Density and Structure of Homomorphism-Critical Graphs

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Let H be a graph. A graph G is H-critical if every proper subgraph of G admits a homomorphism to H, but G itself does not. In 1981, Jaeger made the following conjecture concerning odd-cycle critical graphs: every planar graph of girth at least $4t$ admits a homomorphism to C_{2t+1} (or equivalently, has a $\frac{2t+1}{t}$-circular colouring). The best known result for the $t = 3$ case states that every planar graph of girth at least 18 has a homomorphism to C_7. We improve upon this result, showing that every planar graph of girth at least 16 admits a homomorphism to C_7. This is obtained from a more general result regarding the density of C_7-critical graphs. Our main result is that if G is a C_7-critical graph with $G \notin \{C_3, C_5\}$, then $e(G) \geq \frac{17v(G)-2}{15}$. Additionally, we prove several structural lemmas concerning graphs that are H-critical, when H is a vertex-transitive non-bipartite graph.

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Chapter 1

Introduction

1.1 Introduction and Notation

Let G be a graph, and let k be a positive integer. A *k-colouring* of G is a function ϕ that assigns to each vertex of G a value in the set $\{1, \ldots, k\}$ such that for each edge uv in G, $\phi(u) \neq \phi(v)$. The *chromatic number* of G, $\chi(G)$, is the least k such that G has a k-colouring. For graphs G and H, a *graph homomorphism* $\phi : G \to H$ is a function that maps $V(G)$ to $V(H)$ such that for each edge $uv \in E(G)$, we have $\phi(u)\phi(v) \in E(H)$. Note a graph G has a k-colouring if and only if it has a homomorphism to K_k, and so the question of whether or not a given graph has a k-colouring is in fact a graph homomorphism problem.

Dirac introduced the concept of colour-criticality in 1951 [4], and since then, colour-critical graphs have been widely studied. A graph G is *k-critical* if its chromatic number is k, and the chromatic number of every proper subgraph of G is strictly less than k. As every graph with chromatic number k contains a k-critical subgraph, it is useful to study k-colourability via colour-critical graphs. In the same vein, it is useful to study graph homomorphisms through homomorphism-critical graphs, which we define as follows.

Definition 1.1.1. *Let H be a graph. A graph G is H-*critical *if every proper subgraph of G admits a homomorphism to H, but G itself does not.*

For instance, every odd cycle is K_2-critical, and the graphs found in Figure 4.1 are C_7-critical. Furthermore, if both m and n are odd positive integers and $m < n$, then C_m is C_n-critical.

In the following pages, we will focus the bulk of our attention on graphs that are C_{2t+1}-critical for some integer $t \geq 1$. Much as a graph is K_{k-1}-critical if and only if it is k-critical, graphs that are C-critical for some cycle C are also colour-critical for a variant on ordinary vertex colouring called circular colouring. A short introduction to circular colouring can be found in Section 1.2.

Our main motivation for studying odd-cycle critical graphs is the dual of a conjecture made by Jaeger in 1981. The original conjecture is known as the *circular flow conjecture*; it states that every $4t$-edge-connected graph admits a modulo $(2t+1)$-orientation[1]. Though the conjecture was shown to be false in early 2018 [6], all counterexamples found are non-planar. As such, the conjecture's dual for planar graphs is still open. The dual conjecture is the following

Conjecture 1.1.2. *If G is a planar graph of girth at least $4t$, then G admits a homomorphism to C_{2t+1}.*

(Equivalently, if G is a planar graph of girth at least $4t$, then G admits a $\frac{2t+1}{t}$-circular colouring.) The conjecture has only been confirmed in the $t = 1$ case: this case is equivalent to Grötszch's theorem that every triangle-free planar graph can be 3-coloured. Though considerable progress has been made in the general t case, the girth bound of $4t$ remains elusive. In 1996, Nešetřil and Zhu [11] showed that every planar graph of girth at least $10t - 4$ admits a homomorphism to C_{2t+1}. In 2000, Klostermeyer and Zhang [7] showed that it was sufficient to bound the odd girth of the graph as being at least $10t - 4$. A year later, Zhu [15] showed that a girth bound of at least $8t - 3$ was sufficient, and in 2003, Borodin et al. [2] improved upon this by showing a girth of at least $\frac{20t-3}{3}$ sufficed. Progress stalled for a decade until in 2013, in a much more general paper regarding the primal version of the conjecture, Lovász et al. [10] showed that every planar graph with girth at least $6t$ admits a homomorphism to C_{2t+1}. This is the best known general bound, though in the $t = 2$ case Dvořák and Postle [5] showed that it is enough to bound the odd girth of the graph as being at least 11. These results are summarized in Tabel 1.1. We will show in the following pages that in the $t = 3$ case, it suffices to bound the odd girth of the graph as being at least 17.

For a graph G, we use $e(G)$ and $v(G)$ to refer to the number of edges and vertices, respectively, of G. The degree of a vertex v will be denoted by $\deg(v)$. Our main result is the following:

Theorem 1.1.3. *Let G be a C_7-critical graph. If $G \notin \{C_3, C_5\}$, then $e(G) \geq \frac{17v(G)-2}{15}$.*

[1] That is, an orientation of its edges such that for each vertex, the difference of the in-degree and the out-degree is congruent to 0 modulo $2t + 1$.

Year	Authors	t	Girth Bound
1958	Grötszch	$t = 1$	girth $\geq 4t$
1996	Nešetřil & Zhu [11]	$t \geq 2$	girth $\geq 10t - 4$
2000	Klostermeyer & Zhang [7]	$t \geq 2$	odd girth $\geq 10t - 4$
2001	Zhu [15]	$t \geq 2$	girth $\geq 8t - 3$
2003	Borodin, Kim, Kostochka, & West [2]	$t \geq 2$	girth $\geq \frac{20t-3}{3}$
2013	Lovász, Thomassen, Wu, & Zhang [10]	$t \geq 2$	girth $\geq 6t$
2017	Dvořák & Postle [5]	$t = 2$	odd girth ≥ 11

Table 1.1: Towards Jaeger's conjecture for planar graphs.

This, together with Euler's formula for graphs embedded in surfaces, gives us the following theorem which improves upon the best known result for Conjecture 1.1.2 in the $t = 3$ case, bringing the girth bound down from 18 to 17:

Theorem 1.1.4. *If G is a planar or projective-planar graph with girth at least 17, then G admits a homomorphism to C_7.*

Proof. Suppose not, and let G be a counterexample embedded either in the plane or the projective plane. Note that we may assume that G is connected, as the following argument holds for each component of G. Since G has no homomorphism to C_7, it contains a C_7-critical subgraph G'. Let $f(G')$ denote the number of faces in the embedding of G'. By Euler's formula for graphs embedded on surfaces, if G' is a graph embedded in a surface of Euler genus g, then $v(G') - e(G') + f(G') \geq 2 - g$.

Note if G' has girth 17, then $\frac{2e(G')}{17} \geq f(G')$, and so Euler's formula becomes $v(G') - e(G') + \frac{2e(G')}{17} \geq 2 - g$. Multiplying this by 17 and rearranging, we obtain $17v(G') - 15e(G') \geq 34 - 17g$. Since $g \leq 1$, we have $17v(G') - 15e(G') \geq 17$. But this is a contradiction, as by Theorem 1.1.3, we have $17v(G') - 15e(G') \leq 2$. □

The *odd girth* of a graph is the length of a shortest odd cycle. The following lemma, known as the Folding Lemma, will be used to bring the girth requirement in Theorem 1.1.4 down to 16 for planar graphs.

Folding Lemma (Klostermeyer and Zhang [7]) *Let G be a planar graph with odd girth k. If $C = v_0 \ldots v_{r-1}v_0$ is a cycle in G that bounds a face and $r \neq k$, then there is an integer $i \in \{0, ..., r-1\}$ such that the graph G obtained from G by identifying v_{i-1} and v_{i+1} (mod r) is of odd girth k.*

Using this, we obtain the following theorem:

Theorem 1.1.5. *If G is a planar graph with odd girth at least 17, then G admits a homomorphism to C_7.*

Proof. Suppose not. Let G be a counterexample chosen with $v(G)$ minimum and subject to that, with $e(G)$ minimum. Thus G is C_7-critical, and hence G is 2-connected[2]. It follows that all faces of G are bounded by cycles. Suppose there exists a cycle $C = v_0 \dots v_{k-1}v_0$ with $k < 17$ such that C bounds a face of G. Since the odd girth of G is at least 17, by the Folding Lemma there exists an integer $i \in \{0, \dots, k-1\}$ such that the graph G' obtained from G by identifying v_{i-1} and v_{i+1} (mod k) to a new vertex z is of odd girth at least 17. Since $v(G') < v(G)$ and G is a minimum counterexample, G' admits a homomorphism ϕ to C_7. But ϕ extends to G by setting $\phi(v_{i-1}) = \phi(v_{i+1}) = \phi(z)$, contradicting the fact that G is C_7-critical. Thus we may assume that every face of G has length at least 17. Therefore $f(G) \leq \frac{2e(G)}{17}$. By Euler's formula for planar graphs, $v(G) - e(G) + \frac{2e(G)}{17} \geq 2$. Multiplying by 17 and simplifying, we have that $17v(G) - 15e(G) \geq 34$. But this contradicts Theorem 1.1.3, as $G \notin \{C_3, C_5\}$ is C_7-critical and so $17v(G) - 15e(G) \leq 2$. □

We are thus able to bring the girth bound obtained in Lemma 1.1.4 from 17 down to 16 in the planar case: we obtain as a corollary to Theorem 1.1.5 the following improvement for the $t = 3$ case of Conjecture 1.1.2.

Corollary 1.1.6. *If G is a planar graph with girth at least 16, G admits a homomorphism to C_7.*

1.2 Outline

The remainder of Chapter 1 will provide an overview of the history of the study of circular colouring, and the two main techniques used in the proof of Theorem 1.1.3: the potential method and discharging. In Chapter 2, we will present some general results regarding graphs that are H-critical, where H is any vertex-transitive, non-bipartite graph. In Chapter 3, we will give a proof of the following theorem: if $G \neq C_3$ is a C_5-critical graph, then $e(G) \geq \frac{6v(G)}{5}$. This is included for the sake of exposition: the proof is short and relatively simple as compared to that of Theorem 1.1.3, but nevertheless it serves to illustrate the structure and main techniques used in the proof of Theorem 1.1.3 well. We note a stronger theorem regarding the density of C_5-critical graphs is proved in [5]. The proof of Theorem 1.1.3 will fill the remaining two chapters: Chapter 4 presents the structural properties of a minimum counterexample to the theorem, which will then be used in the discharging portion of the proof, found in Chapter 5.

[2]For a proof of this, see Lemma 2.1.3.

1.3 History

In this section, we will present a brief overview of circular colouring and how it relates to graph homomorphisms. We will then survey the two main techniques used in proving Theorem 1.1.3: the potential method and discharging.

Circular colouring was introduced by Vince in 1988 [13] as a variant of ordinary vertex colouring. Whereas in k-colouring we require adjacent vertices to have colours that differ by at least one, in circular colouring we require the colours of adjacent vertices to differ by at least a given value. Formally, for integers $1 \leq b \leq a$, an $\frac{a}{b}$-*circular colouring* of a graph G is a mapping ϕ of the vertices to the elements of the set $\mathbb{Z}/a\mathbb{Z}$ such that for each edge uv in G, $b \leq |\phi(u) - \phi(v)| \leq a - b$. To see how this relates to graph homomorphisms, let $K_{a:b}$ be the graph given by $V(K_{a:b}) = \{v_0, ..., v_{a-1}\}$ and $E(K_{a:b}) = \{v_i v_j : b \leq |i-j| \leq a-b\}$. Much as a graph G has a k-colouring if and only if G admits a homomorphism to $K_k = K_{k:1}$, the graph G has a $\frac{a}{b}$-circular colouring if and only G admits a homomorphism to $K_{a:b}$. In particular, for $t \in \mathbb{Z}^+$, G has a $\frac{2t+1}{t}$-circular colouring if and only if G admits a homomorphism to $K_{2t+1:t} = C_{2t+1}$. Note every circulant clique of the form $K_{a:b}$ is vertex-transitive, and if $\frac{a}{b} > 2$, $K_{a:b}$ is non-bipartite. The *circular chromatic number* of a graph G (which we will denote by $\chi_c(G)$) is the infimum over all rational numbers $\frac{a}{b}$ such that G has an $\frac{a}{b}$-circular colouring. The circular chromatic number of a graph differs from its chromatic number by at most one, and so χ_c is a refinement of the traditional chromatic number. Indeed, we have the following theorem.

Theorem 1.3.1. *If G is a graph, then $\chi(G) = \lceil \chi_c(G) \rceil$.*

Proof. Let $\chi_c(G) = \frac{a}{b}$, with $\gcd(a, b) = 1$. Note that G admits a homomorphism to $K_{a:b}$. We will first show that $K_{a:b}$ admits a homomorphism to $K_{\lceil \frac{a}{b} \rceil}$, thus showing $\chi(G) \leq \lceil \frac{a}{b} \rceil$. We may assume that $\chi_c(G)$ is not integral, as otherwise $K_{a:b}$ is isomorphic to $K_{\lceil \frac{a}{b} \rceil}$ and we are done.

Let $V(K_{a:b}) = \{v_0, ..., v_{a-1}\}$, with $E(K_{a:b}) = \{v_i v_j : b \leq |i-j| \leq a-b\}$, and let $K_{\lceil \frac{a}{b} \rceil}$ be the complete graph with vertices $\{u_0, ..., u_{\lceil \frac{a}{b} \rceil - 1}\}$. We define a homomorphism $\phi : V(K_{a:b}) \to V(K_{\lceil \frac{a}{b} \rceil})$ as follows: for each $i \in \{0, ..., a-1\}$, let $\phi(v_i) = u_{\lfloor \frac{i}{b} \rfloor}$. We claim ϕ is a homomorphism. To see this, let $v_i v_j$ be an edge in $E(K_{a:b})$ with $j < i$. Since $v_i v_j$ is an edge, we have that $b \leq i - j \leq a - b$, and so $i \geq j + b$. It follows that $\lfloor \frac{i}{b} \rfloor \neq \lfloor \frac{j}{b} \rfloor$. Thus $\phi(v_i) \neq \phi(v_j)$ and so $\phi(v_i)\phi(v_j)$ is an edge in $K_{\lceil \frac{a}{b} \rceil}$. Since $K_{a:b}$ admits a homomorphism to $K_{\lceil \frac{a}{b} \rceil}$, it follows that $\chi(G) \leq \lceil \chi_c(G) \rceil$.

Suppose now $\chi(G) \leq \lceil \frac{a}{b} \rceil - 1$. Since G admits a homomorphism to $K_{\chi(G)}$ and $K_{\chi(G)}$ is isomorphic to $K_{\chi(G):1}$, there exists a mapping ϕ of the vertices of G to the set $\{0, ..., \chi(G) -$

1} such that for each edge $uv \in E(G)$, we have $1 \leq |\phi(u) - \phi(v)| \leq \chi(G) - 1$. Thus $\chi_c(G) \leq \frac{\chi(G)}{1}$. But this is a contradiction, since $\frac{\chi(G)}{1} < \frac{a}{b} = \chi_c(G)$. □

For the remainder of this thesis, we will speak only of circular colouring via homomorphisms to circulant graphs.

The proof of Theorem 1.1.3 uses two main techniques: the potential method and discharging. In this section, we will give a brief overview of the ideas behind the two.

The potential method is used to gain insight into the structure of subgraphs of homomorphism-critical graphs. Our proofs using the potential method will be structured in the following way. The first step is to define the potential of a graph G as being $p(G) = \alpha v(G) - \beta e(G)$, for suitably chosen $\alpha, \beta \geq 0$. Next, we make a hypothesis concerning the potential of the homomorphism-critical graphs. To prove our hypothesis, we suppose the existence of a minimum counterexample to the hypothesis. Using the fact that our graph is critical and a *minimum* counterexample, we are able to bound the potential of subgraphs of the graph. This allows us to uncover the graph's structure, and this structure eventually enables us to dispel the existence of the counterexample through discharging.

The potential method was popularized by Kostochka and Yancey [9, 8], who used it amongst other things used to establish lower bounds for the number of edges in k-critical graphs. Later, Dvořák and Postle [5] used the potential method to show that if $G \neq C_3$ is a C_5-critical graph, then $e(G) \geq \frac{5v(G)-2}{4}$. This together with Euler's formula shows that if G is a planar graph with odd girth at least 11, then G admits a homomorphism to C_5.

Discharging is a much more widely known technique than the potential method. It is typically used for proofs in structural graph theory, perhaps most notably in Appel and Haken's proof of the Four Colour Theorem [1]. Discharging is routinely used to either dispel the existence of —or gain structural insight into —a minimum counterexample to a proposed theorem. The method works in stages: in the first stage, a number (called *charge*) is assigned to substructures of a graph in such a way that the total sum of the charges is known. In the following stages, the charge is redistributed amongst the substructures of the graph according to a set of rules. By examining the resulting charges, relationships between global and local properties of the graph may be inferred.

Our main result concerns the density of C_7-critical graphs. A natural question to wonder is whether or not the density bound obtained is best possible. We suspect not. Kostochka and Yancey [9] showed that if G is k-critical and $k \geq 4$, then $e(G) \geq (\frac{k}{2} - \frac{1}{k-1})v(G) - \frac{k(k-3)}{2(k-1)}$. Later, they showed this is tight for graphs[3] obtained via a construction given by Ore in

[3]Note: they showed further that this bound is tight *only* for the graphs obtained via Ore's construction.

[12]. A k-critical graph given by Ore's construction is called a *k-Ore graph.*

Given a $(2t+2)$-critical graph, there is a seemingly natural way to obtain a C_{2t+1}-critical graph by edge subdivisions. Indeed, we have the following:

Proposition 1.3.2. *If G is a $(2t+2)$-critical graph, then the graph G' obtained from G by subdividing every edge $(2t-2)$ times is C_{2t+1}-critical.*

The proof of this proposition will be delayed until we have built up the necessary machinery in Chapter 2. The general idea is the following: for each edge $uv \in E(G)$, let $P_{uv} \subset G'$ be the (u, v)-path obtained by subdividing uv $(2t-2)$ times. If ϕ is a mapping from u to a vertex c in C_{2t+1}, then there exists an extension of $\phi : P_{uv} \to C_{2t+1}$ with $\phi(v) = c'$ for precisely the set $\{c' : c' \in V(C) - c\}$. In this way, P_{uv} restricts colourings of its endpoints in the same manner as an edge does in ordinary vertex colouring[4]. Thus if ϕ is a proper vertex colouring of some subgraph H of G, then ϕ can be extended to the corresponding subgraph H' of G'. Furthermore, note that G' does not admit a homomorphism to C_{2t+1}; if such a homomorphism (or equivalently, colouring) ϕ exists, the restriction of ϕ to $V(G)$ is a $(2t+1)$-colouring, contradicting the fact that G cannot be $(2t+1)$-coloured.

Since the edge-density obtained by Kostochka and Yancey for k-critical graphs is tight for k-Ore graphs, it seems reasonable that the corresponding density obtained from subdividing a $(2t+2)$-Ore graph could be best possible for C_{2t+1}-critical graphs. This idea motivates the following:

Proposition 1.3.3. *Let $t \geq 1$ be an integer, and let G be a $(2t+2)$-Ore graph. Let G' be the graph obtained from G by subdividing each edge in $E(G)$ $(2t-2)$ times. Then $e(G') = \frac{t(2t+3)v(G')-(t+1)(2t-1)}{2t^2+2t-1}$.*

Proof. Since G' is obtained from G by subdividing each edge $(2t-2)$ times we have that $e(G') = (2t-1)e(G)$ and $v(G') = v(G) + (2t-2)e(G)$. Combining these two, we obtain $v(G') = v(G) + \frac{2t-2}{2t-1}e(G')$; or, rearranging, that $v(G) = v(G') - \frac{2t-2}{2t-1}e(G')$.

By Kostochka and Yancey [9], we have that $e(G) = (t+1-\frac{1}{2t+1})v(G) - \frac{(t+1)(2t-1)}{2t+1}$. Since $e(G') = (2t-1)e(G)$, we obtain $e(G') = (2t-1)(t+1-\frac{1}{2t+1})v(G) - \frac{(t+1)(2t-1)^2}{2t+1}$. Plugging in our expression for $v(G)$ above, we have $e(G') = (2t-1)(t+2-\frac{1}{2t+1})(v(G') - \frac{2t-2}{2t-1}e(G')) - \frac{(t+1)(2t-1)^2}{2t+1}$. Simplifying, we obtain $e(G') = \frac{t(2t+3)v(G')-(t+1)(2t-1)}{2t^2+2t-1}$, as desired. □

We therefore find it reasonable by setting $t = 3$ to conjecture that if G is a C_7-critical graph, then $e(G) \geq \frac{27v(G)-20}{23}$.

[4] This idea is formalized in Lemma 2.1.11

More generally, we ask the following question:

Question 1.3.4. *Let $t \geq 3$. Does there exist a C_{2t+1}-critical graph G such that $e(G) < \frac{t(2t+3)v(G)-(t+1)(2t-1)}{2t^2+2t-1}$?*

We note that the family of graphs described in Proposition 1.3.3 show that it is impossible to prove Jaeger's conjecture for planar graphs using only a density bound. When $t = 3$ for example, the graphs in Proposition 1.3.3 have an asymptotic density of $\frac{27}{23}$. However, using Euler's formula for planar graphs, we have that if G is a planar graph of girth at least g, then $e(G) \leq \frac{g}{g-2}(v(G)-2)$ —or, asymptotically, that $\frac{e(G)}{v(G)} \leq \frac{g}{g-2}$. In order to obtain a density argument that implies a relaxation of Jaeger's conjecture for planar graphs, it follows that the girth bound g chosen in the relaxation will satisfy $\frac{g}{g-2} \leq \frac{27}{23}$ —or in other words, that $g \geq 14$. To give a further example: when $t = 2$, the graphs in Proposition 1.3.3 have an asymptotic density of $\frac{14}{11}$, and so the girth bound g chosen in the relaxation will satisfy $g \geq 11$. A proof of Jaeger's conjecture will thus not be a purely density-based argument: it will require additional tools (for instance planarity).

More generally, we note that a negative answer to Question 1.3.4 implies that if G is a planar graph with girth at least $4t+2$, then G admits a homomorphism to C_{2t+1}. To see this, note that by Euler's formula if G is a planar graph of girth at least $4t+2$, then $e(G) \leq \frac{4t+2}{4t}(v(G)-2)$.

If Question 1.3.4 is answered in the negative, then we have that every C_{2t+1}-critical graph G satisfies $e(G) \geq \frac{t(2t+3)v(G)-(t+1)(2t-1)}{2t^2+2t-1}$. But for all values of $t \geq 1$, we have that $\frac{t(2t+3)v(G)-(t+1)(2t-1)}{2t^2+2t-1} > \frac{4t+2}{4t}(v(G)-2)$.

The girth bound of $4t+2$ is of particular interest as no counterexamples to the primal version of the conjecture[5] with edge-connectivity $4t+2$ have been found. Indeed, all counterexamples found in [6] are at most $(4t+1)$-edge connected.

[5]Recall: every $4t$-edge connected graph admits a modulo (2t+1)-orientation.

Chapter 2

H-Critical Graphs

2.1 General Homomorphism-Critical Graphs

In this chapter, we will present several results concerning general homomorphism-critical graphs.

Definition 2.1.1. *Let H be a graph. A graph G is H-*critical *if every proper subgraph of G admits a homomorphism to H, but G itself does not.*

Chapter 2 will be divided into two sections. In the first section, we will focus on the structure of graphs that are H-critical when H is a vertex-transitive, non-bipartite graph. In the second section, we will turn our attention to graphs that are odd cycle-critical.

