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HILBERT SPACE OPERATORS WITH COMPATIBLE OFF-DIAGONAL CORNERS

L. LIVSHITS, G. MACDONALD[1], L.W. MARCOUX[1], AND H. RADJAVI[1]

Abstract. Given a complex, separable Hilbert space $\mathcal{H}$, we characterize those operators for which $\|PT(I-P)\| = \|(I-P)TP\|$ for all orthogonal projections P on $\mathcal{H}$. When $\mathcal{H}$ is finite-dimensional, we also obtain a complete characterization of those operators for which $\operatorname{rank}(I-P)TP = \operatorname{rank} PT(I-P)$ for all orthogonal projections P. When $\mathcal{H}$ is infinite-dimensional, we show that any operator with the latter property is normal, and its spectrum is contained in either a line or a circle in the complex plane.

1. Introduction

1.1. Let $\mathcal{H}$ be a complex, separable Hilbert space. By $\mathcal{B}(\mathcal{H})$, we denote the algebra of bounded, linear operators on $\mathcal{H}$. If $\dim \mathcal{H} = n < \infty$, then we identify $\mathcal{H}$ with $\mathbb{C}^n$ and $\mathcal{B}(\mathcal{H})$ with $\mathbb{M}_n(\mathbb{C})$.

One of the most important open problems in operator theory is the *Invariant Subspace Problem*, which asks whether or not every bounded, linear operator T acting on a complex, infinite-dimensional, separable Hilbert space $\mathcal{H}$ admits a non-trivial invariant subspace; that is, a closed subspace $\mathcal{M} \notin \{\{0\}, \mathcal{H}\}$ for which $T\mathcal{M} \subseteq \mathcal{M}$.

We say that an operator $T \in \mathcal{B}(\mathcal{H})$ is **(orthogonally) reductive** if for each orthogonal projection $P \in \mathcal{B}(\mathcal{H})$, the condition $PT(I-P) = 0$ implies that $(I-P)TP = 0$. The *Reductive Operator Conjecture* is the assertion that every reductive operator is normal. It was shown by Dyer, Pederson and Procelli [9] that the Invariant Subspace Problem admits a positive solution if and only if the *Reductive Operator Conjecture* is true.

Our goal in this paper is to study two variants of orthogonal reductivity. Let $T \in \mathcal{B}(\mathcal{H})$ and $P \in \mathcal{B}(\mathcal{H})$ be an orthogonal projection. We refer to the operator

$$P^{\perp}TP : P\mathcal{H} \to P^{\perp}\mathcal{H}$$

as an **off-diagonal corner** of T.

Relative to the decomposition $\mathcal{H} = P\mathcal{H} \oplus P^{\perp}\mathcal{H}$, we may write $T = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. We refer to the block-entries of such block-matrices via their geographic positions: NW, NE, SE, SW, and the NE and the SW block-entries are examples of the off-diagonal corners.

In the work below, we shall be interested in two phenomena: firstly, when the operator norm of $B (= B_P)$ coincides with the operator norm of $C (= C_P)$ for all projections P, and secondly, when the rank of B coincides with the rank of C for all projections P. Clearly, any operator which satisfies one of these two conditions is orthogonally reductive. An example is given in Section 5 below to show that the converse to this statement is false.

In the case of normal matrices, some related work has been done by Bhatia and Choi [5]. For instance, if the dimension of the space is $2n < \infty$, and if P is a projection of rank n, it is a consequence of the fact that the Euclidean norm of the k^{th} column of a normal matrix coincides with that of the k^{th} row for all k that the Hilbert-Schmidt (or Frobenius) norm of B always equals that of C. Further, they show that $\|B\| \le \sqrt{n}\|C\|$, and that equality can be achieved for some normal matrix $T \in \mathbb{M}_{2n}(\mathbb{C})$ and some projection P of rank n if and only if $n \le 3$.

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1.2. Definition. *Let $T \in \mathcal{B}(\mathcal{H})$. We say that T has the* ***common norm property*** *(property (CN)) if for any projection $P \in \mathcal{B}(\mathcal{H})$ we have that*

$$\|PTP^{\perp}\| = \|P^{\perp}TP\|.$$

We denote by $\mathfrak{G}_{\mathfrak{norm}}$ the set of operators with property (CN). We say that T has the ***common rank property*** *(property (CR)) if for any projection $P \in \mathcal{B}(\mathcal{H})$ we have that*

$$\operatorname{rank} PTP^{\perp} = \operatorname{rank} P^{\perp}TP.$$

We denote by $\mathfrak{G}_{\mathfrak{rank}}$ the set of operators with property (CR).

As we shall see, our results depend upon whether or not $\mathcal{H}$ is finite-dimensional. When the Hilbert space is finite-dimensional and of dimension at least four, then we shall show that the set of operators satisfying property (CN) coincides with the set of operators satisfying property (CR), and that this consists of those operators which are scalar translates of scalar multiples of hermitian (or of unitary) operators. (See Theorem 3.15 below.)

In the infinite-dimensional setting, we obtain a complete characterization of those operators satisfying property (CN). Again, any scalar translate of a scalar multiple of a hermitian operator will suffice. This time, however, the unitary operators involved must have essential spectrum contained in only half of a circle. (See Theorem 4.13 below.)

The problem of characterizing those operators acting on an infinite-dimensional Hilbert space which enjoy property (CR) is much more delicate. We are able to demonstrate that any operator T satisfying property (CR) must once again be a scalar translate of a scalar multiple of a hermitian (or of a unitary) operator. In particular, such operators are normal. However, an obstruction occurs in that it is not the case that every unitary operator has property (CR). Indeed, as is well-known (see Section 5 for an example) – not every unitary operator is reductive.

1.3. We shall need some standard notations and definitions in what follows.

If $T = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is a block-matrix in $\mathbb{M}_n(\mathbb{C})$, and A is invertible, then the matrix $D - CA^{-1}B$ is said to be the **Schur complement** of A in T and is denoted by $T|A$. In such a case T is invertible if and only if $T|A$ is, and when this happens, the SE block-corner $\left(T^{-1}\right)_{SE}$ of T^{-1} is $(T|A)^{-1}$. Furthermore:

$$\left(T^{-1}\right)_{SW} = -(T|A)^{-1}CA^{-1} \quad \text{and} \quad \left(T^{-1}\right)_{NE} = -A^{-1}B\,(T|A)^{-1}.$$

Similarly, if B is invertible then $C - DB^{-1}A$ is the Schur complement $T|B$ of B in T, and T is invertible if and only if $T|B$ is, in which case

$$\left(T^{-1}\right)_{NE} = (T|B)^{-1}.$$

Corresponding statements and concepts apply to C and D as well.

As always, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. A subset of $\mathbb{C}$ is **circlinear** if it is contained in a circle or a straight line. By $\mathcal{K}(\mathcal{H})$, we denote the closed, two-sided ideal of compact operators in $\mathcal{B}(\mathcal{H})$, and $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ denotes the canonical map from $\mathcal{B}(\mathcal{H})$ into the **Calkin algebra** $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. The **essential spectrum** $\sigma_e(T)$ of $T \in \mathcal{B}(\mathcal{H})$ is the spectrum of $\pi(T)$ in the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, and we say that T is a **Fredholm operator** if $0 \notin \sigma_e(T)$. The **Fredholm domain** of T is $\varrho_F(T) = \mathbb{C} \setminus \sigma_e(T)$. We say that T is a **semi-Fredholm operator** if $\pi(T)$ is either left or right invertible in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, and define the **semi-Fredholm domain** of T to be $\varrho_{sF}(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ is semi-Fredholm}\}$. The complement of $\varrho_{sF}(T)$ is called the **left-right essential spectrum** of T and is denoted by $\sigma_{\ell re}(T)$. If T is semi-Fredholm, we define the **index** of T to be $\operatorname{ind} T = \operatorname{nul} T - \operatorname{nul} T^* \in \mathbb{Z} \cup \{-\infty, \infty\}$. When T is Fredholm, we have that $\operatorname{ind} T \in \mathbb{Z}$.

We say that T is **triangular** if there exists an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ for $\mathcal{H}$ such that the matrix $[T] = [t_{i,j}]$ for T relative to this basis (i.e. $t_{i,j} = \langle Te_j, e_i \rangle$) satisfies $t_{i,j} = 0$ for all $i > j$. The operator is said to be **quasitriangular** if it is of the form $T = T_0 + K$, where T_0 is triangular and K is compact. It was shown by Apostol, Foiaş, and Voiculescu [2] that T is quasitriangular if and only if $\operatorname{ind}(T - \lambda I) \geq 0$ whenever $\lambda \in \varrho_{sF}(T)$. Finally, T is **biquasitriangular** if each of T and T^* is quasitriangular, i.e. if and only if $\operatorname{ind}(T - \lambda I) = 0$ for all $\lambda \in \varrho_{sF}(T)$.

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Recall also that if $T \in \mathcal{B}(\mathcal{H})$, then $|T| = (T^*T)^{1/2}$ denotes the absolute value of T. A unitary operator $U \in \mathcal{B}(\mathcal{H})$ is said to be **absolutely continuous** if the spectral measure for U is absolutely continuous with respect to Lebesgue measure restricted to $\sigma(U)$, while U is said to be **singular** if the spectral measure of U is singular with respect to Lebesgue measure restricted to $\sigma(U)$. These notions will only be used in Section 5.

2. Preliminary results

2.1. We begin with a few simple remarks. Although the proofs are rather elementary, we shall list these in the form of a Proposition so as to be able to more easily refer to them later. The proofs are left to the reader.

2.2. Proposition. *Suppose that $R, T \in \mathcal{B}(\mathcal{H})$ and that R has property* (CR) *and T has property* (CN).

(a) *For all $\lambda, \mu \in \mathbb{C}$, we have that*
- *$\lambda I + \mu R$ and R^* have property* (CR), *while*
- *$\lambda I + \mu T$ and T^* have property* (CN).

(b) *Suppose that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$.*
- *If there exist $A \in \mathcal{B}(\mathcal{H}_1)$ and $D \in \mathcal{B}(\mathcal{H}_2)$ such that $R = A \oplus D$, then A and D both have property* (CR).
- *If there exist $A \in \mathcal{B}(\mathcal{H}_1)$ and $D \in \mathcal{B}(\mathcal{H}_2)$ such that $T = A \oplus D$, then A and D both have property* (CN).

(c) *If $V \in \mathcal{B}(\mathcal{H})$ is unitary, then*
- *V^*RV has property* (CR) *and*
- *V^*TV has property* (CN).

(d) *If $L = L^* \in \mathcal{B}(\mathcal{H})$, then L has both property* (CR) *and property* (CN).

In the case of property (CN), we also observe the following. For $T \in \mathcal{B}(\mathcal{H})$, let us denote by $\mathcal{U}(T)$ the **unitary orbit** of T, i.e. $\mathcal{U}(T) = \{V^*TV : V \in \mathcal{B}(\mathcal{H}) \text{ unitary}\}$. Recall that two operators S and T are said to be **approximately unitarily equivalent** if $S \in \overline{\mathcal{U}(T)}$ (equivalently, $T \in \overline{\mathcal{U}(S)}$). The proofs of the following assertions are elementary and are left to the reader.

2.3. Proposition.

(a) *The set $\mathfrak{G}_{\mathfrak{norm}}$ of operators with property* (CN) *is closed.*

(b) *If $T \in \mathcal{B}(\mathcal{H})$ has property* (CN) *and there exists $S \in \overline{\mathcal{U}(T)}$ of the form $S = A \oplus D$, then A, D have property* (CN).

The following remark, while innocuous in appearance, is actually the key to a number of calculations below.

2.4. Remark. Let $U \in \mathcal{B}(\mathcal{H})$ be a unitary operator and $P \in \mathcal{B}(\mathcal{H})$ be a projection. Write $U = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ relative to $\mathcal{H} = P\mathcal{H} \oplus P^\perp\mathcal{H}$. The fact that U is unitary implies that

$$I = AA^* + BB^* = A^*A + C^*C.$$

Thus $BB^* = I - AA^*$ and $C^*C = I - A^*A$.

It follows that

$$\|B\|^2 = \|BB^*\| = 1 - \min\{\lambda : \lambda \in \sigma(AA^*)\},$$

and similarly

$$\|C\|^2 = \|C^*C\| = 1 - \min\{\mu : \mu \in \sigma(A^*A)\}.$$

However, it is a standard fact that $\sigma(AA^*) \cup \{0\} = \sigma(A^*A) \cup \{0\}$, and thus the only way that we can have $\|B\| \neq \|C\|$ is if either

(I) $0 \in \sigma(AA^*)$ but $0 \notin \sigma(A^*A)$, or

(II) $0 \in \sigma(A^*A)$ but $0 \notin \sigma(AA^*)$.

