Symmetrically-Normed Ideals and Characterizations of Absolutely Norming Operators

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Pure Mathematics

Waterloo, Ontario, Canada, 2018

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This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

I am the sole author of Chapters 1, 2, 4, 5, 6, 7, 8, and 9. Chapter 3 is from joint work with Vern I. Paulsen.

Abstract

The primary objective of this thesis is two fold: first, it is devoted to the study of absolutely norming operators (with respect to various arbitrary symmetric norms on $\mathcal{B}(\mathcal{H})$) with an eye towards the objective of characterizing these classes of operators, and second, it summarizes the first three chapters of the monograph by Gohberg and Krein [GK69] offering an exposition of the theory of symmetrically-normed ideals ("norm ideals" in older literature) in modern terminologies.

Governed by the intention of providing a fairly comprehensive treatment of this theory, independent of the rest of the thesis, we have distributed this "exposition" part of the thesis over two chapters, namely Chapter 2 and Chapter 7. While Chapter 2 concerns the elementary theory of s-numbers and provides the necessary background for the chapters to follow, the goal of the seventh chapter is to discuss the theory of symmetrically-normed ideals of the algebra of operators on a Hilbert space, with most of the attention centered around symmetrically-normed ideals of the algebra of compact operators on a (separable infinite-dimensional) Hilbert space. These chapters are, for all practical purposes, entirely independent of the rest of the thesis; the readers interested in the basic theory of these ideals can go through these chapters and leave everything else.

Chapter 3 is concerned with Chevreau's problem of characterizing the class of absolutely norming operators — operators that attain their norm on every closed subspace. The result of this chapter settles Chevreau's problem by establishing a spectral characterization theorem for such operators.

In Chapters 4 - 6, we first extend the concept of absolutely norming operators to several particular (symmetric) norms (that are equivalent to the operator norm) and then characterize these sets. In particular, we single out three (families of) norms on $\mathcal{B}(\mathcal{H}, \mathcal{K})$: the "Ky Fan k-norm(s)", "the weighted Ky Fan π, k-norm(s)", and the "(p, k)-singular norm(s)", and thereafter define and characterize the set of "absolutely norming" operators with respect to each of these three norms.

In Chapter 8, we restrict our attention to the algebra $\mathcal{B}(\mathcal{H})$ of operators on a separable infinite-dimensional Hilbert space $\mathcal{H}$ and use the theory of symmetrically normed ideals to extend the concept of norming and absolutely norming from the usual operator norm to arbitrary symmetric norms on $\mathcal{B}(\mathcal{H})$. In addition, this chapter presents a constructive method to produce symmetric norm(s) on $\mathcal{B}(\mathcal{H})$ with respect to which the identity operator does not attain its norm.

Finally, in Chapter 9, we introduce and study the notion of "universally symmetric norming operators" and "universally absolutely symmetric norming operators". These refer to the operators that are, respectively, norming and absolutely norming, with respect to

every symmetric norm. The setting of this chapter is again a separable infinite-dimensional Hilbert space. This chapter characterizes such operators: the main result of this chapter states that an operator in $\mathcal{B}(\mathcal{H})$ is universally symmetric norming if and only if it is universally absolutely symmetric norming, which is true if and only if it is compact. In particular, this result provides an alternative characterization theorem for compact operators on a separable Hilbert space.

Acknowledgements

This thesis manifests endeavors of numerous people apart from me to each of whom I would like to acknowledge and express my gratitude.

Most of all, I would like to thank Vern Paulsen, my advisor, for his persistent guidance and tremendous patience. He has been an endless source of helpful suggestions and constructive criticism, and I have immensely benefitted from his knowledge and insight — to the extent that if it were not for him, I wouldn't have made it. As a teacher, his propensity for choosing the most illuminating route to a result has profoundly helped me gain a deeper understanding of the subject. I certainly owe him more than I can possibly express.

I would also like to gratefully acknowledge the multiple stimulating conversations with Laurent Marcoux. I am deeply indebted to him for his valuable feedback on my work. He has been a tremendous source of information and ideas.

Special thanks are due to Mark Tomforde of University of Houston who has always been a constant source of inspiration for me and deserves a major share of the credit for whatever little success I got as a teacher. He showed me ways to incorporate elements in my lectures that would encourage critical thinking.

I would also like to acknowledge with pleasure my debt to Dinesh Singh who encouraged me for higher studies and provided me with financial support during my education. He inculcated in me (and other students) an intelligent questioning attitude — the ability to ask the right questions which immediately led, through their answers, to a better understanding of the matter. He has inspired me in more ways than one and I am forever grateful to him.

My indebtedness remains to the members of the examining committee for this thesis, David R. Pitts, Laurent W. Marcoux, Matthew Kennedy, and Levent Tuncel, whose perceptive comments led this thesis to the present form.

It is a pleasure to thank the Department of Pure Mathematics at the University of Waterloo for providing an unmatched ambience for research. In particular, I would like to mention the Analysis Group at the department which has inspired me more than it is aware of. In addition, I would be remiss not to mention Lis D' Alessio, Nancy Maloney, Pavlina Penk, and Jackie Hilts — our administrative officers — for their help on numerous occasions. They are an integral part of the department and I would like to express my gratitude towards them. Moreover, I would also like to thank the Mathematics Department at University of Houston where portions of this thesis were written.

In addition, I would like to thank many friends and colleagues for enriching my graduate life with their lively conversations that stimulated my research career in myriad of ways and in particular, Jitendra Prakash, Ricky Ng, Da Zheng, Sanat Upadhyay and Mizanur Rahaman with whom I have learnt more mathematics than in class. Jitendra deserves a special mention for countless reasons: he has been a pedantic co-author, a meticuluous critic, a treasured academic brother, an accommodating flatmate, and most of all a reliable friend.

I have been incredibly privileged to have found friends whose company I treasure. My stay at Waterloo owes more than I can say to these friends, especially, Shubham, Ragini, Frozan, Assaimani, and of course Jitendra. I could not have asked for better weekend-evenings than the ones I enjoyed with Ragini's "chai" and Frozan's Hindi — in the most perfect British-Canadian accent. It is widely believed in our group that Shubham knows of more mathematicians than any of us can ever possibly imagine: his stories about renowned mathematicians across the globe kept me motivated and I would like to extend my deepest appreciation to him for this. My acknowledgement will not be complete without mentioning Soham, one of my best friends from Houston, for whom friendship is beyond measure and is never diminished by time or distance. He also deserves the greatest of thanks for being an ardent photographer and sharing his splendid gallery of pictures with me.

It would be remiss of me if I do not acknowledge my friends from India. They are extremely valuable and important to me and words fall short in expressing my gratitude towards them. However, for different reasons, I need to mention some of them. First and foremost are my childhood friends: Chiranjeev, Daljeet, Mayank, and Ashok. They provided a much needed form of escape from my studies and deserve special mention for helping me keep things in perspective. I am also grateful to Amaresh Mishra and Varun Jaiswal, my classmates from the then Ramanujan School of Mathematics (not Super 30), for numerous endless mathematical discussions. I count myself unbelievably fortunate to have made friends with Mrityunjay, Rajwant, and Shyam *bhaiya*. They are friends who have become my family; a support system outside my genetic sphere. I would like to thank them for helping me get through the difficult times, and for all the emotional support they provided.

And finally, I am deeply indebted to my wonderful family members who kept an unflinching faith in me throughout all of my graduate life. It gives me special pleasure to express my gratitude to them for their patience and endurance; for I would not be here if it wasn't for their unconditional help.

Dedication

This thesis is dedicated to my parents Gayatri Pandey and Keshaw Prasad Pandey, and to the memory of my maternal grandfather Ram Bihari Tiwari — to each of whom I owe far beyond my words can possibly express.

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Chapter 1

Introduction

All linear spaces in this thesis are assumed to be over the field $\mathbb{C}$ of complex numbers. A *Banach space* $\mathcal{X}$ is a complex linear space with a norm such that $\mathcal{X}$ is complete (that is, every Cauchy sequence is convergent) in the metric given by this norm. A *Hilbert Space* $\mathcal{H}$ is a complex vector space equipped with an inner product such that $\mathcal{H}$ is complete with respect to the norm induced by the inner product. In particular, a Hilbert space is a Banach space. The setting for our discussion is a Hilbert space, and we are primarily concerned with problems about bounded linear transformations acting on Hilbert spaces (and Banach spaces). We begin by adopting the word *operator* to mean bounded linear transformation.

1.1 Norming operators

Definition 1.1.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. An operator $T : \mathcal{X} \to \mathcal{Y}$ is said to be *norming* or *norm attaining* if there is an element $x \in \mathcal{X}$ with $\|x\| = 1$ such that $\|T\| = \|Tx\|$, where $\|T\| = \sup\{\|Tx\| : x \in \mathcal{X}, \|x\| \leq 1\}$ is the usual operator norm on the Banach space $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ of bounded linear transformations from (the Banach space) $\mathcal{X}$ into (the Banach space) $\mathcal{Y}$. We let $\mathcal{N}(\mathcal{X}, \mathcal{Y})$ denote the set of norming operators in $\mathcal{B}(\mathcal{X}, \mathcal{Y})$.

