# Tsirelson's problems and entanglement breaking rank

by

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This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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## Statement of Contributions

Chapters 3 and 4 are based on joint work with Ken Dykema and Vern I. Paulsen. Chapter 5 is based on my contribution to a joint work with Satish K. Pandey, Vern I. Paulsen, and Mizanur Rahaman.

## Abstract

There are two main themes of this thesis. In the first part, we study the various sets of correlations arising in the study of non-local games and Tsirelson's problems. In the second part we introduce the notion of entanglement breaking rank and study its connection with the SIC POVM existence problem.

Tsirelson's problems have been open for quite some time and continue to be an active area of research. These problems seek to identify relationships between different sets of quantum correlations arising from different quantum models and their relaxations. One of the goals of this thesis is to show that the set of correlations arising from the tensor product model is not topologically closed, and hence is distinct from the set of correlations arising from the commuting tensor model.

We begin our investigation in Chapter 3 by working through an example of a non-local game with three inputs and binary outputs. While this toy example does not help in showing the distinction between the aforementioned sets of correlations, it still provides valuable insights and motivation for a more abstract approach which we take up in the next chapter and which turns out to be fruitful.

In Chapter 4, we observe that given a graph we can associate functions (of a real variable) corresponding to different correlation sets. We study the properties of these functions. We show that for the complete graph on five vertices, the assumption that the correlation set obtained through tensor model is closed forces its corresponding function to be piecewise linear on an interval, while the function corresponding to the commuting model is not. This shows that the quantum correlation set is not closed hence solving one of Tsirelson's problem and also improves the results of Slofstra by reducing the number of experiments and outcomes to five and two, respectively.

Finally, for the second part of the thesis, we introduce the notion of entanglement breaking rank of an entanglement breaking channel. While technically this notion is not new, this point of view helps us in showing that the entanglement breaking rank of a particular channel on $\mathbb{M}_d$ being $d^2$ is equivalent to the existence of a SIC POVM in dimension $d$. This opens up another approach to tackle this existence problem.

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## Dedication

This thesis is dedicated to my parents.

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# Chapter 1

# Introduction

The primary postulate in the axiomatic formalism of quantum mechanics is that the state space of a quantum system is described by a Hilbert space. For describing composite systems, another postulate states that the state space of the composite system is given by the Hilbert space tensor product of the component state spaces [53]. However, in some other formulations of quantum mechanics like the Algebraic Quantum Field Theory (AQFT), it is postulated that there is a single Hilbert space (possibly infinite dimensional) which describes all quantum systems, and the composite system is then described by adding the condition that the observables of individual systems commute [31]. To ascertain whether these two models are same or different, we usually have Alice and Bob run experiments in two labs which are spatially separated but are possibly in an entangled state. Each of them have $n$ experiments to run and each experiment has $k$ outcomes. To study correlations between their outcomes and the model, we let $p(i,j|v,w)$ denote the conditional probability that if the labs run experiment $v$ and $w$, then they get outcomes $i$ and $j$, respectively. This defines a $n^2k^2$-tuple. Assuming different mathematical models we might get different sets of such bipartite conditional probabilities satisfying (see Section 2.1):

$$C_{loc}(n,k) \subseteq C_q(n,k) \subseteq C_{qs}(n,k) \subseteq C_{qa}(n,k) \subseteq C_{qc}(n,k) \subseteq C_{vect}(n,k),$$

where $loc, q, qs, qa, qc, vect$ correspond to the different models. For example, $q$ and $qc$ correspond to the tensor product model and the commuting model described above. The problem then is to determine when equalities occur in the above string of inclusions.

Tsirelson first investigated these sets in [11, 74]. The problem of determining equalities of such sets are now known as Tsirelson's problems [25, 22]. The difference between $C_{loc}(n,k)$ and $C_q(n,k)$ is well encapsulated by the famous Bell's theorem [3]. These problems have become an active area of research. A major development took place when it was shown that one of such equality problem $C_{qa}(n,k) = C_{qc}(n,k)$ for all $n, k$ is equivalent to Connes' embedding problem, a long standing problem in operator algebras [17, 39, 54]. In 2017, William Slofstra showed that $C_q(n,k) \neq C_{qa}(n,k)$ using the theory of finitely presented groups and non-local games [72, 73]. His results require the number of experiments to be on the order of 100 and $k = 8$. One of the objectives of this thesis is to show that this distinction between these two sets can be seen at a much lower number, namely $n = 5$ and $k = 2$.

Moreover, it is widely suspected that the non-local game corresponding to the $I_{3322}$ inequality (based on three experiments and binary outcomes) differentiates between $C_q(3,2)$ and $C_{qa}(3,2)$. This would further improve on Slofstra's and our results [26, 16]. However, since it is highly asymmetric it is quite difficult to analyse. In Chapter 3 we introduce a simpler game (called the Delta game) on three experiments and two outcomes which was brought to our attention by Richard Cleve. Instead of computing the maximum winning probability of the Delta game over a correlation set, we compute the maximum winning probability with some constraints over a correlation set. This results in computing a function of a real variable instead of a number. We get separate functions for separate models. We show that we can detect a difference between the $qc$ and the $vect$ model, but not between the $q$ and $qc$ model using the Delta game. However, this analysis provides us with some key insights which help us to develop this line of thought abstractly continuing in Chapter 4.

The Delta game example showed that equivalently given a graph $G = (V, E)$ one could define a hierarchy of functions on the interval $[0, 1]$ corresponding to each correlation set:

$$f_{loc} \geq f_q \geq f_{qs} \geq f_{qa} \geq f_{qc} \geq f_{vect} \geq 0.$$

We study some basic properties of these functions. For example, these functions are convex and it suffices to describe them on the interval $[0, \frac{1}{2}]$. One of the striking properties that the function $f_q$ satisfies is the following. If we assume that the set $C_q(5,2)$ is closed, then the function $f_q$ is linear on some interval around every irrational number in $[0, 1]$. On the other hand, we could show by comparing $f_q$ with $f_{vect}$ that $f_q$ has to be quadratic on some interval. This evidently presents a contradiction and implies that $C_q(5,2)$ is not closed.

In Chapter 5, the final chapter of the thesis, we embark on a completely different topic: entanglement breaking rank. Entanglement breaking maps always possess a Choi-Kraus representation where the Choi-Kraus operators are rank one [37]. Analogous to the Choi rank, we define the entanglement breaking rank of an entanglement breaking map as the least number of rank one operators required in a Choi-Kraus representation. Viewing an entanglement breaking channel as a separable state (via its Choi matrix), we show that the entanglement breaking rank is nothing but the *length* [9] or the *optimal ensemble cardinality* [18] of the state in hand. While this notion is not new, we show that computing entanglement breaking rank of certain entanglement breaking maps is equivalent to the existence problem of symmetric informationally complete POVMs (SIC POVM). This seems to provide us with a promising approach towards a better understanding of separable states and the problem of SIC POVM existence.

Chapters 3 and 4 are based on [20, 21]. Chapter 5 is a selection of [56].

## 1.1 Notations and conventions

The following notations and conventions will be followed throughout this thesis.

1. We shall let $\mathbb{N}$ denote the set of positive integers $\{1, 2, 3, ...\}$; $\mathbb{R}$ denote the set of real numbers; and $\mathbb{C}$ denote the set of complex numbers.

2. The cardinality of a set $X$ will be denoted by $|X|$. When $i, j \in \{0, 1, ..., k-1\}$, we shall use the notation $i \oplus j$ to mean $i + j$ modulo $k$.

3. All Hilbert spaces will be assumed to be over the field of complex numbers unless specified otherwise. The inner product on a complex Hilbert space is assumed to be linear in the first variable and conjugate linear in the second variable. Moreover, we shall follow the usual mathematical notation $\langle \xi, \eta \rangle$ instead of $\langle \xi | \eta \rangle$ for the inner product of vectors $\xi$ and $\eta$ in some Hilbert space. The algebra of all bounded linear maps (called operators) on a Hilbert space $\mathcal{H}$ will be denoted by $B(\mathcal{H})$. The adjoint of an operator $T \in B(\mathcal{H})$ will be denoted by $T^*$. An operator $P \in B(\mathcal{H})$ is called a projection if $P = P^* = P^2$.

4. Let $d \in \mathbb{N}$. We shall let $\mathbb{C}^d$ denote the Hilbert space of $n$-tuples of complex numbers with the usual inner product. We shall denote the space of $d \times d$ complex matrices by $\mathbb{M}_d$, which may be identified as $B(\mathbb{C}^d)$. The trace of a matrix $A = [a_{i,j}] \in \mathbb{M}_d$ is defined by $\mathrm{Tr}(A) = \sum_{i=1}^d a_{i,i}$. The normalised trace of $A$ is denoted and defined by $\mathrm{tr}_d(A) = \frac{1}{d}\mathrm{Tr}(A)$. The space $\mathbb{M}_d$ has a natural inner product called the *Hilbert-Schmidt inner product* given by $\langle A, B \rangle_2 := \mathrm{Tr}(AB^*)$ for all $A, B \in \mathbb{M}_d$. Set $\|A\|_2^2 := \langle A, A \rangle_2$; for example $\|\mathbb{I}_d\|_2 = \sqrt{d}$.

5. A graph will always be a simple graph (undirected graph with no loops or multiple edges). A graph $G$ is described by $G = (V, E)$, where $V$ is the set of vertices, and

$E \subseteq V \times V$ is the set of edges. If $v$ and $w$ are adjacent vertices of a graph, we shall denote it by $v \sim w$.

# Chapter 2

# Tsirelson's problems for a bipartite system

## 2.1 Non-local games

We outline the general theory of two-player non-local games in this section. Such games have been useful in understanding the power of entanglement in quantum information theory [13].

A *two-player non-local game* is played by two players, Alice ($A$) and Bob ($B$), cooperatively against a referee ($R$). The game has a set of rules given by a function $\lambda$ called the *rule function* or the *predicate function* of the game,

$$\lambda : I_A \times I_B \times O_A \times O_B \to \{0, 1\},$$

where $I_A, I_B, O_A, O_B$ are non-empty finite sets. The sets $I_A$ and $I_B$ are referred to as the *input sets* or *question sets*, while $O_A$ and $O_B$ are called the *output sets* or *answer sets*, and the labels $A$ and $B$ refer to Alice and Bob, respectively. Thus, a two-player non-local game is completely described by the 5-tuple $\mathcal{G} = (I_A, I_B, O_A, O_B, \lambda)$. Since we shall be exclusively focused on two-player games, we shall drop the term "two-player" and call them "non-local games" or simply "games".

For one round of the game, the referee passes an element $v \in I_A$ to Alice and an element $w \in I_B$ to Bob. Alice and Bob do not know what input the other player has received. Alice and Bob "independently" produce outputs $i \in O_A$ and $j \in O_B$, respectively. They "win" the game if $\lambda(v, w, i, j) = 1$, and lose if $\lambda(v, w, i, j) = 0$. Such a round could be repeated a number of times with the referee passing different inputs and the players replying with outputs. Notice that the players could reply with different outputs for the same inputs in two different rounds.

The following are some simple examples of non-local games.

**Example 2.1.1** (The Graph Colouring Game)**.** Let $G = (V, E)$ be a graph. Recall that a *$k$-colouring* of the graph $G$ is a function $f : V \to \{1, ..., k\}$ such that $(v, w) \in E$ implies $f(v) \neq f(w)$.

We define a non-local game based on this graph and a set of $k$ colors. Let $I_A = I_B = V$ be the set of vertices, and let $O_A = O_B = \{1, ..., k\}$ be a set of colors. In this non-local game, Alice and Bob, try to convince the referee that they have a $k$-colouring of the graph. If they are given adjacent vertices in a round, then they win if they produce different colours. On the other hand, they must produce the same colour if they are given the same vertex. In the case when they are given vertices which are neither equal nor adjacent, they can pass arbitrary colours to the referee without losing. Thus the rules of the game may be encoded with the following rule function,

$$\lambda(v, w, i, j) = \begin{cases} 1 & \text{if } v \sim w \text{ and } i \neq j, \\ 0 & \text{if } v \sim w \text{ and } i = j, \\ 1 & \text{if } v = w \text{ and } i = j, \\ 0 & \text{if } v = w \text{ and } i \neq j, \\ 1 & \text{if } v \nsim w \text{ and } v \neq w. \end{cases}$$

**Example 2.1.2** (The CHSH Game)**.** The CHSH game essentially arose from the work in [12]. For the CHSH game, we have binary inputs and binary outputs $I_A = I_B = O_A = O_B = \{0, 1\}$, and the rule function is described by $\lambda(v, w, i, j) = 1 \Leftrightarrow vw = i \oplus j$.

Alice and Bob are aware of the rules (that is, the rule function $\lambda$) of the game and they need to play cooperatively in order to "win" but they are not allowed to communicate during the game. Hence, together they work out a "strategy" before the game begins. Given a non-local game $\mathcal{G} = (I_A, I_B, O_A, O_B, \lambda)$, one of the natural strategies available to Alice and Bob is to use functions $f_A : I_A \to O_A$ and $f_B : I_B \to O_B$. Thus, when the referee passes $v \in I_A$ to Alice and $w \in I_B$ to Bob, they simply apply the functions $f_A$ and $f_B$ to their respective inputs and pass the answers $f_A(v)$ and $f_B(w)$ to the referee. Such a strategy is called a *deterministic strategy.* A *perfect* or *winning* deterministic strategy for Alice and Bob would then be a pair of functions $(f_A, f_B)$ as above such that $\lambda(v, w, f_A(v), f_B(w)) = 1$ for all $v \in I_A, w \in I_B$.

**Example 2.1.3.** Let $G = (V, E)$ be a graph and consider the graph colouring game for the graph with $k$ colours (Example 2.1.1). It is not too hard to verify that there exists a perfect deterministic strategy for the game if and only if there exists a $k$-colouring of the graph. We shall see later (Example 2.1.16), that using certain probabilistic strategies Alice and Bob can win such a game even when there does not exist a perfect deterministic strategy.

More generally, Alice and Bob can employ *probabilistic strategies*. The idea behind probabilistic strategies is that even though there might not exist a perfect deterministic strategy to win the game, the players can generate outputs according to some probability distribution and maximise their chance of winning the game overall by utilising the flexibility offered by the rule function. A probabilistic strategy gives rise to a joint probability distribution

$$(p(i,j|v,w))_{v\in I_A, w\in I_B, i\in O_A, j\in O_B}, \tag{2.1.1}$$

which could be observed by playing multiple rounds of the game and recording the statistics. The number $p(i,j|v,w)$ is the joint conditional probability that Alice and Bob produce outputs $i \in O_A$ and $j \in O_B$, respectively, when given inputs $v \in I_A$ and $w \in I_B$, respectively. Thus, if $|I_A| = n_A, |I_B| = n_B, |O_A| = k_A, |O_B| = k_B$, we have an $n_A n_B k_A k_B$-tuple of non-negative numbers. Observe that a deterministic strategy $(f_A, f_B)$ gives rise to a joint probability distribution trivially: $p(i,j|v,w) = 1$ if and only if $i = f_A(v)$ and $j = f_B(w)$. Henceforth, any tuple

$$(p(i,j|v,w))_{v\in I_A, w\in I_B, i\in O_A, j\in O_B} \in \mathbb{R}^{n_A n_B k_A k_B},$$

satisying

$$p(i,j|v,w) \geq 0, \qquad \text{and} \qquad \sum_{(i,j)\in O_A\times O_B} p(i,j|v,w) = 1, \quad \forall (v,w) \in I_A \times I_B,$$

will be called a *correlation*. Notice that the set of all correlations is a closed convex set in $\mathbb{R}^{n_A n_B k_A k_B}$.

We shall abuse the terminology sometimes by using the terms "strategy" and "correlation" interchangeably. We shall also drop the subscripts in Expression (2.1.1) and simply write it as $(p(i,j|v,w))$.

The condition that the players do not communicate introduces a constraint on the conditional probability densities $p(i,j|v,w)$ in the following way. Let $p_A(i|v)$ be the conditional probability that Alice produces output $i \in O_A$ when she is given input $v \in I_A$. Similarly, let $p_B(j|w)$ be the conditional probability that Bob produces output $j \in O_B$ when he is given input $w \in I_B$. These are called the *marginal densities* for Alice and Bob. Now, if Alice and Bob do not communicate during the game we would expect these marginal densities to be independent of the input that the other player gets. Thus, we would expect $p_A(i|v)$ to be independent of $w$, and similarly, we would expect $p_B(j|w)$ to be independent of $v$. Simple probabilistic considerations tell us that

$$p_A(i|v) = \sum_{j \in O_B} p(i,j|v,w), \qquad p_B(j|w) = \sum_{i \in O_A} p(i,j|v,w). \tag{2.1.2}$$

Hence, the fact that the players do not communicate during the game is reflected by the fact that the sums in Formulae (2.1.2) are well-defined, that is, the above sums do not depend on $w$ and $v$, respectively. *Non-signalling correlations* are exactly the subset of correlations for which there exist such well-defined marginal densities [11, 62]. These are the largest set of correlations which we shall work with in this thesis. We define them formally as follows.

**Definition 2.1.4** (Non-signalling Correlations)**.** A correlation $(p(i,j|v,w))$ is called a non-signalling correlation if the marginal densities $p_A(i|v)$ and $p_B(j|w)$ are well-defined for all $v \in I_A$ and $w \in I_B$, that is,

$$\begin{aligned} p_A(i|v) &= \sum_{j \in O_B} p(i,j|v,w) = \sum_{j \in O_B} p(i,j|v,w'), \qquad w, w' \in I_B, \\ p_B(j|w) &= \sum_{i \in O_A} p(i,j|v,w) = \sum_{i \in O_A} p(i,j|v',w), \qquad v, v' \in I_A. \end{aligned}$$

We let $C_{ns}(n_A, n_B, k_A, k_B)$ denote the set of all such non-signalling correlations. We set $C_{ns}(n,k) = C_{ns}(n,n,k,k)$.

Observe that $\sum_{i \in O_A} p_A(i|v) = \sum_{j \in O_B} p_B(j|w) = 1$, as expected. We again remark that whenever we shall use the term correlation in general, we shall refer to a non-signalling correlation. It is easy to establish that the set $C_{ns}(n_A, n_B, k_A, k_B)$ is a closed convex set in $\mathbb{R}^{n_A n_B k_A k_B}$.

Certain probabilistic strategies chosen by Alice and Bob may always lead to a win, and rightly so are called *winning* or *perfect* strategies. This happens when Alice and Bob never produce outputs for which they lose (i.e. $\lambda(v,w,i,j) = 0$ implies $p(i,j|v,w) = 0$), a requirement for winning. The correlations arising from perfect strategies are formally defined as follows.

**Definition 2.1.5.** Suppose $\mathcal{G} = (I_A, I_B, O_A, O_B, \lambda)$ is a non-local game. A correlation $(p(i, j|v, w))$ is called a *winning* or *perfect correlation* for $\mathcal{G}$ if $\lambda(v, w, i, j) = 0$ implies $p(i, j|v, w) = 0$.

As a trivial example, notice that the correlation arising from a perfect deterministic strategy is a perfect correlation.

More generally, we can associate a probability of winning the game to a probabilistic strategy. To compute the probability the players need to know the probability distribution through which the referee distributes the inputs. Assume that the referee sends inputs $v, w$ to the players according to some probability distribution $\pi : I_A \times I_B \to [0, 1]$. That is, $\pi(v, w) \geq 0$ for all $(v, w) \in I_A \times I_B$ and

$$\sum_{(v,w)\in I_A\times I_B} \pi(v, w) = 1.$$

The players also know this distribution beforehand and can incorporate this information in deciding their strategies.

Given a set of different strategies we can take the supremum of all their winning probabilities leading to a maximum winning probability for that set. Again, we define this winning probability formally for correlations arising from strategies.

**Definition 2.1.6.** Let $\mathcal{G} = (I_A, I_B, O_A, O_B, \lambda)$ be a non-local game. The *value of a correlation* $p = (p(i, j|v, w))$ given the game $\mathcal{G}$ and a distribution $\pi : I_A \times I_B \to [0, 1]$, is defined by

$$V(p, \pi) = \sum_{v\in I_A, w\in I_B, i\in O_A, j\in O_B} \pi(v, w)\lambda(v, w, i, j)p(i, j|v, w).$$

The *value of the game* $\mathcal{G}$ with respect to a distribution $\pi$ and a set $\mathcal{F}$ of correlations is defined by

$$\omega_{\mathcal{F}}(G, \pi) = \sup \{V(p, \pi) : p \in \mathcal{F}\}. \tag{2.1.3}$$

Since the set of all non-signalling correlations is a compact set in a finite-dimensional vector space, whenever $\mathcal{F}$ is closed, the supremum over $\mathcal{F}$ in Equation (2.1.3) will be attained. Also, notice that if $\mathcal{F} \subseteq \mathcal{F}'$ are two sets of correlations, then $\omega_{\mathcal{F}}(G, \pi) \leq \omega_{\mathcal{F}'}(G, \pi)$; if strict inequality holds then we have $\mathcal{F} \subsetneq \mathcal{F}'$. This idea will be one of the ways by which we will show the distinction between two different sets of strategies later (Proposition 2.1.15).

The following easily established lemma shows how a correlation being perfect is related to its value for a non-local game.

**Lemma 2.1.7** ([22])**.** *Let $\mathcal{G} = (I_A, I_B, O_A, O_B, \lambda)$ be a non-local game with a given probability distribution $\pi : I_A \times I_B \to [0,1]$. If $p = (p(i,j|v,w))$ is a perfect correlation for the game, then $V(p,\pi) = 1$. Conversely, if $\pi(v,w) > 0$ for all $v, w$, and a correlation has value 1, then it is a perfect correlation.*

Broadly speaking, Alice and Bob have two distinct ways to devise (probabilistic) strategies: *classical* and *quantum*. One possible way to describe classical strategies is the following. Alice has access to a set of (possibly loaded) dies numbered $1, 2, ..., n_A$, where each die has $k_A$ outputs. Given an input $v \in I_A$, Alice can roll her die numbered $v$, and pass the output $w \in O_A$ to the referee. Bob also performs similar classical experiments to produce outputs. More generally, a classical strategy arises from using *local hidden variables* (or in computer science terminology, *shared randomness*). The correlations arising from such classical strategies are called classical correlations which we formally define now. Thus, classical correlations model the presence of *hidden variables* [5].

**Definition 2.1.8** (Classical Correlations)**.** A correlation $(p(i,j|v,w))$ is called a *classical correlation* or a *local correlation*, if Alice and Bob share a probability space $(\Omega, \mu)$, Alice has a collection of random variables $\{f_v \mid f_v : \Omega \to O_A\}_{v \in I_A}$, and similarly Bob has a collection of random variables $\{g_w \mid g_w : \Omega \to O_B\}_{w \in I_B}$, such that

$$p(i,j|v,w) = \mu\left(\{\omega \in \Omega : f_v(\omega) = i, g_w(\omega) = j\}\right).$$

The set of all correlations $(p(i,j|v,w))$ arising from all choices of the probability space and the random variables is called the set of classical correlations and is denoted by $C_{loc}(n_A, n_B, k_A, k_B)$. We set $C_{loc}(n,k) = C_{loc}(n,n,k,k)$.

It is again not too hard to establish that the set of classical correlations also forms a closed convex set; in fact it is a polytope. We record this fact.

**Proposition 2.1.9** ([5])**.** *For all $n_A, n_B, k_A, k_B \in \mathbb{N}$, the set $C_{loc}(n_A, n_B, k_A, k_B)$ is a closed convex set and $C_{loc}(n_A, n_B, k_A, k_B) \subseteq C_{ns}(n_A, n_B, k_A, k_B)$. Moreover:*

*(a) the set may be realised as the closed convex hull of all correlations $(p(i,j|v,w))$ arising from deterministic strategies. Since there are finitely many deterministic strategies, the set of classical correlations is a polytope.*

*(b) The set may be realised as the closed convex hull of all correlations $(p(i,j|v,w))$ having the property*

$$p(i,j|v,w) = p_1(i|v)p_2(j|w),$$

*where* $p_1(i|v) \geq 0, p_2(j|w) \geq 0, \sum_{i \in O_A} p_1(i|v) = 1$ *and* $\sum_{j \in O_B} p_2(j|w) = 1$. *(In other words,* $\{(p(i|v))_{i \in O_A} : v \in I_A\}$ *and* $\{(p(j|w))_{j \in O_B} : w \in I_b\}$ *are sets of conditional probability distributions with* $k_A$*-outcomes and* $k_B$*-outcomes, respectively.)*

We remark that in the Physics literature, part (b) in Proposition 2.1.9 is usually taken as a definition of classical correlations. In particular, this factorisation property (or locality condition) is seen as manifestations of hidden variable theory. For a survey on this subject we refer the reader to [5].

