Extensions of Galvin's Theorem

by

Maxwell Levit

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

We discuss problems in list coloring with an emphasis on techniques that utilize oriented graphs. Our central theme is Galvin's resolution of the Dinitz problem (Galvin. J. Comb. Theory, Ser. B 63(1), 1995, 153–158).

We survey the related work of Alon and Tarsi (Combinatorica 12(2) 1992, 125–134) and Häggkvist and Janssen (Combinatorics, Probability & Computing 6(3) 1997, 295–313). We then prove two new extensions of Galvin's theorem.

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Chapter 1

Introduction

> It got started when we tried to solve Jeff Dinitz's problem.
>
> *Erdös Rubin and Taylor*
> *Choosability in Graphs*

In 1976, Vizing gave the first published account of list coloring in a Russian journal. His work had not reached Erdös, Rubin, and Taylor by 1979, when the trio introduced the notion of list coloring to the English speaking research community. Their paper, "Choosability in Graphs" incited research on numerous problems in list coloring. See [42], [15].

Probably the most infamous such problem is the so called *edge list coloring conjecture*. This conjecture asserts that the chromatic number and list chromatic number coincide for line graphs. Over the course of the 1980's, Bollobás and Harris, Chetwynd and Häggkvist, and Bollobás and Hind, used probabilistic methods to give successively stronger upper bounds on the list chromatic number of certain very general families of line graphs, see [6], [10], [7].

Following these probabilistic advances, Alon and Tarsi used algebraic techniques to prove that planar bipartite graphs are 3-choosable. In so doing, they codified a powerful polynomial method for proving upper bounds on the list chromatic number. Janssen employed the methods of Alon and Tarsi to show that the edge list coloring conjecture is true for the line graphs of complete bipartite graphs provided that the parts are of unequal size, see [3], [23].

Finally, in 1995, Galvin gave a very surprising proof of the edge list coloring conjecture for the line graph of any bipartite graph. This improved upon Jansen's result, and resolved the so called *Dinitz Problem*, which had served as an impetus for Erdös, Rubin, and Taylor so many years before.

Galvin's proof is neither probabilistic, nor algebraic. Rather, it employs an unpublished lemma of Bondy, Boppana, and Siegel, which gives an upper bound on the list chromatic number under a very special structural assumption of the underlying graph. What is remarkable about Galvin's theorem is that this special structural assumption is precisely guaranteed by the famed Gale-Shapely theorem of the 1960's, see [17], [18].

The contribution of this thesis is to prove new generalizations of Galvin's theorem.

We begin with two expository chapters. In Chapter 2 we discuss the algebraic approach to list coloring pioneered by Alon and Tarsi. We also discuss the application of their technique to the Dinitz problem. In Chapter 3 we discuss a further application of Alon and Tarsi's methods due to Häggkvist and Janssen, see [19]. This work expands upon Janssen's techniques in [23], in order to prove that the list coloring conjecture is true for complete graphs on an odd number of vertices.

In Chapter 4 our main technical contribution is Lemma 33 (page 46), wherein we utilize the matching theory of bipartite graphs to construct special preference systems and stable matchings. This lemma will allow us to prove two new extensions of Galvin's theorem. Theorem 34 (page 47) shows that the coloring guaranteed by Galvin's theorem is not unique by finding two distinct list colorings, and Theorem 35 (page 49) finds a list coloring for a more restrictive type of listing.

In Chapter 5 we introduce a natural generalization of the Dinitz problem by considering the list chromatic number of the Hamming graph $H(n, d)$. To our knowledge, this problem has not previously been studied. In Lemma 44 (page 60) we give a non-trivial upper bound of $2n - 1 \geq \mathrm{ch}(H(n, 3))$ and in Theorem 43 (page 60) we give a non-trivial lower bound of $\mathrm{ch}(H(n, d)) \geq n + 1$ for all $d \geq 3$. We conclude the chapter with Theorem 41 (page 57), where we apply Kahn's result from [24] to prove that for fixed d,

$$\mathrm{ch}(H(n, d)) = n + o(n).$$

In Chapter 6 we discuss the obstacles in improving the results of Chapters 4 and 5. We formulate several related conjectures and open questions.

Before proceeding to our main content, we conclude our survey of related literature. The proof of Galvin's theorem makes use of an oriented digraph which is morally equivalent to a latin square. For our extensions, we will touch upon the theory of latin squares, in particular, the theory of the completions of partial latin squares.

M. Hall initiated this theory by proving that every latin rectangle is completable to a latin square, see [20]. The statement analogous to M. Hall's theorem in higher dimensions is false. Kochol showed this by constructing 3-dimensional examples of non-completable latin cuboids, see [26]. More recently Bryant et al. have generalized this work, see [9]. Although there is no logical dependence between their work and ours, the ideas in Chapter 5 are of a similar flavor. Evans conjectured that any partial latin square with at most $n - 1$ entries can be completed, see [16]. This was proved by Smetaniuk, see [37].

Returning to the topic of list coloring, we note that the resolution of the Dinitz problem did nothing to dissuade the research community from the subject. In fact, the opposite is probably true. For instance, immediately following Galvin's proof, Borodin, Kostochka, and Woodall gave a sharper theorem for bipartite multigraphs that are not necessarily complete, see [8].

Probabilistic methods, predicated strongly on the Rödl nibble and the Lovász local lemma, have continued to prove their worth in the study of list coloring, see [29]. Kahn used these tools to show that the edge list coloring conjecture is asymptotically true, and in fact the analogous statement for hypergraphs of bounded co-degree is also true, see [24]. This result was later improved and simplified by Molloy and Reed, see [29]. In an alternate direction, Reed showed that if each vertex in a graph is given a list of at least $2ek$ acceptable colors, where k is the largest maximum degree of a color subgraph, then the graph may be colored from these lists. The constant $2e$ from Reed's result was then improved to 2 by Haxell (using topological methods), and finally to $(1+o(1))$ by Reed and Sudakov, see [35], [22], [36].

A famous conjecture of Reed states that the chromatic number of a graph is at most the average of its trivial upper and lower bounds see [34]. The most recent stride towards proving this conjecture (also probabilistic) was taken by Delcourt and Postle. Their result is in fact a stronger statement about list coloring, see [11].

Alon and Tarsi's algebraic techniques were extended to a hypergraph coloring variant by Ramamurthi and West, see [33]. In a different algebraic direction, Thomassen considered a list coloring variant of the chromatic polynomial and showed that for sufficiently large lists, the evaluation of this list chromatic polynomial always exceeds the same evaluation of the classical version, see [39]. In a similar but more restricted vein, Haviar and Ivaska conjectured a stronger statement for line graphs of the complete bipartite graph, see [21].

Another well studied problem is to give an upper bound on the list chromatic number of graphs within some minor closed class. Voigt showed that, contrary to the classical case for coloring, there are planar graphs which are not 4-choosable, see [43]. Thomassen later proved that planar graphs are 5-choosable, see [38]. Woodall has worked extensively on questions of this nature, many of which are described in his survey paper, see [44]. More recently, Postle and Thomas have used the ideas from Thomassen's proof to establish many new list coloring results for graphs on surfaces, see [32].

Although the general case of the edge list coloring conjecture lives on in infamy, several other conjectures have been conquered. For instance, the *list square* conjecture of Kostochka and Woodall was disproved by Kim and Park, and a famous eponymous conjecture of Ohba was recently proved by Noel, Reed, and Wu, see [25], [31], [30].

Finally, we must address the myriad generalizations of list coloring that have been studied. In his survey, Woodall discusses several variants such as deficient choosability, $(a : b)$-choosability, and the total chromatic number. There have been fruitful papers on these variants by Borodin, Kostochka, Woodall, Tuza, and Voigt, among others, see [8], [40]. More recently, Dvořák and Postle introduced correspondence coloring, a generalization of list coloring currently being pursued by some members of the research community, see [13], [4], [5].

Chapter 2

The Polynomial Method

In this expository chapter we discuss an algebraic approach to list coloring. We closely follow the pioneering work of Alon and Tarsi in [3]. We will prove the main result, and several of the pertinent corollaries. We then give a short discussion of the application of this theorem to the Dinitz problem. This serves as an appropriate precursor to the results of Häggkivst and Janssen presented in Chapter 3.

2.1 Definitions and Notations

Throughout the following two chapters, the polynomials of interest will be multivariate polynomials over $\mathbb{Z}$. Unless otherwise stated the graphs described will posses the ordered vertex set $V = \{v_1, \ldots, v_n\}$. We take this moment to recall the necessary graph theoretic definitions and notations.

We let $\mathcal{A}$ be a finite subset of $\mathbb{Z}$ whose elements are called **colors**. We will occasionally use α to denote an arbitrary element of $\mathcal{A}$.

A **coloring** of a graph $G = (V, E)$ is a function $c : V \to \mathcal{A}$ so that $c(v) \neq c(u)$ if v and u are adjacent. A family of sets $\mathcal{L} = \{L_v\}_{v \in V}$ with $L_v \subseteq \mathcal{A}$ is a **listing** for G. The elements of $\mathcal{L}$ are called **lists**. If $|L_v| = k$ for all $L_v \in \mathcal{L}$, then $\mathcal{L}$ is a **k-listing**. A coloring is an **$\mathcal{L}$-coloring** if $c(v) \in L_v$ for all $v \in V$. If the listing $\mathcal{L}$ has not not been named, we will implicitly refer to an $\mathcal{L}$-coloring as a **list coloring**. The **chromatic number** $\chi(G)$ is the smallest integer k so that $|\mathcal{A}| = k$ and there exists a coloring $c : V \to \mathcal{A}$. The **list chromatic number** of G, denoted $\mathrm{ch}(G)$ is the smallest integer k so that for each k-listing $\mathcal{L}$ there exists an $\mathcal{L}$-coloring. If $\mathcal{L}$ is a listing for which there are no $\mathcal{L}$-colorings, then $\mathcal{L}$ is an **unsatisfiable** listing. Let G be a graph (or digraph) equipped with a listing $\mathcal{L}$. We define the **color subgraph** G_α to be the subgraph of G induced by the set of vertices $\{v : \alpha \in L_v\}$.

Given a graph $G = (V, E)$, an **orientation** of G is a digraph $D = (V, A)$ such that if

$\{x, y\} \in E$ then exactly one of the arcs (x, y) or (y, x) is in A. Given an orientation D and vertex $v \in V(D)$, the vertices u such that $(v, u) \in A$ are the **out neighbors** of v. The **out degree of v in G** is the number of out neighbors of v in G; it is denoted $\delta^+_G(v)$. We omit the subscript when the underlying graph is clear from the context. We define **in neighbors** and **in degree** analogously.

2.2 The Graph Polynomial

Roughly speaking the strategy of Alon and Tarsi proceeds as follows: Given a graph G and a listing $\mathcal{L}$, we define a multivariate polynomial whose roots correspond to improper list colorings of G. We also require that the domain of the variables corresponds to the list assignment given. If we can argue that this polynomial must take a nonzero value somewhere in a domain, then such a point in the domain will be precisely a list coloring of G.

More formally, given a graph $G = (V, E)$ we define the **Graph Polynomial** $f_G(x_1, \ldots, x_n) = \prod_{(i,j)\in J}(x_i - x_j)$ to be the product of all expressions of the form $(x_i - x_j)$ where $J = \{(i, j) : i < j, \{v_i, v_j\} \in E\}$. Each vertex v_i of G corresponds to a variable x_i in f_G, and each edge in G corresponds to an expression of the form $(x_i - x_j)$ in f_G. It is thus natural to interpret a vertex coloring c as the evaluation $f_G(c) = f_G(c(v_1), c(v_2), \ldots, c(v_n))$ of f_G. If c is a proper coloring, then $f_G(c)$ is a product of non-zero integers, and if c is not a proper coloring, then at least one of the expressions $(x_i - x_j)$ corresponding to an edge of G is zero, and hence, f_G is zero. To address the more restricted notion of list coloring we need only restrict the domain of each variable x_i to some finite set S_i that naturally corresponds to the list L_{v_i} of v_i in a prescribed listing $\mathcal{L} = \{L_{v_i}\}_{v_i \in V}$.

The previous statements can be made more precise after we establish the following lemma.

Lemma 1 *Let $P = P(x_1, \ldots, x_n)$. Suppose that for each $i \in [n]$ the degree of P as a polynomial in x_i is at most d_i and let $S_i \subset \mathbb{Z}$ be a set containing $d_i + 1$ distinct integers. If $P(x_1, \ldots, x_n) = 0$ for each n-tuple $(x_1, \ldots, x_n) \in S_1 \times \cdots \times S_n$ then $P \equiv 0$.*

Proof. We proceed by induction on n. When $n = 1$, this is the statement that a nonzero (single variable) polynomial $P(x)$ has at most d distinct roots. To verify this statement, suppose $P(x)$ is a non-zero polynomial of degree at most d and let $S = \{s_1, \ldots, s_{d_1+1}\}$ be a set of $d + 1$ distinct integers. If $P(s) = 0$ whenever $s \in S$, then $(x - s)$ is a factor of $P(x)$ for each of the $d + 1$ distinct elements of S. But then $P(x)$ has degree at least $d + 1$, a contradiction.

Suppose the statement holds for a polynomial in $n - 1$ variables. Let $P = P(x_1, \ldots, x_n)$ be a polynomial satisfying the hypothesis. Since x_n has degree at most d_n we may write $P = \sum_{i=0}^{d_n} P_i(x_1, \ldots, x_{n-1})x_n^i$, i.e., we view P as a polynomial of the single variable x_n such that the coefficient of x_n^i is a polynomial P_i of the other $n-1$ variables. By assumption, the degree of x_j in P_j is at most d_j, thus the inductive hypothesis applies to each polynomial

P_j. By assumption, $P(s_1, \ldots, s_n) = 0$ for all $(s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$. Hence, for each choice of $(s_1^*, \ldots, s_{n-1}^*) \in S_1 \times \cdots \times S_{n-1}$, the (single-variable) polynomial in x_n given by $P(s_1^*, \ldots, s_{n-1}^*, x_n) = \sum_{i=0}^{d_n} P_i(s_1^*, \ldots, s_{n-1}^*)x_n^i$ vanishes for all values of $s_n \in S_n$. Thus each coefficient of $P = \sum_{i=0}^{d_n} P_i(x_1, \ldots, x_{n-1})x_n^i$, namely, each $P_i(x_1, \ldots, x_{n-1})$, is zero for all values $(s_1^*, \ldots, s_{n-1}^*) \in S_1 \times \cdots \times S_{n-1}$. Hence, by the inductive hypothesis, the P_j are identically zero, and so P is identically zero. $\square$

Now consider the graph polynomial f_G and suppose that the sets S_i have size at most d_i+1. If f_G is not identically zero, then by Lemma 1, there is some n-tuple $(s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$ on which f_G is non-zero. From the definition of the graph polynomial it can be seen that this tuple is a list coloring of G where the list of v_i, L_{v_i}, is taken to be the set S_i.

Notice that the maximum degree of a variable x_i in the graph polynomial f_G is exactly the maximum degree of the vertex v_i in the graph G. Thus, a naive application of Lemma 1 to the graph polynomial does not improve over the easy upper bound of of $\mathrm{ch}(G) \leq \Delta(G) + 1$ discussed in [15].

The innovation of Alon and Tarsi is to relate f_G to orientations of the underlying graph. In certain situations to be discussed, this allows us to improve the bound $\mathrm{ch}(G) \leq \Delta(G) + 1$ by a factor of 2.

2.3 Oriented Graphs

Throughout this section we let G be a graph on n (ordered) vertices, and we let $\mathcal{L} = \{L_{v_1}, \ldots, L_{v_n}\}$ be a fixed listing. Our goal is to describe conditions under which f_G is not identically zero. The method applied by Alon and Tarsi is to interpret f_G in terms of orientations of the graph G

Let $D = (V, A)$ be an orientation of G. For each arc $e = (v_i, v_j) \in A$ define the **weight** of e by $w(e) = x_i$ if $i < j$ and $w(e) = -x_i$ if $i > j$. The weight $w(D)$ of the orientation D is defined as the product $\Pi_{e \in A} w(e)$.

We claim that if f_G is expanded and written as a sum of monomials, these monomials are precisely the weights of orientations of G.

Fact 1 *We have $f_G = \sum w(D)$, where D ranges over all orientations of D.*

Proof. For any given edge $e = \{x_i, x_j\}$ of G with $i < j$, the polynomial f_G can be written as $(x_i - x_j)f_{G'}$ where $G' = G - e$. The fact follows by noting that distributing the monomial $x_i - x_j$ into $f_{G'}$ corresponds directly to considering the two directions in which the edge $\{v_i, v_j\}$ can be oriented in an orientation of G.

More formally, we proceed by induction on the number of edges of G. If G has exactly one edge the claim is immediate. Suppose the claim holds for graphs on $m - 1$ edges, and let G be a graph with m edges. Let $G' = G - e$ for some edge $e = \{x_i, x_j\}$ with $i < j$, and let D'

denote an arbitrary orientation of G'. By the inductive hypothesis, $f_{G'} = \sum w(D')$ where D' ranges over all orientations of G'. Let $\mathcal{D}$ and $\mathcal{D}'$ be the set of all orientations of G and G' respectively. Let $\mathcal{D}^+$ be the set of all orientations of G in which $(x_i, x_j) \in A$, and let $\mathcal{D}^-$ be the set of all orientations of G in which $(x_j, x_i) \in A$. Observe that $\mathcal{D}$ is the disjoint union of $\mathcal{D}^+$ and $\mathcal{D}^-$. We have

$$f_G = (x_i - x_j) \cdot f_{G'} = (x_i - x_j) \sum_{D' \in \mathcal{D}'} w(D') =$$

$$= \sum_{D' \in \mathcal{D}'} \Big(x_i \cdot w(D') + (-x_j) \cdot w(D') \Big) = \sum_{D \in \mathcal{D}^+} w(D) + \sum_{D \in \mathcal{D}^-} w(D) = \sum_{D \in \mathcal{D}} w(D)$$

as desired. $\square$

We can now express the coefficient of each monomial $\prod_{i=1}^n x_i^{d_i}$ in terms of the degrees d_i. If D is an orientation of G we will say that an arc (v_i, v_j) is **decreasing** if $i > j$. We say that an orientation is **even** if it has an even number of decreasing edges, and **odd** otherwise. Let $d_1, \ldots, d_n$ be a sequence of non-negative integers. We denote by $DE(d_1, \ldots, d_n)$ and $DO(d_1, \ldots, d_n)$ respectively the sets of all even and odd orientations of G in which the out degree of v_i is d_i for all $i \in [n]$. Moreover, we denote the sizes of these two sets by $\#DE(d_1, \ldots, d_n)$ and $\#DO(d_1, \ldots, d_n)$, respectively. Notice that for each fixed sequence $d_1, \ldots, d_n$, the coefficient of $\prod_{i=1}^n x_i^{d_i}$ in f_G is the sum of the contributions from even orientations of G with out degree sequence $(d_1, \ldots, d_n)$ minus the contribution from odd orientations of G with out degree sequence $(d_1, \ldots, d_n)$.

More concisely, the remarks above imply the following lemma.

Lemma 2 *We have*

$$f_G(x_1, \ldots, x_n) = \sum_{d_1, \ldots, d_n \geq 0} \big(\#DE(d_1, \ldots, d_n) - \#DO(d_1, \ldots, d_n) \big) \prod_{i=1}^n x_i^{d_i}.$$

If there exists a degree sequence $(d_1, \ldots, d_n)$ such that $\#DE(d_1, \ldots, d_n) \neq \#DO(d_1, \ldots, d_n)$, then by applying Lemma 2 we will be able to show that f_G is not identically zero.

In Chapter 3, a key step will be the construction of orientations in which $\#DE(d_1, \ldots, d_n)$ is not equal to $\#DO(d_1, \ldots, d_n)$. However, for the purposes of Alon and Tarsi's results, an alternate formulation of this condition is more useful.

A (not necessarily connected) subdigraph H of a digraph D is **Eulerian** if the in degree of vertex v is equal to the out degree of vertex v for each vertex v in the subgraph. H is **even** if it has an even number of edges and **odd** otherwise. We let $EE(D)$ and $EO(D)$, respectively, denote the set of even and odd Eulerian subgraphs of D, and we denote the sizes of these two sets by $\#EE(D)$ and $\#EO(D)$, respectively.

Lemma 3 *Let D_1 be an orientation of G in which $\delta^+_{D_1}(v_i) = d_i$, $\forall i \in [n]$, i.e., let $D_1 \in DO(d_1, \ldots, d_n) \cup DE(d_1, \ldots, d_n)$. We have*

$$|\#DE(d_1, \ldots, d_n) - \#DO(d_1, \ldots, d_n)| \; = \; |\#EE(D_1) - \#EO(D_1)|.$$

Proof. For any orientation $D_2 \in DO(d_1, \dots, d_n) \cup DE(d_1, \dots, d_n)$ we denote by $D_1 \setminus D_2$ the set of arcs in D_1 that are not in D_2. This can be thought of as the set of arcs in D_1 that must be "switched" in order to obtain D_2. Note that the out degree of each vertex in D_1 is equal to its out degree in D_2, consequently, each vertex has the same out degree w.r.t. $D_1 \setminus D_2$ as it has in degree w.r.t. $D_1 \setminus D_2$. Hence, $D_1 \setminus D_2$ is Eulerian.

We fix some orientation $D_1 \in DO(d_1, \dots, d_n) \cup DE(d_1, \dots, d_n)$ and consider the map ϕ from $DO(d_1, \dots, d_n) \cup DE(d_1, \dots, d_n)$ to $EE(D_1) \cup EO(D_1)$ given by $\phi(D) = D_1 \setminus D$. We claim that ϕ is a bijection. If D_2 and D_3 are elements of $DO(d_1, \dots, d_n) \cup DE(d_1, \dots, d_n)$ and $D_1 \setminus D_2 = H = D_1 \setminus D_3$ then D_2 and D_3 are both obtained from D_1 by switching the orientation on the same set of edges so $D_2 = D_3$. Hence, ϕ is injective. If H is an Eulerian subgraph of D_1, then H can be obtained by applying ϕ to the graph D' obtained from D_1 by switching the orientation on all edges of H. As H is Eulerian, $D' \in DO(d_1, \dots, d_n) \cup DE(d_1, \dots, d_n)$. Hence, ϕ is surjective.

Recall that an Eulerian digraph is odd exactly if it has an odd number of edges, whereas an orientation is odd if the number of decreasing edges in that orientation is odd.

Claim. $D_1 \setminus D_2$ is an odd Eulerian subgraph if and only if D_1 and D_2 do not have the same parity.

Proof of Claim. Let S be the set of arcs in $D_1 \setminus D_2$ that are decreasing arcs of D_1 and let T be set of arcs in $D_1 \setminus D_2$ that are not decreasing arcs in D_1. The following are equivalent.

(i) $D_1 \setminus D_2$ is an odd Eulerian subgraph.

(ii) An odd number of arcs must be switched to obtain D_2 from D_1.

(iii) $|S|$ and $|T|$ do not have the same parity.

(iv) The orientations D_1 and D_2 do not have the same parity.

The first two statements are equivalent by definition. The third statement is easily seen to be equivalent to the first. Note that regardless of which set, S or T has an even number of arcs, the net effect of reversing the direction of all arcs in S and in T is to reverse the parity of the number of decreasing arcs in the orientation. Thus (iii) implies (iv). Moreover, if D_1 and D_2 have the same parity (as orientations), then the values $|S|$ and $|T|$ must have the same parity. Thus (iv) implies (iii). $\square$

Reframing the previous claim, we note that if D_1 is even, then ϕ maps even orientations to even Eulerian subgraphs and maps odd orientations to odd Eulerian subgraphs. On the other hand, if D_1 is odd, then ϕ maps even orientations to odd Eulerian subgraphs and odd orientations to even Eulerian subgraphs. In either case, since ϕ is a bijection, we have

$$|\#DE(d_1, \dots, d_n) - \#DO(d_1, \dots, d_n)| \;=\; |\#EE(D_1) - \#EO(D_1)|$$

as desired. $\square$

Combining Lemmas 2 and 3 yields the following.

Corollary 4 *Let D be an orientation of G and let $d_i = \delta_D^+(v_i),\ \forall i \in [n]$. If f_G is expanded to the standard representation of a linear combination of monomials, then the absolute value of the coefficient of the monomial $\prod_{i=1}^n x_i^{d_i}$ is $|\#EE(D) - \#EO(D)|$.*

Note that, in particular, if $\#EE(D) \neq \#EO(D)$ then this coefficient is non-zero. We are now prepared to prove the main result of [3].

2.4 The Alon-Tarsi theorem

Theorem 5 *Let $G = (V, E)$ be a graph and let $\mathcal{L} = \{L_v\}_{v\in V}$ be a listing for G. Let D be an orientation of G in which $|L_v| = \delta_D^+(v) + 1,\ \forall v \in V$. If $\#EE(D) \neq \#EO(D)$ then there exists an $\mathcal{L}$-coloring of G.*

Proof. Let G, $\mathcal{L}$ and D be as in the statement and suppose $\#EE(D) \neq \#EO(D)$. Let $V = \{v_1, \ldots, v_n\}$, and let $\bar{d}_i$ denote $\delta_D^+(v_i),\ \forall i \in [n]$. Let $S_i = L_{v_i},\ \forall i \in [n]$.

Assume, for the sake of contradiction, that there is no $\mathcal{L}$ coloring of G. Then $f_G(s_1, \ldots, s_n) = 0$ for every n-tuple $(s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n = L_{v_1} \times \cdots \times L_{v_n}$.

We would like to apply Lemma 1 to show that f_G is identically zero, and derive a contradiction. However, we have no guarantee that the degree of x_i is upper bounded by $\bar{d}_i$. In fact, this is almost surely not the case. (Note that the numbers $\bar{d}_i$ are defined by the particular orientation D of G; there is no relation between these numbers and the degrees d_i of the variables x_i of f_G.)

Our remedy is to consider a related polynomial $\bar{f}_G$ of smaller degree, which is identical to f_G on our domain of interest. To this end we define $Q_i(x_i) = \prod_{s\in S_i}(x_i - s) = x_i^{\bar{d}_i+1} - \sum_{j=0}^{\bar{d}_i} q_{i,j} x_i^j$, where $q_{i,j}$ are simply the requisite coefficients to validate the equation. We note that for $x_i \in S_i$ we have $Q_i(x_i) = 0$ and thus $x_i^{\bar{d}_i+1} = \sum_{j=0}^{\bar{d}_i} q_{i,j} x_i^j$. We may now reduce powers of x_i to a more manageable form. In particular, we let $\bar{f}_G$ be the polynomial obtained from f_G by repeatedly applying the substitutions $x_i^{\bar{d}_i+1} = \sum_{j=0}^{\bar{d}_i} q_{i,j} x_i^j$ to each occurrence of $x_i^{k_i}$ for $k_i > \bar{d}_i$. It is clear that the degree of $\bar{f}_G$ as a polynomial in x_i is at most $\bar{d}_i$, and since our substitutions were valid whenever $(x_1, \ldots, x_n) \in S_1 \times \cdots \times S_n$ it follows that $\bar{f}_G(x_1, \ldots, x_n) = f_G(x_1, \ldots, x_n)$ for all such n-tuples.

