Explorations in black hole chemistry and higher curvature gravity

by

Robie Hennigar

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Physics

Waterloo, Ontario, Canada, 2018

Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

External Examiner: Don Page
Professor, University of Alberta

Supervisor: Robert Mann
Professor, University of Waterloo

Internal Member: Niayesh Afshordi
Professor, University of Waterloo

Internal-External Member: Achim Kempf
Professor, University of Waterloo

Committee Member: Robert Myers
Professor, Perimeter Institute for Theoretical Physics, University of Waterloo

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of contributions

This thesis is based on the following published and forthcoming articles.

- Chapter 3 is based on:
  - R. A. Hennigar, D. Kubiznak and R. B. Mann, *Entropy Inequality Violations from Ultraspinning Black Holes*, *Phys. Rev. Lett.* **115** (2015) 031101, [1411.4309]
  - R. A. Hennigar, D. Kubiznak, R. B. Mann and N. Musoke, *Ultraspinning limits and super-entropic black holes*, *JHEP* **06** (2015) 096, [1504.07529]
  - R. A. Hennigar, D. Kubiznak, R. B. Mann and N. Musoke, *Ultraspinning limits and rotating hyperboloid membranes*, *Nucl. Phys.* **B903** (2016) 400–417, [1512.02293]
- Chapter 4 is based on:
  - R. A. Hennigar and R. B. Mann, *Black holes in Einsteinian cubic gravity*, *Phys. Rev.* **D95** (2017) 064055, [1610.06675]
  - R. A. Hennigar, D. Kubiznak and R. B. Mann, *Generalized quasitopological gravity*, *Phys. Rev.* **D95** (2017) 104042, [1703.01631]
  - J. Ahmed, R. A. Hennigar, R. B. Mann and M. Mir, *Quintessential Quartic Quasi-topological Quartet*, *JHEP* **05** (2017) 134, [1703.11007]
  - R. A. Hennigar, *Criticality for charged black branes*, *JHEP* **09** (2017) 082, [1705.07094]
  - R. A. Hennigar, M. B. J. Poshteh and R. B. Mann, *Shadows, Signals, and Stability in Einsteinian Cubic Gravity*, *Phys. Rev.* **D97** (2018) 064041, [1801.03223]
  - P. Bueno, P. A. Cano, R. A. Hennigar and R. B. Mann, *NUTs and bolts beyond Lovelock*, [1808.01671]
  - P. Bueno, P. A. Cano, R. A. Hennigar and R. B. Mann, *Universality of squashed-sphere partition functions*, [1808.02052]
  - R. A. Hennigar, M. Lu and R. B. Mann, *Five-dimensional generalized quasi-topological gravities*, *In preparation, 2018*
  - M. Mir, R. A. Hennigar, R. B. Mann and J. Ahmed, *Chemistry and holography in generalized quasi-topological gravity*, *In preparation, 2018*

- Chapter 5 is based on:
  - R. A. Hennigar, R. B. Mann and E. Tjoa, *Superfluid Black Holes*, *Phys. Rev. Lett.* **118** (2017) 021301, [1609.02564]
  - R. A. Hennigar, E. Tjoa and R. B. Mann, *Thermodynamics of hairy black holes in Lovelock gravity*, *JHEP* **02** (2017) 070, [1612.06852]
  - H. Dykaar, R. A. Hennigar and R. B. Mann, *Hairy black holes in cubic quasi-topological gravity*, *JHEP* **05** (2017) 045, [1703.01633]
- Contributions not included in the thesis:
  - R. A. Hennigar, W. G. Brenna and R. B. Mann, *Pv criticality in quasitopological gravity*, *JHEP* **07** (2015) 077, [1505.05517]
  - R. A. Hennigar and R. B. Mann, *Reentrant phase transitions and van der Waals behaviour for hairy black holes*, *Entropy* **17** (2015) 8056–8072, [1509.06798]
  - R. A. Hennigar, R. B. Mann and S. Mbarek, *Thermalon mediated phase transitions in Gauss-Bonnet gravity*, *JHEP* **02** (2016) 034, [1512.02611]
  - R. A. Hennigar, F. McCarthy, A. Ballon and R. B. Mann, *Holographic heat engines: general considerations and rotating black holes*, *Class. Quant. Grav.* **34** (2017) 175005, [1704.02314]
  - L. J. Henderson, R. A. Hennigar, R. B. Mann, A. R. H. Smith and J. Zhang, *Harvesting Entanglement from the Black Hole Vacuum*, 1712.10018
  - P. A. Cano, R. A. Hennigar and H. Marrochio, *Complexity Growth Rate in Lovelock Gravity*, 1803.02795

Abstract

This thesis has two goals. The primary goal is to communicate two results within the framework of black hole chemistry, while the secondary goal is concerned with higher curvature theories of gravity.

Super-entropic black holes will be introduced and discussed. These are new rotating black hole solutions that are asymptotically (locally) anti de Sitter with horizons that are topologically spheres with punctures at the north and south poles. The basic properties of the solutions are discussed, including an analysis of the geometry, geodesics, and black hole thermodynamics. It is found that these are the first black hole solutions to violate the reverse isoperimetric inequality, which was conjectured to bound the entropy of anti de Sitter black holes in terms of the thermodynamic volume. Implications of this result for the inequality are discussed.

The second main result is a new phase transition in black hole thermodynamics: the λ-line. This is a line of second order (continuous) phase transitions with no associated first order phase transition. The result is illustrated for black holes in higher curvature gravity — cubic Lovelock theory coupled to real scalar fields. The properties of the black holes exhibiting the transition are discussed and it is shown that there are no obvious pathologies associated with the solutions. The features of the theory that allow for the transition are analyzed and then applied to obtain a further example in cubic quasi-topological gravity.

The secondary goal of the thesis is to discuss higher curvature theories of gravity. This serves as a transition between the discussion of super-entropic black holes and λ-lines but also provides an opportunity to discuss recent work in the area. Higher curvature theories are introduced through a study of general theories on static and spherically symmetric spacetimes. It is found that there are three classes of theories that have a single independent field equation under this restriction: Lovelock gravity, quasi-topological gravity, and generalized quasi-topological gravity, the latter being previously unknown. These theories admit natural generalizations of the Schwarzschild solution, a feature that turns out to be equivalent to a number of other remarkable properties of the field equations and their black hole solutions. The properties of these theories are discussed and their applicability as toy models in gravity and holography is suggested, with emphasis on the previously unknown generalized quasi-topological theories.

Acknowledgements

During my time at the University of Waterloo, I have had the privilege of learning a great deal from very smart people. While I am grateful to all those who have served as a mentor to me in some capacity, I would especially like to single out three individuals. First, is my supervisor Robb Mann who in addition to being an excellent and kind teacher, has always provided superb guidance in both research and professional matters. I owe the very fact that I have been able to study topics in theoretical physics to him — Robb, thanks for responding positively to that email sent to you by a naive little physical chemist who wanted to learn about black holes! Second, I would like to acknowledge Eduardo Martín-Martínez. While I have learned a great deal from Edu concerning relativistic quantum information, I have been particularly inspired by his enthusiasm for science and his dedication to creating a comfortable and effective environment for his students. I hope that if I am lucky enough to one day have a group of my own that I can implement some of the same ideas in my own approach to supervision. Finally, I would like to acknowledge the mythical David Kubizňák. David was my first collaborator when I was still very new in the field of physics and was always kind to me when I made mistakes. I have learned a lot from David — his careful attention to detail in research, his sense of humour, and his positive yet honest outlook have left lasting impressions on me. Also, I think it is only fitting that, in David's honour, I have submitted this thesis on a Friday.

I would like to thank all of the friends, collaborators, and students who have made research and life (though, is there really a distinction at this point?) much more fun: Connor Adair, Jamil Ahmed, Aida Ahmadzadegan, Natacha Altamirano, Alvaro Ballon, Wilson Brenna, Kris Boudreau, Pablo Cano, Alex Chase, Paulina Corona Ugalde, Michael Deveau, Hannah Dykaar, Dan Grimmer, Zhiwei Gu, Laura Henderson, Raines Heath, Christine Kaufhold, Mengqi Lu, Saoussen Mbarek, Fiona McCarthy, Hugo Marrochio, Michael Meiers, Mozhgan Mir, Nathan Musoke, Mohammad Poshteh, Ethan Purdy, Allison Sachs, Turner Silverthorne, Alex Smith, Erickson Tjoa, and Jialin Zhang. Erickson and Hugo get a special bonus thanks for carefully reading drafts of this thesis and providing feedback.

I would like to thank my family for their support. My parents, Sharon and Brady, my brother, William, my uncle, Opie, and my grandparents, Edgar and Hazel, and Ella Mae. You have all provided inspiration for my work.

Finally, I would like to say thank you to my partner, Emily Lavergne. Your support has had an enormous influence on my well-being during my time in Waterloo. Thank you for listening to me babble about exciting new results, and listening to me complain when everything goes wrong. You have enriched my life.

Table of contents

List of Tables

List of Figures

Notation and conventions

- Throughout this thesis, we will use a mostly plus metric signature. Our convention for the curvature is $R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}{}_{\beta\nu,\mu} + \cdots$ with $R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}$.
- We will work in units such that $c = \hbar = k_B = 1$. We will typically keep Newton's constant in expressions unless otherwise indicated.
- The following symbols are used commonly throughout the thesis:

D	Spacetime dimension
L	AdS length
G_N	Newton's constant
$\Omega_{k,n}$	Volume of n-dimensional space of constant curvature k
$\mathbb{S}, \mathbb{R}, \ldots$	Used to denote topological spaces
$\alpha, \beta, \mu, \nu, \ldots$	Spacetime indices
$T_{[\mu\nu]} = \frac{1}{2}\left(T_{\mu\nu} - T_{\nu\mu}\right)$	Anti-symmetric part of tensor
$T_{(\mu\nu)} = \frac{1}{2}\left(T_{\mu\nu} + T_{\nu\mu}\right)$	Symmetric part of tensor

Chapter 1

Introduction

Understanding the quantum nature of the gravitational field remains one of the most profound questions in theoretical physics. While the problem of quantum gravity has its roots in the 19th century attempts to unify electromagnetism and gravitation, there remains today no fully satisfactory or accepted resolution of the problem.

Arguably our best insight into the quantum nature of gravity comes from the thermodynamics of black holes. Quantum effects cause black holes to emit radiation with a temperature proportional to the surface gravity, and lead one to assign to them an entropy proportional to the horizon area. Black holes exhibit incredibly rich thermal structure, and it is even possible to describe phase transitions between different geometries. Studying the thermodynamic aspects of gravity provides important clues about quantum gravity — such a theory should be able to provide a microscopic origin for the black hole entropy, for example. The thermal properties of gravity take on a whole new meaning in light of powerful gauge/gravity dualities that relate gravitational theories to strongly coupled gauge theories.

Black hole thermodynamics is the overarching topic of this thesis. The purpose of this chapter is to provide a high-level introduction and overview of some essential ideas, some of which will be made more concrete in later chapters. The majority of results in this thesis concern asymptotically anti de Sitter spacetimes, and so this chapter begins with a somewhat technical introduction to the anti de Sitter geometry. This is followed by a discussion of black hole thermodynamics in anti de Sitter space, and black hole chemistry is introduced. Black hole chemistry, which forms the context of this thesis, considers the implications of treating the cosmological constant as a pressure in black hole thermodynamics. Next, we provide some motivation for why anti de Sitter space is relevant in physics, highlighting its importance in understanding thermodynamic aspects of black holes, its appearance in many derivations of black hole entropy, and discussing the AdS/CFT correspondence, which provides the strongest motivation for the study of anti de Sitter spaces. With this background material discussed, a final section outlines the structure of the remainder of the thesis.

1.1 Welcome to anti de Sitter space

The geometry of anti de Sitter space is most easily appreciated via the embedding space. Consider a $(D+1)$-dimensional flat spacetime with two time directions. The metric on this space would read

$$ds^2 = -dT_1^2 - dT_2^2 + dX_1^2 + \cdots + dX_{D-1}^2 \,. \tag{1.1}$$

In the same sense that a sphere or hyperboloid can be embedded in Euclidean space, here we can study the D-dimensional surface given by the following constraint [22]

$$-T_1^2 - T_2^2 + X_1^2 + \cdots + X_{D-1}^2 = -L^2 \,. \tag{1.2}$$

The parameter L above has units of length and will be referred to as the AdS length. The induced metric on this surface can be conveniently written in terms of the following coordinates

$$T_1 = \sqrt{r^2+L^2}\cos(t/L)\,, \quad T_2 = \sqrt{r^2+L^2}\sin(t/L)\,, \quad X_i = r\hat{\mu}_i\,, \tag{1.3}$$

where $\hat{\mu}_i$ satisfy $\sum_i \hat{\mu}_i^2 = 1$ and represent the angular coordinates on a sphere. For example, in the standard angular representation we would have

$$\hat{\mu}_1 = \sin\theta\sin\phi_1\cdots\sin\phi_{D-3}\,, \quad \hat{\mu}_2 = \sin\theta\sin\phi_1\cdots\cos\phi_{D-3}\,, \tag{1.4}$$

and so on. In terms of these coordinates, the induced metric becomes

$$ds^2 = -\left(1+\frac{r^2}{L^2}\right)dt^2 + \left(1+\frac{r^2}{L^2}\right)^{-1}dr^2 + r^2 d\Omega_{D-2}^2\,, \tag{1.5}$$

where $d\Omega_{D-2}^2$ is the standard line element on the $(D-2)$-dimensional unit sphere. The line element (1.5) is AdS written in familiar global coordinates. Strictly speaking, this embedding requires that the coordinate t is periodic with period $t \sim t + 2\pi L$, and so AdS has closed timelike curves — the topology is $\mathbb{S}^1 \times \mathbb{R}^{D-1}$. However, as is more common in the literature, throughout this thesis we will use the term "AdS" to refer to the universal cover of this space obtained by "unwrapping" the $\mathbb{S}^1$ time coordinate and letting it take on all values on the real line.

The anti de Sitter spacetime is maximally symmetric with $D(D+1)/2$ Killing vectors that generate the group $SO(2, D-1)$. It is a space of constant negative curvature and solves the vacuum Einstein equations with cosmological constant,

$$\Lambda = -\frac{(D-1)(D-2)}{2L^2} \,. \tag{1.6}$$

Anti de Sitter space has a conformal boundary located at spatial infinity, i.e. $r \to \infty$, that is topologically $\mathbb{R} \times \mathbb{S}^{D-2}$ and timelike. A consequence of the timelike boundary is that AdS is not a globally hyperbolic spacetime. A manifestation of this is that fields living in AdS require boundary conditions specified at infinity [23].

Finally, it is instructive to consider geodesic motion in AdS. It is easy to show that radial timelike geodesics satisfy

$$\left(\frac{dr(\tau)}{d\tau}\right)^2 - \mathcal{E}^2 + 1 + \frac{r(\tau)^2}{L^2} = 0 \,, \tag{1.7}$$

where $\mathcal{E}$ is the conserved energy associated with the Killing vector ∂_t. Differentiating the equation we see that

$$\frac{d^2r(\tau)}{d\tau^2} + \frac{1}{L^2} r(\tau) = 0 \,, \tag{1.8}$$

indicating that test particles following these timelike worldlines execute simple harmonic motion with period $2\pi L$.[1] This allows one to interpret the gravitational potential in AdS as a "confining" potential — particles displaced from the "origin" experience a restoring force. A number of physically interesting phenomena result from the gravitational potential in AdS, and we will describe some below.

1.2 Black hole thermodynamics in anti de Sitter space

When supplemented by a negative cosmological constant, the Einstein equations admit black hole solutions that are asymptotically AdS. The simplest such solution is the spherically symmetric Schwarzschild AdS black hole

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2_{D-2} \,, \tag{1.9}$$

[1] Note the period $2\pi L$ is the same as the period for t in the embedding of AdS in higher dimensional flat space. Despite the absence of CTCs in the universal cover, this simple harmonic motion can be considered a 'relic' of the CTCs in the original space.

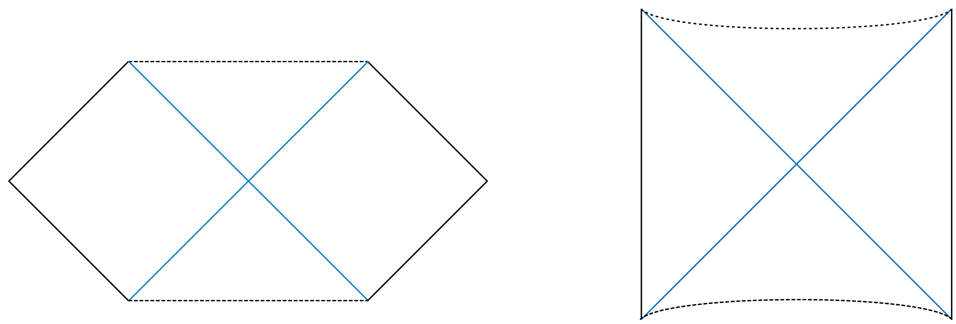

Figure 1.1: **Causal structure of black holes**. *Left*: The causal structure of the maximally extended asymptotically flat Schwarzschild black hole. *Right*: The causal structure of the maximally extended asymptotically AdS Schwarzschild black hole. The dotted lines represent curvature singularities, while the solid diagonal blue lines mark the event horizon.

with

$$f(r) = 1 - \frac{m}{r^{D-3}} + \frac{r^2}{L^2}\,. \tag{1.10}$$

The metric describes a black hole with event horizon located at $r = r_+$ where $f(r_+) = 0$. The parameter m is related to the mass of the black hole, in a way that will be made precise in the following chapter. The causal structure of the black hole is depicted in figure 1.1, where it is compared with the the asymptotically flat case. Some aspects of the causal structure are similar, for example, in both cases the event horizon represents a causal boundary in the spacetime and there is a central singularity located at $r = 0$ where curvature invariants diverge. There are also a number of differences, for example in the AdS case the metric on the conformal boundary is

$$ds^2_{\text{bdry}} = -dt^2 + L^2 d\Omega^2_{D-2} \tag{1.11}$$

which is obtained by first rescaling by L^2/r^2 and then sending $r \to \infty$. The boundary is timelike and topologically $\mathbb{R} \times \mathbb{S}^{D-2}$.

The thermal properties of black holes in AdS differ significantly compared to their asymptotically flat cousins. We will give a more precise discussion of black hole thermodynamics in the following chapter, but here we will simply quote some results. The temperature of a Schwarzschild AdS black hole is given by

$$T = \frac{(D-3)}{4\pi r_+} + \frac{(D-1)r_+}{4\pi L^2}\,, \tag{1.12}$$

where the first term above corresponds to the temperature of the ordinary, asymptotically flat black hole, while the second part is the AdS correction. The implications of the AdS correction are significant. Note that for r_+ large compared to L, the temperature is *proportional* to the horizon radius (and therefore grows with the mass) rather than inversely proportional. This means that large black holes are thermally stable, i.e. have positive specific heat. The thermal stability is a consequence of the gravitational potential of AdS, which effectively corresponds to putting the black hole in a "reflecting box". With positive specific heat, it is possible for the AdS black hole to come to thermal equilibrium with radiation, in constrast to the asymptotically flat case where this equilibrium would be in general unstable [24].

An even more remarkable observation was made by Hawking and Page in 1983 [25]. It is possible, as we will discuss in the following chapter, to assign free energy to black holes, $F = M - TS$ where M is the mass of the black hole, T is the temperature, and S is the entropy. The free energy is measured relative to some specified ground state, which in this case is anti de Sitter space at finite temperature — *thermal AdS*. What Hawking and Page observed is that, at sufficiently high temperatures, the black hole has lower free energy than the thermal AdS space, while the reverse is true at sufficiently low temperatures. Basic thermodynamics reveals that there will be a phase transition between thermal AdS and a large AdS black hole at the temperature where the free energies are equal. This has come to be known as the Hawking-Page transition and is quintessential example of a phase transition in black hole thermodynamics.

The discovery of the AdS/CFT correspondence signaled a boom in studies of the thermodynamics of AdS black holes, e.g. [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36]. This was motivated by possibility of understanding the phase structure of strongly coupled quantum field theories via comparatively simple studies of AdS black holes. A particularly interesting result to emerge during this time was due to Chamblin, Emparan, Johnson, and Myers [30, 31]. These authors studied the thermodynamics of electrically charged AdS black holes. As we will discuss in greater detail in the following chapter, when electric charge is included there exist up to three black holes for a given electric charge and temperature, distinguished by their size. Of these three, two have positive specific heat while the third is thermally unstable. Remarkably, it was found that the thermal behaviour of charged AdS black holes is analogous to the van der Waals model for the liquid-gas system. In the black hole case, the two phases correspond to the large and small thermally stable black holes. These phases are separated by a line of first order phase transitions that terminates at a critical point, where the phase transition is second order. The order parameter, which characterizes the difference between the two phases, is the difference in size of the black holes, $\eta = r_+^{\text{large}} - r_+^{\text{small}}$. As the critical point is approached, the order parameter vanishes and various thermodynamic potentials diverge in power law fashion. The exponents governing the approach to criticality were found to be precisely the mean field theory critical exponents.

More recently, there has been a renewed interest in the thermodynamics and phase structure of black holes. Motivated by basic thermodynamic scaling arguments, it has been realized that when describing AdS black holes, the cosmological constant should be treated as a thermodynamic black hole parameter [37, 38]. The cosmological constant is naturally interpreted as a pressure $P = -\Lambda/(8\pi G_N)$, and its conjugate quantity is known as the thermodynamic volume, V. It was found that within this framework, the critical behaviour of the charged AdS black hole studied in [30] becomes a physical analogy with the van der Waals fluid, with the analog of the liquid/gas transition being a small/large black hole phase transition [39]. Numerous other results have since been obtained, including *triple points* like that of water and *re-entrant phase transitions* like those that occur in nicotine/water mixtures [40, 41, 42]. The thermodynamics of black holes in higher curvature gravity has proven particularly fruitful, including examples of *multiple re-entrant phase transitions* [43], *isolated critical points* [43, 44, 16] and most recently a black hole λ-line — an analog of a superfluid phase transition [13]. These results and their analogy with the thermal behaviour of everyday substances has resulted in the moniker *black hole chemistry* for the field [45]. This programme goes beyond critical behaviour with studies developing entropy inequalities for AdS black holes [46, 1], discussing the notion of holographic heat engines [47], and investigations of holographic implications [48, 49, 50, 51].

1.3 The role of anti de Sitter space in physics

While the best evidence suggests that the cosmological constant in our universe is not negative, studies of gravitation with a negative cosmological constant comprise a large fraction of the literature. There exist many important areas of physics where AdS appears, so let us illustrate several examples.

When quantizing a field theory, a maximally symmetric space is often chosen to be the ground state and the behaviour of perturbations away from the ground state are analyzed. Minkowski, de Sitter, and anti de Sitter spaces exhaust the list of maximally symmetric spaces. Anti de Sitter space appears naturally as the ground state in supergravity and string theories [52].

It is hard to overemphasize the importance of anti de Sitter space in our understanding of black holes and their quantum mechanics. It is often the case that black holes in AdS are thermally stable, in contrast with asymptotically flat black holes that are thermally unstable. Due to this, AdS provides a rich playground for understanding features of black hole thermodynamics. In the most famous example, Hawking and Page demonstrated [25] the existence of a phase transition between thermal AdS and a large AdS black hole, as was described above. In the everyday world, phase transitions and critical behaviour reflect the organization of the microscopic constituents of matter. Applying this same insight in the realm of gravity means that by studying the thermal properties of gravity we are learning, in some limited sense, about its microscopic degrees of freedom — whatever those may be.

The simplest possible black hole solution — the BTZ solution, named for its discoverers, Bañados, Teitelboim, and Zanelli [53, 54] — is a three dimensional black hole obtained by geometric identifications of anti de Sitter space. The solution exhibits many of the features of higher dimensional black holes, including an event horizon, an ergosphere in the rotating case, and black hole thermodynamics. Due to the simplicity of the solution, it has featured prominently in many investigations of the quantum properties of black holes. It has played a key role in the understanding of the microscopic origin of black hole entropy. Strominger was able to reproduce the entropy of the BTZ black hole by building on the work of Brown and Henneaux [55] which demonstrated that the asymptotic symmetry group of three dimensional AdS gravity is generated by two copies of the Virasoro algebra with equal central charges. Application of Cardy's formula [56] then yielded the Bekenstein-Hawking entropy. See [57] and references therein for a review of this argument and its limitations.

Strominger's argument goes beyond the particular example of the three dimensional black hole. In higher dimensions, many black hole solutions exhibit enhanced symmetry in the vicinity of the horizon. For example, the near horizon limit of near extremal static black holes factorizes into two dimensional AdS times spheres. In some cases, the near horizon metric factorizes into a product geometry involving AdS_3 as one of the factors. In these cases, Strominger's argument has been successfully applied to reproduce the entropy of higher dimensional AdS black holes via conformal field theory arguments, e.g. [58, 59, 60, 61].

Perhaps the most remarkable role played by AdS in physics is through the anti de Sitter/conformal field theory (AdS/CFT) correspondence discovered by Maldacena in 1997 [62, 63, 26]. The AdS/CFT correspondence conjectures that gravitational theories in D dimensions are equivalent to quantum field theories in $D-1$ dimensions. The most well-known (and well-tested [64]) example of this duality equates the dynamics of type IIB string theory on $\mathrm{AdS}_5 \times \mathbb{S}^5$ to $\mathcal{N}=4$ super Yang-Mills theory with gauge group $SU(N)$. While it is often not possible to work within a concrete example of the duality, it is expected to be generally true. Moreover, in instances where the field theory is strongly coupled and has a large number of degrees of freedom, i.e. when $N \to \infty$, classical gravitation in AdS is expected to provide a meaningful description. In this way, general relativity and its higher curvature extensions have been fruitfully used to better understand the properties of strongly coupled field theories [65, 66, 67, 68, 69]. For example, the dual theory can be probed at finite temperature by considering the behaviour of black objects in the bulk. In this context, Witten interpreted the Hawking-Page transition for AdS black holes as a confinement/deconfinement transition in gauge theory [26, 27]. On the other hand, since AdS/CFT is a two-way street, it offers the possibility of understanding the resolution to some of the deepest problems in physics, e.g. the black hole information paradox [70, 71] via an understanding of quantum field theory.

1.4 Plan of the thesis

The main goal of this thesis is to present what I regard to be my two most significant contributions to black hole chemistry. Along the way, I will also review some of my more recent work in higher curvature theories of gravity. I have done my best to keep the presentation of the results as concise as possible.

To achieve these goals, this thesis is organized as follows. Chapter 2 deals with the foundations of black hole chemistry. After a more thorough account of the laws of black hole thermodynamics, the thermodynamic status of the cosmological constant is motivated via scaling and geometric derivations of integral mass formulas, i.e. Smarr relations, for black holes. A derivation of the first law of black hole mechanics is presented including variations of the cosmological constant. The remainder of chapter 2 goes into more depth on two important topics in black hole chemistry. First, we review the *reverse isoperimetric inequality*, which is an inequality between the entropy and thermodynamic volume which was conjectured to constrain the entropy of AdS black holes. Second, we review the critical behaviour and van der Waals analogy for the charged AdS black hole, introducing important thermodynamic machinery along the way.

Chapter 3 presents a new class of rotating, asymptotically (locally) AdS black holes known as *super-entropic black holes*. The notion of an ultra-spinning limit is reviewed and then applied to the four-dimensional Kerr-AdS black hole to obtain the super-entropic black hole. The basic properties of the solution are discussed, the horizons of these black holes are topologically punctured spheres and approach Lobachevsky space near the axis. A study of geodesics in the spacetime shows that the symmetry axis is excised from the geometry. The procedure used to obtain the four dimensional solution can be generalized to higher dimensions, and this is demonstrated for a singly spinning Kerr-AdS black hole in all dimensions. Finally, the thermodynamic properties of the solutions are discussed and it is shown that the conjectured reverse isoperimetric inequality is violated — these are the first and so far only AdS black holes in Einstein gravity that violate the conjecture.

Chapter 4 has two purposes. First, it serves as a bridge between chapter 3 and chapter 5, introducing higher curvature theories of gravity, explaining some of their basic properties, and providing details on black hole thermodynamics in higher curvature theories. The path taken to introduce the higher curvature theories is not standard, but it chosen so that some of my recent work on these theories can be highlighted — this is the second purpose. To this effect, certain examples of higher curvature theories are presented based on an exploratory analysis of spherically symmetric solutions. Specifically, it shown that there are three natural classes of higher curvature theories under these symmetry restrictions, including the well-known Lovelock class of theories, quasi-topological gravities, and the recently discovered generalized quasi-topological theories. These classes are distinguished by the fact that they admit natural extensions of the Schwarzschild solution to general relativity. It turns out that this simple fact is unexpectedly equivalent to a number of other properties. For example, all of these theories propagate only the usual massless, spin-2 graviton on constant curvature spaces. The field equations are always total derivatives, when integrated involve at most second derivatives of the metric, and admit vacuum black hole solutions characterized only by their mass. Despite the lack of analytical black hole solutions in some cases, it is always possible to study the thermodynamics of the black holes exactly. These properties make these theories excellent toy models for exploring questions in black hole physics and holography. The properties are demonstrated in detail in the simplest cases, with comments on the general trends.

Chapter 5 returns to the main theme of the thesis and studies the critical behaviour of black holes in higher curvature gravity coupled to scalar fields. It is demonstrated that the black holes in cubic (or higher) order Lovelock gravity exhibit a black hole "λ-line". This is a line of second order (continuous) phase transitions and the example presented in this chapter represents, to the best of our knowledge, the first such example in black hole thermodynamics. The second part of this chapter is dedicated to analyzing the properties of the black holes exhibiting this phase transition, finding no fundamental pathologies. The chapter concludes by presenting a necessary condition for the λ-line required in certain classes of gravity theories, and is applied to find a further example in quasi-topological gravity.

Finally, chapter 6 presents some summarizing thoughts on the research presented in this thesis. A number of appendices collect useful results.

Chapter 2

The foundations of black hole chemistry

The purpose of this chapter is to provide a basic introduction to the foundations of black hole chemistry. To start, we will review fundamental aspects of black hole thermodynamics, and discuss Smarr relations for asymptotically flat and asymptotically AdS black holes. We then motivate the inclusion of the cosmological constant in the first law of black hole mechanics, followed by a derivation. The remainder of the chapter is devoted to discussing topics from black hole chemistry and introducing concepts that are relevant later in the thesis, specifically the reverse isoperimetric inequality and critical behaviour of black holes.

2.1 Black holes and thermodynamics

More than forty years ago, Bardeen, Carter, and Hawking published their now famous account of the laws of black hole mechanics [72], which can be stated succinctly as follows [73].

Zeroth law *The surface gravity* κ *of a stationary black hole is constant everywhere on the surface of the event horizon.*

First law *When the system incorporating a black hole switches from one stationary state to another, the mass of the system changes as*

$$\delta M = \frac{\kappa \delta A}{8\pi G_N} + \Omega_H \delta J + \Phi_H \delta Q + \delta q \,. \tag{2.1}$$

where δJ and δQ are the changes in total angular momentum and electric charge of the black hole, while δq represents the change in mass due to stationary matter surrounding the black hole.

Second law *In any classical process, the area of a black hole A does not decrease.*

Third law *An extremal black hole cannot be created in a finite number of steps.*[2]

The obvious connection with the laws of ordinary classical thermodynamics was not lost on the authors, but they emphasized that this similarity is superficial. It was to be understood that these are the mechanical laws obeyed by black holes, governing their response to the accretion of matter or the extraction of energy, e.g. via the Penrose process [74]. However, as we now know, the laws of black hole mechanics are *precisely* the laws of thermodynamics applied to black holes. Establishing this connection was the result of the pioneering work of Hawking and Bekenstein [75, 76]. Based on the area theorem, Bekenstein argued that black holes should possess an entropy proportional to the area of the event horizon. The constant of proportionality was only fixed after Hawking demonstrated, via a quantum field theory in curved spacetime calculation, that a black hole can radiate. If the black hole is able to radiate, then it can be sensibly assigned a temperature, which turns out to be equal to $T = \kappa/(2\pi)$. In this way, κ is related to the true, physical temperature of the black hole and $A/(4G_N)$ is the entropy.

There are now many different approaches to calculate the temperature of the Hawking radiation showing that it is simply $T = \kappa/(2\pi)$, see e.g. [76, 77, 78, 79]. Perhaps the most practical is the Euclidean method [80, 81], which we illustrate in more detail for a static and spherically symmetric black hole. The most general line element for a static and spherically symmetric black hole can be cast in the form

$$ds^2 = -N(r)^2 f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2_{D-2} \tag{2.2}$$

where D is the spacetime dimension and $d\Omega^2_{D-2}$ is the standard line element on a $(D-2)$-dimensional sphere.[3] We consider the situation where this metric describes a black hole whose horizon radius r_+ is determined as the largest root of $f(r) = 0$. We then Wick

[2] A more complicated but also more technically correct statement of the third law would be: A non-extremal black hole cannot become extremal at finite advanced time in any continuous process in which the stress-energy tensor of accreated matter stays bounded and satisfies the weak energy condition in the neighborhood of the outer apparent horizon [73].

[3] More generally, this could be the line element on any $(D-2)$-dimensional surface of constant curvature $k = -1, 0, +1$.

rotate the time coordinate $t \to -it_E$ such that the metric now has a Euclidean signature and expand $f(r)$ near $r = r_+$. Explicitly, the (t_E, r) sector becomes

$$ds_E^2 = N(r_+)^2 f'(r_+)(r - r_+)dt_E^2 + \frac{dr^2}{f'(r_+)(r - r_+)} \,. \tag{2.3}$$

Now, defining a new coordinate by

$$d\rho^2 = \frac{dr^2}{f'(r_+)(r - r_+)} \tag{2.4}$$

the (t_E, r) sector can be recast as

$$ds_E^2 = d\rho^2 + \left[\frac{N(r_+)f'(r_+)}{2}\right]^2 \rho^2 dt_E^2 \,. \tag{2.5}$$

In this form, we recognize that the (t_E, ρ) sector is just the metric on $\mathbb{R}^2$, written in polar coordinates with the horizon located at the origin. As we know from basic geometry, the azimuthal coordinate must have a period of 2π to avoid conical singularities. So in order for the near horizon metric to be regular, the Euclidean time must be periodically identified as $t_E \sim t_E + 4\pi/\left(N(r_+)f'(r_+)\right)$. Translating back to Lorentzian signature, what we have shown is that regularity has imposed that the time coordinate is periodic in the imaginary direction

$$t \sim t + i\frac{4\pi}{N(r_+)f'(r_+)} \,. \tag{2.6}$$

Now, recalling from quantum field theory that thermal states are defined via the Kubo-Martin-Schwinger (KMS) condition[4] [82, 83, 84] which demands that the thermal Green's functions are periodic in imaginary time

$$G_\beta(t, x; t', x') = G_\beta(t + i\beta, x; t', x') \,, \tag{2.7}$$

with $T = 1/\beta$. Since regularity of the underlying spacetime enforces the imaginary time periodicity, quantum fields placed on the spacetime inherit this periodicity [85]. We conclude that the black hole has a temperature defined by

$$T = \frac{N(r_+)f'(r_+)}{4\pi} = \frac{\kappa}{2\pi} \,. \tag{2.8}$$

[4]The KMS condition can be thought of as the appropriate generalization of a standard Gibbs state to quantum field theory.

This temperature immediately fixes the entropy of the black hole to be

$$S = \frac{A}{4G_N} = \frac{k_B c^3 A}{4 G_N \hbar} \,. \tag{2.9}$$

In the second equality above we have included the relevant constants just to illustrate how many different areas of physics come together in this one simple formula.

Black hole radiance and the resulting black hole thermodynamics is a quantum gravity phenomenon. We can go further and define other thermodynamic potentials for black holes, such as the free energy. An elegant method for doing so was introduced by Gibbons and Hawking [80] and begins with a consideration of the partition function

$$Z = \int \mathcal{D}[g_{\mu\nu}, \phi] e^{-S_E[g_{\mu\nu},\phi]} \,. \tag{2.10}$$

Here $g_{\mu\nu}$ is the metric while ϕ stands for the collection of all matter fields on spacetime. In the case of general relativity, the Euclidean action is defined by

$$S_E = -\frac{1}{16\pi G_N} \int_{\mathcal{M}} d^D x \sqrt{g}\, [R - 2\Lambda] - \frac{1}{8\pi G_N} \int_{\partial\mathcal{M}} d^{D-1} x \sqrt{h}\, K \,, \tag{2.11}$$

where R is the Ricci scalar, K is the extrinsic curvature, and h is the determinant of the induced metric on the boundary. By the standard statistical mechanics argument, we identify the Helmholtz free energy

$$F = -T \log Z \,. \tag{2.12}$$

The standard tools of thermodynamics then allow one to compute thermodynamic potentials

$$M = \langle E \rangle = -\frac{\partial}{\partial \beta} \log Z \,, \quad S = \beta \langle E \rangle + \log Z \,. \tag{2.13}$$

In the absence of chemical potentials like electric charge or angular momentum, we have

$$F = M - TS \,. \tag{2.14}$$

Note that the temperature cannot be determined from the partition function — for a black hole it is determined as described above, by demanding Euclidean regularity. Lastly, we note that in the limit of macroscropic objects, the dominant contribution to the partition function will come from stationary points of the action, i.e. the solutions to the classical theory. Within this saddle point approximation the partition function can be estimated as coming from the metric with the least action

$$\log Z \approx -S_E[g^{\mathrm{cl}}_{\mu\nu}]\,. \tag{2.15}$$

In most cases, the Euclidean action will actually be divergent due to an infinite volume contribution. In those cases, the standard procedure is to subtract off the contribution of empty space. For black holes that are asymptotically AdS, more specialized methods exist such as holographic renormalization which includes the use of counterterms in the action [67, 86].

2.2 Smarr relations for black holes

As we have just discussed, the mechanical laws of black holes were first derived by Bardeen, Carter, and Hawking. A key element in their derivation was the integral mass formula for black holes, which was first discussed by Smarr [87] in the context of Kerr-Newman black holes. Smarr's observation was that the differential mass formula for the Kerr black hole[5]

$$dM = \mathcal{T} dA + \Omega dJ + \Phi dQ\,, \tag{2.16}$$

can be directly integrated to give

$$M = 2TS + 2\Omega J + \Phi Q\,. \tag{2.17}$$

The integration is possible because M is a homogeneous function of the variables (A, J, Q) and proceeds via Euler's homogeneous function theorem, to be described below. The integral mass formula, also widely known as the Smarr formula, is closely analogous to the Gibbs-Duhem relation from ordinary thermodynamics. In that case, one regards the energy as a function of the extensive thermodynamic variables, then by definition the energy must be a homogeneous function of those parameters, and the integration can be performed.

Smarr derived the mass formula for a particular black hole solution, but what Bardeen, Carter, and Hawking demonstrated is that the relationship is in fact general and has its roots in the basic geometry of black holes. The Smarr formula is also not a fluke of four dimensions — it was further established by Myers and Perry [88] that it holds also for higher dimensional black holes:

$$(D-3)M = (D-2)TS + (D-2)\sum_i \Omega_i J_i\,, \tag{2.18}$$

[5] Smarr interpreted the quantity $\mathcal{T}$ to be an effective surface tension for the black hole.

where the coefficients are consistent with the dimensional scaling of the mass, entropy and angular momenta. However, all of these examples involve black holes that are asymptotically flat. The generalization of Smarr's formula to general asymptotically maximally symmetric spaces is more subtle, and it will be the goal to this section to show how the derivation proceeds. We begin by studying in detail that Smarr formula for asymptotically flat black holes, from both a scaling argument and a geometric construction, and then consider how it works for black holes that are asymptotically AdS.

2.2.1 Smarr formula for asymptotically flat black holes

The focus in this subsection will be vacuum, asymptotically flat black holes solutions. Our objective will be to establish the Smarr formula via two methods. First, we will present a simple scaling argument that follows from dimensional analysis and Euler's homogeneous function theorem. Second, we will show that this simple scaling argument gives the same conclusion as a more rigorous geometric derivation based on Komar integrals.

For concreteness and simplicity, let us consider a stationary black hole in D spacetime dimensions with a single angular momentum. Let us consider the mass to be a function of the entropy and angular momentum

$$M = M(S, J)\,. \tag{2.19}$$

From simple dimensional analysis, the parameters have the following length dimensions

$$M \propto [\ell]^{D-3} \quad S \propto [\ell]^{D-2}\,, \quad J \propto [\ell]^{D-2}\,, \tag{2.20}$$

where ℓ is just some length scale, not to be confused with the cosmological constant. Now, consider a scale transformation acting on the system, $\ell \to \alpha\ell$, which results in the following transformation law

$$M \to \alpha^{D-3} M\,, \quad S \to \alpha^{D-2} S\,, \quad J \to \alpha^{D-2} J\,, \tag{2.21}$$

and so the relationship $M(S, J)$ transforms in the following way

$$\alpha^{D-3} M = M\left(\alpha^{D-2} S, \alpha^{D-2} J\right)\,. \tag{2.22}$$

Now, we differentiate this with respect to the transformation parameter, α:

$$(D-3)\alpha^{D-4} M = (D-2)\frac{\partial M}{\partial(\alpha^{D-2} S)}\alpha^{D-3} S + (D-2)\frac{\partial M}{\partial(\alpha^{D-2} J)}\alpha^{D-3} J \tag{2.23}$$

upon setting $\alpha = 1$ we obtain Smarr's formula,

$$(D-3)M = (D-2)TS + (D-2)\Omega J\,, \tag{2.24}$$

where we have recognized that

$$T = \frac{\partial M}{\partial S} \quad \text{and} \quad \Omega = \frac{\partial M}{\partial J}\,. \tag{2.25}$$

The method of obtaining the relationship between a function and its arguments by studying the transformation under scaling is a particular instance of Euler's homogeneous function theorem. In the context of ordinary thermodynamics, the relationship analogous to (2.24) is often called the "Euler integral" of the internal energy.

Let us now establish the relationship (2.24) via a geometric argument. By assumption, we will take the black hole solution to be stationary and axis-symmetric, and so it admits at least two Killing vectors which we will call t^α and ϕ^α. The combination

$$\xi^\alpha = t^\alpha + \Omega \phi^\alpha \tag{2.26}$$

is null on the event horizon, with Ω defining the angular velocity of the black hole. On the horizon, ξ^α satisfies

$$\xi_\beta \xi^{\alpha;\beta} = \kappa \xi^\alpha \tag{2.27}$$

where κ — the surface gravity — is constant over the entire horizon.

Now, we consider a spatial slice Σ, with future-directed normal n_α, that extends from the horizon to spatial infinity — a schematic picture is shown in figure 2.1. Over this slice, we integrate

$$I = \int_\Sigma d\Sigma_\alpha \nabla_\beta \nabla^\beta \xi^\alpha\,. \tag{2.28}$$

We perform this integral in two ways. First, by virtue of the fact that ξ^α is a Killing vector, we can use that $\nabla_\beta \nabla^\beta \xi^\alpha = -R^\alpha{}_\beta \xi^\beta$ which gives

$$\int_\Sigma d\Sigma_\alpha \nabla_\beta \nabla^\beta \xi^\alpha = -\int_\Sigma d\Sigma_\alpha R^\alpha{}_\beta \xi^\beta = 0\,, \tag{2.29}$$

where the last equality follows since we are assuming the vacuum Einstein equations hold. The second method of evaluating the integral makes use of Stokes' theorem, which for an anti-symmetric tensor $B^{\alpha\beta}$ reads [89]

$$\int_\Sigma dA_\alpha \nabla_\beta B^{\alpha\beta} = \frac{1}{2}\oint_{\partial\Sigma} dS_{\alpha\beta} B^{\alpha\beta}\,. \tag{2.30}$$

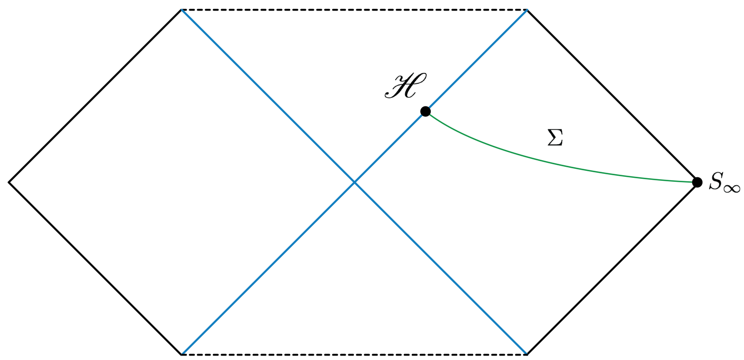

Figure 2.1: **Spacelike hypersurface for asymptotically flat black holes**. A depiction of a spacelike hypersurface Σ extending from a cross-section of the horizon to spatial infinity. While in this diagram the causal structure is particularized to the Schwarzschild black hole, the construction in the text is general.

The boundary of Σ consists of a $(D-2)$-dimensional surface at spatial infinity that we will denote as S_∞, and a $(D-2)$-dimensional cross-section of the horizon which we refer to as $\mathscr{H}$. Combining the two results we have

$$0 = \oint_{\mathscr{H}} dS_{\alpha\beta} \nabla^\alpha \xi^\beta - \oint_{S_\infty} dS_{\alpha\beta} \nabla^\alpha \xi^\beta \,. \tag{2.31}$$

To evaluate the component at infinity, we use the fact that since we are assuming a vacuum solution in D dimensions, the total mass and angular momentum of the spacetime are given by the following Komar integrals [89]

$$\begin{aligned} M &= -\frac{1}{16\pi G_N}\left(\frac{D-2}{D-3}\right)\oint_{S_\infty} dS_{\alpha\beta}\nabla^\alpha t^\beta \,, \\ J &= \frac{1}{16\pi G_N}\oint_{S_\infty} dS_{\alpha\beta}\nabla^\alpha \phi^\beta \end{aligned} \tag{2.32}$$

where S_∞ is a $(D-2)$ dimensional sphere at spacelike infinity. In the Komar integrals, the dimension dependent factors appear so that the conserved energy coincides with the ADM mass when appropriate. Using these definitions for the mass and angular momentum, we recognize that the second integral at S_∞ is

$$\oint_{S_\infty} dS_{\alpha\beta}\nabla^\alpha\xi^\beta = 16\pi\left[\Omega_H J - \left(\frac{D-3}{D-2}\right)M\right] , \tag{2.33}$$

leaving only the integral on the horizon to evaluate. On the horizon, we note that $dS_{\alpha\beta} = 2\xi_{[\alpha}N_{\beta]}\sqrt{\sigma}d^{D-2}x$ where N^β is a null vector normalized so that $\xi_\alpha N^\alpha = -1$ on the horizon and $\sqrt{\sigma}$ is the square root of the determinant of the induced metric on the horizon. Then, we have

$$\begin{aligned}\oint_{\mathscr{H}} dS_{\alpha\beta}\nabla^\alpha\xi^\beta &= 2\oint_{\mathscr{H}}\sqrt{\sigma}d^{D-2}x\xi_\alpha N_\beta\nabla^\alpha\xi^\beta \\ &= 2\kappa\oint_{\mathscr{H}}\sqrt{\sigma}d^{D-2}xN_\beta\xi^\beta \\ &= -2\kappa A\,. \end{aligned} \tag{2.34}$$

Above, we first made use of the fact that ξ^α is a Killing vector, so $\nabla^\alpha\xi^\beta = -\nabla^\beta\xi^\alpha$, to simplify the contraction with the surface element. Then we used that $\xi_\alpha\nabla^\alpha\xi^\beta = \kappa\xi^\beta$ on the horizon, with κ a constant on the horizon. Finally, we used that $N_\beta\xi^\beta = -1$ along with the fact that the integral evaluates to simply the area of the horizon cross-section.

Rearranging the above results we obtain

$$(D-3)M = (D-2)\frac{\kappa A}{8\pi} + (D-2)\Omega_H J \tag{2.35}$$

which, upon recognizing that $T = \kappa/(2\pi)$ and $S = A/4$, gives the same Smarr formula that was derived above via a scaling argument.

2.2.2 Smarr formula for AdS black holes

In the previous subsection, we established the Smarr relation for asymptotically flat black holes via two methods: first via a scaling argument, and then by a geometric argument. The geometric argument, in particular the step in eq. (2.29), made important use of the vacuum Einstein equations with vanishing cosmological constant, suggesting that issues may arise.

To illustrate the difference, let us start by taking a concrete example of the four dimensional Schwarzschild-AdS black hole. The metric is simply

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2\,, \tag{2.36}$$

with

$$f(r) = 1 - \frac{m}{r} + \frac{r^2}{L^2}\,, \quad \Lambda = -\frac{3}{L^2}\,. \tag{2.37}$$

The solution describes a black hole in AdS with event horizon located at $r = r_+$ where $f(r_+) = 0$. The temperature and entropy of the black hole are easily calculated:

$$T = \frac{1}{4\pi r_+}\left[1 + \frac{3r_+^2}{L^2}\right]\,, \quad S = \pi r_+^2\,, \tag{2.38}$$

but the mass merits more detailed consideration.