We note that the study of H-critical graphs where H is bipartite is not of interest, as a graph G admits a homomorphism to a bipartite graph if and only G admits a homomorphism to K_2 (in other words, if and only if G is bipartite itself). In general, it is not interesting to study H-critical graphs when H admits a homomorphism to a proper subgraph H' of itself. The reason for this is that if $H \supset H'$ admits a homomorphism to H', then the graphs that admit a homomorphism to H' are precisely those that admit a homomorphism to H. If H does not admit a homomorphism to any proper subgraph of itself, H is called a *core*. In addition to restricting our study to graphs that are H-critical when H is a core, we will furthermore limit our study to graphs that are H-critical when H is vertex-transitive. As we will see in Lemma 2.1.3, this ensures that the H-critical graphs are 2-connected.

For a set $S \subset V(G)$, the neighbourhood of S, denoted $N(S)$, is the set of vertices that share an edge with a vertex in S. The path with t edges is denoted P_t, and will be referred to as the path of *length* t. The internal vertices of a path are the vertices of the path that are not endpoints. We will also need the following definition:

Definition 2.1.2. *Let H be a graph. We denote by $P_t(H)$ the set of graphs obtained from H by adding a path P of length t joining two distinct vertices of H, such that the internal vertices of P are disjoint from $V(H)$.*

Our first result concerns the connectivity of graphs that are H-critical for a vertex-transitive graph H. It will be useful in Chapters 3 and 4 for establishing the local structure around vertices in odd cycle-critical graphs.

Lemma 2.1.3. *Let H be a vertex-transitive graph, and let G be an H-critical graph. Then G is 2-connected.*

Proof. Suppose not. First, assume G is disconnected. Then G contains two components G' and G''. Let v' be a vertex in $V(G')$, and let v'' be a vertex in $V(G'')$. Since G is H-critical, $G - v'$ admits a homomorphism ϕ to H. Symmetrically, $G - v''$ admits a homomorphism ϕ' to H. But then $\phi(G \setminus G') \cup \phi'(G')$ is a homomorphism of G to H, contradicting the fact that G is H-critical. Therefore G is connected.

Suppose now G contains a cut vertex. Let (G_1, G_2) be a proper separation of G with $G_1 \cap G_2 = v$. Since G is H-critical and for each $i \in \{1, 2\}$, $G_i \subsetneq G$, the subgraph G_i admits a homomorphism ϕ_i to H. Since H is vertex-transitive, there exists an automorphism $\phi_3 : H \to H$ with $\phi_3(\phi_2(v)) = \phi_1(v)$. But then $\phi_1(G_1) \cup \phi_3(\phi_2(G_2))$ is a homomorphism of G to H, contradicting the fact that G is H-critical. □

We note that if G is H-critical for a graph H that is not vertex-transitive, G may contain a cut vertex. To see this, let H be the graph obtained from a 5-cycle $C = abcdea$ by adding a vertex v adjacent to all vertices in C. Note since $\deg(a) = 3$ and $\deg(v) = 5$, there does not exist an automorphism $\phi : H \to H$ with $\phi(a) = v$. Let H' be a copy of H, where for each vertex $u \in V(H)$, the corresponding vertex in $V(H')$ is denoted by u'. Let G be formed from H and H' by identifying $v \in V(H)$ and $a' \in V(H')$ to a new vertex z (see Figure 2.1). We claim G is H-critical: to see this, we first note that G does not admit a homomorphism to H, since every mapping $\phi : H \to H$ has $\phi(v) = v$, and every mapping $\phi' : H' \to H$ has $\phi(a') \neq v$.

To complete our argument, let G' be the graph obtained from G by removing an edge e. We claim G' admits a homomorphism to H. Let $G'_1 = G'[V(H)+z]$ and $G'_2 = G'[V(H')+z]$. Suppose first that $e \in E(G)$ is an edge in $E(G'_1)$. Then G'_1 admits a homomorphism ϕ to $C_3 = zxyz$. But then ϕ enables a homomorphism ϕ' from the resulting graph to H by

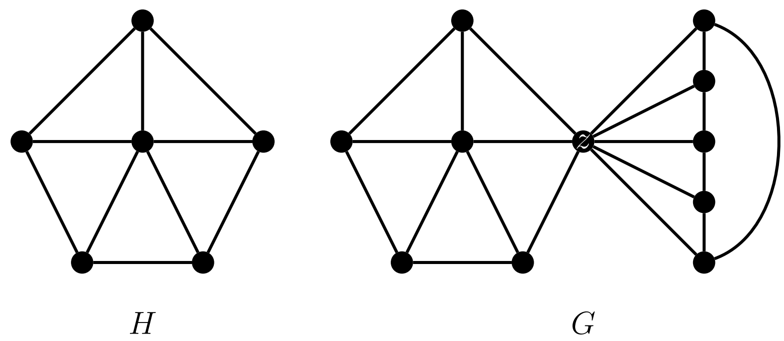

Figure 2.1: The graphs H and G.

setting $\phi(x) = b' \in V(G_2')$ and $\phi(y) = v' \in V(G_2')$. Since G_2' is isomorphic to H, ϕ' is indeed a homomorphism to H. The case where $e \in E(G_2')$ is treated similarly.

More generally, we provide the following construction. Let H be a core that is not vertex transitive and let u and v be two vertices of H that are not in the same orbit under the automorphism group of H. Let H' be a copy of H, and let G be the graph obtained by identifying $u \in V(H)$ and $v' \in V(H')$ to a new vertex z. Then by the same argument as above, G contains an H-critical subgraph that contains z —and so, there exists an H-critical graph that contains a cut vertex.

A lot of the analysis in our study of graph homomorphisms will consist of examining the ways in which partial homomorphisms can be extended to an entire graph. Paths with internal vertices of degree 2 will play an important role in our investigation, as it is easy to see the ways in which a partial homomorphism of a graph can extend along such a path. We therefore define the following terms:

Definition 2.1.4. *Let G be a graph. A* string *in G is a path with internal vertices of degree two and endpoints of degree at least three. A k-*string *is a string with k internal vertices. We say a vertex is* incident *with a string if it is an endpoint of the string. Let $v \in V(G)$ be a vertex of degree $d \geq 3$, and let $k_1, k_2, \ldots, k_d$ be integers with $k_1 \geq \cdots \geq k_d$. If v is incident with d distinct strings $S_1, \ldots, S_d$ where S_i is a k_i-string for each $1 \leq i \leq d$, we say v is of* type *$(k_1, \ldots, k_d)$. If v is a vertex of type $(k_1, ..., k_d)$, we define the* weight *of v as $wt(v) = \sum_{i=1}^{d} k_i$. Two vertices* share *a string if they are the endpoints of that string.*

Note if v is both endpoints of a given string, then the type of v is not defined. This will not be an issue: whenever we will speak of vertex types in the following pages, we will speak of the vertices in an odd cycle-critical graph G. Since odd cycles are vertex-transitive, G

does not contain a cut vertex by Lemma 2.1.3. It follows that each vertex in $V(G)$ has a defined type.

Our analysis will also rely heavily on the insights gained from identifying vertices in a critical graph and examining the resulting graph. Note that we only ever identify non-adjacent vertices, so no loops are created during vertex identifications. It is useful to be able to speak of undoing the identification process; to that end, we define the following term.

Definition 2.1.5. *Let u and v be non-adjacent vertices in a graph G. Let G' be the graph obtained from G by identifying u and v to a new vertex z. Let H be a subgraph of G' that contains z. Given the identification of u and v to z,* splitting z back into u and v *refers to deleting z and adding new vertices u and v to $V(H)$, and for each $x \in \{u, v\}$, adding to $E(H)$ all edges of the form xy such that $y \in V(H)$ and $xy \in E(G)$.*

In both Chapters 3 and 4, we will use the potential method to learn about the density of subgraphs of minimum counterexamples to Theorems 3.0.1 and 1.1.3. To that end, we define the following:

Definition 2.1.6. *Let $\alpha, \beta > 0$ and let G be a graph. The* (α, β)-potential *of G is given by $p_{\alpha,\beta}(G) = \alpha v(G) - \beta e(G)$.*

Note that when α and β are clear from the context, we will omit them and speak only of the *potential* of a graph and its subgraphs.

In the following paragraphs, we will require the following definition.

Definition 2.1.7. *Let $\phi : G \to H$ be a homomorphism, and let $F \subseteq G$. Then $\phi(F) \subseteq H$ is the graph with $V(\phi(F)) = \{\phi(v) : v \in V(F)\}$ and $E(\phi(F)) = \{\phi(u)\phi(v) : uv \in E(F)\}$.*

Let G be an H-critical graph, and let F be a proper subgraph of G. Since G is critical, F has a homomorphism $\phi : F \to H$.

Let G' be the graph with $V(G') = (V(G) \setminus V(F)) \cup V(\phi(F))$, and $E(G') = E(G \setminus F) \cup E(\phi(F))$.

For each $u \in \phi(F)$, let $\phi^{-1}(u)$ be the set of vertices of F with image u under ϕ.

Let $G_F[\phi]$ be the graph obtained from G' by adding an edge vu for each $u \in \phi(F)$ and $v \in V(G) \setminus V(F)$ such that there exists $w \in \phi^{-1}(u)$ with $vw \in E(G)$.

Note $G_F[\phi]$ has no homomorphism to H, as such a homomorphism ϕ' admits an extension to a homomorphism $\phi'' : V(G) \to V(H)$ by setting $\phi''(v) = \phi'(\phi(v))$ for each

$v \in V(F)$, and $\phi''(v) = \phi'(v)$ for each $v \in V(G) \setminus V(F)$. Thus $G_F[\phi]$ contains an H-critical subgraph W. Note if F is not isomorphic to a subgraph of H, then $G_F[\phi]$ contains fewer vertices than F, and hence W contains fewer vertices than G. Furthermore, $W \cap \phi(F) \neq \emptyset$ as otherwise $W \subset G$ and so G contains a proper H-critical subgraph, contradicting the fact that G is H-critical. This motivates the following definitions:

Definition 2.1.8. *The graph F' is an* extension *of $F \subsetneq G$ if there exists a homomorphism $\phi : F \to H$ and an H-critical subgraph W in $G_F[\phi]$ such that $F' = (W \setminus \phi(F)) \cup F$. We call W an* extender *of F, and $W[\phi(F)]$ the* source *of the extension.*

In order to establish certain structural properties of odd cycle-critical graphs in the following chapter, we will make extensive use of the following lemma.

Lemma 2.1.9. *Let G be an H-critical graph with potential $p(G)$. Let F be a proper subgraph of G that is not isomorphic to H. If F' is an extension of F with extender W and source X, then $p(F') = p(F) + p(W) - p(X)$.*

Proof. By the definitions of F', W', and X given in 2.1.8, $v(F') = v(F) + v(W) - v(X)$. Furthermore, $e(F') = e(F) + e(W) - e(X)$. Therefore since $p(F') = \alpha v(F') - \beta e(F')$ and both α and β are greater than 0, $p(F') = p(F) + p(W) - p(X)$. □

In order to effectively qualify the extension of partial homomorphisms of graphs along strings, we will use the following lemma:

Lemma 2.1.10. *Let G be a connected k-regular graph with $V(G) \geq 2$. If S is a proper, nonempty subset of $V(G)$, then exactly one of the following hold:*

(i) $|N(S)| > |S|$, *or*

(ii) $|N(S)| = |S|$ *and G is a bipartite graph with bipartition $(S, N(S))$.*

Proof. Let S be a proper, nonempty subset of $V(G)$. If two vertices in S are adjacent, they are both also in $N(S)$. Therefore for every vertex v in $S \setminus N(S)$, we have that $N(v) \subseteq N(S) \setminus S$. Since G is k-regular, there are exactly $k|S \setminus N(S)|$ edges with one endpoint in $S \setminus N(S)$ and the other in $N(S) \setminus S$. Similarly, there are at most $k|N(S) \setminus S|$ edges with one endpoint in $N(S) \setminus S$ and the other in $S \setminus N(S)$. We have, therefore, that

$$k|S \setminus N(S)| = |E(S \setminus N(S), N(S) \setminus S)| \leq k|N(S) \setminus S|. \tag{2.1}$$

We conclude, dividing by k, that $|S \setminus N(S)| \leq |N(S) \setminus S|$; or equivalently that $|S| \leq |N(S)|$. If $|S| < |N(S)|$, (i) holds as desired.

We may therefore suppose that $|S| = |N(S)|$ (equivalently, that $|S \setminus N(S)| = |N(S) \setminus S|$). Thus equality holds throughout Equation 2.1, and hence we have that $|E(S \setminus N(S), N(S) \setminus S))| = k|N(S) \setminus S|$. In other words, for each vertex v in $N(S) \setminus S$, $N(v) \subseteq S \setminus N(S)$, and as remarked earlier, for every vertex v in $S \setminus N(S)$, $N(v) \subseteq N(S) \setminus S$. Therefore $S \cup N(S)$ induces a bipartite component H of G with bipartition $(S \setminus N(S), N(S) \setminus S)$. Since G is connected, $G = H$. Hence $S \cap N(S) = \emptyset$, and so G is bipartite with bipartition $(S, N(S))$. Finally, since $|S \setminus N(S)| = |N(S) \setminus S|$, (ii) holds as desired. □

The following lemma will be used to restrict the length of strings in an H-critical graph G. We will also use it in order to give a more formal proof of Proposition 1.3.2. Let u and v be vertices on a path P in G such that the internal vertices of P have degree 2 in G. Let $\phi : u \to H$ be a homomorphism. Let Φ be the set of extensions of ϕ to P. We define $B_\phi(v|u, P) := \{\phi'(v) : \phi' \in \Phi\}$. If the choice of ϕ is irrelevant (for instance if we only wish to speak of $|B_\phi(v|u, P)|$), we will sometimes write $B(v|u, P)$.

Lemma 2.1.11. *Let G and H be graphs, and suppose H is connected, regular, and non-bipartite, with $v(H) \geq 2$. Let $P = v_0v_1...v_{k+1}$ be a path with $k + 1$ edges, with $\deg_G(v) = 2$ for each $v \in V(P) \setminus \{v_0, v_{k+1}\}$. Let $\phi : v_0 \to H$ be a homomorphism. Then $|B_\phi(v_{k+1}|v_0, P)| \geq \min(k + 2, v(H))$.*

Proof. We proceed by induction on k.

Suppose first $k = 0$. Note $B_\phi(v_1|v_0, P) = N_H(\phi(v_0))$. Let $S = \{\phi(v_0)\} \subset V(H)$. Since H is connected, regular and non-bipartite, Lemma 2.1.10 (i) applies to S, and so $|B_\phi(v_1|v_0, P)| = |N(\phi(v_0))| \geq 2$.

Thus we may assume $k \geq 1$. Let $Pv_0v_1...v_{k+1}$ be a path with $\deg(v_1) = \cdots = \deg(v_k) = 2$. By induction, $|B_\phi(v_k|v_0, P)| \geq \min(k + 1, v(H))$. Let $S = B_\phi(v_k|v_0, P)$. Note $B_\phi(v_{k+1}|v_0, P) = N_H(S)$. Since H is connected, regular, and non-bipartite, again Lemma 2.1.10 (i) applies to S. Thus either $N(S) = S = V(H)$ or $|N_H(S)| \geq |S| + 1 \geq k + 2$, and so $|B_\phi(v_{k+1}|v_0, P)| \geq \min(k + 2, v(H))$ as desired. □

We now give a more formal proof of Proposition 1.3.2. Recall it said the following.

Proposition 1.3.2. *If G is a $(2t + 2)$-critical graph, then the graph G' obtained from G by subdividing every edge $(2t - 2)$ times is C_{2t+1}-critical.*

Proof. For each edge $uv \in E(G)$, let $P_{uv} \subset G'$ be the (u, v)-path obtained by subdividing uv $(2t-2)$ times. Let uv be an edge in $E(G)$, and let $\phi : u \to C_{2t+1}$ be a homomorphism. By

Lemma 2.1.11, $|B_\phi(v|u, P_{uv})| \geq 2t$. Furthermore, $|B_\phi(v|u, P_{uv})| \leq 2t$ since $B_\phi(v|u, P_{uv}) \subseteq V(C_{2t+1}) \setminus \{u\}$. It follows that $|B_\phi(v|u, P_{uv})| = 2t$.

Thus if ϕ is a $(2t+1)$-vertex colouring of a proper subgraph H of G, then ϕ extends to a homomorphism of the corresponding graph $H' \subsetneq G'$, where $H' = \cup_{uv \in E(H)} P_{uv}$.

Finally, we note that G' does not admit a homomorphism to C_{2t+1}. To see this, suppose to the contrary that there exists a homomorphism $\phi : G' \to C_{2t+1}$. Then $\phi(G'[V(G)])$ is a $(2t+1)$-colouring of $V(G)$, contradicting the fact that G is $(2t+2)$-critical.

□

We are now equipped to restrict the length of strings in H-critical graphs.

Lemma 2.1.12. *Let H be a connected, regular, non-bipartite graph. If G is an H-critical graph, then G does not contain a k-string with $k \geq v(H) - 2$.*

Proof. Suppose not: that is, suppose $P = v_0 v_1 v_2 \cdots v_{v(H)-1}$ is a subpath of a string in G. Since G is H-critical, $G - \{v_1, \cdots v_{v(H)-2}\}$ admits a homomorphism ϕ to H. By Lemma 2.1.11, $|B_\phi(v_{v(H)-1}|v_0, P)| \geq \min(v(H), v(H)) = v(H)$. Hence $\phi(v_{v(H)-1}) \in B_\phi(v_{v(H)-1}|v_0, P)$, and so ϕ extends to G. This contradicts the fact that G is H-critical. □

The following lemma will be used in the following section to provide some insight into the local structure around a vertex in a C_{2t+1}-critical graph.

Lemma 2.1.13. *Let H be a non-bipartite, vertex-transitive graph. Let G be an H-critical graph. If v is a vertex of G, then* $\mathrm{wt}(v) \leq (v(H) - 2)\deg(v) - v(H)$.

Proof. Note since H is vertex-transitive, G does not contain a cut vertex by Lemma 2.1.3. Thus each string in G has two distinct endpoints. Let v be a vertex of type $(k_1, \ldots, k_d)$, incident with a k_i-string S_i with other endpoint v_i, for $1 \leq i \leq d$. Note since G is H-critical, it does not admit a homomorphism to H, and thus if ϕ is a homomorphism from $G \setminus (\cup_{i=1}^d (S_i - v_i))$ to H, we have that $V(H) \setminus (\cap_{i=1}^d B_\phi(v|v_i, S_i)) = V(H)$. Therefore $\sum_{i=1}^d (v(H) - |B_\phi(v|v_i, S_i)|) \geq v(H)$. By Lemma 2.1.11, for each $1 \leq i \leq d$ we have $|B_\phi(v|v_i, S_i)| \geq \min(k_i + 2, v(H))$. By Lemma 2.1.12, $k_i + 2 < v(H)$ for each $1 \leq i \leq d$, and hence $|B_\phi(v|v_i, S_i)| \geq k_i + 2$. Thus $\sum_{i=1}^d (v(H) - (k_i + 2)) \geq v(H)$. Using the fact that $\sum_{i=1}^d k_i = \mathrm{wt}(v)$ and rearranging, we get $d(v(H) - 2) - \mathrm{wt}(v) \geq v(H)$. Therefore $\mathrm{wt}(v) \leq (v(H) - 2)\deg(v) - v(H)$, as desired. □

2.2 Odd Cycle-Critical Graphs

We now present several results and definitions pertaining to graphs that are H-critical when H is an odd cycle.

The following corollary to Lemma 2.1.13 will be used extensively in Chapters 4 and 5.

Corollary 2.2.1. *Let G be a C_{2t+1}-critical graph. If v is a vertex in G, then* $\mathrm{wt}(v) \leq (2t-1)\deg(v) - (2t+1)$.

Definition 2.2.2. *A $(2t+1)$-cycle in a C_{2t+1}-critical graph is called a cell. A cell C is* incident *with a string $S \not\subseteq C$ if one of the endpoints of the string is contained in the cell. The* degree *of C is the number of strings incident with C. Let $\deg(C) = d$, and let $k_1, k_2, ..., k_d$ be integers with $k_1 \geq ... \geq k_d$. If C is incident with d distinct strings $S_1, \ldots, S_d$ where for $1 \leq i \leq d$ S_i is a k_i-string, we say C is a cell of* type $(k_1, \ldots, k_d)$. *If C is a vertex of type $(k_1, ..., k_d)$, we define the* weight of C *as $wt(C) = \sum_{i=1}^{d} k_i$.*

We caution the reader that in defining cells, we do not mean to suggest that there is an intrinsic property of $(2t+1)$-cycles that makes them a vital part of the structural analysis for general C_{2t+1}-critical graphs. For C_5- and C_7-critical graphs, cells as defined proved a useful tool in our analysis. For values of t larger than 3, it seems likely that $(2t+3)$-cycles and perhaps $(2t+5)$-cycles will prove equally useful in establishing the structure of C_{2t+1}-critical graphs.

Note the definition does not preclude a cell C from containing both endpoints of a string S with $S \not\subset C$. We claim, however, that this does not happen. This is addressed in the following lemma.

Lemma 2.2.3. *Let $t \geq 1$ be an integer, and let C be a cell in a C_{2t+1}-critical graph G. Let $S \not\subseteq C$ be a string. At most one of the endpoints of S is contained in $V(C)$.*

Proof. Suppose not: that is, suppose both endpoints u and v of S are contained in $V(C)$. Let P_1 and P_2 be the two distinct (u,v)-paths contained in C. Note since $v(C_{2t+1})$ is odd, exactly one of P_1 and P_2 has an odd number of edges. Without loss of generality, we may assume $e(P_1) \equiv e(S) \mod 2$. Note since G is C_{2t+1} critical, G does not contain an odd cycle with fewer than $2t+1$ edges, as such a cycle is C_{2t+1}-critical itself. It follows that $e(S) \geq e(P_1)$. Since G is C_{2t+1}-critical, $G \setminus (S \setminus \{u,v\})$ has a homomorphism ϕ to C_{2t+1}. But then ϕ extends to G, since S has a homomorphism φ to P_1 with $\phi(u) = \varphi(u)$ and $\phi(v) = \varphi(v)$.

□

In the spirit of Corollary 2.2.1, the following lemma provides some restriction on the local structure surrounding a cell in a C_{2t+1}-critical graph.

Lemma 2.2.4. *Let G be a C_{2t+1}-critical graph. If C is a cell of G, then* $\mathrm{wt}(C) \leq (2t-1)\deg(C) - (2t+1)$.

Proof. Let C be a cell of G with $\deg(C) = d$. By Lemma 2.2.3, C is a cell of type $(k_1, ..., k_d)$, incident with a k_i-string S_i for each $1 \leq i \leq d$. We will denote by c_i and v_i the endpoints of each S_i, with $v_i \notin V(C)$.

