + e_2 + e_4, e_1 + e_2 + e_5, e_1 + e_2 + e_4 + e_5\}$:*

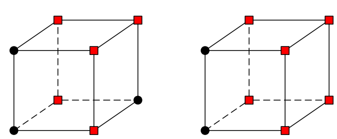


Proof. Let $B := \{\mathbf{0}, e_1, e_2, e_1 + e_2\}$. As B is a bridge and $\mathbf{0} \in S$, $e_1 + e_2 \in S$ and $e_1, e_2 \in \overline{S}$. It follows from Lemma 8.5 (ii) that $e_1 + e_3, e_2 + e_3 \in \overline{S}$. By Lemma 8.5 (iii) and the fact that B has no neighboring bridge, we get that exactly one of $e_3, e_1 + e_2 + e_3$ belongs to S. After twisting coordinates 1 and 2, if necessary, we may assume that $e_3 \in S$ and $e_1 + e_2 + e_3 \in \overline{S}$. Moreover, by Lemma 8.5 (ii), we have that $\{e_1 + e_4, e_2 + e_4\} \subseteq \overline{S}$. Let S' be the 0-restriction of S over coordinate 5, which looks as follows:



Claim 1. *$e_4 \in \overline{S}$ and $e_1 + e_2 + e_4 \in S$.*

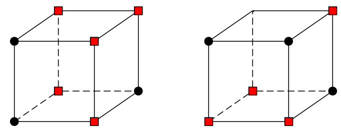
Proof of Claim. Suppose otherwise. Since B has no neighboring bridge in S, it follows from Lemma 8.5 (iii) that $e_4 \in S$ and $e_1 + e_2 + e_4 \in \overline{S}$. If $e_2 + e_3 + e_4 \in S$, then the 0-restriction of S' over coordinate 1 is either F_1 or F_3, which is not the case. Thus, $e_2 + e_3 + e_4 \in \overline{S}$. Since the 0-restriction of S' over coordinate 1 is 1-resistant, it follows that $e_3 + e_4 \in S$. As the 0-restriction of S' over coordinate 2 is not F_3, we have $e_1 + e_3 + e_4 \in \overline{S}$. Since the 1-restriction of S' over coordinate 1 is 1-resistant, it follows that $e_1 + e_2 + e_3 + e_4 \in \overline{S}$, so S' looks as follows:



Observe however now that F_3 is obtained from S' after projecting away coordinate 1, a contradiction. $\diamond$

Claim 2. $\{e_1 + e_3 + e_4, e_2 + e_3 + e_4\} \subseteq \overline{S}$.

Proof of Claim. Suppose otherwise. After interchanging the roles of $1, 2$, if necessary, we may assume that $e_1 + e_3 + e_4 \in S$. If $e_3 + e_4 \in \overline{S}$, then $\{\mathbf{0}, e_3\}$ is a feasible component of S', so the square initiating from e_3 and active in directions e_1, e_4 is a bridge of S' that is not parallel to B, which is contrary to our assumption. Thus, $e_3 + e_4 \in S$. Since $\mathbf{0}, e_1 + e_2$ belong to different feasible components of S, it follows that $e_1 + e_2 + e_3 + e_4 \in \overline{S}$, so S' looks as follows:

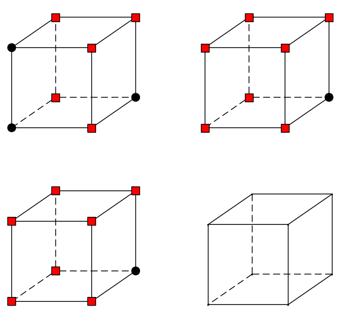


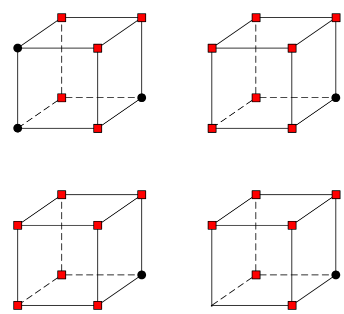
Observe however that S' has two non-parallel bridges, namely B and the square that initiates from $e_1 + e_4$ and is active in directions e_2, e_3, a contradiction. $\diamond$

Claim 3. $\{e_3 + e_4, e_1 + e_2 + e_3 + e_4\} \subseteq \overline{S}$.

Proof of Claim. Since the 0-restriction of S' over coordinate 1 is 1-resistant, it follows that $e_3 + e_4 \in \overline{S}$. Since the 1-restriction of S' over coordinate 1 is also 1-resistant, we see that $e_1 + e_2 + e_3 + e_4 \in \overline{S}$, as required. $\diamond$

We just determined the status of all the points in $\{x : x_5 = 0\}$. A similar argument applied to $\{x : x_4 = 0\}$ gives us the left figure below:





Consider the set obtained from S after 1-restricting over coordinate 1 and 0-restricting over coordinate 3; since this set is 1-resistant and not isomorphic to F_1, F_3, we get that $e_1+e_4+e_5 \in \overline{S}$ and $e_1+e_2+e_4+e_5 \in S$. As the 1-restriction of S over coordinates $1, 2$ is not F_3, we get that $\mathbf{1} \in \overline{S}$. Now consider the set obtained from S after 1-restricting coordinate 2 and 0-restricting over coordinate 3; since this set is not F_3, we get that $e_2+e_4+e_5 \in \overline{S}$. Note that $\{e_1+e_2, e_1+e_2+e_4, e_1+e_2+e_5, e_1+e_2+e_4+e_5\}$ forms a feasible component of S. Hence, as S does not have non-parallel bridges, it follows that $e_2+e_3+e_4+e_5, e_1+e_3+e_4+e_5 \in \overline{S}$, and also that $e_3 + e_4 + e_5 \in \overline{S}$. (See the right figure above.) Once again, as S does not have non-parallel bridges, it follows that $e_4 + e_5 \in \overline{S}$, thereby finishing the proof. $\square$

We are now ready to prove the main result of this section:

Proposition 8.7. *Take an integer $n \geq 3$ and let $S \subseteq \{0,1\}^n$ be a set that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3$ minor. Then every pair of bridges are parallel.*

Proof. Suppose for a contradiction that S has a pair of non-parallel bridges. (In particular, S is not connected.) We may assume that in every proper minor of S, every pair of bridges, if any, are parallel.

Claim 1. *Every direction is active in a bridge.*

Proof of Claim. Suppose for a contradiction that direction e_n is not active in any bridge. For a point $x \in \{0,1\}^n$, denote by $x' \subseteq \{0,1\}^{n-1}$ the point obtained from x after dropping the n^{th} coordinate. Notice first that by Lemma 8.4, the feasible components of S project onto different feasible components of S', the subset of $\{0,1\}^{n-1}$ obtained from S after projecting away coordinate n. We will derive a contradiction to the minimality of S by showing that S' has non-parallel bridges.

We will show that if B is a bridge of S, then $B' := \{x' : x \in B\}$ is still a bridge of S' that is active in the same directions as before. Since e_n is not active in any bridge of S, we may assume that $n \geq 3$ and $B = \{\mathbf{0}, e_1, e_2, e_1+e_2\}$ where $\mathbf{0}, e_1+e_2$ belong to different feasible components of S, and $e_1, e_2 \in \overline{S}$. It follows from Lemma 8.5 (ii) that $\mathbf{0}, e_1+e_2 \in S'$ and $e_1, e_2 \in \overline{S'}$. Moreover, since the feasible components of S project onto different feasible components of S', we see that $\mathbf{0}, e_1+e_2$ belong to different feasible components of S'. Thus, B' is still a bridge of S' that is active in the same directions as before.

As a corollary, S' still has non-parallel bridges, thereby contradicting the minimality of S. $\diamond$

Claim 2. *The following statements hold:*

(i) if B, B' are non-parallel bridges that are not active in direction e_i, then $\{x : x_i = 0\}$ contains one of the bridges and $\{x : x_i = 1\}$ contains the other one,

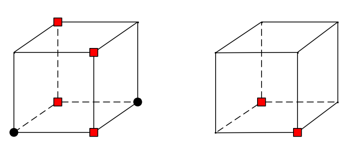
(ii) if B, B', B'' are pairwise non-parallel bridges, then every direction is active in one of the bridges, and

(iii) $n \in \{4,5,6\}$.

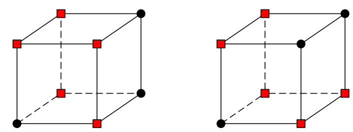
Proof of Claim. **(i)** For if not, then one of the restrictions of S over coordinate i contains B and B', thereby contradicting the minimality of S. **(ii)** Suppose for a contradiction that e_i is not active in either of B, B', B''. Then one of the hyperplanes $\{x : x_i = 0\}, \{x : x_i = 1\}$ contains at least two of B, B', B'', thereby contradicting (i). **(iii)** Let B, B' be non-parallel bridges. It follows from Lemma 8.5 (ii) that $n \geq 4$. If every direction is active in one of B, B', we get that $n = 4$. Otherwise, there is a direction e_i inactive in both B, B'. By Claim 1, there is a bridge B'' active in e_i. Clearly, B, B', B'' are pairwise non-parallel bridges. It now follows from (ii) that $n \leq 6$, as required. $\diamond$

Claim 3. $n \neq 4$.

Proof of Claim. Suppose for a contradiction that $n = 4$. Let B, B' be non-parallel bridges of S. We may assume that $B = \{\mathbf{0}, e_1, e_2, e_1+e_2\}$, $\mathbf{0}, e_1+e_2 \in S$ and $e_1, e_2 \in \overline{S}$. By Lemma 8.5 (ii), $e_1+e_3, e_2+e_3, e_1+e_4, e_2+e_4 \in \overline{S}$:

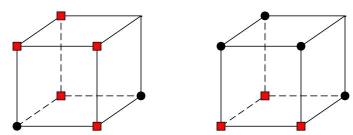


Assume in the first case that B' shares an active direction with B. After possibly relabeling coordinates $1,2$, we may assume that B' is active in directions e_1, e_3. It follows from Claim 2 (i) that B' is contained in $\{x : x_4 = 1\}$. After possibly twisting coordinates $1,2$, we may assume that $B' = \{e_4, e_1+e_4, e_3+e_4, e_1+e_3+e_4\}$. Since $e_1+e_4 \in \overline{S}$, it follows that $e_4, e_1+e_3+e_4 \in S$ and $e_3+e_4 \in \overline{S}$. Applying Lemma 8.5 (ii), we get that $e_3, e_2+e_3+e_4, e_1+e_2+e_4 \in \overline{S}$. Since the two restrictions of S over coordinate 4 are 1-resistant, it follows that $e_1+e_2+e_3, \mathbf{1} \in S$:



Observe, however, that 1-restricting S over coordinate 3 yields a set that is not 1-resistant, a contradiction.

Assume in the remaining case that B' is active in directions e_3, e_4. Observe that B' is not contained in $\{x : x_1 + x_2 = 1\}$. After possibly twisting coordinates $1, 2$, we may assume that that B' initiates from $\mathbf{0}$. This means that $e_3, e_4 \in \overline{S}$ and $e_3 + e_4 \in S$. Applying Lemma 8.5 (iii), we get that $e_1 + e_2 + e_4 \in S$ and $e_1 + e_3 + e_4, e_2 + e_3 + e_4 \in S$:



The 1-restriction of S over coordinate 4, however, is isomorphic to either F_1 or F_3, a contradiction. $\diamond$

Thus, we have that $n \in \{5, 6\}$. It follows from Claim 1 that there are $\lceil \frac{n}{2} \rceil = 3$ pairwise non-parallel bridges B_1, B_2, B_3. We get from Claim 2 (ii) that, after a possible relabeling, B_1 is active in e_1, e_2, B_2 is active in e_3, e_4, and

- if $n = 5$, then B_3 is active in e_3, e_5,
- if $n = 6$, then B_3 is active in e_5, e_6.

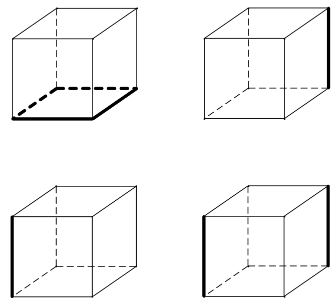
We can further say that,

Claim 4. *If B is a bridge different from B_1, B_2, B_3, then $n = 5$.*

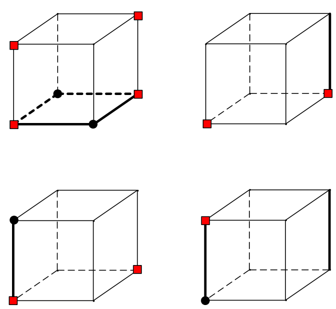
Proof of Claim. Suppose that B is a bridge of S different from B_1, B_2, B_3. It follows from Claim 2 (ii) that B is parallel to one of B_1, B_2, B_3. Consider the bridge B_2. Since B_2, B_3 are inactive in e_1, e_2, it follows from Claim 2 (i) that the hyperplanes $\{x : x_1 = 0\}, \{x : x_2 = 0\}$ split B_2, B_3. Moreover, since B_2, B_1 are inactive in e_5, the hyperplane $\{x : x_5 = 0\}$ splits B_2, B_1. Hence, the residing square of B_2 – and any bridge parallel to it – is determined once B_1 and B_3 are given, implying that B is not parallel to B_2. By the symmetry between B_2 and B_3, we get that B is not parallel to B_3 either. Thus, B is parallel to B_1. This breaks the symmetry between B_1 and B_2, implying in turn that $n \neq 6$. Thus, $n = 5$, as claimed. $\diamond$

Claim 5. *$n \neq 5$.*

Proof of Claim. Suppose for a contradiction that $n = 5$. After twisting coordinates $3, 4, 5$, if necessary, we may assume that B_1 initiates at $\mathbf{0}$. By Claim 2 (i), and after possibly twisting coordinates $1, 2$, we may assume that B_2 initiates at e_5. Another application of Claim 2 (i) tells us that B_3 initiates at $e_1 + e_2 + e_4$:

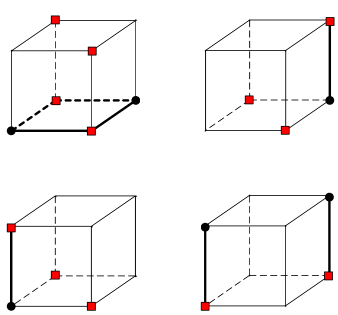


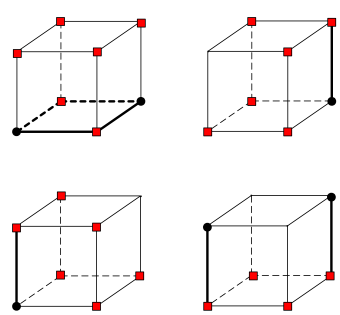
Assume in the first case that $\mathbf{0}, e_1+e_2 \in \overline{S}$ and $e_1, e_2 \in S$. Then a repeated application of Lemma 8.5 (ii) tells us that $e_3, e_1+e_2+e_3, e_5, e_1+e_2+e_5, e_4, e_1+e_2+e_4 \in \overline{S}$. As a result, in the bridge B_2, we have that $e_3+e_5, e_4+e_5 \in S$:



Observe now that the restriction of S obtained after 0-restricting coordinates 1 and 2 is not 1-resistant, a contradiction.

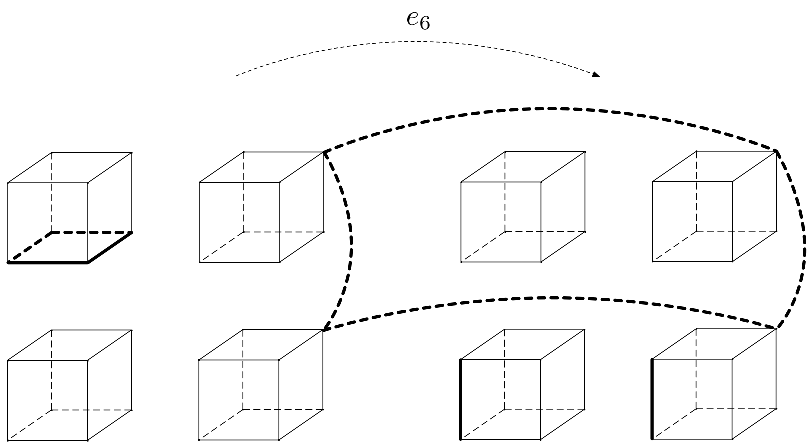
Assume in the remaining case that $\mathbf{0}, e_1+e_2 \in S$ and $e_1, e_2 \in \overline{S}$. A repeated application of Lemma 8.5 (ii) to B_1, followed by an application of it to B_2, B_3 gives us the left figure below:



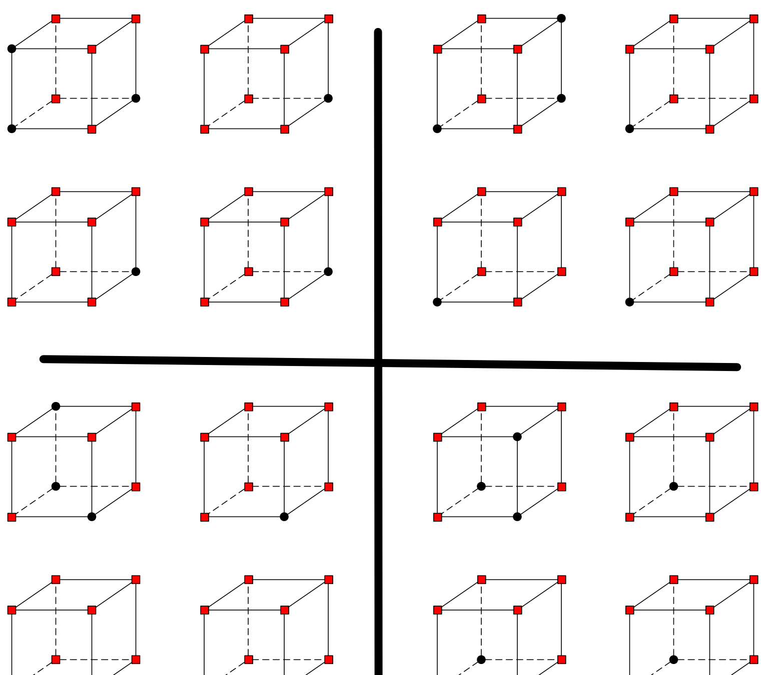


Applying Lemma 8.5 (ii) to B_2, B_3 gives us the following the right figure above, thereby yielding a contradiction as 0-restricting coordinates $4, 5$ of S yields a set that is not 1-resistant. This finishes the proof of the claim. $\diamond$

Thus, $n = 6$. After twisting coordinates $3, 4, 5, 6$, if necessary, we may assume that B_1 initiates at $\mathbf{0}$. Applying Claim 2 (i), we see that after possibly twisting coordinates $1, 2$, we may assume that B_2 initiates at $e_5 + e_6$. Using Claim 2 (i), we see that B_3 must initiate at $e_1 + e_2 + e_3 + e_4$:



Recall from Claim 4 that B_1, B_2, B_3 are the only bridges of S. Let $S' \subseteq \{0,1\}^5$ be the restriction of S obtained after 0-restricting coordinate 6. By assumption, every minor of S' has only parallel bridges. As a bridge in S' is not necessarily a bridge in S, we see that S' may have bridges other than B_1 (that will necessarily be parallel to it).



Claim 6. *B_1 does not have a neighboring bridge in S'.*

Proof of Claim. Suppose for a contradiction that B_1 has a neighboring bridge B in S'. Since B is not a bridge of S by Claim 4, it follows that the points in $B \cap S'$ are in the same feasible component of S. After applying Lemma 8.5 (ii) to B_1, we see that the points in $B_1 \cap S'$ also lie in this feasible component of S, a contradiction. $\diamond$

We may now apply Lemma 8.6 to the bridge B_1 of S'. Depending on which points of B_1 are in S', and how coordinates $1, 2$ are twisted, we get that S' takes on one of the four possibilities shown above. In each one of the four cases, we see that the 3-dimensional restriction of S containing B_2 and $B_2 \triangle e_6$ is either non-1-resistant or isomorphic to F_1, a contradiction. This finally finishes the proof of Proposition 8.7. $\square$

8.2 Proof of Theorem 8.1

Take an integer $n \geq 2$ and a set $S \subseteq \{0,1\}^n$. We say that S is *separable* if there exist a partition of S into nonempty parts S_1, S_2 and distinct coordinates $i, j \in [n]$ such that either $S_1 \subseteq \{x : x_i = 0, x_j = 1\}$ and $S_2 \subseteq \{x : x_i = 1, x_j = 0\}$, or $S_1 \subseteq \{x : x_i = x_j = 0\}$ and $S_2 \subseteq \{x : x_i = x_j = 1\}$. Notice that if S is separable, then it is not connected.