The vector $t^\alpha = \delta_t^\alpha$ is a Killing vector that approaches the natural time translation at the boundary. We would expect the conserved quantity associated with t^α to be the mass, but there is a subtlety. To compute the conserved charge, we can easily verify that the only non-vanishing components of $\nabla^\alpha t^\beta$ are

$$\nabla^t t^r = -\nabla^r t^t = -\frac{m}{2r^2} - \frac{r}{L^2}\,. \tag{2.39}$$

Now, using the definition of the Komar energy, we run into a problem:

$$M = -\frac{1}{8\pi}\oint_{S_\infty} dS_{\alpha\beta}\nabla^\alpha t^\beta = \frac{1}{4\pi}\lim_{R_0\to\infty}\oint_{S_{R_0}} R_0^2\left(\frac{m}{2R_0^2} + \frac{R_0}{L^2}\right)\sin\theta d\theta d\phi \to \infty\,, \tag{2.40}$$

the mass diverges as R_0^3. Of course, the source of the problem is easy to identify, and it is simply due to the non-zero cosmological constant which contributes an infinite amount to the energy. A simple solution is to perform the integral up to some finite cutoff, subtract the $m = 0$ contribution, and then take the cutoff to infinity.[6] This procedure reveals that

$$M = \frac{m}{2}\,. \tag{2.41}$$

Having now expressions for the mass, temperature, and entropy, we can see if the Smarr formula derived in the last section holds. A simple calculation,

$$M - 2TS = -\frac{r_+^3}{L^2} \neq 0\,, \tag{2.42}$$

shows conclusively that it does not.

[6]Let us quickly note that there are sometime subtleties with background subtraction techniques. One has to be careful to correctly match points in the solution of interest and the background. Also, sometimes it is difficult to determine the appropriate background for subtraction — see, e.g., [28]. However, there are no such subtleties in the present case.

The failure of the Smarr formula can be understood with some dimensional analysis. The AdS length has entered into the thermodynamic description of the black holes — this is a new, dimensionful parameter that was not accounted for in the derivation of the Smarr formula via scaling. Repeating the same steps in the scaling argument as in the last section, but now also including the cosmological constant, we can arrive at an 'extended' Smarr formula that applies in the presence of a cosmological constant:

$$(D-3)M = (D-2)TS + (D-2)\Omega J - 2VP \tag{2.43}$$

where we have defined

$$P = -\frac{\Lambda}{8\pi}\,, \tag{2.44}$$

which would be the pressure associated with Λ if viewed as stress-energy. The scaling argument introduces a new quantity

$$V := \frac{\partial M}{\partial P}\,, \tag{2.45}$$

which we refer to as the *thermodynamic volume* due to it having length dimension $(D-1)$ — the same as a spatial volume. Referring back now to the four-dimensional Schwarzschild-AdS black hole, we can confirm that the new Smarr formula holds. Indeed, in that case we have

$$P = \frac{3}{8\pi L^2}\,, \quad V = \frac{4\pi}{3}r_+^3 \tag{2.46}$$

so that

$$M = 2TS - 2PV\,. \tag{2.47}$$

In the flat space limit, $P \to 0$ and we recover the usual Smarr relation.[7]

It seems that the first group to realize the necessity of including a cosmological term in the Smarr formula was Calderelli, Cognola, and Klemm in early work on rotating AdS black holes in the context of the AdS/CFT correspondence [32]. In similar fashion to Smarr, those authors verified the 'extended' Smarr formula for the Kerr-Newman solution, motivated by scaling arguments. However, it then motivates one to ask if the extended Smarr formula also enjoys a geometric foundation of greater generality. The answer is yes, and the key ingredient in the geometric construction is the Killing potential introduced by

[7] There are other interesting limits that one can consider. For example, in the limit of large black holes (i.e. high temperature), it turns out that $-2PV \to -4/3TS$, in which case the Smarr formula becomes, $M = 2/3TS$. This result is precisely that for a three dimensional CFT at finite temperature, and here we can see that the extended Smarr formula naturally reproduces this result in the appropriate limit.

Kastor [37] and then used by Kastor, Ray, and Traschen [38] to derive the extended Smarr formula for AdS black holes.

Our focus will now turn to reviewing the derivation of the extended Smarr formula presented in [38], and for concreteness and simplicity we will consider only static black holes. First, let us note that since $\nabla_\alpha \xi^\alpha = 0$ for a Killing vector ξ^α, the Poincaré lemma implies that, at least locally, ξ^α can be written as

$$\xi^\alpha = \nabla_\beta \omega^{\beta\alpha} \,, \tag{2.48}$$

where $\omega^{\alpha\beta} = \omega^{[\alpha\beta]}$ is the *Killing potential*.[8] If one then considers the following integral over the boundary of some codimension-1 hypersurface Σ:

$$\int_{\partial\Sigma} dS_{\alpha\beta} \left(\nabla^\alpha \xi^\beta + \frac{2}{D-2} \Lambda \omega^{\alpha\beta} \right) = 0 \,, \tag{2.49}$$

where the equality is straightforward to show using the definition of the Killing potential and the (vacuum) Einstein equations. We can then apply the same geometric argument that was used in the previous section — considering the boundary to consist of a cross-section of the black hole horizon, $\mathscr{H}$ and a sphere at infinity, S_∞.

To evaluate the integral at infinity, we need to understand the asymptotic behaviour of the metric there. The falloff conditions for asymptotically AdS spacetimes were discussed in detail by Henneaux and Teitelboim [90] — see also [91, 92]. Here we are considering only static black holes, and so the asymptotic form of the metric must be

$$\begin{aligned} ds^2 = &\left(-f_0 + \frac{c_t}{r^{D-3}} \right) dt^2 + \frac{1}{f_0} \left(1 - \frac{(D-1)(D-2)c_r}{2\Lambda r^{D-1}} \right) dr^2 \\ &+ \left(1 + \frac{2\Lambda}{(D-1)(D-2)} \frac{c_\theta}{r^{D-1}} \right) r^2 d\Omega_{D-2}^2 \end{aligned} \tag{2.50}$$

where

$$f_0 = 1 - \frac{2\Lambda}{(D-1)(D-2)} r^2 \,. \tag{2.51}$$

For solutions of Einstein's equations with negative cosmological constant and (possibly) localized stress energy sources with vanishing angular momentum, the constants in the

[8] Note that the Killing potential is not unique — one can always add to it some divergenceless anti-symmetric 2-tensor, $\omega^{\alpha\beta} \to \omega^{\alpha\beta} + \eta^{\alpha\beta}$ where $\eta^{\alpha\beta} = \eta^{[\alpha\beta]}$ and $\nabla_\alpha \eta^{\alpha\beta} = 0$. In practice, this gauge freedom will not affect any of the results presented in this or the following sections — terms dependent on the gauge choice will cancel between the integrals at the horizon and infinity [38].

above metric will satisfy $c_t = c_r = m$ and $c_\theta = 0$, and the large-r behaviour follows the standard Schwarzschild AdS form [38]

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_{D-2}\,, \tag{2.52}$$

with

$$f(r) = 1 - \frac{m}{r^{D-3}} - \frac{2\Lambda r^2}{(D-1)(D-2)}\,. \tag{2.53}$$

With this asymptotic form of the metric, at large r the Killing potential has components

$$\omega^{rt}_{r\to\infty} = -\omega^{tr}_{r\to\infty} = \frac{r}{D-1}\,, \tag{2.54}$$

while

$$\nabla^r \xi^t = -\nabla^t \xi^r = \frac{(D-3)m}{2r^{D-2}} - \frac{2\Lambda r}{(D-1)(D-2)}\,. \tag{2.55}$$

Evaluating the integral at infinity, we see that the term involving the Killing potential cancels the divergence and we are left with a term proportional to the mass

$$\int_{S_\infty} dS_{\alpha\beta}\left(\nabla^\alpha \xi^\beta + \frac{2}{D-2}\Lambda\omega^{\alpha\beta}\right) = -16\pi\frac{D-3}{D-2}M\,, \tag{2.56}$$

while the integral at the horizon gives

$$\int_{\mathscr{H}} dS_{\alpha\beta}\left(\nabla^\alpha \xi^\beta + \frac{2}{D-2}\Lambda\omega^{\alpha\beta}\right) = -2\kappa A + \frac{2\Lambda}{D-2}\left(\int_{\mathscr{H}} dS_{\alpha\beta}\omega^{\alpha\beta}\right)\,, \tag{2.57}$$

where the first part of the above integral proceeds exactly as in the asymptotically flat case. Combining the expressions, we find the following Smarr formula:

$$(D-3)M = (D-2)TS - 2PV \tag{2.58}$$

where

$$P = -\frac{\Lambda}{8\pi G_N}\,, \quad V = -\frac{1}{2}\int_{\mathscr{H}} dS_{\alpha\beta}\omega^{\alpha\beta}\,, \tag{2.59}$$

define the pressure (as before), and the thermodynamic volume V in terms of an integral of the Killing potential over the horizon. Although we have considered only a static black hole in this setup, the results can be generalized straightforwardly to include angular momentum, giving the same result as the scaling argument above [46].

2.3 The first law with $\delta\Lambda$; mass as spacetime enthalpy

2.3.1 Varying the cosmological constant?

The scaling argument from which we derived the Smarr formula for AdS black holes places the cosmological constant on equal footing with other thermodynamic properties of the black hole, suggesting that the cosmological constant should be considered as a thermodynamic black hole parameter. A bit of thought suggests that this is fundamentally different than the variations of quantities like mass and angular momentum that usually appear in the first law. The distinction arises because while the mass and angular momentum correspond to integration constants characterizing the solution, the cosmological constant is a parameter in the action that characterizes the theory. One way to reconcile this is simply to argue that different Λ do correspond to wholly different theories and variations $\delta\Lambda$ corresponds to moving through this 'theory space'.

While mathematically well defined, viewing variations in Λ as motions in a theory space is somewhat unsatisfying, since in doing so one is giving up on any hope of understanding the phenomena due to variable Λ as physical processes. Luckily, there are alternative ways to view this procedure, and there is a large literature concerned with understanding mechanisms by which the cosmological constant can vary. In this subsection we will briefly review some of the literature on mechanisms of a dynamical cosmological constant.

A simple model for generating a cosmological constant is with scalar fields. Consider the action

$$S = \int d^D x \left[\frac{R}{16\pi G_N} - \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi) \right] , \tag{2.60}$$

which upon variation gives the following field equations for the scalar field and the metric

$$\begin{aligned} \frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right) - V'(\phi) = 0 \, , \\ R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = 8\pi G_N \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial_\sigma \phi \partial^\sigma \phi - g_{\mu\nu} V(\phi) \right) \, . \end{aligned} \tag{2.61}$$

At fixed points of the potential, i.e. where $V'(\phi) = 0$, the scalar field equations admit the simple solution $\phi =$ constant. It is easy to see that for these configurations the stress energy is precisely that one would associate with a cosmological constant:

$$T_{\mu\nu} = -V(\phi) g_{\mu\nu} = -\frac{\Lambda}{8\pi G_N} g_{\mu\nu} \, . \tag{2.62}$$

Understanding (contributions to) the cosmological constant as arising from scalar fields is quite natural in the context of string and supergravity inspired black holes. Here, the effective cosmological constant depends on the asymptotic values of massless scalar fields (the moduli fields). The contributions of these terms to the first law of black hole thermodynamics was first investigated in [93]. A cosmological constant arising from scalar fields is also quite natural in the context of the AdS/CFT correspondence. In this case, one considers Einstein gravity coupled to scalars with potentials; the fixed points of the potentials correspond to AdS geometries. For AdS/CFT, an interesting situation to consider is when the scalars smoothly interpolate between two such fixed points, which corresponds to a holographic representation of renormalization group flow, and gives rise to holographic versions of the c-theorem [94, 95, 96]. As described so far, scalar fields can only lead to discrete values for Λ. For example, in the case where the potential has two fixed points, then there are two possible values for the cosmological constant. For any dynamics that interpolates between these two values, the spacetime would not be asymptotically AdS during the intermediate phases. However, provided that the scalars vary slowly enough so that one is always "approximately" at a fixed point, it is conceivable that the black hole chemistry framework could be understood from this RG flow perspective — it seems likely that Johnson had this in mind when he introduced the notion of holographic heat engines [47]. Precisely this setup, but for the case of *positive* cosmological constant has recently been considered [97, 98]. In these papers, the scalar fields drive the expansion of the universe, i.e. as in inflation. By studying a setup where a scalar field is always approximately at a fixed point, so that the solutions are always approximately asymptotically de Sitter, it was possible to study the dynamics of black hole and cosmological horizons, finding agreement with the black hole chemistry approach.

Let us now discuss a mechanism that allows us to directly interpret the cosmological constant as a constant of integration, rather than a fundamental constant of nature. The method we review here was first seriously studied by Brown and Teitelboim as a proposed solution to the cosmological constant problem [99, 100]. The idea is motivated by supergravity considerations where the inclusion of p-form gauge fields can give rise to a cosmological constant [101]. We consider $(D-1)$-form gauge fields, $A_{\mu_1\dots\mu_{D-1}}$, from which a D-form field strength can be directly constructed

$$F_{\mu_1\dots\mu_D} = (D+1)\partial_{[\mu_1}A_{\mu_2\dots\mu_D]}\,. \tag{2.63}$$

Now, consider the following action:

$$S = \frac{1}{16\pi G_N}\int d^Dx\sqrt{-g}\left[R \mp F^2 \pm \frac{4\nabla_\mu\left(A_{\mu_2\dots\mu_D}F^{\mu\mu_2\dots\mu_D}\right)}{(D-1)!}\right], \tag{2.64}$$

which consists of Einstein gravity, the obvious kinetic term for the gauge field, and a boundary term that will be explained in a moment. Since the field strength is totally anti-symmetric with D indices, it must be proportional to the Levi-Civita tensor. Further, the equations of motion for the gauge field read,

$$\nabla_\mu F^{\mu\mu_2\cdots\mu_D} = 0\,, \tag{2.65}$$

which fix the proportionality to be a constant, i.e.

$$F_{\mu_1\cdots\mu_D} = c\sqrt{-g}\epsilon_{\mu_1\cdots\mu_D}\,. \tag{2.66}$$

The stress energy tensor associated with the gauge field is computed to be

$$T_{\mu\nu} = \pm\frac{1}{4\pi G_N(D-1)!}\left(F_{\mu\rho_2\cdots\rho_D}F_\nu{}^{\rho_2\cdots\rho_D} - \frac{1}{2D}g_{\mu\nu}F^2\right)\,. \tag{2.67}$$

When we substitute the on-shell value of the field strength into the stress energy tensor, we see that it reduces to

$$8\pi G_N T_{\mu\nu} = \mp c^2 g_{\mu\nu}\,, \tag{2.68}$$

which is precisely the stress energy associated with a cosmological constant $\Lambda = \pm c^2$. Going further, if we substitute the on-shell solution for the field strength into the action, thanks to the boundary term, the Lagrangian reduces precisely to Einstein gravity with a cosmological constant! Note the distinction here though: considering D-forms in the action, the cosmological constant is now understood to arise as a constant of integration from the equations of motion of the gauge field, rather than a fundamental constant in the Lagrangian. Since Einstein gravity with a cosmological constant is indistinguishable from Einstein gravity plus a $(D-1)$-form gauge field, there are no conceptual issues with including variations of the cosmological constant in the first law. In the same sense that one could imagine throwing charged matter into a black hole to increase or decrease its electric charge, one could imagine throwing $(D-2)$-dimensional p-form "charged" membranes into the black hole to perturb the cosmological constant.

Generating the cosmological constant (and its dynamics) from $(D-1)$-form gauge fields was first investigated as a possible solution to the cosmological constant problem [102, 99, 100, 103]. The first investigations of the implications for black hole thermodynamics seems to have been by Teitelboim [104], and also by Creighton and Mann [105]. The connection between $(D-1)$-form gauge fields and the 'standard' approach to black hole chemistry was recently discussed in [106], where it was shown that the cosmological constant is the conserved charge associated with the gauge invariance of the $(D-1)$-form, while the volume arises as its conjugate potential, analogous to how the electric potential arises for Maxwell charged black holes.

2.3.2 The first law

We have now seen that for AdS black holes, one is required to include the cosmological constant as a thermodyanmic parameter in order to derive a Smarr relation that is consistent with scaling. Now, following the work of Kastor, Ray and Traschen [38], we will review how the cosmological constant appears in the first law of black hole mechanics. We will consider only Einstein gravity with a cosmological constant for simplicity and clarity, though the method can be generalized to include other sources of stress energy. Our method will be similar to that of Wald and Sudarsky [107], and we will consider the variation of the bulk Hamiltonian under a perturbation, showing its variation can be related to a surface term whose integral gives the first law.

We start by foliating spacetime with a family of hypersurfaces Σ that have a unit timelike normal n_α. The completeness relation for the metric reads

$$g_{\alpha\beta} = -n_\alpha n_\beta + h_{\alpha\beta} \tag{2.69}$$

and we have

$$n_\alpha n^\alpha = -1\,, \quad n^\alpha h_{\alpha\beta} = 0\,. \tag{2.70}$$

If we consider the Hamiltonian formulation of general relativity, then the dynamical variables are the spatial metric $h_{\alpha\beta}$ and its conjugate momentum $\pi_{\alpha\beta}$. A solution of general relativity must satisfy the following constraint equations:

$$H = -16\pi G_N T_{\alpha\beta} n^\alpha n^\beta\,, \quad H_\alpha = -16\pi G_N T_{\sigma\beta} n^\sigma h^\beta{}_\alpha \tag{2.71}$$

where $H = -2G_{\alpha\beta}n^\alpha n^\beta$ and $H_\alpha = -2G_{\sigma\beta}n^\sigma h^\beta{}_\alpha$. In what follows, we will consider the cosmological constant to be the only source of stress energy, which gives $H = -2\Lambda$ and $H_\alpha = 0$ for the constraint equations.

If we have a vector field $\xi^\alpha = Nn^\alpha + N^\alpha$, where $n_\alpha N^\alpha = 0$, then the Hamiltonian density that generates evolution along ξ^α is given by [89]

$$\mathcal{H} = \sqrt{h}\left[N(H + 2\Lambda) + N^\alpha H_\alpha\right]\,. \tag{2.72}$$

Clearly, varying $\mathcal{H}$ with respect to N and N^α produces the constraint equations quoted above; a variation with respect to $h_{\alpha\beta}$ and $\pi^{\alpha\beta}$ produces the evolution equations for $-\dot{\pi}^{\alpha\beta}$ and $\dot{h}_{\alpha\beta}$, respectively, where the overdot denotes Lie differentiation along the vector field ξ^α.

Now, let us consider perturbing a solution $h^{(0)}_{\alpha\beta} \to h^{(0)}_{\alpha\beta} + s_{\alpha\beta}$ and $\pi^{\alpha\beta}_{(0)} \to \pi^{\alpha\beta}_{(0)} + p^{\alpha\beta}$ and also $\Lambda_{(0)} \to \Lambda_{(0)} + \delta\Lambda$. It is assumed that the zeroth order fields satisfy the Einstein equations, while no such assumption is required at this point for the perturbations. Requiring that ξ^α be a Killing vector, Hamilton's equations for the zeroth order spacetime demand that $-\dot{\pi}^{\alpha\beta} = 0$ and $\dot{h}_{\alpha\beta} = 0$, from which it follows after some calculation that the perturbations satisfy

$$N\delta H + N^\alpha \delta H_\alpha = -D_\alpha B^\alpha \tag{2.73}$$

where D_α is the covariant derivative operator on Σ that is compatible with $h_{\alpha\beta}$. The term B^α reads

$$B^\alpha = N(D^\alpha s - D_\beta s^{\alpha\beta}) - sD^\alpha N + s^{\alpha\beta} D_\beta N + \frac{1}{\sqrt{h}} N^\beta \left(\pi_{(0)}^{\sigma\rho} s_{\sigma\rho} h^{(0)}{}^\alpha_\beta - 2\pi_{(0)}^{\alpha\sigma} s_{\beta\sigma} - 2p^\alpha_\beta \right) , \tag{2.74}$$

for arbitrary perturbations $s_{\alpha\beta}$, $p^{\alpha\beta}$ and $\delta\Lambda$ with $s = h^{\alpha\beta} s_{\alpha\beta}$. However, if the perturbations are taken as solutions to the linearized Einstein equations, then they must satisfy the constraint equations and we obtain

$$2N\delta\Lambda = D_\alpha B^\alpha \,. \tag{2.75}$$

The part of this expression involving the cosmological constant can be written as a total derivative by noting that $N = -n_\alpha \xi^\alpha = -n_\alpha \nabla_\beta \omega^{\beta\alpha} = -D_\beta \left(n_\alpha \omega^{\beta\alpha} \right)$ where the last equality can be proven using the completeness relation in the form $\delta^\beta_\alpha = -n_\alpha n^\beta + h^\beta_\alpha$ along with the anti-symmetry of $\omega^{\alpha\beta}$. One then has

$$D_\alpha \left(B^\alpha + 2\omega^{\alpha\beta} n_\beta \delta\Lambda \right) = 0 \Rightarrow I := \oint_{\partial\Sigma} dA_\alpha \left(B^\alpha + 2\omega^{\alpha\beta} n_\beta \delta\Lambda \right) = 0 \,. \tag{2.76}$$

We are interested in evaluating this integral in the case where $\partial\Sigma$ has two components: one corresponding to a black hole horizon, and the other at infinity, as shown schematically in figure 2.2. We will show that this produces the first law of black hole mechanics. As in the case of the extended Smarr relation, for simplicity we will focus on the case where the perturbation is between two static solutions, so that the asymptotic behaviour is again given by (2.50). We suppose that we have a black hole solution with a bifurcation surface. The Killing vector responsible for the Hamiltonian evolution above will be chosen to be the generator of the horizon and we will assume that ξ^α approaches $(\partial/\partial t)^\alpha$ in the asymptotic coordinate system. We consider a hypersurface Σ extending from the bifurcation sphere to infinity. The boundary of Σ consists of two components: the bifurcation sphere of the black hole horizon, B, and a surface at infinity S_∞ chosen so that the unit normal is $n_\alpha = -N\nabla_\alpha t$ at infinity.

First, we consider the integral at infinity,

$$I_\infty = \oint_{S_\infty} dA_\alpha B^\alpha \,. \tag{2.77}$$

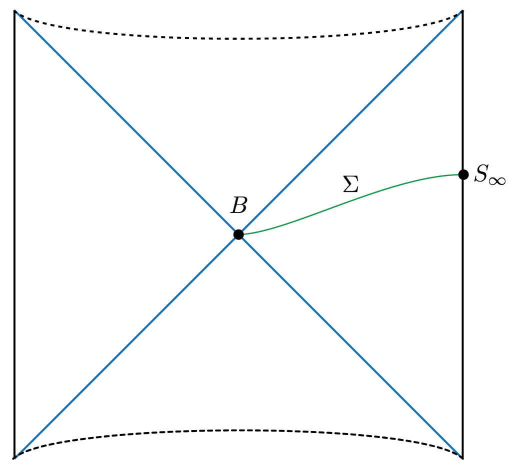

Figure 2.2: **Spacelike surface for AdS black hole**. A schematic picture of the spacelike hypersurface and its boundary for the AdS case. The hypersurface Σ extends from the bifurcation surface to a sphere at spatial infinity. While this figure displays the causal structure of the static AdS black hole, this is just to serve as an illustration — the construction in the text is more general.

Near infinity, the surface element is directed in the r direction, and we need only the r component of B^α. We have $N = \sqrt{f}$ and the metric perturbation

$$s_{rr} = -\frac{1}{f^2}\left(\frac{\delta m}{r^{D-1}} - \frac{2r^2\delta\Lambda}{(D-1)(D-2)}\right), \tag{2.78}$$

which is true to first order in the perturbations δm and $\delta\Lambda$; we neglect all higher order terms. Since the Killing vector approaches $(\partial/\partial t)$ at infinity, the terms involving the shift vector vanish sufficiently rapidly that they do not contribute to the boundary term [38]. A direct computation then gives

$$\begin{aligned}\oint_{S_\infty} dA_\alpha B^\alpha &= -(D-2)\Omega_{D-2}\delta m - \lim_{r\to\infty}\left(\frac{2r^{D-1}\Omega_{D-2}}{D-1}\right)\delta\Lambda \\ &= -16\pi G_N \delta M - \lim_{r\to\infty}\left(\frac{2r^{D-1}\Omega_{D-2}}{D-1}\right)\delta\Lambda\,,\end{aligned} \tag{2.79}$$

while

$$\oint_{S_\infty} dA_\alpha (2\omega^{\alpha\beta} n_\beta \delta\Lambda) = \lim_{r\to\infty}\left(\frac{2r^{D-1}\Omega_{D-2}}{D-1}\right)\delta\Lambda\,. \tag{2.80}$$

Combining these results, we see that the term involving the Killing potential precisely cancels the divergence due to the variation in Λ.

The other component of the boundary is the bifurcation sphere. On this surface, the integral of the boundary term B^α can be shown to be (see appendix A)

$$\oint_B dA_\alpha B^\alpha = -2\kappa\delta A\,, \tag{2.81}$$

where A is the area of the horizon. The best we can do with the term involving the Killing potential is to just re-write it slightly:

$$\oint_B dA_\alpha(2\omega^{\alpha\beta}n_\beta\delta\Lambda) = 2\left(\oint_B dA_\alpha\omega^{\alpha\beta}n_\beta\right)\delta\Lambda\,. \tag{2.82}$$

Putting all of the results together we have

$$\delta M = \frac{\kappa}{2\pi}\frac{\delta A}{4} + V\delta P = T\delta S + V\delta P\,, \tag{2.83}$$

where we have defined

$$P = -\frac{\Lambda}{8\pi G_N}\,,\quad V = -\oint_B dA_\alpha\omega^{\alpha\beta}n_\beta\,, \tag{2.84}$$

with V the thermodymamic volume; this quantity coincides with that introduced earlier in the construction of the Smarr relation.

Our final result in eq. (2.83) is the first law of black hole mechanics if variations of the cosmological constant are allowed. Though we have considered only static black holes, the generalization to the stationary case can be straightforwardly performed [46]. A similar result can be derived for de Sitter black holes, though in that case the integration is performed between the event and cosmological horizons [108].

Recall that in classical thermodynamics, the internal energy satisfies $\delta E = T\delta S - P\delta V$ while the enthalpy $H = E + PV$ satisfies $\delta H = T\delta S + V\delta P$. If we compare with the expression for the first law (2.83), we realize that it is telling us that the mass should be considered the *enthalpy* of spacetime rather than the internal energy! In classical thermodynamics, the enthalpy is the energy of the system plus the amount of work required to place the system in its environment, $H = E + PV$. This is a somewhat intuitive picture for AdS black holes. Loosely speaking, one can consider the $P - V$ terms appearing in the first law and Smarr formula to be the amount of work one would have to do to place a black hole in an otherwise empty AdS spacetime.

The construction of the Smarr relation and first law in spacetimes with a non-zero cosmological constant has introduced two new thermodynamic parameters: the pressure and the thermodynamic volume. The study of the implications of these terms for black hole thermodynamics has been a topic of quite considerable interest for nearly a decade now. In the remainder of this introduction, we will consider in detail two results that arise from this new picture. First, we will discuss the *reverse* isoperimetric inequality, and then discuss $P-V$ criticality of charged AdS black holes.

2.4 The reverse isoperimetric inequality

The volume that was defined in the previous sections was referred to as a 'thermodynamic volume' since it was introduced for thermodynamic rather than geometric reasons. However, a tantalizing geometric interpretation seems to exist. Indeed, we saw that in the case of the four-dimensional Schwarzschild AdS solution that the thermodynamic volume is given by [see eq. (2.46) above]

$$V = \frac{4\pi}{3} r_+^3 \,, \tag{2.85}$$

which is the volume of a Euclidean ball of radius r_+. In fact, this result extends in a simple way to the higher dimensional Schwarzschild-AdS black holes, as we will now demonstrate.

In D-dimensions, the Schwarzschild-AdS solution metric is of the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{D-2}^2 \tag{2.86}$$

where

$$f(r) = 1 - \frac{m}{r^{D-3}} - \frac{2\Lambda r^2}{(D-1)(D-2)} \,. \tag{2.87}$$

The timelike Killing vector $\xi^\alpha = \delta^\alpha_t$ has an associated Killing potential $\omega_{\alpha\beta}$ with components

$$\omega^{rt} = -\omega^{tr} = \frac{r}{(D-1)} \,. \tag{2.88}$$

the integral for the thermodynamic volume then becomes

$$V = -\frac{1}{2}\oint_{\mathscr{H}} dS_{\alpha\beta}\omega^{\alpha\beta} = \frac{r_+}{(D-1)}\oint_B \sqrt{\sigma} d^{D-2}\theta = \frac{r_+ A}{(D-1)} = \frac{\Omega_{D-2} r_+^{D-1}}{D-1} \,. \tag{2.89}$$

We see that the thermodynamic volume for the D-dimensional Schwarzschild solution coincides with the volume of a $(D-1)$-sphere in Euclidean space.

The suggestive fact that the volume of static black holes coincides with the volume of Euclidean spheres motivates the idea that we may expect some geometric properties of the thermodynamic volume after all. A characteristic feature of Euclidean volumes is the *isoperimetic inequality*. For a closed surface with surface area A and enclosed volume V, the isoperimetric inequality states that the ratio

$$\mathcal{R} := \left(\frac{(D-1)V}{\Omega_{D-2}}\right)^{\frac{1}{D-1}} \left(\frac{\Omega_{D-2}}{A}\right)^{\frac{1}{D-2}}, \tag{2.90}$$

always satisfies

$$\mathcal{R} \leq 1 \tag{2.91}$$

where saturation of the inequality occurs for spheres. The authors of [46] considered the applicability of the isoperimetric inequality for a variety of both static and rotating black holes arising as solutions of Einstein gravity and gauged supergravity theories in four and higher dimensions. Using the thermdynamic volume and the area of horizon cross-sections in $\mathcal{R}$, it was found that the thermodynamic volume satisfies not the isoperimetric inequality but rather the reverse of it!

As an example to illustrate the point, we consider the extension to higher dimensional rotating black holes. We will have a more detailed discussion of the properties of these solutions in the following chapter, here let us note that the area and thermodynamic volume of higher dimensional Kerr-AdS black holes are given by [109]

$$A = \frac{\Omega_{D-2}}{r_+^{\epsilon}} \prod_i \frac{r_+^2 + a_i^2}{\Xi_i}, \quad V = \frac{r_+ A}{D-1}\left[1 + \frac{(1 + r_+^2/L^2)}{(D-2)r_+^2} \sum_i \frac{a_i^2}{\Xi_i}\right] \tag{2.92}$$

where $\Xi_i = 1 - a_i^2/L^2$ and a_i with $i = 1, \ldots, \lfloor (D-2)/2 \rfloor$ labeling the spin parameters, and ϵ is equal to zero if the spacetime dimension is even, and unity otherwise. The general expression is complicated, but for insight let us just look at the slow rotation limit of the singly spinning black holes

$$\mathcal{R} = 1 + \frac{(D-3)}{2(D-1)(D-2)^2} \frac{(r_+^2 + L^2)^2}{r_+^4 L^4} a^4 + \mathcal{O}(a^6). \tag{2.93}$$

We see that the correction to the static result is always positive and so $\mathcal{R} > 1$. The same conclusion holds in the general case [46].

Essentially, the result comes from the fact that rotating black holes have less "area per volume" than the static case, explaining why the opposite behaviour is observed. Similar behaviour was observed for all of the black holes studied in [46], which led the authors to conjecture:

Conjecture (Reverse Isoperimetric Inequality)**.** *For an asymptotically AdS black hole solution of general relativity with thermodynamic volume V and horizon area A, the following inequality will be obeyed:*

$$\mathcal{R} := \left(\frac{(D-1)V}{\Omega_{D-2}}\right)^{\frac{1}{D-1}} \left(\frac{\Omega_{D-2}}{A}\right)^{\frac{1}{D-2}} \geq 1\,, \tag{2.94}$$

with saturation occurring for the Schwarzschild-AdS black hole. In other words, the entropy inside of a horizon of thermodynamic volume V is maximized for the Schwarzschild-AdS solution.

Since the initial work in [46], there have been a number of studies focusing on the validity and implications of the reverse isopermetric inequality. While no general proof or list of necessary conditions has been found, there have been some general indications of important features for the validity of the inequality. For example, it seems that the topology of the horizon plays a crucial role. Feng and Lu determined that, for planar black holes, the reverse isoperimetric inequality is equivalent to the null energy condition [110]. Hennigar, Kubiznak, and Mann determined that a class of black holes whose horizons are topologically spheres with two punctures can actually *violate* the reverse isoperimetric inequality. More details about these solutions — named *super-entropic black holes* — will be presented in the following chapter. To date, these are the only black hole solutions that provide a plausible counter-example to the conjectured reverse isoperimetric inequality.

While the reverse isoperimetric inequality was formulated for black holes with AdS asymptotics, there has been some exploration of its applicability beyond this realm with limited success. The case of de Sitter black holes was first studied in [108]. In that case, it is possible to define a thermodynamic volume for both the black hole horizon and the cosmological horizon. It was found that for a wide class of de Sitter black holes, the reverse isoperimetric inequality holds for both the black hole and cosmological thermodynamic volumes, while a true isoperimetric inequality holds for the thermodynamic volume *between* the black hole and cosmological horizons. More recent explorations of the validity of the inequality in the context of dS solitons have found the inequality not applicable [111]. The work considered in [112] studied the applicability of the reverse isoperimetric inequality to black holes that are asymptotically Lifshitz spaces, finding that there are violations in many cases. However, again it should be noted that there is no agreed upon definition for mass in these spacetimes, which is problematic for defining the thermodynamic volume.

Going beyond Einstein gravity and including higher curvature corrections in the gravitational action, the reverse isoperimetric inequality in the form (2.94) loses the nice interpretation as an entropy inequality. This is because the entropy of a black hole in a higher curvature theory of gravity is no longer simply the area of the horizon, but is rather given by the integral of a geometric quantity defined on the horizon [113, 114, 115]. As a result, it is unclear whether the reverse isoperimetric inequality should be extended to higher curvature theories of gravity in the form (2.94), or in the form of an entropy inequality. Nonetheless, some results are known. Static black holes in Lovelock [116] and quasi-topological [117, 118] gravity saturate the reverse isoperimetric inequality in the form presented in (2.94), but would generically violate the inequality in the form of an entropy inequality.[9] Static black holes in Einsteinian cubic gravity [121, 4] can violate the inequality in both forms, provided the spacetime dimension is five or larger.

2.5 $P - V$ criticality

Let us now consider the implications of the new thermodynamic parameters for the critical behaviour of black holes. With notions of pressure and volume for black holes, it is natural to explore questions related to thermodynamic phase transitions within this parameter space. In this section, we will review the initial study of $P - V$ criticality by Kubiznak and Mann [39]. This review will also serve to introduce the standard thermodynamic machinery used in the study of phase transitions, as will appear in the second half of this thesis.

2.5.1 Thermodynamics, equation of state, and specific volume

The starting point is the four dimensional charged AdS black hole which has line element

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \tag{2.95}$$

where $d\Omega^2$ is the standard line element on a two dimensional unit sphere and

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{L^2} \, . \tag{2.96}$$

The (real and positive) zeros of $f(r)$ determine the horizons in this spacetime, with the largest root corresponding to the event horizon and denoted by r_+.

[9] This assumes that the Lovelock couplings can be arbitrary, which is not strictly true. It would be interesting to fully explore the entropy form of the reverse isoperimetric inequality in higher curvature theories when constraints, such as those arising from holography [119, 120], are taken into account.

It is a straightforward matter to compute the thermodynamic parameters associated with this black hole. The results are,

$$T = \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi r_+}\left(1 + \frac{3r_+^2}{L^2} - \frac{Q^2}{r_+^2}\right), \quad S = \pi r_+^2\,, \quad \Phi = \frac{Q}{r_+}\,, \quad V = \frac{4}{3}\pi r_+^3 \tag{2.97}$$

and they satisfy the extended first law of thermodynamics

$$dM = TdS + VdP + \Phi dQ \tag{2.98}$$

and the associated Smarr relation

$$M = 2TS - 2VP + \Phi Q\,. \tag{2.99}$$

We note this this is the Smarr formula consistent with Eulerian scaling for this four-dimensional black hole.

To study the thermodynamic structure of the black hole we require the equation of state. This is obtained very easily by rearranging the expression for the temperature given in eq. (2.97) and isolating for the pressure

$$P = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2} + \frac{Q^2}{8\pi r_+^4}\,. \tag{2.100}$$

The first term appearing in the equation of state is reminiscent of the ideal gas law $P \sim T/v$. This motivates identifying

$$v = 2r_+ = 2\left(\frac{3V}{4\pi}\right)^{1/3} \tag{2.101}$$

as the *specific volume*. In general, the specific volume will be identified as some function of the thermodynamic volume such that, in the appropriate limit, the equation of state takes the form of an ideal gas. In situations where it is possible to do so, it is often more convenient to work with the specific volume than the full thermodynamic volume. Some comments below will be directed at when doing so is — or is not — appropriate. Finally, in terms of the specific volume, the equation of state now reads

$$P = \frac{T}{v} - \frac{1}{2\pi v^2} + \frac{2Q^2}{\pi v^4}\,. \tag{2.102}$$

There are two further points worth mentioning about the specific volume at this stage. First, it might seem offensive to refer to v as a volume since it appears to have dimensions of length, rather than dimensions of volume. However, we have been working in geometric units where $\hbar = c = G_N = 1$ and temporarily restoring these constants is insightful. In terms of the 'geometric' pressure and temperature, we would have

$$\text{Pressure} = \frac{\hbar c}{\ell_{\rm P}^2} P \quad \text{and} \quad \text{Temperature} = \frac{\hbar c}{k} T \,, \tag{2.103}$$

where $\ell_{\rm P}^2 = \frac{\hbar G_N}{c^3}$ is the Planck length. Then, the leading behaviour of the equation of state is seen to be

$$\text{Pressure} = \frac{k\text{Temperature}}{2\ell_{\rm P}^2 r_+} + \cdots \,. \tag{2.104}$$

This reveals the 'missing' dimensionful factors in the specific volume is just $\ell_{\rm P}^2$, which is just unity in geometric units.

Second, it is interesting to note that the specific volume identified by these naive considerations can also be thought of as a ratio of thermodynamic volume per degrees of freedom [109]. That is, we can write

$$v = \frac{V}{N_{\rm dof}} \tag{2.105}$$

where

$$N_{\rm dof} = \frac{D-2}{D-1} \frac{A}{4\ell_{\rm P}^{D-2}} \tag{2.106}$$

with A the area of the horizon and $\ell_{\rm P}$ the Planck length. While an interesting observation, this result is actually only true in a limited sense and does not hold, for example, for rotating black holes or in higher curvature gravity.

It is instructive to consider the behaviour of the equation of state in the $P - v$ plane. Representative isotherms are shown in figure 2.3, where curves of higher opacity correspond to isotherms of higher temperature. For high temperatures, the pressure follows ideal gas law type behaviour and the black hole solutions are unique. That is, for a given pressure there is only a single possibility for v. As the temperature decreases, there is a point $T = \sqrt{6}/(18\pi Q)$ (black curve) after which the isotherms exhibit an inflection point and uniqueness of the solution is lost. In this regime, for a given pressure there can be up to three possible values for v. It is in this regime that phase transitions can occur between the various possible black holes, as we will discuss below. Decreasing the temperature further, there is another distinguished isotherm corresponding to $T = \sqrt{3}/(18\pi Q)$ (orange curve). For temperatures below this, the isotherms are no longer completely positive, with regions of negative pressure appearing. Note that negative pressure corresponds to

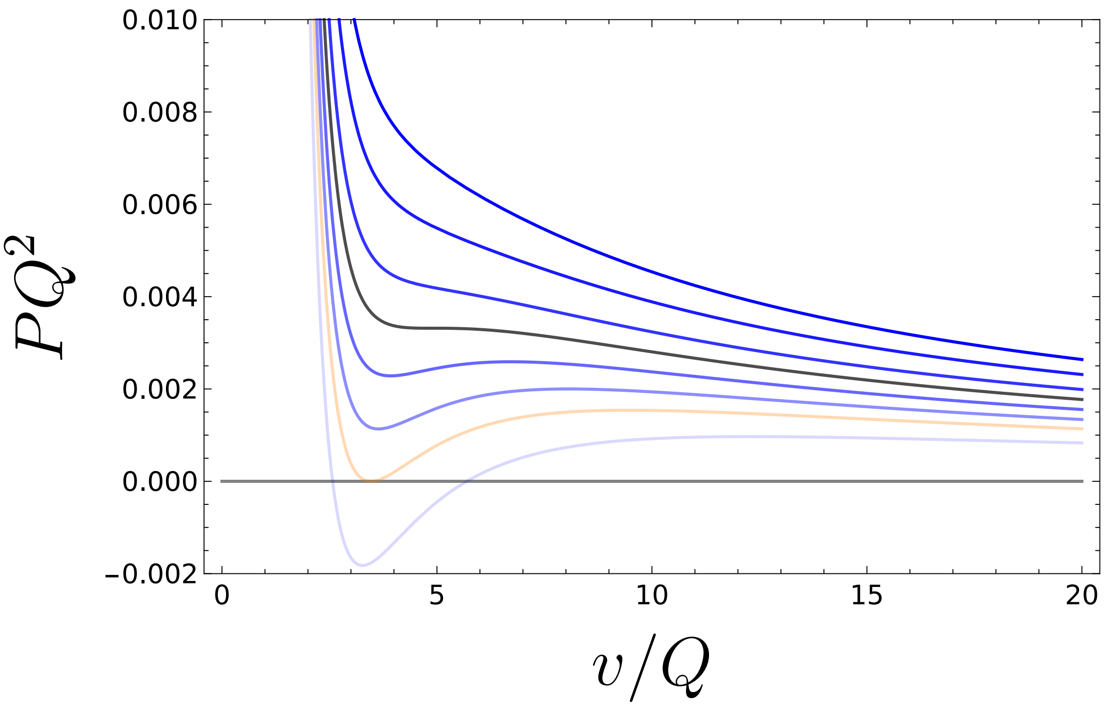

Figure 2.3: **Equation of state for charged AdS black hole**. Here we plot various isotherms for the charged AdS black hole, the curves of higher opacity are isotherms of higher temperature.

positive cosmological constant, these regions must be excised as the whole structure changes dramatically for $P < 0$.

The isotherm which marks the onset of the 'non-uniqueness' of the solutions is very special and actually corresponds to a *critical point*. Recall that a necessary condition for an equation of state to admit a critical point is that

$$\frac{\partial P}{\partial V} = \frac{\partial^2 P}{\partial V^2} = 0\,, \tag{2.107}$$

where the derivatives are computed at fixed temperature. In circumstances where the thermodynamic volume is a simple function of the specific volume, this condition is equivalent to

$$\frac{\partial P}{\partial v} = \frac{\partial^2 P}{\partial v^2} = 0\,. \tag{2.108}$$

Note that if the thermodynamic volume was a generic function of v and P, then the two statements above would not be equivalent.

The charged black hole admits the following solution to these equations

$$P_c = \frac{1}{96\pi Q^2}\,, \quad v_c = 2\sqrt{6}Q\,, \quad T_c = \frac{\sqrt{6}}{18\pi Q}\,, \tag{2.109}$$

where the subscript c denotes 'critical'. The isotherm in figure 2.3 that is shown as a black line corresponds to this critical isotherm.

Our analysis can be summarized in the following way. For temperatures higher than the critical isotherm, there is a unique black hole solution for a given charge and pressure. For temperatures below the critical temperature, there can be up to three possible black hole solutions for a given pressure and charge. Of these possible solutions, the one that is realized in nature will be that which minimizes the free energy. Let us now analyze the free energy.

2.5.2 Free energy and phase structure

The free energy is identified with (regularized) Euclidean action. In the fixed charge ensemble this (complete with boundary terms) is given by

$$\begin{aligned} I_E = &-\frac{1}{16\pi G_N}\int_{\mathcal{M}} d^4x\sqrt{g}\,(R - 2\Lambda - F_{\mu\nu}F^{\mu\nu}) - \frac{1}{8\pi G_N}\int_{\partial\mathcal{M}} d^3x\sqrt{h}K \\ &- \frac{1}{4\pi G_N}\int_{\partial\mathcal{M}} d^3x\sqrt{h}n_\alpha F^{\alpha\beta}A_\beta + I_{ct}\ , \end{aligned} \tag{2.110}$$

where I_{ct} represent the counterterms added to remove the divergences inherent in the action. Specifically, after computing the value of this action, one obtains the free energy

$$G = M - TS = \frac{I_E}{\beta} = \frac{1}{4}\left(r_+ - \frac{8\pi}{3}Pr_+^3 + \frac{3Q^2}{r_+}\right)\,. \tag{2.111}$$

Here we have denoted the free energy by G to make clear that for AdS black holes, the interpretation is that this is the Gibbs free energy.

The state of the physical system is determined by requiring that the Gibbs free energy is minimized at fixed temperature and pressure. The Gibbs free energy can be re-expressed as a function of temperature and pressure alone by solving the equation of state for the specific volume. Since the volume as a function of pressure and temperature is multi-valued, this indicates there will be multiple branches of the free energy, which in turn represent the various possible 'phases' of black hole.

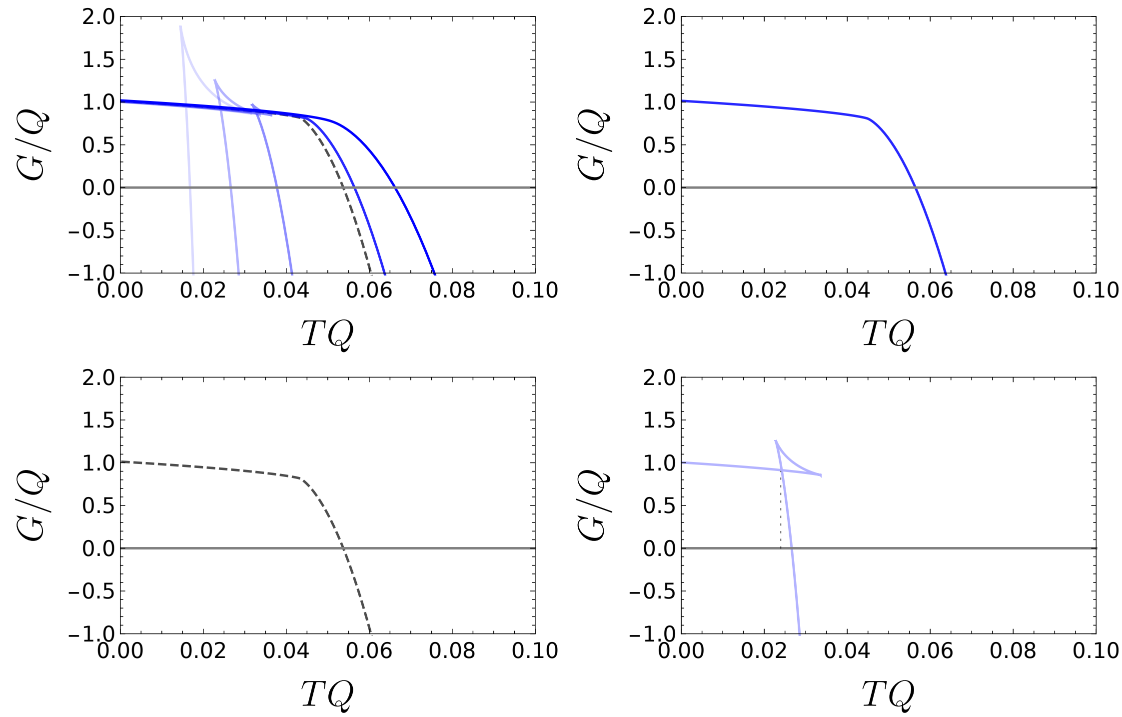

Figure 2.4: **Gibbs free energy for the charged AdS black hole**. *Top left*: A single plot showing various isobars of the Gibbs free energy. The opacity of the blue lines is related to the value of the pressure on the isobar, with higher opacity corresponding to higher pressure. The single black, dashed curve corresponds to the critical pressure. For pressures larger than the critical pressure, the free energy is a smooth single-valued function. At the critical pressure, there is a cusp where the second derivative of the free energy is discontinuous. Below the critical pressure, the free energy exhibits 'swallowtail' behaviour, and is generically multi-valued. In this regime, points where two branches of the free energy intersect corresponds to points where a first order phase transition occurs. *Top right*: Behaviour for $P = 1.1P_c$. *Bottom left*: Behaviour for $P = P_c$. *Bottom right*: Behaviour for $P = 0.25P_c$; the dotted black line shows the location of the first order phase transition.

In the case of the charged AdS black hole, inverting the equation of state for $v = v(P,T)$ can be done analytically. However, the result is the solution of a depressed quartic equation, and is therefore quite messy and not particularly illuminating. The general prescription which we will use here and in more complicated scenarios to follow will be to solve the equation of state for T rather than v, and then plot $G(T,P)$ as a parametric function of

v. Doing so, we show in figure 2.4 representative constant pressure slices of $G(T, P)$.

The behaviour can be understood in the following way. For pressures larger than the critical pressure, the free energy is a smooth, single-valued function (see top right plot). At the critical pressure, a cusp appears in the free energy located at $T = T_c$. This cusp reveals a discontinuity in the second derivative of the free energy, and therefore represents a second order phase transition (see bottom left plot). Below the critical pressure, the free energy exhibits 'swallowtail' behaviour and for a certain range of temperatures, there can be up to three possible black hole solutions, with the one minimizing the free energy being the physically preferred state. Referring to the bottom right plot of figure 2.4, we can see the presence of a first order phase transition. Indeed, if a black hole starts with temperature $T > 0.04/Q$ on this isobar and radiates, cooling down as it does so, then there will be a point (marked with the vertical, dotted line in this plot) where the minimal branch of the Gibbs free energy changes, marking a first order phase transition.[10]

[10]Note that the configurations with positive free energy are possible due to conservation of charge. That is, there is no analog of the Hawking-Page transition between thermal AdS and the charged black hole because such a transition would violate conservation of charge.