Note first there are $2(2t+1)$ homomorphisms of a cell to C_{2t+1}. To see this, choose a vertex $v \in V(C)$ and a vertex $x \in V(C) \cap N(v)$. Note v has $2t+1$ possible images in C_{2t+1}, and for each of those images y, the vertex x has two corresponding possible images: the two vertices in $N_{C_{2t+1}}(y)$. The rest of the vertices in C are determined by the mappings of v and x. Given a homomorphism $\phi : v_i \to C_{2t+1}$, we denote by $B_\phi(C|v_i, S_i)$ the set of possible extensions of ϕ to $S_i \cup C$. Note that $|B_\phi(C|v_i, S_i)| = 2|B_\phi(c_i|v_i, S_i)|$. Since G does not admit a homomorphism to C_{2t+1}, we have $\cap_{i=1}^{d} B(C|v_i, S_i) = \emptyset$. Therefore $\sum_{i=1}^{d}(2(2t+1) - |B(C|v_i, S_i)|) \geq 2(2t+1)$. By Lemma 2.1.11, for each $1 \leq i \leq d$ we have $|B(c_i|v_i, P_i)| \geq \min(k_i + 2, 2t+1)$. From Lemma 2.1.12, since G is C_{2t+1}-critical, $k_i + 2 < 2t+1$. Therefore $|B(c_i|v_i, P_i)| \geq k_i + 2$, and since $|B(C|v_i, S_i)|) | = 2|B(c_i|v_i, P_i)|$ for each i, it follows that $\sum_{i=1}^{d}(2(2t+1) - 2(k_i+2)) \geq 2(2t+1)$. Using the fact that $\sum_{i=1}^{d} k_i = \mathrm{wt}(C)$, dividing by 2, and reorganizing, we obtain $\mathrm{wt}(C) \leq (2t-1)\deg(C) - (2t+1)$, as desired. □

In Chapters 3 and 4, we will make use of the following lemma concerning theta graphs. A *theta graph* is a graph formed by two vertices of degree 3 that share three distinct strings.

Lemma 2.2.5. *If G is a theta graph and t is an integer with $t \geq 1$, then G is not C_{2t+1}-critical.*

Proof. Suppose not, and let t be the least integer such that G is C_{2t+1}-critical. Let u and v denote the two degree 3 vertices in G, and let the three strings incident to v be S_1, S_2, and S_3. If G is bipartite, then G has a homomorphism to $e \in E(C_{2t+1})$, a contradiction. We may therefore assume G contains at least one odd cycle. Let C be a shortest odd cycle in G. Without loss of generality, we may assume $C = S_1 \cup S_2$ and that S_1 has an odd number of edges. For some $i \in \{1, 2\}$, we have that $e(S_3) \equiv e(S_i) \mod 2$. Note that S_3 has at least as many edges as S_i as otherwise $S_3 \cup (\{S_1, S_2\} \setminus S_i)$ contradicts our choice of C. Thus S_3 has a homomorphism ϕ to S_i such that $\phi(u) = u$ and $\phi(v) = v$. But this is equivalent to a homomorphism of G to C. Since $C \subsetneq G$ and G is C_{2t+1}-critical, it follows that C has a homomorphism to C_{2t+1}. Hence G has a homomorphism to C_{2t+1}, contradicting that G is C_{2t+1}-critical. □

Finally, we show that homomorphism-critical graphs do not contain two vertices that shares distinct strings with the same number of vertices modulo 2.

Lemma 2.2.6. *Let H and G be graphs such that G is H-critical. Let k_1 and k_2 be integers with $k_1, k_2 \geq 0$. If $u, v \in V(G)$ are the endpoints of distinct strings S_1 and S_2, where S_1 is a k_1-string and S_2 is a k_2-string, then $k_1 \not\equiv k_2 \mod 2$.*

Proof. Suppose not. Without loss of generality, we may assume $k_2 \geq k_1$. Let $S_1 = ua_1 \dots a_{k_1}v$ and $S_2 = ub_1 \dots b_{k_2}v$. Let G' be the graph obtained from G by deleting $b_1 \dots b_{k_2}$. Since G is H-critical and $G' \subset G$, G' admits a homomorphism ϕ to H. But ϕ extends to G in the following way: for each $1 \leq i \leq k_1$, we set $\phi(b_i) = \phi(a_i)$. For each $k_1 < i \leq k_2$ with $i \equiv k_1 \mod 2$, we set $\phi(b_i) = \phi(a_{k_1})$. For each $k_1 < i \leq k_2$ with $i \equiv k_1 - 1 \mod 2$, we set $\phi(b_i) = \phi(a_{k_1-1})$. This is a contradiction, since G admits no homomorphism to H. □

Chapter 3

C_5-Critical Graphs

In this chapter, we will present a similar result to Theorem 1.1.3 concerning the density of C_5-critical graphs. The proof will serve as an introduction to the proof techniques used in Chapters 4 and 5. We note a stronger theorem is proved by Dvořák and Postle in [5] but as we aim only to demonstrate the proof method, this weaker version will suffice. The theorem we prove is the following:

Theorem 3.0.1. *If $G \neq C_3$ is a C_5-critical graph, then $e(G) \geq \frac{6v(G)}{5}$.*

In order to prove Theorem 3.0.1, we will assume the existence of a minimum counterexample G. For the rest of this chapter, a *minimum counterexample* will be a counterexample to Theorem 3.0.1, minimal with respect to $v(G)$ and subject to that, minimal with respect to $e(G)$. The *potential*[1] of a graph H, denoted $p(H)$, will be defined as $6v(H) - 5e(H)$. In the language of potentials, Theorem 3.0.1 can be restated as follows: if $H \neq C_3$ is a C_5-critical graph, then $p(H) \leq 0$. We will begin by establishing some of the structure of G through what we know of homomorphism-critical graphs and the potential of subgraphs of G. Once we have established enough of the structure of G to complete the proof, we will proceed via discharging.

First, we present a lemma used to uncover lower bounds on the potential of subgraphs of G. Recall from Definition 2.1.2 that if H is a graph, we denote by $P_3(H)$ the set of graphs obtained from H by adding a path P of length 3 joining two distinct vertices of H, such that the internal vertices of P are disjoint from $V(H)$.

Lemma 3.0.2. *Let H be a subgraph of a minimum counterexample G. The following all hold:*

[1] In Chapter 3, *potential* denotes the $(6, 5)$-potential.

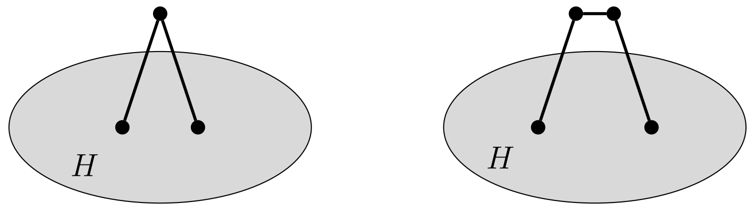

Figure 3.1: G contains a path P of length either 2 or 3, such that the endpoints of P are in H and the internal vertices of P are in $G \setminus V(H)$.

(i) $p(H) \geq 1$ *if* $H = G$,

(ii) $p(H) \geq 4$ *if* $G \in P_3(H)$, *and*

(iii) $p(H) \geq 5$ *otherwise.*

Proof. Suppose not. Let $H \subseteq G$ be a counterexample to the lemma, maximal with respect to $v(H)$ and, subject to that, maximal with respect to $e(H)$. If $H = G$, then (i) holds, a contradiction. Thus we may assume $H \neq G$ and that neither (ii) nor (iii) hold for H. If $H \subseteq C_5$, (iii) holds since $p(C_5) = 5$ and $p(G') = t + 6$ if $G' = P_t$ is a path with t edges. We may therefore assume H is not a subgraph of C_5. Since H is a counterexample to the lemma, it follows that $p(H) \leq 4$.

Suppose first H is not induced. Then $p(G[V(H)]) = p(H) - 5(e(G[V(H)]) - e(H))$, and since $e(G[V(H)]) - e(H) \geq 1$, we have that $p(H) - 5 \geq p(G[V(H)])$. Thus it follows that $p(H) \geq p(G[V(H)]) + 5$. Since $p(H) \leq 4$, we therefore have that $G[V(H)]$ has potential at most -1, and so none of (i)-(iii) hold for $G[V(H)]$. But this is a contradiction, as $G[V(H)]$ contradicts our choice of H.

Thus we may assume H is induced. Since $H \subsetneq G$ and G is C_5-critical, H admits a homomorphism ϕ to C_5. Let W be a C_5-critical subgraph of $G_H[\phi]$. Note W contains at least one vertex from $\phi(H)$ since G contains no proper subgraph that is C_5-critical.

Suppose first that W is a triangle. Since W contains at least one vertex from $G \setminus \phi(H)$ and at least one vertex from $\phi(H)$, we have that G contains a path P with t edges, where $t \in \{2, 3\}$, such that the endpoints of P are in H and the internal vertices of P are in $G \setminus V(H)$ (see Figure 3.1). Note $p(H \cup P) = p(H) + 6(t-1) - 5t$, so $p(H \cup P) = p(H) + t - 6$. If $t = 2$, then $p(H \cup P) = p(H) - 4$. Since $p(H) \leq 4$, we therefore have that $p(H \cup P) \leq 0$. But then none of (i)-(iii) hold for $H \cup P$, and so $H \cup P$ contradicts our choice of H. We may therefore assume $t = 3$, and so $p(H \cup P) = p(H) - 3 \leq 1$. Note by our choice of H, we have that $H \cup P$ is not a counterexample to Lemma 3.0.2, and so (i) holds for $H \cup P$

and $H \cup P = G$. But then $p(H \cup P) = 1$, and so $p(H) = 4$. Since (ii) holds for H, this is a contradiction. We may therefore assume H is not a triangle.

Let H' be an extension of H with extender W and source X (see Definition 2.1.8). By Lemma 2.1.9, we have that $p(H') = p(W) - p(X) + p(H)$. Since H is not a subgraph of a 5-cycle, it follows that $v(\phi(H)) < v(H)$ and so that $v(W) < v(G)$. Since W is not a triangle and G is a minimum counterexample to Theorem 3.0.1, we have that $p(W) \leq 0$. Thus it follows that $p(H') \leq p(H) - p(X)$. Since X is a subgraph of a 5-cycle, we have that $p(X) \geq 5$ and so $p(H') \leq p(H) - 5$. But since $p(H) \leq 4$, it follows that $p(H') \leq -1$. But then none of (i)-(iii) hold for H', contradicting our choice of H. □

We now proceed with establishing the structure of G.

Lemma 3.0.3. *G has girth at least 5.*

Proof. Suppose not. Since 3-cycles are C_5-critical and $G \neq C_3$, it follows that G contains no 3-cycle. We may assume therefore that G contains a cycle C of length 4. Since $p(C) = 4$, by Lemma 3.0.2 $G \in P_3(C)$ and so G is a theta graph. But by Lemma 2.2.5, no theta graph is C_5-critical —a contradiction. □

Lemma 3.0.4. *If C and C' are distinct cells in G, then C and C' are vertex disjoint.*

Proof. Suppose not, and let $H = C \cup C'$ be a subgraph of G. By Lemma 3.0.3, C and C' intersect in a single path P of length k, where $k \in \{0, 1, 2\}$. Note $p(H) = p(C) + p(C') - p(P_k) = 2(6(5) - 5(5)) - (6(k+1) - 5k) = 4 - k$. Since $k \geq 0$, we have $p(H) \leq 4$. Furthermore, H is a theta graph and so by Lemma 2.2.5, $H \neq G$. Thus by Lemma 3.0.2, $p(H) \geq 4$ and so it follows that $k = 0$, Lemma 3.0.2 (ii) holds and $G \in P_3(H)$.

Let $w = C \cap C'$, and let Q be the path such that $G = Q \cup H$. Let u and v be the endpoints of Q. Note that by Lemma 2.1.12, G contains no t-strings with $t \geq 3$. Thus $\{u, v\} \cap \{w\} = \emptyset$. Without loss of generality, we may assume $v \in V(C)$. Note since v and w are the only vertices in $V(C)$ of degree at least three, the cycle C contributes three to the weight of v. Since Q is a 2-string incident with v, we have therefore that $\mathrm{wt}(v) = 5$. This contradicts Corollary 2.2.1. □

The following lemma provides us with local structure for certain types of vertices of degree 3.

Lemma 3.0.5. *Let v be a vertex of degree 3 in G. Suppose that v is incident with strings S_1, S_2 and S_3 such that S_1 is a 2-string. Then $S_2 \cup S_3$ is contained in a cell.*

Proof. Suppose not. For $i \in \{1,2,3\}$, let $v_i \in V(S_i) \cap N(v)$. Let G' be the graph obtained from G by identifying v_2 and v_3 to a new vertex z. Note v_2 and v_3 are not adjacent since G has girth at least 5 by Lemma 3.0.3, and so no loops are created. Furthermore, since the path v_2vv_3 is not contained in a cell, G' is triangle-free. If there exists a homomorphism ϕ of G' to C_5, then ϕ extends to G by setting $\phi(v_2) = \phi(v_3) = \phi(z)$, contradicting the fact that G is C_5-critical. We may therefore assume that G' contains a C_5-critical subgraph G''. Note that G'' is not contained in G, and hence it follows that $z \in V(G'')$. Suppose that $v \in V(G'')$. Since C_5-critical graphs have minimum degree at least 2 by Lemma 2.1.3, then S_1 is contained in G''. This contradicts Lemma 2.1.12, as S_1 is contained in a k-string for some $k \geq 3$. Thus we may assume that $v \notin V(G'')$. Since G'' contains no vertices of degree one, the internal vertices of S_1 are also not contained in $V(G'')$. Since G'' is not a triangle, $v(G'') < v(G)$, and since G is a minimum counterexample to Theorem 3.0.1, it follows that $p(G'') \leq 0$.

Let F be the graph obtained from G'' by splitting z back into v_2 and v_3, and adding the vertex v and the edges vv_2 and vv_3. Note that $v(F) - v(G'') = 2$ and $e(F) - e(G'') = 2$, and so it follows that $p(F) = p(G'') + 6(v(F) - v(G'')) - 5(e(F) - e(G'')) = p(G'') + 2 \leq 2$. But since $F \subsetneq G$, this contradicts Lemma 3.0.2. □

By Lemma 2.1.12, G contains no k-strings with $k \geq 3$. We thus obtain the following corollary to Lemma 3.0.5.

Corollary 3.0.6. *If $v \in V(G)$ is a vertex of degree 3 not contained in a cell, then $wt(v) \leq 3$.*

In order to proceed with discharging, we will need one final lemma regarding the degree of cells in G.

Lemma 3.0.7. *G does not contain a cell of degree at most three.*

Proof. Suppose not, and let $C = v_1v_2v_3v_4v_5v_1$ be a cell of degree at most 3 in G. Note C contains at most three vertices of degree at least 3. Note furthermore that C contains at least two vertices of degree at least 3, since by Lemma 2.1.3 G contains no cut vertices. Suppose C contains exactly two vertices of degree at least 3. By Lemma 2.1.12, C does not contain a k-string with $k \geq 3$, and so C contains a 1-string S_1 and a 2-string S_2. Since C has degree 3, one endpoint of S_2 has degree exactly 3. Without loss of generality, we may assume this endpoint is v_1. Let $S_3 \notin \{S_1, S_2\}$ be a string incident with v_1. By Lemma 3.0.5 applied to v_1 and S_2, we have that the path formed by S_1S_2 is contained in a cell C'. But then C and C' intersect in S_1, contradicting Lemma 3.0.4. Therefore we may assume that C contains exactly three vertices of degree 3. Without loss of generality, we may assume these vertices are v_1, v_2, and v_4.

For $i \in \{1, 2, 4\}$, let $u_i \notin V(C)$ be a neighbour of v_i. Note that u_1 and u_2 are distinct since G is triangle-free, and $u_1u_2 \notin E(G)$ by Lemma 3.0.3. Furthermore, by Lemma 3.0.4 we have that $N(u_1) \cap N(u_2) = \emptyset$ since every two distinct cells in G are vertex-disjoint and $v_1v_2 \in E(C)$. Let G' be the graph obtained from $G \setminus V(C)$ by adding the edge u_1u_2. Note that G' is triangle-free.

We claim that every homomorphism ϕ from G_1 to $C_5 = c_1c_2c_3c_4c_5c_1$ extends to G to C_5. To see this, we may assume without loss of generality that $\phi(u_1) = c_1$ and $\phi(u_2) = c_2$. Choose $\phi(v_4) \in \{c_3, c_4, c_5\}$ adjacent to $\phi(u_4)$. If $\phi(v_4) = c_3$, then let $\phi(v_1) = c_5$, $\phi(v_2) = c_1$, $\phi(v_3) = c_2$ and $\phi(v_5) = c_4$. If $\phi(v_4) = c_4$, then let $\phi(v_1) = c_2$, $\phi(v_2) = c_1$, $\phi(v_3) = c_5$ and $\phi(v_5) = c_3$. If $\phi(v_4) = c_5$, then let $\phi(v_1) = c_2$, $\phi(v_2) = c_3$, $\phi(v_3) = c_4$ and $\phi(v_5) = c_1$. In all cases, we obtain a homomorphism of G to C_5.

We can assume, then, that G' admits no homomorphism to C_5, and thus G' contains a C_5-critical subgraph G''. Since $G'' \neq C_3$, G is a minimum counterexample and $v(G'') < v(G)$, we have $p(G'') \leq 0$. Since G is C_5-critical, G'' is not a subgraph of G and so it follows that $u_1u_2 \in E(G'')$. Let F be the graph obtained from $G'' - u_1u_2$ by adding the path $u_1v_1v_2u_2$. Note that $v(F) - v(G'') = 2$ and $e(F) - e(G'') = 2$, and so it follows that $p(F) = p(G'') + 6(v(F) - v(G'')) - 5(e(F) - e(G'')) = p(G'') + 2 \leq 2$. Furthermore, since $v_4 \notin V(F)$, we have $F \neq G$. But since $F \subset G$, this contradicts Lemma 3.0.2. $\square$

Having established the required structure of G, we are now equipped to prove Theorem 3.0.1.

Proof of Theorem 3.0.1. We proceed via discharging. We assign to each vertex $v \in V(G)$ an initial charge $ch_0(v) = 12 - 5\deg(v)$. Hence $\sum_{v \in V(G)} ch_0(v) = 2p(G)$. Since G is a minimum counterexample and potential is integral, $2p(G) \geq 2$.

We discharge in two steps according to the following rules to obtain a final charge ch_1.

Rule 1. Vertices of degree two send 1 unit of charge to both endpoints of the strings that contain them.

Rule 2. If v is a vertex of degree at least 3 that is contained in a cell, then v sends $ch_0(v)$ units of charge to that cell.

Note by Lemma 3.0.4, cells are vertex disjoint and so Rule 2 is unambiguous.

After discharging, we thus obtain the following charges:

(i) For each v with $\deg(v) = 2$ or such that v is contained in a cell, $ch_1(v) = 0$.

(ii) For each v with $\deg(v) \geq 3$ that is not contained in a cell, $ch_1(v) = 12 - 5\deg(v) + \mathrm{wt}(v)$.

(iii) For each cell K, $ch_1(K) = \sum_{v \in X}(12 - 5\deg(v) + \mathrm{wt}(v))$ where X is the set of vertices in K that have degree at least three.

We now prove that all cells and vertices have non-positive charge, thus arriving at a contradiction.

Claim 3.0.8. *If $v \in V(G)$, then $ch_1(v) \leq 0$.*

Proof. Suppose not. If v has degree 2, then $ch_1(v) = 0$ by Rule 1, a contradiction. If v is contained in a cell, then $ch_1(v) = 0$ by Rule 2 —again, a contradiction. We may therefore assume $\deg(v) \geq 3$ and that v is not contained in a cell. Suppose first $\deg(v) = 3$. Then $ch_1(v) = 12 - 5\deg(v) + \mathrm{wt}(v) > 0$, and so $0 < \mathrm{wt}(v) - 3$. Thus $\mathrm{wt}(v) \geq 4$. This is a contradiction, since by Lemma 3.0.6 $\mathrm{wt}(v) \leq 3$.

We may therefore assume $\deg(v) \geq 4$. By (ii), we have $ch_1(v) = 12 - 5\deg(v) + \mathrm{wt}(v)$. Note by Corollary 2.2.1, $\mathrm{wt}(v) \leq 3\deg(v) - 5$ and so it follows that $0 < ch_1(v) = 12 - 5\deg(v) + \mathrm{wt}(v) \leq 7 - 2\deg(v)$. Thus $\deg(v) \leq 3$, a contradiction. □

Claim 3.0.9. *All cells have non-positive charge.*

Proof. Suppose not, and let K be a cell with positive charge. Let X be the set of vertices of degree at least 3 in K. By (iii), we have

$$\begin{aligned} ch_1(K) &= \sum_{v \in X}(12 - 5\deg(v) + \mathrm{wt}(v)) \\ &= 12|X| - 5\sum_{v \in X}\deg(v) + \sum_{v \in X}\mathrm{wt}(v) \\ &= 12|X| - 5(2|X| + \deg(K)) + 2(5 - |X|) + \mathrm{wt}(K) \\ &= 10 + \mathrm{wt}(K) - 5\deg(K). \end{aligned}$$

By Lemma 2.2.4, $\mathrm{wt}(K) \leq 3\deg(K) - 5$, and so $0 < ch_1(K) \leq 5 - 2\deg(K)$. But this is a contradiction, as by Lemma 3.0.7 $\deg(K) \geq 4$. □

Since the sum of the charges in the graph is equal to $2p(G)$ and all charged elements have non-positive charge, we conclude $p(G) \leq 0$—a contradiction. □

Chapter 4

Structure of C_7-Critical Graphs

We now attack a similar problem using the same general techniques.

Our main result is the following:

Theorem 1.1.3. *Let G be a C_7-critical graph. If $G \notin \{C_3, C_5\}$, then $e(G) \geq \frac{17v(G)-2}{15}$.*

As before, we will restate this theorem in terms of potentials. In this section, the *potential*[1] of a graph G will be defined as $p(G) = 17v(G) - 15e(G)$. We aim to prove that if $G \notin \{C_3, C_5\}$ is a C_7-critical graph, then $p(G) \leq 2$. We note this bound is tight: examples of C_7-critical graphs with potential 2 can be found in Figure 4.1.

Originally, we constructed the proof without knowing the optimal bound for $p(G)$. In order to maintain the spirit of our original work and attempt to provide some intuition of how this threshold of 2 was determined, we will instead show that if G is a C_7-critical graph not isomorphic to C_3 or C_5, then $p(G) \leq T$. T will then be chosen to be the minimal value for which the proof holds. Note since potential is integral, a counterexample to Theorem 1.1.3 has potential at least $T + 1$. A *minimum counterexample* to Theorem 1.1.3 is a C_7-critical graph $G \notin \{C_3, C_5\}$ with $p(G) \geq T + 1$, minimal with respect to $v(G)$, and, subject to that, with respect to $e(G)$.

For the remainder of this thesis, G will be a minimum counterexample to Theorem 1.1.3. A k-string in G will be called *short* if $k \leq 2$.

The following two sections concern the structure of a minimum counterexample to Theorem 1.1.3. Section 4.1 contains general structural results, and Section 4.2 rules out the presence of certain substructures in G.

[1] In Chapters 4 and 5, *potential* denotes the $(17, 15)$-potential.

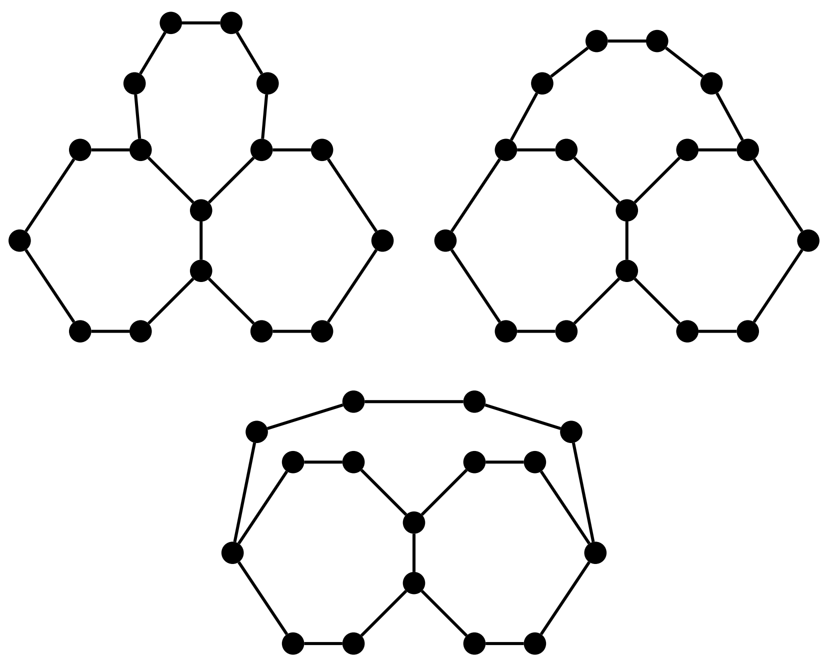

Figure 4.1: Examples of C_7-critical graphs with potential 2.