Remark 8.8. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. If a projection of S is separable, then so is S.*

We will need the following:

Proposition 8.9. *Take an integer $n \geq 2$ and a 1-resistant set $S \subseteq \{0,1\}^n$. Suppose there is a partition of S into nonempty parts S_1, S_2 such that $S_1 \subseteq \{x : x_{n-1} = x_n = 0\}$ and $S_2 \subseteq \{x : x_{n-1} = x_n = 1\}$. Then S_1 and S_2 are hypercubes.*

Proof. The hypercube $\{x : x_{n-1} = 0, x_n = 1\}$ is infeasible. As S is 1-resistant, the Plane Propagation Lemma (7.8) implies that in each of the parallel hypercube $\{x : x_{n-1} = x_n = 0\}$ and $\{x : x_{n-1} = x_n = 1\}$, the feasible points form a hypercube. That is, the two sets

$$S \cap \{x : x_{n-1} = x_n = 0\} = S_1,$$
$$S \cap \{x : x_{n-1} = x_n = 1\} = S_2$$

are hypercubes. $\square$

We are now ready to prove Theorem 8.1, stating the following:

> Take an integer $n \geq 2$ and a set $S \subseteq \{0,1\}^n$ that is 1-resistant, has no $R_{1,1}, F_1, F_2, F_3$ minor and is not connected. Then either
>
> - $S \cong A_k \times \{0,1\}^{n-k}$ for some $k \in \{2, \ldots, n\}$, where $A_k = \{\mathbf{0}^k, \mathbf{1}^k\}$,
> - $S \cong B_k \times \{0,1\}^{n-k}$ for some $k \in \{3, \ldots, n\}$, where $B_k = \{\mathbf{0}^k, e_1, \mathbf{1}^k\}$, or
> - S has a D_3 minor, where $D_3 = \{000, 010, 100, 101\}$.

Proof. Let us start with the following claim:

Claim 1. *S is separable.*

Proof of Claim. Let $k \geq 2$ be the number of feasible components of S. Let $S' \subseteq \{0,1\}^m$ be a projection of S of smallest dimension with exactly k feasible components. It then follows from Lemma 8.4 that every direction of $\{0,1\}^m$ is active in a bridge of S'. However, as S' is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3$ minor, Proposition 8.7 implies that every pair of bridges of S' are parallel. As a result, $m = k = 2$ and S' is either $\{00, 11\}$ or $\{10, 01\}$. In particular, S' is separable, so S is separable by Remark 8.8. $\diamond$

Thus, there is a partition of S into nonempty parts S_1, S_2 such that, after a possible twisting and relabeling, $S_1 \subseteq \{x : x_{n-1} = x_n = 0\}$ and $S_2 \subseteq \{x : x_{n-1} = x_n = 1\}$. As S is 1-resistant, Proposition 8.9 implies that S_1 and S_2 are hypercubes. In particular, since S is not a hypercube, the Plane Propagation Lemma (7.8) implies that the points in S do not agree on a coordinate; notice that this property is preserved in every projection of dimension at least one.

Claim 2. *Either S has a D_3 minor, or one of S_1, S_2 is contained in the antipode of the other.*

Proof of Claim. Suppose that neither of S_1, S_2 is contained in the antipode of the other. We will prove that S has a D_3 projection. Clearly, $n > 2$. We may assume that for each $i \in [n-2]$,

> if S', S_1', S_2' are obtained from S, S_1, S_2 after projecting away coordinate i, then one of S_1', S_2' is contained in the antipode of the other.

As the points in S do not agree on a coordinate, there exists a point $x \in S_1$ such that $\mathbf{1} - x \in S_2$. As neither of S_1, S_2 is contained in the antipode of the other, there exist distinct coordinates $i, j \in [n-2]$ such that $x \triangle e_i \in S_1, x \triangle e_j \notin S_1, \mathbf{1} \triangle x \triangle e_i \notin S_2$ and $\mathbf{1} \triangle x \triangle e_j \in S_2$. Our minimality assumption implies that the only feasible neighbors of $x, \mathbf{1} \triangle x$ are $x \triangle e_i, \mathbf{1} \triangle x \triangle e_j$, respectively. As a result, $S_1 = \{x, x \triangle e_i\}$ and $S_2 = \{\mathbf{1} \triangle x, \mathbf{1} \triangle x \triangle e_j\}$, so $S = \{x, x \triangle e_i, \mathbf{1} \triangle x, \mathbf{1} \triangle x \triangle e_j\}$. Clearly, S has a D_3 projection. $\diamond$

If S has a D_3 minor, then we are done. Otherwise, one of S_1, S_2 is contained in the antipode of the other. After possibly relabeling S_1, S_2, we may assume that S_2 is contained in the antipode of S_1.

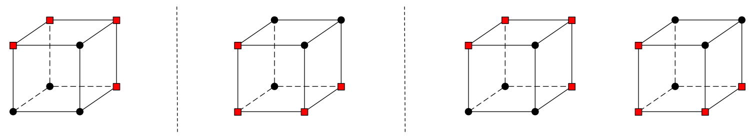
Claim 3. $2|S_2| \geq |S_1| \geq |S_2|$.

Proof of Claim. Clearly, $|S_1| \geq |S_2|$. Suppose for a contradiction that $|S_1| \geq 4|S_2|$. Since S_2 is contained in the antipode of S_1, it can be readily checked that S has an F_3 minor, a contradiction. $\diamond$

As a result, either $|S_1| = |S_2|$ or $|S_1| = 2|S_2|$. It can now be readily checked that either $S \cong A_k \times \{0,1\}^{n-k}$ for some $k \in \{2, \ldots, n\}$, or $S \cong B_k \times \{0,1\}^{n-k}$ for some $k \in \{3, \ldots, n\}$, thereby finishing the proof of Theorem 8.1. $\square$

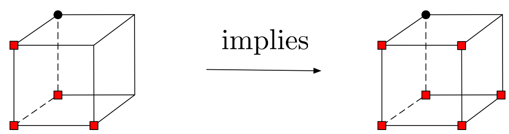
8.3 D_3 minors and proof of Theorem 8.2

To prove Theorem 8.2 we will need three lemmas. Let $D_3^\star := \{010, 011, 111, 101\} \subseteq \{0,1\}^3$. Observe that $D_3^\star$ is a twisting of $D_3 = \{000, 100, 010, 101\}$, and $C_8 = (D_3 \times \{0\}) \cup (D_3^\star \times \{1\})$.



D_3 $D_3^\star$ C_8

In the following lemma, we will use the following implication of Lemma 8.5 (i):



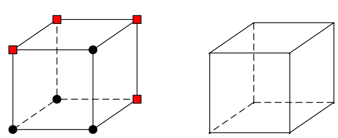
Notice that this is also an application of the Sight Propagation Lemma (7.10) from Chapter 7.

Lemma 8.10. *Let $S \subseteq \{0,1\}^n$ be a set that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3$ minor, where the 0-restriction of S over coordinates $4, \ldots, n$ is either D_3 or $D_3^\star$. Then,*

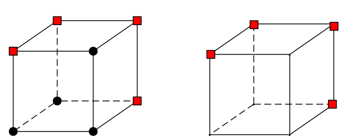
(i) every restriction of S over coordinates $4, \ldots, n$ is either D_3 or $D_3^\star$, and

(ii) either $S \cong D_3 \times \{0,1\}^{n-3}$ or $S \cong C_8 \times \{0,1\}^{n-4}$.

Proof. **(i)** By a recursive argument, it suffices to show that each 3-dimensional restriction of S neighboring a $D_3, D_3^\star$ restriction is also a D_3 or a $D_3^\star$. Thus, we may assume that $n = 4$. After twisting coordinates $1, 2, 3$, if necessary, we may assume that the 0-restriction of S over coordinate 4 is D_3. So $S \cap \{x : x_4 = 0\} = \{0000, 1000, 0100, 1010\}$:

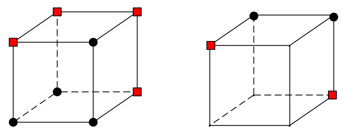


Assume in the first case that $\{0111, 1111\} \cap \overline{S} \neq \emptyset$. After applying Lemma 8.5 (i) (or the Sight Propagation Lemma) twice, we see that $\{0111, 1111, 0011, 1101\} \subseteq \overline{S}$:



Since the two restrictions over coordinate 1 are 1-resistant, $|\{0101, 0001\} \cap S| \neq 1$ and $|\{1001, 1011\} \cap S| \neq 1$. In fact, as S has no F_3 minor, $\{0101, 0001\} \subseteq S$ if and only if $\{1001, 1011\} \subseteq S$. Moreover, as the 0-restriction of S over coordinate 3 is 1-resistant, it follows that $\{0101, 0001, 1001, 1011\} \cap S \neq \emptyset$. As a result, $\{0101, 0001, 1001, 1011\} \subseteq S$, implying in turn that 1-restricting S over coordinate 4 yields D_3.

Assume in the remaining case that $\{0111, 1111\} \cap \overline{S} = \emptyset$. As the 1-restriction of S over coordinate 3 (resp. coordinate 2) is not isomorphic to either of F_1, F_3, we get that $0011 \in \overline{S}$ (resp. $1101 \in \overline{S}$).



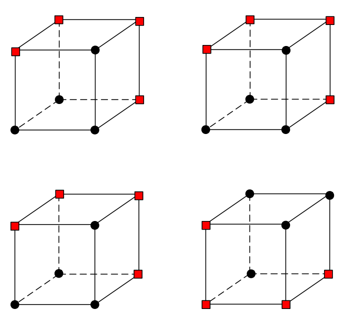
Since S is 1-resistant and has no F_3 restriction, it follows that $0001, 1001 \in \overline{S}$. Since the 0-restriction of S over coordinate 2 (resp. coordinate 3) is 1-resistant, $1011 \in S$ (resp. $0101 \in S$), implying in turn that 1-restricting S over coordinate 4 yeilds $D_3^\star$.

(ii) It follows from (i) that $S = \bigcup_{y \in \{0,1\}^{n-3}} \left(F \times \{y\} : F \in \{D_3, D_3^\star\}\right)$. Let $R \subseteq \{0,1\}^{n-3}$ be the set of points y such that $S \cap \{x : x_i = y_{i-3} \quad 4 \leq i \leq n\} = D_3 \times \{y\}$.

Claim 1. *Every feasible component of R is a hypercube. Similarly, every infeasible component of R is a hypercube.*

Proof of Claim. By Proposition 7.6 from Chapter 7, it suffices to prove that for each $y \in R$ and distinct coordinates $i, j \in [n-3]$, if $y, y\triangle e_i, y\triangle e_j \in R$ then $y\triangle e_i\triangle e_j \in R$.

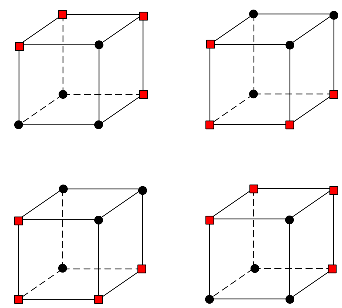
Suppose otherwise. After a possible twisting and relabeling, we may assume that $y = \mathbf{0}, i = 1, j = 2$. Let S' be the 0-restriction of S over coordinates $6, \ldots, n$:



Observe that the 0-restriction of S' over coordinates $1, 2$ is not 1-resistant, a contradiction. $\diamond$

Claim 2. *R is connected. Similarly, $\overline{R}$ is connected.*

Proof of Claim. Suppose for a contradiction that $R \subseteq \{0,1\}^{n-3}$ is not connected. By Claim 1, every feasible component of R is a hypercube, and as there are at least two feasible components, each feasible component is a hypercube of dimension at most $(n-3) - 2 = n - 5$. Thus, there exist $y \in \{0,1\}^{n-3}$ and distinct coordinates $i, j \in [n-3]$ such that $y \in R$ and $y\triangle e_i, y\triangle e_j \in \overline{R}$. Since every infeasible component of R is also a hypercube by Claim 1, it follows that $y\triangle e_i\triangle e_j \in R$. After a possible twisting and relabeling, we may assume that $y = \mathbf{0}, i = 1, j = 2$. Let S' be the 0-restriction of S over coordinates $6, \ldots, n$:



Observe however that the 0-restriction of S' over coordinates $1,2$ is not 1-resistant, a contradiction. $\diamond$

As a result, both $R, \overline{R}$ are hypercubes, implying in turn that $R \cong \emptyset, \{0,1\}^{n-4} \times \{0\}, \{0,1\}^{n-3}$. If $R \cong \emptyset, \{0,1\}^{n-3}$ then $S \cong D_3 \times \{0,1\}^{n-3}$, and if $R \cong \{0,1\}^{n-4} \times \{0\}$ then $S \cong C_8 \times \{0,1\}^{n-4}$, thereby finishing the proof. $\square$

For each $k \geq 4$, recall that $D_k = \{\mathbf{0}, e_2, \mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\} \subseteq \{0,1\}^k$, and let $D_k^\star := D_k \triangle e_k$.

Lemma 8.11. *Take integers $n \geq 3$ and $k \in \{3, \ldots, n\}$. Let $S \subseteq \{0,1\}^{n+1}$ be a set that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3$ minor. Then the following statements hold:*

- *(i) if the projection of S over coordinate $n+1$ is D_n, then S is either $D_{n+1}, D_{n+1}^\star$ or $D_n \times \{0,1\}$,*
- *(ii) if the projection of S over coordinate $k+1$ is $D_k \times \{0,1\}^{n-k}$, then S is either $D_{k+1} \times \{0,1\}^{n-k}, D_{k+1}^\star \times \{0,1\}^{n-k}$ or $D_k \times \{0,1\}^{n-k+1}$.*

Proof. **(i)** Assume that the projection of S over coordinate $n+1$ is D_n. Let

$$S_0 := S \cap \{x : x_i = 0, i \neq 2, 3, n+1\} \subseteq \{0,1\}^{n+1},$$
$$S_1 := S \cap \{x : x_i = 1, i \neq 2, 3, n+1\} \subseteq \{0,1\}^{n+1}.$$

Let $\mathbf{1} := \mathbf{1}^{n+1}$ and $\mathbf{1}' := \mathbf{1}^n$. Then

- $S = S_0 \cup S_1$,
- $S_0 \subseteq \{\mathbf{0}, e_2, e_{n+1}, e_2 + e_{n+1}\}$, and the projection of S_0 over coordinate $n+1$ is $\{\mathbf{0}, e_2\}$, and
- $S_1 \subseteq \{\mathbf{1} - e_2 - e_{n+1}, \mathbf{1} - e_2 - e_3 - e_{n+1}, \mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\}$, and the projection of S_1 over coordinate $n+1$ is $\{\mathbf{1}' - e_2, \mathbf{1}' - e_2 - e_3\}$.

After twisting coordinate $n+1$, if necessary, we may assume that $\mathbf{0} \in S_0$. Then, since S_0 and S_1 are 1-resistant, we get that

$$\begin{aligned} S_0 &= \{\mathbf{0}, e_2\} \quad \text{or} \quad \{\mathbf{0}, e_2, e_{n+1}, e_2 + e_{n+1}\}, \quad \text{and} \\ S_1 &= \{\mathbf{1} - e_2 - e_{n+1}, \mathbf{1} - e_2 - e_3 - e_{n+1}\} \quad \text{or} \quad \{\mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\} \quad \text{or} \\ &\quad \{\mathbf{1} - e_2 - e_{n+1}, \mathbf{1} - e_2 - e_3 - e_{n+1}, \mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\}. \end{aligned}$$

Claim 1. *If $S_0 = \{\mathbf{0}, e_2\}$, then $S = D_{n+1}$.*

Proof of Claim. Suppose that $S_0 = \{\mathbf{0}, e_2\}$.

Assume in the first case that $n = 3$. If $S_1 = \{\mathbf{1} - e_2 - e_4, \mathbf{1} - e_2 - e_3 - e_4\}$, then the 0-restriction of $S = S_0 \cup S_1$ over coordinate 3 is not 1-resistant, which is not the case. If $S_1 = \{\mathbf{1} - e_2 - e_4, \mathbf{1} - e_2 - e_3 - e_4, \mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\}$, then the 0-restriction of $S = S_0 \cup S_1$ over coordinate 2 is isomorphic to F_3, which is again not the case. Therefore, $S_1 = \{\mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\}$, implying in turn that $S = S_0 \cup S_1 = D_4$, as claimed.

Assume in the remaining case that $n \geq 4$. If $S_1 = \{\mathbf{1} - e_2 - e_{n+1}, \mathbf{1} - e_2 - e_3 - e_{n+1}\}$, then the points in $S = S_0 \cup S_1$ all agree on coordinate $n+1$, so by the Plane Propagation Lemma (7.8), S is a hypercube, which is not the case. If $S_1 = \{\mathbf{1} - e_2 - e_{n+1}, \mathbf{1} - e_2 - e_3 - e_{n+1}, \mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\}$, then the projection of $S = S_0 \cup S_1$ over coordinates $[n+1] - \{2, 3, n+1\}$ is isomorphic to F_3, which cannot be the case. Therefore, $S_1 = \{\mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\}$, implying in turn that $S = S_0 \cup S_1 = D_{n+1}$, as claimed. ◇

Claim 2. *If $S_0 = \{\mathbf{0}, e_2, e_{n+1}, e_2 + e_{n+1}\}$, then $S = D_n \times \{0,1\}$.*

Proof of Claim. Suppose that $S_0 = \{\mathbf{0}, e_2, e_{n+1}, e_2 + e_{n+1}\}$. As the projection of $S = S_0 \cup S_1$ over coordinates $[n+1] - \{2, 3, n+1\}$ is not isomorphic to F_3, it follows that $S_1 = \{\mathbf{1} - e_2 - e_{n+1}, \mathbf{1} - e_2 - e_3 - e_{n+1}, \mathbf{1} - e_2, \mathbf{1} - e_2 - e_3\}$, implying in turn that $S = D_n \times \{0,1\}$, as required. ◇

Thus, after twisting coordinate $n+1$, if necessary, S is either D_{n+1} or $D_n \times \{0,1\}$, so (i) holds.

(ii) Assume that the projection of S over coordinate $k+1$ is $D_k \times \{0,1\}^{n-k}$. For each point $y \in \{0,1\}^{n-k}$, let

$$S_y := S \cap \{x : x_{i+k+1} = y_i, i \in [n-k]\} \subseteq \{0,1\}^{n+1}.$$

Notice that $S = \bigcup_{y \in \{0,1\}^{n-k}} S_y$. For each $y \in \{0,1\}^{n-k}$, pick an appropriate $S'_y \subseteq \{0,1\}^{k+1}$ such that $S_y = S'_y \times \{y\}$. Notice that the projection of each S'_y over coordinate $k+1$ is D_k. We therefore get from (i) that each S'_y is either D_{k+1}, $D^\star_{k+1}$ or $D_k \times \{0,1\}$.

Claim 3. *All of $(S'_y : y \in \{0,1\}^{n-k})$ are equal to one another.*

Proof of Claim. Suppose otherwise. Then S has either $S' := (D_{k+1} \times \{0\}) \cup (D_k \times \{01, 11\})$ or $S'' := (D_{k+1} \times \{0\}) \cup (D^\star_{k+1} \times \{1\})$ as a restriction. However, the restriction of S' (resp. S'') obtained after 0-restricting coordinates $[n+1] - \{3, k+1, k+2\}$ is not 1-resistant, so S cannot have either of S', S'' as a restriction, a contradiction. ◇

As a consequence, $S = D_{k+1} \times \{0,1\}^{n-k}, D^\star_{k+1} \times \{0,1\}^{n-k}$ or $D_k \times \{0,1\}^{n-k+1}$, so (ii) holds. □

Lemma 8.12. *Take an integer $n \geq 5$ and a set $S \subseteq \{0,1\}^n$ that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3$ minor. If the projection of S over coordinate n is $C_8 \times \{0,1\}^{n-5}$, then $S = C_8 \times \{0,1\}^{n-4}$.*

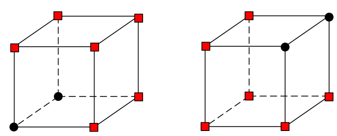
Proof. It suffices to prove this for $n = 5$. Assume that the projection of S over coordinate 5 is $C_8 = (D_3 \times \{0\}) \cup (D^\star_3 \times \{1\})$. For $i, j \in \{0,1\}$, let $S_{ij} \subseteq \{0,1\}^3$ be the restriction of S obtained after i-restricting coordinate 4 and j-restricting coordinate 5. After twisting coordinate 5, if necessary, we may assume that $\mathbf{0} \in S$.

