Action of degenerate Bethe operators on representations of the symmetric group

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Degenerate Bethe operators are elements defined by explicit sums in the center of the group algebra of the symmetric group. They are useful on account of their relation to the Gelfand-Zetlin algebra and the Young-Jucys-Murphy elements, both of which are important objects in the Okounkov-Vershik approach to the representation theory of the symmetric group. We examine all of these results over the course of the thesis.

Degenerate Bethe operators are a new, albeit promising, topic. Therefore, we include proofs for previously-unproven basic aspects of their theory. The primary contribution of this thesis, however, is the computation of eigenvalues and eigenvectors of all the degenerate Bethe operators in sizes 4 and 5, as well as many in size 6. For each partition $\boldsymbol{\lambda} \vdash k$ we compute the operators $B_{\ell j}$, where $\ell + j \leq k$, and give the eigenvalues and their corresponding eigenvectors in terms of standard Young tableaux of shape $\boldsymbol{\lambda}$. The number of terms in the degenerate Bethe operators grows very rapidly so we used a program written in the computer algebra system `SAGE` to compute the eigenvalue-eigenvector pair data. From this data, we observed a number of patterns that we have formalized and proven, although others remain conjectural. All of the data computed is collected in an appendix to this thesis.

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1 INTRODUCTION

This chapter motivates our results by giving a history of related work, thereby explaining how our results fit into the current body of knowledge.

1.1 HISTORY

Building on earlier work of Farahat and Higman [6], Jucys proved in 1974 that the center of the symmetric group ring over the integers is generated by elements of the form

$$X_i = (1, i) + (2, i) + \cdots + (i - 1, i) \in \mathbb{C}\mathfrak{S}_n.$$

These are often referred to as YJM elements [1]. In 1981 Murphy discovered these same elements independently of Jucys and used them to give a new construction of Young's seminormal representation. Since then they have played an important role in our understanding of the representation theory of the symmetric group [24]. These YJM elements are notable because they act diagonally on the Young basis of any irreducible representation V^λ. In fact, the eigenvalue of X_i on e_T for some standard tableau T is simply the content of the cell in T containing i. Okounkov and Vershik were able to leverage this property to completely derive the representation theory of the symmetric group without the use of Young tableaux [29].

Another significant property of YJM elements known to both Jucys and Murphy is that elementary symmetric functions in the YJM elements have a simple expression. For $\alpha \in \mathfrak{S}_n$, if we let $t(\alpha)$ be the number of cycles in α then

$$e_k(X_1, \ldots, X_n) = \sum_{\substack{\alpha \in \mathfrak{S}_n \\ t(\alpha) = n-k}} \sigma.$$

Since cycle type is conjugation invariant, $e_k(X_1, \ldots, X_n)$ is central. In fact, all other symmetric functions in the YJM elements are central. If one recalls that the center of the symmetric group algebra is generated by the class sums, the question of what the class expansion of the YJM elements is arises naturally. To state the question a different way, suppose there is some symmetric function F defined as

$$F(X_1, \ldots, X_n) := \sum_{\lambda \vdash n} a_\lambda^F C_\lambda. \tag{1.1}$$

Can we compute coefficients a_λ^F for this function? Far from contrived, this problem arises in many areas of mathematics. The following examples are presented in [9]:

(i) Suppose F is the power sum symmetric function. It is linked with mathematical physics via vertex operators and Virasoro algebra [17].

(ii) Take F to be a complete symmetric function h_k with the coefficients appearing in Equation (1.1) matching those in the asymptotic expansion of unitary Weingarten functions [25]. The coefficients in the asymptotic expansion are the basic building blocks for computing polynomial integrals over the unitary group.

(iii) Suppose we instead ask, "Given a conjugacy class sum C_λ, how do we write it in terms of YJM elements?" This is equivalent to trying to express character values as a symmetric function of the contents. This has been studied without the use of YJM elements in [5, 18].

The modeling of a completely integrable quantum spin chain was first done by Gaudin in 1976 [12]. When it was originally conceived it was formulated as a spin model related to the Lie algebra $\mathfrak{sl}_2$, but by the 1980s it was discovered that any semisimple complex Lie algebra can have a model of this sort associated with it [13, 16].

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$, and $U(\mathfrak{g})$ its universal enveloping algebra. Fix a basis, $\{J_a\}_{a=1}^{d}$, of $\mathfrak{g}$ and let $\{J^a\}$ be the dual basis with respect to a non-degenerate invariant bilinear form on $\mathfrak{g}$. Let $\{z_k\}_{k=1}^{N}$ be a set of distinct complex numbers. The *Gaudin Hamiltonians* are a family of commuting operators in $U(\mathfrak{g})^{\otimes N}$ defined as

$$\Xi \coloneqq \sum_{j\neq i} \sum_{a=1}^{d} \frac{J_a^{(i)} J^{a(j)}}{z_i - z_j} \quad i = 1, \dots, N.$$

For $A \in \mathfrak{g}$, we denote by $A^{(i)}$ the element of $U(\mathfrak{g})^{\otimes N}$ which is the tensor product of A in the ith factor and 1 in all other factors [10].

The *Bethe ansatz* is a method to find pure states or Bethe vectors in quantum integrable systems. In particular, we're concerned with the common eigenvectors of the Gaudin Hamiltonians. The Gaudin Hamiltonians generate a commutative Bethe algebra $\mathcal{B}$, which acts on irreducible representations of the symmetric group in a way that commutes with the action of $\mathfrak{S}_n$.

Fix a nonzero $r \in \mathbb{N}$. Let $V \subsetneq \mathbb{C}[x]$ be a vector subspace of dimension $r+1$. We refer to the space as being *real* when it has a basis consisting of $\mathbb{R}[x]$. There is a unique differential operator D, corresponding to each V, such that the kernel of D is V. This is called the *fundamental differential operator*:

$$D = \frac{\mathrm{d}^{r+1}}{\mathrm{d}x^{r+1}} + \alpha_1(x)\frac{\mathrm{d}^r}{\mathrm{d}x^r} + \cdots + \alpha_r(x)\frac{\mathrm{d}}{\mathrm{d}x} + \alpha_{r+1}(x).$$

The coefficients of the operator are rational functions in x. The space V is real if and only if the fundamental differential operator has real rational functions for all its coefficients.

The Wronskian of functions $f_1, \ldots, f_N$ in x is the determinant

$$\mathrm{Wr}\left(f_1(u), \ldots, f_N(u)\right) \coloneqq \det\begin{pmatrix} f_1 & f_1' & \cdots & f_1^{(N-1)} \\ f_2 & f_2' & \cdots & f_2^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ f_N & f_N^{(i)} & \cdots & f_N^{(N-1)} \end{pmatrix}.$$

The Wronskian of a basis of a subspace V does not depend on the choice of basis up to multiplication by a nonzero number. We refer to the representative monic polynomial as the *Wronskian of* V, and we denote it by $\mathrm{Wr}_V(x)$.

THEOREM 1.1 (*B. and M. Shapiro Conjecture 1993*)

If all the roots of $\mathrm{Wr}_V(x)$ *are real, then the space* V *is real.*

In 2005 Mukhin, Tarasov, and Varchenko proved the B. and M. Shapiro Conjecture by means of the Bethe Ansatz [23]. In 2007 they published another proof, this time establishing a deep connection between Schubert calculus and the representation theory of $\mathfrak{sl}_{n+1}$ [22]. This theorem arises in many disparate places throughout mathematics, from linear series on the projective line to matrix theory [28]. For example, the following theorem from matrix theory follows from a generalization of Theorem 1.1.

THEOREM 1.2

Let $b_0, \ldots, b_n$ *be distinct real and numbers,* $\alpha_0, \ldots, \alpha_n$ *be complex numbers, and*

$$Z = \det\begin{pmatrix} \alpha_0 & (b_0 - b_1)^{-1} & \cdots & (b_0 - b_n)^{-1} \\ (b_1 - b_0)^{-1} & \alpha_1 & \cdots & (b_1 - b_n)^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ (b_n - b_0)^{-1} & (b_n - b_1)^{-1} & \cdots & \alpha_n \end{pmatrix}.$$

If Z *has only real eigenvalues, then* $\alpha_1, \ldots, \alpha_n$ *are real.*

1.2 OUR CONTRIBUTIONS AND OUTLINE

For this thesis, we investigated whether there are any interesting patterns in eigenvector-eigenvalue pairs of degenerate Bethe operators. Before even beginning to answer this question, however, one must first understand what the degenerate Bethe operators are and where they fit into the representation theory of the symmetric group. Of course, that requires that one have a basic understanding of representation theory. Considering all of this, we will approach the topic as follows:

Chapter 2 provides a brief introduction to the representation theory of abstract groups, for the benefit of those who have not yet been exposed to this material.

Chapter 3 builds upon this by developing the representation theory of the symmetric group using the Okounkov-Vershik approach.

Chapter 4 introduces the degenerate Bethe operators themselves. Over the course of learning this material we discovered multiple interesting properties of the algebra generated by these operators. The proofs of those properties are included here.

Once we have established the significance of the degenerate Bethe operators, we proceed to our primary results. Since these operators can be quite large with many terms, we realized we would need some form of automation in order to efficiently detect patterns within the operators' eigenvector-eigenvalue pairs. Thus, we wrote a program in the computer algebra system `SAGE` to generate a fairly large data set.

2 REPRESENTATIONS OF ABSTRACT GROUPS

2.1 GROUP ACTIONS

DEFINITION 2.1. Let G be a group with identity element 1, and X an arbitrary set. A ***group action*** or ***action*** of G on X is a map

$$*: G \times X \longrightarrow X$$

written as

$$(g, x) \mapsto g * x = gx,$$

which satisfies two conditions.

(i) $g(hx) = (gh)x$ for all $g, h \in G$ and all $x \in X$.

(ii) $1x = x$ for all $x \in X$.

Equivalently, we can characterize a group action as a group homomorphism from G to $\mathrm{Sym}(X)$, where $\mathrm{Sym}(X)$ is the group of all permutations of the set X with functional composition as the group operation.

LEMMA 2.1

An action of G on X is tautologically equivalent to a group homomorphism from G to $\mathrm{Sym}(X)$.

PROOF. Given an action of G on X, there is a natural induced group homomorphism

$$\theta: G \longrightarrow \mathrm{Sym}(X).$$

For $g \in G$, define $\theta(g) = \theta_g \in \mathrm{Sym}(X)$ as the map from X to X given by $x \mapsto gx$. This is clearly a permutation since $\theta_{g^{-1}}$ is an inverse. Furthermore, for all $g, h \in G$,

$$\theta_{gh} = \theta_g \theta_h,$$

because as stated in Item (i), $gh(x) = g(hx)$. Hence θ is indeed a morphism of groups.

Conversely, suppose

$$\pi: G \longrightarrow \mathrm{Sym}(X)$$

is a group homomorphism. Then for each $g \in G$ we have a permutation $\pi(g) \in \mathrm{Sym}(X)$ and $\pi(gh) = \pi(g) \circ \pi(h)$. We then define the action of any $g \in G$ on $x \in X$ as

$$gx := \pi(g)(x) \tag{2.1}$$

It follows immediately that this defines an action. Indeed, using Equation (2.1) the homomorphism properties of π translate directly into the defining properties of an action of G on X. For a very straightforward example of a group action, consider the action of $G = \mathrm{Sym}(X)$ on X: $\square$

EXAMPLE 2.2. We define the action of G on X to be $(\phi, x) \mapsto \phi(x)$. The corresponding homomorphism then is simply the identity map.

As we will soon see, group actions and group representations are intimately related.

2.2 GROUP REPRESENTATIONS

Our focus will be on the structure preserving actions of groups on vector spaces over the complex numbers, $\mathbb{C}$.

DEFINITION 2.3. Let V be a finite-dimensional vector space over a field $\Bbbk$[1], and $\mathrm{GL}(V)$ the set of all invertible linear transformations of V. A (finite-dimensional, linear) ***representation*** of G on V is a group homomorphism

$$\rho = \rho_V : G \longrightarrow \mathrm{GL}(V).$$

We can write ρ_g for $\rho_V(g)$. Hence for each $g \in G$ we have $\rho_g \in \mathrm{GL}(V)$, and by the definition of a group homomorphism $\rho_g \rho_h = \rho_{gh}$ and $\rho_{g^{-1}} = \left(\rho_g\right)^{-1}$.

DEFINITION 2.4. The ***dimension*** or (***degree***) of a representation $\rho\colon G \longrightarrow \mathrm{GL}(V)$ is simply the dimension of V as a $\Bbbk$-vector space.

DEFINITION 2.5. Recall that $\ker\rho \lhd G$ and ${}^{G}\!/_{\ker\rho} \cong \rho(G) \subseteq \mathrm{GL}(V)$. If $\ker\rho = 1$, then the representation is said to be ***faithful***. In other words, a faithful representation is one in which the only element that acts trivially on all of V is the identity.

All of this helps to motivate a more concrete explanation of Definition 2.3. Let Mat_n denote the set of all complex $n \times n$ matrices. The ***complex general linear group of degree n***, denoted by GL_n, is the group of all invertible $n \times n$ matrices with respect to multiplication.

DEFINITION 2.6. A ***matrix representation*** of a group G is a map

$$M\colon G \longrightarrow \mathrm{GL}_n$$

which respects the group structure of G. This means that M is a group homomorphism (therefore $M(1) = I_n$) and

$$M(gh) = M(g)M(h)$$

for all $g, h \in G$. The parameter n here is the dimension, or degree, of the representation.

1 For our purposes $\Bbbk$ will always be $\mathbb{C}$

Notice that given a representation of G on V, the moment we specify a basis for V we get a matrix representation of G by simply using the basis to compute $\rho(g)$ as a matrix in the usual way. The significance of this fact is not to be understated: it allows us to take an abstract group and realize it as a concrete group of matrices where the group operation is simply matrix multiplication.

We can also go from matrix representations to representations as defined in Definition 2.3. Indeed, for any n-dimensional complex vector space V there is a natural bijective correspondence between GL_n and the set of all invertible linear transformations of V, $\mathrm{GL}(V)$.

Definition 2.7. A ***linear action*** of a group G on a vector space V is an action with the additional property that

$$g(\lambda v_1 + v_2) = \lambda(gv_1) + gv_2$$

for all $g \in G$, $v_1, v_2 \in V$, and $\lambda \in \Bbbk$. When we have such an action we say that G acts linearly on V.

Lemma 2.2

Representations are linear actions.

Proof. If g acts linearly on V, the map $G \longrightarrow \mathrm{GL}(V)$, defined by $g \mapsto \rho_g$ with $\rho_g \colon v \mapsto gv$, is a representation. Conversely, given a representation $\rho\colon G \longrightarrow \mathrm{GL}(V)$, we get a linear action of G on V by defining $gv \coloneqq \rho(g)v$. □

Definition 2.8. If there is a linear action of G on V, we say V is a ***G-space*** or ***G-module***[2]

The ways of thinking of representations that we have discussed thus far would likely have existed since the time of the early pioneers of representation theory from the nineteenth-century. In modern times the main approach to representation theory uses ***modules*** of ***group algebras***.

Definition 2.9. Suppose R is a ring and with multiplicative identity 1. A ***(left) R-module***[3] consists of an abelian group M written additively and a map from $R \times M$ to M, with the image of $(r, m) \in R \times M$ written rm, such that for all $r, s \in R$ and all $m, n \in M$ we have:

(i) $r(m + n) = rm + rn$

(ii) $(r + s)m = rm + sn$

(iii) $(rs)m = r(sm)$

(iv) $1m = m$

2 The use of this second terminology will be clear momentarily.

3 There is an analogous definition for a right module. But in our case we will be working exclusively over $\mathbb{C}$ which is commutative meaning there is no meaningful distinction to be made between left and right modules.

If $\Bbbk$ is a field, then the concept of a vector space over $\Bbbk$ ($\Bbbk$-vector space) is identical to that of a $\Bbbk$-module. Therefore, one can think of a module as an abstraction of the concept of a vector space allowing for any ring of scalars, not just fields.

If $\Bbbk$ is a field, then A is an ***algebra*** over $\Bbbk$ (or simply a $\Bbbk$- algebra) if it has a ring structure and a $\Bbbk$-vector space structure that share the same addition operation, with the additional property that for any $\lambda \in \Bbbk$ and any $\boldsymbol{a}, \boldsymbol{b} \in A$ we have

$$(\lambda \boldsymbol{a})\boldsymbol{b} = \lambda(\boldsymbol{a}\boldsymbol{b}) = \boldsymbol{a}(\lambda \boldsymbol{b}).$$

Definition 2.10. Given an arbitrary finite group

$$G = \{\boldsymbol{g_1}, \ldots, \boldsymbol{g_r}\}$$

we can define a corresponding $\Bbbk$-algebra, $\Bbbk G$, by taking all formal linear combinations

$$\alpha_1 \boldsymbol{g_1} + \cdots + \alpha_r \boldsymbol{g_r} \text{ with } \alpha_i \in \Bbbk \; \boldsymbol{i} = 1, \ldots, r.$$

We define multiplication in $\Bbbk G$ by simply extending the multiplication in G:

$$\begin{aligned}\left(\sum_{i=1}^{r} \alpha_i \boldsymbol{g_i}\right)\left(\sum_{j=1}^{r} \beta_j \boldsymbol{g_j}\right) &= \sum_{i,j} (\alpha_i \beta_j) \boldsymbol{g_i} \boldsymbol{g_j} \\ &= \sum_{k=1}^{r} \left(\sum_{\substack{i,j \\ g_i g_j = g_k}} \alpha_i \beta_j\right) \boldsymbol{g_k}.\end{aligned}$$

This is referred to as the ***group algebra***. Since the group algebra is a complex vector space, its basis is G. When writing elements of $\Bbbk G$ we write $\boldsymbol{g_i} \in G$ in bold typeface to emphasize that we are thinking of them as basis vectors.

Let V be a complex vector space and let $\rho\colon G \longrightarrow \mathrm{GL}(V)$ be a linear representation of G in V. Then by Lemma 2.2, V is tautologically a G-space. For $g \in G$ and $\boldsymbol{x} \in V$, we define the action $g\boldsymbol{x} := \rho(g)\boldsymbol{x}$. By linearity, this defines $f\boldsymbol{x}$, for $f \in \Bbbk G$ and $\boldsymbol{x} \in V$. In this way, any G-space V can be given a $\Bbbk G$-module structure. Conversely, given a $\Bbbk G$-module since we can think of elements of G as being elements of the group algebra

$$g = \sum_{g \in G} \delta_{ig} \boldsymbol{g} \in \Bbbk G,$$

where δ_{ig} is the Kronecker delta. Any $\Bbbk G$-module structure gives us a unique G-module structure. Thus, G-modules and $\Bbbk G$-modules are the “same” in so far as they both suffice to define a representation.

We reiterate that all of these ways of defining a representation are in fact equivalent. In Definition 2.3 we explained why using matrix representations and representations given by linear transformations $\rho \in \mathrm{GL}(V)$. The same is true for our module definition.

Fix a n-dimensional matrix representation M as in Definition 2.6. Then take V to be the vector space $\Bbbk^n$ of all column vectors of length n. We can define a $\Bbbk G$-module structure on V by simply defining $gv := M(g)v$, the operation on the right being matrix multiplication. Conversely, if V has the structure of a $\Bbbk G$-module then we have a multiplication $gv = \rho_g(v)$ for all $g \in G$. Thus, given a basis B, for V we can simply compute $M(g)$ for $\rho_g \in \mathrm{GL}(V)$ in terms of B in the usual way.

Example 2.11. Any group G admits the ***trivial representation***. The trivial representation is a degree one representation on the one-dimensional $\Bbbk$-vector space, $V = \Bbbk$. We define

$$\rho\colon G \longrightarrow \mathrm{GL}(V)$$
$$g \mapsto \mathrm{id}_V.$$

Hence,

$$\rho(g) \circ \rho(h) = \mathrm{id}_V \circ \mathrm{id}_V = \mathrm{id}_V = \rho(gh).$$

We use $\mathbb{1}_G$ or just $\mathbb{1}$ to refer to the trivial representation of G.

Example 2.12. A prototypical question in representation theory is to characterize all the representations of some dimension for a given group or class of groups. We can use Definition 2.3 to find all one-dimensional representations of the cyclic group of order n.

Let

$$C_n := \left\{1, g, \ldots, g^{n-1}\right\}$$

be the finite cyclic group of order n generated by g, i.e. $g^n = 1$. Definition 2.3 tells us that a complex representations of G consists of a complex finite-dimensional vector space V and an invertible endomorphism $\rho \in \mathrm{GL}(V)$, with $\rho(1) = \mathrm{id}_V$ and $\rho(g^k) = \rho(g)^k$. In other words, ρ needs to be a group homomorphism from G to the multiplicative group of the complex numbers. Notice that this means the map ρ is completely determined by $\rho(g)$. Hence the one-dimensional representations

$$\rho\colon C_n \longrightarrow \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$$

are completely determined by $\rho(g) = \zeta$, where $\zeta^n = 1$. From this it is clear that ζ is an nth root of unity, and therefore there are exactly n nonisomorphic one-dimensional representations of n.