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This argument demonstrates the rather interesting fact that if $U = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is a unitary operator and $\|B\| \neq \|C\|$, then

$$\min(\|B\|, \|C\|) < 1 = \max(\|B\|, \|C\|).$$

In particular, if $U = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is a unitary such that $\|B\| < \|C\| (= 1)$, then every unitary U' close enough to U has the form $\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right]$ where $\|B'\| < \|C'\| = \mathbf{1}$, which is remarkable.

3. The finite-dimensional setting

3.1. We now turn to the case where the Hilbert space under consideration is finite-dimensional (and complex).

3.2. Proposition. *Let $n \geq 2$ be an integer and $T \in \mathbb{M}_n(\mathbb{C})$. If T has property* (CN) *or property* (CR)*, then T is normal.*

Proof. This is an easy consequence of the fact that given any $T \in \mathbb{M}_n(\mathbb{C})$, there exists an orthonormal basis with respect to which the matrix of T is upper triangular. Either property clearly implies that the matrix of T is in fact diagonal with respect to this basis, and hence that T is normal.

□

3.3. Proposition. *Let $n \geq 2$, and let $U \in \mathbb{M}_n(\mathbb{C})$ be a unitary operator. Then U has both property* (CN) *and property* (CR).

Proof. Let $P \in \mathbb{M}_n(\mathbb{C})$ be a projection, and relative to the decomposition $\mathbb{C}^n = P\mathbb{C}^n \oplus P^\perp \mathbb{C}^n$, let us write

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

As noted in Remark 2.4, since U is unitary, we have that $BB^* = I - AA^*$ and $C^*C = I - A^*A$. Observe, however, that in the finite-dimensional setting we have that A^*A is unitarily equivalent to AA^*, and thus BB^* is unitarily equivalent to C^*C. Thus

- $\|B\| = \|C\|$, and
- $\operatorname{rank} B = \operatorname{rank} BB^* = \operatorname{rank} C^*C = \operatorname{rank} C$.

□

Combining this with Proposition 2.2 (a), we obtain:

3.4. Proposition. *Let $T \in \mathbb{M}_n(\mathbb{C})$. If T is either hermitian or unitary, then for all $\lambda, \mu \in \mathbb{C}$, $\lambda I + \mu T$ has both property* (CN) *and property* (CR).

Our goal is to prove that if $T \in \mathbb{M}_n(\mathbb{C})$ has either property (CN) or property (CR), then it is of the form $\lambda I + \mu X$ where X is either hermitian or unitary.

3.5. Remark. The common link between these two cases is the geometry of the set of eigenvalues of T. If T is normal, then $T = \lambda I + \mu V$ where V is unitary if and only if all of the eigenvalues of T lie on a common circle. If T is normal, then $T = \lambda I + \mu L$ where $L = L^*$ if and only if all of the eigenvalues of T lie on a common line. That is to say, the union of these two sets of operators is precisely the class of normal operators whose spectra are circlinear.

Given a matrix $B \in \mathbb{M}_{n,m}(\mathbb{C})$, we denote by $\|B\|_2 = \operatorname{tr}(B^*B)^{1/2}$ the Hilbert-Schmidt (or Fröbenius) norm of B.

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3.6. Proposition. *Let $k, \ell \geq 1$ be integers, and suppose that $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a normal operator in $\mathcal{B}(\mathbb{C}^k \oplus \mathbb{C}^\ell)$. Then*

$$\|B\|_2 = \|C\|_2.$$

Proof. The fact that T is normal implies that $AA^* + BB^* = A^*A + C^*C$. Using the fact that the trace is linear and that $\mathrm{tr}(XY) = \mathrm{tr}(YX)$ for all $X \in \mathbb{M}_{k,\ell}(\mathbb{C})$, $Y \in \mathbb{M}_{\ell,k}(\mathbb{C})$, we see that

$$\|B\|_2 = \mathrm{tr}(BB^*) = \mathrm{tr}(C^*C) = \|C\|_2.$$

□

We begin by considering the exceptional cases where the dimension of the underlying Hilbert space is too small to allow anything interesting to happen.

3.7. Proposition. *Let $2 \leq n \leq 3$, and let $T \in \mathbb{M}_n(\mathbb{C})$. The following are equivalent.*

(a) *T is normal.*
(b) *T has property* (CN).
(c) *T has property* (CR).

Proof. By Proposition 3.2, both (b) and (c) imply (a).

Conversely, if $T \in \mathbb{M}_n(\mathbb{C})$ is normal and $0 \neq P \neq I$ is a projection in $\mathbb{M}_n(\mathbb{C})$, then $PTP^\perp$ and $P^\perp TP$ both have rank at most one. From this and from Proposition 3.6, we find that

$$\|PTP^\perp\| = \|PTP^\perp\|_2 = \|P^\perp TP\|_2 = \|P^\perp TP\|.$$

Thus T has property (CN); that is, (a) implies (b). This also shows that $PTP^\perp$ and $P^\perp TP$ either both have rank 0 or both have rank 1. Hence (a) implies (c) as well.

□

For the remainder of this section, we shall assume that the dimension n of the underlying Hilbert space is at least 4.

3.8. Remark. Let us now show that the problem of characterizing which operators in $\mathbb{M}_n(\mathbb{C})$ have property (CN) (resp. property (CR)) reduces to the case where $n = 4$. Of course, by Proposition 3.2, we may restrict our attention to normal operators.

Let $n > 4$, and suppose that $T \in \mathbb{M}_n(\mathbb{C})$ is normal.

- As observed in Remark 3.5, if all of the eigenvalues of T are either co-linear or co-circular (i.e. all lie on the same circle), then there exist $\alpha, \beta \in \mathbb{C}$ and either a hermitian operator L or a unitary operator V such that $T = \alpha I + \beta L$, or $T = \alpha I + \beta V$. Either way, by Proposition 3.4, T has property (CN) and property (CR).
- Conversely, suppose that T has property (CN) (resp. T has property (CR)), and *suppose* we know that every $X \in \mathbb{M}_4(\mathbb{C})$ with property (CN) (resp. with property (CR)) has eigenvalues that are either co-linear or co-circular. Given any $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ in $\sigma(T)$, we can write $T = R \oplus Y$, where R is a normal operator in $\mathbb{M}_4(\mathbb{C})$ with $\sigma(R) = \{\theta_1, \theta_2, \theta_3, \theta_4\}$. By Proposition 2.2 (b), R has property (CN) (resp. R has property (CR)). It follows from our hypothesis that $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ are either co-linear or co-circular. Since this is true for an arbitrary collection of four elements from $\sigma(T)$, we conclude that all of the eigenvalues of T are either co-linear or co-circular. As before, this implies the existence of $\alpha, \beta \in \mathbb{C}$ and either a hermitian operator L or a unitary operator V such that $T = \alpha I + \beta L$, or $T = \alpha I + \beta V$.

We now concentrate on proving that a 4×4 matrix T has property (CN) (resp. property (CR)) if and only if the eigenvalues of T are either co-linear or co-circular.

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3.9. Lemma. *Let $X, Y \in \mathbb{M}_2(\mathbb{C})$ and suppose that $\|X\|_2 = \|Y\|_2$. The following are equivalent.*

(a) $\|X\| = \|Y\|$.
(b) $\mathrm{tr}((X^*X)^2) = \mathrm{tr}((Y^*Y)^2)$.
(c) $|\det(X)| = |\det(Y)|$.

Proof. Again, since the Fröbenius norm, the operator norm, and the trace functional are all invariant under unitary conjugation, we may assume without loss of generality that X^*X and Y^*Y are not only positive but diagonal, say

$$X^*X = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}, \qquad Y^*Y = \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix},$$

with $0 \le x_1, x_2, y_1, y_2$.

The hypothesis that $\|X\|_2 = \|Y\|_2$ is the statement that $\varrho = x_1 + x_2 = y_1 + y_2$.

(a) implies (b).

Suppose that $\|X\| = \|Y\|$. Then $\|X\|^2 = \|Y\|^2$ and so $\max\{x_1, x_2\} = \max\{y_1, y_2\}$. By reindexing if necessary, we may assume that $x_1 = y_1$. But we have also assumed that $x_1 + x_2 = y_1 + y_2$, and so $x_2 = y_2$. It follows that

$$\mathrm{tr}((X^*X)^2) = x_1^2 + x_2^2 = y_1^2 + y_2^2 = \mathrm{tr}((Y^*Y)^2).$$

(b) implies (c).

Our current hypotheses are that $x_1 + x_2 = y_1 + y_2$ and that $x_1^2 + x_2^2 = y_1^2 + y_2^2$. Thus

$$\begin{aligned} |\det(Y)|^2 &= \det(Y^*Y) \\ &= y_1 y_2 \\ &= \frac{1}{2}\left((y_1 + y_2)^2 - (y_1^2 + y_2^2)\right) \\ &= \frac{1}{2}\left((x_1 + x_2)^2 - (x_1^2 + x_2^2)\right) \\ &= x_1 x_2 \\ &= \det(X^*X) \\ &= |\det(X)|^2, \end{aligned}$$

from which (c) follows.

(c) implies (a).

Suppose that $|\det(X)| = |\det(Y)|$. Then, as we have just computed, $x_1 x_2 = |\det(X)|^2 = |\det(Y)|^2 = y_1 y_2$.

But then $x_1 + x_2 = y_1 + y_2$ and $x_1 x_2 = y_1 y_2$ together imply that $\{x_1, x_2\} = \{y_1, y_2\}$. In particular,

$$\|X\|^2 = \|X^*X\| = \max\{x_1, x_2\} = \max\{y_1, y_2\} = \|Y^*Y\| = \|Y\|^2.$$

This completes the proof.

□

3.10. Theorem. *Suppose that T is an invertible normal block-matrix in $\mathbb{M}_4(\mathbb{C})$ with 2×2 blocks, and the off-diagonal corners of T have equal rank (respectively equal operator norm), then the same is true for the off-diagonal corners of T^{-1}.*

Proof. Let us start with the case of equal ranks, and employ a proof by contradiction, supposing that $T = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is an invertible normal matrix with $T^{-1} = \left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right]$, and $\operatorname{rank} B = \operatorname{rank} C$, but $\operatorname{rank} B' \neq \operatorname{rank} C'$.

Since T^{-1} is normal, every invariant subspace of T^{-1} is reducing, and so if either B' or C' is zero then both B' and C' are zero, contradicting our hypothesis. Hence we may assume that one of B' and C' has rank 1 and the other has rank equal to 2. Passing to T^* if necessary, we can assume without loss of generality that $\operatorname{rank} C' = 1 < 2 = \operatorname{rank} B'$.

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In particular, B' is invertible, as is T^{-1}, and therefore $B = \left(T^{-1}|B'\right)^{-1}$, from which we conclude that B is invertible. Consequently C has rank 2 and is invertible. Hence $C' = (T|C)^{-1}$, and therefore C' is invertible, i.e. has rank 2, equal to that of B', contradicting our hypothesis.

Next let us deal with the case of equal operator norms. Let us suppose that $\|B\| = \|C\|$, or equivalently, by Lemma 3.9, that $|\det B| = |\det C|$.

First, let us treat the case "A is invertible". In this case

$$\det C' = \det\left(-(T|A)^{-1}CA^{-1}\right) = \frac{\det C}{\det(T|A)\det A}$$

and

$$\det B' = \det\left(-A^{-1}B\,(T|A)^{-1}\right) = \frac{\det B}{\det(T|A)\det A},$$

so that $\det C' = \det B'$, and therefore $\|B'\| = \|C'\|$, again by Lemma 3.9.

Now, the remaining case is "A is not invertible". In this case there is a sequence $[\alpha_k]_{k\in\mathbb{N}}$ convergent to zero and such that each α_k is neither an eigenvalue of T, nor of A. Applying the already settled case "A is invertible" to each (invertible and normal) $T - \alpha_k I$, we can conclude that for each k the off-diagonal corners of $(T - \alpha_k I)^{-1}$ have equal norms. Yet $\lim_{k\to\infty} \|(T - \alpha_k I)^{-1} - T^{-1}\| = 0$, and therefore the off-diagonal corners of $(T - \alpha_k I)^{-1}$ converge to those of T^{-1}, showing that the latter have equal norms as well, and the proof is complete.

□

3.11. Corollary. *Suppose that T is a normal block-matrix in $\mathbb{M}_4(\mathbb{C})$ with 2×2 blocks such that a Möbius map*

$$M(z) = \frac{az+b}{cz+d}$$

is finite at all eigenvalues of T.

If the off-diagonal corners of T have equal rank (respectively, equal operator norm), then the same is true for the off-diagonal corners of $M(T)$.