Claim. *S has a D_3 restriction.*

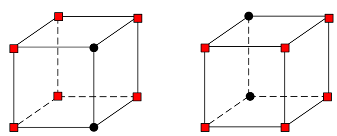
Proof of Claim. Suppose for a contradiction that S does not have a D_3 restriction. In particular, $S_{00}, S_{01} \neq D_3$ and $S_{10}, S_{11} \neq D_3^\star$. Thus by Lemma 8.11 (i),

$$\begin{aligned}(S_{00} \times \{0\}) \cup (S_{01} \times \{1\}) &= D_4 \quad \text{or} \quad D_4^\star,\\ (S_{10} \times \{0\}) \cup (S_{11} \times \{1\}) &= D_4' \quad \text{or} \quad D_4' \triangle e_4,\end{aligned}$$

where $D_4' = \{0100, 0110, 1011, 1111\} \subseteq \{0,1\}^4$. Since $\mathbf{0} \in S$, we must have that $(S_{00} \times \{0\}) \cup (S_{01} \times \{1\}) = D_4$. Thus, $S_{00} = \{000, 010\}$ and $S_{01} = \{100, 101\}$. Since the restriction of S obtained after 0-restricting coordinates 1 and 5 is not isomorphic to D_3, it follows that $(S_{10} \times \{0\}) \cup (S_{11} \times \{1\}) = D_4' \triangle e_4$. So, $S_{10} = \{101, 111\}$ and $S_{11} = \{010, 011\}$:



S_{00} $\quad$ S_{10}



S_{01} $\quad$ S_{11}

Observe however that the 1-restriction of S over coordinates $2, 3$ is not 1-resistant, a contradiction. $\diamond$

Thus, $S \cong D_3 \times \{0,1\}^2$ or $C_8 \times \{0,1\}$ by Lemma 8.10 (ii). It can be readily checked that S must be in fact equal to $C_8 \times \{0,1\}$, as required. $\square$

We are now ready to prove Theorem 8.2, stating the following:

> Take an integer $n \geq 3$ and a 1-resistant set $S \subseteq \{0,1\}^n$ without an $R_{1,1}, F_1, F_2, F_3$ minor. If S has a D_3 minor, then either
>
> - $S \cong C_8 \times \{0,1\}^{n-4}$, or
> - $S \cong D_k \times \{0,1\}^{n-k}$ for some $k \in \{3, \ldots, n\}$.

Proof. Among all projections of S with a D_3 restriction, pick the one $S' \subseteq \{0,1\}^\ell$ of largest dimension $\ell \in \{3, \ldots, n\}$. We may assume, after a possible relabeling, that S' is obtained from S after projecting away coordinates $[n] - [\ell]$. It follows from Lemma 8.10 (ii) that, after a possible twisting and relabeling, $S' = C_8 \times \{0,1\}^{\ell-4}$ or $S' = D_3 \times \{0,1\}^{\ell-3}$.

Claim. *If $S' = C_8 \times \{0,1\}^{\ell-4}$, then $\ell = n$.*

Proof of Claim. This follows immediately from Lemma 8.12 and the maximal choice of S'. $\diamond$

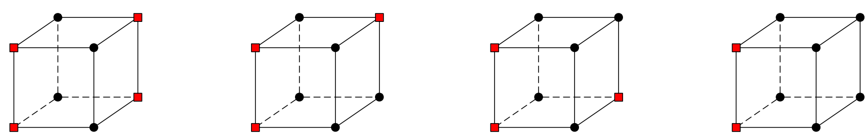
Thus, if $S' = C_8 \times \{0,1\}^{\ell-4}$, then $S \cong C_8 \times \{0,1\}^{n-4}$. Otherwise, $S' = D_3 \times \{0,1\}^{\ell-3}$. In this case, a repeated application of Lemma 8.11 (ii) implies that $S \cong D_k \times \{0,1\}^{n-k}$ for some $k \in \{\ell, \dots, n\}$, thereby finishing the proof of Theorem 8.2. $\square$

8.4 Infeasible hypercubes

Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. In this section, we will prove the following statement:

> Assume that is S is 1-resistant, has no $R_{1,1}, F_1, F_2, F_3$ minor and no D_3 minor. Take a point x and distinct coordinates $i, j \in [n]$ such that x is infeasible while $x \triangle e_i, x \triangle e_j, x \triangle e_i \triangle e_j$ are feasible. Then the infeasible component containing x is a hypercube.

Proving this statement requires three technical lemmas. Let $H_1 := \{100, 010, 101, 011\} \subseteq \{0,1\}^3$, $H_2 := \{100, 010, 101, 011, 110\} \subseteq \{0,1\}^3$, $H_2^\star := \{100, 010, 101, 011, 111\} \subseteq \{0,1\}^3$ and $H_3 := \{100, 010, 101, 011, 110, 111\} \subseteq \{0,1\}^3$, as displayed below:

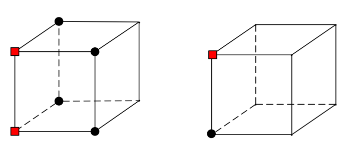


H_1 $\quad$ H_2 $\quad$ $H_2^\star$ $\quad$ H_3

Given $i \in \{0,1\}$, denote by $S_i \subseteq \{0,1\}^{n-1}$ the i-restriction of S over coordinate n.

Lemma 8.13. *Let $S \subseteq \{0,1\}^4$ be a set that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3, D_3$ minor. If $S_0 \in \{H_1, H_2, H_2^\star, H_3\}$, then $|\{000, 001\} \cap S_1| \neq 1$.*

Proof. Suppose, for a contradiction, that $H_1 \subseteq S_0 \subseteq H_3$ and $|\{000, 001\} \cap S_1| = 1$. After twisting coordinate 3, if necessary, we may assume that $000 \in S_1$ and $001 \in \overline{S_1}$. So S may be displayed as below:



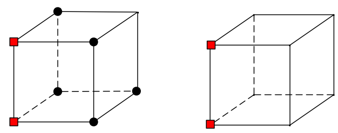
Since the 0-restriction of S over coordinate 1 is not isomorphic to either F_1 or F_3, we get that $011 \in \overline{S_1}$, and since this restriction is not isomorphic to D_3, we get that $010 \in \overline{S_1}$. By the symmetry between coordinates $1, 2$, we get that $\{100, 101\} \subseteq \overline{S_1}$. But then the 0-restriction of S over coordinate 3 is isomorphic to either $P_3, R_{1,1}, F_1$ or F_2, a contradiction. $\square$

Lemma 8.14. *Let $S \subseteq \{0,1\}^4$ be a set that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3, D_3$ minor, where $S_0 \in \{H_2, H_2^\star, H_3\}$ and $\{000, 001\} \cap S_1 = \emptyset$. Then the following statements hold:*

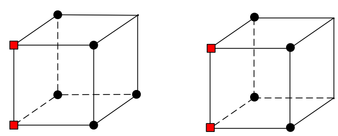
(i) $S_1 \in \{H_1, H_2, H_2^\star, H_3\}$, *and*

(ii) if $S_1 = H_1$, then $S_0 = H_3$.

Proof. **(i)** After twisting coordinate 3, if necessary, we may assume that $S_0 \in \{H_2, H_3\}$. We may therefore display S as:



Since the 0-restriction of S over coordinate 1 is 1-resistant, it follows that $|\{010, 011\} \cap S_1| \neq 1$, and since the 0-restriction of S over coordinate 2 is 1-resistant, it follows that $|\{100, 101\} \cap S_1| \neq 1$. Thus, as the 0-restriction of S over coordinate 3 is 1-resistant, either $\{010, 011\} \subseteq S_1$ or $\{100, 101\} \subseteq S_1$. After relabeling coordinates $1, 2$, if necessary, $\{010, 011\} \subseteq S_1$. Since the 0-restriction of S over coordinate 3 is not isomorphic to D_3 or F_3, it follows that $\{100, 101\} \subseteq S_1$ also:



Hence, $S_1 \in \{H_1, H_2, H_2^\star, H_3\}$. **(ii)** If $S_1 = H_1$, then as the 1-restriction of S over coordinate 1 is not isomorphic to F_3, it follows that $111 \in S_0$, so $S_0 = H_3$, as required. $\square$

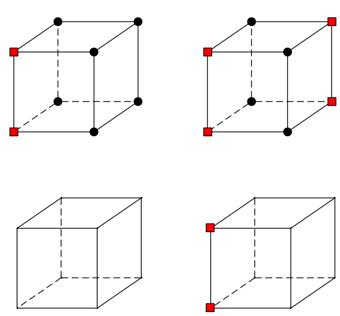
Given that $n \geq 2$ and $i, j \in \{0, 1\}$, denote by $S_{ij} \subseteq \{0, 1\}^{n-2}$ the restriction of S obtained after i-restricting coordinate $n - 1$ and j-restricting coordinate n.

Lemma 8.15. *Let $S \subseteq \{0, 1\}^5$ be a set that is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3, D_3$ minor, where $S_{00} = H_3, S_{10} = H_1$ and $\{000, 001\} \cap S_{11} = \emptyset$. Then the following statements hold:*

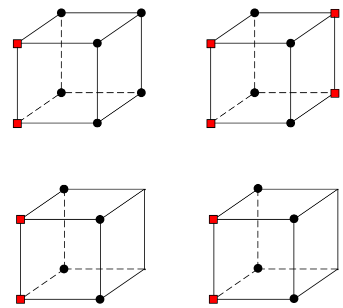
(i) $S_{01}, S_{11} \in \{H_1, H_2, H_2^\star, H_3\}$, and

(ii) if $S_{11} = H_1$ then $S_{01} = H_3$, and therefore $S_1 = S_0$.

Proof. **(i)** For $i, j \in \{0, 1\}$, denote by $R_{ij} \subseteq \{0, 1\}^5$ the restriction of S obtained after i-restricting coordinate 3 and j-restricting coordinate 5.



Notice that $R_{00} = R_{10} = H_2$ and $001 \notin R_{01} \cup R_{11}$. It therefore follows from Lemma 8.13 that $000 \notin R_{01} \cup R_{11}$. We get from Lemma 8.14 (i)-(ii) that $R_{01}, R_{11} \in \{H_2, H_2^\star, H_3\}$:



As a result, $S_{00}, S_{11} \in \{H_1, H_2, H_2^\star, H_3\}$. **(ii)** If $S_{11} = H_1$, then R_{01} and R_{11} must be equal to H_2, implying in turn that $S_{01} = H_3$, as required. $\square$

We are now ready to prove the main result of this section:

Proposition 8.16. *Take an integer $n \geq 1$ and a 1-resistant set $S \subseteq \{0,1\}^n$ that has no $R_{1,1}, F_1, F_2, F_3$ and no D_3 minor. Take a point x and distinct coordinates $i, j \in [n]$ such that x is infeasible while $x \triangle e_i, x \triangle e_j, x \triangle e_i \triangle e_j$ are feasible. Then the infeasible component containing x is a hypercube.*

Proof. We prove this by induction on $n \geq 2$. The base case $n = 2$ holds trivially. For the induction step, assume that $n \geq 3$. Let K be the infeasible component of S containing x. If every neighbor of x belongs to S, then $K = \{x\}$ and we are done. Otherwise, we may assume that $x \in \{\mathbf{0}, e_3\} \subseteq K$ and $i = 1, j = 2$. For each $y \in \{0,1\}^{n-3}$, let $S_y := S \cap \{x : x_{3+i} = y_i, i \in [n-3]\}$ and choose an appropriate $R_y \subseteq \{0,1\}^3$ such that $S_y = R_y \times \{y\}$. Notice that $\{000, 001\} \subseteq \overline{R_\mathbf{0}}$, and either $\{100, 010, 110\} \subseteq R_\mathbf{0}$ or $\{101, 011, 111\} \subseteq R_\mathbf{0}$. Since $R_\mathbf{0}$ is 1-resistant and not isomorphic to D_3, F_3, it follows that $R_\mathbf{0} \in \{H_2, H_2^\star, H_3\}$. In particular, if $n = 3$, then $K = \{\mathbf{0}, e_3\}$ and the induction step is complete. We may therefore assume that $n \geq 4$.

Let S' be the projection of S over coordinate 3. Then S' is 1-resistant and has no $R_{1,1}, F_1, F_2, F_3, D_3$ minor. Hence, since $\mathbf{0} \in \overline{S'}$ and $\{e_1, e_2, e_1 + e_2\} \subseteq S'$, the induction hypothesis implies that the infeasible component of S' containing $\mathbf{0}$, call it K', is a hypercube. Notice that the set of points in $\{0,1\}^n$ projecting onto a point in K' belong to K and form a hypercube whose dimension is larger by one.

Therefore, it suffices to show that K consists precisely of the points in $\{0,1\}^n$ projecting onto K'. Suppose otherwise. Then there must exist points $z, z + e_3 \in \{0,1\}^n$ projecting onto a point $z' \in \{0,1\}^{n-1}$ such that

- z' belongs to S' and is adjacent to a point in K', and
- $|\{z, z + e_3\} \cap S| = 1$.

Notice that $|\{z, z + e_3\} \cap K| = 1$.

Call a point $y \in \{0,1\}^{n-3}$ *involved* if

- $R_y \in \{H_2, H_2^\star, H_3\}$, and
- $00y \in K'$.

Notice that $\mathbf{0} \in \{0,1\}^{n-3}$ is involved. Now, pick a point $t' \in \{0,1\}^{n-1}$ minimizing $\operatorname{dist}(t', z')$ subject to

- $t' \in K'$, and
- there exists an involved $y \in \{0,1\}^{n-3}$ such that $t' = 00y$,

in this order of priority. We may assume that $t' = \mathbf{0} \in \{0,1\}^{n-1}$. Since $z' \notin K'$, we get that $\operatorname{dist}(\mathbf{0}, z') \geq 1$. It follows from Lemma 8.13 that $\operatorname{dist}(\mathbf{0}, z') \geq 2$. Since K' is a hypercube, there exist an integer $d \geq 2$ and distinct coordinates $j_1, j_2, \ldots, j_d \in [n] - \{3\}$ such that $z' = \sum_{i=1}^{d} e_{j_i}$ and

$$\sum_{i=1}^{k} e_{j_i} \in K' \qquad k = 1, \ldots, d-1.$$

Notice that

$$\sum_{i=1}^{k} e_{j_i} \in K \quad \text{and} \quad e_3 + \sum_{i=1}^{k} e_{j_i} \in K \qquad k = 1, \ldots, d-1.$$

Thus, since $R_{\mathbf{0}} \in \{H_2, H_3\}$, we have $j_1 \in [n] - \{1,2,3\}$. We may therefore assume that $j_1 = 4$. Since $R_{\mathbf{0}} \in \{H_2, H_3\}$ and $\{000, 001\} \cap R_{e_1} = \emptyset$, it follows from Lemma 8.14 (i) that $R_{e_1} \in \{H_1, H_2, H_2^\star, H_3\}$. Our minimal choice of $t' = \mathbf{0}$ implies that $R_{e_1} = H_1$ (otherwise, $t' = e_4$ contradicts the minimality of $t' = \mathbf{0}$). We now get from Lemma 8.13 that $d \geq 3$, and from Lemma 8.14 (ii) that $R_{\mathbf{0}} = H_3$. Since $j_2 \in [n] - \{1,2,3,4\}$, we may assume that $j_2 = 5$. So $e_4 + e_5 \in K'$. As $\mathbf{0}, e_4, e_4 + e_5 \in K'$ and K' is a hypercube, it follows that $e_5 \in K'$. Since $\{000, 001\} \cap R_{e_1+e_2} = \emptyset$, we get from Lemma 8.15 that either

- $R_{e_1+e_2} \in \{H_2, H_2^\star, H_3\}$, or
- $R_{e_2} = H_3$ and $R_{e_1+e_2} = H_1$.

The first case is not possible as it contradicts the minimal choice of $t' = \mathbf{0}$, for $t' = e_4 + e_5$ would be a better choice. However, the second case is not possible either as it also contradicts the minimal choice of $t' = \mathbf{0}$, for $t' = e_5$ would be a better choice. This finishes the proof of Proposition 8.16. $\square$

8.5 Proof of Theorem 8.3

We are now ready to prove Theorem 8.3, stating that

> Take an integer $n \geq 1$ and a 1-resistant set $S \subseteq \{0,1\}^n$ without an $R_{1,1}, F_1, F_2, F_3$ minor. If S is connected and has no D_3 minor, then either
>
> - S is a hypercube, or

- every infeasible component of S is a hypercube.

Proof. Assume that there is an infeasible component K that is not a hypercube.

Claim 1. *Take a point x and distinct coordinates $i, j \in [n]$ such that $x \in K$ and $x \triangle e_i \in S$. If $x \triangle e_i \triangle e_j \in S$, then $x \triangle e_j \in K$.*

Proof of Claim. For if not, $x \triangle e_j \in S$, so by Proposition 8.16, the infeasible component of S containing x, which is K, is a hypercube, a contradiction. $\diamond$

This claim has the following subtle implication:

Claim 2. *The points in S agree on a coordinate.*

Proof of Claim. Take a point $y \in K$ and a direction $i \in [n]$ such that $y \triangle e_i \in S$. We may assume that $y = \mathbf{0}$ and $i = 1$. As S is connected, it follows from Claim 1 that $S \subseteq \{x : x_1 = 1\}$, as required. $\diamond$

As S is 1-resistant, it follows from the Plane Propagation Lemma (7.8) that S is a hypercube, thereby proving Theorem 8.3. $\square$

8.6 ± 1-Resistance implies strict polarity.

Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Recall that S is polar if either there are antipodal feasible points, or the feasible points all agree on a coordinate. Recall further that S is strictly polar if every restriction, including S itself, is polar. We will need the following immediate remark:

Remark 8.17. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. If S is strictly polar, then so is $S \times \{0,1\}$.*

We will also need the following variant of Lemma 6.8 from Chapter 6:

Lemma 8.18 ([1])**.** *Take an integer $n \geq 1$ and a nonempty set $S \subseteq \{0,1\}^n$ where every infeasible component is a hypercube. Assume that S has no $R_{1,1}$ restriction. Then*

- *$|S| \geq 2^{n-1}$, and*
- *if $|S| = 2^{n-1}$, then S is either a hypercube of dimension $n-1$ or the union of antipodal hypercubes of dimension $n-2$.*

In particular, S is strictly polar.

Proof. We will prove this by induction on $n \geq 1$. The base cases $n \in \{1,2\}$ are clear. For the induction step, assume that $n \geq 3$. For $i \in \{0,1\}$, let $S_i \subseteq \{0,1\}^{n-1}$ be the i-restriction of S over coordinate n. If one of S_0, S_1 is empty, then the other one must be $\{0,1\}^{n-1}$, so S is a hypercube of dimension $n-1$ and the induction step is complete. We may therefore assume that S_0, S_1 are nonempty. Since every infeasible component of both S_0, S_1 is a hypercube, we may apply the induction hypothesis. Thus, $|S_0| \geq 2^{n-2}$ and $|S_1| \geq 2^{n-2}$, implying in turn that $|S| = |S_0| + |S_1| \geq 2^{n-1}$. Assume next that $|S| = 2^{n-1}$. Then

$|S_0| = |S_1| = 2^{n-2}$, so by the induction hypothesis, each S_i is either a hypercube of dimension $n-2$ or the union of antipodal hypercubes of dimension $n-3$. If one of S_0, S_1 is a hypercube, then as every infeasible component of S is either a hypercube, S is either a hypercube of dimension $n-1$ or the union of antipodal hypercubes of dimension $n-2$. Otherwise, each one of S_0, S_1 is the union of two antipodal hypercubes of dimension $n-3$. As S has no $R_{1,1}$ restriction, it must be that $S_0 = S_1$, implying in turn that S is the union of antipodal hypercubes of dimension $n-2$, thereby completing the induction step. ☐

As a consequence,

Theorem 8.19 ([1]). *A ± 1-resistant set is strictly polar.*

Proof. Take an integer $n \geq 1$ and a ± 1-resistant set $S \subseteq \{0,1\}^n$. Then by Theorem 1.45, either

(i) $S \cong A_k \times \{0,1\}^{n-k}$ for some $k \in \{2, \ldots, n\}$,

(ii) $S \cong B_k \times \{0,1\}^{n-k}$ for some $k \in \{3, \ldots, n\}$,

(iii) $S \cong C_8 \times \{0,1\}^{n-4}$,

(iv) $S \cong D_k \times \{0,1\}^{n-k}$ for some $k \in \{3, \ldots, n\}$,

(v) S is a hypercube, or

(vi) every infeasible component of S is a hypercube.