2.3 Mappings of Representations

Definition 2.13. Fix a group G and a field $\Bbbk$. Let V and W be finite dimensional vector spaces over $\Bbbk$ and

$$\rho\colon G \longrightarrow \mathrm{GL}(V) \text{ and } \psi\colon G \longrightarrow \mathrm{GL}(W)$$

be representations of G. Then the linear map

$$T\colon V \longrightarrow W$$

$$\begin{tikzcd} V \arrow[r, "\rho_g"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ W \arrow[r, "\psi_g"] & W \end{tikzcd}$$

Figure 2.1: Commutative diagram of homomorphism of representations.

is a ***G-homomorphism*** if

$$T \circ \rho(g) = \psi(g) \circ T. \tag{2.2}$$

Figure 2.1 shows this relationship pictorially in the form of a commutative diagram.[4] The map T is said to ***intertwine*** ρ and ψ and is referred to as an ***intertwining operator***. The space of all such maps is denoted $\mathrm{Hom}_G(V, W)$.

Definition 2.14. A G-homomorphism is a ***G-isomorphism*** if T is bijective.

Definition 2.15. Two representations ρ and ψ are ***equivalent*** or ***isomorphic*** if there is a G-isomorphism between them.

If T is a G-isomorphism, then we can write Equation (2.2) as

$$\psi = T\rho T^{-1}. \tag{2.3}$$

In this case, we write $\rho \cong \psi$.

Other equivalent notions of representations can similarly provide equivalent formulations of G-isomorphisms.

Earlier we described how Definition 2.3 and Definition 2.6 are equivalent and how one can move back and forth between them. Another way of saying this is that every representation is isomorphic to a matrix representation. Suppose there is a group G, a field $\Bbbk$, an n-dimensional $\Bbbk$-vector space, and a representation $\rho\colon G \longrightarrow \mathrm{GL}(V)$. Fix a basis $\mathcal{B}$ of V. Write $\boldsymbol{v} \in V$ as a column vector with respect to $\mathcal{B}$,:

$$T\colon V \longrightarrow \Bbbk^n$$

$$\boldsymbol{v} \longmapsto [\boldsymbol{v}]_{\mathcal{B}}$$

Now we have a linear G-isomorphism. From there, we get a representation $\psi\colon G \longrightarrow \mathrm{GL}(\Bbbk^n)$ isomorphic to ρ:

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ \Bbbk^n \arrow[r, "\psi"] & \Bbbk^n. \end{tikzcd}$$

4 This diagram demonstrates that no matter which way the diagram is traversed from V in the top left to W in the bottom right, the composition of the maps will be equal.

Thus, in terms of matrices, two representations M and M' are G-isomorphic if there exists some non-singular matrix X such that for all $g \in G$,

$$M'(g) = XM(g)X^{-1}.$$

Finally, in terms of G-actions, we say that the actions of G on V and W are G-isomorphic provided there is some isomorphism $T\colon V \longrightarrow W$ such that

$$gT(\boldsymbol{v}) = T(g\boldsymbol{v})$$

for all $g \in G$ and $\boldsymbol{v} \in V$. This situation is also often described by saying the isomorphism T commutes with the action of G.

2.4 REDUCIBILITY

This section we will examine how representations break down into their constituent parts. To do this, we begin by defining the relevant notion of a subobject.

DEFINITION 2.16. Let $\rho\colon G \longrightarrow \mathrm{GL}(V)$ be a representation. We say $W \subseteq V$ is a ***G-subspace*** if it is a subspace that is $\rho(G)$-invariant, i.e.

$$\rho_g(W) \subseteq W$$

for all $g \in G$.

DEFINITION 2.17. Every G-space V has the trivial G-subspaces $\{0\}$ and V. A representation is ***irreducible*** or ***simple*** if it has no non-trivial G-subspaces. Otherwise it is said to be ***reducible***.

DEFINITION 2.18. A representation $\rho\colon G \longrightarrow \mathrm{GL}(V)$ is said to be ***decomposable*** if there are non-trivial G-invariant subspaces $U, W \subseteq V$ such that

$$V = U \oplus W.$$

We say that ρ is a direct sum $\rho_U \oplus \rho_W$. If no such decomposition exists then ρ is ***indecomposable***.

DEFINITION 2.19. Let $\rho_1\colon G \longrightarrow \mathrm{GL}(V_1)$ and $\rho_2\colon G \longrightarrow \mathrm{GL}(V_2)$ be representations of G with dimensions k_1 and k_2 respectively. Then their ***direct sum*** is the representation

$$\rho = \rho_1 \oplus \rho_2\colon G \longrightarrow GL(V_1 \oplus V_2)$$

of dimension $k_1 + k_2$ given by

$$(\rho_1 \oplus \rho_2)(g)(\boldsymbol{v}_1 + \boldsymbol{v}_2) = \rho(g)\boldsymbol{v} + \rho_2(g)\boldsymbol{v}_2.$$

In matrix form, we write this as

$$M(g) = \begin{pmatrix} M_1(g) & 0 \\ 0 & M_2(g) \end{pmatrix}.$$

Note here that $V_1 \oplus V_2$ is a direct sum of vector spaces, whereas $\rho_1 \oplus \rho_2$ is a direct sum of representations.

Definition 2.20. A representation $\rho : G \to \mathrm{GL}(V)$ is ***completely reducible*** or ***semisimple*** if it is the direct sum of irreducible representations.

Theorem 2.3 (*Maschke's theorem*)

> *Let G be a finite group, and $\rho : G \to \mathrm{GL}(V)$ a representation over a finite-dimensional vector space V over a field $\Bbbk$ with characteristic zero. If W is a G-subspace of V, then there exists a G-subspace U of V such that $V = W \oplus U$.*

Proof. Let W' be any vector subspace complement of W in V, i.e. $V = W \oplus W'$ *as vector spaces*[5].

Let $\pi : V \to W$ be the projection of V onto W along W'. That is, if $\mathbf{v} = \mathbf{w} + \mathbf{w}'$ with $\mathbf{w} \in W$ and $\mathbf{w}' \in W'$, then $\pi(\mathbf{v}) = \mathbf{w}$.

The key step is to "average" π over G to construct a G-linear projection:

$$\bar{\pi} : \mathbf{v} \mapsto \frac{1}{|G|} \sum_{g \in G} \rho(g)\pi(\rho(g^{-1})\mathbf{v}).$$

Here we need the field to have characteristic zero such that $1/|G|$ is well-defined. In fact, this theorem holds as long as the characteristic of $\Bbbk$ does not divide $|G|$.

For simplicity of expression, we drop the ρs, and simply write

$$\bar{\pi} : \mathbf{v} \mapsto \frac{1}{|G|} \sum_{g \in G} g\pi(g^{-1}\mathbf{v}).$$

Observe that $\bar{\pi}$ has an image in W. Indeed, for $\mathbf{v} \in V$, $\pi(g^{-1}\mathbf{v}) \in W$ and $gW \le W$.

Furthermore, for $\mathbf{w} \in W$, we have $\bar{\pi}(\mathbf{w}) = \mathbf{w}$. This follows from the fact that π itself fixes W. Since W is G-invariant, we have $g^{-1}\mathbf{w} \in W$ for all $\mathbf{w} \in W$. Hence,

$$\bar{\pi}(\mathbf{w}) = \frac{1}{|G|} \sum_{g \in G} g\pi(g^{-1}\mathbf{w}) = \frac{1}{|G|} \sum_{g \in G} gg^{-1}\mathbf{w} = \frac{1}{|G|} \sum_{g \in G} \mathbf{w} = \mathbf{w}.$$

Putting these together, we have that $\bar{\pi}$ is a projection onto W.

Finally, for $h \in G$ we have $h\bar{\pi}(\mathbf{v}) = \bar{\pi}(h\mathbf{v})$, meaning it is invariant under the G-action. This follows easily from the definition:[6]

$$\begin{aligned}
h\bar{\pi}(\mathbf{v}) &= h\frac{1}{|G|} \sum_{g \in G} gq(g^{-1}\mathbf{v}) \\
&= \frac{1}{|G|} \sum_{g \in G} hgq(g^{-1}\mathbf{v}) \\
&= \frac{1}{|G|} \sum_{g \in G} (hg)q((hg)^{-1}h\mathbf{v}) \\
&= \frac{1}{|G|} \sum_{g' \in G} g'q(g'^{-1}(h\mathbf{v})) \\
&= \bar{\pi}(h\mathbf{v})
\end{aligned}$$

5 The existence of such a subspace is a standard fact from linear algebra.

6 For the fourth equality put $g' = hg$. Since h is invertible, summing over all g is the same as summing over all g'.

The final step is to show $\ker \bar{\pi}$ is G-invariant. If $\mathbf{v} \in \ker \bar{\pi}$ and $h \in G$, then $\bar{\pi}(h\mathbf{v}) = h\bar{\pi}(\mathbf{v}) = 0$. So $h\mathbf{v} \in \ker \bar{\pi}$.

Thus,

$$V = \operatorname{im} \bar{\pi} \oplus \ker \bar{\pi} = W \oplus \ker \bar{\pi}$$

is a G-subspace decomposition. □

Theorem 2.4

Every finite-dimensional representation V of a finite group over a field of characteristic zero is completely reducible, namely

$$V \cong V_1 \oplus \cdots \oplus V_r \tag{2.4}$$

is a direct sum of irreducible representations.

This decomposition is not unique. There is a sense, however, in which the decomposition can be made unique. A standard result in representation theory tells us that we can combine isomorphic copies of the same irreducible representation in the decomposition given in Equation (2.4). The set of irreducible representations that appear in the decomposition and the multiplicities with which they appear are unique. Hence we have

$$V \cong V_1 \oplus \cdots \oplus V_r \cong W_1^{\oplus m_1} \oplus \cdots \oplus W_t^{\oplus m_t}, \tag{2.5}$$

where the W_i are irreducible and $W_i \not\cong W_j$ for all $i \neq j$. This is referred to as the ***isotypic decomposition*** of V, and the W_i are the ***isotypic components***. The existence of an isotypic decomposition for any complex representation is why much of representation theory involves studying the irreducible representations of groups.

We conclude our discussion of the representation theory of abstract groups with Schur's Lemma, a simple but powerful result we will need later. We then proceed to our main group of interest, the symmetric group.

Lemma 2.5 (*Schur's Lemma*)

Let V and W be irreducible G-modules over a field $\Bbbk$.

(i) Any G-homomorphism $\varphi\colon V \longrightarrow W$ is either zero or an isomorphism.

(ii) If $\Bbbk$ is algebraically closed and V is an irreducible G-module, then any G-endomorphism of V is a scalar multiple of the identity map on V.

Proof.

(i) Fix a G-homomorphism $\varphi\colon V \longrightarrow W$. Since the kernel of φ is a submodule of V and V is irreducible, it follows that either $\ker \varphi = 0$ or $\ker \varphi = V$. *Mutatis mutandis* we must also have $\operatorname{im} \varphi = 0$ or $\operatorname{im} \varphi = W$. Hence if $\ker \varphi$ is not V - meaning φ is not the zero map - then it must be the case that $\ker \varphi = 0$ and $\operatorname{im} \varphi = W$, making φ a bijection.

(ii) Since $\Bbbk$ is algebraically closed any G-endomorphism $\varphi\colon V \longrightarrow V$ has some eigenvalue λ. It follows then that $\varphi - \lambda \mathrm{id}_V$ is a singular G-endomorphism of V. By Item (i), $\varphi - \lambda(\mathrm{id}_V)$ is therefore the zero map. Thus, $\varphi = \lambda(\mathrm{id}_V)$.

$\square$

3 | REPRESENTATIONS OF THE SYMMETRIC GROUPS

3.1 SYMMETRIC GROUPS, PARTITIONS, AND YOUNG DIAGRAMS

DEFINITION 3.1. The ***symmetric group*** on n elements $\mathfrak{S}_n$ consists of all bijections $\pi\colon [n] \longrightarrow [n]$, with normal composition of functions as the group operation. Each such $\pi \in \mathfrak{S}_n$ is referred to as a ***permutation***.

We can write $\pi \in \mathfrak{S}_n$ in ***cycle notation***. To do this, we partition the elements of $[n]$ into cycles based on their images under π. If $i \in [n]$, we have the cycle

$$\left(i, \pi(i), \pi^2(i), \dots, \pi^{p-1}(i)\right)$$

where $\pi^p = i$. This means that π sends i to $\pi(i)$, $\pi(i)$ to $\pi^2(i)$, etc. We then proceed by selecting an element of $[n]$ not contained in the cycle containing i and iterate the aforementioned process until every element of $[n]$ is contained in some cycle. Collectively these cycles uniquely determine π.

EXAMPLE 3.2. Suppose $\pi \in \mathfrak{S}_n$ is given by

$$\pi(1) = 2,\ \pi(2) = 3,\ \pi(3) = 1,\ \pi(4) = 4,\ \pi(5) = 5.$$

Then in cycle notation we would write $\pi = (1, 2, 3)(4)(5)$. When writing the permutation in cycle notation, we omit the cycles corresponding to fixed points of π for brevity. Thus, the final expression would be $\pi = (1, 2, 3)$.

Note that the permutation is invariant under cyclic permutations of the elements within the cycles as well as permutations of the cycles themselves.

A ***k-cycle*** or cycle of length k is simply a cycle containing k elements from $[n]$.

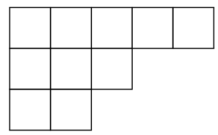
DEFINITION 3.3. The ***cycle type*** of $\pi \in \mathfrak{S}_n$, written $\mathrm{cyc}(\pi)$, is an expression of the form

$$\left(1^{m_1}, 2^{m_2}, \dots, n^{m_n}\right),$$

where m_k is the number of cycles of length k in π.

DEFINITION 3.4. Let G be a group. We say g and $h \in G$ are ***conjugates*** if and only if there exists some $k \in G$ such that $g = khk^{-1}$. The set of all elements of G which are conjugate to g is referred to as the ***conjugacy class*** of g. Conjugacy is an equivalence relation and so the set G can be partitioned into disjoint conjugacy classes.

Figure 3.1: Young diagram of $\boldsymbol{\lambda} = (5, 3, 2)$



We care about the cycle types of permutations when talking about $\mathfrak{S}_n$ because the conjugacy classes of $\mathfrak{S}_n$ are determined by cycle type. Indeed, if

$$\pi = (i_1, i_2, \ldots, i_\ell) \cdots (i_m, \ldots, i_n),$$

then for any $\sigma \in \mathfrak{S}_n$

$$\sigma\pi\sigma^{-1} = \big(\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_\ell)\big) \cdots \big(\sigma(i_m), \ldots, \sigma(i_n)\big).$$

Note that if we take the cycles of $\pi \in \mathfrak{S}_n$ and arrange the cycles by decreasing length, the lengths of the cycles form an integer partition of n.

Definition 3.5. A ***partition*** of $n \in \mathbb{N}$ is a k-tuple

$$\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$$

such that

$$\lambda_1 \geq \cdots \geq \lambda_k$$
$$\lambda_1 + \cdots + \lambda_k = n.$$

We write this as $\boldsymbol{\lambda} \vdash n$ and refer to the λ_i as the ***parts*** of the partition. A part λ_i of $\boldsymbol{\lambda}$ is ***non-trivial*** if $\lambda_i \geq 2$. Let $\#\boldsymbol{\lambda}$ be the sum of its nontrivial parts.

Conversely, note that if we have a partition $\boldsymbol{\lambda} \vdash n$ we can think of each part of the partition as giving the length of a cycle in a permutation. In this way we see that there is a natural bijective correspondence between cycle types (hence conjugacy classes) of $\mathfrak{S}_n$ and integer partitions of n. The significance of the conjugacy classes is made clear by the following theorem.

Theorem 3.1

Up to isomorphism, the number of irreducible representations of a group G is equal to the number of distinct conjugacy classes in G.

The ***Young diagram*** of the partition $\boldsymbol{\lambda}$ is an array of boxes (called *cells*) with λ_i cells in row i (indexed from the top), for each positive λ_i, aligned at the left. For example, the Young diagram of $(5, 3, 2)$ is given in Figure 3.1. By Theorem 3.1 we can therefore conclude that the irreducible representations of $\mathfrak{S}_k$ are in bijective correspondence with Young diagrams containing k cells.

Example 3.6. The ***defining representation*** or ***standard representation*** of $\mathfrak{S}_n$ sends $\pi \in \mathfrak{S}_n$ to the permutation matrix of π. As a concrete example, if $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{C}^n$, then $\mathfrak{S}_n$ acts in the standard representation by permuting coordinates $M(\pi)e_i = e_{\pi(i)}$.

This is not irreducible. Consider the one-dimensional subspace spanned by the sum of the basis vectors. Since any permutation of this sum still returns the sum, this one-dimensional subspace is an invariant subspace of the standard representation. Furthermore, since every $\pi \in \mathfrak{S}_n$ sends the sum to itself, the corresponding permutation matrix is simply the identity. Thus, this subrepresentation is a "copy" of the trivial representation. To be more precise, one of the isotypic components of this representation is the trivial representation.

Example 3.7. A one-dimensional representation need not be trivial. Consider the ***sign representation*** (also called the ***alternating representation***), an irreducible representation of $\mathfrak{S}_n$ which sends $\pi \in \mathfrak{S}_n$ to the map *scalar multiplication* by $\operatorname{sgn} \pi$. Indeed,

$$\operatorname{sgn}(\pi\sigma) = \operatorname{sgn}(\pi)\operatorname{sgn}(\sigma).$$

3.2 Young symmetrizers and Specht modules

As stated previously, much of representation theory amounts to the study of irreducible representations because they can be combined to form all other representations. The traditional method of constructing the irreducible representations of $\mathfrak{S}_n$ is based on the combinatorics of Young diagrams and tableaux.

Definition 3.8. A ***Young tableau*** T_λ, corresponding to a partition $\boldsymbol{\lambda} \vdash k$, is a Young diagram with the numbers $1, \ldots, k$ placed in the boxes without repetition. A tableau is called ***standard*** if the numbers in the boxes increase top to bottom and left to right.

Definition 3.9. Fix a Young tableau T_λ. We now define two subgroups of $\mathfrak{S}_k$ that correspond to T_λ:

$$P = P_\lambda = \left\{\sigma \in \mathfrak{S}_k : \sigma \text{ preserves each row of } T_\lambda\right\}$$
$$Q = Q_\lambda = \left\{\sigma \in \mathfrak{S}_k : \sigma \text{ preserves each column of } T_\lambda\right\}$$

Clearly, $P_\lambda \cap Q_\lambda = \{1\}$.

Definition 3.10. The ***Young projectors*** are defined as

$$a_\lambda := \frac{1}{|P_\lambda|} \sum_{g \in P_\lambda} g$$
$$b_\lambda := \frac{1}{|Q_\lambda|} \sum_{g \in Q_\lambda} \operatorname{sgn}(g) g$$

The ***Young symmetrizer*** is defined as

$$c_\lambda := a_\lambda b_\lambda.$$

Since $P_\lambda \cap Q_\lambda = \{1\}$, $c_\lambda \neq 0$.

EXAMPLE 3.11. If $\boldsymbol{\lambda} = (3, 2)$ and the tableau is

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array},$$

then

$$\begin{aligned} a_\lambda &= \big(1 + (13) + (14) + (34) + (134) + (143)\big)\big(1 + (25)\big) \\ b_\lambda &= \big(1 - (12)\big)\big(1 - (35)\big). \end{aligned}$$

The irreducible representations of $\mathfrak{S}_k$ are described by the following theorem from [11, §4.2].

THEOREM 3.2

The subspace $V_\lambda := (\mathbb{C}\mathfrak{S}_k)c_\lambda$ of $\mathbb{C}\mathfrak{S}_k$ is an irreducible representation of $\mathfrak{S}_k$ under left multiplication. Furthermore, every irreducible representation of $\mathfrak{S}_k$ is isomorphic to V_λ for a unique $\boldsymbol{\lambda}$. These modules V_λ are called the **Specht modules**.

3.3 OKOUNKOV-VERSHIK APPROACH

Given a representation $\rho\colon G \longrightarrow \mathrm{GL}(V)$ of a group G and a subgroup $H \leq G$, we can restrict the representation to H by simply taking $\rho|_H$. Conversely, given a representation of H we can induce a representation of G.

DEFINITION 3.12. If V is a representation of G, then $\mathrm{Res}^G_H V = \mathrm{Res}\, V$ is the ***restriction*** of V to the subgroup H.

DEFINITION 3.13. Suppose V is a representation of G and $W \subsetneq V$ is a subspace which is H-invariant for some subgroup H of G. For any $g \in G$, the subspace $g \cdot W = \left\{g \cdot w : w \in W\right\}$ depends only on the left coset gH of g modulo H, since $gh \cdot W = g \cdot (h \cdot W) = g \cdot W$. For a coset $\sigma \in {}^{G}\!/_{H}$, we write $\sigma \cdot W$ for this subspace of V. We say V is ***induced*** by W if every element in V can be written uniquely as a sum of elements in such translates of W, i.e.,

$$V = \bigoplus_{\sigma \in {}^{G}\!/_{H}} \sigma \cdot W.$$

In this case we write $V = \mathrm{Ind}^G_H W = \mathrm{Ind}\, W$.