Proof. The claim is obviously true if $c = 0 \neq d$. Let us consider the case $c \neq 0$. In this case, $\frac{-d}{c}$ is not an eigenvalue of T, and

$$M(z) = \frac{a}{c} + \left(\frac{b - \frac{ad}{c}}{c}\right)\cdot\frac{1}{z + \frac{d}{c}}.$$

If the off-diagonal corners of T have equal rank (respectively, equal operator norm), then the same is true for $T + \frac{d}{c}I$. Then, by Theorem 3.10, the off-diagonal corners of $\left(T + \frac{d}{c}I\right)^{-1}$ have equal rank (respectively, equal operator norm), and thus the same can be said about the off-diagonal corners of $M(T)$.

□

3.12. Corollary. *If $T \in \mathbb{M}_4(\mathbb{C})$ has property* (CN) *or if T has property* (CR)*, then T is normal and $M(T)$ has the same property for any Möbius map M that is finite on the spectrum of T.*

Proof. This is the consequence of Proposition 3.2, Corollary 3.11 and the standard analytic functional-calculus fact that

$$M\left(U^* TU\right) = U^* M(T)U.$$

□

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3.13. Proposition. *If T is a normal block-matrix in $\mathbb{M}_4(\mathbb{C})$ with 2×2 blocks, and the spectrum of T is $\{0,1,2,\delta\}$, where $\delta \notin \mathbb{R}$, then there exists a unitary block-matrix U in $\mathbb{M}_4(\mathbb{C})$ such that, with respect to the 2×2 block partitioning,*

$$\operatorname{rank}\left(U^* TU\right)_{NE} < 2 = \operatorname{rank}\left(U^* TU\right)_{SW}.$$

Proof. Every complex number δ other than 2 can be expressed as $2-8/(6+\beta)$ for a unique $\beta \neq -6$. Furthermore, δ is real exactly when β is real.

Hence, after applying a unitary similarity we can assume without loss of generality that

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2-\frac{8}{\beta+6} \end{pmatrix}$$

The unitary U shall be the product VW of the unitaries

$$V = \begin{pmatrix} \sigma & 0 & \gamma & 0 \\ 0 & \gamma & 0 & \sigma \\ \gamma & 0 & -\sigma & 0 \\ 0 & \sigma & 0 & -\gamma \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{e^{i\theta}}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{e^{i\theta}}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix},$$

where σ, γ and θ, to be specified later, are subject to the conditions:

$$\sigma \neq \gamma, \quad \sigma^2+\gamma^2 = 1, \quad 0<\sigma, \quad 0<\gamma, \quad \text{and} \quad \theta \text{ is not an integer multiple of } \pi.$$

A direct calculation shows that

$$\left(U^* TU\right)_{NE} = \begin{pmatrix} -\frac{(3\beta+10)\gamma\sigma}{2(\beta+6)} & -\frac{4\gamma^2}{\beta+6} - \frac{1}{2}e^{i\theta}\sigma^2 \\ -\frac{1}{2}e^{-i\theta}\gamma^2 - \frac{4\sigma^2}{\beta+6} & -\frac{(3\beta+10)\gamma\sigma}{2(\beta+6)} \end{pmatrix}$$

and

$$\left(U^* TU\right)_{SW} = \begin{pmatrix} -\frac{(3\beta+10)\gamma\sigma}{2(\beta+6)} & -\frac{1}{2}e^{i\theta}\gamma^2 - \frac{4\sigma^2}{\beta+6} \\ -\frac{4\gamma^2}{\beta+6} - \frac{1}{2}e^{-i\theta}\sigma^2 & -\frac{(3\beta+10)\gamma\sigma}{2(\beta+6)} \end{pmatrix},$$

and that

$$\det\left(U^* TU\right)_{NE} = \left(\frac{-2\gamma^2\sigma^2}{\beta+6}\right)\left(\frac{1}{\frac{e^{i\theta}\sigma^2}{\gamma^2}} + \frac{e^{i\theta}\sigma^2}{\gamma^2} - \beta\right),$$

while

$$\det\left(U^* TU\right)_{SW} = \left(\frac{-2}{\beta+6}\right)\left(\gamma^4 e^{i\theta} + e^{-i\theta}\sigma^4 - \gamma^2\sigma^2\beta\right).$$

Now one can see that

$$\det\left(U^* TU\right)_{NE} - \det\left(U^* TU\right)_{SW} = \frac{4i\left(\gamma^4-\sigma^4\right)\sin(\theta)}{\beta+6} \neq 0,$$

because of the conditions that we have imposed on σ, γ and θ.

Note that $\dfrac{e^{i\theta}\sigma^2}{\gamma^2}$ can take on any non-real complex value even when σ, γ are restricted to be distinct positive numbers whose squares add up to 1, and θ is not an integer multiple of π.

It is also easy to see that the equation $\zeta + \dfrac{1}{\zeta} = \beta$ has a complex solution for ζ, and since $\beta \notin \mathbb{R}$, the solution cannot be real.

It follows that there exist σ, γ and θ satisfying the conditions:

$$\sigma \neq \gamma, \quad \sigma^2+\gamma^2 = 1, \quad 0<\sigma, \quad 0<\gamma, \quad \text{and} \quad \theta \text{ is not an integer multiple of } \pi,$$

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as well as the condition

$$\frac{1}{\frac{e^{i\theta}\sigma^2}{\gamma^2}} + \frac{e^{i\theta}\sigma^2}{\gamma^2} = \beta.$$

These are the σ, γ and θ that we use in the construction of U, and it is now clear that such a U is the one we seek, since for this U:

$$0 = \det\left(U^* TU\right)_{NE} \neq \det\left(U^* TU\right)_{SW}.$$

□

3.14. Corollary. *If $T \in \mathbb{M}_4(\mathbb{C})$ has property* (CN) *or if T has property* (CR)*, then T is normal and the spectrum of T is circlinear.*

Proof. Such T has to be normal by Proposition 3.2. To verify the rest of the claim we proceed by contradiction. Suppose that the eigenvalues $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ of T are not circlinear. Then they are all distinct.

By the spectral mapping theorem, given a Möbius map M that is finite on the spectrum of T, the eigenvalues of $M(T)$ are the images of the eigenvalues of T under M. It is well-known that Möbius maps take circlines to circlines, and exhibit sharp three-fold transitivity.

In particular there is a unique Möbius map M_o such that

$$M_o(\lambda_i) = i, \quad \text{for } i = 0, 1, 2.$$

If M_o sends λ_3 to z_o that is a real number or "∞", then the inverse of M_o sends $0, 1, 2, z_o$ to $\lambda_0, \lambda_1, \lambda_2, \lambda_3$, indicating that the latter set is part of the image (under M_o) of the extended real line, and hence must be circlinear, contrary to our hypothesis. Therefore M_o sends λ_3 to some complex non-real number δ.

Applying Corollary 3.12 and Proposition 3.13 to $M_o(T)$ yields a contradiction, and the proof is complete.

□

By combining Remark 3.8 and Corollary 3.14, we obtain the main theorem of this section.

3.15. Theorem. *Let $n \geq 4$ be an integer and $T \in \mathbb{M}_n(\mathbb{C})$. The following are equivalent.*

(a) *T has property* (CN).
(b) *T has property* (CR).
(c) *One of the following holds:*
   (i) *there exist $\lambda, \mu \in \mathbb{C}$ and $V \in \mathbb{M}_n(\mathbb{C})$ unitary such that $T = \lambda I + \mu V$.*
   (ii) *There exist $\lambda, \mu \in \mathbb{C}$ and $L = L^* \in \mathbb{M}_n(\mathbb{C})$ such that $T = \lambda I + \mu L$.*

3.16. There is also an alternative proof for Theorem 3.15 that does not involve Möbius maps, and while we have chosen not to include it here, we will gladly share it with an interested reader. Clearly the invariance of property (CN) and property (CR) under Möbius maps (as in Corollary 3.12) can be inferred from Theorem 3.15.

4. The infinite-dimensional setting – property (CN)

4.1. Let us now consider the case where the underlying Hilbert space is infinite-dimensional and separable. We begin by studying operators with property (CN). In the finite-dimensional setting, we saw that any such operator is normal. While this is also true in the infinite-dimensional setting, the proof is rather different.

Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be **strongly reductive** if, whenever $(P_n)_{n=1}^\infty$ is a sequence of orthogonal projections such that $\lim_n \|P_n^\perp T P_n\| = 0$, it follows that $\lim_n \|P_n T P_n^\perp\| = 0$ (or equivalently, $\lim_n \|P_n T - T P_n\| = 0$). Let us say that a compact set $\Omega \subseteq \mathbb{C}$ is **Lavrentiev** if it has empty interior and if $\mathbb{C} \setminus \Omega$ is connected.

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4.2. Proposition. *Let $\mathcal{H}$ be an infinite-dimensional, separable Hilbert space. If $T \in \mathcal{B}(\mathcal{H})$ has property* (CN)*, then T is normal and has Lavrentiev spectrum.*

Proof. It is an immediate consequence of the definition that if T has property (CN), then T is strongly reductive. It was shown by Harrison [12] that any strongly reductive operator has Lavrentiev spectrum. Apostol, Foiaş and Voiculescu [3] showed that any strongly reductive operator is normal.

It is also easy to obtain the normality of T which enjoys property (CN) directly. Suppose that $T \in \mathcal{B}(\mathcal{H})$ has property (CN), and let $e \in \mathcal{H}$ be an arbitrary vector of norm one. Then $P_e(x) = \langle x, e\rangle e$, $x \in \mathcal{H}$ defines a rank one projection. By hypothesis, $\|P_e^\perp T^* P_e\| = \|P_e T P_e^\perp\| = \|P_e^\perp T P_e\|$.

Now

$$\begin{aligned}\langle TT^* e, e\rangle = \|T^* e\|^2 &= \|P_e T^* e\|^2 + \|P_e^\perp T^* e\|^2 \\ &= \|P_e T^* P_e e\|^2 + \|P_e^\perp T^* P_e e\|^2 \\ &= \|P_e T^* P_e\|^2 + \|P_e^\perp T^* P_e\|^2.\end{aligned}$$

Similarly,

$$\langle T^* T e, e\rangle = \|P_e T P_e\|^2 + \|P_e^\perp T P_e\|^2.$$

Now

$$\|P_e T P_e\| = \|P_e T^* P_e\|,$$

and combining this with the fact that $\|P_e^\perp T^* P_e\| = \|P_e^\perp T P_e\|$ from above, we see that

$$\langle TT^* e, e\rangle = \langle T^* T e, e\rangle.$$

Since e was an arbitrary norm-one vector, we conclude that $TT^* = T^*T$; i.e. that T is normal.

□

4.3. In Section 2, we noted that if $L = L^* \in \mathcal{B}(\mathcal{H})$ and $\lambda, \mu \in \mathbb{C}$, then $\lambda I + \mu L$ has property (CN). Although in the finite-dimensional setting every unitary operator V also has property (CN), this is no longer true in the infinite-dimensional setting, as the following counterexample shows.

Let $\{e_n\}_{n\in\mathbb{Z}}$ be an orthonormal basis for our Hilbert space $\mathcal{H}$, and consider the **bilateral shift** operator W determined by $We_n = e_{n-1}$, $n \in \mathbb{Z}$. Then W is unitary and $\sigma(W) = \mathbb{T}$. By Proposition 4.2, W does not have property (CN).

This can be seen directly as well. If P is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M} = \overline{\mathrm{span}}\{e_n : n \leq 0\}$, then $\mathcal{M}$ is invariant for W, so that $P^\perp W P = 0$. However, $0 \neq e_0 = We_1 = PWP^\perp e_1$, so that $PWP^\perp \neq 0$. *A fortiori*, W has neither property (CN) nor property (CR). Furthermore, since the set of operators having property (CN) is clearly (norm-)closed, no operator close enough to W has property (CN).

For $n \geq 3$, let $C_n \in \mathbb{M}_n(\mathbb{C})$ denote an n-cycle; that is, there is an orthonormal basis $\{e_k\}_{k=1}^n$ of $\mathbb{C}^n$, such that $C_n e_k = e_{k+1}$, $1 \leq k \leq n-1$, and $C_n e_n = e_1$. It follows easily from the results in [7] that there exists a sequence $(V_n)_{n=1}^\infty$ of unitary operators with $V_n \simeq C_n \otimes I$ for all $n \geq 1$ such that $\lim_n V_n = W$. Thus V_n does not have property (CN) for all sufficiently large n. That is, for sufficiently large n, $C_n \otimes I$ fails to have property (CN), despite the fact that $C_n \in \mathbb{M}_n(\mathbb{C})$ has property (CN) by Proposition 3.3. In fact, as we shall soon see, $C_n \otimes I$ fails to have property (CN) for all $n \geq 3$.