Thus Lemma 1 implies that $\bar{f}_G \equiv 0$.

On the other hand, since $\#EE(D) \neq \#EO(D)$, Corollary 4 implies that the coefficient of $\prod_{i=1}^n x_i^{\bar{d}_i}$ in f_G is nonzero. The degree of each x_i in this monomial is $\bar{d}_i$, and thus this monomial was not altered by any of our substitutions of large degree variables. Moreover, each application of our substitution strictly reduces the total degree of each monomial involved. As f_G is homogeneous, this implies that our substitutions do not introduce any new multiples of $\prod_{i=1}^n x_i^{\bar{d}_i}$ that were not present in f_G. Hence the coefficient of $\prod_{i=1}^n x_i^{\bar{d}_i}$ is the same in f_G and $\bar{f}_G$. This contradicts the fact that $\bar{f}_G \equiv 0$. And so G must have an $\mathcal{L}$-coloring. $\square$

By a similar argument we obtain the following.

Theorem 6 *Let $G = (V, E)$ be a graph, where $V = \{v_1, \ldots, v_n\}$, and let $\mathcal{L} = \{L_{v_i}\}_{v_i \in V}$ be a listing for G. Let $(d_1, \ldots, d_n)$ be a fixed sequence of non-negative integers so that $\#DE(d_1, \ldots, d_n) - \#DO(d_1, \ldots, d_n) \neq 0$. If $|L_{v_i}| = d_i + 1$, $\forall i \in [n]$, then there is an $\mathcal{L}$-coloring of G.*

Proof. By Lemma 2, the condition $\#DE(d_1, \ldots, d_n) - \#DO(d_1, \ldots, d_n) \neq 0$ implies f_G is not identically zero. In particular, the coefficient of $\prod_{i=1}^{n} x_i^{d_i}$ is non-zero. W assume that G does not have an $\mathcal{L}$-coloring and proceed as above to derive a contradiction. □

We will return to this formulation in Chapter 3.

2.5 Applications to list coloring

After presenting their proof of Theorem 5 in [3] Alon and Tarsi go on to give the first application of the result to list coloring. In particular, they prove that planar bipartite graphs have list chromatic number at most 3. We now present this result.

For a graph G define $L(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|}$ where H ranges over all subgraphs of G. It can be seen that $2L(G)$ is the maximum value of the average degree of a subgraph of G.

As is to be expected, there is a strong relationship between $L(G)$ and the maximum out degree of an orientation of G.

Lemma 7 *The graph $G = (V, E)$ has an orientation D in which every out degree is at most d if and only if $L(G) \leq d$.*

Proof. Suppose D has such an orientation and let H be any subgraph of G. We have

$$|E(H)| = \sum_{v \in V(H)} \delta_H^+(v) \leq \sum_{v \in V(H)} \delta_D^+(v) \leq d|V(H)|.$$

The first equality is a consequence of the handshaking lemma and the fact that the sum of the out degrees in a subgraph is half the sum of the degrees in that subgraph. The middle inequality is immediate, and the last inequality follows since d is an upper bound on $\delta_D^+(v)$ for all $v \in V$. Thus we have $|E(H)|/|V(H)| \leq d$ and so $L(G) \leq d$.

Suppose now that $L(G) \leq d$. We must construct an orientation with maximum out degree at most d. That is, we must choose an orientation for each edge of G without "overloading" any vertex with too many out neighbors. A well known way to do this starts by constructing a bipartite graph F whose vertices fall into classes A and B where $A = E$ and B is the union of d disjoint copies of V. In F we join the element $e = \{u, v\}$ of A to the d copies of u in B and the d copies of v in B. If F contains a matching which saturates all of A, then we create an orientation of G by orienting each edge $e = \{u, v\}$ of E away from the vertex $m(e)$

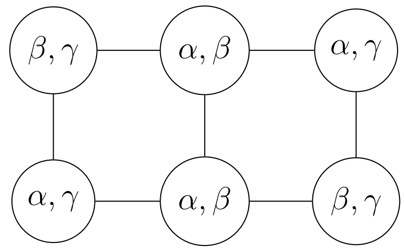


Figure 2.1: A planar bipartite graph exhibiting an unsatisfiable 2-listing.

(where $m(e)$ denotes u or v) such that the vertex e of A is matched with the vertex $m(e)$ of B. Each copy of the vertex u in B will be matched to at most one vertex in A (i.e., edge of E), and thus the out degree of u in our corresponding orientation will be at most d. Thus the orientation is as desired.

We now prove that F has such a matching saturating A, thus concluding the proof. Any subset $E' \subseteq E$ induces a subgraph $H[E']$ of G; note that the vertex set of $H[E']$, denoted $V(H)$, consists of the end vertices of the edges in E'. Hence, by the definition of F, the set E' (as a set of vertices in F) has exactly $d|V(H)|$ neighbors. By the definition of $L(G)$ we have $|E'|/|V(H)| \leq L(G) \leq d$, and so $d|V(H)| \geq |E'|$. As this argument holds for each subset E', Hall's theorem implies that the desired matching saturating A exists in F. $\square$

We may now give an upper bound on the list chromatic number of bipartite graphs in terms of $L(G)$.

Theorem 8 *Every bipartite graph G has list chromatic number at most $\lceil L(G) \rceil + 1$.*

Proof. Set $d = \lceil L(G) \rceil$. By Lemma 7 there is an orientation D of G in which the maximum out degree is at most d. G is bipartite, and thus D contains no odd directed cycles. We claim that this implies that $\#EO(D) = 0$. It can be seen by induction on the number of cycles in the underlying undirected graph that an Eulerian subgraph is the edge disjoint union of directed cycles. Hence, if H is an odd Eulerian subgraph, then at at least one of these edge disjoint directed cycles must be odd. Thus $\#EO(D) = 0$. On the other hand, $\#EE(D) \geq 1$ since, the empty subgraph is Eulerian and it is even (by definition). Thus $\#EE(D) \neq \#EO(D)$, and the result follows from Theorem 5. $\square$

This implies the following.

Corollary 9 *Every bipartite planar graph G is 3-choosable.*

Proof. If G is a planar bipartite graph with the maximum number of edges possible, then each facial cycle in an embedding of G is a 4-cycle. Hence we may deduce from Euler's formula that the maximum number of edges in a planar bipartite graph on n vertices is $2n-4$. Hence $L(G) \leq 2$ for all planar bipartite graphs. The result follows from Theorem 8.

We note that this result is tight as evidenced by the listing given in Figure 2.1.

This is also an appropriate time to note that the absence of odd directed cycles is a recurring theme in the literature of list coloring. A result of Richardson says that a digraph which contains no odd directed cycles is kernel perfect. This result, in conjunction with the "Kernel Lemma" of Bondy, Boppana, and Siegel (See Chapter 4, section 4.1.3) allows another approach to list coloring. However, the methods presented in this chapter are more general than Richardson's theorem. As we will see in Chapter 3, Theorem 6 can be used to address interesting families of graphs for which the orientations of note may well contain odd directed cycles.

2.6 The Alon-Tarsi conjecture

The work of Alon and Tarsi predates Galvin's proof of the Dinitz conjecture by several years. In Chapter 4 we will address Galvin's theorem; at present we focus on Alon and Tarsi's attempt to employ Theorem 6 to solve the problem. Although they were unsuccessful, the attempt that we outline in this section gave rise to an interesting conjecture that remains open to this day.

A **latin square** of order n is an $n \times n$ array of cells in which each cell contains an integer from the set $\{0, \ldots, n-1\}$ so that each number occurs exactly once in each row and each column of the array. Let $\mathfrak{L}(n)$ denote the set of all latin squares of order n. Let L be a latin square of order n. The rows of L can be thought of as permutations $\pi_1, \ldots, \pi_n$ of the numbers 0 through $n-1$ by reading the entries of a given row from left to right. Similarly, the columns of L can be thought of as permutations $\pi_{n+1}, \ldots, \pi_{2n}$ of the numbers 0 through $n-1$ by reading the entries top to bottom. Let $\epsilon(L)$ denote $\prod_{i=1}^{2n} sgn(\pi_i)$.

Conjecture 1 (Alon-Tarsi) *If n is even then*

$$\sum_{L \in \mathfrak{L}(n)} \epsilon(L) \neq 0.$$

We note that the statement is false for all odd values of n. In this section we explicate the connection between Conjecture 1 and (the statement of) Galvin's Theorem.

Here we interpret Galvin's Theorem as the statement that n is the list chromatic number of the line graph of $K_{n,n}$. In the rest of this section, we use G to denote the line graph of $K_{n,n}$; note that $|V(G)| = n^2$. Suppose we wish to apply Theorem 6 to prove this statement. Then we must construct a sequence $(d_1, \ldots, d_{n^2})$ satisfying the following conditions:

(i) $d_i \leq n-1$ for all $i \in [n^2]$.

(ii) There exists an orientation D of G with out degree sequence $(d_1, \ldots, d_{n^2})$ (i.e., $\delta_D^+(v_i) = d_i, \ \forall i \in [n^2]$) such that $\#DO(d_1, \ldots, d_{n^2}) \neq \#DE(d_1, \ldots, d_{n^2})$.

Observe that the degree of each vertex in G is $2n-2$, and thus the inequality in (i) must hold as an equation for each $i \in [n^2]$.

There is a natural correspondence between latin squares and certain orientations of G satisfying condition (i).

In particular, a **normal** orientation of G is an orientation in which the subgraph induced by each maximal clique is a transitive tournament. We consider the vertices of G to be the (empty) cells of an $n \times n$ grid where two vertices are adjacent if they are in the same row or in the same column. Let L be a latin square of order n whose cells contain the symbols $0 \ldots n-1$. We obtain a normal orientation of G from L as follows.

For each pair of cells that occupy the same row, we orient the edge of G between those cells towards the cell containing the larger entry. For each pair of cells that occupy the same column, we orient the edge of G between those cells towards the cell containing the smaller entry.

This orientation is normal since each entry occurs exactly once in each row and each column. Moreover this orientation satisfies property (i) above because a cell whose entry is k has exactly $n-1-k$ out neighbors in its row and k out neighbors in its column. In Chapter 4 we will discuss this correspondence more formally.

Although these normal orientations that arise from latin squares satisfy property (i), they are not the only orientations which do. An important idea in this section and throughout Chapter 3 is that these "latin" orientations are the only ones which contribute to the quantity $\#DO(n-1,\ldots,n-1) \; - \; \#DE(n-1,\ldots,n-1)$.

Lemma 10 *Let G be the line graph of $K_{n,n}$, and let $\mathcal{D}$ be the set of all orientations of G which are not normal orientations and in which $\delta^+(v) = n-1$ for all $v \in V$. The number of even orientations in $\mathcal{D}$ is equal to the number of odd orientations in $\mathcal{D}$.*

Alon and Tarsi do not comment on this idea or its proof. Fortunately, Janssen gives a nice argument in [23]. A generalization of this argument will be a key first step in the work of Chapter 3, and so we defer the proof until then. We require one more idea to motivate the formulation of the Alon-Tarsi conjecture.

Lemma 11 *Let G be the line graph of $K_{n,n}$. Let L be a latin square of order n and let D be the corresponding orientation of G. $\epsilon(L) = 1$ if and only if D is an even orientation.*

Proof. We label the vertices of $K_{n,n}$ left to right row-wise and top to bottom column-wise. Recall that we also read the permutations given by the latin square L from left to right and from top to bottom. For a given row i, the sign of the corresponding permutation π_i is $(-1)^k$ where k is the number of inversions in that row. Since the rows are labeled lexicographically, the number of inversions is also the number of decreasing arcs in the sub-digraph of D induced by vertices in the given row. Thus the (additive) contribution of this row to the number of decreasing arcs in D is k and the (multiplicative) contribution of this row to the value of $\epsilon(L)$ is $(-1)^k$. As this is true for each row and column, the result follows. $\square$

We can now conclude that the Alon-Tarsi conjecture implies Galvin's theorem for even values of n. If the conjecture holds, then $\sum_{L \in \mathfrak{L}(n)} \epsilon(L) \neq 0$ and hence by Lemma 11 the contribution to $\#DO(n-1,\ldots,n-1) - \#DE(n-1,\ldots,n-1)$ from orientations that correspond to latin squares is nonzero. On the other hand, by Lemma 10 we see that the contribution from all other orientations satisfying the degree sequence $(n-1,\ldots,n-1)$ is zero. Thus $\#DO(n-1,\ldots,n-1) - \#DE(n-1,\ldots,n-1)$ is non-zero, and so Theorem 6 implies the desired result.

2.7 The Combinatorial Nullstellensatz

For our purposes, Theorems 5 and 6 are an entirely adequate set of tools. However, it is worth mentioning that there are many generalizations of these theorems. Chief among these generalizations is the so called "Combinatorial Nullstellensatz", which earns its name by being a generalization of a special case of Hilbert's Nullstellensatz, see [14].

Alon formulates the Combinatorial Nullstellensatz as follows, see [1].

Theorem 12 *Let F be an arbitrary field, and let $f = f(x_1,\ldots,x_n)$ be a polynomial in $F[x_1,\ldots,x_n]$. Suppose the degree of f is $\sum_{i=1}^n t_i$ where each t_i is a nonnegative integer and suppose the coefficient of $\prod_{i=1}^n x_i^{t_i}$ in f is nonzero. Then if $S_1,\ldots,S_n$ are subsets of F with $|S_i| > t_i$, there are $s_i \in S_i$ so that $f(s_1,\ldots,s_n) \neq 0$.*

It is not too difficult to see that Theorem 5 follows from Theorem 12 when the polynomial f is taken to be the graph polynomial f_G. Moreover, the proof of Theorem 12 is quite similar to the proof of Theorem 5. The main difference is of course that the existence of a non-zero coefficient for the specified monomial is now an explicit assumption, rather than a consequence of the assumption that $\#EE(D) \neq \#EO(D)$.

The generality of this theorem is such that the polynomial method has been applied to a host of problems in combinatorics, many of which are outside the realm of list coloring. Alon surveys many applications in [1]. This survey is already quite old, and many more applications of the Combinatorial Nullstellensatz have been found. We mention one particularly concrete example: An extension of Theorem 5 to hypergraph coloring proved by Ramamurthi and West, see [33]. Here, a k-uniform hypergraph H is to be list colored under the proviso that no hyperedge is monochromatic.

Chapter 3

A result of Häggkvist and Janssen

In this expository chapter we present Häggkvist and Janssen's application of Theorem 6 to edge list coloring found in [19]. We present their first main result: The list chromatic index of the complete graph K_n is at most n. We then go on to sketch the second main result which is an upper bound on the list chromatic number of all simple graphs.

3.1 The list chromatic index of the complete graph

Given a graph G with vertex set V, we consider a map $\rho : V \to \mathbb{N}$. We think of ρ as specifying an out degree to each vertex of V, and we say that an orientation D of G **obeys** ρ if $\rho(v) = \delta_D^+(v)$ for all v. If G is a graph with vertices $v_1, \ldots, v_n$ we write $DO_G(\rho) = DO(\rho(v_1), \ldots, \rho(v_n))$, and $\#DO_G(\rho) = \#DO(\rho(v_1), \ldots, \rho(v_n))$. We adopt the same notation for DE and $\#DE$.

Rather than explicitly address the list chromatic index of K_n, we will instead consider the list chromatic number of the line graph $\Lambda(K_n)$. In this language, the goal of this section is to apply Theorem 6 to $G = \Lambda(K_n)$ by constructing an appropriate map ρ so that $\#DE_G(\rho) \neq \#DO_G(\rho)$. We begin by reducing the orientations that must be considered to the class of clique transitive orientations (defined below). This requires a series of three lemmas regarding orientations of the complete graph.

Lemma 13 *Let G be the complete graph on n vertices and let D be an orientation of G. D contains a directed 3-cycle if and only if there are two vertices of D with the same out degree.*

Proof. Let D be an orientation of G in which vertices u and v both have out degree k. Without loss of generality, assume that the edge between u and v is directed towards v. From among the $n-2$ other vertices of G there are $n-1-k$ in neighbors of u and there are k out neighbors of v. As $n-1-k+k > n-2$, there must be at least one vertex that

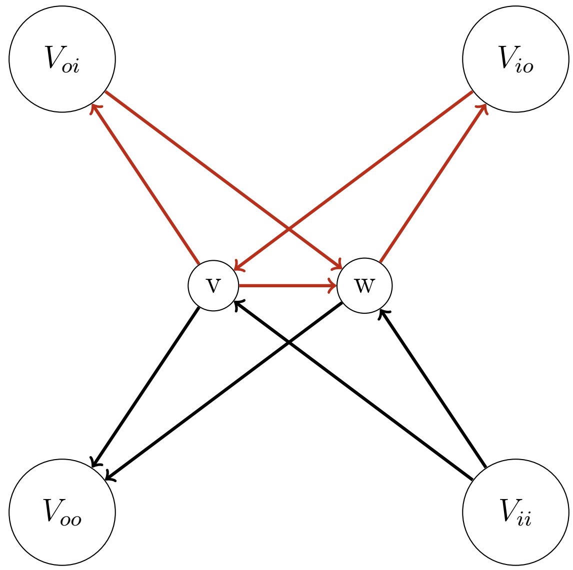


Figure 3.1: A subgraph of K_n induced by edges incident with v and w. Edges are oriented according to D. To obtain $\phi(D)$, the direction of the red edges is reversed.

is an out neighbor of v and an in neighbor of u. This vertex together with u and v induces a directed 3-cycle.

For the converse direction, suppose that the out degrees of vertices in D are distinct. The possible out degrees of vertices in D are $0, \ldots, n-1$, and there are n such vertices. It is easy to verify by induction that the only orientation of the complete graph in which the out degrees of the vertices are $0, \ldots, n-1$ is a transitive tournament. Such an orientation does not contain a directed 3-cycle. $\square$

Lemma 14 *Let $G = K_n$ and let $\mathcal{D}$ be the set of all orientations of G in which there is at least one directed 3-cycle. The number of even orientations in $\mathcal{D}$ is equal to the number of odd orientations in $\mathcal{D}$.*

Proof. We will construct a bijection $\phi : \mathcal{D} \to \mathcal{D}$ that maps odd orientations to even orientations and vice versa.

For each orientation $D \in \mathcal{D}$ let (v, w) be the first pair of vertices (lexicographically, according to the ordering of V) so that v and w have the same out degree in G. We assume without loss of generality that the edge between v and w is directed towards w. Let $\delta_D^+(v) = \delta_D^+(w) = k$ be the out degree of v and w. We partition the other vertices of D into four sets, V_{oo}, V_{oi}, V_{io}, and V_{ii} as follows. See Figure 3.1.

V_{oo} contains those vertices u that are out neighbors of both v and w.

V_{oi} contains those vertices u that are out neighbors of v and in neighbors of w.

V_{io} contains those vertices u that are out neighbors of w and in neighbors of v.

V_{ii} contains those vertices u that are in neighbors of both v and w.

By counting the number of edges in D that are directed out of v and out of w respectively we can see that $|V_{oi}| + |V_{oo}| + 1 = k = |V_{io}| + |V_{oo}|$, and thus $|V_{io}| = |V_{oi}| + 1$.

We define the image of D under ϕ by the following procedure. See Figure 3.1.

Reverse the direction of the arc (v, w).

Reverse the direction of arcs that are incident to v and to a vertex in $V_{io} \cup V_{oi}$.

Reverse the direction of arcs that are incident to w and to a vertex in $V_{io} \cup V_{oi}$.

We can see that in $\phi(D)$ the out degree of v is $|V_{oo}| + |V_{io}| = k$ and the out degree of w is $|V_{oo}| + |V_{oi}| + 1 = k$. Each vertex in $V_{oi} \cup V_{io}$ has had one outgoing arc changed to an incoming arc and one incoming arc to an outgoing arc, thus the degrees of such vertices are unchanged. So, since the degrees of all vertices in D are the same under D and under $\phi(D)$, we see that $\phi(D) \in \mathcal{D}$.

The out degree of each vertex is the same in $\phi(D)$ as in D, and so (v, w) is the first pair of vertices in $\phi(D)$ with the same out degree. Moreover, since the arc between v and w has switched direction, the set V_{oi} relative to D is the set V_{io} relative to $\phi(D)$ and vice versa. Thus $V_{oi} \cup V_{io}$ is the same set of vertices for both orientations. It follows that ϕ is "self inverse," i.e., $\phi(D) = \phi(\phi(D))$. Moreover, ϕ reverses the direction of exactly $2|V_{oi} \cup V_{io}| + 1$ arcs. Since this number is odd, ϕ maps even orientations to odd orientations. Thus ϕ is a bijection between even and odd orientations within $\mathcal{D}$ and the two sets are equicardinal. $\square$

We remark that Lemma 14 appears as stated in [19]. However, the proof we have presented follows [23].

We now extend the result of Lemma 14 to edge disjoint unions of complete graphs. This requires some definitions. A **clique decomposition** $G = G_1 \oplus \cdots \oplus G_\ell$ is a partition of the graph G into edge disjoint cliques $G_1, \ldots, G_\ell$. A **clique transitive orientation** of G is an orientation of G in which each G_i is a transitive tournament. Note that a clique transitive orientation can only be defined relative to a particular clique decomposition.

Lemma 15 *Let $G = G_1 \oplus \cdots \oplus G_\ell$ be a clique decomposition of G. For any map $\rho : V(G) \to \mathbb{N}$, the number of even orientations obeying ρ that are not clique transitive is equal to the number of odd orientations obeying ρ that are not clique transitive.*

Proof. From Lemma 2 we know that in the graph polynomial f_G, the (absolute value of the) coefficient of $\prod_{i=1}^{n} x_i^{\rho}(v_i)$ is $\big|\#DE_G(\rho) - \#DO_G(\rho)\big|$. Since G is an edge disjoint union of the subgraphs G_i, we may write $f_G = \prod_{j=1}^{\ell} f_{G_j}$. Let $\rho^i : V(G_i) \to \mathbb{N}$ be some degree map for the subgraph G_i. We may similarly write the coefficient (in f_{G_i}) of the monomial $\prod_{i=1}^{|V(G_i)|} x_i^{\rho}(v_i)$ as $\big|\#DO_{G_i}(\rho^i) - \#DE_{G_i}(\rho^i)\big|$. This gives us an alternate expression for the coefficient of $\prod_{i=1}^{n} x_i^{\rho}(v_i)$. Namely, we obtain

$$\big|\#DE_G(\rho) - \#DO_G(\rho)\big| = \sum_{\rho^1,\rho^2,\ldots,\rho^n} \Big((\big|\#DE_{G_1}(\rho^1) - \#DO_{G_1}(\rho^1)\big|)\ldots(\big|\#DE_{G_n}(\rho^n) - \#DO_{G_n}(\rho^n)\big|)\Big),$$

where the sum is taken over all maps ρ^i satisfying $\sum_i \rho^i(v) = \rho(v)$ for each vertex $v \in V(G)$.

If D is an orientation obeying ρ that is not clique transitive, then there is at least one clique G_i so that ρ^i assigns the same out degree to two vertices in G_i. But then, by Lemma 14 we have $\big|\#DE_{G_i}(\rho^i) - \#DO_{G_i}(\rho^i)\big| = 0$ for all such orientations, thus the contribution to $\big|\#DE_G(\rho) - \#DO_G(\rho)\big|$ is zero from any orientation that is not clique transitive. $\square$

Given a clique decomposition of G, in order to determine $\big|\#DE_G(\rho) - \#DO_G(\rho)\big|$, we now only need to consider the clique transitive orientations of G that obey ρ.

We remark that in the case of $G = \Lambda(K_{n,n})$, there is a natural clique decomposition in which each clique has size n and is induced by the set of edges incident to a given vertex in $K_{n,n}$. The application of Lemmas 13 and 15 to $\Lambda(K_{n,n})$ with this decomposition verifies Lemma 10 from the previous chapter.

Returning to the line graph of the complete graph, note that $\Lambda(K_n)$ is $2n-2$ regular, and thus any orientation of $\Lambda(G)$ contains at least one vertex of out degree $n-1$. Therefore, if we wish to apply Theorem 6 to achieve an upper bound on $\mathrm{ch}(\Lambda(G))$, the best bound attainable is n. Moreover, such a bound can only be obtained if ρ is identically $n-1$, that is, if $\#DE(n-1,\ldots,n-1) \neq \#DO(n-1,\ldots,n-1)$.

The simplest way to prove that $\#DE(\rho) \neq \#DO(\rho)$ is to show that there is a unique clique transitive orientation that obeys ρ. When ρ is identically $n-1$, and $G = \Lambda(K_n)$ this is patently false. This is easy to see explicitly in the case of $\Lambda(K_4)$ where any clique transitive orientation with out degree 2 at each vertex can be "rotated" 120 or 240 degrees to obtain two additional distinct orientations.

Therefore we must either contend with analyzing a potentially very large number of clique transitive orientations obeying ρ, or augment ρ in some way so that a unique clique transitive orientation obeys ρ.

Häggkvist and Janssen take the latter approach and circumvent the issue of symmetry by "blocking" certain out degrees from specified vertices. As an example, to block out the value 5 from some clique K in G, one adds a new vertex v to G and attaches it to each vertex in K. We extend ρ so that $\rho(v) = 5$. There is a natural clique decomposition of this new graph formed by extending clique K to include v and keeping the rest of the decomposition unchanged. We will always implicitly reference this decomposition when adding blocking vertices.

Let D be a clique transitive orientation of this new graph. Then v is a member of exactly one clique in the decomposition, and so the out degree of v in D must be 5. Moreover, since $K \cup \{v\}$ is transitively oriented, no vertex of K has out degree 5 in the subgraph $K \cup \{v\}$. Thus 5 is "blocked" from K.

More formally, let G be a graph with a given clique decomposition let ρ be a map as above and let $\bar{G}$ be the graph formed by extending some clique K by vertices $u_1, \ldots, u_m$. An orientation D of G obeying ρ with values $\{b_1, \ldots, b_m\}$ **blocked out** in clique K is a orientation that can be embedded in an orientation $\bar{D}$ of $\bar{G}$ that has the following properties:

1. $\bar{D}$ obeys ρ at each vertex of $V(G)$.