Even though we have resorted to using probabilities, the following proposition says that classical strategies are as good as deterministic strategies in winning a game. This follows readily from Proposition 2.1.9.

**Proposition 2.1.10** ([24])**.** *Let* $\mathcal{G} = (I_A, I_B, O_A, O_B, \lambda)$ *be a non-local game with a probability distribution* $\pi : I_A \times I_B \to [0, 1]$*. The game has a perfect deterministic strategy if and only if it has a perfect classical strategy.*

Quantum strategies, on the other hand, are remarkable as they can be used to win games which may be impossible to win with classical strategies (Example 2.1.16). Even in the simple case of CHSH game in Example 2.1.2, the players can have a higher probability of winning if they employ quantum strategies than using classical ones (Proposition 2.1.15). That quantum probabilities fare better was essentially first shown by John Bell in his seminal paper [3].

To explain quantum strategies we shall assume familiarity with the basics of quantum information, for example, Section I.2.2 of [53] will suffice. To devise a quantum strategy, Alice and Bob share an entangled state and have access to quantum measurement systems which they use to generate probabilities (Section I.2.2.3 of [53]). More precisely, let $\mathcal{G} = (I_A, I_B, O_A, O_B, \lambda)$ be a non-local game. For each input $v \in I_A$, Alice has a quantum measurement system $M_v$ which has outcomes in $O_A$. So, when she gets input $v \in I_A$, she runs the measurement system $M_v$, and returns the output $i \in O_A$ obtained from the experiment to the referee. Similarly, Bob has a quantum measurement system $N_w$ for each input $w \in I_B$ having outcomes in the set $O_B$. Different trials of the same experiment can result in different outcomes.

Since we shall be working with probabilities, it is enough to consider POVM measurements (Section I.2.2.6 of [53]) which we quickly recall.

**Definition 2.1.11.** A set $\{R_k\}_{k=1}^K$ of operators on some Hilbert space $\mathcal{H}$ is called a *positive operator-valued measure* (POVM) if $R_k \geq 0$, for each $k$, and $\sum_{k=1}^K R_k = I_{\mathcal{H}}$. A set of projections $\{P_k\}_{k=1}^K$ on some Hilbert space $\mathcal{H}$ is called a *projection-valued measure* if $\sum_{k=1}^K P_k = I_{\mathcal{H}}$.

Notice that every PVM is a POVM. Also, if a set of projections $\{P_k\}_{k=1}^K$ form a PVM, then $P_iP_j = 0$ for all $i \neq j$. In other words, they correspond to mutually orthogonal closed subspaces of the Hilbert space $\mathcal{H}$.

We get (possibly) different sets of quantum strategies depending upon whether we allow the state spaces of Alice and Bob to be finite dimensional or infinite dimensional. When we consider finite-dimensionality, we get the set of quantum correlations.

**Definition 2.1.12** (Quantum Correlations)**.** A correlation $(p(i,j|v,w))$ is called a *quantum correlation* if

(a) Alice's and Bob's state spaces are given by finite-dimensional Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively,

(b) for each input $v \in I_A$, Alice has a POVM $\{P_{v,i}\}_{i\in O_A}$ on $\mathcal{H}_A$, and similarly for each input $w \in I_B$, Bob has a POVM $\{Q_{w,j}\}_{j\in O_B}$ on $\mathcal{H}_B$, and

(c) they share a state (possibly entangled) $h \in \mathcal{H}_A \otimes \mathcal{H}_B$,

such that $p(i,j|v,w) = \langle (P_{v,i} \otimes Q_{w,j})h, h\rangle$.

The set of all correlations $(p(i,j|v,w))$ arising from all choices of finite-dimensional Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$, all POVMs and all states $h \in \mathcal{H}_A \otimes \mathcal{H}_B$ is called the set of quantum correlations, and is denoted by $C_q(n_A, n_B, k_A, k_B)$. We set $C_q(n,k) = C_q(n,n,k,k)$.

We may relax Definition 2.1.12 above by allowing the Hilbert spaces appearing therein to be infinite dimensional. This gives rise to the set of spatial quantum correlations.

**Definition 2.1.13** (Spatial Quantum Correlations)**.** A correlation $(p(i,j|v,w))$ is called a *spatial quantum correlation* if

(a) Alice's and Bob's state spaces are given by (possibly infinite dimensional) Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively,

(b) for each input $v \in I_A$, Alice has a POVM $\{P_{v,i}\}_{i\in O_A}$ on $\mathcal{H}_A$, and similarly for each input $w \in I_B$, Bob has a POVM $\{Q_{w,j}\}_{j\in O_B}$ on $\mathcal{H}_B$, and

(c) they share a state (possibly entangled) $h \in \mathcal{H}_A \otimes \mathcal{H}_B$,

such that $p(i,j|v,w) = \langle (P_{v,i} \otimes Q_{w,j})h, h\rangle$.

The set of all correlations $(p(i,j|v,w))$ arising from all choices of Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$, all POVMs and all states $h \in \mathcal{H}_A \otimes \mathcal{H}_B$ is called the set of spatial quantum correlations, and is denoted by $C_{qs}(n_A, n_B, k_A, k_B)$. We set $C_{qs}(n,k) = C_{qs}(n,n,k,k)$.

We have the following proposition relating the correlation sets discussed so far. The first inclusion was shown in [61]. The second inclusion follows trivially from the definition and the third inclusion is easy to check. The convexity of the sets of quantum correlations and spatial quantum correlations is shown in Lemma 2.1 in [25].

**Proposition 2.1.14** ([61, 25])**.** *For all $n_A, n_B, k_A, k_B \in \mathbb{N}$, we have*

$$C_{loc}(n_A, n_B, k_A, k_B) \subseteq C_q(n_A, n_B, k_A, k_B) \subseteq C_{qs}(n_A, n_B, k_A, k_B) \subseteq C_{ns}(n_A, n_B, k_A, k_B),$$

*and $C_q(n_A, n_B, k_A, k_B)$ and $C_{qs}(n_A, n_B, k_A, k_B)$ are convex sets in $\mathbb{R}^{n_A n_B k_A k_B}$.*

The first inclusion in Proposition 2.1.14 is usually a proper inclusion as the following two examples demonstrate. The third inclusion is also a proper inclusion [41, 62].

**Proposition 2.1.15** ([13, 12, 41, 11])**.** *Consider the CHSH game described in Example 2.1.2 and assume that the referee passes on the inputs $(v, w)$ with a uniform distribution. The maximum winning probability when the players restrict to classical strategies is 0.75, while the maximum winning probability when the players use quantum strategies is $\cos^2(\pi/8) \approx 0.85$. In particular, $C_{loc}(2, 2) \subsetneq C_q(2, 2)$.*

**Example 2.1.16** ([4, 29, 2])**.** There exists a graph (specifically, the Hadamard graph $G_{12}$) for which the associated graph colouring game with 12 colours has a perfect quantum strategy but no perfect classical strategy. (The referee passes the vertices to the players according to the uniform distribution $\pi : V \times V \to [0, 1]$ on $\{(v, v) : v \in V\} \cup E$.)

However, for the second inclusion in Proposition 2.1.14, it was not known whether it is proper or not for all $n_A, n_B, k_A, k_B \in \mathbb{N}$ until very recently through the work of Coladangelo and Stark [15]. Moreover, for around two and a half decades it was not known whether these two quantum correlation sets are closed or not till the recent work of William Slofstra [72, 73]. This thesis also arrives at the same conclusion but with considerably smaller input and output sets and with entirely new methods. It is not obvious why these results matter unless we see it in a broader context which we describe next.

In defining the quantum and spatial quantum correlation sets in Definitions 2.1.12 and 2.1.13, respectively, we are assuming a fundamental axiom that is used almost always in quantum information theory: given two individual quantum systems with state spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, the state space of the composite system is described by the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ (for more details see Section I.2.2.8 in [53]). However, as Tobias Fritz points out in the introduction of [25], this axiom is open to challenge by not having a strong justification. In some more general axiomatic formulations (more general in the sense that they incorporate general relativity) of quantum mechanics, like the Algebraic Quantum Field Theory (AQFT), it is posited that there is a universal Hilbert space (possibly infinite dimensional) for Alice and Bob (as opposed to having possibly different Hilbert spaces for Alice and Bob in the usual formulation) and the measurement operators corresponding to both players commute. Then the question that arises is whether these two formulations yield the same correlation sets. For a more detailed discussion we refer the reader to the introduction by Fritz in his paper [25] (and also [39] by Junge et. al.). We introduce the set of correlations obtained via this commuting framework.

**Definition 2.1.17** (Commuting Quantum Correlations)**.** A correlation $(p(i,j|v,w))$ is called a *commuting quantum correlation* if

(a) Alice's and Bob's state space is a common (possibly infinite dimensional) Hilbert space $\mathcal{H}$,

(b) for each input $v \in I_A$, Alice has a POVM $\{P_{v,i}\}_{i\in O_A}$ and for each input $w \in I_B$, Bob has a POVM $\{Q_{w,j}\}_{j\in O_B}$ on the Hilbert space $\mathcal{H}$, such that $P_{v,i}Q_{w,j} = Q_{w,j}P_{v,i}$ for all $i,j,v,w$ (which explains the name commuting), and

(c) they share a state $h \in \mathcal{H}$,

such that $p(i,j|v,w) = \langle P_{v,i}Q_{w,j}h, h\rangle$.

The set of all correlations $(p(i,j|v,w))$ arising from all choices of Hilbert space $\mathcal{H}$, all POVMs and all states $h \in \mathcal{H}$ is called the set of commuting quantum correlations, and is denoted by $C_{qc}(n_A, n_B, k_A, k_B)$. We set $C_{qc}(n,k) = C_{qc}(n,n,k,k)$.

Thus observe that the tensor product model gives rise to the set of quantum correlations and the commuting model gives rise to the set of commuting quantum correlations.

We now record some properties of the set of commuting quantum correlation and its relations with the previous ones.

**Theorem 2.1.18.** *For all $n_A, n_B, k_A, k_B \in \mathbb{N}$, the set $C_{qc}(n_A, n_B, k_A, k_B)$ is a closed convex set, and*

$$C_{qs}(n_A, n_B, k_A, k_B) \subseteq C_{qc}(n_A, n_B, k_A, k_B) \subseteq C_{ns}(n_A, n_B, k_A, k_B).$$

*Proof.* The first inclusion follows from the observation

$$\langle (P_{v,i} \otimes Q_{w,j})\, h, h\rangle = \langle (P_{v,i} \otimes I_{\mathcal{H}_B})(I_{\mathcal{H}_A} \otimes Q_{w,j})h, h\rangle\,.$$

To show the second inclusion, let $(p(i,j|v,w)) \in C_{qc}(n_A, n_B, k_A, k_B)$. Then, by Definition 2.1.17, there exist POVMs $\{P_{v,i} : i \in O_A\}$ and $\{Q_{w,j} : j \in O_B\}$ for each $v \in I_A$ and $w \in I_B$, in some Hilbert space $\mathcal{H}$, with the condition $[P_{v,i}, Q_{w,j}] = 0$ for all $v, w, i, j$, and such that $p(i,j|v,w) = \langle P_{v,i} Q_{w,j} h, h \rangle$. The commutativity condition implies that for all $i, j, v, w$,

$$\left[P_{v,i}^{\frac{1}{2}}, Q_{w,j}^{\frac{1}{2}}\right] = 0.$$

Using this, it follows that

$$p(i,j|v,w) = \langle P_{v,i} Q_{w,j} h, h \rangle = \left\langle P_{v,i}^{\frac{1}{2}} Q_{w,j}^{\frac{1}{2}} h, P_{v,i}^{\frac{1}{2}} Q_{w,j}^{\frac{1}{2}} h \right\rangle \geq 0.$$

Moreover, using the POVM condition, we obtain

$$\sum_{\substack{i \in O_A \\ j \in O_B}} p(i,j|v,w) = \sum_{\substack{i \in O_A \\ j \in O_B}} \langle P_{i,j} Q_{w,j} h, h \rangle = \left\langle \sum_{j \in O_B} Q_{w,j} h, \sum_{i \in O_A} P_{v,i} h \right\rangle = \langle h, h \rangle = 1.$$

Thus $(p(i,j|v,w))$ is indeed a correlation. That this correlation satisfies the non-signalling condition follows from

$$p_A(i|v) = \sum_{j \in O_B} p(i,j|v,w) = \sum_{j \in O_B} \langle P_{v,i} Q_{w,j} h, h \rangle = \left\langle P_{v,i} h, \sum_{j \in O_B} Q_{w,j} h \right\rangle = \langle P_{v,i} h, h \rangle,$$

which is clearly independent from $w$. Similarly, the other non-signalling condition is also fulfilled.

Finally, that the set $C_{qc}(n_A, n_B, k_A, k_B)$ is closed and convex is proved in Proposition 3.4 in [25]. □

Since it is not known whether $C_q(n_A, n_B, k_A, k_B)$ is closed or not for all $n_A, n_B, k_A, k_B$, we introduce the (topological) closure of the set,

$$C_{qa}(n_A, n_B, k_A, k_B) := \overline{C_q(n_A, n_B, k_A, k_B)}.$$

As usual, we let $C_{qa}(n,k) = C_{qa}(n,n,k,k)$. We have the following relation.

**Proposition 2.1.19** ([68]). *For all $n_A, n_B, k_A, k_B \in \mathbb{N}$, $C_{qa}(n_A, n_B, k_A, k_B)$ is closed convex set and*

$$C_q(n_A, n_B, k_A, k_B) \subseteq C_{qs}(n_A, n_B, k_A, k_B) \subseteq C_{qa}(n_A, n_B, k_A, k_B) \subseteq C_{qc}(n_A, n_B, k_A, k_B).$$

*Moreover, $C_{qa}(n_A, n_B, k_A, k_B) = \overline{C_{qs}(n_A, n_B, k_A, k_B)}$.*

Boris Tsirelson was the first to investigate these correlation sets arising from different quantum models in [74] in which he assumed that the tensor product model and the commuting model yield the same set of correlations implying that these two models are essentially the same. That is, Tsirelson originally wanted to show that $C_q(n,k) = C_{qc}(n,k)$ for all $n,k \in \mathbb{N}$. However, when the authors of [52] requested a proof from Tsirelson, he could not supply one except in the case of finite-dimensional Hilbert spaces. He then published this as an open question on the website [75]. In the absence of a proof, one can ask several weaker questions.

**Problem 2.1.20** (Tsirelson's Problems)**.** For which $r, s \in \{q, qs, qa, qc\}$ do we have the equality $C_r(n,k) = C_s(n,k)$ for all $n,k$? In particular, for which $r \in \{q, qs, qa\}$ do we have $C_r(n,k) = C_{qc}(n,k)$ for all $n,k$?

Depending upon what $r$ we choose we get different versions of Tsirelson's problems. In Vern Paulsen's terminology [22], the problem whether $C_q(n,k) = C_{qc}(n,k)$ for all $n,k$ is referred to as the strong Tsirelson's problem, while the problem whether $C_{qa}(n,k) = C_{qc}(n,k)$ for all $n,k$ is referred to as the weak Tsirelson's problem.

Incidentally, when the number of inputs and outputs is binary, all the quantum correlation sets coincide.

**Proposition 2.1.21** ([25, 23, 50])**.** *We have* $C_q(2,2) = C_{qs}(2,2) = C_{qa}(2,2) = C_{qc}(2,2)$*.*

There has been a renewed interest in the study of these problems since it was shown that one of Tsirelson's problems is equivalent to a long standing problem in Operator Algebras: Connes' embedding problem [25, 39, 54].

**Conjecture 2.1.22** (Connes' Embedding Conjecture, [17])**.** Any finite von Neumann algebra $(\mathcal{M}, \tau)$ with separable predual is embeddable into the ultrapower $R^\omega$ of the hyperfinite $II_1$-factor $R$.

In [39], it was proved that if Connes' embedding conjecture is true then $C_{qa}(n,k) = C_{qc}(n,k)$ for all $n,k \in \mathbb{N}$. The converse was shown in [54].

**Theorem 2.1.23** ([39, 54])**.** *Connes' embedding conjecture is true if and only if* $C_{qa}(n,k) = C_{qc}(n,k)$ *for all* $n,k \in \mathbb{N}$*.*

Intimately related to Tsirelon's problems are the following questions. Are the sets $C_q(n,k)$ and $C_{qs}(n,k)$ closed? For example, if $C_q(n,k)$ is not closed then $C_q(n,k) \subsetneq C_{qc}(n,k)$ solving one of Tsirelson's problems. Indeed, recently Slofstra showed the following using the representation theory of finitely presented groups.

**Theorem 2.1.24** ([72])**.** *There is a non-local game for which there exists a perfect correlation in $C_{qc}(n_A, n_B, k_A, k_B)$ for some $n_A, n_B, k_A, k_B \in \mathbb{N}$, but no perfect correlation exists in $C_{qs}(n_A, n_B, k_A, k_B)$.*

**Theorem 2.1.25** ([73])**.** *There is a non-local game $\mathcal{G}$ for which there exists a perfect correlation in $C_{qa}(184, 235, 8, 2)$, but no perfect correlation exists in $C_{qs}(184, 235, 8, 2)$. In particular, the correlation sets $C_q(184, 235, 8, 2)$ and $C_{qs}(184, 235, 8, 2)$ are not closed.*

We shall show in Chapter 4 that in fact the input and output sets can be made even smaller. We use methods from the theory of $C^*$-algebras which are completely different from Slofstra to show the following.

**Theorem 2.1.26** ([21])**.** *The correlation sets $C_q(5, 2)$ and $C_{qs}(5, 2)$ are not closed.*

However, it still remains an open question whether the input sets can be made even smaller. It is conjectured [55] that $C_q(3, 2)$ is not closed and can be exhibited by the so-called $I_{3322}$-game.

**Open Problem 2.1.27.** Are the correlation sets $C_q(n, 2)$ and $C_{qs}(n, 2)$ closed for $n = 3$ and $n = 4$?

In general, the following conjecture is due to Vern Paulsen.

**Conjecture 2.1.28.** We have $C_q(n, k) \neq C_{qc}(n, k)$ for some $n, k \in \mathbb{N}$ if and only if there is some non-local game with $n$ inputs and $k$ outputs which can be played perfectly using a limit of finite-dimensional quantum strategies but which cannot be played perfectly if the dimension of Hilbert space is fixed.

Very recently, Coladangelo and Stark showed using the notion of *self testing* that in general the quantum correlation and spatial quantum correlation sets are not equal.

**Theorem 2.1.29** ([15])**.** *We have $C_q(4, 5, 3, 3) \neq C_{qs}(4, 5, 3, 3)$.*

These results settle all the Tsirelson's problems except the part which is equivalent to Connes'. There are some variants of non-local games such as ones with "quantum questions" or with infinite output/input sets, in which the analogue of these questions have been studied, for example see [47, 49, 65, 14]. Quantum correlation sets have also been looked through the lens of *completely positive semidefinite cone* [6]. An analogous theory of *unitary correlations sets* has been developed by Sam Harris in [32, 33]. He showed that if

$UC_{qc}(n,n)$ and $UC_q(n,n)$ denote the sets of unitary correlations in commuting model and tensor product model, respectively, then $UC_{qc}(n,n) = \overline{UC_q(n,n)}$ for all $n \geq 2$ is equivalent to Connes' embedding conjecture.

There is yet another set of correlations which first arose in [52] where they form the first level of the *NPA hierarchy* called $Q^1$ (and also as *almost quantum correlations* in [51]). They have been used to approximate the quantum value for *unique games* in [40]. They have been also studied in the context of *vectorial chromatic number* [60].

**Definition 2.1.30** (Vector Correlations)**.** A correlation $(p(i,j|v,w))$ is called a *vector correlation* if there are sets of vectors $\{x_{v,i} : v \in I_A, i \in O_A\}, \{y_{w,j} : w \in I_B, j \in O_B\}$ in some Hilbert space $\mathcal{H}$ and a unit vector $h \in \mathcal{H}$, which satisfy

1. $x_{v,i} \perp x_{v,j}$ for all $v \in I_A$ and $i \neq j \in O_A$,
2. $y_{w,i} \perp y_{w,j}$ for all $w \in I_B$ and $i \neq j \in O_B$,
3. $\sum_{i \in O_A} x_{v,i} = h = \sum_{j \in O_B} y_{w,j}$ for all $v, w$,
4. $\langle x_{v,i}, y_{w,j} \rangle \geq 0$ for all $v, w, i, j$,

and such that $p(i,j|v,w) = \langle x_{v,i}, y_{w,j} \rangle$.

The set of all correlations $(p(i,j|v,w))$ arising from all choices of Hilbert space and vectors therein satisfying above conditions is called the set of vector correlations and is denoted by $C_{vect}(n_A, n_B, k_A, k_B)$. We set $C_{vect}(n,k) = C_{vect}(n,n,k,k)$.

Since all of the inner products appearing in Definition 2.1.30 are real-valued, there is no generality lost in requiring $\mathcal{H}$ to be a real Hilbert space. Vector correlations are also non-signalling (Definition 2.1.4) and form a closed convex set.

**Proposition 2.1.31** ([11, 58])**.** *For all $n_A, n_B, k_A, k_B$, the set $C_{vect}(n_A, n_B, k_A, k_B)$ is a closed convex set and*

$$C_{qc}(n_A, n_B, k_A, k_B) \subseteq C_{vect}(n_A, n_B, k_A, k_B) \subseteq C_{ns}(n_A, n_B, k_A, k_B).$$

*Moreover, $C_{qc}(2,2) = C_{vect}(2,2)$, extending the set equalities in Proposition 2.1.21.*

Before ending this section, we state two results which we shall use in the next section. One of the first simplifications towards understanding the different quantum correlation sets is the following proposition. If we modify Definitions 2.1.12, 2.1.13, and 2.1.17 by replacing "POVM" with "PVM" we get essentially the same sets of correlations. This gives us the benefit of working with simpler operators and makes the analysis easier.

**Proposition 2.1.32.** *Let $\widetilde{C}_q(n,k), \widetilde{C}_{qs}(n,k), \widetilde{C}_{qc}(n,k)$ be the sets obtained by replacing "POVM" with "PVM" in Definitions 2.1.12, 2.1.13, and 2.1.17. Then, for every $n, k \in \mathbb{N}$, we have*

*(a)* $C_q(n,k) = \widetilde{C}_q(n,k)$,

*(b)* $C_{qs}(n,k) = \widetilde{C}_{qs}(n,k)$,

*(c)* $C_{qc}(n,k) = \widetilde{C}_{qc}(n,k)$.

*Proof.* Part (a) and (b) are non-trivial. The core idea is to use Naimark's dilation theorem [57]. On the Hilbert space $\mathcal{H}_A$, we use the dilation theorem to dilate one set of POVMs on $\mathcal{H}_A$ to a set of PVMs on a larger Hilbert space $\mathcal{K}_A$. Then we lift the remaining POVMs to this larger space and iterate. Similarly, we dilate Bob's POVMs to another larger Hilbert space $\mathcal{K}_B$, and then one considers the tensor products of these PVMs on $\mathcal{K}_A \otimes \mathcal{K}_B$. For a detailed proof we refer the reader to [58].

Part (c) is considerably more difficult and we refer the reader to Proposition 3.4 in [25], or Remark 10 in [39], or [60]. □

The next proposition characterises correlations in $C_{loc}(n,k)$ and $C_q(n,k)$ as members of $C_{qc}(n,k)$. Relevant literature in this context are [74, 19, 50].

**Proposition 2.1.33** (Theorem 5.3 and Remark 5.4 in [59])**.** *A correlation $(p(i,j|v,w))$ belongs to the set $C_q(n,k)$ if and only if $(p(i,j|v,w)) \in C_{qc}(n,k)$ and such that there exists a finite dimensional Hilbert space (as in Definition 2.1.17) which realises the correlation. Similarly, a correlation $(p(i,j|v,w))$ belongs to $C_{loc}(n,k)$ if and only if $(p(i,j|v,w)) \in C_{qc}(n,k)$ and such that all the operators in its realisation commute.*

We indicated in the paragraph after Definition 2.1.6 that the notion of value of a non-local game can be used to distinguish two sets of correlations. Since we shall be interested in the correlation sets $C_r(n_A, n_B, k_A, k_B)$ for various values of $r$ described earlier, we introduce a notation to talk about the values of a game with respect to these sets.