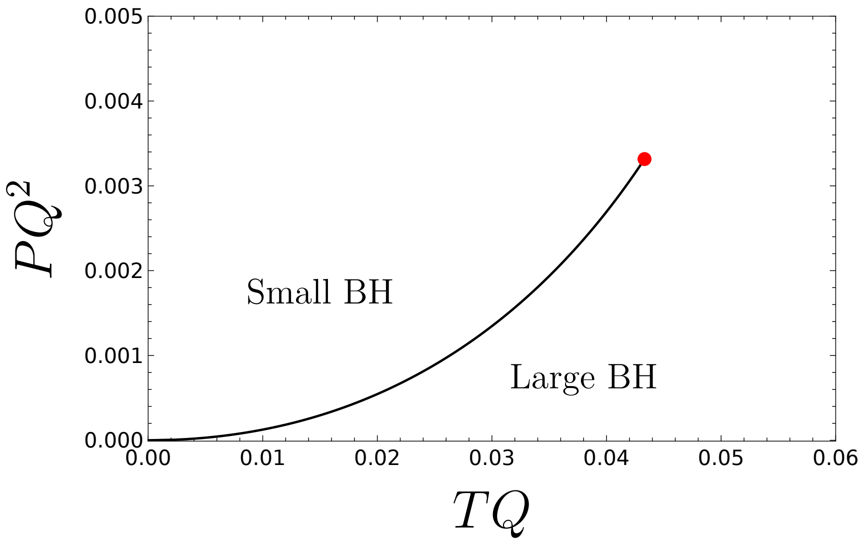

Figure 2.5: **Phase diagram for the charged AdS black hole**. The solid black line denotes the line of coexistence; along this line the small and large black hole phases have the same free energy. Crossing the coexistence line indicates a first order phase transition between the small and large black hole phases. The coexistence curve terminates at the critical point, which is the only point in this parameter space for which a second order (continuous) phase transition occurs. Beyond the critical point, there are no distinguished phases.

The information revealed in the structure of the Gibbs free energy is nicely captured in a phase diagram, which is shown in figure 2.5. Here we see a line which marks the first order phase transition between small and large black holes. At any point along this line, there are two black holes with different volumes that have the same free energy. The line originates at $(T, P) = (0, 0)$ and terminates at the critical point, denoted here by the solid red circle. Beyond the critical point, there are no distinguished phases.

2.5.3 Critical points and universality

The presence of the critical point results in several examples of universal behaviour, which we will now describe. The first and simplest observation of universality is that the critical ratio

$$\frac{P_c v_c}{T_c} = \frac{3}{8}\,, \tag{2.112}$$

is independent of the specific details (i.e. the charge) of the black hole. This situation is similar to what is observed for the van der Waals fluid, where one obtains exactly the same value for the ratio of critical values. So, this ratio is the same for the charged AdS black hole and for any fluid for which the van der Waals equation is a good approximate description.

The second example of universal behaviour is the law of corresponding states. We can introduce the following new variables

$$\hat{\tau} = \frac{T}{T_c}\,, \quad \hat{\nu} = \frac{v}{v_c}\,, \quad \hat{p} = \frac{P}{P_c}\,. \tag{2.113}$$

In terms of these variables, the equation of state can be recast into the following form

$$\hat{p} = \frac{8\hat{\tau}}{3\hat{\nu}} - \frac{2}{\hat{\nu}^2} + \frac{1}{3\hat{\nu}^4}\,. \tag{2.114}$$

This 'law of corresponding states' applies for *black holes of any charge*. In other words, written down in these rescaled variables, the behaviour is independent of the specific details of the particular black hole.

The third example of universal behavior (and the one most important for this thesis) is captured by critical exponents. Near the critical point, various thermodynamic potentials and susceptibilities diverge in a power law fashion. The exponents characterizing the divergence is known as the *universality class* of the critical point. If two substances share the same universality class, then near the critical point the physical description of both substances will be identical.

To calculate the critical exponents of the charged AdS black hole, we proceed by first performing an expansion of the equation of state near the critical point in terms of the following parameters

$$\rho = \frac{P - P_c}{P_c}\,, \quad \tau = \frac{T - T_c}{T_c}\,, \quad \omega = \frac{V - V_c}{V_c}\,, \tag{2.115}$$

where we note that the thermodynamic volume V has been used in the definition of ω rather than the specific volume. Then, near-critical equation of state is given by

$$\rho = \frac{8}{3}\tau - \frac{8}{9}\tau\omega - \frac{4}{81}\omega^3 + \mathcal{O}(\tau\omega^2, \omega^4)\,. \tag{2.116}$$

Here terms of order $\mathcal{O}(\tau\omega^2)$ can be dropped as their contributions are sub-leading in the determination of the critical exponents.

Now, let us discuss briefly the critical exponents that we are interested in here.

- The exponent α governs the behaviour of the specific heat capacity at constant volume

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_V \propto |\tau|^{-\alpha}\,. \tag{2.117}$$

- The exponent β governs the difference in volume between the two phases along an isotherm, a quantity known as the *order parameter* η. That is

$$\eta = V_1 - V_2 \propto |\tau|^{\beta} \tag{2.118}$$

  where the subscripts 1 and 2 denote the individual phases.

- The exponent γ governs the behaviour of the isothermal compressibility

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T \propto |\tau|^{-\gamma}\,. \tag{2.119}$$

- Lastly, the exponent δ governs the behaviour of the pressure along the critical isotherm $T = T_c$:

$$|P - P_c| \propto |V - V_c|^{\delta}\,. \tag{2.120}$$

In general, the critical exponents are not independent and obey scaling laws (or more generally, scaling inequalities). The scaling relations follow from the homogeneous scaling of extensive thermodynamic potentials. Two examples of scaling relations are the Widom relation

$$\gamma = \beta(\delta - 1)\,, \tag{2.121}$$

and the Rushbrooke inequality

$$\alpha + 2\beta + \gamma \geq 2\,. \tag{2.122}$$

Now, let us calculate the critical exponents for the charged AdS black hole. First, we note that the entropy is independent of temperature. As a result, the specific heat at constant volume vanishes identically. Thus, there is no temperature dependence and we therefore conclude that $\alpha = 0$. The exponent δ can be computed with similar ease. Along the critical isotherm we have $\tau = 0$ and there immediately conclude from eq. (2.116) that $\delta = 3$.

Next, we compute β making use of the fact that, during the phase transition, the pressure remains constant and Maxwell's equal area law holds. We denote the volume of the large black hole phase by V_L and use V_S to denote the small black hole phase. In terms of the dimensionless quantity ω these translate to ω_L and ω_S, respectively. We then have the following two equations:

$$\begin{aligned} 1+\frac{8}{3}\tau-\frac{8}{9}\tau\omega_L-\frac{4}{81}\omega_L^3 &= 1+\frac{8}{3}\tau-\frac{8}{9}\tau\omega_S-\frac{4}{81}\omega_S^3\,, \\ 0 &= \int_{\omega_S}^{\omega_L}\left[\omega\left(6\tau+\omega^2\right)\right]d\omega\,, \end{aligned} \tag{2.123}$$

where the first equation is due to the constancy of the pressure at the phase transition and the second is Maxwell's equal area law, $0=\int(dP/dV)dV$. These equations have a unique non-trivial solution,

$$\omega_L=-\omega_S=3\sqrt{-2\tau}\,, \tag{2.124}$$

which imply that

$$\eta=V_c(\omega_L-\omega_S)=6V_c\sqrt{-2\tau}\Rightarrow\beta=1/2\,. \tag{2.125}$$

Finally, we compute γ by first noting that the derivative required in the isothermal compressibility is given by

$$\left.\frac{\partial V}{\partial P}\right|_T=-\frac{9V_c}{8P_c}\frac{1}{\tau}+\mathcal{O}(\omega)\,. \tag{2.126}$$

Thus, near criticality, the isothermal compressibility is

$$\kappa_T=-\frac{1}{V}\left.\frac{\partial V}{\partial P}\right|_T\propto\frac{9}{8P_c}\frac{1}{\tau}\Rightarrow\gamma=1\,. \tag{2.127}$$

We have now computed the critical exponents associated with the critical point of the charged AdS black hole. Collecting them in one place, they are

$$\alpha=0\,,\quad\beta=\frac{1}{2}\,,\quad\gamma=1\,,\quad\delta=3\,. \tag{2.128}$$

These critical exponents are exactly those expected from mean field theory. Mean field theory critical exponents are remarkably robust for second order phase transitions in gravitational theory. To the best of the author's knowledge, the only examples of critical exponents that are not the mean field theory values occur for black holes in certain highly constrained higher curvature theories [44, 16].

Chapter 3

Super-entropic black holes

The purpose of this chapter is to introduce and study the properties of the four dimensional super-entropic black hole and its singly-spinning generalization to D-dimensions. These metrics represent new rotating solutions in Einstein gravity that are asymptotically locally AdS. We will present an analysis of the basic properties of the solutions and their thermodynamics.

The super-entropic solutions are interesting for at least two reasons. First, black holes in general relativity are highly constrained objects and so solutions that present novel features are useful in understanding general properties of black holes. A fundamental result in the study of black holes is Hawking's theorem concerning the topology of black hole horizons [122]. Hawking showed that the two-dimensional event horizon cross sections of four-dimensional asymptotically flat stationary black holes satisfying the dominant energy condition necessarily have topology $\mathbb{S}^2$. More interesting black objects are permitted in four and higher dimensions if one relaxes some of the assumptions going into Hawking's theorem. For example, since Hawking's argument relies on the Gauss–Bonnet theorem, it does not directly extend to higher dimensions. It is then not so surprising that higher-dimensional spacetimes permit a much richer variety of black hole topologies. The most famous example of this type is the black ring solution of Emparan and Reall which has horizon topology $\mathbb{S}^2 \times \mathbb{S}^1$ [123]. Another possibility is to relax asymptotic flatness. For example, in four-dimensions with a negative cosmological constant, the Einstein equations admit black hole solutions with the horizons being Riemann surfaces of any genus g [124, 125, 126, 127, 128]. Higher-dimensional asymptotically AdS spacetimes are also known to yield interesting horizon topologies, for example, black rings with horizon topology $\mathbb{S}^1 \times \mathbb{S}^{D-3}$ [129] and rotating black hyperboloid membranes with horizon topology $\mathbb{H}^2 \times \mathbb{S}^{D-4}$

[130]. More generally, event horizons which are Einstein manifolds of positive, zero, or negative curvature are possible in D-dimensional asymptotically AdS space [125, 131].

The super-entropic black hole solutions, which exist in any dimension $D \geq 4$, have horizons that are topologically spheres with two punctures. These analytical spacetimes therefore provide further examples of black holes with interesting topological features, and may provide valuable testing grounds for various gauge/gravity calculations — indeed, in many cases they already have, e.g. [132, 133, 134, 135].

A second reason why super-entropic black holes are interesting is because they provide the first — and so far only — plausible counter-example to the reverse isoperimetric inequality. As we will discuss below, despite some subtleties in understanding the thermodynamics of the solutions, these black holes have more entropy per "unit thermodynamic volume" than the reverse isoperimetric inequality would suggest possible. It is from this property that these solutions derive their name.

The organization of this chapter is as follows. First, we review the Kerr-Myers-Perry solutions in four and higher dimensions before introducing the Kerr-AdS class of metrics. A key point in this discussion is the notion of an 'ultra-spinning limit', since it is through this type of procedure that the super-entropic black hole is derived. The four-dimensional super-entropic black hole is then introduced, followed by a general analysis of its properties. We then perform a short study of geodesics in this spacetime, concluding that the axis of symmetry is in fact excised from the geometry. Finally, we present singly-spinning D-dimensional generalization of the super-entropic black hole and discuss the thermodynamics of the solutions. We conclude with some summarizing remarks.

3.1 Kerr-AdS black holes and ultra-spinning limits

Our goal in this section will be to introduce some examples of rotating AdS black holes and highlight relevant properties for understanding the super-entropic limit that will follow. In particular, we will motivate the notion of an 'ultra-spinning' limit for rotating black holes.

The study of rotating black holes has a long history, with the first solution describing a four-dimensional asymptotically flat rotating black hole found by Kerr in 1963 [136]. Let us begin by recalling the four-dimensional asymptotically flat Kerr solution. The metric reads

$$ds^2 = -dt^2 + \frac{2mr}{\rho_a^2}\left(dt + a\sin^2\theta d\phi\right)^2 + \frac{\rho_a^2}{\Delta_a}dr^2 + \rho_a^2 d\theta^2 + (r^2+a^2)\sin^2\theta d\phi^2 \tag{3.1}$$

with

$$\rho_a^2 = r^2 + a^2 \cos^2\theta\,, \quad \Delta_a = r^2 - 2Mr + a^2\,. \tag{3.2}$$

The solution has two horizons: an event horizon located at the largest root of Δ, and an inner Cauchy horizon located at the smaller root. It is instructive to consider the explicit expressions for the location of the horizons:

$$r_\pm = M \pm \sqrt{M^2 - a^2}\,. \tag{3.3}$$

Now we can note an interesting property of the four-dimensional Kerr metric: there is a maximum spin parameter that is consistent with the metric describing a black hole. Equivalently, this can be stated in terms of the angular momentum of the Kerr solution. Here, let us just note that the mass and angular momentum are given simply by the parameters M and $J = Ma$. For the Kerr solution to represent a black hole, we must have $J^2 \leq M^4$, otherwise the solution would represent a naked singularity.

The "Kerr bound" limiting the angular momentum of the four-dimensional rotating black hole hole does not extend generally to higher dimensions. Higher dimensional generalizations were obtained in 1983 by Myers and Perry [88]. In higher dimensions, rotating black holes are complicated beasts, and in D spacetime dimensions can have up to $\lfloor (D-1)/2 \rfloor$ independent angular momenta.[11] Rather than considering the most general Myers-Perry solution, let us just focus on the case of an asymptotically flat black hole in D dimensions with a single spin parameter. In this restricted case, the metric is given by

$$\begin{aligned} ds^2 = -dt^2 + \frac{2mr^{5-D}}{\rho_a^2}\left(dt + a\sin^2\theta d\phi\right)^2 + \frac{\Sigma_a}{\Delta_a}dr^2 + \Sigma_a d\theta^2 \\ + (r^2 + a^2)\sin^2\theta d\phi^2 + r^2\cos^2\theta d\Omega_{D-4}^2\,, \end{aligned} \tag{3.4}$$

where

$$\Delta_a = r^2 + a^2 - 2mr^{5-D}\,, \tag{3.5}$$

and ρ_a^2 is the same as in the Kerr case. The mass and angular momentum can be computed to be

$$M = \frac{(D-2)\Omega_{D-2}m}{8\pi G_N}\,, \quad J = \frac{2}{D-2}Ma\,, \tag{3.6}$$

[11] This is a consequence of the fact that the rotation group $SO(D-1)$ has rank $r = \lfloor (D-1)/2 \rfloor$, which is equal to the number of independent Casimir invariants. These group invariants are naturally associated with the conserved charges. Another way to see this same fact is to note that the number of independent angular momenta is equal to the number of independent spatial 2-planes.

the second relationship above is sometimes referred to as the 'chirality condition'. The location of the event horizon is again determined by the largest root of Δ. It is not hard to see that for both $D = 4$ and $D = 5$, extremality places an upper-bound on a. However, for $D \geq 6$ we can see that there is no upper-bound on a. Note that, for $D \geq 6$, at small r the $-2m$ term dominates and Δ is negative, while at large r the r^2 term dominates and Δ is positive. Simple calculus then implies that Δ has a single positive root, independent of the value of a. So, Myers-Perry black holes in $D \geq 6$ can (in principle) have arbitrarily large angular momentum.

There is, however, a limit on the angular momentum of Myers-Perry black holes that is imposed by physical considerations. Emparan and Myers [137] considered the geometry of a higher dimensional black hole with arbitrarily large angular momentum relative to the mass, which they termed the 'ultra-spinning' limit. Roughly speaking, the idea is to keep the mass finite while taking the angular momentum to be infinite. Mathematically, this corresponds to taking $a \to \infty$ while holding $\hat{m} = m/a^2$ fixed. In this limit, the geometry limits to a black membrane,

$$ds^2 = -\left(1 - \frac{2\hat{m}}{r^{D-5}}\right) dt^2 + \left(1 - \frac{2\hat{m}}{r^{D-5}}\right)^{-1} dr^2 + r^2 d\Omega^2_{D-4} + d\sigma^2 + \sigma^2 d\phi^2 \,. \tag{3.7}$$

Due to the work of Gregory and Laflamme [138], black branes are known to be unstable. Therefore it is expected that at some sufficiently large angular momentum, an instability will present itself, and an effective "Kerr Bound" will be imposed dynamically.

Besides revealing an instability of higher dimensional black holes with large angular momentum, the ultra-spinning limit is interesting for another reason. Note that the topology of the horizon of the Myers-Perry black hole (3.4) is $\mathbb{S}^{D-2}$, while the black membrane has topology $\mathbb{R}^2 \times \mathbb{S}^{D-4}$. Generating black objects with novel horizon topology is quite generally true for ultra-spinning limits.

Let us now turn to a discussion of Kerr-AdS black holes and their properties, since these asymptotically AdS solutions will feature prominently in the following sections. The four dimensional version of this solution was found by Carter [139] just five years after Kerr's discovery of the asymptotically flat case. Motivated by the AdS/CFT correspondence, in 1999 Hawking, Hunter and Taylor-Robinson constructed the five dimensional Kerr-AdS solution, and also provided an example of a singly-spinning solution in all dimensions [29]. The most general solution with cosmological constant was obtained by Gibbons, Lu, Page and Pope in 2004 [140]. For our purposes, it will suffice to consider a singly-spinning

Kerr-AdS black hole in general dimensions. The metric reads

$$ds^2 = -\frac{\Delta_a}{\rho_a^2}\left[dt - \frac{a}{\Xi}\sin^2\theta d\phi\right]^2 + \frac{\rho_a^2}{\Delta_a}dr^2 + \frac{\rho_a^2}{S_a}d\theta^2 + \frac{S_a\sin^2\theta}{\rho_a^2}\left[adt - \frac{r^2+a^2}{\Xi}d\phi\right]^2 + r^2\cos^2\theta d\Omega_{D-4}^2\,, \tag{3.8}$$

where

$$\Delta_a = (r^2+a^2)\left(1+\frac{r^2}{L^2}\right) - 2mr^{5-D}\,, \quad S_a = 1 - \frac{a^2}{L^2}\cos^2\theta\,, \quad \Xi = 1 - \frac{a^2}{L^2}\,, \tag{3.9}$$

and ρ_a^2 is the same as in the asymptotically flat cases. With m non-zero the solution describes a black hole with horizon located at the largest root of Δ. The solution with $m = 0$ is simply AdS, but written in non-standard coordinates. If the following change of coordinates is used,

$$\Xi\hat{r}^2\sin^2\hat{\theta} = (r^2+a^2)\sin^2\theta\,, \qquad \hat{\phi} = \phi + \frac{a}{L^2}t\,, \tag{3.10}$$

with the other coordinates remaining the same, the relationship to AdS in global coordinates is manifest. This change of coordinates, when applied to the spacetime with $m \neq 0$ has the effect of bringing the solution into a frame that does not rotate at infinity.

A full list of the thermodynamic quantities for the singly-spinning AdS black hole can be found in [109], here let us just record a couple of the more relevant results for the discussion:

$$M = \frac{\Omega_{D-2}}{4\pi}\frac{m}{\Xi^2}\left[1 + \frac{(D-4)\Xi}{2}\right]\,, \quad J = \frac{\Omega_{D-2}}{4\pi}\frac{ma}{\Xi^2}\,,$$
$$A = 4S = \Omega_{D-2}\frac{\left(a^2+r_+^2\right)r_+^{D-4}}{\Xi}\,, \quad V = \frac{r_+A}{D-1}\left[1 + \frac{a^2}{\Xi}\frac{1+r_+^2/L^2}{(D-2)r_+^2}\right]\,. \tag{3.11}$$

From the above, we can draw a few important conclusions. First, it follows that these black holes satisfy the reverse isoperimetric inequality with $\mathcal{R} > 1$ for any non-zero a — equality only occurs when $a = 0$, i.e. when the solution is static. Second, unlike the asymptotically flat Myers-Perry solution, here there is not a simple proportionality between the angular momentum and the mass. Further, since the term Ξ enters into the definitions of the physical mass and angular momentum, both these quantities blow up in the limit $a \to L$ while satisfying

$$J \leq ML \tag{3.12}$$

which is reminiscent of the "Kerr bound" for asymptotically flat black holes. In this sense, Caldarelli, Emparan, and Rodríguez argued that the limit $a \to L$ can be thought of as the AdS analog of the ultra-spinning limit studied in the asymptotically flat case.

For AdS solutions, there are a number of ways in which the ultra-spinning limit can be taken, with each yielding new and interesting examples of black objects with various horizon topologies. Caldarelli, Emparan, and Rodríguez [130] studied the *black brane limit* wherein the physical mass is held fixed and the $a \to L$ limit is taken while simultaneously zooming in to the pole. This limit is sensible only in $D \geq 6$ and yields a static, asymptotically flat black brane. For this reason, the procedure is analogous to the limit first studied by Emparan and Myers. While ref. [130] considered only the singly-spinning Kerr-AdS geometry, the limit was generalized to the full multi-spinning Kerr-AdS solution in the appendix of [2]. In the same work, and again in a more recent paper [141], Caldarelli and collaborators studied a different approach to the $a \to L$ limit that we will call the *hyperboloid membrane limit*. In this case, the horizon radius, r_+, is held fixed while zooming in to the pole and taking $a \to L$. The limit, valid for any $D \geq 4$, yields a rotating AdS hyperboloid membrane with horizon topology $\mathbb{H}^2 \times \mathbb{S}^{D-4}$. These ultra-spinning limits are reviewed in appendix B.

The super-entropic black hole corresponds to another example of an ultra-spinning limit for the Kerr-AdS black hole. The procedure consists of the following steps. i) We start from a given rotating AdS black hole and, to eliminate any possible divergent terms in the metric that would prevent us from taking the $a \to L$ limit, recast it in a rotating-at-infinity coordinate system that allows one to introduce a rescaled azimuthal coordinate. ii) We then take the $a \to L$ limit, effectively 'boosting' the asymptotic rotation to the speed of light. iii) Finally, we compactify the corresponding azimuthal direction. In so doing we qualitatively change the structure of the spacetime since it is no longer possible to return to a frame that does not rotate at infinity. The obtained black holes have non-compact horizons that are topologically spheres with two punctures.

The four-dimensional super-entropic black hole was first discovered by Klemm [142, 143] in the course of classifying various rotating solutions of four-dimensional Fayet-Iliopoulos gauged supergravities. His approach takes the Carter-Plebański solution as a starting point, and then constrains the possible roots of the angular structure function — this construction is reviewed in appendix C. Since there is no known higher dimensional analog of the Carter-Plebański solution, our approach has the advantage of being directly applicable to any rotating black hole geometry, allowing us to obtain generalizations of Klemm's solution in higher dimensions, and for multi-spinning black holes.

3.2 The super-entropic limit

We begin our discussion of super-entropic black holes by considering the simplest possible example in $D = 4$. Let us first demonstrate this procedure on the Kerr–Newman-AdS black hole in four dimensions [139]. We write the metric and electromagnetic 1-form $\mathcal{A}$ in the 'standard Boyer–Lindquist form' [29]

$$\begin{aligned} ds^2 &= -\frac{\Delta_a}{\Sigma_a}\left[dt - \frac{a\sin^2\theta}{\Xi}d\phi\right]^2 + \frac{\Sigma_a}{\Delta_a}dr^2 + \frac{\Sigma_a}{S}d\theta^2 \\ &\quad + \frac{S\sin^2\theta}{\Sigma_a}\left[adt - \frac{r^2+a^2}{\Xi}d\phi\right]^2, \\ \mathcal{A} &= -\frac{qr}{\Sigma_a}\left(dt - \frac{a\sin^2\theta}{\Xi}d\phi\right), \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} \Sigma_a &= r^2 + a^2\cos^2\theta\,, \quad \Xi = 1 - \frac{a^2}{L^2}\,, \quad S = 1 - \frac{a^2}{L^2}\cos^2\theta\,, \\ \Delta_a &= (r^2+a^2)\Big(1 + \frac{r^2}{L^2}\Big) - 2mr + q^2\,, \end{aligned} \tag{3.14}$$

with the horizon r_h defined by $\Delta_a(r_h) = 0$. As written, the coordinate system rotates at infinity with an angular velocity $\Omega_\infty = -a/L^2$ and the azimuthal coordinate ϕ is a compact coordinate with range 0 to 2π. The choice of coordinates (3.13), while convenient, is not necessary to obtain the metric (3.15) below, as we demonstrate in the following subsection.

We now want to take the limit $a \to L$. To avoid a singular metric in this limit, we need only define a new azimuthal coordinate $\psi = \phi/\Xi$ (the metric is already written in coordinates that rotate at infinity) and identify it with period $2\pi/\Xi$ to prevent a conical singularity. After this coordinate transformation the $a \to L$ limit can be straightforwardly taken and we get the following solution:

$$\begin{aligned} ds^2 &= -\frac{\Delta}{\Sigma}\left[dt - L\sin^2\theta d\psi\right]^2 + \frac{\Sigma}{\Delta}dr^2 + \frac{\Sigma}{\sin^2\theta}d\theta^2 \\ &\quad + \frac{\sin^4\theta}{\Sigma}\left[Ldt - (r^2+L^2)d\psi\right]^2, \\ \mathcal{A} &= -\frac{qr}{\Sigma}\left(dt - L\sin^2\theta d\psi\right), \end{aligned} \tag{3.15}$$

where

$$\Sigma = r^2 + L^2\cos^2\theta\,, \quad \Delta = \Big(L + \frac{r^2}{L}\Big)^2 - 2mr + q^2\,. \tag{3.16}$$

Note that coordinate ψ is now a noncompact azimuthal coordinate, which we now choose to compactify by requiring that $\psi \sim \psi + \mu$. The result is equivalent to the metric presented by Klemm in [143] for the case of vanishing magnetic and NUT charges, as can be seen directly using the following coordinate transformation:

$$\tau = t\,, \quad p = L\cos\theta\,, \quad \sigma = -\psi/L\,, \quad L_{\text{Klemm}} = \mu/L\,. \tag{3.17}$$

Originally, this solution was found as a limit of the Carter–Plebański solution and corresponds to the case where the angular quartic structure function has two double roots [142, 143]. Klemm's construction is reviewed in appendix C. The advantage of our construction is that it immediately generalizes to the entire class of rotating AdS black holes.

3.2.1 Properties of the rotating frame

The super-entropic limit employed above required that the Kerr-AdS metric be written in a form that rotates at infinity. Here we explore that requirement in more detail, understanding exactly what the restrictions on the rotating frame are and how the results relate to the super-entropic solution presented just above.

Let us begin with the Kerr-AdS solution written in the standard Boyer–Lindquist form, given by (3.13), (3.14) above. In this form, the metric is already written in 'rotating coordinates', characterized by $\Omega_\infty = -a/L^2$. The fact that these coordinates are 'rotating' is crucial for the super-entropic limit—working in non-rotating coordinates leads to a singular limit, as we will see. We can ask, though, what restrictions (if any) are there on the rotating coordinates we use? That is, are there other frames besides that characterized by $\Omega_\infty = -a/L^2$ in which it is possible to perform the super-entropic limit? Let us begin to answer this question by writing the metric in 'non-rotating coordinates' by transforming

$$\Phi = \phi + \frac{a}{L^2}t\,, \tag{3.18}$$

where Φ is the non-rotating coordinate. We find

$$\begin{aligned} ds^2 &= -\frac{\Delta_a}{\Sigma_a}\left[\left(1+\frac{a^2\sin^2\!\theta}{L^2\Xi}\right)dt - \frac{a\sin^2\!\theta}{\Xi}d\Phi\right]^2 + \frac{\Sigma_a}{\Delta_a}dr^2 + \frac{\Sigma_a}{S}d\theta^2 \\ &+ \frac{S\sin^2\!\theta}{\Sigma_a}\left[\left(a+\frac{a}{L^2}\frac{r^2+a^2}{\Xi}\right)dt - \frac{r^2+a^2}{\Xi}d\Phi\right]^2 . \end{aligned}$$

It is now clear that the limit cannot be directly taken in the non-rotating coordinates: the g_{tt} and $g_{t\Phi}$ components of the metric are singular in the $a \to L$ limit and cannot be made finite through our rescaling of ϕ. There appears to be two possible ways to fix this. First, one could simply re-scale t as $t \to \Xi t$ while simultaneously taking $\phi \to \Xi\phi$. The second possibility (which was the one implemented in the previous section) involves transforming to a rotating frame and then taking $\phi \to \Xi\phi$. It turns out that the first method does not work (it leads to a singular metric) and so transforming to a rotating frame is essential.

Now, starting from the non-rotating metric let us transform to an *arbitrary* rotating frame via the transformation

$$\varphi = \Phi - x\frac{a}{L^2}t\,, \tag{3.19}$$

where x is (for now) an arbitrary parameter. Note that with the choice $x = 1$ eq. (3.13) is recovered. We then have for the metric in rotating-at-infinity coordinates,

$$\begin{aligned} ds^2 \;&=\; -\frac{\Delta_a}{\Sigma_a}\left[\left(1+\frac{a^2\sin^2\!\theta}{L^2\Xi}(1-x)\right)dt - \frac{a\sin^2\!\theta}{\Xi}d\varphi\right]^2 \\ &+\; \frac{S\sin^2\!\theta}{\Sigma_a}\left[\left(1+\frac{r^2+a^2}{L^2\Xi}(1-x)\right)adt - \frac{r^2+a^2}{\Xi}d\varphi\right]^2 \\ &+\; \frac{\Sigma_a}{\Delta_a}dr^2 + \frac{\Sigma_a}{S}d\theta^2\,. \end{aligned} \tag{3.20}$$

Considering this metric we see that the g_{tt} and $g_{t\varphi}$ metric components can be made finite with the choice

$$x = 1 + y\Xi + o(\Xi)\,, \tag{3.21}$$

where y is a constant with $y = 0$ yielding the coordinates we have used earlier, and $o(\Xi)$ denotes terms of higher order in Ξ. We then have, in these coordinates,

$$\Omega_\infty = -\frac{a}{L^2}\left(1+\Xi y\right)\,. \tag{3.22}$$

This result tells us that we do face some restrictions in our choice of coordinates. For example, it is not possible to perform the super-entropic limit if one begins in coordinates that rotate at infinity with $\Omega_\infty = -2a/L^2$ since this would require $y = 1/\Xi$, which is not valid. Now we must ask: when we perform the super-entropic limit in coordinates with an arbitrary (but valid) choice of y, how is the result related to our standard choice of $y = 0$?

The answer is that different values of y correspond simply to coordinate transformations of the solutions discussed earlier – there is nothing qualitatively different about the solution. To see this consider the transformation we made to the rotating frame

$$\varphi = \Phi - x\frac{a}{L^2}t = \Phi - \frac{a}{L^2}t - y\Xi\frac{a}{L^2}t\,. \tag{3.23}$$

Now recall that, at this point, when taking the super-entropic limit, we rescale φ via $\varphi = \Xi\psi$ and then take $a \to L$. So, with a non-vanishing y we have:

$$\psi = \frac{\varphi}{\Xi} = \frac{\Phi - \frac{a}{L^2}t}{\Xi} - y\frac{a}{L^2}t \overset{a \to L}{=} \psi_{\text{SE}} - \frac{y}{L}t\,, \tag{3.24}$$

where ψ_{SE} denotes the azimuthal coordinate from the super-entropic solutions. So beginning in other rotating-coordinate systems just turn out to yield a simple coordinate transformation applied to the solution we have already obtained.

We need to move to a rotating coordinate system because otherwise we will have a divergence in g_{tt} and $g_{t\phi}$. While there is some freedom in the choice of starting frame, we cannot perform the super-entropic limit from any rotating frame whatsoever. When an appropriate coordinate system is chosen, however, we always recover the 'standard' super-entropic solution, up to a simple coordinate transformation.

3.3 Basic properties

In this section we will explore some of the basic properties of the super-entropic black hole. We note that some of the results presented here were also considered by Klemm [143]; here the treatment follows the more detailed account given in [2].

First, let us note that the $m = q = 0$ form of the metric is a space of constant negative curvature with curvature scale L. For non-zero m, q we find that the metric (3.15) describes a black hole, with horizon at $r = r_+$ — the largest root of $\Delta(r_+) = 0$.

We first note that there is a minimum value of the mass required for horizons to exist. Examining the roots of Δ in eq. (3.16) we find

$$m \geq m_0 \equiv 2r_0\Big(\frac{r_0^2}{L^2} + 1\Big)\,, \tag{3.25}$$

where

$$r_0^2 \equiv \frac{L^2}{3}\left[-1 + \left(4 + \frac{3q^2}{L^2}\right)^{\frac{1}{2}}\right]. \tag{3.26}$$

For $m > m_0$ horizons exist while for $m < m_0$ there is a naked singularity. When $m = m_0$ the two roots of Δ coincide and the black hole is extremal.

To gain a deeper understanding of the spacetime, let us consider the geometry of constant (t, r) surfaces. The induced metric on such a surface reads

$$ds^2 = \frac{r^2 + L^2\cos^2\theta}{\sin^2\theta}d\theta^2 + \frac{L^2\sin^4\theta(2mr - q^2)}{r^2 + L^2\cos^2\theta}d\psi^2\,. \tag{3.27}$$

Since

$$g_{\psi\psi} = \frac{L^4 \sin^4\theta}{L^2\cos^2\theta + r^2}\left(2mr - q^2\right) \,, \tag{3.28}$$

it follows (using $m > m_0$ and $r_+ > r_0$) that $g_{\psi\psi}$ is strictly positive outside the horizon, indicating that the spacetime is free of closed timelike curves.

The metric (3.27) appears to be ill-defined for $\theta = 0, \pi$. To ensure there is nothing pathological occurring near these points let us examine the metric in the small θ limit (due to symmetry, the $\theta = \pi$ limit will be identical). We introduce the change of variables

$$\kappa = L(1 - \cos\theta)\,, \tag{3.29}$$

and examine the metric for small κ. This yields

$$ds^2 = (r^2 + L^2)\left[\frac{d\kappa^2}{4\kappa^2} + \frac{4(2mr - q^2)}{(r^2+L^2)^2}\kappa^2 d\psi^2\right] \,, \tag{3.30}$$

and the associated curvature tensor is just

$$R_{\mu\nu}{}^{\rho\sigma} = -\frac{4}{L^2 + r^2}\delta^{[\rho}_{[\mu}\delta^{\sigma]}_{\nu]}\,. \tag{3.31}$$

So the metric on these slices is nothing but a metric of constant negative curvature on the hyperbolic space $\mathbb{H}^2$. This implies that the $t, r = const.$ slices are non-compact manifolds and that the space is free from pathologies near the poles.[12] In particular, this analysis applies to the case of the black hole horizon, for which

$$ds_h^2 = (r_+^2 + L^2)\left[\frac{d\kappa^2}{4\kappa^2} + \frac{4\kappa^2}{L^2}d\psi^2\right] \,, \tag{3.32}$$

showing that the horizon is non-compact.

The above argument has allowed us to conclude that, near the poles, the spacetime is free of pathologies. However, using this argument alone we cannot conclude anything definitive about what happens precisely at $\theta = 0, \pi$. In the next section, we will consider the motion of test particles in the spacetime and argue that the symmetry axis is actually excised from the spacetime.

[12]The statement that these surfaces are non-compact should not be confused with the idea that they extend to $r = \infty$: they are, after all, a surface at $r = const.$. The notion is better understood as meaning that there is infinite proper distance between any fixed $\theta \in (0, \pi)$ and either pole.

To visualize the geometry of the horizon, we embed it in Euclidean 3-space. The induced metric on the horizon is

$$ds_h^2 = g_{\psi\psi} d\psi^2 + g_{\theta\theta} d\theta^2 \Big|_{r=r_+} . \tag{3.33}$$

We identify this line element with the line element in cylindrical coordinates

$$ds_3^2 = dz^2 + dR^2 + R^2 d\phi^2 ,$$

yielding

$$R^2(\theta) = \left(\frac{\mu}{2\pi}\right) g_{\psi\psi} , \tag{3.34}$$

$$\left(\frac{dz(\theta)}{d\theta}\right)^2 = g_{\theta\theta} - \left(\frac{dR(\theta)}{d\theta}\right)^2 , \tag{3.35}$$

where the prefactor in eq. (3.34) comes from the manner in which we have compactified ψ. Unfortunately, the resulting equations cannot be solved analytically. However it is straightforward to integrate them numerically for various values of r_+, L and q, as shown in figure 3.1. We stress that the reader should not confuse the fact that $z(\theta)$ extends to $\pm\infty$ at the poles with the horizon extending to spatial infinity in the bulk spacetime — it is just that the transverse sections are non-compact.

The ergosphere is the region for which the Killing vector ∂_t is no longer timelike, given by

$$\Delta - L^2 \sin^4\theta \leq 0 , \tag{3.36}$$

with equality corresponding to its outer boundary. Although at $\theta = 0, \pi$ the ergosphere appears to touch the horizon, this does not take place since this axis is excised from the spacetime as we shall see.

On the conformal boundary the metric (3.15) takes the following form (the conformal factor being given by L^2/r^2)

$$ds^2_{\text{bdry}} = -dt^2 - 2L \sin^2\theta dt d\psi + \frac{L^2}{\sin^2\theta} d\theta^2 , \tag{3.37}$$

and we see that ψ becomes a null coordinate there. Writing again $\kappa = L(1 - \cos\theta)$, the small κ limit gives

$$\begin{aligned} ds^2_{\text{bdry}} &= -dt^2 - 4\kappa d\psi dt + \frac{L^2}{4\kappa^2} d\kappa^2 , \\ &= -(dt + 2\kappa d\psi)^2 + L^2 \left(\frac{d\kappa^2}{4\kappa^2} + \frac{4\kappa^2}{L^2} d\psi^2\right) . \end{aligned} \tag{3.38}$$

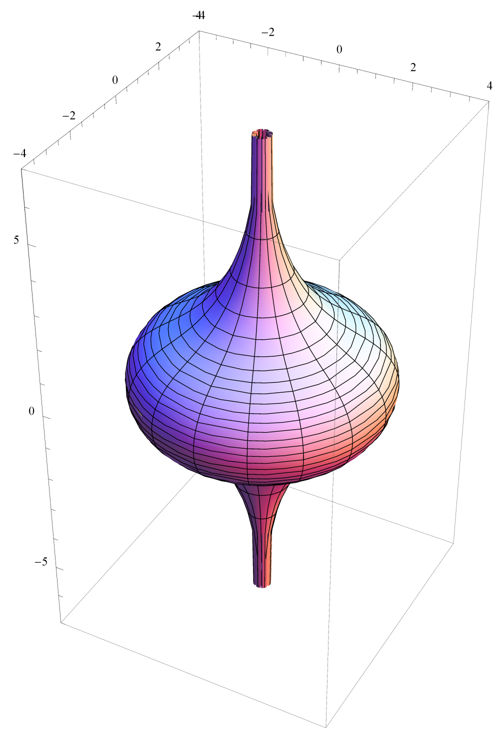

Figure 3.1: **Horizon embedding.** The horizon geometry of a 4D super-entropic black hole is embedded in $\mathbb{E}^3$ for the following choice of parameters: $q = 0$, $L = 1$, $r_+ = \sqrt{10}$ and $\mu = 2\pi$.

As the second expression makes manifest, the boundary is AdS_3 written as a Hopf-like fibration over $\mathbb{H}^2$. Due to the symmetry of the metric, an identical result holds for $\theta = \pi$. The structure of the conformal boundary reveals why the spacetime is asymptotically *locally* AdS. In somewhat formal treatments, the terminology 'asymptotically AdS' is reserved for solutions that approach a constant negative curvature space that is topologically $\mathbb{R} \times \mathbb{S}^{D-2}$. Here, due to the punctures along the symmetry axis (see also next section), these solutions do not meet this topological criterion.

Lastly, let us make an interesting remark about the full spacetime geometry considered in the vicinity of the symmetry axis. Taking (for simplicity) $q = 0$ and implementing the following coordinate transformation [143] [again after taking the limit $\kappa \to 0$ with $\kappa = L(1 - \cos(\theta))$]:

$$r = 2\rho\,, \quad \frac{4\kappa/L(\psi - i) + i}{4\kappa/L(\psi + i) + i} = e^{i\phi}\tanh\frac{\sigma}{2}\,, \quad 2t = \tau + iL\ln\frac{e^{i\phi}\tanh\frac{\sigma}{2} - 1}{e^{-i\phi}\tanh\frac{\sigma}{2} + 1}\,, \tag{3.39}$$

the metric is cast into the form

$$ds^2 = -V(\rho)\left[dt^2 + 2L\sinh^2\frac{\sigma}{2}d\phi\right]^2 + \frac{d\rho^2}{V(\rho)} + (\rho^2 + L^2/4)\left(d\sigma^2 + \sinh^2\sigma d\phi^2\right) \tag{3.40}$$

with

$$V(\rho) = \frac{L^4 - 4\rho(m-2\rho)L^2 + 16\rho^2}{16L^2}\,. \tag{3.41}$$

This is a (Lorentzian) AdS Taub-NUT geometry with an $\mathbb{H}^2$ base space and the NUT parameter equal to $n = L/2$. See, e.g., [28, 144, 145] for a discussion of these solutions. Note that for the case where the base space is $\mathbb{H}^2$ the fibration is trivial — there are no Misner strings [146], and as a result no periodicity is enforced on t.

3.4 Geodesics and the symmetry axis

In order to understand the role of the symmetry axis $\theta = 0, \pi$, we shall now study the geodesics. The geometry admits a closed conformal Killing–Yano 2-form, $h = db$,

$$b = (L^2\cos^2\theta - r^2)dt - L(L^2\cos^2\theta - r^2\sin^2\theta)d\psi\,, \tag{3.42}$$

inherited from the Kerr-AdS spacetime. Such an object guarantees separability of the Hamilton–Jacobi, Klein–Gordon, and Dirac equations in this background. In particular, it generates a Killing tensor $k_{\alpha\beta} = (*h)_{\alpha\mu}(*h)^{\mu}{}_{\beta}$, $\nabla_{(\alpha}k_{\beta\mu)} = 0$, whose existence implies a Carter constant of motion [147], $k_{\alpha\beta}u^\alpha u^\beta$, rendering geodesic motion (with 4-velocity u^α) completely integrable.

The fastest way to obtain the explicit expressions for the 4-velocity is to separate the Hamilton–Jacobi equation [147]

$$\frac{\partial S}{\partial\lambda} + g^{\alpha\beta}\frac{\partial S}{\partial x^\alpha}\frac{\partial S}{\partial x^\beta} = 0\,, \tag{3.43}$$

where the inverse metric to (3.15) reads

$$\begin{aligned}\partial_s^2 \;=\; & -\frac{1}{\Sigma\Delta}\left[(r^2+L^2)\partial_t + L\partial_\psi\right]^2 + \frac{\Delta}{\Sigma}\partial_r^2 + \frac{\sin^2\theta}{\Sigma}\partial_\theta^2 \\ & + \frac{1}{\Sigma\sin^4\theta}\left[L\sin^2\theta\partial_t + \partial_\psi\right]^2\end{aligned} \tag{3.44}$$

and where one can identify ∂S with the momentum 1-form u

$$\partial_a S = u_a\,. \tag{3.45}$$

We seek an additive separated solution (with the constants $\mathcal{E}, h, \sigma = -u^2$ corresponding to explicit symmetries)

$$S = \sigma\lambda - \mathcal{E}t + h\psi + R(r) + \Lambda(\theta)\,, \tag{3.46}$$

giving from (3.43)

$$\begin{aligned}
&\sigma - \frac{1}{\Sigma\Delta}\left[-(r^2+L^2)\mathcal{E} + Lh\right]^2 + \frac{\Delta}{\Sigma}R'^2 + \frac{\sin^2\!\theta}{\Sigma}\Lambda'^2 \\
&+\frac{1}{\Sigma\sin^4\!\theta}\left[h - L\sin^2\!\theta\mathcal{E}\right]^2 = 0\,,
\end{aligned} \tag{3.47}$$

where $R' = dR/dr$ and $\Lambda' = d\Lambda/d\theta$. Multiplying by Σ and reshuffling the terms, we obtain

$$\begin{aligned}
C &= -\sigma r^2 + \frac{1}{\Delta}\left[-(r^2+L^2)\mathcal{E} + Lh\right]^2 - \Delta R'^2 \\
&= \sin^2\!\theta\Lambda'^2 + \sigma L^2\cos^2\!\theta + \frac{1}{\sin^4\!\theta}\left[h - L\sin^2\!\theta\mathcal{E}\right]^2,
\end{aligned} \tag{3.48}$$

where C is *Carter's constant*, the additional (hidden) integral of geodesic motion.

Hence the geodesic 4-velocity ($u_t = -\mathcal{E}, u_\psi = h$) is given by

$$\begin{aligned}
\dot{t} &= \frac{\mathcal{E}(2mr - q^2)L^2}{\Sigma\Delta} + \frac{Lh\big(\Delta - \sin^2\!\theta(r^2+L^2)\big)}{\Sigma\Delta\sin^2\!\theta}\,, \\
\dot{\psi} &= \frac{h\big(\Delta - \sin^4\!\theta L^2\big)}{\Sigma\Delta\sin^4\!\theta} - \frac{L\mathcal{E}\big(\Delta - \sin^2\!\theta(r^2+L^2)\big)}{\Sigma\Delta\sin^2\!\theta}\,, \\
\dot{r} &= \frac{\sigma_r}{\Sigma}\sqrt{[Lh - (r^2+L^2)\mathcal{E}]^2 - \Delta C - \sigma\Delta r^2}\,, \\
\dot{\theta} &= \frac{\sigma_\theta\sin\theta}{\Sigma}\sqrt{C - \frac{1}{\sin^4\!\theta}\left[h - L\sin^2\!\theta\mathcal{E}\right]^2 - \sigma L^2\cos^2\!\theta}\,,
\end{aligned} \tag{3.49}$$

where $\sigma_r = \pm$ and $\sigma_\theta = \pm$ are independent signs.

We do not provide a complete analysis of the geodesics, leaving an analysis similar to [148] for future study. In what follows we limit ourselves to presenting an argument showing that the symmetry axis $\theta = 0, \pi$ cannot be reached by null geodesics ($\sigma = 0$) emanating from the bulk in a finite affine parameter. This indicates that the axis is some kind of a 'boundary' that is to be excised from the spacetime.

Let us probe the behavior close to $\theta = 0$ (the discussion for $\theta = \pi$ is due to the symmetry analogous). Consider 'ingoing' null geodesics for which θ decreases. For any finite value of C, it is obvious from the expression underneath the square root in the last equation (3.49) that when $h \neq 0$, $\theta = 0$ cannot be reached (the term $[h - l\mathcal{E}\sin^2\theta]^2/\sin^4\theta$ dominates for small θ driving the square root imaginary).

Consider next $h = 0$, then we have

$$\dot{\theta} = -\frac{\sin\theta}{\Sigma}\sqrt{C - L^2\mathcal{E}^2}\,. \tag{3.50}$$

It is straightforward to show from the third equation in (3.49) that there exists a constant $C = C_* > 0$ and $r = r_* > r_+$ such that $\dot{r}(r_*) = 0$; or in other words there exists a constant-r surface along which such photons are confined. Such geodesics will spiral towards $\theta = 0$ with $\dot{\psi} \neq 0$. For small θ we obtain $\dot{\theta}/\theta \approx -b^2 = -\sqrt{C_* - L^2\mathcal{E}^2}/r_*^2 =$ constant, i.e., $\theta \to e^{-b^2\tau}$. Photons moving on constant $r = r_*$ surfaces spiral toward $\theta = 0$ in infinite affine parameter. Moreover, using the first equation (3.49) together with (3.50), we have

$$\frac{d\theta}{dt} = -k\sin\theta\,, \quad k = \frac{\Delta_*\sqrt{C_* - L^2\mathcal{E}^2}}{\mathcal{E}L^2(2mr_* - q^2)} > 0\,. \tag{3.51}$$

Hence, starting from some finite θ_0, we have

$$t = -\frac{1}{k}\int\frac{d\theta}{\sin\theta} = -\frac{1}{k}\ln\Bigl(\tan\frac{\theta}{2}\Bigr) + \text{const.} \tag{3.52}$$

Evidently, as θ approaches zero, $t \propto -\frac{1}{k}\ln\theta \to \infty$; the axis is reached in infinite coordinate time t. Hence photons of this type can never reach the symmetry axis.[13]

The final possibility is that (while $h = 0$) the coordinate r changes as the photon approaches $\theta = 0$. Dividing the last two equation in (3.49) and introducing the following dimensionless quantities:

$$x = \frac{r}{l}\,, \quad A = \frac{2m}{L(1 - L^2\mathcal{E}^2/C)} > 0\,, \quad B = \frac{q^2}{2mL}\,, \tag{3.54}$$

we find that

$$\int\frac{d\theta}{\sin\theta} = \ln\left(\tan\frac{\theta}{2}\right) = -\sigma_r\int\frac{dx}{\sqrt{P(x)}}\,, \tag{3.55}$$

[13] For comparison, let us review here the behavior of radial geodesics in AdS space. Writing the metric in static coordinates, $ds^2 = -fdt^2 + dr^2/f$, $f = 1 + r^2/L^2$, we have 2 constants of motion $u^2 = -\sigma$ and $u_t = -\epsilon$, giving

$$\dot{t} = \frac{\epsilon}{f}\,, \quad \dot{r} = \pm\sqrt{\epsilon^2 - \sigma f}\,. \tag{3.53}$$

Specifically, radial null geodesics ($\sigma = 0$) starting from $r = 0$ reach the AdS boundary situated at $r = \infty$ in infinite affine parameter, $\tau = r/\epsilon \to \infty$, but (integrating $dr/dt = f$) at finite coordinate time $t = L\arctan(r/L) = \pi L/2$.

where $P(x)$ is the fourth-order polynomial given by

$$P(x) = A(x - B) - (1 + x^2)^2 \,. \tag{3.56}$$

It is easy to see that $P(x)$ can have at most 2 positive roots $0 < x_1 < x_2$ and that geodesic motion occurs for $r = xL$ obeying $x_1 \le x \le x_2$. The case $x_1 = x_2$ corresponds to motion on fixed $r = r_*$ discussed in the previous paragraph. To reach $\theta = 0$, the l.h.s. of eq. (3.55) diverges as $\ln\theta$. However, in the region of allowed motion, the r.h.s. of (3.55) remains finite (as only simple roots of $P(x)$ occur). This excludes the final possibility that the axis $\theta = 0$ can be reached by null geodesics emanating from some finite θ_0 in the bulk.