2. For each i, u_i has out degree b_i in $\bar{D}_i$.

Note that D is clique transitive relative to the decomposition $G = H_1 \oplus \cdots \oplus H_m$ if and only if $\bar{D}$ is clique transitive relative to $\bar{G} = (H_1 \cup B_1) \oplus \cdots \oplus (H_m \cup B_m)$. Moreover, if D is clique transitive then the out degrees in D of vertices in the clique induced by $K \cup \{u_1, \ldots, u_m\}$ take on exactly the values $\{0, 1, \ldots, |K| + m + 1\} \smallsetminus \{b_1, \ldots, b_m\}$.

For convenience, we now codify the notion of blocking vertices and the conditions of Theorem 6 into a key lemma.

Lemma 16 *Let $G = H_1 \oplus \cdots \oplus H_m$ be a clique decomposition of G. Let $B_1, \ldots, B_m$ be sets of non-negative integers. Let $\mathcal{L}$ be a listing and let $\rho : V(G) \to \mathbb{N}$ be such that $\rho(v) < |L_v|$ for all $v \in V(G)$. If there exists a unique clique transitive orientation D of G with the values in B_i blocked out on clique H_i for each $i \in [m]$ then there exists an $\mathcal{L}$-coloring of G.*

Proof. We let B_i have size m_i and set $B_i = \{b_{i1}, \ldots, b_{im_i}\}$ for each i. Let $\bar{G}$ be the graph containing G formed by extending each clique H_i by vertices $v_{i1}, \ldots, v_{im_i}$. Define the map $\bar{\rho} : V(\bar{G}) \to \mathbb{N}$ as follows:

$$\bar{\rho}(v) = \begin{cases} \rho(v) & \text{if } v \in V(G) \\ b_{ij} & \text{if } v = v_{ij} \text{ for some } i, j. \end{cases}$$

We extend $\mathcal{L}$ to a listing $\bar{\mathcal{L}}$ for $\bar{G}$ by assigning each v_{ij} an arbitrary list of size at least $b_{ij} + 1$. For each clique transitive orientation D on G with values B_i blocked out on H_i there is a corresponding clique transitive orientation $\bar{D}$ of $\bar{G}$ obeying $\bar{\rho}$.

Moreover, since the only orientations considered are clique transitive, this correspondence is one to one. To see this, note that in a clique transitive orientation of $H_i \cup \{v_{i1}, \ldots, v_{im_i}\}$ the subgraph induced by $\{v_{i1}, \ldots, v_{im_i}\}$ is a transitive tournament whose arcs are determined by the set $\{b_{i1}, \ldots, b_{im_i}\}$.

So, since G has a unique clique transitive orientation obeying ρ, $\bar{G}$ has a unique clique transitive orientation obeying $\bar{\rho}$, hence by Lemma 15, $\left|\#DE_{\bar{G}}(\rho) - \#DO_{\bar{G}}(\rho)\right| = \pm 1$. So

Theorem 6 implies that there is an $\bar{\mathcal{L}}$-coloring of $\bar{G}$. Since $\mathcal{L}$ and $\bar{\mathcal{L}}$ agree on vertices of G, this coloring restricts to an $\mathcal{L}$-coloring of G. $\square$

We are now ready to prove the main result of this section. We note that this result is tight for odd n.

Theorem 17 *For each n, $ch(\Lambda(K_n)) \leq n$.*

Proof. We let $G = K_n$ and we label the vertices of G as $\{0, 1, \ldots, n-1\}$. We let $H = \Lambda(K_n)$ and consider $V(H) = \{(i, j) : 0 \leq i < j < n\}$. Thus (i, j) and (i', j') are adjacent if and only if $i = i'$, $i = j'$, $j = i'$, or $j = j'$. Moreover, $H = C_0 \oplus C_1 \oplus \cdots \oplus C_{n-1}$ where C_i is the clique in H of size $n - 1$ whose vertices are edges of G incident with vertex $i \in V(K_n)$.

We wish to apply Lemma 16, and so we must show that there is some map ρ and set of blocking values for which there is a unique orientation. To this end, we define $\rho : V(H) \to \mathbb{N}$ as follows:

$$\rho(i, j) = \begin{cases} n-1 & \text{if } i + j \geq n-1 \\ n-2 & \text{otherwise.} \end{cases}$$

We also define values b_i, $0 \leq i < n$:

$$b_i = \begin{cases} n-2-i & \text{if } i < \lfloor \frac{n}{2} \rfloor \\ n-1-i+\lfloor \frac{n}{2} \rfloor & \text{if } i \geq \lfloor \frac{n}{2} \rfloor. \end{cases}$$

We will soon show that there is a unique clique transitive orientation of H that obeys ρ with value b_i blocked out in clique C_i. We first note the structure of H relative to C_i. Each vertex (i, j) in H is a member of exactly two cliques C_i and C_j. Thus the assignment $\rho(i, j)$ is equal to an assignment $\rho_i(i, j) + \rho_j(i, j)$ where ρ_i and ρ_j are the out degrees of (i, j) in C_i and C_j respectively. If $\rho_i(i, j)$ is assigned, then $\rho_j(i, j)$ is called the **complementary degree** at (i, j) and vice versa.

In order to prove the theorem, we describe an algorithm that assigns values $\rho_i(v)$ and $\rho_j(v)$ sequentially to the vertices of H in such a way that $\rho(v) = \rho_i(v) + \rho_j(v)$ and C_i contains each out degree $\{0, 1, \ldots, |C_i|\} \setminus \{b_i\}$ on exactly one vertex.

Before we do this, however, we present two frameworks for understanding and exemplifying the procedure. The first framework is the "obvious" one: a visual representation of the graph $\Lambda(K_n)$ containing the information specified by the map ρ. See Figure 3.2.

The second framework we present follows Häggkvist and Janssen's treatment of their algorithm. Their strategy is to associate $G = \Lambda(K_n)$ to an $n \times n$ matrix whose cells represent the vertices of G. See Figures 3.3 through 3.9.

We index the rows and columns by vertices of K_n; thus a cell and vertex are denoted by (i, j). An orientation of $\Lambda(K_n)$ then corresponds to a labeling of the remaining cells of this

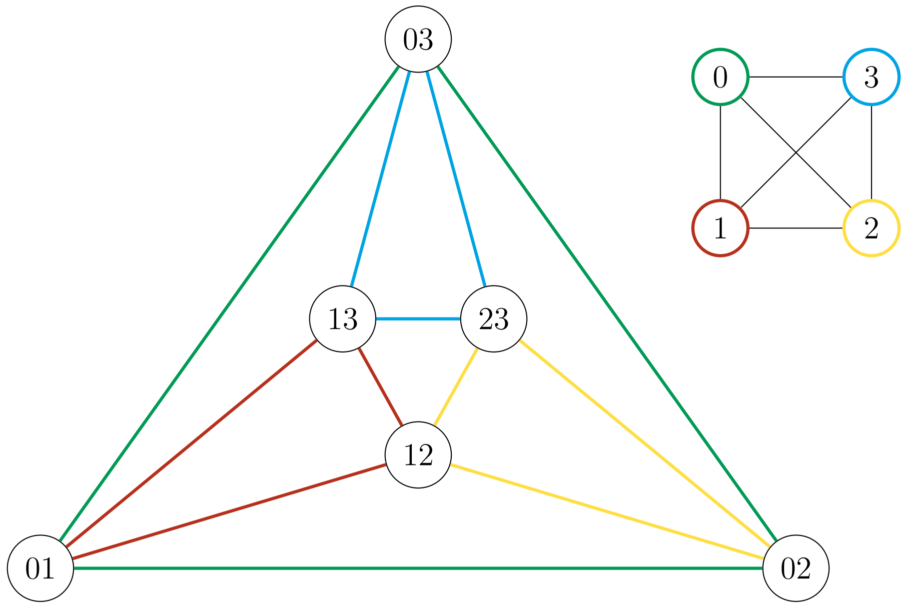


Figure 3.2: The line graph of K_4 (left) and K_4 (right) with our labeling scheme.

matrix by ordered pairs $[x, y]$. Given an orientation, a cell (i, j) is labeled $[x, y]$ exactly if the out degree of vertex (i, j) in C_i is x and the out degree of vertex (i, j) in C_j is y. Note that the cells on the main diagonal are empty. Moreover, all cells below the main diagonal are empty.

We will think of a specific map ρ by specifying which cells of the matrix are to receive which out degrees. We will also denote which values b_i are blocked out on clique C_i by placing the label $[b_i, b_i]$ in cell (i, i). As we are interested in maps ρ where the vertices take only one of two values, it is sufficient for our purposes to shade the cells (i, j) for which $\rho(i, j) = n - 2$ and leave the other cells (for which $\rho(i, j) = n - 1$) unshaded.

We may now interpret an orientation of G obeying ρ with b_i blocked out on C_i as a labeling of the cells of the matrix described above by ordered pairs satisfying the following pair of properties.

(1) If (i, j) is a shaded cell with label $[x, y]$ then $x + y = n - 2$, and if (i, j) is an unshaded cell with label $[x, y]$ then $x + y = n - 1$.

(2) For each row, the x-coordinates in the labels $[x, y]$ of all cells in that row are distinct entries from $\{0, \ldots, n - 1\}$. For each column, the y-coordinates in the labels $[x, y]$ of all cells in that column are distinct entries from $\{0, \ldots, n - 1\}$.

Property (1) is equivalent to the fact that the orientation specified obeys ρ. Property (2) is equivalent to the fact that b_i is blocked out on C_i and the orientation is clique transitive.

In order to prove the theorem, we will now describe an algorithm that labels the entries of this matrix in adherence with properties 1 and 2. In other words, the algorithm assigns values $\rho_i(v)$ and $\rho_j(v)$ sequentially to the vertices of H in such a way that $\rho(v) = \rho_i(v)+\rho_j(v)$ and C_i contains each out degree $\{0, 1, \ldots, |C_i|\} \setminus \{b_i\}$ on exactly one vertex. We then prove that at each step, there is exactly one valid choice for the next step in the algorithm. The only steps where this is not immediate are 4,7,10 and 14. In these steps we specify a vertex to receive a certain degree and it is not clear, a priori, that these choices are always unique. We will prove that they are. We now present the algorithm:

1. Set $i = 0$.
2. Set $t = 0$.
3. If $t = \lfloor \frac{n}{2} \rfloor - i$ then go to step 6.
4. Assign degree $n - i - 1$ to a vertex in clique C_{t+2i}. The unique choice is vertex $(t + 2i, n - 1 - t)$. The complementary degree of this vertex becomes i.
5. Set $t = t + 1$. Go to step 3.
6. If $t = n - 1 - 2i$ then go to step 9.
7. Assign degree $n - 1 - i$ to a vertex in clique C_{t+2i+1}. The unique choice is vertex $(t + n - 1 - t, t + 2i + 1)$. The complementary degree of this vertex becomes i.
8. Set $t = t + 1$. Go to step 6.
9. If $t = n - 1 - i$ then go to step 12.
10. Assign degree $n - 2 - i$ to a vertex in clique $C_{t+2i-(n-1)}$. The unique choice is vertex $(t + 2i - (n - 1), n - 1 - t)$. The complementary degree of this vertex becomes i.
11. Set $t = t + 1$ go to step 9.
12. If n is even and $i + 1 \geq \lfloor \frac{n}{2} \rfloor$ then go to step 17.
13. If $t > n - 1$ then go to step 16.
14. Assign degree $n - 2 - i$ to a vertex in clique $C_{t+2i+1-(n-1)}$. The unique choice is vertex $(n - 1 - t, t + 2i + 1 - (n - 1))$. The complementary degree of this vertex becomes i.
15. Set $t = t + 1$. Go to step 13.
16. Set $i = i + 1$. If $i < \lfloor \frac{n}{2} \rfloor$ then go to step 2.
17. End.

Before we prove the correctness of this algorithm, we present an example of its execution. Our example will be on the line graph of K_4. See Figures 3.3 through 3.9.

We set $i = 0$, $t = 0$ and as $t \neq 2$ we go to step 4. We assign degree 3 to a vertex in clique C_0. By construction of ρ, all but one of the vertices in C_0 have a specified out degree (in G) of at most 2, hence there is a unique choice of vertex to obtain out degree 3, namely $(0, 3)$.

	C_0	C_1	C_2	C_3
C_0	[2,2]			
C_1		[1,1]		
C_2			[3,3]	
C_3				[2,2]

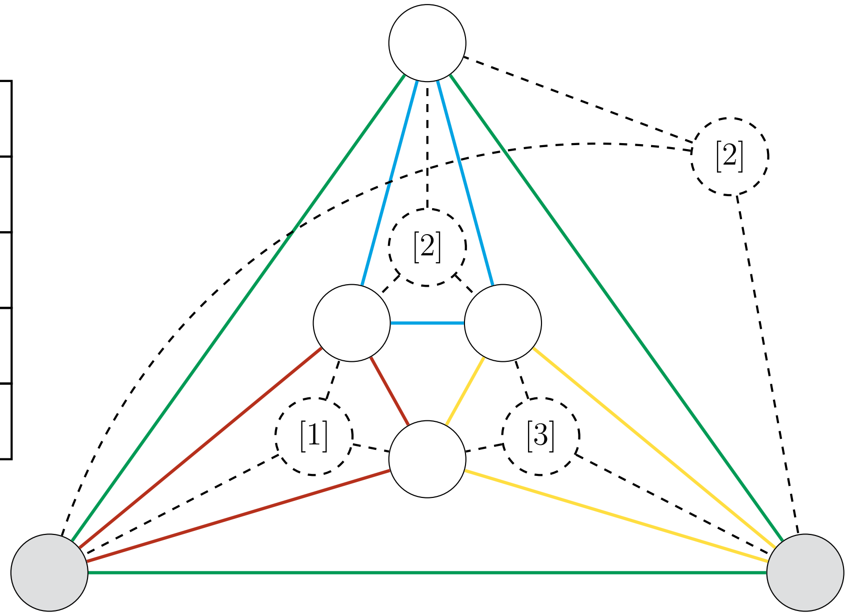


Figure 3.3: The initial state for the example. Blocking value b_i is placed on a (dashed) vertex adjoined to C_i and is denoted by $[b_i]$.

	C_0	C_1	C_2	C_3
C_0	[2,2]			[3,0]
C_1		[1,1]		
C_2			[3,3]	
C_3				[2,2]

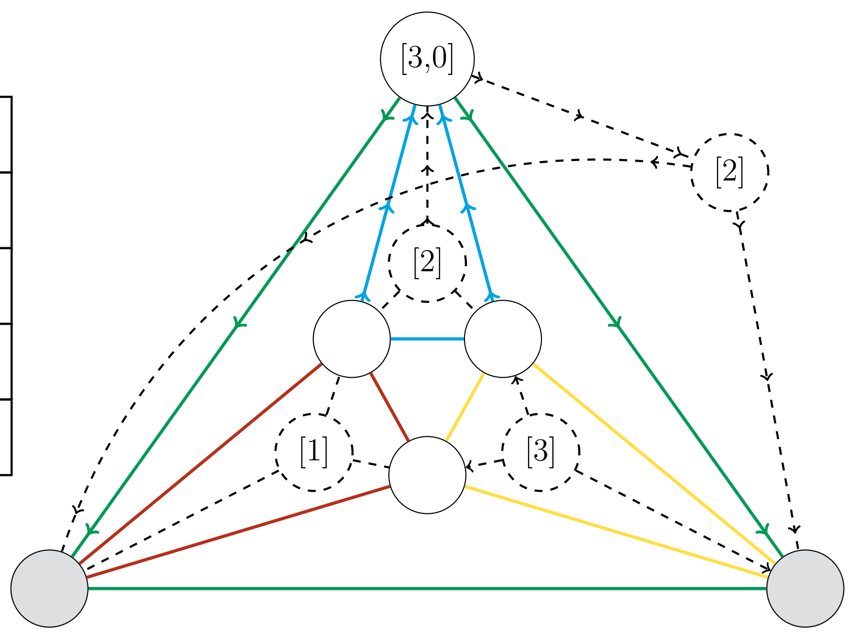


Figure 3.4: After the first out degree is assigned.

	C_0	C_1	C_2	C_3
C_0	[2,2]			[3,0]
C_1		[1,1]	[3,0]	
C_2			[3,3]	
C_3				[2,2]

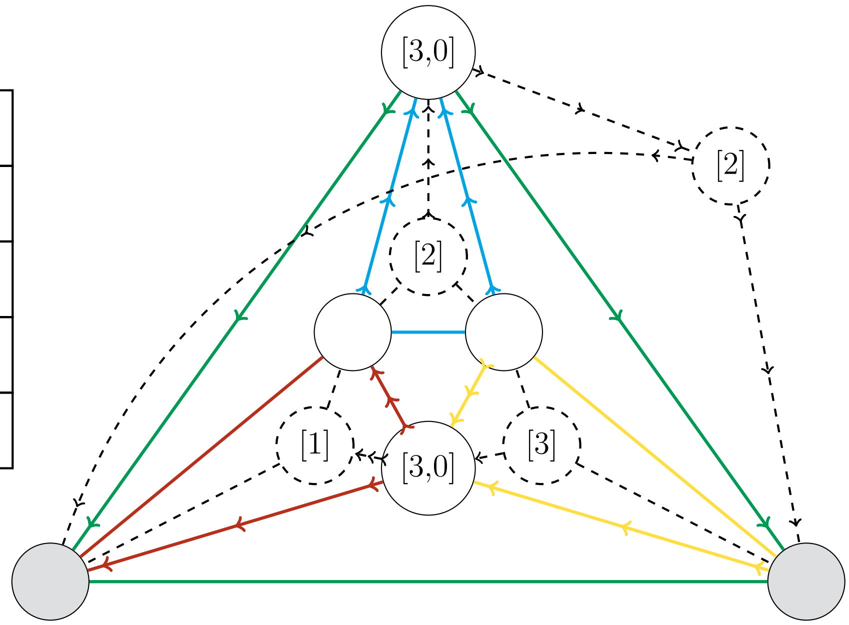


Figure 3.5: After the second out degree is assigned.

	C_0	C_1	C_2	C_3
C_0	[2,2]			[3,0]
C_1		[1,1]	[3,0]	[0,3]
C_2			[3,3]	
C_3				[2,2]

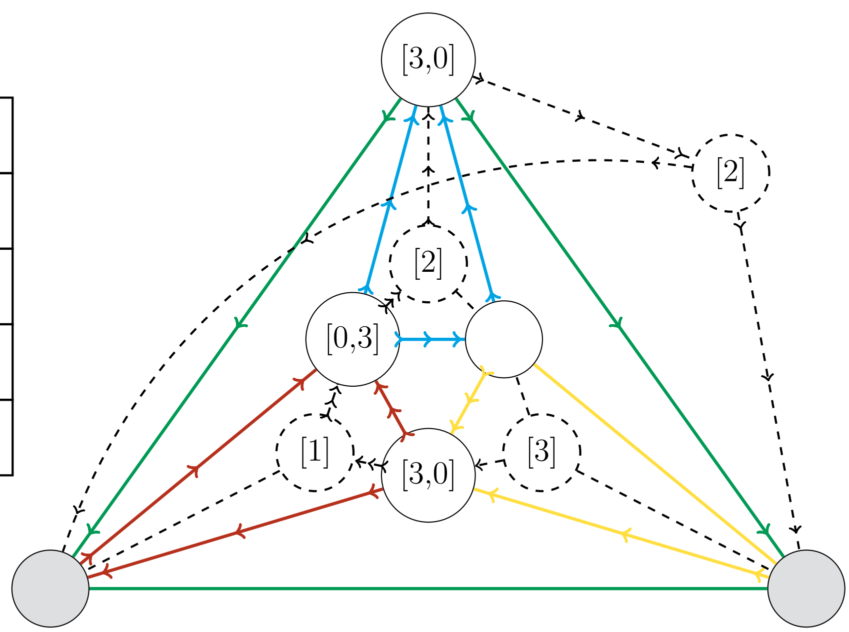


Figure 3.6: After the third out degree is assigned.

	C_0	C_1	C_2	C_3
C_0	[2,2]	[2,0]		[3,0]
C_1		[1,1]	[3,0]	[0,3]
C_2			[3,3]	
C_3				[2,2]

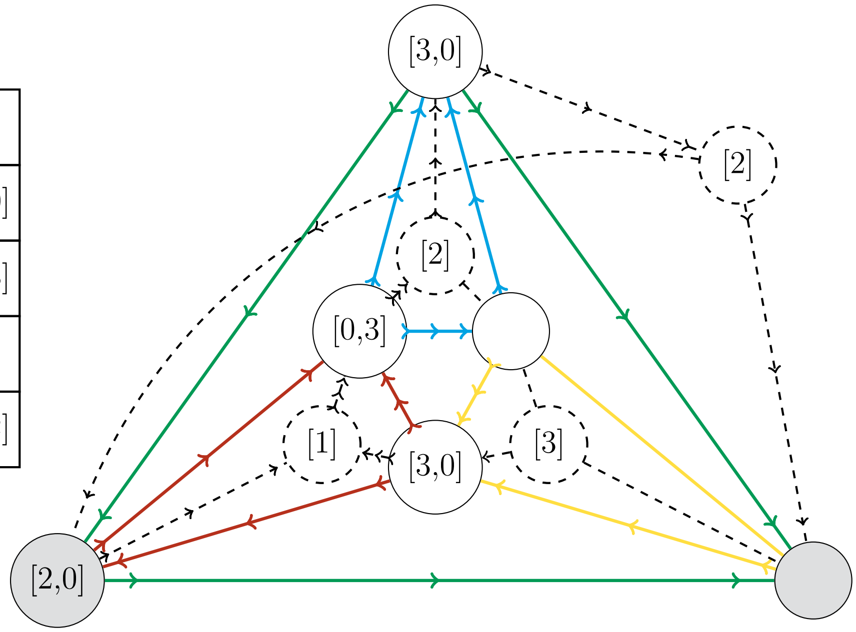


Figure 3.7: After the fourth out degree is assigned.

	C_0	C_1	C_2	C_3
C_0	[2,2]	[2,0]		[3,0]
C_1		[1,1]	[3,0]	[0,3]
C_2			[3,3]	[2,1]
C_3				[2,2]

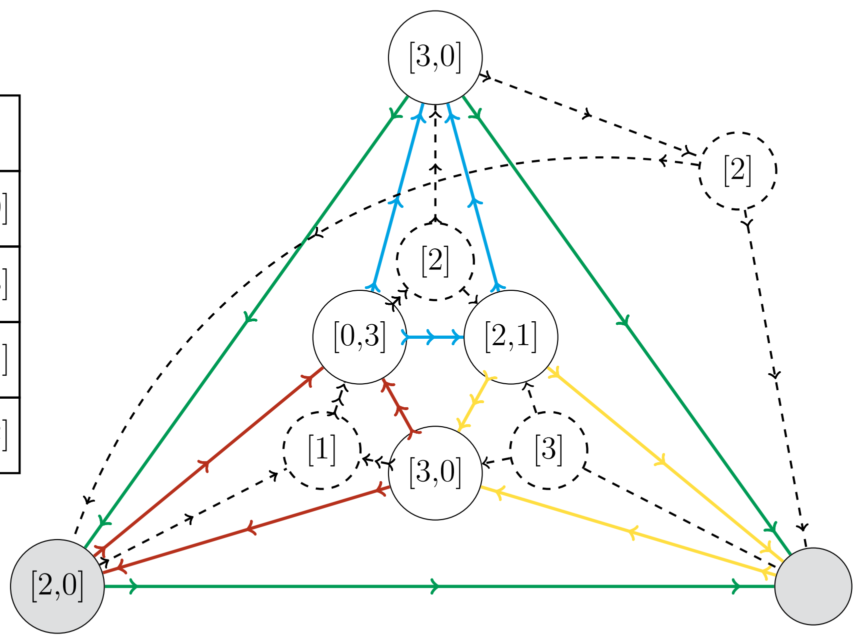


Figure 3.8: After the fifth out degree is assigned.

	C_0	C_1	C_2	C_3
C_0	[2,2]	[2,0]	[1,1]	[3,0]
C_1		[1,1]	[3,0]	[0,3]
C_2			[3,3]	[2,1]
C_3				[2,2]

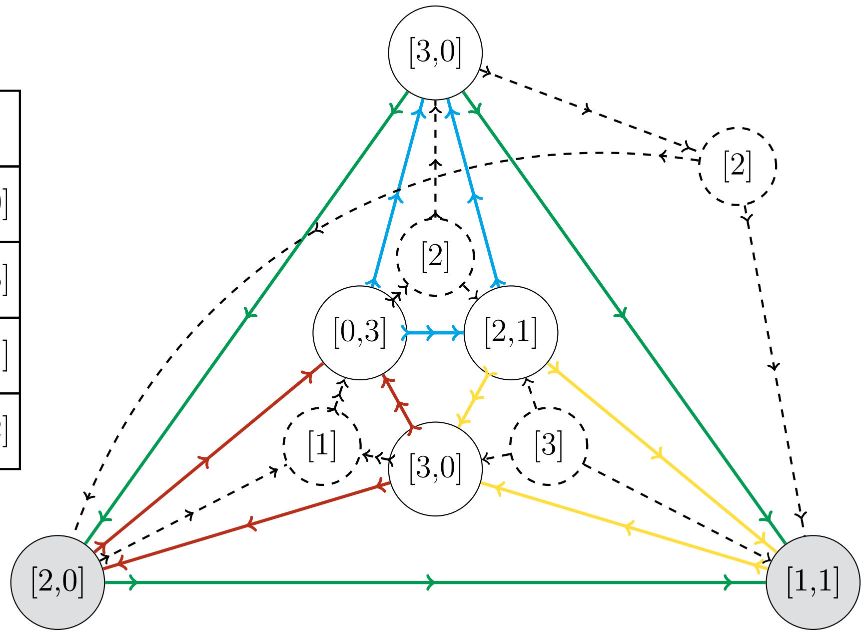


Figure 3.9: The completed orientation.

We set $t = 1$ and return to step 3, which sends us to step 4. We now must assign out degree 3 to some vertex in C_1. The potential choices are vertices $(1, 2)$ and vertices $(1, 3)$. However, in our previous application of step 4 we have assigned out degree 0 to a vertex in C_3. If we were to assign out degree 3 to $(1, 3)$ in C_1 we would simultaneously be assigning out degree 0 to $(1, 3)$ in C_3, this contradicts the fact that we are constructing a clique transitive orientation.

Note that the discussion of the previous paragraph is most easily read off of the matrix interpretation of the problem, wherein the fact that vertex $(1, 3)$ cannot receive out degree $[3, 0]$ is an immediate consequence of property 2.