For further details, including a proof that V exists and is unique up to isomorphism given a representation W of H, the reader is referred to [11, §3.3]. There the reader will also find the following result relating induction and restriction.

Proposition 3.3

Let W be a representation of H, U a representation of G, and suppose $V = \operatorname{Ind} W$. Then any H-module homomorphism $\varphi\colon W \longrightarrow U$ extends uniquely to a G-module homomorphism $\widetilde{\varphi}\colon V \longrightarrow U$, i.e.

$$\operatorname{Hom}_H(W, \operatorname{Res} U) = \operatorname{Hom}_G(\operatorname{Ind} W, U).$$

Frobenius reciprocity is a straightforward corollary of this. While we do not give the result here, we will note what it says about irreducible representations: if W and U are irreducible, then the number of times U appears in the isotypic decomposition of $\operatorname{Ind} W$ is the same as the number of times W appears in the isotypic decomposition of $\operatorname{Res} U$.

In the previous section, one may have noted that there is no immediate connection between tableaux and irreducible representations, and as such introducing tableaux to construct the irreducible representations of $\mathfrak{S}_n$ may seem unmotivated or unexpected. It naturally raises the question of whether one can arrive at the combinatorial objects of the theory in a more natural way. Okounkov's and Vershik's modern approach to the theory in [29] does exactly this. Their criteria for a satisfying representation theory of the symmetric group are as follows:

(1) Symmetric groups form a natural sequence and their representation theory should be recursive with respect to the sequence, which means that the representation theory of $\mathfrak{S}_n$ should rely on representation theory of $\mathfrak{S}_{n-1}$ for all $n \in \mathbb{N}$.

(2) The combinatorics of the Young diagrams and Young tableaux, which reflects the branching rule[1] for restriction

$$\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}$$

must not be introduced as an auxiliary tool of construction, not a priori: it should be deduced starting from the inside structure of the symmetric groups. In this case the branching rule (which is one of the main theorems of the theory) will appear naturally and not as a last corollary after developing the whole theory.

(3) Symmetric groups are Coxeter groups. The methods of their representation should apply to all classical sequences of Coxeter groups.

Okounkov and Vershik thus develop an approach to the representation theory of the symmetric group satisfying these principles in [29]. Critical to their approach are the following concepts, which we will discuss in some detail:

(1) Gelfand-Zetlin (GZ) algebra and basis

(2) Young-Jucys-Murphy (YJM) elements

(3) Algebras with a local system of the generators (ALSG)

1 This is a result that describes how irreducible representations of $\mathfrak{S}_n$ can be broken down as irreducible representations of $\mathfrak{S}_{n-1}$. We build to this result throughout the section and prove it in Proposition 3.28

3.3.1 Gelfand-Zetlin basis for inductive families

Let

$$\{1\} = G_0 \subset G_1 \subset G_2 \subset \cdots \tag{3.1}$$

be a chain of finite groups. We denote the set of isomorphism classes of irreducible complex representations of G_n as $\widehat{G_n}$.

DEFINITION 3.14. The ***branching graph*** or ***Bratelli diagram*** of the series given in Equation (3.1) is a directed graph where:

(i) The vertices consist of the elements of the set

$$\bigsqcup_{n \geq 0} \widehat{G_n}.$$

(ii) Two vertices λ and μ are joined by k directed edges from λ to μ whenever $\mu \in \widehat{G_{n-1}}$ and $\lambda \in \widehat{G_n}$ for some n, and the multiplicity of μ in the restriction of λ to G_{n-1} is k.

The elements of the set $\widehat{G_n}$ constitute the *nth level* of the branching graph (Bratelli diagram). We write $\lambda \longrightarrow \mu$ to indicate that (λ, μ) is an edge of the branching graph.

We say that the series in Equation (3.1) has ***simple branching*** or ***multiplicity free branching*** if we always have $k \in \{0, 1\}$. Equivalently, let there be an irreducible representation of G_{n-1} at the $n-1$th level of the branching graph. Its multiplicity in the isotypical decomposition of an irreducible representation at the nth level of the branching graph will be at most one.

Now suppose we have a series as in Equation (3.1) with simple branching, and let V_λ be a G_n-module. Then by simple branching, the decomposition $V_\lambda = \oplus_\mu V_\mu$, where the sum is over all $\mu \in \widehat{G_{n-1}}$ with $\lambda \longrightarrow \mu$, is *canonical*. It follows then that repeating this decomposition iteratively gives a canonical decomposition of V_λ into irreducible G_1-modules, or one-dimensional subspaces. Hence,

$$V_\lambda = \bigoplus_T V_T, \tag{3.2}$$

where the sum is over all possible chains

$$T = \lambda^{(n)} \longrightarrow \cdots \longrightarrow \lambda^{(1)} \tag{3.3}$$

with $\lambda^{(i)} \in \widehat{G_i}$ and $\lambda^{(n)} = \lambda$.

DEFINITION 3.15. Choosing a nonzero vector v_T in each one-dimensional space V_T gives a unique canonical (up to scalar multiplication) basis $\{v_T\}$ of V_λ. This basis is called the ***Gelfand-Zetlin basis*** (GZ-basis).

REMARK 3.16. Note that Definition 3.15 immediately implies that $(\mathbb{C}G_i)v_T = V_{\lambda^{(i)}}$ for all i. Indeed, $v_T \in V_{\lambda^{(i)}}$ is an irreducible representation which implies that the action of $\mathbb{C}G_i$ on v_T recovers the entirety of the irreducible representation $V_{\lambda^{(i)}}$.

DEFINITION 3.17. We let $Z_n \coloneqq Z(\mathbb{C}G_n)$ be the center of the group algebra. Let

$$GZ_n \coloneqq \langle Z_1, \ldots, Z_n \rangle,$$

where $\langle Z_1, \ldots, Z_n \rangle$ is used to denote the subalgebra generated by the Z_i. This is a commutative $\mathbb{C}$-subalgebra of $\mathbb{C}G_n$ and is referred to as the ***Gelfand-Zetlin algebra*** of the inductive chain of subgroups from Equation (3.1).

Consider the following map:

$$\varphi\colon \mathbb{C}G_n \longrightarrow \bigoplus_{\lambda \in \widehat{G_n}} \operatorname{End} V_\lambda \tag{3.4}$$

$$g \longmapsto \left(V_\lambda \xrightarrow{g} V_\lambda \colon \boldsymbol{\lambda} \in \widehat{G_n} \right).$$

Observe first that φ is an isomorphism of algebras. If $\varphi(x) = \varphi(y)$, then x and y act the same on every irreducible representation, including the regular representation, and $x = y$. Surjectivity then follows easily from dimension counting. First, $\operatorname{End} V_\lambda$ has dimension $(\dim V_\lambda)^2$. Since $\mathbb{C}G_n$ has its basis indexed by the elements of G_n, the desired result follows from the standard result from character theory [27] that

$$\sum_{\lambda \in \widehat{G_n}} (\dim V_\lambda)^2 = |G_n|.$$

DEFINITION 3.18. Let $D(V_\lambda)$ be the set of operators on V_λ, diagonal in the GZ-basis of V_λ.

The following theorem demonstrates the significance of the Gelfand-Zetlin algebra.

THEOREM 3.4

The GZ-algebra GZ_n is the image of $\oplus D(V_\lambda)$ under the isomorphism given in Equation (3.4). It consists of those elements of $\mathbb{C}G_n$ that act diagonally in the GZ-basis in every irreducible representation of G_n. Thus, GZ_n is a maximal commutative subalgebra of $\mathbb{C}G_n$ where

$$\dim GZ_n = \sum_{\lambda \in \widehat{G_n}} \dim V_\lambda.$$

PROOF. Consider the chain T from Equation (3.3). Let $p_{\lambda^{(i)}} \in Z_i$, be the central idempotent corresponding to the representation defined by $\boldsymbol{\lambda}^i \in \widehat{G}_i$. Define

$$p_T = \prod_{i=1}^{n} p_{\lambda^{(i)}} \in GZ_n.$$

The image of p_T under the map from Equation (3.4) is $(f_\mu \colon \boldsymbol{\mu} \in \widehat{G_n})$, where $f_\mu = 0$ if $\boldsymbol{\mu} \neq \boldsymbol{\lambda}$, and f_λ is a projection onto V_T with respect to the decomposition in Equation (3.2).

Thus, the image of GZ_n under φ from Equation (3.4) includes $\oplus_{\lambda \in \widehat{G_n}} D(V_\lambda)$, which is a commutative maximal subalgebra of $\oplus_{\lambda \in \widehat{G_n}} \mathrm{End}(V_\lambda)$. Since GZ_n is itself commutative, the desired result follows. ◻

Corollary 3.4.1

(i) Let $v \in V_\lambda, \boldsymbol{\lambda} \in \widehat{G_n}$. If v is an eigenvector for the action of every operator in GZ_n, then a scalar multiple of v is a GZ-basis vector of V_λ.

(ii) Let u, v be two GZ-vectors. If u, v have the same eigenvalues for every element of GZ_n, then $u = v$.

Proof.

(i) Let
$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$
be an eigenvector for the action of every operator in GZ_n. Fix distinct $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. By Theorem 3.4,
$$\begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix} \tag{3.5}$$
is in GZ_n. Thus, we have
$$\begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \lambda_1 v_1 \\ \lambda_2 v_2 \\ \vdots \\ \lambda_n v_n \end{pmatrix}. \tag{3.6}$$
Since the λ_i are all distinct the only way for the right hand side of Equation (3.7) to be a multiple of the right hand side of Equation (3.6) is for v to be a multiple of a basis vector.

(ii) Fix distinct $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. By Theorem 3.4,
$$\begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix} \tag{3.7}$$

is in GZ_n. Assume for the sake of contradiction that $u \neq v$. If v is a multiple of u then they cannot both be basis vectors. Therefore, the only way for u and v to have the same eigenvalues under the action of this matrix would be for two of its diagonal entries to be the same. This is a contradiction as the λ_i are distinct by hypothesis.

□

3.3.2 Simple branching in symmetric groups

In order for the theory outlined in the previous section to apply we must show that the symmetric groups have simple branching. To prove this we need two results from the theory of semisimple algebras[2], stated here without proof. For further details, the reader is referred to [15, Chapter 5].

Theorem 3.5 (*Artin-Wedderburn Theorem*)

If A is a semisimple $\mathbb{C}$-algebra, then A decomposes as a direct sum of matrix algebras over $\mathbb{C}$.

Theorem 3.6 (*Double Centralizer Theorem*)

Let A be a finite-dimensional **central** *($Z(A) = \mathbb{C}$) algebra and $B \subseteq A$ a simple subalgebra. For centralizer $C = Z(A, B)$, C is simple and $Z(A, C) = B$. Moreover,*

$$\dim_{\mathbb{C}}(A) = \dim_{\mathbb{C}}(B) = \dim_{\mathbb{C}}(C).$$

We now proceed with showing that the symmetric groups have simple branching.

Theorem 3.7

Let M be a finite-dimensional semisimple complex algebra and N a semisimple subalgebra. If $Z(M, N)$ be the centralizer of the pair (M, N) such that

$$Z(M, N) = \left\{ m \in M | mn = nm \ \forall n \in N \right\},$$

then $Z(M, N)$ is semisimple.

Proof. By Theorem 3.5,

$$M = \bigoplus_{i=1}^{\ell} M_i$$

where each summand M_i is some matrix algebra. Let N_i be the image of N under the projection $M \longrightarrow M_i$. Since N_i is the homomorphic image of a semisimple algebra, it

2 The precise definition of semisimple in this context is not that pertinent to us. It is sufficient to know that group algebras over the complex numbers are semisimple, and that homomorphisms preserve semisimplicity.

follows that N_i is itself semisimple. Observe that

$$Z(M,N) = \bigoplus_{i=1}^{k} Z(M_i, N_i).$$

By Theorem 3.6, $Z(M_i, N_i)$ is semisimple for all i, and thus $Z(M,N)$ is also semisimple. $\square$

Our goal is to use this theorem in the case $M = \mathbb{C}\mathfrak{S}_n$ and $N = \mathbb{C}\mathfrak{S}_{n-1}$.

Theorem 3.8

The following are equivalent:

(i) The restriction of any finite-dimensional complex irreducible representation of M to N is multiplicity free.

(ii) $Z(M,N)$ is commutative.

Proof. Let

$$V_i := \left\{ (m_1, \ldots, m_k) \in M \;\middle|\; \begin{array}{l} m_j = \delta_{ij} \\ [m_i]_{pq} = 0 \; \forall q \neq 1 \end{array} \right\}.$$

The δ_{ij} in the first condition is the Kronecker delta with i the same as in V_i. The second condition simply states that any entry of m_i not in the first column is zero. The V_i each represent a distinct irreducible M-module. The decomposition of V_i into irreducible N-modules is equivalent to its decomposition into irreducible N_i-modules.

The next step is to observe that $Z(M,N)$ will be commutative if and only if for all i, $Z(M_i, N_i)$ is commutative. In turn, this is true if and only if every irreducible representation of $Z(M_i, N_i)$ has dimension one. This is analogous to the result for irreducible representations of abelian groups. Thus, to prove the desired result it suffices to show that that $\mathrm{Res}^{M_i}_{N_i} U$ is multiplicity free for any irreducible representation of $Z(M,N)$, U.

Suppose that all irreducible representations of $Z(M_i, N_i)$ have dimension one. Now let U and V be irreducible representations of M_i and N_i respectively. This means $\mathrm{Hom}_{N_i}(V,U)$ is an irreducible representation and so has dimension one. Indeed, $\mathrm{Hom}_{N_i}(V,U)$ as a representation of N_i is some number of copies of V. This is by Lemma 2.5, which states that the image of every nonzero map in $\mathrm{Hom}_{N_i}(V,U)$ is isomorphic to V. These isomorphic copies of V must be nonintersecting because otherwise the intersection would be N_i-invariant, contradicting the irreducibility of V. Given any two submodules of U isomorphic to V, there exists some N_i-equivariant automorphism of U sending one to the other. Hence, any map in $\mathrm{Hom}_{N_i}(V,U)$ can be taken to any other by composing with the appropriate N_i-equivariant automorphism of U, thus precluding the possibility of a proper nonzero submodule. By Lemma 2.5, this is the multiplicity of V in $\mathrm{Res}^{M_i}_{N_i} U$, so the branching is multiplicity-free.

We prove the converse by contraposition. Suppose that $Z(M_i, N_i)$ has an irreducible representation of dimension strictly greater than one. Now let U and V

once again be irreducible representations of M_i and N_i respectively. By Lemma 2.5, $\mathrm{End}_{\mathbb{C}}(U) \cong M$, so $\mathrm{End}_{N_i}(U) \cong Z(M_i, N_i)$. Therefore, if $\mathrm{Res}^{M_i}_{N_i} U = \oplus_j W_j$ is the decomposition of U into simple N_i-modules, then

$$Z(M_i, N_i) \cong \mathrm{End}_{N_i}(U) \cong \bigoplus_{j,k} \mathrm{Hom}_{N_i}(W_j, W_k)$$

is a decomposition of $Z(M_i, N_i)$ into irreducible representations. Hence the existence of an irreducible representation of $Z(M_i, N_i)$ with dimension strictly greater than one implies $W_j \cong W_k$ for some $j \neq k$. This irreducible representation of N_i therefore occurs with multiplicity strictly greater than one in $\mathrm{Res}^{M_i}_{N_i} U$ and the branching is not simple. □

By Theorem 3.8, to prove the symmetric groups have simple branching it suffices to show that the centralizer $Z(\mathbb{C}\mathfrak{S}_n, \mathbb{C}\mathfrak{S}_{n-1})$ is commutative. We do this using the Young-Jucys-Murphy elements of the group algebra.

3.3.3 Young-Jucys-Murphy elements (YJM elements)

Definition 3.19. Define

$$P_n := \left\{(\boldsymbol{\mu}, i) : \boldsymbol{\mu} \vdash n \text{ and } i \text{ is a part of } \boldsymbol{\mu}\right\}.$$

Definition 3.20. For $2 \leq i \leq n$, define Y_i to be the sum of all i-cycles in $\mathfrak{S}_{n-1}$. By convention, $Y_1 = 0$. Define Y_i' to be the sum of all i-cycles in $\mathfrak{S}_n$ containing n.

Definition 3.21. For $(\boldsymbol{\mu}, i) \in P_n$, let $c_{(\boldsymbol{\mu},i)} \in \mathbb{C}\mathfrak{S}_n$ be the sum of permutations $\pi \in \mathfrak{S}_n$ such that the cycle type of π is $\boldsymbol{\mu}$ and the size of the cycle containing n is i.

Remark 3.22. Observe that each of $Y_2, \ldots, Y_{n-1}, Y_2', \ldots, Y_n'$ equals $c_{(\boldsymbol{\mu},i)}$ for a suitable choice of $\boldsymbol{\mu}$ and i. In particular,

$$Y_j = c_{(\boldsymbol{\mu},1)} \text{ for } \boldsymbol{\mu} = (j, 1, \ldots, 1)$$
$$Y_j' = c_{(\boldsymbol{\mu},j)} \text{ for } \boldsymbol{\mu} = (j, 1, \ldots, 1)$$

Lemma 3.9

The set

$$\left\{c_{(\boldsymbol{\mu},i)} : (\boldsymbol{\mu}, i) \in P_n\right\}$$

is a basis of $Z(\mathbb{C}\mathfrak{S}_n, \mathbb{C}\mathfrak{S}_{n-1})$. Therefore, we have that

$$\langle Y_2, \ldots, Y_{n-1}, Y_2', \ldots, Y_n'\rangle \subseteq Z(\mathbb{C}\mathfrak{S}_n, \mathbb{C}\mathfrak{S}_{n-1}).$$

Proof. We begin by first observing that $c_{(\boldsymbol{\mu},i)}$ is invariant under conjugation by elements of $\mathfrak{S}_{n-1}$. This is because conjugating by elements of $\mathfrak{S}_{n-1}$ does not change the length of the cycle that any element of any term of $c_{(\boldsymbol{\mu},i)}$ appears in. It follows that

$c_{(\mu,i)} \in Z(\mathbb{C}\mathfrak{S}_n, \mathbb{C}\mathfrak{S}_{n-1})$. Since there is no way to add permutations where n appears in different cycle lengths and get zero, the $c_{(\mu,i)}$ are linearly independent. All that remains to be shown is that the $c_{(\mu,i)}$ span $Z(\mathbb{C}\mathfrak{S}_n, \mathbb{C}\mathfrak{S}_{n-1})$.

Fix $f \in Z(\mathbb{C}\mathfrak{S}_n, \mathbb{C}\mathfrak{S}_{n-1})$. By definition of $Z(\mathbb{C}\mathfrak{S}_n, \mathbb{C}\mathfrak{S}_{n-1})$, we have that for any $\tau \in \mathfrak{S}_{n-1}$, $\tau f \tau^{-1} = f$. So if we write

$$f = \sum_{\sigma \in \mathfrak{S}_n} \alpha_\sigma \sigma, \tag{3.8}$$

then $\alpha_\sigma = \alpha_{\tau\sigma\tau^{-1}}$. Hence α_σ is invariant under conjugation by permutations in $\mathfrak{S}_{n-1}$. Furthermore, since conjugation by an element of $\mathfrak{S}_{n-1}$ cannot change the cycle length of the cycle in which n appears, by grouping the terms of the sum by the cycle type of the permutation along with the length of the cycle in which n appears we may write

$$f = \sum_{(\mu,i) \in P_n} \alpha_{(\mu,i)} c_{(\mu,i)}.$$

Thus, the $c_{(\mu,i)}$ are spanning. The desired result then follows immediately by Remark 3.22. ◻

Lemma 3.10

$$c_{(\mu,i)} \in \langle Y_2, \ldots, Y_k, Y'_2 \ldots, Y'_k \rangle \textit{ for } k = \#\mu.$$

Proof. We proceed via induction on $\#\mu$.

BASE CASE

Suppose $\#\mu = 0$. Then $c_{(\mu,i)}$ is simply the identity permutation, which lies in the subalgebra $\langle Y_2, \ldots, Y_k, Y'_2, \ldots, Y'_k \rangle$.

INDUCTION HYPOTHESIS

Assume that the desired result holds for all $\#\mu \leq m$:

$$c_{(\mu,i)} \in \langle Y_2, \ldots, Y_k, Y'_2 \ldots, Y'_k \rangle \text{ for } k = \#\mu.$$

INDUCTIVE STEP

Fix $(\boldsymbol{\mu}, i) \in P_n$ with $\#\mu = m + 1$. Let the nontrivial parts of $\boldsymbol{\mu}$ be $\mu_1, \ldots, \mu_\ell$ in some order. We now proceed via case analysis.