Finally, a much simpler argument, based on studying null geodesics on the conformal boundary, indicates that the axis of symmetry is in fact removed from the spacetime. Writing $\sin\theta = e^{-y}$, the metric on the conformal boundary reads

$$ds^2 = -dt^2 + L^2 dy^2 + 2Le^{-2y} dt d\psi \,. \tag{3.57}$$

The geodesic motion on this space admits 3 constants of motion $u^2 = -\sigma$, $u_t = -\mathcal{E}$ and $u_\psi = h$, giving the following 3 equations for null geodesics:

$$\begin{aligned} \dot{t} &= \frac{h}{L} e^{2y} \,, \quad \dot{\psi} = \frac{e^{4y}}{L^2}(h - \mathcal{E} L e^{-2y}) \,, \\ \dot{y} &= \pm \frac{e^{2y}}{L^2} \sqrt{h(2\mathcal{E} L e^{-2y} - h)} \,. \end{aligned} \tag{3.58}$$

From the last equation it is obvious that no null geodesic emanating from finite y_0 can reach the pole $y = \infty$ ($\theta = 0$) on the conformal boundary.

To summarize, the above arguments clearly demonstrate that the symmetry axis $\theta = 0, \pi$ is actually not part of the spacetime and represents instead some kind of a boundary. It is an interesting question as to whether such a boundary has similar properties to those of the boundary of AdS space.

3.5 Singly spinning super-entropic black holes in all dimensions

As was commented at the beginning of this chapter, the super-entropic limit generalizes to higher dimensional rotating black holes. The general solution for rotating AdS black holes is very complicated [140], and so here we simply present the generalization to a singly-spinning AdS black hole — the full generalization for the Kerr-AdS black holes, as well as charged and rotating black holes of $5D$ minimal gauged supergravity, can be found in [2].

To generalize the super-entropic black hole solution to higher dimensions, we start from the singly spinning D-dimensional Kerr-AdS geometry [29]

$$\begin{aligned} ds^2 &= -\frac{\Delta_a}{\rho_a^2}\left[dt - \frac{a}{\Xi}\sin^2\theta d\phi\right]^2 + \frac{\rho_a^2}{\Delta_a}dr^2 + \frac{\rho_a^2}{\Sigma_a}d\theta^2 \\ &+ \frac{\Sigma_a \sin^2\theta}{\rho^2}\left[adt - \frac{r^2+a^2}{\Xi}d\phi\right]^2 + r^2\cos^2\theta d\Omega_{D-4}^2\,, \end{aligned} \tag{3.59}$$

where

$$\begin{aligned} \Delta_a &= (r^2+a^2)(1+\frac{r^2}{L^2}) - 2mr^{5-D}, \quad \Sigma_a = 1 - \frac{a^2}{L^2}\cos^2\theta\,, \\ \Xi &= 1 - \frac{a^2}{L^2}, \quad \rho_a^2 = r^2 + a^2\cos^2\theta\,. \end{aligned} \tag{3.60}$$

Replacing $\phi = \psi\Xi$ everywhere and then taking the limit $a \to L$ we obtain

$$\begin{aligned} ds^2 &= -\frac{\Delta}{\rho^2}(dt - L\sin^2\theta d\psi)^2 + \frac{\rho^2}{\Delta}dr^2 + \frac{\rho^2}{\sin^2\theta}d\theta^2 \\ &+ \frac{\sin^4\theta}{\rho^2}[Ldt - (r^2+L^2)d\psi]^2 + r^2\cos^2\theta d\Omega_{D-4}^2\,, \end{aligned} \tag{3.61}$$

where

$$\Delta = \left(L + \frac{r^2}{L}\right)^2 - 2mr^{5-D}\,,\ \rho^2 = r^2 + L^2\cos^2\theta\ . \tag{3.62}$$

As before, ψ is a noncompact coordinate, which we now compactify via $\psi \sim \psi + \mu$. It is straightforward to show that the metric (3.61) satisfies the Einstein-AdS equations. Horizons exist in any dimension $D > 5$ provided $m > 0$ and in $D = 5$ provided $m > L^2/2$.

Similar to the 4-dimensional case, the solution inherits a closed conformal Killing–Yano 2-form from the Kerr-AdS geometry, $h = db$, where

$$b = (L^2\cos^2\theta - r^2)dt - L(L^2\cos^2\theta - r^2\sin^2\theta)d\psi\,. \tag{3.63}$$

This object guarantees complete integrability of geodesic motion as well as separability of the Hamilton–Jacobi, Klein–Gordon, and Dirac equations in this background; see [149] for analogous results in the Kerr-AdS case. In particular, the geodesics can be discussed in a way analogous to the previous subsection.

The arguments concerning the behavior near the symmetry axis at $\theta = 0, \pi$ for the 4-dimensional case can be repeated here. The induced metric on the horizon is

$$\begin{aligned} ds_h^2 &= \frac{r_+^2 + L^2\cos^2\theta}{\sin^2\theta} d\theta^2 + \frac{\sin^4\theta (r_+^2 + L^2)^2}{L^2\cos^2\theta + r_+^2} d\psi^2 \\ &+ r_+^2 \cos^2\theta d\Omega_{D-4}^2\,, \end{aligned} \tag{3.64}$$

and introducing as before $\kappa = L(1 - \cos\theta)$ we find

$$ds_h^2 = (r_+^2 + L^2)\left[\frac{d\kappa^2}{4\kappa^2} + \frac{4\kappa^2}{L^2} d\psi^2\right] + r_+^2 d\Omega_{D-4}^2\,. \tag{3.65}$$

This is a product geometry $\mathbb{H}^2 \times \mathbb{S}^{D-4}$ of two constant curvature spaces; the horizons of these black holes are non-compact and have finite horizon area. Similar to the four-dimensional case, they have topology of a cylinder as the actual axis is excised from the spacetime.

3.6 Thermodynamics and the reverse isoperimetric inequality

Let us now directly consider the thermodynamics of the super-entropic black hole. We present detailed calculations for the four-dimensional solution, and quote the results in the higher dimensional setting.

As written, ∂_t and ∂_ψ are Killing vectors for the metric. The linear combination

$$\xi = \partial_t + \Omega \partial_\psi \tag{3.66}$$

is the null generator of the horizon with

$$\Omega = \frac{L}{r_+^2 + L^2}\,. \tag{3.67}$$

The surface gravity can be obtained through the identity

$$\kappa = \lim_{r \to r_+} \sqrt{-\frac{1}{2}\xi^{\alpha;\beta}\xi_{\alpha;\beta}} \tag{3.68}$$

which yields

$$\kappa = \frac{1}{2r_+}\left[3\frac{r_+^2}{L^2} - 1 - \frac{q^2}{r_+^2 + L^2}\right]. \tag{3.69}$$

The temperature is then given directly by $T = \kappa/(2\pi)$.

The conserved mass associated with the Killing vector ∂_t can be computed via a variety of techniques. Here we will provide an explicit calculation using the conformal method of Ashtekar, Magnon, and Das [91, 92, 150], which has the advantage of requiring no background subtraction. To facilitate the calculation of the mass, we note the leading order behaviour of the Weyl tensor component

$$C^t{}_{rtr} = \frac{2L^2 m}{r^5} + \cdots \tag{3.70}$$

at large distance. We then obtain the electric component $\bar{\mathcal{E}}^t{}_t$ via a conformal rescaling to be

$$\bar{\mathcal{E}}^t{}_t = \frac{2m}{L^3}. \tag{3.71}$$

The volume element of a $t =$ constant hypersurface lying in the boundary is given by

$$d\bar{\Sigma}_t = L^2 \sin\theta d\theta d\psi. \tag{3.72}$$

Thus we obtain for the mass

$$M = Q[\partial_t] = \frac{L}{8\pi}\oint_\Sigma \bar{\mathcal{E}}^t{}_t \left(L^2 \sin\theta d\theta d\psi\right) = \frac{\mu m}{2\pi}. \tag{3.73}$$

We note that this result for the mass also follows from the Komar definition of the mass (2.32), using a naive background subtraction of the $m = 0$ solution.

The angular momentum can be computed unambiguously via a number of methods, with no need for background subtraction, and we will perform the calculation with a Komar integration. The surface element is given by

$$dS_{\alpha\beta} = -2r_{[\alpha}n_{\beta]}\sqrt{\sigma}d\theta d\psi, \tag{3.74}$$

where σ is the determinant of the induced metric on the surfaces of constant t and r, r^α is a unit normal vector in the radial direction, and n^β is the unit normal to the surface of constant t

$$n^\alpha = \left[-g_{tt} + 2\frac{g_{t\psi}^2}{g_{\psi\psi}} - g_{t\psi}\right]^{-1/2}\left[\delta^\alpha_t - \frac{g_{t\psi}}{g_{\psi\psi}}\delta^\alpha_\psi\right]. \tag{3.75}$$

A direct computation then yields

$$J = \frac{1}{16\pi}\oint_{S_\infty} dS_{\alpha\beta}\nabla^\alpha \phi^\beta = \frac{\mu m L}{2\pi}\,. \tag{3.76}$$

The thermodynamic quantities

$$\begin{aligned} &M = \frac{\mu m}{2\pi}\,, \quad J = ML\,, \quad \Omega = \frac{L}{r_+^2 + L^2}\,, \quad T = \frac{1}{4\pi r_+}\left(3\frac{r_+^2}{L^2} - 1 - \frac{q^2}{L^2 + r^2}\right), \\ &S = \frac{\mu}{2}(L^2 + r_+^2) = \frac{A}{4}, \quad \Phi = \frac{q r_+}{r_+^2 + L^2}\,, \quad Q = \frac{\mu q}{2\pi}\,, \quad V = \frac{r_+ A}{3} = \frac{2}{3}\mu r_+\left(r_+^2 + l^2\right) \end{aligned} \tag{3.77}$$

satisfy the relation

$$\delta M = T\delta S + \Omega \delta J + V\delta P + \Phi \delta Q\,. \tag{3.78}$$

However, note that this cannot be the correct first law for these black holes. The reason is that, due to the chirality condition $J = ML$, the relation (3.78) is not of the correct cohomogeneity.[14] As a result, eq. (3.78) does not correctly define the conjugate quantities because it is not possible to hold constant all of the parameters (M, S, J, P, Q) independently. A correct approach requires implementing the chirality condition at the level of the first law. Doing so, treating (S, J, P, Q) as the independent variables[15] we obtain

$$0 \;\; = \;\; \Omega' \delta J + V' \delta P + \Phi \delta Q \tag{3.79}$$

where

$$\Omega' = \Omega - 1/L\,, \quad V' = \frac{\omega_{D-2}(r_+ - L)(r_+ + L)(r_+^2 + L^2)}{6r_+}\,. \tag{3.80}$$

The Smarr relation that follows from scaling holds,

$$0 = 2\Omega' J - 2V' P + \Phi Q\,. \tag{3.81}$$

Note that this reduction of the thermodynamic phase space can be viewed equivalently as transforming to a frame in which the mass vanishes (see below).

[14] This problem was noted in [1], but was not fully appreciated until the writing of this thesis.

[15] One could also choose (M, S, P, Q) as the independent variables, and while this would lead to a different answer for the thermodynamic volume, the conclusions would be the same. This possibility was considered in [1]; it has the unpleasing feature that the mass is no longer a homogeneous function of the other variables. The remaining option would be to use (M, S, J, Q) as the independent variables, in which case the thermodynamic volume would not even be defined. This option is also unpalatable, since again the mass would not be a homogeneous function of the remaining parameters.

Note that the thermodynamic volume V' is not always positive — similar results have also been observed for AdS Taub-NUT and bolt solutions by Johnson [151]. The thermodynamic volume will be positive provided that $r_+ > L$. Restricting to this parameter region, it is easy to see that the reverse isoperimetric inequality is always violated. Bearing in mind that our space is compactified according to $\psi \sim \psi + \mu$, the orthogonal 2-dimensional surface area takes the form $\omega_2 = 2\mu$. Consequently, the isoperimetric ratio reads (with $x = r_+/L$)

$$\mathcal{R} = \left(\frac{3V'}{2\mu}\right)^{1/3} \left(\frac{2\mu}{A}\right)^{1/2} = \left(\frac{(x^2-1)^2}{4x^2(x^2+1)}\right)^{1/6} < 1\,. \tag{3.82}$$

Hence these black holes have more entropy 'per unit thermodynamic volume' than the Schwarzschild AdS solution, and so are *super-entropic*.

This result stands in contrast to the 'usual' ultra-spinning limit of Kerr-AdS black holes in which, as $a \to L$, the isoperimetric ratio approaches infinity, maximally satisfying the reverse isoperimetric inequality. The distinction arises because of the nature of the ultra-spinning limit we are taking. Rather than keeping M fixed and letting the horizon area approach zero as $a \to L$ [137, 109], here we require this limit be taken whilst demanding the horizon area remain finite.

Unfortunately this class of charged black holes does not have interesting phase behaviour or critical phenomena. This is clear since solving the equation defining the temperature for r_+ produces only a single real branch for r_+.

The thermodynamic considerations above can be extended straightforwardly to the higher dimensional generalizations of the super-entropic black hole. In the singly spinning case presented above, the thermodynamic quantities now read

$$\begin{aligned} M &= \frac{\omega_{D-2}}{8\pi}(D-2)\,m\,, \quad J = \frac{2}{D-2}ML\,, \quad \Omega = \frac{L}{r_+^2+L^2}\,, \\ T &= \frac{1}{4\pi r_+ L^2}\Big[(D-5)L^2 + r_+^2(D-1)\Big]\,, \\ S &= \frac{\omega_{D-2}}{4}(L^2+r_+^2)r_+^{D-4} = \frac{A}{4}\,, \quad V = \frac{r_+ A}{D-1}\,, \end{aligned} \tag{3.83}$$

where ω_D given by

$$\omega_D = \frac{\mu \pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D+1}{2}\right)} \tag{3.84}$$

is the volume of the D-dimensional unit 'sphere'. Here Ω is the angular velocity of the horizon and J and M have been computed via the method of conformal completion as the conserved quantities associated with the ∂_ψ and ∂_t Killing vectors, respectively. Note also that the chirality condition relating the mass and angular momentum here is reminiscent of that for the singly-spinning Myers-Perry solution (3.6).

Again, the thermodynamic quantities satisfy the 'naive' extended first law, but due to the chirality condition $J = 2ML/(D-2)$ the quantities are not independent. As a result, we repeat the procedure illustrated in the four-dimensional case, treating (J, P, S) as the independent quantities. We find that the following first law holds

$$0 = T\delta S + \Omega'\delta J + V'\delta P \tag{3.85}$$

with

$$\Omega' = \Omega - \frac{D-2}{2L}\,, \quad V' = \omega_{D-2}\frac{r_+^{D-1}\left(1 - L^4/r_+^4\right)}{2(D-1)}\,. \tag{3.86}$$

These quantities satisfy the Smarr relation that follows directly from scaling

$$0 = (D-2)TS + (D-2)\Omega' J - 2PV'\,. \tag{3.87}$$

Restricting to the cases where $r_+ > L$ so that the volume is positive, the isoperimetric ratio for these black holes reads (defining $x = r_+/L$)

$$\begin{aligned}\mathcal{R} &= \left(\frac{(D-1)V'}{\omega_{D-2}}\right)^{\frac{1}{D-1}} \left(\frac{\omega_{D-2}}{A}\right)^{\frac{1}{D-2}} \\ &= \left(\frac{(x^4-1)^{D-2}}{2^{D-2}x^{2D-6}(x^2+1)^{D-1}}\right)^{\frac{1}{(D-1)(D-2)}}\,.\end{aligned} \tag{3.88}$$

The ratio is clearly monotonic in x, and so the maximum value occurs in the limit $x \to \infty$. Thus we see that, the isoperimetric ratio is bounded above by

$$\mathcal{R} < \left(\frac{1}{2}\right)^{1/(D-1)}\,, \tag{3.89}$$

and so, similar to their 4-dimensional cousins, these black holes are also super-entropic.

3.6.1 Subtleties in defining the conserved charges

Let us close the discussion of the thermodynamics by noting a subtlety in the definition of the conserved mass and angular momentum that arises for the super-entropic black hole.

This was only noticed in the preparation of this chapter, and certainly merits further study. The subtlety arises from the chirality condition and also the non-trivial topology of the spacetime. It is clear that the operations defining the conserved quantities are linear, and so if K_1 and K_2 are Killing vectors, then

$$Q[c_1 K_1 + c_2 K_2] = c_1 Q[K_1] + c_2 Q[K_2] \,. \tag{3.90}$$

As a result there is always a freedom to shift a conserved quantity by some amount proportional to another conserved quantity. For example, we can define a new conserved mass by adding to ∂_t some multiple of ∂_ψ which would yield

$$M_{\text{new}} = M + \Omega' J \,, \tag{3.91}$$

for some (possibly L-dependent) constant Ω'. Physically, this can be viewed as a coordinate transformation that mixes the Killing coordinates (t, ϕ).

In general, this ambiguity should be fixed by choosing some privileged observer at infinity. In the context of rotating AdS black holes, this point is subtle. Indeed, the expressions given by Hawking, Hunter and Taylor-Robinson in [29] for the Kerr-AdS black hole do not even satisfy the first law. In that case, the correct answer can be obtained by transforming the Kerr-AdS solution to coordinates that do not rotate at infinity and then calculating the conserved quantities. The appropriate transformation — which ensures that the Killing coordinates coincide with the standard generators of the AdS group — was first presented in [90] for the four dimensional Kerr-AdS black hole. The case for general dimensions was discussed in [152].

In the case of the super-entropic black hole, the situation is more subtle, but the resolution of the problem would be useful for deciding which set of thermodynamic parameters $[(M, S, P)$ vs. $(J, S, P)]$ are the appropriate choices in the first law and, therefore, the correct thermodynamic volume. In the previous section, the choice (J, S, P) was motivated since it respects the homogeneity of the Smarr relation, though in either case the black hole is super-entropic. Unsurprisingly, the super-entropic limit renders singular the coordinate transformation that brings the Kerr-AdS solution into global AdS form, and so this approach cannot be used. In the absence of a simpler alternative, it may be that the only way to fix this ambiguity for the super-entropic black hole would be to determine the algebra generated by the Killing vectors of the $m = 0$ solution and determine which combination t and ψ lead to the 'standard' commutation relations, similar to what was done in [90]. We hope to return to this problem in the future.

3.7 Summary remarks

In this chapter, we have described the super-entropic limit and applied it to singly-spinning AdS black holes in all dimensions. The resulting metrics describe rotating black holes whose horizons are topologically spheres with two punctures. The thermodynamic analysis implies that these black holes violate the conjectured reverse isoperimetric inequality, providing the first plausible counter-example to this conjecture. Let us close by remarking on some further properties and problems for future study.

While the discussion in this chapter focused on singly-spinning black holes, the procedure generalizes in the natural way to multi-spinning black holes — see [2] for the details. However, let us note that in the multi-spinning case it is only ever possible to take a single $a_i \to L$ using the super-entropic limit. If one tries to take the limit for multiple a_i, a degenerate metric is obtained. A consequence of this is that it is not possible to apply this limit to higher dimensional Kerr-AdS black holes with equal rotation parameters, i.e. $a_i = a$. Naively, one would expect that if, for example, a 'double super-entropic limit' could be taken, the resulting black hole would have a horizon topologically a sphere with four punctures. Attempts to construct black holes with horizons with multiple punctures starting from the more general Kerr-NUT-AdS class of solutions [153] was attempted but was not successful, suggesting that there is possibly some topological obstruction to their existence.

Another point not addressed here but considered in [2, 3] is the notion of taking ultra-spinning limits of the super-entropic black holes themselves. For example, since the super-entropic limit can only ever be applied in a single direction, one could imagine taking the super-entropic limit for one rotation parameter and a different type of ultra-spinning limit for the remaining parameters. The answer is that the super-entropic limit can be combined with the hyperboloid membrane limit described in the first section of this chapter and in appendix B, but is not compatible with the black brane limit. The reason seems to be that while the hyperboloid membrane limit maintains the asymptotically AdS properties of the spacetime, the black brane limit produces an asymptotically flat solution. This latter point is in tension with the fact that the super-entropic black hole appears to have no simple asymptotically flat limit — similar to the AdS black holes constructed in [154] via an $\mathbb{S}^4$ reduction of 11D supergravity. Further, an interesting observation was made in [3] regarding the combination of the super-entropic and hyperboloid membrane limits. If one starts with a rotating black hole with N rotation parameters, takes the super-entropic limit for one of these parameters and then the hyperboloid membrane limit for the remaining $N - 1$, the final solution has no punctures. That is, taking the maximum number of hyperboloid membrane limits 'removes' the punctures that the super-entropic limit introduced.

Let us note some directions that may merit further investigation in the future. First, the subtleties in defining the conserved charges should be addressed — this is an important point for understanding what is the 'true' mass of the super-entropic black hole. Second, the Euclidean version of these solutions should be scrutinized — naively, it seems that no (positive definite) Euclidean sector exists. The reason is that the standard Wick rotation trick for rotating black holes requires not only $t \to -it_E$ but also a wick rotation of the rotation parameters. Here, the rotation parameter is the AdS length, and Wick rotation of L would change not only the structure of the metric, but would also correspond to changing both the field equations and asymptotic structure. The situation is somewhat analogous to the Eguchi-Hanson metrics [155], for which a Euclidean sector exists, but no clear Lorentzian sector does. In fact, it seems that in this same sense it should be possible to construct topologically non-trivial asymptotically dS instantons via a super-entropic limit applied to the Euclidean Kerr-dS solutions.

Chapter 4

Generalized theories of gravity

In this chapter, we will introduce higher curvature theories of gravity and study some of their basic properties. Higher curvature theories in the context of black hole chemistry will appear in the following chapter, and this chapter is to serve as an introduction to these theories through the lens of the author's recent work. After providing a brief introduction to some general features of higher curvature theories, the focus will shift to studying these theories for the case of static and spherically symmetric black hole solutions, as in [5, 6]. The idea will be to study which theories of gravity admit 'nice' field equations under the restriction to spherically symmetric metrics. It turns out there are three such classes: Lovelock gravity [116], quasi-topological gravity [118], and generalized quasi-topological gravity [5]. The latter class of theories was recently discovered by the author and their properties have been developed in a series of papers by the author and also Bueno and Cano, e.g. [156]. All of these theories possess a number of interesting — and surprising — properties that make them excellent toy models for exploring questions in black hole thermodynamics and holography. The primary references for this chapter are [4, 157, 6, 158, 7, 12, 11, 9, 10], including some unpublished work.

4.1 Higher curvature theories: an overview

Here we will discuss some basic properties of higher curvature theories. The literature on this subject is vast, and we do not attempt to offer a complete picture here. Rather, the goal will be to motivate the study of higher curvature theories in general, and then discuss the propagating degrees of freedom and black hole thermodynamics.

4.1.1 Introduction and motivation

There can be no doubt that general relativity is a tremendously successful physical theory, having been vindicated in all experimental tests to date [159]. However, despite this success, there are good reasons for studying theories of gravity beyond general relativity. One class of modifications is *higher curvature gravity*, where one supplements the Einstein-Hilbert action by terms that are higher-order in the curvature tensor.

Perhaps the most prominient reason for interest in higher curvature theories in modern times comes from quantum gravity. It is generally expected that in a quantum theory of gravity the Einstein-Hilbert action will be modified by the addition of higher curvature terms. This follows from an effective field theory type argument, and can already been seen from the renormalization of quantum fields on curved spacetime [160]. It is also possible, via the addition of terms quadratic in curvature, to construct renormalizable theories of quantum gravity [161]. However, this brings with it other problems, such as the existence of ghosts in the spectrum, as will be discussed further below. Within the context of string theory, higher curvature corrections appear in the low energy effective action. However, the precise terms that appear in the four-dimensional world depend on the string theory under consideration and the compactification [162, 163, 164, 165].

The discovery of the AdS/CFT correspondence [62, 26, 63] around 20 years ago gives rise to new motivations for the study of higher curvature theories. In Einstein gravity, the only scales are the AdS length L and the Planck length ℓ_P, and so the only dimensionless ratio is L/ℓ_P. Higher curvature theories introduce new scales via their couplings, providing additional parameters that can appear in the results of holographic calculations. In this way, degeneracy is broken and one is able to make contact with a wider class of CFTs. This has been used with great success in uncovering holographic results which are particular to theories with Einstein gravity duals and results that are universal. For example, the Kovtun–Son–Starinets (KSS) bound on the viscosity/entropy density ratio [68] was conjectured to be a lower bound for all theories in nature, but it was later found that CFTs dual to higher curvature theories can in fact violate this bound [166]. Another example would be the contribution to the entanglement entropy that arises from a sharp corner in the entangling surface. In this case, the contribution depends on a function of the corner opening angle, whose ratio with the central charge is effectively universal for a wide class of holographic theories [167, 168, 169].

Perhaps the simplest reason for interest in higher curvature theories is simply to understand which features of general relativity are special and which are robust. For example, black holes in general relativity obey the laws of black hole thermodynamics. A huge success of work on black holes in higher curvature theories was in showing that this is not a fluke — black hole solutions in any diffeomorphism invariant theory will obey the laws of black hole thermodynamics [170, 113, 114, 171, 115]. This is an incredibly profound result, not only in terms of the implications for higher curvature corrections to black holes in nature, but also because of the deep connections it suggests between geometry, thermodynamics, and information theory. However, it should be noted that there are differences in the black hole thermodynamics: for example, the black hole entropy in a higher curvature theory is no longer simply proportional to the area, as we will discuss in more detail below. These differences often lead to interesting consequences for black hole thermodynamics. In the context of black hole chemistry, studies of higher curvature theories have provided examples of triple points, (multiple) re-entrant phase transitions, and even an example of a λ-line for black holes [172, 173, 174, 175, 176, 177, 178, 179, 180, 43, 44, 181, 182, 183, 184, 16, 185, 186, 187, 188, 189, 190, 191, 4, 43, 13].

Having motivated the study of higher curvature theories, we now turn to discuss the linearization of general higher-order gravities, and their black hole thermodynamics.

4.1.2 Linear spectrum of higher curvature theories

Now let us consider the linearized spectrum of higher curvature theories. Our interest here will be to review the equations of motion for a metric perturbation $h_{\mu\nu}$ from a maximally symmetric geometry. In doing this, it is possible to identify the propagating degrees of freedom in the theory of gravity. Since a general higher curvature theory can propagate additional pathological degrees of freedom, it will be important to identify when these problematic modes are — or are not — present will be important in the study of these models. The primary references for this section are [192, 193, 194, 195].

We will consider a general theory of gravity that has a Lagrangian built from contractions of the Riemann tensor,

$$S = \frac{1}{16\pi G_N} \int d^D x \mathcal{L}\left(g_{\alpha\beta}, R_{\alpha\beta\mu\nu}\right) . \tag{4.1}$$

Here, and in the remainder of the chapter, we will focus on the specific case where the Lagrangian density is a polynomial in the curvature tensor. The field equations (derived in appendix D) that follow from the action are most conveniently written as [196]

$$\mathcal{E}_{\alpha\beta} = P_{\alpha}{}^{\sigma\rho\delta} R_{\beta\sigma\rho\delta} - \frac{1}{2} g_{\alpha\beta}\mathcal{L} - 2\nabla^{\rho}\nabla^{\sigma} P_{\alpha\rho\sigma\beta} = 8\pi G_N T_{\alpha\beta} \tag{4.2}$$

where

$$P^{\alpha\beta\rho\sigma} := \frac{\partial \mathcal{L}}{\partial R_{\alpha\beta\rho\sigma}} , \tag{4.3}$$

will sometimes be referred to as the 'entropy tensor' and inherits the symmetries of the Riemann tensor. Note that due to diffeomorphism invariance of the action, the generalized Einstein tensor $\mathcal{E}_{\alpha\beta}$ obeys the Bianchi identity,

$$\nabla^{\alpha}\mathcal{E}_{\alpha\beta} = 0 \, . \tag{4.4}$$

Now we will consider a maximally symmetric space with metric $\bar{g}_{\alpha\beta}$ that solves the field equations. For a maximally symmetric space, the Riemann tensor has the following simple form

$$\bar{R}_{\alpha\beta\mu\nu} = 2K\bar{g}_{\alpha[\mu}\bar{g}_{\nu]\beta} \, , \tag{4.5}$$

where K is a constant of length dimension -2. In this case, the equations of motion greatly simplify — since the background is maximally symmetric, the derivatives of the entropy tensor that appear in the equations of motion vanish. Further, since the only objects available are $\bar{g}_{\alpha\beta}$, $\bar{g}^{\alpha\beta}$ and δ_{α}^{β}, the form of the entropy tensor is completely fixed by symmetry to be

$$\bar{P}^{\mu\alpha\beta\nu} = 2e\bar{g}^{\mu[\beta}\bar{g}^{\nu]\alpha} \, , \tag{4.6}$$

where e is some theory-dependent constant. Using this, and the background Riemann tensor, we can show that the trace of the field equations reduces to

$$\bar{\mathcal{L}}[K] = 4(D-1)eK \, , \tag{4.7}$$

where we write $\bar{\mathcal{L}}[K]$ to clarify that this is the Lagrangian density evaluated on the background spacetime $\bar{g}_{\alpha\beta}$ and will be some polynomial in K. At the same time, we also have

$$\frac{d\bar{\mathcal{L}}[K]}{dK} = \bar{P}^{\alpha\beta\mu\nu}\left(2\bar{g}_{\alpha[\mu}\bar{g}_{\nu]\beta}\right) = 2eD(D-1) \, , \tag{4.8}$$

which when combined with the previous result yields the following algebraic equation:

$$D\bar{\mathcal{L}}[K] - 2K\frac{d\bar{\mathcal{L}}[K]}{dK} = 0 \, , \tag{4.9}$$

which determines the maximally symmetric vacua of the theory. That is, this equation will be some polynomial in K, and will depend on the couplings of the theory. The solutions will determine how the curvature scales of the maximally symmetric vacua depend on the coupling constants of the theory and the cosmological constant length scale. Note that theories having up to n powers of curvature in the Lagrangian will have up to n different maximally symmetric vacua.

Without getting into too much detail, we note that to study the equations of motion for perturbations away from a maximally symmetric background requires introducing another tensor

$$C^{\sigma\rho\lambda\eta}_{\mu\alpha\beta\nu} := g_{\mu\xi}g_{\alpha\delta}g_{\beta\gamma}g_{\nu\chi}\frac{\partial^2\mathcal{L}}{\partial R_{\sigma\rho\lambda\eta}R_{\xi\delta\gamma\chi}} \,. \tag{4.10}$$

For a maximally symmetric background, symmetry completely fixes this object up to three theory-dependent constants [195]

$$\begin{aligned} C^{\sigma\rho\lambda\eta}_{\mu\alpha\beta\nu} &= a\left[\delta^{[\sigma}_{\mu}\delta^{\rho]}_{\alpha}\delta^{[\lambda}_{\beta}\delta^{\eta]}_{\nu} + \delta^{[\lambda}_{\mu}\delta^{\eta]}_{\alpha}\delta^{[\sigma}_{\beta}\delta^{\rho]}_{\nu}\right] + b\left[\bar{g}_{\mu\beta}\bar{g}_{\alpha\nu} - \bar{g}_{\mu\nu}\bar{g}_{\alpha\beta}\right]\left[\bar{g}^{\sigma\lambda}\bar{g}^{\rho\eta} - \bar{g}^{\sigma\eta}\bar{g}^{\rho\lambda}\right] \\ &\quad + 4c\delta^{[\sigma}_{(\tau}\bar{g}^{\rho][\lambda}\delta^{\eta]}_{\epsilon)}\delta^{\tau}_{[\mu}\bar{g}_{\alpha][\beta}\delta^{\epsilon}_{\nu]} \,. \end{aligned} \tag{4.11}$$

In general computing this tensor for a theory of gravity is a laborious task. However, there are efficient mechanisms for computing the constants directly from the Lagrangian [121] or using computer algebra software like Mathematica [197], but we will not discuss those techniques here.

Now, given the above, one can show that on general grounds a perturbation away from a maximally symmetric solution,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \,, \tag{4.12}$$

will satisfy the following equation at linear order [195],

$$\begin{aligned} \mathcal{E}^L_{\mu\nu} &= 2\left[e - 2K\left(a(D-1) + c\right) + (2a+c)\bar{\Box}\right]G^L_{\mu\nu} + 2\left[a + 2b + c\right]\left[\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu\right]R^L \\ &\quad - 2K\left[a(D-3) - 2b(D-1) - c\right]\bar{g}_{\mu\nu}R^L = 8\pi G_N T^L_{\mu\nu} \,, \end{aligned} \tag{4.13}$$

where a, b, c and e are constants that depend on the theory under consideration and also the spacetime dimension, and are fixed by the structure of the tensors defined above. The terms with superscript "L" correspond to the following:

$$\begin{aligned} R^L &= \bar{\nabla}^\mu\bar{\nabla}^\nu h_{\mu\nu} - \bar{\Box}h - (D-1)Kh \,, \\ R^L_{\mu\nu} &= \bar{\nabla}_{(\mu|}\bar{\nabla}_\sigma h^\sigma_{|\nu)} - \frac{1}{2}\bar{\Box}h_{\mu\nu} - \frac{1}{2}\bar{\nabla}_\mu\bar{\nabla}_\nu h + DKh_{\mu\nu} - Kh\bar{g}_{\mu\nu} \,, \\ G^L_{\mu\nu} &= R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^L - (D-1)Kh_{\mu\nu} \,. \end{aligned} \tag{4.14}$$

For a general theory of gravity, the linearized equations of motion are fourth order, which is related to the fact that theories propagate additional modes as well. Beyond the usual massless graviton, there is also an additional scalar mode and massive graviton. In general, the coupling of the massive graviton to matter has the wrong sign, indicating that this term is a ghost. The masses of the scalar and ghost-like graviton can be given directly in terms of the same parameters appearing in the linearized equations [195],

$$
\begin{aligned}
m_s^2 &= \frac{e(D-2) - 4K(a + bD(D-1) + c(D-1))}{2a + Dc + 4b(D-1)}\,, \\
m_g^2 &= \frac{-e + 2K(D-3)a}{2a + c}\,.
\end{aligned}
\tag{4.15}
$$

Now, if we want the spectrum of the theory to contain only the usual massless graviton (which will ensure the absence of the Boulware-Deser ghost instability [198]), then we must require that the masses of the scalar and ghost-like graviton are infinite. Doing so places the following constraints on the parameters:

$$
2a + c = 0\,, \quad 4b + c = 0\,. \tag{4.16}
$$

Note that these two conditions are also sufficient to remove all of the four derivative terms that appear in the linear equations of motion. The conclusion is that if the linearized equations are second order, then the theory will not possess the extra scalar or ghost-like graviton modes. Further, note that when these conditions are imposed, the (now second-order) linearized equations reduce to

$$
\mathcal{E}^L_{\mu\nu} = 2\left[e - 2Ka(D-3)\right] G^L_{\mu\nu} = 8\pi G_N T^L_{\mu\nu}\,, \tag{4.17}
$$

and we see that the higher curvature corrections have resulted in an effective renormalization of Newton's constant

$$
G_{N\mathrm{eff}} = \frac{G_N}{2e - 4K(D-3)a}\,. \tag{4.18}
$$

4.1.3 Black hole thermodynamics

As was mentioned in the motivation, the fact that black holes obey the laws of thermodynamics is not a peculiarity of general relativity — it is a generic result for black objects derived from a diffeomorphism invariant Lagrangian. However, there are differences in the quantities that appear in the first law. There is a great deal that could be said about the thermodynamics of black holes in higher curvature theories, but here we will just focus on defining entropy and conserved charges for black holes in a higher curvature theory, and mention how these results extend to black hole chemistry at the end of the section.

Black hole entropy in general relativity is simply proportional to the area of horizon cross-sections. In higher curvature theories of gravity, the area law receives corrections. Wald provided a derivation of black hole entropy in an arbitrary theory of gravity [114, 115] that gives a simple recipe for calculating the corrections. The derivation, which we will not present here, is in the same spirit as the Hamiltonian method described in the introduction for general relativity. The result is that black hole entropy is the Noether charge associated with the diffeomorphism invariance of the Lagrangian. Explicitly, one can directly calculate the entropy in the following way:[16]

$$S = -\frac{2\pi}{G_N}\oint_B d^{D-2}x\sqrt{-\sigma}P^{\alpha\beta\mu\nu}\hat{\varepsilon}_{\alpha\beta}\hat{\varepsilon}_{\mu\nu}\,, \tag{4.19}$$

which explains why we have referred to $P^{\alpha\beta\mu\nu}$ as the 'entropy tensor'. Here the integral is performed over a cross-section of the horizon, and $\hat{\varepsilon}_{\mu\nu}$ is the horizon binormal, normalized so that $\hat{\varepsilon}_{\mu\nu}\hat{\varepsilon}^{\mu\nu} = -2$; for a static black hole we will just have $\hat{\varepsilon}_{\mu\nu} = 2r_{[\mu}n_{\nu]}$ with n^ν the unit timelike normal to the horizon and r^μ the unit spacelike normal to the horizon.

The problem of conserved charges in generic higher curvatures theories was nicely addressed by Deser and Tekin [199] and Sentürk, Şişman, and Tekin [200]. The approach takes advantage of the results of [199] which addresses the problem of conserved charges in gravity theories involving up to quadratic powers of curvature. To define the conserved charges in higher-order theories, the following fact is used. When the action of an arbitrary higher-order theory is expanded to second order in a metric perturbation, the theory can be mapped into an equivalent action that involves only up to quadratic powers of curvature. The quadratic theory is taken to have an action

$$S = \int d^Dx\sqrt{-g}\left[\lambda(R-2\Lambda_0) + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma\left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right)\right]\,, \tag{4.20}$$

and the couplings in this action can be matched directly to the constants that characterize an arbitrary higher curvature theory at the linear level [195]

$$\lambda = 2e - 4K\left[a + bD(D-1) + c(D-1)\right]\,,\quad \alpha = 2b - a\,,\quad \beta = 4a + 2c\,,\quad \gamma = a\,. \tag{4.21}$$

If $Q_{\rm Ein}[\xi]$ is a conserved charge in Einstein gravity, then the conserved charge in the higher-order gravity will be given by

$$Q_{\mathcal{L}}[\xi] = \left(\lambda + 2D(D-1)K\alpha + 2(D-1)K\beta + 2(D-3)(D-4)K\gamma\right)Q_{\rm Ein}[\xi]\,, \tag{4.22}$$

[16]Note that while, for a given Lagrangian, the Wald entropy assigns a unique entropy to a black hole, there would be ambiguities applying it more generally. For example, any term added to the Wald entropy proportional to the extrinsic curvature would yield the same black hole entropy (the extrinsic curvature of the horizon vanishes), but would differ elsewhere.

where K is the curvature length scale of a maximally symmetric vacuum in the higher curvature theory determined by the polynomial (4.9). In the case where the theory propagates neither the scalar mode nor the massive, ghost-like graviton, then the relationship between the charges becomes quite simple:

$$Q_{\mathcal{L}}[\xi] = \frac{G_N}{G_{N\text{eff}}} Q_{\text{Ein}}[\xi] \,. \tag{4.23}$$

In other words, essentially the only change is that Newton's constant is replaced with the effective Newton's constant for the theory.

With the appropriately defined conserved charges, black holes in a higher curvature theory will satisfy the first law of black hole mechanics. The temperature that appears in the first law can be obtained in the same way as before: either by identifying it as the surface gravity divided by 2π, or, equivalently, obtaining it via regularity of the Euclidean sector. Smarr formulae can also be defined for higher curvature theories, but there are differences. Since the coupling constants of the higher curvature terms are dimensionful, a consistent scaling relation demands they must appear in the Smarr formula. This 'black hole chemistry' approach for a higher-order gravity results in a first law and Smarr formula of the following schematic form:

$$\begin{aligned} \delta M &= T\delta S + V\delta P + \sum_n \sum_i \Psi_n^{(i)} \delta\lambda_n^{(i)} \,, \\ (D-3)M &= (D-2)TS - 2PV + \sum_n \sum_i (2n-2)\lambda_n^{(i)} \Psi_n^{(i)} \,, \end{aligned} \tag{4.24}$$

where $\lambda_n^{(i)}$ are the coupling constants for terms of order n in the curvature, and $\Psi_n^{(i)}$ are the conjugate potentials. In the same sense as the thermodynamic volume, the quantities that are conjugate to the couplings can be understood as arising from higher-order Komar potentials [37, 201].

4.2 Classification of theories: spherical symmetry

Having provided some essential background material, in this section we are going to address a simple question: when does a theory of gravity admit solutions that are natural extensions of the Schwarzschild solution? We will see that the answer leads naturally to gravitational theories that are particularly well-suited as toy models for addressing questions about black hole physics, thermodynamics and holography. Unless otherwise indicated, for the remainder of this chapter, it should be assumed that we are always discussing spherically symmetric metrics, as described below.

First, let us clarify what is meant by 'natural extensions of the Schwarzschild solution'. We are interested in vacuum, static and spherically symmetric solutions. The most general static, spherically symmetric spacetime can be cast in the form[17]

$$ds^2 = -N^2(r)f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma^2_{k,D-2} \tag{4.25}$$

where $d\Sigma^2_{k,D-2}$ is the metric on a $(D-2)$-dimensional space of constant curvature k.[18] The Schwarzschild solution has the property that one is free to set $N(r) = 1$. Mathematically, this freedom can be traced back to Einstein gravity having a single independent field equation under the constraint of spherical symmetry [202]. So a more mathematically precise phrasing of the question is: "What theories of gravity have a single independent field equation for a static, spherically symmetric vacuum?" Further, we will restrict our attention to theories that reduce to Einstein gravity as some parameter in the action is sent to zero.

The question as now posed eliminates certain examples that spring to mind. For example, it is well-known that the standard Schwarzschild solution can be embedded in $f(R)$ and Weyl^2 gravity. However, neither of these theories meet our criteria. In the case of $f(R)$ gravity, there are two independent field equations in the most general case — it is only after one places some constraints on the metric function that the $N = 1$ solution is permitted. Second, in the case of Weyl^2, the field equations for pure Weyl^2 gravity do admit the single metric function solution in general [203]. However, in this case the property is spoiled by demanding an Einstein gravity limit, i.e. by adding the Ricci scalar to the Lagrangian.

However, this does not mean that, for example, Weyl^2 gravity is a theory of type we are looking for. This theory has two independent field equations, and moreover does not reduce to Einstein gravity in any limit.

[17] Let us clarify a minor technicality regarding spherically symmetric spacetimes. The metric on a spherically symmetric spacetime can always be decomposed as

$$ds^2 = d\gamma^2 + r^2 d\Omega^2_{D-2}$$

where $d\Omega^2_{D-2}$ is a metric on a $(D-2)$ dimensional sphere and $d\gamma^2$ is a metric on a two-dimensional manifold. The parameter r that appears multiplying the metric of the sphere is a scalar function on the two-dimensional manifold with metric $d\gamma^2$. Provided r is a good coordinate, i.e. that $\nabla r \neq 0$, the metric on $d\gamma^2$ can be decomposed as described. We will assume that this is possible throughout this chapter.

[18] This is an abuse of language that is often used in the literature — referring to this type of metric as 'spherically symmetric' even when the transverse geometry is not a sphere.

To address the question, we consider a theory of gravity that is built from the metric and Riemann tensor with bulk action

$$S = \frac{1}{16\pi G_N} \int d^D x \mathcal{L}\left(g_{\mu\nu}, R_{\mu\nu\rho\sigma}\right) \,, \tag{4.26}$$

where we allow for the possibility of matter contributions but do not write them explicitly.

We will refer to the explicit field equations on several occasions below, but let us note that there is a much more efficient method for obtaining field equations under certain symmetry restrictions. This method, which is sometimes referred to as the 'Weyl method' in the literature, involves evaluating the action for a specific choice of metric and then varying the action with respect to the metric functions to obtain the field equations [204, 205]. Substituting the ansatz (4.25) into the action, it is possible to integrate over the transverse directions giving the following schematic result

$$S = \frac{\Omega_{k,D-2}}{16\pi G_N} \int dt dr r^{D-2} N \mathcal{L}\left[f, N\right] \tag{4.27}$$

where the Lagrangian density is now identified as a functional of f and N, and we have suppressed their r-dependence for convenience. The field equations can be obtained simply by varying the action with respect to N and f. The results are

$$\frac{\delta S}{\delta N} = \frac{\Omega_{k,D-2} r^{D-2}}{16\pi G_N} \frac{2\mathcal{E}_{tt}}{f N^2} \,, \quad \frac{\delta S}{\delta f} = \frac{\Omega_{k,D-2} r^{D-2}}{16\pi G_N} \frac{\mathcal{E}^t{}_t - N \mathcal{E}^r{}_r}{f} \,. \tag{4.28}$$

The second variation above gives a simple criteria for when the theory has a single independent field equation: If $\delta S/(\delta f) = 0$ upon setting $N = 1$ there is only single independent field equation. To concretely illustrate the idea, let us now apply this procedure to theories up to cubic order in curvature. We will do this in an 'exploratory' manner, mentioning the various theories that result as we go along. At the end, we will summarize and provide greater detail on the particular theories that have resulted.

The first interesting case is quadratic gravity, where there are three invariants that can be combined to build an action

$$\mathcal{Q}_1 = R^2 \,, \quad \mathcal{Q}_2 = R_{\alpha\beta} R^{\alpha\beta} \,, \quad \mathcal{Q}_3 = R_{\alpha\beta\sigma\rho} R^{\alpha\beta\sigma\rho} \,. \tag{4.29}$$

The Lagrangian density of the quadratic part of the theory would then be given by

$$\mathcal{L}_{\rm quad} = c_1 \mathcal{Q}_1 + c_2 \mathcal{Q}_2 + c_3 \mathcal{Q}_3 \,, \tag{4.30}$$

for some constants c_1, c_2 and c_3. To implement the procedure just described, we need these invariants in the case where $N=1$. They are quite easily computed to be

$$
\begin{aligned}
\mathcal{Q}_1 &= \left[f'' + \frac{2(D-2)f'}{r} - \frac{(D-2)(D-3)(k-f)}{r^2}\right]^2 ,\\
\mathcal{Q}_2 &= \frac{1}{2}\left(f'' + \frac{(D-2)f'}{r}\right)^2 + (D-2)\left(-\frac{f'}{r} + \frac{(D-3)(k-f)}{r^2}\right)^2 ,\\
\mathcal{Q}_3 &= (f'')^2 + 2(D-2)\left(\frac{f'}{r}\right)^2 + 2(D-2)(D-3)\left(\frac{k-f}{r^2}\right)^2 .
\end{aligned}
\tag{4.31}
$$

We now must compute the Euler-Lagrange derivatives of $r^{D-2}\mathcal{Q}_i$, with the results,

$$
\begin{aligned}
\delta_f\left(r^{D-2}\mathcal{Q}_1\right) =& \, r^{D-6}\left[2r^4 f^{(4)} + (D-2)\left(4r^3 f^{(3)} + 2r^2(D-7)f'' - 8r(D-4)f'\right.\right.\\
&\left.\left. -12(D-3)(k-f)\right)\right] ,\\
\delta_f\left(r^{D-2}\mathcal{Q}_2\right) =& \, r^{D-6}\left[r^4 f^{(4)} + (D-2)\left(2r^3 f^{(3)} + r^2(D-6)f'' - 3r(D-4)f'\right.\right.\\
&\left.\left. -4(D-3)(k-f)\right)\right] ,\\
\delta_f\left(r^{D-2}\mathcal{Q}_3\right) =& \, r^{D-6}\left[2r^4 f^{(4)} + (D-2)\left(4r^3 f^{(3)} + 2r^2(D-5)f'' - 4r(D-4)f'\right.\right.\\
&\left.\left. -4(D-3)(k-f)\right)\right] .
\end{aligned}
\tag{4.32}
$$

To ensure that the theory possesses a single independent field equation, we must demand that

$$
\delta_f\left(r^{D-2}\left[c_1\mathcal{Q}_1 + c_2\mathcal{Q}_2 + c_3\mathcal{Q}_3\right]\right) = 0 \tag{4.33}
$$

for all possible choices of f. Explicitly, this constraint takes the form

$$
\begin{aligned}
0 =& -4(D-2)(D-3)\left(3c_1 + c_2 + c_3\right)(k-f) - (D-2)(D-4)\left(8c_1 + 3c_2 + 4c_3\right) r f'\\
&+ 2(D-2)\left((D-7)c_1 + \frac{1}{2}(D-6)c_2 + (D-5)c_3\right) r^2 f''\\
&+ 2(D-2)(2c_1 + c_2 + 2c_3) r^3 f^{(3)} + (2c_1 + c_2 + 2c_3) r^4 f^{(4)} .
\end{aligned}
\tag{4.34}
$$

Demanding that the coefficients of each term in the above vanishes, we see that there are two possible solutions:

$$
\{c_1 = c_3\,, c_2 = -4c_3\} \quad \text{and} \quad \{D = 2\,, c_2 = -2(c_1 + c_3)\} . \tag{4.35}
$$

The first solution, which has a single free parameter c_3, is valid in all dimensions and gives (unsurprisingly) the Gauss-Bonnet density, $\mathcal{Q}_{\rm GB} = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 4R_{\mu\nu}R^{\mu\nu} + R^2$. The second solution is valid only in $D = 2$, and it corresponds to a 'trivial theory' that vanishes identically for the static spherically symmetric ansatz. While uninteresting, the second solution highlights the fact that in general we can expect the results to depend on the particular spacetime dimension.