1.2 Notation and conventions

We use $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ to denote generic (complex) Hilbert spaces. As mentioned in the introduction of this chapter, we limit ourselves to complex spaces throughout this thesis and hence by a Hilbert space we mean a complex Hilbert space. Let W be a subspace of $\mathcal{H}$. We use $\operatorname{clos}[W]$ to denote the norm closure of W in $\mathcal{H}$. The n-dimensional Hilbert space is usually denoted by ℓ^2_n, while ℓ^2 is the (concrete) separable, infinite-dimensional Hilbert space. We let $\mathcal{B}(\mathcal{H},\mathcal{K})$ (respectively $\mathcal{B}(\mathcal{H})$) denote the set of bounded linear transformations (henceforth called "operators") from $\mathcal{H}$ to $\mathcal{K}$ (respectively from $\mathcal{H}$ to $\mathcal{H}$). We recall that $\mathcal{B}(\mathcal{H},\mathcal{K})$ is a complex Banach space with respect to the operator norm $\|T\| = \sup\{\|Tx\| : x \in \mathcal{H}, \|x\| \leq 1\}$. We let $\mathcal{B}_{00}(\mathcal{H},\mathcal{K})$, respectively, $\mathcal{B}_0(\mathcal{H},\mathcal{K})$ denote the set of finite rank operators from $\mathcal{H}$ to $\mathcal{K}$, respectively, the set of compacts. We reserve $\mathcal{B}_1(\mathcal{H})$ for the set of trace class operators with the trace norm $\|\cdot\|_1$.

Let I be any set. (Give I the discrete topology so that it becomes locally compact. Also, any function on I is continuous.) We use $\ell^\infty(I)$ to denote the collection of all bounded complex functions $f : I \to \mathbb{C}$ with $\|f\| = \sup\{\|f(i)\| : i \in I\}$. Further, we reserve $c_0(I)$ to denote the set of all functions $f : I \to \mathbb{C}$ in $\ell^\infty(I)$ such that for every $\epsilon > 0$, the set $\{i \in I : |f(i)| \geq \epsilon\}$ is finite; $c_0(I)$ is thus referred to as the set of continuous functions on I which *vanish at infinity*. If $I = \mathbb{N}$, the usual notation for these spaces are simply ℓ^∞ and c_0; ℓ^∞ consists of all bounded sequences of scalars (complex numbers) and c_0 consists of all sequences of scalars that converge to zero. In addition, let $\ell^1(\mathbb{N})$ (or, simply ℓ^1) denote the collection of all complex functions φ on $\mathbb{N}$ such that $\sum_{n=1}^\infty |\varphi(n)| < \infty$, with the norm defined by $\|\varphi\|_1 = \sum_{n=1}^\infty |\varphi(n)|$. Alternatively, ℓ^1 consists of all sequences $x = (x_n)_{n\in\mathbb{N}}$ of scalars (complex numbers) such that $\sum_{n=1}^\infty |x_n| < \infty$, with the norm defined by $\|x\|_1 = \sum_{n=1}^\infty |x_n|$.

The closed unit ball of a Banach space $\mathcal{X}$ is the set $\{x \in \mathcal{X} : \|x\| \leq 1\}$ and is denoted by $(\mathcal{X})_1$ in the sequel.

1.3 Absolutely norming operators

Definition 1.3.1. Suppose that $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces and $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$. We say that T is *absolutely norming* if for every nontrivial closed subspace $\mathcal{M}$ of $\mathcal{H}$, $T|_\mathcal{M}$ is *norming*, that is, there exists an element $x \in \mathcal{M}$ with $\|x\|_\mathcal{H} = 1$ such that $\|T|_\mathcal{M}\| = \|Tx\|_\mathcal{K}$.

These operators, when restricted to any nontrivial closed subspace of $\mathcal{H}$, attain their (operator) norm on that closed subspace, and are hence referred to as *absolutely norming* operators. Compact operators are the prototypical examples of such operators. We let $\mathcal{N}(\mathcal{H},\mathcal{K})$ and $\mathcal{AN}(\mathcal{H},\mathcal{K})$ respectively denote the set of norming and absolutely norming operators in $\mathcal{B}(\mathcal{H},\mathcal{K})$. The class of norming operators has been extensively studied and there is a plethora of information on these operators; see, for instance, [JW77, JW79, JW82, AR02, ARG98, Iwa79, AAP96, Aco99, Agu98, Par82, Sch83a, Sch83b, Shk09] and references therein. The class of absolutely norming operators on Hilbert spaces, however, was introduced recently in [CN12]. In what follows we discuss a complete historical account of these operators.

One of the most interesting problems concerning any class of operators is to characterize them. Bernard Chevreau was the first to ask this question for the class of absolutely norming operators in 1995. About 15 years after the dissemination of the question Carvajal and Neves [CN12] proved a partial structure theorem [CN12, Theorem 3.25] for the set of positive absolutely norming operators on Hilbert spaces that included an uncharacterized "remainder" operator. This theorem motivated Ramesh [Ram14] to claim a full characterization theorem [Ram14, Theorem 2.3] without remainder, for positive absolutely norming operators on separable Hilbert spaces.

However, in [PP17] we presented a counterexample to Ramesh's characterization theorem [Ram14, Theorem 2.3]. We then gave a full spectral characterization of the class of positive absolutely norming operators on complex Hilbert spaces of arbitrary dimensions. Earlier results needed to assume separability. The correct characterization requires more terms than were used in [Ram14] and [CN12]. Using this theorem, we have given a complete characterization theorem for the set $\mathcal{AN}(\mathcal{H},\mathcal{K})$ of absolutely norming operators on complex Hilbert spaces of arbitrary dimension. The following theorem the main result of our work [PP17] and it settles the original problem asked by Bernard Chevreau. Chapter 3 of this thesis entirely centers around this theorem and is written in considerable detail to give a smooth treatment of the solution to Chevreau's problem.

Theorem 3.5.3 ([PP17])**.** *Let $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$, and let $T = U|T|$ be its polar decomposition. Then $T \in \mathcal{AN}(\mathcal{H},\mathcal{K})$ if and only if $|T|$ is of the form $|T| = \alpha I + F + K$, where $\alpha \geq 0, K$ is a positive compact operator and F is self-adjoint finite-rank operator.*

1.4 Detailed overview of the thesis

One of the intentions of this thesis is to offer an exposition of the theory of symmetrically-normed ideals ("norm ideals" in older literature) in a fairly reasonable manner; specifically, to develop the main ideas and their interconnections in a minimal amount of time, and yet be essentially elementary in the sense of being accessible to a more general audience. These ideals were first introduced by Schatten [Sch60] and studied extensively by the Russian school, particularly by Gohberg and Kreĭn [GK69]. Dictated by the intention of providing a fairly comprehensive treatment of this theory, independent of the rest of the thesis, we have distributed this "exposition" part of the thesis over two chapters, namely Chapter 2 and Chapter 7. Chapter 2 concerns the elementary theory of s-numbers and provides the necessary background for the chapters to follow. The notion of a symmetric norm is introduced in the final section of this chapter but an extensive treatment of this concept awaits until Chapter 7. That accomplished, the goal of the seventh chapter is to discuss the theory of symmetrically-normed ideals of the algebra of operators on a Hilbert space, with most of the attention centered around symmetrically-normed ideals of the algebra of compact operators on a (separable infinite-dimensional) Hilbert space. These two chapters together summarize the first three chapters of the monograph by Gohberg and Kreĭn [GK69], and include the basic content of the classic monograph by Schatten [Sch60]. In writing these chapters, care has been taken to keep them entirely independent of the rest of the thesis so that those who are interested in the basic theory of these ideals can go through Chapters 2 and 7 and leave everything else.

This thesis is devoted to the study of absolutely norming operators (with respect to various arbitrary symmetric norms on $\mathcal{B}(\mathcal{H})$) with an eye towards the objective of characterizing these classes of operators. Chapter 3 is concerned with Chevreau's problem of characterizing the class of absolutely norming operators — operators that attain their norm on every closed subspace. The following spectral characterization theorem for such operators is the main result of this chapter.

Theorem 3.5.3 ([PP17]). *Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, and let $T = U|T|$ be its polar decomposition. Then $T \in \mathcal{AN}(\mathcal{H}, \mathcal{K})$ if and only if $|T|$ is of the form $|T| = \alpha I + K + F$, where $\alpha \geq 0, K$ is a positive compact operator and F is self-adjoint finite-rank operator.*

The above result of this chapter settles Chevreau's problem and serves to be the first hint to a more general situation. Suppose $\mathcal{B}(\mathcal{H}, \mathcal{K})$ is equipped with a norm $|||\cdot|||$ equivalent to the usual operator norm and let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. What does it mean to say that T is norming or absolutely norming in this setting? What about characterizing these operators?

In Chapters 4 - 6, the underlying purpose is to first extend the concept of absolutely norming operators to several particular (symmetric) norms (that are equivalent to the operator norm) and then characterize these sets. In particular, we single out three norms on $\mathcal{B}(\mathcal{H}, \mathcal{K})$: the "Ky Fan k-norm", "the weighted Ky Fan π, k-norm", and the "(p, k)-singular norm", and thereafter define and characterize the set of "absolutely norming" operators with respect to each of these three norms.

Chapter 4 gives a detailed treatment of the theory of absolutely norming operators with respect to the Ky Fan k-norm and give a spectral characterization theorem for the set of such operators.

Chapter 5 is devoted to the study of the operators in $\mathcal{B}(\mathcal{H},\mathcal{K})$ that are absolutely norming with respect to the weighted Ky Fan π, k-norm. The central goal of this chapter is to present a spectral characterization theorem for the set of such operators. These results parallel those for norming and absolutely norming operators on $\mathcal{B}(\mathcal{H},\mathcal{K})$, and can be viewed as an appropriate generalization of the absolutely norming property from the set of Ky Fan k-norms to the set $\{\|\cdot\|_{[\pi,k]} : \pi \in \Pi, k \in \mathbb{N}\}$ of weighted Ky Fan π, k-norms (and hence may possibly at first sight render Chapter 4 redundant — but that is not the case as we will see shortly).