CASE 1

We first consider the case where $i = 1$. By Lemma 3.9,

$$\prod_{j=1}^{\ell} Y_{\mu_j} = \alpha_{(\mu,1)} c_{(\mu,1)} + \sum_{(\boldsymbol{\tau},1)} \alpha_{(\boldsymbol{\tau},1)} c_{(\boldsymbol{\tau},1)}$$

where $\alpha_{(\mu,1)} \neq 0$ and the sum is over all $(\boldsymbol{\tau}, 1)$ with $\#\boldsymbol{\tau} < \#\boldsymbol{\mu}$. The desired result then follows from the induction hypothesis.

CASE 2

Let us now consider the case where $i > 1$. Without loss of generality, assume that $i = \mu_1$. Then

$$Y'_{\mu_1} \prod_{j=2}^{\ell} Y_{\mu_j} = \alpha_{(\mu,i)} c_{(\mu,i)} + \sum_{(\tau,i)} \alpha_{(\tau,j)} c_{(\tau,j)}$$

where $\alpha_{(\mu,i)} \neq 0$ and the sum is over all $(\boldsymbol{\tau}, j)$ with $\#\boldsymbol{\tau} < \#\boldsymbol{\mu}$.

□

Lemma 3.11

$$Z(\mathbb{C}\mathfrak{S}_n, \mathbb{C}\mathfrak{S}_{n-1}) = \langle Y_2, \ldots, Y_{n-1}, Y'_2, \ldots, Y'_n \rangle$$

Proof. This follows immediately from Lemmas 3.9 and 3.10. □

Lemma 3.12

$$Z(\mathbb{C}\mathfrak{S}_{n-1}) = \langle Y_2, \ldots, Y_{n-1} \rangle$$

Proof. Observe that

$$Z(\mathbb{C}\mathfrak{S}_{n-1}) \subseteq Z(\mathbb{C}\mathfrak{S}_n, \mathbb{C}\mathfrak{S}_{n-1}) = \langle Y_2, \ldots, Y_{n-1}, Y'_2, \ldots, Y'_n \rangle.$$

Simply omitting the generators that permute n then gives the desired result. □

Definition 3.23. For $1 \leq i \leq n$, the ***Young-Jucys-Murphy elements*** (YJM elements) are defined as

$$X_i = (1, i) + (2, i) + \cdots + (i-1, i) \in \mathbb{C}\mathfrak{S}_n.$$

By convention, $X_1 = 0$. Note that this is the sum of all 2-cycles in $\mathfrak{S}_i$ minus the sum of all 2-cycles in $\mathfrak{S}_{i-1}$.

Theorem 3.13 (*Okounkov-Vershik*)

$$Z(\mathbb{C}\mathfrak{S}_n, \mathbb{C}\mathfrak{S}_{n-1}) = \langle Z_{n-1}, X_n \rangle$$

Proof. By Lemma 3.11, we have that

$$\langle Z_{n-1}, X_n \rangle \subseteq Z(\mathbb{C}\mathfrak{S}_n, \mathbb{C}\mathfrak{S}_{n-1})$$

because $X_n = Y'_2$. Since $Y_k \in Z_{n-1}$ for all k, to prove the desired result it suffices to show that $Y'_2, \ldots, Y'_n \in \langle Z_{n-1}, X_n \rangle$. We proceed via induction on the indices.

BASE CASE

Since $Y'_2 = X_n$, this forms the base case of our induction.

INDUCTION HYPOTHESIS

For the inductive hypothesis, assume that $Y'_2, \ldots, Y'_{k+1} \in \langle Z_n, X_n \rangle$.

INDUCTIVE STEP

We wish to prove $Y'_{k+2} \in \langle Z_{n-1}, X_n \rangle$. By definition,

$$Y'_{k+1} = \sum_{i_1,\dots,i_k} (i_1, \dots, i_k, n)$$

summed over all distinct $i_1, \dots, i_k \in \{1, \dots, n\}$. Consider the element $Y'_{k+1}X_n \in \langle Z_{n-1}, X_n \rangle$:

$$Y'_{k+1}X_n = \left(\sum_{i_1,\dots,i_k} (i_1, \dots, i_k, n) \right) \left(\sum_{i=1}^{n-1} (i, n) \right). \tag{3.9}$$

Fix an arbitrary term $(i_1, \dots, i_k)$ of the product in Equation (3.9). There are two possibilities. Either $i \neq i_j$ for all j, or there exists a j such that $i = i_j$. The products in each instance will have the forms $(i, i_1, \dots, i_k)$ and $(i_1, \dots, i_j)(i_{j+1}, i_{j+2}, \dots, n)$. Hence, we can rewrite the right side of Equation (3.9) as

$$\sum_{i_1,\dots,i_k} (i, i_1, \dots, i_k, n) + \sum_{i_1,\dots,i_k} \sum_{j=1}^{k} (i_1, \dots, i_j)(i_{j+1}, \dots, i_k, n). \tag{3.10}$$

The first sum is over all distinct $i, i_1, \dots, i_k \in [n-1]$, while the outer sum on the right is over all distinct $i_1, \dots, i_k \in [n-1]$. Then Equation (3.10) becomes

$$Y'_{k+2} + \sum_{(\mu,i)} \alpha_{(\mu,i)} c_{\mu,i}$$

where the sum is over all (μ, i) such that $\#\mu \leq k+1$. The desired result then follows from induction and Lemma 3.10.

□

Theorem 3.14 (*Okounkov-Vershik*)

$$GZ_n = \langle X_1, \dots, X_n \rangle$$

Proof. We proceed via induction on n. The cases for $n = 1$ and $n = 2$ are trivial.

For the inductive hypothesis, assume that GZ_{n-1} is generated by $\langle X_1, \dots, X_{n-1} \rangle$. We wish to show $GZ_n = \langle GZ_{n-1}, X_n \rangle$. One inclusion, $\langle GZ_{n-1}, X_n \rangle \subseteq GZ_n$, is immediate since X_n is the difference between an element of Z_n and Z_{n-1}. To complete the proof, it suffices to show that $Z_n \subseteq \langle GZ_{n-1}, X_n \rangle$. This follows from Lemma 3.9 as

$$Z_n \subseteq Z(\mathbb{C}\mathfrak{S}_n, \mathbb{C}\mathfrak{S}_{n-1}) \subseteq \langle Z_{n-1}, X_n \rangle \subseteq \langle GZ_{n-1}, X_n \rangle.$$

□

Proposition 3.15 (*Okounkov-Vershik*)

$Z(\mathbb{C}\mathfrak{S}_n, \mathbb{C}\mathfrak{S}_{n-1})$ is commutative.

Proof. By Theorem 3.13, $Z(\mathbb{C}\mathfrak{S}_n, \mathbb{C}\mathfrak{S}_{n-1}) = \langle Z_{n-1}, X_n \rangle$. The desired result then follows immediately from the fact that X_n commutes with every element of Z_{n-1}. □

3.3.4 Eigenvalues of YJM elements

The significance of the YJM elements, as we shall eventually see, is that they have known eigenvalues when acting on the irreducible representations of the symmetric group [24].

DEFINITION 3.24. The GZ-basis for $G = \mathfrak{S}_n$ is called the ***Young basis***. By Corollary 3.4.1, the Young/GZ-vectors are common eigenvectors for GZ_n.

DEFINITION 3.25. Let v be a Young vector for $\mathfrak{S}_n$. Then define

$$\alpha(v) := (a_1, \ldots, a_n) \in \mathbb{C}^n,$$

where a_i is the eigenvalue of X_i on v. We call $\alpha(v)$ the ***weight*** of v. Note that $a_1 = 0$ since $X_1 = 0$

DEFINITION 3.26. Let

$$\operatorname{Spec}(n) := \left\{\alpha(v) : v \text{ is a Young vector}\right\}.$$

This is the ***spectrum*** of YJM-elements.

By Corollary 3.4.1,

$$\left|\operatorname{Spec}(n)\right| = \dim GZ_n = \sum_{\lambda \in \widehat{\mathfrak{S}_n}} \dim \boldsymbol{\lambda}$$

.

LEMMA 3.16

Let $\mathcal{T}$ be the set of chains in the Bratelli diagram as in Equation (3.3). Then

$$\operatorname{Spec}(n) \rightleftharpoons \mathcal{T}.$$

PROOF. For $\alpha \in \operatorname{Spec}(n)$, we denote by v_α the Young vector of weight α and T_α the chain in the Bratelli diagram corresponding to v_α. Given a $T \in \mathcal{T}$, let $\alpha(v_T)$ be denoted by $\alpha(T)$. Hence our one-to-one correspondence is given by

$$T \mapsto \alpha(T)$$
$$\alpha \mapsto T_\alpha.$$

□

We can also define the following natural equivalence relation $\sim$ on $\operatorname{Spec}(n)$.

DEFINITION 3.27. Let $\alpha, \beta \in \operatorname{Spec}(n)$. Then the following are equivalent:

(i) $\alpha \sim \beta$

(ii) v_α, v_β belong to the same irreducible module for $\mathfrak{S}_n$

(iii) T_α and T_β start at the same vertex of the Bratelli diagram.

Note that

$$\#\left(\mathrm{Spec}(n)/\sim\right)$$

is the number of paths in the Bratelli diagram from level n to level 0.

The Coxeter generators for the symmetric group are $s_i = (i, i+1)$ for $1 \le i \le n$. They commute with one another except for when the absolute value of the difference of their indices is strictly less than two. Understanding the action of s_i on v_T in terms of the weights $\alpha(T)$ will be key to understanding how the YJM elements relate to partitions and standard tableaux. Observe that we have the following relations:

$$s_i X_j = X_j s_i \qquad j \neq i, i+1 \tag{3.11}$$

$$s_i^2 = 1 \qquad X_i X_{i+1} = X_{i+1} X_i, \qquad s_i X_i + 1 = X_{i+1} s_i \tag{3.12}$$

Given T as in Equation (3.3), let $\alpha(T) = (a_1, \ldots, a_n)$ and let V be a subspace of $V_{\lambda^{(i+1)}}$ generated by v_T and $s_i v_T$. Observe that the dimension of V is at most two. Furthermore, the relations given in Equation (3.12) allow us to conclude that V is invariant under the action of s_i, X_i and X_{i+1}.

DEFINITION 3.28. The ***degenerate affine Hecke algebra*** $H(n)$ is generated by commuting variables $H_1, \ldots, H_n$ and Coxeter involutions $s_1, \ldots, s_{n-1}$ with relations

$$s_i H_j = H_j s_i \text{ for } j \neq i, i+1 \qquad s_i H_i + 1 = H_{i+1} s_i \tag{3.13}$$

Notice that the H_i in Definition 3.28 satisfy a subset of the relations that the YJM elements do in Equations (3.11) and (3.12).

LEMMA 3.17

All irreducible representations of $H(2)$ are at most 2-dimensional.

PROOF. Let V be an irreducible $H(2)$-module. Since H_1, H_2 commute they have a common eigenvector v. Let $W := \mathrm{Span}(v, sv)$. Then the dimension of W is at most two. By the relations given in Equation (3.13), we can conclude that W is a submodule. It follows immediately that $W = V$ as V is irreducible by hypothesis. □

LEMMA 3.18

For $1 \le i \le n-1$, the image of $H(2)$ in $\mathbb{C}\mathfrak{S}_n$ obtained by setting

$$s = s_i = (i, i+1) \quad H_1 = X_i, \text{ and } H_2 = X_{i+1}$$

is semisimple, i.e. the subalgebra M of $\mathbb{C}\mathfrak{S}_n$ generated by s_i, X_i, X_{i+1} is semisimple.

PROOF. Let $\mathrm{Mat}_{n!}$ be the algebra of $n! \times n!$ complex matrices where the rows and columns of these matrices are indexed by permutations in $\mathfrak{S}_n$. If we now look at the regular representations of $\mathfrak{S}_n$ we get an embedding of $\mathfrak{S}_n$ into $\mathrm{Mat}_{n!}$. Transposition matrices are real and symmetric, so as sums of transposition matrices X_i and X_{i+1} are also real and symmetric. It follows that the subalgebra M is closed under conjugate transposition. Thus, M is a $\mathbb{C}^*$-subalgebra of the $\mathbb{C}^*$-algebra $\mathrm{Mat}_{n!}$, making it semisimple as was desired. □

For our next theorem we will use the following result from linear algebra, stated without proof:

Lemma 3.19

Matrices of the form

$$\begin{pmatrix} a & \pm 1 \\ 0 & b \end{pmatrix}$$

are diagonalizable if and only if $a \neq b$. If a matrix is diagonalizable, then the eigenvalue a has eigenvector

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and the eigenvalue b has eigenvector

$$\begin{pmatrix} \pm 1/(b-a) \\ 1 \end{pmatrix}.$$

We will now use this result to give the action of s_i on the Young basis in terms of weights.

Theorem 3.20

Fix a chain T as in Equation (3.3) such that $\alpha(T) = (a_1, \ldots, a_n) \in \mathrm{Spec}(n)$. Take a Young vector $v_\alpha = v_T$. Then

(i) $a_i \neq a_{i+1}$ *for all* i.

(ii) The following are equivalent:

  a) $a_{i+1} = a_i \pm 1$.

  b) $s_i v_\alpha = \pm v_\alpha$.

  c) $s_i v_\alpha$ *and* v_α *are linearly dependent.*

(iii) for $1 \leq i \leq n-2$, *the following cannot occur:* $a_i = a_{i+1} + 1 = a_{i+2}$ *and* $a_i = a_{i+1} - 1 = a_{i+2}$.

(iv) if $a_{i+1} \neq a_i \pm 1$, *then* $\alpha' = s_i \alpha = (a_1, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots, a_n)$ *belongs to* $\mathrm{Spec}(n)$ *and* $\alpha \sim \alpha'$, *where* $\sim$ *is the relation from Definition 3.27. Moreover,*

$$v := \left(s_i - \frac{1}{a_{i+1} - a_i} \right) v_\alpha$$

is a scalar multiple of $v_{\alpha'}$. *Thus, in the basis* $\{v_\alpha, v_{\alpha'}\}$, *the actions of* X_i, X_{i+1} *and* s_i *are given by the matrices*

$$X_i \mapsto \begin{bmatrix} a_i & 0 \\ 0 & a_{i+1} \end{bmatrix} \qquad X_{i+1} \mapsto \begin{bmatrix} a_{i+1} & 0 \\ 0 & a_i \end{bmatrix} \qquad s_i \mapsto \begin{bmatrix} 1/a_{i+1} - a_i & 1 - 1/(a_{i+1} - a_i)^2 \\ 1 & -1/a_{i+1} - a_i \end{bmatrix}.$$

PROOF. Our first step will be to establish the invariance of the span of v_α and $s_i v_\alpha$ under the actions of X_i, X_{i+1} and s_i. By doing so, we will also establish the invariance under the action of the algebra they generate. This is a result of applying the relation $s_i X_{i+1} - 1 = X_i s_i$ and using the fact that by definition $X_i v_\alpha = a_i v$ and $X_{i+1} v_\alpha = a_{i+1} v_\alpha$. We now proceed through the claims of the theorem.

(i) If v_α and $s_i v_\alpha$ are linearly dependent, then $s_i v_\alpha = \lambda v_\alpha$. Since $s_i^2 = 1$, we know $\lambda^2 = 1$. Hence, $\lambda = \pm 1$. It follows that $s_i v_\alpha = \pm v_\alpha$. By the relation from Equation (3.12), we get

$$s_i X_i S_i = X_{i+1} \tag{3.14}$$

which means $a_i v_\alpha \pm v_\alpha = a_{i+1} v_\alpha$. Therefore, $s_i v_\alpha = \pm v_\alpha$, and $a_{i+1} = a_i \pm 1$

If v_α and $s_i v_\alpha$ are linearly independent, let V be the subspace of $V_{\lambda^{(i+1)}}$ they span. Then, V is invariant under the action of the algebra generated by X_i, X_{i+1} and s_i. In the basis $\{v_\alpha, s_i v_\alpha\}$, we have the matrices

$$X_i \mapsto \begin{bmatrix} a_i & -1 \\ 0 & a_{i+1} \end{bmatrix} \qquad X_{i+1} \mapsto \begin{bmatrix} a_{i+1} & 1 \\ 0 & a_i \end{bmatrix} \qquad s_i \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The action of X_i on $V_{\lambda^{(i+1)}}$ is diagonizable. Furthermore, the action of X_i on V is diagonizable owing to V being invariant under the action. Thus, by Lemma 3.19 $a_i \neq a_{i+1}$ as desired.

(ii) One direction of this argument was given in Item (i). Suppose that $a_{i+1} = a_i + 1$. Assume for the sake of contradiction that v_α and $s_i v_\alpha$ are linearly independent (if they are dependent, then the argument from (i) applies). Let

$$V := \operatorname{Span}(v_\alpha, s_i v_\alpha) \subseteq V_{\lambda^{(i+1)}},$$

a semisimple M-module. To complete the proof it suffices to show that V has a single one-dimensional M-invariant subspace. Indeed, since they are semisimple, representations of M are completely reducible. Therefore, if V has a one-dimensional M-invariant subspace W, then its complement should be another M-invariant one-dimensional subspace.

Suppose $W \subseteq V$ is a one-dimensional subspace invariant under the action of M. Let W be spanned by $\ell v_\alpha + m s_i v_\alpha$. Since W is invariant under the action of $s_i \in M$, we can conclude that $\ell, m \neq 0$. Without loss of generality, set $b = 1$. Then

$$s_i(v_\alpha + m s_i v_\alpha) = s_i v_\alpha + m v_\alpha.$$

Now $s_i^2 = 1$ implies that $v_\alpha + m s_i v_\alpha = \pm(m v_\alpha + s_i v_\alpha)$. It follows that $m = -1$, since a simple computation shows that the subspace spanned by $v_\alpha + s_i v_\alpha$ is not M-invariant. Thus, $W = \operatorname{Span}(v_\alpha - s_i v_\alpha)$. A similar argument holds for $a_{i+1} = a_i - 1$.

(iii) Assume $a_i = a_{i+1} - 1 = a_{i+2}$. By Item (ii),

$$s_i v_\alpha = v_\alpha \text{ and } s_{i+1} v_\alpha = -v_\alpha.$$

If both sides of the Coxeter relation $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ act on v_α, then $v_\alpha = -v_\alpha$ which is a contradiction. The other case is similar.

(iv) By Item (ii), v_α and $s_i v_\alpha$ are linearly independent. For $j \neq i, i+1, X_j v = a_j v$. Furthermore, by Equation (3.12), $X_i v = a_{i+1} v$ and $X_{i+1} v = a_i v$. By Corollary 3.4.1, we have that $\alpha' \in \mathrm{Spec}(n)$ and that v is a scalar multiple of $v_{\alpha'}$. Thus, $\alpha \sim \alpha'$ as $v \in V_{\lambda^{(n)}}$.

□

Thus far in using the Okounkov-Vershik approach to the representation theory of $\mathfrak{S}_n$ we have yet to see any connection with Young tableaux. This connection is established through the use of the ***content vector***.

Definition 3.29. Call $\alpha = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ a ***content vector*** if

(i) $a_1 = 0$.

(ii) for all $i > 1$, $\{a_i - 1, a_i + 1\} \cap \{a_1, \ldots, a_{i-1}\} \neq \varnothing$.

(iii) if $a_i = a_j = a$ for some $i < j$, then $\{a_i - 1, a_i + 1\} \subseteq \left\{a_{i+1}, \ldots, a_{j-1}\right\}$

Definition 3.30. We denote the set of all content vectors of length n by $\mathrm{Cont}(n)$

Example 3.31. $\mathrm{Cont}(1) = \{0\}$ and $\mathrm{Cont}(2) = \left\{(0,1), (0,-1)\right\}$.

Proposition 3.21

The condition in Definition 3.29 Item (ii) can be strengthened as follows: for all $i > 1$, if $a_i > 0$, then $a_j = a_i - 1$ for some $j < i$. If $a_i < 0$ then $a_j = a_i + 1$ for some $j < i$.

Proof. Suppose $a_i > 0$. By iteratively applying Definition 3.29 Item (ii) to Definition 3.29 Item (i), we can construct a sequence

$$a_i = a_{i_0}, a_{i_1}, \ldots, a_{i_k} = 0$$

such that $i = i_0 > i_1 > \cdots > i_k \geq 1$ with $a_{i_\ell} > 0$, and $\left|a_{i_\ell} - a_{i_{\ell+1}}\right| = 1$ for all $0 \leq h \leq k-1$. Observe that as h varies from 0 to $k-1$, a_{i_ℓ} attains all integer values between 0 and a_i. Indeed, a similar argument will also work in the case where $a_i < 0$. □

Proposition 3.22

The condition in Definition 3.29 Item (iii) can be strengthened as follows: if $i < j$, $a_i = a_j$, and $a_k \neq a_j$ for all $k = i+1, \ldots, j-1$, then there exists some unique $\ell_-, \ell_+ \in \left\{i+1, \ldots, j-1\right\}$ such that $a_{\ell_-} = a_j - 1$ and $a_{\ell_+} = a_j + 1$.