Having determined the constraints on the constants appearing in the Lagrangian density, we can now go back and compute the field equations. To obtain the non-trivial field equation, we need the curvature tensors with N explicitly included. These are a bit messier, and so we do not explicitly present them. Adding back in the Einstein-Hilbert term, so that the Lagrangian density reads $\mathcal{L} = -2\Lambda + R + c_3 (\mathcal{Q}_1 - 4\mathcal{Q}_2 + \mathcal{Q}_3)$, we find the following,

$$\delta_N \left(\frac{r^{D-2} N \mathcal{L}}{16\pi G_N} \right) \Big|_{N=1} = \frac{(D-2)}{16\pi G_N} \Bigg[r^{D-1} \Bigg(\frac{-2\Lambda}{(D-1)(D-2)} + \frac{(k-f)}{r^2} + c_3 (D-3)(D-4) \left(\frac{k-f}{r^2} \right)^2 \Bigg) \Bigg]' , \tag{4.36}$$

where the prime denotes a derivative with respect to r. The fact that the field equation is a total derivative is not an accident — as we will discuss below, this is one of the properties that are generic to this class of gravity theories.

The quadratic theory is the only one for which it is useful to discuss the construction explicitly — at higher orders in the curvature, the expressions are much messier and very little insight can be gained from the intermediate steps. So let us now consider this construction at cubic order in curvature, presenting only the most relevant details. At cubic order, there are eight possible independent invariants that can be constructed from the metric and Riemann tensor, given by [206]

$$\begin{aligned} \mathcal{C}_1 &= R_{\alpha}{}^{\sigma}{}_{\beta}{}^{\rho} R_{\sigma}{}^{\mu}{}_{\rho}{}^{\nu} R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta} , \quad \mathcal{C}_2 = R_{\alpha\beta}{}^{\sigma\rho} R_{\sigma\rho}{}^{\mu\nu} R_{\mu\nu}{}^{\alpha\beta} , \quad \mathcal{C}_3 = R_{\alpha\beta\sigma\rho} R^{\alpha\beta\sigma}{}_{\mu} R^{\rho\mu} , \\ \mathcal{C}_4 &= R_{\alpha\beta\sigma\rho} R^{\alpha\beta\sigma\rho} R , \quad \mathcal{C}_5 = R_{\alpha\beta\sigma\rho} R^{\alpha\sigma} R^{\beta\rho} , \quad \mathcal{C}_6 = R_{\alpha}{}^{\beta} R_{\beta}{}^{\sigma} R_{\sigma}{}^{\alpha} , \\ \mathcal{C}_7 &= R_{\alpha}{}^{\beta} R_{\beta}{}^{\alpha} R , \quad \mathcal{C}_8 = R^3 . \end{aligned} \tag{4.37}$$

These curvature invariants can be computed in arbitrary dimensions,[19] which we do first with $N = 1$. We consider the following cubic Lagrangian density

$$\mathcal{L}_{\text{cubic}} = \sum_{i=1}^{8} c_i \mathcal{C}_i , \tag{4.38}$$

[19] Efficient techniques for doing this computation by hand are given in [207].

and perform the variation with respect to f, yielding the constraint equations to enforce a single independent field equation. The number of parameters fixed by the constraint equations depends on the spacetime dimension. Specifically, the situation differs depending on whether $D > 4$ or $D \leq 4$. Here we will discuss the four-dimensional case, and then proceed to the more general case in higher dimensions.

In four dimensions there is the following four-parameter family of solutions

$$\begin{aligned} c_4 &= \frac{3c_1 - 36c_2 - 14c_3}{56}, \quad c_5 = -\frac{3c_1 + 48c_2 + 14c_3}{7}, \\ c_7 &= \frac{6c_1 + 96c_2 + 14c_3 - 21c_6}{28}, \quad c_8 = \frac{-3c_1 - 20c_2 + 7c_6}{56}. \end{aligned} \tag{4.39}$$

What this means is that there is a four-parameter family of gravitational theories that admit single metric function solutions; we now chose a convenient 'basis' for this 'theory space'. We find that the choice

$$c_1 = -8, \quad c_2 = 4, \quad c_3 = -24, \quad c_6 = 16 \tag{4.40}$$

gives the six-dimensional Euler density (i.e. the cubic Lovelock density), which vanishes identically in $D < 6$ due to the Schouten identities. The remaining three interactions are given by the following convenient choices of coefficients:

$$\begin{aligned} \mathcal{P}: &\quad c_1 = 12, \ c_2 = 1, \quad c_3 = 0, \ c_6 = 8. \\ \mathcal{T}_1: &\quad c_1 = 0, \quad c_2 = 0, \quad c_3 = 1, \quad c_6 = 0. \\ \mathcal{T}_2: &\quad c_1 = 0, \quad c_2 = 0, \quad c_3 = 0, \quad c_6 = 1. \end{aligned} \tag{4.41}$$

Here the various terms are given by the following cubic densities:

$$\begin{aligned} \mathcal{P} &= 12R_{\alpha}{}^{\beta}{}_{\mu}{}^{\nu}R_{\beta}{}^{\sigma}{}_{\nu}{}^{\rho}R_{\sigma}{}^{\alpha}{}_{\rho}{}^{\mu} + R_{\alpha\beta}^{\mu\nu}R_{\mu\nu}^{\sigma\rho}R_{\sigma\rho}^{\alpha\beta} - 12R_{\alpha\beta\mu\nu}R^{\alpha\mu}R^{\beta\nu} + 8R_{\alpha}^{\beta}R_{\beta}^{\mu}R_{\mu}^{\alpha}, \\ \mathcal{T}_1 &= \tfrac{1}{2}R_{\alpha}{}^{\beta}R_{\beta}{}^{\alpha}R - 2R^{\alpha\mu}R^{\beta\nu}R_{\alpha\beta\mu\nu} - \tfrac{1}{4}RR_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} + R^{\nu\sigma}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu}{}_{\sigma}, \\ \mathcal{T}_2 &= R_{\alpha}{}^{\beta}R_{\beta}{}^{\mu}R_{\mu}{}^{\alpha} - \tfrac{3}{4}R_{\alpha}{}^{\beta}R_{\beta}{}^{\alpha}R + \tfrac{1}{8}R^3. \end{aligned} \tag{4.42}$$

Here $\mathcal{T}_1$ and $\mathcal{T}_2$ are two 'trivial theories', in the sense that they produce vanishing field equations under the constraint of spherical symmetry. The $\mathcal{P}$ term is quite interesting — it coincides with the Einsteinian Cubic Gravity (ECG) term that was recently constructed by Bueno and Cano [121] by completely different methods.[20] While the Lagrangian density

[20] In that work, Bueno and Cano were constructing the most general theory of gravity that propagates only the usual massless spin-2 graviton on the vacuum and has dimension independent coefficients in the Lagrangian. The fact that it admits four-dimensional single metric function solutions under the constraint of spherical symmetry was only noticed afterward [4, 157]

for ECG is the same in all dimensions, it is only in four dimensions that the theory admits single metric function solutions, as was noticed in [4, 157]. In four-dimensions, ECG can be considered the most general (non-trivial) cubic theory that has this property, and it is also the simplest — we will have more to say about ECG below.

As mentioned, the situation is different for $D \geq 4$. We find that only three couplings c_i are independent; we choose these to be c_1, c_2 and c_3, while the others are constrained in the following way:

$$\begin{aligned} c_4 &= \frac{3}{8(2D-1)}c_1 + \frac{6+6D-3D^2}{2(D-2)(2D-1)}c_2 + \frac{1+2D-D^2}{2(D-2)(2D-1)}c_3\,, \\ c_5 &= \frac{-3}{(2D-1)}c_1 + \frac{-48+36D-12D^2}{(D-2)\,(2\,D-1)}c_2 + -\frac{4(3-3D+D^2)}{(D-2)\,(2\,D-1)}c_3\,, \\ c_6 &= \frac{4}{2D-1}c_1 + \frac{8(8-5\,D)}{(D-2)\,(2\,D-1)}c_2 + \frac{2(18-7\,D-2\,D^2)}{3\,(D-2)\,(2\,D-1)}c_3\,, \\ c_7 &= \frac{-3}{2\,(2\,D-1)}c_1 + \frac{2(D^2+6D-12)}{(D-2)\,(2\,D-1)}c_2 + \frac{2(D^2-2)}{(D-2)\,(2\,D-1)}c_3\,, \\ c_8 &= \frac{1}{8(2D-1)}c_1 + \frac{6-6\,D-D^2}{2\,(D-2)\,(2\,D-1)}c_2 - \frac{(D-1)\,(D+3)}{6\,(D-2)\,(2\,D-1)}c_3\,. \end{aligned} \tag{4.43}$$

Since there are three free parameters, $\{c_1, c_2, c_3\}$, the resulting theory is a linear combination of three independent cubic densities, for which we now seek a convenient basis. We find that choosing

$$c_1 = -8\,, \quad c_2 = 4\,, \quad c_3 = -24 \tag{4.44}$$

produces the six-dimensional Euler density, $\mathcal{X}_6$, which is topological in six dimensions and vanishes identically for $D \leq 5$. Meanwhile choosing

$$c_1 = 1\,, \quad c_2 = 0\,, \quad c_3 = -\frac{3(D-2)}{(2D-3)(D-4)}\,, \tag{4.45}$$

produces the quasi-topological density

$$\begin{aligned} \mathcal{Z}_D &= R_{\alpha}{}^{\beta}{}_{\mu}{}^{\nu} R_{\beta}{}^{\sigma}{}_{\nu}{}^{\rho} R_{\sigma}{}^{\alpha}{}_{\rho}{}^{\mu} + \frac{1}{(2D-3)(D-4)}\Big(\frac{3(3D-8)}{8} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} R \\ &\quad -\frac{3(3D-4)}{2} R_{\alpha}{}^{\mu} R_{\mu}{}^{\alpha} R - 3(D-2) R_{\alpha\mu\beta\nu} R^{\alpha\mu\beta}{}_{\sigma} R^{\nu\sigma} + 3D R_{\alpha\mu\beta\nu} R^{\alpha\beta} R^{\mu\nu} \\ &\quad +6(D-2) R_{\alpha}{}^{\mu} R_{\mu}{}^{\beta} R_{\beta}{}^{\alpha} + \frac{3D}{8} R^3\Big)\,. \end{aligned} \tag{4.46}$$

The gravitational theory described by this term was discovered independently by Oliva and Ray [117] and Myers and Robinson [118]. We will discuss some of its properties below.[21]

Since there is an extra free parameter, we obtain an additional independent cubic density which shares the property of permitting a solution with a single metric function. The new term could be obtained by setting $c_3 = 1$ and $c_1 = c_2 = 0$, but a more convenient choice (motivated by the Einsteinian theory in four dimensions) is

$$c_1 = 14\,, \quad c_2 = 0\,, \quad c_3 = 2\,, \tag{4.49}$$

which selects the following Lagrangian density,

$$\begin{aligned}\mathcal{S}_D \;=\;& 14 R_{\alpha\ \ \mu}^{\ \sigma\ \ \rho} R^{\alpha\beta\mu\nu} R_{\beta\sigma\nu\rho} + 2 R^{\alpha\beta} R_{\alpha}{}^{\mu\nu\sigma} R_{\beta\mu\nu\sigma} - \frac{4(66 - 35D + 2D^2)}{3(D-2)(2D-1)} R_{\alpha}{}^{\mu} R^{\alpha\beta} R_{\beta\mu} \\ & - \frac{2(-30 + 9D + 4D^2)}{(D-2)(2D-1)} R^{\alpha\beta} R^{\mu\nu} R_{\alpha\mu\beta\nu} - \frac{(38 - 29D + 4D^2)}{4(D-2)(2D-1)} R R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \\ & + \frac{(34 - 21D + 4D^2)}{(D-2)(2D-1)} R_{\alpha\beta} R^{\alpha\beta} R - \frac{(30 - 13D + 4D^2)}{12(D-2)(2D-1)} R^3\,. \end{aligned} \tag{4.50}$$

The choices (4.49) lead to the following relationship in four dimensions:

$$\mathcal{S}_4 + \frac{1}{4}\mathcal{X}_6 + 4\mathcal{T}_1 = \mathcal{P}\,. \tag{4.51}$$

Since $\mathcal{X}_6$ vanishes identically in four dimensions and $\mathcal{T}_1$ makes no contribution to the field equations, we see that in four dimensions the theory given by the Lagrangian density $\mathcal{S}_D$ (4.50) yields the same field equations as Einsteinian Cubic Gravity. However, we note that there is no choice of c_1, c_2 and c_3 such that the theory reduces precisely to ECG in four dimensions — mathematically, this is related to the fact that the constraint equations are fundamentally different in structure in four compared to higher dimensions.

The discussion of the cubic theory provides all the additional insight required to make some general comments. In general we can expect three types of theory to appear in this

[21] Note that the expression $\mathcal{Z}'_D$ from [118] can be obtained by choosing

$$c_1 = 0\,, \quad c_2 = 1\,, \quad c_3 = -\frac{12(D^2 - 5D + 5)}{(2D-3)(D-4)}\,. \tag{4.47}$$

However, this term is not independent from the quasi-topological term and the six-dimensional Euler density, but rather [118]

$$\mathcal{X}_6 = 4\mathcal{Z}'_D - 8\mathcal{Z}_D\,, \tag{4.48}$$

and so $\mathcal{Z}'_D$ does not provide the third independent invariant we are searching for here.

construction: the Lovelock class of theories, quasi-topological gravities, and a new class of Lagrangians that generalize some of the nice properties that Lovelock and quasi-topological gravities share — we will call these generalized quasi-topological gravities. While we have only presented the analysis for the quadratic and cubic class of theories, this classification seems general. It was shown to be true for quartic theories in arbitrary dimension in [6] and (at least) up to tenth order in curvature in four dimensions in [158]. In forthcoming work, examples up to tenth order in curvature are presented in five dimensions [11]. In the following subsections we will discuss at more length the three classes of theories, along with their field equations and general properties.

4.2.1 Lovelock gravity

We have seen that in both the quadratic and cubic cases, one class of objects selected by the single metric function criterion is the Euler densities. These objects comprise Lovelock gravity [116], which is the most well-known and well-studied higher curvature theory of gravity. In some ways, Lovelock gravity can be considered the natural generalization of Einstein gravity to higher dimensions. As we will describe, Lovelock theory is constructed from the Euler densities. The n^{th} Euler density involves n powers of curvature and integrates to the Euler characteristic of $2n$-dimensional manifolds. In this sense, n^{th} Euler density is a topological term in $D = 2n$ and does not contribute to the equations of motion. However, if we *dimensionally continue* the Euler density, and include it in the action for $D > 2n$ then it contributes non-trivially to the dynamics.

In D dimensions, the Lovelock action will include terms up to $\lfloor (D-1)/2 \rfloor$ in curvature. Since the Ricci scalar is the Euler density for two-dimensional manifolds, in four dimensions the Lovelock action reduces to the Einstein-Hilbert action with a cosmological constant, while in higher dimensions it will include additional terms. A particularly nice feature of Lovelock gravity is that the equations of motion are always second order for any metric. In fact, Lovelock theory is the unique theory that has this property for any metric. It is in these aspects that Lovelock theory can be considered the natural extension of general relativity to higher dimensions. Let us consider spherically symmetric solutions of the theory in more detail.

The bulk action of Lovelock gravity is given by

$$S = \frac{1}{16\pi G_N} \int_{\mathcal{M}} d^D x \sqrt{|g|} \mathcal{L} \tag{4.52}$$

where $\mathcal{L}_{\text{Lovelock}}$ is the Lovelock Lagrangian density [116],

$$\mathcal{L} = \left[\frac{(D-1)(D-2)}{L^2} + R + \sum_{n=2}^{\lfloor (D-1)/2 \rfloor} \lambda_n \frac{(D-2n-1)!}{(D-3)!} (-1)^n \mathcal{X}_{2n} \right] \tag{4.53}$$

where the λ_n are coupling constants with length dimension $2n-2$. The Euler densities $\mathcal{X}_{2n}$ are given by

$$\mathcal{X}_{2n} = \frac{1}{2^n} \delta^{\mu_1 \dots \mu_{2n}}_{\nu_1 \dots \nu_{2n}} R^{\nu_1 \nu_2}_{\mu_1 \mu_2} \dots R^{\nu_{2n-1} \nu_{2n}}_{\mu_{2n-1} \mu_{2n}} \,, \tag{4.54}$$

with the generalized Kronecker symbol given as

$$\delta^{\mu_1 \mu_2 \dots \mu_r}_{\nu_1 \nu_2 \dots \nu_r} = r! \delta^{[\mu_1}_{\nu_1} \delta^{\mu_2}_{\nu_2} \dots \delta^{\mu_r]}_{\nu_r} \,. \tag{4.55}$$

Vacuum spherically symmetric black holes in Lovelock gravity, first studied by Wheeler, are now well-known, see for example [208, 209, 43], and in general the solution takes the form

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma^2_{k,D-2} \,, \tag{4.56}$$

i.e. is characterized by a single metric function. There is a single independent field equation and it is a total derivative,

$$\left[r^{D-1} h \left(\frac{L^2 (f(r) - k)}{r^2} \right) \right]' = 0 \,, \tag{4.57}$$

which can be directly integrated. A remarkable fact about Lovelock gravity is that, after integrating the field equation, the function f satisfies an algebraic rather than differential equation,

$$h \left(\frac{L^2 (f(r) - k)}{r^2} \right) = \frac{\omega^{D-3} L^2}{r^{D-1}} \,, \tag{4.58}$$

with $h(x)$ given by the polynomial function

$$h(x) = 1 - x + \sum_{n=2}^{\lfloor (D-1)/2 \rfloor} \frac{\lambda_n}{L^{2n-2}} x^n \,. \tag{4.59}$$

In these expressions, ω is an integration constant related to the conserved mass M

$$\omega^{D-3} = \frac{16 \pi G_N M}{(D-2) \Omega_{k,D-2}} \tag{4.60}$$

where $\Omega_{k,D-2}$ is the (dimensionless) volume of the transverse space. Note that since the field equations are algebraic and the mass is the only constant of integration, Lovelock black holes are characterized only by their mass.

An immediate consequence of the fact that the field equations are second order for any metric is that Lovelock gravity matches Einstein gravity at the linearized level. For example, the equations of motion for a perturbation about a maximally symmetric background take the form,

$$\mathcal{E}^L_{\alpha\beta} = \frac{1}{2}h'(f_\infty)G^L_{\alpha\beta} \tag{4.61}$$

where $G^L_{\alpha\beta}$ is the Einstein tensor linearized on a space of constant curvature, while $h'(f_\infty)$ is the derivative of the polynomial $h(x)$ evaluated for a space of curvature radius $L/\sqrt{f_\infty}$. The metric function for such a space, at large r, reads

$$f(r) \sim f_\infty \frac{r^2}{L^2}\,. \tag{4.62}$$

Since the higher derivative contributions to the linearized equations vanish, we conclude that Lovelock theory propagates only the usual massless graviton.

A nice aspect of Lovelock gravity is that we have an expression for the Lagrangian density in arbitrary dimensions. However, it is useful to think for a moment of how one could 'discover' Lovelock gravity if this expression was not known. Recalling the general structure of the field equations for a generalized theory of gravity,

$$\mathcal{E}_{\alpha\beta} = P_\alpha{}^{\sigma\rho\delta}R_{\beta\sigma\rho\delta} - \frac{1}{2}g_{\alpha\beta}\mathcal{L} - 2\nabla^\rho\nabla^\sigma P_{\alpha\rho\sigma\beta} = 8\pi G_N T_{\alpha\beta}\,, \tag{4.63}$$

the Lovelock theories could be constructed in the following way. Since these theories have second order equations of motion for any metric, it must be the case that $\nabla^\rho\nabla^\sigma P_{\alpha\rho\sigma\beta} = 0$. In fact, it turns out that Lovelock theories satisfy the condition that $P_{\alpha\rho\sigma\beta}$ is divergenceless on any index, and imposing the slightly weaker condition involving second derivatives leads to nothing new [210]. We will apply some of this insight for quasi-topological gravity in the next section.

4.2.2 Quasi-topological gravity

Another class of theories that have been selected by the single metric function condition is quasi-topological gravity. Cubic quasi-topological gravity was discovered independently by Oliva and Ray [211] and Myers and Robinson [118]. The theory as presented by Myers and Robinson is non-trivial in $D = 5$ and also in $D \geq 7$; in six dimensions, the theory is trivial.[22] A quartic generalization was provided shortly after [212], and more recently a quintic theory valid in $D = 5$ was presented [213].[23]

The advantage of quasi-topological gravity is that theories of any order in curvature exist in $D = 5$ and have second order equations of motion for spherical geometries. Contrast this with Lovelock gravity, where to have curvature of order n the spacetime dimension must be at least $2n+1$. As a result, the quasi-topological theories provide simple theories to study the effects of higher curvature corrections in five dimensions. Of course, nothing comes for free and the tradeoff here is that if the assumption of spherical symmetry is relaxed, then the field equations will generically be fourth order. Nonetheless, quasi-topological gravity has been used successfully as a toy model in holography and beyond [214, 96, 16].

Let us consider the properties of cubic quasi-topological gravity before making some general remarks. The action for cubic quasi-topological gravity is

$$S = \frac{1}{16\pi G_N} \int d^D x \sqrt{|g|} \left[\frac{(D-1)(D-2)}{L^2} + R - \frac{8(2D-3)}{(D-6)(D-3)(3D^2-15D+16)} \mu \mathcal{Z}_D \right] , \tag{4.64}$$

with $\mathcal{Z}_D$ given by eq. (4.46) above. The field equations are most easily computed by substituting a general spherically symmetric metric ansatz and varying the action with respect to N. This produces the following equation of motion:

$$\left[r^{D-1} \left(1 - \frac{L^2(f-k)}{r^2} + \frac{\mu}{L^4} \left(\frac{L^2(f-k)}{r^2} \right)^3 \right) \right]' = 0, . \tag{4.65}$$

The field equation can be directly integrated, giving

$$\tilde{h} \left(\frac{L^2(f-k)}{r^2} \right) = \frac{\omega^{D-3} L^2}{r^{D-1}} . \tag{4.66}$$

where we have defined

$$\tilde{h}(x) = 1 - x + \frac{\mu}{L^4} x^3 , \tag{4.67}$$

and ω is an integration constant related to the mass:

$$\omega^{D-3} = \frac{16\pi G_N M}{(D-2)\Omega_{k,D-2}} . \tag{4.68}$$

[22] It is for this reason that the theory was called 'quasi-topological' gravity, since in six dimensions it behaves like a topological invariant for certain metrics. Of course, it is not actually a topological invariant, and does not vanish for sufficiently complicated six dimensional geometries.

[23] Examples of quasi-topological theories up to 10th order in curvature in five dimensions have been found in unpublished work by the author [11].

The most remarkable property of the field equation is that it is *identical* to the equivalent equation in cubic Lovelock gravity, but here it is also valid in $D = 5$. As a consequence, we see that similar properties to the Lovelock case carry over: the black holes are described entirely by their mass.

An unexpected and non-trivial property of quasi-topological gravity that was noted in [118] is that the linearized equations of motion for graviton perturbations about a maximally symmetric space are second order. This is quite remarkable since the theory itself generically has fourth order equations. Explicitly, the linearized equations of motion are of the same form as in Lovelock gravity,

$$\mathcal{E}^L_{\alpha\beta} = \frac{1}{2}\tilde{h}'(f_\infty)G^L_{\alpha\beta}\,, \tag{4.69}$$

where f_∞ satisfies the embedding equation

$$1 - f_\infty + \frac{\mu}{L^4}f_\infty^3 = 0\,. \tag{4.70}$$

Our discussion has focused on the field equations of cubic quasi-topological gravity, but the similarities with Lovelock theory generalizes. In all known examples of quasi-topological theories, the metric function satisfies a polynomial equation identical to the Lovelock polynomial of the same order. Quasi-topological theories linearized about a maximally symmetric background give rise to second order equations of motion for the perturbation, and so propagate only the usual massless graviton.

Unfortunately, unlike Lovelock theory, very little is known about the general form of quasi-topological Lagrangians. For example, it is known that the density

$$\begin{aligned}\mathcal{QT}_n = {}& \frac{1}{2^n}\frac{1}{D-2n+1}\delta^{\mu_1\dots\mu_{2n}}_{\nu_1\dots\nu_{2n}}\left(C^{\nu_1\nu_2}_{\mu_1\mu_2}\dots C^{\nu_{2n-1}\nu_{2n}}_{\mu_{2n-1}\mu_{2n}} - R^{\nu_1\nu_2}_{\mu_1\mu_2}\dots R^{\nu_{2n-1}\nu_{2n}}_{\mu_{2n-1}\mu_{2n}}\right)\\ &- \alpha_n C^{\mu_n\nu_n}_{\mu_1\nu_1}C^{\mu_1\nu_1}_{\mu_2\nu_2}\cdots C^{\mu_{n-1}\nu_{n-1}}_{\mu_n\nu_n}\end{aligned} \tag{4.71}$$

where

$$\alpha_n = \frac{(D-4)!}{(D-2k+1)!}\frac{n(n-2)D(D-3)+n(n+1)(D-3)+(D-2n)(D-2n-1)}{(D-3)^{n-1}(D-2)^{n-1}+2^{n-1}-2(3-D)^{n-1}} \tag{4.72}$$

is a quasi-topological theory in $D = 2n-1$ [211]. However, beyond this no general closed form expressions are known. Some progress can be made by noting the following. All examples of quasi-topological theories presented to date satisfy the condition $\nabla_\alpha P^{\alpha\beta\mu\nu} = 0$ for a spherically symmetric metric. This condition is identical to that which defines Lovelock gravity, but restricted to spherically symmetric metrics. This motivates the following definition for quasi-topological theories:

Definition. *A theory of gravity with Lagrangian density* $\mathcal{L} = \mathcal{L}\left(g_{\alpha\beta}, R_{\alpha\beta\mu\nu}\right)$ *and action* $S = \int d^D x \sqrt{|g|}\mathcal{L}$ *belongs to the quasi-topological class if it is not a Lovelock theory and it satisfies both:*

$$\left.\frac{\delta S}{\delta f}\right|_{N=1} = 0 \qquad \nabla_\alpha P^{\alpha\beta\mu\nu} = 0 \tag{4.73}$$

for a spherically symmetric metric.

Using the definition, there is a simple check that one can do to see if a theory is quasi-topological: evaluate the action for a black brane (i.e. $k = 0$), and verify that

$$\begin{aligned} 0 &= \frac{\partial}{\partial r}\frac{\partial \mathcal{L}}{\partial f''} + \frac{(D-2)}{r}\frac{\partial \mathcal{L}}{\partial f''} - \frac{1}{2}\frac{\partial \mathcal{L}}{\partial f'}\,, \\ 0 &= \frac{\partial}{\partial r}\left(\frac{f}{r}\frac{\partial \mathcal{L}}{\partial f'}\right) + \left[\frac{(D-1)f}{r^2} - \frac{f'}{r}\right]\frac{\partial \mathcal{L}}{\partial f'} - \frac{2f}{r}\frac{\partial \mathcal{L}}{\partial f} \end{aligned} \tag{4.74}$$

which are just the trt and $x_i r x_i$ components of $\nabla_\alpha P^{\alpha\beta\mu\nu}$ (the other components vanish identically) written in terms of direct variations of the on-shell Lagrangian. Note that in deriving these equations we have implicitly used the condition $\delta S/(\delta f)|_{N=1} = 0$ to justify setting $N = 1$. The constraints in eq. (4.74) can be used to efficiently derive quasi-topological theories, and may be useful in determining a closed form expression for the quasi-topological Lagrangians.

4.2.3 Generalized quasi-topological gravity

In our construction of the most general cubic theory of gravity that satisfied the single metric function criterion, we found in addition to the Lovelock and quasi-topological theories a new Lagrangian density which we called $\mathcal{S}_D$. While this was just one example, carrying out the same procedure at higher order in curvature yields new Lagrangian densities that share a number of properties with $\mathcal{S}_D$, suggesting that these theories form a class of their own. We shall refer to these theories as 'generalized quasi-topological gravities' since they share a number of the same surprising properties of quasi-topological gravity, but are a bit more complicated. A particularly nice property of the generalized quasi-topological theories is that they are non-trivial even in $D = 4$, and as we will see, they are in some sense the 'nicest' higher curvature theories in four-dimensions. Going forward, we will use the following definition:

Definition. *A theory of gravity with Lagrangian density* $\mathcal{L} = \mathcal{L}\left(g_{\alpha\beta}, R_{\alpha\beta\mu\nu}\right)$ *and action* $S = \int d^D x \sqrt{|g|}\mathcal{L}$ *is a generalized quasi-topological gravity if it satisfies*

$$\left.\frac{\delta S}{\delta f}\right|_{N=1} = 0\,, \tag{4.75}$$

for a static, spherically symmetric metric and has non-trivial field equations, but is not a Lovelock or quasi-topological theory.

Next, let us consider the field equations for the cubic generalized quasi-topological theory. The action is

$$S = \frac{1}{16\pi G_N}\int d^D x\sqrt{-g}\Big(R - 2\Lambda + \frac{12\lambda(D-2)(2D-1)}{(D-3)(184-514D+291D^2-49D^3+4D^4)}\mathcal{S}_D\Big)\,, \tag{4.76}$$

which when varied with respect to N yields a field equation of the form

$$\mathcal{F}' = 0\,, \tag{4.77}$$

where $\mathcal{F}$ is a functional of f and its first two derivatives,

$$\begin{aligned}
\mathcal{F} \;=\; & (D-2)r^{D-3}\Big(k - f + \frac{r^2}{L^2}\Big) - \frac{12\lambda(D-2)}{(184-514D+291D^2-49D^3+4D^4)}\Big[(D^2+5D-15) \\
& \times\Big(\frac{4}{3}r^{D-4}f'^3 - 8r^{D-5}ff''\big(\frac{rf'}{2} + k - f\big) - 2r^{D-5}((D-4)f - 2k)f'^2 \\
& + 8(D-5)r^{D-6}ff'(f-k)\Big) \\
& - \frac{1}{3}(D-4)r^{D-7}(k-f)^2\Big(\big(-D^4 + \frac{57}{4}D^3 - \frac{261}{4}D^2 + 312D - 489\big)f \\
& + k\big(129 - 192D + \frac{357}{4}D^2 - \frac{57}{4}D^3 + D^4\big)\Big)\Big]\,.
\end{aligned} \tag{4.78}$$

The field equation can be directly integrated, giving

$$\mathcal{F} = (D-2)\omega^{D-3} \tag{4.79}$$

where ω is again a constant of integration that is related to mass in the following way:

$$\omega^{D-3} = \frac{16\pi G_N M}{(D-2)\Omega_{k,D-2}}\,. \tag{4.80}$$

Despite the very complicated structure of the field equations, let us note two remarkable properties. First, it is a total derivative that can be integrated once for free giving a constant of integration related to the mass. Second, the field equations are no longer algebraic: they reduce (after integration) to a non-linear second-order differential equation for the metric function f.

As mentioned, the theory defined by $\mathcal{S}_D$ shares a number of properties with the quasi-topological and Lovelock theories, including being ghost free, having black hole solutions characterized only by mass, field equations that can be directly integrated, and non-perturbative black hole thermodynamics. In the following section, we will show in detail that these properties, that are somewhat trivially true for Lovelock and quasi-topological gravity, are indeed true for the generalized quasi-topological class as well.

4.3 Properties of the theories

We will now turn to a discussion of some basic properties of these theories. In particular, we will discuss how the theories selected by the single field equation criterion are (perturbatively) ghost free, the field equations reduce to (at most) second-order differential equations, black holes are described only by their mass, and black hole thermodynamics can be studied non-perturbatively in the higher curvature couplings. Since these properties have been solidly established for Lovelock and quasi-topological gravity, in these subsections the focus will be showing these properties are valid also for the generalized quasi-topological models as well.

4.3.1 No ghosts

Theories that have a single independent field equation for spherically symmetric metrics are (perturbatively) ghost free about a maximally symmetric background. By 'ghost-free' we actually mean something slightly stronger: that the theory propagates only the usual massless spin-2 graviton. In fact, one can view the single metric function property as a sufficient condition for a theory to propagate only the massless graviton. However, it is not a necessary condition — there exist (perturbatively) ghost free theories that do not admit single metric function solutions. A simple example is ECG in dimensions larger than four [4]. This can be summarized in the following theorem:

Theorem. *If a theory of gravity with Lagrangian density* $\mathcal{L}_{\text{grav}} = \mathcal{L}\left(g_{\alpha\beta}, R_{\alpha\beta\mu\nu}\right)$ *has a single independent field equation under the constraint of a vacuum static and spherically* *symmetric metric, then the theory propagates only the usual massless spin-2 graviton on a maximally symmetric background.*

In exploring cubic and quartic theories, this result was first conjectured in [5], and then was subsequently proven in [156]. We will demonstrate its validity for the new term $\mathcal{S}_D$ that was presented above, and then sketch how the general proof works. Recall that the action is

$$S = \frac{1}{16\pi G_N} \int d^D x \sqrt{-g} \Big(R - 2\Lambda + \frac{12\lambda(D-2)(2D-1)}{(D-3)(184 - 514D + 291D^2 - 49D^3 + 4D^4)} \mathcal{S}_D \Big) , \tag{4.81}$$

and we parameterize the cosmological constant in the standard way as

$$\Lambda = -\frac{(D-1)(D-2)}{2L^2} . \tag{4.82}$$

To linearize the theory, let us first seek the appropriate vacuum background. It is easy to verify that the Minkowski background provides a vacuum for the theory with $\Lambda = 0$. Slightly more generally, considering a nontrivial Λ and imposing the maximal symmetry condition

$$\bar{R}_{\alpha\beta\sigma\rho} = -\frac{2f_\infty}{L^2} \, \bar{g}_{\alpha[\sigma}\bar{g}_{\rho]\beta} \tag{4.83}$$

upon evaluating the field equations for this choice of background, we find the following constraint:

$$h(f_\infty) := 1 - f_\infty + (D-6)\frac{\lambda}{L^4} f_\infty^3 = 0 , \tag{4.84}$$

which determines f_∞. This in turn is related to the effective cosmological constant of the theory

$$\Lambda_{\text{eff}} = -\frac{(D-1)(D-2)f_\infty}{2L^2} . \tag{4.85}$$

Due to the cubic nature of this condition, the theory will generically have three distinct vacua, with one having a smooth limit to the Einstein case as $\lambda \to 0$.

Next we consider a perturbation around the obtained maximally symmetric backgrounds

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + h_{\alpha\beta} . \tag{4.86}$$

The linear equations of motion can be efficiently obtained using the method introduced in [121, 195]—they are given by

$$\mathcal{E}^L_{\alpha\beta} = -\frac{1}{2} h'(f_\infty) G^L_{ab} , \tag{4.87}$$

where the prime denotes differentiation with respect to f_∞ and $G^L_{\alpha\beta}$ is the Einstein tensor linearized on the background (4.83). The remarkable fact that the linearized equations of motion coincide with the linearized Einstein equations indicates that the massive and scalar modes are suppressed (i.e. they are infinitely heavy) and the theory propagates the same transverse massless graviton as Einstein's gravity.

The fact that the theory matches Einstein gravity at the linear level is not a fluke — as summarized at the beginning of this section, it is a general property of the theories that admit single metric function solutions. The fact that this happens at the cubic and quartic levels [6] in curvature, motivated the conjecture presented in [5]. A proof of this was first provided by Bueno and Cano in [156]. Their proof proceeds in the following way. First, a perturbative solution for the metric outside of a static and spherically symmetric mass distribution is obtained. In this solution, the masses of the ghost-like graviton and scalar mode appear, and it is found that only when both $m_s^2 = m_g^2 = \infty$ is the solution characterized by a single metric function.

4.3.2 Second-order integrated field equations

A remarkable property of all of the theories defined by the single metric function condition is that the field equations for a spherically symmetric metric is always a total derivative and can be integrated to yield a differential equation that is at most a second-order differential equation for the metric function. Using the same argument, it is possible to prove that when the theory is either Lovelock or quasi-topological, then the equations are algebraic. (Note that part of this proof is inspired by [156], though the approach here is somewhat different and closer to the understanding developed in [5]).

As mentioned near the start of this chapter, a general theory of gravity built from contractions of Riemann tensors will give rise to fourth-order equations of motion. Under the constraint of spherical symmetry, the number of independent field equations is reduced to two. These correspond to the $\mathcal{E}_t{}^t$ and $\mathcal{E}_r{}^r$ components, while the Bianchi identity relates the remaining components of the field equations (which are all equal when written with one index down and one index up). For theories that admit single metric function solutions, the Bianchi identity reduces the order of the single independent field equation by one. To see this, note that for these theories the r-component of the Bianchi identity reads (for arbitrary k)

$$\frac{d\mathcal{E}_r{}^r}{dr} + \frac{(D-2)}{r}\mathcal{E}_r{}^r - \frac{(D-2)}{r}\mathcal{E}_i{}^i = 0\,, \tag{4.88}$$

where $\mathcal{E}_i{}^i$ stands for one of the transverse components of the field equations (they are all the same), and no summation is implied. Since any component of the $\mathcal{E}_\alpha{}^\beta$ contains at most four derivatives of $f(r)$, we can conclude that $\mathcal{E}_r{}^r$ contains at most *three* derivatives of $f(r)$, i.e. the Bianchi identity reduces its order by one.

To see that the field equations are actually total derivatives and can be integrated once more, we turn to a more in-depth discussion of the on-shell action for these theories. The defining condition of the single metric function theories is that

$$\left.\frac{\delta S}{\delta f}\right|_{N=1} = 0\,, \tag{4.89}$$

which is equivalent to demanding that the Lagrangian, when evaluated on a spherically symmetric metric with $N = 1$, is a total derivative. More generally, the on-shell action will have the general structure [5]

$$\sqrt{|g|}\mathcal{L} = N\mathcal{F}_1' + N'\mathcal{F}_2 + N''\mathcal{F}_3 + \text{ terms non-linear in derivatives of } N \tag{4.90}$$

where $\mathcal{F}_i$ are functionals of f and its first two derivatives. The schematic structure of the terms involving non-linear derivatives of the lapse is completely fixed by the time reparameterization invariance of the theory, as noted in [156]. That is, since $t \to \alpha t$ is equivalent to $N \to \alpha N$, the Lagrangian must be a homogeneous function of N of degree 1. This means the only possible non-linear contributions come in the form N'^2/N, $N'N''/N$, and so on. As noted in [5], such terms cannot be eliminated using integration by parts. As a result, successive integration by parts can reduce the action to the following form:

$$\sqrt{|g|}\mathcal{L} = N\left[\mathcal{F}_1 - \mathcal{F}_2 + \mathcal{F}_3'\right]' + \text{ terms non-linear in derivatives of } N. \tag{4.91}$$

A variation with respect to N then setting $N = 1$ yields the only independent field equation:

$$\mathcal{E}_r{}^r = \mathcal{F}' = \left[\mathcal{F}_1 - \mathcal{F}_2 + \mathcal{F}_3'\right]'\,, \tag{4.92}$$

which we now directly see must be a total derivative. In vacuum (or in the presence of suitable matter), this equation can be integrated for free, yielding an equation that is at most second-order determining the metric function.

Now, let us restrict to Lovelock and quasi-topological theories, where we can actually do better than the above argument. As defined in the previous section, these theories both satisfy $\nabla_\alpha P^{\alpha\beta\mu\nu} = 0$ for a spherically symmetric metric. As a result, the field equations for these theories take the simplified form

$$\mathcal{E}_{\alpha\beta} = R_{\alpha\sigma\mu\nu}P_\beta{}^{\sigma\mu\nu} - \frac{1}{2}g_{\alpha\beta}\mathcal{L}\,. \tag{4.93}$$

Note that because the terms involving derivatives of the entropy tensor vanish, these equations contain at most second order derivatives of the metric. Now, by the same argument above the Bianchi identity implies that ${\mathcal{E}_r}^r$ contains at most a single derivative of the metric function, and then the fact that ${\mathcal{E}_r}^r = \mathcal{F}'$ reveals that the integrated field equations must be algebraic in $f(r)$. Thus, we have proven that Lovelock and quasi-topological theories will always have field equations that can be integrated to give a polynomial determining the metric function $f(r)$.

4.3.3 Black holes have no hair

It has been rigorously established that single metric function theories do not propagate ghosts and have second order integrated field equations. The remaining two properties seem to hold completely generally, but at this point we do not have rigorous proofs of them for the generalized quasi-topological theories.

The next interesting property that we will discuss for these theories is that black hole solutions have no *higher derivative hair* [215]. Spherically symmetric vacuum solutions in general relativity are characterized by mass alone. This property also extends to Lovelock and quasi-topological gravity. Here we will argue that it is true for the entire family of single metric function theories by studying the generalized quasi-topological class of theories. For simplicity of presentation, we will focus on asymptotically flat black holes in four-dimensional ECG, but note that this property extends generally to higher dimensions and different maximally symmetric asymptotics [4, 157, 6, 158, 7, 12, 11].

In general, a differential equation of order n requires n constants of integration. For black hole solutions, additional constants of integration that arise in this way are often called 'higher derivative hair'. It is obvious that spherically symmetric vacuum black holes in Lovelock and quasi-topological gravity have no higher derivative hair, since in these cases the field equations are algebraic and the only constant of integration is the mass. In the generalized quasi-topological case, however, after integrating the field equations once, the field equation is still a (non-linear) second order differential equation. Thus, in general, one would expect the black hole solutions in these theories to be characterized by three integration constants — the mass, and two additional parameters. However, by fixing the asymptotic structure of the solution and demanding a regular horizon, there appear to be no such terms.

To illustrate this in a concrete fashion, we will focus on four-dimensional vacuum black holes in ECG. The action for ECG reads

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[R - \frac{\lambda}{6}\mathcal{P}\right], \tag{4.94}$$

where R is the usual Ricci scalar and

$$\mathcal{P} = 12 R_{\alpha\ \mu}^{\ \beta\ \ \nu} R_{\beta\ \nu}^{\ \sigma\ \ \rho} R_{\sigma\ \rho}^{\ \alpha\ \ \mu} + R_{\alpha\beta}^{\mu\nu} R_{\mu\nu}^{\sigma\rho} R_{\sigma\rho}^{\alpha\beta} - 12 R_{\alpha\beta\mu\nu} R^{\alpha\mu} R^{\beta\nu} + 8 R_{\alpha}^{\beta} R_{\beta}^{\mu} R_{\mu}^{\alpha}, \tag{4.95}$$

is the ECG Lagrangian density. Note that throughout this subsection we will set $G_N = 1$ to simplify the equations. We restrict ourselves to asymptotically flat, static and spherically symmetric vacuum black holes. In this case, the only independent field equation is

$$-(f-1)r - \lambda\left[\frac{f'^3}{3} + \frac{1}{r}f'^2 - \frac{2}{r^2}f(f-1)f' - \frac{1}{r}ff''(rf' - 2(f-1))\right] = 2M, \tag{4.96}$$

where a prime denotes differentiation with respect to r. The quantity M appearing on the right-hand side of the equation is the ADM mass of the black hole [157, 5], and we will assume $\lambda > 0$ in what follows. Unfortunately the field equations cannot be solved analytically (except in certain special cases [216]), and either numerical or approximate solutions (or some combination) must be computed to make progress. We will review the construction of a numerical solution.

We begin by solving the field equations via a series expansion near the horizon using the ansatz

$$f_{\rm nh}(r) = 4\pi T(r - r_+) + \sum_{n=2} a_n (r - r_+)^n, \tag{4.97}$$

which ensures that the metric function vanishes linearly at the horizon ($r = r_+$), and $T = f'(r_+)/4\pi$ is the Hawking temperature. Substituting this ansatz into the field equations (4.96) allows one to solve for the temperature and mass in terms of r_+ and the coupling λ:

$$\begin{aligned} M &= \frac{r_+^3}{12\lambda^2}\left[r_+^6 + (2\lambda - r_+^4)\sqrt{r_+^4 + 4\lambda}\right], \\ T &= \frac{r_+}{8\pi\lambda}\left[\sqrt{r_+^4 + 4\lambda} - r_+^2\right]. \end{aligned} \tag{4.98}$$

One then finds that a_2 is left undetermined by the field equations, while all a_n for $n > 2$ are determined by messy expressions involving T, M, r_+, and a_2.

We now consider an expansion of the solution in the large-r asymptotic region. To obtain this, we linearize the field equations about the Schwarzschild background:

$$f_{\text{asymp}} = 1 - \frac{2M}{r} + \epsilon h(r), \tag{4.99}$$

where $h(r)$ is to be determined by the field equations, and we linearize the differential equation by keeping terms only to order ϵ, before setting $\epsilon = 1$. The resulting differential equation for $h(r)$ takes the form

$$h'' + \gamma(r)h' - \omega(r)^2 h = g(r), \tag{4.100}$$

where

$$\begin{aligned}\gamma(r) &= -\frac{2(M-r)}{(2M-r)r}\,, \quad \omega^2(r) = \frac{r^6 + 56M^2\lambda - 12Mr\lambda}{6Mr^2(r-2M)\lambda}\,,\\ g(r) &= -\frac{2M(46M-27r)}{9(2M-r)r^3}\,.\end{aligned} \tag{4.101}$$

In the large r limit, the homogeneous equation reads

$$h_h'' - \frac{2}{r}h_h - \frac{r^3}{6M\lambda}h_h = 0\,, \tag{4.102}$$

and can be solved exactly in terms of modified Bessel functions:

$$h_h = r^{3/2}\left[\tilde{A}I_{-\frac{3}{5}}\left(\frac{2r^{5/2}}{5\sqrt{6M\lambda}}\right) + \tilde{B}K_{\frac{3}{5}}\left(\frac{2r^{5/2}}{5\sqrt{6M\lambda}}\right)\right], \tag{4.103}$$

where $I_\nu(x)$ and $K_\nu(x)$ are the modified Bessel functions of the first and second kinds, respectively. To leading order in large r, this can be expanded as

$$h_h(r) \approx Ar^{1/4}\exp\left[\frac{2r^{5/2}}{5\sqrt{6M\lambda}}\right] + Br^{1/4}\exp\left[\frac{-2r^{5/2}}{5\sqrt{6M\lambda}}\right] \tag{4.104}$$

where we have absorbed various constants into the definitions of A and B (compared to $\tilde{A}$ and $\tilde{B}$). Thus, the homogeneous solution consists of a growing mode and a decaying mode. Asymptotic flatness demands that we set $A = 0$, while the second term decays super-exponentially and can therefore be neglected.[24] In this way, one of the possible integration constants has been uniquely fixed by the boundary conditions.

[24] This assumes that $M\lambda > 0$. In cases where $M\lambda < 0$, the homogeneous solution contains oscillating terms that spoil the asymptotic flatness. The only viable solution in this case is to set the homogenous solution to zero.

The particular solution, which reads

$$h_{\mathrm{p}} = -\frac{36\lambda M^2}{r^6} + \frac{184}{3}\frac{\lambda M^3}{r^7} + \mathcal{O}\left(\frac{M^3\lambda^2}{r^{11}}\right), \tag{4.105}$$

clearly dominates over the super-exponentially decaying homogenous solution at large r, thereby giving

$$f(r) \approx 1 - \frac{2M}{r} + h_{\mathrm{p}}\,. \tag{4.106}$$

Neither the near horizon approximation nor the asymptotic solution is valid in the entire spacetime outside of the horizon. One means to bridge this gap is to numerically solve the equations of motion in the intermediate regime. The idea is quite simple: For a given choice of M and λ, pick a value for the free parameter a_2. Use these values in the near horizon expansion to obtain initial data for the differential equation just outside the horizon:

$$\begin{aligned} f(r_+ + \epsilon) &= 4\pi T\epsilon + a_2\epsilon^2\,, \\ f'(r_+ + \epsilon) &= 4\pi T + 2a_2\epsilon\,, \end{aligned} \tag{4.107}$$

where ϵ is some small, positive quantity. A generic choice of a_2 will excite the exponentially growing mode in (4.104). Thus, a_2 must be chosen extremely carefully and with high precision to obtain the asymptotically flat solution. A satisfactory solution will be obtained if for some value of r that is large (compared with the other scales in the problem), the numeric solution agrees with the asymptotic expansion to a high degree of precision. In practice, we find that there is a unique value of a_2 for which this occurs. Of course, since the differential equation is very stiff, the numerical scheme will ultimately fail at some radius, r_{max}. The point at which this failure occurs can be pushed to larger distance by choosing a_2 more precisely and increasing the working precision, but this comes at the cost of increased computation time.[25] In this way, we fix two integration constants via boundary conditions — the only remaining constant is the mass, and so it seems these black holes have no higher derivative hair.