There is another set of norms on $\mathcal{B}(\mathcal{H},\mathcal{K})$, the (p,k)-singular norms, which are a generalization of the set of Ky Fan k-norms. In Chapter 6 these norms are introduced and the operators which are absolutely norming with respect to these norms are studied. Continuing in the same spirit, as that of the previous two chapters, a spectral characterization theorem for such operators presented. Chapter 4 thus serves to be, in a certain sense, an illuminating framework which allows us to appreciate the results in the following two chapters. These three chapters 4 - 6 not only provide motivation for why the norms considered therein are introduced and important, but also offer the usage of these norms in applications. Even more, each of these chapters serves as an exposition of new techniques introduced to work with the respective norms taken up for study.

In Chapter 8, we restrict our attention to the algebra $\mathcal{B}(\mathcal{H})$ of operators on a separable infinite-dimensional Hilbert space $\mathcal{H}$ and use the theory of symmetrically normed ideals to extend the concept of norming and absolutely norming from the usual operator norm to arbitrary symmetric norms on $\mathcal{B}(\mathcal{H})$. The subsequent discussion in this chapter involves positive operators of the form of a nonnegative scalar multiple of identity plus a positive compact plus a self-adjoint finite-rank. It is not clear, *a priori*, if the operators of this form are absolutely norming with respect to *every* symmetric norm on $\mathcal{B}(\mathcal{H})$. It turns out that there exists a symmetric norm on $\mathcal{B}(\ell^2)$ such that the identity operator does not attain its norm. The following theorem presents this nonintuitive result which renders the identity operator nonnorming.

Theorem 8.3.1 ([Pan17a])**.** *There exists a symmetric norm $\|\cdot\|_{\Phi_\pi^*}$ on $\mathcal{B}(\ell^2)$ such that $I \notin \mathcal{N}_{\Phi_\pi^*}(\ell^2)$.*

Chapter 8, in fact, presents a family of symmetric norms on $\mathcal{B}(\mathcal{H})$ with respect to each of which the identity operator is rendered nonnorming.

Finally, in Chapter 9, we introduce and study the notion of "universally symmetric norming operators" (u.s.n. operators) and "universally absolutely symmetric norming operators" (u.a.s.n. operators). These refer to the operators that are, respectively, norming and absolutely norming, with respect to every symmetric norm. The goal of this chapter is to characterize such operators and the setting of our discussion in this chapter is again a separable infinite-dimensional Hilbert space.

One of the results from Chapter 8 states that a compact operator in $\mathcal{B}(\mathcal{H})$ is universally absolutely symmetric norming (and hence universally symmetric norming): this provides compact operators as prototypical examples of such operators. So, we have

$$\text{compact operators} \subseteq \text{u.a.s.n. operators} \subseteq \text{u.s.n. operators}.$$

It would be desirable to know whether an u.s.n. operator is compact. In Chapter 9 we answer this question affirmatively. The following is the main result of this chapter which essentially states that an operator in $\mathcal{B}(\mathcal{H})$ is universally symmetric norming if and only if it is compact.

Theorem 9.2.6 ([Pan17b])**.** *Let $T \in \mathcal{B}(\mathcal{H})$ and let Φ_1 denote the maximal s.n.function. Then the following statements are equivalent.*

1. *$T \in \mathcal{B}_0(\mathcal{H})$.*
2. *T is universally absolutely symmetric norming, that is, $T \in \mathcal{AN}_{\Phi^*}(\mathcal{H})$ for every s.n.function Φ equivalent to Φ_1.*
3. *T is universally symmetric norming, that is, $T \in \mathcal{N}_{\Phi^*}(\mathcal{H})$ for every s.n.function Φ equivalent to Φ_1.*

We hence establish a characterization theorem for such operators on $\mathcal{B}(\mathcal{H})$. In particular, this result provides an alternative characterization theorem for compact operators on a separable Hilbert space.

Chapter 2

Preliminaries I: Theory of s-numbers

The primary objective of this chapter is to give a fairly comprehensive treatment of the theory of s-numbers which were first introduced by E. Schmidt in the study of integral equations with nonsymmetric (nonhermitian) kernels. The purpose of this chapter is two-fold. First, it serves as the prerequisite for Chapters 3 - 6, and second, as mentioned in the introduction of this thesis, it summarizes the first two chapters of the monograph by Gohberg and Kreĭn [GK69] offering an exposition of the theory of s-numbers in modern terminologies.

2.1 Completely continuous operators

We begin by revisiting completely continuous operators and their relation with compact operators on Banach spaces. Recall that if $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and $T : \mathcal{X} \to \mathcal{Y}$ is a linear transformation, then T is *compact* if $T[(\mathcal{X})_1]$ has a compact closure in $\mathcal{Y}$. If $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, then T is *completely continuous* if for any sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{X}$ which converges weakly to x it follows that $\{Tx_n\}_{n=1}^{\infty}$ converges in norm to Tx.

The following proposition provides most of the elementary facts about completely continuous operators.

Proposition 2.1.1 ([Con90]). *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and let $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$.*

(a) If T is compact, then T is completely continuous.

(b) If $\mathcal{X}$ is reflexive and T is completely continuous, then T is compact.

Whereas in the setting of Banach spaces the set of compact operators may be properly contained in the set of completely continuous operators, this is not true for Hilbert spaces where these sets are identical.

2.2 The spectral representation of a positive compact operator

For the rest of this chapter we return to the Hilbert space situation. We briefly review the theory of compact operators at least up to the spectral theorem of compact normal operators and thereby deduce the spectral representation of positive compact operators. Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is *diagonalizable* if there exists an orthonormal basis $\{e_j | j \in J\}$ for $\mathcal{H}$ and a set $\{\lambda_j | j \in J\}$ in $\mathbb{C}$ such that

$$Tx = \sum_{j \in J} \lambda_j \langle x, e_j \rangle e_j \qquad \text{for every } x \in \mathcal{H}. \tag{2.2.1}$$

In this case the numbers $\langle x, e_j \rangle$ are the coordinates for x in the basis $\{e_j | j \in J\}$ and each λ_j is an eigenvalue for T corresponding to the eigenvector e_j. It is well known that a diagonalizable operator $T \in \mathcal{B}(\mathcal{H})$ is compact if and only if its eigenvalues $\{\lambda_j : j \in J\}$ corresponding to an orthonormal basis $\{e_j : j \in J\}$ vanish at infinity, that is, $\{\lambda_j : j \in J\} \in c_0(J)$. The spectral theorem per se states that a compact operator on $\mathcal{H}$ is diagonalizable if and only if it is normal. This result is so remarkable and yet fundamental that we state it in an alternative way that is relevant to our present interest.

Theorem 2.2.1 ([Ped89])**.** *If $T \in \mathcal{B}(\mathcal{H})$, then the following statements are equivalent.*

1. *T is compact normal.*
2. *T is diagonalizable and its eigenvalues (counted with multiplicity) vanish at infinity.*

Before proceeding, it will be convenient to introduce the notation $y \otimes x$ for the rank one operator in $\mathcal{B}(\mathcal{H}, \mathcal{K})$ determined by the vectors $x \in \mathcal{H}$ and $y \in \mathcal{K}$ by the formula

$$y \otimes x(z) := \langle z, x \rangle y \qquad \text{for every } z \in \mathcal{H}.$$

In Dirac's 'bra-ket' notation we write the operator $y \otimes x$ above as $|y\rangle\langle x|$. It is easy to verify that for $x \in \mathcal{H}$ and $y \in \mathcal{K}$ we have $(y \otimes x)^* = x \otimes y$, $T(y \otimes x) = (Ty) \otimes x$ for every

$T \in \mathcal{B}(\mathcal{K}, \mathcal{L})$, and $(y \otimes x)R = y \otimes (R^*x)$ for every $R \in \mathcal{B}(\mathcal{L}, \mathcal{H})$. Moreover if $e \in \mathcal{H}$ and $\|e\| = 1$, then $e \otimes e$ is the rank one orthogonal projection of $\mathcal{H}$ onto $\mathbb{C}e$. Every compact normal operator T on $\mathcal{H}$ can now by (2.2.1) and the above theorem be written in the form

$$T = \sum_{j \in J} \lambda_j e_j \otimes e_j,$$

for a suitable orthonormal basis $\{e_j : j \in J\}$, or alternatively, every compact normal operator on a Hilbert space can be written as a norm convergent sum of rank one orthogonal projections. Moreover, the series $\sum_j \lambda_j e_j \otimes e_j$ converges to T in the metric defined by the norm on $\mathcal{B}(\mathcal{H})$. Indeed, the set $\{\lambda_j : j \in J\}$ vanish at infinity which implies that either the set $J_0 = \{j \in J : \lambda_j \neq 0\}$ is finite, in which case $T \in \mathcal{B}_{00}(\mathcal{H})$ or else is countably infinite, in which case the sequence $\{\lambda_j\}_{j \in J_0}$ converges to zero. We say that the compact set $\sigma(T) = \{\lambda_j : j \in J_0\} \cup \{0\}$ is the *spectrum* of T. So, the above theorem tells us that compact normal operators are essentially completely described in terms of the nonzero numbers in the spectrum (and their multiplicity, that is, the number of times they appear in the spectrum), which in this case consists of the nonzero eigenvalues.

We make the following convention: Unless the contrary is explicitly stated, it is assumed that in every expression $\sum_{j \in J} \lambda_j e_j \otimes e_j$ considered in the sequel, $\{e_j : j \in J\}$ is an orthonormal set of vectors and all the scalars λ_j are nonzero.