After we have assigned out degree 3 to both $(0, 3)$ and $(1, 2)$, we now have $t = 2$ and find ourselves at step 7. Here we must assign degree 3 to some vertex in clique C_3. Since no vertex has out degree greater than four, the complementary degree at this vertex must be zero. Out degree 0 has already been assigned to cliques C_3 and C_2, thus the out degree $\delta^+_{C_2}(2, 3)$ is not 0. Thus $\delta^+_{C_3}(2, 3)$ is not 3. The only remaining vertex in C_3 that could be assigned out degree 3, is $(1, 3)$.

We set $t = 3$ in step 8 and then pass to steps 6, 9, 12, and 13, to arrive at step 14 with $t = 3$. We must assign out degree 2 to some vertex in C_1. The only valid choice is $(0, 1)$. This is again because if we assigned out degree 2 to $(0, 2)$ then the complementary out degree would be 0. We would then have two vertices in C_2 with out degree 0, a contradiction.

We now set $t = 4$ in step 15 and pass through steps 13, 16, 2, 3, and arrive at step 4 with $i = 1$ and $t = 0$. We must now assign out degree 2 to some vertex in C_2. Vertex $(2, 3)$ is the only vertex in C_2 that has not been assigned, and so we choose it.

Finally, we set $t = 1$ in step 4 and traverse steps 3, 6, 9 and arrive at step 10. Here we must assign out degree 1 to a vertex in C_3. The only choice is vertex $(0, 2)$.

We then set $t = 2$ in step 11, pass to steps 9, and 12, and then end at step 17.

We now verify the correctness of the algorithm. Note that for each $i \leq \lfloor \frac{n}{2} \rfloor$, the out degree $n-1-i$ is assigned to all cliques C_j except those where $j = \lfloor \frac{n}{2} \rfloor + i$. Moreover, $b_j = n-1-i$ precisely when $j = \lfloor \frac{n}{2} \rfloor + i$. Similarly, for each $i \leq \lfloor \frac{n}{2} \rfloor$ the out degree $n-2-i$ is assigned to all cliques except to C_i, and $b_i = n-2-i$. Thus each out degree is assigned to each clique C_i with the exception of the out degree blocked out on that clique.

We now prove that the "choices" made in steps 4, 7, 10, and 14 are unique. The method of proof is to verify that a certain predicate holds before and after each iteration of the given step. We then apply the predicate to show the uniqueness within each step. The predicate **P** is as follows. We note that this is exactly the strategy followed within the example.

P:

(i) The out degrees $0, \ldots, i-1$ have been assigned in all cliques.

(ii) The out degree i has been assigned in cliques $C_{n-1}, \ldots, C_{n-t}$.

(iii) The following sets of vertices have all been assigned an out degree and corresponding complementary out degree.

  (a) Vertices (x, y) with $x + y = d \pmod{n-1}$, where $0 \leq d < 2i$

  (b) Vertices (x, y) with $x + y = 2i \pmod{n-1}$ satisfying one of $2i \leq x < t + 2i$ or $0 \leq x < t + 2i - (n-1)$.

  (c) Vertices (x, y) with $x + y = 2i + 1 \pmod{n-1}$ and $\lfloor \frac{n}{2} \rfloor + i + 1 \leq y < t + 2i + 1$.

The statement of parts (i) and (ii) are clear from the example. Part (iii) specifies the order in which vertices are assigned their out degrees. This is easiest to interpret in terms of the matrix representation described above. In this interpretation, property (a) states that vertices on the main anti-diagonal of this matrix are filled first, and that after this main anti-diagonal is filled. The next anti-diagonals to be filled are those directly above and below. Parts (b) and (c) address the "zigzagging" behavior of the subsequently filled cells. This is illustrated in Figure 3.10.

We now show that predicate **P** is satisfied before and after each iteration of steps 4, 7, 10, and 14, and that this predicate implies each step has a unique choice. (In what follows, we refer to (i) of **P** as property (i), and similarly for (ii) and (iii) of **P**.)

Suppose **P** holds before step 4. Certainly this is true when $i = 0$ and $t = 0$. The out degree $n-1-i$ must be assigned to one of the vertices $(\ell, t+2i)$ or $(t+2i, \ell)$ in clique C_{t+2i}. If $\ell + t + 2i < n-1$ then the complementary out degree at this vertex will be $\rho(\ell, t+2i) -$

[7,7]	9	16	18	23	27	30	36	1
	[6,6]	17	24	26	31	35	2	8
		[5,5]	25	32	34	3	7	10
			[4,4]	33	4	6	11	15
				[8,8]	5	12	14	19
					[7,7]	13	20	22
						[6,6]	21	28
							[5,5]	29
								[4,4]

Figure 3.10: The matrix for K_9 with blocking values on the main diagonal. Cells above the main diagonal are labeled in the order that they are assigned out degrees by the algorithm.

$(n-1-i) = i-1$, but this would contradict property (i), and so $\ell + t + 2i \geq n-1$ and the complementary out degree is i. By property (ii) the out degree i has been assigned in cliques $C_{n-1}, \dots, C_{n-t}$, so ℓ is at most $n-1-t$. Since $\ell + t + 2i \geq n-1$, the representative of $\ell + t + 2i$ when taken modulo $n-1$ is $\ell + t + 2i - (n-1)$. By property (iii)(a), all vertices with $x + y = d \pmod{n-1}$ where $d < 2i$ have already received an out degree. Thus $\ell + t + 2i - (n-1) \geq 2i$, or equivalently, $\ell \geq n-1-t$. Since (i, j) is only a vertex when $(i < j)$, the only vertex in C_{t+2i} that could receive out degree $n-1-i$ is $(t+2i, n-1-t)$. It is easy to see that properties (i), (ii) and (iii) are again satisfied once t is augmented to $t+1$ in the application of step 5 that follows each application of step 4.

Suppose **P** holds before step 7. The out degree $n-1-i$ must be assigned to one of the vertices $(\ell, t+2i+1)$ or $(t+2i+1, \ell)$ in clique C_{t+2i+1}. Again, the out degree $i-1$ has already been assigned in all cliques, and so the complementary out degree at our vertex of interest must be i. The definition of ρ again implies that $\ell + t + 2i + 1 \geq n-1$. Property (ii) implies that out degree i has been assigned in cliques C_{n-1} through C_{n-t}, thus $\ell \leq n-1-t$. Since $\ell + t + 2i + 1 \geq 2i$, the representative of $\ell + t + 2i$ when taken modulo $n-1$ is $\ell + t + 2i + 1 - (n-1)$. By property (iii)(a), all vertices with $x + y = d \pmod{n-1}$ where $d < 2i$ have already received an out degree. Thus $\ell + t + 2i + 1 - (n-1) \geq 2i$, or equivalently, $\ell \geq n-2-t$. Suppose $\ell = n-2-t$, then $\ell + t + 2i + 1 = 2i \pmod{n-1}$. But this vertex already has had its out degree assigned by property (iii)(c). This is because $\lfloor \frac{n}{2} \rfloor - i \leq t < n-1-2i$ implies $2i \leq \ell < \lfloor \frac{n}{2} \rfloor + i \leq t+2i$. Thus the only vertex in C_{t+2i+1} that could receive out degree $n-1-i$ is $(n-1-t, t+2i+1)$. It is easy to see that properties (i), (ii) and (iii) are again satisfied once t is augmented to $t+1$ in the application of step 8 that follows each application of step 7.

Suppose **P** holds before step 10. Now out degree $n-2-i$ must be assigned to either $(\ell, t+2i-(n-1))$ or $(t+2i-(n-1), \ell)$ within clique $C_{t+2i-(n-1)}$. By a similar calculation as in the previous steps, it can be seen that if $t+2i-(n-1)+\ell \geq n-1$ then the representative of this number modulo $n-1$ is at most $2i$. Again, by property (iii)(a), all such vertices have already been assigned out degrees, thus $n-1 > t+2i-(n-1)+\ell$. Moreover, $t+2i-(n-1)+\ell \geq 2i$ and hence $\ell \geq n-1-t$. Since $t+2i-(n-1)+\ell < n-1$, the complementary degree of our vertex is $n-2-(n-2-i) = i$. Out degree i has been assigned in cliques $C_{n-1}, \dots, C_{n-t}$, so $\ell \leq n-1-t$ as well. Thus $\ell = n-1-t$ and the only vertex in $C_{t+2i-(n-1)}$ that could receive out degree $n-2-i$ is $(t+2i-(n-1), n-1-t)$. It is easy to see that properties (i), (ii) and (iii) are again satisfied once t is augmented to $t+1$ in the application of step 11 that follows each application of step 10.

Suppose **P** holds before step 14. Then out degree $n-2-i$ must be assigned to either $(\ell, t+2i+1-(n-1))$ or $(t+2i+1-(n-1), \ell)$ within clique $C_{t+2i+1-(n-1)}$. By a similar calculation as in the previous steps, it can be seen that if $t+2i+1-(n-1)+\ell \geq n-1$ then the representative of this number modulo $n-1$ is upper bounded by $2i+1$. By property (iii)(a) all vertices (x,y) with $x+y = d \pmod{n-1}$ where $d < 2i$ or $d = 2i$ already have assigned out degrees. If equality holds, then $\ell = t = n-1$, and so $\lfloor \frac{n}{2} \rfloor + i + 1 \leq \ell < t+2i+1$ and thus property (iii)(c) implies that this vertex has already been assigned an out degree. Thus $n-1 > t+2i-(n-1)+\ell$ and so $t+2i+1-(n-1)+\ell \geq 2i$. This implies $\ell \geq n-1-t-1$. Suppose for the sake of contradiction that equality holds, then $t+2i+1-(n-1)+\ell = 2i$ or $t+2i+1-(n-1) = 2i-\ell$. Now since $n-1-i \leq t \leq n-1$ we obtain $t+2i+1-n > n-1-t-1 = \ell$. Since $\ell \geq 0$ this gives $t+2i+1 > 0$, hence, this vertex has already been assigned an out degree, a contradiction. Thus $\ell \geq n-1-t$. Now since $t+2i+1-(n-1)+\ell < n-1$, we again see that the complementary degree of our vertex is $n-2-(n-2-i) = i$. Out degree i has been assigned in cliques $C_{n-1}, \dots, C_{n-t}$, so $\ell \leq n-1-t$ as well. Thus $\ell = n-1-t$ and the only vertex in $C_{t+2i+1-(n-1)}$ that could receive out degree $n-2-i$ is $(n-1-t, t+2i+1-(n-1))$. It is easy to see that properties (i), (ii) and (iii) are again satisfied once t is augmented to $t+1$ in the application of step 15 that follows each application of step 14. Moreover, it can be seen that after the augmentation of i to $i+1$ in step 16, the predicate holds as well.

Thus predicate **P** holds throughout, and the orientation constructed by the algorithm is the unique clique transitive orientation obeying ρ with b_i blocked out on C_i. Thus Lemma 16 implies that the list chromatic number of $\Lambda(K_n)$ is at most n. $\square$

3.2 A bound for simple graphs

In this section we outline Häggkvist and Janssen's use of Theorem 5 to prove the following upper bound on the list chromatic index of any simple graph of maximum degree d.

Theorem 18 *Let G be a simple graph of maximum degree d. The list chromatic index of G satisfies $ch'(G) \leq d + O(d^{2/3}\sqrt{\log d})$.*

Very roughly, the strategy is as follows. We use Theorem 5 and ideas from the proof of Theorem 17 to obtain an upper bound on a class of graphs obtained from the complete graph (Theorem 19). We then show that any graph which may be written as the edge disjoint union of graphs from the aforementioned class has a suitably bounded list chromatic index (Theorem 20). We then employ a probabilistic lemma (Lemma 21), to show that any simple graph can be embedded in another graph that can be written as such an edge disjoint union (Theorem 22). We combine these results to prove Theorem 18.

Our first step is to extend Theorem 17. Let $G = (V, E)$ be a graph, and let $\mathcal{H} = \{H_e : e \in E\}$ be a family of bipartite graphs H_e, wherein each H_e has bipartition (W_e^1, W_e^2), and $|W_e^1| = |W_e^2| = m$ for all $e \in E$. We define the **edge composition graph** $G\langle\mathcal{H}\rangle$ to be the graph obtained from G by replacing each vertex with m copies of that vertex, and replacing edge e with the bipartite graph H_e.

More formally, let $W_e^1 = \{w_{e,1}^1, \ldots, w_{e,m}^1\}$ and $W_e^2 = \{w_{e,1}^2, \ldots, w_{e,m}^2\}$ for each $e \in E$ and define a set $S_v = \{s_{v,1}, \ldots, s_{v,m}\}$ for each $v \in V$. We may then characterize $G\langle\mathcal{H}\rangle = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \cup_{v \in V} S_v$, and $(s_{u,i}, s_{v,j}) \in \mathcal{E}$ exactly when $(u, v) = e \in E$ and $w_{e,i}^1$ is adjacent to $w_{e,j}^2$ in H_e; we assume that some ordering has been imposed on V and $u < v$ in this ordering.

For our purposes, the graph G will be a complete graph, and the family $\mathcal{H}$ will contain only 1-regular graphs.

Theorem 19 *Let $\mathcal{H}$ be a family of 1-regular graphs on $2m$ vertices and let $\mathcal{G} = K_n\langle\mathcal{H}\rangle$. Then $ch'(\mathcal{G}) \leq n$.*

Proof Sketch. As before, we label the vertices of K_n by $0, \ldots, n-1$ and the edges of K_n by (i, j) with $i < j$. We let $\mathcal{H} = \{H_{i,j}\}$ where $H_{i,j}$ is the 1-regular graph that replaces edge (i, j). We label the vertices of $\mathcal{G}$ so that all vertices which originate from a single vertex of K_n are sequentially labeled. That is to say, vertex v_{im+k} is the k-th vertex in the set of vertices that replace vertex i of K_n.

Notice that the structure of $\Lambda(\mathcal{G})$ is similar to the structure of $\Lambda(K_n)$. As before, we may denote $V(\Lambda(\mathcal{G})) = \{(i, j) : i < j, (v_i, v_j) \in \mathcal{E}\}$ and $(i, j) \sim (i', j')$ if and only if $i = i'$, $i = j'$, $j = i'$, or $j = j'$.

Now, when we consider the line graph $\Lambda(\mathcal{G})$ we note that the edges of $\Lambda(\mathcal{G})$ are partitioned into nm edge disjoint cliques each of size $n - 1$. We label by C_{im+k} the clique in $\Lambda(\mathcal{G})$ which corresponds to vertex v_{im+k} in $\mathcal{G}$.

The idea of the proof is to utilize the map ρ and blocking values b_i for $0 \leq i < n$ defined in the proof of Theorem 17. In particular, we use ρ to define a map $\bar{\rho}$ on the vertex set of $\Lambda(\mathcal{G})$, and we use the blocking values b_i to define blocking values $\bar{b}_j$ for $0 \leq j < nm$. More concretely, the out degree that ρ assigns to vertex (i,j) in $\Lambda(G)$ is assigned to all vertices of $\Lambda(\mathcal{G})$ which correspond to edges of H_{ij}. The blocking value b_i is assigned to all cliques C_{mi+k} for $0 \leq k < m-1$ that arise as copies of the clique C_i in $\Lambda(G)$.

To prove that there is a unique clique transitive orientation of $\Lambda(\mathcal{G})$ obeying $\bar{\rho}$ with blocking values $\bar{b}_j$, we augment the algorithm so that whenever it assigns an out degree to a clique C_s it also assigns an out degree to a vertex in the cliques $C_{sm}, C_{sm+1}, \ldots, C_{sm+(m-1)}$ that correspond to copies of C_s.

We alter predicate $\mathbf{P}$ so that the conditions which hold for clique C_s after a certain step in the original algorithm also hold for the cliques $C_{sm}, C_{sm+1}, \ldots, C_{sm+(m-1)}$ after the same step in our modified algorithm.

Suppose that at a certain step the original algorithm assigns out degree $n-1-i$ to a vertex in C_s, then by predicate $\mathbf{P}$ there is a unique vertex (s,t) (or (t,s)) to which this out degree is assigned.

Since $\bar{\rho}$ and $\bar{b}_j$ are defined analogously on each clique C_{sm+r} with $0 \leq r < m$, the same argument shows there is a unique vertex $(sm+r, tm+r')$ (or $(tm+r', sm+r)$) for each such clique which must receive out degree $n-1-i$.

Since $H_{s,t}$ is one-regular, there is exactly one vertex $v_{tm+r'}$ in $\mathcal{G}$ that is incident to v_{sm+r} and hence within $\Lambda(\mathcal{G})$ there is exactly one clique $C_{tm+r'}$ that intersects C_{sm+r}. The unique vertex at the intersection of these cliques receives out degree $n-i-1$. Hence the "choices" made at each step of the modified algorithm are unique.

So the orientation given by the modified algorithm obeying $\bar{\rho}$ with prescribed blocking values is the unique such clique transitive orientation. Since the maximal value that $\bar{\rho}$ takes is $n-1$, we may apply Lemma 16 to obtain the desired bound. $\square$

The next step is to extend the bound of Theorem 19 to the edge disjoint union of the edge composition graphs discussed above. We note that in this context, an edge disjoint union of graphs is obtained by identifying vertices but not edges; e.g., the edge disjoint union of two copies of K_2 on the same vertex set is the multigraph on two vertices with two edges. This is the only place throughout this thesis where multigraphs make an appearance.

Theorem 20 *Let G be a regular graph of degree d which is the edge disjoint union of q graphs, each of which is an edge composition graph $K_n\langle\mathcal{H}^i\rangle$, where $\mathcal{H}^i$ is a family of one-regular graphs each on the same number of vertices. Then $ch'(G) \leq d + 2q - 1$.*

Proof sketch. The idea of the proof is again to define a carefully chosen map ρ and sets of blocking values and to prove that there is a unique clique transitive orientation satisfying this assignment.

The graph G is a disjoint union of graphs $G_i = K_n\langle\mathcal{H}^i\rangle$. Denote by E_s the edge set of G_s. Thus we may write the vertex set of the line graph $\Lambda(G)$ as $V(\Lambda(G)) = \cup_{s=0}^{q-1} V_s^\Lambda$ where $V_s^\Lambda = \{(i,j,s) : i < j, (v_i, v_j) \in E_s\}$.

We define a map ρ on $V(\Lambda(G))$ and a set B_ℓ of k blocking values for each clique C_j with $0 \leq j < nm$. We also define a family of functions $\beta_s : V_s^\Lambda \to \mathbb{N} \times \mathbb{N}$ which we interpret as a specific orientation of G_s. That is to say, in this specified orientation, if $\beta_s(i,j) = (r, \ell)$ then vertex (i,j,s) has out-degree r in clique C_i and out degree ℓ in clique C_j.

The maps β_s and ρ are defined so that the restriction of the specified orientation to the subgraph $\Lambda(G_s)$ is the unique clique transitive orientation obeying the restriction of ρ to $\Lambda(G_s)$. Using the argument from Theorem 19 it is then shown that the orientation of $\Lambda(G)$ specified by the maps β_s is the unique clique transitive orientation satisfying ρ.

A heuristic argument for the additive factor of $2q$ in the bound obtained is that in order to define a map ρ which admits a unique orientation we must sufficiently distinguish the k subgraphs $\Lambda(G_s)$ from one another. To do this, ρ must take on a maximum value of $d+2q-2$. The details are given in [19]. $\square$

In order to use these results to achieve a bound on the list chromatic index of any simple graph G, we must represent G as a subgraph of some graph satisfying the conditions of Theorem 20. In order for the bound $\mathrm{ch}'(G) \leq d + 2k - 1$ to be meaningful, both the number of graphs k and the degree d should be relatively small.

We will show that any simple graph can be suitably embedded in such a graph $\mathcal{J} \cup \Gamma$ via a two step process. The first step is to apply a probabilistic lemma, found in [10]. If S is a subset of the vertices of a graph, let $d(v, S)$ denote the number of neighbors of v in S.

Lemma 21 (Chetwynd and Häggkvist) *Let G be a simple graph of maximal degree d. Then, for every integer $p = 2^k$ such that $\frac{d}{60 \log 3d} > 2^k$ there exists a partition of the vertices of G into p parts $V_1, \ldots, V_p$ such that the degree $d(v, V_i)$ from any vertex v to any set V_i, $i \in [p]$, differs from its expected value of $\delta_G(v)/p$ by at most $3\sqrt{\frac{d}{p}\log 3d}$.*

Lemma 21 states that if we wish to partition the vertex set of a graph of maximum degree d into parts that have about the expected number of edges between them, then we may do so provided the number of parts is small enough as a function of d.

The proof of Lemma 21 is probabilistic and employs the Lovász local lemma, along with some standard concentration bounds for the binomial distribution. See [10].

Let G be a simple graph. We now use the partition $\{V_1, \ldots, V_p\}$ from Lemma 21 in order to embed G in the union of s edge composition graphs $K_p\langle\mathcal{H}_s\rangle$, where p is as given in Lemma 21 and s is to be determined later.

Note that G cannot be embedded in a single edge composition graph $K_p\langle\mathcal{H}\rangle$ since the bipartite subgraph of G induced by any two of the sets V_i and V_j is not necessarily a union of 1-regular graphs. Note also that the sets V_i are not necessarily independent sets of G.

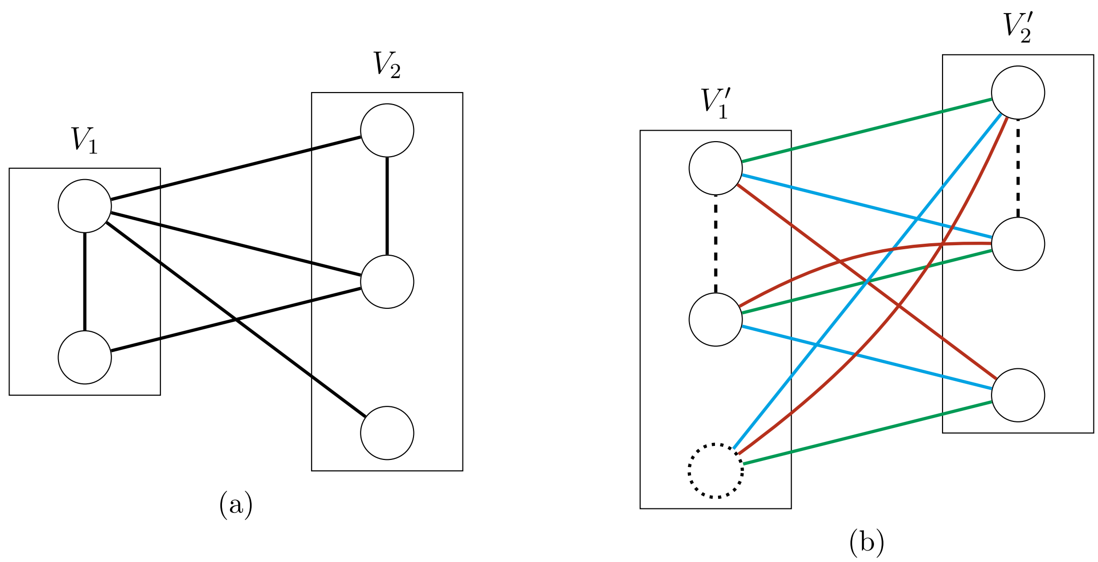


Figure 3.11: The simple graph pictured in (a) can be embedded in the graph pictured in (b). This latter graph is the edge disjoint union of three perfect matchings (Red, Blue, Green) along with two extra edges (dashed) that disobey the bipartition. Hence the graph from (a) can be embedded into the disjoint union of three edge compositions of K_2 along with two additional edges.

(In fact, by Lemma 21, there are roughly $\delta(v)/p$ edges between a vertex $v \in V_i$ and other vertices in V_i.)

We now show that G can be embedded in a graph Γ which is a union of edge compositions of K_p along with a sufficiently small set of extra edges $\mathcal{J}$; the latter set is used to embed the edges of G within each of the sets $V_i,\ i \in [p]$. See Figure 3.11.

Theorem 22 *Let G be a simple graph of maximum degree d on n vertices. Then for every integer $p = 2^k$ such that $\frac{d}{60 \log 3d} > 2^k$, G can be embedded in a graph $\mathcal{J} \cup \Gamma$, where Γ is the edge disjoint union of $s = d/p + 3\sqrt{\frac{d}{p} \log 3d}$ graphs, each of which is the edge composition of the complete graph K_p with a family of one-regular graphs, and $\mathcal{J}$ is a graph of maximum degree at most s.*

Proof. We let the graph G and integer p be as in the statement. From Lemma 21 and the fact that each vertex of G has degree at most d, there exists a partition of the vertex set V of G into p parts, $V_1, \ldots, V_p$, in which $d(v, V_i) \leq s$ for each $v \in V$ and for all $i \in [p]$. Let t denote the maximum size of any of the sets V_i. We form a graph G' by adding vertices to each of the sets $V_i,\ i \in [p]$, until all the sets have size t; we denote these "padded" sets by $V_i',\ i \in [p]$.

We let $G'_{i,j}$ be the subgraph of G' induced by all edges between V_i' and V_j'. Since each G'_{ij} is bipartite and has maximum degree at most s, Hall's theorem implies that G'_{ij} can be edge colored with s colors. We use the same set of colors $\{c_\ell : 0 \le \ell < s\}$ to properly edge color each subgraph G'_{ij}. We now "sort" our colored edges into subgraphs G'_b for $b \in [s]$ where G'_b is induced by the set of edges of G' colored c_b.

For any given pair i, j, the set of edges of G'_b whose end vertices are in both V'_i and V'_j is some matching of the vertices in V'_i with the vertices in V'_j. We may extend this matching to a perfect matching $M'_{b,ij}$ between V'_i and V'_j. We denote by Γ_b the graph given by the union over all pairs i, j of the graphs $M'_{b,ij}$. We extend the coloring so that all edges in Γ_b receive color c_b.

Hence, for each color c_b, and pair i, j, the edges of Γ_b contained in the subgraph induced by $V'_i \cup V'_j$ yield a 1-regular subgraph. So Γ_b is an edge composition of K_p and $\Gamma = \cup_b \Gamma_b$ is an edge disjoint union of edge compositions of K_p. Moreover, each edge of G with ends in distinct sets V_i, V_j is an edge of Γ.

To address edges between two vertices within V_i, we define $\mathcal{J}$ to be the graph with vertex set $\cup_i V'_i$; the edge set of $\mathcal{J}$ consists of all edges $\{v, w\}$ of G for which v and w are in the same set V_i. $\mathcal{J}$ is a union of disconnected components, each of which is contained within the vertex set of some V_i. We have chosen V_i so that $d(v, V_i) \le s$ for all i and all $v \in V$, so the maximum degree of any such induced subgraph is at most s as desired. It follows that G', and thus also G, can be embedded in $\mathcal{J} \cup \Gamma$. □

We may now prove a slightly more specific instantiation of Theorem 18 by embedding any simple graph in an appropriate union of edge compositions and applying Theorem 20.