4.1 General Structure Results

The lemmas in this section provide us with a general framework for G. Lemma 4.1.1 concerns the potential of subgraphs of G, and will be useful in proving further structural lemmas. Lemma 4.1.2 establishes a lower bound for the girth of G. The proofs of Lemmas 4.1.3, 4.1.4, and 4.1.3 will establish the neighbouring structure of vertices incident with long strings. Finally, with Lemmas 4.1.6, 4.1.7, and 4.1.8 we will characterize the intersections of distinct 7-cycles and 9-cycles in G.

Lemma 4.1.1. *Let H be a subgraph of G. Then the following all hold:*

(i) $p(H) \geq T + 1$ *if* $H = G$,

(ii) $p(H) \geq T + 8$ *if* $G \in P_5(H)$,

(iii) $p(H) \geq T + 10$ *if* $G \in P_4(H)$,

(iv) $p(H) \geq T + 12$ *if* $G \in P_3(H)$,

(v) $p(H) = 14$ *if* $H = C_7$, *and*

(vi) $p(H) \geq 15$ *otherwise.*

Proof. Suppose not. Let H be a counterexample to Lemma 4.1.1, maximal with respect to $v(H)$, and subject to that, with respect to $e(H)$. Since G is a minimum counterexample to Theorem 1.1.3 and potential is integral, if $H = G$, then (i) holds—a contradiction. If H is isomorphic to C_7, then (v) holds, a contradiction. We may therefore assume $H \notin \{C_7, G\}$.

First suppose that H is not induced. Then $p(G[V(H)]) = p(H) - 15(e(G[V(H)]) - e(H))$. As H is not induced, $e(G[V(H)]) - e(H) \geq 1$ and so it follows that $p(G[V(H)]) \leq p(H) - 15 \leq -1$. But then $G[V(H)]$ is a counterexample to Lemma 4.1.1, contradicting our choice of H.

We may therefore assume H is induced. Note every proper subgraph H of C_7 has potential at least 17, since $p(H) = 2t + 17$ if H is a path with t edges. Thus if H is a proper subgraph of C_7, (vi) holds, a contradiction. Since G is C_7-critical and $H \subsetneq G$, H has a homomorphism ϕ to a subgraph of C_7. Let H' be an extension of H with extender W and source X (see Definition 2.1.8). By Lemma 2.1.9, $p(H') = p(H) + p(W) - p(X)$.

Suppose first that W is a triangle. Since $W \not\subset G$, W contains at least one vertex in $\phi(H)$ (see Definition 2.1.7). Similarly, since $W \not\subset \phi(H)$, W contains at least one vertex in $V(G) \setminus V(H)$. This gives rise to at least two edges in $E(H')$ from vertices in $V(G) \setminus V(H)$

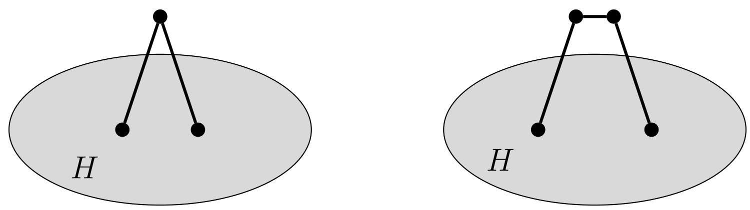

Figure 4.2: G contains a path P of length either 2 or 3, such that the endpoints of P are in H and the internal vertices of P are in $G \setminus V(H)$.

to $V(H)$. Thus G has a path P_k, $k \in \{2,3\}$, with endpoints in H and internal vertices in $G \setminus V(H)$ (see Figure 4.2). Since $e(H \cup P_k) - e(H) = k$ and $v(H \cup P_k) - v(H) = k-1$, we have that $p(H \cup P_k) = p(H) + 17(k-1) - 15k$, and so $p(H \cup P_k) = p(H) + 2k - 17 \leq 2k-3$. Since H is maximal, $H \cup P_k$ is not a counterexample to Lemma 4.1.1. Thus $p(H \cup P_k) \geq 1 + T$ and so $k = 3$. As $p(H \cup P_3) \leq 3$, we have $G = H \cup P_3$. But then $T + 1 \leq p(H \cup P_3) = p(H) - 11$, and so $p(H) \geq T + 12$. But now $G = (P_3 \cup H) \in \mathcal{P}_3(H)$ and so (iv) holds —a contradiction.

Suppose next that W is a 5-cycle. Since $W \not\subset G$, W contains at least one vertex in $\phi(H)$. Note there are at most two components in $W \setminus V(H)$, since each such component gives rise to at least two edges in W and $e(W) = 5$. Thus each component in $W \setminus V(H)$ is a path, and so at least one of the following cases hold (see Figure 4.3):

(1) G contains a path P_k, $k \in \{2,3,4,5\}$, joining two distinct vertices of H such that the internal vertices of P_k are not in H, or

(2) G contains a path P_2 joining two distinct vertices of H such that the internal vertex of P_2 is not in H, and a second path Q_k, $k \in \{2,3\}$, joining distinct vertices of H such that the internal vertices of Q_k are not in H.

Suppose that G contains a path P_2 as described in (1) or (2). Note that $p(H \cup P_2) = p(H) - 13 \leq 1$. But since H is maximal and we may choose $T \geq 2$, this contradicts our choice of H. If G does not contain a path P_2 as described, then it contains a path P_k, with $k \in \{3,4,5\}$ as described in (1). We have $p(H \cup P_k) = p(H) + 17(k-1) - 15k = p(H) + 2k - 17$.

- If $k = 3$, then $p(H \cup P_k) = p(H) - 11 \leq 3$. Since H is maximal, $H \cup P_3$ is not a counterexample and so $H \cup P_3 = G$. But then $T + 1 \leq p(H \cup P_3) = p(H) - 11$, and so $p(H) \geq T + 12$. But then (iv) holds —a contradiction.

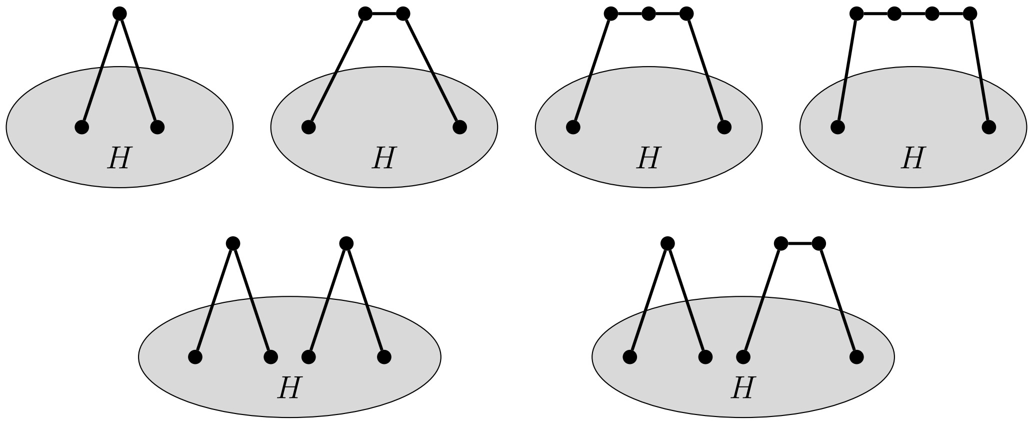

Figure 4.3: G contains a path P_k, $k \in \{2, 3, 4, 5\}$, joining two distinct vertices of H such that the internal vertices of P_k are not in H, or G contains a path P_2 joining two distinct vertices of H such that the internal vertex of P_2 is not in H, and a second path Q_k, $k \in \{2, 3\}$, joining distinct vertices of H such that the internal vertices of Q_k are not in H.

- If $k = 4$, then $p(H \cup P_k) = p(H) - 9 \leq 5$. Since H is maximal, $H \cup P_4$ is not a counterexample and so $H \cup P_4 = G$. But then $T + 1 \leq p(H \cup P_4) = p(H) - 9$, and so $p(H) \geq T + 10$. But then (iii) holds —a contradiction.
- If $k = 5$, then $p(H \cup P_k) = p(H) - 7 \leq 7$. Since H is maximal, $H \cup P_5$ is not a counterexample and so $H \cup P_5 = G$. But then $T + 1 \leq p(H \cup P_5) = p(H) - 7$, and so $p(H) \geq T + 8$. But then (ii) holds —a contradiction.

We can therefore assume $W \notin \{C_3, C_5\}$. Recall H' is an extension of H with extender W and source X. Note since $H \not\subseteq C_7$, we have that $v(\phi(H)) < v(H)$, and consequently that $v(W) < v(G)$. Since G is a minimum counterexample to Theorem 1.1.3, and W is neither a 3-cycle nor a 5-cycle, it follows that $p(W) \leq T$. Since X is a subgraph of a 7-cycle, $p(X) \geq 14$. We have therefore that $p(H') = p(W) + p(H) - p(X) \leq T + p(H) - 14$. Since $p(H) \leq 14$ we have $p(H') \leq T$. But then H' is a counterexample to Lemma 4.1.1, contradicting our choice of H. □

The following lemma gives a lower bound for the girth of G.

Lemma 4.1.2. *G has girth at least 7.*

Proof. Suppose not. Note G does not contain a 5-cycle or a triangle, as these are C_7-critical themselves and $G \notin \{C_3, C_5\}$. It follows that G contains a cycle C of length $2t$, where

$t \in \{2, 3\}$. But then $p(C) = 17(2t) - 15(2t) = 2t \leq 12$, so $G \in \{P_4(C), P_5(C)\}$ by lemma 4.1.1. But then G is a theta graph, and by Lemma 2.2.5, no such graph is C_{2t+1}-critical —a contradiction. $\square$

Note that by Lemma 4.1.2, there are no chords in cells or 9-cycles of G.

The following three lemmas are used to show that certain types of vertices of degree three and four are contained either in cells or in cycles of length nine.

Lemma 4.1.3. *If v is a vertex of degree three incident with strings S_1, S_2, and S_3 such that S_3 is a 4-string, then $S_1 \cup S_2$ is contained in a cell.*

Proof. Let $\{a_1, a_2\} = N(v) \setminus V(S_3)$. It suffices to show that the path a_1va_2 is contained in a cell, since the internal vertices of S_1 and S_2 (if they exist) have degree 2. Suppose this is not the case. Let G' be the graph obtained from G by identifying a_1 and a_2 to a new vertex z. Note since a_1va_2 is not contained in a cell and G has girth at least 7 by Lemma 4.1.2, G' contains no triangle nor 5-cycle. Let $x \neq v$ be an endpoint of S_3 in G, and let $S = S_3 - x$. If G' admits a homomorphism ϕ to C_7, then ϕ extends to G by setting $\phi(a_1) = \phi(a_2) = \phi(z)$. Thus there does not exist a homomorphism of G' to C_7, and so G' contains a C_7-critical subgraph G''. Since $G'' \notin \{C_3, C_5\}$, since $v(G'') < v(G)$, and since G is a minimum counterexample, we have that $p(G'') \leq T$. Since $G'' \nsubseteq G$, it follows that $z \in V(G'')$. Furthermore, S is not contained in G'', since by Lemma 2.1.12 G'' does not contain the 5-string S_3z and the minimum degree of G'' is at least 2. Let F be the graph obtained from G'' by splitting z back into vertices a_1 and a_2, and adding the path a_1va_2. The potential of F is given by $p(F) = p(G'') + 17(2) - 15(2) \leq T + 4$. By Lemma 4.1.1, $F = G$. But this is a contradiction, since $S - v$ is not contained in F. $\square$

Lemma 4.1.4. *If v is a vertex of degree three incident with strings S_1, S_2, and S_3 such that S_3 is a 3-string and both S_1 and S_2 contain at least two edges, then $S_1 \cup S_2$ is contained in a cell or a 9-cycle.*

Proof. Let $\{a_1, b_1\} = N(v) \setminus V(S_3)$. It suffices to show the path a_1vb_1 is contained in a cell or 9-cycle, since the internal vertices of S_1 and S_2 have degree 2. Suppose this is not the case. Let $a_2 = N(a_1) - v$, and let $b_2 = N(b_1) - v$. Let G' be the graph obtained from G by identifying a_1 and b_1 to a new vertex z_1, and identifying a_2 and b_2 to a new vertex z_2. Note since a_1vb_1 is not contained in a cell or 9-cycle, G' contains no 3- nor 5-cycle. Let $x \neq v$ be an endpoint of S_3 in G, and let $S = (S_3 \cup z_1) \setminus \{x\}$. If G' admits a homomorphism ϕ to C_7, then ϕ extends to G by setting $\phi(a_1) = \phi(b_1) = \phi(z_1)$ and $\phi(a_2) = \phi(b_2) = \phi(z_2)$. Thus there does not exist a homomorphism of G' to C_7, and so G' contains a C_7-critical subgraph G''. Since $G'' \notin \{C_3, C_5\}$, since $v(G'') < v(G)$, and since G is a minimum counterexample, it follows that $p(G'') \leq T$. Furthermore, S is not contained in G'' since by Lemma 2.1.12 G'' does not contain a 5-string $S_3 z_1 z_2$ and the minimum degree of G'' is at least 2. Since $G'' \nsubseteq G$ and $z_1 \notin V(G'')$, we have that $z_2 \in V(G'')$.

Let F be the graph obtained from G by splitting z_1 and z_2 back into a_1, b_1 and a_2, b_2, respectively, and adding the path $b_2 b_1 v a_1 a_2$. The potential of F is given by $p(F) = p(G'') + 17(4) - 15(4) \leq T + 8$. By Lemma 4.1.1, either $F = G$ or $G \in P_5(F)$. Since $S_3 \setminus \{x, v\}$ is not contained in F, we have that $F \neq G$ and so $G = F \cup P$ for a path P of length 5. But again since $S_3 \setminus \{x, v\}$ is not contained in F, P contains S_3. But by definition, S_3 is a path of length 4 with endpoints of degree three —a contradiction. □

Lemma 4.1.5. *If v is a vertex of degree 4 incident with strings S_1, S_2, S_3, and S_4 such that S_4 is a 4-string, then there exists $\{i, j\} \subset \{1, 2, 3\}$ with $i \neq j$ such that $S_i \cup S_j$ is contained in a cell.*

Proof. Let $\{u_1, u_2, u_3\} = N(v) \setminus V(S_3)$. Note $u_1 \neq u_2 \neq u_3$ by Lemma 4.1.2. It suffices to show one of the paths $u_1 v u_2$, $u_1 v u_3$ or $u_2 v u_3$ is contained in a cell, since the internal vertices of S_1, S_2 and S_3 (if they exist) have degree 2. Suppose this is not the case. Let G' be the graph obtained from G by identifying u_1, u_2 and u_3 to a new vertex z. Note G' contains no 3- nor 5-cycle. Let $x \neq v$ be an endpoint of S_4 in G, and let $S = S_3 - x$. If G' admits a homomorphism ϕ to C_7, then ϕ extends to G by setting $\phi(u_1) = \phi(u_2) = \phi(u_3) = \phi(z)$. Thus there does not exist a homomorphism of G' to C_7, and so G' contains a C_7-critical subgraph G''. Since $G'' \notin \{C_3, C_5\}$, since $v(G'') < v(G)$, and since G is a minimum counterexample, it follows that $p(G'') \leq T$. Since $G'' \nsubseteq G$, we have $z \in V(G'')$. Furthermore, S is not contained in G'', since by Lemma 2.1.12 G'' does not contain a 5-string and the minimum degree of G'' is at least 2. Let F be the graph obtained from G'' by splitting z back into vertices u_1, u_2 and u_3, and adding the path $u_1 v u_2$ and the edge $v u_3$. The potential of F is given by $p(F) = p(G'') + 17(3) - 15(3) \leq T + 6$. By Lemma 4.1.1, $F = G$. But this is a contradiction, since $S - v$ is not contained in F. □

Finally, the last three lemmas in Section 4.1 characterize the intersection of distinct 7- and 9-cycles in G. Together with Lemmas 4.1.3, 4.1.4, and 4.1.5, the following lemmas will allow us to rule out the existence of certain types of vertices in Section 4.2.

Lemma 4.1.6. *Let C and C' be distinct 7-cycles in G. Then C and C' are vertex-disjoint.*

Proof. Suppose not, and let $H = C \cup C'$. First suppose $C \cap C'$ has two components P and P'. Since paths have potential at least 17, $p(H) \leq p(C) + p(C') - p(P) - p(P') \leq 14 + 14 - 17 - 17 \leq -6$. This contradicts Lemma 4.1.1.

Thus we may assume the cycles C and C' intersect in a single path P of length k. Note $0 \leq k \leq 3$, as otherwise H (and thus G) contains a cycle of length at most six, contradicting Lemma 4.1.2. The potential of H is given by $p(H) = p(C) + p(C') - p(P) = 14 + 14 - (2k + 17) = 11 - 2k$. Note $H \neq G$ since no theta graph is C_7-critical by Lemma 2.2.5. By Lemma 4.1.1 and since $T \geq 2$, we have that $p(H) \geq 10$. Since $p(H) \leq 11$ we have $k = 0$, and by Lemma 4.1.1 $G \in P_5(H)$. Since $k = 0$, P is a single vertex v. Let Q be the path of length five such that $G = H \cup Q$. Since G is C_7-critical, $Q \cap (C - v) \neq \emptyset$ and $Q \cap (C' - v) \neq \emptyset$ as otherwise v is a cut vertex, and no C_7-critical graph contains a cut-vertex by Lemma 2.1.3. The cell C has degree three, and is incident with strings S_1, S_2 and Q, where $S_1 \cup S_2 = C'$. Note Q is a 4-string, and S_1 and S_2 together contribute 5 to the weight of C. Thus $\text{wt}(C) = 9$. But this is a contradiction, since by Lemma 2.2.4 $\text{wt}(C) \leq 8$. □

Lemma 4.1.7. *Let C and C' be cycles of length seven and nine, respectively, in G. Then C and C' are edge-disjoint.*

Proof. Suppose not. Let C and C' be the cycles of length 7 and 9, respectively, of G, chosen such that their intersection is maximal. Let $H = C \cup C'$. First suppose $C \cap C'$ has at least two components P and P'. Since paths have potential at least 17, $p(H) \leq p(C) + p(C') - p(P) - p(P') \leq 14 + 18 - 17 - 17 = -2$, contradicting Lemma 4.1.1.

Thus we may assume $C \cap C'$ is a single path P of length k. Note since C and C' share an edge, $k \geq 1$. Furthermore, $k \leq 6$, as otherwise H (and thus G) contains a cycle of length at most six, contrary to Lemma 4.1.2. The potential of H is therefore given by $p(H) = p(C) + p(C') - p(P) = 18 + 14 - (2k + 17) = 15 - 2k$. Note $H \neq G$, since H is a theta graph and no such graph is C_7-critical by Lemma 2.2.5. By Lemma 4.1.1 and since $T \geq 2$, we have that $p(H) \geq 10$. Since $p(H) \leq 15$, we have $k \in \{1, 2\}$. We now break into cases according to the value of k.

Case 1: $k = 1$. By Lemma 4.1.1, $G \in P_4(H) \cup P_5(H)$. Let Q_i be the path of length $i \in \{4, 5\}$ with $G = H \cup Q_i$. Note since G is C_7-critical, $Q_i \cap P = \emptyset$. To see this, suppose not. Then at least one of $(C \setminus V(P)) \cap Q_i$ and $(C' \setminus V(P)) \cap Q_i$ is the empty set. But then C or C' contains a j-string with $j \geq 5$, contradicting Lemma 2.1.12. Thus $Q_i \cap P = \emptyset$ and both $Q_i \cap C \neq \emptyset$ and $Q_i \cap C' \neq \emptyset$. The cycle C has degree three, and its incident strings are S_1, S_2, and Q_i, where $S_1 \cup S_2 \cup P = C'$. Q_i contributes $i - 1$ to the weight of C, and S_1 and S_2 together contribute 6. Thus $\text{wt}(C) = i + 5$. Since $i \geq 4$, we have that $\text{wt}(C) \geq 9$, contradicting Lemma 2.2.4.

Case 2: $k = 2$. By Lemma 4.1.1, $G \in P_5(H)$. Let Q be the path of length five with $G = H \cup Q$. Note $(C' \setminus V(P)) \cap Q \neq \emptyset$ as otherwise C' contains a 6-string, contrary to Lemma

2.1.12. Note therefore at least one vertex in P has degree exactly three in H, and hence at least one of the endpoints of P is not in $V(Q)$. Let v be an endpoint of P with $v \notin V(Q)$. Let $u \in V(C') \setminus V(C)$ be adjacent to v. Suppose $(C \setminus V(P)) \cap Q = \emptyset$. Then v is incident with a 4-string of G contained in C, and so by Lemma 4.1.3, Pu is contained in a cell $C'' \neq C$. Since $P \subset C$, this contradicts Lemma 4.1.6, since distinct 7-cycles in G are vertex-disjoint. Thus we may assume $Q \cap P = \emptyset$, and both $Q \cap C \neq \emptyset$ and $Q \cap C' \neq \emptyset$. Let w be the endpoint of Q contained in C'. By Lemma 4.1.3, since Q is a 4-string, the path formed by w and w's neighbours in C' is contained in a cell C''. Since every vertex in $C' \setminus V(P)$ except w has degree exactly 2, it follows that $|E(C') \cap E(C'')| \geq |E(C')| - |E(P)| = 7$. But this is a contradiction, since C' and C were chosen to be the cycle of length 9 and 7 that have the largest intersection. □

Lemma 4.1.8. *Let C and C' be distinct 9-cycles in G, with $V(C) \cap V(C') \neq \emptyset$. Their intersection is a path of length at most 2.*

Proof. Suppose not, and let $H = C \cup C'$. First suppose the cycles C and C' intersect in at least two paths P and P'. Since paths have potential at least 17, we have $p(H) = p(C) + p(C') - p(P) - p(P') \leq 2$, contradicting Lemma 4.1.1.

Thus we may assume the cycles C and C' intersect in a single path P of length $k \geq 3$. The potential of H is given by $p(H) = p(C) + p(C') - p(P) = 18 + 18 - (2k + 17) = 19 - 2k$. Note $H \neq G$ since by Lemma 2.2.5 no theta graph is C_7-critical. Thus by Lemma 4.1.1, $p(H) \geq 10$ since $T \geq 2$. Since $p(H) \leq 19$, we have $k \leq 4$. Note by assumption $k \geq 3$. By Lemma 4.1.1, $G \in P_5(H) \cup P_4(H)$.

Suppose first $G \in P_5(H)$. Let Q_5 be the path with $G = H \cup Q_5$. Let a and b be the endpoints of the path $P = C \cap C'$. Suppose first $(C \setminus V(P)) \cap Q_5 = \emptyset$. Let $a_1 a_2 ... a_{9-k-1} = C \setminus V(P)$, and let $b_1 ... b_{9-k-1} = C' \setminus V(P)$, labeled so that $a a_1 ... a_{9-k-1} b b_{9-k-1} ... b_1 a$ forms a cycle of length $2(9 - k)$. Since G is C_7-critical, $C' \cup Q_5$ has a homomorphism ϕ to C_7. But then ϕ extends to a homomorphism of G by setting $\phi(a_i) = \phi(b_i)$ for each $i \in \{1, \ldots, 9 - k - 1\}$.