4.4. Let us recall that the **numerical range** of $T \in \mathcal{B}(\mathcal{H})$ is the set $W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}$, and the **numerical radius** of T is $w(T) = \sup\{|\lambda| : \lambda \in W(T)\}$. It is known that the numerical range of an operator is always a convex set (this is the classical Toeplitz-Hausdorff Theorem), and that the closure of the numerical range of T always contains $\sigma(T)$ (see, for example, Problem 214 of [11]).

In trying to characterize operators with property (CN), we know that we may restrict our attention to normal operators. For a normal M, it is known that $\overline{W(M)} = \mathrm{co}(\sigma(M))$, that is, the closure of the numerical range of M is the convex hull of the spectrum of M. If $\sigma(M)$ happens to be finite, then (by the Spectral Theorem) $\sigma(M)$ consists of the eigenvalues of M, and these belong to the numerical range of M, as does their convex hull. Hence in such a case $W(M) = \mathrm{co}(\sigma(M))$.

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The numerical radius defines a norm on $\mathcal{B}(\mathcal{H})$ which is equivalent to the operator norm, because $\frac{1}{2}\|T\| \leq w(T) \leq \|T\|$ for all $T \in \mathcal{B}(\mathcal{H})$. (See, e.g. [11], Chapter 22 for all of these results.)

A **state** on $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is a positive linear functional of norm one. For $T \in \mathcal{B}(\mathcal{H})$, the **essential numerical range** $W_e(T)$ of T is the set $\{\varphi(\pi(T)) : \varphi$ is a state on $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})\}$. It is known that $W_e(T)$ is closed and convex, and so it follows that $W_e(T) = \mathrm{co}(\sigma_e(T))$, whenever T is normal.

4.5. Theorem. (***Fillmore-Stampfli-Williams****; Theorem 5.1 of* [10]) *For $T \in \mathcal{B}(\mathcal{H})$, the following conditions are equivalent.*

(a) $0 \in W_e(T)$.

(b) *There exists an orthonormal sequence $(e_n)_{n=1}^\infty$ in $\mathcal{H}$ such that* $\lim_n \langle Te_n, e_n\rangle = 0$.

(c) $0 \in \cap\{\overline{W(T+F)} : F$ *is of finite rank*$\}$.

From this it easily follows that $W_e(T) = \cap\{\overline{W(T+F)} : F$ is of finite rank$\}$.

4.6. If $R \in \mathcal{B}(\mathcal{K})$ where $\mathcal{K} \subseteq \mathcal{H}$ is a subspace of $\mathcal{H}$, then T is said to be a **dilation** of R if, relative to the decomposition $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^\perp$, we may write

$$T = \begin{bmatrix} R & B \\ C & D \end{bmatrix}$$

for some choice of B, C and D.

Recall that if $\{e_n\}_{n=1}^\infty$ is an orthonormal basis for $\mathcal{H}$, then the **unilateral forward shift** on $\mathcal{H}$ is the operator $S \in \mathcal{B}(\mathcal{H})$ satisfying $Se_n = e_{n+1}$ for all $n \geq 1$.

The following result of Choi and Li will be useful.

4.7. Theorem. (***Choi-Li****; Theorem 4.3 of* [6]) *Suppose that $A \in \mathcal{B}(\mathcal{H})$, and $T \in \mathbb{M}_3(\mathbb{C})$ has a non-trivial reducing subspace. Then A has a dilation that is unitarily equivalent to $T \otimes I$ if and only if $W(A) \subseteq W(T)$.*

4.8. Theorem. *Suppose that $V \in \mathbb{M}_3(\mathbb{C})$ is a unitary operator and that 0 lies in the interior of $W(V)$. Then $V \otimes I$ does not have property* (CN).

Proof. Clearly every unitary operator V in $\mathbb{M}_3(\mathbb{C})$ has a non-trivial reducing subspace. Note also that if $S \in \mathcal{B}(\mathcal{H})$ is the unilateral forward shift, $\mathrm{spr}(S) \leq \|S\| = 1$. (In fact, $\mathrm{spr}(S) = 1$, but that is not important here.)

Thus $\mathrm{spr}(\varepsilon S) \leq \varepsilon$ for all $\varepsilon > 0$. Since 0 lies in the *interior* of $W(V)$, there exists $\varepsilon_0 > 0$ such that $W(\varepsilon_0 S) \subseteq W(V)$. Let $A = \varepsilon_0 S$. By Theorem 4.7 above, we may write

$$V \otimes I \simeq \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Let $P = SS^*$, so that P is an orthogonal projection of co-rank one (i.e. the rank of $(I-P)$ is equal to one). In particular, $\sigma(P) = \{0, 1\}$.

Then, since $V \otimes I$ is unitary, we have that

$$BB^* = I - AA^* = I - \varepsilon_0^2 P$$
$$C^*C = I - A^*A = I - \varepsilon_0^2 I.$$

It follows that

$$\|B\|^2 = \|BB^*\| = \|I - \varepsilon_0^2 P\| = 1,$$

while

$$\|C\|^2 = \|C^*C\| = \|I - \varepsilon_0^2 I\| = 1 - \varepsilon_0^2.$$

In particular, $\|B\| \neq \|C\|$, so that $V \otimes I$ does not have property (CN).

□

It follows from Theorem 4.8 that $C_n \otimes I$ does not have property (CN) for $n \geq 3$, since in such a case 0 lies in the interior of $W(C_n)$.

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4.9. Corollary. *Suppose that $U \in \mathcal{B}(\mathcal{H})$ is a unitary operator that has property* (CN). *Then* 0 *does not lie in the interior of $W_e(U)$.*

Proof. We prove a contrapositive implication. Suppose that 0 lies in the interior of the essential numerical range of U, that is in the interior of the convex hull of the essential spectrum of U. Then there exist α, β and γ in the essential spectrum of U such that 0 lies in the interior of the convex hull of $\{\alpha, \beta, \gamma\}$. Let $V \in \mathbb{M}_3(\mathbb{C})$ be a unitary operator with spectrum $\{\alpha, \beta, \gamma\}$. Then, as noted in Section 4.4, $W(V)$ is closed and $W(V) = \text{co}\{\alpha, \beta, \gamma\}$. Hence 0 lies in the interior of $W(V)$. By Theorem 4.8, $V \otimes I$ does not have property (CN).

Since α, β and γ lie in the essential spectrum of the normal operator U, U is approximately unitarily equivalent to $U \oplus (V \otimes I)$. (This is a consequence of the Weyl-von Neumann-Berg Theorem for normal operators – see, e.g. Theorem II.4.4 of [8] – and can also be deduced from the results of [7].) It now follows from Proposition 2.3 (b) that U does not have property (CN).

$\square$

4.10. Theorem. *Suppose that $U \in \mathcal{B}(\mathcal{H})$ is a unitary operator and* 0 *does not lie in $W_e(U)$. Then U has property* (CN).

Proof. We prove a contrapositive implication. Let U be a unitary operator which fails to have property (CN). Then there exists a projection $P \in \mathcal{B}(\mathcal{H})$ such that with respect to the decomposition $\mathcal{H} = P\mathcal{H} \oplus P^\perp\mathcal{H}$, we may write

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $\|B\| \neq \|C\|$.

As noted in Remark 2.4, this can only happen if one of the following holds:

(a) either $0 \in \sigma(AA^*)$ but $0 \notin \sigma(A^*A)$,
(b) $0 \in \sigma(A^*A)$ but $0 \notin \sigma(AA^*)$.

Since an operator T has property (CN) if and only if T^* has property (CN), by replacing U by U^* if necessary (which does not affect the conclusion, as $0 \in W_e(U)$ if and only if $0 \in W_e(U^*)$), we may assume without loss of generality that $0 \in \sigma(AA^*)$ but $0 \notin \sigma(A^*A)$. In particular, A is not invertible.

Since A^*A is invertible, we see that A is bounded below. Hence $\text{ran}\, A$ is closed and $\text{nul}\, A = 0$. Therefore A is semi-Fredholm. If $\text{nul}\, A^* = 0$, then $\text{ran}\, A = P\mathcal{H}$, so that A is invertible, which is a contradiction. Thus $\text{nul}\, A^* > 0$, and so $\text{ind}\, A < 0$.

Since $\text{ind}\,(A + F) = \text{ind}\, A < 0$ for all finite-rank operators $F \in \mathcal{B}(P\mathcal{H})$, and since $\sigma(T) \subseteq \overline{W(T)}$ for all operators T, we may apply Theorem 4.5 of Fillmore, Stampfli and Williams to obtain:

$$\begin{aligned} 0 &\in \cap\{\sigma(A + F) : F \in \mathcal{B}(P\mathcal{H}),\ F \text{ finite-rank}\} \\ &\subseteq \cap\{\overline{W(A + F)} : F \in \mathcal{B}(P\mathcal{H}),\ F \text{ finite-rank}\} \\ &= W_e(A), \end{aligned}$$

Since $W_e(A) \subseteq W_e(U)$, the result follows.

$\square$

4.11. Theorem. *Suppose that $U \in \mathcal{B}(\mathcal{H})$ is unitary and that* 0 *lies on the boundary of $W_e(U)$. Then U has property* (CN).

Proof. The hypotheses of the theorem imply that $\sigma_e(U)$ lies on a closed half-circle of $\mathbb{T}$, and includes two diametrically opposite points. By multiplying U by an appropriate $\mu \in \mathbb{T}$ (which does not affect the conclusion of the Theorem), we may assume without loss of generality that $\sigma_e(U) \subseteq \mathbb{T} \cap \{z \in \mathbb{C} : \text{Re} z \geq 0\}$, and that $\{i, -i\} \subseteq \sigma_e(U)$.

For each $n \geq 1$, let $\mathcal{C}_n = \{z \in \mathbb{T} : z = e^{i\theta}, \frac{\pi}{2} - \frac{1}{n} \leq \theta \leq \frac{\pi}{2} + \frac{1}{n}\}$, and let Q_n be the spectral projection for U corresponding to $\mathcal{C}_n$. Then $U = X_n + Y_n$, where $X_n = UQ_n^\perp$, and where $Y_n = UQ_n$ is a unitary with $\sigma(Y_n) \subseteq \mathcal{C}_n$. Set $V_n = X_n + e^{i(\frac{\pi}{2} - \frac{1}{n})}Q_n$.

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It is reasonably straightforward to check that $\|U - V_n\| = \|Y_n - e^{i(\frac{\pi}{2}-\frac{1}{n})}Q_n\| \le \frac{4\pi}{n}$ and that $\sigma_e(V_n) \subseteq \Omega_n = \{z = e^{i\theta} \in \mathbb{T} : -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} - \frac{1}{n}\}$.

Thus 0 does not lie in the closed, convex hull of $\sigma_e(V_n)$, and in particular, 0 does not lie in $W_e(V_n)$ for any $n \ge 1$. By Theorem 4.10, V_n has property (CN). But as we saw in Proposition 2.3, the set $\mathfrak{G}_{\mathfrak{norm}}$ of operators with property (CN) is closed, and thus U has property (CN).

□

Combining these results, and keeping in mind that $W_e(T) = \mathrm{co}(\sigma_e(T))$ for normal T, we obtain the following.

4.12. Corollary. *The following are equivalent for a unitary operator $U \in \mathcal{B}(\mathcal{H})$.*

(a) *U has property* (CN)*.*
(b) *0 does not lie in the interior of $W_e(U)$.*
(c) *There exists a half-circle $\mathcal{C}$ of $\mathbb{T}$ such that $\sigma_e(U) \subseteq \mathcal{C}$*
(i.e. there exists $\mu \in \mathbb{T}$ such that $\sigma_e(U) \subseteq \mathbb{T} \cap \{\mu z \in \mathbb{C} : \mathrm{Re}(z) \ge 0\}$).

We are now ready to state and prove the main theorem of this section.

4.13. Theorem. *Let $T \in \mathcal{B}(\mathcal{H})$. The following conditions are equivalent.*

(a) *T has property* (CN)*.*
(b) *One of the following holds.*
  (i) *There exist $\lambda, \mu \in \mathbb{C}$ and $L = L^* \in \mathcal{B}(\mathcal{H})$ such that $T = \lambda I + \mu L$.*
  (ii) *There exist $\lambda, \mu \in \mathbb{C}$ with $\mu \neq 0$ and a unitary operator $U \in \mathcal{B}(\mathcal{H})$ with $\sigma_e(U) \subseteq \mathbb{T} \cap \{z \in \mathbb{C} : \mathrm{Re}(z) \ge 0\}$ such that $T = \lambda I + \mu U$.*

Proof. Suppose first that (a) holds, and recall that this implies that T is normal.