**Definition 2.1.34.** Let $\mathcal{G} = (I_A, I_B, O_A, O_B, \lambda)$ be a non-local game with a probability distribution $\pi : I_A \times I_B \to [0,1]$. Let $r \in \{loc, q, qs, qa, qc, vect, ns\}$. A perfect correlation $(p(i,j|v,w))$ is called a perfect $r$-correlation if $(p(i,j|v,w)) \in C_r(n_A, n_B, k_A, k_B)$. A strategy which yields a perfect $r$-correlation is called a perfect $r$-strategy. The $r$-value of $\mathcal{G}$ is defined as $\omega_r(\mathcal{G}, \pi) = \omega_{\mathcal{F}}(\mathcal{G}, \pi)$, where $\mathcal{F} = C_r(n_A, n_B, k_A, k_B)$.

For example, for the CHSH game in Proposition 2.1.15 we may write $\omega_{loc}(CHSH) = 0.75$, whereas $\omega_q(CHSH) \approx 0.85$. Observe that $\omega_q(\mathcal{G}, \pi) = \omega_{qs}(\mathcal{G}, \pi) = \omega_{qa}(\mathcal{G}, \pi)$ for all non-local games $\mathcal{G}$ and all distributions $\pi$. Therefore, just computing these numbers won't imply that $C_q(n, k) \subsetneq C_{qa}(n, k)$. Moreover, computing these numbers is hard.

## 2.2 Synchronous correlations

In this section, we focus on an another major simplification towards understanding the correlation sets described in the previous section. To do so we study the subset of synchronous correlations for each correlation set. The notion of synchronous subsets was first formulated by Vern Paulsen et. al. in [59, 22]. The motivation for this arises from instances of a special kind of non-local games like the graph colouring games. Notice that in the graph colouring game, the rules dictate that whenever the referee passes the same input to the players, they must return the same output in order to win the game. In contrast, the CHSH game does not share this property, since if the referee passes 1 to both Alice and Bob, they cannot both respond with 0 (or 1) to win.

**Definition 2.2.1.** A non-local game $\mathcal{G} = (I_A, I_B, O_A, O_B, \lambda)$ is called *synchronous* if $I_A = I_B = I$ and $O_A = O_B = O$ and $\lambda(v, v, i, j) = 0$ for all $v \in I$ and $i \neq j \in O$.

It is then a simple exercise to show the following lemma which motivates the following definition.

**Lemma 2.2.2.** *Let $\mathcal{G} = (I, I, O, O, \lambda)$ be a synchronous non-local game. If $(p(i, j|v, w))$ is a perfect correlation for the game then $p(i, j|v, v) = 0$ for all $v \in I$ and $i \neq j \in O$.*

**Definition 2.2.3.** Let $v, w \in I$ and $i, j \in O$. A correlation $(p(i, j|v, w))$ is called a *synchronous correlation* if $p(i, j|v, v) = 0$ for all $v \in I$ and $i \neq j \in O$. This means that whenever Alice and Bob get the same input they always produce the same outputs.

If $\mathcal{F}$ is a set of correlations, we let $\mathcal{F}^s \subseteq \mathcal{F}$ denote the subset of synchronous correlations in $\mathcal{F}$. Notice that if $\mathcal{F}$ is closed then the subset $\mathcal{F}^s$ is also closed. Also, if $\mathcal{F}_1 = \mathcal{F}_2$ are two sets of correlations, then $\mathcal{F}_1^s = \mathcal{F}_2^s$. However, the converse need not hold. Let $r \in \{loc, q, qs, qa, qc, vect, ns\}$. We let $C_r^s(n, k) \subseteq C_r(n, k)$ denote the subset of synchronous correlations of the respective correlation set. Using propositions from Section 2.1, it is then straightforward to establish the following proposition.

**Proposition 2.2.4.** *For all $n, k \in \mathbb{N}$, we have*

$$C^s_{loc}(n,k) \subseteq C^s_q(n,k) \subseteq C^s_{qs}(n,k) \subseteq C^s_{qa}(n,k) \subseteq C^s_{qc}(n,k) \subseteq C^s_{vect}(n,k) \subseteq C^s_{ns}(n,k). \tag{2.2.1}$$

*All these synchronous correlation subsets are convex. The synchronous correlation subsets $C^s_{loc}(n,k), C^s_{qa}(n,k), C^s_{qc}(n,k), C^s_{vect}(n,k)$ and $C^s_{ns}(n,k)$ are closed as well.*

Notice that $C^s_r(n,k) \subseteq C^s_{r'}(n,k)$ does not necessarily imply that $C_r(n,k) \subseteq C_{r'}(n,k)$. We can ask the same kind of questions for the hierarchy of the synchronous correlation subsets as we did in the previous section.

**Problem 2.2.5** (Synchronous versions of Tsirelson's problems)**.** For which $r \in \{q, qs, qa\}$, do we have $C^s_r(n,k) = C^s_{qc}(n,k)$ for all $n, k \in \mathbb{N}$?

Again because it was not known whether the sets $C_q(n,k)$ and $C_{qs}(n,k)$ were closed or not, it was not known whether $C^s_q(n,k)$ and $C^s_{qs}(n,k)$ were closed or not. We shall prove the following theorem in Chapter 4. Notice that Theorem 2.2.6 implies Theorem 2.1.26.

**Theorem 2.2.6.** *The synchronous correlation sets $C^s_q(5,2)$ and $C^s_{qs}(5,2)$ are not closed.*

It has also been shown that the Connes embedding problem is related to the synchronous version of Theorem 2.1.23.

**Theorem 2.2.7** ([22, 43])**.** *Connes' embedding conjecture is true if and only if $C^s_{qa}(n,k) = C^s_{qc}(n,k)$ for all $n, k \in \mathbb{N}$.*

Working with synchronous correlation subsets adds a great deal of simplification as the following theorems show. Notice that in Theorem 2.2.8, PVM is used instead of POVM as mentioned in Theorem 2.1.32. Also, the remarkable thing about this theorem is that we need PVMs of only one player to generate the correlation.

Recall that a linear functional $\varphi$ on a $C^*$-algebra $\mathcal{A}$ is called a *state* if $\varphi$ is positive and $\varphi(1) = 1$. A state $\varphi : \mathcal{A} \to \mathbb{C}$ is called *tracial* if $\varphi(ab) = \varphi(ba)$ for all $a, b \in \mathcal{A}$.

**Theorem 2.2.8** (Theorem 5.5, [59])**.** *Let $(p(i,j|v,w)) \in C^s_{qc}(n,k)$ be realised with PVMs $\{P_{v,i} : i \in O\}_{v \in I}$ and $\{Q_{w,j} : j \in O\}_{w \in I}$ on some Hilbert space $\mathcal{H}$ satisfying $P_{v,i}Q_{w,j} = Q_{w,j}P_{v,i}$ and with some unit vector $h \in \mathcal{H}$ such that $p(i,j|v,w) = \langle P_{v,i}Q_{w,j}h, h\rangle$. Then,*

*(a) $P_{v,i}h = Q_{v,i}h$ for all $v, i$, and*

*(b)* $p(i,j|v,w) = \langle (P_{v,i}P_{w,j})h, h\rangle = \langle (Q_{w,j}Q_{v,i})h, h\rangle = p(j,i|w,v)$.

*(c)* *Let $\mathcal{A}$ be the $C^*$-algebra in $B(\mathcal{H})$ generated by the family $\{P_{v,i} : v \in I, i \in O\}$ and define $\tau : \mathcal{A} \to \mathbb{C}$ by $\tau(X) = \langle Xh, h\rangle$. Then $\tau$ is a tracial state on $\mathcal{A}$ and $p(i,j|v,w) = \tau(P_{v,i}P_{w,j})$.*

*Conversely, let $\mathcal{A}$ be a unital $C^*$-algebra equipped with a tracial state $\tau$ and having a family of projections $\{e_{v,i} : v \in I, i \in O\} \subset \mathcal{A}$ such that $\sum_{i\in O} e_{v,i} = 1$ for all $v$. Then $(p(i,j|v,w))$ defined by $p(i,j|v,w) = \tau(e_{v,i}e_{w,j})$ is an element of $C^s_{qc}(n,m)$. That is, there exists a Hilbert space $\mathcal{H}$, a unit vector $h \in \mathcal{H}$ and mutually commuting PVMs $\{P_{v,i} : i \in O\}_{v\in I}$ and $\{Q_{w,j} : j \in O\}_{w\in I}$ on $\mathcal{H}$ such that*

$$p(i,j|v,w) = \langle (P_{v,i}Q_{w,j})h, h\rangle = \langle (P_{v,i}P_{w,j})h, h\rangle = \langle (Q_{w,j}Q_{v,i})h, h\rangle.$$

Theorem 2.2.8 and Proposition 2.1.33 lead to the following nice characterisation of the synchronous subsets $C^s_{loc}(n,k)$ and $C^s_q(n,k)$.

Recall that a finite dimensional $C^*$-algebra may be expressed as a direct sum of matrix algebras. That is, if $\mathcal{A}$ is a finite dimensional $C^*$-algebra then there exists $k_1, ..., k_n \in \mathbb{N}$ such that $\mathcal{A} = \mathbb{M}_{k_1} \oplus \cdots \oplus \mathbb{M}_{k_n}$. In contrast to a full matrix algebra, notice that there can be infinitely many tracial states on a finite dimensional $C^*$-algebra. Indeed, if $\mathcal{A} = \mathbb{M}_{k_1} \oplus \cdots \oplus \mathbb{M}_{k_n}$ and $\tau : \mathcal{A} \to \mathbb{C}$ is a tracial state, then there exists $n$ scalars $\lambda_i \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1$ and $\tau = \lambda_1 \mathrm{tr}_{k_1} \oplus \cdots \oplus \lambda_n \mathrm{tr}_{k_n}$, where $\mathrm{tr}_{k_i}$ is the normalised tracial state on $\mathbb{M}_{n_i}$.

**Corollary 2.2.9** (Corollary 5.6, [59]). *We have that $(p(i,j|v,w)) \in C^s_q(n,k)$ (respectively, $C^s_{loc}(n,k)$) if and only if there exists a finite dimensional (respectively, abelian) $C^*$-algebra $\mathcal{A}$ with a tracial state $\tau$ and with a generating family $\{e_{v,i} : v \in I, i \in O\} \subseteq \mathcal{A}$ of projections such that $\sum_{i\in O} e_{v,i} = 1$ for all $v$ and $p(i,j|v,w) = \tau(e_{v,i}e_{w,j})$ for all $v, w, i, j$. Moreover, when $\mathcal{A}$ is abelian and $\tau$ is tracial, then $\tau$ is a $*$-homormorphism.*

The subset of synchronous vector correlations also witnesses this kind of simplification.

**Proposition 2.2.10** ([60]). *Suppose $(p(i,j|v,w)) \in C^s_{vect}(n,k)$ is generated by a set of vectors $\{x_{v,i}, y_{w,j}, h : v, w \in I, i, j \in O\} \subseteq \mathcal{H}$ satisfying the conditions in Definition 2.1.30. Then $x_{v,i} = y_{v,i}$ for all $v \in I$ and $i \in O$.*

The notion of $r$-value of a non-local game (as in Definition 2.1.34) for the correlation sets $C_r(n_A, n_B, k_A, k_B)$ may also be defined for synchronous subsets.

**Definition 2.2.11.** Let $\mathcal{G} = (I, I, O, O, \lambda)$ be a non-local game with a probability distribution $\pi : I \times I \to [0,1]$. Let $r \in \{loc, q, qs, qa, qc, vect, ns\}$. The synchronous $r$-value of $\mathcal{G}$ is defined as $\omega_r^s(\mathcal{G}, \pi) = \omega_{\mathcal{F}}(\mathcal{G}, \pi)$, where $\mathcal{F} = C_r^s(n, k)$.

The following proposition relates the values of a non-local game to Connes' embedding conjecture [22]. However, it is not known whether the converse holds true.

**Proposition 2.2.12** ([22])**.** *If Connes' embedding conjecture is true, then* $\omega_q(\mathcal{G}, \pi) = \omega_{qc}(\mathcal{G}, \pi)$ *and* $\omega_q^s(\mathcal{G}, \pi) = \omega_{qc}^s(\mathcal{G}, \pi)$ *for every non-local game* $\mathcal{G}$ *and every distribution* $\pi : I \times I \to [0,1]$.

From the work in [43], it follows that $\overline{C_q^s(n,k)} = C_{qa}^s(n,k)$ for all $n, k \in \mathbb{N}$, and therefore $\omega_q^s(\mathcal{G}, \pi) = \omega_{qs}^s(\mathcal{G}, \pi) = \omega_{qa}^s(\mathcal{G}, \pi)$ for all games $\mathcal{G}$ and all distributions $\pi$. Thus as mentioned before in the previous section these numbers do not highlight any difference between the sets $C_q^s(n,k)$ and $C_{qa}^s(n,k)$. Moreover, even though working with synchronous correlations brings significant simplifications, it is still difficult to compute these numbers for arbitrary games. To work around this, we shall try to obtain these numbers under some constraints and analyse them so as to detect any differences. We shall go over this idea for a specific game in Chapter 3.

# Chapter 3

# The Delta game

In this chapter we investigate a synchronous non-local game having three inputs and binary outputs with the aim being to understand the synchronous correlation sets $C_r^s(3,2)$ for $r \in \{q, qc, vect\}$. The game we discuss here is called the Delta game (stylised as the $\Delta$ game) which was brought to our attention by Richard Cleve. We show that studying the maximum winning probability for the Delta game by introducing a parameter is helpful in distinguishing the vector correlation set $C_{vect}^s(3,2)$ and $C_{qc}^s(3,2)$. This method also provides us with some clues to extend this analysis for larger number of quantum experiments, and indeed, we use this approach in Chapter 4.

## 3.1 Description of the Delta game

Let $\Delta = (I, I, O, O, \lambda)$ represent the Delta game where $I = \{0, 1, 2\}$ is the input set and $O = \{0, 1\}$ is the output set. To describe the rule function notice that there are 36 possible tuples. The tuples $(v, w, i, j) \in I \times I \times O \times O$ for which $\lambda(v, w, i, j) = 0$ are given by

$$\begin{array}{cccccc} (0,0,0,1), & (0,1,0,0), & (1,1,0,1), & (1,2,0,0), & (2,2,0,1), & (2,0,0,0), \\ (0,0,1,0), & (0,1,1,1), & (1,1,1,0), & (1,2,1,1), & (2,2,1,0), & (2,0,1,1). \end{array}$$

For the remaining 24 tuples $(v, w, i, j)$, we always have $\lambda(v, w, i, j) = 1$. A glance at the rule function tells us that the game is indeed synchronous.

While the rule function defined above might seem arbitrary, here is an easy way to visualise it (see Figure 3.1). The edges $(0, 0), (1, 1), (2, 2)$ are shown with dashed lines while $(0,1),(1,2),(2,0)$ are shown with solid lines. The dashed lines are "even" while the solid lines are "odd". This means that if Alice and Bob are given inputs joined by dashed lines then to win they must return outputs with even sum; and in the other case they return outputs with odd sum. For the edges which are not connected by any lines, they are free to output anything.

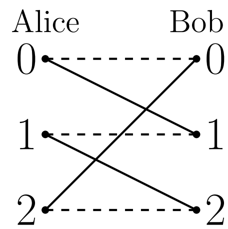


Figure 3.1: The $\Delta$ game rule function.

Alice and Bob receive inputs according to the uniform distribution $\pi = (\pi(v,w))$ on the set of inputs

$$E = \{(0,0),(1,1),(2,2),(0,1),(1,2),(2,0)\},$$

that is, $\pi(v,w) = \frac{1}{6}$ for all $(v,w) \in E$ (and zero otherwise). This completes the description of the game.

## 3.2 Constrained synchronous value of the Delta game

In this section we compute the expressions for the synchronous $r$-values (see Definition 2.2.11) of the $\Delta$ game for $r \in \{q, qc, vect\}$. In what follows, we would usually have $v \in \{0,1,2\}$ and $i \in \{0,1\}$. Thus the expressions $v+1$ and $i+1$ will always mean $v+1 \bmod 3$ and $i+1 \bmod 2$, respectively.

Let $p = (p(i,j|v,w))$ be a synchronous correlation. The synchronous value of $p$ for the $\Delta$ game given the distribution $\pi$ as above is then

$$\begin{aligned} V(p,\pi) &= \sum_{v,w=0}^{2} \sum_{i,j=0}^{1} \pi(v,w)\lambda(v,w,i,j)p(i,j|v,w) \\ &= \frac{1}{6}\left(\sum_{v=0}^{2}\sum_{i=0}^{1} p(i,i|v,v) + p(i,i+1|v,v+1)\right), \end{aligned}$$

so that the synchronous $r$-value of the $\Delta$ game becomes,

$$\omega_r^s(\mathcal{G},\pi) = \sup\left\{\frac{1}{6}\left(\sum_{v=0}^{2}\sum_{i=0}^{1} p(i,i|v,v) + p(i,i+1|v,v+1)\right) : p(i,j|v,w) \in C_r^s(3,2)\right\},$$

where $r \in \{q, qc, vect\}$. Let us denote the expression inside the braces by $\widetilde{\theta}$, that is,

$$\widetilde{\theta} = \frac{1}{6}\left(\sum_{v=0}^{2}\sum_{i=0}^{1} p(i,i|v,v) + p(i,i+1|v,v+1)\right).$$

We shall use Theorem 2.2.8 and Proposition 2.2.10 to simplify $\widetilde{\theta}$ and to obtain expressions involving projections and vectors in the case of $r = qc$ and $r = vect$, respectively. Moreover, when $r = q$, by Corollary 2.2.9 it suffices to proceed as in the case $r = qc$ using Theorem 2.2.8 to simplify $\widetilde{\theta}$, but restricting to the case of projections in finite dimensional $C^*$-algebras.

We first handle the $r = qc$ case. By Theorem 2.2.8, a correlation $(p(i,j|v,w))$ is in $C_{qc}^s(3,2)$ if and only if there exists a $C^*$-algebra $\mathcal{A}$ of $B(\mathcal{H})$ generated by a family of projections $\{A_{v,i} : i = 0,1 \text{ and } v = 0,1,2\}$ satisfying $A_{v,0} + A_{v,1} = I_{\mathcal{H}}$ for $v \in \{0,1,2\}$ and a tracial state $\tau : \mathcal{A} \to \mathbb{C}$ such that $p(i,j|v,w) = \tau(A_{v,i}A_{w,j}) = \langle (A_{v,i}A_{w,j})h, h\rangle$, for some unit vector $h \in \mathcal{H}$. For notational convenience we define

$$A_0 = A_{0,0}, \qquad A_1 = A_{1,0}, \qquad A_2 = A_{2,0}.$$

Then $A_{v,1} = I_{\mathcal{H}} - A_{v,0} = I_{\mathcal{H}} - A_v$ for $v \in \{0,1,2\}$. Using this we can rewrite $\widetilde{\theta}$ as

$$\begin{aligned}
\widetilde{\theta} &= \frac{1}{6}\left(\sum_{v=0}^{2}\sum_{i=0}^{1} p(i,i|v,v) + p(i,i+1|v,v+1)\right) \\
&= \frac{1}{6}\left(\sum_{v=0}^{2}\sum_{i=0}^{1} \tau(A_{v,i}A_{v,i}) + \tau(A_{v,i}A_{v+1,i+1})\right) \\
&= \frac{1}{2} + \frac{1}{3}\tau(A_0 + A_1 + A_2) - \frac{1}{3}\tau(A_0A_1 + A_1A_2 + A_2A_0). \qquad (3.2.1)
\end{aligned}$$

Thus, to compute the synchronous $qc$-value (respectively, synchronous $q$-value) of the game we simply take the supremum of the above quantity over all possible $C^*$-algebras $\mathcal{A}$ (respectively, finite dimensional $C^*$-algebras $\mathcal{A}$) equipped with a tracial state $\tau$ and having three projections $A_0, A_1$ and $A_2$. However as mentioned in Section 2.2, it is difficult to compute the actual number and also the number itself provides little help to gather anything. Therefore, we introduce a "parameter" so that the synchronous $r$-value becomes a function of the parameter instead of simply a number. A cleaner way to visualise this approach is provided in the summary of this chapter which also paves way for a more abstract approach in Chapter 4.

Revisiting the expression of $\widetilde{\theta}$ in Equation (3.2.1), we define a "parameter" $\theta$ by setting

$$\theta = \frac{1}{3}\tau(A_0 + A_1 + A_2),$$

which enables us to write $\widetilde{\theta}$ as

$$\widetilde{\theta} = \frac{1}{2} + \theta - \frac{1}{3}\sum_{v=0}^{2} \tau(A_v A_{v+1}). \tag{3.2.2}$$

Notice that

$$\begin{aligned}\theta &= \frac{1}{3}\tau(A_0 + A_1 + A_2) = \frac{1}{3}\tau(A_{0,0}^2 + A_{1,0}^2 + A_{2,0}^2) \\ &= \frac{1}{3}\left(p(0,0|0,0) + p(0,0|1,1) + p(0,0|2,2)\right),\end{aligned}$$

so that $\theta$ is independent of the $C^*$-algebra $\mathcal{A}$ and projections therein. Thus beginning with $(p(i,j|v,w)) \in C_{qc}^s(3,2)$ we compute the tuple $(\theta, \widetilde{\theta})$ using Theorem 2.2.8. Notice that since $A_v$'s are projections and $\tau$ is a state we get, $0 \leq \frac{1}{3}\tau(A_0 + A_1 + A_2) \leq 1$. Hence $0 \leq \theta \leq 1$. Conversely, if $\theta \in [0,1]$, then we can always find projections $A_0, A_1$ and $A_2$ in some $C^*$-algebra with a tracial state $\tau$, such that $\frac{1}{3}\tau(A_0 + A_1 + A_2) = \theta$, for example using the next lemma.

**Lemma 3.2.1.** *Let $\theta \in [0,1]$, then there exists a projection $P$ in some (finite dimensional) $C^*$-algebra $\mathcal{A}$ with some tracial state $\tau$ such that $\tau(P) = \theta$.*

*Proof.* First suppose that $\theta$ is rational with $\theta = \frac{p}{q}$ $(0 < p < q)$. Consider the $q \times q$ matrix

$$P = \begin{bmatrix} I_p & 0 \\ 0 & 0_{q-p} \end{bmatrix},$$

then $\tau(P) = \frac{p}{q} = \theta$. Next let $0 < \theta < 1$ be irrational. Let $\mathcal{A} = \mathbb{M}_n \oplus \mathbb{M}_n$ and define a tracial state on $\mathcal{A}$ by $\tau(X_1 \oplus X_2) = \theta\tau(X_1) + (1-\theta)\tau(X_2)$, for all $X_1, X_2 \in \mathbb{M}_n$. If $P = I_n \oplus 0_n$, then clearly $\tau(P) = \theta$. $\square$

For the $r = vect$ case, using Proposition 2.2.10 we know that a correlation $(p(i,j|v,w))$ belongs to $C^s_{vect}(3,2)$ if and only if there exist a set of vectors $\{x_{v,0}, x_{v,1} : 0 \le v \le 2\}$ in some Hilbert space $\mathcal{H}$ and a unit vector $h$ satisfying conditions in Definition 2.1.30 and such that $p(i,j|v,w) = \langle x_{v,i}, x_{w,j}\rangle$. As before, for notational convenience we let

$$x_0 = x_{0,0}, \qquad x_1 = x_{1,0}, \qquad x_2 = x_{2,0}.$$

Then $\widetilde{\theta}$ becomes

$$\widetilde{\theta} = \frac{1}{2} + \frac{1}{3}\langle x_0 + x_1 + x_2, h\rangle - \frac{1}{3}\sum_{v=0}^{2}\langle x_v, x_{v+1}\rangle.$$

Again letting $\theta = \frac{1}{3}\langle x_0 + x_1 + x_2, h\rangle$, we may write

$$\widetilde{\theta} = \frac{1}{2} + \theta - \frac{1}{3}\sum_{v=0}^{2}\langle x_v, x_{v+1}\rangle. \tag{3.2.3}$$

Again, it is is easy to show that $\theta$ is independent of the Hilbert space $\mathcal{H}$ and the vectors chosen therein. Moreover, by the Cauchy-Schwarz inequality we get $0 \le \frac{1}{3}\langle x_0 + x_1 + x_2, h\rangle \le 1$, and conversely, given $\theta \in [0,1]$ it is easy to find vectors $x_1, x_2, x_3, h$ in some Hilbert space $\mathcal{H}$ with required properties and such that $\frac{1}{3}\langle x_0 + x_1 + x_2, h\rangle = \theta$.