Theorem 23 *For all simple graphs G of maximal degree d where d is such that $d^{2/3} > 60 \log 3d$, the list chromatic index satisfies $ch'(G) \le d + 23d^{2/3}\sqrt{\log 3d}$.*

Proof. Let G be a simple graph of maximal degree d. By Theorem 22, for every integer $p = 2^k$ for which $\frac{d}{60 \log 3d} > 2^k$, we know that G can be embedded in a graph $\mathcal{J} \cup \Gamma$ where $\mathcal{J}$ has maximum degree at most s and Γ is the edge disjoint union of $s = d/p + 3\sqrt{\frac{d}{p} \log 3d}$ graphs, each one an edge composition of the complete graph K_p with a family of one-regular graphs. Note that Γ has degree ps.

By Theorem 20, we have $\mathrm{ch}'(\Gamma) \le ps + 3s - 1 = d + 3\sqrt{dp \log 3d} + 3(d/p + 3\sqrt{\frac{d}{p} \log 3d}) - 1$. Moreover, since $\mathcal{J}$ has maximum degree s, we may greedily list color the edges of $\mathcal{J}$ provided each edge has a list of size at least $2s$. Thus, if we supply the edges of $\mathcal{J} \cup \Gamma$ with lists of size at least $\mathrm{ch}'(\Gamma) + 2s$, then we may greedily edge color $\mathcal{J}$ with enough colors left over to edge color Γ. Thus $\mathrm{ch}'(\mathcal{J} \cup \Gamma) \le ps + 3s - 1 + 2s$. Since G is a subgraph of $\mathcal{J} \cup \Gamma$ we have $\mathrm{ch}'(G) \le ps + 3s - 1 + 2s$. This expression is minimized if we take p to be the power of 2 closest to $d^{1/3}$, in which case $ps + 5s - 1 = d + 3\sqrt{d^{4/3} \log 3d} + 5(d^{2/3} + 3\sqrt{d^{2/3} \log 3d}) - 1 \le 23d^{2/3}\sqrt{\log 3d}$. □

Chapter 4

Extensions of Galvin's Theorem

In this chapter we will introduce and expand upon Galvin's proof of the Dinitz conjecture. All pertinent definitions and notations are found in Section 2.1.

Informally, we will think of Galvin's theorem as an algorithm. The input to this algorithm is an edge n-list assignment $\mathcal{L}$ for the complete bipartite graph $G = K_{n,n}$. The output of the algorithm is an $\mathcal{L}$-coloring of G.

The algorithm proceeds as follows. Choose some color α and consider the color subgraph G_α. Choose an appropriate subset M_α from $E(G_\alpha)$. Color M_α with color α, and then remove α from the lists of all other edges in G_α. Repeat with a new color β. Continue repeating this process until all edges are colored.

We will sometimes refer to this procedure as Galvin's algorithm, and refer to the process of choosing a color β, and coloring the edges of M_β as one iteration of the algorithm. Notice that each iteration involves several non-trivial choices. First, some new color β must be selected. Second, an appropriate subset of the edges of G_β must be colored.

The main technique introduced in this chapter is to choose the color used in the first iteration and the "appropriate subset" to be colored with some care. By doing this we are able to obtain several extensions of Galvin's theorem.

Our first result is Theorem 34 (page 47). Here we exhibit two distinct list colorings for any given n-listing. This expands upon the single coloring guaranteed by Galvin's theorem, and generalizes the main result of [21].

Our second result is Theorem 35 (page 49). Here we construct an $\mathcal{L}$-coloring for a more restrictive type of listing: Namely, a listing where all lists save for one have size n and a special "weakened" list has size $n-1$.

Lemma 33 (page 46) is the main technical lemma of this chapter. It establishes the existence of special latin squares that facilitate our careful choices of colors and subsets. Lemma 33 is morally a sort of converse of the Gale-Shapley theorem: It states that if we have a bipartite graph with a certain specified matching, we can construct a preference system on the vertices for which that matching is stable.

Throughout this chapter, and particularly in the proof of Lemma 33, we will employ classical results from the theory of bipartite matchings. We will also require several results about completions of partial latin squares. For the most part we include the proofs of these results. We will also make extensive use of the Gale-Shapely theorem and the so called *Kernel Lemma* of Bondy, Boppana, and Siegel.

4.1 Preliminaries

We now introduce the notations and definitions pertinent to this chapter. Our discussion of coloring in Subsection 4.1.1 follows [12]. Our description of oriented graphs and the kernel lemma follows Chapter 17 of [41]. Theorem 24 originates in [17] where the terminology of stable matchings is introduced. The terminology of Subsection 4.1.5 is nonstandard but is encompassed by the material of Chapter 17 in [41]. The definition of the Hungarian forest and the statement and proof of Lemma 31 are reproduced from [27].

4.1.1 Coloring

$\mathcal{A}$ is a finite set whose elements are called **colors**. We will denote these colors by the greek letters α, β, and γ.

A **coloring** of a graph $G = (V, E)$ is a function $c : V \to \mathcal{A}$ so that $c(v) \neq c(u)$ if v and u are adjacent. A family of sets $\mathcal{L} = \{L_v\}_{v \in V}$ with $L_v \subseteq \mathcal{A}$ is a **listing** for G. The elements of $\mathcal{L}$ are called **lists**. If $|L_v| = k$ for all $L_v \in \mathcal{L}$, then $\mathcal{L}$ is a **k-listing**. A coloring is an **$\mathcal{L}$-coloring** if $c(v) \in L_v$ for all $v \in V$. If the listing $\mathcal{L}$ has not not been named, we will implicitly refer to an $\mathcal{L}$-coloring as a **list coloring**. The **chromatic number** $\chi(G)$ is the smallest integer k so that $|\mathcal{A}| = k$ and there exists a coloring $c : V \to \mathcal{A}$. The **list chromatic number** of G, denoted $\mathrm{ch}(G)$ is the smallest integer k so that for each k-listing $\mathcal{L}$ there exists an $\mathcal{L}$-coloring. If $\mathcal{L}$ is a listing for which there are no $\mathcal{L}$-colorings, then $\mathcal{L}$ is an **unsatisfiable** listing. Let G be a graph (or digraph) equipped with a listing $\mathcal{L}$. We define the **color subgraph** G_α to be the subgraph of G induced by the set of vertices $\{v : \alpha \in L_v\}$.

We give a set of completely analogous definitions for edge coloring.

An **edge coloring** of a graph $G = (V, E)$ is a function $c : E \to \mathcal{A}$ so that $c(e) \neq c(f)$ if e and f are incident. A family of sets $\mathcal{L} = \{L_e\}_{e \in E}$ with $L_e \subseteq \mathcal{A}$ is an **edge listing** for G. The elements of $\mathcal{L}$ are called **lists**. If $|L_e| = k$ for all $L_e \in \mathcal{L}$, then $\mathcal{L}$ is a **edge k-listing**. An edge coloring is an **$\mathcal{L}$-edge coloring** if $c(e) \in L_e$ for all $e \in E$. If the listing $\mathcal{L}$ has not not been named, we will implicitly refer to an $\mathcal{L}$-coloring as a **edge list coloring**. The **chromatic index** $\chi'(G)$ is the minimum, over all colorings of G, of $|c(E)|$. The **list**

chromatic index of G, denoted $\mathrm{ch}'(G)$ is the smallest integer k so that for each edge k-listing $\mathcal{L}$ there exists an $\mathcal{L}$-coloring. If $\mathcal{L}$ is an edge listing for which there are no $\mathcal{L}$-colorings, then $\mathcal{L}$ is an **unsatisfiable** edge listing. Let G be a graph equipped with an edge listing $\mathcal{L}$. We define the **edge color subgraph** G_α to be the subgraph of G induced by the set of edges $\{e : \alpha \in L_e\}$.

When it is is contextually clear that we are working with an edge list assignment we will sometimes omit the prefix "edge" from the above definitions.

Finally, in Chapter 5 we will make use of a result from [24] that is phrased in terms of an analogous definition for hypergraph edge coloring. If $\mathcal{H}$ is a hypergraph we denote by $\chi\mathcal{H}$ the edge list chromatic index of a hypergraph H, which is defined to be the minimum cardinality k so that for every assignment of k colors to the edges of $\mathcal{H}$ there is a valid assignment of one color from each list to the corresponding edge, so that no two incident edges receive the same color.

4.1.2 Stable Matchings

Whenever $G = (V, E)$ is a bipartite graph we let X and Y denote the color classes of the bipartition. A vertex named x or x_i is always an element of X. A vertex named y or y_i is always an element of Y. All figures are drawn so that vertices of X are at the top of the figure and vertices of Y are at the bottom.

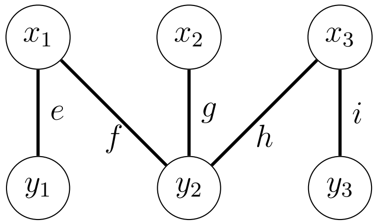
Let $E_G(v)$ denote the set of all edges in G incident with vertex v. A **preference graph** (G, P) is a bipartite graph G, together with a collection P of (reflexive) total orders $>_v$ on the sets $E_G(v)$. If $e >_v f$ we say that v **favors** e over f.

Given a preference graph (G, P) and a matching $M \subseteq E$, we say that an edge $e \in E(G)$ is **stable** if at least one end (say v) of e favors an edge of M over e. In this case, we say that the edge e is M**-stable**, and that v **stabilizes** e. If e is not M-stable then e is M**-blocking**. A matching M is **stable** if every edge of G is M-stable. Note that the total orders imposed by the preference system P are reflexive, and so each edge of M is stabilized by both of its ends regardless of the choice of G, P, or M. The main result of [17] is to introduce the notion of stable matchings and prove the following.

Theorem 24 (Gale-Shapley) *There exists a stable matching for any preference graph.*

4.1.3 Oriented graphs

Given a graph $G = (V, E)$, an **orientation** of G is a digraph $D = (V, A)$ such that if $\{x, y\} \in E$ then exactly one of the arcs (x, y) or (y, x) is in A. Given an orientation D and vertex $v \in V(D)$, the vertices u such that $(v, u) \in A$ are the **out-neighbors** of v. The



v	$>_v$
x_1	$e > f$
x_2	g
x_3	$h > i$
y_1	e
y_2	$f > g > h$
y_3	i

Figure 4.1: A bipartite graph G and preference system P. The matching $\{e, g, i\}$ is stable for this preference system. In particular, f is stabilized by x_1, and h is stabilized by y_2.

out-degree of v **in** G is the number of out-neighbors of v in G; it is denoted $\delta_G^+(v)$. We omit the subscript when the underlying graph is clear from context.

A **kernel** of D is a set $K \subseteq V$ satisfying two properties:

1) Independence: $v, u \in K$ implies neither $(v, u) \in A$ nor $(u, v) \in A$.

2) Absorption: For all $w \in V \smallsetminus K$, there is some $x \in K$ with $(w, x) \in A$.

A digraph D is **kernel perfect** if each induced subgraph of D contains a kernel.

Lemma 25 (Bondy-Boppana-Siegel) *If D is kernel perfect, and $\mathcal{L}$ is a listing for D in which $|L_v| \geq \delta_D^+(v) + 1$ for all $v \in V$, then there is an $\mathcal{L}$-coloring of D.*

Proof. We proceed by induction on $n = |V(D)|$. The cases $n = 1, 2, 3$ are easy to verify. Suppose $|L_v| \geq \delta_D^+(v) + 1$ for all $v \in V(D)$ and let α be some color occurring in at least one list. Consider the color sub(di)graph D_α induced by the set of vertices $\{v : \alpha \in L_v\}$. Since D is kernel perfect, D_α contains a kernel K_α. Let D_1 be the graph induced by $V(D) \smallsetminus K_\alpha$ and define the listing $\mathcal{L}' = \{L'_v\}_{v \in V(D_1)}$ by

$$L'_v = \begin{cases} L_v \smallsetminus \{\alpha\} & \text{if } v \in D_\alpha \\ L_v & \text{otherwise} \end{cases}$$

We claim that $|L'_v| \geq \delta_{D_1}^+(v) + 1$ for all $v \in V(D_1)$.

Certainly this is true for all $v \in V(D_1) \smallsetminus V(D_\alpha)$ since for these vertices, $|L'_v| = |L_v|$, and since $\delta_{D_1}^+(v) \leq \delta_D^+(v)$.

Since K_α is a kernel of D_α, we have $\delta_D^+(u) - 1 \geq \delta_{D_1}^+(u)$ for each vertex $u \in V(D_\alpha) \smallsetminus K_\alpha$. Moreover, $|L_u| - 1 = |L'_u|$ for such vertices. Thus the hypothesis of the lemma is satisfied for all vertices in D_1. By induction, there is an $\mathcal{L}'$-coloring c_1 of D_1. Since α is not in any list in $\mathcal{L}'$, the map c defined by

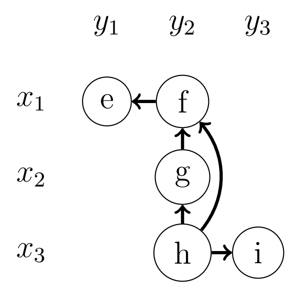


Figure 4.2: The orientation $\Lambda(G, P)$ of the preference graph (G, P) shown in Figure 4.1.

$$c(v) = \begin{cases} \alpha & \text{if } v \in K_\alpha \\ c_1(v) & \text{if } v \in V(D) \smallsetminus K_\alpha \end{cases}$$

is an $\mathcal{L}$-coloring as desired. $\square$

4.1.4 A convenient correspondence

Galvin's resolution of the Dinitz problem is an elegant synthesis of Theorem 24 and Lemma 25. The key element in combining these results can be articulated as the following correspondence.

Let $G = (V, E)$ be a bipartite graph and let $\Lambda(G)$ be the line graph of G. Let P be a preference system for G.

Each vertex $v \in V(G)$ corresponds to a clique C_v in $\Lambda(G)$ whose vertices are the members of $E_G(v)$. Thus, since G is bipartite, the preference system P induces a total order on the vertices of each clique C_v in $\Lambda(G)$.

We define the orientation $\Lambda(G, P) = (E, A)$ to be the digraph with vertex set $E(G)$ and arc set $A = \{(e, f) : \{e, f\} \in C_v, e >_v f\}$. See Figure 4.2.

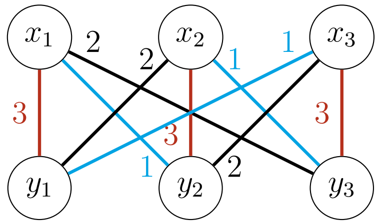
Lemma 26 *S is stable with respect to (G, P) if and only if S is a kernel of $\Lambda(G, P)$.*

Proof. From the definition of $\Lambda(G, P)$, the following are equivalent. The result follows.

(1) S is a matching and for each edge $e = (x, y) \in E(G) \smallsetminus S$, there is an edge $f \in S$ for which either $f >_x e$ or $f >_y e$.

(2) S is an independent set and for each vertex $e \in V(\Lambda(G, P)) \smallsetminus S$, there is a vertex $f \in S$ for which $(e, f) \in A(\Lambda(G, P))$. $\square$

Armed with this correspondence Galvin's approach is as follows. Choose a preference system P so that in the orientation $\Lambda(G, P)$, we have $\delta^+(v) = n - 1$ for all v. By Theorem 24, this orientation is kernel perfect. Since $\delta^+(v) + 1 = n$, any n-listing satisfies the inequality



v	$>_v$
x_1	$x_1y_1 > x_1y_3 > x_1y_2$
x_2	$x_2y_2 > x_2y_1 > x_2y_3$
x_3	$x_3y_3 > x_3y_2 > x_3y_1$
y_1	$x_3y_1 > x_2y_1 > x_1y_1$
y_2	$x_1y_2 > x_3y_2 > x_2y_2$
y_3	$x_2y_3 > x_1y_3 > x_3y_3$

Figure 4.3: A preference map Φ on $K_{3,3}$ and the preference system $[\Phi]$. Notice that vertices in X favor edges assigned larger numbers, whereas vertices in Y favor edges assigned smaller numbers.

$|L_v| \geq \delta^+(v) + 1 = n$ for all v. Lemma 25 implies that there is a list coloring for any such n-listing.

The final piece of the puzzle is of course to construct an appropriate preference system P. Galvin's approach is to use a preference system naturally associated to a 1-factorization of $K_{n,n}$. This is equivalent, via the correspondence just described, to the use a latin square of order n.

For our purposes we must either translate the idea of latin squares to the setting of bipartite graphs or translate the matching theory of bipartite graphs to setting of their line graphs. We have chosen the former. For this reason the definitions and notations in subsection 4.1.7 are quite standard, whereas those in the following section are not.

4.1.5 Preference maps and completions

Like Galvin, we will construct our preference systems from 1-factorizations of $K_{n,n}$. Because we wish to impose extra conditions on these preference systems, it will be useful to build our 1-factorizations via a two step process.

Let $[n]$ denote the set $\{1, 2, ...n\}$, let $G = (V, E)$ be a subgraph of $K_{n,n}$, and let S be a subset of E. A **partial preference map** $\phi : S \to [n]$ is an edge coloring of the subgraph (V, S) whose set of colors is $[n]$. If $S = E$ then ϕ is a **preference map**. We reiterate that if $G = K_{n,n}$ then a partial preference map on G and a partial latin square of order n are essentially the same notion.

If Φ is a preference map, the **preference system induced by** Φ is denoted $[\Phi]$ and defined as follows. See Figure 4.3.

For $x \in X$, and each pair of edges $e = \{x, y\}$, $f = \{x, y'\}$

$e >_x e'$ if and only if $\Phi(e) \geq \Phi(f)$.

For $y \in Y$, and each pair of edges $e = \{x, y\}$, $g = \{x', y\}$

(a) A completable partial preference map. (b) An incompletable partial preference map.

Figure 4.4: The 1-factorization in Figure 4.3 is a completion for the partial preference map shown in (a). Note that the preference system induced by this map is the system given in Figure 1. In (b), the only 1-factor of $K_{3,3}$ that contains both Red edges also contains the blue edge, thus no 1-factorization completes this partial preference map.

$e >_y e'$ if and only if $\Phi(e) \leq \Phi(g)$.

A **1-factorization** $F = (F_1, \ldots, F_n)$ of the complete bipartite graph $K_{n,n}$ is a partition of the edges into n perfect matchings F_i, called **1-factors**. We may also consider F to be a preference map for $K_{n,n}$ given by the function $F(e) = i$ if $e \in F_i$.

A partial preference map ϕ is **completable** if there is a 1-factorization $(F_1, \ldots, F_n)$ of $K_{n,n}$ such that for all $i \in [n]$ we have $e \in F_i$ implies $\phi(e) = i$. In this case, we call $(F_1, \ldots, F_n)$ a **completion** of ϕ. We will denote partial preference systems by the lower case greek letters ϕ and ψ. We denote their completions by Φ and Ψ respectively. If ϕ is a completable partial preference map, and M is a matching that is stable with respect to any completion of ϕ, we say that M is stable with respect to ϕ.

If ϕ is a completable partial preference map, and Φ is a completion of ϕ, then we say that the preference system induced by Φ is **latin**. Most of the work in this chapter will be in constructing completable preference maps with certain special properties and then applying Galvin's theorem using the associated latin preference systems.

In Section 4.2 we will employ the following result from the theory of completable partial preference maps.

Theorem 27 (Smetaniuk) *If G is a bipartite graph, $S \subseteq E(G)$, $\phi : S \to [n]$ a partial preference map, and $|S| \leq n - 1$, then ϕ is completable.*

In reality, we will only require the much weaker statement that for $n \geq 4$, if $|S| \leq 3$ then ϕ is completable. For another application we require a different special case of Smetaniuk's theorem: If ϕ is a partial preference map whose domain is a matching of size at most $n - 1$, and if ϕ assigns at most two colors to the edges of M, then ϕ is completable. The proof of this statement is pertinent to our discussion in Section 4.3, and therefore included.

Lemma 28 *Let M be some matching of size at most $n - 1$ in $K_{n,n}$ and let $\phi : M \to \{i, j\}$ be a partial preference map. If $n \geq 3$ then ϕ is completable.*

Figure 4.5: The two cases from Lemma 28. The 1-factor F consists of all vertical edges. Boldface vertical edges are in M and dashed vertical edges are not. The sets T_i and T_j are colored Green and Purple respectively.

Proof. We show that there are disjoint 1-factors F_i containing $\phi^{-1}(i)$ and F_j containing $\phi^{-1}(j)$. The result then follows from Hall's theorem.

Let F be a 1-factor of $K_{n,n}$ that contains M. If one of $\phi^{-1}(i)$ or $\phi^{-1}(j)$ is empty, then F is a 1-factor containing the other. If both are empty, there is nothing to show. Thus we assume neither is empty.

Case 1. $|M| \leq n-2$. Then there is some partition of F of the form $F = S_i \dot{\cup} S_j$ where $\phi^{-1}(i) \subseteq S_i$ and $\phi^{-1}(j) \subseteq S_j$ and both S_i and S_j have size at least 2. The subgraphs G_i and G_j induced by S_i and S_j are regular complete bipartite graphs with parts of size at least 2. Thus, by Hall's theorem, G_i contains a perfect matching T_i disjoint from S_i and G_j contains a perfect matching T_j disjoint from S_j. The sets $F_i = S_i \cup T_j$ and $F_j = S_j \cup T_i$ are the desired 1-factors. See Figure 4.5 (a).

Case 2. $|M| = n-1$. Then there is a single edge $e \in F$ that is not in the image of ϕ. Let G_i be the subgraph induced by the edges $\phi^{-1}(i) \cup \{e\}$ and G_j the subgraph induced by the edges $\phi^{-1}(j) \cup \{e\}$. Again, G_i and G_j are regular complete bipartite graphs with parts of size at least 2. Thus G_i contains a perfect matching T_i disjoint from $\phi^{-1}(i) \cup \{e\}$ and G_j contains a perfect matching T_j disjoint from $\phi^{-1}(j) \cup \{e\}$. Note that e is the only edge in $E(G_i) \cap E(G_j)$ and that e is not in $T_i \cup T_j$. Thus the sets $F_i = \phi^{-1}(i) \cup T_j$ and $F_j = \phi^{-1}(j) \cup T_i$ are the desired 1-factors. See Figure 4.5 (b). □

In Section 4.3 we will require an easy corollary of the proof of lemma 28.

Corollary 29 *Let M be some matching in $K_{n,n}$ and let $\phi : M \to \{i, j\}$ be a partial preference map. If $n \geq 3$ and $\min\{|\phi^{-1}(i)|, |\phi^{-1}(j)|\} \geq 2$ then ϕ is completable.*

Proof. As in Case 1 above, we have sets S_i and S_j of size at least two. The same construction applies. □

4.1.6 Galvin's Theorem

We have now amassed the components requisite to prove Galvin's theorem and take this opportunity to do so.

Theorem 30 (Galvin) *For any n-listing $\mathcal{L}$ of $\Lambda(K_{n,n})$ there is an $\mathcal{L}$-coloring.*

Proof. Let Φ be any 1-factorization of $K_{n,n}$ and let $D = \Lambda(K_{n,n}, [\Phi])$ be the associated orientation. Then $\delta^+(v) = n-1$ for each vertex $v \in V(D)$. By Theorem 24, any subgraph of $K_{n,n}$ contains a stable matching, and thus by Lemma 26, each induced subgraph of D contains a kernel. Thus the conditions of Lemma 25 are satisfied and there exists an $\mathcal{L}$-coloring for any n-listing $\mathcal{L}$ of $\Lambda(K_{n,n})$. $\square$

4.1.7 The Hungarian Forest

The main results of this chapter will require the construction of a certain partial preference map in a bipartite graph G relative to a maximum cardinality matching M. To construct this preference map, we will use a classical structural characterization of bipartite graphs.

Let $G = (V, E)$ be a bipartite graph with color classes X and Y. Let M be a maximum matching of G. Let A_0 and B_0 respectively denote the sets of vertices in X and Y that are not M-saturated. A **Hungarian forest rooted at** A_0 denoted F_{A_0} is a forest in G with the following properties.

(i) For each $y \in Y \cap V(F)$, the degree of y in F is 2 and there is an edge of $E(F) \cap M$ incident with y.

(ii) Each component of F contains a vertex in A_0.

(iii) F_{A_0} is a maximal forest with properties (i) and (ii). That is, for each $x \in X \cap V(F)$, all edges incident to x are in F.

The impetus for this definition is that the set $F_{A_0} \cap X$ consists of all vertices of X that can be reached from A_0 by traversing an alternating path. The following characterization can be found in [27].

Lemma 31 *If $F = F_{A_0}$ is a Hungarian forest rooted at A_0, then M is maximum if and only if no vertex in F is adjacent to a vertex in B_0.*

Proof. Suppose there is a vertex $x \in F$ adjacent to some $b \in B_0$. Since $x \in X$, properties (ii) and (iii) imply that that there is an alternating path P from A_0 to x. But then $P \cup \{y\}$ is an M-augmenting path, and M is not maximum.

For the converse direction, suppose no vertex of F is adjacent to a vertex of B_0. Let $S = X \setminus V(F)$ and let $T = Y \cap V(F)$. We claim that $S \cup T$ is a vertex cover of G and that $|S \cup T| = |M|$.

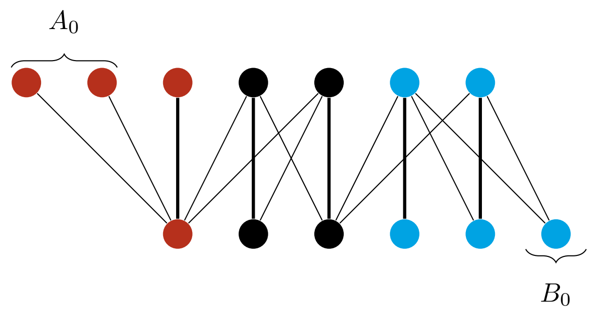


Figure 4.6: The Hungarian forests rooted at A_0 and B_0 and the induced M-partition. M consists of the five bold vertical edges. Vertices of A, B, and C are colored Red, Blue, and Black, respectively.

Each element of S is M-saturated since S is a subset of X disjoint from A_0. Each element of T is M-saturated since B_0 is disjoint from F. Moreover, if $\{s,t\} \in M$ and $t \in F(V) \cap Y$, then by (i), $s \in V(F)$, and so $s \notin S$. Thus no edge of M contains two vertices of $X \cup Y$. Hence $|M| = |S \cup T|$.