Proof. If $i < j$, then ℓ_- and ℓ_+ exist by Definition 3.29 Item (iii). To show uniqueness, assume for the sake of contradiction that there exists ℓ_-' such that $a_{\ell_-'} = a_j - 1$. Without loss of generality let $\ell_- < \ell_-'$, then by Definition 3.29 Item (iii) there exists ℓ between ℓ_- and ℓ_-' such that $a_\ell = (a_j - 1) + 1 = a_j$. This is a contradiction. □

Theorem 3.23

For all $n \geq 1$, $\mathrm{Spec}(n) \subseteq \mathrm{Cont}(n)$.

Proof. We proceed by induction on n.

BASE CASE

For $n = 1$ the desired result holds trivially.

INDUCTIVE HYPOTHESIS

Suppose that $\text{Spec}(n-1) \subseteq \text{Cont}(n-1)$.

INDUCTIVE STEP

Fix $\alpha \in \text{Spec}(n)$ with $\alpha = (a_1, \ldots, a_n)$. Since the first YJM element $X_1 = 0$, clearly $a_1 = 0$, thus satisfying Definition 3.29 Item (i). Our induction hypothesis gives us that $\alpha' = (a_1, \ldots, a_{n-1}) \in \text{Spec}(n-1) \subseteq \text{Cont}(n-1)$. To complete the proof it suffices to check that Definition 3.29 Items (ii) and (iii) hold upon the addition of a_n.

Assume for the sake of contradiction that

$$\{a_n - 1, a_n + 1\} \cap \{a_1, \ldots, a_{n-1}\} = \varnothing. \tag{3.15}$$

By Theorem 3.5 Item (iv), the transposition $(n, n-1)$ is admissible for α, which means that $(a_1, \ldots, a_{n-2}, a_n, a_{n-1}) \in \text{Spec}(n)$. Hence,

$$(a_1, \ldots, a_{n-2}, a_n) \in \text{Spec}(n-1) \subseteq \text{Cont}(n-1).$$

But $\{a_n - 1, a_n + 1\} \cap \{a_1, \ldots, a_{n-2}\} = \varnothing$ by Equation (3.15), which contradicts Definition 3.29 Item (ii) since $(a_1, \ldots, a_{n-2}, a_n) \in \text{Cont}(n-1)$. Thus, α satisfies Definition 3.29 Item (ii).

Again for the sake of contradiction, assume that α fails to satisfy Definition 3.29 Item (iii). This means that $a_i = a_n = a$ for some $i < n$. In addition, assume that i is the largest index such that

$$a \notin \{a_{i+1}, \ldots, a_{n-1}\}. \tag{3.16}$$

There are two cases: $a - 1 \notin \{a_{i+1}, \ldots, a_{n-1}\}$ and $a + 1 \notin \{a_{i+1}, \ldots, a_{n-1}\}$. These are quite similar so we will only discuss the first.

Since $(a_1, \ldots, a_{n-1}) \in \text{Cont}(n-1)$, by our inductive hypothesis $a+1$ can occur at most once in $\{a_{i+1}, \ldots, a_{n-1}\}$. if it occurred twice, then by induction a would also occur, thereby contradicting the maximality of the index i satisfying Equation (3.16). There are now two cases to consider. The first is that $(a_i, \ldots, a_n) = (a, *, \ldots, *, a)$, where $* \notin \{a-1, a, a+1\}$. A sequence of $n - i + 1$ admissible transpositions will then give us α' such that $\alpha \sim \alpha'$. $\alpha' = (\ldots, a, a, \ldots) \in \text{Spec}(n)$, which contradicts Theorem 3.20 Item (i).

The other possibility is that we have $(a_i, \ldots, a_n) = (a, *, \ldots, *, a)$, where $* \notin \{a-1, a, a+1\}$. In this case we can apply a sequence of admissible transpositions to get $\alpha' \sim \alpha$ such that $\alpha' = (\ldots, a, a+1, a, \ldots) \in \text{Spec}(n)$, thus contradicting Theorem 3.20 Item (iii).

$\square$

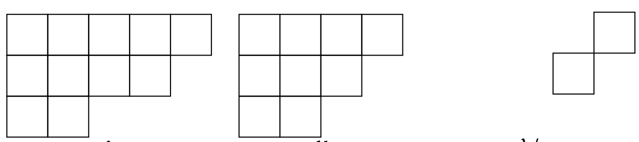
Definition 3.32. If $\alpha = (a_1, \ldots, a_n) \in \mathrm{Cont}(n)$ and $a_i \neq a_{i+1} \pm 1$, we say that the transposition s_i is ***admissible*** for α. These transpositions define an equivalence relation on $\mathrm{Cont}(n)$ where $\alpha \approx \beta$ if β can be obtained from α by a sequence of admissible transpositions.

We now arrive at the portion of the theory that ties irreducible representations of $\mathfrak{S}_n$ to Young diagrams and tableaux.

Definition 3.33. The ***Young poset*** is the set $\mathbb{Y} := \{\boldsymbol{\lambda} : \lambda \vdash n, n \in \mathbb{N}\}$ of all partitions with the poset structure as follows. Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k) \vdash n$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_h) \vdash m$ be partitions in $\mathbb{Y}$. We can state that the Young diagram corresponding to $\boldsymbol{\mu}$ is a subdiagram of the one corresponding to $\boldsymbol{\lambda}$ using the following expression:

$$\boldsymbol{\mu} \preccurlyeq \boldsymbol{\lambda} \iff m \geq n, h \geq k, \text{ and } \lambda_j \geq \mu_j \forall j = 1, \ldots, k. \tag{3.17}$$

Example 3.34. If $\boldsymbol{\mu} = (4, 3, 2)$ and $\boldsymbol{\lambda} = (5, 4, 2)$, then $\boldsymbol{\mu} \preccurlyeq \boldsymbol{\lambda}$.



$\boldsymbol{\lambda}$ $\qquad$ $\boldsymbol{\mu}$ $\qquad$ ${}^{\lambda}/_{\mu}$.

The notation ${}^{\lambda}/_{\mu}$ is used to denote the squares that remain after removing $\boldsymbol{\mu}$ from $\boldsymbol{\lambda}$.

Definition 3.35. The Young graph is a directed graph that has elements of $\mathbb{Y}$ as its vertices. The vertices $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ are connected by a directed edge from $\boldsymbol{\lambda}$ to $\boldsymbol{\mu}$ if and only if $\boldsymbol{\mu} \subseteq \boldsymbol{\lambda}$ and ${}^{\lambda}/_{\mu}$ consists of a single box. When this happens we say that $\boldsymbol{\lambda}$ ***covers*** $\boldsymbol{\mu}$, written as $\boldsymbol{\lambda} \longrightarrow \boldsymbol{\mu}$ or $\boldsymbol{\lambda} \nearrow \boldsymbol{\mu}$.

Definition 3.36. The ***content*** $c(\square)$ of a box $\square$ in a Young tableau is the y-coordinate minus the x-coordinate.

Example 3.37. Suppose we have the standard tableau

1	2	4	5	7
3	6	10	11	
8	9			

.

The contents of the boxes of this tableau are

0	1	2	3	4
−1	0	1	2	
−2	−1			

.

Definition 3.38. The ***content vector*** of a tableau T is the vector

$$\alpha = C(T) = (a_1, \ldots, a_n),$$

where the ith entry is given by the content of the cell of T in which i appears. For example, the content vector of the tableau in Example 3.37 is

$$(0, 1, -1, 2, 3, 0, 4, -2, -1, 1, 2).$$

Definition 3.39. A ***path*** in the Young graph is a sequence

$$\pi = \left(\boldsymbol{\lambda}^{(n)} \longrightarrow \boldsymbol{\lambda}^{(n-1)} \longrightarrow \cdots \longrightarrow \boldsymbol{\lambda}^{(1)}\right)$$

of partitions $\boldsymbol{\lambda}^{(\ell)} \vdash \ell$ such that $\boldsymbol{\lambda}^{(\ell-1)}$ for $2 \leq j \leq n$. A path always ends at the trivial partition $\boldsymbol{\lambda}^{(1)} = (1) \vdash 1$.

Observe that we can associate to any path in the Young graph starting at $\boldsymbol{\lambda} \vdash n$ a unique standard tableau of shape $\boldsymbol{\lambda}$. We do this by placing the integer $k \in \{1, \ldots, n\}$ in the box $\lambda^{(k)}/\lambda^{(k-1)}$.

Example 3.40. Consider a path π:

$$\underbrace{(4,3,1)}_{\boldsymbol{\lambda}^{(8)}} \longrightarrow (4,3) \longrightarrow (3,3) \longrightarrow (3,2) \longrightarrow (2,2) \longrightarrow (2,1) \longrightarrow (2) \longrightarrow \underbrace{(1)}_{\boldsymbol{\lambda}^{(1)}}.$$

To construct the corresponding standard tableau, simply add the boxes back one at a time starting at $\boldsymbol{\lambda}^{(1)}$. As we do this, we fill in the numbers in increasing order as illustrated here:

$$\begin{ytableau} 1 \end{ytableau} \quad \begin{ytableau} 1 & 2 \end{ytableau} \quad \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} \quad \begin{ytableau} 1 & 2 \\ 3 & 4 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 5 \\ 3 & 4 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 5 \\ 3 & 4 & 6 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 5 & 7 \\ 3 & 4 & 6 \end{ytableau}$$

$$\begin{ytableau} 1 & 2 & 5 & 7 \\ 3 & 4 & 6 \\ 8 \end{ytableau}$$

Definition 3.41. Suppose T_1 is a standard Young tableau of shape $\boldsymbol{\lambda}$. Assume that i and $i+1$ do not appear in the same row or column of T_1. This is sufficient to guarantee that switching the cells in which i and $i+1$ appear in T_1 will result in another standard tableau T_2, also of shape $\boldsymbol{\lambda}$. In this case we say that T_2 can be obtained from T_1 by an ***admissible*** transposition. As was the case with content vectors, this defines an equivalence relation where $T_1 \approx T_2$ if T_2 can be obtained from T_1 by a sequence of admissible transpositions.

We now state key lemmas without proof, followed by an outline of the main theorem of the Okounkov-Vershik approach tying together what we have discussed thus far. For full details the reader is referred to [29].

Let $\mathrm{Tab}(\boldsymbol{\lambda})$ denote the set of Young diagrams of shape $\boldsymbol{\lambda}$. Then

$$\mathrm{SYT}(n) = \mathrm{Tab}(n) := \bigcup_{\lambda \vdash n} \mathrm{Tab}(\boldsymbol{\lambda}).$$

LEMMA 3.24

Let

$$\Phi \colon \mathrm{Tab}(n) \longrightarrow \mathrm{Cont}(n)$$

be defined as follows: given a tableau

$$T = \left(\boldsymbol{\lambda}^{(n)} \longrightarrow \boldsymbol{\lambda}^{(n-1)} \longrightarrow \cdots \longrightarrow \boldsymbol{\lambda}^{(1)} = (1)\right),$$

define

$$\Phi(T) = C(T) = \left(c\left(\boldsymbol{\lambda}^{(1)}\right), c(\lambda^{(2)}/\lambda^{(1)}), \ldots, c(\lambda^{(n)}/\lambda^{(n-1)})\right)$$

to be the content of the tableau T. Then Φ is a bijection that takes $\approx$-equivalent standard Young tableaux to $\approx$-equivalent content vectors.

The idea is that the content vector of any standard Young tableau will satisfy the conditions of being a content vector. In fact, these conditions uniquely determine the tableau as a sequence of boxes of the Young diagram.

LEMMA 3.25

Suppose $T_1, T_2 \in \mathrm{Tab}(n)$. Then $T_1 \approx T_2$ if and only if the Young diagram of T_1 and T_2 have the same shape.

We now proceed to the main theorem of [29].

THEOREM 3.26

(i) $\mathrm{Spec}(n) = \mathrm{Cont}(n)$, *and the equivalence relations $\sim$ and $\approx$ coincide.*

(ii) Φ^{-1} *is a bijection. For $\alpha, \beta \in \mathrm{Spec}(n)$, $\alpha \sim \beta$ if and only if their images under Φ^{-1} have the same Young diagram.*

(iii) The Bratelli diagram or branching graph of a chain of symmetric groups is the Young graph. Furthermore, the spectrum of the Gelfand-Tsetlin algebra GZ_n is the space of paths in the space of standard Young tableaux with n boxes.

PROOF SKETCH.

(i)
- We have the inclusion $\mathrm{Spec}(n) \subseteq \mathrm{Cont}(n)$ already by Theorem 3.23.
- Suppose $\alpha \in \mathrm{Spec}(n)$ and $\beta \in \mathrm{Cont}(n)$. Then

$$\alpha \approx \beta \implies \beta \in \mathrm{Spec}(n) \text{ and } \alpha \sim \beta.$$

- The previous item implies that given a $\sim$-equivalence class $\mathcal{C}$ and a $\approx$-equivalence class $\mathcal{D}$, either $\mathcal{C} \cap \mathcal{D} = \varnothing$ or $\mathcal{D} \subseteq \mathcal{C}$.
- It is easy to show that

$$\left| {}^{\operatorname{Spec}(n)}/_{\sim} \right| = \left| {}^{\operatorname{Cont}(n)}/_{\approx} \right| = \# \left\{ \text{partitions of } n \right\}.$$

  The desired result then follows.

(ii) This follows from Lemma 3.25

(iii) We have a bijection between paths in the Bratelli diagram and $\operatorname{Spec}(n)$. Combining the bijection between paths in the Young graph and tableaux with Φ from Lemma 3.24 then gives a bijection between paths in the Young graph and $\operatorname{Cont}(n)$.

Furthermore, if $\alpha, \beta \in \operatorname{Cont}(n)$ correspond to paths starting at $\boldsymbol{\lambda}^{(n)}$ and $\boldsymbol{\mu}^{(n)}$ respectively, then by Lemma 3.25 $\alpha \approx \beta$ if and only if $\boldsymbol{\lambda}^{(n)} = \boldsymbol{\mu}^{(n)}$. Thus we have a bijective correspondence between the vertices of these two graphs, which clearly gives a graph isomorphism.

□

Given $\boldsymbol{\lambda} \vdash n$, denote by $S_{\boldsymbol{\lambda}}$ the irreducible representation of $\mathfrak{S}_n$ spanned by vectors $\{v_\alpha\}$ with $\alpha \in \operatorname{Spec}(n) = \operatorname{Cont}(n)$ corresponding to the standard tableaux of shape $\boldsymbol{\lambda}$. The dimension of $S_{\boldsymbol{\lambda}}$ is equal to the number of standard $\boldsymbol{\lambda}$-tableaux. Theorem 3.26 then implies the following:

Proposition 3.27

Let $0 \leq k < n$, $\boldsymbol{\lambda} \vdash n$, and $\boldsymbol{\mu} \vdash k$. Then the multiplicity $m_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ of $S_{\boldsymbol{\mu}}$ in $\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} S_{\boldsymbol{\lambda}}$ is equal to zero if $\boldsymbol{\mu} \not\preccurlyeq \boldsymbol{\lambda}$ and it equals the number of paths in $\mathbb{Y}$ from $\boldsymbol{\lambda}$ to $\boldsymbol{\mu}$.

Proof. We have

$$\operatorname{Res}^{\mathfrak{S}_k}_{\mathfrak{S}_n} S_{\boldsymbol{\lambda}} = \operatorname{Res}^{\mathfrak{S}_{k+1}}_{\mathfrak{S}_k} \operatorname{Res}^{\mathfrak{S}_{k+2}}_{\mathfrak{S}_{k+1}} \cdots \operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} S_{\boldsymbol{\lambda}}.$$

At each step of the consecutive restrictions, the decomposition is multiplicity-free and according to the branching graph $\mathbb{Y}$. Hence the multiplicity of $S_{\boldsymbol{\mu}}$ in $\operatorname{Res}^{\mathfrak{S}_k}_{\mathfrak{S}_n} S_{\boldsymbol{\lambda}}$ is equal to the number of paths in $\mathbb{Y}$ that start at $\boldsymbol{\lambda}$ and end at $\boldsymbol{\mu}$. In turn, the number of paths is equal to the number of ways to obtain the Young diagram of $\boldsymbol{\lambda}$ from that of $\boldsymbol{\mu}$. This is done by successively adding $n-k$ cells in such a way that we get a Young diagram at each stage. □

Proposition 3.28 (*Branching Rule*)

For every $\boldsymbol{\lambda} \vdash n$, we have

$$\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} S_{\boldsymbol{\lambda}} = \bigoplus_{\substack{\boldsymbol{\mu} \vdash n-1 \\ \boldsymbol{\lambda} \longrightarrow \boldsymbol{\mu}}} S_{\boldsymbol{\mu}}. \tag{3.18}$$

The sum in Equation (3.18) runs over all partitions $\boldsymbol{\mu} \vdash n-1$ that may be obtained *from $\boldsymbol{\lambda}$ by removing a single box. Furthermore, for every $\boldsymbol{\mu} \vdash n-1$,*

$$\operatorname{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} S_{\mu} = \bigoplus_{\substack{\lambda \vdash n \\ \lambda \longrightarrow \mu}} S_{\lambda}. \tag{3.19}$$

PROOF. Equation (3.18) is a particular case of Proposition 3.27, and Equation (3.19) is equivalent to Equation (3.18) by Frobenius reciprocity. □

We can now characterize the map which associates an irreducible representation of $\mathfrak{S}_n$ to any $\boldsymbol{\lambda} \vdash n$.

PROPOSITION 3.29

For all $n \geq 1$, let $\{V_\lambda\}_{\lambda \vdash n}$ be a family of representations of $\mathfrak{S}_n$ such that

(i) $V_{(1)} \cong S_{(1)}$ *(The unique trivial representation of $\mathfrak{S}_1$)*

(ii) $V_{(2)}$ *and* $V_{(1,1)}$ *are the trivial and alternating representations of $\mathfrak{S}_2$, respectively*

(iii)

$$\operatorname{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} V_{\mu} = \bigoplus_{\substack{\lambda \vdash n \\ \lambda \longrightarrow \mu}} V_{\lambda}$$

for every $\boldsymbol{\mu} \vdash n$ such that $n \geq 2$.

Then V_λ is irreducible and isomorphic to S_λ for every $\boldsymbol{\lambda} \vdash n$.

PROOF. Observe that any partition $\boldsymbol{\lambda} \vdash n$ is uniquely determined by the set

$$\left\{\boldsymbol{\mu} \vdash n-1 : \boldsymbol{\lambda} \longrightarrow \boldsymbol{\mu}\right\}.$$

The desired result then follows by induction. □

In [26, Theorem 2.8.3], it is shown that the ***Specht modules*** from Theorem 3.2 have these same properties. Hence, Proposition 3.29 relates the S_λ to the Young symmetrizer construction from Section 3.2 by telling us that S_λ is in fact isomorphic to the Specht module from Theorem 3.2.

4 DEGENERATE BETHE OPERATORS AND THE SYMMETRIC GROUP ALGEBRA

4.1 CURRENT ALGEBRA

With the prerequisite background sufficiently established, we are now ready to discuss the central objects of this thesis: the degenerate Bethe operators and the associated algebra.

It is worth taking a moment to note that these topics are important to numerous fields of study. Given the prevalence of symmetry in nature and its fundamental importance in physics, it should come as no surprise that the representation theory of the symmetric group would arise in mathematical physics. One such place is in the study of quantum integrable systems. The quantum Gaudin model, which describes a completely integrable quantum spin chain, has the ***Bethe algebra*** as its main object of study.

This commutative algebra is also of independent mathematical interest, specifically in the area of Schubert calculus. This is because the Bethe algebra acts on the space of invariant vectors in a suitable tensor product of finite-dimensional irreducible representations of $\mathfrak{gl}_N$. The dimension of this space of vectors is the intersection index of the Schubert varieties in the Grassmannian of the N-dimensional planes.

In order to further contextualize the significance of the Bethe algebra, we now provide a brief, informal discussion of Schubert varieties.

4.1.1 Schubert Varieties

The study of enumerative geometry began approximately 150 years ago with the work of Grassmann, Schubert, Pieri, Giambelli, Severi, and others. Early questions in this field included:

(i) What is the dimension of the intersection between two general lines in $\mathbb{R}^2$?

(ii) How many lines intersect two given lines and a given point in $\mathbb{R}^3$?

(iii) How many lines intersect four given lines in $\mathbb{R}^3$?

Algebraic geometry is primarily interested in objects called algebraic varieties. These are classically defined as the zero locus of a system of polynomial equations over the real or complex numbers. Modern questions in enumerative geometry concern a particular type of variety referred to as a Schubert variety. These include but are not limited to questions like "For $n \in \mathbb{N}$, what is the number of points of intersection between n Schubert varieties in general position?"

4.1.2 Lie Algebra

DEFINITION 4.1. A ***Lie algebra*** $\mathfrak{g}$ is a vector space over a field $\Bbbk$ together with a binary operation $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathfrak{g}$ called the Lie bracket that satisfies the following axioms.