For each $r \in \{q, qc, vect\}$, let $\Theta^s_r$ denote the set of all points $(\theta, \widetilde{\theta}) \in \mathbb{R}^2$ that can be obtained from correlations $(p(i,j|v,w)) \in C^s_r(3,2)$ in the manner described above. Our goal is to examine how $\Theta^s_r$ behaves under different values of $r$. Since $C^s_q(3,2) \subseteq C^s_{qc}(3,2) \subseteq C^s_{vect}(3,2)$, it follows that

$$\Theta^s_q \subseteq \Theta^s_{qc} \subseteq \Theta^s_{vect}. \tag{3.2.4}$$

It is easy to verify that $\Theta^s_r$ is a convex set since it is the affine image of the convex set $C^s_r(n,m)$. To find $\Theta^s_r$, it is enough to compute the following two functions,

$$f^u_r(\theta) = \sup\{\widetilde{\theta} : (\theta, \widetilde{\theta}) \in \Theta^s_r\}, \qquad f^l_r(\theta) = \inf\{\widetilde{\theta} : (\theta, \widetilde{\theta}) \in \Theta^s_r\}, \tag{3.2.5}$$

where "u" and "l" stand for "upper" and "lower", respectively. We also need to determine if the supremum and the infimum are attained or not. Notice that in the $r = qc$ case, in order to find the supremum (resp., infimum) of $\widetilde{\theta} = \frac{1}{2} + \theta - \frac{1}{3}\sum_{v=0}^{2}\tau(A_v A_{v+1})$, we need to find the infimum (resp., supremum) of the quantity $\sum_{v=0}^{2}\tau(A_v A_{v+1})$. A similar statement holds for the $r = vect$ case.

## 3.3 The *vect* case

We compute the functions $f^l_{vect}$ and $f^u_{vect}$ (Definition 3.2.5) in this section. We will employ the symmetrisation provided by the next lemma.

**Lemma 3.3.1.** *The set $\Theta^s_{vect}$ is equal to the set of pairs $(\theta, \widetilde{\theta})$ with $0 \le \theta \le 1$, such that there exist vectors $x_0, x_1, x_2, h$ in a Hilbert space $\mathcal{H}$ with the properties:*

*(a)* $\|h\| = 1$,

*(b)* $\langle x_v, h\rangle = \langle x_v, x_v\rangle = \theta$, *for all* $0 \le v \le 2$

*(c)* $\langle x_v, x_{v+1}\rangle = \beta$ *for all* $0 \le v \le 2$, *where* $\widetilde{\theta} = \frac{1}{2} + \theta - \beta$ *and* $2\theta - 1 \le \beta \le \theta$.

*Proof.* By Proposition 2.4, and the discussion in Section 3.2, $\Theta^s_{vect}$ is the set of pairs $(\theta, \widetilde{\theta})$ such that there exist vectors $x_0, x_1, x_2, h$ in a Hilbert space $\mathcal{H}$ with the properties that $\|h\| = 1$, and for all $v$ and $w$, we have

$$\langle x_v, h\rangle = \langle x_v, x_v\rangle, \quad \langle x_v, x_w\rangle \ge 0, \quad \langle x_v, h - x_w\rangle \ge 0, \quad \langle h - x_v, h - x_w\rangle \ge 0$$

and, moreover,

$$\frac{1}{3}\sum_{v=0}^{2}\langle x_v, h\rangle = \theta, \qquad \frac{1}{3}\sum_{v=0}^{2}\langle x_v, x_{v+1}\rangle = \beta,$$

where $\widetilde{\theta} = \frac{1}{2} + \theta - \beta$. The conditions appearing in the lemma are precisely these, but with the additional requirement that the quantities $\langle x_v, h\rangle$ and $\langle x_v, x_{v+1}\rangle$ are the same for all $v \in \{0, 1, 2\}$. However, given $x_0, x_1, x_2, h$ satisfying these weaker conditions and considering

$$\widetilde{h} = \frac{1}{\sqrt{3}}(h \oplus h \oplus h), \qquad \widetilde{x}_v = \frac{1}{\sqrt{3}}(x_v \oplus x_{v+1} \oplus x_{v+2}), \qquad 0 \le v \le 2,$$

in the Hilbert space $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$, we see that $\widetilde{x}_0, \widetilde{x}_1, \widetilde{x}_2, \widetilde{h}$ satisfy the stronger conditions and yield the same pair $(\theta, \widetilde{\theta})$. □

We now describe the functions $f^l_{vect}$ and $f^u_{vect}$ in the next theorem.

**Theorem 3.3.2.** *The functions $f^l_{vect}$ and $f^u_{vect}$ are given by*

$$f^l_{vect}(\theta) = \frac{1}{2}, \qquad f^u_{vect}(\theta) = \begin{cases} \frac{1}{2} + \theta & \text{for } 0 \le \theta \le \frac{1}{3} \\ \frac{1+3\theta-3\theta^2}{2} & \text{for } \frac{1}{3} \le \theta \le \frac{2}{3} \\ \frac{3}{2} - \theta & \text{for } \frac{2}{3} \le \theta \le 1. \end{cases} \tag{3.3.1}$$

*Moreover, both the infimum and supremum are attained, for all values of $\theta \in [0,1]$.*

*Proof.* Fix $\theta \in [0,1]$. By Lemma 3.3.1, we are interested in the set of $\beta$ such that there exist vectors $x_0, x_1, x_2, h$ in some Hilbert space satisfying the conditions listed there. Let $y_v = h - x_v$. Consider the Gramian matrix $G$ associated with the seven vectors $h, x_0, x_1, x_2, y_0, y_1, y_2$. Note that

$$\langle y_v, y_{v+1} \rangle = \langle h, h \rangle - \langle h, x_{v+1} \rangle - \langle x_v, h \rangle + \langle x_v, x_{v+1} \rangle = 1 - 2\theta + \beta \ge 0,$$

where the inequality follows from Lemma 3.3.1. This and the conditions of Lemma 3.3.1 imply that $G$ is the $7 \times 7$ matrix

$$G = \begin{bmatrix} 1 & \theta & \theta & \theta & 1-\theta & 1-\theta & 1-\theta \\ \theta & \theta & \beta & \beta & 0 & \theta-\beta & \theta-\beta \\ \theta & \beta & \theta & \beta & \theta-\beta & 0 & \theta-\beta \\ \theta & \beta & \beta & \theta & \theta-\beta & \theta-\beta & 0 \\ 1-\theta & 0 & \theta-\beta & \theta-\beta & 1-\theta & 1+\beta-2\theta & 1+\beta-2\theta \\ 1-\theta & \theta-\beta & 0 & \theta-\beta & 1+\beta-2\theta & 1-\theta & 1+\beta-2\theta \\ 1-\theta & \theta-\beta & \theta-\beta & 0 & 1+\beta-2\theta & 1+\beta-2\theta & 1-\theta \end{bmatrix},$$

and furthermore, that $G$ is positive semidefinite and since each entry of $G$ is non-negative,

$$\max(0, 2\theta - 1) \le \beta \le \theta. \tag{3.3.2}$$

Conversely, given any such $7 \times 7$ positive semidefinite matrix and with the additional condition (3.3.2), we can construct seven such vectors in a Hilbert space. Indeed, assume that we are given a positive semidefinite $7 \times 7$ matrix. Then it is the Gramian of some set of vectors, $h, x_0, x_1, x_2, y_0, y_1, y_2$ and we claim that these vectors satisfy the relations given in Lemma 3.3.1. To see this note that $\|h\|^2 = 1$, while $\|x_v\|^2 = \theta$ and $\|y_v\|^2 = 1 - \theta$. The zeros in the matrix yield that $x_v \perp y_v$. Thus, $\|x_v + y_v\| = 1$. The fact that $\langle h, x_v + y_v \rangle = \theta + (1-\theta) = 1$, together with Cauchy-Schwarz inequality yields that $h = x_v + y_v$. The rest of the relations follow similarly. Thus, we are interested in the set of $\beta$ that satisfy (3.3.2) and yield a positive semidefinite matrix $G$ given above.

We apply one step of the Cholesky algorithm [35], and conclude that the $7 \times 7$ matrix $G$ is positive semidefinite if and only if the following $6 \times 6$ matrix $G'$ is positive semidefinite:

$$G' = \begin{bmatrix} \theta-\theta^2 & \beta-\theta^2 & \beta-\theta^2 & \theta^2-\theta & \theta^2-\beta & \theta^2-\beta \\ \beta-\theta^2 & \theta-\theta^2 & \beta-\theta^2 & \theta^2-\beta & \theta^2-\theta & \theta^2-\beta \\ \beta-\theta^2 & \beta-\theta^2 & \theta-\theta^2 & \theta^2-\beta & \theta^2-\beta & \theta^2-\theta \\ \theta^2-\theta & \theta^2-\beta & \theta^2-\beta & \theta-\theta^2 & \beta-\theta^2 & \beta-\theta^2 \\ \theta^2-\beta & \theta-\theta^2 & \theta^2-\beta & \beta-\theta^2 & \theta-\theta^2 & \beta-\theta^2 \\ \theta^2-\beta & \theta^2-\beta & \theta^2-\theta & \beta-\theta^2 & \beta-\theta^2 & \theta-\theta^2 \end{bmatrix}.$$

This matrix $G'$ partitions into a block matrix of the form $\begin{bmatrix} A & -A \\ -A & A \end{bmatrix}$, where

$$A = \begin{bmatrix} a & x & x \\ x & a & x \\ x & x & a \end{bmatrix},$$

with $a = \theta - \theta^2$ and $x = \beta - \theta^2$. Thus the matrix $G'$ is positive semidefinite if and only if $A \geq 0$. Using the determinant criteria we see that $A \geq 0$ if and only if $|x| \leq a$ and $2x^3 - 3ax^2 + a^3 \geq 0$. Simplifying we see that $A \geq 0$ if and only if $-\frac{a}{2} \leq x \leq a$. Substituting the values of $a$ and $x$, we find that the Gramian matrix $G$ is positive semidefinite if and only if

$$\frac{3\theta^2 - \theta}{2} \leq \beta \leq \theta.$$

Thus, the set of all possible $\beta$ is the set satisfying

$$\max\left\{\frac{3\theta^2 - \theta}{2}, 2\theta - 1, 0\right\} \leq \beta \leq \theta.$$

This becomes

$$\begin{aligned} 0 \leq \beta \leq \theta \quad & \text{for } 0 \leq \theta \leq \tfrac{1}{3} \\ \tfrac{3\theta^2-\theta}{2} \leq \beta \leq \theta \quad & \text{for } \tfrac{1}{3} \leq \theta \leq \tfrac{2}{3} \\ 2\theta - 1 \leq \beta \leq \theta \quad & \text{for } \tfrac{2}{3} \leq \theta \leq 1. \end{aligned}$$

Thus, using these inequalities we obtain the values (3.3.1) and we have that the infimum and supremum in (3.2.5) are attained. $\square$

## 3.4 The $q$ and $qc$ cases

In this section, we compute $f_r^l$ and $f_r^u$ when $r \in \{q, qc\}$. We begin with a symmetrisation lemma, analogous to Lemma 3.3.1.

**Lemma 3.4.1.** *The set $\Theta_{qc}^s$ (resp., $\Theta_q^s$), is equal to the set of pairs $(\theta, \widetilde{\theta})$ with $0 \leq \theta \leq 1$, such that there exists a $C^*$-algebra $\mathcal{A}$ (resp., a finite dimensional $C^*$-algebra, $\mathcal{A}$) with a faithful tracial state $\tau$ and with projections $A_0, A_1, A_2 \in \mathcal{A}$ such that for all $0 \leq v \leq 2$,*

$$\tau(A_v) = \theta, \qquad \tau(A_v A_{v+1}) = \beta, \tag{3.4.1}$$

*where $\widetilde{\theta} = \frac{1}{2} + \theta - \beta$.*

*Proof.* By Theorem 2.2.8 and the discussion in Section 3.2, $(\theta, \widetilde{\theta})$ belongs to $\Theta_{qc}^s$ (respectively, $\Theta_q^s$) if and only if there is a $C^*$-algebra $\mathcal{A}$ (respectively, a finite dimensional $C^*$-algebra $\mathcal{A}$), with a faithful tracial state $\tau$ and projections $A_0, A_1, A_2$ such that

$$\frac{1}{3}\sum_{v=0}^{2} \tau(A_v) = \theta, \qquad \frac{1}{3}\sum_{v=0}^{2} \tau(A_v A_{v+1}) = \beta,$$

where $\widetilde{\theta} = \frac{1}{2} + \theta - \beta$. But if such projections exist, then we can consider the $C^*$-algebra $\widetilde{\mathcal{A}} = \mathcal{A} \oplus \mathcal{A} \oplus \mathcal{A}$ with the trace $\widetilde{\tau} = \frac{1}{3}\tau \oplus \frac{1}{3}\tau \oplus \frac{1}{3}\tau$, and projections $\widetilde{A}_v = A_v \oplus A_{v+1} \oplus A_{v+2}$ that satisfy the stronger requirements of the lemma that include Equations (3.4.1). $\square$

We now show that if there exists a $C^*$-algebra $\mathcal{A}$ with a tracial state $\tau$ and projections $A_0, A_1, A_2 \in \mathcal{A}$, such that these projections attain the infimum, then the projections satisfy a nice relation, viz., each projection commutes with the sum of the other two. To prove this, we begin with a $C^*$-algebra result.

**Proposition 3.4.2.** *Let $\mathcal{A}$ be a unital $C^*$-algebra with a faithful tracial state $\tau$. Let $A$ and $P$ be hermitian elements in $\mathcal{A}$. If $AP - PA \neq 0$, then there exists $H = H^* \in \mathcal{A}$ such that, letting $f(t) = \tau(A(e^{iHt}Pe^{-iHt}))$ for $t \in \mathbb{R}$ , we have $f'(0) > 0$.*

*Proof.* If $H \in \mathcal{A}$ is hermitian, then

$$f'(0) = i\tau(AHP - APH) = i\tau((PA - AP)H),$$

where we used the fact that $\tau$ is a tracial state. Supppose $AP - PA \neq 0$. Let $H = i(PA - AP)$. Then $H$ is hermitian and $f'(0) = \tau(|PA - AP|^2) > 0$, where the strict inequality follows because $AP - PA \neq 0$ and $\tau$ is a faithful state. $\square$

**Proposition 3.4.3.** *Let $\mathcal{A}$ be a unital $C^*$-algebra with a faithful tracial state $\tau$. Fix $\theta \in [0,1]$. Let*

$$\beta = \inf\left\{\frac{1}{3}\tau\left(AB+BC+CA\right) : A, B, C \in \mathcal{A} \textit{ projections}, \tau(A) = \tau(B) = \tau(C) = \theta\right\}.$$

*If there exist projections $A_0, B_0, C_0$ in $\mathcal{A}$ such that $\tau(A_0) = \tau(B_0) = \tau(C_0) = \theta$ and $\beta = \frac{1}{3}\tau(A_0B_0 + B_0C_0 + C_0A_0)$, then*

$$[A_0, B_0 + C_0] = [B_0, C_0 + A_0] = [C_0, A_0 + B_0] = 0.$$

*Proof.* We will show that $A_0$ commutes with $B_0+C_0$ and the other commutation relations follow by symmetry. Let $P = B_0+C_0$. Suppose, for contradiction, that $[A_0, P] \neq 0$. Then, by Proposition 3.4.2, there exists $H = H^* \in \mathcal{A}$ such that if $f(t) = \tau(A_0(e^{iHt}Pe^{-iHt}))$, then $f'(0) > 0$. Fix some small and negative $t$ such that $f(t) < f(0)$. Letting $B_t = e^{iHt}B_0e^{-iHt}$ and $C_t = e^{iHt}C_0e^{-iHt}$, we see that $B_t$ and $C_t$ are themselves projections in $\mathcal{A}$ and $\tau(B_t) = \tau(C_t) = \theta$. But then for our value of $t$,

$$\begin{aligned}\tau(A_0B_t + B_tC_t + C_tA_0) &= \tau(A_0(B_t + C_t) + B_tC_t)\\ &= \tau(A_0(e^{iHt}Pe^{-iHt})) + \tau((e^{iHt}B_0e^{-iHt})(e^{iHt}C_0e^{-iHt}))\\ &= f(t) + \tau(B_0C_0) < f(0) + \tau(B_0C_0) = 3\beta,\end{aligned}$$

which implies that $\beta$ is not the infimum, contrary to hypothesis. Thus, $A_0$ commutes with $B_0 + C_0$. □

Proposition 3.4.3 provides a nice constraint on the projections which achieve the infimum. This leads to the question of understanding the universal $C^*$-algebra $\mathfrak{A}$ generated by three projections which satisfy the constraint in Proposition 3.4.3. This is the $C^*$-algebra that is obtained in the following manner. First form the universal unital complex algebra $\mathcal{A}$ generated by three non-commuting variables $A$, $B$ and $C$. Each time that we have a set of three projections on a Hilbert space $\mathcal{H}$ satisfying the above equations, we have a representation of this algebra, $\pi : \mathcal{A} \to B(\mathcal{H})$. Setting $|||u||| = \sup \|\pi(u)\|$, where the supremum is over all such representations, defines a seminorm on $\mathcal{A}$. The norm-zero elements form a 2-sided ideal, $\mathcal{J}$, and this seminorm induces a norm on $\mathcal{A}/\mathcal{J}$. The completion of $\mathcal{A}/\mathcal{J}$ is what we mean by $\mathfrak{A}$. It has the universal property that given three projections $P_A, P_B, P_C$ on a Hilbert space, $\mathcal{H}$, satisfying the above relations, then there exists a unique $*$-homomorphism $\pi : \mathfrak{A} \to B(\mathcal{H})$ with

$$\pi(A + \mathcal{J}) = P_A, \qquad \pi(B + \mathcal{J}) = P_B, \qquad \pi(C + \mathcal{J}) = P_C.$$

**Proposition 3.4.4.** *Let $\mathfrak{A}$ be the universal unital $C^*$-algebra generated by three projections $A, B$ and $C$ satisfying the commutator relations*

$$[A, B+C] = [B, A+C] = [C, A+B] = 0. \tag{3.4.2}$$

*Then $\mathfrak{A}$ is isomorphic to $\mathbb{C}^8 \oplus \mathbb{M}_2$, wherein*

$$A = 0 \oplus 0 \oplus 0 \oplus 0 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$
$$B = 0 \oplus 0 \oplus 1 \oplus 1 \oplus 0 \oplus 0 \oplus 1 \oplus 1 \oplus \begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix},$$
$$C = 0 \oplus 1 \oplus 0 \oplus 1 \oplus 0 \oplus 1 \oplus 0 \oplus 1 \oplus \begin{pmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix}.$$

*Proof.* We will describe all irreducible $*$-representations of $\mathfrak{A}$ on Hilbert spaces. Let

$$Y = 2(B+C) - (B+C)^2 \in \mathfrak{A}.$$

By the commutation relations (3.4.2), $Y$ commutes with $A$. We also note that $Y = B + C - BC - CB$ and $BY = B - BCB = YB$, namely, that $Y$ commutes with $B$. Similarly, $Y$ commutes with $C$. Hence $Y$ lies in the center of $\mathfrak{A}$. Thus, under any irreducible $*$-representation $\pi : \mathfrak{A} \to B(\mathcal{H}_\pi)$, $Y$ must be sent to a scalar multiple of the identity operator. In other words, we have

$$\pi(B + C - BC - CB) = \pi(Y) = \lambda\pi(1)$$

for some $\lambda \in \mathbb{C}$, so that

$$\pi(CB) \in \operatorname{span} \pi\big(\{1, B, C, BC\}\big).$$

Similarly, we have

$$\pi(CA) \in \operatorname{span} \pi\big(\{1, A, C, AC\}\big), \qquad \pi(BA) \in \operatorname{span} \pi\big(\{1, A, B, AB\}\big).$$

Since $\mathfrak{A}$ is densely spanned by the set of all words in the idempotents $A$, $B$ and $C$, we see

$$\pi(\mathfrak{A}) = \operatorname{span} \pi\big(\{1, A, B, C, AB, AC, BC, ABC\}\big).$$

This implies that $\dim \pi(\mathfrak{A}) \leq 8$. Since $\pi(\mathfrak{A})$ is finite dimensional and acts irreducibly on a Hilbert space $\mathcal{H}_\pi$, it must be equal to a full matrix algebra. Considering dimensions, we must have $\dim \mathcal{H}_\pi \leq 2$.

The irreducible representations $\pi$ of $\mathfrak{A}$ for which $\dim \mathcal{H}_\pi = 1$ are easy to describe. They are the eight representations that send $A$, $B$ and $C$ variously to 0 and 1. We will now characterise the irreducible representations $\pi$ of $\mathfrak{A}$ for which $\dim \mathcal{H}_\pi = 2$, up to unitary equivalence. Let $\pi$ be such a representation. From the commutation relations (3.4.2), we see that, if $\pi(A)$ and $\pi(B)$ commute, then also $\pi(C)$ commutes with $\pi(A)$ and with $\pi(B)$, and the entire algebra $\pi(\mathfrak{A})$ is commutative. This would require $\dim \mathcal{H}_\pi = 1$. By symmetry we conclude that no two of $\pi(A)$, $\pi(B)$ and $\pi(C)$ can commute. In particular, each must be a projection of rank 1. After conjugation with a unitary, we must have

$$\pi(A) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \pi(B) = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix},$$

for some $0 < t < 1$. Since $\pi(B) + \pi(C)$ must commute with $\pi(A)$, we must have

$$\pi(C) = \begin{pmatrix} c_{11} & -\sqrt{t(1-t)} \\ -\sqrt{t(1-t)} & c_{22} \end{pmatrix},$$

for some $c_{11}, c_{22} \geq 0$. Since $\pi(C)$ is a projection, the only possible choices are (i) $c_{11} = t$ and $c_{22} = 1-t$ and (ii) $c_{11} = 1-t$ and $c_{22} = t$. But in Case (ii), we have $\pi(C) = I_{\mathcal{H}_\pi} - \pi(B)$, which violates the prohibition against $\pi(C)$ and $\pi(B)$ commuting. Thus, we must have

$$\pi(C) = \begin{pmatrix} t & -\sqrt{t(1-t)} \\ -\sqrt{t(1-t)} & 1-t \end{pmatrix}.$$

Now, using that $\pi(A) + \pi(B)$ and $\pi(C)$ commute, we see that we must have $t = \frac{1}{4}$ and we easily check that this does provide an irreducible representation of $\mathfrak{A}$.

To summarise, up to unitary equivalence, there are exactly nine different irreducible representations of $\mathfrak{A}$, one of them is two-dimensional and the others are one-dimensional. Thus, $\mathfrak{A}$ is finite dimensional and is isomorphic to the direct sum of the images of its irreducible representations, namely to $\mathbb{C}^8 \oplus \mathbb{M}_2$, with $A$, $B$ and $C$ as indicated. $\square$

We now compute the functions $f_r^l$ and $f_r^u$ for $r \in \{q, qc\}$.

**Theorem 3.4.5.** *For $r \in \{q, qc\}$, the functions $f_r^l$ and $f_r^u$ are given by*

$$f_r^l(\theta) = \frac{1}{2}, \qquad f_r^u(\theta) = \begin{cases} \frac{1}{2} + \theta & \text{for } 0 \le \theta \le \frac{1}{3} \\ \frac{3+\theta}{4} & \text{for } \frac{1}{3} \le \theta \le \frac{1}{2} \\ \frac{4-\theta}{4} & \text{for } \frac{1}{2} \le \theta \le \frac{2}{3} \\ \frac{3}{2} - \theta & \text{for } \frac{2}{3} \le \theta \le 1. \end{cases} \tag{3.4.3}$$

*Moreover, both the infimum and supremum are attained, for all values of $\theta \in [0, 1]$.*

*Proof.* Fix $\theta \in [0, 1]$. From the inclusions (3.2.4), we conclude

$$f_{qc}^l(\theta) \le f_q^l(\theta) \le f_q^u(\theta) \le f_{qc}^u(\theta).$$

To find $f_{qc}^l(\theta)$, by Lemma 3.4.1, we should find the supremum of values $\beta$ such that there exists a $C^*$-algebra $\mathcal{A}$ with faithful tracial state $\tau$ and with projections $A_0, A_1, A_2$ such that

$$\forall v, \quad \tau(A_v) = \theta, \qquad \tau(A_v A_{v+1}) = \beta. \tag{3.4.4}$$

By Cauchy-Schwarz, $\beta \le \theta$. But taking $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}$ with $A_v = 1 \oplus 0$ and an appropriate trace $\tau$ shows that $\beta = \theta$ occurs, and in a finite dimensional example. Thus, we find $f_{qc}^l(\theta) = f_q^l(\theta) = \frac{1}{2}$.