Observe that $\{A_k : k \geq 2\}, \{B_k, D_k : k \geq 3\}$ and C_8 are strictly polar sets. As a result, in cases (i)-(iv), the set S is strictly polar by Remark 8.17. A hypercube is strictly polar, so in case (v), S is also strictly polar. For the last case (vi), as S is ± 1-resistant, it has no $R_{1,1}$ restriction, so by Lemma 8.18, S is strictly polar. ☐

This theorem, together with Theorem 6.4, has the following immediate consequence:

Corollary 8.20 ([1]). *The cuboid of a ± 1-resistant set has the packing property.*

Chapter 9

The minimally non-ideal binary clutters with a member of cardinality three

Recall that a clutter is binary if the symmetric difference of any odd number of members contains a member. When is a binary clutter ideal? If a clutter is binary, then so is every minor of it [69]. We may therefore ask the following question instead: what are the minimally non-ideal binary clutters?

The f-Flowing Conjecture (Seymour [70, 66])**.** *Up to isomorphism, $\mathbb{L}_7, \mathbb{O}_5, b(\mathbb{O}_5)$ are the only minimally non-ideal binary clutters.*

Recall that $\mathbb{L}_7$ is the Fano clutter, whose elements and members are the points and lines of the Fano plane; and over ground set $E(K_5)$, $\mathbb{O}_5$ and $b(\mathbb{O}_5)$ are the clutters of odd circuits and cut complements of K_5, respectively. In this chapter, we will prove that $\mathbb{L}_7, \mathbb{O}_5$ are, up to isomorphism, the only minimally non-ideal binary clutters with a member of cardinality three.

Representing binary clutters

Take a binary matroid M over ground set $E(M)$. Let $\Sigma \subseteq E(M)$. The pair (M, Σ) is called a *signed binary matroid* over *ground set* $E(M)$. A subset $\Gamma \subseteq E(M)$ is a *signature* of (M, Σ) if $\Sigma \triangle \Gamma$ is a cocycle of M. A signature is *minimal* if it does not properly contain another signature. Observe that the symmetric difference of an odd number of signatures is another signature. For a signature Γ, the operation of replacing (M, Σ) by (M, Γ) is called *resigning*. A subset $S \subseteq E(M)$ is said to be *odd* (resp. *even*) if $|S \cap \Sigma|$ is odd (resp. even). An element $f \in E(M)$ is *odd* (resp. *even*) if $\{f\}$ is odd (resp. even). Observe that resigning a signed binary matroid preserves the parity of every cycle. Signed binary matroids represent binary clutters:

Proposition 9.1 ([48, 59], also see [23, 41]). *A clutter $\mathcal{C}$ over ground set E is binary if, and only if, the members of $\mathcal{C}$ are the odd circuits of a signed binary matroid over ground set E. Moreover, assuming $\mathcal{C}$ is the clutter of odd circuits of a signed binary matroid, then $b(\mathcal{C})$ is the clutter of minimal signatures.*

Denote by F_7 the Fano matroid. Then the Fano clutter $\mathbb{L}_7$ is represented as the signed binary matroid $(F_7, E(F_7))$. For a graph G, denote by cycle(G) the cycle matroid of G, i.e. the binary matroid whose cycles are exactly the cycles of the graph G. Then $\mathbb{O}_5$ is represented as the signed binary matroid $(\text{cycle}(K_5)\,, E(K_5))$.

Finding a suitable representation

Let $r : 2^{E(M)} \to \{0, 1, 2, \ldots\}$ be the rank function of M. The *connectivity function* $\lambda_M : 2^{E(M)} \to \{0, 1, 2, \ldots\}$ is defined, for each $X \subseteq E(M)$, as $\lambda_M(X) := r(X) + r(\overline{X}) - r(E(M))$. Here, $\overline{X} := E(M) - X$. Take an integer $k \geq 1$. We say that $X \subseteq E(M)$ is *k-separating* if $\lambda_M(X) \leq k - 1$. A *k-separation* is a pair $(X, \overline{X})$, where X is k-separating and $\min\{|X|, |\overline{X}|\} \geq k$. We say M is *$(k+1)$-connected* if, for each $r \in [k]$, M has no r-separation. A binary matroid is *internally 4-connected* if it is 3-connected, and for every 3-separation $(X, \overline{X})$, either $|X| = 3$ or $|\overline{X}| = 3$.

Theorem 9.2. *Let $\mathcal{C}$ be a minimally non-ideal binary clutter over ground set E with a member of cardinality three. Then $\mathcal{C}$ is the clutter of odd circuits of a signed binary matroid (M, E) where the following statements hold:*

(a) M is internally 4-connected,

(b) every element in E is contained in exactly three triangles of M,

(c) if $|E| \leq 12$, then $\mathcal{C} \cong \mathbb{L}_7$ or $\mathcal{C} \cong \mathbb{O}_5$,

(d) if M is graphic, then $\mathcal{C} \cong \mathbb{O}_5$, and

(e) if M has an induced K_4, then $\mathcal{C} \cong \mathbb{L}_7$ or $\mathcal{C} \cong \mathbb{O}_5$.

What is an induced K_4? Let M be a binary matroid over ground set $E(M)$. For $R \subseteq E(M)$, we write $M|R := M \setminus (E(M) - R)$. We say that $\{e_1, \ldots, e_6\} \subseteq E(M)$ is an *induced K_4* of M if $M|\{e_1, \ldots, e_6\}$ is isomorphic to cycle(K_4).

Working with the representation

Take disjoint subsets $I, J \subseteq E(M)$. If I contains an odd circuit, we define $(M, \Sigma)/I \setminus J := (M/I \setminus J, \emptyset)$. Otherwise, by Proposition 9.1, there is a signature Σ' that is disjoint from I, and we define $(M, \Sigma)/I \setminus J := (M/I \setminus J, \Sigma' - J)$. We call $(M, \Sigma)/I \setminus J$ a *minor* of (M, Σ). Notice that minors are defined only up to resigning. We have the following relation between minors of binary clutters and minors of signed binary matroids:

Remark 9.3 (see [23]). *Let $\mathcal{C}$ be a binary clutter over ground set E represented as the signed binary matroid (M, Σ). Take disjoint subsets $I, J \subseteq E$. Then $\mathcal{C}/I \setminus J$ is represented as the signed binary matroid $(M, \Sigma)/I \setminus J$.*

Theorem 9.4. *Let M be an internally 4-connected binary matroid where every element is contained in exactly three triangles. Then at least one of the following statements holds:*

(i) $|E(M)| \leq 11$,

(ii) *M is graphic,*

(iii) *M has an induced K_4, or*

(iv) *the signed binary matroid $(M, E(M))$ has $(F_7, E(F_7))$ as a minor.*

Putting it all together

Theorem 1.34 is a rather immediate consequence of Theorems 9.2 and 9.4:

> **Theorem 1.34 ([6]).** *Up to isomorphism, $\mathbb{L}_7, \mathbb{O}_5$ are the only minimally non-ideal binary clutters with a member of cardinality three.*

Proof. Let $\mathcal{C}$ be a minimally non-ideal binary clutter over ground set E with a member of cardinality three. By Theorem 9.2, $\mathcal{C}$ is represented as a signed binary matroid (M, E), where M is an internally 4-connected binary matroid and every element is contained in exactly three triangles. If either $|E| \leq 12$, M is graphic, or M has an induced K_4, then by Theorem 9.2 (c)-(e), $\mathcal{C} \cong \mathbb{L}_7$ or $\mathcal{C} \cong \mathbb{O}_5$. Otherwise, by Theorem 9.4, the signed binary matroid (M, E) has $(F_7, E(F_7))$ as a minor, so the minimally non-ideal $\mathcal{C}$ has the non-ideal $\mathbb{L}_7$ as a minor by Remark 9.3, implying in turn that $\mathcal{C} \cong \mathbb{L}_7$, thereby finishing the proof. □

The rest of this chapter is dedicated to proving Theorems 9.2 and 9.4. In §9.1 we prove Theorem 9.2, while the proof of Theorem 9.4 is outlined in §9.2 and spans §9.3-§9.6.

9.1 Proof of Theorem 9.2

We will need the following three results:

Theorem 9.5 (Cornuéjols and Guenin [23], Remark 5.3, Propositions 6.1 and 7.1)**.** *Let $\mathcal{C}$ be a minimally non-ideal binary clutter represented as the signed binary matroid (M, Σ). Then M is internally 4-connected.*

Theorem 9.6 (Guenin [40], also see Schrijver [64])**.** *Let $\mathcal{C}$ be a minimally non-ideal binary clutter represented as the signed binary matroid (M, Σ). If M is graphic, then $\mathcal{C} \cong \mathbb{O}_5$.*

Remark 9.7 (see [54])**.** *Let $\mathcal{C}$ be a minimally non-ideal binary clutter with a member of cardinality three. If $\tau(\mathcal{C}) = 3$ then $\mathcal{C} \cong \mathbb{L}_7$, and if $\tau(\mathcal{C}) = 4$ then $\mathcal{C} \cong \mathbb{O}_5$.*

Given a clutter $\mathcal{C}$, denote by $\bar{\mathcal{C}}$ the clutter of the minimum cardinality members. For an integer $k \geq 1$, a $0,1$ matrix is *k-regular* if each row and each column has exactly k ones. We will also need the following powerful result of Lehman (combined with a result of Bridges and Ryser) stated only for binary clutters:

Theorem 9.8 (Lehman, Bridges and Ryser [50, 16], see [67])**.** *Let $\mathcal{C}$ be a minimally non-ideal binary clutter over ground set E. Then $\mathcal{B} := b(\mathcal{C})$ is also minimally non-ideal, and the following statements hold:*

- *$M(\bar{\mathcal{C}})$ and $M(\bar{\mathcal{B}})$ are square matrices,*
- *for some integers $r \geq 3$ and $s \geq 3$, $M(\bar{\mathcal{C}})$ is r-regular and $M(\bar{\mathcal{B}})$ is s-regular,*
- *for $n := |E|$, $rs - n$ is an even integer such that $2 \leq rs - n \leq \min\{r-1, s-1\}$, and*
- *after possibly rearranging the rows of $M(\bar{\mathcal{B}})$, we have*
$$M(\bar{\mathcal{C}})M(\bar{\mathcal{B}})^\top = J + (rs-n)I = M(\bar{\mathcal{B}})^\top M(\bar{\mathcal{C}}).$$
Here, J is the all-ones matrix and I is the identity matrix.

Notice that if $\mathcal{C}$ is a minimally non-ideal binary clutter with a member of cardinality three, then $r = 3$ and $3s - n = 2$. We are now ready to prove Theorem 9.2:

Proof of Theorem 9.2. Let $\mathcal{C}$ be a minimally non-ideal binary clutter over ground set E with a member of cardinality three. Let us refer to the minimum cardinality members as *triangles of* $\mathcal{C}$. Set $n := |E|$. Let $\mathcal{B} := b(\mathcal{C})$ and denote by s the minimum cardinality of a set in $\mathcal{B}$. By Theorem 9.8, after possibly rearranging the rows of $M(\bar{\mathcal{B}})$,

$$r = 3 \quad \text{and} \quad s \geq 3 \quad \text{and} \quad 3s - n = 2 \quad \text{and} \quad M(\bar{\mathcal{C}})M(\bar{\mathcal{B}})^\top = J + 2I = M(\bar{\mathcal{B}})^\top M(\bar{\mathcal{C}}). \tag{$\star$}$$

Moreover, $M(\bar{\mathcal{C}})$ is 3-regular, so every element in E is contained in exactly 3 triangles of $\mathcal{C}$. Label the rows of $M(\bar{\mathcal{C}})$ as $S_1, \ldots, S_n \in \bar{\mathcal{C}}$, and the rows of $M(\bar{\mathcal{B}})$ as $R_1, \ldots, R_n \in \bar{\mathcal{B}}$. Then the last equation implies, for all $i, j \in [n]$, that

$$|S_i \cap R_j| = \begin{cases} 3 & \text{if } i = j, \\ 1 & \text{if } i \neq j. \end{cases}$$

For each $i \in [n]$, we say that S_i and R_i are *mates* of one another. Thus, a triangle of $\mathcal{C}$ is contained in its mate, and it intersects all the other triangle mates exactly once.

Claim 1. *Take an element $e \in E$, denote by S, S', S'' the triangles of $\mathcal{C}$ containing e, and by R, R', R'' their respective mates in $\bar{\mathcal{B}}$. Then*

- *$R \cap R' = R' \cap R'' = R'' \cap R = \{e\}$,*
- *$R \cup R' \cup R'' = E$, and*
- *$S \cap S' = S' \cap S'' = S'' \cap S = \{e\}$.*

Proof of Claim. By $(\star)$, $M(\bar{\mathcal{B}})^\top M(\bar{\mathcal{C}}) = J+2I$. Denote by c_e the column of $M(\bar{\mathcal{C}})$ corresponding to e, and for each $f \in E$, denote by c'_f the column of $M(\bar{\mathcal{B}})$ corresponding to f. Then the matrix equation implies that $c_e^\top c'_e = 3$ and, for each $f \in E(\mathcal{C}) - e$, that $c_e^\top c'_f = 1$; the first and second lines follow.[1] Since $S \subseteq R$, $S' \subseteq R'$ and $S'' \subseteq R''$, the third line follows. $\diamond$

Since $\mathcal{C}$ is a binary clutter over ground set E, we get from Proposition 9.1 that $\mathcal{C}$ is the clutter of odd circuits of a signed binary matroid (M, Σ) over ground set E.

Claim 2. *E is a signature of (M, Σ).*

Proof of Claim. Take $e \in E$ and let R, R', R'' be the mates of the triangles of $\mathcal{C}$ containing e. Since R, R', R'' belong to $b(\mathcal{C})$, they are signatures of (M, Σ) by Proposition 9.1. So their symmetric difference $R \triangle R' \triangle R''$ is also a signature. However, Claim 1 implies that $R \triangle R' \triangle R'' = E$, so E is a signature. $\diamond$

Thus, $\mathcal{C}$ is the clutter of odd circuits of the signed binary matroid (M, E). It follows from Theorem 9.5 that M is internally 4-connected, so **(a)** holds.

Claim 3. *Every element in E is contained in exactly 3 triangles of M, so* **(b)** *holds.*

Proof of Claim. Since $\mathcal{C}$ is the clutter of odd circuits of (M, E), the triangles of $\mathcal{C}$ are precisely the triangles of M. Since every element in E is contained in exactly 3 triangles of $\mathcal{C}$, the claim follows. $\diamond$

Claim 4. *If $|E| \leq 12$, then $\mathcal{C} \cong \mathbb{L}_7$ or $\mathcal{C} \cong \mathbb{O}_5$, so* **(c)** *holds.*

Proof of Claim. By $(\star)$, $3s - 2 = n = |E| \leq 12$ and $s \geq 3$, so $s \in \{3, 4\}$ and by Remark 9.7, we get that $\mathcal{C} \cong \mathbb{L}_7$ or $\mathcal{C} \cong \mathbb{O}_5$. $\diamond$

It follows from Theorem 9.6 that if M is graphic, then $\mathcal{C} \cong \mathbb{O}_5$, so **(d)** holds. It remains to prove (e). To this end, assume that M has an induced K_4, that is, there are elements $e_1, \ldots, e_6 \in E$ such that $M|\{e_1, \ldots, e_6\} \cong \text{cycle}(K_4)$. As the triangles of M are precisely the triangles of $\mathcal{C}$, we may assume that S_1, S_2, S_3, S_4 are the four triangles of $M|\{e_1, \ldots, e_6\}$.

Claim 5. *For all distinct $i, j \in [4]$, $R_i \cap R_j \subseteq \{e_1, \ldots, e_6\}$.*

Proof of Claim. As S_i, S_j are distinct triangles of K_4, there is an $e \in \{e_1, \ldots, e_6\}$ such that $S_i \cap S_j = \{e\}$. It now follows from Claim 1 that $R_i \cap R_j = \{e\} \subseteq \{e_1, \ldots, e_6\}$. $\diamond$

Claim 6. *For all $i \in [4]$, $R_i \cap \{e_1, \ldots, e_6\} = S_i$ and $|R_i - \{e_1, \ldots, e_6\}| = s - 3$.*

Proof of Claim. Since R_i is the mate of S_i, we have $S_i \subseteq R_i$. As R_i intersects every other triangle exactly once, and $|S_i \cap S_j| = 1$ for each $j \in [4] - i$, we get that $R_i \cap \{e_1, \ldots, e_6\} = S_i$. $\diamond$

Putting Claim 5 and Claim 6 together, we get that $|E| \geq 6 + 4(s - 3)$. From $(\star)$ we have that $s \geq 3$, and also that $|E| = n = 3s - 2$, so $3s - 2 \geq 6 + 4(s - 3)$, implying in turn that $s \in \{3, 4\}$. It now follows Remark 9.7 that $\mathcal{C} \cong \mathbb{L}_7$ or $\mathcal{C} \cong \mathbb{O}_5$, thereby proving **(e)**. This finishes the proof of Theorem 9.2. $\square$

[1] For a set A and an element $a \in A$, denote $A - a := A - \{a\}$.

9.2 An outline of the proof of Theorem 9.4

Let M be a binary matroid where the following assumptions hold:

Common hypotheses

(h1) M is an internally 4-connected binary matroid,

(h2) every element in $E(M)$ is contained in exactly three triangles of M.

Since M is internally 4-connected, M is a simple (and cosimple) binary matroid. In particular, the three triangles containing an element are otherwise pairwise disjoint. Take an element $\Omega \in E(M)$. Denote the three triangles of M containing Ω by $\{\Omega, f, f'\}, \{\Omega, g, g'\}, \{\Omega, h, h'\}$. Since M is simple, M/Ω does not have a loop, and $\{f, f'\}, \{g, g'\}, \{h, h'\}$ are *the* non-trivial parallel classes of M/Ω. It follows that the simplification $\mathrm{si}(M/\Omega)$ is obtained from M/Ω by deleting one element from each one of $\{f, f'\}, \{g, g'\}, \{h, h'\}$. If f, g, h are the elements left in $\mathrm{si}(M/\Omega)$, we write $\Lambda(\Omega) := \{f, g, h\}$.

The proof of Theorem 9.4 relies on the following four propositions, as well as two theorems proved by others:

Proposition 9.9. *Suppose (h1)-(h2) hold and let $\Omega \in E(M)$. If $\Lambda(\Omega)$ is a cocycle of $si(M/\Omega)$, then M has an induced K_4.*

Proof. Suppose that $\Lambda(\Omega)$ is a cocycle of $\mathrm{si}(M/\Omega)$. Denote by $\{\Omega, f, f'\}, \{\Omega, g, g'\}, \{\Omega, h, h'\}$ the triangles of M containing Ω where $\Lambda(\Omega) = \{f, g, h\}$. Since $\{f, g, h\}$ is a cocycle of $\mathrm{si}(M/\Omega)$, $D := \{f, f', g, g', h, h'\}$ is a cocycle of M/Ω and hence of M. As f is in three triangles of M, it is contained in a triangle C that is different from $\{\Omega, f, f'\}$. For D is a cocycle, $|C \cap D|$ is even, and because $f \in C \cap D$, $|C \cap D| = 2$. Moreover, $C \cap D \neq \{f, f'\}$, for otherwise $C \triangle \{f, f', \Omega\}$ would be a cycle of cardinality two, which cannot be the case as M is simple. Hence, we may assume that $C \cap D = \{f, g\}$ or $C \cap D = \{f, g'\}$. In either case, $C \cup \{\Omega, f, f', g, g'\}$ is an induced K_4 of M, as required. $\square$

Recall that a graft is a pair (G, T), where G is a graph and $T \subseteq V(G)$ is of even cardinality. Recall that vertices in T are called terminals. Take a subset $J \subseteq E(G)$. Denote by $\mathrm{odd}(J) \subseteq V(G)$ the vertices incident with an odd number of non-loop edges in J. Recall that if $\mathrm{odd}(J) = T$, then we say J a T-join. Start with the vertex-edge incidence matrix of G, and add the vertex-incidence vector of T as a column; call this matrix A. Let M be the binary matroid whose binary representation is A, and denote by t the element of M corresponding to column T. Then $C \subseteq E(M)$ is a cycle of M if, and only if, one of the following holds:

- $t \notin C$ and C is a cycle of G,
- $t \in C$ and $C - t$ is a T-join of G.