Corollary 2.2.2. *If $T \in \mathcal{B}(\mathcal{H})$ is a compact normal operator, then*

$$T = \sum_{j \in J} \lambda_j e_j \otimes e_j,$$

where the series converges in norm, $\{e_j : j \in J\}$ is an at most countable orthonormal basis of $\operatorname{clos}[\operatorname{ran} T] \subseteq \mathcal{H}$ *consisting entirely of eigenvectors of T, and for every $j \in J$, $Te_j = \lambda_j e_j$ with $\lambda_j \neq 0$. Moreover, $(\lambda_j)_{j \in J}$ is either a finite m-tuple (for some $m \in \mathbb{N}$) or a (countably infinite) sequence converging to zero.*

In order to maintain rigor with brevity we reserve the term "finite sequence" for a finite m-tuple (for some $m \in \mathbb{N}$), while the term "sequence" necessarily means a countably infinite sequence. For emphasis, we will, occasionally, write countably infinite sequence instead of sequence.

If in the above corollary $T \in \mathcal{B}_0(\mathcal{H})$ is positive, then the eigenvalues λ_j of T are nonnegative (in fact strictly positive; for it is assumed that each λ_j in the expression of the above corollary is nonzero). Moreover, in this case $\|T\| = \max\{\lambda_j : j \in J\}$. This allows us to reorder the strictly positive eigenvalues λ_j of T in a nonincreasing manner with each eigenvalue appearing as many times as the dimension of the corresponding eigenspace. By abuse of notation let us continue to write $(\lambda_j)_{j\in J}$ for the (reordered) nonincreasing finite or countably infinite sequence of strictly positive eigenvalue of T counting multiplicities. It is worth noticing that the limit of the reordered sequence remains unaltered.

Corollary 2.2.3. *If $T \in \mathcal{B}(\mathcal{H})$ is a positive compact operator, then*

$$T = \sum_{j\in J} \lambda_j e_j \otimes e_j,$$

where

(i) the series converges to T in the metric defined by the norm on $\mathcal{B}(\mathcal{H})$;

(ii) $\{e_j : j \in J\}$ is an at most countable orthonormal basis of clos[ran T] $\subseteq \mathcal{H}$ *consisting entirely of eigenvectors of T so that for every $j \in J$, $Te_j = \lambda_j e_j$ with $\lambda_j > 0$; and*

(iii) $(\lambda_j)_{j\in J} \in c_0(J)$ is a nonincreasing finite or countably infinite sequence of strictly positive eigenvalues of T counting multiplicities.

The representation $T = \sum_{j\in J} \lambda_j e_j \otimes e_j$, derived above for a positive compact operator T is referred to as its *spectral representation*.

We say a few words about the uniqueness of the spectral representation of positive compact operators. Of course we could have chosen a different orthonormal basis for clos[ran T] consisting of entirely of eigenvectors of T and in that case the spectral representation would have been different from the one we obtained above. However, the (finite or countably infinite) sequence $(\lambda_j)_{j\in J} \in c_0(J)$ appearing in the spectral representation of T is unique in the sense that λ_j's are necessarily all the strictly positive eigenvalues of T enumerated in a nonincreasing order with each of them appearing as many times as is its multiplicity.

2.3 s-Numbers of compact operators

Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. We define $|T| := \sqrt{T^*T}$ — this is conventionally known as the absolute value (or modulus) of the operator T and $|T|^2 = T^*T$.

Let $T \in \mathcal{B}_0(\mathcal{H}, \mathcal{K})$. Then $|T|$ is a positive compact operator on $\mathcal{H}$. Let $(\lambda_j(|T|))_{j\in J} \in c_0(J)$ be the (unique) nonincreasing finite or countably infinite sequence of strictly positive eigenvalues of $|T|$ counting multiplicities. If $T \notin \mathcal{B}_{00}(\mathcal{H}, \mathcal{K})$, that is, if T is a compact operator that is not of finite rank, then the sequence $(\lambda_n(|T|))_{j\in J}$ is countably infinite in which case the index set J is safely replaced by $\mathbb{N}$ and we define a sequence $(s_n(T))_{n\in\mathbb{N}}$ via

$$s_n(T) := \lambda_n(|T|) \qquad \text{for every } n \in \mathbb{N}.$$

If $T \in \mathcal{B}_{00}(\mathcal{H}, \mathcal{K})$, then the sequence $(\lambda_j(|T|))_{j\in J}$ is finite, that is, there exists $N \in \mathbb{N}$ such that the sequence is of the form $(\lambda_1(|T|), ..., \lambda_N(|T|))$, in which case we define a sequence $(s_n(T))_{n\in\mathbb{N}}$ via

$$\begin{aligned} &s_n(T) := \lambda_n(|T|) \qquad \text{for every } 1 \leq n \leq N; \text{ and} \\ &s_n(T) := 0 \qquad \text{for every } n \geq N. \end{aligned}$$

The countably infinite sequence $s(T) := (s_n(|T|))_{n\in\mathbb{N}}$ we obtain in this way for every $T \in \mathcal{B}_0(\mathcal{H})$ is what we shall call the *sequence of singular numbers* of T and the elements of this sequence are called the *singular numbers* (or *s-numbers*) of T. The n-th s-number of T is denoted by $s_n(T)$.

Here and subsequently, $\operatorname{rank}(T)$ denotes the dimension of the range of $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, that is, $\operatorname{rank}(T) = \dim[\operatorname{ran} T]$. Of course $\operatorname{rank} T$ either a natural number or the first transfinite cardinal number whenever T is compact. The following proposition is elementary.

Proposition 2.3.1. *Let $T \in \mathcal{B}_0(\mathcal{H})$.*

1. *Then $s_1(T) = \|T\|$.*
2. *For any scalar c, $s_n(cT) = |c|s_n(T)$ for every $n \in \mathbb{N}$.*
3. *If T is normal, then $s_n(T) = |\lambda_n(T)|$ for every $1 \leq n \leq \operatorname{rank}(|T|)$, where $\{\lambda_n(T)\}$ are ordered such that $|\lambda_n(T)| \geq |\lambda_{n+1}(T)|$.*

2.4 Variational characterizations for positive compact operators

In listing a complete account of the basic properties of s-numbers of compact operators, frequent use is made of the following celebrated Courant-Fischer-Weyl min-max principle.

Theorem 2.4.1 (Courant-Fischer-Weyl for Positive Compact Operators)**.** *Let $P \in \mathcal{B}(\mathcal{H})$ be a positive compact operator and let $\{e_j : j \in J\}$ be an orthonormal basis of $\mathcal{H}$ consisting entirely of eigenvectors of P so that for every j, $Pe_j = \lambda_j(P)e_j$, where the eigenvalues $\lambda_j(P)$ of P are ordered in nonincreasing sense, taking account of their multiplicities. Let $k \in \mathbb{N}$ and let S denote a subspace of $\mathcal{H}$. Then*

$$\lambda_1(P) = \max_{\{x: x\in\mathcal{H} \text{ and } \|x\|=1\}} \langle Px, x\rangle \tag{2.4.1}$$

$$\lambda_{k+1}(P) = \min_{\{S:\dim(S)=k\}} \left(\max_{\{x: x\in S^{\perp} \text{ and } \|x\|=1\}} \langle Px, x\rangle \right) \tag{2.4.2}$$

where the maximum in (2.4.1) is attained only at those eigenvectors of P which correspond to $\lambda_1(P)$ and the minimum in (2.4.2) is attained when S coincides with the k-dimensional subspace spanned by the eigenvectors $\{u_j : 1 \le j \le k\}$ of P corresponding to the eigenvalues $\{\lambda_j : 1 \le j \le k\}$, so that

$$\lambda_{k+1}(P) = \max_{\{x: x\in S^{\perp} \text{ and } \|x\|=1\}} \langle Px, x\rangle \,. \tag{2.4.3}$$

Remark 2.4.2. From the assertion (2.4.1), it is evident that the maximum in (2.4.3) is attained only at the eigenvectors of P corresponding to the eigenvalue $\lambda_{k+1}(P)$. Moreover, as is well known, this result can be extended to the positive eigenvalues $\lambda_1^+(P) \ge \lambda_2^+(P) \ge$ ... of any self-adjoint compact operator in $\mathcal{B}(\mathcal{H})$.

The following lemma is then obvious.

Lemma 2.4.3. *Let $P_1, P_2 \in \mathcal{B}_0(\mathcal{H})$. If $0 \le P_1 \le P_2$, then for every j, we have*

$$\lambda_j(P_1) \le \lambda_j(P_2).$$

Furthermore, the equality holds simultaneously for all j if and only if $P_1 = P_2$.

Suppose that $A \in \mathcal{B}_0(\mathcal{H})$ is a self-adjoint operator with spectral decomposition $A = \sum_{j=1}^{\mathrm{rank}(T)} \lambda_j(A) e_j \otimes e_j$. Then we form the positive operators

$$A_+ = \sum_{\lambda_j>0} \lambda_j(A) e_j \otimes e_j \text{ and } A_- = -\sum_{\lambda_j<0} \lambda_j(A) e_j \otimes e_j.$$

Obviously then, $A = A_+ - A_-$. This is one way in which a self-adjoint compact operator can be written as the difference of two positive compact operators. If on the other hand a self-adjoint compact operator is represented as the difference of two positive compact operators, how are the eigenvalues of these positive compacts related to that of the given operator? The following lemma answers this question.

Lemma 2.4.4. *Let $A \in \mathcal{B}_0(\mathcal{H})$ be a self-adjoint operator and let $P_1, P_2 \in \mathcal{B}_0(\mathcal{H})$ be positive operators such that $A = P_1 - P_2$. Then*

$$\lambda_j(A_+) \leq \lambda_j(P_1) \text{ and } \lambda_j(A_-) \leq \lambda_j(P_2).$$

2.5 Schmidt expansion of compact operators

Using the spectral theorem for compact normal operators on $\mathcal{H}$ and the polar decomposition of an element in $\mathcal{B}(\mathcal{H}, \mathcal{K})$ it is possible to write every compact operator in $\mathcal{B}(\mathcal{H}, \mathcal{K})$ as a norm convergent sum of rank one operators. Before we show this, we state the Polar Decomposition theorem.