To find $f_{qc}^u(\theta)$, again using Lenma 3.4.1, we should find the infimum $\beta_0$ of values $\beta$ as described above. Since $\Theta_{qc}^s$ is closed, this infimum is attained. Thus, there exists a $C^*$-algebra $\mathcal{A}$ with tracial state $\tau$ and projections $A_0, A_1, A_2$ such that (3.4.4) holds with $\beta = \beta_0$. Morover, by the proof of Lemma 3.4.1, we have that $\beta_0$ equals the infimum of $\frac{1}{3}\tau(AB + BC + CA)$ over all projections $A, B, C$ in some $C^*$-algebra with faithful tracial state $\tau$ such that $\tau(A) = \tau(B) = \tau(C) = \theta$. Thus, Proposition 3.4.3 applies and the commutation relations

$$[A_0, A_1 + A_2] = [A_1, A_0 + A_2] = [A_2, A_0 + A_1] = 0$$

hold. Thus, there is a representation of the universal $C^*$-algebra $\mathfrak{A}$ considered in Proposition 3.4.4, sending $A$ to $A_0$, $B$ to $A_1$ and $C$ to $A_2$. So, using Gelfand–Naimark–Segal representations, in order to find $\beta_0$, it suffices to consider tracial states (faithful or not) on $\mathfrak{A}$. In particular, $\beta_0$ is the minimum of all values of $\beta \ge 0$ for which there exists a tracial state $\tau$ on $\mathfrak{A}$ satisfying

$$\tau(A) = \tau(B) = \tau(C) = \theta, \quad \tau(AB) = \tau(AC) = \tau(BC) = \beta. \tag{3.4.5}$$

Since $\mathfrak{A}$ is finite dimensional, we get $f_{qc}^u(\theta) = f_q^u(\theta)$.

An arbitrary tracial state of $\mathfrak{A}$ is of the form

$$\tau\left(\lambda_1 \oplus \cdots \oplus \lambda_8 \oplus \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right) = \left(\sum_{j=1}^{8} t_j \lambda_j\right) + \frac{s}{2}(x_{11} + x_{22}),$$

for some $t_1, \ldots, t_8, s \geq 0$ satisfying $t_1 + \cdots + t_8 + s = 1$. The conditions (3.4.5) become

$$t_5 + t_6 + t_7 + t_8 + \frac{s}{2} = t_3 + t_4 + t_7 + t_8 + \frac{s}{2} = t_2 + t_4 + t_6 + t_8 + \frac{s}{2} = \theta,$$
$$t_7 + t_8 + \frac{s}{8} = t_6 + t_8 + \frac{s}{8} = t_4 + t_8 + \frac{s}{8} = \beta.$$

These are equivalent to

$$\begin{aligned} t_1 &= 1 + 3\beta - 3\theta + \frac{s}{8} - t_8 \\ t_2 = t_3 = t_5 &= \theta - 2\beta - \frac{s}{4} + t_8 \\ t_4 = t_6 = t_7 &= \beta - \frac{s}{8} - t_8. \end{aligned}$$

Thus, writing $t = t_8$, $\beta_0$ is the minimum value of $\beta$ such that there exist $s, t \geq 0$ such that the inequalities

$$1 + 3\beta - 3\theta + \frac{s}{8} - t \geq 0, \qquad \theta - 2\beta - \frac{s}{4} + t \geq 0, \qquad \beta - \frac{s}{8} - t \geq 0$$

hold. This is a standard linear programming problem solvable by the simplex method and whose solution is,

$$\beta_0 = \begin{cases} 0, & 0 \leq \theta \leq \frac{1}{3} \\ \frac{3\theta - 1}{4}, & \frac{1}{3} \leq \theta \leq \frac{1}{2} \\ \frac{5\theta - 2}{4}, & \frac{1}{2} \leq \theta \leq \frac{2}{3} \\ 2\theta - 1, & \frac{2}{3} \leq \theta \leq 1, \end{cases}$$

which, using $f_{qc}^u(\theta) = \frac{1}{2} + \theta - \beta_0$, yields the values given in (3.4.3). $\square$

## 3.5 Summary

We may summarise Theorem 3.3.2 and Theorem 3.4.5 as follows. The functions as obtained in Theorem 3.5.1 are shown in Figure 3.2.

**Theorem 3.5.1.** *For $r \in \{q, qc\}$, we have*

$$f_r^l(\theta) = \frac{1}{2}, \qquad f_r^u(\theta) = \begin{cases} \frac{1}{2} + \theta & \text{for } 0 \le \theta \le \frac{1}{3}, \\ \frac{3+\theta}{4} & \text{for } \frac{1}{3} \le \theta \le \frac{1}{2}, \\ \frac{4-\theta}{4} & \text{for } \frac{1}{2} \le \theta \le \frac{2}{3}, \\ \frac{3}{2} - \theta & \text{for } \frac{2}{3} \le \theta \le 1. \end{cases} \tag{3.5.1}$$

*For $r = vect$, we have*

$$f_{vect}^l(\theta) = \frac{1}{2}, \qquad f_{vect}^u(\theta) = \begin{cases} \frac{1}{2} + \theta & \text{for } 0 \le \theta \le \frac{1}{3}, \\ \frac{1+3\theta-3\theta^2}{2} & \text{for } \frac{1}{3} \le \theta \le \frac{2}{3}, \\ \frac{3}{2} - \theta & \text{for } \frac{2}{3} \le \theta \le 1. \end{cases} \tag{3.5.2}$$

*In all of these cases, the infimum and supremum as in Equation (3.2.5) are attained by both $f_r^l$ and $f_r^u$. Since $(\theta, \widetilde{\theta}) \in \Theta_r^s$ if and only if $0 \le \theta \le 1$ and $f_r^l(\theta) \le \widetilde{\theta} \le f_r^u(\theta)$, we see that $\Theta_r^s$ is a closed set in $\mathbb{R}^2$ for each $r \in \{q, qc, vect\}$. In particular, we have*

$$\Theta_q^s = \Theta_{qc}^s \subsetneq \Theta_{vect}^s. \tag{3.5.3}$$

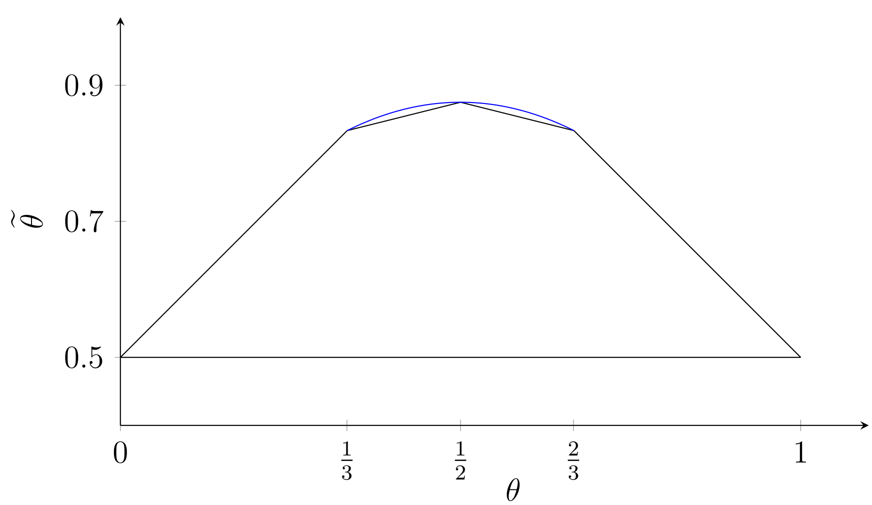


Figure 3.2: Plots of $f_r^l$ and $f_r^u$ for $r \in \{q, qc, vect\}$.

The fact that $\Theta_{qc}^s \subsetneq \Theta_{vect}^s$ allows us to deduce the following.

**Corollary 3.5.2.** *We have that $C_{qc}^s(3, 2) \subsetneq C_{vect}^s(3, 2)$.*

We may analogously define the set $\Theta_{ns}^s$ corresponding to the set of synchronous non-signalling correlations, and similarly the functions $f_{ns}^l$ and $f_{ns}^u$. Since $C_{ns}^s(3,2)$ is a polytope (for example, see [45]), it implies that that $f_{ns}^l$ and $f_{ns}^u$ would be piecewise linear. Since $f_{vect}^u$ is seen to be quadratic on $\left[\frac{1}{3}, \frac{2}{3}\right]$, we get $f_{vect}^u \neq f_{ns}^u$ and we can conclude that $C_{vect}^s(3,2) \subsetneq C_{ns}^s(3,2)$.

**Corollary 3.5.3.** *We have that* $C_{vect}^s(3,2) \subsetneq C_{ns}^s(3,2)$.

Essentially, what we did in this chapter was to minimise, for example, the expression $\frac{1}{3}\tau(A_0A_1 + A_1A_2 + A_2A_0)$ over all $C^*$-algebras $\mathcal{A}$ with a tracial state $\tau$ and projections $A_0, A_1$ and $A_2$ subject to the condition that $\frac{1}{3}\tau(A_0 + A_1 + A_2) = \theta$ for a fixed $\theta \in [0,1]$. As we saw earlier, we may write

$$\frac{1}{3}\tau(A_0A_1 + A_1A_2 + A_2A_0) = \frac{1}{3}\left(p(0,0|0,1) + p(0,0|1,2) + p(0,0|2,0)\right). \tag{3.5.4}$$

Now, $(0,1), (1,2)$ and $(2,0)$ may be thought of as edges of the complete graph on three vertices, $K_3$. Thus the right-hand side of Equation (3.5.4) may be interpreted as

$$\frac{1}{6}\sum_{(v,w)\in E(K_3)} p(0,0|v,w),$$

where $E(K_3) = \{(0,1),(1,0),(1,2),(2,1),(0,2),(2,0)\}$. The additional factor $\frac{1}{2}$ arises due to $p(0,0|0,1) = \tau(A_0A_1) = \tau(A_1A_0) = p(0,0|1,0)$. The constraint in hand is

$$\theta = \frac{1}{3}\tau(A_0 + A_1 + A_2) = \frac{1}{3}\left(p(0,0|0,0) + p(0,0|1,1) + p(0,0|2,2)\right),$$

which may be interpreted as

$$\frac{1}{3}\sum_{v\in V(K_3)} p(0,0|v,v) = \theta.$$

In general, given a graph $G = (V(G), E(G))$, we aim to minimise the expression

$$\frac{1}{|E(G)|}\sum_{(v,w)\in E} p(0,0|v,w),$$

subject to the set of correlations $(p(i,j|v,w)) \in C_r^s(n,2)$ satisfying the constraint

$$\frac{1}{|V(G)|}\sum_{v\in E} p(0,0|v,v) = \theta.$$

By symmetrisation process as done in Lemma 3.3.1 and Lemma 3.4.1, we may simply demand $p(0,0|v,v) = \theta$ for all $v \in E$. But, now notice that because the correlation is synchronous, $p(0,0|v,v) = p(0,0|v,v)+p(0,1|v,v) = p_A(0|v)$. Thus we may define a subset for each $r \in \{loc, q, qs, qa, qc, vect, ns\}$ and for each $\theta \in [0,1]$ as

$$\Gamma_r(\theta) = \{(p(i,j|v,w)) \in C_r^s(n,2) : p_A(0|v) = p_B(0|w) = \theta, \forall v, w\},$$

and we try to minimise the function

$$\sum_{(v,w)\in E} p(0,0|v,w)$$

with the constraint $p \in \Gamma_r(\theta)$ for each graph $G = (V,E)$. This provides a cleaner approach and generalises to the study of such functions for different graphs. This will be taken up in the next chapter.

# Chapter 4

# Graph correlation functions

In this chapter we develop the idea outlined in Section 3.5. We introduce the graph correlation functions and study their general properties. We develop some of the basic theory in the first section and apply these general results in the special case of complete graphs to prove the promised non-closure Theorems 2.1.26 and 2.2.6.

## 4.1 Basic properties of the graph correlation functions

We begin by defining the constraint set over which we shall be working. The motivation behind this set is outlined in Section 3.5.

**Definition 4.1.1.** Let $r \in \{loc, q, qs, qa, qc, vect, ns\}$. For each $t \in [0,1]$, define a slice

$$\Gamma_r(t) = \{(p(i,j|v,w)) \in C_r^s(n,2) : p_A(0|v) = p_B(0|w) = t, 1 \leq v,w \leq n\} \subseteq C_r^s(n,2),$$

where $p_A$ and $p_B$ are the marginal densities from Equations (2.1.2).

As usual, we have the following result which is straightforward to prove.

**Proposition 4.1.2.** *Let $r \in \{loc, q, qs, qa, qc, vect, ns\}$. Then for each $t \in [0,1]$, the set $\Gamma_r(t)$ is non-empty and convex, and*

$$\Gamma_{loc}(t) \subseteq \Gamma_q(t) \subseteq \Gamma_{qs}(t) \subseteq \Gamma_{qa}(t) \subseteq \Gamma_{qc}(t) \subseteq \Gamma_{vect}(t) \subseteq \Gamma_{ns}(t). \tag{4.1.1}$$

*Moreover, for $r \in \{loc, qa, qc, vect, ns\}$, the set $\Gamma_r(t)$ is closed for all $t \in [0,1]$.*

Using these sets we can now define our graph correlation functions.

**Definition 4.1.3** (Graph Correlation Functions for a Graph)**.** Given a graph $G = (V, E)$ on $n$ vertices, we consider the affine function $F : C_{ns}(n, 2) \to \mathbb{R}$ given by

$$F((p(i, j|v, w))) = \sum_{(v,w)\in E} p(0, 0|v, w).$$

For each $r \in \{loc, q, qs, qa, qc, vect, ns\}$ and $t \in [0, 1]$, we let

$$f_r(t) = \inf\{F(p) : p \in \Gamma_r(t)\}. \tag{4.1.2}$$

The functions $f_r$ are called the graph correlation functions for the graph $G$.

We begin with the following elementary proposition.

**Proposition 4.1.4.** *If $G = (V, E)$ is a graph on $n$ vertices, then for all $t \in [0, 1]$,*

$$f_{loc}(t) \geq f_q(t) \geq f_{qs}(t) \geq f_{qa}(t) \geq f_{qc}(t) \geq f_{vect}(t) \geq f_{ns}(t) \geq 0. \tag{4.1.3}$$

*For $r \in \{loc, q, qa, vect, ns\}$, the infimum in (4.1.2) is attained for all $0 \leq t \leq 1$. Moreover,*

$$f_{ns}(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ |E|(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

*Proof.* The first two claims of the proposition follow from Proposition 4.1.2. We prove the expression for $f_{ns}$.

Since $\Gamma_{ns}(t)$ is a closed set for each $t \in [0, 1]$, there exists a correlation $(p(i, j|v, w)) \in \Gamma_{ns}(t)$ such that $p_A(0|v) = p_B(0|w) = t$ for all $v, w \in V$ and $f_{ns}(t) = \sum_{(v,w)\in E} p(0, 0|v, w)$. Since $p_A(0|v) = t$, $\sum_{i,j=0}^{1} p(i, j|v, v) = 1$, and using the fact that the correlation is synchronous, we have

$$p(0, 0|v, v) = t, \qquad p(0, 1|v, v) = p(1, 0|v, v) = 0, \qquad p(1, 1|v, v) = 1 - t. \tag{4.1.4}$$

If $(v, w) \in E$, then using the non-signalling conditions with Equations (4.1.4) we get the equations

$$\begin{aligned} p(0, 0|v, w) + p(0, 1|v, w) = p(0, 0|v, w) + p(1, 0|v, w) = t, \\ p(1, 0|v, w) + p(1, 1|v, w) = p(0, 1|v, w) + p(1, 1|v, w) = 1 - t, \end{aligned}$$

which have the solution,

$$\begin{aligned} p(0,1|v,w) = p(1,0|v,w) = t - p(0,0|v,w), \\ p(1,1|v,w) = 1 - 2t + p(0,0|v,w). \end{aligned} \tag{4.1.5}$$

Since these are probabilities we must also have

$$p(0,0|v,w) \geq 0, \qquad t - p(0,0|v,w) \geq 0, \qquad 1 - 2t + p(0,0|v,w) \geq 0,$$

which yields

$$\max\{0, 2t-1\} \leq p(0,0|v,w) \leq t. \tag{4.1.6}$$

Furthermore, choosing any values for $p(0,0|v,w)$ such that (4.1.6) and (4.1.4) are satisfied and then assigning the other values of $p(i,j|v,w)$ using (4.1.5), we do get an element of $C_{ns}^s(n,2)$. This shows that the choice

$$p(0,0|v,w) = \max\{0, 2t-1\} = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{1}{2} \\ 2t-1 & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases},$$

yields an element of $C_{ns}^s(n,2)$, whereby the desired value of $f_{ns}(t)$ is attained. □

Next, we prove that the correlation functions $f_r$ enjoy a certain symmetry, in the sense that if $f_r$ is known on the half-interval $\left[0, \frac{1}{2}\right]$, then it can be described on rest of the interval $\left[\frac{1}{2}, 1\right]$. To prove this we define some actions on the correlation sets $C_r(n,k)$.

**Definition 4.1.5.** Let $S_n$ and $S_k$ denote the symmetric groups on $n$ and $k$ elements, respectively. For each permutation $\pi \in S_n$, we define an affine self-map $\beta_\pi : C_{ns}(n,k) \to C_{ns}(n,k)$ given by

$$\beta_\pi\left(\left(p(i,j|v,w)\right)\right) = \left(p(i,j|\pi^{-1}(v), \pi^{-1}(w))\right). \tag{4.1.7}$$

For each permutation $\sigma \in S_k$, we define another affine self-map $\gamma_\sigma : C_{ns}(n,k) \to C_{ns}(n,k)$ given by

$$\gamma_\sigma\left(\left(p(i,j|v,w)\right)\right) = \left(p(\sigma^{-1}(i), \sigma^{-1}(j)|v,w)\right). \tag{4.1.8}$$

**Proposition 4.1.6.** *Let $\pi \in S_n$ and $\sigma \in S_k$. Then the affine maps $\beta_\pi$ and $\gamma_\sigma$ as in Definition 4.1.5 define actions $\beta : S_n \times C_{ns}(n,k) \to C_{ns}(n,k)$ and $\gamma : S_k \times C_{ns}(n,k) \to C_{ns}(n,k)$, respectively, in a canonical way. These actions restrict to actions on $C_r(n,k)$ and $C_r^s(n,k)$ for each $r \in \{loc, q, qs, qa, qc, vect\}$.*

*Proof.* It is easy to verify that $\beta$ and $\gamma$ are actions. To see that the restrictions are indeed self-maps, say for $r = vect$, suppose that the set $\{x_{v,i}, y_{w,j}, h : v, w \in I, i, j \in O\}$ of vectors satisfying Definition 2.1.30 realise a given vector correlation $p = (p(i, j|v, w))$. Then it is easy to check that the (re-indexed) set $\{x_{\pi^{-1}(v),i}, y_{\pi^{-1}(w),j}, h\}$ of vectors realise $\beta_\pi(p)$. Similarly, for $r \in \{loc, q, qs, qc\}$, applying permutations to systems of projections that realise a given $p \in C_r(n, k)$ show $\beta_\pi(p) \in C_r(n, k)$. The case of $r = qa$ now follows by taking closures. Restrictions to synchronous subsets also follow similarly. $\square$

We will use this only in the case $k = 2$, when for the order-two transposition $\sigma : 0 \leftrightarrow 1$, we get the reflection $R = \gamma_\sigma$. We now prove that it suffices to describe the functions $f_r$ on the interval $\left[0, \frac{1}{2}\right]$.

**Proposition 4.1.7.** *Let $G = (V, E)$ be a graph on $n$ vertices. Then $f_r$ is a convex function for all $r \in \{loc, q, qa, qc, vect, ns\}$, and*

$$f_r(1 - t) = |E|(1 - 2t) + f_r(t), \qquad t \in [0, 1]. \tag{4.1.9}$$

*Proof.* By the convexity of $C_r^s(n, 2)$, for each $t_1, t_2, \lambda \in [0, 1]$, we have

$$\lambda\Gamma_r(t_1) + (1 - \lambda)\Gamma_r(t_2) \subseteq \Gamma_r(\lambda t_1 + (1 - \lambda)t_2).$$

Applying $F$, we have

$$\lambda F(\Gamma_r(t_1)) + (1 - \lambda)F(\Gamma_r(t_2)) = F(\lambda\Gamma_r(t_1) + (1 - \lambda)\Gamma_r(t_2)) \subseteq F(\Gamma_r(\lambda t_1 + (1 - \lambda)t_2)).$$

Taking infima implies

$$\lambda f_r(t_1) + (1 - \lambda)f_r(t_2) = \inf(F(\lambda\Gamma_r(t_1) + (1 - \lambda)\Gamma_r(t_2))) \geq f_r(\lambda t_1 + (1 - \lambda)t_2),$$

namely, that $f_r$ is convex.

To prove Equation (4.1.9), we use the reflection map $R : C_r^s(n, 2) \to C_r^s(n, 2)$ described above. Using (4.1.4) we see that $R$ maps $\Gamma_r^s(t)$ onto $\Gamma_r^s(1 - t)$ and using (4.1.5) we see $F \circ R(p) = |E|(1 - 2t) + F(p)$ for every $p \in C_r^s(n, 2)$. Equation (4.1.9) then follows by taking infima on both sides. $\square$

Let $k, n \in \mathbb{N}$, and let $\mathbb{Z}_k = \{0, 1, ..., k - 1\}$ be the cyclic group of order $k$. Define

$$\mathbb{F}(n, k) = \underbrace{\mathbb{Z}_k * ... * \mathbb{Z}_k}_{n \text{ times}}$$

to be the free product of $n$ copies of $\mathbb{Z}_k$. Since the left regular representation of $\mathbb{F}(n, k)$ is faithful on the group algebra, we may think of $C^*(\mathbb{F}(n, k))$ as the universal $C^*$-algebra generated by the elements $\{e_{v,i} : 1 \leq v \leq n, 1 \leq i \leq k\}$ satisfying the following relations:

(a) $e_{v,i}^2 = e_{v,i}^* = e_{v,i}$, for all $1 \leq v \leq n$ and $1 \leq i \leq k$,

(b) $e_{v,i}e_{v,j} = 0$, for all $1 \leq v \leq n$ and $i \neq j$, and

(c) $\sum_{i=1}^k e_{v,i} = 1$, for all $1 \leq v \leq n$.

Every state $\tau$ on $C^*(\mathbb{F}(n,k))$ has Gelfand-Naimark-Segal (GNS) representation, that is, there exists a Hilbert space $\mathcal{H}$, a unital $*$-homomorphism $\pi : C^*(\mathbb{F}(n,k)) \to B(\mathcal{H})$, and a unit vector $\psi \in \mathcal{H}$ such that $\tau(a) = \langle \pi(a)\psi, \psi \rangle$ for all $a \in C^*(\mathbb{F}(n,k))$. We shall call a state $\tau$ on $C^*(\mathbb{F}(n,k))$ finite-dimensional provided that the Hilbert space in the GNS representation is finite dimensional. We shall call a state abelian if the image of $C^*(\mathbb{F}(n,k))$ under the GNS representation is commutative.

In what follows, we shall be primarily working with $C^*(\mathbb{F}(n,2))$, which is the universal $C^*$-algebra generated by $\{e_{v,0}, e_{v,1} : 0 \leq v \leq n-1\}$ satisfying Relations (a), (b) and (c) in the previous paragraph. To keep subscripts uncluttered, we shall adopt the notation $e_v := e_{v,0}$, for all $0 \leq v \leq n-1$.

**Corollary 4.1.8.** *Let $G = (V, E)$ be a graph on $n$ vertices. Then we have*

$$f_{qc}(t) = \inf \left\{ \sum_{(v,w)\in E} \tau(e_v e_w) : \tau \textit{ is a tracial state on } C^*(\mathbb{F}(n,2)), \tau(e_v) = t, \ \forall\, v \in V \right\},$$

*and $f_q(t)$ can be obtained by the same equation but now the infimum runs over all finite-dimensional tracial states $\tau$.*

*Proof.* This follows from Theorem 2.2.8 and Corollary 2.2.9. □

We establish a result similar to that of Proposition 3.4.3. While this result is not used anywhere in this chapter, it led to the question of realising scalar multiples of the identity as sums of projections as described in [44], which is crucial in the proof of Theorem 4.3.5.