Thus we may assume $(C \setminus V(P)) \cap Q_5 \neq \emptyset$, and symmetrically, $(C' \setminus V(P)) \cap Q_5 \neq \emptyset$. Let $q \in V(Q_5 \cap C)$. Let $v_1 \neq v_2$ be neighbours of q such that $\{v_1, v_2\} \subset V(C)$. Let G' be the graph obtained from G by identifying v_1 and v_2 to a new vertex v. Note if G' admits a homomorphism ϕ to C_7, then ϕ extends to G by setting $\phi(v_1) = \phi(v_2) = \phi(v)$. Therefore G' contains a C_7-critical subgraph G''. Note there exists an edge in the 5-string formed by $Q_5 v$ that is not contained in $E(G'')$ by Lemma 2.1.12. Since C_7-critical graphs have minimum degree two by Lemma 2.1.3, it follows that $E(Q_5 v) \cap E(G'') = \emptyset$. Thus G'' is a subgraph of a theta graph H'. By Lemma 2.2.5, no theta graph is C_7-critical. Since H' has girth at least 7, we have that $G'' \notin \{C_3, C_5\}$. But then H' does not contain a C_7-critical subgraph, a contradiction.

We may therefore assume $G \notin P_5(H)$, and so by Lemma 4.1.1, we have $p(H) \geq T + 10$. Since $T \geq 2$ and $p(H) = 19 - 2k$, it follows that $k = 3$. By Lemma 4.1.1, $G \in P_4(H)$. Let $Q_4 = q_0q_1q_2q_3q_4$ be the path such that $G = H \cup Q_4$. As above, $(C \setminus V(P)) \cap Q_4 \neq \emptyset$ as otherwise a homomorphism $\phi : Q_4 \cup C' \to C_7$ extends to G, a contradiction. Symmetrically, $(C \setminus V(P)) \cap Q_4 \neq \emptyset$. Let $q \in V(C \cap Q_4)$, and let $q' \in V(C' \cap Q_4)$. Let v_1 and v_2 neighbour q, with $\{v_1, v_2\} \subset V(C)$. Similarly, let $u_1 \neq u_2$ be neighbours of q' such that $\{u_1, u_2\} \subset V(C')$. Let G' be the graph obtained from G by both identifying v_1 and v_2 to a new vertex v, and identifying u_1 and u_2 to a new vertex u. Note if G' admits a homomorphism ϕ to C_7, then ϕ extends to a homomorphism of G by setting $\phi(v_1) = \phi(v_2) = \phi(v)$, and $\phi(u_1) = \phi(u_2) = \phi(u)$. Therefore G' contains a C_7-critical subgraph G''. Note that there exists an edge in the 5-string formed by uQ_4v that is not contained in G'' by Lemma 2.1.12. Since C_7-critical graphs have minimum degree two by Lemma 2.1.3, it follows $E(uQ_4v) \cap E(G'') = \emptyset$. Thus G'' is a subgraph of a theta graph H'. By Lemma 2.2.5, no theta graph is C_7-critical. Since H' has girth at least 7, we have that $G'' \notin \{C_3, C_5\}$. But then H' does not contain a C_7-critical subgraph, a contradiction. □

4.2 Forbidden Structures

The lemmas in this section are used to rule out the existence of certain configurations in G, and to establish the neighbouring structure of others. Lemmas 4.2.1, 4.2.2, and 4.2.3 rule out the existence of certain types of vertices of degree three. In Lemma 4.2.4, we show that G does not contain cells of low degree. Finally, Lemmas 4.2.5, 4.2.6, and 4.2.7 establish the neighbouring structure of certain types of vertices not contained in cells.

Given the structure established in the previous section, we are now equipped to rule out several types of degree three vertices. We note that in the discharging portion of the proof of Theorem 1.1.3, the problematic structures will be degree three vertices with weight at least six. In ruling out a subset of these types of vertices, we therefore shorten and simplify the discharging portion of the proof of Theorem 1.1.3.

Lemma 4.2.1. *G does not contain a vertex of type $(4, 4, k)$, where $0 \leq k \leq 4$.*

Proof. Suppose not. Then there exists a vertex v of type $(4, 4, k)$ with neighbours a, b, and c, where a is contained in a 4-string S_a, and b is contained in a 4-string $S_b \neq S_a$ by Lemma 2.1.3. Lemma 4.1.3 applied to v and S_a implies the edge vc is contained in a cell C.

Lemma 4.1.3 applied to v and S_b implies vc is contained in a cell $C' \neq C$. This contradicts Lemma 4.1.6, since distinct cells are vertex-disjoint. □

Lemma 4.2.2. *G does not contain a vertex of type $(4, 3, k)$, where $1 \leq k \leq 3$.*

Proof. Suppose not. Then for some $k \in \{1, 2, 3\}$, there exists a vertex v of type $(4, 3, k)$. Let a be a vertex in $N(v)$ that is contained in a 4-string S_a, and let b be a vertex in $N(v)$ contained in a 3-string S_b. By applying Lemma 4.1.3 to the v and S_a, we have that the path bvc is contained in a cell C. By applying Lemma 4.1.4 to v and S_b, we have that the path avc is contained in a cycle C' of length either seven or nine. In particular, the edge vc is contained in $E(C' \cap C)$. First suppose C' is of length seven. This contradicts Lemma 4.1.6 as distinct cells are vertex-disjoint. Thus we may assume C' is of length nine. This contradicts Lemma 4.1.7, since 7-cycles and 9-cycles are edge-disjoint. □

Lemma 4.2.3. *G does not contain a vertex of type $(3, 3, 2)$.*

Proof. Suppose not. Then there exists a vertex v of type $(3, 3, 2)$. Let a be a neighbour of v that is contained in a 3-string S_b. Let $b \neq a$ be a neighbour of v contained in a 3-string S_b. Finally, let c be the neighbour of v contained in a 2-string S_c. By applying Lemma 4.1.4 to the v and S_a, we have that S_c is contained in a cycle C of length either seven or nine. By applying Lemma 4.1.4 to v and S_b, we have that the path S_c is contained in a cycle $C' \neq C$ of length either seven or nine. Suppose first C and C' are both cells. Since $S_c \in C \cap C'$, this contradicts Lemma 4.1.6 as cycles of length seven are vertex disjoint. Suppose next that one of C and C' is a 9-cycle, and the other is a cell. Since $S_c \in C \cap C'$, this contradicts Lemma 4.1.7 as 7-cycles and 9-cycles are edge disjoint. Thus we may assume both C and C' are 9-cycles. But this contradicts Lemma 4.1.8, as distinct 9-cycles intersect in a path of length at most two. □

The following lemma is used to lower-bound the degree of cells in G. This will be useful in the discharging portion of the proof.

Lemma 4.2.4. *G does not contain a cell of degree at most two.*

Proof. Suppose not. Note since G is C_7-critical, G is not a cell. Thus if G contains a cell of degree 0, we have that G is disconnected, contradicting Lemma 2.1.3. If G contains a cell of degree one, then G contains a cut vertex, contradicting Lemma 2.1.3.

We may therefore assume G contains a cell C of degree two. Let P be a longest string contained in C. Note P is a k-string with $k \geq 3$ since $\deg(C) = 2$. Note that $k \leq 4$ by Lemma 2.1.12. Suppose first that P is a 4-string, and let v be an endpoint of P. Note since C has degree 2, it follows that v has degree 3. Let $u_1 \neq u_2$ be neighbours of v, such that $P \cap \{u_1, u_2\} = \emptyset$. By Lemma 4.1.3, the path u_1vu_2 is contained in a cell C'. But since v is also contained in C, we have that $V(C) \cap V(C') \neq \emptyset$, contradicting Lemma 4.1.6.

Thus we may assume P is a 3-string. Let $u \neq v$ be the endpoints of P. Let u_1 be a vertex in $V(C) \cap N(u)$, with $u_1 \notin V(P)$. Similarly, let v_1 be a vertex in $V(C) \cap N(v)$, with $v_1 \notin V(P)$. Let u_2 be a vertex in $N(u) \setminus V(C)$, and let v_2 be a vertex in $N(v) \setminus V(C)$. Note $v_2 \neq u_2$, as otherwise v_2Pv_2 is a cycle of length 6 in G, contradicting Lemma 4.1.2. Furthermore, v_2 and u_2 are not adjacent as otherwise the cell $v_2Pu_2v_2 \neq C$ intersects C contradicting Lemma 4.1.6.

Let G' be the graph obtained from G by both identifying u_1 and u_2 to a new vertex z_u, and identifying v_1 and v_2 to a new vertex z_v. Note since v_2 and u_2 are not adjacent, no loop is created. Moreover, we have the following two claims.

Claim 1. *G' does not contain a triangle.*

Proof. Suppose not. Since G does not contain a 5-cycle, a triangle in G' contains both z_u and z_v. But then the path $u_2uu_1v_1vv_2$ is contained in a cell $C' \neq C$. This contradicts Lemma 4.1.6. □

Claim 2. *G' does not contain a 5-cycle.*

Proof. Suppose not. Let K be a 5-cycle contained in G'. Since $K \not\subset G$, we have that at least one of z_u and z_v is contained in $V(K)$. Suppose first exactly one of z_u and z_v is contained in $V(K)$. Without loss of generality, suppose $z_u \in V(K)$. Then the path u_1uu_2 is contained in a cell C' in G. Since $u \in V(C \cap C')$, this contradicts Lemma 4.1.6.

Thus we may assume both z_u and z_v are contained in $V(C')$, and that the path $u_2uu_1v_1vv_2$ is contained in a 9-cycle C' in G. But then the path uu_1v_1v is contained in both C' and C, contradicting Lemma 4.1.7. □

Note G' does not admit a homomorphism ϕ to C_7, as any such homomorphism extends to G by setting $\phi(u_1) = \phi(u_2) = \phi(z_u)$, and $\phi(v_1) = \phi(v_2) = \phi(v)$. Thus G' contains a C_7-critical subgraph G''. By Claims 1 and 2, we have that $G'' \notin \{C_3, C_5\}$. Since $v(G'') < v(G)$ and G is a minimum counterexample, $p(G'') \leq T$. Note by Lemma 2.1.12, $P \nsubseteq G''$ since z_uPz_v is a 5-string. Furthermore, G'' contains at least one of $\{z_u, z_v\}$ since $G'' \nsubseteq G$.

Suppose first G'' contains exactly one of $\{z_u, z_v\}$, and without loss of generality suppose $z_v \in V(G'')$. Let F be the graph obtained from G'' by splitting z_v back into v_1 and v_2 and adding v_1vv_2. We have $p(F) \leq p(G'') + 17(2) - 15(2) \leq T + 4$, and so F contradicts Lemma 4.1.1.

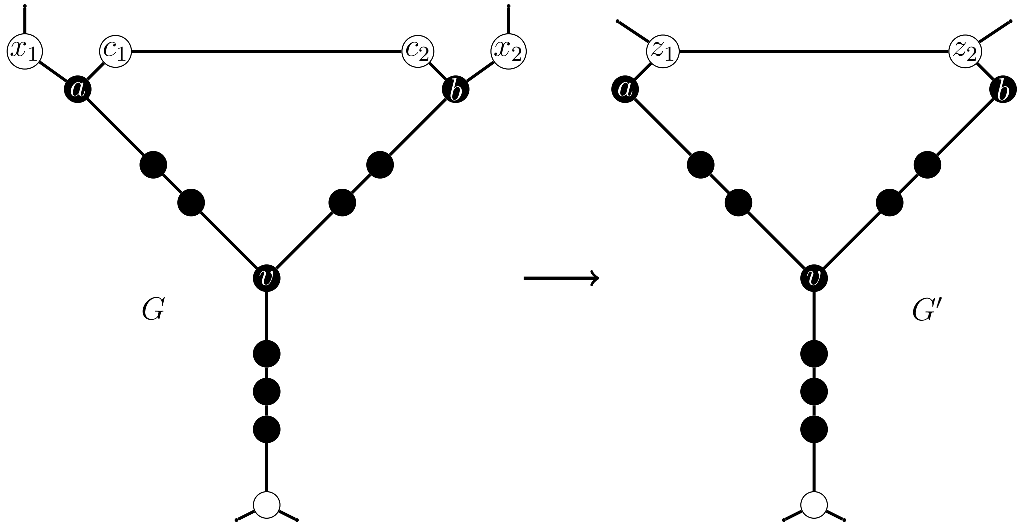

Figure 4.4: Obtaining G' from G by identifying x_i and c_i to a new vertex z_i, for each $i \in \{1, 2\}$. The black vertices' degrees are as illustrated. The white vertices' degrees are at least that illustrated.

Thus we may assume G'' contains both of $\{z_u, z_v\}$. Let F be the graph obtained from G'' by splitting z_v back into v_1 and v_2 and adding the path v_1vv_2. Let F' be obtained from F by splitting z_u back into u_1 and u_2 and adding the path u_1uu_2. We have $p(F') \leq p(G'') + 17(4) - 15(4) \leq T + 8$. By Lemma 4.1.1, $G \in P_5(F')$ or $F' = G$. But since $P \setminus \{u, v\} \notin F'$ and P is not a 4-string, this is a contradiction. □

Finally, the last three lemmas in this section establish the neighbouring structure of certain types of vertices not contained in cells. These lemmas will be useful in the discharging portion of the proof of Lemma 1.1.3.

Lemma 4.2.5. *Let $v \in V(G)$ be a vertex of type (3,2,2) that is not contained in a cell. If $a \neq b$ are the vertices that share a 2-string with v and $\deg(a) = \deg(b) = 3$, then at least one of a and b is contained in a cell.*

Proof. Suppose not. Let $S_a = aa_1a_2v$ be the 2-string shared by a and v, and let $S_b = bb_1b_2v$ be the 2-string shared by b and v. Let S be the 4-string incident with v. By Lemma 4.1.4 applied to v, since v is not contained in a cell there exists a 9-cycle $C = S_a \cup S_b \cup bc_2c_1a$ in G. Let $x_1 \in N(a) \setminus V(C)$, and let $x_2 \in N(b) \setminus V(C)$.

Note since C is a 9-cycle, by Lemma 4.1.8 the path $x_1ac_1c_2bx_2$ is not contained in a 9-cycle. Furthermore, since cells and 9-cycles are edge-disjoint by Lemma 4.1.7, ac_1 and

c_2b are each not contained in a cell. Let G' be the graph obtained from G by identifying x_i and c_i to a new vertex z_i, for each $i \in \{1, 2\}$ (see Figure 4.4). Note that G' does not contain a triangle or 5-cycle. Furthermore, if G' admits a homomorphism ϕ to C_7, then ϕ extends to G by setting $\phi(x_1) = \phi(c_1) = \phi(z_1)$ and $\phi(x_2) = \phi(c_2) = \phi(z_2)$, contradicting the fact that G is C_7-critical. Thus G' does not admit a homomorphism to C_7. Furthermore, G' is not C_7-critical, as v is a vertex of degree 3 and weight 9 in G', contradicting Corollary 2.2.1. Thus G' contains a proper C_7-critical subgraph G''. Note since G'' does not contain a vertex of degree 3 with weight at least 9, at least one edge in one of the strings S^* incident with v is not contained in G''. Since G'' has minimum degree 2 by Lemma 2.1.3, we have $E(S) \cap E(G'') = \emptyset$. Let S' and S'' be the strings in $\{S_a, S_b, S\} \setminus \{S^*\}$. Since $S' \cup S''$ is a k-string with $k \geq 5$, at least one of the edges in $S' \cup S''$ is not contained in $E(G'')$. Since G'' has minimum degree 2, it follows that $E(S' \cup S'') \cap E(G'') = \emptyset$. In particular, it follows that $v \notin V(G'')$.

Since $G'' \not\subseteq G$, it follows that G'' contains at least one of z_1 and z_2. Furthermore, since G'' is not a triangle or a 5-cycle and $v(G'') < v(G)$, we have $p(G'') \leq T$.

Suppose first exactly one of $\{z_1, z_2\}$ is contained in $V(G'')$. Without loss of generality, we may assume $z_1 \in V(G'')$. Let F be the graph obtained from G'' by splitting z_1 back into c_1 and x_1, and adding the path x_1ac_1. Then $p(F) = p(G'') + 17(2) - 15(2) \leq T + 4$. Since $F \neq G$, this contradicts Lemma 1.1.3.

Thus we may assume both of $\{z_1, z_2\}$ are contained in $V(G'')$. Let F be the graph obtained from G'' by splitting z_1 back into c_1 and x_1, splitting z_2 back into c_2 and x_2, and adding the paths x_1ac_1 and x_2bc_2. Then $p(F) = p(G'') + 17(4) - 15(4) \leq T + 8$. By Lemma 4.1.1, either $F = G$ or $G \in P_5(H)$. But since $v \notin V(F)$ is a vertex of degree 3, this is a contradiction. □

Lemma 4.2.6. *Let $v \in V(G)$ be a vertex of type (2,2,2) not contained in a cell. If a, b, and c are the vertices that share a 2-string with v and* $\deg(a) = \deg(b) = \deg(c) = 3$*, then at least one of a, b and c is contained in a cell.*

Proof. Suppose not. Let S_a, S_b, and S_c be the 2-strings shared by v with a, b, and c, respectively. Let $N(a) \setminus V(S_a) = \{a_1, a_2\}$, let $N(b) \setminus V(S_b) = \{b_1, b_2\}$, and similarly let $N(c) \setminus V(S_c) = \{c_1, c_2\}$. Note first a, b, and c are all distinct vertices, since G does not contain a 6-cycle by Lemma 4.1.2. Furthermore, $\{a_1, a_2\} \cap \{b, c\} = \emptyset$ since v is not contained in a cell. Similarly, $\{b_1, b_2\} \cap \{a, c\} = \{c_1, c_2\} \cap \{a, b\} = \emptyset$.

Let G' be the graph obtained from G by (see Figure 4.5):

– identifying a_1 and a_2 to a new vertex z_a,

– identifying b_1 and b_2 to a new vertex z_b, and

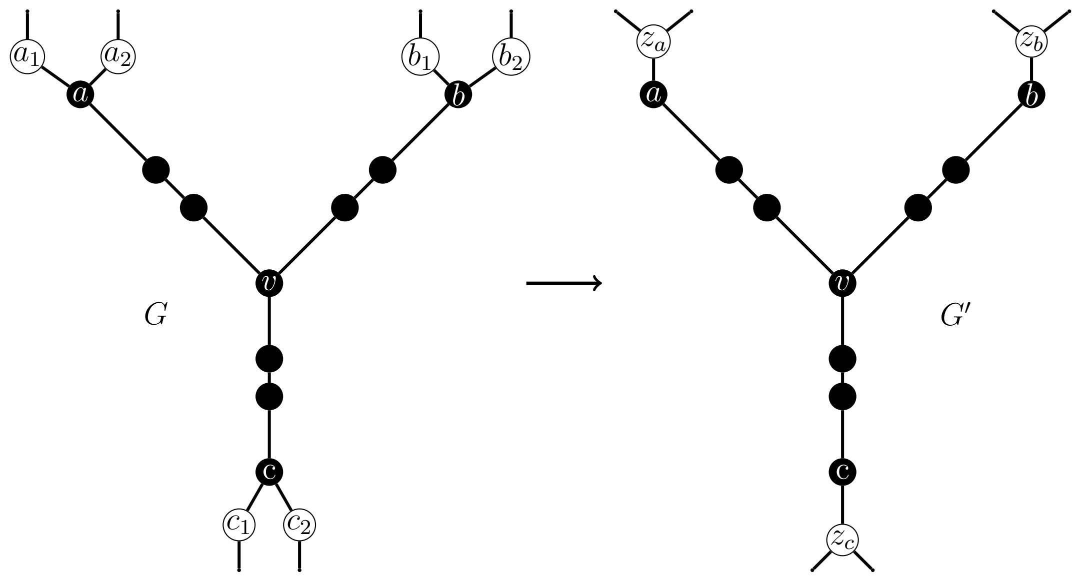

Figure 4.5: Obtaining G' from G by identifying x_1 and x_2 to a new vertex z_x, for each $x \in \{a, b, c\}$. The black vertices' degrees are as illustrated. The white vertices' degrees are at least that illustrated.

– identifying c_1 and c_2 to a new vertex z_c.

If G' admits a homomorphism ϕ to C_7, then ϕ extends to G by setting $\phi(x_1) = \phi(x_2) = \phi(z_x)$ for each $x \in \{a, b, c\}$, contradicting the fact that G is C_7-critical. Therefore G' contains a C_7-critical subgraph G''. Note by Lemma 2.2.1, at least one vertex in $V(S_a) \cup V(S_b) \cup V(S_c)$ is not in $V(G'')$ as otherwise v has weight nine in G'', contradicting Corollary 2.2.1. Without loss of generality, suppose there is a vertex in $V(S_a)$ not contained in $V(G'')$. Since G'' is C_7-critical, by Lemma 2.1.3 G'' has minimum degree 2, and so $(V(S_a) \setminus \{v\}) \cap V(G'') = \emptyset$. Suppose $S_a \cup S_b$ is contained in G''. Since $E(S_a) \cap E(G'') = \emptyset$, it follows that $S_a \cup S_b$ is a 7-string in G'', contradicting Lemma 2.1.12. Thus at least one edge in $E(S_a \cup S_b)$ is not contained in $E(G'')$. Since G'' has minimum degree 2 by 2.1.3, it follows that $E(S_a \cup S_b) \cap E(G'') = \emptyset$, and furthermore that $V(G'') \cap V(S_a \cup S_b \cup S_c) = \emptyset$.

Claim 1. *G'' is neither a triangle nor a 5-cycle.*

Proof. Suppose not, and suppose first G'' is a triangle. Since neither a, b, nor c is contained in a 5-cycle in G by Lemma 4.1.2, G'' we have that $|\{z_a, z_b, z_c\} \cap V(G'')| \geq 2$. Since neither a, b, nor c is contained in a cell, it follows that $|\{z_a, z_b, z_c\} \cap V(G'')| = 3$ and that the paths a_1aa_2, b_1bb_2, and c_1cc_2 are contained in a 9-cycle C. Let $F = S_a \cup S_b \cup S_c \cup C$. The potential of F is given by

$$\begin{aligned} p(F) &= p(C) + p(S_a) + p(S_b) + p(S_c) - p(a) - p(b) - p(c) - 2p(v) \\ &= 18 + 3(23) - 3(17) - 2(17) \\ &= 2. \end{aligned}$$

Since $T \geq 2$, this contradicts Lemma 4.1.1.

Suppose next G'' is a 5-cycle. Since neither a, b, nor c is contained in a cell, $|\{z_a, z_b, z_c\} \cap V(G'')| \geq 2$. First suppose that $|\{z_a, z_b, z_c\} \cap V(G'')| = 2$, and so that two of the paths a_1aa_2, b_1bb_2, and c_1cc_2 are contained in a 9-cycle C. Without loss of generality, assume the paths are a_1aa_2 and b_1bb_2. Let F be the graph formed by $C \cup S_a \cup S_b$. The potential of F is given by

$$\begin{aligned} p(F) &= p(C) + p(S_a) + p(S_b) - p(a) - p(b) - p(v) \\ &= 18 + 2(23) - 3(17) \\ &= 13. \end{aligned}$$

By Lemma 4.1.1, since $T \geq 2$, we have $F = G$ or $G \in P_5(F) \cup P_4(F)$. But this is a contradiction, since S_c is a 2-string and $S_c \not\subset F$.