If $\sigma(T)$ has at most three points, then those points are either co-linear or co-circular. It is routine to check from this that T is either of the form of (b) (i), or there exist $\lambda, \mu \in \mathbb{C}$ with $\mu \neq 0$ and a unitary $U \in \mathcal{B}(\mathcal{H})$ such that $T = \lambda I + \mu U$. In this case, since $\mu \neq 0$, T has property (CN) if and only if U has property (CN). But then U has property (CN) by our hypothesis on T, and so the spectral conditions on U follow from Corollary 4.12.

Thus we assume that $\sigma(T)$ has cardinality at least 4, and we let $\{\alpha, \beta, \gamma, \delta\}$ be four distinct points in $\sigma(T)$. By Theorem II.4.4 in [8], T is approximately unitarily equivalent to an operator of the form $A \oplus D$, where $D = \mathrm{diag}(\alpha, \beta, \gamma, \delta) \in \mathbb{M}_4(\mathbb{C})$. By Proposition 2.3 (b), D has property (CN). By Corollary 3.14, the eigenvalues of D are either co-linear or co-circular. Since this is true for any choice of four distinct points of $\sigma(T)$, we see that $\sigma(T)$ is either contained in a line – in which case it is easily seen that there exist λ, μ and L as in (b) (i) such that $T = \lambda I + \mu L$, or $\sigma(T)$ lies on a proper circle, i.e. there exist $\lambda, \mu \in \mathbb{C}$ with $\mu \neq 0$ and a unitary $U \in \mathcal{B}(\mathcal{H})$ such that $T = \lambda I + \mu U$. We argue as in the previous paragraph to obtain the spectral conditions on U.

Suppose next that (b) holds.

If there exist $\lambda, \mu \in \mathbb{C}$ and $L = L^* \in \mathcal{B}(\mathcal{H})$ such that $T = \lambda I + \mu L$, then T has property (CN) by Proposition 2.2 (d).

If there exist $\lambda, \mu \in \mathbb{C}$ with $\mu \neq 0$ and a unitary operator $U \in \mathcal{B}(\mathcal{H})$ with $\sigma_e(U) \subseteq \mathbb{T} \cap \{z \in \mathbb{C} : \mathrm{Re}(z) \ge 0\}$ such that $T = \lambda I + \mu U$. Then T has property (CN) if and only if U has property (CN) by Proposition 2.2 (a). But U has property (CN) by Corollary 4.12, whence T has property (CN).

□

4.14. It is natural to consider a weakening of the property (CN) obtained by restricting our attention to the finite-rank projections $P \in \mathcal{B}(\mathcal{H})$ in the case when $\mathcal{H}$ is infinite-dimensional. As the reader can easily check, the results and proofs presented in this section readily demonstrate that such a "weakening" of the property (CN) is in fact equivalent to the original property (CN).

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5. The infinite-dimensional setting – property (CR)

5.1. We next turn our attention to the study of operators with property (CR), acting on an infinite-dimensional Hilbert space. Although we have not been able to obtain a complete classification of such operators, we will mention a number of interesting facts.

We recall from above that an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be (orthogonally) reductive if for each projection $P \in \mathcal{B}(\mathcal{H})$, the condition $PTP^{\perp} = 0$ implies that $P^{\perp}TP = 0$. It is clear that if T has property (CR), then T must be reductive.

It should be noted that not every normal operator is reductive. Sarason [18] has shown that a normal operator N is reductive if and only if N^* lies in the weak operator topology closure of the set of polynomials in N. As a concrete example, let $W \in \mathcal{B}(\mathcal{H})$ be the bilateral shift; i.e. let $\{e_n\}_{n\in\mathbb{Z}}$ be an orthonormal basis for $\mathcal{H}$ and let W be defined by $We_n = e_{n-1}$ for all $n \in \mathbb{Z}$. It is well-known that W is unitary with $\sigma(W) = \mathbb{T}$. If P is the orthogonal projection onto $\mathcal{M} = \overline{\operatorname{span}}\{e_n : n \le 0\}$, then clearly $\mathcal{M}$ is invariant for W – so $\operatorname{rank} P^{\perp}WP = 0$, but it is easily verified that $PWP^{\perp}$ has rank 1. Thus W fails to be reductive.

The condition that an operator has property (CR) is strictly stronger than asking that it be orthogonally reductive – see Example 5.9 below. It is worth observing that there is one inherent weakness in the definition of orthogonally reductive operators: it is entirely possible that there might exist an operator with no non-trivial closed, invariant subspace, in which case the operator is reductive for trivial reasons. On the other hand, it was shown by Popov and Tcaciuc [16] that given any operator T acting on an infinite-dimensional, complex, separable Hilbert space $\mathcal{H}$, there exists an orthogonal projection P of infinite rank and co-rank such that $\operatorname{rank} PTP^{\perp} \le 1$. (Their result actually holds for operators acting on reflexive Banach spaces and beyond, but we do not require that here.) As such, property (CR) always has significance for Hilbert space operators.

We begin with some observations regarding the general class $\mathfrak{G}_{\mathfrak{rank}}$ of operators with property (CR).

5.2. Proposition. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ has property* (CR). *Then T is biquasitriangular; that is,* $\operatorname{ind}(T - \lambda I) = 0$ *for all* $\lambda \in \varrho_{sF}(T)$.

Proof. It is clear that if $\lambda \in \mathbb{C}$, then $T - \lambda I$ and $(T - \lambda I)^*$ also have property (CR).

Suppose that $\lambda \in \varrho_{sF}(T)$ and that $\operatorname{ind}(T - \lambda I) \neq 0$. By considering $(T - \lambda I)^*$ if necessary, we may assume that $\operatorname{ind}(T - \lambda I) > 0$ (it is possibly infinite).

Thus $\operatorname{nul}(T - \lambda I) > \operatorname{nul}(T - \lambda I)^*$. Write $\mathcal{H} = \ker(T - \lambda I) \oplus (\ker(T - \lambda I))^{\perp}$, and write

$$(T - \lambda I) = \begin{bmatrix} 0 & B \\ 0 & D \end{bmatrix}$$

relative to this decomposition. If $B = 0$, then $(T - \lambda I)^* = \begin{bmatrix} 0 & 0 \\ 0 & D^* \end{bmatrix}$, showing that $\operatorname{nul}(T - \lambda I)^* \ge \operatorname{nul}(T - \lambda I)$, a contradiction. But then $B \neq 0$ implies that $T - \lambda I$ does not have property (CR), and hence neither does T.

The contrapositive is the statement that if T has property (CR), then T is biquastriangular.

□

5.3. Let $T \in \mathcal{B}(\mathcal{H})$ and suppose that T has *property*(CR). It is clear that if λ is an eigenvalue for T, then it is a reducing eigenvalue for T; that is, we may write $T \simeq \lambda Q \oplus T_0$, where Q is an orthogonal projection and λ is no longer an eigenvalue for T_0 (though it may be an approximate eigenvalue for T_0).

Let $\Lambda := \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue for } T\}$. Suppose that $\lambda_1 \neq \lambda_2 \in \Lambda$ and that $T = \lambda_1 Q_1 \oplus T_1$, where Q_1 is an orthogonal projection and λ_1 is not an eigenvalue of T_1. Let x be an eigenvector T corresponding to the eigenvalue λ_2, and write $x = y + z$, where $y = Q_1 x$ and $z = (I - Q_1)x$. Then

$$\lambda_2 y + \lambda_2 z = \lambda_2 x = Tx = \lambda_1 y + T_1 z.$$

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Since $z, Tz \in (I - Q_1)\mathcal{H}$ while $y \in Q_1\mathcal{H}$, and since $\lambda_1 \neq \lambda_2$, we see that $Q_1 x = y = 0$. In other words, eigenvectors of T corresponding to distinct eigenvalues are mutually orthogonal. Since $\mathcal{H}$ is assumed to be a separable Hilbert space, this implies that T admits at most countably many eigenvalues, say $\Lambda = \{\lambda_n\}_{n\in\Gamma}$ (where Γ is some countable set).

For each $n \in \Gamma$, let Q_n be the orthogonal projection of $\mathcal{H}$ onto the (reducing) space $\ker(T - \lambda_n I)$, and set $\mathcal{H}_0 = \overline{\text{span}}\{Q_n\mathcal{H} : n \in \Gamma\}$. Then $\mathcal{H}_0$ is reducing for T (since each $Q_n\mathcal{H}$ is), and the compression M of T to $\mathcal{H}_0$ is diagonalizable with eigenvalues precisely equal to Λ. Furthermore, the compression T_0 of T to $\mathcal{H}_0^\perp$ has no eigenvalues.

For each $n \in \Gamma$, choose a norm-one eigenvector $w_n \in \ker(T - \lambda_n I)$. Given any four eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and λ_4 of M, the compression of M to the 4-dimensional reducing subspace $\text{span}\{w_1, w_2, w_3, w_4\}$ of M still has property (CR) (it is, after all, a direct summand of T - which has property (CR)). From the results of Section 3, we may conclude that $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ are either co-linear or co-circular. Since this is true of *any* four elements of Λ, it follows that Λ either lies on a line or on a circle in $\mathbb{C}$. But M is diagonalizable, hence normal, and $\sigma(M) = \overline{\Lambda}$. Thus $\sigma(M)$ is either co-linear or co-circular.

Summarizing the information we shall require, *the eigenvalues of any operator T with* property (CR) *are either co-linear or co-circular, and they are reducing eigenvalues for T.*

Our next goal is to prove that every operator which satisfies property (CR) is normal with circlinear spectrum. We shall accomplish this through a sequence of lemmas. It is worth noting that we shall not invoke the full strength of the property (CR) hypothesis. Indeed, for the next few results, we only require a weaker form of property (CR) that requires that T be reductive and that if P is a projection for which $\text{rank}\, P^\perp TP = 1$, then $\text{rank}\, PTP^\perp = 1$. It can in fact be shown that Proposition 5.2 also holds under this weaker hypothesis, though we shall not need that here.

5.4. Proposition. *Let $T \in \mathcal{B}(\mathcal{H})$ and suppose that T has property* (CR)*. Then there exist α, β, γ and $\delta \in \mathbb{C}$, not all equal to zero, and an operator $F \in \mathcal{B}(\mathcal{H})$ of rank at most three such that*

$$\alpha I + \beta T + \gamma T^* + \delta T^*T + F = 0.$$

Proof. Fix $0 \neq \xi \in \mathcal{H}$. We first claim that the set $S_\xi = \{\xi, T\xi, T^*\xi, T^*T\xi\}$ is linearly dependent.

Let $\mathcal{M}_\xi = \text{span}\,\{\xi, T\xi\}$. If $\dim \mathcal{M}_\xi = 1$, then clearly $\{\xi, T\xi\}$ is linearly dependent, whence S_ξ is linearly dependent and we are done.

Suppose therefore that $\dim \mathcal{M}_\xi = 2$ and let P_ξ denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}_\xi$. Note that $T\xi \in \mathcal{M}_\xi$ implies that $\text{rank}\, P_\xi^\perp TP_\xi \in \{0, 1\}$. From our hypothesis, $\text{rank}\, P_\xi^\perp T^* P_\xi \in \{0, 1\}$.

But then

$$\begin{aligned}\dim(P_\xi\mathcal{H} + P_\xi^\perp T^* P_\xi\mathcal{H}) &\le \dim(P_\xi\mathcal{H}) + \dim(P_\xi^\perp T^* P_\xi\mathcal{H})\\ &\le 2 + 1 = 3.\end{aligned}$$

Since $S_\xi \subseteq P_\xi\mathcal{H} + P_\xi^\perp T^* P_\xi\mathcal{H}$, our claim follows.

As $0 \neq \xi \in \mathcal{H}$ was arbitrary, we see that the set $\{I, T, T^*, T^*T\}$ is *locally linearly dependent* in the sense of [1, 4] and [14]. By Theorem 2 of [4], there exist α, β, γ, and $\delta \in \mathbb{C}$, not all equal to zero, such that

$$\text{rank}\,(\alpha I + \beta T + \gamma T^* + \delta T^*T) \le 3.$$

This clearly implies the statement of the proposition.

□

We begin by dealing with the case where δ above is equal to zero.

5.5. Lemma. *Let $\mathcal{H}$ be a complex Hilbert space and suppose that $T \in \mathcal{B}(\mathcal{H})$. If there exist complex numbers α, β and γ, not all equal to zero, and $F \in \mathcal{B}(\mathcal{H})$ of rank at most $m < \frac{1}{2}\dim \mathcal{H}$ such that*

$$\alpha I + \beta T + \gamma T^* + F = 0,$$

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then there exist a hermitian operator R, a finite-rank operator L of rank at most $2m$, and $\mu, \lambda \in \mathbb{C}$ such that

$$T = \lambda(R + L) + \mu I.$$

Proof. Case 1. Suppose that $\gamma = 0$.