**Proposition 4.1.9.** *Let $G = (V, E)$ be a graph on $n$ vertices, and assume that $\tau : C^*(\mathbb{F}(n,2)) \to \mathbb{C}$ is a faithful tracial state (respectively, faithful finite dimensional tracial state) such that $\tau(e_v) = t$ for all $v \in V$ and $f_{qc}(t)$ (respectively, $f_q(t)$) is equal to $\sum_{(v,w)\in E} \tau(e_v e_w)$. Set $p_v = \sum_{\{w : (v,w)\in E\}} e_w$. If $\pi : C^*(\mathbb{F}(n,2)) \to B(\mathcal{H})$ is the GNS representation of $\tau$, then $\pi(e_v)\pi(p_v) = \pi(p_v)\pi(e_v)$.*

*Proof.* Fix $v \in V$. Let $\pi : C^*(\mathbb{F}(n,2)) \to B(\mathcal{H})$ be the GNS representation of $\tau$ with $\tau(a) = \langle \pi(a)\psi, \psi\rangle$ for all $a \in C^*(\mathbb{F}(n,2))$ and for some cyclic vector $\psi \in \mathcal{H}$. Let $\mathcal{B} = \pi(C^*(\mathbb{F}(n,2))) \subseteq B(\mathcal{H})$ be the image $C^*$-algebra. Suppose, for contradiction, that $\pi(e_v)$ and $\pi(p_v)$ do not commute. Then by Proposition 3.4.2, there exists $H = H^* \in \mathcal{B}$ (therefore $H = \pi(h)$, $h \in C^*(\mathbb{F}(n,2))$) such that if

$$f(t) = \left\langle \pi(e_v)(e^{iHt}\pi(p_v)e^{-iHt})\psi, \psi\right\rangle = \tau(e_v(e^{iht}\pi(p_v)e^{-iht})),$$

then $f'(0) > 0$. Fix some small and negative $t_0$ such that $f(t_0) < f(0)$.

Define for $y \in V$,

$$F_y = \begin{cases} \pi(e_v) & \text{if } y = v, \\ e^{iHt_0}\pi(e_y)e^{-iHt_0} & \text{if } y \neq v. \end{cases}$$

Then each $F_y$ is a projection in $\mathcal{B}$ and

$$\langle F_y\psi, \psi\rangle = \left\langle \left(e^{iHt_0}\pi(e_y)e^{-iHt_0}\right)\psi, \psi\right\rangle = \tau\left(e^{iht_0}\pi(e_y)e^{-iht_0}\right) = \tau(e_y) = t.$$

But for this new set of projections, we have that

$$\begin{aligned}
&\sum_{(x,y)\in E} \langle F_xF_y\psi, \psi\rangle \\
&\quad = \sum_{\{w\,:\,(v,w)\in E\}} \langle F_vF_w\psi, \psi\rangle + \sum_{\{w\,:\,(w,v)\in E\}} \langle F_wF_v\psi, \psi\rangle + \sum_{\{(x,y)\in E\,:\,x\neq v,\, y\neq v\}} \langle F_xF_y\psi, \psi\rangle \\
&\quad = \left\langle F_v\left(\sum_{\{w\,:\,(v,w)\in E\}} F_w\right)\psi, \psi\right\rangle + \left\langle \left(\sum_{\{w\,:\,(w,v)\in E\}} F_w\right)F_v\psi, \psi\right\rangle \\
&\qquad\qquad + \sum_{\{(x,y)\in E\,:\,x\neq v,\, y\neq v\}} \langle F_xF_y\psi, \psi\rangle \\
&\quad = 2\,\mathrm{Re}\left\langle \pi(e_v)\left(\sum_{\{w\,:\,(v,w)\in E\}} e^{iHt_0}\pi(e_w)e^{-iHt_0}\right)\psi, \psi\right\rangle \\
&\qquad + \sum_{\{(x,y)\in E\,:\,x\neq v,\, y\neq v\}} \left\langle \left(e^{iHt_0}\pi(e_x)e^{-iHt_0}\right)\left(e^{iHt_0}\pi(e_y)e^{-iHt_0}\right)\psi, \psi\right\rangle \\
&\quad = 2\,\mathrm{Re}\left\langle \pi(e_v)\left(e^{iHt_0}\pi(p_v)e^{-iHt_0}\right)\psi, \psi\right\rangle + \sum_{\{(x,y)\in E:x\neq v,y\neq v\}} \tau(e^{iht_0}e_xe_ye^{-iht_0})
\end{aligned}$$

$$
\begin{aligned}
&= 2f(t_0) + \sum_{\{(x,y)\in E\,:\,x\neq v,\,y\neq v\}} \tau(e_x e_y) < 2f(0) + \sum_{\{(x,y)\in E\,:\,x\neq v,\,y\neq v\}} \tau(e_x e_y) \\
&= \tau(e_v p_v) + \tau(p_v e_v) + \sum_{\{(x,y)\in E\,:\,x\neq v,\,y\neq v\}} \tau(e_x e_y) = \sum_{(x,y)\in E} \tau(e_x e_y) = f_{qc}(t),
\end{aligned}
$$

where we have used that $(v,w)\in E$ if and only if $(w,v)\in E$. This contradicts the definition of $f_{qc}$. $\square$

**Proposition 4.1.10.** *Let $G=(V,E)$ be a graph on $n$ vertices. Then*

$$
f_q\left(\frac{1}{2}\right) = f_{qa}\left(\frac{1}{2}\right) = f_{qc}\left(\frac{1}{2}\right) = f_{vect}\left(\frac{1}{2}\right).
$$

*Proof.* From the relations (4.1.3), it is sufficient to show that $f_q\left(\frac{1}{2}\right) = f_{vect}\left(\frac{1}{2}\right)$. In fact, we show that $\Gamma_{vect}\left(\frac{1}{2}\right) = \Gamma_q\left(\frac{1}{2}\right)$.

Let $(p(i,j|v,w)) \in \Gamma_{vect}\left(\frac{1}{2}\right)$. By Proposition 2.2.10, there exist vectors $\{x_{v,0}, x_{v,1}, h : v\in V\} \subseteq \mathcal{H}$ such that $p(i,j|v,w) = \langle x_{v,i}, x_{w,j}\rangle$. Without loss of generality, we may assume that $\mathcal{H}$ is a finite-dimensional real Hilbert space, say of dimension $m$. Set $x_v = x_{v,0}$ for all $v\in V$. Then $\frac{1}{2} = p_A(0|v) = \langle x_v, h\rangle$, and non-signalling conditions yield,

$$
\begin{aligned}
p(0,0|v,w) &= p(1,1|v,w) = \langle x_v, x_w\rangle, \\
p(0,1|v,w) &= p(1,0|v,w) = \frac{1}{2} - \langle x_v, x_w\rangle.
\end{aligned}
$$

Define $\widetilde{x}_v = 2x_v - h$ for all $v\in V$. It is easy to verify that each $\widetilde{x}_v$ is a unit vector, and

$$
p(i,j|v,w) = \frac{1}{4}\left(1 + (-1)^{i+j}\langle \widetilde{x}_v, \widetilde{x}_w\rangle\right).
$$

Recall the representation of the Clifford algebra [34] that is determined by a real linear map $\mathcal{H} \ni x \mapsto C(x) \in \mathbb{M}_d$ for some $d$, where each $C(x)$ is self-adjoint and has trace zero and where they satisfy $C(x)C(y) + C(y)C(x) = 2\langle x,y\rangle \mathbb{I}_d$. Thus, when $x$ is a unit vector, $C(x)$ is a symmetry. We let

$$
P_{v,i} = \frac{\mathbb{I}_d + (-1)^i C(\widetilde{x}_v)}{2}.
$$

Then each $P_{v,i}$ is a projection and computation shows that

$$
\mathrm{tr}_d(P_{v,i}P_{w,j}) = \frac{1}{4}\left(1 + (-1)^{i+j}\langle \widetilde{x}_v, \widetilde{x}_w\rangle\right) = p(i,j|v,w).
$$

Therefore $(p(i,j|v,w)) \in \Gamma_q\left(\frac{1}{2}\right)$ as well and the proposition follows. $\square$

The graph correlation functions of a graph are also related to the fractional chromatic number and its generalisations. For completeness sake we include the definitions.

**Definition 4.1.11** (Fractional Chromatic Number)**.** A graph $G = (V, E)$ is said to have an $a/b$-colouring if to each vertex we can assign a subset of $\{1, ..., a\}$ having cardinality $b$ such that whenever two vertices are adjacent, their corresponding subsets are disjoint. The fractional chromatic number of $G$ is then defined by

$$\chi_f(G) = \inf\left\{\frac{a}{b} : G \text{ has an } a/b\text{-colouring}\right\}.$$

**Definition 4.1.12** (Mančinska-Roberson's projective rank, [48])**.** Let $G = (V, E)$ be a graph. Let $d \in \mathbb{N}$ and $r \in \mathbb{Z}^+$. We say that the graph $G$ admits a $d/r$-representation if there exists a collection of projections $\{E_v : v \in V\} \subseteq \mathbb{M}_d$ such that $\mathrm{Tr}(E_v) = r$ for all $v \in V$ and $E_vE_w = 0$ for all $(v, w) \in E$. Define the *projective rank* to be

$$\xi_f(G) = \inf\left\{\frac{d}{r} : G \text{ has a } d/r\text{-representation}\right\}.$$

**Definition 4.1.13** (Tracial Rank, [59])**.** Let $G = (V, E)$ be a graph. We define the *tracial rank* $\xi_{tr}(G)$ to be the infimum of the the set of all real numbers $s$ such that there exists a $C^*$-algebra $\mathcal{A}$ with a tracial state $\tau$ and projections $\{p_v : v \in V\} \subseteq \mathcal{A}$ such that $\tau(p_v) = s$, and $\tau(p_vp_w) = 0$ for all $(v, w) \in E$.

We remark that if we restrict the infimum in Definition 4.1.13 over finite dimensional $C^*$-algebras we get the projective rank of the graph [59, Proposition 5.11]. Moreover, it is well-known that the fractional chromatic number gives a lower bound on the chromatic number. In fact, $\xi_f(G) \leq \chi_f(G) \leq \chi(G)$, for all graphs $G$ [30]. The other two parameters provide lower bounds for quantum versions of chromatic numbers [48, 60, 59]

The following two propositions follow from the definitions.

**Proposition 4.1.14.** *For a graph $G$, we have*

$$\begin{aligned}
\chi_f(G)^{-1} &= \sup\{t : f_{loc}(t) = 0\},\\
\xi_f(G)^{-1} &= \sup\{t : f_q(t) = 0\},\\
\xi_{tr}(G)^{-1} &= \sup\{t : f_{qc}(t) = 0\},
\end{aligned}$$

*where $\chi_f(G), \xi_f(G)$ and $\xi_{tr}(G)$ as defined in Definitions 4.1.11, 4.1.12, and 4.1.13, respectively.*

**Proposition 4.1.15.** *Let $G$ be a graph on $n$ vertices and let $\overline{G}$ be its complement. If $K_n$ denotes the complete graph on $n$ vertices, then*

$$f_r(G) + f_r(\overline{G}) \leq f_r(K_n),$$

*for all $r \in \{loc, q, qs, qa, qc, vect\}$.*

*Proof.* Let $r \in \{loc, q, qs, qa, qc, vect\}$ and let $t \in [0,1]$. Let $p = (p(i,j|v,w)) \in C_r^s(n,2)$ be such that $p_A(0|v) = p_B(0|w) = t$ for all $1 \leq v, w \leq n$. Letting $E(G)$ and $E(\overline{G})$ denote the edge sets of $G$ and $\overline{G}$, respectively, we see that

$$f_r(G)(t) + f_r(\overline{G})(t) \leq \sum_{(v,w)\in E(G)} p(0,0|v,w) + \sum_{(v,w)\in E(\overline{G})} p(0,0|v,w) = \sum_{(v,w)\in E(K_n)} p(0,0|v,w).$$

Since this holds for all $p = (p(i,j|v,w)) \in \Gamma_r(t)$, we get the desired result. $\square$

## 4.2 Vertex and edge transitive graphs

With these basic properties of graph correlation functions in place, it would be beneficial to have some concrete examples in hand. The class of vertex transitive and edge transitive graphs provide examples of graphs with ample symmetry to exploit. We begin with the definition of such graphs.

**Definition 4.2.1.** A graph automorphism of a graph $G = (V, E)$ is a bijective function $\pi : V \to V$ such that $(v, w) \in E$ if and only if $(\pi(v), \pi(w)) \in E$. We let $\mathrm{Aut}(G)$ denote the group of all graph automorphisms of $G$. A graph is called vertex transitive if for every $v, w \in V$ there is a graph automorphism $\pi$ with $\pi(v) = w$. A graph is called edge transitive if for every $(v, w), (x, y) \in E$, there is a graph automorphism $\pi$ with $(\pi(v), \pi(w)) = (x, y)$.

The symmetries of a vertex and edge transitive graph allow us to restrict our attention to even smaller sets of correlations with higher degrees of symmetry.

**Proposition 4.2.2.** *If $G = (V, E)$ is a vertex and edge transitive graph on $n$ vertices, then for every $r \in \{loc, q, qa, qc, vect, ns\}$ and every $t \in [0,1]$, we have $f_r(t) = \inf\{F(p) : p \in \widetilde{\Gamma}_r(t)\}$, where*

$$\widetilde{\Gamma}_r(t) = \{(p(i,j|v,w)) \in \Gamma_r(t) : p(0,0|v,w) = p(0,0|x,y),\ \forall\,(v,w),\ (x,y) \in E\}.$$

*Proof.* Using the convexity of $\Gamma_r(t)$ and the vertex and edge transitivity of the graph $G$, it is not too hard to show that the map

$$p \mapsto \frac{1}{|\mathrm{Aut}(G)|} \sum_{\pi \in \mathrm{Aut}(G)} \beta_\pi(p),$$

maps $\Gamma_r(t)$ into $\widetilde{\Gamma}_r(t)$. Since $\beta_\pi$ leaves the function $F$ invariant, it follows that the above map also leaves $F$ invariant. But then by Equation (4.1.2) we get $f_r(t) = \inf\{F(p) : p \in \widetilde{\Gamma}_r(t)\}$. $\square$

**Corollary 4.2.3.** *Let $G = (V, E)$ be a vertex and edge transitive graph. Then for all $t \in [0, 1]$, $f_{qc}(t)$ is the infimum over all real numbers $s$ such that there exists a $C^*$-algebra $\mathcal{A}$ with a tracial state $\tau$ and projections $\{P_v : v \in V\} \subseteq \mathcal{A}$ such that $\tau(P_v) = t$ for all $v \in V$ and $\tau(P_v P_w) = \frac{s}{|E|}$ whenever $(v, w) \in E$. Moreover, $f_q$ can be computed in an identical manner but with the infimum restricted to finite dimensional $C^*$-algebras.*

*Proof.* This follows from Proposition 4.2.2 and Corollary 4.1.8. $\square$

**Remark 4.2.4.** Let $r \in \{loc, q, qa, qc, vect, ns\}$ and let $(p(i, j|v, w)) \in C_r^s(n, 2)$ be such that $p_A(0|v) = p_B(0|w) = t$ for all $v, w \in V$ and $p(0, 0|v, w) = \frac{s}{|E|}$ for all $(v, w) \in E$. The synchronous condition implies $t = p_A(0|v) = p(0, 0|v, v) + p(0, 1|v, v) = p(0, 0|v, v)$, so that

$$p(0, 0|v, v) = t, \qquad p(0, 1|v, v) = p(1, 0|v, v) = 0, \qquad p(1, 1|v, v) = 1 - t. \tag{4.2.1}$$

If $(v, w) \in E$, then using the non-signalling conditions (Definition 2.1.4) with Equations (4.2.1) we must have

$$p(0, 0|v, w) = \frac{s}{|E|}, \quad p(0, 1|v, w) = p(1, 0|v, w) = t - \frac{s}{|E|}, \quad p(1, 1|v, w) = 1 - 2t + \frac{s}{|E|}.$$

Since these are probabilities, we must have

$$0 \leq \max\{0, 2t - 1\} \leq \frac{s}{|E|} \leq t. \tag{4.2.2}$$

The following proposition states that the graph correlation function $f_q$ for a vertex and edge transitive graph is "piecewise" linear, and it may certainly contain infinitely many linear parts.

**Proposition 4.2.5.** *Let $G = (V, E)$ be a vertex and edge transitive graph on $n$ vertices and let $t \in [0, 1]$ be irrational. Suppose that the value of $f_q(t)$ is attained in the infimum (4.1.2) defining it. Then there is a non-degenerate interval $[r, s]$ having rational endpoints such that $t \in [r, s]$ and the restriction of $f_q$ to $[r, s]$ is linear.*

*Proof.* Since the value $f_q(t)$ is attained, there is a finite dimensional $C^*$-algebra $\mathcal{A}$ generated by projections $\{P_v : v \in V\}$ and equipped with a trace $\tau : \mathcal{A} \to \mathbb{C}$ with $\tau(P_v) = t$ for all $v \in V$ and such that

$$f_q(t) = \sum_{(v,w)\in E} \tau(P_v P_w).$$

Since $\mathcal{A}$ is finite dimensional, we may write $\mathcal{A} = \bigoplus_{l=1}^m \mathbb{M}_{n_l}$ and $\tau = \oplus_{l=1}^m \lambda_l \mathrm{tr}_{n_l}$, where $\lambda_l > 0$ with $\sum_{l=1}^m \lambda_l = 1$, and where $\mathrm{tr}_{n_l} : \mathbb{M}_{n_l} \to \mathbb{C}$ denotes the normalised trace on matrices. Moreover, we have $P_v = \oplus_{l=1}^m P_{v,l}$ for projections $P_{v,l} \in \mathbb{M}_{n_l}$. Let $\mathrm{Aut}(G)$ denote the group of graph automorphisms of the graph $G$ and set $N = |\mathrm{Aut}(G)|$. For $v \in V$ and $1 \le l \le m$, set

$$\widetilde{P}_{v,l} = \oplus_{\pi\in\mathrm{Aut}(G)} P_{\pi(v),l} \in \bigoplus_{\pi\in\mathrm{Aut}(G)} \mathbb{M}_{n_l} =: \mathcal{A}_l.$$

Define a trace, $\tau_l : \mathcal{A}_l \to \mathbb{C}$, by

$$\tau_l\left(\oplus_{\pi\in\mathrm{Aut}(G)} X_\pi\right) = \frac{1}{N} \sum_{\pi\in\mathrm{Aut}(G)} \mathrm{tr}_{n_l}(X_\pi).$$

Given any $v, w \in V$ if we fix $\rho \in \mathrm{Aut}(G)$ such that $\rho(v) = w$, then

$$\tau_l(\widetilde{P}_{w,l}) = \frac{1}{N} \sum_{\pi\in\mathrm{Aut}(G)} \mathrm{tr}_{n_l}(P_{\pi(w),l}) = \frac{1}{N} \sum_{\pi\in\mathrm{Aut}(G)} \mathrm{tr}_{n_l}(P_{\pi\rho(v),l}) = \tau_l(\widetilde{P}_{v,l}),$$

which is some fixed rational number $r_l$. After a permutation we may assume that these rational numbers $r_l$ are arranged in non-decreasing order.

Thus, $\{\widetilde{P}_{v,l} : v \in V\}$ is a feasible set for the definition of $f_q(r_l)$ and hence we have that

$$f_q(r_l) \le \sum_{(v,w)\in E} \tau_l(\widetilde{P}_{v,l}\widetilde{P}_{w,l}).$$

Now, we set $\widetilde{\mathcal{A}} = \oplus_{l=1}^m \mathcal{A}_l$, and define a normalised trace $\widetilde{\tau} : \widetilde{\mathcal{A}} \to \mathbb{C}$ by $\widetilde{\tau}(\oplus_{l=1}^m Y_l) = \sum_{l=1}^m \lambda_l \tau_l(Y_l)$. Define projections $\widetilde{P}_v$ in $\widetilde{\mathcal{A}}$ by $\widetilde{P}_v = \oplus_{l=1}^m \widetilde{P}_{v,l}$. Then we have that

$$\widetilde{\tau}(\widetilde{P}_v) = \sum_{l=1}^m \lambda_l \tau_l(\widetilde{P}_{v,l}) = \frac{1}{N} \sum_{l=1}^m \sum_{\pi\in\mathrm{Aut}(G)} \lambda_l \mathrm{tr}_{n_l}(P_{\pi(v),l}) = \frac{1}{N} \sum_{\pi\in\mathrm{Aut}(G)} \tau(P_{\pi(v)}) = t,$$

while a similar calculation shows that $\sum_{(v,w)\in E} \widetilde{\tau}(\widetilde{P}_v \widetilde{P}_w) = f_q(t)$. Thus,

$$f_q(t) = \sum_{(v,w)\in E} \sum_{l=1}^{m} \lambda_l \tau_l(\widetilde{P}_{v,l} \widetilde{P}_{w,l}) \geq \sum_{l=1}^{m} \lambda_l f_q(r_l).$$

By Proposition 4.1.7, $f_q$ is a convex function and so we have

$$f_q(t) = \sum_{l=1}^{m} \lambda_l f_q(r_l),$$

and so we must have that $f_q(r_l) = \sum_{(v,w)\in E} \tau_l(\widetilde{P}_{v,l} \widetilde{P}_{w,l})$.

But this is exactly the equality case of Jensen's inequality, which holds if and only if either all the points in the convex combination are the same or the function is piecewise linear on an interval containing the points. Since $t$ is irrational, the points $r_l$ cannot all be same and this forces the function $f_q$ to be linear on an interval containing the points $r_l$. □

**Theorem 4.2.6.** *Let $G = (V, E)$ be a vertex and edge transitive graph on $n$ vertices and let $t \in [0, 1]$. Then $f_{vect}(t) = s$, where $s$ is the smallest real number satisfying Equation (4.2.2) and for which there exists an $(n + 1) \times (n + 1)$ positive semidefinite matrix $P = [p_{i,j}]_{i,j=0}^{n}$ satsifying*

*(a)* $p_{i,j} \geq 0$ *for all* $0 \leq i, j \leq n$

*(b)* $p_{0,0} = 1$ *and* $p_{i,i} = t$ *for all* $1 \leq i \leq n$,

*(c)* $p_{0,j} = p_{j,0} = t$ *for all* $1 \leq j \leq n$

*(d)* $p_{i,j} = \frac{s}{|E|}$ *for all* $(i, j) \in E$.

*Proof.* Fix $t \in [0, 1]$ and let $f_{vect}(t) = s$. Then $s$ must satisfy Equation (4.2.2). Since $C_{vect}^{s}(n, 2)$ is closed, by Proposition 4.2.2 there exists $(p(i, j|v, w)) \in C_{vect}^{s}(n, 2)$ such that $p_A(0|v) = p_B(0|w) = t$ for all $v, w \in V$ and $p(0, 0|v, w) = \frac{s}{|E|}$ for all $(v, w) \in E$. By Proposition 2.2.10 there exist vectors $\{x_{v,0}, x_{v,1}, h : v \in V\} \subseteq \mathcal{H}$ in some Hilbert space $\mathcal{H}$ such that

$$\|h\| = 1, \qquad \langle x_{v,0}, x_{v,1}\rangle = 0, \qquad h = x_{v,0} + x_{v,1}, \qquad p(i, j|v, w) = \langle x_{v,i}, x_{w,j}\rangle.$$

Set $x_v = x_{v,0}$ and $y_v = x_{v,1}$. Let $x_0 = h$ and let $P = [p_{v,w}]_{v,w=0}^n$ be the Gramian of vectors $\{x_0, x_1, \ldots, x_n\}$. Then this matrix is positive semidefinite and satisfies the properties stated in theorem. For notice that for all $v \in V$ we have

$$\langle x_v, h\rangle = \langle x_{v,0}, x_{v,0} + x_{v,1}\rangle = p(0,0|v,v) + p(0,1|v,v) = p_A(0|v) = t,$$
$$\|x_v\|^2 = \langle x_{v,0}, x_{v,0}\rangle = \langle x_{v,0}, h - x_{v,1}\rangle = \langle x_{v,0}, h\rangle = t,$$

and for all $(v,w) \in E$ we have

$$\langle x_v, x_w\rangle = \langle x_{v,0}, x_{w,0}\rangle = p(0,0|v,w) = \frac{s}{|E|}.$$

Conversely, given such a matrix $P$ there are vectors $\{x_0, \ldots, x_n\}$ such that $P$ is the Gramian of these vectors. Set $h = x_0$ and $y_v = x_0 - x_v$ for all $1 \le v \le n$ and observe that $\langle x_v, y_v\rangle = \langle x_v, x_0 - x_v\rangle = p_{0,v} - p_{v,v} = t - t = 0$, from which it is easy to construct a synchronous vector correlation. □

## 4.3 Complete graphs

In this section, we compute the function $f_{vect}$ explicitly for the complete graph $K_n$ when $n \ge 3$. We shall then compare the function $f_{vect}$ with the function $f_q$ for $K_5$ to deduce that the set $C_q(5,2)$ is not closed.