It remains to show that $S \cup T$ is a cover of G. Suppose some edge $\{a,b\}$ is not covered by $S \cup T$. Then $a \in V(F)$ and $b \notin V(F)$. Moreover, by hypothesis, $b \notin B_0$. Thus M covers b, with some edge $e = \{a', b\}$. But now F can be extended to a larger forest containing the path $\{a, b, a'\}$, contradicting (iii).

Thus $S \cup T$ is a cover of size $|M|$ and M is maximum. $\square$

If F_{B_0} is a Hungarian forest rooted at B_0, then it can be seen from a symmetric argument that M is maximum if and only if no vertex in F_{B_0} is adjacent to a vertex in A_0. For our purposes, it will be useful to consider forests F_{A_0} and F_{B_0} simultaneously, and to partition the vertices of G according to these forests.

Let G be a bipartite graph, let M be a maximum matching in G, let $A = V(F_{A_0})$, and let $B = V(F_{B_0})$. Note that A and B must be disjoint, since any element of their intersection would be a member of an augmenting path from A_0 to B_0. Thus (A, B, C) where $C = V(G) \setminus (A \cup B)$ is a partition of $V(G)$. We call this the M**-partition** of G.

If $\mathcal{P} = (P_1, P_2, \ldots, P_k)$ is partition of the vertex set of a bipartite graph G, we say that $e = \{x, y\} \in E(G)$ is **type** P_iP_j if $x \in P_i$ and $y \in P_j$. We now note several important properties of M-partitions.

Lemma 32 *If (A, B, C) is the M-partition of a bipartite graph G*

(a) (A, B, C) is a partition of $V(G)$.

(b) No edge of G is type AB, AC, or CB.

(c) Each edge of M is type AA, BB, or CC.

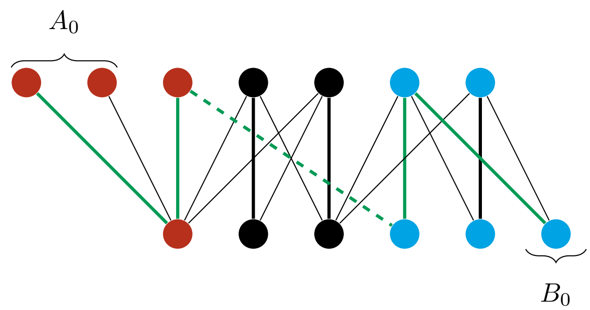


Figure 4.7: If the graph from Figure 4.6 were to contain the dashed edge of type AB, then this edge, together with the other green edges would form an augmenting path with respect to the matching formed by the vertical edges.

Proof of (a). This was noted above.

Proof of (b). Let $f = \{x, y\}$ be an edge of G. If f is type AB then there are alternating paths from A_0 to x and from B_0 to y whose union with f is M-augmenting. This contradicts the maximality of M. See Figure 4.7.

If f is type AC, then there is an alternating path from A_0 to x. By property (iii), $y \in A$. This contradicts (a) since $y \in C$.

If f is type CB, then there is an alternating path from B_0 to y. By property (iii), $x \in B$. This contradicts (a) since $x \in C$.

$\square$

Proof of (c). Observe that the restriction of M to the induced subgraph $H[C]$ is a perfect matching of $H[C]$. Thus there are no edges of M of type AC, CA, BC, or CB. Moreover, by (b), there are no edges (at all) of type AB.

Suppose $e = \{x, y\}$ in M is of type BA. Since $y \in A$, by property (ii) of the Hungarian forest, y is incident to an edge of M that is in F_{A_0}. This edge must be e and so $x \in A \cap B$. This contradicts (a). $\square$

4.2 Non-uniqueness of Galvin colorings

In this section we develop our main technical lemma and use it to show that Galvin's algorithm is always capable of producing at least two list colorings.

Note that the proof of Lemma 25 (page 38) did not make any particular demands on the 1-factorization of $K_{n,n}$ used to obtain a latin orientation. Moreover, in that proof we may choose any color α from the set $\mathcal{A}$, and then any kernel for the subgraph G_α.

The key observation of this chapter is that making these choices carefully yields stronger results. To this end we formulate exactly what is meant by these choices. We define the procedure in terms of edge listings of $K_{n,n}$ so that we may employ the matching theory of bipartite graphs described in Section 4.1.7.

Given an edge-listing $\mathcal{L} = \{L_e\}_{e \in E}$ for $K_{n,n}$, let α be some color in $\mathcal{A}$, let ϕ be a partial preference map for G_α, and let Φ be a completion of ϕ.

Define the **initial choice** $([\Phi], \alpha, M_\alpha)$ to be a triple consisting of the latin preference system $[\Phi]$, the color α, and a stable matching M_α for the preference graph $(G_\alpha, [\phi])$.

Observation 1 *Given an edge-listing $\mathcal{L}$ for $K_{n,n}$, if $([\Phi], \alpha, M_\alpha)$ is any initial choice, then there is an $\mathcal{L}$-coloring $c : E \to \mathcal{A}$ of $K_{n,n}$ so that $c(e) = \alpha$ for all $e \in M_\alpha$.*

Proof. Apply Galvin's theorem using the latin preference system $[\Phi]$, the color α, and the stable matching M_α. □

The main result of this section will be to show that one can always construct two distinct list colorings for a given listing. To do so, we will prove that there are two initial choices $([\Phi], \alpha, M_\alpha)$ and $([\Psi], \alpha, N_\alpha)$ so that $M_\alpha \neq N_\alpha$. The $\mathcal{L}$-colorings that arise from Galvin's algorithm with these initial choices will necessarily be distinct.

We remark that the preference systems $[\Phi]$ and $[\Psi]$ must be chosen with some care. For example, if Φ and Ψ are both completions of the preference map in Figure 4.1, and G_α is the subgraph given in the figure, then there is a unique matching M_α that is stable for this color subgraph. We are thus tasked with constructing completable preference systems systematically. Our main tool in doing so is the following sufficient condition for a matching to be stable.

Lemma 33 *Let H be a subgraph of $K_{n,n}$ and let M be a maximum matching in H. Then M is stable with respect to some latin preference system.*

Proof. We assume H is connected. If not, we apply the following argument to each component. Let (A, B, C) be the M-partition of H. By Lemma 32 (c) we see that M is partitioned into three sets M_A, M_B, and M_C whose members are edges with ends in A, B, and C respectively.

Define the partial preference map $\psi : M \to \{1, n\}$ by

$$\psi(e) = \begin{cases} 1 & \text{if } e \in M_A \\ n & \text{if } e \in M_B \\ n & \text{if } e \in M_C. \end{cases}$$

We first verify that ψ is completable. If $|M| = n$ then $M = M_C$ and the domain of ψ is a 1-factor. Hall's theorem implies that ψ is completable. If $|M| < n$ then the hypothesis of Lemma 28 (page 41) is satisfied, and a completion exists.

Let Ψ be some completion of ψ. We now show that the restriction of Ψ to $E(H)$ induces a preference system on H for which M is stable. Certainly all edges of M are M-stable. Let $f = \{x, y\}$ be an edge of $E(H) \setminus M$.

If f is type BA, BB, or BC, then x is incident with an edge $e_B \in M_B$. Any completion of ψ assigns e_B the value n. Thus e_B is favored by x over all other edges. This implies that f is M-stable.

If f is type AA, BA, or CA, then y is incident with an edge $e_A \in M_A$. Any completion of ψ assigns e_A the value 1. Thus e_A is favored by y over all other edges. This implies that f is M-stable.

If f is type CC, then x is incident with an edge $e_C \in M_C$. Any completion of ψ assigns e_C the value n. Thus e_C is favored by x over all other edges. This implies that f is M-stable.

By Lemma 32 (b) each edge of G is of one of the types addressed above. Thus M is stable in $(H, [\Psi])$. $\square$

We call the partial preference map ψ in the proof of Lemma 33 the **M-stabilizing map**. We will make use of this map presently, as well as in Section 4.3. In Section 4.3 we will also make use of a very similar map

$$\psi'(e) = \begin{cases} 1 & \text{if } e \in M_A \\ n & \text{if } e \in M_B \\ 1 & \text{if } e \in M_C, \end{cases}$$

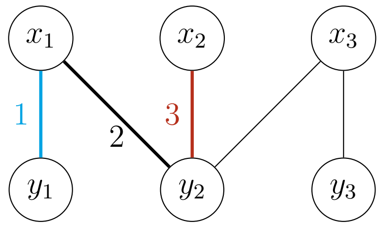
which we call the **alternate M-stabilizing map**.

Note that we may have just as easily carried out the proof of Lemma 33 using the alternate M-stabilizing map. Lemma 28 implies that ψ' is completable. Moreover, it can be seen that the only edges that could be stable with respect to ψ and blocking with respect to ψ' are type CB. We have argued in Lemma 32 (b) that there are no such edges.

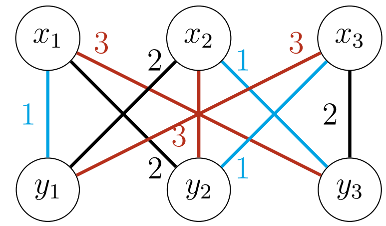
We are ready to prove the main result of this section.

For any color α, if G_α has two (or more) maximum matchings, then Lemma 33 allows us to exhibit two initial choices as desired. On the other hand, if G_α has a unique maximum matching M we will require a different approach. In this case we will construct a completable partial preference map for which M cannot be stable. See Figure 4.8. We then conclude from Theorem 24 that some stable matching N must exist. This allows us to construct two initial choices leading to two distinct colorings: one in which M is colored α, and one in which N is colored α.

Theorem 34 *Suppose $n \geq 3$. If $\mathcal{L}$ is an edge n-listing of $K_{n,n}$ then Galvin's algorithm exhibits two distinct $\mathcal{L}$-colorings of $K_{n,n}$.*



(a) A partial preference map.



(b) A completion of that map.

Figure 4.8: With respect to the partial preference map in (a) the edge x_1y_2 is blocking for any matching that contains both x_1y_1 and x_2y_2. Such a matching is not stable in any preference system induced by a completion of this partial preference map. A completion of this map is given in (b).

Proof: Suppose first that there is some color α in $\mathcal{A}$ so that G_α contains two distinct maximum matchings M_1 and M_2. By Lemma 33, there are latin preference systems $[\Phi]$ and $[\Psi]$ for which M_1 and M_2 are stable respectively. Thus $([\Phi], \alpha, M_1)$ and $([\Psi], \alpha, M_2)$ are initial choices that yield distinct $\mathcal{L}$-colorings.

Suppose now that each color subgraph has a unique maximum matching.

Case 1. There is no color β so that G_β contains a path of length three.

Then G_β is the disjoint union of components isomorphic to $K_{1,k}$ for various values of k. Since M is a unique maximum matching, k is 0 or 1 for each component, and for all choices of β, G_β is a matching. Thus for any function that assigns each edge a color from its list, it is never the case that two incident edges are assigned the same color, so all $(n^2)^n$ such functions are $\mathcal{L}$-colorings.

Case 2. There is a color β such that G_β contains a path of length three.

Let H be a component of G_β containing a path of length three and let f be an edge of M in H, and let e be an edge incident with f. Since e is not in M, the uniqueness of M implies there is some third edge g of M incident with e. Without loss of generality, suppose f and e are incident at a vertex $x \in X$ and e and g are indecent at a vertex $y \in Y$. Consider the partial preference system ϕ defined by $\phi(f) = 1, \phi(e) = 2, \phi(g) = n$.

If $n = 3$, there is a (unique) completion of ϕ, see Figure 4.8 (b). If $n \geq 4$ then ϕ is completable by Theorem 27.

Let Φ be any completion of ϕ. M cannot be a stable matching in the latin preference system $[\Phi]$. In particular, e is M-blocking since x favors e over f and y favors e over g. However Theorem 24 implies the existence of some stable matching N for $[\Phi]$.

By Lemma 33 there is also a latin preference system $[\Psi]$ for which the maximum matching M is stable. Thus $([\Psi], \alpha, M)$ and $([\Phi], \alpha, N)$ are initial choices that yield distinct $\mathcal{L}$-colorings.
$\square$

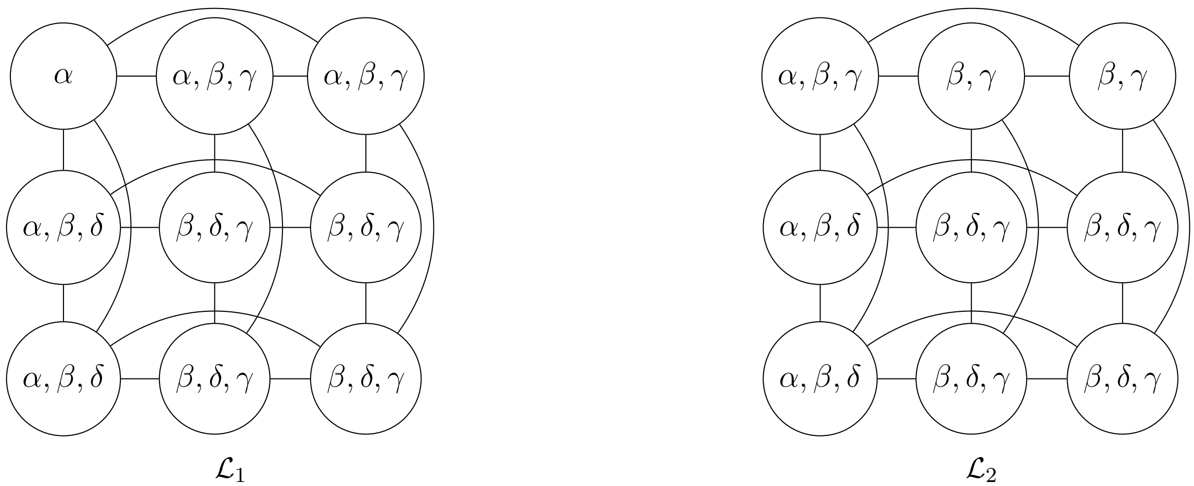


Figure 4.9: Two copies of $\Lambda(K_{3,3})$, equipped with unsatisfiable listings. In each example, the top left vertex must receive color α. The neighbors to the right of this vertex must each receive one of β and γ and the neighbors below must receive one of β and δ. The vertices of the remaining 2×2 sub-graph cannot be properly colored from their lists.

4.3 Weakening the hypothesis

In this section we show that Galvin's theorem can be applied to a more restrictive kind of listing.

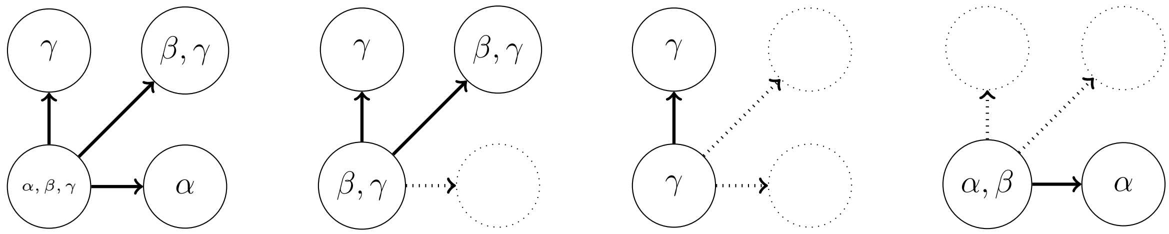
We say that a listing $\mathcal{L}$ of $\Lambda(K_{n,n})$ is **weakened** if there is a single vertex v_0 so that $|L_{v_0}| = n-1$ and $|L_v| = n$ for all other vertices. We call v_0 the **weak** vertex in this listing. The aim of this section is to prove the following.

Theorem 35 *Let $n \geq 3$. Let $\mathcal{L}$ be a weakened listing for $\Lambda(K_{n,n})$. There is an initial choice for which Galvin's algorithm exhibits an $\mathcal{L}$-coloring.*

Before proceeding we note that for $n = 3$, Theorem 35 is tight in two different ways. In Figure 4.9 we demonstrate two listings $\mathcal{L}_1$ and $\mathcal{L}_2$ for $\Lambda(K_{3,3})$. $\mathcal{L}_1$ contains a "doubly weakened" list of size $n-2$, and $\mathcal{L}_2$ contains two weakened lists of size $n-1$. It is not difficult to check that there are no $\mathcal{L}_1$-colorings or $\mathcal{L}_2$-colorings. We will return to these examples in the subsequent chapters.

To prove Theorem 35 we would like to apply Lemma 25 (page 38). We are still free to choose a kernel perfect orientation as in the proof of Galvin's Theorem. However, in any such orientation the inequality $|L_v| \geq |\delta^+(v)| + 1$ will now be violated at the weak vertex.

The key observation of this section is that the violation of this inequality is not fatal to the argument of Lemma 25 so long as it can be addressed in some iteration of Galvin's algorithm before the violating vertex runs out of available colors. We begin with an example and then articulate this idea formally.



(a) (b) (c) (d)

Figure 4.10: The graphs in (a), (b), and (c) violate $|L_v| \geq |\delta^+(v)| + 1$ and illustrate an execution of Galvin's algorithm that does not produce a list coloring. The graph in (d) satisfies the inequality at all vertices.

Suppose we wish to apply Lemma 25 to the (kernel perfect) digraph in Figure 4.10 (a). The inequality $|L_v| \geq |\delta^+(v)| + 1$ is violated at the vertex of degree three. Lemma 25 does not apply. In fact, the following sequence of choices for Galvin's algorithm does not produce a list coloring: We choose the color α and then color the (unique) kernel of D_α, then the digraph induced by the uncolored vertices is as in (b). If we then choose color β and color the (unique) kernel D_β, then the remaining digraph is as in (c). Again, the inequality $|L_v| \geq |\delta^+(v)| + 1$ is violated, and only one of the remaining vertices can receive a color from its list.

Suppose instead that we first choose color γ and then color the (unique) kernel of D_γ. The subgraph remaining is as in (d). This subgraph satisfies $|L_v| \geq |\delta^+(v)| + 1$ for all v. Thus Lemma 25 yields a list coloring of the remaining vertices. Choosing γ first is effective because both vertices of the kernel of D_γ are out-neighbors of v_0. This example illustrates a general observation, which we now formalize.

Observation 2 *Let $\mathcal{L}$ be a weakened listing for $G = \Lambda(K_{n,n})$ with weak vertex v_0. Let D be a latin orientation of G, and let $\alpha \in L_{v_0}$.*

(a) If there is a kernel K of G_α containing v_0, then there is an $\mathcal{L}$-coloring of G.

(b) If there is a kernel K of G_α that does not contain v_0, but K contains two neighbors u, w of v_0, so that (v_0, u) and (v_0, w) are arcs of D, then there is an $\mathcal{L}$-coloring of G.

Proof. In either case, we consider $D \setminus K$ and remove α from the lists of any vertices therein.

If we are in case (a), then since v_0 was the only weak vertex and it is not in $D \setminus K$ then Lemma 25 applies to produce an $\mathcal{L}$-coloring c_1 of $D \setminus K$.

If we are in case (b), then by assumption u and w are in K, and both are out-neighbors of v_0. Thus $\delta^+_{D\setminus K}(v_0) = n - 3$. After removing α from L_{v_0} we have $|L_{v_0}| = n - 2$. Thus $|L_v| \geq |\delta^+_{D\setminus K}(v)| + 1$ for all $v \in V(D) \setminus K$ and Lemma 25 produces an $\mathcal{L}$-coloring c_1 of $D \setminus K$.

In either case

$$c(v) = \begin{cases} \alpha & \text{if } v \in K \\ c_1(v) & \text{if } v \in V(D) \setminus K \end{cases}$$

is an $\mathcal{L}$-coloring of D. $\square$

The details used to prove Theorem 35 will again be most easily addressed in the setting of edge colorings of the complete bipartite graph. Thus we translate the definition of weak listing and the content of Observation 2 into this setting.

We say that an edge listing $\mathcal{L}$ of $K_{n,n}$ is **weakened** if there is a single edge e_0 so that $|L_{e_0}| = n-1$ and $|L_e| = n$ for all other edges. We call e_0 the **weak** edge in this edge listing.

Lemma 36 (Observation 2 for edge coloring) *Let $\mathcal{L}$ be a weakened edge listing for $G = K_{n,n}$ with weak edge $e_0 = \{x_0, y_0\}$. Let P be a latin preference system for G, and let $\alpha \in L_{e_0}$.*

- *(a) If there is a stable matching M of G_α containing e_0, then there is an $\mathcal{L}$-coloring of G.*
- *(b) If there is a stable matching M of G_α that does not contain e_0, but M contains two edges f, g incident to e_0, so that $f >_{x_0} e_0$ and $g >_{y_0} e_0$ in P, then there is an $\mathcal{L}$-coloring of G.*

The first step in applying Lemma 36 is to refine our analysis of M-partitions in the presence of a special edge that occurs in no maximum matching.

Lemma 37 *Let G be a connected bipartite graph and let e_0 be an edge of G that is in no maximum matching of G. Let M be some fixed maximum matching of G and let (A, B, C) be the M-partition of G. Then e_0 is type BA, CA, BC, or CC.*

Proof. We have already argued in the proof of Lemma 32 (b) that there are no edges of type AB, AC, or CB. Thus we must only show that e_0 is not type AA or type BB.

Suppose e_0 is type AA, then by definition there is an alternating path P_a in F_A that contains e_0 and has one of its ends in A_0. The matching $M \Delta E(P_a)$ is a maximum matching of G containing e_0, a contradiction.

Similarly, suppose e_0 is type BB, then there is an alternating path P_b in F_B that contains e_0 and has one of its ends in B_0. The matching $M \Delta E(P_b)$ is a maximum matching of G containing e_0, a contradiction. $\square$

We now show that if $e_0 = \{x_0, y_0\}$ is in no maximum matching then there is a preference system satisfying case (b) of Lemma 36. Note that since e_0 is in no maximum matching, then any maximum matching M must saturate both ends of e_0. In the following lemma we fix a maximum matching M. We also let $f = \{x_0, w_0\}$ denote the edge of M incident with e_0 in X and let $g = \{y_0, z_0\}$ denote the edge of M incident with e_0 in Y.

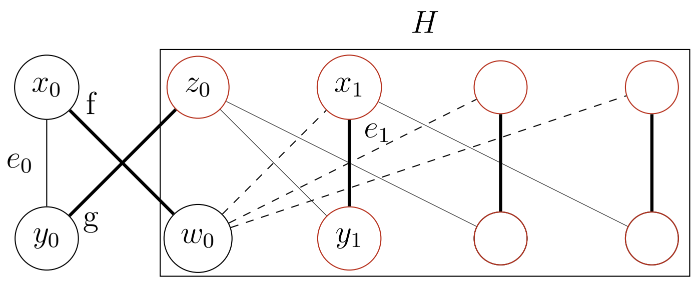


Figure 4.11: The graph G with perfect matching M (bold) but no perfect matching containing e_0. Tthe subgraph H is also pictured. Note that the within H, the Hungarian forest rooted at $X_0 = \{z_0\}$ (vertices in red) contains $M \smallsetminus \{f, g\}$. By Lemma 31 (page 43), w_0 is disjoint from $H \cap X$.

Lemma 38 *Let G be a subgraph of $K_{n,n}$ for $n \geq 3$ and let, M, e_0, f, and g be as above. There is a completable partial preference map ϕ for which $\phi(f) = n$ and $\phi(g) = 1$. Moreover M is stable with respect to any completion of ϕ.*

Proof. From Lemma 37 we see that e_0 is of one of four types. For each type, we give a construction of an appropriate partial preference map based on the maps ψ and ψ' from the proof of Lemma 33 (page 46).

Case 1. e_0 is type BA. Then $f \in M_B$ and $g \in M_A$, thus for the M-stabilizing map ψ defined in Lemma 33, we have $\psi(f) = n$ and $\psi(g) = 1$ as desired.

Case 2. e_0 is type CA. Then $f \in M_C$ and $g \in M_A$, thus for the M-stabilizing map ψ defined in Lemma 33, we have $\psi(f) = n$ and $\psi(g) = 1$ as desired.

Case 3. e_0 is type BC. Then $f \in M_B$ and $g \in M_C$. Here, ψ does not have the desired property. However, for the alternate M-stabilizing map ψ' defined after Lemma 33, we have $\psi'(f) = n$ and $\psi'(g) = 1$ as desired.

Case 4. e_0 is type CC. Then $f \in M_C$ and $g \in M_C$. The strategy of the previous cases is not immediately applicable. This is because the maps ψ and ψ' are constant on M_C. Moreover, each of the previous cases implicitly required the Hungarian forests F_{A_0} and F_{B_0} to be non-empty. This is no longer a valid assumption. Fortunately, the extra structure imposed by the fact that e_0 is in no maximum matching is enough to address both issues.

Let $\nu(G)$ denote the size of a maximum matching in G and let H be the subgraph of G induced by $V(G) \smallsetminus \{x_0, y_0\}$. See Figure 4.11. H has no matching of size $\nu(G) - 1$ since e_0 is in no maximum matching of G. So $M_H = M \smallsetminus \{f, g\}$ is a maximum matching of H.

Let A_1 and B_1 respectively denote the sets of vertices in $X \smallsetminus \{x_0\}$ and $Y \smallsetminus \{y_0\}$ that are not M_H-saturated. We consider the M_H-partition of H, and construct M_H-stabilizing map ψ_H for H as in in Lemma 33. Let M_{A_1} and M_{B_1} be the parts of the partition of M_H containing edges of F_{A_1} and F_{B_1} respectively.

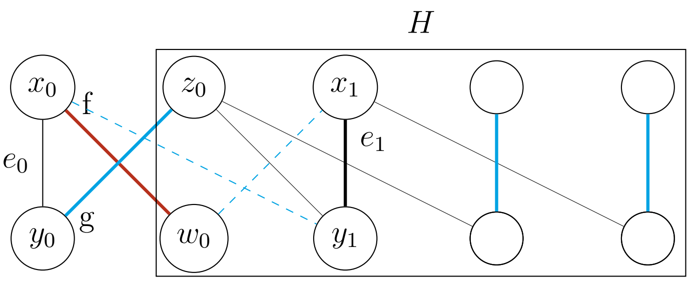


Figure 4.12: The graph G from Figure 4.11 with the partial preference system ψ. Blue, Black, and Red edges take values 1,2, n respectively. The dashed blue edges are not in G; These edges must be blue in any completion of ϕ.

Subcase A. The sets M_{A_1} and M_{B_1} are non-empty. In this case, ψ_H is surjective onto $\{1, n\}$. We extend ψ_H to the partial preference map $\psi_G : M \to \{1, n\}$ by setting $\psi_G(f) = n$ and $\psi_G(g) = 1$.