[25] A solution for $r < r_+$ can be obtained by choosing ϵ to be small and negative in (4.107). The numerical scheme encounters no issues in this case.

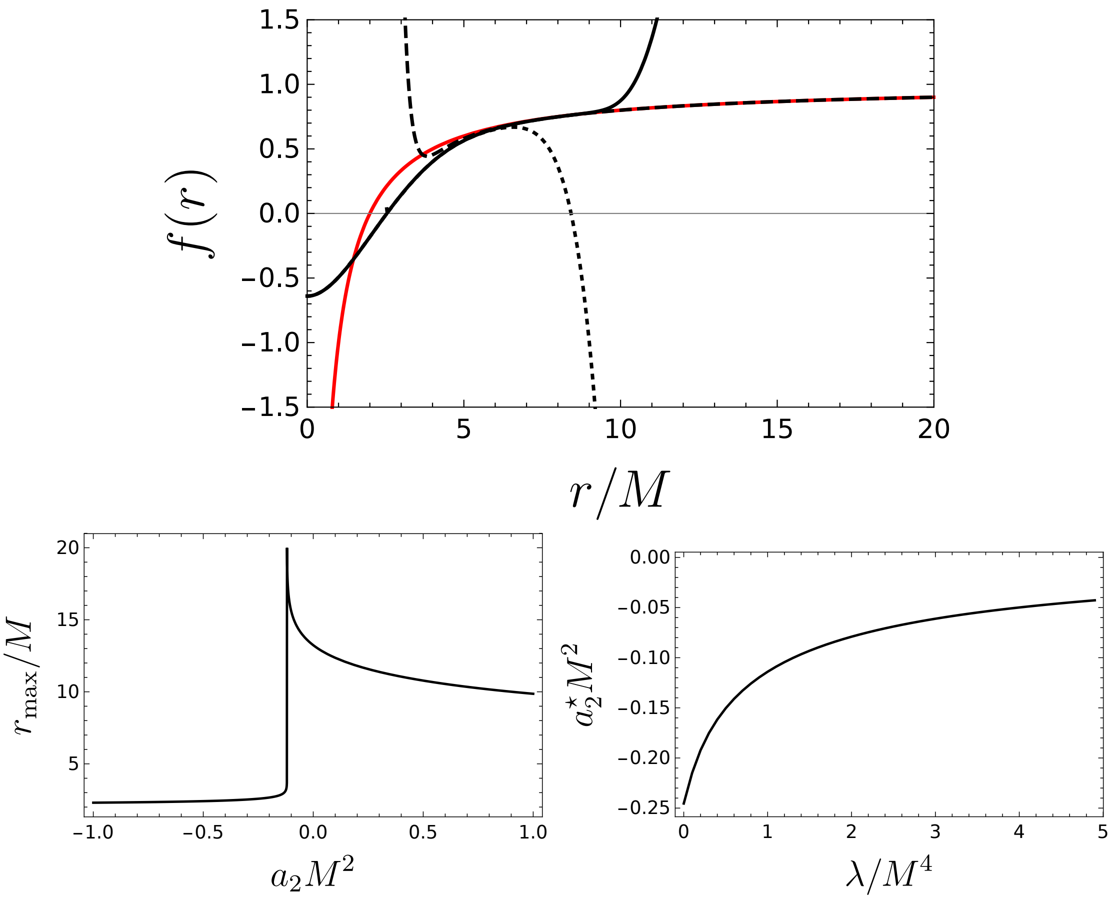

Figure 4.1: **Numerical scheme**. *Top*: Numerical solution for $\lambda/M^4 = 10$ and a shooting parameter $a_2^\star = -0.022853992336918507$. The solid, red curve is the Schwarzschild solution of Einstein gravity. The black, dotted curve is the near horizon approximation, including terms up to order $(r-r_+)^8$. The dashed, black curve is the asymptotic solution, including terms up to order r^{-12}. The solid black line is the numeric solution. *Bottom left*: A plot of $r_{\max}$ (where the numerical solution breaks down) vs. a_2 for the case $\lambda = 1$. The peak corresponds to the value of a_2 that gives an asymptotically flat solution. *Bottom right*: A plot of the value of a_2 giving an asymptotically flat solution vs. λ. Note that in the limit $\lambda \to 0$ we have $a_2M^2 \to -1/4$, which coincides with the Einstein gravity result. In all cases, $\epsilon = 10^{-6}$ was used in eq. (4.107) to obtain the initial data.

In figure 4.1 we highlight some representative numerical results. The bottom left plot displays $r_{\max}$ vs. a_2, revealing a prominent peak at a point $a_2^\star$. The peak coincides with the value of a_2 that produces the asymptotically flat solution. If a_2 is chosen different from this value, it results in the excitation of the growing mode and the solution is not asymptotically flat. In the bottom right plot, we show the value of $a_2^\star$ plotted against the coupling, λ. Notably, $a_2^\star$ limits to the Schwarzschild value of $a_2^\star M^2 = -1/4$ when $\lambda \to 0$. While we have not been able to deduce a functional form for $a_2^\star$ from first principles, it is possible to perform a fit of the numeric results giving

$$a_2^\star\left(x = \lambda/M^4\right) \approx -\frac{1}{M^2}\frac{1 + 2.1347x + 0.0109172x^2}{4 + 15.5284x + 8.03479x^2}, \tag{4.108}$$

which is accurate to three decimal places or better on the interval $\lambda/M^4 \in [0, 5]$.

In the top plot of figure 4.1 we show a numerical solution for $\lambda/M^4 = 10$ and compare it with the Schwarzschild solution, as well as the near horizon and asymptotic approximate solutions. For the same physical mass, the ECG black hole has a larger horizon radius than the Schwarzschild solution. Note that the near horizon solution provides an accurate approximation from $r = 0$ to about $r = 5M$, but then rapidly diverges to $f \to -\infty$. The numeric solution begins to rapidly converge to the asymptotic solution near $r = 4M$, but near $r = 10M$ it breaks down: the stiff system causes the integrated solution to rapidly diverge to $f \to +\infty$. This is just a consequence of not choosing $a_2^\star$ to high enough precision in the numeric method, and the exponentially growing mode has been excited. Before the numeric solution breaks down, the asymptotic solution (dashed line) is accurate to better than 1 part in 1,000 and so it can be used to continue the solution to infinity.

The numerical results suggest that there is a unique value of a_2 such that the solution is asymptotically flat. We can provide further evidence for this by considering the field equations near the horizon and demanding that the solution limits to the Einstein gravity result as the ECG coupling is set to zero. Recall that, near the horizon, the metric function is expanded as

$$f_{\text{nh}}(r) = 4\pi T(r - r_+) + a_2^\star(r - r_+)^2 + \sum_{i=3} a_i(a_2^\star)(r - r_+)^i, \tag{4.109}$$

where the constants a_n with $n > 2$ are determined by the field equations in terms of the parameter $a_2^\star$ and M, T and r_+. We will demand that this expansion has a smooth $\lambda \to 0$ limit. It turns out that this constraint is also enough to ensure that the near horizon expansion limits to that for the Schwarzschild solution

$$f_{\text{nh}}^{\text{Ein}} = \sum_{i=1}(-1)^{i-1}\frac{(r - r_+)^i}{r_+^i}\,. \tag{4.110}$$

We proceed by writing $a_2^\star = a_2^\star(\lambda)$ and expand each of $a_n(a_2^\star)$ to lowest order in λ. For example, the expansion for the first two terms is

$$
\begin{aligned}
a_3(a_2^\star) &= \frac{a_2(0)r_+^3 + r_+}{9\lambda} + \frac{3r_+^6 a_2'(0) - 6a_2(0)^2 r_+^4 + 34a_2(0)r_+^2 - 14}{27r_+^3} + \mathcal{O}(\lambda)\,, \\
a_4(a_2^\star) &= +\frac{a_2(0)r_+^6 + r_+^4}{216\lambda^2} + \frac{3r_+^6 a_2'(0) - 60a_2(0)^2 r_+^4 + 89a_2(0)r_+^2 + 68}{648\lambda} \\
&+ \frac{3r_+^{10}a_2''(0) - 240a_2(0)r_+^8 a_2'(0) + 2r_+^6\left(89a_2'(0) + 72a_2(0)^3\right) - 968a_2(0)^2 r_+^4 + 1040a_2(0)r_+^2 - 278}{1296r_+^4} \\
&+ \mathcal{O}(\lambda).
\end{aligned} \tag{4.111}
$$

Clearly, for a_3 to have a smooth $\lambda \to 0$ limit, we must take

$$
a_2^\star(0) = -\frac{1}{r_+^2}, \tag{4.112}
$$

which also cures the λ^{-2} divergence in a_4. Then, for a_4 to have a smooth $\lambda \to 0$ limit, we must take

$$
a_2^{\star\prime}(0) = \frac{27}{r_+^6}\,. \tag{4.113}
$$

Interestingly, this choice for $a_2^{\star\prime}(0)$ also ensures that

$$
a_3(a_2^\star) = \frac{1}{r_+^3} + \mathcal{O}(\lambda), \tag{4.114}
$$

which is precisely the value expected from the Schwarzschild solution. This procedure continues in the obvious way: The expansion of a_n for small λ fixes $a_2^{\star(n-3)}(0)$, which in turn guarantees that the term a_{n-1} limits to the Schwarzschild value from (4.110).

It is straight-forward, but computationally costly, to do this to arbitrary order. We have computed $a_2^{\star(n)}(0)$ up to $n = 15$, finding the results presented in table 4.1. While $a_2^{\star(n)}(0) \propto 1/r_+^{4n+2}$, we were not able to deduce the dependence of the coefficients of $a_2^{\star(n)}(0)$ on n. The fact that these coefficients grow unboundedly indicates that a Taylor series expansion of $a_2^\star(\lambda)$ has a small or vanishing radius of convergence. Thus the function is not analytic, and the Taylor series cannot be expected to provide a good approximation. However, rather than a Taylor series we can use a Padé approximant to reconstruct the form of $a_2^\star(\lambda)$, and we show this in figure 4.2. The basic conclusion is that, as more terms are included in the Padé approximant, the form of $a_2^\star(\lambda)$ converges to the results of our numerical scheme presented in figure 4.1. While the convergence is fast for small λ, more

n	$(-1)^{n+1} r_+^{4n+2} a_2^{\star(n)}(0)$
0	1
1	27
2	3384
3	1320534
4	1151833248
5	1875967406160
6	5107532147380800
7	21544624968666695280
8	133135416924677418585600
9	1154324990320626883159054080
10	13568049825205878205542081792000
11	210227289858470130670513367566041600
12	4194920428540096167815139429105212006400
13	105700177837430847101072792547386798551142400
14	3306987976911675043248786217918581692121564979200
15	126609498143560198473638841716966388468374445902592000

Table 4.1: **Series coefficients for ECG shooting parameter**. Here we display the first fifteen derivatives of the shooting ECG parameter $a_2^\star$ computed by demanding that the near horizon solution limits smoothly to the Einstein gravity result.

terms are required to obtain good convergence for larger λ. With the fifteen derivatives presented in table 4.1, it is not possible to accurately match $a_2^\star$ over the full domain of λ, and the fit to the numerical data (4.108) is more accurate for larger λ. If the functional dependence of $a_2^{\star(n)}(0)$ on n could be deduced, then this would allow $a_2^\star$ to be determined to arbitrary precision.

Let us summarize: the existence of a numerical asymptotically flat solution seems to fix the value of a_2 uniquely. Further, we have also seen that demanding that the near horizon solution limits smoothly to the Einstein gravity result uniquely determines all of the derivatives of a_2 when regarded as a function of λ. Further, these two methods seem to agree on what the value of a_2 should be: by fitting the derivatives of a_2 to a Padé approximant, the derivative expansion appears to converge to the same value required by the numerical scheme as more terms are included in the series expansion. Now, let us note that while we have presented this discussion for four-dimensional ECG, the same

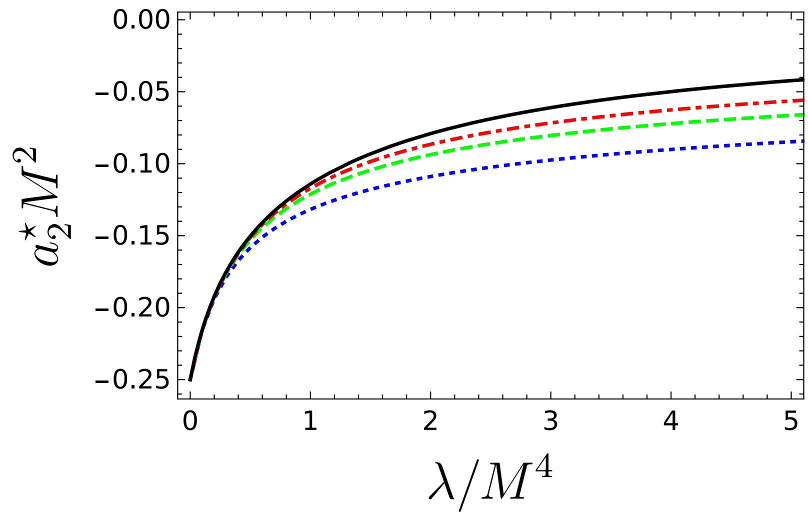

Figure 4.2: **Analytical approach for shooting parameter**. Here the solid black line denotes the value of $a_2^\star$ as determined through the numerical scheme. The remaining curves denote Padé approximants built from the derivatives presented in table 4.1. Specifically, the dotted, blue curve corresponds to a $[2, 2]$-order Padé approximant, the dashed, green curve corresponds to a $[4, 4]$ order Padé approximant and the dot-dashed, red curve corresponds to a $[7, 7]$ order Padé approximant. For $\lambda/M^4 < 1$ convergence to the numerical result is rapid, but convergence for larger values would require more derivatives than we were able to reasonably compute.

thing happens for the higher dimensional theories and also when the asymptotics are dS or AdS. Therefore, we have strong evidence that the vacuum black holes in generalized quasi-topological gravity are characterized simply by their mass, and there is no higher derivative hair that is often present in generic higher-order theories [217, 215].

4.3.4 Non-perturbative thermodynamics

In the case of Lovelock and quasi-topological black holes, the temperature, entropy and mass can be obtained for arbitrary couplings. It appears that this extends to the entire class of theories defined by the single metric function condition. This is in contrast to an arbitrary theory of gravity, where the best one could hope to do would be to work perturbatively in the couplings to obtain corrections [193]. While the result is again somewhat trivially true for Lovelock and quasi-topological gravity, it is not obviously true for the generalized quasi-topological class. In these theories, we do not have an analytical expression for $f(r)$, and so it is somewhat surprising that, for example, an expression for the black hole temperature can be obtained analytically. Our focus here will be to show how this plays out for four-dimensional asymptotically flat black holes in ECG, and comment on how the result generalizes [4, 157, 6, 158, 7, 12, 11].

Again, the action for ECG (with $G_N = 1$) reads

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[R - \frac{\lambda}{6}\mathcal{P}\right], \tag{4.115}$$

and we will be interested in the near-horizon expansion of the field equations,

$$f_{\rm nh}(r) = 4\pi T(r-r_+) + \sum_{n=2} a_n (r-r_+)^n\,. \tag{4.116}$$

The lowest order field equations take the form:

$$\begin{aligned} 2M - r_+ + 4\pi T\lambda\left(\frac{4\pi T}{r_+}\right)\left(\frac{4\pi T r_+}{3}+1\right) &= 0\,, \\ 1 - 4\pi T r_+ - \lambda\left(\frac{4\pi T}{r_+}\right)^2 &= 0\,, \end{aligned} \tag{4.117}$$

while at the next order a_3 and a_2 appear, and so on. It is a remarkable fact that the two lowest order terms do not involve the undetermined constant a_2. This allows us to exactly determine the temperature and mass simply by solving a system of polynomial equations, even though we lack an analytical solution for $f(r)$. The correct choice of branch is that which limits smoothly to the Einstein gravity result when $\lambda \to 0$,

$$\begin{aligned} M &= \frac{r_+^3}{12\lambda^2}\left[r_+^6 + (2\lambda - r_+^4)\sqrt{r_+^4+4\lambda}\right]\,, \\ T &= \frac{r_+}{8\pi\lambda}\left[\sqrt{r_+^4+4\lambda} - r_+^2\right]\,. \end{aligned} \tag{4.118}$$

Using Wald's formula, one can show that the entropy of the black holes is given by

$$S = \pi r_+^2\left[1 - 2\lambda\left(\frac{4\pi T}{r_+}\right)^2\left[\frac{1}{2}+\frac{1}{2\pi T r_+}\right]\right] + 4\pi\sqrt{\lambda}\,, \tag{4.119}$$

where the last term ensures that $S \to 0$ as $M \to 0$. Identifying the coupling potential

$$\Psi = \frac{8\pi^2 T^2(3+2\pi r_+ T)}{3r_+} - \frac{2\pi T}{\sqrt{\lambda}}\,, \tag{4.120}$$

the first law and extended Smarr formula hold:[26]

$$\begin{aligned} \delta M &= T\delta S + \Psi\delta\lambda \,, \\ M &= 2TS + 4\lambda\Psi \,. \end{aligned} \tag{4.121}$$

Einsteinian Cubic Gravity is the simplest example of the generalized quasi-topological theories, but it highlights all of the essential points. In more general cases, one proceeds in the same way. Expanding the field equations near a black hole horizon, it is found that the first two equations determine the mass and temperature as solutions of (often complicated) polynomial equations. No perturbative expansions in the coupling are required, and as a result these theories provide excellent toy models whenever thermodynamics is important. Because the field equations depend on second derivatives of the metric, it is in principle possible that the 'arbitrary' parameter a_2 could appear in these lowest order equations, and this would spoil the property. However, it seems that the theories selected by the single metric function criterion have field equations with a structure that gives a precise cancellation of these possible terms. It would be desirable to better understand how this is connected with the single metric function condition.

4.4 Summary remarks

In this chapter we have introduced and studied higher curvature theories of gravity, focusing on the case of static and spherically symmetric metrics. We found that in this case, the theories decouple into three groups — Lovelock, quasi-topological, and generalized quasi-topological gravities — distinguished by the structure of their field equations and the dimensions in which the theory is non-trivial. The Lovelock and quasi-topological theories have been known for some time [116, 118], while the generalized quasi-topological theories are relatively new [5]. The theories are united in that they share certain desirable and somewhat unexpected properties: they are ghost-free on constant curvature backgrounds, the integrated field equations are at most second order differential equations, and they admit black hole solutions characterized by mass alone with thermodynamics that can be studied exactly.

The properties of these theories make them excellent toy models for exploring questions in black hole thermodynamics and holography [218, 219, 120, 214, 43, 16]. The generalized quasi-topological theories are particularly interesting in this regard, since they are

[26]Note that, for small black holes we have $4\lambda\Psi \approx -4TS/3$ and the extended Smarr formula reduces to $M = 2TS/3$, which coincides with what one would expect for a three-dimensional CFT at finite temperature. This intriguing limit was first noted in [158] via a different approach.

non-trivial in all dimensions $D \geq 4$, they allow for the holographic studies in four and six dimensions, in which both (cubic) Lovelock and quasi-topological gravity are trivial. To date, a number of interesting results have been found through studies of these theories. It was found that in four dimensions, small asymptotically flat black holes become thermally stable [158]. Studies of AdS solutions revealed examples of critical behaviour in the phase space of electrically charged black branes, a result that is observed in neither Lovelock nor quasi-topological theories [7]. In the context of holography, it is interesting to note that simply by demanding that the black hole solutions are well-behaved asymptotically the KSS bound [68] on the ratio of shear viscosity to entropy density is enforced in both four [220] and higher dimensions [12]. In ongoing work, it has been found that the generalized quasi-topological theories admit NUT charged solutions in four and higher dimensions. In addition to providing the first examples of higher curvature theories admitting NUT solutions in four dimensions, this study has allowed for the identification of simple and universal results for the free energy of holographic CFTs defined on odd dimensional squashed spheres [9, 10].

Of course, there remains much about these theories to study. One obvious problem to attack would be the dynamical stability of black holes. While the theories are free from ghosts on constant curvature spaces, since the field equations for all but the Lovelock class of theories are higher-order for a generic metric, we do expect that at least some solutions of the theories will be unstable.[27] Determining if and for what metrics the solutions are unstable would be an important step, especially if one wished to use the four-dimensional theories for more than just toy models [158]. However, this is in general a very complicated problem, and there is still lots of work to be done even in the simplest case of Lovelock theory [223, 224, 225].

[27] Note that higher order equations of motion are not always fatal for black hole stability — see, e.g. [221, 222].

Chapter 5

Black hole λ-lines

Having introduced higher curvature gravity models in the previous chapter, we now consider their relevance in black hole chemistry. In particular, here we will focus on a class of exact, spherically symmetric black holes in Lovelock gravity coupled to real scalar fields. Black hole chemistry was first studied for this model in [17] in the case where the gravitational sector consisted only of Einstein gravity. In [13, 14], the study was generalized to higher-order Lovelock theories. The most interesting result to come from those investigations is that, in third and higher order Lovelock gravity, there are examples of black hole 'λ-lines'. These are lines of second order phase transitions in the thermal phase space, and the example presented in [13] represents, to the best of our knowledge, the first such example in black hole physics.

The purpose of this chapter will be to explain this result in greater detail. We begin by providing a brief overview of the theory, the motivation for it, and the black hole solutions. After presenting the thermodynamic quantities, the existence of a black hole λ-line is demonstrated. The remainder of the chapter is spent discussing the necessary conditions for this phenomena, and any possible pathological behaviour of the black holes exhibiting it. Our discussion will mostly follow [13].

5.1 Lovelock black holes with scalar fields

Recently, Oliva and Ray have constructed a simple recipe that allows for the conformal coupling of a scalar field to higher curvature terms [226]. The field equations of the theory are second order for both the metric and the scalar field and admit exact black hole solutions with the scalar field is regular everywhere outside of — and on — the horizon and the back-reaction of the scalar field onto the metric is captured analytically [227, 228, 229, 229, 230, 231]. This work provided the first example of black holes with conformal scalar hair in $D > 4$ where no-go results had been reported previously [232]. The obtained solutions are valid for positive, negative and vanishing cosmological constant; however, the AdS case has received the most attention due to hopeful applications in the AdS/CFT correspondence.

The model we consider consists of Lovelock gravity, a Maxwell field, and a real scalar field coupled conformally to the dimensionally extended Euler densities,

$$\mathcal{I} = \frac{1}{16\pi G_N} \int d^D x \sqrt{-g} \left(\sum_{k=0}^{n_{\max}} \mathcal{L}^{(n)} - 4\pi G_N F_{\mu\nu} F^{\mu\nu} \right) \tag{5.1}$$

where

$$\mathcal{L}^{(n)} = \frac{1}{2^n} \delta^{(n)} \left(a_n \prod_r^n R^{\alpha_r \beta_r}_{\mu_r \nu_r} + b_n \phi^{D-4n} \prod_r^n S^{\alpha_r \beta_r}_{\mu_r \nu_r} \right) \tag{5.2}$$

with $\delta^{(n)} = \delta^{\alpha_1 \beta_1 \cdots \alpha_n \beta_n}_{\mu_1 \nu_1 \cdots \mu_n \nu_n}$ the generalized Kronecker tensor, a_n and b_n are coupling constants, and $n_{\max} \leq (D-1)/2$. Here the tensor $S_{\mu\nu}{}^{\gamma\delta}$ describes how the scalar field couples to gravity:

$$S_{\mu\nu}{}^{\gamma\delta} = \phi^2 R_{\mu\nu}{}^{\gamma\delta} - 2\delta^{[\gamma}_{[\mu} \delta^{\delta]}_{\nu]} \nabla_\rho \phi \nabla^\rho \phi - 4\phi \delta^{[\gamma}_{[\mu} \nabla_{\nu]} \nabla^{\delta]} \phi + 8 \delta^{[\gamma}_{[\mu} \nabla_{\nu]} \phi \nabla^{\delta]} \phi \,, \tag{5.3}$$

and transforms homogeneously under the conformal transformation, $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ and $\phi \to \Omega^{-1} \phi$ as $S_{\mu\nu}{}^{\gamma\delta} \to \Omega^{-4} S_{\mu\nu}{}^{\gamma\delta}$.

The theory has stress-energy associated with both the scalar and Maxwell fields, with the former given by

$$(T_1)^\nu_\mu = \sum_{n=0}^{n_{\max}} \frac{b_n}{2^{n+1}} \phi^{D-4n} \delta^{\nu \lambda_1 \cdots \lambda_{2n}}_{\mu \rho_1 \cdots \rho_{2n}} S^{\rho_1 \rho_2}{}_{\lambda_1 \lambda_2} \cdots S^{\rho_{2k-1} \rho_{2n}}{}_{\lambda_{2n-1} \lambda_{2n}} \tag{5.4}$$

and the latter,

$$(T_2)^\nu_\mu = \left(F_{\mu\rho} F^{\nu\rho} - \frac{1}{4} F_{\lambda\rho} F^{\lambda\rho} \delta^\nu_\mu \right) . \tag{5.5}$$

The gravitational field equations then read

$$\mathcal{E}_{\mu\nu} = (T_1)_{\mu\nu} + 8\pi G_N (T_2)_{\mu\nu} \,, \tag{5.6}$$

where

$$\mathcal{E}_\mu^\nu = -\sum_{n=0}^{n_{\max}} \frac{a_n}{2^{n+1}} \delta^{\nu\lambda_1\cdots\lambda_{2n}}_{\mu\rho_1\cdots\rho_{2n}} R^{\rho_1\rho_2}{}_{\lambda_1\lambda_2} \cdots R^{\rho_{2n-1}\rho_{2n}}{}_{\lambda_{2n-1}\lambda_{2n}} \,. \tag{5.7}$$

is the generalized Einstein tensor.

By varying the action with respect to the scalar field, one can show that the scalar field must obey the following equation of motion:

$$\sum_{n=0}^{n_{\max}} \frac{(D-2n)b_n}{2^n} \phi^{D-4n-1} \delta^{(n)} S^{(n)} = 0 \,. \tag{5.8}$$

Note that the above equation of motion ensures that the trace of the stress-energy tensor of the scalar field vanishes on shell, as expected for this conformally invariant theory. Similarly, varying the action with respect to the Maxwell gauge field A_μ, we obtain the Maxwell equations

$$\nabla_\mu F^{\mu\nu} = 0 \,. \tag{5.9}$$

We take a line element of the form

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Sigma^2_{k,D-2} \tag{5.10}$$

where $d\Sigma^2_{k,D-2}$ (its volume denoted $\Omega_{k,D-2}$) is the line element on a surface of constant curvature k with $k = +1, 0, -1$ corresponding to spherical, flat and hyperbolic geometries; in the latter cases, the space is compact via identification [128]. For this ansatz, the field equations for the metric reduce to

$$\begin{aligned} \sum_{n=0}^{n_{\max}} \alpha_n \left(\frac{k-f}{r^2} \right)^n =& \frac{16\pi G_N M}{(D-2)\Omega_{k,D-2} r^{D-1}} \\ &+ \frac{H}{r^D} - \frac{8\pi G_N}{(D-2)(D-3)} \frac{Q^2}{r^{2D-4}} \end{aligned} \tag{5.11}$$

where

$$\begin{aligned} \alpha_0 &= \frac{1}{L^2} = \frac{a_0}{(D-1)(D-2)} \,, \quad \alpha_1 = a_1 \,, \\ \alpha_n &= a_n \prod_{j=3}^{2n} (D-j) \text{ for } n \geq 2 \,, \end{aligned} \tag{5.12}$$

and

$$H = \sum_{n=0}^{n_{\max}} \frac{(D-3)!}{(D-2(n+1))!} b_n k^n N^{D-2n} \tag{5.13}$$

is the "hair parameter". For this configuration, the electromagnetic field strength is given by

$$F = \frac{Q}{r^{D-2}} dt \wedge dr \tag{5.14}$$

where Q is the conserved electric charge. The scalar field takes the form

$$\phi = \frac{N}{r} \tag{5.15}$$

and its equations of motion reduce to the following constraints:

$$\begin{aligned} \sum_{n=1}^{n_{\max}} n b_n \frac{(D-1)!}{(D-2k-1)!} k^{n-1} N^{2-2n} &= 0\,, \\ \sum_{n=0}^{n_{\max}} b_n \frac{(D-1)!\,(D(D-1)+4n^2)}{(d-2n-1)!} k^n N^{-2n} &= 0\,. \end{aligned} \tag{5.16}$$

These equations also ensure that the trace of the scalar field stress energy tensor vanishes. Since these are two equations in a single unknown (N), one equation enforces a constraint on the allowed coupling constants, b_n. Further, these equations can only be solved in the cases $k = \pm 1$, otherwise the only solution is $N = 0$. This means that the scalar field configuration is completely fixed in terms of the coupling constants of the theory, and for a given set of couplings, there are only a finite number of possible values for N corresponding to the various roots of a polynomial. Because of this, the black holes are not technically 'hairy' since that would require there to be an additional, free constant of integration appearing in the equations of motion.

Asymptotically, the field equations reduce to the following embedding equation:

$$h(f_\infty) = 1 - f_\infty + \sum_{n=2}^{n_{\max}} \frac{\alpha_n f_\infty^n}{L^{2n-2}} = 0\,, \tag{5.17}$$

where f_∞ is defined such that the leading order behaviour of the metric at large r is

$$f(r) \sim f_\infty \frac{r^2}{L^2}\,. \tag{5.18}$$

This equation determines the maximally symmetric vacua of the theory, characterized by the curvature

$$R_{\alpha\beta}{}^{\mu\nu} = -\frac{2f_\infty}{L^2}\delta^{[\mu}_{[\alpha}\delta^{\nu]}_{\beta]} \,. \tag{5.19}$$

For the space to be asymptotically AdS, we require that $f_\infty > 0$. Further, since the derivative of (5.17) with respect to f_∞ appears as a prefactor in the linearized equations of motion for graviton perturbations about the vacuum, we also require that

$$h'(f_\infty) < 0 \tag{5.20}$$

to ensure that the graviton is not a ghost or, equivalently, that Newton's constant has the correct sign.

The thermodynamics of the black holes can be studied using the standard methods. Computing the temperature by requiring the absence of conical singularities in the Euclidean sector and the entropy using the Iyer-Wald formalism [115], we find the thermodynamic quantities for this solution are

$$\begin{aligned}
M &= \frac{(D-2)\Omega_{k,D-2}}{16\pi G_N}\sum_{n=0}^{n_{\max}} \alpha_n k^n r_+^{D-2n-1} - \frac{(D-2)\Omega_{k,D-2}H}{16\pi G_N r_+} + \frac{\Omega_{k,D-2}Q^2}{2(D-3)r_+^{D-3}} \\
T &= \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi r_+ \mathcal{D}(r_+)}\left[\sum_n k\alpha_n(D-2n-1)\left(\frac{k}{r_+^2}\right)^{n-1} + \frac{H}{r_+^{D-2}} - \frac{8\pi G_N Q^2}{(D-2)r_+^{2(D-3)}}\right] \\
S &= \frac{\Omega_{k,D-2}}{4G_N}\left[\sum_{n=1}^{n_{\max}} \frac{(D-2)nk^{n-1}\alpha_n}{D-2n} r_+^{D-2n} - \frac{D}{2k(D-4)}H\right] \quad \text{if } b_n = 0 \ \ \forall n > 2.
\end{aligned} \tag{5.21}$$

In the above, $\mathcal{D}(r_+) = \sum_{n=1}^{n_{\max}} n\alpha_n (k r_+^{-2})^{n-1}$. Employing the extended first law

$$\delta M = T\delta S + \Phi\delta Q + \sum_k \Psi^{(n)}\delta\alpha_n + \sum_n \mathcal{K}^{(n)}\delta b_n \tag{5.22}$$

we find it is satisfied provided

$$\begin{aligned}
\Psi^{(n)} &= \frac{\Omega_{k,D-2}(D-2)}{16\pi G_N} k^{n-1} r_+^{D-2n}\left[\frac{k}{r_+} - \frac{4\pi n T}{D-2n}\right] \,, \\
\mathcal{K}^{(n)} &= -\frac{\Omega_{k,D-2}(D-2)!}{16\pi G_N} k^{n-1} N^{D-2n}\left[\frac{k}{(D-2(n+1))! r_+} + \frac{4\pi n T}{(D-2n)!}\right]
\end{aligned} \tag{5.23}$$

and the Smarr relation which follows from scaling

$$(D-3)M = (D-2)TS + (D-3)\Phi Q + \sum_n 2(n-1)\Psi^{(n)}\alpha_n + (D-2)\sum_n \mathcal{K}^{(n)} b_n \,, \tag{5.24}$$

also holds. We point out that in the situation when black hole solutions are considered, the couplings b_k are not all independent, but are constrained by eq. (5.16). As a result, in these cases, one must keep in mind that the variations of b_k in the first law above are not all independent. Henceforth we shall set $\alpha_1 = 1$ so that we recover general relativity in the limit $\alpha_k \to 0$ for $k > 1$ and we will also set $G_N = 1$.

This completes the basic set-up of black holes in this theory, our focus will now turn to understanding the phase structure.

5.2 Black hole λ-line

An exhaustive analysis of the thermodynamics of these black holes was performed in [14], where it was found that the black holes exhibit a rich thermodynamic structure, including examples of van der Waals behaviour, re-entrant phase transitions, and triple points in their thermal phase space. Here we will focus on the most interesting result to come from that work (also reported in [13]) — an example of a black hole λ-line.

To present the simplest possible example, in what follows we consider $\alpha_n = 0\,\forall n > 3$ and $b_n = 0\,\forall n > 2$. This last condition is for simplicity: the falloff in the metric function is the same for all b_n and the contribution to the entropy is always just a constant; so only the first three b_n's are required to see all the physics of the scalar hair.

Introducing the dimensionless parameters

$$\begin{aligned}
r_+ &= v\alpha_3^{1/4}\,, \quad T = \frac{t\alpha_3^{-1/4}}{D-2}\,, \quad H = \frac{4\pi h}{D-2}\alpha_3^{\frac{D-2}{4}} \\
Q &= \frac{q}{\sqrt{2}}\alpha_3^{\frac{D-3}{4}}\,, \quad m = \frac{16\pi M}{(D-2)\Sigma^k_{D-2}\alpha_3^{\frac{D-3}{4}}} \\
p &= \frac{\alpha_0(D-1)(D-2)\sqrt{\alpha_3}}{4\pi}\,, \quad \alpha = \frac{\alpha_2}{\sqrt{\alpha_3}}\,, \\
G &= M - TS = \alpha_3^{\frac{(D-3)}{4}}\Omega_{k,D-2}\,g
\end{aligned} \tag{5.25}$$

the dimensionless equation of state (obtained by solving the expression for the temperature in eq. (5.21) for the pressure) reads

$$\begin{aligned}
p =& \frac{t}{v} - \frac{\sigma(D-3)(D-2)}{4\pi v^2} + \frac{2\alpha k t}{v^3} - \frac{\alpha(D-2)(D-5)}{4\pi v^4} + \frac{3t}{v^5} \\
&- \frac{k(D-7)(D-2)}{4\pi v^6} + \frac{q^2}{v^{2(D-2)}} - \frac{h}{v^D}
\end{aligned} \tag{5.26}$$

where the quantity p represents the pressure and g the dimensionless Gibbs free energy. At equilibrium, the state of the system is that which minimizes the Gibbs free energy.

Let us now consider the behaviour of this equation of state. Noting that the conditions for a critical point are

$$\frac{\partial p}{\partial v} = \frac{\partial^2 p}{\partial v^2} = 0 \tag{5.27}$$

we find that for $\alpha = \sqrt{5/3}$ if h and q are set to

$$h = \frac{4(2D-5)(D-2)^2 v_c^{D-6}}{\pi D(D-4)}, \;\; q^2 = \frac{2(D-1)(D-2)v_c^{2D-10}}{\pi(D-4)}\,, \tag{5.28}$$

and $k = -1$, eq. (5.27) will be satisfied by $v_c = 15^{1/4}$ and

$$p_c = \left[\frac{8}{225}(15)^{\frac{3}{4}}\right] t_c + \frac{\sqrt{15}(11D-40)(D-1)(D-2)}{900\pi D} \tag{5.29}$$

for all temperatures t_c! In other words, this system exhibits *infinitely many critical points* with critical volume $v_c = 15^{1/4}$. In the $p - v$ plane, every isotherm is a critical isotherm, i.e. has an inflection point at $v = 15^{1/4}$. In the variables (t, p) there is no first order phase transition but rather a line of second order phase transitions, characterized by a diverging specific heat $c_p = -t\,\partial^2 g/\partial t^2$ at the critical values. We show representative thermodynamic behaviour in figure 5.1 for $D = 7$. We note in passing that the line of second order transitions given in (5.29) exhibits dimension dependence only in the zero temperature intercept — the slope is the same in all dimensions.

Let us pause for a moment to discuss where, in condensed matter systems, similar lines of second order phase transitions occur. In that context, lines of second order phase transitions correspond to, for example, fluid/superfluid transitions [233], the onset of superconductivity [234], and paramagentism/ferromagnetism transitions [235]. In these cases, lines of second order phase transitions are often called 'λ-lines' since the divergence of the specific heat across the phase transition resembles the Greek letter 'λ' (see below).

To the best of our knowledge, this is the first example of a of a λ transition in black hole thermodynamics. Building on the black hole/van der Waals fluid analogy [39], the natural interpretation here is that this second order phase transition between small/large black holes is analogous to a fluid/superfluid type transition. The resemblance to the fluid/superfluid λ-line transition of ^{4}He (c.f. figure 2 in [13]) is striking. In each case, a line of critical points separates the two phases of 'fluid' where specific heat takes on the same qualitative "λ" structure. The phase diagram for helium is more complicated, including solid and gaseous states. This is to be expected since helium is a complicated system, while these hairy black hole solutions are comparatively simple being characterized by only four numbers: v, h, q and α. However, it is remarkable that with so few parameters the essence of the λ-line can be captured. Most of the interesting properties of a superfluid are either dynamical or require a full quantum description to understand (see, e.g. [236, 237] for an introduction and review). Since we do not have access to a model of the underlying quantum degrees of freedom it is not possible to explore the black hole analogues of these properties at a deeper level. However, an interesting direction for future investigation would be to compute transport properties in the holographically dual theories and see if they exhibit any interesting behaviour near the λ-line.

5.3 Further properties and a necessary condition

While the specific heat is positive — indicating thermal stability — it is natural to wonder if there is any pathological behaviour hiding behind the scenes here. We will consider several possibilities below, in most cases particularized to the case of $d = 7$ for the purpose of clarity.

First, we consider the Kretschmann scalar evaluated on the horizon:

$$K = R_{abcd}R^{abcd} = \left[\left(\frac{d^2 f}{dr^2}\right)^2 + \frac{2(D-2)}{r^2}\left(\frac{df}{dr}\right)^2 + \frac{2(D-2)(D-3)}{r^4}\right]_{r=r_+} . \tag{5.30}$$

The first derivative of f is clearly finite for any finite temperature, so we need only consider f''. For simplicity we consider the case $D = 7$ where the λ-line solution was first observed. Expanding near v_c, p_c we see that

$$f''(v_c+dv, p_c+dp) = 200\pi^2 t_c^2 + \frac{172\pi(15)^{1/4}}{75} t_c + \left[750(15)^{1/4}\pi^2 t_c + \frac{129\pi}{2}\right] dp - \frac{172\sqrt{15}\pi}{75} t_c dv \, . \tag{5.31}$$

This is completely finite at the horizon both at the critical point and near it; there are no curvature singularities associated with this thermodynamic behaviour. For thoroughness, we have also examined the explicit solution to the field equations in detail. Outside the horizon the metric function is well-behaved and the Kretschmann scalar is everywhere finite. There is a curvature singularity that occurs within the horizon and before $r = 0$, but this type of behaviour is nothing new —similar behaviour occurs for both neutral and electrically charged black holes in Gauss-Bonnet and cubic Lovelock black holes [170, 43].

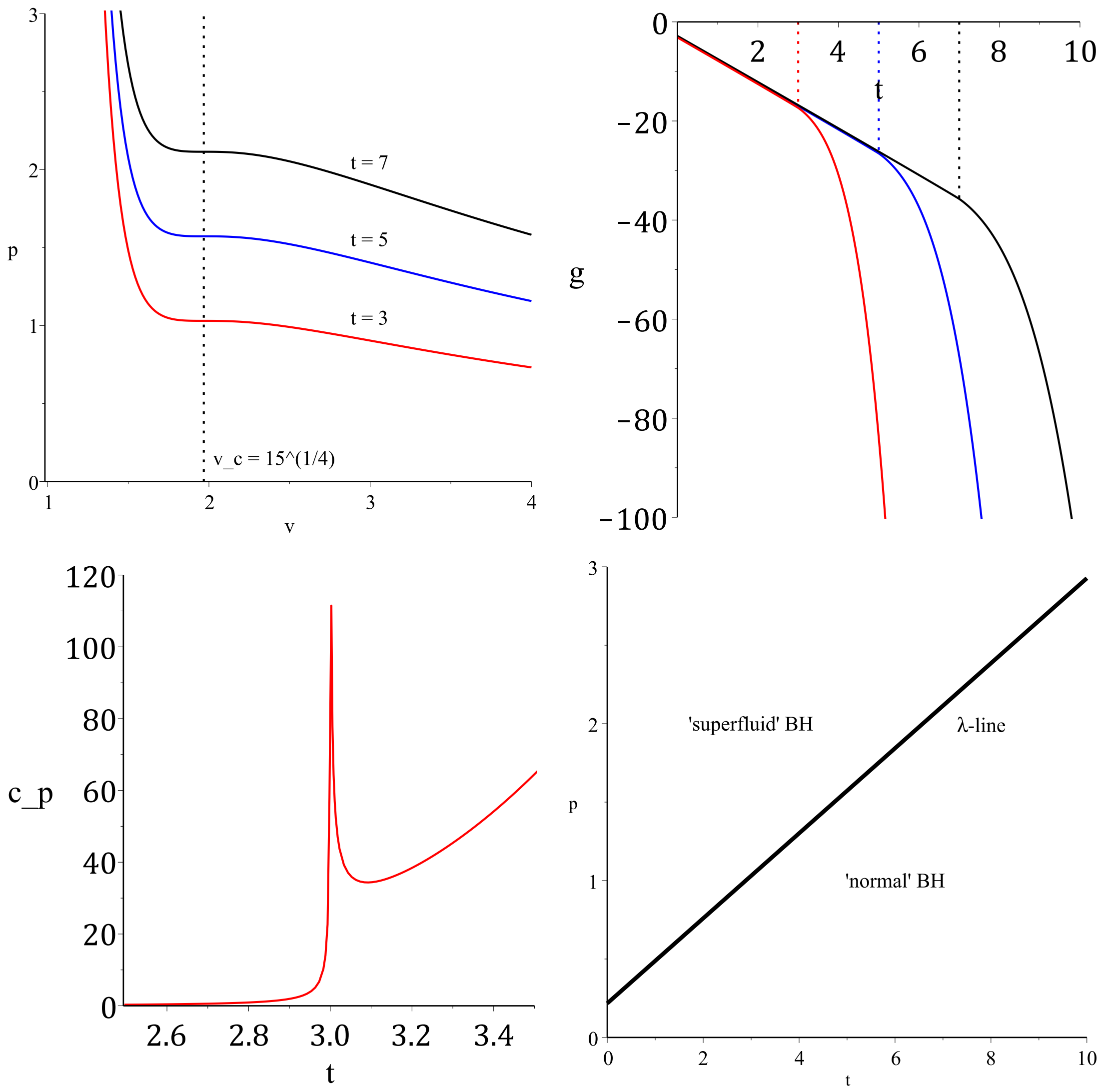

Figure 5.1: **Thermodynamic behaviour near λ transition**. *Top left*: the $p-v$ diagram for various temperatures. Note that all the isotherms here are critical isotherms with an inflection point. *Top right*: A plot of the Gibbs free energy vs. temperature for three distinct pressures chosen so that critical temperatures are $t_c = 3, 5, 7$ corresponding to the red, blue and black curves. The dotted lines highlight the points where the second derivative of the Gibbs free energy diverges. *Bottom left*: A plot of the specific heat $c_p = -t\frac{\partial^2 g}{\partial t^2}$ for the case $t_c = 3$. *Bottom right*: p–t parameter space. The black line shows the locus of critical points, i.e. a line of second-order phase transitions known as the 'lambda' line in the context of superfluidity. These plots are for $D = 7$.

Next, we note that the couplings are such that the vacuum is free of ghosts. It can be verified that, along the λ-line, the conditions are such that $h'(f_\infty) < 0$ for all temperatures. Indeed, it can be shown that in any dimension, the maximum value of $h'(f_\infty)$ is $-4/9$ which occurs at the temperature

$$t = \frac{(16D^3 + 87D^2 - 373D + 270)15^{7/4}}{108\pi D}, \tag{5.32}$$

which is positive in all cases of interest. Therefore, theses choices of couplings correspond to vacua with a positive Newton constant. Further, since the only constraint is that $\alpha_3 = (3/5)\alpha_2^2$ it seems that the black holes satisfy known constraints arising from the physicality conditions imposed on the dual CFT, e.g. positivity of energy flux and causality constraints [218]. These checks suggest that there is nothing obviously wrong with the black hole solutions exhibiting these transitions. A more thorough analysis (e.g. a study of the dynamics of perturbations) would be required to make conclusive statements about the stability.

The λ-line is a line of critical points, and it would be desirable to determine the critical exponents along this line. The standard method of computing the critical exponents ultimately fails in this case, as we will now explicitly highlight. To see this, suppose we proceed to calculate the critical exponents in the naive way (we do this for $D = 7$ for concreteness). We can expand the equation of state near any one of the infinitely many critical points to obtain

$$\frac{p}{p_c} = 1 + \frac{112(15)^{1/4}\pi t_c}{112(15)^{1/4}\pi t_c + 555}\tau - \frac{280(15)^{1/4}\pi t_c}{112(15)^{1/4}\pi t_c + 555}\tau\omega^3 - \frac{140(85 + 2\pi(15)^{1/4}t_c)}{112(15)^{1/4}\pi t_c + 555}\omega^3 . \tag{5.33}$$

Proceeding naively, we find that $\alpha = 0$ governs the behaviour of the specific heat at constant volume near the critical point and $\delta = 3$ governs the behaviour of $|p - p_c| \propto |v - v_c|^\delta$ along any critical isotherm. For β, we evaluate the following expressions:

$$\frac{p}{p_c}\Big|_{\omega=\omega_1} = \frac{p}{p_c}\Big|_{\omega=\omega_2} \qquad 0 = \int_{\omega_1}^{\omega_2} \omega \frac{d(p/p_c)}{d\omega} \tag{5.34}$$

These equations can only be solved by the trivial solution, $\omega_1 = \omega_2$, or if τ is constrained to be

$$\tau = -\frac{15^{1/4}}{30\pi t_c}\left[85\sqrt{15} + 2\pi(15)^{3/4}t_c\right] \tag{5.35}$$

with ω_1 and ω_2 free. This latter case is not a sensible solution since in the limit of $\tau \to 0$, we are left with a negative t_c, while for the trivial solution we have that the order parameter,

$\eta = v_c(\omega_2 - \omega_1)$ vanishes, suggesting that $\beta = 0$. This result is unchanged by the inclusion of higher order terms in the expansion of the equation of state. The exponent γ governs the behaviour of the isothermal compressibility near criticality

$$\kappa_T = -\frac{1}{v}\frac{\partial v}{\partial P}\Big|_T \propto |\tau|^{-\gamma} \,. \tag{5.36}$$

Computing this for the above expansion we find

$$\kappa_T = \frac{1}{420\omega^2}\left[\frac{112\pi(15)^{1/4}t_c + 555}{2\pi(15)^{1/4}t_c\tau + 2\pi(15)^{1/4}t_c + 85}\right] \tag{5.37}$$

which in the limit of the critical point is independent of τ, suggesting that $\gamma = 0$. Therefore, by this argument, it seems that each critical point on this line of criticality is characterized by the critical exponents

$$\alpha = 0\,, \quad \beta = 0\,, \quad \gamma = 0\,, \quad \delta = 3\,. \tag{5.38}$$

These critical exponents trivially satisfy the Widom relation

$$\gamma = \beta(\delta - 1)\,, \tag{5.39}$$

but violate the Rushbrooke inequality

$$\alpha + 2\beta + \gamma \ngeq 2\,. \tag{5.40}$$

While these are manifestly 'non-standard' critical exponents, these bizarre results signal that something has gone wrong in the approach. The problem lies in the assumption that the pressure is still the ordering field. Here, this is not the case—changing the pressure merely changes the temperature at which the second order phase transition occurs. Thus, pressure is no longer the appropriate ordering field, a situation similar to that in liquid He^4 at the λ line [233].