BILINEARITY

$$\begin{aligned} [ax+by, z] &= a[x,z] + b[y,z] \\ [z, ax+by] &= a[z,x] + b[z,y] \end{aligned}$$

for all scalars $a, b \in \Bbbk$ and all elements $x, y, z \in \mathfrak{g}$.

ALTERNATIVITY

$$[x,x] = 0$$

for all $x \in \mathfrak{g}$.

JACOBI IDENTITY

$$\big[x, [y,z]\big] + \big[z, [x,y]\big] + \big[y, [z,x]\big] = 0$$

for all $x, y, z \in \mathfrak{g}$.

The general idea behind the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ is to embed $\mathfrak{g}$ into an associative algebra $\mathcal{A}$ with an identity element. There is an additional requirement that the abstract bracket operation in $\mathfrak{g}$ corresponds to the commutator $xy - yx$ in $\mathcal{A}$. There may be many ways to make such an embedding, but there is one "largest" such $\mathcal{A}$, called the universal enveloping algebra of $\mathfrak{g}$, denoted $U(\mathfrak{g})$.

4.1.3 Current Algebra $\mathfrak{gl}_N[t]$

Let e_{ij}, $i, j = 1, \ldots, N$ be the standard generators of the complex Lie algebra $\mathfrak{gl}_N$ satisfying the relations

$$[e_{ij}, e_{sk}] = \delta_{js}e_{ik} - \delta_{ik}e_{sj}.$$

We identify the Lie algebra $\mathfrak{sl}_N$ with the subalgebra in $\mathfrak{gl}_N$ generated by the elements $e_{ii} - e_{jj}$ and e_{ij} for $i \neq j$, $i, j = 1, \ldots, N$. The subalgebra $\mathfrak{z}_N \subsetneq \mathfrak{gl}_N$ generated by the element $\sum_{i=1}^{N} e_{ii}$ is central. The Lie algebra $\mathfrak{gl}_N$ is canonically isomorphic to the direct sum $\mathfrak{sl}_N \oplus \mathfrak{z}_N$.

DEFINITION 4.2. Let

$$\mathfrak{gl}_N[t] := \mathfrak{gl}_N \otimes \mathbb{C}[t]$$

be the complex Lie algebra of $\mathfrak{gl}_N$-valued polynomials with the pointwise commutator. We call it the ***current algebra***. We identify the Lie algebra $\mathfrak{gl}_N$ with the subalgebra $\mathfrak{gl}_N \otimes 1$ of constant polynomials in $\mathfrak{gl}_N[t]$ Hence any $\mathfrak{gl}_N[t]$-module has the canonical structure of a $\mathfrak{gl}_N$-module.

For convenience we collect the elements of $\mathfrak{gl}_N[t]$ in a generating series of a formal variable u. For $g \in \mathfrak{gl}_N$, set

$$g(u) = \sum_{s=0}^{\infty} (g \otimes t^s) u^{-s-1}.$$

For any $a \in \mathbb{C}$, there is a corresponding automorphism of $\mathfrak{gl}_n[t]$:

$$\begin{aligned} \rho_a \colon \mathfrak{gl}_n[t] &\longrightarrow \mathfrak{gl}_n[t] \\ g(u) &\mapsto g(u-a). \end{aligned}$$

The pullback of a $\mathfrak{gl}_n[t]$-module, M, through ρ_a is denoted $M(a)$.

4.1.4 Weight and Weyl modules

Definition 4.3. Let M be a $\mathfrak{gl}_N$-module. A vector $v \in M$ has ***weight*** $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N) \in \mathbb{C}^n$ if $e_{ii}v = \lambda_i v$ for $1 \leq i \leq N$. A vector v is called ***singular*** if $e_{ij}v = 0$ for all $1 \leq i < j \leq N$.

By convention, we use the following notation:

(i) $(M)_{\boldsymbol{\lambda}}$: the subspace of M of weight $\boldsymbol{\lambda}$

(ii) M^{sing}: the subspace of M consisting of all singular vectors

(iii) $(M)^{\text{sing}}_{\boldsymbol{\lambda}}$: the subspace of M of all singular vectors of weight $\boldsymbol{\lambda}$.

Definition 4.4. Let W_m be the $\mathfrak{gl}_N[t]$ module generated by a vector v_m with the defining relations:

$$\begin{aligned} e_{ii}(u)v_m &= \delta_{1i}\frac{m}{u}v_m, \quad i = 1, \ldots, N \\ e_{ij}(u)v_m &= 0, \quad 1 \leq i < j \leq N \\ \left(e_{ji} \otimes 1\right)^{m\delta_{1i}+1} v_m &= 0, \quad 1 \leq i < j \leq N \end{aligned}$$

Considered as a $\mathfrak{sl}_N[t]$ module, the module W_m is isomorphic to the Weyl module from [4, 3].

Definition 4.5. Given sequences $\boldsymbol{n} = (n_1, \ldots, n_k)$ of natural numbers and $\boldsymbol{b} = (b_1, \ldots, b_k)$ of distinct complex numbers, we refer to the $\mathfrak{gl}_N[t]$-module

$$\bigotimes_{s=1}^{k} W_{n_s}(b_s)$$

as the ***Weyl module*** associated with $\boldsymbol{n}$ and $\boldsymbol{b}$.

Definition 4.6. Let V be the standard N-dimensional vector representation of $\mathfrak{gl}_N$. Let $\mathcal{V}$ be space of polynomials in $z_1, \ldots, z_n$ with coefficients in $V^{\otimes n}$:

$$\mathcal{V} := V^{\otimes n} \otimes_{\mathbb{C}} \mathbb{C}[z_1, \ldots, z_n].$$

The space $V^{\otimes n}$ is embedded in $\mathcal{V}$ as the subspace of constant polynomials. Abusing notations we write

$$p(z_1, \ldots, z_n)v$$

instead of

$$v \otimes p(z_1, \ldots, z_n)$$

for all $v \in V^{\otimes n}$ and $p(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n]$. We can define an action of $\mathfrak{S}_n$ on $\mathcal{V}$ by taking

$$\sigma(p(z_1, \ldots, z_n)v_1 \otimes \cdots \otimes v_n)) = p(z_{\sigma(1)}, \ldots, z_{\sigma(n)})v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)} \quad \sigma \in \mathfrak{S}_n.$$

Finally, we denote by $\mathcal{V}^{\mathfrak{S}}$ the subspace of $\mathfrak{S}_n$-invariants in $\mathcal{V}$.

4.1.5 Bethe Algebra

Definition 4.7. Given a $N \times N$ matrix A with possibly non-commuting entries a_{ij}, we define its ***row determinant*** as

$$\operatorname{rdet} A = \sum_{\sigma \in \mathfrak{S}_N} (-1)^{\sigma} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{N\sigma(N)}.$$

Let ∂ be the operator of formal differentiation in the variable u. Then the ***universal differential operator*** is

$$\mathcal{D}^{\mathcal{B}} := \operatorname{rdet} \begin{pmatrix} \partial - e_{11}(u) & -e_{21}(u) & \cdots & -e_{N1}(u) \\ -e_{12}(u) & \partial - e_{22}(u) & \cdots & -e_{N2}(u) \\ \vdots & \vdots & \ddots & \vdots \\ e_{1N}(u) & -e_{2N}(u) & \cdots & \partial - e_{NN}(u) \end{pmatrix}.$$

It is a differential operator in the variable u, whose coefficients are formal power series in u^{-1} with coefficients in $U\left(\mathfrak{gl}_N[t]\right)$:

$$\mathcal{D}^{\mathcal{B}} := \partial^N + \sum_{i=1}^{N} B_i(u)\partial^{N-i}, \tag{4.1}$$

where

$$B_i(u) := \sum_{j=1}^{\infty} B_{ij} u^{-j} \tag{4.2}$$

and

$$B_{ij} \in U\left(\mathfrak{gl}_N[t]\right) \text{ for } i = 1, \ldots, N \text{ and } j \in \mathbb{Z}_{>0}$$

Clearly, $B_{ij} = 0$ for $j < i$.

DEFINITION 4.8. The ***Bethe algebra*** $\mathcal{B}$ is the unital subalgebra of $U\left(\mathfrak{gl}_N[t]\right)$ generated by B_{ij}, $i = 1, \ldots, N$, $j \in \mathbb{Z}_{>0}$.

The important detail about the Bethe algebra is that as a subalgebra of $U\left(\mathfrak{gl}_N[t]\right)$, it acts on any $\mathfrak{gl}_N[t]$-module M. If $K \subsetneq M$ is a $\mathcal{B}$-invariant subspace, then we call the image of $\mathcal{B}$ in $\mathrm{End}(K)$ the *Bethe algebra associated with* K.

For our purposes, we are interested in the action of the Bethe algebra $\mathcal{B}$ on the following $\mathcal{B}$-modules:

$$\mathcal{M}_{\lambda} = \left(\mathcal{V}^{\mathfrak{S}}\right)_{\lambda}^{\mathrm{sing}}$$

$$\mathcal{M}_{\lambda,a} = \left(\bigotimes_{s=1}^{k} W_{n_s}(b_s)\right)_{\lambda}^{\mathrm{sing}}.$$

We denote the associated Bethe algebras by $\mathcal{B}_{\lambda}$ and $\mathcal{B}_{\lambda,a}$, respectively.

4.2 SCHUBERT CELLS AND WRONSKIANS

4.2.1 *Schubert cells and flags*

Let $N, d \in \mathbb{N}$, with $N \leq d$. Let $\mathbb{C}_d[u]$ be the space of polynomials in u of degree less than d. Let $\mathrm{Gr}(N,d)$ be the space of all N-dimensional subspaces in $\mathbb{C}_d[u]$, referred to as the ***Grassmannian***.

DEFINITION 4.9. For a ***complete flag***

$$\mathcal{F} = \left\{0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_d = \mathbb{C}_d[u]\right\},$$

where the F_i are subspaces, and a partition $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)$ such that $\lambda_1 \leq d - N$, the ***Schubert cell*** $\Omega_{\lambda}(\mathcal{F}) \subset \mathrm{Gr}(N,d)$ is given by

$$\Omega_{\lambda}(\mathcal{F}) = \left\{X \in \mathrm{Gr}(N,d) : \dim(X \cap F_{d-j-\lambda_j}) = N - j, \dim(X \cap F_{d-j-\lambda_j-1}) = N - j - 1\right\}.$$

We have $\mathrm{codim}\ \Omega_{\lambda}(\mathcal{F}) = \left|\boldsymbol{\lambda}\right|$.

Given any complete flag $\mathcal{F}$ we can associate a Schubert cell decomposition. For example, in [14] we see

$$\mathrm{Gr}(N,d) = \bigsqcup_{\lambda, \lambda_1 \leq d-N} \Omega_{\lambda}(\mathcal{F}). \tag{4.3}$$

Given a partition $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)$ where $\lambda_1 \leq d - N$, we can define

$$P = \{d_1, \ldots, d_N\},\ d_i = \lambda_i + N - i,$$

as well as a partition

$$\overline{\boldsymbol{\lambda}} = (d - N - \lambda_N, d - N - \lambda_{N-1}, \ldots, d - N - \lambda_1). \tag{4.4}$$

Then $d_1 > d_2 > \cdots > d_n$ and

$$\begin{aligned}\left|\lambda\right| &= N(d-N) - \left|\overline{\lambda}\right| \\ &= \sum_{i=1}^{N} d_i - \frac{N(N-1)}{2}.\end{aligned}$$

Let $\mathcal{F}(\infty)$ be the complete flag given by

$$\mathcal{F}(\infty) = \left\{0 \subset \mathbb{C}_1[u] \subset \mathbb{C}_2[u] \subset \cdots \subset \mathbb{C}_d[u]\right\}.$$

We denote the Schubert cell $\mathcal{O}_{\overline{\lambda}}\left(\mathcal{F}(\infty)\right)$ by $\mathcal{O}_{\overline{\lambda}}(\infty)$, with $\dim \mathcal{O}_{\overline{\lambda}}(\infty) = \left|\lambda\right|$ [22].

Consider the space of all N-dimensional subspaces $X \subsetneq \mathbb{C}_d[u]$, which have a basis $\left\{f_1(u), \ldots, f_N(u)\right\}$ of the form

$$f_i(u) = u^{d_i} + \sum_{j=1, d_i - j \notin P}^{d_i} f_{ij} u^{d_i - j}. \tag{4.5}$$

These subspaces make up $\mathcal{O}_{\overline{\lambda}}(\infty) \subsetneq \mathrm{Gr}(N, d)$. Let $\mathcal{O}_\lambda$ denote the algebra of regular functions on this space of N-dimensional subspaces. Define the differential operator $\mathcal{D}_\lambda^{\mathcal{O}}$ as

$$\mathcal{D}_\lambda^{\mathcal{O}} := \frac{1}{\mathrm{Wr}\left(f_1(u), \ldots, f_N(u)\right)} \mathrm{rdet} \begin{pmatrix} f_1(u) & f_1'(u) & \cdots & f_1^{(N)}(u) \\ f_2(u) & f_2'(u) & \cdots & f_2^{(N)}(u) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \partial & \cdots & \partial^N \end{pmatrix}. \tag{4.6}$$

It is a differential operator in the variable u, whose coefficients are rational functions with coefficients in $\mathcal{O}_\lambda$. Hence

$$\mathcal{D}_\lambda^{\mathcal{O}} = \partial^N + \sum_{i=1}^{N} F_i(u) \partial^{N-i}. \tag{4.7}$$

The $F_i(u)$ from Equation (4.7) can be used to define a set of generators, F_{ij}, of $\mathcal{O}_\lambda$.

4.2.2 Wronski map

Fix a point X in $\mathrm{Gr}(N, d)$. Suppose that we have a basis of subspace X; the Wronskian of that basis does not depend on the choice of basis up to multiplication by a nonzero number. We refer to the monic polynomial representing the Wronskian as the *Wronskian of* X, and denote it by $\mathrm{Wr}_X(u)$.

Definition 4.10. Fix a partition λ. The partition $\overline{\lambda}$ is then given by Equation (4.4). The ***Wronski map*** is a map

$$\mathrm{Wr}_\lambda : \mathcal{O}_{\overline{\lambda}}(\infty) \longrightarrow \mathbb{C}^n.$$

It maps

$$X \mapsto (a_1, \ldots, a_n).$$

It can be shown that given any $\boldsymbol{a} \in \mathbb{C}^n$, we can construct the algebra of functions on the preimage $\mathrm{Wr}_{\lambda}^{-1}(\boldsymbol{a})$ as a quotient of $\mathcal{O}_\lambda$ and $\mathcal{O}_{\lambda,\boldsymbol{a}}$ [22].

4.3 ALGEBRA ISOMOPRHISMS

The main results of [22] are the following:

(i) The Bethe algebra $\mathcal{B}$ associated with the space $\mathcal{M}_\lambda = \left(\mathcal{V}^{\mathfrak{S}}\right)_\lambda^{\text{sing}}$ is isomorphic to the algebra $\mathcal{O}_\lambda$ of regular functions of the Schubert cell $\Omega_{\overline{\lambda}}(\infty)$. Furthermore, under this isomorphism the $\mathcal{B}_\lambda$-module $\mathcal{M}_\lambda$ is isomorphic to the regular representation of $\mathcal{O}_\lambda$.

(ii) Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be a sequence of complex numbers. Let distinct complex numbers $b_1, \ldots, b_k$ and integers $n_1, \ldots, n_k$ be given as in [22, (2.6)]. Then the Bethe algebra $\mathcal{B}_{\lambda,\boldsymbol{a}}$ associated with the space $\mathcal{M}_{\lambda,\boldsymbol{a}} = \left(\otimes_{s=1}^k W_{n_s}(b_s)\right)_\lambda^{\text{sing}}$ is isomorphic to the algebra $\mathcal{O}_{\lambda,\boldsymbol{a}}$ of functions on the preimage $\mathrm{Wr}_{\lambda}^{-1}(\boldsymbol{a})$. In addition, under this isomorphism the $\mathcal{B}_{\lambda,\boldsymbol{a}}$-module $\mathcal{M}_{\lambda,\boldsymbol{a}}$ is isomorphic to the regular representation of $\mathcal{O}_{\lambda,\boldsymbol{a}}$.

These connections to $\mathcal{O}_\lambda$ and the preimage $\mathrm{Wr}^{-1}(\boldsymbol{a})$ are what motivates our study of the Bethe algebra.

4.4 DEGENERATE BETHE OPERATORS

We now discuss the Bethe subalgebras of $\mathbb{C}\mathfrak{S}_N$. This family, $\mathcal{B}_n^{\mathfrak{S}}(z_1, \ldots, z_n)$, of commutative subalgebras corresponds to the Bethe algebras for the Gaudin model via the Schur-Weyl duality.

4.4.1 *Schur Weyl Duality*

In this subsection we describe the classical Schur-Weyl duality as presented in [8, Section 2].

Consider the tensor space

$$\underbrace{V \otimes V \otimes \cdots \otimes V}_{k \text{ factors}},$$

where V is an n-dimensional vector space. The symmetric group $\mathfrak{S}_k$ acts on this space by permuting the k factors as shown in the following equation:

$$\sigma(v_1 \otimes \cdots v_k) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}.$$

This action commutes with the diagonal action of GL(V). The Schur-Weyl theorem states that since $V^{\otimes k}$ is a GL$(V) \times \mathfrak{S}_N$ module, it has a decomposition into the direct sum

$$V^{\otimes k} \cong \bigoplus_{\lambda} M^{\lambda} \otimes W^{\lambda},$$

where the M^{λ} are inequivalent irreducible GL(V)-modules, the W^{λ} are inequivalent irreducible $\mathfrak{S}_N$ modules, and the sum is over partitions of k with at most n parts.

4.4.2 Bethe Operators

In this subsection we explore why Bethe operators have an expression in terms of elements of the symmetric group and compare our degenerate Bethe operators to the $\Phi_{i,j}^{[n]}$ of [21].

For distinct $r_1, \ldots, r_m \in \{1, \ldots, n\}$, we denote the embedding induced by the correspondence $i \mapsto r_i$ by the expression

$$\pi_{r_1,\ldots,r_m}^{[n]} : \mathbb{C}\mathfrak{S}_m \longrightarrow \mathbb{C}\mathfrak{S}_n.$$

Let

$$A^{[m]} = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} (\operatorname{sgn} \sigma)\sigma \tag{4.8}$$

be the antisymmetrizer in $\mathfrak{S}_m$. Given complex numbers $z_1, \ldots, z_n$, consider polynomials $\Phi_1^{[n]}(u), \ldots, \Phi_n^{[n]}(u)$ in one variable with coefficients $\mathbb{C}\mathfrak{S}_n$:

$$\Phi_i^{[n]}(u) = \sum_{1 \le r_1 < \cdots < r_i \le n} i!\pi_{r_1,\ldots,r_i}^{[n]}\left(A^{[i]}\right) \prod_{\substack{a=1 \\ a \notin \{r_1,\ldots,r_i\}}}^{n} (u - z_a) = \sum_{j=0}^{n-i} \Phi_{i,j}^{[n]} u^{n-i-j}. \tag{4.9}$$

Example 4.11. By Equation (4.9), we have

$$\Phi_1^{[n]}(u) = \sum_{r=1}^{n} \prod_{a \ne r} (u - z_a)$$

and

$$\Phi_{i,0}^{[n]} = \sum_{1 \le r_1 < \cdots < r_i \le n} i!\pi_{r_1,\ldots,r_i}^{[n]}\left(A^{[i]}\right),\ i = 1, \ldots, n.$$

Denote by $\mathcal{B}_n^{\mathfrak{S}}(z_1, \ldots, z_n)$ the subalgebra of $\mathbb{C}\mathfrak{S}_n$ generated by all $\Phi_{i,j}^{[n]}$ for $i = 1, \ldots, n, j = 0, \ldots, n-i$. This subalgebra depends on the $\{z_i\}_{i=1}^n$ as parameters. Clearly,

$$\mathcal{B}_n^{\mathfrak{S}}(z_1, \ldots, z_n) = \sigma \mathcal{B}_n^{\mathfrak{S}}\left(z_{\sigma(1)}, \ldots z_{\sigma(n)}\right)\sigma^{-1}, \tag{4.10}$$

for any $\sigma \in \mathfrak{S}_n$. The subalgebras $\mathcal{B}_n^{\mathfrak{S}}(z_1, \ldots, z_n)$ are referred to as the *Bethe subalgebras* of $\mathfrak{S}_n$ of *Gaudin type.*

Let $V = \mathbb{C}^N$ and identify elements of $\mathrm{End}(V)$ with $N \times N$ complex matrices. Let $E_{i,j}$ be the matrix with 1 in the (i, j) entry and zeroes elsewhere. Now consider the first-order differential operators in u with $\mathrm{End}(V^{\otimes n})$-valued coefficients:

$$X_{i,j} = \delta_{i,j}\partial_u - \sum_{a=1}^{n} \frac{1^{\otimes(a-1)} \otimes E_{i,j} \otimes 1^{\otimes(n-a)}}{u - z_a} \quad i, j = 1, \ldots, N.$$

We can also define an nth-order differential operator in u, given by the formula

$$\mathcal{D} = \prod_{a=1}^{n}(u - z_a) \sum_{\sigma \in \mathfrak{S}_N} \operatorname{sgn}\sigma X_{\sigma(1),1} X_{\sigma(2),2} \cdots X_{\sigma(N),N}.$$

By [20], $\mathcal{D}$ is a polynomial differential operator:

$$\mathcal{D} = \sum_{i=1}^{n} \sum_{j=0}^{n-i} (-1)^i C_{i,j}^{[n]} u^{n-i-j} \partial_u^{N-i}.$$

Denote by $\mathcal{B}_{n,N}(z_1, \ldots, z_n)$ the subalgebra of $\mathrm{End}(V^{\otimes n})$ generated by all $C_{i,j}^{[n]}$, for $i = 1, \ldots, n$, $j = 0, \ldots, n - i$. This algebra depends on the $\{z_i\}_{i=1}^n$ as parameters. This subalgebra is referred to as the Bethe algebra for the Gaudin model with parameters $z_1, \ldots, z_n$.