We may therefore assume that $|\{z_a, z_b, z_c\} \cap V(G'')| = 3$, and so that the paths a_1aa_2, b_1bb_2 and c_1cc_2 are contained in an 11-cycle C. Let $F = C \cup S_a \cup S_b \cup S_c$. The potential of F is given by

$$\begin{aligned} p(F) &= p(C) + p(S_a) + p(S_b) + p(S_c) - p(a) - p(b) - p(c) - 2p(v) \\ &= 22 + 3(23) - 17(5) \\ &= 6. \end{aligned}$$

By Lemma 4.1.1, $F = G$. Let P_{ab} be the (a, b)-path in C that does not contain c. Similarly, let P_{bc} and P_{ac} be the (b, c)- and (a, c)-paths along C that do not contain a and b, respectively. Note since neither a, b, nor c is contained in a cell by assumption, neither P_{ab}, P_{bc}, nor P_{ac} is a 4-string by Lemma 4.1.3. Since together the three paths form an 11-cycle, we may assume without loss of generality that each of P_{ab} and P_{bc} is a 3-string, and that P_{ac} is a 2-string. Note $S_a \cup S_c \cup P_{ac}$ forms a 9-cycle C'. Let a' and c' be v's neighbours in S_a and S_c, respectively. Let F' be the graph obtained from G by identifying a' and c'. Note F' does not admit a homomorphism to C_7 as such a homomorphism extends to G. Thus F' contains a C_7-critical subgraph F''. Note C' is a cell of weight 9 in F'', and so F'' does not contain at least one string in or incident with C'. But then F'' is a subgraph of a theta graph. By Lemma 2.2.5 no theta graph is C_7-critical; since F'' has girth at least 7, it follows that F'' is not C_7-critical. □

By Claim 1, $G'' \notin \{C_3, C_5\}$, and since G is a minimum counterexample and $v(G'') < V(G)$, it follows that $p(G'') \leq T$. Since $G'' \not\subset G$, at least one of $\{z_a, z_b, z_c\}$ is contained in $V(G'')$. Let $I = \{x : z_x \in V(G'')\}$. Let F be the graph obtained from G'' by splitting z_x back into x_1 and x_2 and adding the path $x_1 x x_2$, for each $i \in I$. Let $k = |I| \leq 3$. We have $p(F) \leq p(G'') + 17(2k) - 15(2k)$. Note this does not necessarily hold with equality, since $\{x_1, x_2\}$ and $\{y_1, y_2\}$ are not necessarily disjoint for $x \neq y$, $\{x, y\} \subseteq I$. Simplifying, $p(F) \leq p(G'') + 2(2k) \leq T + 12$. By Lemma 4.1.1, either $F = G$ or $G \in P_5(F) \cup P_4(F) \cup P_3(F)$. But since $v \notin V(F)$ and $\deg(v) = 3$, this is a contradiction. □

Lemma 4.2.7. *Let $v \in V(G)$ be a vertex of type (3,3,0) that is not contained in a cell. Let S be the 0-string incident with v, and let $u \neq v$ be an endpoint of S. If u is not contained in a cell, u is not of type (3,3,0).*

Proof. Suppose not. Let S_a and S_b be the 3-strings incident with v, and let S_c and S_d be the 3-strings incident with u. Let $a \neq v$ and $b \neq v$ be endpoints of S_a and S_b, respectively. Let $c \neq u$ and $d \neq u$ be endpoints of S_c and S_d, respectively. Note $a \neq b$ and $c \neq d$ by Lemma 2.2.6.

In this proof, we consider all numerical indices to be taken modulo 7. We aim to show $\{a, b\} \cap \{c, d\} \neq \emptyset$. To see this, suppose not. Let ϕ be a homomorphism from $G \setminus (V(S_a \cup S_b \cup S_c \cup S_d) \setminus \{a, b, c, d\})$ to $C = c_0 c_1 .. c_6 c_0$. Let $I_1 = B_\phi(v|a, S_a) \cap B_\phi(v|b, S_b)$. Let $I_2 = B_\phi(u|c, S_c) \cap B_\phi(u|d, S_d)$. Note since G is C_7-critical, we have $N_C(I_1) \cap I_2 = \emptyset$; otherwise, ϕ extends to G, a contradiction. First, we will show the following:

Claim 1. *Given ϕ as described, $\{\phi(c), \phi(d)\} \cap \phi(a) \neq \emptyset$.*

Proof. By Lemma 2.1.11, since each of S_a, S_b, S_c, and S_d is a 3-string, each of $|B_\phi(v|a, S_a)|$, $|B_\phi(v|b, S_b)|$, $|B_\phi(u|c, S_c)|$, and $|B_\phi(u|d, S_d)|$ is at least 5. Note since I_1 and I_2 are each the intersection of two subsets of $V(C)$ of size at least 5, it follows that $|I_1| \geq 3$ and $|I_2| \geq 3$. Suppose $|I_1| \geq 4$. Then $|N_C(I_1)| \geq 5$ by Lemma 2.1.10, and so $N_C(I_1) \cap I_2 \neq \emptyset$. But then ϕ extends to G, a contradiction. Therefore I_1 (and symmetrically, I_2) is a set of size 3.

We may assume without loss of generality that $\phi(u) = c_1$. Then $B_\phi(v|a, S_a) = V(C) \setminus \{c_0, c_2\}$. Let $\phi(b) = c_j$, with $j \in \{0, \ldots, 6\}$. In order to have $|I_2| = |B_\phi(v|b, S_b) \cap B_\phi(v|a, S_a)| = 3$, we therefore have $\{c_0, c_2\} \cap \{c_{j-1}, c_{j+1}\} = \emptyset$. Thus $j \in \{4, 5, 2, 0\}$. Suppose $j = 2$. Then $I_1 = \{c_4, c_5, c_6\}$. But then $|N_{C_7}(I_1)| = 5$, and so since $N_C(I_1) \cap I_2 \neq \emptyset$, ϕ extends to G, a contradiction. The same is symmetrically true if $j = 0$. We may therefore assume $j \in \{4, 5\}$. Without loss of generality, we will take $j = 4$, as the $j = 5$ case corresponds to simply renaming the vertices along the cycle C in the opposite orientation. Similarly, if $\phi(c) = c_k$, then $\phi(d) \in \{c_{k+3}, c_{k+4}\}$. Note since $\phi(a) = c_1$ and $\phi(b) = c_4$, we have $I_1 = \{c_1, c_4, c_6\}$. Thus $N_C(I_1) = \{c_0, c_2, c_3, c_5\}$. Since ϕ does not extend to G, we have that $I_2 = \{c_1, c_4, c_6\}$, and so without loss of generality, $\phi(c) = c_1$ and $\phi(d) = c_4$. Since $\phi(a) = \phi(c)$, this is a contradiction. $\square$

Let G' be the graph obtained from $G \setminus (V(S_a \cup S_b \cup S_c \cup S_d) \setminus \{a, b, c, d\})$ by adding a 4-string S_{ac} with endpoints a and c, and a 4-string S_{ad} with endpoints a and d. Note by assumption assumption $a \neq c$ and $a \neq d$. It follows that G' does not contain a cycle of length three or five, since a cycle containing either S_{ad} or S_{ac} has length at least six. Note if G' admits a homomorphism ϕ to C, then ϕ extends to G by Claim 1, since $\phi(a) \notin \{\phi(d), \phi(c)\}$. Thus G' is not homomorphic to C_7, and so it contains a C_7-critical subgraph G''. Since $G'' \notin \{C_3, C_5\}$, since $v(G'') < v(G)$ and since G is a minimum counterexample, it follows that $p(G'') \leq T$. Note since G'' has minimum degree two and $G'' \not\subset G$, we have that G'' contains at least one of S_{ac} and S_{ad}.

Suppose first G'' contains exactly one of S_{ac} and S_{ad}; without loss of generality, assume $S_{ac} \subset G''$. Let F be the graph obtained from G'' by deleting S_{ac} and adding $S_a uv S_c$. Then $F \subset G$, and $p(F) = p(G'') + 17(4) - 15(4) \leq T + 8$. By Lemma 4.1.1, either $F = G$ or $G \in P_5(F)$. But since $S_d \not\subset F$ and $S_b \not\subset F$, this is a contradiction.

We may therefore assume G'' contains both S_{ac} and S_{ad}. Let F be the graph obtained from G'' by deleting S_{ac} and S_{ad}, and adding in S_a, S_c, S_d, and the edge uv. Since this is a net addition of 3 vertices and 3 edges, we have $p(F) = p(G'') + 17(3) - 15(3) \leq T + 6$. By Lemma 4.1.1, $F = G$. But since $S_b \not\subset F$, this is a contradiction.

Thus we may assume $a \in \{c, d\}$. Without loss of generality, assume $a = c$. We now break into two cases depending on whether or not $b = d$.

Case 1: $b = d$. Let $G_1 = G \setminus (V(S_a \cup S_b \cup S_c \cup S_d) \setminus \{a, b\})$. Note since G is C_7-critical and $G_1 \subset G$, G_1 admits a homomorphism ϕ to $C_7 = c_1 ... c_7 c_1$, with $\phi(a) = c_1$ and $\phi(b) \in \{c_1, c_2, c_3, c_4\}$. Note $\phi(b) = c_4$, as otherwise ϕ extends to G, a contradiction. To see this, see Figure 4.6.

Let $G_2 \in P_2(G_1)$ be the graph obtained from G_1 by adding an (a, b)-path P of length 2. Note if G_2 admits a homomorphism ϕ to C_7, then ϕ extends to G since there does not exist a homomorphism $\phi' : G_2 \to C_7$ with $\phi'(a) = c_1$ and $\phi'(b) = c_4$. Thus G_2 contains a C_7-critical subgraph G_2'. Suppose G_2' is a triangle. Then ab is an edge in G. But then $C = S_a \cup S_c \cup uv$ and $C' = S_b \cup S_a \cup ab$ are two 9-cycles that intersect in a 3-string S_a, contradicting Lemma 4.1.1. Suppose next that G_2' is a 5-cycle. Then there exists an (a, b)-path Q of length 3 in G. Let $F = S_a \cup S_b \cup S_c \cup S_d \cup uv \cup Q$. Since $v(F) = 18$ and $e(F) = 20$, the potential of F is given by $p(F) = 17(18) - 15(20) = 6$. By Lemma 4.1.1, $F = G$. But since there exists a homomorphism ϕ of Q to C_7 with $\phi(a) = c_1$ and

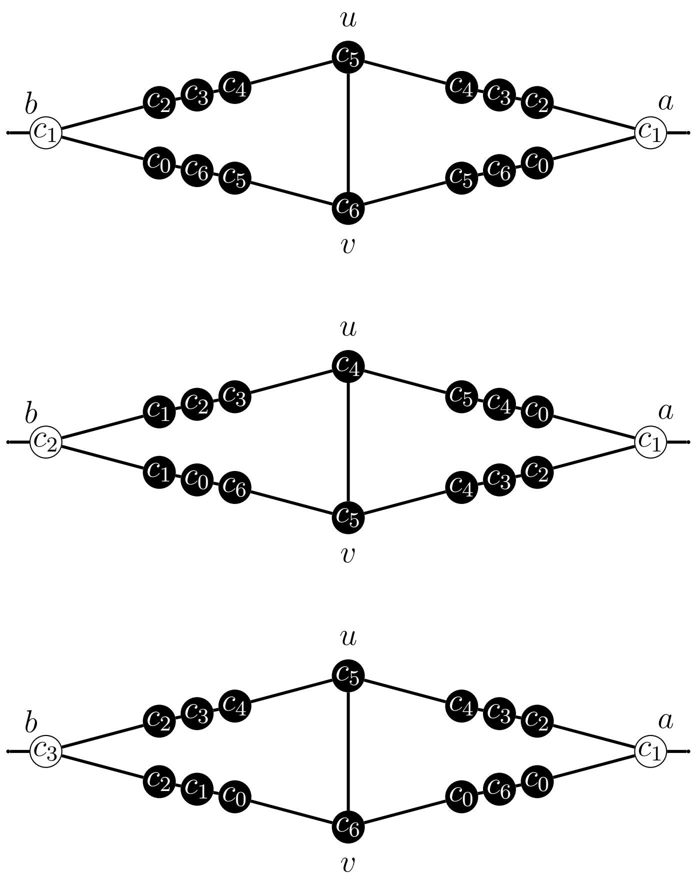

Figure 4.6: Figure for Lemma 4.2.7. Extensions of ϕ to G. The white vertices are of unknown degree, though their degree is at least that shown. The black vertices' degrees are as illustrated.

$\phi(b) = c_2$, we have that ϕ extends to a homomorphism of G to C_7, contradicting the fact that G is C_7-critical. Thus we may assume $G'_2 \notin \{C_3, C_5\}$, and since $v(G'_2) < v(G)$ and G is a minimum counterexample, it follows that $p(G'_2) < T$. Note since $G'_2 \not\subset G$ and G'_2 has minimum degree at least 2 by Lemma 2.1.3, the path P is contained in G'_2. Let F be the graph obtained from G'_2 by deleting $V(P) \setminus \{a, b\}$ and adding $S_a \cup S_b$. The potential of F is given by $p(F) = p(G'_2) + 17(6) - 15(6) \leq T + 12$. By Lemma 4.1.1, either $F = G$ or $G \in P_5(F) \cup P_4(F) \cup P_3(F)$. But since one of u and v is not contained in F and $\deg(u) = \deg(v) = 3$, this is a contradiction.

Case 2: $b \neq d$. Let G' be the graph obtained from $G \setminus (V(S_a \cup S_b \cup S_c \cup S_d) \setminus \{a, b, d\})$ by adding a 4-string S_{bd} with endpoints b and d. Note since $b \neq d$, G' does not contain a cycle of length 3 or 5, since a cycle containing S_{bd} has length at least 6.

Suppose G' admits a homomorphism ϕ to C_7. Without loss of generality, we may assume $\phi(a) = c_1$ and $\phi(b) \in \{c_1, c_2, c_3, c_4\}$. Note $\phi(b) \neq \phi(d)$, since $B_\phi(b|d, S_{bd}) = V(C_7) - \phi(b)$.

Claim 2. *The homomorphism ϕ extends to G.*

Proof. Let $I_1 = B_\phi(v|a, S_a) \cap B_\phi(v|b, S_b)$ and $I_2 = B_\phi(u|a, S_c) \cap B_\phi(u|d, S_d)$. Note since G is C_7 critical, it follows that $I_1 \cap N_{C_7}(I_2) = \emptyset$ as otherwise ϕ extends to G. Since each of S_a, S_b, S_c, and S_d is a 3-string, by Lemma 2.1.11 each of $|B_\phi(v|a, S_a)|$, $|B_\phi(v|b, S_b)|$, $|B_\phi(u|a, S_c)|$, and $|B_\phi(u|d, S_d)|$ is at least 5. Note since I_1 and I_2 are each the intersection of two subsets of $V(C_7)$ of size at least 5, it follows that $|I_1| \geq 3$ and $|I_2| \geq 3$. Suppose $|I_1| \geq 4$. Then $|N_{C_7}(I_1)| \geq 5$ by Lemma 2.1.10, and so $N_{C_7}(I_1) \cap I_2 \neq \emptyset$. But then ϕ extends to G, a contradiction. Therefore I_1 (and symmetrically, I_2) is a set of size exactly 3.

Since $\phi(a) = c_1$, it follows that $B_\phi(v|a, S_a) = \{c_1, c_3, c_4, c_5, c_6\}$. Since $\phi(b) \in \{c_1, c_2, c_3, c_4\}$ and $B_\phi(v|b, S_b) = \{c_i : i \in [7], i \neq \phi(b) \pm 1\}$, it follows that $\phi(b) \in \{c_2, c_4\}$.

Suppose $\phi(b) = c_2$. Then we have

$$\begin{aligned} I_1 &= B_\phi(v|a, S_a) \cap B_\phi(v|b, S_b) \\ &= \{c_1, c_3, c_4, c_5, c_6\} \cap \{c_2, c_4, c_5, c_6, c_7\} \\ &= \{c_4, c_5, c_6\}. \end{aligned}$$

But then $|N_{C_7}(I_1)| = 5$, and so it follows that $I_2 \cap N_{C_7}(I_1) \neq \emptyset$ since I_2 is a set of size 3. This contradicts the fact that ϕ does not extend to G. Thus we may assume $\phi(b) = c_4$.

Similarly, we may assume $\phi(d) \neq c_2$, and symmetrically, $\phi(d) \neq c_7$. Thus $\phi(d) \in \{c_4, c_5\}$. Since $\phi(d) \neq \phi(b)$, we have that $\phi(d) = c_5$. But then $I_1 = \{c_1, c_4, c_6\}$ and so $N(I_1) = \{c_2, c_3, c_5, c_7\}$. Since $I_2 = \{c_1, c_3, c_5\}$, we have that $N(I_1) \cap I_2 \neq \emptyset$, and so that ϕ extends to G. ☐

Since G is C_7-critical, Claim 2 is a contradiction and so we may assume that G' does not admit a homomorphism ϕ to C_7. Thus G' contains a C_7-critical subgraph G''. Note since G' does not contain a cycle of length 3 or 5, it follows that $G'' \notin \{C_3, C_5\}$. Furthermore, since $v(G'') < v(G)$ and G is a minimum counterexample, we have that $p(G'') \leq 2$. Note at least one edge in $E(S_{bd})$ is contained in $E(G'')$ since $G'' \not\subset G$. Furthermore, since G'' has minimum degree 2 by Lemma 2.1.3, it follows that $S_{bd} \subset G''$. Let F be the graph obtained from G'' by deleting $S_{bd} \setminus \{b, d\}$ and adding $S_b uv S_d$. Since this is a net addition of 4 vertices and 4 edges, it follows that $p(F) = p(G'') + 17(4) - 15(4) \leq 10$. By Lemma 4.1.1, since $T \geq 2$ it follows that either $F = G$ or $G \in P_5(F)$. But this is a contradiction, since $S_a \not\subset F$ and $S_c \not\subset F$.

□

Chapter 5

Discharging

Now that we have established the required structure of a minimum counterexample, the remainder of this thesis will be dedicated to proving Theorem 1.1.3 via discharging. The discharging will proceed in five stages: in each stage, charge will only be sent to structures that have not received charge in previous stages. It follows, then, that a structure in need of charge will only ever receive charge in a single stage.

Proof of Theorem 1.1.3. Suppose not. Let G be a minimum counterexample, and let $X \subseteq V(G)$ be the set of vertices of degree at least three. We assign an initial charge of $ch_0(v) = 15\deg(v) - 2wt(v) - 34$ to each vertex $v \in X$, and $ch_0(v) = 0$ for each $v \in V(G) \setminus X$. Note $\sum_{v \in X}(15\deg(v) - 2wt(v) - 34) = \sum_{v \in X}(15\deg(v) - 34) - \sum_{v \in V(G)\setminus X} 4$, since every vertex v of degree 2 contributes to the weight of two distinct vertices in X (namely, the endpoints of the string that contain v). Since $\sum_{v \in V(G)\setminus X} 4 = \sum_{v \in V(G)\setminus X} 34 - 15\deg(v)$, we have

$$\begin{aligned}\sum_{v \in V(G)} ch_0(V) &= \sum_{v \in V(G)} (15\deg(v) - 34) \\ &= 15 \sum_{v \in V(G)} \deg(v) - \sum_{v \in V(G)} 34 \\ &= 30e(G) - 34v(G) \\ &= -2p(G) \\ &\leq -2(T+1).\end{aligned}$$

Note since $T \geq 2$, this is at most -6. We will redistribute the charge amongst the vertices and cells until every vertex and cell has non-negative charge, contradicting the fact that

Weight six	Weight seven	Weight eight
$(4,2,0)$	$(4,3,0)$	$(4,2,2)$
$(4,1,1)$	$(4,2,1)$	
$(3,3,0)$	$(3,3,1)$	
$(3,2,1)$	$(3,2,2)$	
$(2,2,2)$		

Table 5.1: The types of poor vertices in G.

the sum of the charges is at most $-2(T+1)$. The proof will be done in two sections: in Section 5.1, we will show that after discharging no structures that start with non-negative charge end with negative charge. In Section 5.2, we will show that all structures that start with negative charge end with non-negative charge.

For simplicity, we define the following term.

Definition 5.0.1. *A vertex is poor if it has negative charge.*

Note by Corollary 2.2.1, if v is a vertex in $V(G)$, then $\mathrm{wt}(v) \leq 5\deg(v) - 7$. For a vertex $v \in X$, we therefore have $ch_0(v) \geq 15\deg(v) - 2(5\deg(v) - 7) - 34 = 5\deg(v) - 20$. Therefore the only possibly poor vertices are vertices of degree three. If v has degree three and is poor, then it has weight at least six since $ch_0(v) = 11 - 2\mathrm{wt}(v)$. By Corollary 2.2.1, vertices of degree three (and thus poor vertices) have weight at most eight. By Lemmas 4.2.1, 4.2.2, and 4.2.3 the only poor vertices of weight eight are of type (4,2,2). The poor vertices of weight seven are of type (4,3,0), (4,2,1), (3,3,1), or (3,2,2), and the poor vertices of weight six are of type (4,2,0), (4,1,1), (3,3,0), or (3,2,1). This is summarized in Table 5.1.

We will discharge in steps: each step consists of a single rule that will be carried out instantaneously throughout the graph. For convenience, since a single rule is carried out in each step, we will refer to the rules and steps interchangeably. At the end of Step i., the resulting charge of each cell and vertex will be denoted by ch_i.

Rule 1. Each vertex contained in a cell sends all of its charge to the cell that contains it. (Since cells are disjoint by Lemma 4.1.6, this is unambiguous.)

Rule 2. Let u and v share a short string. If u is in a cell C and v is poor after Step 1, C sends $-ch_1(v)$ charge to v.

Rule 3. Let u and v share a short string with $\deg(u) \geq 4$. If v is poor after Step 2, u sends $-ch_2(v)$ charge to v.

Rule 4. Let u and v share a short string with $\deg(u) = 3$ and $\text{wt}(u) \leq 4$. If v is poor after Step 3, u sends $-ch_3(v)$ charge to v.

Rule 5. Let u and v share a short string with $\deg(u) = 3$ and $\text{wt}(u) = 5$. If v is the only poor vertex that shares a short string with u after Step 4, then u sends $-ch_4(v)$ charge to v.

Before proceeding with the proof, we note two important facts regarding the discharging rules. First, the rules are performed sequentially. This ensures that in the later steps of the discharging process, we will have uncovered a significant amount of information regarding the local structure of the vertices and cells receiving charge. For instance, if a vertex v receives charge in Step 4, then v is not contained in a cell and does not share a short string with a vertex in a cell or a vertex of degree at least 4.

Second, vertices and cells only send charge along short strings. If a vertex or cell sends charge to many poor structures, this means it is incident with many short strings. It follows that the vertex or cell sending charge has relatively low weight, and so consequently has a large amount of charge to spare.

5.1 No New Negative Structures are Created

In this section, we will show that no cell or vertex x with initial charge $ch_0(x) \geq 0$ has negative final charge after discharging. First we will show that all cells have non-negative charge at the end of the discharging process (Lemma 5.1.1). We will then prove that no vertex with degree at least four is poor after Step 5 (Lemma 5.1.2). Finally with Lemmas 5.1.3 and 5.1.4, we will prove that all vertices of degree 3 and weight at most 5 have non-negative final charge. In Section 5.2, we will prove that all vertices of degree 3 and weight at least 6 have non-negative final charge. As cells and vertices are the only structures that carry charge at any point during the discharging, this will show that the sum of the charges is non-negative, contradicting our initial assumption and completing the proof of Theorem 1.1.3.