We claim that ψ_G is completable and that M is stable for any such completion. Since ψ_G restricts to ψ_H on H, each edge of $E(H)$ is M-stable by the argument in the proof of Lemma 33. Each edge of $E(G) \smallsetminus E(H)$ is either incident with f at x_0 or incident with g at y_0. Such edges are stabilized by x_0 and y_0 respectively. We also have that $|\psi_G^{-1}(n)| \geq 2$, and $|\psi_G^{-1}(1)| \geq 2$, thus ψ_G is completable, by Corollary 29.

Subcase B. At least one of M_{A_1} or M_{B_1} is empty. If M_{B_1} is empty then we will consider the alternate M_H-stabilizing map on H, ψ'_H. The case when M_{A_1} is empty is similar except that we consider ψ_H instead of ψ'_H. Since M_{B_1} is empty, $\psi'_H(e) = 1$ for all $e \in M_H$. If $|M| < n$ then we may extend to ψ'_G as above. The domain of ψ'_G has size at most $n-1$. Thus Lemma 28 implies ψ'_G is completable

If $|M| = n$, then ψ'_G assigns 1 to $n-1$ edges of M and n to the remaining edge. Such a map is not completable. Thus we must amend our partial preference map slightly. Let $e_1 = \{x_1, y_1\} \in M_H$, and define a new partial preference map ϕ on G by "swapping" the value on e_1 from 1 to 2. That is

$$\phi(e) = \begin{cases} 1 & \text{if } e \in M \smallsetminus \{e_1, f\} \\ 2 & \text{if } e \in \{e_1\} \\ n & \text{if } e \in \{f\} \end{cases}$$

See Figure 4.12. To see that ϕ is completable, note that there is a unique choice for a 1-factor F_1 of $K_{n,n}$ containing $\phi^{-1}(1)$: namely, the union of $\phi^{-1}(1)$, with the edges $\{x_0, y_1\}$ and $\{w_0, x_1\}$. Since $n \geq 3$, it is possible to construct a 1-factor F_2 disjoint from F_1 that contains e_1 and does not contain f. Hall's theorem then implies that there is a 1-factorization $F = (F_1, F_2, \ldots, F_n)$ such that $\phi^{-1}(1) \subseteq F_1$, $e_1 \in F_2$, and $f \in F_3$. Hence ϕ is completable.

We conclude by showing that M is stable in any completion of ϕ. As in the previous subcase, any edge of $E(G) \setminus E(H)$ is incident with x_0 or y_0 is stabilized by that vertex. Since ψ'_G and ϕ differ only in the value that they assign to e_1, each edge of $E(H)$ that is not incident to e_1 is M-stable with respect to ϕ. Let e_2 be an an edge of H incident with e_1.

Suppose e_2 is incident to x_1. By assumption, F_{B_1} is empty. w_0 is an element of B_1, thus w_0 is not the other end of e_2. Each vertex of $Y \setminus w_0$ is incident with an edge of M that is labeled 1. Thus e_2 is stabilized by that vertex.

Suppose e_2 is incident to y_1. Let Φ be a completion of ϕ. If $\Phi(e_2) > 2$ then e_2 is stabilized by e_1. Thus the only issue arises if $\Phi(e_2) = 1$. But we have noted that in any completion of ϕ, the edge of $K_{n,n}$ incident to y_1 and labeled 1 is the edge $\{x_0, y_1\}$. This edge is stabilized by x_0.

Thus each edge of G is M-stable with respect to any completion of ϕ. $\square$

We now combine Lemmas 36 and 38 to prove our main result of this section.

Proof of Theorem 35.

Let $\mathcal{L}$ be some weak edge listing of $K_{n,n}$ with $n \geq 3$. Let e_0 be the weak edge for $\mathcal{L}$, and choose a color $\alpha \in L_{e_0}$. Suppose there is some maximum matching M for G_α so that e_0 is in M. Then by Lemma 33 there is a latin preference system $[\Phi]$ for which M is stable. Applying Galvin's theorem with initial choice $([\Phi], \alpha, M)$ yields an $\mathcal{L}$-coloring, as noted in case (a) of Lemma 36.

Suppose now that e_0 lies in no maximum matching of G_α. Fix a maximum matching N of G_α and let f and g be as in Lemma 38. By Lemma 38 there is a completable partial preference map ϕ with $\phi(f) = n$ and $\phi(g) = 1$. Moreover, N is stable with respect to $(G_\alpha, [\phi])$. Applying Galvin's theorem with initial choice $([\Phi], \alpha, N)$ yields an $\mathcal{L}$-coloring as noted in case (b) of Lemma 36. $\square$

4.4 An application to minimal listings

We conclude this chapter with two applications of Theorem 35. The first is to give a short alternative proof of Theorem 34.

Proof of Theorem 34. Let $\mathcal{L}$ be an n-listing for $\Lambda(K_{n,n})$. By Galvin's theorem there is some $\mathcal{L}$-coloring c. Choose some vertex v_0 and consider the listing $\mathcal{L}'$ which is identical to $\mathcal{L}$ except that the color $c(v_0)$ has been removed from L_{v_0}. This is a weakened listing, and so Theorem 35 implies that there exists some $\mathcal{L}'$-coloring c'. This coloring is also an $\mathcal{L}$-coloring and is distinct from c. $\square$

For the second application, let $\mathcal{L}$ be a listing for $\Lambda(K_{n,n})$ and define $\mathcal{N}(\mathcal{L})$ to be the set of distinct $\mathcal{L}$-colorings of $\Lambda(K_{n,n})$. We say that an n-listing $\mathcal{L}'$ is **minimal** if $|\mathcal{N}(\mathcal{L}')| \leq |\mathcal{N}(\mathcal{L})|$

for all n-listings $\mathcal{L}$.

It is of interest to determine the minimal n-listings of $\Lambda(K_{n,n})$. We defer further motivation to subsection 6.1.1. In this section we show that the color subgraphs induced by minimal listings do not contain isolated vertices. Our strategy is to show that a listing with such an isolated vertex can be transformed into a listing that admits fewer colorings. To do this, we employ a structural claim that follows easily from Theorem 35.

Lemma 39 *If $\mathcal{L}$ is an n-listing and there exists a color α such that G_α contains an isolated vertex, then $\mathcal{L}$ is not minimal.*

Proof. Let v_0 be some vertex with $L_{v_0} = \{\alpha_1, \alpha_2, ...\alpha_n\}$ such that v_0 is isolated in G_α.

Claim. There is some neighbor u_0 of v_0, some color $\beta \in L_{u_0} \setminus L_{v_0}$ and some $\mathcal{L}$-coloring c so that $c(u_0) = \beta$.

Proof of Claim. Suppose not. Then in each $\mathcal{L}$-coloring, each neighbor of v_0 is colored with an element of L_{v_0}. Let S be one of the cliques of size n containing v_0. Since $\alpha_1 \notin L_s$ for any $s \in S \setminus \{v_0\}$, each of the $n-1$ vertices of $S \setminus \{v_0\}$ receives one of the $n-1$ colors in $\{\alpha_2, ...\alpha_n\}$. Thus, in each $\mathcal{L}$-coloring c, we have $c(v_0) = \alpha_1$. Consider the listing $\mathcal{L}'$ obtained by removing α from L_{v_0}. This is a weakened listing, and so Theorem 35 implies that there is some $\mathcal{L}'$-coloring c'. But such a coloring is an $\mathcal{L}$-coloring for which $c'(v_0) \neq \alpha$, a contradiction.

We now define a new listing $\mathcal{M}$ which is identical to $\mathcal{L}$ with the exception that $M_{v_0} = \{\beta, \alpha_2, ...\alpha_n\}$. We will show that $|\mathcal{N}(\mathcal{M})| < |\mathcal{N}(\mathcal{L})|$ by exhibiting an injection Ω from $\mathcal{N}(\mathcal{M})$ into $\mathcal{N}(\mathcal{L})$. The Claim will show that Ω is not surjective.

The function Ω will map a coloring $f \in \mathcal{N}(\mathcal{M})$ to the identical coloring in $\mathcal{N}(\mathcal{L})$ unless $f(v_0) = \beta$. If $f(v_0) = \beta$, then $\Omega(f)$ will map v_0 to α_1, and change no other color. More formally, for each coloring $f \in \mathcal{N}(\mathcal{M})$ we define the function $g_f(v)$ as

$$g_f(v) = \begin{cases} f(v) & \text{if } v \neq v_0 \\ \alpha_1 & \text{if } v = v_0 \end{cases}$$

and the function $\Omega : \mathcal{N}(\mathcal{M}) \to \mathcal{N}(\mathcal{L})$ as

$$\Omega(f) = \begin{cases} f & \text{if } f(v_0) \neq \beta \\ g_f & \text{if } f(v_0) = \beta \end{cases}.$$

We first note that $\Omega(f)$ is a proper coloring. f is a proper coloring, and $\Omega(f)(v) = f(v)$ except possibly at v_0. Thus the only possible "impropriety" in $\Omega(f)$ is $\Omega(f)(u) = \Omega(f)(v_0)$ for some $u \in N(v_0)$. But again, $\Omega(f)(v_0) = f(v_0)$ unless $f(v_0) = \beta$, so the only possible impropriety occurs when $f(v_0) = \beta$, $\Omega(f)(v_0) = \alpha_1$, and $\Omega(f)(u) = \alpha_1$. But $\Omega(f)(u) = f(u)$ and $\alpha \notin M_u$ by assumption. Thus $\Omega(f)$ is a proper coloring.

Next we show that Ω is injective. Let $\Omega(f_1) = \Omega(f_2)$ be an element of $\mathcal{N}(\mathcal{L})$.

If $\Omega(f_1)(v_0) = \alpha_1 = \Omega(f_2)(v_0)$, then, by the definition of Ω and g_f, we have $f_1(v_0) = \beta = f_2(v)$. This is the only vertex on which f_1 and f_2 could possibly differ, thus $f_1 = f_2$.

If $\Omega(f_1) = \Omega(f_2) = \gamma$ for some $\gamma \neq \alpha$ then $f_1(v_0) = \gamma = f_2(v_0)$ and Ω is the identity on both colorings. Again, $f_1 = f_2$.

On the other hand, Ω is not surjective. By the Claim, there is some $\mathcal{L}$-coloring c so that $c(u_0) = \beta$, for some $u_0 \in N(v_0)$. Since α_1 does not occur in the list of any neighbor of v_0, the coloring c' obtained from c by setting $c'(v_0) = \alpha_1$ is an $\mathcal{L}$-coloring. If $\Omega^{-1}(c')$ were defined, it would color both u_0 and v_0 with color β, a contradiction. Thus $|\mathcal{N}(\mathcal{M})| < |\mathcal{N}(\mathcal{L})|$.

$\square$

Chapter 5

The List Chromatic Number of the Hamming Graph

For integers $n \geq 2$ and $d \geq 1$ define the **Hamming Graph** $H(n,d)$ as follows: The vertex set of $H(n,d)$ consists of all d-tuples with entries in $[n]$. The edge set of $H(n,d)$ consists of all pairs of tuples that differ in exactly one coordinate.

Note that $H(n,2)$ is the graph $\Lambda(K_{n,n})$ discussed in Chapter 4. Galvin's theorem is thus the statement that the list chromatic number of $H(n,2)$ is n. The purpose of this chapter is to give non-trivial bounds on the list chromatic number of $H(n,d)$ for $d \geq 3$. In Section 5.2 we prove that

Theorem 40 *For all $n \geq 2, d \geq 3$,*

$$ch(H(n,d)) \geq n+1.$$

In Section 5.3 we note that $2n-1 \geq \text{ch}(H(n,3))$ by an iterated application of Galvin's theorem. We then apply Kahn's theorem from [24] and prove

Theorem 41 *For all $n \geq 2, d \geq 3$,*

$$n + o(n) \geq ch(H(n,d)).$$

5.1 A Geometric Viewpoint

Our strategy throughout this chapter is to consider the graph $H(n,d)$ geometrically. $H(n,2)$ can be visualized as an $n \times n$ grid of vertices in which two vertices are adjacent exactly if they appear in the same row or the same column. See Figure 4.9 (page 49). This viewpoint extends easily to $H(n,3)$.

For fixed $c_1, c_2, c_3 \in [n]$ we define the **line**, $[c_1, c_2, *]$ to be the subset of triples in $V(H(n,3))$ whose first two coordinates are c_1 and c_2. Thus $[c_1, c_2, *] = \{(c_1, c_2, x) : x \in [n]\}$. We define $[c_1, *, c_3]$ and $[*, c_2, c_3]$ analogously. Note that the subgraphs induced by lines of $H(n,3)$ are precisely the maximal cliques.

Similarly, we define the **plane** $[c_1, *, *]$ to be the subset of $V(H(n,3))$ consisting of all triples whose first coordinate is c_1. We define $[*, c_2, *]$ and $[*, *, c_3]$ analogously. As above, the subgraphs induced by planes are isomorphic to $H(n,2)$.

In the more general setting of $H(n,d)$ we define a k-**flat**, for $k \le d$ to be the subset $S \subseteq V(H(n,d))$ with $d-k$ fixed coordinates and k free coordinates. Again, the subgraphs induced by a k-flats are isomorphic to $H(n,k)$.

5.2 Lower bounds

Both [42] and [15] note that $\mathrm{ch}(G) \ge \chi(G)$, and thus a lower bound on $\mathrm{ch}(H(n,d))$ is immediate from the following fact.

Fact 2 *For all n, d, $\chi(H(n,d)) = n$.*

Proof. We have $\chi(H(n,d)) \ge n$ since $H(n,d)$ contains a clique of size n. Let $c : V \to [n]$ be defined by $c((i_1, ... i_d)) = i_1 + ... + i_d \pmod n$. Each line is an n-clique, and therefore is assigned n distinct colors. Since every edge of $H(n,d)$ is in some line, no pair of adjacent vertices may be assigned conflicting colors. $\square$

In particular, $\chi(H(n,2)) = n$, so Galvin's theorem shows that the trivial lower bound on the list chromatic number is tight. Thus the most optimistic conjecture would be that $\mathrm{ch}(H(n,d)) = n$ as well. This is not the case. In this section we establish $\mathrm{ch}(H(n,3)) > n$ by exhibiting an unsatisfiable n-listing for $H(n,3)$.

Our strategy will be to define an n-listing $\mathcal{L}'$ on $H = H(n,2)$ in which there is a particular vertex v_0, and color $a_0 \in L_{v_0}$ such that no $\mathcal{L}'$-coloring of H assigns a_0 to v. We have already encountered such a listing in disguise: There is no $\mathcal{L}$-coloring for the listing $\mathcal{L}_1$ given in Figure 4.9 (page 49). If we arbitrarily add any two colors to the list of size 1 we obtain an n-listing for which α is in no $\mathcal{L}$-colorings.

Once we have constructed our special listing $\mathcal{L}'$ we imagine $H(n,3)$ as n copies of H, each assigned $\mathcal{L}'$, stacked on top of one another. The vertex (i,j) in $H(n,2)$ will be replaced by an n-clique $[i,j,*]$ in $H(n,3)$, and each vertex in $[i,j,*]$ has list $L_{(i,j)}$.

In this listing we do not know that any particular copy of v_0 is colored a_0. But since the n copies of v_0 form a clique in $H(n,3)$, and since their lists are identical, we are assured that some copy of v_0 is colored a_0. This implies that the n-listing of $H(n,3)$ is unsatisfiable.

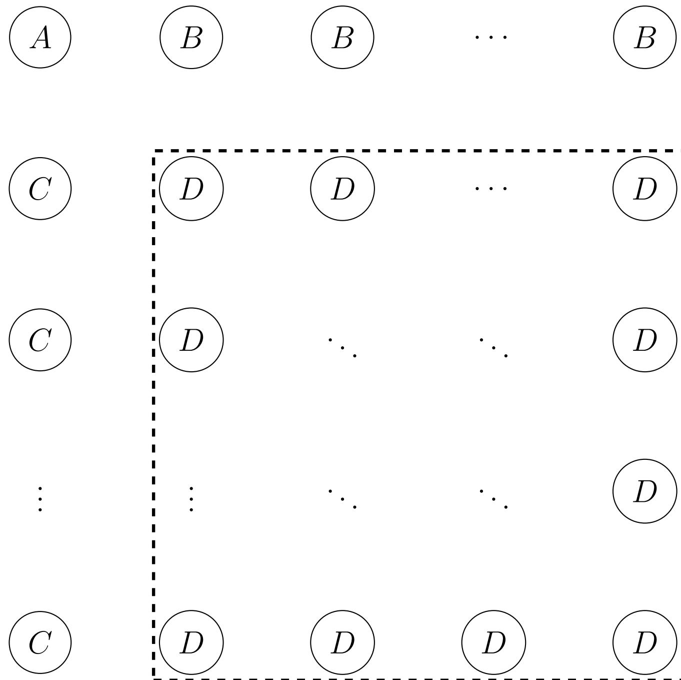


Figure 5.1: The vertices of $H(n,2)$ equipped with the listing $\mathcal{L}'$ given in Lemma 42. Vertices with $L_v = D$ induce a subgraph isomorphic to $H(n-1,2)$.

Our main lemma establishes the existence of an n-listing $\mathcal{L}'$ for $H(n,2)$ in which, upon precoloring vertex $(1,1)$ with α_0, there is no $\mathcal{L}'$-coloring. We let $\mathcal{A} = \{\alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2, ...\beta_{n-1}\}$ be our set of colors. The reader should focus on the colors $\alpha_0, \alpha_1, \alpha_2$ and think of the β_i as "filling in" the rest of the lists as needed.

Lemma 42 *Given a graph $H = H(n,2)$, there exists an n-listing $\mathcal{L}'$ on H so that for some vertex v and color $\alpha_0 \in L_v$, no $\mathcal{L}'$-coloring of $H(n,2)$ assigns α_0 to v.*

Proof. Let $A = \{\alpha_0, \beta_1, ...\beta_{n-1}\}$, $B = \{\alpha_0, \alpha_1, \beta_1, ...\beta_{n-2}\}$, $C = \{\alpha_0, \alpha_2, \beta_1, ...\beta_{n-2}\}$, $D = \{\alpha_1, \alpha_2, \beta_1, ...\beta_{n-2}\}$ and define $\mathcal{L}'$ as follows. See Figure 5.1.

$$\mathcal{L}'(v) = \begin{cases} A & \text{if } v = (1,1) \\ B & \text{if } v \in [*,1] \setminus (1,1) \\ C & \text{if } v \in [1,*] \setminus (1,1) \\ D & \text{for all other } v. \end{cases}$$

Suppose for sake of contradiction that c is an $\mathcal{L}'$-coloring for which $c((1,1)) = \alpha_0$.

Since the line $[*,1]$ is a clique in H, c must assign each element of $B \setminus \{\alpha_0\}$ to exactly one vertex of $[*,1] \setminus (1,1)$. Similarly c must assign each element of $C \setminus \{\alpha_0\}$ to exactly one vertex of $[1,*] \setminus (1,1)$.

Note that D=$(B \cup C) \setminus \{\alpha_0\}$. Let H_D be the subgraph of H induced by the set of vertices $\{v : L_v = D\}$. There are $(n-1)^2$ vertices in H_D and n colors in D. Since $\frac{(n-1)^2}{n} = n-2+\frac{1}{n} > n-2$ there must be some color $\gamma \in D$ that c assigns to at least $n-1$ of these vertices.

Since H_D is isomorphic to $H(n-1,2)$, no two vertices colored γ may appear in the same row or column. Thus each row and each column of H_D contains a vertex colored γ. But each color from D already occurs on some row or column of $H(n,2)$, and this occurrence is not at the vertex $(1,1)$ since that vertex was colored α_0. Therefore c is not a proper coloring. $\square$

Now that we have defined our problematic listing for $H(n,2)$ we complete our argument by "stacking" n copies of $H(n,2)$, each one given this same listing.

Theorem 43 *There is an n-listing $\mathcal{L}$ on $H(n,3)$ for which no coloring is an $\mathcal{L}$-coloring.*

Proof. Define A, B, C, D as in the lemma. For each plane $[*,*,k]$ in $H(n,3)$ define $\mathcal{L}$ by:

$$\mathcal{L}(v) = \begin{cases} A & \text{if } v = (1,1,k) \\ B & \text{if } v \in [*,1,k] \setminus (1,1,k) \\ C & \text{if } v \in [1,*,k] \setminus (1,1,k) \\ D & \text{for all other } v \in [*,*,k]. \end{cases}$$

The line $[1,1,*]$ contains n vertices, each of which has the list A. This line is a clique of $H(n,3)$, so in any proper coloring of $H(n,3)$ the color α_0 must appear on exactly one vertex in $[1,1,*]$, say on the vertex $(1,1,k^*)$. But since the plane $[*,*,k^*]$ has the problematic listing $\mathcal{L}'$ given in the proof of Lemma 42, there is no proper coloring which assigns α_0 to $(1,1,k^*)$. Thus there is no $\mathcal{L}$-coloring. $\square$

Theorem 40 is an immediate corollary since $H(n,d)$ contains a subgraph isomorphic to $H(n,3)$ for $d > 3$.

5.3 Upper bounds

We now consider upper bounds on $\mathrm{ch}(H(n,d))$. We begin with a non-trivial bound on $\mathrm{ch}(H(n,3))$.

Lemma 44 *We have $2n-1 \geq ch(H(n,3))$.*

Proof. Let $\mathcal{L}$ be a $(2n-1)$-listing for $H(n,3)$. Since the plane $[*,*,1]$ induces a subgraph isomorphic to $H(n,2)$, we may apply Galvin's theorem to color this induced subgraph. For each pair (i,j), let $\alpha_{i,j}$ be the color assigned to $(i,j,1)$ and remove $\alpha_{i,j}$ from the list of each vertex in $[1,1,*]$ if it appears.

We then apply Galvin's theorem again to color $[*,*,2]$ and remove colors from lists as above. We continue in this way until we have colored the plane $[*,*,n]$. Since our initial lists were of size $2n-1$, at each step of the procedure we will have at least n available colors for each vertex, thus Galvin's theorem is applicable at each step and produces an $\mathcal{L}$-coloring. $\square$

It seems quite difficult to improve this upper bound for all n. However, a much stronger asymptotic result follows from a powerful theorem of Kahn, see [24].

A hypergraph has **max degree** k if each vertex is in at most k edges. It is **d-uniform** if each edge has size d. It is **linear** if every pair of edges intersect in at most one vertex.

Theorem 45 (Kahn) *If ξ is a d-uniform linear hypergraph with max degree n then $n + o(n) \geq ch'(\xi)$.*

We now use this result to show that when d is fixed, $n + o(n) \geq \mathrm{ch}(H(n,d))$.

Let ξ be a hypergraph and define the line graph $\Lambda(\xi)$ follows. The vertex set of $\Lambda(\xi)$ is the edge set of ξ. A pair (e_1, e_2) is an edge of $\Lambda(\xi)$ if and only if e_1 and e_2 are edges of ξ with non-empty intersection. Note that that $\mathrm{ch}'(\xi) = \mathrm{ch}(\Lambda(\xi))$.

In order to apply Kahn's theorem, we must define some hypergraph $\xi(n,d)$, with $\Lambda(\xi(n,d)) = H(n,d)$.

The most natural choice for $\xi(n,2)$ is $K_{n,n}$. Each vertex of $K_{n,n}$ corresponds to an n-clique in $H(n,2)$. The vertices in $K_{n,n}$ in one color class correspond to the horizontal lines of $H(n,2)$, while the vertices of the other color class correspond to the vertical lines. Each pair of vertices from different color classes uniquely determines an edge of $K_{n,n}$. Analogously, each pair of lines in $H(n,2)$, one horizontal and one vertical, uniquely determine a vertex in $H(n,2)$. This highlights that vertices in $K_{n,n}$ and lines in $H(n,2)$ play an analogous role. It is this phenomenon we preserve in our extension to $\xi(n,d)$.

For ease of explanation we restrict our attention to the graph $H(n,3)$ and think of its vertex set $V = V(H(n,3))$ as the set of points $[n] \times [n] \times [n]$ in three dimensions. We shall refer to the vertices of $H(n,3)$ as **points**. For each point $v \in V$, define $\phi(v)$ to be the set of all lines of V that contain that point. For each point (c_1, c_2, c_3) in V, there are exactly three lines that contain it, namely $[c_1, c_2, *]$, $[c_1, *, c_3]$ and $[*, c_2, c_3]$. So, $|\phi(v)| = 3$ for all $v \in V$.

We now define $\xi(n,d) = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is the set of all lines in V and $\mathcal{E} = \{\phi(v) : v \in V\}$. See Figure 5.2.

Lemma 46 *The hypergraph $\xi(n,3)$ is an n-regular, 3-uniform, linear hypergraph and $\Lambda(\xi(n,3))$ is isomorphic to $H(n,3)$*

Proof. It is immediate from the definition that $\xi(n,3)$ is 3-uniform. It is easy to verify that $\xi(n,3)$ is n-regular.

To see that $\xi(n,3)$ is linear, let $e = \phi(u)$ and $f = \phi(v)$ be edges of $\mathcal{E}$. Suppose $|e \cap f| \geq 2$. Then there is some pair of lines $\ell, m \in \mathcal{V}$ contained in both e and f. It is clear that ℓ and m

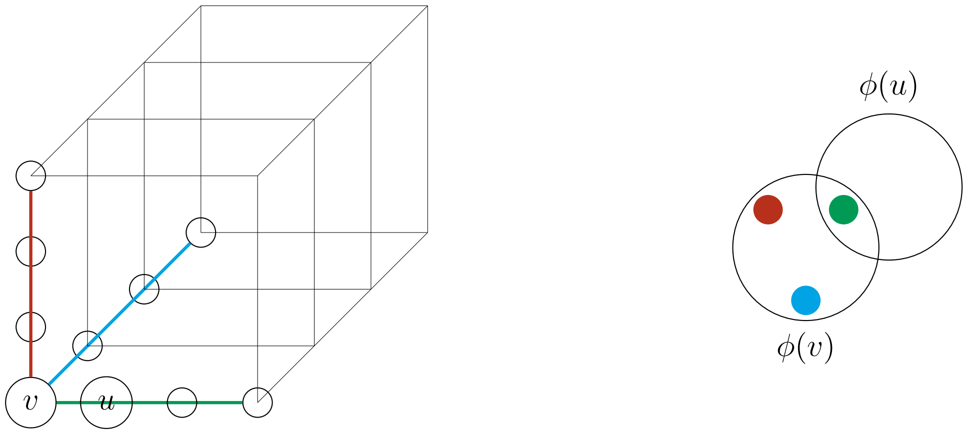


Figure 5.2: Three lines (Red, Blue, Green) intersecting at vertex v in $H(4,3)$, and the corresponding hyperedge of $\xi(4,3)$.

intersect in exactly one point $p = (c_1, c_2, c_3)$. So, since $\ell, m \in e$ we have $e = \phi(p)$ and since $\ell, m \in f$ we have $f = \phi(p)$ and thus $e = f$. Hence $\xi(n,3)$ is linear.