**Proposition 4.3.1.** *For the complete graph $K_n$ on $n \ge 3$ vertices, we have that*

$$f_{vect}(t) = \begin{cases} 0, & \text{if } 0 \le t \le \frac{1}{n}, \\ nt(nt-1), & \text{if } \frac{1}{n} \le t \le \frac{n-1}{n}, \\ (n^2-n)(2t-1), & \text{if } \frac{n-1}{n} \le t \le 1. \end{cases}$$

*Proof.* We seek the smallest $s$ for which the $(n+1)\times(n+1)$ matrix satisfying the conditions of Theorem 4.2.6 is positive semidefinite. Applying one step of the Cholesky algorithm, this is equivalent to the $n \times n$ matrix $Q = [q_{i,j}]$ being positive semidefinite, where $q_{i,i} = t - t^2$ and $q_{i,j} = \frac{s}{|E|} - t^2$ for $i \ne j$. Let $J$ be the $n \times n$ matrix of all 1's, then

$$Q = \left(t - \frac{s}{|E|}\right) I + \left(\frac{s}{|E|} - t^2\right) J$$

which has eigenvalues,

$$\left\{t - \frac{s}{|E|}, (n-1)\frac{s}{|E|} + t - nt^2\right\}.$$

Thus, $Q$ is positive semidefinite if and only if

$$\frac{nt^2 - t}{n-1} \leq \frac{s}{|E|} \leq t.$$

Combining this condition with the constraint in (4.2.2) and observing that $\frac{nt^2-t}{n-1} \leq t$ for $0 \leq t \leq 1$, we arrive at

$$\max\left\{0, \frac{nt^2 - t}{n-1}\right\} \leq \frac{s}{|E|}, \text{ when } 0 \leq t \leq \frac{1}{2},$$
$$\max\left\{2t-1, \frac{nt^2 - t}{n-1}\right\} \leq \frac{s}{|E|}, \text{ when } \frac{1}{2} \leq t \leq 1.$$

Simplifying this proves the proposition. □

Before we prove our main result of this chapter, we digress to an interesting problem which arose while trying to understand certain universal $C^*$-algebras related to the quantum correlation sets. Understanding this problem (and its solution) is key to Theorem 4.3.5.

**Problem 4.3.2.** Fix $n \in \mathbb{N}$. Determine the set $\Sigma_n$ of all real numbers $\alpha$ such that there exist $n$ projections $P_1, ..., P_n$ in some Hilbert space $\mathcal{H}$ such that $P_1 + ... + P_n = \alpha I_{\mathcal{H}}$.

It is not hard to establish some elementary properties of the set $\Sigma_n$. For example,

(a) $\{0, 1, ..., n\} \subseteq \Sigma_n \subseteq [0, n]$, and,

(b) $\alpha \in \Sigma_n$ if and only if $n - \alpha \in \Sigma_n$.

It is also pretty straightforward to find $\Sigma_n$ for $n = 1, 2, 3$. In fact,

$$\Sigma_1 = \{0, 1\}, \qquad \Sigma_2 = \{0, 1, 2\}, \qquad \Sigma_3 = \left\{0, 1, \frac{3}{2}, 2, 3\right\}.$$

However, it is not so simple to find what $\Sigma_n$ is for $n \geq 4$. It turns out that this problem has been studied deeply in [44, 63, 64] together with representations of algebras generated by such projections. For our purpose, we merely state the results that we shall use. It is fascinating to notice that for $1 \le n \le 3$, the set $\Sigma_n$ is finite; for $n = 4$ the set is countably infinite; however, for $n \ge 5$, the set $\Sigma_n$ contains a non-degenerate interval, and this is why Theorem 4.3.5 works.

**Theorem 4.3.3** ([44]). *Let $n \ge 4$. Then*

$$\Sigma_n = \left\{ \Lambda_n^1, \Lambda_n^2, \left[ \frac{n - \sqrt{n(n-4)}}{2}, \frac{n + \sqrt{n(n-4)}}{2} \right], n - \Lambda_n^1, n - \Lambda_n^2 \right\}, \tag{4.3.1}$$

*where $\Lambda_n^1, \Lambda_n^2$ are discrete sets which lie in the interval $\left[0, \frac{n-\sqrt{n(n-4)}}{2}\right)$. In particular, when $n \ge 5$, the interval $\left[\frac{n-\sqrt{n(n-4)}}{2}, \frac{n+\sqrt{n(n-4)}}{2}\right]$ is non-degenerate.*

**Theorem 4.3.4** ([44]). *Let $n \ge 5$ and let $\alpha \in \Sigma_n \cap \left[\frac{n-\sqrt{n(n-4)}}{2}, \frac{n+\sqrt{n(n-4)}}{2}\right]$. Then there exist projections $P_1, ..., P_n$ in a finite-dimensional Hilbert space $\mathcal{H}$ such that $\sum_{i=1}^n P_i = \alpha I_{\mathcal{H}}$ if and only if $\alpha$ is rational.*

With these results in hand we now proceed to prove our key result.

**Theorem 4.3.5.** *The synchronous correlation set $C_q^s(5,2)$ is not closed.*

*Proof.* Consider the complete graph $G = K_5$ on five vertices. By Proposition 4.3.1 we know that

$$f_{vect}(t) = \begin{cases} 0, & \text{if } 0 \le t \le \frac{1}{5}, \\ 5t(5t-1), & \text{if } \frac{1}{5} \le t \le \frac{4}{5}, \\ 20(2t-1), & \text{if } \frac{4}{5} \le t \le 1. \end{cases}$$

Notice that $f_{vect}(t)$ is quadratic in $t$ on the interval $\left[\frac{1}{5}, \frac{4}{5}\right]$. We show that $f_q(t) = f_{vect}(t) = 5t(5t-1)$ for all rational $t \in \left[\frac{\sqrt{5}-1}{2\sqrt{5}}, \frac{\sqrt{5}+1}{2\sqrt{5}}\right] \subset \left[\frac{1}{5}, \frac{4}{5}\right]$. This will imply that $f_q$ cannot be linear on any non-degenerate subinterval of $\left[\frac{\sqrt{5}-1}{2\sqrt{5}}, \frac{\sqrt{5}+1}{2\sqrt{5}}\right]$, so that, by Proposition 4.2.5, it will follow that the value of $f_q(t)$ is not attained for any irrational $t$ in that interval. In this case, $C_q^s(5,2)$ cannot be closed.

From (4.1.3), we have $f_q(t) \ge f_{vect}(t) = 5t(5t-1)$ when $t \in [\frac{1}{5}, \frac{4}{5}]$. Suppose $t \in \left[\frac{\sqrt{5}-1}{2\sqrt{5}}, \frac{\sqrt{5}+1}{2\sqrt{5}}\right]$ and $t$ is rational. We will show $f_q(t) \le 5t(5t-1)$. Since $5t \in \left[\frac{5-\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}\right] \cap \mathbb{Q}$, by Theorem 4.3.4, it follows that there exist five projections $P_1, \dots, P_5 \in \mathbb{M}_k$ for some natural number $k$, such that $P_1 + \cdots + P_5 = 5t\mathbb{I}_k$. Define

$$\widetilde{P}_i = P_i \oplus P_{i+1} \oplus \cdots \oplus P_{i+4} \in \mathbb{M}_k \oplus \mathbb{M}_k \oplus \cdots \oplus \mathbb{M}_k \subseteq \mathbb{M}_{5k}.$$

Clearly $\sum_{j=1}^5 \widetilde{P}_j = 5t\mathbb{I}_{5k}$, and also notice that if $\mathrm{tr}_{5k}$ denotes the normalised trace on $\mathbb{M}_{5k}$, then

$$\mathrm{tr}_{5k}(\widetilde{P}_i) = \frac{1}{5k}\mathrm{Tr}(\widetilde{P}_i) = \frac{1}{5k}\sum_{j=1}^{5}\mathrm{Tr}(P_j) = \frac{1}{5k}\mathrm{Tr}\left(\sum_{j=1}^{5} P_j\right) = \frac{1}{5k}(5tk) = t.$$

Therefore, we have five projections $\widetilde{P}_1, \dots, \widetilde{P}_5 \in \mathbb{M}_{5k}$ such that $\mathrm{tr}_{5k}(\widetilde{P}_i) = t$, for all $1 \le i \le 5$, and $\sum_{j=1}^5 \widetilde{P}_j = 5t\mathbb{I}_{5k}$. Squaring the sum, we get $\sum_{i \neq j} \widetilde{P}_i\widetilde{P}_j = 5t(5t-1)\mathbb{I}_{5k}$, which, upon taking the normalised trace, yields

$$\sum_{i \neq j} \mathrm{tr}_{5k}(\widetilde{P}_i\widetilde{P}_j) = 5t(5t-1).$$

This implies $f_q(t) = 5t(5t-1)$ for all $t \in \left[\frac{\sqrt{5}-1}{2\sqrt{5}}, \frac{\sqrt{5}+1}{2\sqrt{5}}\right] \cap \mathbb{Q}$, completing the proof. □

**Remark 4.3.6.** Examining the above proof, we can write down an explicit element of $C_{qa}(5,2)$ that is not an element of $C_q(5,2)$. Indeed, let $t$ be an irrational element of the interval $\left[\frac{\sqrt{5}-1}{2\sqrt{5}}, \frac{\sqrt{5}+1}{2\sqrt{5}}\right]$. Working with the complete graph $K_5$, since $f_{qa}(t) = f_q(t) = 5t(5t-1)$, by Proposition 4.2.2 and since $\widetilde{\Gamma}_{qa}(t)$ is closed, there exists

$$p = (p(i,j|v,w)) \in \widetilde{\Gamma}_{qa}(t) \subseteq C_{qa}(5,2),$$

such that $p_A(0|v) = p_B(0|w) = t$ for all $v, w \in V$ and $p(0,0|v,w) = \frac{t}{4}(5t-1)$ for all $v, w \in V$ with $v \neq w$. Now using Remark 4.2.4, we calculate: if $v = w$, then

$$p(0,0|v,w) = t, \qquad p(0,1|v,w) = p(1,0|v,w) = 0, \qquad p(1,1|v,w) = 1-t,$$

while if $v \neq w$, then

$$p(0,0|v,w) = \frac{1}{4}t(5t-1), \qquad p(0,1|v,w) = p(1,0|v,w) = \frac{5}{4}t(1-t),$$
$$p(1,1|v,w) = \frac{1}{4}(1-t)(4-5t).$$

However, since the value $f_q(t)$ is not attained in the infimum defining it, we have $p \notin C_q(5,2)$.

**Corollary 4.3.7.** *The sets $C_q(5,2)$ and $C_{qs}(5,2)$ are not closed, and $C_{qs}(5,2) \neq C_{qa}(5,2)$.*

*Proof.* It is easily seen that if $C_q(5,2)$ were closed then necessarily the subset of synchronous quantum correlations would be closed. Hence, $C_q(5,2)$ is not closed.

Similar reasoning shows that if $C_{qs}(5,2)$ were closed, then $C^s_{qs}(5,2)$ would be closed. But Theorem 3.10 of [43] shows that $C^s_{qs}(5,2) = C^s_q(5,2)$, and so $C_{qs}(5,2)$ is not closed.

The last claim follows from the fact that $C_{qa}(5,2)$ is closed. $\square$

# Chapter 5

# Entanglement breaking rank

In this chapter we move away from the study of quantum correlations and focus on a parameter of entanglement breaking maps called the entanglement breaking rank, and the existence problem of symmetric informationally complete POVMs. The results mentioned in this chapter became the basis of [56]. For a background on quantum channels and their representations, we refer the reader to [76, Chapter 2].

The study of separable states is an important topic in quantum information theory and helps to shed light on the nature of entanglement. Recall that, a state $\rho \in \mathbb{M}_m \otimes \mathbb{M}_n$ is called *separable* if it can be written as a finite convex combination $\rho = \sum_i \lambda_i \sigma_i \otimes \delta_i$, where $\sigma_i$ and $\delta_i$ are pure states. In general, a separable state can have many such representations [38]. It is then natural to introduce the notion of *optimal ensemble cardinality* [18] or *length* [9] of a separable state by defining it to be the minimum number $\ell(\rho)$ of pure states $\sigma_i \otimes \delta_i$ required to write the separable state $\rho$ as their convex combination.

The notion of entanglement breaking maps was introduced and studied in [37, 67]. We say that a linear map $\Phi : \mathbb{M}_d \to \mathbb{M}_m$ is *entanglement breaking* if the tensor product $\Phi \otimes \mathcal{I}_n$ maps states of $\mathbb{M}_d \otimes \mathbb{M}_n$ to separable states in $\mathbb{M}_m \otimes \mathbb{M}_n$, for all $n \in \mathbb{N}$. An equivalent criterion of entanglement breaking [37] is that $\Phi$ admits a Choi-Kraus representation of the form

$$\Phi(X) = \sum_{k=1}^{K} R_k X R_k^*, \qquad X \in \mathbb{M}_d, \tag{5.0.1}$$

where the $R_k$'s are rank one matrices in $\mathbb{M}_{m,d}$. Since Choi-Kraus representations are not unique, we define the *entanglement breaking rank* of $\Phi$ to be the minimum $K$ required in such an expression (5.0.1), and denote it by $\mathrm{ebr}(\Phi)$. Clearly, the entanglement breaking rank of $\Phi$ is never less than its Choi rank, since the Choi rank is the minimum $K$ required but without the restriction on $R_k$'s being rank one.

Another equivalent criterion [37] for a map to be entanglement breaking is that its Choi-matrix $C_\Phi := [\Phi(E_{i,j})] \in \mathbb{M}_d \otimes \mathbb{M}_m$ should be separable, where $\{E_{i,j} : 1 \leq i, j \leq d\}$ is the set of canonical matrix units of $\mathbb{M}_d$. If $\Phi$ is entanglement breaking, then it is easy to see that $\ell(C_\Phi) = \mathrm{ebr}(\Phi)$ so that studying entanglement breaking rank is the same as studying length. However, for our purposes it is more natural to study channels instead of states, so we will think in terms of entanglement breaking rank instead of length.

On the other hand, the existence of a *symmetric informationally complete POVM* (SIC POVM) for an arbitrary dimension $d \geq 2$ is a major unsolved problem in quantum information theory and is an active area of research, beginning with [77, 78, 66]. SIC POVMs have found applications in quantum state tomography, quantum cryptography and foundations of quantum mechanics. For a sample and the extent of numerical evidence we refer the reader to [8, 27, 28, 71, 66, 1, 69]. Stated in terms of equiangular lines, the problem asks about the existence of $d^2$ unit vectors $\{v_i : 1 \leq i \leq d^2\} \subseteq \mathbb{C}^d$ such that $|\langle v_i, v_j\rangle|^2 = \frac{1}{d+1}$ for all $i \neq j$.

Assuming the existence of a SIC POVM $\{v_i\}_{i=1}^{d^2} \subset \mathbb{C}^d$ in a dimension $d$, it can be shown [7] that if $P_i$ are the rank one projections onto the span of $v_i$, then,

$$\frac{1}{d}\sum_{i=1}^{d^2} P_i X P_i = \frac{1}{d+1}(X + \mathrm{Tr}(X)\mathbb{I}_d), \qquad X \in \mathbb{M}_d.$$

Denoting this channel by $\mathfrak{Z}_d$, it is independently known to be entanglement breaking and its Choi-rank is $d^2$ [46, 36]. Thus, if a SIC POVM exists in dimension $d$, then the entanglement breaking rank of the channel $\mathfrak{Z}_d$ is $d^2$.

We prove the converse: if $\mathrm{ebr}(\mathfrak{Z}_d) = d^2$, then there exists a SIC POVM in dimension $d$. This is a slight relaxation of the SIC POVM conditions, since the $d^2$ rank one matrices are not a priori assumed to be positive. Although our proof will show that they are necessarily positive. This leads to the conclusion (Corollary 5.2.7) that for $d \geq 2$, $\mathrm{ebr}(\mathfrak{Z}_d) = d^2$ if and only if a SIC POVM exists in dimension $d$.

We collect necessary definitions and results in Section 5.1 and prove the equivalence result in Section 5.2.

We close this discussion with a remark pointed out by John Watrous. A similar kind of study was undertaken in [42], where the author studies a relationship between the quantumness of a set of states with the cardinality of the set. The notion of quantumness for a set of states was defined by [27] which we reproduce here. If $\{\psi_i : 1 \leq i \leq N\} \subseteq \mathbb{C}^d$ is a set of states in a $d$-dimensional Hilbert space, we define its quantumness as

$$Q_{\{\psi_i:1\leq i\leq N\}} = \inf_{\pi} \sup \sum_{i,j} \pi_i \langle E_j \psi_i, \psi_i \rangle \langle \sigma_j \psi_i, \psi_i \rangle ,$$

where the infimum runs over all probability distributions $\pi$, and the supremum runs over all over POVMs $\{E_j : 1 \leq j \leq J\}$ and density matrices $\{\sigma_j : 1 \leq j \leq J\}$. It was shown in [27], that

$$\frac{2}{d+1} \leq Q_{\{\psi_i:1\leq i\leq N\}} \leq 1,$$

and moreover, that a SIC POVM in dimension $d$ achieves the lower bound. The authors asked whether this lower bound can be achieved by a set of states with cardinality less than $d^2$. Kim [42] showed that the answer is no by showing that at least $d^2$ states are always needed to achieve the minimal quantumness, and the states in that collection must form a SIC POVM.

## 5.1 Preliminaries

We begin with the definition of a symmetric informationally complete POVM (SIC POVM). Recall the definition of a POVM in Definition 2.1.11.

**Definition 5.1.1.** A POVM $\{R_i\}_{i=1}^{d^2} \subseteq \mathbb{M}_d$ is called a *symmetric informationally complete POVM* if it satisfies the following conditions:

(a) The POVM is symmetric: $\langle R_i, R_i \rangle_2$ is constant for all $1 \leq i \leq d^2$, and $\langle R_i, R_j \rangle_2$ is constant for all $i \neq j$. (Recall that $\langle ., . \rangle_2$ is the Hilbert-Schmidt inner product on $\mathbb{M}_d$.)

(b) The POVM is informationally complete: $\text{span}\{R_i : 1 \leq i \leq d^2\} = \mathbb{M}_d$, and,

(c) $\text{rank}(R_i) = 1$ for all $1 \leq i \leq d^2$.

The following proposition gives a characterisation of SIC POVMs in terms of equiangular unit vectors.

**Proposition 5.1.2** ([78])**.** *A set of operators* $\{R_i\}_{i=1}^{d^2} \subseteq \mathbb{M}_d$ *is a SIC POVM if and only if there exist* $d^2$ *unit vectors* $\{w_i\}_{i=1}^{d^2} \subseteq \mathbb{C}^d$ *such that* $R_i = \frac{1}{d} w_i w_i^*$ *for each* $1 \le i \le d^2$*, and* $|\langle w_i, w_j \rangle|^2 = \frac{1}{d+1}$ *for all* $i \neq j$*.*

Because of Proposition 5.1.2, we shall call a set $\{w_i\}_{i=1}^{d^2} \subseteq \mathbb{C}^d$ of $d^2$ unit vectors a SIC POVM if $|\langle w_i, w_j \rangle|^2 = \frac{1}{d+1}$ for all $i \neq j$.

The constant $\frac{1}{d+1}$ appearing in Proposition 5.1.2 can be derived independently.

**Proposition 5.1.3** ([78])**.** *If* $\{w_i\}_{i=1}^{d^2} \subseteq \mathbb{C}^d$ *is a set of unit vectors such that* $|\langle w_i, w_j \rangle|^2 = k$ *for all* $i \neq j$*, then* $k = \frac{1}{d+1}$*.*

We now state the SIC POVM existence conjecture in terms of equiangular vectors, which we shall call *Zauner's conjecture*.

**Conjecture 5.1.4** (Zauner's conjecture)**.** For each positive integer $d \ge 2$, there exist $d^2$ unit vectors $\{w_i\}_{i=1}^{d^2}$ in $\mathbb{C}^d$ such that $|\langle w_i, w_j \rangle|^2 = \frac{1}{d+1}$ for all $i \neq j$.

The existence of the SIC POVM for all dimensions $d \ge 2$ is still an open question. However, in some specific dimensions ($d = 1 - 21, 24, 28, 30, 31, 35$ etc.), analytic solutions have been found. See [70] for the latest progress in finding these solutions.

We now collect some facts about quantum channels. We refer the reader to [10, 76] for more details. Recall that a linear map $\Omega : \mathbb{M}_d \to \mathbb{M}_m$ is completely positive and trace preserving (hence a *quantum channel*) if and only if $\Omega$ can be expressed as

$$\Omega(X) = \sum_{k=1}^{K} B_k X B_k^*, \qquad X \in \mathbb{M}_d, \tag{5.1.1}$$

for some $B_1, ..., B_K \in \mathbb{M}_{m,d}$ with $\sum_{k=1}^{K} B_k^* B_k = \mathbb{I}_d$. The expression (5.1.1) is called a *Choi-Kraus representation* of the quantum channel $\Omega$. Choi-Kraus representations of a quantum channel are not unique, and therefore the minimum $K$ possible in (5.1.1) is called the *Choi-rank* of $\Omega$. The Choi-rank of $\Omega$ is equal to the rank of the $dm \times dm$ *Choi-matrix* $[\Omega(E_{i,j})]_{i,j=1}^d$, where $\{E_{i,j} : 1 \le i, j \le d\}$ are the canonical matrix units of $\mathbb{M}_d$. If the value of $K$ in expression (5.1.1) equals the Choi-rank of $\Omega$, then $\{B_k\}_{k=1}^K \subseteq \mathbb{M}_{m,d}$ is a linearly independent set.

We identify two simple quantum channels for our purpose.

**Definition 5.1.5.** The quantum channel $\mathcal{I}_d : \mathbb{M}_d \to \mathbb{M}_d$ defined by $\mathcal{I}_d(X) = X$, for all $X \in \mathbb{M}_d$, is called the *identity channel*. The quantum channel $\Psi_d : \mathbb{M}_d \to \mathbb{M}_d$ defined by $\Psi_d(X) = \frac{1}{d}\operatorname{Tr}(X)\mathbb{I}_d$, for all $X \in \mathbb{M}_d$, is called the *completely depolarizing channel*. For $t \in [0,1]$, we define $\Phi_t : \mathbb{M}_d \to \mathbb{M}_d$ to be $\Phi_t = t\mathcal{I}_d + (1-t)\Psi_d$. When $t = \frac{1}{d+1}$, we set,

$$\mathfrak{Z}_d := \Phi_{\frac{1}{d+1}} = \frac{1}{d+1}\mathcal{I}_d + \frac{d}{d+1}\Psi_d.$$

A Choi-Kraus representation of the identity channel $\mathcal{I}_d$ is simply $\mathcal{I}_d(X) = \mathbb{I}_d X \mathbb{I}_d$, and hence its Choi-rank is 1. Since $C_{\Psi_d} = [\Psi_d(E_{i,j})] = \frac{1}{d}\mathbb{I}_{d^2}$, it follows that the Choi-rank of the completely depolarizing channel $\Psi_d$ is $d^2$.

Since the set of quantum channels $\Omega : \mathbb{M}_d \to \mathbb{M}_d$ forms a convex set in the space $\mathcal{L}(\mathbb{M}_d, \mathbb{M}_d)$, it follows that the map $\Phi_t = t\mathcal{I}_d + (1-t)\Psi_d$ is also a quantum channel for every $t \in [0,1]$. When $t \in (0,1)$, it is easy to see that the Choi-rank of $\Phi_t$ is $d^2$. Indeed, notice that the Choi-matrix of $\Phi_t$ is $C_{\Phi_t} = t[E_{i,j}] + \frac{1-t}{d}\mathbb{I}_{d^2}$. Let $\xi \in \mathbb{C}^{d^2}$. If $C_{\Phi_t}(\xi) = 0$, then $[E_{i,j}]\xi = -\frac{1-t}{td}\xi$. But the only eigenvalues of $[E_{i,j}]$ are 0 and $d$, which implies that $\xi = 0$. Thus $\operatorname{rank}(C_{\Phi_t}) = d^2$.