Lemma 5.1.1. *Let C be a cell in G. At the end of Step 2, $ch_2(C) \geq 0$.*

Proof. Let X be the set of vertices in C of degree at least three. At the end of Step 1, $ch_1(C) = \sum_{v \in X}(15\deg(v) - 2\text{wt}(v) - 34)$. Rewriting,

$$\begin{aligned} ch_1(C) &= \sum_{v \in X}((15\deg(v) - 30) - (2\mathrm{wt}(v) + 4)) \\ &= 15\sum_{v \in X}(\deg(v) - 2) - 2\sum_{v \in X}(\mathrm{wt}(C) + 2). \end{aligned}$$

But since $\sum_{v\in X}(\mathrm{wt}(v) + 2) = \mathrm{wt}(C) + 14$ and $\sum_{v\in X}(\deg(v) - 2) = \deg(C)$, we have $ch_1(C) = 15\deg(C) - 28 - 2\mathrm{wt}(C)$. We will split our further analysis into two cases depending on the degree of the cell C. Note $\deg(C) \geq 3$ by Lemma 2.2.3.

Case 1: $\deg(C) \geq 4$. Suppose that immediately after Step 1, there are p poor vertices that are each the endpoint of a short string whose other endpoint is in C. Note each of these p strings contributes at most 2 to the weight of C. For each poor vertex u we have $ch_1(u) \geq -3$, and so $ch_2(C) \geq ch_1(C) - 3p = 15\deg(C) - 28 - 2\mathrm{wt}(C) - 3p$. Since $\mathrm{wt}(C) \leq 4(\deg(C) - p) + 2p$, we have

$$\begin{aligned} ch_2(C) &\geq 15\deg(C) - 28 - 2(4(\deg(C) - p) + 2p) - 3p \\ &= 7\deg(C) - 28 + p. \end{aligned}$$

Since $\deg(C) \geq 4$, $ch_2(C) \geq p \geq 0$, as desired.

Case 2: $\deg(C) = 3$. Suppose for a contradiction that $ch_2(C) < 0$. Suppose first $|X| = 2$. Then X contains a single vertex of degree 3, and a single vertex of degree 4. Note all vertices that are poor immediately after Step 1 have weight at most seven, since vertices of type $(4, 4, 2)$ are contained in cells and so send their charge to the cells that contain them in Step 1. Let v be the vertex of degree 3 in X. Let the three strings incident with v be S_1, S_2 and S_3, named such that S_1 and S_2 are contained in C. Suppose that S_1 is a 4-string. By Lemma 4.1.3, $S_2 \cup S_3$ is contained in a cell $C' \neq C$, contradicting Lemma 4.1.6. Thus we may assume S_1 is not a 4-string. Symmetrically, S_2 is not a 4-string.

Since $|X| = 2$ by assumption, we may therefore assume without loss of generality that S_1 is a 3-string and S_2 is a 2-string. Note S_3 is therefore a 0-string. To see this, suppose not. Then by Lemma 4.1.4 applied to v and S_1, we have that $S_2 \cup S_3$ (and in particular, S_2) is contained in either a cell or a 9-cycle $C' \neq C$. Suppose first C' is a cell. Since $S_2 \in C \cap C'$, this contradicts Lemma 4.1.6 as distinct cells in G are vertex-disjoint. Thus we may assume C' is a 9-cycle. Since $S_2 \in C \cap C'$, this contradicts Lemma 4.1.7 since cells and 9-cycles are edge-disjoint.

Thus we may assume S_3 is a 0-string. Let v_1 be the vertex that shares S_3 with v. Note if $ch_2(v_1) < 0$, then v_1 is a vertex of degree 3 not contained in a 7-cycle. By Lemma 4.1.3, v_1 is thus not incident with a 4-string. Since v_1 is incident with a 0-string S_3, we have that $\text{wt}(v_1) \leq 6$. Thus if $ch_2(v_1) < 0$, it follows that $ch_2(v_1) = -1$. Let u be the vertex of degree 4 in X, and let $S_4 \nsubseteq C$ and $S_5 \nsubseteq C$ be the strings incident with u, with endpoints $u_1 \neq u$ and $u_2 \neq u$, respectively. Note since $ch_2(C) = 17 - 2\text{wt}(C)$, if $\text{wt}(C) \leq 5$, then $ch_2(C) \geq 0$ since C pays at most 1 to v_1 in Step 2, and at most 3 to each of u_1 and u_2 in Step 2. Thus we may assume $\text{wt}(C) \geq 6$. Note also if neither u_1 nor u_2 is poor immediately after Step 1, then $ch_2(C) \geq 0$ since $ch_2(C) \geq 17 - 2\text{wt}(C) - ch_1(v_1)$, and $\text{wt}(C) \leq 8$ by Lemma 2.2.4. Thus we may assume at least one of u_1 and u_2 receives charge from C in Step 2, and so that at least one of S_4 and S_5 is a short string. Note that G does not contain a k-string with $k \geq 5$ by Lemma 2.1.12. Furthermore, since at least one of S_4 and S_5 is short, S_4 and S_5 together contribute at most 6 to the weight of C. Thus $\text{wt}(v) \leq 6$, and since $\text{wt}(C) \geq 6$, it follows that $\text{wt}(C) = 6$. Since S_3 is a 0-string and at least one of S_4 and S_5 is a short string, it follows that exactly one of S_4 and S_5 is a short string, and so that C sends charge to exactly one of u_1 and u_2. But then $ch_2(C) \geq 0$, since $ch_2(C) = 17 - 2\text{wt}(C) = 5$ and C sends at most 1 to v_1 and 3 to one of u_1 and u_2.

Thus we may assume $|X| = 3$. Since $\deg(C) = 3$, it follows that X contains three vertices of degree three. Let S_0 be a k-string with $k \leq 2$ and endpoints u, v such that $v \in V(C)$. Suppose that u is poor after Step 1. Then u is a degree three vertex not contained in a cell. Let S_1 and S_2 be the other strings incident with u.

Claim 1. *The weight of u is at most six.*

Proof. Suppose not. First suppose either of S_1 or S_2 is a 4-string. By Lemma 4.1.3 applied to u, v is contained in a cell $C' \neq C$, contradicting Lemma 4.1.6. Thus u is not contained in a 4-string. Suppose now $k \in \{1, 2\}$ (i.e. that S_0 is either a 1- or 2-string). First suppose that S_1 is a 3-string and that S_2 is not a 0-string. Then by Lemma 4.1.4 applied to u and S_1, there exists an edge $vv_1 \in E(C)$ contained in a cell or 9-cycle $C' \neq C$. If C' is a 9-cycle, this is a contradiction since 9-cycles and cells are edge-disjoint by Lemma 4.1.7. If C' is a cell, this too is a contradiction since distinct cells are vertex-disjoint by Lemma 4.1.6. Thus if S_1 is a 3-string, we may assume S_2 is a 0-string. Symmetrically, if S_2 is a 3-string, we may assume S_1 is a 0-string.

If $k \geq 1$, then S_1 and S_2 therefore contribute at most 4 to the weight of C. If $k = 0$, since neither S_1 nor S_2 is a 4-string, they contribute at most 6 to the weight of C. Note by assumption $k \leq 2$. Thus u has weight at most six. Since by assumption $ch_2(u) < 0$, it follows that u has weight exactly six. □

Since u has weight exactly six, we have that $ch_2(u) = -1$.

Let A be the set of vertices that are poor immediately after Step 1, and that are incident with a short string incident with C. Let $|A| - p$. Note that since C has degree 3, it follows that $p \leq 3$. After Step 2, we have $ch_2(C) = ch_1(C) - p = 17 - 2\text{wt(C)} - p$. Note if $\text{wt}(C) \leq 7$, then since $p \leq 3$ we have that $ch_2(C) \geq 0$, a contradiction. Thus we may assume that $\text{wt}(C) \geq 8$. Note also that by Lemma 2.2.4, $\text{wt}(C) \leq 8$, and so we may assume $\text{wt}(C) = 8$. Note also since $ch_2(C) < 0$, we have $p \geq 2$. Thus at least two of the strings incident with C each contribute at most 2 to the weight of C. Since C has weight 8, it follows that C is incident with a 4-string and so that $p \leq 2$. Thus we may assume $p = 2$, and since $\text{wt}(C) = 8$, it follows that C is a $(4, 2, 2)$-cell.

Let $S = u_0u_1u_2u_3$ be a 2-string incident with C, such that $u_0 \in V(C)$. Note since u_3 is poor after Step 2, it is not contained in a cell. Let u_4 and u_5 be u_3's neighbours not contained in S.

Let G' be the graph obtained from G by identifying u_4 and u_5 to a new vertex z. Note G' contains a cell of weight nine, and so G' is not C_7-critical by Lemma 2.2.4. Furthermore, G' admits no homomorphism to C_7, as any such homomorphism ϕ extends to G by setting $\phi(u_4) = \phi(u_5) = \phi(z)$. Therefore G' contains a C_7-critical subgraph G'', and since $G'' \not\subset G$, we have $z \in V(G'')$.

Note G' contains no 5-cycles nor triangles, since $u_3 \in V(G)$ is not contained in a 7-cycle by assumption. Since $v(G'') \leq v(G)$, G'' is not a counterexample to Theorem 1.1.3 and thus $p(G'') \leq T$. Note at least one string incident with C or at least one string $S' \subset C$ is not contained in G'', as otherwise C has weight 9 in G'', contradicting Lemma 2.2.4.

Suppose first $u_3 \notin V(G'')$. Since G'' has minimum degree at least 2 by Lemma 2.1.3, it follows that $u_2 \notin V(G'')$. Let F be the graph obtained from G'' by splitting z back to u_4, u_5 and adding in the path $u_4u_3u_5$. Then $p(F) = p(G'') + 17(2) - 15(2) \leq T + 4$. Since $u_2 \notin V(G'')$ and since $F \subset G$, this contradicts Lemma 4.1.1.

We may therefore assume that $u_3 \in V(G'')$, and so there exists a string $S' \neq S$ whose internal vertices are not contained in $V(G'')$. Note either S' is incident with C, in which case it has at least 2 internal vertices, or $S' \subset C$, in which case $V(C) \cap V(G'') = \emptyset$ since G'' contains no k-string with $k \geq 5$ by Lemma 2.1.12. Thus we have $V(G) \setminus V(G'') \neq \emptyset$. Let F be the graph obtained from G'' by splitting z back to u_4 and u_5, and adding edges to create the path $u_4u_3u_5$. Then $p(G) = p(G'') + 17(1) - 15(1) \leq T + 2$. Since $F \subset G$ but $F \subsetneq G$, again this contradicts Lemma 4.1.1. □

Lemma 5.1.2. *Let $v \in V(G)$ be a vertex of degree at least four. At the end of Step 3, $ch_3(v) \geq 0$.*

Proof. Suppose not. Let A be the set of vertices that are poor immediately after Step 2 and that each share a short string with v. Let $p = |A|$. Note none of the p vertices in A are contained in cells, as otherwise they have charge at least 0 at the end of Step 1. Furthermore, v is not contained in a cell, as this cell would send charge to the vertices in A in Step 2. Note since all vertices of degree 3 and weight 8 are contained in cells by Lemma 4.1.3, each vertex in A has weight at most 7, and so v sends at most $3p$ units of charge in Step 3.

First, suppose $\deg(v) \geq 5$. At the end of Step 3, we have $ch_3(v) \geq ch_2(v) - 3p = 15\deg(v) - 2\mathrm{wt}(v) - 34 - 3p$. Note since v is incident with p short strings and the remaining $\deg(v) - p$ strings incident with v each contribute at most 4 to the weight of v, it follows that $\mathrm{wt}(v) \leq 4(\deg(v) - p) + 2p$. Thus $ch_3(v) \geq 15\deg(v) - 2(4\deg(v) - 2p) - 34 - 3p = 7\deg(v) - 34 + p \geq 1 + p$, since $\deg(v) \geq 5$. Since p is non-negative, $ch_3(v) \geq 0$, a contradiction.

Thus we may assume $\deg(v) = 4$. Since v is not contained in a cell, by Lemma 4.1.5 v is not incident with a 4-string. Since v is incident with at least p short strings and at most $(\deg(v) - p)$ k-string with $k = 3$, it follows that v has weight at most $3(\deg(v) - p) + 2p = 12 - p$, and so

$$\begin{aligned} ch_3(v) \geq ch_2(v) - 3p &= 15\deg(v) - 2\mathrm{wt}(v) - 34 - 3p \\ &\geq 60 - 2(12 - p) - 34 - 3p \\ &= 2 - p. \end{aligned}$$

Thus if $p \leq 2$, we have $ch_3(v) \geq 0$, a contradiction.

Since $p \leq \deg(v) = 4$, we may therefore assume $p \in \{3, 4\}$. Note if $\mathrm{wt}(v) \leq 7$, since

$$\begin{aligned} ch_3(v) &\geq 15\deg(v) - 2\mathrm{wt}(v) - 34 - 3p \\ &\geq 60 - 14 - 34 - 3p \\ &= 12 - 3p \end{aligned}$$

it follows that $ch_3(v) \geq 0$, a contradiction.

Thus v has weight at least 8 and is incident with at least three short strings. It follows that v is either a vertex of type $(3, 2, 2, 2)$, $(2, 2, 2, 2)$, or $(3, 2, 2, 1)$. Note if v shares a short string with a poor vertex v' of weight 6, then v pays v' only 1 and each of the other $(p-1)$ vertices in A at most 3. Thus v pays the vertices in A at most $3(p - 1) + 1$, and so it follows that $ch_3(v) \geq 4 - p$. Since $p \in \{3, 4\}$, this is non-negative. Thus we may assume that v pays at least three poor vertices of weight 7. We finish the proof with the following three claims.

Claim 1. *v is not a vertex of type* $(2, 2, 2, 2)$.

Proof. Suppose not. Recall v only pays a subset of the vertices of degree 3 that are not contained in cells. Furthermore, since v is of type $(2,2,2,2)$, v only pays vertices that are incident with 2-strings. Since the only vertices of degree 3 and weight 7 not contained in cells and incident with 2-strings are vertices of type $(3,2,2)$, we may assume v pays at least three vertices of type $(3,2,2)$. Since each vertex of type $(3,2,2)$ in A is not in a cell, by Lemma 4.1.4 they are each contained in 9-cycles. Thus there exist vertices $a \neq b$ in A of type $(3,3,2)$, such that $S_a = aa_1a_2v$ is a 2-string shared by v and a, S_b is a 2-string shared by v and b, and S_{ab} is a 2-string shared by a and b. Similarly, there exist vertices $c \neq d$ in A such that c is of type $(3,2,2)$, S_c is a 2-string shared by v and c, S_d is a 2-string shared by v and d, and S_{cd} is a 2-string shared by c and d. Let C be the 9-cycle formed by $S_a \cup S_b \cup S_{ab}$. Let G' be the graph obtained from G by identifying v and a_1 to a new vertex z, and deleting a_2. Note G' does not admit a homomorphism ϕ to C_7 as otherwise ϕ extends to G by setting $\phi(v) = \phi(a_1) = \phi(z)$ and $\phi(a_2) \in N_{C_7}(\phi(z))$. Furthermore, G' itself is not C_7-critical, as the cycle C' obtained from C with the identification of v and a_1 is a cell of weight 10 in G', contradicting Lemma 2.2.4. Thus G' contains a C_7-critical subgraph G''.

Note G'' does not contain at least one string incident with C' or one string in $E(C')$. We claim one of these strings not contained in G'' contains at least one internal vertex. To see this, suppose not. Note C' is incident with four strings: S_c, S_d, and two 3-strings S'_a and S'_b incident with a and b, respectively. Since these four strings all have internal vertices, we may assume they are all contained in G''. Since S_{ab} and S_b are both 2-strings, we may assume they are both contained in G''. Thus G' does not contain the edge az. But then $S_{ab} \cup S'_a$ is a 6-string contained in G'', contradicting Lemma 2.1.12.

Thus we may assume there is a vertex in $V(G) \setminus \{a_1, a_2, v\}$ that is not contained in $V(G'')$.

Since $G'' \not\subset G$, it follows that G'' contains the new vertex z. Note G'' does not contain a triangle or 5-cycle, since S_a is contained in a 9-cycle in G and so is not contained in a cell by Lemma 4.1.7. Since $v(G'') < v(G')$, it follows that $p(G'') \leq T$.

Let F be the graph obtained from G'' by splitting z back into the vertices v and a_1 and adding in the path va_2a_1. Then $p(F) = p(G'') + 17(2) - 15(2) \leq T + 4$. But since $F \neq G$, this contradicts Lemma 4.1.1. □

Claim 2. *v is not a vertex of type* $(3,2,2,2)$.

Proof. Suppose not. Since v is of type $(3,2,2,2)$, v only pays vertices that are incident with 2-strings. Since the only vertices of degree 3 and weight 7 not contained in cells and incident with 2-strings are vertices of type $(3,2,2)$, we may assume v pays three vertices of type $(3,2,2)$. Since each vertex of type $(3,2,2)$ in A is not in a cell, by Lemma 4.1.4 they are each contained in 9-cycle. Let $a \neq b \neq c$ be the three vertices of type $(3,2,2)$ in A. Let S_a, S_b, and S_c be the 2-strings shared by v with a, b, and c, respectively. Let S'_a be the other 2-string incident with a. By Lemma 4.1.4, $S'_a \cup S_a$ is contained in a 9-cycle C. Since v has degree exactly 4 and is incident only with 2-strings and a 3-string, one of S_b and S_c is contained in C. Without loss of generality, we may assume $C = S_a \cup S'_a \cup S_b$, and so that S'_a is shared by b and a. Similarly, let $S'_c \neq S_c$ be a 2-string incident with c. By Lemma 4.1.4, $S_c \cup S'_c$ is contained in a 9-cycle $C' \neq C$. Since v has degree exactly 4 and is incident only with 2-strings and a 3-string, it follows that there exists a 2-string $S \neq S_c$ incident with v that is contained in C'. Thus S_a or S_b is contained in C'. But this contradicts Lemma 4.1.8, since $S_b \cup S_a \subset C$. □

The only remaining possibility is then that v is of type (3,2,2,1). Since $|A| \geq 3$ and each vertex in A has weight 7, we may assume v shares its incident 2-strings S_1 and S_2 with two poor $(3,2,2)$ vertices u_1 and u_2, and its incident 1-string S_3 with a poor $(3,3,1)$ vertex u_3. By applying Lemma 4.1.4 to u_1, u_2, and u_3, we find each of S_1 and S_2 is contained in two 9-cycles, contradicting Lemma 4.1.8. □

Lemma 5.1.3. *Let $v \in V(G)$ be a vertex of degree 3 and weight at most 4. At the end of Step 4, $ch_4(v) \geq 0$.*

Proof. Suppose not. Let A be the set of vertices that are poor immediately after Step 4 and that each share a short string with v. Let $p = |A|$. Note none of the p vertices in A are contained in cells, as otherwise they have charge at least 0 at the end of Step 1. Furthermore, v is not contained in a cell, as otherwise this cell sends charge to the vertices in A in Step 2. Note since all vertices of degree 3 and weight 8 are contained in cells by Lemma 4.1.3, each vertex in A has weight at most 7, and so v sends at most $3p$ units of charge in Step 3.

Note if A contains $p \leq 3$ vertices of weight 6, then v sends only 3 units of charge in Step 4, and so

$$\begin{aligned} ch_4(v) &\geq ch_3(v) - 3 \\ &= 15\deg(v) - 2\mathrm{wt}(v) - 34 - 3 \\ &\geq 45 - 8 - 34 - 3 \\ &= 0, \text{ a contradiction.} \end{aligned}$$

We may therefore assume A contains a vertex u of weight at least 7. Since A only contains vertices of weight at most seven, we may assume u has weight exactly 7. Furthermore, if $p = 1$ (and so v sends charge to only one vertex in Step 4), then $ch_4(v) = ch_3(v) - 3 = 0$, a contradiction. Thus we may assume $p \geq 2$. Since vertices of type $(4, 3, 0)$ and $(4, 2, 1)$ are contained in cells and thus have charge 0 after Step 1, we may assume u is either of type $(3, 3, 1)$ or of type $(3, 2, 2)$.

First suppose u is of type $(3, 2, 2)$. Let u_1 be the other vertex that shares a 2-string with u. Note if u_1 is contained in a cell, then $ch_4(u) = 0$ by Step 2, and so $u \notin A$, a contradiction. Furthermore, if $\deg(u_1) \geq 4$, then $ch_4(u) = 0$ by Step 3, and so again $u \notin A$. We may therefore assume u_1 has degree three. But then this contradicts Lemma 4.2.5, as neither v nor u_1 is contained in a cell.

Thus we may assume u is of type $(3, 3, 1)$. Let S_1 and S_3 be the two 3-strings incident with u, and let S_2 be the 1-string shared by u and v (see Figure 5.1. By Lemma 4.1.4, since u is a $(3, 3, 1)$-vertex not contained in a cell, we have that $S_2 \cup S_3$ is contained in a 9-cycle C. Similarly, $S_1 \cup S_2$ is contained in a 9-cycle $C' \neq C$. Since both 9-cycles contain S_2, we have that $C' \cap C = S_2$ by Lemma 4.1.8. Let S_4 and S_5 be the other two strings incident with v. Without loss of generality, suppose $S_4 \subset C'$ and $S_5 \subset C$. Note since $p \geq 2$, one of S_4 and S_5 is shared by v with a vertex in A. Without loss of generality, we may assume S_4 is shared by v and a vertex w in A.

Since v has weight at most four and is incident with a 1-string S_2, it follows that at least one of S_4 and S_5 is not a 2-string.

First suppose S_4 is not a 2-string. Since $w \in A$, we have that w has degree three and $\mathrm{wt}(w) \geq 6$. Furthermore, since w is not contained in a cell, w is not incident with a 4-string by Lemma 4.1.3. Since the internal vertices of two of the strings incident with w are contained in $C' \setminus V(S_1 \cup S_2)$, together these two strings contribute at most 1 to the weight of w. But then w has weight at most 4, a contradiction.

Thus we may assume S_4 is a 2-string, and since w has weight at least 6, we have that w is a vertex of type $(3, 2, k)$ with $k \geq 1$.

Let S_6 denote the third string incident with w (so w is incident with S_1, S_4, and S_6). By Lemma 4.1.4 applied to w, S_4 and S_6 are contained in a 9-cycle C''. Since S_4 is contained in $C' \cap C''$, this contradicts Lemma 4.1.8. □

Lemma 5.1.4. *Let $v \in V(G)$ be a vertex of degree 3 and weight 5 that shares a short string with only one poor vertex at the end of Step 4. At the end of Step 5, $ch_5(v) \geq 0$.*

Proof. Suppose not. Let u be the vertex with $ch_4(u) < 0$ that shares a short string with

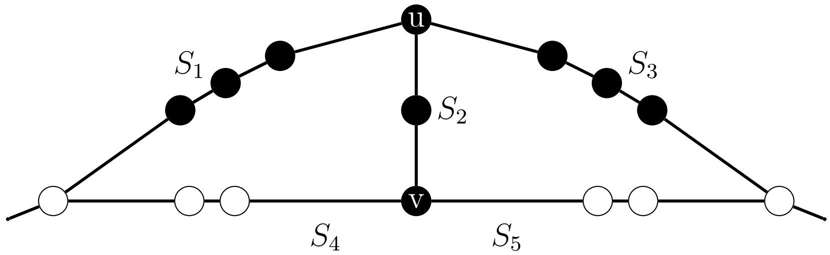

Figure 5.1: Figure for Lemma 5.1.3. v has weight at most four and degree three. The white vertices are of unknown degree, though their degree is at least that shown. The black vertices' degrees are as illustrated.

v. Suppose $\text{wt}(u) = 6$. Then $ch_4(u) = -1$, and so

$$\begin{aligned} ch_5(v) &= ch_4(v) - ch_4(u) \\ &= 15\deg(v) - 2\text{wt}(v) - 34 - 1 \\ &= 45 - 10 - 34 - 1 \\ &= 0, \text{ a contradiction.} \end{aligned}$$

We may therefore assume u has weight at least seven.