Proposition 2.5.1 ([Con00] Polar Decomposition)**.** *If $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then there exists a unique partial isometry $U : \mathcal{H} \to \mathcal{K}$ with final space clos*[ran T] *and initial space clos*[ran $|T|$] *such that $T = U|T|$ and $|T| = U^*T$. If T is invertible, then U is unitary.*

Now we are prepared to deduce the desired 'spectral decomposition' of compact operators. If $T \in \mathcal{B}_0(\mathcal{H}, \mathcal{K})$, then $|T|$ is a positive compact operator on $\mathcal{H}$ and by Corollary 2.2.3, it admits a spectral representation

$$|T| = \sum_{j \in J} s_j(T) \phi_j \otimes \phi_j,$$

where the series converges to $|T|$ in norm, $\{\phi_j : j \in J\}$ is an at most countable orthonormal basis of clos[ran $|T|$] consisting entirely of eigenvectors of T so that for every j, $|T|\phi_j = s_j(T)\phi_j$ with $s_j(T) > 0$.

Moreover, $(s_j(T))_{j \in J} \in c_0(J)$ is a nonincreasing finite or countably infinite sequence of strictly positive s-numbers of T counting multiplicities. Since the Polar Decomposition theorem guarantees the existence of a unique partial isometry $U \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ with final space clos[ran T] and initial space clos[ran $|T|$] such that $T = U|T|$, we have

$$T = U|T| = \sum_{j \in J} s_j(T)(U\phi_j) \otimes \phi_j.$$

Since the isometries preserve dimension, we have dim[ran $|T|$] = dim[ran T]. Also notice that $\langle U\phi_i, U\phi_j \rangle = \langle U^*U\phi_i, \phi_j \rangle = \langle \phi_i, \phi_j \rangle = 0$ whenever $i \neq j$ and 1 otherwise, which makes the set $\{U\phi_j : j \in J\}$ orthonormal. This implies that the set $\{U\phi_j : j \in J\}$ is an orthonormal basis of $\operatorname{ran} T$. If we define $\psi_j := U\phi_j$, then the above equation says, in effect, that every $T \in \mathcal{B}_0(\mathcal{H}, \mathcal{K})$ admits a *Schmidt expansion*

$$T = \sum_{j=1}^{\mathrm{rank}(T)} s_j(T)\psi_j \otimes \phi_j.$$

Since the above representation is derived from the spectral representation via the polar decomposition, it is sometimes also referred to as the *polar representation* of $T \in \mathcal{B}_0(\mathcal{H}, \mathcal{K})$. We can summarize the result of the above discussion in the following proposition.

Proposition 2.5.2 (Schmidt Expansion)**.** *Every $T \in \mathcal{B}_0(\mathcal{H}, \mathcal{K})$ admits a Schmidt expansion*

$$T = \sum_{j=1}^{rank(T)} s_j(T)\psi_j \otimes \phi_j, \tag{2.5.1}$$

where

(i) the series converges to T in the norm topology on $\mathcal{B}(\mathcal{H}, \mathcal{K})$;

(ii) $\{\phi_j : j \in J\}$ is an at most countable orthonormal basis of clos$[\operatorname{ran}|T|] \subseteq \mathcal{H}$ consisting entirely of eigenvectors of T so that for every j, $|T|\phi_j = s_j(T)\phi_j$ with $s_j(T) > 0$;

(iii) $\{\psi_j : j \in J\}$ is an at most countable orthonormal basis of clos$[\operatorname{ran} T] \subseteq \mathcal{K}$; and

(iv) $(s_j(T))_{j\in J} \in c_0(J)$ is a nonincreasing finite or countably infinite sequence of strictly positive s-numbers of T counting multiplicities.

Remark 2.5.3. Uniqueness of Schmidt expansion: notice that for a given compact operator, its Schmidt expansion is not unique for we could have chosen a different orthonormal basis for clos$[\operatorname{ran}|T|]$ consisting entirely of eigenvectors of $|T|$. However, the nonincreasing (finite or countably infinite) sequence $(\lambda_j)_{j\in J} \in c_0(J)$ of strictly positive s-numbers (counting multiplicities) of T appearing in the spectral representation of T is unique. See Proposition 2.5.7.

Remark 2.5.4. Note that T^* is compact whenever T is compact. So in order to deduce a Schmidt expansion of T^* from that of T, recall that $(\psi_j \otimes \phi_j)^* = \phi_j \otimes \psi_j$ for every j. It follows then that

$$T^* = \sum_{j\in J} s_j(T)\phi_j \otimes \psi_j. \tag{2.5.2}$$

Again $\operatorname{rank}(T)$ denotes the dimension of the range of $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, that is, $\operatorname{rank}(T) = \dim[\operatorname{ran} T]$ and is either a natural number or the first transfinite cardinal number whenever T is compact.

Proposition 2.5.5. *If $T \in \mathcal{B}_0(\mathcal{H}, \mathcal{K})$, then $s_j(T) = s_j(T^*)$ for every $1 \leq j \leq \operatorname{rank}(T)$.*

Proof. Using Equations (2.5.1) and (2.5.2) we obtain

$$T^*T(\phi_j) = s_j^2(T)\phi_j \text{ and } TT^*(\psi_j) = s_j^2(T)\psi_j \text{ for all } j.$$

Clearly then T^*T (respectively TT^*) can be viewed as a diagonal operator by extending $\{\phi_j\}_j$ (respectively $\{\psi_j\}_j$) to an orthonormal basis of $\mathcal{H}$ (respectively $\mathcal{K}$). Consequently, $\lambda_j(T^*T) = \lambda_j(TT^*)$ for every j, and the result follows trivially from the following equation:

$$s_j^2(T) = \lambda_j^2(|T|) = \lambda_j(|T|^2) = \lambda_j(T^*T) = \lambda_j(TT^*) = \lambda_j(|T^*|^2) = \lambda_j^2(|T^*|) = s_j^2(T^*).$$

□

Proposition 2.5.6. *If $T \in \mathcal{B}_0(\mathcal{H})$, then for every operator $R \in \mathcal{B}(\mathcal{H})$ we have*

$$s_j(RT) \leq \|R\| s_j(T) \text{ and } s_j(TR) \leq \|R\| s_j(T), \qquad \text{for } 1 \leq j \leq \operatorname{rank}(T).$$

We have seen in Proposition 2.5.2 that the singular values appear in a specific decomposition of a compact operator. In the result that follows next we show that the singular values appear in *any* such decomposition.

Proposition 2.5.7. *Let $T \in \mathcal{B}_0(\mathcal{H}, \mathcal{K})$ and suppose*

$$T = \sum_{j \in J} t_j \psi_j \otimes \phi_j,$$

where the series converges to T in norm, $(t_j)_{j \in J} \in c_0(J)$, i.e., is either a nonincreasing finite m-tuple of strictly positive numbers (for some $m \in \mathbb{N}$) or a nonincreasing (countably infinite) sequence of strictly positive numbers converging to zero, and $\{\phi_j\}_j \subseteq \mathcal{H}$ and $\{\psi_j\}_j \subseteq \mathcal{K}$ are at most countable orthonormal sets. Then $t_j = s_j(T)$ for every $j \in J$. Moreover, $\|T\| = \sup_{j \in J}\{t_j\}$.

Proof. The goal is to determine the s-numbers of T. To do so we compute T^*T. Since $T = \sum_{j \in J} t_j \psi_j \otimes \phi_j$, we have $T^* = \sum_{j \in J} t_j \phi_j \otimes \psi_j$. Thus for every $x \in \mathcal{H}$, we get

$$T^*Tx = \left(\sum_{j \in J} t_j^2 \phi_j \otimes \phi_j \right) x.$$

Consequently, $T^*T = \sum_{j\in J} t_j^2 \phi_j \otimes \phi_j$ as this sum clearly converges in the norm topology of $\mathcal{B}(\mathcal{H})$ due to the fact that $(t_j^2)_{j\in J} \in c_0(J)$. Moreover, since $\{\phi_j : j \in J\} \subseteq \mathcal{H}$ is an orthonormal set, T^*T can be viewed as a diagonal operator by extending $\{\phi_j : j \in J\}$ to an orthonormal basis of $\mathcal{H}$. Next, by the same argument as above, the operator $K := \sum_{j\in J} t_j \phi_j \otimes \phi_j$ defines an element of $\mathcal{B}(\mathcal{H})$ which can be viewed as a positive diagonal operator with respect to the same orthonormal basis of $\mathcal{H}$ which we obtained by extending the orthonormal set $\{\phi_j : j \in J\}$ in case of the operator T^*T. It is then clear that $K^2 = T^*T$ which implies that $K = |T|$ because of the uniqueness of the positive square root of a positive compact operator. Thus the strictly positive eigenvalues of $|T|$ enumerated in a nonincreasing order with each of them appearing as many times as is its multiplicity is precisely the (finite or countably infinite) sequence $(t_j)_{j\in J} \in c_0(J)$. Then by the definition of the s-numbers, we get $s_j(T) = t_j$ for every $j \in J$.

The final assertion follows from the fact that $\|T\| = \|\,|T|\,\|$ and that the spectral representation of $|T|$ is given by $|T| = \sum_{j\in J} t_j \phi_j \otimes \phi_j$ which can be viewed as a diagonal operator in $\mathcal{B}(\mathcal{H})$ with diagonal entries contained in the set $\{t_j : j \in J\} \cup \{0\}$. □

Remark 2.5.8. The above result, together with Equations (2.5.1) and (2.5.2) immediately implies Proposition 2.5.5.