In this case, we have that $\alpha I + \beta T + F = 0$. If $\beta = 0$, then the fact that F has finite rank $\operatorname{rank} F = m < \dim \mathcal{H} = \operatorname{rank} I$ implies that $\alpha = 0 (= \beta = \gamma)$, contradicting our hypothesis. Hence $\beta \neq 0$.

But then

$$T = -\alpha\beta^{-1}I - \beta^{-1}F.$$

- If $\alpha = 0$, then
$$T = -\beta^{-1}F = -\beta^{-1}(0 + F) + 0I$$
expresses T in the desired form.
- If $\alpha \neq 0$, then writing
$$T = -\alpha\beta^{-1}(I - \alpha^{-1}F) + 0I$$
expresses T in the desired form.

Case 2. Suppose that $\beta = 0$.

Then $\alpha I + \gamma T^* + F = 0$, and arguing as before, $\gamma \neq 0$. Thus $T^* = -\alpha\gamma^{-1}I - \gamma^{-1}F$.

- If $\alpha = 0$, then
$$T^* = -\gamma^{-1}(0 + F) + 0I$$
means that
$$T = -\overline{\gamma}^{-1}(0 + F^*) + 0I$$
expresses T in the desired form.
- If $\alpha \neq 0$, then writing
$$T^* = -\alpha\gamma^{-1}(I - \alpha^{-1}F) + 0I$$
means that
$$T = -\overline{\alpha\gamma}^{-1}(I - \overline{\alpha}^{-1}F^*) + 0I$$
expresses T in the desired form.

Case 3. Suppose that $\beta \neq 0 \neq \gamma$.

We have that $\alpha I + \beta T + \gamma T^* + F = 0$, whence $\overline{\alpha}I + \overline{\beta}T^* + \overline{\gamma}T + F^* = 0$. Set $\varrho = (\alpha + \overline{\alpha})$, $\theta = \beta + \overline{\gamma}$ and $F_0 = F + F^*$. Adding the two previous equations involving T yields:

$$\varrho I + \theta T + \overline{\theta}T^* + F_0 = 0,$$

and $\operatorname{rank} F_0 \leq 2 \operatorname{rank} F \leq 2m < \dim \mathcal{H}$.

Subcase 3.A. $\theta = 0$.

If $\theta = 0$, then $\varrho I + F_0 = 0$, combined with the fact that $\dim \mathcal{H} > \operatorname{rank} F_0$ implies that $\varrho = 0 = F_0$. That is, $\alpha \in i\mathbb{R}$ and $\gamma = -\overline{\beta} \neq 0$, so that

$$\alpha I + \beta T - \overline{\beta}T^* + F = 0.$$

Let $A = \beta T + \frac{\alpha}{2}I$, and let $A = R + iB$ be the Cartesian decomposition of A, so that R and B are hermitian. The above equation shows that $0 = (A - A^*) + F = 2iB + F$, and thus B has finite rank at most m and

$$T = \beta^{-1}(R + iB) - \frac{\alpha\beta^{-1}}{2}I$$

expresses T in the desired form.

Subcase 3.B. $\theta \neq 0$.

We have

$$\varrho I + \theta T + \overline{\theta}T^* + F_0 = 0,$$

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where rank $F_0 \leq 2m$ and $\varrho \in \mathbb{R}$.

Let $\kappa = \dfrac{\varrho i}{2|\theta|^2} \in i\mathbb{R}$, and $A = (\overline{\theta} i)^{-1} T - \kappa I$. Then $T = \overline{\theta} i(A + \kappa I)$ and our equation $\varrho I + \theta T + \overline{\theta} T^* + F_0 = 0$ implies that

$$0 = |\theta|^2 i(A - A^*) + F_0.$$

In particular, $A - A^*$ has rank at most $2m$. Again, we write $A = R + iB$ where $R = (A + A^*)/2$ and $B = (A - A^*)/2i$. Then B has rank at most rank $F_0 \leq 2m$ and

$$T = \overline{\theta} i(R + iB) + \overline{\theta} i\kappa I$$

expresses T in the desired form.

□

5.6. Lemma. *Let $\mathcal{H}$ be a complex Hilbert space and suppose that $T \in \mathcal{B}(\mathcal{H})$ satisfies*

$$\operatorname{rank}(\alpha I + \beta T + \gamma T^* + \delta T^* T) \leq 3$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, where $\delta \neq 0$. Then there exist a unitary operator V, a finite-rank operator L of rank at most 6, and $\mu, \lambda \in \mathbb{C}$ such that

$$T = \lambda(V + L) + \mu I.$$

Proof. It is clear that there is no loss of generality in assuming that $\delta = \frac{1}{2}$. Choose $F \in \mathcal{B}(\mathcal{H})$ with rank $F \leq 3$ such that

$$\alpha I + \beta T + \gamma T^* + \frac{1}{2} T^* T + F = 0.$$

This trivially implies that $\overline{\alpha} I + \overline{\gamma} T + \overline{\beta} T^* + \frac{1}{2} T^* T + F^* = 0$. As before, we set $\varrho = \alpha + \overline{\alpha}$, $\theta = \beta + \overline{\gamma}$ and $F_0 = F + F^*$, and note that rank $F_0 \leq 6$. Then

$$\varrho I + \theta T + \overline{\theta} T + T^* T + F_0 = 0.$$

A routine calculation shows that

$$(\varrho - |\theta|^2) I + (T + \overline{\theta} I)^* (T + \overline{\theta} I) + F_0 = 0.$$

Of course, $(T + \overline{\theta} I)^* (T + \overline{\theta} I) \geq 0$, and so - by considering this equation modulo the compact operators, we conclude that $|\theta|^2 - \varrho \geq 0$.

We again consider two cases.

Case 1. $|\theta|^2 = \varrho$.

Then

$$(T + \overline{\theta} I)^* (T + \overline{\theta} I) = -F_0.$$

But then $|T + \overline{\theta} I|$ has finite rank, so that $G = T + \overline{\theta} I$ has finite rank. More specifically, rank $G =$ rank $G^* G =$ rank $F_0 \leq 6$. Thus

$$T = -\overline{\theta}(I - \overline{\theta}^{-1} G) + 0I$$

expresses T in the desired form.

Case 2. $|\theta|^2 > \varrho$.

Set $V_0 = (|\theta|^2 - \varrho)^{-1/2} (T + \overline{\theta} I)$ and $F_2 = (|\theta|^2 - \varrho)^{-1} F_0$, so that rank $F_2 =$ rank $F_0 \leq 6$. Then

$$\begin{aligned} V_0^* V_0 &= (|\theta|^2 - \varrho)^{-1} (T + \overline{\theta} I)^* (T + \overline{\theta} I) \\ &= (|\theta|^2 - \varrho)^{-1} ((|\theta|^2 - \varrho) I - F_0) \\ &= I - F_2. \end{aligned}$$

That is, $\pi(V_0)$ is an isometry in the Calkin algebra. In particular, V_0 is semi-Fredholm, and therefore $(T + \overline{\theta} I)$ is semi-Fredholm. But $T + \overline{\theta} I$ has property (CR), so $T + \overline{\theta} I$ is biquasitriangular, by Proposition 5.2.

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Hence V_0 is Fredholm with index 0. Using the polar decomposition and the fact that V_0 has index 0, we may find a unitary operator U such that $V_0 = U|V_0| = U(I - F_2)^{1/2}$. Thus $U - V_0 = U(I - (I - F_2)^{1/2})$ is of finite rank at most 6, and

$$V_0 = U - (U - V_0) = (|\theta|^2 - \varrho)^{-1/2}(T + \overline{\theta} I).$$

In other words,

$$T = (|\theta|^2 - \varrho)^{1/2}(U + (V_0 - U)) - \overline{\theta} I$$

again expresses T in the desired form.

□

5.7. Proposition. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and $F \in \mathcal{B}(\mathcal{H})$ be a finite-rank operator.*

(a) *Suppose that $V \in \mathcal{B}(\mathcal{H})$ is unitary. If $W = V + F$ has property* (CR)*, then W is normal and $\sigma(W)$ is circlinear.*
(b) *Suppose that $R \in \mathcal{B}(\mathcal{H})$ is hermitian. If $L = R + F$ has property* (CR)*, then R is normal and $\sigma(L)$ is circlinear.*

Proof.

(a) It is obvious that $I - W^*W$ is of finite rank. By Corollary 6.17 of [17], W must have a non-trivial invariant subspace, which must – by virtue of property (CR) – in fact be an orthogonally reducing subspace for W. Thus we may write $W \simeq W_1 \oplus W_2$, and it is clear that each W_k must satisfy property (CR) (by Proposition 2.2) and be of the form $V_k + F_k$ for some unitary operator V_k and some finite-rank operator F_k, $k = 1, 2$. At least one of these summands acts on an infinite-dimensional space, and thus we may again apply Theorem 6.17 of [17] to find non-trivial invariant – hence reducing – subspaces for that summand.

Repeating this process, we see that for any $n \geq 1$, we can find n summands $X_{n,1}, X_{n,2}, \ldots, X_{n,n}$ of W such that

$$W = X_{n,1} \oplus X_{n,2} \oplus \cdots \oplus X_{n,n}.$$

Furthermore, a moments' thought will convince the reader that at most $\operatorname{rank} F$ of these summands can fail to be unitary themselves, and hence when $n > \operatorname{rank} F$, at least one of the $X_{n,k}$'s is a unitary operator.

Let

$$\mathcal{J} = \{(U, \mathcal{M}) : U \text{ is a unitary direct summand of } W \text{ acting on the subspace } \mathcal{M} \text{ of } \mathcal{H}\}.$$

The above paragraph shows that $\mathcal{J}$ is non-empty. Partially order $\mathcal{J}$ by setting $(U_1, \mathcal{M}_1) \leq (U_2, \mathcal{M}_2)$ if $\mathcal{M}_1 \subseteq \mathcal{M}_2$. (Note that this automatically implies that U_1 is a direct summand of U_2.) If $\mathcal{C} = \{(U_\nu, \mathcal{M}_\nu) : \nu \in \Gamma\}$ is a chain in $\mathcal{C}$, then by setting $\mathcal{M} = \overline{\cup_{\nu \in \Gamma} \mathcal{M}_\nu}$, we see that $\mathcal{M}$ is a reducing subspace for W (as each $\mathcal{M}_\nu$ is), and $U = W|_{\mathcal{M}}$ is unitary (since it is clearly unitary on the dense submanifold $\cup_{\nu \in \Gamma} \mathcal{M}_\nu$ of $\mathcal{M}$). It follows from Zorn's Lemma that $\mathcal{J}$ admits a maximal element $(U_0, \mathcal{M}_0)$. If $\mathcal{M}_0^\perp$ is infinite-dimensional, then the argument of the first two paragraphs can be used to show that $W|_{\mathcal{M}_0^\perp}$ admits a unitary direct summand, contradicting the maximality of $(U_0, \mathcal{M}_0)$. Thus $m = \dim \mathcal{M}_0^\perp < \infty$.

Write $W = U_0 \oplus Y$, where Y acts on $\mathcal{M}_0^\perp$, and note that Y has property (CR). We may view Y as an element of $\mathbb{M}_m(\mathbb{C})$, so that Y can be upper triangularized with respect to some orthonormal basis. The fact that Y has property (CR) implies that it is reductive, and is therefore normal. This forces W to be normal as well. There remains to show that $\sigma(W)$ is circlinear. Note that if $\sigma(W)$ is finite, then all elements of $\sigma(W)$ are eigenvalues, and so $\sigma(W)$ is circlinear by the comments of Section 5.3.

Hence we may assume that $\sigma(W)$ is infinite, which is equivalent to assuming that $\sigma(U_0)$ is infinite. In this case, we shall prove that W is unitary. We argue by contradiction. Suppose otherwise, and let $\tau \in \sigma(Y)$ with $|\tau| \neq 1$. Let $\mathcal{N} \subseteq \ker(W - \tau I) \subseteq \mathcal{M}_0^\perp$ be a one-dimensional

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subspace. We see that the operator $Z = U_0 \oplus \tau$, being a direct summand of W, also satisfies property (CR). With respect to the decomposition $\mathcal{M}_0 \oplus \mathcal{N}$, we may write

$$Z = \begin{bmatrix} U_0 & 0 \\ 0 & \tau \end{bmatrix}.$$

Let $x \in \mathcal{M}_0$ be a unit vector such that $\{x, U_0 x, U_0^2 x\}$ is linearly independent. Such a vector must exist, otherwise U_0 is *boundedly locally linearly dependent*, which – by Kaplansky's Theorem [13], Lemma 14 – implies that U_0 is algebraic, and therefore has finite spectrum, a contradiction of our current assumption.