Note since $ch_4(u) < 0$, u is not contained in a cell by the discharging rules. By Lemma 4.1.3, it follows that u is not of type (4,2,2), (4,3,0), or (4,2,1). Thus we may assume u is a vertex of type either (3,3,1), or (3,2,2). By Lemma 4.2.5, if u is a (3,2,2)-vertex it has charge at least 0 by either Step 1, 2, or 3.

Therefore we may assume u is a vertex of type (3,3,1), and hence the string shared by u and v is a 1-string. Note v is not incident with a 4-string as otherwise it is contained in a cell by Lemma 4.1.5 and so $ch_2(u) \geq 0$, contradicting the fact that $u \in A$. Thus since v has degree 3 and weight 5 and is incident with a 1-string, it is either a vertex of type (3,1,1) or of type (2,2,1).

First suppose v is of type $(3,1,1)$. Let S_1 and S_2 be the two 1-strings incident with v, named such that S_1 is shared by u and v. Let S_3 be the 3-string incident with v. Let the two 3-strings incident with u be named S_4 and S_5. Note S_4 and S_5 do not have the same two endpoints by Lemma 2.2.6. Furthermore v and u share only S_1 by Lemma 2.2.6. By Lemma 4.1.4 applied to u and S_4, since u is not contained in a cell $S_1 \cup S_5$ is contained in a 9-cycle C. Note since v has degree 3, it follows that either S_2 or S_3 is contained in C. Since $v(S_1 \cup S_5 \cup S_3) = 11$, we may assume that $S_2 \subset C$.

Similarly, by Lemma 4.1.4 applied to u and S_5, we have that $S_4 \cup S_1$ is contained in a 9-cycle C'. Note since v has degree 3, it follows that either S_2 or S_3 is contained in C'. Since $v(S_1 \cup S_4 \cup S_3) = 11$, it follows that $S_2 \subset C'$.

Since $S_1 \cup S_2 \subset C \cap C'$, this contradicts Lemma 4.1.8.

Therefore we may assume v is a $(2,2,1)$-vertex. Let a and b be the two vertices that share a 2-string with v. Note $a \neq b$ by Lemma 2.2.6. Let S_a be the string shared by a and v, and let S_b be the 2-string shared by b and v. Let S_1, S_2 and S_3 be the three strings incident with u, named such that S_1 is shared by u and v. Note S_2 and S_3 do not have the same endpoints by Lemma 2.2.6. By Lemma 4.1.4 applied to u and S_2, since u is not contained in a cell we have that $S_3 \cup S_1$ is contained in a 9-cycle C. Similarly, by applying Lemma 4.1.4 to u and S_3, we have that $S_2 \cup S_1$ is contained in a 9-cycle C'. Thus without loss of generality, we may assume S_2 is shared by a and u, and that S_3 is shared by b and u.

Let G' be the graph obtained from G by deleting u, v, and all of the internal vertices of their incident strings. Since G is C_7-critical, G' has a homomorphism ϕ to a cycle $C = c_1c_2c_3c_4c_5c_6c_7c_1$. Without loss of generality, we may assume $\phi(a) = c_1$ and $\phi(b) \in \{c_1, c_2, c_3, c_4\}$. Since ϕ does not extend to G, $\phi(b) = c_4$. (To see the extensions of all other homomorphisms to G, see Figure 5.2.) Let G'' be the graph obtained from G' by adding a new vertex z and edges az and bz. Note now the following:

Claim 1. *There does not exist a homomorphism $\phi : G'' \to C$ with $\phi(a) = c_1$ and $\phi(b) = c_4$.*

Proof. Suppose $\phi : G'' \to C$ is such that $\phi(a) = c_1$. Note $\phi(b) \in B_\phi(b|a, azb)$. But $B_\phi(b|a, azb) = N_C(N_C(c_1)) = \{c_1, c_3, c_6\}$. □

Thus if G'' admits a homomorphism to C, this homomorphism extends to a homomorphism of G to C, since by Claim 1 there does not exist a homomorphism ϕ from G'' to C with $\phi(a) = c_1$ and $\phi(b) = c_4$. We may thus assume G'' contains a C_7-critical subgraph G''', and since $G''' \not\subset G$, we have $z \in V(G''')$. Furthermore, since G''' has minimum degree at least two, $\{az, zb\} \in E(G''')$.

Suppose G''' is a triangle. Then ab is an edge in $E(G)$. But then abS_aS_b is a cell C'' with $S_a \subset C'' \cap C'$, contradicting Lemma 4.1.7. Suppose now G''' is a 5-cycle. Then there exists an (a,b)-path P of length 3. But then $P \cup S_a \cup S_b$ is a 9-cycle C'' with $S_a \subset C'' \cap C'$, contradicting Lemma 4.1.8.

Thus we may assume that G''' is not a triangle or 5-cycle. Since $v(G''') < v(G)$, it follows that G''' is not a counterexample to Theorem 1.1.3, and so $p(G''') \leq T$. Let F be the graph

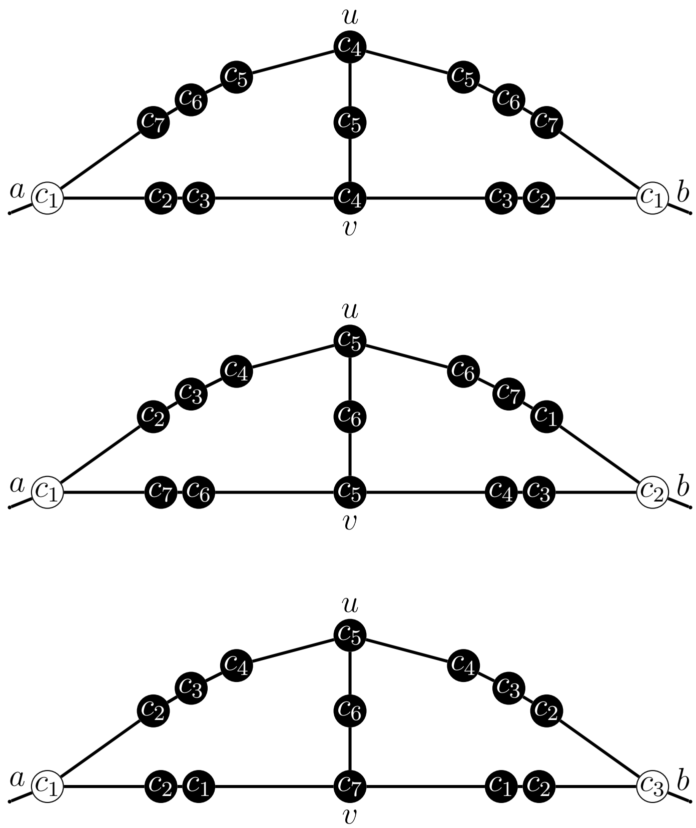

Figure 5.2: Figure for Claim 5.1.4. Extensions of ϕ to G. The white vertices are of unknown degree, though their degree is at least that shown. The black vertices' degrees are as illustrated.

obtained from G''' by deleting z and adding $S_a \cup S_b$. Then $p(F) = p(G''') + 17(4) - 15(4) \leq T + 8$. But since u is a vertex of degree 3 and $u \notin V(F)$, this contradicts Lemma 4.1.1. $\square$

5.2 Every Initially Poor Structure Receives Charge

We have shown that no new poor vertices or cells are created through discharging. Since no cell is initially poor, it remains to show only the following lemma.

Lemma 5.2.1. *Let $v \in V(G)$ have $ch_0(v) < 0$. At the end of Step 5, $ch_5(v) \geq 0$.*

Proof. Suppose not. Since $ch_0(v) < 0$, we may assume v is a vertex of degree 3 and weight at least 6. We may assume v is not contained in a cell, as otherwise v receives $-ch_0(v)$ charge from its cell in Step 1, resulting in $ch_5(v) = ch_1(v) = 0$. Since v is not in a cell, by Lemma 4.1.3 v is not incident with a 4-string. It follows that v is not of type $(4, 2, 2)$, $(4, 3, 0), (4, 2, 1), (4, 2, 0)$, or $(4, 1, 1)$. We may also assume v does not share a short string with a vertex in a cell or a vertex of degree at least 4, as otherwise v receives charge at least $-ch_0(v)$ in either Step 2 or Step 3, a contradiction. Furthermore, we may assume v does not share a short string with a vertex of degree 3 of weight at most 4, as otherwise $ch_5(v) \geq 0$ by Step 4. Finally, by Step 5 we may assume that if v shares a short string with a vertex u of degree 3 and weight 5, then v is not the only poor vertex that shares a short string with u.

Note v is not a vertex of type $(3, 2, 2)$. To see this, suppose not. Let a and b be the vertices that share a short string with v. Note $a \neq b$ by Lemma 2.2.6. Since $ch_5(v) < 0$, neither a nor b is contained in a cell, and both a and b have degree 3. But by Lemma 4.2.5, since v is not contained in a cell and $\deg(a) = \deg(b) = 3$, at least one of a and b is contained in a cell, a contradiction.

Note furthermore the following claim.

Claim 5.2.2. *v is not of type* $(2, 2, 2)$.

Proof. Suppose not. By Lemma 4.2.6, either v is contained in a cell, v shares a short string with a cell, or v shares a short string with a vertex of degree at least 4. But then v receives charge in either Step 1, Step 2, or Step 3, contradicting that $ch_5(v) < 0$. □

Thus v is either a vertex of type (3,3,0), (3,2,1), or (3,3,1).

Claim 5.2.3. *v is not of type (3,3,1).*

Proof. Suppose not, and let u be the vertex with which v shares its short string. Since $ch_5(v) < 0$, it follows that u is either a vertex of degree 3 and weight at least 6, or a vertex of degree 3 and weight 5 that shares short strings with at least two vertices that are poor after Step 5. Let A be the set of vertices that share a short string with u and that have negative charge after Step 5.

Let S_1 and S_3 be the 3-strings incident with v, and let S_2 be the 1-string shared by u and v. Let a and b be the other endpoints of the strings S_1 and S_3, respectively. Note $a \neq b$ by Lemma 2.2.6. By Lemma 4.1.4 applied to v, since v is not contained in a cell by assumption, $S_2 \cup S_3$ is contained in a 9-cycle C_1. Similarly, $S_2 \cup S_1$ is contained in a 9-cycle C_2. Let $S_4 \neq S_2$ and $S_5 \notin \{S_2, S_4\}$ be the other two strings adjacent with u, such that at least one edge from S_5 contained in C_1. Note C_2 does not contain an edge e in S_5 as otherwise $eS_2 \subset C_1 \cap C_2$, contradicting Lemma 4.1.8. Since u has degree 3, and the distance from each of a and b to u along C_2 and C_1, respectively, is at most three, we have that u has weight at most five. Since $\text{wt}(u) \geq 5$, we have therefore that $\text{wt}(u) = 5$, and so that S_4 and S_5 are 2-strings with endpoints a, u and b, u, respectively. Furthermore, since $|A| \geq 2$, at least one of a and b has negative final charge. Without loss of generality, we may assume $a \in A$, and so that a has degree 3 and $\text{wt}(a) \geq 6$. Let S_4 be the 2-string shared with u by a. Let $S_6 \notin \{S_1, S_4\}$ be the third string incident with a. Since a has weight at least six, since S_1 is a 3-string, and since S_4 is a 2-string, we have that S_6 is a k-string with $k \geq 1$. But then Lemma 4.1.4 applies to a, and so S_6 and S_4 are contained in a 9-cycle C_3. Since $S_4 \subset C_3 \cap C_1$, this contradicts Lemma 4.1.8. We may therefore assume v is not a vertex of type $(3, 3, 1)$. □

Claim 5.2.4. *v is not a vertex of type (3,2,1).*

Proof. Suppose not. Let a, b, and c be the vertices that share a 1-string, 2-string, and 3-string, respectively, with v. Let S_a, S_b and S_c be the three strings incident with v, such that S_a is incident with a, S_b is incident with b, and S_c is incident with c. Note $a \neq b$ and $a \neq c$ since G has girth at least 7 by Lemma 4.1.2. Furthermore, $b \neq c$ since v is not contained in a cell by assumption. By Lemma 4.1.4, $S_b \cup S_a$ is contained in a 9-cycle C. Let S_{ab} be the (a, b)-path of length 4 in C with $v \notin V(S_{ab})$.

Note since $ch_5(v) < 0$, it follows from the discharging rules that each of a and b has degree 3, is not contained in a cell, and has weight at least 5. Suppose first that S_{ab} is a 3-string. Since a has weight at least 5, it is incident with a k-string S_d, with $k \geq 1$. By Lemma 4.1.4, $S_a \cup S_d$ is contained in a 9-cycle $C' \neq C$. Note $E(S_b) \cap E(C') = \emptyset$, as otherwise $S_a \cup S_b \subset C \cap C'$, contradicting Lemma 4.1.8. Thus we may assume S_c is contained in C', and so $k \leq 2$.

Suppose first that $k = 2$. Let $G' = G \setminus (V(S_a \cup S_b \cup S_c \cup S_d \cup S_{ab}) \setminus \{b, c\})$. Since $G' \subsetneq G$ and G is C_7-critical, G' admits a homomorphism ϕ to $C = c_1c_2c_3c_4c_5c_6c_7c_1$. Without loss of generality, we may assume $\phi(b) = c_1$ and $\phi(c) \in \{c_1, c_2, c_3, c_4\}$. Note $\phi(c) = c_3$ as otherwise ϕ extends to G. To see this, see Figure 5.3. Let $G_1 \in P_3(G')$ be the graph obtained from G' by adding a (b, c)-path P of length 3.

Claim 1. *There does not exist a homomorphism $\phi : G_1 \to C$ with $\phi(b) = c_1$ and $\phi(c) = c_3$.*

Proof. Suppose $\phi : G_1 \to C$ is such that $\phi(b) = c_1$. Note $\phi(c) \in B_\phi(c|b, P)$. But $B_\phi(c|b, P) = N_C(N_C(N_C(c_1))) = \{c_2, c_7, c_4, c_5\}$. □

By Claim 1, it follows that G_1 does not admit a homomorphism to C_7. Therefore G_1 contains a C_7-critical subgraph G_2. Since $G_2 \not\subset G$ and G_2 has minimum degree 2, $P \subset G_2$. Note since $b \neq c$, it follows that G_2 is not a triangle.

Suppose G_2 is a 5-cycle. Then there exists a (b,c)-path Q of length 2 in G. Let $F = Q \cup S_a \cup S_b \cup S_c \cup S_d \cup S_{ab}$. Note $v(F) = 16$ and $e(F) = 18$, and so it follows that $p(F) = 17(16) - 15(18) = 2$. This contradicts Lemma 4.1.1, since $T \geq 2$.

We may therefore assume that G_2 is not a triangle or 5-cycle. Since $v(G_2) < v(G)$ and G is a minimum counterexample, it follows that $p(G_2) \leq T$. Let F be the graph obtained from G_2 by deleting $P \setminus \{a,b\}$ and adding $S_d \cup S_{ab}$. Then $p(F) = p(G_2) + 17(4) - 15(4) \leq T + 8$. By Lemma 4.1.1, either $F = G$ or $G \in P_5(F)$. But since v is not contained in F and $\deg(v) = 3$, this is a contradiction.

We may therefore assume that $k = 1$. Since a is not contained in a cell, a it not incident with a 4-string by Lemma 4.1.3. Since a is incident with two 1-strings and has weight at least 5, it follows that a is a vertex of type (3,1,1). Note since a has weight 5, it follows from the discharging rules that a shares each of its short strings with a vertex that is poor immediately after Step 4. Otherwise a sends charge to v in Step 5. Let $d \neq a$ be an endpoint of S_d. Note d is adjacent to c which has degree at least 3 since it is the endpoint of a string. Thus d is adjacent to a 0-string and a 1-string. Since $ch_5(d) < 0$, it follows that d has degree 3 and weight at least 6. But then d is adjacent to a k-string with $k \geq 5$, contradicting Lemma 2.1.12.

We may therefore assume S_{ab} is not a 3-string. But then S_{ab} contributes at most 2 to $\text{wt}(a) + \text{wt}(b)$. Let S_1 be the third string incident with a, with $S_1 \not\subset S_{ab}$ and $S_1 \neq S_a$. Similarly, let S_2 be the third string incident with b, with $S_2 \not\subset S_{ab}$ and $S_2 \neq S_b$. Let m_a and m_b be integers chosen such that S_1 is an m_a-string, and S_2 is an m_b-string. Since a is incident with a 1-string S_a and b is incident with a 2-string S_2, we have $\text{wt}(a) + \text{wt}(b) \leq 2 + 2 + 1 + m_a + m_b$. Since each of a and b has weight at least 5, it follows that $m_a + m_b \geq 5$. Hence at least one of m_a and m_b is at least three. Suppose first $m_b \geq 3$. Note $m_b \leq 3$, since otherwise by Lemma 4.1.3 b is contained in a cell, contrary to assumption.

First suppose b is a vertex of type $(3,2,0)$. Then since b has weight 5, it shares each of its short strings with a poor vertex as otherwise b sends charge to v in Step 5. Thus b shares its 0-string with a vertex w of degree 3 and weight at least six. Note $w \in S_{ab}$, and S_{ab} contributes at most 2 to the weight of w by assumption. But then since w has weight at least 6, it is the endpoint of a 4-string and so is contained in a cell. This is a contradiction, as vertices contained in cells are not poor after Step 1.

We may therefore assume that b is either of type $(3,2,1)$ or of type $(3,2,2)$. But then S_{ab} contributes at most 1 to the weight of a. Since a has weight at least 5 and a is not

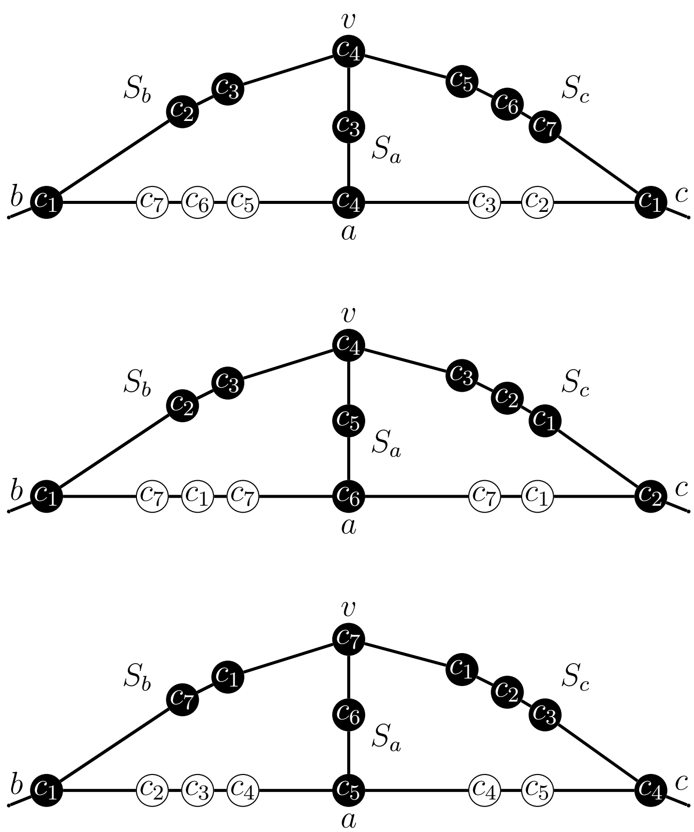

Figure 5.3: Figure for Lemma 5.2.4. Extensions of ϕ to G. The white vertices are of unknown degree, though their degree is at least that shown. The black vertices' degrees are as illustrated.

contained in a cell, we have that $m_a = 3$, and S_{ab} contributes 1 to the weight of each of a and b. Note $\mathrm{wt}(a) = 5$. By assumption, a shares both its short strings with vertices that are poor after Step 5. Let $w' \in S_{ab}$ be the vertex of degree at least 3 that shares a 1-string with each of a and b. Since w' is poor after Step 5, it has degree 3 and weight at least 6. But then w' is incident with a 4-string, and so by Lemma 4.1.3 it is contained in a cell. This is a contradiction, as vertices contained in cells are not poor after Step 1.

Thus we may assume $m_b \leq 2$, and so $m_a \geq 3$. Since a is not contained in a cell, a is not incident with a 4-string by Lemma 4.1.3. Thus $m_a = 3$. Note since $\mathrm{wt}(a) \geq 5$, we have that S_{ab} contributes at least 1 to the weight of a. Thus a is either of type $(3,1,1)$ or (3,2,1).

Suppose first a is of type (3,1,1). By the discharging rules, since $ch_5(v) < 0$ it follows that a shares a 1-string with a vertex $w'' \in V(S_{ab})$ such that $ch_5(w'') < 0$. But then $\mathrm{wt}(w'') \geq 6$ and w'' has degree 3. Since S_{ab} contributes at most 2 to the weight of w'', it follows that w'' is incident with an r-string with $r \geq 4$. But this is a contradiction, as by Lemma 4.1.3 vertices of degree 3 incident with 4-strings are contained in cells.

Thus we may assume a is of type (3,2,1). But then S_{ab} contributes 0 to the weight of b. Since b is not contained in a cell, b is not incident with a 4-string by Lemma 4.1.3. Thus it follows that since $\mathrm{wt}(b) \geq 5$, b is of type $(3,2,0)$. Note since $ch_5(v) < 0$ and $\mathrm{wt}(b) = 5$, it follows from Rule 5 that b shares its 0-string with a vertex w^* of degree 3 and weight at least 6, such that $ch_5(w^*) < 0$. Thus w^* is not contained in a cell. But since S_{ab} contributes at most 2 to the weight of w^* and w^* has weight at least 6, it follows that w^* is incident with a 4-string. By Lemma 4.1.3, w^* is contained in a cell —a contradiction. □

The only remaining possibility is then that v is a vertex of type (3,3,0). Let u be the vertex that shares a 0-string with v. Since $ch_5(v) < 0$, we have that u has degree 3 and weight at least 5. Note u is not incident with a 4-string as otherwise by Lemma 4.1.3 v is contained in a cell, a contradiction.

Thus since u has degree 3 and weight at least 5 and is incident with a 0-string, it follows that u is either of type (3,3,0) or (3,2,0). Suppose first u is of type (3,2,0). Since $ch_5(v) < 0$, and $\mathrm{wt}(u) = 5$, it follows that u shares its incident 2-string with another vertex w with $ch_5(w) < 0$. Otherwise, u sends $-ch_5(v)$ to v in Step 5, contradicting that $ch_5(v) < 0$. Since w is not contained in a cell, by Lemma 4.1.3 w is not incident with a 4-string. It follows that w is not of type $(4,2,2)$, $(4,2,1)$, or $(4,2,0)$. Furthermore, by Lemma 4.2.5, w is not a vertex of type $(3,2,2)$. Since w is the endpoint of a 2-string, it is thus of type (3,2,1) or (2,2,2). But by Claim 5.2.4, if w is of type (3,2,1) then $ch_5(w) \geq 0$. Similarly, by Claim 5.2.2 w is not of type (2,2,2).

Therefore we may assume u is of type $(3, 3, 0)$. But then by Lemma 4.2.7, u is contained in a cell C. Since v receives charge from C in Step 2, $ch_5(v) \geq 0$ —a contradiction. □

Thus by Lemma 5.1.1 and since the charge of cells only changes in Steps 1 and 2, all cells have non-negative charge at the end of Step 5. By Lemmas 5.1.2, 5.1.3, and 5.1.4, all vertices of degree at least 4 and all vertices of degree 4 and weight at most 5 have non-negative charge at the end of Step 5. By Lemma 5.2.1, all degree 3 vertices with weight at least 6 have non-negative charge at the end of Step 5. Since vertices and cells are the only structures in G that carry charge, the sum of the charges is therefore non-negative. But since the total charge did not change, the total charge carried by the graph is at most $-2(T + 1)$. Since $T \geq 2$, this is a contradiction. □

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