2.6 An approximation property of s-numbers

Consider the set $\mathcal{B}_0(\mathcal{H}, \mathcal{K})$ of compact operators and recall that we use $\mathcal{B}_{00}(\mathcal{H}, \mathcal{K})$ to denote the set of finite rank operators from $\mathcal{H}$ to $\mathcal{K}$ with $\mathcal{B}_{00}(\mathcal{H}, \mathcal{H})$ abbreviated $\mathcal{B}_{00}(\mathcal{H})$. If $n \in \mathbb{Z}^+$, we let $\mathcal{B}_{00}^n(\mathcal{H})$ denote the set of finite rank operators on $\mathcal{H}$ with rank less than or equal to n.

The following proposition gives a geometrical insight of the nth singular value of a compact operator. The formula therein illustrates that the $(n+1)$th singular value $s_{n+1}(T)$ of $T \in \mathcal{B}(\mathcal{H})$ is the distance of the compact operator T from the set $\mathcal{B}_{00}^n(\mathcal{H})$. This result can be considered an alternative definition of the s-numbers.

Proposition 2.6.1. *If $T \in \mathcal{B}_0(\mathcal{H})$, then for every nonnegative integer $n \in \mathbb{Z}^+$,*

$$s_{n+1}(T) = \min\{\|T - F\| : F \in \mathcal{B}_{00}^n(\mathcal{H})\}.$$

Proof. When $n = 0$, there is nothing to show. Suppose that $n \in \mathbb{N}$. From Equation (2.4.2) of Theorem 2.4.1, we have

$$s_{n+1}(T) = \min_{\{S:\dim(S)=n\}} \left(\max_{\{x: x\in S^\perp \text{ and } \|x\|=1\}} \langle |T|x, x\rangle \right),$$

where S is a subspace of $\mathcal{H}$. Therefore for any n-dimensional subspace $S \subseteq \mathcal{H}$, we have

$$s_{n+1}(T) = \max\{\langle |T|x, x\rangle : x \in S^{\perp} \text{ and } \|x\| = 1\}.$$

Let F be a finite rank operator in $\mathcal{B}(\mathcal{H})$ with n-dimensional range and set $S = [\ker F]^{\perp}$ so that S is n-dimensional and $S^{\perp} = \ker F$. Then we get

$$\begin{aligned} s_{n+1}(T) &\leq \max\{\langle |T|x, x\rangle : x \in \ker F \text{ and } \|x\| = 1\} \\ &\leq \max\{\| \, |T|x \, \| : x \in \ker F \text{ and } \|x\| = 1\} \\ &\leq \max\{\|Tx\| : x \in \ker F \text{ and } \|x\| = 1\} \\ &\leq \max\{\|(T - F)x\| : x \in \ker F \text{ and } \|x\| = 1\} \\ &\leq \|T - F\|, \end{aligned}$$

which implies that

$$s_{n+1}(T) \leq \min\{\|T - F\| : F \in \mathcal{B}_{00}^{n}(\mathcal{H})\}. \tag{2.6.1}$$

Further let the nth partial sum of the Schmidt expansion (2.5.1) of T be denoted by T_n and is given by the formula

$$T_n = \sum_{j=1}^{n} s_j(T)\psi_j \otimes \phi_j. \tag{2.6.2}$$

That $T_n \in \mathcal{B}_{00}^{n}(\mathcal{H})$ is a trivial observation. Also notice that

$$T - T_n = \sum_{j=n+1}^{\operatorname{rank} T} s_j(T)\psi_j \otimes \phi_j,$$

which implies that

$$\begin{aligned} s_{n+1}(T) &= s_1(T - T_n) \\ &= \|T - T_n\| \\ &\geq \min\{\|T - F\| : F \in \mathcal{B}_{00}^{n}(\mathcal{H})\}. \end{aligned} \tag{2.6.3}$$

The inequalities (2.6.1) and (2.6.3) yield $s_{n+1}(T) = \min\{\|T - F\| : F \in \mathcal{B}_{00}^{n}(\mathcal{H})\}$. Since $n \in \mathbb{N}$ is arbitrary, the assertion follows. $\square$

Corollary 2.6.2. *Let $T \in \mathcal{B}_0(\mathcal{H})$ and let $F \in \mathcal{B}_{00}(\mathcal{H})$ with* $\operatorname{rank}(F) = r$. *Then for every nonnegative integer $n \geq r + 1$, we have*

$$s_{n+r}(T) \leq s_n(T + F) \leq s_{n-r}(T).$$

Proof. Let, as in the preceding result, T_n denote the nth partial sum of the Schmidt expansion of T. Obviously the operator $F + T_n \in \mathcal{B}_{00}^{n+r}(\mathcal{H})$. Consequently, we have

$$\begin{aligned}
s_{n+1}(T) &= \|T - T_n\| \\
&= \|(T + F) - (F + T_n)\| \\
&\geq \min\{\|(T + F) - K\| : K \in \mathcal{B}_{00}^{n+r}(\mathcal{H})\} \\
&= s_{n+r+1}(T + F).
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
s_{n+1}(T + F) &= \|(T + F) - (T + F)_n\| \\
&= \|T + (F - (T + F)_n)\| \\
&\geq \min\{\|T + K\| : K \in \mathcal{B}_{00}^{n+r}(\mathcal{H})\} \\
&= s_{n+r+1}(T),
\end{aligned}$$

where $(T+F)_n$ is the nth partial sum of the Schmidt expansion of $T+F$ (hence its rank is at most n) and the inequality in the above expression is due to the fact that $F - (T+F)_n$ has rank at most $n + r$. We have thus shown that for every $n \in \mathbb{Z}^+$,

$$s_{n+1}(T) \geq s_{n+r+1}(T + F) \text{ and} \tag{2.6.4}$$

$$s_{n+1}(T + F) \geq s_{n+r+1}(T), \tag{2.6.5}$$

which yields the required inequality. □

Corollary 2.6.3 ([Fan51])**.** *Let $T_1, T_2 \in \mathcal{B}_0(\mathcal{H})$. Then for every $m, n \in \mathbb{N}$, we have*

$$\begin{aligned}
s_{m+n-1}(T_1 + T_2) &\leq s_m(T_1) + s_n(T_2), \text{ and} \\
s_{m+n-1}(T_1 T_2) &\leq s_m(T_1) s_n(T_2).
\end{aligned}$$

Proof. Since $s_m(T_1) = \min\{\|T_1 - F\| : F \in \mathcal{B}_{00}^{m-1}(\mathcal{H})\}$, there exists $F_1 \in \mathcal{B}_{00}^{m-1}(\mathcal{H})$ such that $s_m(T_1) = \|T_1 - F_1\|$. In fact, it is not too hard to see that rank $F_1 = m - 1$. Similarly, there exists an operator F_2 on $\mathcal{H}$ with rank $F_2 = n - 1$. Since $\operatorname{rank}(F_1 + F_2) \leq m + n - 2$, we have

$$\begin{aligned}
s_{m+n-1}(T_1 + T_2) &= \min\{\|T_1 + T_2 - K\| : K \in \mathcal{B}_{00}^{m+n-2}(\mathcal{H})\} \\
&= \|T_1 + T_2 - (F_1 + F_2)\| \\
&\leq \|T_1 - F_1\| + \|T_2 - F_2\| \\
&= s_m(T_1) + s_n(T_2).
\end{aligned}$$

This proves the first inequality. To see the second inequality notice that $(T_1-F_1)(T_2-F_2) = T_1T_2-T_1F_2-F_1(T_2-F_2)$ and that $\mathrm{rank}(T_1F_2+F_1(T_2-F_2)) \le (n-1)+(m-1) = n+m-2$. This yields

$$\begin{aligned}
s_{m+n-1}(T_1T_2) &= \min\{\|T_1T_2 - K\| : K \in \mathcal{B}_{00}^{m+n-2}(\mathcal{H})\} \\
&\le \|T_1T_2 - (T_1F_2 + F_1(T_2 - F_2))\| \\
&= \|(T_1 - F_1)(T_2 - F_2)\| \\
&\le \|(T_1 - F_1)\|\|(T_2 - F_2)\| \\
&= s_m(T_1)s_n(T_2).
\end{aligned}$$

□

Corollary 2.6.4. *If $T_1, T_2 \in \mathcal{B}_0(\mathcal{H})$ and $n \in \mathbb{N}$, then*

$$|s_n(T_1) - s_n(T_2)| \le \|T_1 - T_2\|.$$

Proof. For every $n \in \mathbb{Z}^+$, we have

$$\begin{aligned}
s_{n+1}(T_1) &= \min\{\|T_1 - F\| : F \in \mathcal{B}_{00}^n(\mathcal{H})\} \\
&= \min\{\|T_2 - F + T_1 - T_2\| : F \in \mathcal{B}_{00}^n(\mathcal{H})\} \\
&\le \min\{\|T_2 - F\| + \|T_1 - T_2\| : F \in \mathcal{B}_{00}^n(\mathcal{H})\} \\
&= \min\{\|T_2 - F\| : F \in \mathcal{B}_{00}^n(\mathcal{H})\} + \|T_1 - T_2\| \\
&= s_{n+1}(T_2) + \|T_1 - T_2\|,
\end{aligned}$$

which implies

$$s_{n+1}(T_1) - s_{n+1}(T_2) \le \|T_1 - T_2\|.$$

Similarly, by interchanging the roles of operators T_1 and T_2, we obtain

$$s_{n+1}(T_2) - s_{n+1}(T_1) \le \|T_2 - T_1\| = \|T_1 - T_2\|.$$

Consequently, from these two inequalities, we have

$$|s_n(T_1) - s_n(T_2)| \le \|T_1 - T_2\|.$$

□

Remark 2.6.5. Note that the above corollary reduces to $|\,\|T_1\| - \|T_2\|\,| \le \|T_1 - T_2\|$, in case of $n = 1$, and hence it generalizes the reverse triangle inequality in the present context.