We call M the *graft matroid* of (G, T). By convention, t will always be the element of M corresponding to the terminals T. Notice that if $|T| \leq 2$, then the graft matroid of (G, T) is graphic. The next folklore remark states that graft matroids are precisely those binary matroids that are one deletion away from being graphic (see for instance [60], Lemma 10.3.8):

Remark 9.10. *Take a binary matroid M and an element $t \in E(M)$ such that $M \setminus t = cycle(G)$, for some graph G. If C is a cycle of M containing t, then M is the graft matroid of the graft $(G, \mathrm{odd}(C - t))$.*

We are now ready for the next proposition:

Proposition 9.11. *Suppose (h1)-(h2) hold and let $\Omega \in E(M)$. If $M \setminus \Omega$ is graphic, then M is graphic or has an induced K_4.*

Proof. Suppose $M \setminus \Omega$ is graphic. By Remark 9.10, there is a graft (G, T) whose graft matroid is M, where $t = \Omega$. If $|T| \leq 2$, then M is graphic, so we are done. Otherwise, $|T| \geq 4$. Denote the three triangles of M containing Ω by $\{\Omega, f, f'\}, \{\Omega, g, g'\}, \{\Omega, h, h'\}$. Then $\{f, f'\}, \{g, g'\}$ and $\{h, h'\}$ are T-joins of G. Since M is simple, we see that G does not have parallel edges. As a result, $|T| = 4$ and $\{f, f', g, g', h, h'\}$ is an induced K_4 of M, as required. ☐

We will prove the following two propositions in the forthcoming sections:

Proposition 9.12. *Suppose (h1)-(h2) hold and let $\Omega \in E(M)$. If $\Lambda(\Omega)$ is contained in a circuit of $si(M/\Omega)$, then either M has an induced K_4 or $(M, E(M))$ has $(F_7, E(F_7))$ as a minor.*

Proposition 9.13. *Suppose (h1)-(h2) hold and let e_1, e_2, e_3, e_4 be distinct elements of M such that, for every $i \in [4]$, $si(M/e_i)$ is internally 4-connected and is the cycle matroid of a graph where the three edges of $\Lambda(e_i)$ are incident to the same vertex. Then either $|E(M)| \leq 11$, or there is an element $\Omega \in E(M)$ such that $M \setminus \Omega$ is graphic.*

We will also need the following result of Seymour that characterizes, under appropriate connectivity conditions, when three distinct elements of a binary matroid are contained in a circuit:

Theorem 9.14 (Seymour [71])**.** *Let M be an internally 4-connected binary matroid, and let f, g, h be distinct elements. Then one of the following statements holds:*

- *$\{f, g, h\}$ is contained in a circuit of M,*
- *$\{f, g, h\}$ is a cocycle of M, or*
- *M is the cycle matroid of a graph where edges f, g, h are incident to the same vertex.*

The following result of Chun and Oxley on internally 4-connected binary matroids is the last needed ingredient:

Theorem 9.15 (Chun and Oxley [18])**.** *Let M be an internally 4-connected binary matroid where every element is in exactly three triangles. Then there exist distinct elements $e_1, e_2, e_3, e_4 \in E(M)$ such that, for each $j \in [4]$, $si(M/e_j)$ is internally 4-connected.*

We are now ready to prove Theorem 9.4:

Proof of Theorem 9.4. Suppose (h1)-(h2) hold. By Theorem 9.15, there exist distinct elements e_1, e_2, e_3, e_4 of M such that, for each $j \in [4]$, $\mathrm{si}(M/e_j)$ is internally 4-connected. For $j \in [4]$,

- if $\Lambda(e_j)$ is contained in a circuit of $\mathrm{si}(M/e_j)$, then by Proposition 9.12, either M has an induced K_4 and so (iii) holds, or $(M, E(M))$ has $(F_7, E(F_7))$ as a minor and so (iv) holds,
- if $\Lambda(e_j)$ is a cocycle of $\mathrm{si}(M/e_j)$, then by Proposition 9.9, M has an induced K_4, so (iii) holds.

Otherwise, it follows from Theorem 9.14 that, for each $j \in [4]$, $\mathrm{si}(M/e_j)$ is the cycle matroid of a graph where the three edges in $\Lambda(e_j)$ are incident to the same vertex. By Proposition 9.13, either $|E(M)| \leq 11$ and so (i) holds, or there is an element $\Omega \in E(M)$ such that $M \setminus \Omega$ is graphic. By Proposition 9.11, either M is graphic and so (ii) holds, or M has an induced K_4 and so (iii) holds. In all cases, one of (i)-(iv) holds, and so we are done. □

In §9.3 we introduce *signed grafts* and present two instances that have $(F_7, E(F_7))$ as a minor. In §9.4 we leverage these results to prove Proposition 9.12. In §9.5 we introduce *even cycle matroids* and prove several relevant results, which in turn lead to a proof of Proposition 9.13 in §9.6.

9.3 Quadrums and trifolds

Representations of the Fano matroid

A *plain quadrum* is the graft $(K_4, V(K_4))$. A *plain trifold* is the graft for which the graph has vertex set $[5]$ and edges $\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}$, and the terminals are $\{2,3,4,5\}$. Drawings of the plain quadrum and the plain trifold are given in Figure 9.1.

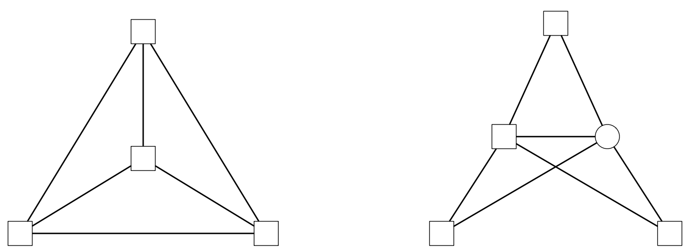


Figure 9.1: Left: plain quadrum, right: plain trifold. Square vertices are terminals.

Remark 9.16. *Let (G,T) be a graft, and N its graft matroid. Then the following statements hold:*

(a) if (G,T) is a plain quadrum, then $N \cong F_7$,

(b) if (G,T) is a plain trifold, then $N/t \cong F_7$.

Proof. Notice that a binary matroid is determined by the set of its circuits. **(a)** Consider Figure 9.2 (a). We assign t and each edge of the plain quadrum to an element of F_7. It now suffices to observe that the circuits of N correspond to the circuits of F_7, i.e. to the lines and the line complements of the Fano plane. **(b)** Consider Figure 9.2 (b). We assign each edge of the plain trifold to an element of F_7. Observe that the circuits of N/t, which are the circuits and T-joins of G, correspond to the circuits of F_7. □

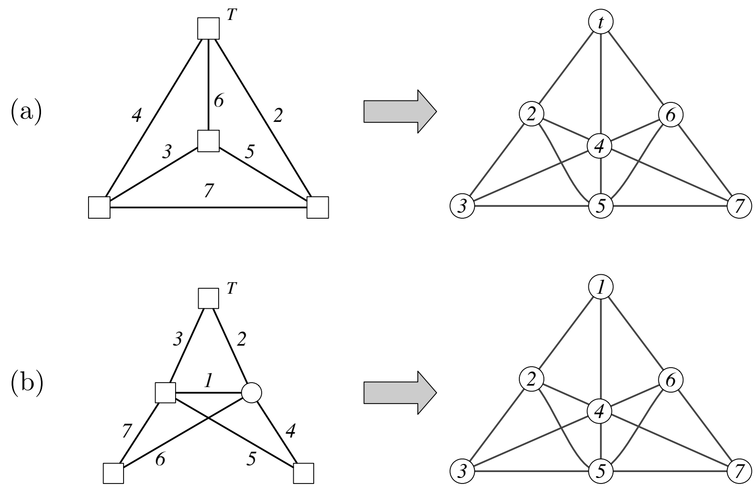


Figure 9.2: The Fano matroid in disguise

Signed grafts: quadrums and trifolds

A *signed graft* is a triple (G, T, Γ), where (G, T) is a graft and $\Gamma \subseteq E(G) \cup \{t\}$. Note that we assign parity to each edge as well as to the set of terminals.

A *quadrum* is the signed graft (G, T, Γ) where (G, T) is a plain quadrum and $\Gamma = E(G) \cup \{t\}$. A *super quadrum* is the signed graft displayed in Figure 9.3 (a) which is obtained as follows: start with a plain quadrum, take a set S of four edges that contain a triangle, the element t and the two edges outside S can have either parities, and replace each edge of S by a pair of parallel edges of distinct parities.

A *trifold* is the signed graft (G, T, Γ) where (G, T) is a plain trifold and $\Gamma = E(G)$. A *super trifold* is the signed graft displayed in Figure 9.3 (b) which is obtained as follows: start with a plain trifold, take two triangles and the two edges S disjoint from them, the edges outside S are odd, replace each edge of S by a pair of parallel edges of distinct parities, and element t can have either parities.

Finding an $(F_7, E(F_7))$ minor

Let G be a graph. For a vertex $v \in V(G)$, we denote by $\delta_G(v)$ the set of non-loop edges of G that are incident with v. Take a signed graft (G, T, Γ) and a terminal $v \in T$. Let $B := \delta_G(v) \cup \{t\}$. We say that $(G, T, \Gamma \triangle B)$ is obtained from (G, T, Γ) by *resigning on the terminal* v.

Remark 9.17. *Let (G, T, Γ) be a signed graft, and N the graft matroid of the graft (G, T). If (G, T, Γ') is obtained from (G, T, Γ) by resigning on a terminal, then Γ' is a signature of the signed binary matroid (N, Γ).*

Proof. It suffices to show that, for each terminal $v \in T$, the set $B := \delta_G(v) \cup \{t\}$ is a cocycle of N. To this end, let C be a cycle of N. If $t \notin C$, then C is a cycle of G, and so $|C \cap B| = |C \cap \delta_G(v)|$ even. Otherwise,

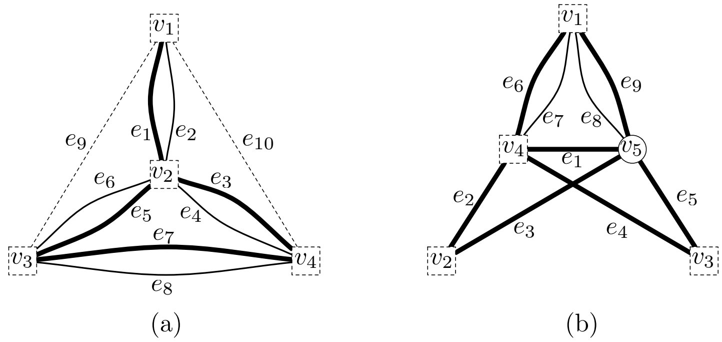


Figure 9.3: (a) Super quadrum, (b) super trifold. Square vertices are terminals. Bold edges are odd. Thin edges are even. Dashed edges can be either odd or even. In (a) and (b), t can be odd or even.

$t \in C$ and $C - t$ is a T-join of G. Since $v \in T$, $|(C - t) \cap \delta_G(v)|$ must be odd, implying in turn that $|C \cap B|$ is even. In both cases, for every cycle C of N, $|C \cap B|$ is even, which means that B is a cocycle of N. $\square$

Proposition 9.18. *Let (G, T, Γ) be a signed graft, and N the graft matroid of the graft. If (G, T, Γ) is a quadrum, a super quadrum, a trifold, or a super trifold, then (N, Γ) has $(F_7, E(F_7))$ as a minor.*

Proof. **Case 1:** (G, T, Γ) is a quadrum. By definition, (G, T) is a plain quadrum and $\Gamma = E(N)$. Remark 9.16 states that $N \cong F_7$. Hence, $(N, \Gamma) \cong (F_7, E(F_7))$. **Case 2:** (G, T, Γ) is a super quadrum. Label the vertices and edges of G as in Figure 9.3 (a). After possibly resigning on terminals v_2, v_3, v_4, we may assume by Remark 9.17 that $\Gamma = \{e_1, e_3, e_5, e_7, e_9, e_{10}, t\}$. Let $(N', \Gamma) := (N, \Gamma) \setminus \{e_2, e_4, e_6, e_8\}$ and $G' := G \setminus \{e_2, e_4, e_6, e_8\}$. Then N' is the graft matroid of (G', T), and (G', T, Γ) is a quadrum. It therefore follows from Case 1 that $(N, \Gamma) \setminus \{e_2, e_4, e_6, e_8\} = (N', \Gamma) \cong (F_7, E(F_7))$. **Case 3:** (G, T, Γ) is a trifold. By definition, (G, T) is a plain trifold and $\Gamma = E(N) - t$. Remark 9.16 states that $N/t \cong F_7$, implying in turn that $(N, \Gamma)/t \cong (F_7, E(F_7))$. **Case 4:** (G, T, Γ) is a super trifold. Label the vertices and edges of G as in Figure 9.3 (b). After possibly resigning on terminal v_1, we may assume by Remark 9.17 that $\Gamma = \{e_1, e_2, e_3, e_4, e_5, e_6, e_9\}$. Let $(N', \Gamma) := (N, \Gamma) \setminus \{e_7, e_8\}$ and $G' := G \setminus \{e_7, e_8\}$. Then N' is the graft matroid of (G', T), and (G', T, Γ) is a trifold. It therefore follows from Case 3 that $(N, \Gamma) \setminus \{e_7, e_8\}/t = (N', \Gamma)/t \cong (F_7, E(F_7))$. $\square$

9.4 Proposition 9.12

Suppose (h1)-(h2) hold, that is, M is an internally 4-connected binary matroid where every element is in exactly three triangles. Let $\Omega \in E(M)$. We would like to show that if $\Lambda(\Omega)$ is contained in a circuit of $\mathrm{si}(M/\Omega)$, then either M has an induced K_4 or $(M, E(M))$ has an $(F_7, E(F_7))$ minor. Let us make the following assumptions:

Further hypotheses

(h3) $\Omega \in E(M)$ is contained in the triangles $\{\Omega, f, f'\}$, $\{\Omega, g, g'\}$, $\{\Omega, h, h'\}$ where $\Lambda(\Omega) = \{f, g, h\}$,

(h4) $M_\Omega := M/\Omega \setminus \{f', g', h'\}$ and $(M_\Omega, \Sigma_\Omega) := (M, E(M))/\Omega \setminus \{f', g', h'\}$,

(h5) C is a circuit in M_Ω of minimum cardinality that contains $\{f, g, h\}$,

(h6) M does not have an induced K_4.

Note that $M_\Omega = \mathrm{si}(M/\Omega)$. We leave the following as an easy exercise for the reader:

Remark 9.19. *Take a binary matroid N, an element $e \in E(N)$, and a subset $D \subseteq E(N)$. Then the following statements hold:*

(a) if D is a circuit of N/e, then exactly one of $D, D \cup \{e\}$ is a circuit of N,

(b) if D is a cycle of N/e, then at least one of $D, D \cup \{e\}$ is a cycle of N,

(c) if D is a cycle of N and $e \in D$, then $D - e$ is a cycle of N/e, and

(d) if D is a cycle of N and $e \notin D$, then D is a cycle of N/e.

A key milestone in the proof is the proposition below:

Proposition 9.20. *Suppose (h1)-(h6) hold. Let D be a circuit of M_Ω containing $\{f, g, h\}$. Then,*

(a) the elements of $D - \{f, g, h\}$ are in series in $M|(D \cup \{f', g', h', \Omega\})$,

(b) $D - \{f, g, h\} \neq \emptyset$, and for each $t \in D - \{f, g, h\}$, $M|\big(D \cup \{f', g', h', \Omega\}\big)/(D - \{f, g, h, t\})$ is the graft matroid of a plain trifold,[2] and

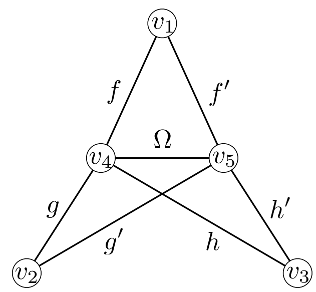
(c) if $|D|$ is odd, then $(M, E(M))$ has an $(F_7, E(F_7))$ minor.

Proof. Let $M' := M|D \cup \{f', g', h', \Omega\}$.

(a) Note that M'/Ω consists of the circuit D together with the elements f', g', h'. Since $\{f, f'\}, \{g, g'\}, \{h, h'\}$ are parallel classes of M'/Ω, it follows that the elements of $D - \{f, g, h\}$ are in series in M'/Ω, so they are also in series in M'. **(b)** If $\{f, g, h\}$ or $\{f, g, h, \Omega\}$ is a circuit of M, then $\{\Omega, f, f', g, g', h\}$ would be an induced K_4 of M, which cannot occur by (h6). Hence, neither $\{f, g, h\}$ nor $\{f, g, h, \Omega\}$ is a circuit of M – this has two consequences. (1) Since one of $D, D \cup \{\Omega\}$ is a circuit of M by Remark 9.19 (a), we get that $D \neq \{f, g, h\}$. (2) The set $\{f, g, h, \Omega\}$ is independent in the binary matroid M, and this is in turn implies that $M|\{f, f', g, g', h, h', \Omega\}$ is the cycle matroid of the graph G displayed below on vertices $\{v_1, v_2, v_3, v_4, v_5\}$ and edges

$$\Omega = \{v_4, v_5\}, \quad f = \{v_1, v_4\}, \quad f' = \{v_1, v_5\}, \quad g = \{v_2, v_4\}, \quad g' = \{v_2, v_5\}, \quad h = \{v_3, v_4\}, \quad h' = \{v_3, v_5\}.$$

[2] $M|I/J$ is short-hand notation for $(M|I)/J$.



Let $t \in D - \{f, g, h\}$ and

$$N := M'/(D - \{f, g, h, t\}).$$

By (a), the elements of $D - \{f, g, h\}$ are in series in M', so $M'/(D - \{f, g, h, t\}) \setminus t = M' \setminus (D - \{f, g, h\})$, implying in turn that $N \setminus t = M|\{f, f', g, g', h, h', \Omega\} = \text{cycle}(G)$. It therefore follows from Remark 9.10 that, for some $T \subseteq V(G)$ of even cardinality, N is the graft matroid of the graft (G, T). Since one of $D, D \cup \{\Omega\}$ is a circuit of M by Remark 9.19 (a), it follows that one of $\{f, g, h, t\}, \{f, g, h, t, \Omega\}$ is a circuit of N, implying in turn that one of $\{f, g, h\}, \{f, g, h, \Omega\}$ is a T-join of G. This means that $T = \{v_1, v_2, v_3, v_4\}$ or $T = \{v_1, v_2, v_3, v_5\}$. Either way, we see that (G, T) is a plain trifold. **(c)** Assume that $|D|$ is odd. Let $\Gamma := \{\Omega, f, f', g, g', h, h'\}$. Notice that (G, T, Γ) is a trifold. Thus, by Proposition 9.18, (N, Γ) has an $(F_7, E(F_7))$ minor. It therefore suffices to show that (N, Γ) is a minor of $(M', E(M'))$, which itself is a minor of $(M, E(M))$. Since $|D|$ is odd, $D - \{f, g, h\}$ has an even number of elements, all of which are in series in M' by (a), so $D - \{f, g, h\}$ is a cocycle of M'. As a result, $E(M') \triangle (D - \{f, g, h\}) = \Gamma$ is a signature of $(M', E(M'))$. However, $(M', \Gamma)/(D - \{f, g, h, t\}) = (N, \Gamma)$, so (N, Γ) is a minor of $(M', E(M'))$, as required. $\square$

We may therefore make the following assumption:

Further hypothesis

(h7) every circuit of M_Ω containing $\{f, g, h\}$ has even cardinality.

In particular, $|C|$ is even and so $C - \{f, g, h\} \neq \emptyset$. Let S be a triangle of M_Ω containing an element of $C - \{f, g, h\}$. We say that S is *f-splitting* if either $S \cap \{f, g, h\} = \{f\}$ or the following statements hold:

- $|S \cap C| = 1$, and
- $S \triangle C$ is the union of two disjoint circuits of M_Ω, one of which contains f and the other contains g, h.

Similarly, we can define *g-splitting* and *h-splitting* triangles.