Thus $\{U_0^* x, x, U_0 x\}$ is again linearly independent, as U_0 is unitary. We shall now find vectors y and z in $\mathcal{M}_0 \oplus \mathcal{N}$ such that $\mathcal{E}_1 = \{y, z, Zy, Zz\}$ is linearly independent, but $\mathcal{E}_2 = \{y, z, Z^* y, Z^* z\}$ is not. This will yield the desired contradiction, by implying that $(I-P)ZP$ and $(I-P)Z^*P$ have ranks two and one respectively.

Let $y = \begin{bmatrix} x \\ 1 \end{bmatrix}$ and $z = \begin{bmatrix} U_0 x \\ \xi \end{bmatrix}$, with $\xi \in \mathbb{C}$ to be determined shortly. (Here, we have identified $\mathcal{N}$ with $\mathbb{C}$.) Now

$$\mathcal{E}_1 = \left\{ \begin{bmatrix} x \\ 1 \end{bmatrix}, \begin{bmatrix} U_0 x \\ \xi \end{bmatrix}, \begin{bmatrix} U_0 x \\ \tau \end{bmatrix}, \begin{bmatrix} U_0^2 x \\ \tau\xi \end{bmatrix} \right\},$$

and

$$\mathcal{E}_2 = \left\{ \begin{bmatrix} x \\ 1 \end{bmatrix}, \begin{bmatrix} U_0 x \\ \xi \end{bmatrix}, \begin{bmatrix} U_0^* x \\ \overline{\tau} \end{bmatrix}, \begin{bmatrix} x \\ \overline{\tau}\xi \end{bmatrix} \right\}.$$

Let $\xi = \tau$. Then $\mathcal{E}_1$ is linearly dependent, but $\mathcal{E}_2$ is not, because $\overline{\tau}\xi = |\tau|^2 \neq 1$.

(b) The proof of this result is similar. We may use Corollary 6.15 of [17] to assert that if $L - L^*$ has finite rank, then L has a non-trivial invariant subspace, which is again orthogonally reducing by our hypothesis that L satisfies property (CR). One then looks for a maximal hermitian direct summand, and separately argues the cases where that summand has finite or infinite spectrum. The details are left to the reader.

□

5.8. Theorem. *Let $\mathcal{H}$ be an infinite-dimensional, complex Hilbert space, and let $T \in \mathcal{B}(\mathcal{H})$. If T satisfies property* (CR)*, then there exist $\lambda, \mu \in \mathbb{C}$ and $A \in \mathcal{B}(\mathcal{H})$ with A either selfadjoint or an orthogonally reductive unitary operator such that $T = \lambda A + \mu I$.*

In particular, if T satisfies property (CR)*, then T is normal with circlinear spectrum.*

Proof. By combining Lemma 5.5 and Lemma 5.6, we can assume without loss of generality that $T = X + F$, where F is of finite rank and X is either selfadjoint or unitary.

Either way, by Proposition 5.7, we see that T is normal with circlinear spectrum. From this it is easy to verify that T is of the form $\lambda A + \mu I$ for some $\lambda, \mu \in \mathbb{C}$ with A either selfadjoint or unitary. The fact that T is orthogonally reductive implies that A is as well. (This last argument is superfluous when considering the case where A is selfadjoint.)

□

5.9. Example. We mention in passing that property (CR) is a strictly stronger condition than that of being orthogonally reductive. Indeed, suppose that $N \in \mathcal{B}(\mathcal{H})$ is a normal operator with $\sigma(N) = \{1, 2, 3, 4 + i\}$. Thus the eigenvalues of N are neither co-linear nor co-circular, and so N does not have property (CR), by Corollary 3.14. However, N is orthogonally reductive, as N^* is a polynomial function of N, combined with Sarason's result [18].

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5.10. It would be interesting to know whether or not the converse of Theorem 5.8 holds.

On the one hand, suppose that $N \in \mathcal{B}(\mathcal{H})$ is normal and has co-linear spectrum. Arguing as before, we have that there exist scalars $\lambda, \mu \in \mathbb{C}$ and a hermitian operator L such that $N = \lambda I + \mu L$. It is routine to verify that N has property (CR).

On the other hand, for normal operators with co-circular spectrum, the problem is a bit more complicated.

5.11. Proposition. *Let $U \in \mathcal{B}(\mathcal{H})$ be unitary and suppose that $\sigma(U) \neq \mathbb{T}$. Then U has property* (CR).

Proof. Let $0 \neq P \neq I$ be a projection in $\mathcal{B}(\mathcal{H})$, and relative to $\mathcal{H} = P\mathcal{H} \oplus P^{\perp}\mathcal{H}$, write

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Our goal is to show that $\operatorname{rank} B = \operatorname{rank} C$. As always, we have

$$BB^* = I - AA^*$$
$$C^*C = I - A^*A.$$

If B and C are both of infinite rank, then there is nothing to prove. Thus we may suppose that either B or C is of finite rank. Now, since U has property (CR) if and only if U^* has property (CR), we may suppose – by taking adjoints if necessary – that C is of finite rank and that $\operatorname{rank} C \leq \operatorname{rank} B$.

Case 1. B is compact.

Then A and D are essentially unitary. Since $\sigma_e(A) \subseteq \sigma_e(U) \neq \mathbb{T}$, it follows that $\operatorname{ind} A = 0$. (That is, in order for A to have non-zero index, 0 must lie in a bounded component of $\mathbb{C} \setminus \sigma_e(A)$, of which there are none.)

Write $A = V|A|$, and note that as $\operatorname{ind} A = 0$, we may assume without loss of generality that V is unitary. Thus $AA^* = V|A||A|V^* = V(A^*A)V^*$. That is, AA^* and A^*A are unitarily equivalent.

But then BB^* and C^*C are unitarily equivalent, whence $\operatorname{rank} B = \operatorname{rank} BB^* = \operatorname{rank} C^*C = \operatorname{rank} C$.

Case 2. B is not compact.

We shall show that under the hypothesis that $\sigma(U) \neq \mathbb{T}$, this cannot happen. Indeed, the equation $C^*C = I - A^*A$ with $\operatorname{rank} C < \infty$ implies that

$$1 = \pi(I) = \pi(A)^*\pi(A).$$

Thus $\pi(A)$ is a partial isometry in the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, which implies that $\pi(A)\pi(A)^*$ is a projection. The fact that B is not compact, combined with the fact that $BB^* = I - AA^*$ shows that

$$1 = \pi(I) \neq \pi(A)\pi(A)^*.$$

Thus $\pi(A)$ is not unitary. Choose a projection $R \in \mathcal{B}(\mathcal{H})$ such that $\pi(R) = \pi(A)\pi(A)^*$. By Lemma V.6.4 of [8], there exists a partial isometry $W \in \mathcal{B}(\mathcal{H})$ such that $W = RW$ and $\pi(W) = \pi(A)$. Moreover, by that same result, the integer

$$\xi = \operatorname{rank}(I - W^*W) - \operatorname{rank}(R - WW^*)$$

is defined independent of the choice of W.

In our case, $\operatorname{rank}(I - W^*W) < \infty$ while $\operatorname{rank}(I - R) = \infty$ and $\operatorname{rank}(R - WW^*) < \infty$. Hence $\operatorname{rank}(I - WW^*) = \infty$.

Thus we have that W is a partial isometry with initial space $W^*W\mathcal{H}$, and final space $W\mathcal{H}$, and

(i) $\dim (W^*W\mathcal{H})^{\perp} < \infty$; and

(ii) $\dim (W\mathcal{H})^{\perp} = \infty$.

It is routine to produce a partial isometry W_0 with initial space $(W^*W\mathcal{H})^{\perp} < \infty$ and final space contained in $(W\mathcal{H})^{\perp}$, and to verify that $V = W + W_0$ is an isometry on $\mathcal{H}$.

By the Wold Decomposition, V is unitarily equivalent to $S^{(\kappa)} \oplus Y$, where S denotes the unilateral forward shift, Y is a unitary operator, and $\kappa \in \mathbb{N} \cup \{0, \infty\}$.

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If $\kappa = 0$, then V is unitary. But then $\pi(V) = \pi(W) = \pi(A)$ is also unitary, a contradiction. Thus $\kappa \neq 0$. But then $\sigma_e(A) = \sigma_e(V) \supseteq \sigma_e(S) = \mathbb{T}$.

On the other hand, it is not too hard to show that $\partial(\sigma_e(A)) \subseteq \sigma_{\ell re}(A) \subseteq \sigma_e(U)$. (For example, by the Corollary to Theorem 4.3 of [10], there exists a compact operator $K_1 \in \mathcal{B}(P\mathcal{H})$ such that $A + K_1 = \begin{bmatrix} \lambda I & 0 \\ 0 & A_4 \end{bmatrix}$ with respect to the decomposition $P\mathcal{H} = \mathcal{M} \oplus (P\mathcal{H} \ominus \mathcal{M})$, for an appropriate subspace $\mathcal{M} \subseteq P\mathcal{H}$ satisfying $\dim \mathcal{M} = \dim (P\mathcal{H} \ominus \mathcal{M}) = \infty$. Letting $K = K_1 \oplus 0$ yields that $U + K = \begin{bmatrix} \lambda I & 0 & B_1 \\ 0 & A_4 & B_2 \\ 0 & 0 & D \end{bmatrix}$. Thus $\lambda \in \sigma_e(U + K) = \sigma_e(K)$.) But $\partial(\sigma_e(A)) = \mathbb{T}$, which contradicts our hypothesis that $\sigma(U) \neq \mathbb{T}$.

This shows that the case where C is of finite rank and B is not compact cannot happen, and completes the proof.

□

Having seen that the bilateral shift W is a unitary operator with $\sigma(W) = \mathbb{T}$ which is not reductive, we now show that there exists a unitary operator whose spectrum is the unit circle $\mathbb{T}$, but which nonetheless has property (CR).

Before embarking upon the proof of this, we first require a result due to Wu and Takahashi [20]. Recall that if $X \in \mathcal{B}(\mathcal{H})$ is an operator, then we define the **defect indices** of X to be

$$d_X = \dim(\overline{\operatorname{ran}}(I - X^*X)^{1/2}) \text{ and}$$
$$d_{X^*} = \dim(\overline{\operatorname{ran}}(I - XX^*)^{1/2}).$$

5.12. Proposition. (***Wu-Takahashi****; Theorem 3.5 of* [20]) *Let $X \in \mathcal{B}(\mathcal{H})$ be a contraction and suppose that $d_X \neq d_{X^*}$. Then X does not admit a singular unitary dilation.*

5.13. Proposition. *Let $(d_n)_{n=1}^\infty$ be a sequence in $\mathbb{T}$ and let $V = \operatorname{diag}(d_n)_{n=1}^\infty$ be a corresponding diagonal unitary operator in $\mathcal{B}(\mathcal{H})$. Then V has property* (CR).

Proof. Suppose that we can find a projection $P \in \mathcal{B}(\mathcal{H})$ such that with respect to the decomposition $\mathcal{H} = P\mathcal{H} \oplus P^\perp\mathcal{H}$ we may write

$$V = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

with $\operatorname{rank} C < \operatorname{rank} B$.

(In particular, we must have $\operatorname{rank} C < \infty$). Then

$$BB^* = I - AA^* \text{ and}$$
$$C^*C = I - A^*A$$

have different ranks. Since C is of finite rank, A is essentially isometric and thus is a semi-Fredholm operator - in particular, both A and A^* have closed range. Also,

$$\operatorname{rank} C = \operatorname{rank} C^*C = \operatorname{rank}(I - AA^*) = d_{A^*} < \infty, \text{ while}$$
$$\operatorname{rank} B = \operatorname{rank} BB^* = \operatorname{rank}(I - A^*A) = d_A.$$

In other words, A is a contraction and the defect indices of A are unequal. By Proposition 5.12 above, A does not admit unitary dilation. But U is diagonal, and is therefore a singular unitary dilation of A, which is obviously a contradiction.

□

If, in Proposition 5.13 we choose $\{d_n\}_n$ to be dense in $\mathbb{T}$, we immediately obtain the following consequence:

5.14. Corollary. *There exists a unitary operator V with $\sigma(V) = \mathbb{T}$ which has property* (CR).

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5.15. Wermer [19] has shown that a unitary operator U fails to be reductive if and only if Lebesgue measure is absolutely continuous with respect to the spectral measure μ for U. Since any operator with property (CR) is necessarily reductive, this provides a measure-theoretic obstruction to property (CR) for unitary operators.