2.7 Fundamental inequalities concerning s-numbers

Here we collect some essential inequalities for the s-numbers of sums of compact operators. We provide almost no proofs in this section.

Lemma 2.7.1 ([Wey49];[HLP52]). *Let $\Phi(x)$, $-\infty \leq x < \infty$ be a convex function vanishing at $x = -\infty$ (that is, $\Phi(-\infty) = \lim_{x\to\infty} \Phi(-x) = 0$), and let $\{a_j\}_{j=1}^{\omega}$ and $\{b_j\}_{j=1}^{\omega}$ ($\omega \leq \infty$) be nonincreasing sequence of real numbers such that*

$$\sum_{j=1}^{k} a_j \leq \sum_{j=1}^{k} b_j \qquad \text{for every } j \in \{1, ..., \omega\}.$$

Then

$$\sum_{j=1}^{k} \Phi(a_j) \leq \sum_{j=1}^{k} \Phi(b_j) \qquad \text{for every } j \in \{1, ..., \omega\}.$$

In particular,

$$\sum_{j=1}^{\infty} \Phi(a_j) \leq \sum_{j=1}^{\infty} \Phi(b_j).$$

If in addition the function $\Phi(x)$ is strictly convex, then the equality

$$\sum_{j=1}^{\omega} \Phi(a_j) = \sum_{j=1}^{\omega} \Phi(b_j)\,(< \infty)$$

will hold if and only if

$$a_j = b_j \qquad \text{for every } j \in \{1, ..., \omega\}.$$

Lemma 2.7.2 ([Fan51]). *Let $T \in \mathcal{B}_0(\mathcal{H})$. Then for any $n \in \mathbb{N}$ we have*

$$\max \left| \sum_{j=1}^{n} \langle UT\phi_j, \phi_j \rangle \right| = \sum_{j=1}^{n} s_j(T),$$

where the maximum is taken over all unitary operators U on $\mathcal{H}$ and all orthonormal sets $\{\phi_j\}_{j=1}^{n}$ in $\mathcal{H}$. In particular,

$$\sum_{j=1}^{n} |\langle T\phi_j, \phi_j \rangle| \leq \sum_{j=1}^{n} s_j(T).$$

Lemma 2.7.3 ([Fan51]). *If $T_1, T_2 \in \mathcal{B}_0(\mathcal{H})$, then*

$$\sum_{j=1}^{n} s_j(T_1 + T_2) \leq \sum_{j=1}^{n} s_j(T_1) + s_j(T_2) \qquad \textit{for every } n \in \mathbb{N},$$

Proof. By Lemma 2.7.2 there exists an orthonormal set $\{\phi_j\}_{j=1}^{n} \subseteq \mathcal{H}$ of vectors and a unitary operator U on $\mathcal{H}$ such that

$$\left| \sum_{j=1}^{n} \langle U(T_1 + T_2)\phi_j, \phi_j \rangle \right| = \sum_{j=1}^{n} s_j(T_1 + T_2).$$

It follows then that

$$\sum_{j=1}^{n} s_j(T_1 + T_2) \leq \left| \sum_{j=1}^{n} \langle UT_1\phi_j, \phi_j \rangle \right| + \left| \sum_{j=1}^{n} \langle UT_2\phi_j, \phi_j \rangle \right| \leq \sum_{j=1}^{n} s_j(T_1) + \sum_{j=1}^{n} s_j(T_2),$$

where the last inequality is again due to the Lemma 2.7.2. $\square$

Theorem 2.7.4 ([Fan51]). *If $T_1, T_2 \in \mathcal{B}_0(\mathcal{H})$ and $f(x)\,(0 \leq x < \infty)$ is a nondecreasing convex function which vanishes at $x = 0$, then*

$$\sum_{j=1}^{n} f(s_j(T_1 + T_2)) \leq \sum_{j=1}^{n} f(s_j(T_1)) + f(s_j(T_2)) \qquad \textit{for every } n \in \mathbb{N},$$

and consequently

$$\sum_{j=1}^{\infty} f(s_j(T_1 + T_2)) \leq \sum_{j=1}^{\infty} f(s_j(T_1)) + f(s_j(T_2)).$$

2.8 s-Numbers of operators

Following [GK69] we generalize the concept of s-numbers from compact operators to *any* operator (that is, bounded linear transformation). Since the n-th singular number of a compact operator T is defined to be $\lambda_n(|T|)$, the generalization requires us to define the numbers $\lambda_n(|T|)$ for $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ for this concept to parallel the definition in the case when $T \in \mathcal{B}_0(\mathcal{H}, \mathcal{K})$. After recalling the following definition and stating an elementary proposition, we define the numbers $\lambda_n(P)$ for a positive operator $P \in \mathcal{B}(\mathcal{H})$.

Definition 2.8.1. Let $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{B}_0(\mathcal{H})$ be the canonical quotient map from $\mathcal{B}(\mathcal{H})$ to the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{B}_0(\mathcal{H})$. If $T \in \mathcal{B}(\mathcal{H})$, the *essential spectrum* $\sigma_e(T)$ of T is defined to be the spectrum of $\pi(T)$ in $\mathcal{B}(\mathcal{H})/\mathcal{B}_0(\mathcal{H})$, that is, $\sigma_e(T) = \sigma(\pi(T))$.

Proposition 2.8.2 ([Con90])**.** *Let $N \in \mathcal{B}(\mathcal{H})$ be a normal operator and let $\lambda \in \sigma(N)$. Then $\lambda \in \sigma_e(N)$ if and only if at least one of the following two conditions hold.*

(a) The point λ is an accumulation point of $\sigma(N)$.

(b) The point λ is an isolated point of $\sigma(N)$ that is an eigenvalue with infinite multiplicity.

Definition 2.8.3. Let $P \in \mathcal{B}(\mathcal{H})$ be a positive operator and let $\mu = \sup\{\nu : \nu \in \sigma(P)\}$. If $\mu \in \sigma_e(P)$ we define

$$\lambda_n(P) := \mu, \qquad \text{for every } n \in \mathbb{N}.$$

If $\mu \notin \sigma_e(P)$ then it is an eigenvalue of P with finite multiplicity, say M. In this case, we define

$$\lambda_n(P) := \mu, \qquad \text{for } 1 \leq n \leq M.$$
$$\lambda_{M+n}(P) := \lambda_n(P_1), \qquad \text{for } n \in \mathbb{N},$$

where $P_1 = P - \mu P_{E_\mu}$ with P_{E_μ} being the orthogonal projection of $\mathcal{H}$ onto the eigenspace E_μ corresponding to the eigenvalue μ and $\lambda_n(P_1)$ is defined similarly.

If $\text{rank}(P) < \infty$ we define

$$\lambda_n(P) = 0 \qquad \text{for } n > \text{rank}(P).$$

This notion agrees with the original definition if P is a positive compact operator. Here we get a sequence $(\lambda_n(P))_{n\in\mathbb{N}}$ corresponding to every positive operator $P \in \mathcal{B}(\mathcal{H})$. In the light of the above definition, the following definition makes sense.

Definition 2.8.4 (s-numbers of bounded linear transformation)**.** The n-th *s-number* of an arbitrary operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is defined by

$$s_n(T) = \lambda_n(|T|), \qquad \text{for every } n \in \mathbb{N}.$$

If $\text{rank}(T) < \infty$ we define

$$s_n(T) = 0 \qquad \text{for } n > \text{rank}(T).$$

The sequence $s(T) := (s_n(|T|))_{n\in\mathbb{N}}$ obtained in this way for every $T \in \mathcal{B}(\mathcal{H})$ is what we call the sequence of *s-numbers* of T.

This completes the formal description of the s-numbers of bounded linear transformations. We devote rest of this section to list a few of their properties which are relevant to our present interest.

The following result is an immediate generalization of Theorem 2.4.1 with min and max replaced by inf and sup in the appropriate relations.

Theorem 2.8.5 (Courant-Fischer-Weyl for Positive Operators)**.** *Let $P \in \mathcal{B}(\mathcal{H})$ be a positive operator, $k \in \mathbb{N}$, and let S denote a subspace of $\mathcal{H}$. Then*

$$\lambda_1(P) = \sup_{\{x : x \in \mathcal{H} \text{ and } \|x\|=1\}} \langle Px, x\rangle \tag{2.8.1}$$

$$\lambda_{k+1}(P) = \inf_{\{S : \dim(S)=k\}} \left(\sup_{\{x : x \in S^{\perp} \text{ and } \|x\|=1\}} \langle Px, x\rangle \right). \tag{2.8.2}$$

Corollary 2.8.6. *Let $P_1, P_2 \in \mathcal{B}(\mathcal{H})$. If $0 \leq P_1 \leq P_2$, then for every $n \in \mathbb{N}$, we have*

$$\lambda_n(P_1) \leq \lambda_n(P_2).$$

Let $P \in \mathcal{B}(\mathcal{H})$ be a positive operator. Then the sequence $(\lambda_n(P))_{j=1}^{\infty}$ is nonincreasing and thus has a limit. The following corollary states that this limit is the supremum of the essential spectrum of P.