If one wishes to assign critical exponents to the λ-line, the thermodynamic parameter space must be enlarged. That is, we must choose an additional parameter such that, in the larger phase space, the λ-line is approached along a first order coexistence curve. There are three choices here for the ordering field, Θ, and they are q, h or α. It turns out that the obtained critical exponents are the same regardless of which choice is made, but the electric charge q is in some sense the most natural choice since it is easy to imagine adjusting q by throwing charged material into the black hole. To calculate the critical exponents, we proceed as usual, expanding the ordering field near any of the critical points in terms of τ and ω. We find

$$\frac{\Theta}{\Theta_c} = 1 - A\tau + B\tau\omega - C\omega^3 + \mathcal{O}(\tau\omega^2, \omega^4)\,, \tag{5.41}$$

where the values of A, B, C depend on both the pressure, p, and will be different (but non-zero) depending on which choice is made for the ordering field. This expansion yields the following critical exponents

$$\tilde{\alpha} = 0\,, \quad \tilde{\beta} = \frac{1}{2}\,, \quad \tilde{\gamma} = 1\,, \quad \tilde{\delta} = 3\,, \tag{5.42}$$

which govern the behaviour of the specific heat at constant volume, $C_V \propto |\tau|^{-\tilde{\alpha}}$, the order parameter $\omega \propto |\tau|^{\tilde{\beta}}$, the susceptibility/compressibility $(\partial\omega/\partial\Theta)|_\tau \propto |\tau|^{-\tilde{\gamma}}$ and the ordering field $|\Theta - \Theta_c| \propto |\omega|^{\delta}$ near a critical point. These results coincide with the mean field theory values, which agree with those for a superfluid in a $D > 5$ (cf. Table I of [233]), though this is not particularly insightful since essentially all critical phenomena in black hole physics falls into this universality class.

One way to visualize this result is that the line of critical points in the $p - t$ plane represents a line where a surface of first order phase transitions terminates in some larger space (p, t, Θ). Our calculation of critical exponents then represents the behaviour of the system as the line of criticality is approached, not in (p, t) space, but rather in this larger space. We highlight this in figure 5.2 for the case (p, t, q^2).

The λ-line is an interesting new result in the thermodynamics of black holes, and it would be desirable to find further examples where it occurs. With some hindsight, we can return to the Lovelock case and see if anything general can be learned. The necessary feature of our result is that the conditions for a critical point are satisfied irrespective of the temperature. Thus, consider a general black hole equation of state of the form

$$P = a_1(r_+, \varphi_i)T + a_2(r_+, \varphi_i) \tag{5.43}$$

where φ_i represent additional constants in the equation of state (here they would correspond to α, q and h), and the equation of state is linear in these parameters. The equation of state considered above is of this form, and it is more generally true for a wide class of higher curvature theories. Given this equation of state, a λ-line will occur provided that the following system of equations is satisfied:

$$\frac{\partial a_1}{\partial r_+} = 0\,, \quad \frac{\partial^2 a_1}{\partial r_+^2} = 0\,, \quad \frac{\partial a_2}{\partial r_+} = 0\,, \quad \frac{\partial^2 a_2}{\partial r_+^2} = 0\,, \tag{5.44}$$

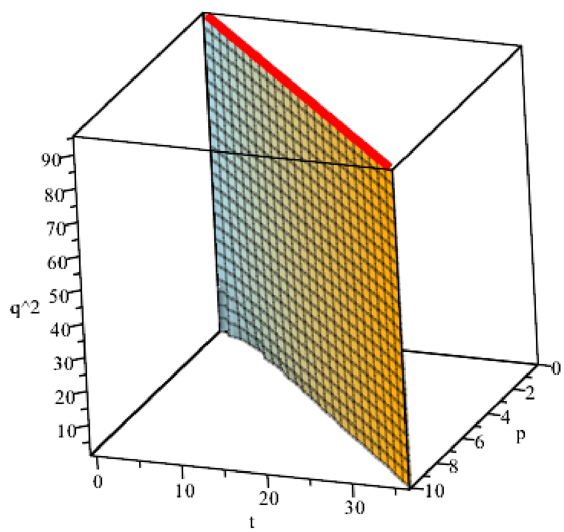

Figure 5.2: **Line of criticality in** (p, t, q) **space**. If the space of thermodynamic parameters is enlarged, the line of critical points in the $p - t$ plane (the bold red line here) can be thought of as the critical line at which a surface of first order phase transitions terminates. For each constant p slice, this is a first order small/large black hole phase transition as temperature increases.

with both a_1 and a_2 non-trivial. From this perspective, it makes sense that we found the behaviour that we did: here we have four equations, and in the case considered here there are a total of four variables. It is natural then to wonder if other systems exhibit this behaviour. We have checked this for the rotating black hole of $5D$ minimal gauged super-gravity [238] which has four parameters, but have found that no solution to the above equations exist. Furthermore, in the case of higher order Lovelock gravity with electric charge (but not hair) solving the four equations forces $a_1 = 0$. Hence such a line of critical points does not occur (or, if something similar does, it happens under a different configuration).

5.3.1 An example in quasi-topological gravity

As an example, we can apply condition eq. 5.44 in quasi-topological gravity coupled to scalar fields in the same manner described above. The details of this calculation were presented in [15], and here we simply quote some results for the purposes of illustration. Since quasi-topological gravity is non-trivial in $D = 5$, this provides a means of constructing the λ-line in fewer than seven dimensions.

The equation of state for cubic quasi-topological black holes with scalar hair takes the form [15]

$$p = \frac{t}{v} - \frac{3k}{2\pi v^2} + \frac{2\alpha k t}{v^3} + \epsilon\frac{3k^2 t}{v^5} + \epsilon\frac{3k^3}{2\pi v^6} + \frac{q^2}{v^6} - \frac{h}{v^5}\,, \tag{5.45}$$

where ϵ represents the sign of the quasi-topological coupling, and the various lower-case quantities correspond to dimensionless thermodynamic parameters in the same way as above. Notice that with $\epsilon = +1$ this is identical to the equation of state presented for the Lovelock black holes with $D = 5$. The equation of state has the form of eq. (5.43) with

$$a_1 = \frac{1}{v} + \frac{2\alpha k}{v^3} + \epsilon\frac{3k^2}{v^5}\,, \quad a_2 = -\frac{3k}{2\pi v^2} + \epsilon\frac{3k^3}{2\pi v^6} - \frac{h}{v^5} + \frac{q^2}{v^6}\,. \tag{5.46}$$

The condition for a black hole λ-line is given by the simultaneous solution of

$$\frac{\partial a_i}{\partial v} = 0\,, \quad \frac{\partial^2 a_i}{\partial v^2} = 0 \text{ for } i = 1, 2 \tag{5.47}$$

which amounts to solving the conditions for a critical point without placing any restrictions on what t_c should be. Here we find that a solution to this system exists, but only for the following parameters:

$$\epsilon = +1\,, \quad k = -1\,, \quad \alpha = \sqrt{\frac{5}{3}}\,, \quad v_c = 15^{1/4}\,, \quad h = \frac{12(15)^{3/4}}{5\pi}\,, \quad q^2 = \frac{24}{\pi}\,. \tag{5.48}$$

The value of α above agrees with the result obtained by setting $D = 5$ in the expressions from the Lovelock case above, but the values of q and h are different. This is simply because there is an additional term in a_2 for the equation of state when $D > 5$.

For the above parameter values we have a line of critical points with the critical values

$$v_c = 15^{1/4}\,, \quad p_c = \frac{8(15)^{3/4}}{225}t_c + \frac{\sqrt{15}}{25\pi}\,, \quad t_c \in \mathbb{R}^+\,. \tag{5.49}$$

We emphasize that there is no first order phase transition associated with this line of critical points: there is simply a line of second order (continuous) phase transitions. The entropy of the black holes which possess this superfluid-like transition is positive. Once again, by enlarging the phase space, one can show that the critical exponents are the mean field theory values.

5.4 Summary remarks

In this chapter, we have presented the first example of a λ-line in black hole thermodynamics. This line of second order phase transitions was observed in a class of Lovelock black holes coupled to scalar fields in seven and higher dimensions. The λ-line resembles those that occur in various condensed matter models such as, for example, superfluidity.

An analysis of the black hole solutions was given indicating they are free of any obvious pathologies. Reflecting on the structure of the equation of state that permitted this solution, it was possible to determine a necessary condition for λ-line transitions to occur. This was successfully used to identify a second example of a λ-line in quasi-topological gravity, also coupled to scalar fields. Going forward, it would be desirable to understand what properties distinguish the black holes across the λ-line. In the case of a first order phase transition, the order parameter is simply the difference in size between the two phases of black holes. However, since this is a second order phase transition, the size varies continuously across the phase transition. It is possible some insight could be gained from holography, e.g. by studying the behaviour of transport properties in the dual theory across the λ transition.

It may be helpful in this endeavor to find further examples of the black hole λ-line beyond the Lovelock and quasi-topological gravity examples presented here. One possibility for this would be to explore black holes that have an equation of state that is not linear in temperature. In this direction, there has been a recent example found in Hořava gravity [239, 240] for four-dimensional black holes with spherical horizon topology.

Chapter 6

Final thoughts

The primary purpose of this thesis was to present two novel results in black hole chemistry: super-entropic black holes and black hole λ-lines. The secondary purpose was to highlight recent work on higher curvature theories of gravity that have particularly nice properties when restricted to spherically symmetric metrics. Before concluding the thesis, let us reflect on the results presented here, what they have taught us, and what meaning they may hold for future directions.

Super-entropic black holes are rotating, asymptotically (locally) AdS solutions of the Einstein equations. The horizons of these black holes are topologically spheres with punctures at the north and sole poles. This result is interesting in part because black holes in general relativity are highly constrained objects, and the topology of the event horizon is no exception. This is especially true in the four-dimensional, asymptotically flat case where horizon must have spherical topology, provided the dominant energy condition is satisfied. The super-entropic black holes therefore provide new examples, in four and higher dimensions, of the possibilities for event horizon topology when a cosmological constant is included. While intrinsically interesting for this reason in the context of gravity, the solutions are further valuable since they are asymptotically AdS and so can be studied in the AdS/CFT correspondence.

As was discussed, the super-entropic black holes also violate the conjectured reverse isoperimetric inequality. This inequality (which is the "opposite" of the standard reverse isoperimetric inequality) suggests that the entropy of asymptotically AdS black holes should be bounded from above in a way that depends on the thermodynamic volume. So far, the super-entropic black holes provide the only examples of black holes in Einstein gravity that violate the conjecture. The necessary and sufficient conditions for the inequality are at present unknown, but the fact that the super-entropic black hole violates the inequality may provide some insight. For example, two non-trivial aspects of these solutions is their non-compact horizon and their asymptotic structure — the super-entropic black holes are only *locally* asymptotically AdS since the boundary metric is not topologically $\mathbb{R} \times \mathbb{S}^{D-2}$. Perhaps, then, the reverse isoperimetric inequality requires that the horizon be compact and/or that the black hole be asymptotically AdS in the strictest sense.

The second key result concerned phase transitions of black hole spacetimes. Since the discovery of the AdS/CFT correspondence, there has been considerable effort dedicated to the investigation of the phase transitions of AdS black holes since this thermal behaviour can in some sense be mapped to that of the dual theory. When the advent of black hole chemistry, research on black hole critical behaviour has further intensified, and numerous new and interesting results have been obtained. For example, van der Waals behaviour, triple points, and (multiple) re-entrant phase transitions have been found for black holes involving various matter fields, and also in various theories of gravity. In this thesis, we have discussed a new addition to this list — the black hole λ-line. In contrast to more ordinary van der Waals behaviour where a line of first order phase transitions terminates at a critical point (which corresponds to a second-order phase transition), λ-lines are *lines* of second order phase transitions. That is, the conditions for a critical point are satisfied at infinitely many points in the thermodynamic parameter space and there is no associated first-order phase transition. In some sense, this could be viewed as a second-order (or continuous) analog of the Hawking-Page transition. While this result is intrinsically interesting insofar as it deepens our understanding of the types of thermal behaviour accessible to AdS black holes, it would be desirable to better understand the 'microscopics' of the λ-transition or its implications for holographic field theories. In the gravitational picture outlined in this thesis, it was not possible to know precisely what this underlying description is. In ordinary thermal systems, λ-lines can represent fluid/superfluid transitions, mark the onset of superconductivity, or mark paramagnetism/ferromagnetism transitions. Perhaps a future investigation involving holography could help to shed light on whether any of these descriptions provide a decent picture for what this phase transition could be describing.

In the context of higher curvature theories of gravity we discussed how, under the restriction to spherically symmetric metrics, there are three particularly natural classes of gravitational theories, one of which — generalized quasi-topological gravity — was previously unknown. When constructing the theories, we posed an intentionally naive question: "what theories of gravity have a single independent field equation for spherically symmetric metrics?" However, it turns out that this property — which is true for Einstein gravity and Lovelock gravity, but often taken for granted — implies a number of interesting and unexpected results for the gravity theories that possess it. We saw that theories meeting this single field equation criterion have second-order linearized equations of motion on constant curvature spaces, and so do not propagate ghost modes. For a spherically symmetric metric, the single independent field equation is a total derivative and so can be integrated to give an equation that is at most a second-order differential equation for the metric function. Further, black hole solutions of the theories seem to be characterized by mass alone, and the thermodynamics of black holes can be studied exactly despite the lack of an analytical form for the metric function.

It is the hope that this class of theories will provide useful toy models in the context of black hole thermodynamics and holography. One promising direction that is currently under investigation is Taub-NUT type metrics [9, 10]. In that work, we demonstrate that the generalized quasi-topological theories allow for Taub-NUT and Bolt solutions characterized by a single metric function in four and six dimensions (and almost certainly in higher dimensions) for a variety of base spaces. This work represents, to the best of our knowledge, the first example of four-dimensional Taub-NUT metrics in a higher curvature theory of gravity. Further, the higher curvature terms have allowed us to identify what appear to be universal results for the energy and free energy of these solutions in terms of the embedding equation and the Lagrangian evaluated on an AdS space. Since Taub-NUT metrics with complex projective spaces as the base describe holographic theories on squashed spheres, these results have allowed us to identify a universal structure in how the free energy of CFTs on squashed spheres behaves for small squashings. As this is a work in progress, further scrutiny will be required before a conclusive statement can be made.

References

[1] R. A. Hennigar, D. Kubiznak and R. B. Mann, *Entropy Inequality Violations from Ultraspinning Black Holes*, *Phys. Rev. Lett.* **115** (2015) 031101, [1411.4309].

[2] R. A. Hennigar, D. Kubiznak, R. B. Mann and N. Musoke, *Ultraspinning limits and super-entropic black holes*, *JHEP* **06** (2015) 096, [1504.07529].

[3] R. A. Hennigar, D. Kubiznak, R. B. Mann and N. Musoke, *Ultraspinning limits and rotating hyperboloid membranes*, *Nucl. Phys.* **B903** (2016) 400–417, [1512.02293].

[4] R. A. Hennigar and R. B. Mann, *Black holes in Einsteinian cubic gravity*, *Phys. Rev.* **D95** (2017) 064055, [1610.06675].

[5] R. A. Hennigar, D. Kubiznak and R. B. Mann, *Generalized quasitopological gravity*, *Phys. Rev.* **D95** (2017) 104042, [1703.01631].

[6] J. Ahmed, R. A. Hennigar, R. B. Mann and M. Mir, *Quintessential Quartic Quasi-topological Quartet*, *JHEP* **05** (2017) 134, [1703.11007].

[7] R. A. Hennigar, *Criticality for charged black branes*, *JHEP* **09** (2017) 082, [1705.07094].

[8] R. A. Hennigar, M. B. J. Poshteh and R. B. Mann, *Shadows, Signals, and Stability in Einsteinian Cubic Gravity*, *Phys. Rev.* **D97** (2018) 064041, [1801.03223].

[9] P. Bueno, P. A. Cano, R. A. Hennigar and R. B. Mann, *NUTs and bolts beyond Lovelock*, [1808.01671].

[10] P. Bueno, P. A. Cano, R. A. Hennigar and R. B. Mann, *Universality of squashed-sphere partition functions*, [1808.02052].

[11] R. A. Hennigar, M. Lu and R. B. Mann, *Five-dimensional generalized quasi-topological gravities*, *In preparation, 2018* .

[12] M. Mir, R. A. Hennigar, R. B. Mann and J. Ahmed, *Chemistry and holography in generalized quasi-topological gravity*, *In preparation, 2018* .

[13] R. A. Hennigar, R. B. Mann and E. Tjoa, *Superfluid Black Holes*, *Phys. Rev. Lett.* **118** (2017) 021301, [1609.02564].

[14] R. A. Hennigar, E. Tjoa and R. B. Mann, *Thermodynamics of hairy black holes in Lovelock gravity*, *JHEP* **02** (2017) 070, [1612.06852].

[15] H. Dykaar, R. A. Hennigar and R. B. Mann, *Hairy black holes in cubic quasi-topological gravity*, *JHEP* **05** (2017) 045, [1703.01633].

[16] R. A. Hennigar, W. G. Brenna and R. B. Mann, *Pv criticality in quasitopological gravity*, *JHEP* **07** (2015) 077, [1505.05517].

[17] R. A. Hennigar and R. B. Mann, *Reentrant phase transitions and van der Waals behaviour for hairy black holes*, *Entropy* **17** (2015) 8056–8072, [1509.06798].

[18] R. A. Hennigar, R. B. Mann and S. Mbarek, *Thermalon mediated phase transitions in Gauss-Bonnet gravity*, *JHEP* **02** (2016) 034, [1512.02611].

[19] R. A. Hennigar, F. McCarthy, A. Ballon and R. B. Mann, *Holographic heat engines: general considerations and rotating black holes*, *Class. Quant. Grav.* **34** (2017) 175005, [1704.02314].

[20] L. J. Henderson, R. A. Hennigar, R. B. Mann, A. R. H. Smith and J. Zhang, *Harvesting Entanglement from the Black Hole Vacuum*, 1712.10018.

[21] P. A. Cano, R. A. Hennigar and H. Marrochio, *Complexity Growth Rate in Lovelock Gravity*, 1803.02795.

[22] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time*. Cambridge, England, 1973.

[23] S. J. Avis, C. J. Isham and D. Storey, *Quantum Field Theory in anti-De Sitter Space-Time*, *Phys. Rev.* **D18** (1978) 3565.

[24] S. W. Hawking, *Black Holes and Thermodynamics*, *Phys. Rev.* **D13** (1976) 191–197.

[25] S. Hawking and D. N. Page, *Thermodynamics of Black Holes in anti-De Sitter Space*, Commun.Math.Phys. **87** (1983) 577.

[26] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2** (1998) 253–291, [hep-th/9802150].

[27] E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories*, Adv. Theor. Math. Phys. **2** (1998) 505–532, [hep-th/9803131].

[28] A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, *Large N phases, gravitational instantons and the nuts and bolts of AdS holography*, Phys. Rev. **D59** (1999) 064010, [hep-th/9808177].

[29] S. Hawking, C. Hunter and M. Taylor, *Rotation and the AdS / CFT correspondence*, Phys.Rev. **D59** (1999) 064005, [hep-th/9811056].

[30] A. Chamblin, R. Emparan, C. Johnson and R. Myers, *Charged AdS black holes and catastrophic holography*, Phys.Rev. **D60** (1999) 064018, [hep-th/9902170].

[31] A. Chamblin, R. Emparan, C. Johnson and R. Myers, *Holography, thermodynamics and fluctuations of charged AdS black holes*, Phys.Rev. **D60** (1999) 104026, [hep-th/9904197].

[32] M. Caldarelli, G. Cognola and D. Klemm, *Thermodynamics of Kerr-Newman-AdS black holes and conformal field theories*, Class.Quant.Grav. **17** (2000) 399–420, [hep-th/9908022].

[33] R. Emparan, *AdS / CFT duals of topological black holes and the entropy of zero energy states*, JHEP **06** (1999) 036, [hep-th/9906040].

[34] M. Cvetic and S. Gubser, *Phases of R charged black holes, spinning branes and strongly coupled gauge theories*, JHEP **9904** (1999) 024, [hep-th/9902195].

[35] S. Nojiri and S. D. Odintsov, *Anti-de Sitter black hole thermodynamics in higher derivative gravity and new confining deconfining phases in dual CFT*, Phys. Lett. **B521** (2001) 87–95, [hep-th/0109122].

[36] R.-G. Cai, *Gauss-Bonnet black holes in AdS spaces*, Phys. Rev. **D65** (2002) 084014, [hep-th/0109133].

[37] D. Kastor, *Komar Integrals in Higher (and Lower) Derivative Gravity*, Class.Quant.Grav. **25** (2008) 175007, [0804.1832].

[38] D. Kastor, S. Ray and J. Traschen, *Enthalpy and the Mechanics of AdS Black Holes*, Class.Quant.Grav. **26** (2009) 195011, [0904.2765].

[39] D. Kubiznak and R. B. Mann, *P-V criticality of charged AdS black holes*, JHEP **1207** (2012) 033, [1205.0559].

[40] C. Hudson, *The mutual solubility of nicotine in water*, Zeit Phys Chem **47** (1904) 113.

[41] N. Altamirano, D. Kubiznak, R. B. Mann and Z. Sherkatghanad, *Kerr-AdS analogue of triple point and solid/liquid/gas phase transition*, Class. Quant. Grav. **31** (2014) 042001, [1308.2672].

[42] N. Altamirano, D. Kubiznak and R. B. Mann, *Reentrant phase transitions in rotating antide Sitter black holes*, Phys. Rev. **D88** (2013) 101502, [1306.5756].

[43] A. M. Frassino, D. Kubiznak, R. B. Mann and F. Simovic, *Multiple Reentrant Phase Transitions and Triple Points in Lovelock Thermodynamics*, 1406.7015.

[44] B. P. Dolan, A. Kostouki, D. Kubiznak and R. B. Mann, *Isolated critical point from Lovelock gravity*, Class. Quant. Grav. **31** (2014) 242001, [1407.4783].

[45] D. Kubiznak and R. B. Mann, *Black Hole Chemistry*, 1404.2126.

[46] M. Cvetic, G. Gibbons, D. Kubiznak and C. Pope, *Black Hole Enthalpy and an Entropy Inequality for the Thermodynamic Volume*, Phys.Rev. **D84** (2011) 024037, [1012.2888].

[47] C. V. Johnson, *Holographic Heat Engines*, Class.Quant.Grav. **31** (2014) 205002, [1404.5982].

[48] A. Karch and B. Robinson, *Holographic Black Hole Chemistry*, JHEP **12** (2015) 073, [1510.02472].

[49] E. Caceres, P. H. Nguyen and J. F. Pedraza, *Holographic entanglement entropy and the extended phase structure of STU black holes*, JHEP **09** (2015) 184, [1507.06069].

[50] B. P. Dolan, *Pressure and compressibility of conformal field theories from the AdS/CFT correspondence*, Entropy **18** (2016) 169, [1603.06279].

[51] J. Couch, W. Fischler and P. H. Nguyen, *Noether charge, black hole volume and complexity*, 1610.02038.

[52] G. W. Gibbons, *Anti-de-Sitter spacetime and its uses*, in *Mathematical and quantum aspects of relativity and cosmology. Proceedings, 2nd Samos Meeting on cosmology, geometry and relativity, Pythagoreon, Samos, Greece, August 31-September 4, 1998*, pp. 102–142, 2011. 1110.1206.

[53] M. Banados, C. Teitelboim and J. Zanelli, *The Black hole in three-dimensional space-time*, *Phys. Rev. Lett.* **69** (1992) 1849–1851, [hep-th/9204099].

[54] M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, *Geometry of the (2+1) black hole*, *Phys. Rev.* **D48** (1993) 1506–1525, [gr-qc/9302012].

[55] J. D. Brown and M. Henneaux, *Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity*, *Commun. Math. Phys.* **104** (1986) 207–226.

[56] J. L. Cardy, *Operator Content of Two-Dimensional Conformally Invariant Theories*, *Nucl. Phys.* **B270** (1986) 186–204.

[57] S. Carlip, *What we don't know about BTZ black hole entropy*, *Class. Quant. Grav.* **15** (1998) 3609–3625, [hep-th/9806026].

[58] M. Z. Iofa and L. A. Pando Zayas, *Statistical entropy of magnetic black holes from near horizon geometry*, *Phys. Lett.* **B434** (1998) 264–268, [hep-th/9803083].

[59] M. Z. Iofa and L. A. Pando Zayas, *Statistical entropy of Calabi-Yau black holes*, *Phys. Rev.* **D59** (1999) 064023, [hep-th/9804129].

[60] N. Kaloper, *Entropy count for extremal three-dimensional black strings*, *Phys. Lett.* **B434** (1998) 285–293, [hep-th/9804062].

[61] M. Cvetic and F. Larsen, *Statistical entropy of four-dimensional rotating black holes from near-horizon geometry*, *Phys. Rev. Lett.* **82** (1999) 484–487, [hep-th/9805146].

[62] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231–252, [hep-th/9711200].

[63] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett.* **B428** (1998) 105–114, [hep-th/9802109].

[64] E. D'Hoker and D. Z. Freedman, *Supersymmetric gauge theories and the AdS / CFT correspondence*, in *Strings, Branes and Extra Dimensions: TASI 2001: Proceedings*, pp. 3–158, 2002. hep-th/0201253.

[65] J. M. Maldacena, *Wilson loops in large N field theories*, *Phys. Rev. Lett.* **80** (1998) 4859–4862, [hep-th/9803002].

[66] M. Henningson and K. Skenderis, *The Holographic Weyl anomaly*, *JHEP* **07** (1998) 023, [hep-th/9806087].

[67] V. Balasubramanian and P. Kraus, *A Stress tensor for Anti-de Sitter gravity*, *Commun. Math. Phys.* **208** (1999) 413–428, [hep-th/9902121].

[68] P. Kovtun, D. T. Son and A. O. Starinets, *Viscosity in strongly interacting quantum field theories from black hole physics*, *Phys. Rev. Lett.* **94** (2005) 111601, [hep-th/0405231].

[69] S. Ryu and T. Takayanagi, *Holographic derivation of entanglement entropy from AdS/CFT*, *Phys. Rev. Lett.* **96** (2006) 181602, [hep-th/0603001].

[70] S. W. Hawking, *Breakdown of Predictability in Gravitational Collapse*, *Phys. Rev.* **D14** (1976) 2460–2473.

[71] A. Almheiri, D. Marolf, J. Polchinski and J. Sully, *Black Holes: Complementarity or Firewalls?*, *JHEP* **02** (2013) 062, [1207.3123].

[72] J. Bardeen, B. Carter and S. Hawking, *The Four laws of black hole mechanics*, *Commun.Math.Phys.* **31** (1973) 161–170.

[73] V. P. Frolov and I. D. Novikov, eds., *Black hole physics: Basic concepts and new developments*. 1998.

[74] R. Penrose and R. M. Floyd, *Extraction of rotational energy from a black hole*, *Nature* **229** (1971) 177–179.

[75] J. D. Bekenstein, *Black holes and entropy*, *Phys. Rev.* **D7** (1973) 2333–2346.

[76] S. W. Hawking, *Particle Creation by Black Holes*, *Commun. Math. Phys.* **43** (1975) 199–220.

[77] J. B. Hartle and S. W. Hawking, *Path Integral Derivation of Black Hole Radiance*, *Phys. Rev.* **D13** (1976) 2188–2203.

[78] R. Brout, S. Massar, R. Parentani and P. Spindel, *A Primer for black hole quantum physics*, *Phys. Rept.* **260** (1995) 329–454, [0710.4345].

[79] M. Visser, *Essential and inessential features of Hawking radiation*, *Int. J. Mod. Phys.* **D12** (2003) 649–661, [hep-th/0106111].

[80] G. W. Gibbons and S. W. Hawking, *Action Integrals and Partition Functions in Quantum Gravity*, *Phys. Rev.* **D15** (1977) 2752–2756.

[81] G. W. Gibbons and S. W. Hawking, *Cosmological event horizons, thermodynamics, and particle creation*, .

[82] R. Kubo, *Statistical mechanical theory of irreversible processes. 1. General theory and simple applications in magnetic and conduction problems*, *J. Phys. Soc. Jap.* **12** (1957) 570–586.

[83] P. C. Martin and J. S. Schwinger, *Theory of many particle systems. 1.*, *Phys. Rev.* **115** (1959) 1342–1373.

[84] R. Haag, N. M. Hugenholtz and M. Winnink, *On the Equilibrium states in quantum statistical mechanics*, *Commun. Math. Phys.* **5** (1967) 215–236.

[85] G. W. Gibbons and M. J. Perry, *Black Holes and Thermal Green's Functions*, *Proc. Roy. Soc. Lond.* **A358** (1978) 467–494.

[86] R. Emparan, C. V. Johnson and R. C. Myers, *Surface terms as counterterms in the AdS / CFT correspondence*, *Phys. Rev.* **D60** (1999) 104001, [hep-th/9903238].

[87] L. Smarr, *Mass formula for Kerr black holes*, *Phys. Rev. Lett.* **30** (1973) 71–73.

[88] R. Myers and M. Perry, *Black Holes in Higher Dimensional Space-Times*, *Annals Phys.* **172** (1986) 304.

[89] E. Poisson, *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics*. Cambridge University Press, 2004, 10.1017/CBO9780511606601.

[90] M. Henneaux and C. Teitelboim, *Asymptotically anti-De Sitter Spaces*, *Commun. Math. Phys.* **98** (1985) 391–424.

[91] A. Ashtekar and A. Magnon, *Asymptotically anti-de Sitter space-times*, *Class.Quant.Grav.* **1** (1984) L39–L44.

[92] A. Ashtekar and S. Das, *Asymptotically Anti-de Sitter space-times: Conserved quantities*, *Class.Quant.Grav.* **17** (2000) L17–L30, [hep-th/9911230].

[93] G. W. Gibbons, R. Kallosh and B. Kol, *Moduli, scalar charges, and the first law of black hole thermodynamics*, *Phys.Rev.Lett.* **77** (1996) 4992–4995, [hep-th/9607108].

[94] D. Z. Freedman, S. S. Gubser, K. Pilch and N. P. Warner, *Renormalization group flows from holography supersymmetry and a c theorem*, *Adv. Theor. Math. Phys.* **3** (1999) 363–417, [hep-th/9904017].

[95] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, *Novel local CFT and exact results on perturbations of N=4 superYang Mills from AdS dynamics*, *JHEP* **12** (1998) 022, [hep-th/9810126].

[96] R. C. Myers and A. Sinha, *Holographic c-theorems in arbitrary dimensions*, *JHEP* **01** (2011) 125, [1011.5819].

[97] R. Gregory, D. Kastor and J. Traschen, *Black Hole Thermodynamics with Dynamical Lambda*, *JHEP* **10** (2017) 118, [1707.06586].

[98] R. Gregory, D. Kastor and J. Traschen, *Evolving Black Holes in Inflation*, 1804.03462.

[99] J. D. Brown and C. Teitelboim, *Dynamical Neutralization of the Cosmological Constant*, *Phys. Lett.* **B195** (1987) 177–182.

[100] J. D. Brown and C. Teitelboim, *Neutralization of the Cosmological Constant by Membrane Creation*, *Nucl. Phys.* **B297** (1988) 787–836.

[101] A. Aurilia, H. Nicolai and P. K. Townsend, *Hidden Constants: The Theta Parameter of QCD and the Cosmological Constant of N=8 Supergravity*, *Nucl. Phys.* **B176** (1980) 509–522.

[102] M. Henneaux and C. Teitelboim, *The Cosmological Constant as a Canonical Variable*, *Phys. Lett.* **143B** (1984) 415–420.

[103] R. Bousso and J. Polchinski, *Quantization of four form fluxes and dynamical neutralization of the cosmological constant*, *JHEP* **06** (2000) 006, [hep-th/0004134].

[104] C. Teitelboim, *The Cosmological Constant as a Thermodynamic Black Hole Parameter*, *Phys. Lett.* **158B** (1985) 293–297.

[105] J. Creighton and R. B. Mann, *Quasilocal thermodynamics of dilaton gravity coupled to gauge fields*, Phys.Rev. **D52** (1995) 4569–4587, [gr-qc/9505007].

[106] D. Chernyavsky and K. Hajian, *Cosmological constant is a conserved charge*, 1710.07904.

[107] D. Sudarsky and R. M. Wald, *Extrema of mass, stationarity, and staticity, and solutions to the Einstein Yang-Mills equations*, Phys.Rev. **D46** (1992) 1453–1474.

[108] B. P. Dolan, D. Kastor, D. Kubiznak, R. B. Mann and J. Traschen, *Thermodynamic Volumes and Isoperimetric Inequalities for de Sitter Black Holes*, Phys.Rev. **D87** (2013) 104017, [1301.5926].

[109] N. Altamirano, D. Kubiznak, R. B. Mann and Z. Sherkatghanad, *Thermodynamics of rotating black holes and black rings: phase transitions and thermodynamic volume*, Galaxies **2** (2014) 89–159, [1401.2586].

[110] X.-H. Feng and H. Lu, *Butterfly Velocity Bound and Reverse Isoperimetric Inequality*, Phys. Rev. **D95** (2017) 066001, [1701.05204].

[111] S. Mbarek and R. B. Mann, *Thermodynamic Volume of Cosmological Solitons*, Phys. Lett. **B765** (2017) 352–358, [1611.01131].

[112] W. G. Brenna, R. B. Mann and M. Park, *Mass and Thermodynamic Volume in Lifshitz Spacetimes*, Phys. Rev. **D92** (2015) 044015, [1505.06331].

[113] T. Jacobson and R. C. Myers, *Black hole entropy and higher curvature interactions*, Phys. Rev. Lett. **70** (1993) 3684–3687, [hep-th/9305016].

[114] R. M. Wald, *Black hole entropy is the Noether charge*, Phys. Rev. **D48** (1993) 3427–3431, [gr-qc/9307038].

[115] V. Iyer and R. Wald, *Some properties of Noether charge and a proposal for dynamical black hole entropy*, Phys.Rev. **D50** (1994) 846–864, [gr-qc/9403028].

[116] D. Lovelock, *The Einstein tensor and its generalizations*, J.Math.Phys. **12** (1971) 498–501.

[117] J. Oliva and S. Ray, *A new cubic theory of gravity in five dimensions: Black hole, Birkhoff's theorem and C-function*, Class. Quant. Grav. **27** (2010) 225002, [1003.4773].

[118] R. C. Myers and B. Robinson, *Black Holes in Quasi-topological Gravity*, *JHEP* **1008** (2010) 067, [1003.5357].

[119] A. Buchel, J. Escobedo, R. C. Myers, M. F. Paulos, A. Sinha and M. Smolkin, *Holographic GB gravity in arbitrary dimensions*, *JHEP* **03** (2010) 111, [0911.4257].

[120] X. O. Camanho, J. D. Edelstein and J. M. Snchez De Santos, *Lovelock theory and the AdS/CFT correspondence*, *Gen. Rel. Grav.* **46** (2014) 1637, [1309.6483].

[121] P. Bueno and P. A. Cano, *Einsteinian cubic gravity*, *Phys. Rev.* **D94** (2016) 104005, [1607.06463].

[122] S. W. Hawking, *Black holes in general relativity*, *Commun. Math. Phys.* **25** (1972) 152–166.

[123] R. Emparan and H. S. Reall, *A Rotating black ring solution in five-dimensions*, *Phys.Rev.Lett.* **88** (2002) 101101, [hep-th/0110260].

[124] L. Vanzo, *Black holes with unusual topology*, *Phys.Rev.* **D56** (1997) 6475–6483, [gr-qc/9705004].

[125] R. B. Mann, *Pair production of topological anti-de Sitter black holes*, *Class.Quant.Grav.* **14** (1997) L109–L114, [gr-qc/9607071].

[126] J. Lemos, *Cylindrical black hole in general relativity*, *Phys.Lett.* **B353** (1995) 46–51, [gr-qc/9404041].

[127] R.-G. Cai and Y.-Z. Zhang, *Black plane solutions in four-dimensional space-times*, *Phys.Rev.* **D54** (1996) 4891–4898, [gr-qc/9609065].

[128] R. B. Mann, *Topological black holes: Outside looking in*, *Annals Israel Phys.Soc.* **13** (1997) 311, [gr-qc/9709039].

[129] P. Figueras and S. Tunyasuvunakool, *Black rings in global anti-de Sitter space*, 1412.5680.

[130] M. M. Caldarelli, R. Emparan and M. J. Rodriguez, *Black Rings in (Anti)-deSitter space*, *JHEP* **0811** (2008) 011, [0806.1954].

[131] D. Birmingham, *Topological black holes in Anti-de Sitter space*, *Class.Quant.Grav.* **16** (1999) 1197–1205, [hep-th/9808032].

[132] M. Sinamuli and R. B. Mann, *Super-Entropic Black Holes and the Kerr-CFT Correspondence*, *JHEP* **08** (2016) 148, [1512.07597].

[133] S. M. Noorbakhsh and M. Ghominejad, *Ultra-Spinning Gauged Supergravity Black Holes and their Kerr/CFT Correspondence*, *Phys. Rev.* **D95** (2017) 046002, [1611.02324].

[134] S. M. Noorbakhsh and M. Ghominejad, *Higher Dimensional Charged AdS Black Holes at Ultra-spinning Limit and Their 2d CFT Duals*, 1702.03448.

[135] S. M. Noorbakhsh and M. H. Vahidinia, *Extremal Vanishing Horizon Kerr-AdS Black Holes at Ultraspinning Limit*, *JHEP* **01** (2018) 042, [1708.08654].

[136] R. P. Kerr, *Gravitational field of a spinning mass as an example of algebraically special metrics*, *Phys. Rev. Lett.* **11** (1963) 237–238.

[137] R. Emparan and R. C. Myers, *Instability of ultra-spinning black holes*, *JHEP* **0309** (2003) 025, [hep-th/0308056].

[138] R. Gregory and R. Laflamme, *Black strings and p-branes are unstable*, *Phys. Rev. Lett.* **70** (1993) 2837–2840, [hep-th/9301052].

[139] B. Carter, *Hamilton-Jacobi and Schrodinger separable solutions of Einstein's equations*, *Commun. Math. Phys.* **10** (1968) 280.

[140] G. Gibbons, H. Lu, D. N. Page and C. Pope, *The General Kerr-de Sitter metrics in all dimensions*, *J.Geom.Phys.* **53** (2005) 49–73, [hep-th/0404008].

[141] M. M. Caldarelli, R. G. Leigh, A. C. Petkou, P. M. Petropoulos, V. Pozzoli et al., *Vorticity in holographic fluids*, *PoS* **CORFU2011** (2011) 076, [1206.4351].

[142] A. Gnecchi, K. Hristov, D. Klemm, C. Toldo and O. Vaughan, *Rotating black holes in 4d gauged supergravity*, *JHEP* **1401** (2014) 127, [1311.1795].

[143] D. Klemm, *Four-dimensional black holes with unusual horizons*, *Phys.Rev.* **D89** (2014) 084007, [1401.3107].

[144] D. Astefanesei, R. B. Mann and E. Radu, *Nut charged space-times and closed timelike curves on the boundary*, *JHEP* **01** (2005) 049, [hep-th/0407110].

[145] R. B. Mann and C. Stelea, *New multiply nutty spacetimes*, *Phys. Lett.* **B634** (2006) 448–455, [hep-th/0508203].

[146] C. W. Misner, *The Flatter regions of Newman, Unti and Tamburino's generalized Schwarzschild space*, *J. Math. Phys.* **4** (1963) 924–938.

[147] B. Carter, *Global structure of the Kerr family of gravitational fields*, *Phys.Rev.* **174** (1968) 1559–1571.

[148] E. Hackmann, C. Lammerzahl, V. Kagramanova and J. Kunz, *Analytical solution of the geodesic equation in Kerr-(anti) de Sitter space-times*, *Phys.Rev.* **D81** (2010) 044020, [1009.6117].

[149] V. P. Frolov and D. Kubiznak, *Higher-Dimensional Black Holes: Hidden Symmetries and Separation of Variables*, *Class.Quant.Grav.* **25** (2008) 154005, [0802.0322].

[150] S. Das and R. Mann, *Conserved quantities in Kerr-anti-de Sitter space-times in various dimensions*, *JHEP* **0008** (2000) 033, [hep-th/0008028].

[151] C. Johnson, *Thermodynamic Volumes for AdS-Taub-NUT and AdS-Taub-Bolt*, 1405.5941.

[152] G. W. Gibbons, M. J. Perry and C. N. Pope, *The First law of thermodynamics for Kerr-anti-de Sitter black holes*, *Class. Quant. Grav.* **22** (2005) 1503–1526, [hep-th/0408217].

[153] W. Chen, H. Lu and C. Pope, *General Kerr-NUT-AdS metrics in all dimensions*, *Class.Quant.Grav.* **23** (2006) 5323–5340, [hep-th/0604125].

[154] M. Cvetic, M. J. Duff, P. Hoxha, J. T. Liu, H. Lu, J. X. Lu et al., *Embedding AdS black holes in ten-dimensions and eleven-dimensions*, *Nucl. Phys.* **B558** (1999) 96–126, [hep-th/9903214].

[155] T. Eguchi and A. J. Hanson, *Asymptotically Flat Selfdual Solutions to Euclidean Gravity*, *Phys. Lett.* **74B** (1978) 249–251.

[156] P. Bueno and P. A. Cano, *On black holes in higher-derivative gravities*, 1703.04625.

[157] P. Bueno and P. A. Cano, *Four-dimensional black holes in Einsteinian cubic gravity*, *Phys. Rev.* **D94** (2016) 124051, [1610.08019].

[158] P. Bueno and P. A. Cano, *Universal black hole stability in four dimensions*, *Phys. Rev.* **D96** (2017) 024034, [1704.02967].

[159] C. M. Will, *The Confrontation between General Relativity and Experiment*, *Living Rev. Rel.* **17** (2014) 4, [1403.7377].

[160] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space*. Cambridge Monographs on Mathematical Physics. Cambridge Univ. Press, Cambridge, UK, 1984, 10.1017/CBO9780511622632.

[161] K. S. Stelle, *Renormalization of Higher Derivative Quantum Gravity*, *Phys. Rev.* **D16** (1977) 953–969.

[162] B. Zwiebach, *Curvature Squared Terms and String Theories*, *Phys. Lett.* **B156** (1985) 315–317.

[163] R. R. Metsaev and A. A. Tseytlin, *Curvature Cubed Terms in String Theory Effective Actions*, *Phys. Lett.* **B185** (1987) 52–58.

[164] D. J. Gross and J. H. Sloan, *The Quartic Effective Action for the Heterotic String*, *Nucl. Phys.* **B291** (1987) 41–89.

[165] R. C. Myers, *Superstring Gravity and Black Holes*, *Nucl. Phys.* **B289** (1987) 701–716.

[166] M. Brigante, H. Liu, R. C. Myers, S. Shenker and S. Yaida, *Viscosity Bound Violation in Higher Derivative Gravity*, *Phys. Rev.* **D77** (2008) 126006, [0712.0805].

[167] P. Bueno, R. C. Myers and W. Witczak-Krempa, *Universality of corner entanglement in conformal field theories*, *Phys. Rev. Lett.* **115** (2015) 021602, [1505.04804].

[168] P. Bueno and R. C. Myers, *Corner contributions to holographic entanglement entropy*, *JHEP* **08** (2015) 068, [1505.07842].

[169] P. Bueno, R. C. Myers and W. Witczak-Krempa, *Universal corner entanglement from twist operators*, *JHEP* **09** (2015) 091, [1507.06997].

[170] R. C. Myers and J. Z. Simon, *Black Hole Thermodynamics in Lovelock Gravity*, *Phys. Rev.* **D38** (1988) 2434–2444.

[171] T. Jacobson, G. Kang and R. C. Myers, *On black hole entropy*, *Phys. Rev.* **D49** (1994) 6587–6598, [gr-qc/9312023].

[172] S.-W. Wei and Y.-X. Liu, *Critical phenomena and thermodynamic geometry of charged Gauss-Bonnet AdS black holes*, Phys.Rev. **D87** (2013) 044014, [1209.1707].

[173] R.-G. Cai, L.-M. Cao, L. Li and R.-Q. Yang, *P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space*, JHEP **1309** (2013) 005, [1306.6233].

[174] W. Xu, H. Xu and L. Zhao, *Gauss-Bonnet coupling constant as a free thermodynamical variable and the associated criticality*, 1311.3053.

[175] J.-X. Mo and W.-B. Liu, *$P-V$ criticality of topological black holes in Lovelock-Born-Infeld gravity*, Eur.Phys.J. **C74** (2014) 2836, [1401.0785].

[176] S.-W. Wei and Y.-X. Liu, *Triple points and phase diagrams in the extended phase space of charged Gauss-Bonnet black holes in AdS space*, 1402.2837.

[177] J.-X. Mo and W.-B. Liu, *Ehrenfest scheme for $P-V$ criticality of higher dimensional charged black holes, rotating black holes and Gauss-Bonnet AdS black holes*, Phys.Rev. **D89** (2014) 084057, [1404.3872].

[178] D.-C. Zou, Y. Liu and B. Wang, *Critical behavior of charged Gauss-Bonnet AdS black holes in the grand canonical ensemble*, 1404.5194.

[179] A. Belhaj, M. Chabab, H. E. Moumni, K. Masmar and M. Sedra, *Ehrenfest Scheme of Higher Dimensional Topological AdS Black Holes in Lovelock-Born-Infeld Gravity*, 1405.3306.

[180] H. Xu, W. Xu and L. Zhao, *Extended phase space thermodynamics for third order Lovelock black holes in diverse dimensions*, 1405.4143.

[181] A. Belhaj, M. Chabab, H. El moumni, K. Masmar and M. B. Sedra, *Maxwell's equal-area law for Gauss-Bonnet-Anti-de Sitter black holes*, Eur. Phys. J. **C75** (2015) 71, [1412.2162].

[182] Z. Sherkatghanad, B. Mirza, Z. Mirzaeyan and S. A. H. Mansoori, *Critical behaviors and phase transitions of black holes in higher order gravities and extended phase spaces*, 1412.5028.

[183] S. H. Hendi and R. Naderi, *Geometrothermodynamics of black holes in Lovelock gravity with a nonlinear electrodynamics*, Phys. Rev. **D91** (2015) 024007, [1510.06269].

[184] S. H. Hendi, S. Panahiyan and M. Momennia, *Extended phase space of AdS Black Holes in Einstein-Gauss-Bonnet gravity with a quadratic nonlinear electrodynamics*, *Int. J. Mod. Phys.* **D25** (2016) 1650063, [1503.03340].

[185] S. H. Hendi and A. Dehghani, *Thermodynamics of third-order Lovelock-AdS black holes in the presence of Born-Infeld type nonlinear electrodynamics*, *Phys. Rev.* **D91** (2015) 064045, [1510.06261].

[186] Z.-Y. Nie and H. Zeng, *P-T phase diagram of a holographic s+p model from Gauss-Bonnet gravity*, *JHEP* **10** (2015) 047, [1505.02289].

[187] S. H. Hendi, S. Panahiyan and B. Eslam Panah, *Charged Black Hole Solutions in Gauss-Bonnet-Massive Gravity*, *JHEP* **01** (2016) 129, [1507.06563].

[188] S. H. Hendi, S. Panahiyan and B. Eslam Panah, *Extended phase space of Black Holes in Lovelock gravity with nonlinear electrodynamics*, *PTEP* **2015** (2015) 103E01, [1511.00656].

[189] C. V. Johnson, *Gauss-Bonnet Black Holes and Holographic Heat Engines Beyond Large N*, 1511.08782.

[190] S. H. Hendi, S. Panahiyan, B. Eslam Panah, M. Faizal and M. Momennia, *Critical behavior of charged black holes in Gauss-Bonnet gravitys rainbow*, *Phys. Rev.* **D94** (2016) 024028, [1607.06663].

[191] X.-X. Zeng, S. He and L.-F. Li, *Holographic Van der Waals-like phase transition in the Gauss-Bonnet gravity*, 1608.04208.

[192] T. C. Sisman, I. Gullu and B. Tekin, *All unitary cubic curvature gravities in D dimensions*, *Class. Quant. Grav.* **28** (2011) 195004, [1103.2307].

[193] J. Smolic and M. Taylor, *Higher derivative effects for 4d AdS gravity*, *JHEP* **06** (2013) 096, [1301.5205].

[194] B. Tekin, *Particle Content of Quadratic and $f(R_{\mu\nu\sigma\rho})$ Theories in (A)dS*, *Phys. Rev.* **D93** (2016) 101502, [1604.00891].

[195] P. Bueno, P. A. Cano, V. S. Min and M. R. Visser, *Aspects of general higher-order gravities*, *Phys. Rev.* **D95** (2017) 044010, [1610.08519].

[196] T. Padmanabhan, *Some aspects of field equations in generalised theories of gravity*, *Phys. Rev.* **D84** (2011) 124041, [1109.3846].

[197] D. Brizuela, J. M. Martin-Garcia and G. A. Mena Marugan, *xPert: Computer algebra for metric perturbation theory*, Gen. Rel. Grav. **41** (2009) 2415–2431, [0807.0824].

[198] D. G. Boulware and S. Deser, *String Generated Gravity Models*, Phys.Rev.Lett. **55** (1985) 2656.