Corollary 9.21. *Suppose (h1)-(h7) hold. Then every triangle of M_Ω containing an element of $C - \{f, g, h\}$ is a splitting triangle.*

Proof. Take an element $e \in C-\{f,g,h\}$ and a triangle S of M_Ω such that $e \in S$. Notice that $|S\cap\{f,g,h\}| \leq 1$. So if $S \cap \{f,g,h\} \neq \emptyset$, then S is a splitting triangle. We may therefore assume that $S \cap \{f,g,h\} = \emptyset$. Clearly, $1 \leq |S \cap C| \leq 2$. Note that $|S \cap C| = 1$; for if not, then $S \triangle C$ would be an odd-length circuit of M_Ω containing $\{f,g,h\}$, which cannot occur by (h7). Consider now the odd-length cycle $S\triangle C$, which is either a circuit or the disjoint union of two circuits. However, it follows from (h7) that $S\triangle C$ is the union of two disjoint circuits, both of which contain elements from $\{f,g,h\}$. This implies that R is a splitting triangle. □

The rest of this section is organized as follows: we will show that

- unless $(M, E(M))$ has an $(F_7, E(F_7))$ minor, every element of $C-\{f,g,h\}$ is in three otherwise disjoint triangles of M_Ω, one of which is f-splitting, the second one is g-splitting, and the third one is h-splitting (§9.4.2),
- the circuit C, together with its splitting triangles, gives rise to a so-called Type I or a Type II configuration in $(M_\Omega, \Sigma_\Omega)$ (§9.4.3),
- a Type I configuration gives a super trifold minor in $(M, E(M))$, and a Type II configuration gives a super quadrum minor in $(M, E(M))$ (§9.4.1),

and by Proposition 9.18, the last step leads to an $(F_7, E(F_7))$ minor, thereby finishing the proof of Proposition 9.12.

9.4.1 Type I and Type II configurations

In M_Ω, take an element $p \in E(M_\Omega) - C$ that is spanned by C. Then $C \cup \{p\}$ contains exactly three circuits, one of which is C, the other two contain p and their symmetric difference is C; notice that the other two circuits either separate f,g,h or not. We say that p is *f-splitting* if there is a circuit in $C\cup\{p\}$ that contains f and none of g,h. Observe that if S is an f-splitting triangle, then each element of $S-C$ is f-splitting. Similarly, we can define *g-splitting* and *h-splitting* elements. If p is e-splitting, for some $e \in \{f,g,h\}$, we denote by $\Theta(p)$ the circuit contained in $C \cup \{p\}$ such that $\Theta(p) \cap \{f,g,h\} = \{e\}$.

In this subsection, we identify two configurations of splitting elements and show that their presence implies the existence of a super trifold or super quadrum minor in $(M, E(M))$.

We say $\{p_1,p_2\} \subseteq E(M_\Omega)$ is a *Type I configuration* if the following statements hold:

- p_1 and p_2 are e-splitting, for some $e \in \{f,g,h\}$,
- $C - \big(\Theta(p_1) \cup \Theta(p_2) \cup \{f,g,h\}\big) \neq \emptyset$,
- $\Theta(p_1) \triangle \Theta(p_2)$ is an odd cycle of $(M_\Omega, \Sigma_\Omega)$,
- $|\Theta(p_1)|$ and $|\Theta(p_2)|$ are odd.

We will show that a Type I configuration leads to a super trifold in $(M, E(M))$. To prove this, however, we need an ingredient. Recall that by Remark 9.19 (a), if D is a circuit of M/Ω, then exactly one of $D, D \cup \{\Omega\}$ is a circuit of M; the following proposition characterizes when D is the circuit in M:

Remark 9.22. *Suppose (h1)-(h7) hold. Let D be a circuit of M/Ω. Then D is a circuit of M if, and only if, the parity of $|D|$ is equal to the parity of D in $(M, E(M))/\Omega$. In particular, if D is a circuit of M_Ω, then the following statements are equivalent:*

- *D is a circuit of M,*
- *$|D|$ and $|D \cap \Sigma_\Omega|$ have the same parity.*

Proof. Let D be a circuit of M/Ω. Assume that D is a circuit of M. Then the parity of $|D|$ is equal to the parity of D in $(M, E(M))$, which is equal to the parity of D in $(M, E(M))/\Omega$. Conversely, assume that the parity of $|D|$ is equal to the parity of D in $(M, E(M))/\Omega$. Suppose, for a contradiction, that D is not a circuit of M. By Remark 9.19 (a), $D \cup \{\Omega\}$ is a circuit of M, and moreover, the parity of $D \cup \{\Omega\}$ in $(M, E(M))$ is equal to the parity of D in $(M, E(M))/\Omega$, which by assumption is equal to the parity of $|D|$. However, the parity of $D \cup \{\Omega\}$ in $(M, E(M))$ is equal to the parity of $|D \cup \{\Omega\}| = |D| + 1$, a contradiction. $\square$

Proposition 9.23. *Suppose (h1)-(h7) hold. If there is a Type I configuration, then $(M, E(M))$ has $(F_7, E(F_7))$ as a minor.*

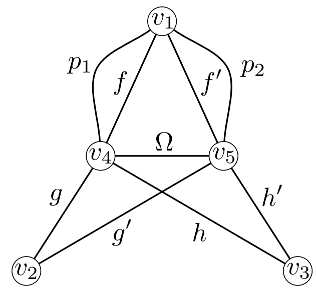
Proof. Assume that there is a Type I configuration $\{p_1, p_2\}$. After possibly interchanging the roles of f, g, h, we may assume that p_1, p_2 are f-splitting, and after possibly interchanging the roles of p_1, p_2, we may assume that $\Theta(p_1)$ is an odd circuit and $\Theta(p_2)$ an even circuit of $(M_\Omega, \Sigma_\Omega)$. Since $|\Theta(p_1)|$ and $|\Theta(p_2)|$ are odd, it follows from Remark 9.22 that $\Theta(p_1)$ is an odd circuit of $(M, E(M))$, and together with Remark 9.19 (a), that $\Theta(p_2) \cup \{\Omega\}$ is an even circuit of $(M, E(M))$. Now take an element $t \in C - \big(\Theta(p_1) \cup \Theta(p_2) \cup \{f, g, h\}\big)$. Consider the following minor of $(M, E(M))$:

$$(N, \Gamma) := (M, E(M))|(C \cup \{f', g', h', \Omega, p_1, p_2\})/(C - \{f, g, h, t\}).$$

We will show that (N, Γ) corresponds to a super trifold. By Proposition 9.20 (b), $N \setminus \{p_1, p_2\}$ is the graft matroid of a plain trifold (G', T), where $V(G') = \{v_1, v_2, v_3, v_4, v_5\}$, $E(G')$ consists of

$$\Omega = \{v_4, v_5\}, \quad f = \{v_1, v_4\}, \quad f' = \{v_1, v_5\}, \quad g = \{v_2, v_4\}, \quad g' = \{v_2, v_5\}, \quad h = \{v_3, v_4\}, \quad h' = \{v_3, v_5\},$$

and either $T = \{v_1, v_2, v_3, v_4\}$ or $T = \{v_1, v_2, v_3, v_5\}$. (See below for an illustration.) Since the triangles $\{\Omega, f, f'\}, \{\Omega, g, g'\}, \{\Omega, h, h'\}$ are odd in the signed binary matroid $(M, E(M))$, they are odd also in the minor $(N, \Gamma) \setminus \{p_1, p_2\} = \big(N \setminus \{p_1, p_2\}, \Gamma - \{p_1, p_2\}\big)$. We may therefore assume that $\{\Omega, f, f', g, g', h, h'\} \subseteq \Gamma - \{p_1, p_2\}$. Notice that we do not know whether or not t belongs to $\Gamma - \{p_1, p_2\}$. Since $\Theta(p_1)$ is a circuit of M containing $\{f, p_1\}$ and all of its other edges belong to $C - \{f, g, h, t\}$, it follows that $\{f, p_1\}$ is a circuit of N. Similarly, since $\Theta(p_2) \cup \{\Omega\}$ is a circuit of M containing $\{f, p_2, \Omega\}$ and all of its other edges belong to $C - \{f, g, h, t\}$, we get that $\{f, p_2, \Omega\}$ is a triangle of N, which in turn implies that $\{f', p_2\}$ is a circuit of N. As a consequence, N is the graft matroid of the graft (G, T) obtained from (G', T) after adding edge p_1 parallel to f, and edge p_2 parallel to f'. Since $\Theta(p_1)$ is an odd circuit of $(M, E(M))$, we get that $\{f, p_1\}$ is an odd circuit of (N, Γ), so $p_1 \notin \Gamma$. Similarly, as $\Theta(p_2) \cup \{\Omega\}$ is an even circuit of $(M, E(M))$, we get that $\{f, p_2, \Omega\}$ is an even triangle of (N, Γ), and as $\{f, f', \Omega\}$ is an odd triangle, we have that $\{f', p_2\}$ is also an odd circuit of (N, Γ). Hence, $p_2 \notin \Gamma$. Therefore, the signed graft (G, T, Γ) is a super trifold.



It follows from Proposition 9.18 that (N, Γ), and therefore $(M, E(M))$, has an $(F_7, E(F_7))$ minor, as desired. $\square$

We say $\{p_1, p_1', p_2, p_3\} \subseteq E(M_\Omega)$ is a *Type II configuration* if the following statements hold:

- p_1 and p_1' are e_1-splitting, p_2 is e_2-splitting, and p_3 is e_3-splitting, for a permutation e_1, e_2, e_3 of f, g, h,
- $\Theta(p_1) \cap \Theta(p_1') \cap \Theta(p_2) \cap \Theta(p_3) \neq \emptyset$,
- $\Theta(p_1) \triangle \Theta(p_1')$ is an odd cycle of $(M_\Omega, \Sigma_\Omega)$.

We will show that a Type II configuration leads to a super quadrum in $(M, E(M))/\Omega$, and therefore, in $(M, E(M))$:

Proposition 9.24. *Suppose (h1)-(h7) hold. If there is a Type II configuration, then* $(M, E(M))$ *has* $(F_7, E(F_7))$ *as a minor.*

Proof. Assume that there is a Type II configuration $\{p_1, p_1', p_2, p_3\}$. By symmetry, we may assume that p_1, p_1' are f-splitting, p_2 is g-splitting, and p_3 is h-splitting. Take an element $t \in \Theta(p_1) \cap \Theta(p_1') \cap \Theta(p_2) \cap \Theta(p_3)$. Observe that $(M', \Sigma') := (M, E(M))/\Omega$ is obtained from $(M_\Omega, \Sigma_\Omega)$ after adding elements f', g', h' parallel of different parity to f, g, h, respectively. Consider now the minor

$$(N, \Gamma) := (M', \Sigma')|\big(C \cup \{p_1, p_1', p_2, p_3\}\big)/\big(C - \{f, g, h, t\}\big).$$

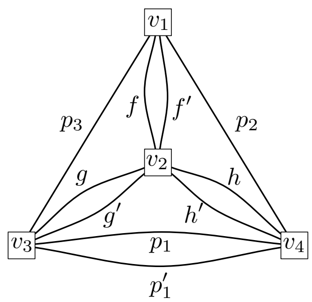
We will show that (N, Γ) corresponds to a super quadrum.

Since C is a circuit of M', it follows that $\{f, g, h, t\}$ is a circuit of N. Start with the graft (G'', T) on vertices $\{v_1, v_2, v_3, v_4\}$ and edges $f = \{v_2, v_1\}, g = \{v_2, v_3\}, h = \{v_2, v_4\}$, where $T = \{v_1, v_2, v_3, v_4\}$. Note that $N|\{f, g, h, t\}$ is the graft matroid of (G'', T). Since $\Theta(p_1)$ is a circuit of M_Ω, it is also a circuit of M', and as it contains $\{f, p_1, t\}$ and all of its other edges belong to $C - \{f, g, h, t\}$, it follows that $\{f, p_1, t\}$ is a cycle of N. Similarly, $\{g, p_2, t\}$ and $\{h, p_3, t\}$ are also cycles of N. As a consequence, $N|\{f, g, h, t, p_1, p_2, p_3\}$ is the graft matroid of the plain quadrum (G', T) obtained from (G'', T) after adding $p_1 = \{v_3, v_4\}$, $p_2 = \{v_4, v_1\}$ and $p_3 = \{v_1, v_3\}$.

Notice that N has no loop, because $\big(C - \{f, g, h, t\}\big) \cup \{e\}$ contains no circuit of M', for each $e \in E(N)$. Therefore, since $\{f, f'\}, \{g, g'\}, \{h, h'\}$ are odd circuits in (M', Σ'), they are also odd circuits in (N, Γ). Moreover, as $\Theta(p_1) \triangle \Theta(p_1')$ is an odd cycle of $(M_\Omega, \Sigma_\Omega)$, it is also an odd cycle of (M', Σ'), and because it contains $\{p_1, p_1'\}$ and all of its other edges belong to $C - \{f, g, h, t\}$, it follows that $\{p_1, p_1'\}$ is an odd circuit of (N, Γ). Thus, N is the graft matroid of the graft (G, T) obtained from (G', T) after adding edges f', g', h', p_1' parallel to f, g, h, p_1, respectively. (See below for an illustration.) Moreover,

$$|\Gamma \cap \{f, f'\}| = |\Gamma \cap \{g, g'\}| = |\Gamma \cap \{h, h'\}| = |\Gamma \cap \{p_1, p_1'\}| = 1,$$

and we do not know whether or not p_2, p_3, t belong to Γ. This means that (G, T, Γ) is a super quadrum.



From Proposition 9.18 we get that (N, Γ), and therefore $(M, E(M))$, has an $(F_7, E(F_7))$ minor. ☐

9.4.2 Splitting triangles

Take a signed binary matroid (N, Γ). For $R \subseteq E(N)$, we write $(N, \Gamma)|R := (N, \Gamma) \setminus (E(N) - R)$. We say that $\{e_1, \ldots, e_6\}$ is in *induced odd* K_4 of (N, Γ) if $N|\{e_1, \ldots, e_6\}$ is an induced K_4 in which every triangle is odd in $(N, \Gamma)|\{e_1, \ldots, e_6\}$. A consequence of Remark 9.22 is the following:

Corollary 9.25. *Suppose (h1)-(h7) hold. Then $(M, E(M))/\Omega$ does not have an induced odd K_4.*

Proof. Suppose, for a contradiction, that $\{e_1, \ldots, e_6\}$ is an induced odd K_4 of $(M, E(M))/\Omega$, whose odd triangles are $\{e_1, e_2, e_3\}, \{e_1, e_4, e_5\}, \{e_2, e_4, e_6\}, \{e_3, e_5, e_6\}$. It then follows from Remark 9.22 that these are also triangles of M, implying in turn that $\{e_1, \ldots, e_6\}$ is an induced K_4 of M, thereby contradicting (h6). ☐

Remark 9.26. *Suppose (h1)-(h7) hold. If S is a triangle of M_Ω, then $|S \cap \{f, g, h\}| \leq 1$.*

Proof. Let S be a triangle of M_Ω. It follows from Proposition 9.20 (b) that $|S \cap \{f, g, h\}| \leq 2$. Suppose, for a contradiction, that $|S \cap \{f, g, h\}| = 2$. We may assume that $S \cap \{f, g, h\} = \{f, g\}$. By Remark 9.19 (a), one of $S, S \cup \{\Omega\}$ is a circuit of M, so one of $S, S \triangle \{f, f'\}$ is a triangle of M. But then $S \cup \{\Omega, f, f', g, g'\}$ is an induced K_4 of M, a contradiction to (h6). ☐

We are now ready to prove the main result of this section:

Proposition 9.27. *Suppose (h1)-(h7) hold, and $(M, E(M))$ does not have $(F_7, E(F_7))$ as a minor. Then for every element $e \in C - \{f, g, h\}$, there exist triangles S_f, S_g, S_h of M_Ω such that $S_f \cap S_g = S_g \cap S_h = S_h \cap S_f = \{e\}$, and for each $z \in \{f, g, h\}$,*

- *S_z is z-splitting, and*
- *if $S_z \cap \{f, g, h\} = \emptyset$, then S_z is odd in $(M_\Omega, \Sigma_\Omega)$.*

Proof. Take an element $e \in C - \{f, g, h\}$. Denote by T_1, T_2, T_3 the three triangles of M containing e, whose existence is guaranteed by (h2). Recall that $T_1 \cap T_2 = T_2 \cap T_3 = T_3 \cap T_1 = \{e\}$. Since $e \notin \{\Omega, f, f', g, g', h, h'\}$, $\Omega \notin T_1 \cup T_2 \cup T_3$. Therefore, since M is a simple binary matroid and M/Ω is a loopless binary matroid whose non-trivial parallel classes are precisely $\{f, f'\}, \{g, g'\}, \{h, h'\}$, it follows that T_1, T_2, T_3 are also triangles of M/Ω; note that they are odd triangles of $(M, E(M))/\Omega$. For each $i \in [3]$, let S_i be the triangle corresponding to T_i in the simplification M_Ω. We will show that, after a relabeling, S_1, S_2, S_3 are the desired three triangles.

Claim 1. *$S_1 \cap S_2 = S_2 \cap S_3 = S_3 \cap S_1 = \{e\}$. Moreover, for each $i \in [3]$, S_i is a splitting triangle, and if $S_i \cap \{f, g, h\} = \emptyset$, then S_i is odd in $(M_\Omega, \Sigma_\Omega)$.*

Proof of Claim. Suppose, for a contradiction, that $\{e\} \subsetneq S_1 \cap S_2$. Since $\{e\} = T_1 \cap T_2$, we may assume that $f \in S_1 \cap S_2$, $f \in T_1$ and $f' \in T_2$. However, since $\{f, f', \Omega\}$ is a triangle of M, it follows that $T_1 \cup T_2 \cup \{\Omega\}$ is an induced K_4 of M, a contradiction to (h6). Thus, $S_1 \cap S_2 = \{e\}$ and similarly, $S_2 \cap S_3 = S_3 \cap S_1 = \{e\}$. Take an index $i \in [3]$. Clearly, if $S_i \cap \{f, g, h\} = \emptyset$, then $S_i = T_i$ and therefore S_i is an odd triangle of $(M_\Omega, \Sigma_\Omega)$. Moreover, since $e \in S_i$, we get from Corollary 9.21 that S_i is a splitting triangle. $\diamond$

It therefore suffices to show that no two of S_1, S_2, S_3 split the same element of $\{f, g, h\}$. Suppose, for a contradiction, that S_1, S_2 are f-splitting. Since these triangles are f-splitting, it follows that $S_1 \cap \{g, h\} = S_2 \cap \{g, h\} = \emptyset$, and by Claim 1, $S_1 \cap S_2 = \{e\}$.

Fix an index $i \in [2]$. Let us carefully label the elements of $S_i - e$. If $f \in S_i$, then let $p_i := f$ and q_i the element in $S_i - \{e, f\}$. Otherwise, $f \notin S_i$. Because S_i is f-splitting, $S_i \cap C = \{e\}$ and $S_i \triangle C$ is the union of two disjoint circuits of M_Ω. That is, the elements of $S_i - \{e\}$ are f-splitting, and for a labeling p_i, q_i of these elements, $S_i \triangle C$ is the disjoint union of $\Theta(p_i)$ and $C \triangle \Theta(q_i)$.

Since M_Ω is a simple binary matroid, when $f \notin \{p_1, p_2\}$, we get that $\Theta(p_1) - p_1 \neq \Theta(p_2) - p_2$. We may therefore assume that $f \neq p_2$ and, if $f \neq p_1$, $(\Theta(p_2) - p_2) - (\Theta(p_1) - p_1) \neq \emptyset$.

Claim 2. *$M_\Omega|\big(C \cup \{p_1, q_1, p_2, q_2\}\big)$ is the cycle matroid of a simple graph G described as follows: for some integers n, k such that $n - 2 \geq k \geq 3$,*

- *$V(G) = \{v_1, \ldots, v_n\}$,*
- *$C = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\big\}$,*
- *$e = \{v_1, v_2\}$, $p_1 = \{v_2, v_k\}$, $q_1 = \{v_1, v_k\}$, $p_2 = \{v_2, v_{k+1}\}$, $q_2 = \{v_1, v_{k+1}\}$, and*
- *$f \in \big\{\{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\}\big\}$ and $g, h \in \big\{\{v_{k+1}, v_{k+2}\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\big\}$.*

Proof of Claim. Let $n := |C|$. Clearly, $M_\Omega|C$ is the cycle matroid of the simple graph G_1 on vertices $\{v_1, \ldots, v_n\}$ whose edges are $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$. We assume that $e = \{v_1, v_2\}$ and the edges of $\Theta(q_1) - q_1$ appear consecutively on the graph circuit. Then there is an integer $k \geq 3$ such that $\Theta(q_1) - q_1 = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\}\big\}$. Note that $f \in \big\{\{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\}\big\}$, and $f = p_1$ if and only if $k = 3$.