Finally, we must verify that $\Lambda(\xi(n,3)) = H(n,3)$. It is clear that ϕ is a bijection from V to $\mathcal{E}$. We claim that ϕ is a graph isomorphism from $H(n,3)$ to $\Lambda(\xi(n,3))$. The simplest way to prove this is to show that the bijection $\phi^{-1} : V(\Lambda(\xi(n,3))) \to V(H(n,3))$ preserves adjacency.

Let e, f be adjacent vertices in $\Lambda(\xi(n,3))$. Then e and f are incident edges in $\xi(n,3)$. Since $\xi(n,3)$ is linear, $e \cap f = \{\ell\}$ for some line ℓ in $V(H(n,3))$. Thus $\phi^{-1}(e) = v$ and $\phi^{-1}(f) = u$ are points on the line ℓ in $V(H(n,3))$, and so u and v are adjacent vertices in $V(H(n,3))$. Thus ϕ^{-1} is a graph isomorphism. $\square$

Thus Theorem 45 immediately implies the following.

Fact 3 *For all n, $n + o(n) \geq ch(H(n,3))$.*

We remark that no aspect of the proof of Lemma 46 requires that $d = 3$. Thus our argument shows that $\xi(n,d)$ is a d-uniform, n-regular, linear hypergraph whose line graph is $H(n,d)$ for all $d \geq 2$. This establishes Theorem 41 (page 57).

Chapter 6

Conjectures and Open Problems

The contributions of Chapters 4 and 5 probably raise more questions than they answer. In this chapter we address some of those questions explicitly.

We begin by addressing the extensions given in Chapter 4. In particular, we address the limitations that seem to prevent our methods from extending further. Along the way we discuss several conjectures and open questions. We then turn to the content of Chapter 5 and consider open questions in improving the bounds on $\mathrm{ch}(H(n, d))$, and in particular the hypercube $H(2, d)$.

6.1 Further extension of Galvin's theorem

6.1.1 Enumeration of list colorings

In Section 4.4 (page 55) we discussed the structure of minimal k-listings of $H(n, 2)$. Recall that $\mathcal{N}(\mathcal{L})$ is the set of all $\mathcal{L}$-colorings of some fixed graph G.

In [38], Thomassen defines $P_l(G, k) = \min\{|\mathcal{N}(\mathcal{L})|\}$ taken over all k-listings $\mathcal{L}$ of G. He compares this function with the classical chromatic polynomial $P(G, k)$ which evaluates the number of colorings of G from an alphabet of size k.

By considering the k-list assignment in which all lists are identical we see that $P_l(G, k) \leq P(G, k)$. Thomassen asks if there exists a universal constant α so that when $k \geq \mathrm{ch}(G) + \alpha$, we have $P_l(G, k) = P(G, k)$. He notes that it is possible that $\alpha = 0$ when G is $H(n, 2)$.

In [21], Haviar and Ivaska make the explicit conjecture that $\alpha = 0$ when G is $H(n, 2)$, and show that $P_l(H(3, 2), 3) \geq 2$. Their result is subsumed by Theorem 34, where we have shown that $P_l(H(3, 2), n) \geq 2$ for all $n \geq 3$. Our theorem is the furthest progress (to our knowledge) towards resolving their conjecture.

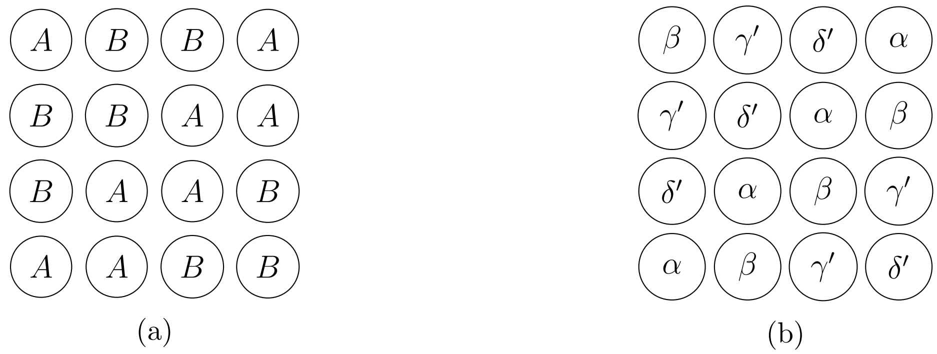


Figure 6.1: A listing (a) for which Galvin's theorem is not capable of producing many $\mathcal{L}$-colorings. In particular, any coloring produced by Galvin's Theorem will be congruent to the coloring in (b).

Note that $P(H(n,2),n)$ is equal to the number of latin squares of order n. This quantity is well understood for small values of n, as well as asymptotically. See [28] and [41] respectively.

Since $P(H(3,2),3) = 12$ there is plenty of room for improvement of Theorem 34 even for small values of n. Unfortunately, our methods do not seem particularly well suited to improving this lower bound further. The primary reason for this is the existence of list assignments for which the application of Lemma 25 yields very restrictive colorings.

Lemma 33 gave us some control over the execution of Galvin's algorithm, but almost all of our control was over the initial choice. Moreover, we were only able to ensure the existence of two kernels for any given color. We begin with an example where this is the best we can do with any initial choice.

Let $A = \{\alpha, \beta, \gamma, \delta\}$ and $B = \{\alpha', \beta', \gamma', \delta'\}$. Consider the 4-listing for $H(4,2)$ in Figure 6.1 (a). Each vertex is assigned either A or B as its list. Note that the set of vertices $\{v : L_v = A\}$ induces an 8-cycle. The set $\{v : L_v = B\}$ does the same. For any latin orientation of $H(n,2)$, the orientation of one of these induced 8-cycles contains a kernel of size four and no smaller kernel. (This is not a difficult fact to check. The details are omitted.) We see that the two kernels exhibited in Lemma 33 are the only kernels that can be found in any application of Galvin's algorithm.

This example is actually far more insidious. By the remark of the previous paragraph, it can be seen that each iteration of the algorithm colors an independent set of size 4. The resulting $\mathcal{L}$-coloring is thus a 4-coloring such as that in Figure 6.1 (b). One can find many other $\mathcal{L}$-colorings that use five or more colors, yet none of these colorings are attainable from Galvin's algorithm. This shows that there are list colorings which Galvin's theorem is incapable of constructing. Moreover, in this example the non-constructible list colorings are much more plentiful.

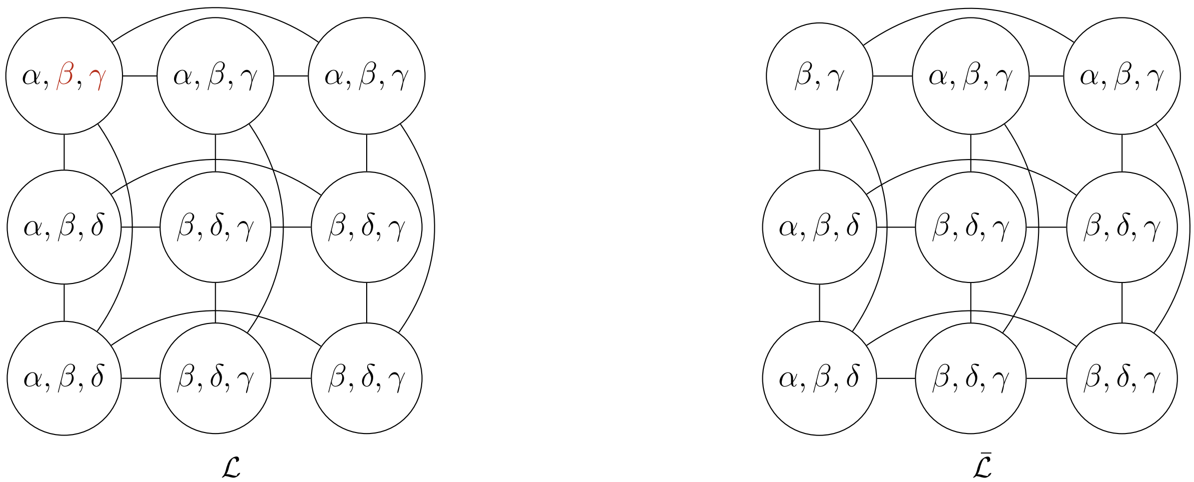


Figure 6.2: A 3-listing and its compression.

6.1.2 A coarser equivalence

The example above illustrates another limitation of our methods. One natural strengthening of Theorem 34 would be to find two different $\mathcal{L}$-colorings that are somehow "more distinct" from one another than those we have exhibited. There are plenty of notions of "more distinct" available for vertex colored graphs, most of them will be at least as coarse as the following.

We say that two $\mathcal{L}$-colorings of $H(n, 2)$ are **congruent** if one can be obtained from the other by a permutation of the names of the colors. One might hope that our technique can be extended to show that each n-listing can produce multiple non-congruent colorings. This is not generally possible. All $\mathcal{L}$-colorings for the listing Figure 6.1 (a) that are obtained from Galvin's theorem are congruent to the coloring given in Figure 6.1 (b).

6.1.3 Compression of lists

Another interesting question is most easily stated from a reformulation of Theorem 35.

If $\mathcal{L}$ is a listing for $H(n, d)$, let $\bar{L}_v \subseteq L_v$ denote the subset of L_v consisting of colors that are used to color v in some $\mathcal{L}$-coloring. We define the **compression of $\mathcal{L}$**, to be the list assignment $\bar{\mathcal{L}} = \{\bar{L}_v : v \in V\}$. Intuitively, the compression of a listing, is the result of "throwing out" any colors that couldn't have been used.

It is immediate that c is an $\mathcal{L}$-coloring if and only if c is an $\bar{\mathcal{L}}$-coloring. Note that the compression of a k-listing is not, in general, a k-listing. Consider the listing from Figure 4.9 (a), redrawn below in Figure 6.2 with symbols β and γ added to the list in the top left. This is a 3-listing, but as discussed in Chapter 5, there is no $\mathcal{L}$-coloring for which the top left

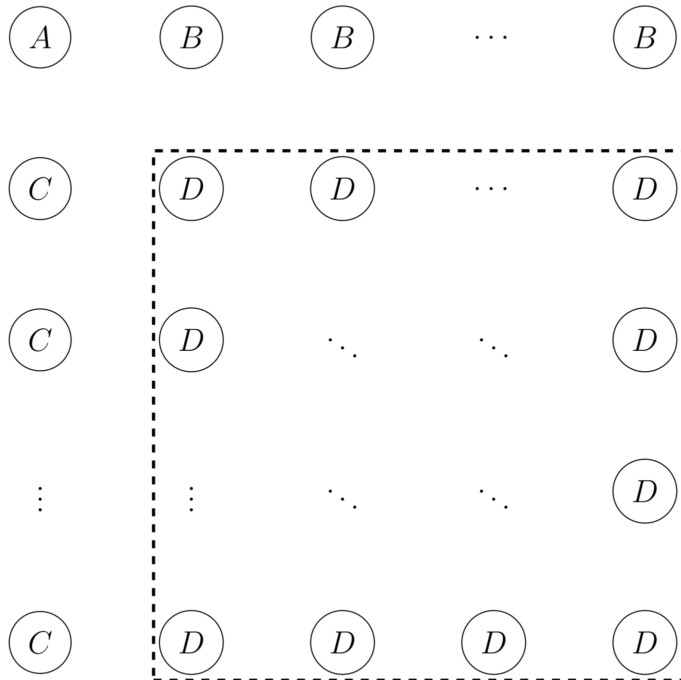


Figure 6.3: Illustration for Fact 4.

vertex is colored α. Thus α is not in $\bar{L}_v$.

Theorem 35 (page 49) implies the following corollary on compressions of listings.

Corollary 47 *For $n \geq 3$, if $\mathcal{L}$ is an n-listing for $H(n,2)$ then each element of $\bar{\mathcal{L}}$ has size at least two.*

Proof. Suppose we have some v_0 with $\bar{L}_{v_0} \in \bar{\mathcal{L}}$ with $|\bar{L}_{v_0}| \leq 1$. If $|\bar{L}_{v_0}| = 0$ then there is no $\mathcal{L}$-coloring, contradicting Galvin's theorem. Suppose $\bar{L}_{v_0} = \{\alpha\}$. Then for any $\mathcal{L}$-coloring, c, so that $c(v_0) = \alpha$ onsider the listing $\mathcal{L}'$ obtained by removing α from L_{v_0}. This is a weakened listing, and so Theorem 35 implies that there is some $\mathcal{L}'$-coloring c'. But such a coloring is an $\mathcal{L}$-coloring for which $c'(v_0) \neq \alpha$, a contradiction. $\square$

One may ask more generally for bounds on the size of elements of the compression. We can augment Lemma 5.3 (page 60) to show the following.

Fact 4 *There is an n-listing $\mathcal{L}$ for $H(n,2)$ whose compression contains a list of size at most $\lceil \frac{n}{2} \rceil$.*

Proof. Set $A = \{\alpha_1, \alpha_2, ...\alpha_{\lfloor \frac{n}{2} \rfloor}, \delta_1, \delta_2, ...\delta_{\lceil \frac{n}{2} \rceil}\}$

$B = \{\alpha_1, \alpha_2, ...\alpha_{\lfloor \frac{n}{2} \rfloor}, \beta_1, \beta_2, ...\beta_{\lceil \frac{n}{2} \rceil}\}$

$C = \{\alpha_1, \alpha_2, ...\alpha_{\lfloor \frac{n}{2} \rfloor}, \gamma_1, \gamma_2, ...\gamma_{\lceil \frac{n}{2} \rceil}\}$

$D = \{\beta_1, \beta_2, ...\beta_{\lceil \frac{n}{2} \rceil}, \gamma_1, \gamma_2, ...\gamma_{\lfloor \frac{n}{2} \rfloor}\}$

Proof. Assign these 4 types of list in the same configuration as Lemma 5.3. If v_0 is the vertex with $L_{v_0} = A$ we claim that $\bar{L}_{v_0}$ does not contain any of the α_i. The argument is similar to that in Lemma 42. If c is a list coloring with $c(v_0) = \alpha_i$ then each color from D occurs on some vertex in the first row and some vertex in the first column. By a pigeonhole argument, some color from D must appear on $n-1$ of the vertices in the subgraph induced by vertices with list D. But this color already occurs in either the first row or first column, and does not occur on v_0. This precludes the existence of an $\mathcal{L}$-coloring. See Figure 6.3. □

We conjecture that this upper bound on the size of the smallest compressed list is tight.

Conjecture 2 *If $\mathcal{L}$ is an n-listing for $H(n,2)$, then each element of $\bar{\mathcal{L}}$ has size at least $\lceil \frac{n}{2} \rceil$.*

A related conjecture comes from the fact that these pathological listings require a great deal of "concentrated effort" at a particular vertex. We conjecture that there is no way to compress two different lists simultaneously.

Conjecture 3 *If $\mathcal{L}$ is an n-listing for $H(n,2)$, then there is at most one list in $\bar{\mathcal{L}}$ of size less than n.*

6.2 k-sheets

We now consider extensions of the work in Chapter 5. The first open problem of note is to improve the inequality $2n-1 \geq \text{ch}(H(n,3)) \geq n+1$. This seems at least as difficult as addressing the problem for an an intermediary class of graphs between $H(n,2)$ and $H(n,3)$ which we call k-sheets.

Define the **k-sheet** $S(n,k)$ to be the subgraph of $H(n,3)$ induced by the vertices of planes $[*,*,1]$, $[*,*,2]$, ...$[*,*,k]$.

As in the proof of Lemma 5.3, we may employ Galvin's theorem iteratively to see that $\text{ch}(S(n,k)) \leq n+k-1$. Moreover, if $\text{ch}(S(n,k+1)) > \text{ch}(S(n,k))$ then $\text{ch}(S(n,k+1)) = \text{ch}(S(n,k))+1$. In particular, $\text{ch}(S(n,2))$ is either n or $n+1$. We conjecture that for $n \geq 3$, the former is correct.

Conjecture 4 *For $n \geq 3$, $ch(S(n,2)) = n$.*

This would imply the following weaker conjecture. We say that two colorings c_1, c_2 are **disjoint** if $c_1(v) \neq c_2(v)$ for all $v \in V(G)$.

Conjecture 5 *For each n-listing of $H(n,2)$, there exist two disjoint list colorings.*

Another question in the same vein comes from the result of Theorem 5.2. Since $H(n,3) \geq n+1$ and $H(n,2) = n$ there is some "threshold value" k_0, where $\text{ch}(H(n,k_0)) = n$, and $\text{ch}(H(n,k_0+1)) = n+1$. It would be very interesting to determine this threshold value.

6.3 The list chromatic number of the hypercube

Another direction is to give bounds on $H(n,d)$ in terms of d rather than n. The simplest instance of this problem is to determine the list chromatic number of $H(2,d)$, which is the d-dimensional hypercube. This problem is open. We write Q_d for $H(2,d)$ throughout this section.

6.3.1 A simple upper bound

We first note an easy upper bound on $\mathrm{ch}(Q_d)$. This follows from an elementary fact about orienting hypercubes, combined with one of the first results on kernel perfect graphs.

Fact 5 *There is an orientation of Q_d so that $\delta^+(v) \leq \lceil \frac{d}{2} \rceil$ for all $v \in V$.*

This follows easily by induction and by considering the hypercube Q_d as four copies of Q_{d-2} joined by a pair of matchings whose union is a set of 2^{d-2} disjoint 4-cycles. The second relevant result is due to Richardson.

Theorem 48 (Richardson) *If D is an oriented digraph containing no odd directed cycles, then D is kernel perfect.*

Putting these results together gives the following.

Lemma 49 *$ch(Q_d) \leq \lceil \frac{d}{2} \rceil + 1$.*

Proof. Let D be an orientation of Q_d as in Fact 4. Since Q_d is bipartite, D contains no odd directed cycles and is thus kernel perfect by Theorem 48. Thus Lemma 25 implies $\mathrm{ch}(Q_d) \leq \lceil \frac{d}{2} \rceil + 1$. □

Since Q_3 is not 2-choosable, we have as an immediate corollary that $\mathrm{ch}(Q_3) = 3 = \mathrm{ch}(Q_d)$. Past this point, the degree of Q_d outstrips the lower bound, and we are again left with ambiguity.

6.3.2 An asymptotic lower bound in d

We have shown in Chapter 5 that $\mathrm{ch}(H(n,d)) \geq n+1$ when $d \geq 3$. Thus the most optimistic conjecture would be that $\mathrm{ch}(H(n,d)) = n+1$ for all $d \geq 3$. This is false. A theorem of Alon shows that $\mathrm{ch}(H(n,d))$ is not independent of d. In particular, it is shown that

$$ch(G) \geq (\frac{1}{2} - o(1)) \log_2 \delta$$

for a graph of minimum degree δ, see [2]. Since $H(n,d)$ is $d(n-1)$ regular, Alon's theorem implies that $\mathrm{ch}(H(n,d))$ increases as a function of d.

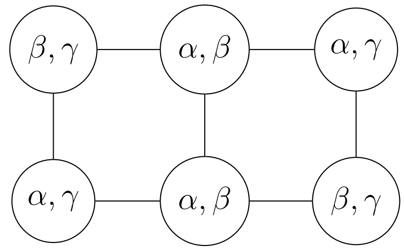


Figure 6.4: A graph exhibiting an unsatisfiable 2-listing.

On the other hand, the rate at which the list chromatic number increases as a function of δ is not necessarily very fast. For instance, in order to certify that a graph of minimum degree δ has list chromatic number at least 4, it can be seen that Alon's result requires

$$\delta > \frac{400}{(\log_2 e)^2} 2^6 \approx 12800$$

(details omitted).

We conclude this section by showing that for the particular instance of the hypercube, the jump to list chromatic number 4 occurs at a much smaller degree.

Fact 6 *If $d \geq 63$ then $ch(Q_d) > 3$.*

Proof. We exhibit an unsatisfiable 3-listing $\mathcal{L}$ for a subgraph $\mathcal{S}$ of Q_{63}. The result follows.

First consider the graph S in Figure 6.4. The 2-listing shown in the figure is not satisfiable. Note also that S is a subgraph of the cube Q_3.

We now choose some subgraph of Q_{63} that is isomorphic to S and call it S_0. Assign to each vertex of S_0 the list $\{\alpha', \beta', \gamma'\}$. By a simple exhaustive search it can be seen that there are exactly sixty list colorings of S_0, which we label $c_1, \ldots, c_{60}$.

We may think of Q_{63} as the cartesian product $Q_{60} \Box Q_3$. That is, we consider Q_{63} to be Q_{60} with each vertex replaced with a copy of Q_3. See Figure 6.5. Thus each vertex of our chosen subgraph S_0 is adjacent to sixty other copies of that vertex each contained within one of sixty copies of S. We label these copies $S_1, \ldots, S_{60}$.

We define a list assignment $\mathcal{L}$ on the graph $\mathcal{S}$ induced by these sixty-one subgraphs as follows. For each coloring c_i, if c_i is the coloring used on S_0, then the vertices of S_0 each "eliminate" one color from the list of the corresponding vertex in S_i. We assign the lists of S_i so that the remaining colors will be as in Figure 6.4 if c_i is the coloring of S_0. See Figure 6.6.

Suppose c is an $\mathcal{L}$-coloring of $\mathcal{S}$, then the restriction of c to S_0 is c_i for some i. But then the restriction of c to S_i is a list coloring of the unsatisfiable 2-listing of S given in Figure 6.4, a contradiction. $\square$

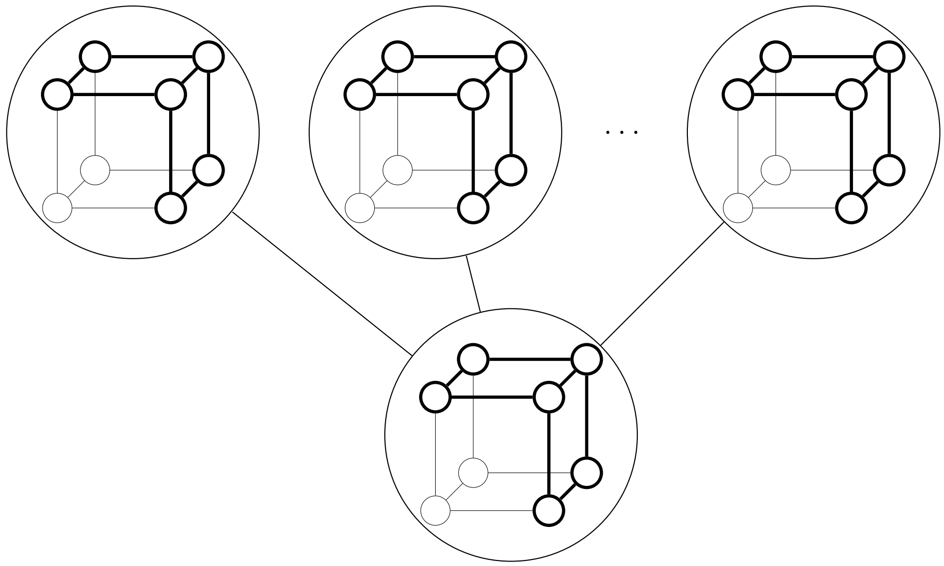


Figure 6.5: A subgraph of Q_{63} consisting of a cube and all of its neighbors. The subgraph $\mathcal{S}$ is induced by the bold vertices.

Note that we have not come close to needing the whole of Q_{63} in our argument. This suggests that perhaps the first value d_0 so that $\mathrm{ch}(Q_{d_0}) > 3$ is much smaller than 63. Lemma 49 implies $d_0 \geq 5$. Tightening this "threshold value" for d_0 is our final open problem.

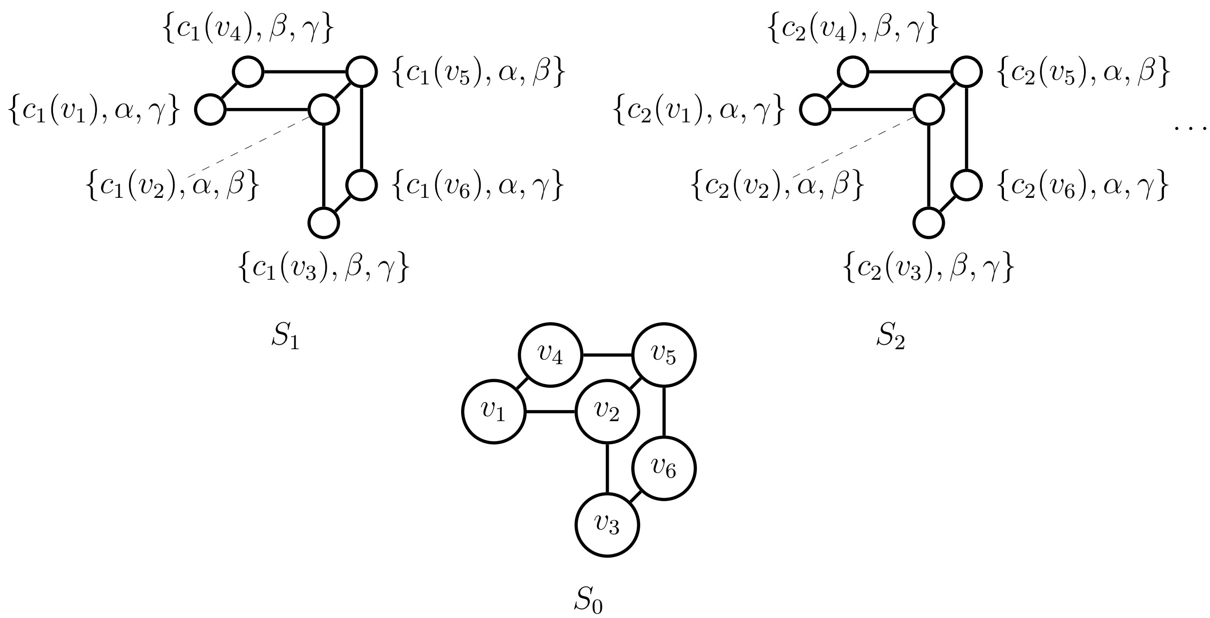


Figure 6.6: Part of the subgraph $\mathcal{S}$ exhibiting the unsatisfiable listing described in Fact 6. Vertices v_1 through v_6 each have list $\{\alpha', \beta', \gamma'\}$.

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