We now define an entanglement breaking map. Recall that a state $\rho \in \mathbb{M}_m \otimes \mathbb{M}_n$ is called *separable* if it can be expressed as $\rho = \sum_{i=1}^r \lambda_i x_i x_i^* \otimes y_i y_i^*$, where $x_i$ and $y_i$ are unit vectors in $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively, and $\lambda_i \geq 0$ with $\sum_{i=1}^r \lambda_i = 1$. If a state is not separable it is called *entangled*.

**Definition 5.1.6** ([37])**.** A linear map $\Phi : \mathbb{M}_d \to \mathbb{M}_m$ is called *entanglement breaking* if for all $n \in \mathbb{N}$, the tensor product map

$$\Phi \otimes \mathcal{I}_n : \mathbb{M}_d \otimes \mathbb{M}_n \to \mathbb{M}_m \otimes \mathbb{M}_n$$

maps all states (entangled or not) in $\mathbb{M}_d \otimes \mathbb{M}_n$ to separable states in $\mathbb{M}_m \otimes \mathbb{M}_n$.

The set of all entanglement breaking maps $\Phi : \mathbb{M}_d \to \mathbb{M}_m$ forms a convex set in the space of all linear maps $\mathcal{L}(\mathbb{M}_d, \mathbb{M}_m)$. The following theorem lists some equivalent conditions on a map to be entanglement breaking.

**Theorem 5.1.7** ([37])**.** *Let $\Phi : \mathbb{M}_d \to \mathbb{M}_m$ be a linear map. Then the following are equivalent.*

*(a) The map $\Phi$ is entanglement breaking.*

*(b) The Choi matrix of $\Phi$ is separable.*

*(c) The map $\Phi$ has a Choi-Kraus representation*

$$\Phi(X) = \sum_{i=1}^{I} A_i X A_i^*, \qquad X \in \mathbb{M}_d,$$

*where each $A_i$ is a rank-one operator.*

We end this section with the following known result which characterizes when the maps $\Phi_t = t\mathcal{I}_d + (1-t)\Psi_d$ as in Definition 5.1.5 are entanglement breaking.

**Proposition 5.1.8** ([36, 46])**.** *Let $t \in \mathbb{R}$. The map $\Phi_t = t\mathcal{I}_d + (1-t)\Psi_d$ is entanglement breaking if and only if $\frac{-1}{d^2-1} \leq t \leq \frac{1}{d+1}$.*

## 5.2 Entanglement breaking rank and SIC POVMs

**Definition 5.2.1.** Let $\Phi : \mathbb{M}_d \to \mathbb{M}_m$ be an entanglement breaking map. The *entanglement breaking rank* of $\Phi$, denoted by $\mathrm{ebr}(\Phi)$, is the minimum number of rank-one operators $A_k$ required when $\Phi$ is written in the form $\Phi(X) = \sum_{k=1}^{K} A_k X A_k^*$.

Let $\mathrm{cr}(\Phi)$ denote the Choi-rank of a entanglement breaking channel $\Phi : \mathbb{M}_d \to \mathbb{M}_m$. We have the following simple estimate: $\mathrm{cr}(\Phi) \leq \mathrm{ebr}(\Phi)$.

By Proposition 5.1.8, it follows that the channel $\mathfrak{Z}_d$ is an entanglement breaking channel, and hence it has a Choi-Kraus representation consisting of rank-one Choi-Kraus operators. Zauner's conjecture can then be related to the problem of obtaining a *minimal* Choi-Kraus representation of the quantum channel $\mathfrak{Z}_d$ consisting of rank-one operators. First we establish a weaker proposition.

**Proposition 5.2.2.** *Zauner's conjecture is true if and only if for each positive integer $d \geq 2$, the quantum channel $\mathfrak{Z}_d$ has a Choi-Kraus representation $\mathfrak{Z}_d(X) = \sum_{i=1}^{d^2} R_i X R_i$, for all $X \in \mathbb{M}_d$, where each $R_i$ is a rank-one positive operator.*

*Proof.* The forward implication is a known result [7]. However, we include a proof for the sake of completeness.

Suppose that Zauner's conjecture is true. Then for each positive integer $d \geq 2$, there exist $d^2$ unit vectors $\{w_i\}_{i=1}^{d^2}$ in $\mathbb{C}^d$ such that $|\langle w_i, w_j\rangle|^2 = \frac{1}{d+1}$ for all $i \neq j$. By Proposition

5.1.2, the set $\left\{\frac{1}{d}w_iw_i^*\right\}_{i=1}^{d^2} \subseteq \mathbb{M}_d$ forms a SIC POVM. Set $R_i = \frac{1}{d}w_iw_i^*$ for each $1 \leq i \leq d^2$, and define a map $\Phi : \mathbb{M}_d \to \mathbb{M}_d$ by

$$\Phi(X) = d\sum_{i=1}^{d^2} R_iXR_i, \qquad X \in \mathbb{M}_d.$$

Then $\Phi$ is a unital quantum channel since

$$d\sum_{i=1}^{d^2} R_i^*R_i = d\sum_{i=1}^{d^2} \frac{1}{d}R_i = \sum_{i=1}^{d^2} R_i = \mathbb{I}_d.$$

The set $\{R_j\}_{j=1}^{d^2}$ being informationally complete spans $\mathbb{M}_d$, and so each $X \in \mathbb{M}_d$ may be written as $X = \sum_{j=1}^{d^2} r_jR_j$ for some unique scalars $r_j$. Taking trace on both sides of $X = \sum_{j=1}^{d^2} r_jR_j$ yields

$$\sum_{j=1}^{d^2} r_j = d\,\mathrm{Tr}(X). \tag{5.2.1}$$

Next observe that

$$\begin{aligned}\Phi(R_j) &= d\left(R_j^3 + \sum_{\substack{1\leq i\leq d^2\\ i\neq j}} R_iR_jR_i\right) = d\left(\frac{1}{d^2}R_j + \frac{1}{d^2(d+1)}\sum_{\substack{1\leq i\leq d^2\\ i\neq j}} R_i\right)\\ &= d\left(\frac{1}{d^2}R_j + \frac{1}{d^2(d+1)}(\mathbb{I}_d - R_j)\right) = \frac{1}{d+1}\left(R_j + \frac{1}{d}\mathbb{I}_d\right),\end{aligned}$$

which implies

$$\begin{aligned}\Phi(X) &= \sum_{j=1}^{d^2} r_j\Phi(R_j) = \frac{1}{d+1}\sum_{j=1}^{d^2} r_jR_j + \frac{\mathbb{I}_d}{d(d+1)}\sum_{j=1}^{d^2} r_j\\ &= \frac{1}{d+1}X + \frac{\mathbb{I}_d}{d(d+1)}d\,\mathrm{Tr}(X) = \frac{1}{d+1}\mathcal{I}_d(X) + \frac{d}{d+1}\Psi_d(X),\end{aligned}$$

where we used Equation (5.2.1) in the third equality. Hence $\Phi = \mathfrak{Z}_d$.

Conversely, suppose that $\mathfrak{Z}_d$ has a Choi-Kraus representation, $\mathfrak{Z}_d(X) = \sum_{i=1}^{d^2} R_i X R_i$, where each $R_i$ is a rank-one positive operator, $R_i = v_i v_i^*$ for some $v_i \in \mathbb{C}^d$. Since the channel is unital, we have

$$\mathbb{I}_d = \sum_{i=1}^{d^2} R_i^2 = \sum_{i=1}^{d^2} \|v_i\|^2 R_i.$$

Using this, on one hand we have

$$\begin{aligned}\mathfrak{Z}_d(R_j) &= \frac{1}{d+1}(R_j + \|v_j\|^2 \mathbb{I}_d) = \frac{1}{d+1}\left(R_j + \sum_{i=1}^{d^2} \|v_j\|^2 \|v_i\|^2 R_i\right) \\ &= \frac{1}{d+1}\left((1+\|v_j\|^4)R_j + \sum_{\substack{1\le i\le d^2 \\ i\neq j}} \|v_i\|^2\|v_j\|^2 R_i\right),\end{aligned}$$

and on the other hand,

$$\mathfrak{Z}_d(R_j) = \sum_{i=1}^{d^2} R_i R_j R_i = R_j^3 + \sum_{\substack{1\le i\le d^2 \\ i\neq j}} R_i R_j R_i = \|v_j\|^4 R_j + \sum_{\substack{1\le i\le d^2 \\ i\neq j}} |\langle v_i, v_j\rangle|^2 R_i.$$

Comparing $\mathfrak{Z}_d(R_j)$ obtained in two ways, we get

$$\left(\frac{1+\|v_j\|^4}{d+1} - \|v_j\|^4\right) R_j + \sum_{\substack{1\le i\le j \\ i\neq j}} \left(\frac{\|v_i\|^2\|v_j\|^2}{d+1} - |\langle v_i, v_j\rangle|^2\right) R_i = 0.$$

Since $\{R_i\}_{i=1}^{d^2}$ is a linearly independent set (because number of Choi-Kraus operators equals the Choi-rank; see the discussion after Definition 5.1.5), we must have

$$\frac{1+\|v_j\|^4}{d+1} - \|v_j\|^4 = 0, \qquad \frac{\|v_i\|^2\|v_j\|^2}{d+1} - |\langle v_i, v_j\rangle|^2 = 0.$$

The first one yields, $\|v_j\|^4 = \frac{1}{d}$, which is constant for all $1 \le j \le d^2$, and using this the second one yields $|\langle v_i, v_j\rangle|^2 = \frac{1}{d(d+1)}$, for all $i \neq j$. Then it is easy to see that the normalised vectors $w_i = \frac{v_i}{\|v_i\|}$ satisfy $|\langle w_i, w_j\rangle|^2 = \frac{1}{d+1}$ for all $i \neq j$, so that Zauner's conjecture holds. $\square$

We can drop the positivity condition on $R_i$ in Proposition 5.2.2.

**Theorem 5.2.3.** *Zauner's conjecture is true if and only if for each positive integer $d \geq 2$, the quantum channel $\mathfrak{Z}_d$ has a Choi-Kraus representation $\mathfrak{Z}_d(X) = \sum_{i=1}^{d^2} R_i X R_i^*$, for all $X \in \mathbb{M}_d$, where each $R_i$ is a rank-one operator.*

*Proof.* We only have to show the converse part. Suppose that the quantum channel $\mathfrak{Z}_d$ has a Choi-Kraus representation given by $\mathfrak{Z}_d(X) = \sum_{i=1}^{d^2} R_i X R_i^*$, where each $R_i$ is a rank one operator, $R_i = x_i y_i^*$, for some vectors $x_i, y_i \in \mathbb{C}^d$. Without loss of generality, we may assume that each $y_i$ is a unit vector. Since $\mathfrak{Z}_d$ is unital, we have

$$\mathbb{I}_d = \mathfrak{Z}_d(\mathbb{I}_d) = \sum_{i=1}^{d^2} R_i R_i^* = \sum_{i=1}^{d^2} (x_i y_i^*)(x_i y_i^*)^* = \sum_{i=1}^{d^2} x_i y_i^* y_i x_i^* = \sum_{i=1}^{d^2} x_i x_i^*. \tag{5.2.2}$$

Since the Choi rank of $\mathfrak{Z}_d$ is $d^2$, it follows that the set $\{R_i\}_{i=1}^{d^2}$ is a basis for $\mathbb{M}_d$. We prove the following claims.

**Claim 5.2.4.** *The set of matrices $\{x_i x_i^*\}_{i=1}^{d^2}$ is also a basis for $\mathbb{M}_d$.*

*Proof.* Since the set $\{x_i x_i^*\}_{i=1}^{d^2}$ contains $d^2$ elements and since $\{R_j\}_{j=1}^{d^2}$ form a basis for $\mathbb{M}_d$, it is enough to show that each $R_j$ can be expressed as a linear combination of elements in $\{x_i x_i^*\}_{i=1}^{d^2}$. To this end, notice that for $1 \leq j \leq d^2$,

$$\mathfrak{Z}_d(R_j) = \frac{1}{d+1}\left(R_j + \mathrm{Tr}(R_j)\mathbb{I}_d\right) = \frac{1}{d+1}\left(R_j + \langle x_j, y_j\rangle \left(\sum_{i=1}^{d^2} x_i x_i^*\right)\right),$$

where we used Equation (5.2.2). On the other hand, we have

$$\mathfrak{Z}_d(R_j) = \sum_{i=1}^{d^2} R_i R_j R_i^* = \sum_{i=1}^{d^2} (x_i y_i^*)(x_j y_j^*)(x_i y_i^*)^* = \sum_{i=1}^{d^2} \langle x_j, y_i\rangle \langle y_i, y_j\rangle x_i x_i^*.$$

Comparing the two expressions of $\mathfrak{Z}_d(R_j)$, we get

$$\frac{1}{d+1}\left(R_j + \langle x_j, y_j\rangle \left(\sum_{i=1}^{d^2} x_i x_i^*\right)\right) = \sum_{i=1}^{d^2} \langle x_j, y_i\rangle \langle y_i, y_j\rangle x_i x_i^*,$$

which implies

$$R_j = \sum_{i=1}^{d^2} \left((d+1)\langle x_j, y_i\rangle \langle y_i, y_j\rangle - \langle x_j, y_j\rangle\right) x_i x_i^*,$$

completing the proof of the claim. $\square$

**Claim 5.2.5.** *For all $1 \le j \le d^2$, we have $\|x_j\|^2 = \frac{1}{d}$.*

*Proof.* For $1 \le j \le d^2$, using Equation (5.2.2) we have

$$\mathfrak{Z}_d(x_j x_j^*) = \frac{1}{d+1}\left(x_j x_j^* + \mathrm{Tr}(x_j x_j^*)\mathbb{I}_d\right) = \frac{1}{d+1}\left(x_j x_j^* + \|x_j\|^2 \left(\sum_{i=1}^{d^2} x_i x_i^*\right)\right)$$
$$= \frac{1}{d+1}\left((1+\|x_j\|^2)x_j x_j^* + \sum_{\substack{1\le i\le d^2\\ i\ne j}} \|x_j\|^2 x_i x_i^*\right),$$

and on the other hand, we have

$$\mathfrak{Z}_d(x_j x_j^*) = \sum_{i=1}^{d^2} R_i(x_j x_j^*)R_i^* = \sum_{i=1}^{d^2} x_i y_i^* x_j x_j^* y_i x_i^* = \sum_{i=1}^{d^2} |\langle y_i, x_j\rangle|^2 x_i x_i^*.$$

Comparing the two expressions of $\mathfrak{Z}_d(x_j x_j^*)$, we get

$$\frac{1}{d+1}\left((1+\|x_j\|^2)x_j x_j^* + \sum_{\substack{1\le i\le d^2\\ i\ne j}} \|x_j\|^2 x_i x_i^*\right) = \sum_{i=1}^{d^2} |\langle y_i, x_j\rangle|^2 x_i x_i^*.$$

Because of linear independence of the set $\{x_i x_i^*\}_{i=1}^{d^2}$ from Claim 5.2.4, we have

$$(d+1)|\langle y_j, x_j\rangle|^2 = 1 + \|x_j\|^2, \qquad (d+1)|\langle y_i, x_j\rangle|^2 = \|x_j\|^2, \qquad \forall i \ne j. \tag{5.2.3}$$

Using the Cauchy-Schwarz inequality in the first equation of (5.2.3),

$$1 + \|x_j\|^2 = (d+1)|\langle y_j, x_j\rangle|^2 \le (d+1)\|y_j\|^2\|x_j\|^2 = (d+1)\|x_j\|^2,$$

which implies $\frac{1}{d} \le \|x_j\|^2$. Taking the trace of Equation (5.2.2), we get $\sum_{i=1}^{d^2} \|x_i\|^2 = d$, using which we have

$$\sum_{i=1}^{d^2}\left(\|x_i\|^2 - \frac{1}{d}\right) = \sum_{i=1}^{d^2}\|x_i\|^2 - \sum_{i=1}^{d^2}\frac{1}{d} = d - d^2\frac{1}{d} = 0.$$

Thus $\|x_j\|^2 = \frac{1}{d}$, for all $1 \le j \le d^2$. □

**Claim 5.2.6.** *The unit vectors $\{y_i\}_{i=1}^{d^2}$ satisfy Zauner's conjecture.*

*Proof.* For $1 \leq j \leq d^2$, the first equation of (5.2.3) together with Claim 5.2.5 yields

$$(d+1)|\langle y_j, x_j\rangle|^2 = 1 + \|x_j\|^2 = 1 + \frac{1}{d} = \frac{d+1}{d},$$

which implies that $|\langle y_j, x_j\rangle|^2 = \frac{1}{d}$. Also for $i \neq j$, using the second equation of (5.2.3), we get

$$(d+1)|\langle y_i, x_j\rangle|^2 = \|x_j\|^2 = \frac{1}{d},$$

which implies that $|\langle y_i, x_j\rangle|^2 = \frac{1}{d(d+1)}$. Therefore, for $1 \leq i, j \leq d^2$,

$$|\langle x_i, y_j\rangle|^2 = \begin{cases} \frac{1}{d}, & \text{if } i = j, \\ \frac{1}{d(d+1)}, & \text{if } i \neq j. \end{cases}$$

But

$$\frac{1}{d} = |\langle x_i, y_i\rangle|^2 \leq \|x_i\|^2\|y_i\|^2 = \frac{1}{d},$$

so that equality holds everywhere. This implies (from the equality case in Cauchy-Schwarz) that $x_i = \lambda_i y_i$, for some $\lambda_i \in \mathbb{C}$. Then

$$\frac{1}{d} = |\langle x_i, y_i\rangle|^2 = |\langle \lambda_i y_i, y_i\rangle|^2 = |\lambda_i|^2.$$

Using this when $i \neq j$, we get

$$\frac{1}{d(d+1)} = |\langle x_i, y_j\rangle|^2 = |\lambda_i|^2 |\langle y_i, y_j\rangle|^2 = \frac{1}{d}|\langle y_i, y_j\rangle|^2,$$

which implies $|\langle y_i, y_j\rangle|^2 = \frac{1}{d+1}$. Therefore the unit vectors $\{y_i\}_{i=1}^{d^2}$ satisfy Zauner's conjecture. □

Finally, note that

$$\mathfrak{Z}_d(X) = \sum_{i=1}^{d^2}(x_i y_i^*)X(x_i y_i^*)^* = \sum_{i=1}^{d^2}|\lambda_i|^2(y_i y_i^*)X(y_i y_i^*) = \sum_{i=1}^{d^2}\frac{1}{d}(y_i y_i^*)X(y_i y_i^*),$$

so that $\mathfrak{Z}_d$ has a Choi-Kraus representation with Choi-Kraus operators being rank-one positive operators. □

**Corollary 5.2.7.** *Zauner's conjecture holds if and only if* $\mathrm{ebr}(\mathfrak{Z}_d) = d^2$ *for all* $d \geq 2$.

*Proof.* This follows from Proposition 5.2.2 and Theorem 5.2.3, together with the fact that $\mathrm{cr}(\Phi) = d^2$. □

Interestingly, for $t \in \left[\frac{-1}{d^2-1}, \frac{1}{d+1}\right)$, (see Proposition 5.1.8) the channels $\Phi_t : \mathbb{M}_d \to \mathbb{M}_d$ defined by $\Phi_t = t\mathcal{I}_d + (1-t)\Psi_d$ cannot have a Choi-Kraus representation with $d^2$ *positive* rank-one Choi-Kraus operators, which we prove next.

**Proposition 5.2.8.** *Let* $t \in \left[\frac{-1}{d^2-1}, \frac{1}{d+1}\right)$. *Let* $\Phi_t : \mathbb{M}_d \to \mathbb{M}_d$ *be the quantum channel given by* $\Phi_t = t\mathcal{I}_d + (1-t)\Psi_d$. *Then* $\Phi_t$ *cannot have a Choi-Kraus representation,* $\Phi_t(X) = \sum_{i=1}^{d^2} R_i X R_i^*$, *where each* $R_i$ *is a positive rank-one operator.*

*Proof.* We shall follow the arguments as in Proposition 5.2.2. Suppose $\Phi_t$ has a Choi-Kraus representation, $\Phi_t(X) = \sum_{i=1}^{d^2} R_i X R_i^*$, where $R_i = v_i v_i^*$ for some vector $v_i \in \mathbb{C}^d$. Since $\Phi_t$ is unital, we have $\mathbb{I}_d = \sum_{i=1}^{d^2} \|v_i\|^2 R_i$. Applying $\Phi_t$ on $R_j$,

$$\Phi_t(R_j) = tR_j + \frac{1-t}{d}\,\mathrm{Tr}(R_j)\mathbb{I}_d = \left(t + \frac{1-t}{d}\|v_j\|^4\right) R_j + \sum_{\substack{1\leq i\leq d^2 \\ i\neq j}} \frac{1-t}{d}\|v_j\|^2\|v_i\|^2 R_i,$$

and also

$$\Phi_t(R_j) = \sum_{i=1}^{d^2} R_i R_j R_i^* = \|v_j\|^4 R_j + \sum_{\substack{1\leq i\leq d^2 \\ i\neq j}} |\langle v_i, v_j\rangle|^2 R_i.$$

Comparing both the expressions of $\Phi_t(R_j)$ and using the linear independence of $R_i$'s, we get

$$t + \frac{1-t}{d}\|v_j\|^4 = \|v_j\|^4, \qquad \frac{1-t}{d}\|v_j\|^2\|v_i\|^2 = |\langle v_i, v_j\rangle|^2, \quad \forall i \neq j.$$

This implies

$$\|v_j\|^4 = \frac{dt}{d+t-1}, \qquad |\langle v_i, v_j\rangle|^2 = \frac{t(1-t)}{d+t-1}, \quad \forall i \neq j.$$

Let $w_i = \frac{v_i}{\|v_i\|}$. Then for all $i \neq j$, we have

$$|\langle w_i, w_j\rangle|^2 = \frac{1}{\|v_i\|^2\|v_j\|^2}|\langle v_i, v_j\rangle|^2 = \frac{d+t-1}{dt}\frac{t(1-t)}{d+t-1} = \frac{1-t}{d}.$$

But by Corollary 5.1.3, we must have $\frac{1-t}{d} = \frac{1}{1+d}$, which implies that $t = \frac{1}{d+1}$, which is a contradiction. □

## 5.3 Summary

In this summary we outline the work in [56] which developed from the work here.

In Section 5.2 we showed that computing the entanglement breaking rank of the channel $\mathfrak{Z}_d$ for all $d \geq 2$ is equivalent to Zauner's problem. It is natural to ask whether $\mathfrak{Z}_d$ is the only channel which has this property. If $X^T$ denotes the transpose of $X$, we prove in [56] that having $\mathrm{ebr}(\mathfrak{Z}_d^T) = d^2$ for all $d \geq 2$ is also equivalent to Zauner's conjecture. The channel $\mathfrak{Z}_d^T$ is an example of Werner-Holevo channels. What is remarkable with the channel $\mathfrak{Z}_d^T$ is that it is also an example of a map for which the Choi rank is strictly smaller than its entanglement breaking rank [56].

The channel $\mathfrak{Z}_d$ is a particular convex combination of the identity channel and the completely depolarizing channel. We conjecture that, for all $t$ with $0 \leq t \leq \frac{1}{d+1}$, the entanglement breaking rank of each of these channels is $d^2$. In [56], we verify this stronger conjecture in dimensions $d = 2$ and $d = 3$. Moreover, we show, in these dimensions, that there is a continuous family of $d^2$ rank one matrices $R_i : \left[0, \frac{1}{d+1}\right] \to \mathbb{M}_d$ for $1 \leq i \leq d^2$ such that

$$(t\mathcal{I}_d + (1-t)\Psi_d)(X) = \sum_{i=1}^{d^2} R_i(t) X R_i(t)^*, \qquad X \in \mathbb{M}_d.$$

In particular, when $t = \frac{1}{d+1}$, we get a Choi-Kraus representation of $\mathfrak{Z}_d$ consisting of $d^2$ rank one matrices, so that (by Theorem 5.2.3) we get the existence of a SIC POVM in that dimension. We conjecture that such continuous families of rank one matrices exist in all dimensions. Finally, we also show that more generally, the ebr is a lower semicontinuous function.

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