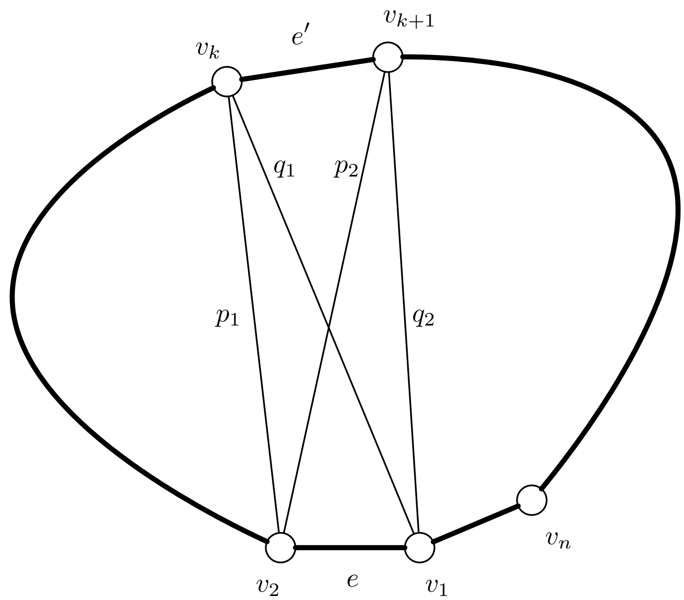


Figure 9.4: An illustration of graph G, where the edges in C are bold.

Let G_2 be the graph obtained from G_1 after adding the edge $q_1 = \{v_1, v_k\}$, and if $k > 3$, the edge $p_1 = \{v_2, v_k\}$. Note that $M_\Omega|\big(C \cup \{p_1, q_1\}\big)$ is the cycle matroid of the simple graph G_2. Consider the set $\Theta(p_2) - p_2$. As $f \in \Theta(p_2)$, we have $(\Theta(p_2)-p_2) \cap \big\{\{v_2, v_3\}, \dots, \{v_{k-1}, v_k\}\big\} \neq \emptyset$, and as $(\Theta(p_2)-p_2)-(\Theta(p_1)-p_1) \neq \emptyset$ when $k > 3$, we have $(\Theta(p_2)-p_2) \cap \big\{\{v_k, v_{k+1}\}, \dots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\big\} \neq \emptyset$. After possibly rearranging the edges of G_2 within series classes $\big\{\{v_2, v_3\}, \dots, \{v_{k-1}, v_k\}\big\}$ and $\big\{\{v_k, v_{k+1}\}, \dots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\big\}$, we may assume that the edges of $\Theta(p_2) - p_2$ appear consecutively on the circuit C. So there are indices $i, j \in [n]$ such that

$$\Theta(p_2) - p_2 = \big\{\{v_i, v_{i+1}\}, \dots, \{v_{j-1}, v_j\}\big\}$$

where $k - 1 \geq i \geq 2$ and $n - 1 \geq j \geq k + 1$.

Let G_3 be the graph obtained from G_2 after adding the edge $p_2 = \{v_i, v_j\}$. Note that $M_\Omega|\big(C \cup \{p_1, q_1, p_2\}\big)$ is the cycle matroid of the simple graph G_3. We will show that $i = 2$ and $j = k + 1$. Consider the following circuit of G_3:

$$\{p_2\} \cup \big\{\{v_i, v_{i+1}\}, \dots, \{v_{k-1}, v_k\}\big\} \cup \{q_1\} \cup \big\{\{v_1, v_n\}, \{v_n, v_{n-1}\}, \dots, \{v_{j+1}, v_j\}\big\}.$$

This circuit contains edges f, g, h and has $n - (i-1) - (j-k) + 2$ many edges. It therefore follows from the minimality of C in (h5) that $n - (i-1) - (j-k) + 2 \geq n$, implying in turn that $i = 2$ and $j = k + 1$. Now let G be the graph obtained from G_3 after adding the edge $q_2 = \{v_1, v_{k+1}\}$. It is clear that $M_\Omega|\big(C \cup \{p_1, q_1, p_2, q_2\}\big)$ is the cycle matroid of the simple graph G, which is the desired graph. $\diamond$

By Remark 9.26, edges g, h do not lie in a triangle of G, so in fact $n - 3 \geq k$. Let $e' := \{v_k, v_{k+1}\} \in$

$E(G) = E(M_\Omega)$ and note that $\{e, p_1, q_1, p_2, q_2, e'\}$ is an induced K_4 of M_Ω. Let

$$(N, \Gamma) := (M_\Omega, \Sigma_\Omega)|\{e, p_1, q_1, p_2, q_2, e'\}.$$

The triangle $S_2 = \{e, p_2, q_2\}$, being disjoint from $\{f, g, h\}$, is odd in (N, Γ), and if $f \neq p_1$, then the triangle $S_1 = \{e, p_1, q_1\}$ would also be odd in (N, Γ). Since M has no induced K_4 by (h6), it follows from Corollary 9.25 that exactly two of $\{e, p_1, q_1\}, \{e', p_1, p_2\}, \{e', q_1, q_2\}$ are even in (N, Γ). Thus, if $f \neq p_1$, then $\{e\}$ is a signature for (N, Γ), and if $f = p_1$, then one of $\{e\}, \{e, f\}, \{e, q_1\}$ is a signature for (N, Γ).

Claim 3. $f \neq p_1$.

Proof of Claim. Suppose, for a contradiction, that $f = p_1$. We will show that $\{p_2, q_1\}$ is a Type I configuration. Recall that p_2, q_1 are f-splitting, and since $n-3 \geq k$, it follows that $C - \big(\Theta(p_2) \cup \Theta(q_1) \cup \{f, g, h\}\big) \neq \emptyset$. Moreover, $|\Theta(p_2)| = |\Theta(q_1)| = 3$. If $\{e, q_1\}$ is a signature for (N, Γ), then $\{e, f, q_1, p_2, q_2, e'\} \triangle \{f, f'\}$ is an induced odd K_4 of $(M, E(M))/\Omega$, which cannot occur by Corollary 9.25. Thus, one of $\{e\}, \{e, f\}$ is a signature for (N, Γ). Either way, we see that $\Theta(p_2) \triangle \Theta(q_1)$ is odd cycle of $(M_\Omega, \Sigma_\Omega)$. Thus, $\{p_2, q_1\}$ is a Type I configuration. But then Proposition 9.23 implies that $(M, E(M))$ has an $(F_7, E(F_7))$ minor, a contradiction to our hypothesis. ◇

Recall that p_1, q_1, p_2, q_2 are f-splitting, and since $n - 3 \geq k$,

$$C - \big(\Theta(p_2) \cup \Theta(q_1) \cup \{f, g, h\}\big) = C - \big(\Theta(p_1) \cup \Theta(q_2) \cup \{f, g, h\}\big) \neq \emptyset.$$

We also know that $\{e\}$ is a signature for (N, Γ). It can be readily seen that if $|\Theta(p_1) - p_1|$ is odd, then $\{p_2, q_1\}$ is a Type I configuration, and otherwise, $\{p_1, q_2\}$ is a Type I configuration. Either way, we get from Proposition 9.23 that $(M, E(M))$ has an $(F_7, E(F_7))$ minor, thereby contradicting our hypothesis. Therefore, S_1 and S_2 cannot both be f-splitting. Similarly, no two of S_1, S_2, S_3 split the same element. Among these triangles, let S_f be the f-splitting one, S_g the g-splitting one, and S_h the h-splitting one. These are the desired triangles, and the proof of Proposition 9.27 is finished. □

9.4.3 Proof of Proposition 9.12

We may assume that (h1)-(h7) hold. To remind the reader why, assume that (h1)-(h2), as well as the setup conditions (h3)-(h5), hold. If M has an induced K_4, then we are done. Otherwise, (h6) holds. We will prove that $(M, E(M))$ has $(F_7, E(F_7))$ as a minor, thereby finishing the proof of Proposition 9.12. Suppose otherwise. It then follows from Proposition 9.20 (c) that (h7) holds.

Claim 1. $|C| \geq 6$.

Proof of Claim. Suppose otherwise. By (h7), $|C|$ is even. Thus, for some $t \in E(M_\Omega)$, $C = \{f, g, h, t\}$. By Proposition 9.27, there is an h-splitting triangle $\{t, h, p\}$, where p is h-splitting. But then $\{p, f, g\}$ is a triangle of M_Ω, thereby contradicting Remark 9.26. ◇

Claim 2. *There is an f-splitting triangle S where $|S \cap C| = 1$ and S is odd in $(M_\Omega, \Sigma_\Omega)$.*

Proof of Claim. Suppose otherwise. By Claim 1, there are distinct elements $e_1, e_2, e_3 \in C - \{f, g, h\}$. Fix an index $i \in [3]$. Then by Proposition 9.27 and our contrary assumption, e_i is contained in an f-splitting triangle S_i such that $S_i \cap C = \{e_i, f\}$. By Remark 9.19 (a), one of $S_i, S_i \cup \{\Omega\}$ is a circuit of M, implying in turn that one of $S_i, S_i \triangle \{f, f'\}$ is a triangle of M. By (h2), f and f' are each in exactly 3 triangles of M, a common one being $\{\Omega, f, f'\}$. Hence, it cannot be that each one of S_1, S_2, S_3 is a triangle of M or that each one of $S_1 \triangle \{f, f'\}, S_2 \triangle \{f, f'\}, S_3 \triangle \{f, f'\}$ is a triangle of M. We may therefore assume that $S_1, S_2 \triangle \{f, f'\}$ are triangles of M. In other words, $S_1, S_2 \cup \{\Omega\}$ are circuits of M, so by Remark 9.22, S_1 is an odd triangle and S_2 is an even triangle of $(M_\Omega, \Sigma_\Omega)$. Let p_1 be the element in $S_1 - \{e_1, f\}$ and p_2 the element in $S_2 - \{e_2, f\}$. Then p_1, p_2 are f-splitting elements for which $\Theta(p_1) = S_1$ and $\Theta(p_2) = S_2$. Since $e_3 \in C - (S_1 \cup S_2 \cup \{f, g, h\})$, it follows that $\{p_1, p_2\}$ is a Type I configuration. By Proposition 9.23, $(M, E(M))$ has an $(F_7, E(F_7))$ minor, a contradiction. ◇

Write $S = \{e, p_1, p'_1\}$ where $C \cap S = \{e\}$ and $e \in \Theta(p'_1)$. Note that $\Theta(p'_1) \triangle \Theta(p_1) = S$, so it is an odd cycle of $(M_\Omega, \Sigma_\Omega)$. As M_Ω is simple, there is an element $t \in \Theta(p_1) - \{f, p_1\}$. Note that $t \in \Theta(p'_1)$. By Proposition 9.27, t is contained in a g-splitting triangle S_2 and an h-splitting triangle S_3. Pick the g-splitting element $p_2 \in S_2$ for which $t \in \Theta(p_2)$ and the h-splitting element $p_3 \in S_3$ for which $t \in \Theta(p_3)$. Then $\{p_1, p'_1, p_2, p_3\}$ is a Type II configuration. By Proposition 9.24, $(M, E(M))$ has an $(F_7, E(F_7))$ minor, which is a contradiction. This finishes the proof of Proposition 9.12. □

9.5 Even cycle matroids

Let G be a graph and $\Gamma \subseteq E(G)$. The signed binary matroid $(\mathrm{cycle}(G), \Gamma)$ is identified as (G, Γ) and is simply referred to as a *signed graph*. Zaslavsky [75] proved that the even cycles of (G, Γ) are the cycles of a binary matroid that we call the *even cycle matroid* of (G, Γ) and denote by $\mathrm{ecycle}(G, \Gamma)$. Notice that every signature of (G, Γ) is a cocycle of $\mathrm{ecycle}(G, \Gamma)$.

Given a graph H and a new edge label e, denote by $H + e$ any graph obtained from H after adding e as a loop. The following folklore result states that binary matroids one contraction away from being graphic are even cycle matroids:

Remark 9.28. *Take a binary matroid M and an element $e \in E(M)$ such that $M/e = \mathrm{cycle}(H)$, for some graph H. If Γ is a cocycle of M containing e, then $M = \mathrm{ecycle}(H + e, \Gamma)$.*

Proof. Let Γ be a cocycle of M containing e. Let $C \subseteq E(M)$. We need to show that C is a cycle of M if, and only if, C is an even cycle of $(H + e, \Gamma)$. ($\Rightarrow$) Suppose first that C is a cycle of M. Since $|C \cap \Gamma|$ is even, it suffices to show that C is a cycle of $H + e$. If $e \in C$, then $C - e$ is a cycle of M/e by Remark 9.19 (c), so it is also a cycle of H, implying in turn that C is a cycle of $H + e$. Otherwise, $e \notin C$, so C is a cycle of M/e by Remark 9.19 (d), implying in turn that C is a cycle of H, and therefore, of $H + e$. ($\Leftarrow$) Suppose conversely that C is an even cycle of $(H + e, \Gamma)$. Assume first that $e \notin C$. Then C is a cycle of H, so it is also a cycle of M/e, implying by Remark 9.19 (b) that either C or $C \cup \{e\}$ is a cycle of M. Since $|C \cap \Gamma|$ is even, $e \in \Gamma$ and $e \notin C$, it follows that $|(C \cup \{e\}) \cap \Gamma|$ is odd, and because Γ is a cocycle of M, $C \cup \{e\}$ cannot be a cycle of M. Thus, C is a cycle of M. Assume in the remaining case that $e \in C$. Then $C - e$ is a cycle of H, so it is also a cycle of M/e, and thus by Remark 9.19 (b), one of $C - e, C$ is a cycle of M. Since $e \in C \cap \Gamma$ and $|C \cap \Gamma|$ is even, it follows that $|(C - e) \cap \Gamma|$ is odd. Therefore, because Γ is a cocycle of M, $C - e$ cannot be a cycle of M, and as a result, C is a cycle of M. □

Even cycle matroids and connectivity

Let G be a graph. For a subset $X \subseteq E(G)$, we denote by $V_G(X)$ the ends of the edges in X, and by $G[X]$ the subgraph on vertices $V_G(X)$ and edges X.

Let (G, Γ) be a signed graph, where G is connected. If (G, Γ) has no odd circuit, then $\mathrm{ecycle}(G, \Gamma) = \mathrm{cycle}(G)$ and therefore, any spanning tree of G is a basis for $\mathrm{ecycle}(G, \Gamma)$. Otherwise, when (G, Γ) has an odd circuit, $T \cup \{e\}$ is a basis for $\mathrm{ecycle}(G, \Gamma)$, where T is a spanning tree of G, and $e \in E(G) - T$ is chosen so that $T \cup \{e\}$ contains an odd circuit of (G, Γ).

The next remark describes the connectivity function for even cycle matroids.

Remark 9.29 ([43])**.** *Let (G, Γ) be a signed graph, where G is connected. Take a nonempty and proper subset $X \subseteq E(G)$ where both $G[X]$ and $G[\overline{X}]$ are connected. Then*

$$\lambda_{ecycle(G,\Gamma)}(X) \leq \lambda_{cycle(G)}(X) + 1 = \left|V_G(X) \cap V_G(\overline{X})\right|.$$

Proof. The equality is a routine exercise (see [60] Lemma 8.1.7 for details). To prove the inequality, let $E := E(G)$, $M := \mathrm{cycle}(G)$ and $M' := \mathrm{ecycle}(G, \Gamma)$. Denote by r, r' the rank functions of M, M', respectively. If (G, Γ) has no odd circuit, then $M = M'$, so $r = r'$, implying in turn that $\lambda_M = \lambda_{M'}$. We may therefore assume that (G, Γ) has an odd circuit. What we argued above implies that $r'(E) = r(E)+1$, $r'(X) \in \{r(X), r(X)+1\}$ and $r'(\overline{X}) \in \{r(\overline{X}), r(\overline{X})+1\}$, so

$$\lambda_{M'}(X) = r'(X) + r'(\overline{X}) - r'(E) \leq r(X) + 1 + r(\overline{X}) + 1 - r(E) - 1 = \lambda_M(X) + 1,$$

as required. □

A connected graph on at least 3 vertices is 2-*connected* if it remains connected after deleting any vertex. (This definition applies to this chapter only.) A 2-connected graph on at least 4 vertices is 3-*connected* if it remains connected after deleting any pair of vertices. For a graph G, denote the set of all loops by $\mathrm{loops}(G)$. Given a signed graph (G, Γ), denote by $\mathrm{si}(G, \Gamma)$ the signed graph obtained after deleting all even loops, deleting all odd loops except for one, and deleting all but one edge from each class of parallel edges in G of the same parity in (G, Γ).

Proposition 9.30. *Let (G, Γ) be a signed graph that has an odd loop e, and let $N := ecycle(G, \Gamma)$. Then,*

(a) if N is simple and cosimple, e is the unique loop of G, parallel edges of G have distinct parities in (G, Γ), and G does not have edges in series,

(b) if N is internally 4-connected and $|E(N)| \geq 8$, then $G \setminus e$ is 3-connected,

(c) $si(N) = ecycle(si(G, \Gamma))$, and

(d) if $si(N)$ is internally 4-connected and $|E(si(N))| \geq 8$, then $G \setminus loops(G)$ is 3-connected.

Proof. **(a)** Suppose N is simple and cosimple. Then N has no cycle of size at most 2 and no two elements in series. In particular, (G, Γ) does not have an even loop or an even cycle of size two, and G does not have two edges in series. Since e is an odd loop, there cannot be another odd loop. As a result, e is the unique loop of G and parallel edges of G have distinct parities in (G,Γ). **(b)** Suppose N is internally 4-connected. In particular, N is simple and cosimple. Thus, (a) implies that in G, no two edges are in series, e is the unique loop, and every parallel class has size at most two. Since $|E(G\setminus e)|\geq 7$, we get that $G\setminus e$ has at least 4 vertices. It suffices to show that when $G\setminus e$ is connected, then it is 3-connected. We first show that $G\setminus e$ is 2-connected. Suppose, for a contradiction, that there is a non-trivial partition X,Y of $E(G\setminus e)$ such that $|V_{G\setminus e}(X)\cap V_{G\setminus e}(Y)|=1$. Since $|X|+|Y|\geq 7$, after possibly interchanging the roles of X and Y, we may assume that $|X|\geq 2$. Let $\bar{X}:=Y\cup\{e\}$. Then $|\bar{X}|\geq 2$. Assuming the end of e belongs to $V_{G\setminus e}(Y)$, we see that $G,G[X],G[\bar{X}]$ are connected. Hence, Remark 9.29 implies that $\lambda_N(X)\leq|V_G(X)\cap V_G(\bar{X})|=1$, so $(X,\bar{X})$ is a 2-separation of N, a contradiction as N is 3-connected. It remains to show that $G\setminus e$ is 3-connected. Suppose, for a contradiction, that there is a non-trivial partition X,Y of $E(G\setminus e)$ such that $|V_{G\setminus e}(X)\cap V_{G\setminus e}(Y)|=2$, $|V_{G\setminus e}(X)|\geq 3$ and $|V_{G\setminus e}(Y)|\geq 3$. Since $G\setminus e$ is 2-connected, it follows that $(G\setminus e)[X],(G\setminus e)[Y]$ are connected, implying in turn that $|X|\geq 2$ and $|Y|\geq 2$. In fact, since $G\setminus e$ does not have two edges in series, we have $|X|\geq 3$ and $|Y|\geq 3$. Because $|X|+|Y|\geq 7$, we may assume that $|X|\geq 4$. Let $\bar{X}:=Y\cup\{e\}$. Then $|\bar{X}|\geq 4$. Assuming the end of e belongs to $V_{G\setminus e}(Y)$, we see that $G,G[X],G[\bar{X}]$ are connected. Thus, by Remark 9.29, we get that $\lambda_N(X)\leq|V_G(X)\cap V_G(\bar{X})|=2$, so $(X,\bar{X})$ is a 3-separation of N, a contradiction as N is internally 4-connected. **(c)** is immediate. **(d)** Let $(G',\Gamma'):=\mathrm{si}(G,\Gamma)$. We may assume that e is also an odd loop of (G',Γ'). By (c), $\mathrm{si}(N)=\mathrm{ecycle}(G',\Gamma')$, so from (b) we get that $G'\setminus e$ is 3-connected. As G is obtained from G' by adding loops and edges parallel to existing ones, we get that $G\setminus\mathrm{loops}(G)$ is also 3-connected. □

Even cycle matroids that are graphic

Here we characterize, under relevant conditions, when an even cycle matroid is graphic. A complete and technical answer to this problem was obtained by Shih in her PhD thesis [72] but was never published in a refereed journal – our arguments will not rely on this characterization. We will need the following seminal result of Whitney [74]:

Theorem 9.31. *Let G,G' be graphs over the same edge set such that $\mathrm{cycle}(G)=\mathrm{cycle}(G')$. If $G\setminus\mathrm{loops}(G)$ is 3-connected, then $G\setminus\mathrm{loops}(G)=G'\setminus\mathrm{loops}(G')$ and $\mathrm{loops}(G)=\mathrm{loops}(G')$.*

Let (G,Γ) be a signed graph, and take a vertex $v\in V(G)$. We say v is a *blocking vertex* if every non-loop odd circuit of (G,Γ) uses v. It follows from Proposition 9.1 that v is a blocking vertex if, and only if, there is a signature contained in $\delta_G(v)\cup\mathrm{loops}(G)$.