Our interest in $\mathcal{B}_{n,N}(z_1, \ldots, z_n)$ is due to the fact that it is the image of the Bethe subalgebra of $U(\mathfrak{gl}_N[t])$ in a tensor product of $\mathfrak{gl}_N[t]$ modules. A detailed argument for this can be found in [21, Theorem 3.1], along with the following result.

THEOREM 4.1

The algebra $\mathcal{B}_{n,N}(z_1, \ldots, z_n)$ is commutative and commutes with the action of GL_N on $V^{\otimes n}$.

Let the symmetric group $\mathfrak{S}_n$ act naturally on $V^{\otimes n}$ by permuting the factors. Denote the corresponding homomorphism by

$$\varphi \colon \mathbb{C}\mathfrak{S}_N \longrightarrow \mathrm{End}(V^{\otimes n}).$$

By Theorem 4.1, the algebra $\mathcal{B}_{n,N}(z_1, \ldots, z_n)$ is contained in the image of φ because of Schur-Weyl duality. Hence, we have the following theorem and corollary from [21, Theorem 3.2, Corollary 3.3].

THEOREM 4.2

$$\varphi(\mathcal{B}_n^{\mathfrak{S}}(z_1, \ldots, z_n)) = \mathcal{B}_{n,N}(z_1, \ldots, z_n)$$

$$\varphi\left(\Phi_{i,j}^{[n]}\right) = C_{i,j}^{[n]} \textit{ for any } i = 1, \ldots, n, j = 1, \ldots, n - i.$$

COROLLARY 4.2.1

The algebra $\mathcal{B}_{n,N}(z_1, \ldots, z_n)$ for $N \geq n$ is isomorphic to $\mathcal{B}_n^{\mathfrak{S}}(z_1, \ldots, z_n)$.

This is what allows us to write operators in $\mathcal{B}_{n,N}(z_1, \dots, z_n)$ in terms of the group algebra $\mathbb{C}\mathfrak{S}_n$. Thus, we are interested in the $\Phi^{[n]}_{i,j}$ from Equation (4.9), and in particular their leading coefficients of the universal differential equation. These leading coefficients are the degenerate Bethe operators.

DEFINITION 4.12. The ***degenerate Bethe operators*** are linear operators in $\mathbb{C}\mathfrak{S}_k$ defined as

$$B_{\ell j} := \sum_{\substack{A \subseteq [k-j] \\ |A| = \ell}} \sum_{\sigma \in \mathfrak{S}_A} \operatorname{sgn}(\sigma)\sigma \tag{4.11}$$

There are remarkable similarities between Equation (4.9) and Equation (4.11). In fact, the right side of the latter is the leading coefficient of the right side of the former. To see this, consider

$$\Phi^{[k-j]}_{\ell}(u) = \sum_{1 \le r_1 < \cdots < r_\ell \le k-j} \ell! \pi^{[k-j]}_{r_1,\dots,r_i} \left(A^{[\ell]}\right) \prod_{\substack{a=1 \\ a \notin \{r_1,\dots,r_i\}}}^{k-j} (u - z_a) = \sum_{\ell=0}^{k-j-\ell} \Phi^{[k-j]}_{\ell,j} u^{k-j-\ell}.$$

Using Equation (4.8), we get

$$\begin{aligned} \Phi^{[k-j]}_{\ell}(u) &= \sum_{1 \le r_1 < \cdots < r_\ell \le k-j} \pi^{[k-j]}_{r_1,\dots,r_\ell} \left(\sum_{\sigma \in \mathfrak{S}_\ell} (\operatorname{sgn} \sigma)\sigma \right) \prod_{\substack{a=1 \\ a \notin \{r_1,\dots,r_i\}}}^{k-j} (u - z_a) \\ &= \sum_{\ell=0}^{k-j-\ell} \Phi^{[k-j]}_{\ell,j} u^{k-j-\ell}. \end{aligned}$$

Now, observe that for distinct $1 \le r_1 < \cdots < r_\ell \le k-j$, the embedding

$$\pi^{[k-j]}_{r_1,\dots,r_\ell} : \mathbb{C}\mathfrak{S}_\ell \longrightarrow \mathbb{C}\mathfrak{S}_{k-j}$$

is equivalent to choosing a subset of ℓ elements from $[k-j]$.

LEMMA 4.3

For a finite group

$$G = \{g_1, \dots, g_r\}$$

, the complex group algebra $\mathbb{C}G$ can be identified with the space of functions on G with codomain $\mathbb{C}$.

PROOF. Fix any

$$a = \alpha_1 g_1 + \cdots + \alpha_r g_r \in \mathbb{C}G.$$

We can associate with a the function $f : G \longrightarrow \mathbb{C}$, where $f(g_i) = \alpha_i$. Conversely, given any such function f we can define a unique element of $\mathbb{C}G$ by letting α_i be the value the function takes at g_i. The sum of two functions, f and g, on G is defined by $x \mapsto f(x) + g(x)$ for all $x \in G$. The product is defined by

$$x \mapsto \sum_{uv=x} f(u)g(v) = \sum_{u \in G} f(u)g(u^{-1}x).$$

An easy calculation then shows these definitions to be consistent with those given in Definition 2.10. □

Lemma 4.4

The center of a complex group algebra $\mathbb{C}G$ consists of all functions in $\mathbb{C}G$ which are constant on the conjugacy classes of G.

Proof. By linearity, f is central in $\mathbb{C}G$ if and only if $f(gh) = f(hg)$ for all $g, h \in G$. Substituting hg^{-1} for h, this condition can be restated as

$$f(ghg^{-1}) = f(h) \text{ for all } g, h \in G.$$

□

4.5 Our results

We believe that a better understanding of the Bethe operators might allow one to come to the same conclusions as in the previous section without the need for all the constructions. While we did not conclusively show this, we were able to develop some of the theory of this rather new family of operators.

4.5.1 General results on Degenerate Bethe operators

Through the course of our investigation, we proved a number of interesting properties of the degenerate Bethe operators.

Lemma 4.5

$$B_{\ell j} := \sum_{\substack{A \subseteq [k-j] \\ |A|=\ell}} \sum_{\sigma \in \mathfrak{S}_A} \operatorname{sgn}(\sigma)\sigma \in Z\left(\mathbb{C}\mathfrak{S}_{k-j}\right)$$

Proof. The cycle type of a permutation is conjugation-invariant in any symmetric group. Hence, fixing some $k - \ell - j$ element subset of $[k - j]$ is a conjugation-invariant property. In other words, conjugation of $B_{\ell j}$ by any permutation in $\mathfrak{S}_{k-j}$ merely permutes the terms of the outermost sum defining it. The desired result then follows from Lemma 4.4. □

Lemma 4.6

The degenerate Bethe operators generate a commutative subalgebra of $\mathbb{C}\mathfrak{S}_k$.

Proof. Note that if we have $B_{\ell_1 j}$ and $B_{\ell_2 j}$, then these must commute because they are both contained in $Z\left(\mathbb{C}\mathfrak{S}_{k-j}\right)$. Now, suppose we have $B_{\ell_1 j_1}$ and $B_{\ell_2 j_2}$. Without loss of generality, let $j_2 > j_1$. Since $B_{\ell_1 j_1}$ is contained in the center of $\mathbb{C}\mathfrak{S}_{k-j_1}$, it commutes with $B_{\ell_2 j_2}$ because $B_{\ell_2 j_2}$ is a signed sum of permutations in $\mathbb{C}\mathfrak{S}_{k-j_2} \subsetneq \mathbb{C}\mathfrak{S}_{k-j_1}$. It follows that the degenerate Bethe operators are pairwise commutative. □

Lemma 4.7

The Young-Jucys-Murphy elements can be expressed in terms of degenerate Bethe operators.

Proof. Observe that B_{2j} is the signed sum of $\binom{k-j}{2}$ copies of the identity and all transpositions whose largest element is less than or equal to $k-j$. In other words,

$$B_{2j} \coloneqq \binom{k-j}{2} - \left(\sum_{i=1}^{k-j} X_i\right),$$

where X_i is the ith YJM element. Since $B_{1k-1} = 1$, it follows that

$$B'_{2j} \coloneqq B_{2j} - \binom{k-j}{2} B_{1k-1} = -\left(\sum_{i=1}^{k-j} X_i\right)$$

can be written in terms of the degenerate Bethe operators:

$$B'_{2j} - B'_{2j-1} = X_j.$$

Thus, the YJM elements can be expressed in terms of the degenerate Bethe operators as was desired. □

Proposition 4.8

The degenerate Bethe operators generate the Gelfand-Zetlin algebra.

Proof. By Lemma 4.7, the algebra generated by the YJM elements is contained in the algebra generated by the degenerate Bethe operators. The desired result then follows immediately by the well known result that the Gelfand-Zetlin algebra is a maximal commutative subalgebra generated by the YJM elements. □

4.5.2 Eigenvector-eigenvalue pair results

We also investigated the action of these degenerate Bethe operators on irreducible representations of the symmetric group. In particular, we wanted to see if any significant patterns could be found in eigenvalue-eigenvector pairs. Observe that since $B_{\ell j}$ is contained in the center of $\mathbb{C}\mathfrak{S}_{k-j}$, it follows that $B_{\ell j}$ is contained in GZ_n. Therefore, the $B_{\ell j}$ have eigenvectors that are indexed by the standard tableaux with k cells.

To calculate the eigenvalues of the degenerate Bethe operators, we use a classical construction from Alfred Young to create irreducible representations of the symmetric group. We first construct an irreducible representation of $\mathfrak{S}_n$ corresponding to the partition $\boldsymbol{\lambda}$ of n, for each such $\boldsymbol{\lambda}$. The dimension of this representation is denoted $f(\boldsymbol{\lambda})$ and is equal to the number of Young tableaux of shape $\boldsymbol{\lambda}$. Suppose S and T are two different Young tableaux of the same shape. We say that i is a disagreeing number between S and T if i appears in a different cell in S than it does in T. The ***last-letter order*** for the Young tableaux of the same shape is defined in terms of disagreeing numbers. If l is the largest disagreeing number between S and T, and l appears in a lower indexed row of S than T , then we say that $S < T$.

Let the vector space V_λ have ordered basis $T_1, \ldots, T_{f(\lambda)}$, arranged from left to right in increasing last-letter order. For example, if $\lambda = (3, 2)$ then there are five Young tableaux of shape λ, and their last-letter order is displayed below:

$$\underset{T_1}{\begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4 \\ \cline{1-2}\end{array}} < \underset{T_2}{\begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4 \\ \cline{1-2}\end{array}} < \underset{T_3}{\begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5 \\ \cline{1-2}\end{array}} < \underset{T_4}{\begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5 \\ \cline{1-2}\end{array}} < \underset{T_5}{\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5 \\ \cline{1-2}\end{array}} .$$

The reason for this choice of basis is that it is known that the standard tableaux index an eigenbasis of GZ-vectors.

For a Young tableau with n cells and $k = 1, \ldots, n-1$, we define an action of the adjacent transposition $(k, k+1)$ on T by saying that $(k, k+1)T$ is the object obtained from T by interchanging the places in which k and $k+1$ appear. Note that this is a Young tableau if k and $k+1$ do not appear in the same row or column of T, but otherwise it is not a Young tableau.

Finally, for $a, b \in \{1, \ldots, n\}$ and a Young tableau T with n cells, we define

$$d_T(a, b) = c(y) - c(x), \tag{4.12}$$

where a appears in cell x, b appears in cell y, and $c(x)$ and $c(y)$ are the contents of cells x and y, respectively. Now we define a linear transformation $\tau_\lambda((k, k+1))$ for each $k = 1, \ldots, n-1$ and each basis element T:

$$\tau^{(\lambda)}((k, k+1))T = \begin{cases} a(T,k)T + b(T,k)T' & \text{if } T' = (k, k+1)T \quad \text{is a SYT} \\ a(T,k)T & \text{otherwise} \end{cases} \tag{4.13}$$

where $a(T, k)$ and $b(T, k)$ are scalars.

Now that we have all these definitions we can proceed to Theorem 4.9 by Alfred Young.

Theorem 4.9 (*Young*)

A representation of $\mathfrak{S}_n$ is obtained from Equation (4.13) in the following case:

$$a(T,k) = \frac{1}{d_T(k, k+1)} \text{ and } b(T,k) = \begin{cases} 1 - a(T,k)^2 & \text{if } T < T' \\ 1 & \text{otherwise} \end{cases}$$

This is referred to as **Young's seminormal form**.

We used Equation (4.13) and Theorem 4.9 to write a program in a computer algebra system SAGE to compute eigenvectors and corresponding eigenvalues for our degenerate Bethe operators. The steps of the program were as follows:

First, we explicitly computed the matrices corresponding to each adjacent transposition. Then, since any permutation can be written as a product of adjacent transpositions, we computed the matrix corresponding to any permutation. Finally, we took the prescribed linear combinations in Equation (4.11) in terms of these matrices and computed the eigenvalue-eigenvector pairs of the resulting matrix.

By writing such a program, we were able to collect eigenvalue-eigenvector data for the action of the degenerate Bethe operators on a wide range of irreducible representations of the symmetric groups. Some of this data is shown in the appendix.

A number of patterns appeared in the data, some of which we were able to prove. One such pattern is that the first row of each table seems to be completely predictable and only depends upon the partition's size and number of Young tableaux. This is statement is formally captured in the following proposition.

Proposition 4.10

The degenerate Bethe operator B_{1j} will have a single eigenvalue, $k-j$, which will have associated with it every basis vector.

Proof. By the formula for the degenerate Bethe operators given in Equation (4.11), we have that B_{1j} is equal to $k-j$ times the identity. Since the identity elements must act by the identity map in any representation, the desired result follows. □

Another pattern observed in the data is that for $\boldsymbol{\lambda} = (k)$, all eigenvalues in rows $2, \ldots, k-1$ are zero.

Proposition 4.11

If the eigenvalue-eigenvector pair data for a partition $\boldsymbol{\lambda} = (k)$ were displayed as described in Section A.1, then it would have the form

ℓ/j	1	2	...	$k-2$	$k-1$
1	(k - 1, [1], 1)	(k - 2, [1], 1)	...	(2, [1], 1)	(1, [1], 1)
2	(0, [1], 1)	(0, [1], 1)	...	(0, [1], 1)	
⋮	⋮	⋮	⋮		
$k-2$	(0, [1], 1)	(0, [1], 1)			
$k-1$	(0, [1], 1)				

Proof. The form of the first row follows from Proposition 4.10. From [2, Section 3.4] we know that for the partition $\boldsymbol{\lambda} = (k)$, Young's seminormal form gives the trivial representation. Consider the formula for the degenerate Bethe operators:

$$B_{\ell j} \coloneqq \sum_{\substack{A \subseteq [k-j] \\ |A| = \ell}} \sum_{\sigma \in \mathfrak{S}_A} \operatorname{sgn}(\sigma)\sigma.$$

The inner sum is a signed sum. Since in the trivial representation every permutation acts by the identity, the desired result follows from the fact that for any A, $\mathfrak{S}_A$ has the same number of even permutations as it does odd. □

One notable pattern found across all the irreducible representations is that all eigenvalues appear to be integers. This will always be the case, as seen in the proof we have included in Proposition 4.14. Before we can do this, though, we will require a number of intermediate results. The following result, stated without proof, is from Farahat-Higman in [6].

Theorem 4.12 (*Farahat-Higman, 1959*)

For $\alpha \in \mathfrak{S}_n$, define $t(\alpha)$ to be the number of cycles (including 1-cycles) in α. Then, the center of the integral group ring of $\mathfrak{S}_n$ is generated by $C_1, C_2, \ldots, C_n$, where

$$C_i := \sum_{t(\alpha)=i} \alpha. \tag{4.14}$$

The second result we shall need to prove Proposition 4.14 is from Jucys. We state the result and give the original proof from [1] in Theorem 4.13 after the following brief review of the theory of symmetric functions.

The ring of symmetric polynomials in $n-1$ variables has infinite basis

$$e_1^{k_1} e_2^{k_2} \cdots e_{n-1}^{k_{n-1}} \ (k_i = 0, 1, \ldots; i = 1, 2, \ldots, n-1), \tag{4.15}$$

where the e's are ***elementary symmetric functions***[1] [30]:

$$\begin{aligned} e_1 &:= e_1(x_1, x_2, \ldots, x_{n-1}) = x_1 + x_2 + \cdots + x_{n-1} \\ e_2 &:= e_2(x_1, x_2, \ldots, x_{n-1}) = x_1x_2 + x_1x_3 + \cdots + x_2x_3 + \cdots + x_{n-2}x_{n-1} \\ &\ \vdots \\ e_{n-1} &:= e_{n-1}(x_1, x_2, \ldots, x_{n-1}) = x_1x_2\cdots x_{n-1}. \end{aligned} \tag{4.16}$$

The unique element of the basis in Equation (4.15) with all k's vanishing is the identity of the ring. We take it for the elementary symmetric function of zero degree. It may be written as

$$e_0 := e_0(x_1, x_2, \ldots, x_{n-1}) = e_1^0 e_2^0 \cdots e_{n-1}^0 = 1.$$

The n symmetric functions $e_0, e_1, \ldots, e_{n-1}$ are the generators of the basis in Equation (4.15), and therefore the symmetric polynomials in $n-1$ variables. Referred to as the "fundamental theorem" of the theory of symmetric functions, this result can be found in [30].

Theorem 4.13 (*Jucys, 1974*)

The center of the symmetric group algebra is generated by elementary symmetric functions in the YJM-elements.

Proof. Let X_i be the ith YJM-element. Since the C_i from Theorem 4.12 generate the center of the symmetric group algebra, it suffices to prove that

$$C_{n-p} = e_p(X_1, X_2, \ldots, X_{n-1}) \text{ for } p = 0, 1, \ldots, n-1. \tag{4.17}$$

We proceed via induction.

1 These are not to be confused with the e's from the definition of the universal differential operator.

BASE CASE

Performing the corresponding operation on the X_i involved in the definition of the symmetric functions from Equation (4.16), one can check that the desired result holds for $\mathfrak{S}_2, \mathfrak{S}_3$ and $\mathfrak{S}_4$.

INDUCTION HYPOTHESIS

Assume the desired result holds for $\mathfrak{S}_{n-1}$.

INDUCTION STEP

Consider the expression

$$e_p(X_1, X_2, \dots X_{n-1}) = e_p(X_1, X_2, \dots, X_{n-2}) + e_{p-1}(X_1, X_2, \dots, X_{n-2})X_{n-1} \tag{4.18}$$

where $p = 0, 1, \dots, n-1$. By the induction hypothesis considered as an element of the group ring of $\mathfrak{S}_{n-1}$,

$$e_p(X_1, X_2, \dots, X_{n-2}) = C_{n-1-p}.$$

If we instead consider it as an element of the group ring of $\mathfrak{S}_n$, then it is the sum of all permutations from C_{n-p} which fix n. Furthermore observe that $e_{p-1}(X_1, X_2, \dots, X_{n-2})$ is, by the induction hypothesis, the sum of all permutations of $\mathfrak{S}_{n-1}$ having $n-p$ cycles. When multiplied by X_{n-1}, the resulting permutations of $\mathfrak{S}_n$ have the same number of cycles with the symbol n not occurring in the 1-cycle and each such permutation occurring only once. Thus the expression in Equation (4.18) is the sum of all permutations of $\mathfrak{S}_n$ having $n-p$ cycles.

This completes the proof. □

PROPOSITION 4.14

The degenerate Bethe operators have integer eigenvalues when acting on complex irreducible representations of $\mathfrak{S}_k$.

PROOF. Theorem 4.12 tells us that

$$C_i \quad 1 \leq i \leq n$$

generate $Z(\mathbb{Z}\mathfrak{S}_n)$. By Theorem 4.13, we have that

$$C_{n-p} \quad 0 \leq p \leq n-1$$

is an integer polynomial in the YJM-elements. But we know that the YJM-elements have integer eigenvalues, and also that $B_{\ell j} \in Z(\mathbb{Z}\mathfrak{S}_{n-j})$. Thus, the $B_{\ell j}$ are integer polynomials in operators with integer eigenvalues, meaning they too must have integer eigenvalues. □

4.5.3 Future inquiry

A number of other patterns have been observed for which we have no formal proof. These represent potentially profitable future lines of inquiry.