Another consequence of the above analysis is that it proves that the set $\mathfrak{G}_{\mathfrak{rank}}$ of operators with property (CR) is not closed. Indeed, it follows easily from [7] that the bilateral shift W is a limit of unitary operators V_n such that $\sigma(V_n) \neq \mathbb{T}$. (The V_n's can in fact be chosen to be unitary operators with spectrum $\Gamma_n = \{e^{2\pi i\theta} : 0 \leq \theta \leq 1 - \frac{1}{n}\}$.) As we saw in Proposition 5.11, each V_n has property (CR), but $W = \lim_n V_n$ does not.

Alternatively, the Weyl-von Neumann-Berg Theorem (see, e.g., [8], Theorem II.4.4) shows that there exists a sequence $(W_n)_{n=1}^{\infty}$ of diagonal unitary operators such that $\sigma(W_n) = \mathbb{T}$ for all $n \geq 1$, such that $W = \lim_n W_n$.

We now investigate a consequence of property (CR) which relates to cyclic subspaces for operators.

5.16. Proposition. *Suppose that T is reductive. Then T and T^* have the same cyclic subspaces. In particular, if T has property* (CR)*, then T and T^* have the same cyclic subspaces.*

Proof. Suppose that $0 \neq \mathcal{M} \subseteq \mathcal{H}$ is a cyclic subspace for T, and let $0 \neq x \in \mathcal{M}$ be a cyclic vector for T in $\mathcal{M}$, so that $\mathcal{M} = \overline{\text{span}}\{x, Tx, T^2x, \ldots\}$. If P is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$, then $P^{\perp}TP = 0$, so by reductivity, $PTP^{\perp} = 0$, which implies that $\mathcal{M}$ is invariant for T^*.

Now let $\mathcal{N} = \overline{\text{span}}\,\{x, T^*x, (T^*)^2x, \ldots\}$ be the cyclic subspace for T^* generated by x. Since $x \in \mathcal{M}$ and $\mathcal{M}$ is invariant for T^*, we see that $\mathcal{N} \subseteq \mathcal{M}$. Also, as T^* is also reductive, the argument of the first paragraph shows that $x \in \mathcal{N}$ is invariant for T. But then $\mathcal{N} \supseteq \mathcal{M}$, whence $\mathcal{N} = \mathcal{M}$, completing the proof.

□

There exists a variant of this result which is somewhat interesting.

5.17. Proposition. *Let $T \in \mathcal{B}(\mathcal{H})$ and suppose that for each orthogonal projection $P \in \mathcal{B}(\mathcal{H})$, the off-diagonal corner $P^{\perp}TP$ has rank one if and only if $PTP^{\perp}$ has rank one. A subspace $\mathcal{M}$ of $\mathcal{H}$ of dimension at least 3 is cyclic for T if and only if it is cyclic for T^*.*

Proof. Given T as in the statement of the Proposition, it is clear that T^* also has this property. The argument used to prove Proposition 5.16 shows that it suffices to show that the cyclic subspace $\mathcal{M} = \overline{\text{span}}\{x, Tx, T^2x, \ldots\}$ for T generated by a non-zero vector x is invariant for T^*.

We consider first the case where $\mathcal{M}$ is infinite-dimensional, as it is the easier of the two.

For each $n \geq 1$, let P_n denote the orthogonal projection of $\mathcal{H}$ onto $\overline{\text{span}}\{x, Tx, T^2x, \ldots, T^nx\}$, and let P denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. It is clear that $(P_n)_{n=1}^{\infty}$ is an increasing sequence which converges in the strong operator topology to P. An easy calculation then shows that the sequence $(P_n^{\perp}TP_n)_{n=1}^{\infty}$ converges in the strong operator topology to $P^{\perp}TP$.

As x is a cyclic vector for T in $\mathcal{M}$, we have that $\text{rank}\,(P_n^{\perp}TP_n) = 1$ for all $n \geq 1$, and our hypothesis then asserts that $\text{rank}\,(P_n^{\perp}T^*P_n) = 1$ for all $n \geq 1$. But rank is lower-semicontinuous with respect to the strong operator topology, and thus $\text{rank}\,P^{\perp}T^*P \leq 1$. If $\text{rank}\,P^{\perp}T^*P = 1$, then the hypothesis on T implies that $\text{rank}\,P^{\perp}TP = 1$, contradicting the fact that $\mathcal{M}$ is invariant for T. Hence $P^{\perp}T^*P = 0$, proving that $\mathcal{M}$ is invariant for T^*.

Next we suppose that $\mathcal{M}$ is finite-dimensional with $\dim \mathcal{M} = N \geq 3$, and we find a cyclic vector x for T so that $\mathcal{M} = \text{span}\{x, Tx, T^2x, \ldots, T^{N-1}x\}$. Let $\{e_1, e_2, \ldots, e_N\}$ be the orthonormal basis obtained from $\{x, Tx, T^2x, \ldots, T^{N-1}x\}$ by applying the Gram-Schmidt process, so that $\text{span}\{e_1, e_2, \ldots, e_k\} = \text{span}\{x, Tx, \ldots, T^{k-1}x\}$ for $1 \leq k \leq N$. Let Q_k denote the orthogonal projection of $\mathcal{H}$ onto $\mathbb{C}e_k$, $1 \leq k$, and define $P_k = Q_1 + Q_2 + \cdots + Q_k$, $1 \leq k \leq N$. Finally, extend $\{e_k\}_{k=1}^{N}$ to an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ for $\mathcal{H}$.

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Note that the fact that x is cyclic for $\mathcal{M}$, combined with our hypothesis, implies that $\operatorname{rank} P_k^\perp T P_k = 1 = \operatorname{rank} P_k T P_k^\perp$, $1 \le k \le N-1$. Moreover, $\mathcal{M}$ is invariant for T, whence $P_N^\perp T P_N = P^\perp T P = 0$. By hypothesis, $\operatorname{rank} PTP^\perp \neq 1$.

But

$$\begin{aligned} PTP^\perp &= P_N T P_N^\perp \\ &= P_{N-1} T P_{N-1}^\perp P_N^\perp + Q_N T P_N^\perp, \end{aligned}$$

so that $\operatorname{rank} PTP^\perp \le \operatorname{rank} P_{N-1} T P_{N-1}^\perp + \operatorname{rank} Q_N T P_N^\perp \le 1 + 1 = 2$.

Thus $\operatorname{rank} PTP^\perp \in \{0, 2\}$, and our goal is to show that $\operatorname{rank} PTP^\perp \neq 2$.

Suppose, to the contrary, that $\operatorname{rank} PTP^\perp = 2$. It follows that $\operatorname{rank} P_{N-1} T P_{N-1}^\perp = 1 = \operatorname{rank} Q_N T P_N^\perp$. Thus there exists $1 \le k \le N-1$ such that $Q_k T P_N^\perp \neq 0$, and $\operatorname{rank}(Q_k + Q_N) T P^\perp = \operatorname{rank}(Q_k + Q_N) T P_N^\perp = 2$.

Case 1. $k = N-1$. Let us reorder the basis for $P\mathcal{H}$ as $\{e_{N-1}, e_N, e_1, e_2, \dots, e_{N-2}\}$. The matrix for T relative to $P\mathcal{H} \oplus P^\perp\mathcal{H}$ is:

$$[T] = \begin{bmatrix} t_{N-1,N-1} & t_{N-1,N} & t_{N-1,1} & t_{N-1,2} & \dots & t_{N-1,N-3} & t_{N-1,N-2} & Q_{N-1}TP^\perp \\ t_{N,N-1} & t_{N,N} & t_{N,1} & t_{N,2} & \dots & t_{N,N-3} & t_{N,N-2} & Q_N TP^\perp \\ \vdots & & & & \dots & & \vdots & \\ t_{N-3,N-1} & t_{N-3,N} & t_{N-3,1} & t_{N-3,2} & \dots & t_{N-3,N-3} & t_{N-3,N-2} & Q_{N-3}TP^\perp \\ t_{N-2,N-1} & t_{N-2,N} & t_{N-2,1} & t_{N-2,2} & \dots & t_{N-2,N-3} & t_{N-2,N-2} & Q_{N-2}TP^\perp \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & P^\perp TP^\perp \end{bmatrix}$$

Let $R = P - Q_{N-2}$. Since $t_{N-2,N-3} \neq 0$ (as x is a cyclic vector for $\mathcal{M}$), it follows that $\operatorname{rank} R^\perp T R = 1$.

Thus

$$\operatorname{rank} \begin{bmatrix} t_{N-1,N-2} & Q_{N-1}TP^\perp \\ t_{N,N-2} & Q_N TP^\perp \\ \vdots & \vdots \\ t_{N-3,N-2} & Q_{N-3}TP^\perp \end{bmatrix} = \operatorname{rank} RTR^\perp = 1,$$

and so $\operatorname{rank}(Q_{N-1} + Q_N)TP^\perp = \operatorname{rank} \begin{bmatrix} Q_{N-1}TP^\perp \\ Q_N TP^\perp \end{bmatrix} \le 1$, a contradiction. Thus in this case, $PTP^\perp = 0$, so $\mathcal{M}$ is invariant for T^*.

Case 2. $1 \le k < N-1$.

This time we reorder the basis for $P\mathcal{H}$ as $\{e_k, e_{k+2}, \dots, e_N, e_1, \dots, e_{k-1}, e_{k+1}\}$. The matrix for T relative to $P\mathcal{H} \oplus P^\perp\mathcal{H}$ is then:

$$[T] = \begin{bmatrix} t_{k,k} & t_{k,k+2} & \dots & t_{k,N} & t_{k,1} & \dots & t_{k,k-1} & t_{k,k+1} & Q_k TP_N^\perp \\ t_{k+2,k} & t_{k+2,k+2} & \dots & t_{k+2,N} & t_{k+2,1} & \dots & t_{k+2,k-1} & t_{k+2,k+1} & Q_{k+2}TP_N^\perp \\ \vdots & & & & \dots & & \vdots & & \\ t_{k-1,k} & t_{k-1,k+2} & \dots & t_{k-1,N} & t_{k-1,1} & \dots & t_{k-1,k-1} & t_{k-1,k+1} & Q_{k-1}TP_N^\perp \\ t_{k+1,k} & t_{k+1,k+2} & \dots & t_{k+1,N} & t_{k+1,1} & \dots & t_{k+1,k-1} & t_{k+1,k+1} & Q_{k+1}TP_N^\perp \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & P_N^\perp TP_N^\perp \end{bmatrix}$$

Let $R = P - Q_{k+1}$. Since $t_{k+1,k} \neq 0$ (as x is a cyclic vector for $\mathcal{M}$), it follows that $\operatorname{rank} R^\perp T R = 1$. By hypothesis, $\operatorname{rank} RTR^\perp = 1$.

Thus

$$\operatorname{rank}(Q_k + Q_N)TP^\perp = \operatorname{rank}(Q_k + Q_N)[RTR^\perp]P^\perp \le \operatorname{rank} RTR^\perp = 1,$$

a contradiction. Thus in this case as well, $PTP^\perp = 0$, so $\mathcal{M}$ is invariant for T^*.

The remainder of the proof is identical to the second paragraph of the proof of Proposition 5.16.

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6. Essentially reductive operators with property (CR)

6.1. In [12], Ken Harrison introduced the notion of **essentially reductive** operators: we say that $T \in \mathcal{B}(\mathcal{H})$ is **essentially reductive** if for each projection P we have that $PTP^{\perp}$ compact if and only if $P^{\perp}TP$ is compact. (One can view this as $\pi(T)$ having property (CR) in the Calkin algebra.)

In the paper [15] (Theorem 2), Moore shows that every essentially reductive operator T is essentially normal – i.e. $\pi(T)$ is normal in the Calkin algebra. Earlier, Harrison ([12], Theorem 4.5) had characterized all essentially normal operators which are essentially reductive. Combining these results, one obtains the following.

6.2. Theorem. (***Moore**; Corollary 1 of* [15]) *Let $T \in \mathcal{B}(\mathcal{H})$. The following are equivalent.*

(a) *T is essentially reductive.*
(b) *T is essentially normal and $\sigma_e(T)$ is Lavrentiev.*

The next result is a simple consequence of Moore's Theorem together with Theorem 5.8 and Proposition 5.11.

6.3. Corollary. *Let $T \in \mathcal{B}(\mathcal{H})$. The following are equivalent.*

(a) *T is essentially reductive and has property* (CR).
(b) *One of the following holds.*
  (i) *There exist $\lambda, \mu \in \mathbb{C}$ and a hermitian operator R such that $T = \lambda R + \mu I$.*
  (ii) *There exist $\lambda, \mu \in \mathbb{C}$ and a unitary operator V with $\sigma(V) \neq \mathbb{T}$ such that $T = \lambda V + \mu I$.*

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