Remark 9.32. *Let (G,Γ) be a signed graph. If (G,Γ) has a blocking vertex, then $\mathrm{ecycle}(G,\Gamma)$ is graphic.*

Proof. Let v be a blocking vertex. After possibly resigning, we may assume that $\Gamma\subseteq\delta_v(G)\cup\mathrm{loops}(G)$. We may also assume that every odd loop is incident to v and every even loop is incident to another vertex. Let H be the graph obtained from G after splitting v into vertices v_1,v_2 such that every edge in $\delta_G(v)\cap\Gamma$ is incident with v_1, every edge in $\delta_G(v)-\Gamma$ is incident with v_2, and every odd loop has ends v_1,v_2. It can be readily checked that $\mathrm{ecycle}(G,\Gamma)=\mathrm{cycle}(H)$. □

Provided an odd loop and 3-connectedness, we can guarantee the converse also holds:

Proposition 9.33. *Let (G,Γ) be a signed graph that has an odd loop and $G\setminus\mathrm{loops}(G)$ is 3-connected. If $\mathrm{ecycle}(G,\Gamma)$ is graphic, then (G,Γ) has a blocking vertex.*

Proof. Set $E := E(G)$ and let $e \in E$ be an odd loop of (G, Γ). Let H be a graph with edge set E such that $\text{ecycle}(G, \Gamma) = \text{cycle}(H)$. As e is not an even loop of (G, Γ), e is not a loop of H; let v_1, v_2 be the ends of e in H. Since the even circuits of (G, Γ) are precisely the circuits of H we have, for $C \subseteq E$, the following correspondence:

- C is an odd circuit of (G, Γ) if, and only if, C is a v_1v_2-path of H,
- C is an even circuit of (G, Γ) if, and only if, C is a circuit of H.

Let G' be the graph obtained from H after identifying vertices v_1 and v_2; call the identified vertex v. Let $\Gamma' := \delta_H(v_1)$. Then the correspondence above implies that $\text{cycle}(G') = \text{cycle}(G)$ and $\text{ecycle}(G', \Gamma') = \text{ecycle}(G, \Gamma)$. Since $G \setminus \text{loops}(G)$ is 3-connected, it follows from Theorem 9.31 that $G' \setminus \text{loops}(G') = G \setminus \text{loops}(G)$ and $\text{loops}(G) = \text{loops}(G')$. After changing the ends of the loops of G', if necessary, we may assume that $G' = G$. Since $\text{ecycle}(G, \Gamma') = \text{ecycle}(G, \Gamma)$, Γ' is a signature of (G, Γ) and as $\Gamma' \subseteq \delta_G(v) \cup \text{loops}(G)$, we see that v is a blocking vertex of (G, Γ). ☐

Blocking pairs

Let (G, Γ) be a signed graph. Take disjoint $I, J \subseteq E(G)$. If I contains an odd circuit, we define $(G, \Gamma)/I \setminus J := (G/I \setminus J, \emptyset)$. Otherwise, by Proposition 9.1, there is a signature Γ' that is disjoint from I, and we define $(G, \Gamma)/I \setminus J := (G/I \setminus J, \Gamma' - J)$. We call $(G, \Gamma)/I \setminus J$ a *minor* of (G, Γ). Notice that minors are defined only up to resigning, and since $\text{cycle}(G)/I \setminus J = \text{cycle}(G/I \setminus J)$, the signed graph $(G, \Gamma)/I \setminus J$ represents $(\text{cycle}(G), \Gamma)/I \setminus J$. We also have the following relationship:

Remark 9.34 ([62], page 21)**.** *Take a signed graph (G, Γ) and disjoint $I, J \subseteq E(G)$. Then ecycle$(G, \Gamma)/I \setminus J$ = ecycle$((G, \Gamma)/I \setminus J)$.*

Take vertices u, v of G. We say u and v form a *blocking pair* if every non-loop odd circuit of (G, Γ) uses either u or v. We see from Proposition 9.1 that u and v form a blocking pair if, and only if, there is a signature contained in $\delta_G(u) \cup \delta_G(v) \cup \text{loops}(G)$.

Proposition 9.35. *Let (G, Γ) be a signed graph with an odd loop and without a blocking vertex, and let $N :=$ ecycle(G, Γ). If e is a non-loop edge of G such that*

- *$|E(\text{si}(N/e))| \geq 8$,*
- *si(N/e) is internally 4-connected,*
- *si(N/e) is graphic,*

then the ends of e form a blocking pair.

Proof. Let $(G', \Gamma') := (G, \Gamma)/e$. By Remark 9.34, $N/e = \text{ecycle}(G', \Gamma')$. Notice that (G', Γ') also has an odd loop. Therefore, as $\text{si}(N/e)$ is internally 4-connected and $|E(\text{si}(N/e))| \geq 8$, it follows from Proposition 9.30 (d) that $G' \setminus \text{loops}(G')$ is 3-connected. Since $\text{si}(N/e)$ is graphic, so is N/e, and so $\text{ecycle}(G', \Gamma')$ is graphic. Putting these together, we get from Proposition 9.33 that (G', Γ') has a blocking vertex w, that is, every non-loop odd circuit of (G', Γ') uses w. Since (G, Γ) does not have a blocking vertex, w is the vertex in $G' = G/e$ obtained after identifying the ends of e in G. Thus, every non-loop odd circuit of (G, Γ) uses one of the ends of e, implying that the ends of e form a blocking pair of (G, Γ), as required. ☐

9.6 Proof of Proposition 9.13

Suppose (h1)-(h2) hold and there are distinct elements e_1, e_2, e_3, e_4 of M such that, for each $i \in [4]$, $\mathrm{si}(M/e_i)$ is internally 4-connected and is the cycle matroid of a graph where the three edges $\Lambda(e_i)$ are incident to the same vertex. Assuming $|E(M)| \geq 12$, we need to show M is one deletion away from being graphic. Recall that, for each $i \in [4]$, M/e_i is a loopless binary matroid with exactly three non-trivial parallel classes, and these classes have cardinality two, so $|E(\mathrm{si}(M/e_i))| = |E(M)| - 4 \geq 12 - 4$:

Claim 1. *For each $i \in [4]$, $|E(si(M/e_i))| \geq 8$.*

By (h1), M is simple, so we may assume that $\{e_1, e_2, e_3\}$ is not a triangle of M. Denote the three triangles of M containing e_1 by $\{e_1, f, f'\}, \{e_1, g, g'\}, \{e_1, h, h'\}$ where $\Lambda(e_1) = \{f, g, h\}$; the existence of these triangles is guaranteed by (h2). Recall that the non-trivial parallel classes of M/e_1 are $\{f, f'\}, \{g, g'\}, \{h, h'\}$. As it is the case for $\mathrm{si}(M/e_1)$, we know that M/e_1 also is the cycle matroid of a graph H where f, g, h are incident to the same vertex, say $v \in V(H)$. Notice that H is a loopless graph with exactly three non-trivial parallel classes $\{f, f'\}, \{g, g'\}, \{h, h'\}$. In particular, v is the only vertex common to any two of f, g, h. Let Γ be a cocycle of M that contains e_1. By Remark 9.28, $M = \mathrm{ecycle}(H + e_1, \Gamma)$. Clearly e_1 is an odd loop of $(H + e_1, \Gamma)$, and therefore, $\{f, f'\}, \{g, g'\}, \{h, h'\}$ are odd circuits of this signed graph. As a result, if $(H + e_1, \Gamma)$ has a blocking vertex, then v must be the one, and if it has a blocking pair, then v must belong to the pair. If v is a blocking vertex, then M is graphic by Remark 9.32, and we are done. Otherwise,

Claim 2. *$(H + e_1, \Gamma)$ does not have a blocking vertex.*

Consider the two edges e_2, e_3 of H. Since $\{e_1, e_2, e_3\}$ is not a triangle of M, edges e_2, e_3 are not parallel. Since Claims 1 and 2 hold, we may use Proposition 9.35 to conclude that, for $j \in \{2, 3\}$, the ends of e_j form a blocking pair of $(H + e_1, \Gamma)$. In particular, $e_2 \cap e_3 = \{v\}$. Write $e_2 = \{v, u\}$ and $e_3 = \{v, w\}$.

Claim 3. *$H \setminus \{u, v, w\}$ is connected.*

Proof of Claim. Let $H' := H/e_2$ and $(H' + e_1, \Gamma') := (H + e_1, \Gamma)/e_2$. By Remark 9.34, $\mathrm{ecycle}(H' + e_1, \Gamma') = M/e_2$. Since Claim 1 holds, we may use Proposition 9.30 (d) to conclude that $H' \setminus \mathrm{loops}(H')$ is 3-connected. In particular, if uv is the vertex of H' corresponding to the ends of e_2, the graph $H' \setminus \mathrm{loops}(H') \setminus \{uv, w\}$ is connected. As a result, $H \setminus \{u, v, w\}$ is connected. $\diamond$

Since $\{v, u\}$ and $\{v, w\}$ are blocking pairs, every non-loop odd circuit of $(H + e_1, \Gamma)$ uses either v or both u, w. As the non-trivial parallel classes of H are incident with v, there is at most one edge with ends u, w.

Claim 4. *H has an edge Ω with ends u, w, and every non-loop odd circuit of $(H + e_1, \Gamma)$ uses v or the edge Ω.*

Proof of Claim. Let C be a non-loop odd circuit C of $(H + e_1, \Gamma)$ such that $v \notin V(C)$. Then $\{u, w\} \subseteq V(C)$. It suffices to show that C contains an edge whose ends are u and w. Suppose otherwise. Let x, y be the two neighbors of u in $H[C]$ – note $x, y \in V(H) - \{u, v, w\}$ by our contrary assumption. Thus by Claim 3, there is an xy-path $P \subseteq E(H)$ that is disjoint from $\{u, v, w\}$. Consider the two cycles $C_1 := \{u, x\} \cup \{u, y\} \cup P$ and $C_2 := C \triangle C_1$. Since C_1 is disjoint from the blocking pair $\{v, w\}$, and C_2 is disjoint from the blocking pair $\{v, u\}$, it follows that both C_1, C_2 are even in $(H + e_1, \Gamma)$, implying in turn that $C = C_1 \triangle C_2$ is also even in $(H + e_1, \Gamma)$, contradicting our choice of C. $\diamond$

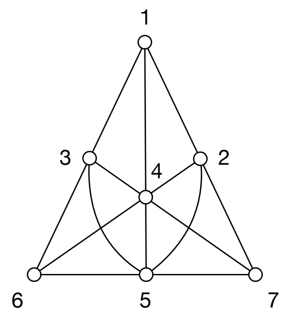


Figure 9.5: The Fano plane.

Therefore, $(H + e_1, \Gamma) \setminus \Omega$ has v as a blocking vertex. By Remark 9.34, we have that

$$M \setminus \Omega = \mathrm{ecycle}((H + e_1, \Gamma) \setminus \Omega)\,,$$

so it follows from Remark 9.32 that $M \setminus \Omega$ is graphic, as required. □

9.7 Further notes

What about the f-Flowing Conjecture in general? We have shown the following partial result addressing the remaining case:

Theorem 9.36 ([7, 8])**.** *Let $\mathcal{C}, \mathcal{B}$ be a blocking pair of minimally non-ideal binary clutters over ground set E, neither of which has a member of cardinality three. Let (M, Σ) represent $\mathcal{C}$ and let (N, Γ) represent $\mathcal{B}$. Then, for each $e \in E$, there exist disjoint $I, J \subseteq E - e$ such that either*

$$(M, \Sigma) \setminus I/J \cong (F_7, E(F_7) - \omega)$$

or

$$(N, \Gamma) \setminus I/J \cong (F_7, E(F_7) - \omega).$$

Here, ω is an arbitrary element of the Fano matroid F_7.

The *two-point Fano clutter* is the clutter over ground set [7] whose members are the lines, and their complements, of the Fano plane displayed in Figure 9.5 that intersect $\{1, 5\}$ exactly once:

$$\mathbb{LC} := \big\{\{1,3,6\},\{2,4,5,7\},\{1,2,7\},\{3,4,5,6\},\{5,6,7\},\{1,2,3,4\},\{2,3,5\},\{1,4,6,7\}\big\}.$$

(Notice that changing the two points $1, 5$ yields an isomorphic clutter.) It can be readily checked that $\mathbb{LC}$ is an ideal binary clutter represented by $(F_7, E(F_7) - \omega)$. As a consequence of Theorems 1.34 and 9.36, we get the following:

Corollary 9.37 ([7, 8])**.** *Let $\mathcal{C}$ be a non-ideal binary clutter. Then $\mathcal{C}$ or $b(\mathcal{C})$ has one of $\mathbb{L}_7, \mathbb{O}_5, \mathbb{LC}$ as a minor.*

Chapter 10

Conclusion

Let us conclude this thesis by reiterating the major conjectures about ideal clutters. It is the study of these conjectures that has shaped and will continue to shape the field.

The oldest conjecture of the field is due to Paul Seymour from 1977 [70, 66]:

> **The f-Flowing Conjecture.** *A binary clutter is ideal if, and only if, it has none of $\mathbb{L}_7, \mathbb{O}_5, b(\mathbb{O}_5)$ as a minor.*

The next major conjecture is due to Michele Conforti and Gérard Cornuéjols from 1993 [19]:

> **The Replication Conjecture.** *If a clutter has the packing property, then so does any replication of it.*

In an attempt to resolve this conjecture, Gérard Cornuéjols, Bertrand Guenin and Fronçois Margot made the following stronger conjecture in 2000 [24]:

> **The $\tau = 2$ Conjecture.** *Every ideal minimally non-packing clutter has covering number two.*

A major objective predating all of these conjectures, however, has been to come up with a *good characterization* of ideal clutters. This pursuit, however, was crushed by the following surprising result of Guoli Ding, Li Feng, Wenan Zang in 2008 [27]:

Theorem 1.1. *Let $\mathcal{C}$ be a clutter over ground set E. Then the problem "Is $\mathcal{C}$ ideal?" is co-NP-complete.*

The bottleneck was the following, they showed:

Theorem 2.12. *Let $\mathcal{C}$ be a clutter over ground set E where every element is used in exactly two members. Then the problem "Does $\mathcal{C}$ have an odd hole minor?" is NP-complete.*

In spite of these results, we conjecture the following two statements:

Conjecture 2.13. *There exists a finite family $\mathcal{F}$ of non-ideal clutters and a polynomial $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ such that the following statement holds:*

> *Let $\mathcal{C}$ be a clutter over ground set V without a minor in $\mathcal{F}$. Then in time $f(|V|, |\mathcal{C}|)$, one can find a delta or the blocker of an extended odd hole minor, or certify that none exists.*

Conjecture 1.23. *There exists an algorithm that given an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$ determines in time polynomial in n and $|S|$ whether or not S is cube-ideal.*

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Glossary of Notation

(G, Γ)	signed graph with signing Γ, page 155
(G, T)	graft with graph G and terminals T, page 15
(G, T, Γ)	signed graft with signing Γ, page 144
(M, Σ)	signed binary matroid with signing Σ, page 136
$(M, \Sigma)/I \setminus J$	the minor of (M, Σ) obtained after contracting I and deleting J, page 137
$[n]$	the set $\{1, \dots, n\}$, page 6
χ_C	the characteristic vector of C, page 13
$\cong$	symbol for isomorphism, page 6
$\mathrm{conv}(S)$	the convex hull of set S, page 12
$\mathrm{cuboid}(S)$	the cuboid of set S, page 11
$\mathrm{cycle}(G)$	the cycle matroid of graph G, page 137
Δ_n	the delta of dimension n, page 6
$\mathrm{dist}(a, b)$	the Hamming distance between points a, b, page 70
$\mathrm{ecycle}(G, \Gamma)$	the even cycle matroid of signed graph (G, Γ), page 155
$\mathrm{ind}(S \triangle x)$	the induced clutter of S with respect to x, page 13
λ_M	the connectivity function of binary matroid M, page 137
$\mathbb{L}_7$	the Fano clutter, page 8
$\mathbb{O}_5$	the clutter of odd circuits of K_5, page 16
$\mathcal{A} \vee \mathcal{B}$	the join of clutters $\mathcal{A}, \mathcal{B}$, page 38
$\mathcal{C} \setminus I/J$	the minor of $\mathcal{C}$ obtained after deleting I and contracting J, page 6
$\mathcal{C}\|_{e=f}$	the clutter obtained from $\mathcal{C}$ after identifying elements e, f, page 9

$\mathcal{C}_n^2$	the odd hole of dimension n, page 7
$\nu(\mathcal{C})$	the packing number of $\mathcal{C}$, page 9
$\mathrm{odd}_G(J)$	the set of odd-degree vertices of edge-set J in graph G, page 141
$\overline{X}$	the complement of X in the host ground set, page 17
$\mathrm{si}(M)$	the simplification of binary matroid M, page 141
$\tau(\mathcal{C})$	the covering number of $\mathcal{C}$, page 9
$A - a$	shorthand notation for $A - \{a\}$, page 140
$A - B$	the set of elements in set A that are not in set B, page 6
$A \triangle B$	the symmetric difference $(A \cup B) - (A \cap B)$, page 31
$a \triangle b$	the coordinatewise sum of a, b modulo 2, page 12
$b(\mathbb{O}_5)$	the clutter of the cut complements of K_5, page 16
$b(\mathcal{C})$	the blocker of clutter $\mathcal{C}$, page 2
$E(G)$	the edge set of graph G, page 27
F_7	the Fano matroid, page 137
$F_7^\star$	the dual of the Fano matroid, page 15
$G[U]$	the subgraph of G induced on vertex (or edge) set U, page 21
G_n	the skeleton graph of $\{0,1\}^n$, page 18
I	the identity matrix, page 139
J	the all-ones matrix, page 139
$M(\mathcal{C})$	the incidence matrix of clutter $\mathcal{C}$, page 2
$M(K_5)^\star$	the cut matroid of K_5, page 15
$P(\mathcal{C})$	the set covering polytope associated with $\mathcal{C}$, page 13
P_3	the set $\{110, 011, 101\}$, page 20
$Q(\mathcal{C})$	the set covering polyhedron associated with $\mathcal{C}$, page 5
Q_6	the clutter $\{\{1,3,6\},\{1,4,5\},\{2,3,5\},\{2,4,6\}\}$, page 10
$Q_{r,t}$	an ideal minimally non-packing clutter over $2r+2t+2$ elements, page 11
R_5	the set $\{0000, 1000, 1100, 1110, 1111, 0111, 0011, 0001\} * \{0\}$, page 19
$R_{1,1}$	the set $\{000, 110, 011, 101\}$ whose cuboid is $Q_{1,1} = Q_6$, page 12

R_{10}	an excluded minor for the sums of circuits property, page 15
$R_{k,1}$	the set $\{\mathbf{0}^{k+1}, \mathbf{1}^{k+1}\} * \{0\}$, page 20
$S\triangle a$	the set $\{y\triangle a : y \in S\}$, page 12
$S_1 * S_2$	the set $(S_1 \times S_2) \cup (\overline{S_1} \times \overline{S_2})$, page 112
$S_1 \times S_2$	the set $\{(x, y) : x \in S_1, y \in S_2\}$, page 21
S_3	the set $\{110, 011, 101, 111\}$, page 20
$V(G)$	the vertex set of graph G, page 27
$x(C)$	shorthand notation for $\sum_{e\in C} x_e$, page 5
$\mathbf{0}^n, \mathbf{1}^n$	the all-zeros and all-ones vectors of dimension n, respectively, page 20

Index