(i) For $\boldsymbol{\lambda} = (1, 1, \ldots, 1)$, the operator $B_{\ell j}$ has eigenvector-eigenvalue pair data of the form

$$((k-1)(k-2)\cdots(k-j-\ell+1), [1], 1).$$

(ii) If $\boldsymbol{\lambda}$ has i parts, then for $\ell \leq i$ there exists $B_{\ell j}$ with at least one nonzero eigenvalue. In contrast, for $\ell > i$ the degenerate Bethe operators have only zero eigenvalues.

(iii) There appears to be some weak order-reversing map between the eigenvalues and the indices of the eigenvectors.

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Appendices

A | EIGENVALUE–EIGENVECTOR PAIR DATA

A.1 READING THE DATA

The numbers down the side of each table represent the possible values of ℓ, while those across the top represent the possible values of j. Each entry in the table is a list of tuples. The first number in each tuple is an eigenvalue of the operator $B_{\ell j}$. The second entry of the tuple is a list of indices - these are the indices of the standard tableaux that have the first entry of the tuple as an eigenvalue. The final entry of the tuple is simply the number of standard tableaux that have the first entry of the tuple as an eigenvalue.

Alternatively, the last entry of the tuple is simply the cardinality of the list in the second entry of the tuple. We illustrate this with an example. Suppose that $\boldsymbol{\lambda} = (2, 1, 1)$ and that we are interested in the boxed entry of the following table.

$$\underset{T_1}{\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}} < \underset{T_2}{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}} < \underset{T_3}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}}$$

ℓ/j	1	2	3
1	$(3, [1, 2, 3], 3)$	$(2, [1, 2, 3], 3)$	$(1, [1, 2, 3], 3)$
2	$(6, [1], 1), (3, [2, 3], 2)$	$\boxed{(0, [3], 1), (2, [1, 2], 2)}$	
3	$(6, [1], 1), (0, [2, 3], 2)$		

From the boxed entry we can see that the operator $B_{2,2}$ has two eigenvalues, $\{0, 2\}$. Furthermore, the eigenvalue 2 has two eigenvectors, T_1 and $T2$, while the eigenvalue 3 has one, T_3.

A.2 PARTITIONS OF FOUR

$\boldsymbol{\lambda} = (1, 1, 1, 1)$

$$\underset{T_1}{\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}}$$

ℓ/j	1	2	3
1	$(3, [1], 1)$	$(2, [1], 1)$	$(1, [1], 1)$
2	$(6, [1], 1)$	$(2, [1], 1)$	
3	$(6, [1], 1)$		

$\boldsymbol{\lambda} = (2, 1, 1)$

$$\underset{T_1}{\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}} < \underset{T_2}{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}} < \underset{T_3}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}}$$

ℓ/j	1	2	3
1	$(3, [1, 2, 3], 3)$	$(2, [1, 2, 3], 3)$	$(1, [1, 2, 3], 3)$
2	$(6, [1], 1), (3, [2, 3], 2)$	$(0, [3], 1), (2, [1, 2], 2)$	
3	$(6, [1], 1), (0, [2, 3], 2)$		

$\boldsymbol{\lambda} = (2, 2)$

$$\underset{T_1}{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}} < \underset{T_2}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}}$$

ℓ/j	1	2	3
1	$(3,[1,2],2)$	$(2,[1,2],2)$	$(1,[1,2],2)$
2	$(3,[1,2],2)$	$(2,[1],1),(0,[2],1)$	
3	$(0,[1,2],2)$		

$\boldsymbol{\lambda} = (3,1)$

$$\underset{T_1}{\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}} < \underset{T_2}{\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}} < \underset{T_3}{\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}}$$

ℓ/j	1	2	3
1	$(3,[1,2,3],3)$	$(2,[1,2,3],3)$	$(1,[1,2,3],3)$
2	$(0,[3],1),(3,[1,2],2)$	$(2,[1],1),(0,[2,3],2)$	
3	$(0,[1,2,3],3)$		

$\boldsymbol{\lambda} = (4)$

$$\underset{T_1}{\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline\end{array}}$$

ℓ/j	1	2	3
1	$(3,[1],1)$	$(2,[1],1)$	$(1,[1],1)$
2	$(0,[1],1)$	$(0,[1],1)$	
3	$(0,[1],1)$		

A.3 PARTITIONS OF FIVE

$\boldsymbol{\lambda} = (1, 1, 1, 1, 1)$

$$\underset{T_1}{\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline 5 \\ \hline \end{array}}$$

ℓ/j	1	2	3	4
1	$(4, [1], 1)$	$(3, [1], 1)$	$(2, [1], 1)$	$(1, [1], 1)$
2	$(12, [1], 1)$	$(6, [1], 1)$	$(2, [1], 1)$	
3	$(24, [1], 1)$	$(6, [1], 1)$		
4	$(24, [1], 1)$			

$\boldsymbol{\lambda} = (2, 1, 1, 1)$

$$\underset{T_1}{\begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}} < \underset{T_2}{\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array}} < \underset{T_3}{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array}} < \underset{T_4}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array}}$$

ℓ/j	1	2	3	4
1	$(4, [1,2,3,4], 4)$	$(3, [1,2,3,4], 4)$	$(2, [1,2,3,4], 4)$	$(1, [1,2,3,4], 4)$
2	$(12, [1], 1), (8, [2,3,4], 3)$	$(6, [1,2], 2), (3, [3,4], 2)$	$(0, [4], 1), (2, [1,2,3], 3)$	
3	$(24, [1], 1), (8, [2,3,4], 3)$	$(6, [1,2], 2), (0, [3,4], 2)$		
4	$(24, [1], 1), (0, [2,3,4], 3)$			

$\boldsymbol{\lambda} = (2, 2, 1)$

$$\underset{T_1}{\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}} < \underset{T_2}{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}} < \underset{T_3}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}} < \underset{T_4}{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}} < \underset{T_5}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}}$$

ℓ/j	1	2	3	4
1	$(4,[1,2,3,4,5],5)$	$(3,[1,2,3,4,5],5)$	$(2,[1,2,3,4,5],5)$	$(1,[1,2,3,4,5],5)$
2	$(6,[4,5],1),(8,[1,2,3],3)$	$(6,[1],1),(3,[2,3,4,5],4)$	$(0,[3,5],2),(2,[1,2,4],3)$	
3	$(0,[4,5],2),(8,[1,2,3],3)$	$(6,[1],1),(0,[2,3,4,5],4)$		
4	$(0,[1,2,3,4,5],5)$			

$\boldsymbol{\lambda} = (3, 1, 1)$

$$\underset{T_1}{\begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}} < \underset{T_2}{\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}} < \underset{T_3}{\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}} < \underset{T_4}{\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}} < \underset{T_5}{\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}} < \underset{T_6}{\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}}$$

ℓ/j	1	2	3	4
1	$(4, [1, 2, 3, 4, 5, 6], 6)$	$(3, [1, 2, 3, 4, 5, 6], 6)$	$(2, [1, 2, 3, 4, 5, 6], 6)$	$(1, [1, 2, 3, 4, 5, 6], 6)$
2	$(8, [1, 2, 3], 3), (4, [4, 5, 6], 3)$	$(6, [1], 1), (0, [6], 1), (3, [2, 3, 4, 5], 4)$	$(2, [1, 2, 4], 3), (0, [3, 5, 6], 3)$	
3	$(8, [1, 2, 3], 3), (0, [4, 5, 6], 3)$	$(6, [1], 1), (0, [2, 3, 4, 5, 6], 5)$		
4	$(0, [1, 2, 3, 4, 5, 6], 6)$			

$\boldsymbol{\lambda} = (3, 2)$

$$\underset{T_1}{\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}} < \underset{T_2}{\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}} < \underset{T_3}{\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}} < \underset{T_4}{\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}} < \underset{T_5}{\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}}$$

ℓ/j	1	2	3	4
1	$(4, [1, 2, 3, 4, 5], 5)$	$(3, [1, 2, 3, 4, 5], 5)$	$(2, [1, 2, 3, 4, 5], 5)$	$(1, [1, 2, 3, 4, 5], 5)$
2	$(6, [1, 2], 2), (4, [3, 4, 5], 3)$	$(0, [5], 1), (3, [1, 2, 3, 4], 4)$	$(2, [1, 3], 2), (0, [2, 4, 5], 3)$	
3	$(0, [1, 2, 3, 4, 5], 5)$	$(0, [1, 2, 3, 4, 5], 5)$		
4	$(0, [1, 2, 3, 4, 5], 5)$			

$\boldsymbol{\lambda} = (4, 1)$

$$\underset{T_1}{\begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 2 \\ \cline{1-1}\end{array}} \quad < \quad \underset{T_2}{\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}} \quad < \quad \underset{T_3}{\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}} \quad < \quad \underset{T_4}{\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}}$$

ℓ/j	1	2	3	4
1	$(4, [1, 2, 3, 4], 4)$	$(3, [1, 2, 3, 4], 4)$	$(2, [1, 2, 3, 4], 4)$	$(1, [1, 2, 3, 4], 4)$
2	$(0, [4], 1), (4, [1, 2, 3], 3)$	$(3, [1, 2], 2), (0, [3, 4], 2)$	$(2, [1], 1), (0, [2, 3, 4], 3)$	
3	$(0, [1, 2, 3, 4], 4)$	$(0, [1, 2, 3, 4], 4)$		
4	$(0, [1, 2, 3, 4], 4)$			

$\boldsymbol{\lambda} = (5)$

$$\underset{T_1}{\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline\end{array}}$$

ℓ/j	1	2	3	4
1	$(4,[1],1)$	$(3,[1],1)$	$(2,[1],1)$	$(1,[1],1)$
2	$(0,[1],1)$	$(0,[1],1)$	$(0,[1],1)$	
3	$(0,[1],1)$	$(0,[1],1)$		
4	$(0,[1],1)$			

A.4 PARTITIONS OF SIX

$\lambda = (1,1,1,1,1,1)$

$$\underset{T_1}{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline 5\\\hline 6\\\hline\end{array}}$$

ℓ/j	1	2	3	4	5
1	$(5,[1],1)$	$(4,[1],1)$	$(3,[1],1)$	$(2,[1],1)$	$(1,[1],1)$
2	$(20,[1],1)$	$(12,[1],1)$	$(6,[1],1)$	$(2,[1],1)$	
3	$(60,[1],1)$	$(24,[1],1)$	$(6,[1],1)$		
4	$(120,[1],1)$	$(24,[1],1)$			
5	$(120,[1],1)$				

$\lambda = (2,1,1,1,1)$

$$\underset{T_1}{\begin{array}{|c|c|}\hline 1 & 6 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}} < \underset{T_2}{\begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} 6 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}} < \underset{T_3}{\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 5 & \multicolumn{1}{c}{} \\ \cline{1-1} 6 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}} < \underset{T_4}{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} 5 & \multicolumn{1}{c}{} \\ \cline{1-1} 6 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}} < \underset{T_5}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} 5 & \multicolumn{1}{c}{} \\ \cline{1-1} 6 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}}$$

ℓ/j	1	2	3	4	5
1	(5, [1, 2, 3, 4, 5], 5)	(4, [1, 2, 3, 4, 5], 5)	(3, [1, 2, 3, 4, 5], 5)	(2, [1, 2, 3, 4, 5], 5)	(1, [1, 2, 3, 4, 5], 5)
2	(20, [1], 1), (15, [2, 3, 4, 5], 4)	(12, [1, 2], 2), (8, [3, 4, 5], 3)	(3, [4, 5], 2), (6, [1, 2, 3], 3)	(0, [5], 1), (2, [1, 2, 3, 4], 4)	
3	(60, [1], 1), (30, [2, 3, 4, 5], 4)	(24, [1, 2], 2), (8, [3, 4, 5], 3)	(0, [4, 5], 2), (6, [1, 2, 3], 3)		
4	(120, [1], 1), (30, [2, 3, 4, 5], 4)	(24, [1, 2], 2), (0, [3, 4, 5], 3)			
5	(120, [1], 1), (0, [2, 3, 4, 5], 4)				

$\boldsymbol{\lambda} = (2, 2, 1, 1)$

$$\underset{T_1}{\begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 & 6 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}} < \underset{T_2}{\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 6 \\ \hline 3 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}} < \underset{T_3}{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 6 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}} < \underset{T_4}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 6 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}} < \underset{T_5}{\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}} < \underset{T_6}{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}} < \underset{T_7}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}} < \underset{T_8}{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}} < \underset{T_9}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}}$$

ℓ/j	1	2	3
1	(5, [1, 2, 3, 4, 5, 6, 7, 8, 9], 9)	(4, [1, 2, 3, 4, 5, 6, 7, 8, 9], 9)	(3, [1, 2, 3, 4, 5, 6, 7, 8, 9], 9)
2	(15, [1, 2, 3, 4], 4), (12, [5, 6, 7, 8, 9], 5)	(12, [1], 1), (6, [8, 9], 2), (8, [2, 3, 4, 5, 6, 7], 6)	(6, [1, 2, 5], 3), (3, [3, 4, 6, 7, 8, 9], 6)
3	(30, [1, 2, 3, 4], 4), (12, [5, 6, 7, 8, 9], 5)	(24, [1], 1), (0, [8, 9], 2), (8, [2, 3, 4, 5, 6, 7], 6)	(6, [1, 2, 5], 3), (3, [3, 4, 6, 7, 8, 9], 6)
4	(30, [1, 2, 3, 4], 4), (0, [5, 6, 7, 8, 9], 5)	(24, [1], 1), (0, [2, 3, 4, 5, 6, 7, 8, 9], 8)	
5	(0, [1, 2, 3, 4, 5, 6, 7, 8, 9], 9)		

4	5
(2, [1, 2, 3, 4, 5, 6, 7, 8, 9], 9)	(1, [1, 2, 3, 4, 5, 6, 7, 8, 9], 9)
(0, [4, 7, 9], 3), (2, [1, 2, 3, 5, 6, 8], 6)	

$\boldsymbol{\lambda} = (2, 2, 2)$

$$\underset{T_1}{\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline\end{array}} < \underset{T_2}{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline\end{array}} < \underset{T_3}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & 6 \\ \hline\end{array}} < \underset{T_4}{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 6 \\ \hline\end{array}} < \underset{T_5}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline\end{array}}$$

ℓ/j	1	2	3	4	5
1	$(5, [1, 2, 3, 4, 5], 5)$	$(4, [1, 2, 3, 4, 5], 5)$	$(3, [1, 2, 3, 4, 5], 5)$	$(2, [1, 2, 3, 4, 5], 5)$	$(1, [1, 2, 3, 4, 5], 5)$
2	$(12, [1, 2, 3, 4, 5], 5)$	$(6, [4, 5], 2), (8, [1, 2, 3], 3)$	$(6, [1], 1), (3, [2, 3, 4, 5], 4)$	$(0, [3, 5], 2), (2, [1, 2, 4], 3)$	
3	$(12, [1, 2, 3, 4, 5], 5)$	$(0, [4, 5], 2), (8, [1, 2, 3], 3)$	$(6, [1], 1), (0, [2, 3, 4, 5], 4)$		
4	$(0, [1, 2, 3, 4, 5], 5)$	$(0, [1, 2, 3, 4, 5], 5)$			
5	$(0, [1, 2, 3, 4, 5], 5)$				

$\boldsymbol{\lambda} = (3, 3)$

$$\underset{T_1}{\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline\end{array}} < \underset{T_2}{\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline\end{array}} < \underset{T_3}{\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline\end{array}} < \underset{T_4}{\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline\end{array}} < \underset{T_5}{\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline\end{array}}$$

ℓ/j	1	2	3	4	5
1	$(5, [1, 2, 3, 4, 5], 5)$	$(4, [1, 2, 3, 4, 5], 5)$	$(3, [1, 2, 3, 4, 5], 5)$	$(2, [1, 2, 3, 4, 5], 5)$	$(1, [1, 2, 3, 4, 5], 5)$
2	$(8, [1, 2, 3, 4, 5], 5)$	$(6, [1, 2], 2), (4, [3, 4, 5], 3)$	$(0, [5], 1), (3, [1, 2, 3, 4], 4)$	$(2, [1, 3], 2), (0, [2, 4, 5], 3)$	
3	$(0, [1, 2, 3, 4, 5], 5)$	$(0, [1, 2, 3, 4, 5], 5)$	$(0, [1, 2, 3, 4, 5], 5)$		
4	$(0, [1, 2, 3, 4, 5], 5)$	$(0, [1, 2, 3, 4, 5], 5)$			
5	$(0, [1, 2, 3, 4, 5], 5)$				

$\boldsymbol{\lambda} = (4, 2)$

$$\underset{T_1}{\begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}} < \underset{T_2}{\begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}} < \underset{T_3}{\begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}} < \underset{T_4}{\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 6 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}} < \underset{T_5}{\begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 6 \\ \hline 4 & 4 \\ \cline{1-2}\end{array}} < \underset{T_6}{\begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 2 & 6 \\ \cline{1-2}\end{array}} <$$

$$\underset{T_7}{\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 3 & 6 \\ \cline{1-2}\end{array}} < \underset{T_8}{\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 \\ \cline{1-2}\end{array}} < \underset{T_9}{\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2}\end{array}}$$

ℓ/j	1	2	3	4
1	(5, [1, 2, 3, 4, 5, 6, 7, 8, 9], 9)	(4, [1, 2, 3, 4, 5, 6, 7, 8, 9], 9)	(3, [1, 2, 3, 4, 5, 6, 7, 8, 9], 9)	(2, [1, 2, 3, 4, 5, 6, 7, 8, 9], 9)
2	(5, [6, 7, 8, 9], 4)(8, [1, 2, 3, 4, 5], 5)	(0, [9], 1), (6, [1, 2], 2), (4, [3, 4, 5, 6, 7, 8], 6)	(0, [5, 8, 9], 3), (3, [1, 2, 3, 4, 6, 7], 6)	(2, [1, 3, 6], 3), (0, [2, 4, 5, 7, 8, 9], 6)
3	(0, [1, 2, 3, 4, 5, 6, 7, 8, 9], 9)	(0, [1, 2, 3, 4, 5, 6, 7, 8, 9], 9)	(0, [1, 2, 3, 4, 5, 6, 7, 8, 9], 9)	
4	(0, [1, 2, 3, 4, 5, 6, 7, 8, 9], 9)	(0, [1, 2, 3, 4, 5, 6, 7, 8, 9], 9)		
5	(0, [1, 2, 3, 4, 5, 6, 7, 8, 9], 9)			

5	(1, [1, 2, 3, 4, 5, 6, 7, 8, 9], 9)

$\boldsymbol{\lambda} = (5, 1)$

$$\underset{T_1}{\begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 5 & 6 \\ \hline 2 \\ \cline{1-1}\end{array}} < \underset{T_2}{\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 5 & 6 \\ \hline 3 \\ \cline{1-1}\end{array}} < \underset{T_3}{\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 5 & 6 \\ \hline 4 \\ \cline{1-1}\end{array}} < \underset{T_4}{\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 6 \\ \hline 5 \\ \cline{1-1}\end{array}} < \underset{T_5}{\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 \\ \cline{1-1}\end{array}}$$

ℓ/j	1	2	3	4	5
1	$(5, [1, 2, 3, 4, 5], 5)$	$(4, [1, 2, 3, 4, 5], 5)$	$(3, [1, 2, 3, 4, 5], 5)$	$(2, [1, 2, 3, 4, 5], 5)$	$(1, [1, 2, 3, 4, 5], 5)$
2	$(0, [5], 1), (5, [1, 2, 3, 4], 4)$	$(0, [4, 5], 2), (4, [1, 2, 3], 3)$	$(3, [1, 2], 2), (0, [3, 4, 5], 3)$	$(2, [1], 1), (0, [2, 3, 4, 5], 4)$	
3	$(0, [1, 2, 3, 4, 5], 5)$	$(0, [1, 2, 3, 4, 5], 5)$	$(0, [1, 2, 3, 4, 5], 5)$		
4	$(0, [1, 2, 3, 4, 5], 5)$	$(0, [1, 2, 3, 4, 5], 5)$			
5	$(0, [1, 2, 3, 4, 5], 5)$				

$\boldsymbol{\lambda} = (6)$

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 & 6 \\ \hline\end{array}$$

T_1

ℓ/j	1	2	3	4	5
1	(5, [1], 1)	(4, [1], 1)	(3, [1], 1)	(2, [1], 1)	(1, [1], 1)
2	(0, [1], 1)	(0, [1], 1)	(0, [1], 1)	(0, [1], 1)	
3	(0, [1], 1)	(0, [1], 1)	(0, [1], 1)		
4	(0, [1], 1)	(0, [1], 1)			
5	(0, [1], 1)				