[199] S. Deser and B. Tekin, *Energy in generic higher curvature gravity theories*, Phys. Rev. **D67** (2003) 084009, [hep-th/0212292].

[200] C. Senturk, T. C. Sisman and B. Tekin, *Energy and Angular Momentum in Generic F(Riemann) Theories*, Phys. Rev. **D86** (2012) 124030, [1209.2056].

[201] D. Kastor, S. Ray and J. Traschen, *Mass and Free Energy of Lovelock Black Holes*, Class.Quant.Grav. **28** (2011) 195022, [1106.2764].

[202] T. Jacobson, *When is g(tt) g(rr) = -1?*, Class. Quant. Grav. **24** (2007) 5717–5719, [0707.3222].

[203] R. J. Riegert, *Birkhoff's Theorem in Conformal Gravity*, Phys. Rev. Lett. **53** (1984) 315–318.

[204] R. S. Palais, *The principle of symmetric criticality*, Comm. Math. Phys. **69** (1979) 19–30.

[205] S. Deser and B. Tekin, *Shortcuts to high symmetry solutions in gravitational theories*, Class. Quant. Grav. **20** (2003) 4877–4884, [gr-qc/0306114].

[206] S. A. Fulling, R. C. King, B. G. Wybourne and C. J. Cummins, *Normal forms for tensor polynomials. 1: The Riemann tensor*, Class. Quant. Grav. **9** (1992) 1151–1197.

[207] S. Deser and A. V. Ryzhov, *Curvature invariants of static spherically symmetric geometries*, Class. Quant. Grav. **22** (2005) 3315–3324, [gr-qc/0505039].

[208] J. T. Wheeler, *Symmetric Solutions to the Maximally Gauss-Bonnet Extended Einstein Equations*, Nucl. Phys. **B273** (1986) 732–748.

[209] M. Cvetic, S. Nojiri and S. D. Odintsov, *Black hole thermodynamics and negative entropy in de Sitter and anti-de Sitter Einstein-Gauss-Bonnet gravity*, Nucl. Phys. **B628** (2002) 295–330, [hep-th/0112045].

[210] T. Padmanabhan and D. Kothawala, *Lanczos-Lovelock models of gravity*, *Phys. Rept.* **531** (2013) 115–171, [1302.2151].

[211] J. Oliva and S. Ray, *Classification of Six Derivative Lagrangians of Gravity and Static Spherically Symmetric Solutions*, *Phys. Rev.* **D82** (2010) 124030, [1004.0737].

[212] M. H. Dehghani and M. H. Vahidinia, *Quartic Quasi-topological Gravity, Black Holes and Holography*, *JHEP* **10** (2013) 210, [1307.0330].

[213] A. Cisterna, L. Guajardo, M. Hassaine and J. Oliva, *Quintic quasi-topological gravity*, *JHEP* **04** (2017) 066, [1702.04676].

[214] R. C. Myers, M. F. Paulos and A. Sinha, *Holographic studies of quasi-topological gravity*, *JHEP* **08** (2010) 035, [1004.2055].

[215] K. Goldstein and J. J. Mashiyane, *Ineffective Higher Derivative Black Hole Hair*, 1703.02803.

[216] X.-H. Feng, H. Huang, Z.-F. Mai and H. Lu, *Bounce Universe and Black Holes from Critical Einsteinian Cubic Gravity*, *Phys. Rev.* **D96** (2017) 104034, [1707.06308].

[217] H. L, A. Perkins, C. N. Pope and K. S. Stelle, *Spherically Symmetric Solutions in Higher-Derivative Gravity*, *Phys. Rev.* **D92** (2015) 124019, [1508.00010].

[218] X. O. Camanho, J. D. Edelstein and M. F. Paulos, *Lovelock theories, holography and the fate of the viscosity bound*, *JHEP* **1105** (2011) 127, [1010.1682].

[219] X. O. Camanho and J. D. Edelstein, *A Lovelock black hole bestiary*, *Class.Quant.Grav.* **30** (2013) 035009, [1103.3669].

[220] P. Bueno, P. A. Cano and A. Ruiprez, *Holographic studies of Einsteinian cubic gravity*, *JHEP* **03** (2018) 150, [1802.00018].

[221] D. Ayzenberg, K. Yagi and N. Yunes, *Linear Stability Analysis of Dynamical Quadratic Gravity*, *Phys. Rev.* **D89** (2014) 044023, [1310.6392].

[222] H. L, A. Perkins, C. N. Pope and K. S. Stelle, *Lichnerowicz Modes and Black Hole Families in Ricci Quadratic Gravity*, *Phys. Rev.* **D96** (2017) 046006, [1704.05493].

[223] T. Takahashi and J. Soda, *Stability of Lovelock Black Holes under Tensor Perturbations*, *Phys. Rev.* **D79** (2009) 104025, [0902.2921].

[224] R. A. Konoplya and A. Zhidenko, *Quasinormal modes of Gauss-Bonnet-AdS black holes: towards holographic description of finite coupling*, *JHEP* **09** (2017) 139, [1705.07732].

[225] N. Tanahashi, H. S. Reall and B. Way, *Causality, Hyperbolicity and Shock Formation in Lovelock Theories*, in *Proceedings, 2nd LeCosPA Symposium: Everything about Gravity, Celebrating the Centenary of Einstein's General Relativity (LeCosPA2015): Taipei, Taiwan, December 14-18, 2015*, pp. 380–385, 2017. DOI.

[226] J. Oliva and S. Ray, *Conformal couplings of a scalar field to higher curvature terms*, *Class. Quant. Grav.* **29** (2012) 205008, [1112.4112].

[227] G. Giribet, M. Leoni, J. Oliva and S. Ray, *Hairy black holes sourced by a conformally coupled scalar field in D dimensions*, *Phys. Rev.* **D89** (2014) 085040, [1401.4987].

[228] G. Giribet, A. Goya and J. Oliva, *Different phases of hairy black holes in AdS5 space*, *Phys. Rev.* **D91** (2015) 045031, [1501.00184].

[229] M. Galante, G. Giribet, A. Goya and J. Oliva, *Chemical potential driven phase transition of black holes in AdS space*, 1508.03780.

[230] M. Chernicoff, M. Galante, G. Giribet, A. Goya, M. Leoni, J. Oliva et al., *Black hole thermodynamics, conformal couplings, and R^2 terms*, *JHEP* **06** (2016) 159, [1604.08203].

[231] M. Chernicoff, G. Giribet and J. Oliva, *Hairy Lovelock black holes and Stueckelberg mechanism for Weyl symmetry*, 2016. 1608.05000.

[232] C. Martinez, *Black holes with a conformally coupled scalar field*, in *Quantum Mechanics of Fundamental Systems: The Quest for Beauty and Simplicity.*

[233] P. B. Weichman, A. W. Harter and D. L. Goodstein, *Criticality and superfluidity in liquid* ^{4}He *under nonequilibrium conditions*, *Rev. Mod. Phys.* **73** (Jan, 2001) 1–15.

[234] C. Dasgupta and B. I. Halperin, *Phase transition in a lattice model of superconductivity*, *Phys. Rev. Lett.* **47** (Nov, 1981) 1556–1560.

[235] R. Pathria and P. D. Beale, *12 - phase transitions: Criticality, universality, and scaling*, in *Statistical Mechanics (Third Edition)* (R. Pathria and P. D. Beale, eds.), pp. 401 – 469. Academic Press, Boston, third edition ed., 2011. DOI.

[236] T. Guénault, *Basic Superfluids*. CRC Press, 2002.

[237] A. J. Leggett, *Superfluidity*, Rev. Mod. Phys. **71** (Mar, 1999) S318–S323.

[238] Z.-W. Chong, M. Cvetic, H. Lu and C. Pope, *General non-extremal rotating black holes in minimal five-dimensional gauged supergravity*, Phys.Rev.Lett. **95** (2005) 161301, [hep-th/0506029].

[239] M.-S. Ma and R.-H. Wang, *Peculiar $P-V$ criticality of topological Hoava-Lifshitz black holes*, Phys. Rev. **D96** (2017) 024052, [1707.09156].

[240] W. Xu, *λ phase transition in Horava gravity*, 1804.03815.

[241] J. D. Brown, *Black hole entropy and the Hamiltonian formulation of diffeomorphism invariant theories*, Phys. Rev. **D52** (1995) 7011–7026, [gr-qc/9506085].

[242] R. M. Wald, *General Relativity*. The University of Chicago Press, Chicago and London, 1984.

[243] J. Armas and N. Obers, *Blackfolds in (Anti)-de Sitter Backgrounds*, Phys.Rev. **D83** (2011) 084039, [1012.5081].

[244] R. C. Myers, *Higher Derivative Gravity, Surface Terms and String Theory*, Phys. Rev. **D36** (1987) 392.

[245] C. Teitelboim and J. Zanelli, *Dimensionally continued topological gravitation theory in Hamiltonian form*, Class.Quant.Grav. **4** (1987) L125.

Appendix A

Derivation of Eq. (2.81)

Here we provide some additional calculations that are important to establishing the first law via the Wald-Sudarsky argument. Namely, we provide a derivation of eq. (2.81). We closely follow [241].

We consider a black hole spacetime with bifurcate Killing horizon. As in the main text, we denote the bifurcation surface as B; the Killing field ξ^α vanishes on B. We are considering Hamiltonian evolution along ξ^α we have $\xi^\alpha = N n^\alpha + N^\alpha$ where n^α is the unit timelike normal to the family of hypersurfaces that foliate spacetime. To evaluate the integral appearing in eq. (2.81), we consider B to be approached as a limit within a slice Σ.

First, note that since ξ^α vanishes on B it follows by taking projections of ξ^α that both N and N^α vanish on B. As a result, the boundary terms proportional to the lapse and shift vanish and we need only consider those that are proportional to the derivative of the lapse. Our goal now is to evaluate

$$\lim D_\alpha N \tag{A.1}$$

where lim denotes the limit where B is approached from within Σ. The calculation is somewhat subtle, but the final result will be that this derivative is proportional to the surface gravity, and so that is where we begin.

Let us note that the surface gravity can be defined by [242]

$$\kappa = \lim \sqrt{\frac{\left(\xi^\alpha \nabla_\alpha \xi^\beta\right)\left(\xi^\lambda \nabla_\lambda \xi_\beta\right)}{-\xi_\sigma \xi^\sigma}} = \lim \sqrt{\nabla_\alpha |\xi| \nabla^\alpha |\xi|}\,, \tag{A.2}$$

where in the second equality we defined the quantity $|\xi| := \sqrt{-\xi^\alpha \xi_\alpha}$. Therefore, to determine the surface gravity, we must work out the components of $\nabla_\alpha |\xi|$.

We decompose the metric on Σ as

$$h_{\alpha\beta} = r_\alpha r_\beta + \sigma_{\alpha\beta} \tag{A.3}$$

where $r_\alpha r^\alpha = 1$ and $r_\alpha \sigma^{\alpha\beta} = 0$. In the limit where we approach B, $\sigma_{\alpha\beta}$ will be the induced metric on B and r^α will be the spacelike normal of B pointing into Σ. It is easy to see that

$$\lim \sigma_\alpha^\beta \nabla_\alpha |\xi| = 0\,, \tag{A.4}$$

since this represents the derivative of $|\xi|$ along B, it vanishes because $|\xi| = \sqrt{-\xi^\alpha \xi_\alpha} = \sqrt{N^2 - N_\alpha N^\alpha} = 0$ on B. Our second observation is that

$$\lim \xi^\alpha \nabla_\alpha |\xi| = 0\,. \tag{A.5}$$

This result follows because

$$\xi^\alpha \nabla_\alpha |\xi| = \frac{\xi^\alpha \xi^\lambda \nabla_\alpha \xi_\lambda}{\sqrt{-\xi^\sigma \xi_\sigma}} = \frac{\xi^\alpha \xi^\lambda \nabla_{[\alpha} \xi_{\lambda]}}{\sqrt{-\xi^\sigma \xi_\sigma}} = 0 \tag{A.6}$$

where the third inequality makes use of the fact that ξ^α is a Killing vector, i.e. $\nabla_{(\alpha} \xi_{\beta)} = 0$. Combining these two observations, we conclude that any non-vanishing component of $\nabla_\alpha |\xi|$ must be in the r_α direction. Further, since the surface gravity of a bifurcate Killing horizon does not vanish, we conclude that $r^\alpha \nabla_\alpha |\xi| \neq 0$. Our goal now is to determine an expression for that component.

First we produce a result that will be useful below. We consider

$$\lim \frac{N^\alpha}{|\xi|}\,, \tag{A.7}$$

which is an indeterminant form of the type 0/0, and so we can determine the limit using l'Hopital's rule. We perform some straightforward manipulations:

$$\begin{aligned} r^\alpha \nabla_\alpha N^\beta &= r^\alpha \nabla_\alpha \left(h^\beta_\sigma \xi^\sigma \right) \\ &= r^\alpha \xi^\sigma \nabla_\alpha h^\beta_\sigma + r^\alpha h^\beta_\sigma \nabla_\alpha \xi^\sigma \\ &= r^\alpha \xi^\sigma \nabla_\alpha \left[g^\beta_\sigma + n^\beta n_\sigma \right] + r^\alpha h^\beta_\sigma \nabla_\alpha \xi^\sigma \\ &= r^\alpha \xi^\sigma n_\sigma \nabla_\alpha n^\beta + r^\alpha \xi^\sigma n^\beta \nabla_\alpha n_\sigma + r^\alpha h^\beta_\sigma \nabla_\alpha \xi^\sigma \\ &= -N r^\alpha K^\beta_\alpha + n^\beta r^\alpha N^\sigma K_{\alpha\sigma} + r^\alpha \left[\sigma^{\beta\sigma} + r^\beta r^\sigma \right] \nabla_\alpha \xi_\sigma \\ &= -N r^\alpha K^\beta_\alpha + n^\beta r^\alpha N^\sigma K_{\alpha\sigma} - r^\alpha \sigma^{\beta\sigma} \nabla_\sigma \xi_\alpha \end{aligned} \tag{A.8}$$

We made use of the fact ξ^α is a Killing vector and that $r^\alpha \nabla_\alpha n^\sigma = r^\beta h^\alpha_\beta \nabla_\alpha n^\sigma = r^\alpha K^\sigma_\alpha$ since r^α lies entirely within Σ and $n^\alpha n_\alpha = -1$ always. From the last line, we can conclude that

$$\lim r^\alpha \nabla_\alpha N^\beta = 0\,, \tag{A.9}$$

since the lapse and shift (not to mention also the extrinsic curvature) vanish at B, and $\lim r^\alpha \sigma^{\beta\sigma} \nabla_\sigma \xi_\alpha = 0$ since the projection ensures the derivative is acting only on B where ξ is a constant (namely zero). This ensures that

$$\lim \frac{N^\alpha}{|\xi|} = 0\,, \tag{A.10}$$

and as a byproduct we also see that,

$$0 = \lim \frac{N^\alpha}{|\xi|} = \lim \frac{N^\alpha}{N} \frac{1}{\sqrt{1 - N^{-2} N_\sigma N^\sigma}} \Rightarrow \lim \frac{N^\alpha}{N} = 0\,. \tag{A.11}$$

Finally, let us consider $\lim \nabla_\alpha |\xi|$. By direct computation we have

$$\nabla_\alpha |\xi| = \nabla_\alpha \sqrt{N^2 - N_\sigma N^\sigma} = \frac{N \nabla_\alpha N}{|\xi|} - \frac{N_\sigma \nabla_\alpha N^\sigma}{|\xi|}\,. \tag{A.12}$$

The second term above vanishes under the limit because of (A.11). Then we have,

$$\frac{N \nabla_\alpha N}{|\xi|} = \frac{\nabla_\alpha N}{\sqrt{1 - \frac{N_\sigma N^\sigma}{N^2}}} \tag{A.13}$$

so that

$$\lim \nabla_\alpha |\xi| = \lim \frac{N \nabla_\alpha N}{|\xi|} = \lim \nabla_\alpha N\,, \tag{A.14}$$

again by eq. (A.11). Going back again to the expression for the surface gravity we can now see that

$$\kappa = \lim |r^\alpha \nabla_\alpha N|\,. \tag{A.15}$$

Taking the minus sign (to ensure an outward pointing normal) we then have

$$\lim \nabla_\alpha N = -\lim \kappa r_\alpha \Rightarrow D_\alpha N = -\kappa r_\alpha \quad \text{on } B. \tag{A.16}$$

This important result will be crucial to showing eq. (2.81). To proceed, let us note that, on B, the integral (2.81) can be recast into the following form:

$$\oint_B dA_\alpha B^\alpha = \oint_B dA_\alpha D_\beta N \left[h^{\alpha\sigma} h^{\beta\lambda} - h^{\alpha\beta} h^{\sigma\lambda} \right] \delta h_{\sigma\lambda}\,. \tag{A.17}$$

The surface element on B is given by, $dS_\alpha = -r_\alpha \sqrt{\sigma} d^{D-2}\theta$ where θ^α are the coordinates on B and $\sigma = \det\left(\sigma^{\alpha\beta}\right)$ is the determinant of the induced metric. Substituting our result above for the derivative of the lapse and noting that the surface gravity is constant over the horizon, we can re-express the above in the following manner:

$$\oint_B dA_\alpha D_\beta N \left[h^{\alpha\sigma}h^{\beta\lambda} - h^{\alpha\beta}h^{\sigma\lambda}\right]\delta h_{\sigma\lambda} = \kappa \oint_B \sqrt{\sigma} d^{D-2}\theta r_\alpha r_\beta \left[h^{\alpha\sigma}h^{\beta\lambda} - h^{\alpha\beta}h^{\sigma\lambda}\right]\delta h_{\sigma\lambda}\,. \tag{A.18}$$

Now performing some simple manipulations on the integral:

$$\begin{aligned}
\oint_B \sqrt{\sigma} d^{D-2}\theta r_\alpha r_\beta \left[h^{\alpha\sigma}h^{\beta\lambda} - h^{\alpha\beta}h^{\sigma\lambda}\right]\delta h_{\sigma\lambda} &= \oint_B \sqrt{\sigma} d^{D-2}\theta r_\alpha r_\beta \left[h^{\alpha\sigma}h^{\beta\lambda} - h^{\alpha\beta}h^{\sigma\lambda}\right]\delta h_{\sigma\lambda} \\
&= \oint_B \sqrt{\sigma} d^{D-2}\theta \left[r^\sigma r^\lambda - h^{\sigma\lambda}\right]\delta h_{\sigma\lambda} \\
&= -\oint_B \sqrt{\sigma} d^{D-2}\theta \sigma^{\sigma\lambda}\delta h_{\sigma\lambda} \\
&= -\oint_B \sqrt{\sigma} d^{D-2}\theta \sigma^{\sigma\lambda}\delta \sigma_{\sigma\lambda} \\
&= -2\delta \oint_B \sqrt{\sigma} d^{D-2}\theta \\
&= -2\delta A\,.
\end{aligned} \tag{A.19}$$

In this derivation we have used the completeness relation for $h_{\alpha\beta}$, as well as the fact that $\sigma^{\alpha\beta}$ projects the variations into B, i.e. $\sigma^{\alpha\beta}\delta h_{\alpha\beta} = \sigma^{\alpha\beta}\delta\left(r_\alpha r_\beta + \sigma_{\alpha\beta}\right) = \sigma^{\sigma\lambda}\delta\sigma_{\sigma\lambda}$, and finally the formula for the variation of the metric determinant $\delta\sigma = \sigma\sigma^{\rho\lambda}\delta\sigma_{\rho\lambda}$. This result establishes eq. (2.81).

Appendix B

Review of ultra-spinning limits for AdS black holes

In this appendix, we discuss the ultra-spinning limits of rotating AdS black holes. Specifically, we discuss the black brane limit and the hyperboloid membrane limit. In each case, we illustrate the idea of the limit for a single rotation parameter — additional details can be found in [2, 3].

B.1 Black brane limit

Ultraspinning black holes were first studied by Emparan and Myers [137] in an analysis focusing on the stability of Myers–Perry black holes [88] in the limit of large angular momentum. For AdS black holes several physically distinct ultraspinning limits are possible. In this appendix we review the first type—the black brane ultraspinning limit—first studied by Caldarelli et al. [130] for Kerr-AdS black holes. The procedure consists of taking a limit where one or more rotation parameters, a_i, approach the AdS radius, L, $a_i \to L$, keeping the physical mass M of the black hole spacetime fixed while simultaneously zooming in to the pole. This limit is sensible only for $D \geq 6$ and yields a vacuum solution of Einstein equations (with zero cosmological constant) describing a static black brane. Armas and Obers later showed that the same solution can be obtained by taking $a \to \infty$ while keeping the ratio a/L fixed, their approach having the advantage of being directly applicable to dS solutions as well [243].

In this appendix we follow the original reference [130] and demonstrate the procedure for the multiply spinning Kerr-AdS black hole spacetimes discussed in section. We also comment on an (im)possibility of taking the black brane limit starting from the newly constructed super-entropic black holes.

First, let us introduce the general Kerr-AdS metrics. In the generalized Boyer–Lindquist coordinates the metric takes the following form:

$$ds^2 = d\gamma^2 + \frac{2m}{U}\omega^2 + \frac{Udr^2}{F-2m} + d\Omega^2\,, \tag{B.1}$$

where we have defined

$$\begin{aligned} d\gamma^2 &= -\frac{W\rho^2}{L^2}dt^2 + \sum_{i=1}^{N}\frac{r^2+a_i^2}{\Xi_i}\mu_i^2 d\phi_i^2\,, \\ d\Omega^2 &= \sum_{i=1}^{N+\varepsilon}\frac{r^2+a_i^2}{\Xi_i}d\mu_i^2 - \frac{1}{W\rho^2}\Big(\sum_{i=1}^{N+\varepsilon}\frac{r^2+a_i^2}{\Xi_i}\mu_i d\mu_i\Big)^2\,, \\ \omega &= Wdt - \sum_{i=1}^{N}\frac{a_i\mu_i^2 d\phi_i}{\Xi_i}\,, \end{aligned} \tag{B.2}$$

and, as usual $\rho^2 = r^2 + L^2$, while

$$\begin{aligned} W &= \sum_{i=1}^{N+\varepsilon}\frac{\mu_i^2}{\Xi_i}\,, \quad U = r^\varepsilon \sum_{i=1}^{N+\varepsilon}\frac{\mu_i^2}{r^2+a_i^2}\prod_j^N(r^2+a_j^2)\,, \\ F &= \frac{r^{\varepsilon-2}\rho^2}{L^2}\prod_{i=1}^{N}(r^2+a_i^2)\,, \quad \Xi_i = 1-\frac{a_i^2}{L^2}\,. \end{aligned} \tag{B.3}$$

To treat even ($\varepsilon = 1$) odd ($\varepsilon = 0$) spacetime dimensionality D simultaneously, we have parametrized

$$D = 2N + 1 + \varepsilon \tag{B.4}$$

and in even dimensions set for convenience $a_{N+1} = 0$. The coordinates μ_i are not independent, but obey the following constraint:

$$\sum_{i=1}^{N+\varepsilon}\mu_i^2 = 1\,. \tag{B.5}$$

In general the spacetime admits N independent angular momenta J_i, described by N rotation parameters a_i. Namely, the mass M, the angular momenta J_i, and the angular velocities of the horizon Ω_i read [152]

$$\begin{aligned}
M &= \frac{m\omega_{D-2}}{4\pi(\prod_j \Xi_j)}(\sum_{i=1}^{N}\frac{1}{\Xi_i}-\frac{1-\varepsilon}{2})\,,\\
J_i &= \frac{a_i m\omega_{D-2}}{4\pi\Xi_i(\prod_j \Xi_j)}\,,\quad \Omega_i=\frac{a_i(1+\frac{r_+^2}{L^2})}{r_+^2+a_i^2}\,,
\end{aligned} \tag{B.6}$$

Let us first discuss how to take the black brane limit in one direction, associated with the j 2-plane. Starting from the Kerr-AdS metric (B.1) we perform the following scaling:

$$t=\epsilon^2\hat{t}\,,\quad r=\epsilon^2\hat{r}\,,\quad \mu_j=\epsilon^{\frac{D-1}{2}}\sigma/L\,, \tag{B.7}$$

where

$$\epsilon=\Xi_j^{\frac{1}{D-5}}\to 0\quad \text{as}\quad a_j\to L\,. \tag{B.8}$$

Since we want to keep the physical mass M and angular momenta J_i finite for all i, we have to have $m\sim\epsilon^{2(D-5)}$. Namely, we set

$$m\to\epsilon^{2(D-5)}\hat{m}L^2\,, \tag{B.9}$$

where the factor L^2 was chosen to cancel a factor of L^2 in U after the rescaling. For the limit to work, we must have also keep m/U finite. Using the scalings (B.7),

$$\begin{aligned}
U &= r^\varepsilon\sum_{i=1}^{N+\varepsilon}\frac{\mu_i^2}{r^2+a_i^2}\prod_k^N(r^2+a_k^2) \qquad\qquad (B.10)\\
&= \epsilon^{2\varepsilon}\hat{r}^\varepsilon\Big(\frac{\sigma^2}{L^2}\epsilon^{D-1}+\sum_{i\neq j}^{N+\varepsilon}\frac{\mu_i^2(\epsilon^4\hat{r}^2+a_j^2)}{\epsilon^4\hat{r}^2+a_i^2}\Big)\prod_{k\neq j}^{N}(\epsilon^4\hat{r}^2+a_k^2)\,.
\end{aligned}$$

We see from this that we will not have $U\sim\epsilon^{2(D-5)}\hat{U}$ unless we rescale the a_i's so that

$$a_i\to\epsilon^2\hat{a}_i\quad\text{for}\quad i\neq j\,. \tag{B.11}$$

Let us define the following two functions for future reference:

$$\begin{aligned}
\hat{U} &= \hat{r}^\varepsilon\Big(\sum_{i\neq j}^{N+\varepsilon}\frac{\mu_i^2}{\hat{r}^2+\hat{a}_i^2}\Big)\prod_{k\neq j}^{N}(\hat{r}^2+\hat{a}_k^2)\,,\\
\hat{F} &= \hat{r}^{\varepsilon-2}\prod_{i\neq j}^{N}(\hat{r}^2+\hat{a}_i^2)\,.
\end{aligned} \tag{B.12}$$

Then we find

$$
\begin{aligned}
U &= \epsilon^{4N-8+2\varepsilon}\hat{r}^{\varepsilon}\Big(\frac{\sigma^2}{l^2}\epsilon^{D+3}+\sum_{i\neq j}^{N+\varepsilon}\frac{\mu_i^2(\epsilon^4\hat{r}^2+a_j^2)}{\hat{r}^2+\hat{a}_i^2}\Big)\prod_{k\neq j}^{N}(\hat{r}^2+\hat{a}_k^2) \\
&= \epsilon^{2(D-5)}a_j^2\hat{U}+\mathcal{O}(\epsilon^{2D-6})\,,
\end{aligned}
\tag{B.13}
$$

giving (in the limit $\epsilon \to 0$)

$$
\frac{m}{U} \quad \sim \quad \frac{\epsilon^{2(D-5)}\hat{m}L^2}{\epsilon^{2(D-5)}a_j^2\hat{U}} \to \frac{\hat{m}}{\hat{U}}\,. \tag{B.14}
$$

Also the limits of W and F are now easy to take

$$
\begin{aligned}
F &= r^{\varepsilon-2}\Big(1+\frac{r^2}{L^2}\Big)\prod_{i=1}^{N}(r^2+a_i^2) = \epsilon^{2(D-5)}a_j^2\hat{F}+\mathcal{O}(\epsilon^{2D-6})\,, \\
W &= \sum_{i=1}^{N+\varepsilon}\frac{\mu_i^2}{\Xi_i} = \epsilon^4\frac{\sigma^2}{L^2}+\sum_{i\neq j}^{N+\varepsilon}\frac{\mu_i^2}{(1-\epsilon^4\hat{a}_i^2/L^2)} \\
&= \sum_{i\neq j}^{N+\varepsilon}\mu_i^2+\mathcal{O}(\epsilon^4) = 1+\mathcal{O}(\epsilon^4)\,.
\end{aligned}
\tag{B.15}
$$

Hence we get the correct scaling of F to keep $U/(F-2m)$ finite. We also have

$$
\begin{aligned}
&\sum_{i=1}^{N}\frac{a_i\mu_i^2 d\phi_i}{\Xi_i} = \epsilon^4\frac{a_j\sigma^2 d\phi_j}{L^2}+\sum_{i\neq j}^{N}\frac{\epsilon^2\hat{a}_i\mu_i^2 d\phi_i}{(1-\epsilon^4\hat{a}_i^2/L^2)} \\
&\quad = \epsilon^2\sum_{i\neq j}^{N}\hat{a}_i\mu_i^2 d\phi_i+\mathcal{O}(\epsilon^4)\,, \\
&\sum_{i=1}^{N}\frac{r^2+a_i^2}{\Xi_i}\mu_i^2 d\phi_i^2 \\
&\quad = \frac{\epsilon^8 r^2+\epsilon^4 a_j^2}{L^2}\sigma^2 d\phi_j^2+\epsilon^4\sum_{i\neq j}^{N}\frac{\hat{r}^2+\hat{a}_i^2}{(1-\epsilon^4\hat{a}_i^2/L^2)}\mu_i^2 d\phi_i^2 \\
&\quad = \epsilon^4\frac{a_j^2}{L^2}\sigma^2 d\phi_j^2+\epsilon^4\sum_{i\neq j}^{N}(\hat{r}^2+\hat{a}_i^2)\mu_i^2 d\phi_i^2+\mathcal{O}(\epsilon^8)\,,
\end{aligned}
\tag{B.16}
$$

and the $d\mu_i$ terms give

$$\begin{aligned}
&\sum_{i=1}^{N+\varepsilon}\frac{r^2+a_i^2}{\Xi_i}d\mu_i^2 = \frac{\epsilon^8\hat{r}^2+\epsilon^4 a_j^2}{L^2}d\sigma^2+\epsilon^4\sum_{i\neq j}^{N+\varepsilon}\frac{\hat{r}^2+\hat{a}_i^2}{(1-\epsilon^4\hat{a}_i^2/L^2)}d\mu_i^2 \\
&\quad = \epsilon^4\frac{a_j^2}{L^2}d\sigma^2+\epsilon^4\sum_{i\neq j}^{N+\varepsilon}(\hat{r}^2+\hat{a}_i^2)d\mu_i^2+\mathcal{O}(\epsilon^8)\,, \\
&\Big(\sum_{i=1}^{N+\varepsilon}\frac{r^2+a_i^2}{\Xi_i}\mu_i d\mu_i\Big)^2 \\
&\quad = \Big(\frac{\epsilon^8\hat{r}^2+\epsilon^4 a_j^2}{L^2}\sigma d\sigma+\epsilon^4\sum_{i\neq j}^{N+\varepsilon}\frac{\hat{r}^2+\hat{a}_i^2}{(1-\epsilon^4\hat{a}_i^2/L^2)}\mu_i d\mu_i\Big)^2 \\
&\quad = \mathcal{O}(\epsilon^8)\,.
\end{aligned} \tag{B.17}$$

Now that we know how all the components of the metric scale at lowest order as we take the black brane ultraspinning limit, we can set $\phi_j=\varphi$, and rescale the metric by a constant conformal factor, $s=\epsilon^2\hat{s}$. There are no components of order less than 4 in the rescaled metric, so we may cancel the ϵ^4 and complete the limit $a_j\to L$.

The obtained metric is a vacuum solution of Einstein equations with zero cosmological constant that describes a (static in the original 2-plane) black brane

$$\begin{aligned}
d\hat{s}^2 &= -d\hat{t}^2+\frac{2\hat{m}}{\hat{U}}\Big(d\hat{t}-\sum_{i\neq j}^{N}\hat{a}_i\mu_i^2 d\phi_i\Big)^2+\frac{\hat{U}d\hat{r}^2}{\hat{F}-2\hat{m}} \\
&+ d\sigma^2+\sigma^2 d\varphi^2+\sum_{i\neq j}^{N+\varepsilon}(\hat{r}^2+\hat{a}_i^2)d\mu_i^2 \\
&+ \sum_{i\neq j}^{N}(\hat{r}^2+\hat{a}_i^2)\mu_i^2 d\phi_i^2\,.
\end{aligned} \tag{B.18}$$

Here, the metric functions $\hat{F}$ and $\hat{U}$ are given by (B.12), and the coordinates μ_i are bound to satisfy the following constraint:

$$\sum_{i\neq j}^{N+\varepsilon}\mu_i^2=1\,. \tag{B.19}$$

Note that in the process of taking the black brane limit we have 'lost' the AdS radius L and no longer have an asymptotically AdS space. This is in contrast to the super-entropic and hyperboloid membrane limits which retain their asymptotic AdS structure. Another difference is that the black brane limit can be simultaneously taken in several directions[28], whereas this is impossible for the super-entropic limit.

B.2 Hyperboloid membrane limit

In this appendix, we examine another type of the ultraspinning limit—the hyperboloid membrane limit. The hyperboloid membrane limit was first studied in [130, 141], where it was found applicable to the Kerr-AdS spacetime for $D \geq 4$. In this limit, one lets the rotation parameter a approach the AdS radius L, $a \to L$, while scaling the polar angle $\theta \to 0$ in a way so that the coordinate σ defined by

$$\sin\theta = \sqrt{\Xi}\sinh(\sigma/2) \tag{B.20}$$

remains fixed. Contrary to the super-entropic limit, this limit does not require any special rotating frame. We shall now demonstrate how this works for black holes in four dimensions.

In four dimensions, applying the coordinate transformation (B.20) to the Kerr–Newman-AdS metric (3.13) and taking the limit $a \to L$, we find

$$\begin{aligned} ds^2 &= -f\big(dt - L\sinh^2(\sigma/2)\, d\phi\big)^2 + \frac{dr^2}{f} \\ &\quad + \frac{\rho^2}{4}\big(d\sigma^2 + \sinh^2\sigma d\phi^2\big)\,, \end{aligned} \tag{B.21}$$

where

$$f = 1 - \frac{2mr}{\rho^2} + \frac{r^2}{L^2}\,, \quad \rho^2 = r^2 + L^2\,. \tag{B.22}$$

Note that whereas the black brane limit discussed previously yield asymptotically flat metrics, this limit retains the asymptotically AdS structure of the spacetime.

[28] Since the result of the black brane limit is no longer AdS it is not possible to take several such limits successively.

Appendix C

Klemm's construction of the super-entropic black hole

In this appendix, we review Klemm's construction of the super-entropic black hole in four dimensions [143]. Klemm begins with the Carter-Plebański solution of Einstein-Maxwell-Λ theory, which has the following metric and field strength:

$$ds^2 = -\frac{X(x)}{p^2+q^2}(d\tau - y^2 d\sigma)^2 + \frac{y^2+x^2}{X(x)}dq^2 + \frac{y^2+x^2}{Y(y)}dy^2 + \frac{Y(y)}{y^2+x^2}(d\tau + x^2 d\sigma)^2\,, \quad \text{(C.1)}$$

$$F = \frac{Q(y^2-x^2)+2Pxy}{(y^2+x^2)^2}dx\wedge(d\tau - y^2 d\sigma) + \frac{P(y^2-x^2)-2\mathsf{Q}xy}{(p^2+q^2)^2}dy\wedge(d\tau + q^2 d\sigma)\,. \quad \text{(C.2)}$$

The Einstein-Maxwell equations demand that the functions $X(x)$ and $Y(y)$ are quartic polynomials with the following forms:

$$\begin{aligned} X(x) &= \alpha + Q^2 - 2mx + \varepsilon x^2 + (-\Lambda/3)x^4\,, \\ Y(y) &= \alpha - P^2 + 2ny - \varepsilon y^2 + (-\Lambda/3)y^4\,. \end{aligned} \quad \text{(C.3)}$$

Here, Q, P and n denote the electric, magnetic and NUT-charge respectively, m is the mass parameter, while α and ε are additional non-dynamical constants. Also, Λ is the cosmological constant, which we take to be negative, $\Lambda = -3/L^2$.

The solution possesses a scaling symmetry

$$\begin{aligned} y &\to \lambda y\,, \quad x \to \lambda x\,, \quad \tau \to \tau/\lambda\,, \quad \sigma \to \sigma/\lambda^3\,, \\ \alpha &\to \lambda^4\alpha\,, \quad P \to \lambda^2 P\,, \quad Q \to \lambda^2 Q\,, \\ m &\to \lambda^3 m\,, \quad n \to \lambda^3 n\,, \quad \varepsilon \to \lambda^2\varepsilon\,. \end{aligned} \quad \text{(C.4)}$$

that leaves (C.1) and (C.2) invariant. This scaling symmetry can be used at leisure to adjust the non-dynamical constants α and ε. From now on will we assume that the magnetic and NUT charges vanish, $\mathsf{P} = n = 0$.

The Carter-Plebański solution reduces to various more familiar solutions upon assuming certain explicit forms for the polynomials $X(x)$ and $Y(y)$. For the case of black hole solutions, roughly speaking, we can think of the roots of $X(x)$ as determining the locations of horizons, while the roots of $Y(y)$ determine the transverse geometry; the range of y should be constrained to regions where $Y(y)$ is positive to ensure the absence of closed timelike curves and proper signature.

A particularly interesting case arises when multiple roots of $Y(y)$ coincide — this yields the super-entropic black hole. Under the constraint

$$L^2\varepsilon^2 = 4\alpha\,, \tag{C.5}$$

the polynomial $Y(y)$ has two double roots at

$$y = \pm y_a := \frac{\varepsilon L^2}{2}\,. \tag{C.6}$$

Because of the scaling symmetry, without loss of generality we can set $\varepsilon = 2$ yielding $y_a = L$. In this case, the polynomials read

$$X(x) = \left(L + \frac{x^2}{L}\right)^2 + Q^2 - 2mx\,, \tag{C.7}$$

and the metric describes the super-entropic black hole, with the largest root of $X(x)$ giving the location of the horizon.

The equivalence between Klemm's solution and the four-dimensional super-entropic black hole presented in chapter 3 is observed after performing the coordinate transformation given in eq. (3.17).

Appendix D

Field equations in a general theory of gravity

In this appendix, we provide a derivation of the field equations in a higher curvature theory of gravity. The primary reference for this appendix is [196].

We consider a Lagrangian of the form

$$L = \sqrt{-g}\mathcal{L}(g_{\alpha\beta}, R_{\alpha\beta\mu\nu}) \tag{D.1}$$

and vary the action to produce the field equations and isolate the boundary term. Here we do not consider the possibility that the Lagrangian depends on derivatives of the Riemann tensor.

We have

$$\delta L = \sqrt{-g}\left(\frac{\partial \mathcal{L}}{\partial g_{\alpha\beta}}\delta g_{\alpha\beta} + \frac{1}{2}\mathcal{L}g^{\alpha\beta}\delta g_{\alpha\beta} + P^{\alpha\beta\mu\nu}\delta R_{\alpha\beta\mu\nu}\right) \tag{D.2}$$

where we have defined

$$P^{\alpha\beta\mu\nu} := \frac{\partial \mathcal{L}}{\partial R_{\alpha\beta\mu\nu}}\,, \tag{D.3}$$

which, by definition, has the same symmetries as the Riemann tensor. We will now focus on the two "non-trivial" terms in the variation.

First, we focus on the variation of the Riemann term. Using $R^{\alpha}{}_{\beta\mu\nu} = g^{\alpha\sigma}R_{\sigma\beta\mu\nu}$ we have

$$\begin{aligned}
\delta\left(g^{\alpha\sigma}R_{\sigma\beta\mu\nu}\right) &= g^{\alpha\sigma}\delta R_{\sigma\beta\mu\nu} + R_{\sigma\beta\mu\nu}\delta g^{\alpha\sigma} = \delta R^{\alpha}{}_{\beta\mu\nu} \\
&\Rightarrow g^{\alpha\sigma}\delta R_{\sigma\beta\mu\nu} = \nabla_{\mu}\left(\delta\Gamma^{\alpha}_{\beta\nu}\right) - \nabla_{\nu}\left(\delta\Gamma^{\alpha}_{\beta\mu}\right) - R_{\sigma\beta\mu\nu}\delta g^{\alpha\sigma} \\
&\Rightarrow \delta R_{\alpha\beta\mu\nu} = g_{\alpha\rho}\nabla_{\mu}\left(\delta\Gamma^{\rho}_{\beta\nu}\right) - g_{\alpha\rho}\nabla_{\nu}\left(\delta\Gamma^{\rho}_{\beta\mu}\right) + g^{\rho\sigma}R_{\sigma\beta\mu\nu}\delta g_{\alpha\rho}
\end{aligned} \tag{D.4}$$

Now, we simplify the contraction with $P^{\alpha\beta\mu\nu}$:

$$\begin{aligned}
P^{\alpha\beta\mu\nu}\delta R_{\alpha\beta\mu\nu} &= P^{\alpha\beta\mu\nu}R^{\rho}{}_{\beta\mu\nu}\delta g_{\alpha\rho} + P^{\alpha\beta\mu\nu}g_{\alpha\rho}\nabla_{\mu}\left(\delta\Gamma^{\rho}_{\beta\nu}\right) - P^{\alpha\beta\mu\nu}g_{\alpha\rho}\nabla_{\nu}\left(\delta\Gamma^{\rho}_{\beta\mu}\right) \\
&= P^{\alpha\beta\mu\nu}R^{\rho}{}_{\beta\mu\nu}\delta g_{\alpha\rho} + 2P^{\alpha\beta\mu\nu}g_{\alpha\rho}\nabla_{\mu}\left(\delta\Gamma^{\rho}_{\beta\nu}\right) \\
&= P^{\alpha\beta\mu\nu}R^{\rho}{}_{\beta\mu\nu}\delta g_{\alpha\rho} + 2\nabla_{\mu}\left(P^{\alpha\beta\mu\nu}g_{\alpha\rho}\delta\Gamma^{\rho}_{\beta\nu}\right) + 2\nabla_{\beta}\left(\nabla_{\nu}P^{\alpha\beta\mu\nu}\delta g_{\alpha\mu}\right) \\
&\quad + 2\nabla_{\beta}\nabla_{\mu}P^{\alpha\beta\mu\nu}\delta g_{\alpha\nu}
\end{aligned} \tag{D.5}$$

In going from the second line to the third line, we used

$$\delta\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(\nabla_{\alpha}\delta g_{\beta\nu} + \nabla_{\beta}\delta g_{\alpha\nu} - \nabla_{\nu}g_{\alpha\beta}\right) \tag{D.6}$$

and made plenty of use of the symmetries of $P^{\alpha\beta\mu\nu}$.

To compute

$$\frac{\partial\mathcal{L}}{\partial g_{\alpha\beta}} \tag{D.7}$$

we will study how the Lagrangian changes under diffeomorphisms $x^{\alpha} \to x^{\alpha} + \xi^{\alpha}$ in two different ways. First, since $\mathcal{L}$ is a function of the metric and Riemann tensor we have

$$\begin{aligned}
\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial g_{\alpha\beta}}\mathscr{L}_{\xi}g_{\alpha\beta} + P^{\alpha\beta\mu\nu}\mathscr{L}_{\xi}R_{\alpha\beta\mu\nu} \\
&= 2\frac{\partial\mathcal{L}}{\partial g_{\alpha\beta}}\nabla_{\alpha}\xi_{\beta} + P^{\alpha\beta\mu\nu}\xi^{\sigma}\nabla_{\sigma}R_{\alpha\beta\mu\nu} + 4P^{\alpha\beta\mu\nu}R^{\sigma}{}_{\beta\mu\nu}\nabla_{\alpha}\xi_{\sigma}
\end{aligned} \tag{D.8}$$

To go from the first line to the second line, we exploited the symmetry of $P^{\alpha\beta\mu\nu}$.

Second, since $\mathcal{L}$ is a scalar, we have:

$$\begin{aligned}
\delta\mathcal{L} &= \mathscr{L}_{\xi}\mathcal{L} \\
&= \xi^{\sigma}\nabla_{\sigma}\mathcal{L} \\
&= \xi^{\sigma}\left(\frac{\partial\mathcal{L}}{\partial g_{\alpha\beta}}\nabla_{\sigma}g_{\alpha\beta} + P^{\alpha\beta\mu\nu}\nabla_{\sigma}R_{\alpha\beta\mu\nu}\right) \\
&= P^{\alpha\beta\mu\nu}\xi^{\sigma}\nabla_{\sigma}R_{\alpha\beta\mu\nu}
\end{aligned} \tag{D.9}$$

Now, putting the two together we have:

$$\begin{aligned}
&2\frac{\partial\mathcal{L}}{\partial g_{\alpha\beta}}\nabla_{\alpha}\xi_{\beta} + P^{\alpha\beta\mu\nu}\xi^{\sigma}\nabla_{\sigma}R_{\alpha\beta\mu\nu} + 4P^{\alpha\beta\mu\nu}R^{\sigma}{}_{\beta\mu\nu}\nabla_{\alpha}\xi_{\sigma} = P^{\alpha\beta\mu\nu}\xi^{\sigma}\nabla_{\sigma}R_{\alpha\beta\mu\nu} \\
\Rightarrow &\frac{\partial\mathcal{L}}{\partial g_{\alpha\beta}} = -2P^{\alpha\sigma\mu\nu}R^{\beta}{}_{\sigma\mu\nu}
\end{aligned} \tag{D.10}$$

where the last implication follows because this must be true for any vector ξ^α.

Combining the various results above, we have

$$\begin{aligned}\delta L &= \sqrt{-g}\left(\frac{\partial\mathcal{L}}{\partial g_{\alpha\beta}}\delta g_{\alpha\beta} + \frac{1}{2}\mathcal{L}g^{\alpha\beta}\delta g_{\alpha\beta} + P^{\alpha\beta\mu\nu}\delta R_{\alpha\beta\mu\nu}\right)\\ &= \sqrt{-g}\left(-2P^{\alpha\sigma\mu\nu}R^{\beta}{}_{\sigma\mu\nu} + \frac{1}{2}g^{\alpha\beta}\mathcal{L} + P^{\alpha\sigma\mu\nu}R^{\beta}{}_{\sigma\mu\nu} + 2\nabla_\sigma\nabla_\mu P^{\alpha\sigma\mu\beta}\right)\delta g_{\alpha\beta}\\ &+ 2\sqrt{-g}\nabla_\mu\left(P^{\alpha\beta\mu\nu}g_{\alpha\rho}\delta\Gamma^{\rho}_{\beta\nu} + \nabla_\nu P^{\alpha\mu\sigma\nu}\delta g_{\alpha\sigma}\right)\\ &= \sqrt{-g}\left(-P^{\alpha\sigma\mu\nu}R^{\beta}{}_{\sigma\mu\nu} + \frac{1}{2}g^{\alpha\beta}\mathcal{L} + 2\nabla_\sigma\nabla_\mu P^{\alpha\sigma\mu\beta}\right)\delta g_{\alpha\beta}\\ &+ \partial_\mu\left[2\sqrt{-g}\left(P^{\alpha\beta\mu\nu}g_{\alpha\rho}\delta\Gamma^{\rho}_{\beta\nu} + \nabla_\nu P^{\alpha\mu\sigma\nu}\delta g_{\alpha\sigma}\right)\right]\end{aligned} \tag{D.11}$$

From the above, we can read off the generalized Einstein tensor:

$$\mathcal{E}^{\alpha\beta} = -P^{\alpha\sigma\mu\nu}R^{\beta}{}_{\sigma\mu\nu} + \frac{1}{2}g^{\alpha\beta}\mathcal{L} + 2\nabla_\sigma\nabla_\mu P^{\alpha\sigma\mu\beta} \tag{D.12}$$

with indices up, or (noting that $\delta g_{\sigma\rho} = -g_{\alpha\sigma}g_{\beta\rho}\delta g^{\alpha\beta}$)

$$\mathcal{E}_{\alpha\beta} = P_{\alpha}{}^{\sigma\mu\nu}R_{\beta\sigma\mu\nu} - \frac{1}{2}g_{\alpha\beta}\mathcal{L} - 2\nabla^\sigma\nabla^\mu P_{\alpha\sigma\mu\beta} \tag{D.13}$$

with indices down. By definition

$$T_{\alpha\beta} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{\mathrm{M}})}{\delta g^{\alpha\beta}} \tag{D.14}$$

so the full field equations read

$$\mathcal{E}_{\alpha\beta} = \kappa T_{\alpha\beta}\,. \tag{D.15}$$

Written in the action form, we see that:

$$\delta S = \int_{\mathcal{V}} d^dx\sqrt{-g}\mathcal{E}_{\alpha\beta}\delta g^{\alpha\beta} + \oint_{\partial\mathcal{V}} d\Sigma_\mu\delta v^\mu \tag{D.16}$$

where

$$\delta v^\mu = 2\left(P^{\alpha\beta\mu\nu}g_{\alpha\rho}\delta\Gamma^{\rho}_{\beta\nu} + \nabla_\nu P^{\alpha\mu\sigma\nu}\delta g_{\alpha\sigma}\right)\,. \tag{D.17}$$

If Dirichlet boundary conditions are enforced on the metric, then the term involving the variation of the metric above will vanish, giving

$$\delta v^\mu = 2P^{\alpha\beta\mu\nu}g_{\alpha\rho}\delta\Gamma^{\rho}_{\beta\nu}\,. \tag{D.18}$$

This term should be recast in terms of extrinsic and intrinsic curvatures of the boundary giving generalizations of the Gibbons-Hawking-York term for Einstein gravity. In the case of Lovelock theory, these terms were worked out independently by Myers [244] and Teitelboim and Zanelli [245].