Corollary 2.8.7. *Let $P \in \mathcal{B}(\mathcal{H})$ be a positive operator and let us define $\lambda_\infty(P) := \lim_{n\to\infty}(\lambda_j(P))_n$. Then*

$$\lambda_\infty(P) = \sup \sigma_e(P).$$

Remark 2.8.8. From Definition 2.8.4 and Corollary 2.8.7, it is easy to see that if $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then the sequence $(s_n(T))_{n\in\mathbb{N}}$ of s-numbers of T is nonincreasing and that

$$\lim_{j\to\infty} \{s_j(T)\}_j = s_\infty(T), \tag{2.8.3}$$

where $s_\infty(T)$ is the number $\lambda_\infty(|T|)$. This observation parallels that of the compact operators; for the essential spectrum of a positive compact operator is the singleton set $\{0\}$.

Proposition 2.8.9. *Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$.*

(a) Then $s_1(T) = \|T\|$.

(b) For every $n \in \mathbb{N}$, we have $s_n(T) = s_n(T^*)$.

(c) For any scalar c, $s_n(cT) = |c| s_n(T)$ for every $n \in \mathbb{N}$.

(d) If $R \in \mathcal{B}(\mathcal{H})$, then $s_n(RT) \leq \|R\| s_n(T)$ and $s_n(TR) \leq \|R\| s_n(T)$ for every $n \in \mathbb{N}$.

Proposition 2.6.1 provides a geometrical insight of the singular values of a compact operator thereby providing an alternative definition of the s-numbers of compact operators on $\mathcal{H}$. What can be said about operators in $\mathcal{B}(\mathcal{H})$ in the similar vein? The following result is an extension of Proposition 2.6.1 to the the s-numbers of operators in $\mathcal{B}(\mathcal{H})$.

Proposition 2.8.10. *If $T \in \mathcal{B}(\mathcal{H})$, then for every nonnegative integer $n \in \mathbb{Z}^+$,*

$$s_{n+1}(T) = \min\{\|T - F\| : F \in \mathcal{B}_{00}^n(\mathcal{H})\}.$$

The Corollaries 2.6.2 and 2.6.3 extend word for word to an arbitrary operator in $\mathcal{B}(\mathcal{H})$. After stating the following two analogous corollaries, we will discuss yet another corollary of the above proposition.

Corollary 2.8.11. *Let $T \in \mathcal{B}(\mathcal{H})$ and let $F \in \mathcal{B}_{00}(\mathcal{H})$ with $\operatorname{rank}(F) = r$. Then for every nonnegative integer $n \geq r + 1$, we have*

$$s_{n+r}(T) \leq s_n(T + F) \leq s_{n-r}(T).$$

Corollary 2.8.12. *Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$. Then for every $m, n \in \mathbb{N}$, we have*

$$s_{m+n-1}(T_1 + T_2) \leq s_m(T_1) + s_n(T_2), \text{ and}$$
$$s_{m+n-1}(T_1 T_2) \leq s_m(T_1) s_n(T_2).$$

Corollary 2.8.13. *If $T \in \mathcal{B}(\mathcal{H})$, then for every nonnegative integer $n \in \mathbb{Z}^+$,*

$$\min\{\|T - K\| : K \in \mathcal{B}_0(\mathcal{H})\} = s_\infty(T).$$

We finish this section by mentioning, without going into details, that Lemma 2.7.2 carries over word for word to operators in $\mathcal{B}(\mathcal{H})$, with the sole difference that the the "max" is replaced by "sup".

Lemma 2.8.14 ([GK69])**.** *Let $T \in \mathcal{B}(\mathcal{H})$. Then for any $n \in \mathbb{N}$ we have*

$$\sup \left| \sum_{j=1}^{n} \langle UT\phi_j, \phi_j \rangle \right| = \sum_{j=1}^{n} s_j(T),$$

where the maximum is taken over all unitary operators U on $\mathcal{H}$ and all orthonormal sets $\{\phi_j\}_{j=1}^n$ in $\mathcal{H}$. In particular,

$$\sum_{j=1}^{n} |\langle T\phi_j, \phi_j \rangle| \leq \sum_{j=1}^{n} s_j(T).$$

From this lemma, the following can be easily deduced.

Lemma 2.8.15 ([GK69])**.** *If $T_1, T_2 \in \mathcal{B}(\mathcal{H})$, then*

$$\sum_{j=1}^{n} s_j(T_1 + T_2) \leq \sum_{j=1}^{n} s_j(T_1) + s_j(T_2) \qquad \textit{for every } n \in \mathbb{N},$$

2.9 Symmetric norms

We now turn our attention to ideals of the algebra $\mathcal{B}(\mathcal{H})$ of operators on $\mathcal{H}$ and define various norms on it. In particular, we are interested in "symmetric norms", which are essential for the study of "symmetrically-normed ideals" ("norm ideals" in older literature). This section mostly contains definitions and easy propositions and serves as a prerequisite for the study of "symmetrically-normed ideals" of compact operators on a Hilbert space which we explore in Chapter 7.

Throughout this exposition the term "ideal" will always mean a "two-sided" ideal. The *trivial ideals* in the algebra $\mathcal{B}(\mathcal{H})$ are the *zero ideal* $\{0\}$ consisting of the zero element alone, and the full algebra $\mathcal{B}(\mathcal{H})$ itself. We see from this that every algebra with nonzero elements has at least two distinct ideals.

Definition 2.9.1. Let $\mathfrak{I}$ be an ideal in $\mathcal{B}(\mathcal{H})$. A *norm* on $\mathfrak{I}$ is a function $\|.\|_{\mathfrak{I}} : \mathfrak{I} \to [0, \infty)$ which satisfies the following conditions:

(1) $\|X\|_{\mathfrak{I}} \geq 0$ for each $X \in \mathfrak{I}$;

(2) $\|X\|_{\mathfrak{I}} = 0$ if and only if $X = 0$;

(3) $\|\lambda X\|_{\mathfrak{I}} = |\lambda| \|X\|_{\mathfrak{I}}$ for every $X \in \mathfrak{I}$ and for every $\lambda \in \mathbb{C}$; and

(4) $\|X + Y\|_{\mathfrak{I}} \leq \|X\|_{\mathfrak{I}} + \|Y\|_{\mathfrak{I}}$ for every $X, Y \in \mathfrak{I}$.

The usual operator norm $\|\cdot\|$ is of course a norm on $\mathfrak{I}$. The norm $\|.\|_{\mathfrak{I}}$ on $\mathfrak{I}$ is a *crossnorm* if it also possesses the "cross property", that is, if

(5) $\|X\|_{\mathfrak{I}} = \|X\|$ for every rank one operator $X \in \mathfrak{I}$.

We say that the norm $\|.\|_{\mathfrak{I}}$ on $\mathfrak{I}$ is *unitarily invariant* if

(6) $\|UXV\|_{\mathfrak{J}} = \|X\|_{\mathfrak{J}}$ for every $X \in \mathfrak{J}$ and for every pair U, V of unitary operators in $\mathcal{B}(\mathcal{H})$.

We define $\|.\|_{\mathfrak{J}}$ as *uniform* if

(7) $\|AXB\|_{\mathfrak{J}} \leq \|A\|\|X\|_{\mathfrak{J}}\|B\|$ for every $X \in \mathfrak{J}$ and for every pair A, B of operators in $\mathcal{B}(\mathcal{H})$.

A crossnorm $\|.\|_{\mathfrak{J}}$ is termed unitarily invariant (respectively uniform) if in addition to properties $(1) - (5)$, it also satisfies property (6)(respectively (7)).

Definition 2.9.2 (Symmetric Norm)**.** Let $\mathfrak{J}$ be an ideal in $\mathcal{B}(\mathcal{H})$. A norm on $\mathfrak{J}$ is *symmetric* if it is a uniform crossnorm.

Remark 2.9.3. In the definition of symmetric norm, if we consider the ideal $\mathfrak{J}$ to be $\mathcal{B}(\mathcal{H})$, then it is said to be a *symmetric norm on* $\mathcal{B}(\mathcal{H})$. That is, this definition can be extended to the trivial ideals as well. Moreover, the following observations are obvious:

(a) the usual operator norm on any ideal $\mathfrak{J}$ of $\mathcal{B}(\mathcal{H})$, including the trivial ideals, is a symmetric norm; and

(b) every symmetric norm on $\mathcal{B}(\mathcal{H})$ is topologically equivalent to the ordinary operator norm.

After stating the following elementary proposition which gives an alternative definition of unitarily invariant crossnorm on an ideal $\mathfrak{J}$ of $\mathcal{B}(\mathcal{H})$, we move on to establish a relation between symmetric norms (uniform crossnorm) and unitarily invariant crossnorms.

Proposition 2.9.4. *Let $\mathfrak{J}$ be an ideal of the algebra $\mathcal{B}(\mathcal{H})$ and let $\|.\|_s$ be a symmetric norm (uniform crossnorm) defined on $\mathfrak{J}$. Then the following statements are equivalent.*

(a) $\|UXV\|_s = \|X\|_s$ *for every* $X \in \mathfrak{J}$ *and for every pair* U, V *of unitary operators in* $\mathcal{B}(\mathcal{H})$.

(b) $\|UX\|_s = \|XU\|_s = \|X\|_s$ *for every* $X \in \mathfrak{J}$ *and for every unitary operator* $U \in \mathcal{B}(\mathcal{H})$.

Lemma 2.9.5. *Every symmetric norm is unitarily invariant.*

Proof. Indeed, for any unitary operators U, V, we have, via uniformity of $\|\cdot\|_s$,

$$\|UXV\|_s \leq \|U\|\|X\|_s\|V\| = \|X\|_s,$$

while on the other hand, we have

$$\|X\|_s = \|U^{-1}UXVV^{-1}\|_s \leq \|U^{-1}\|\|UXV\|_s\|V^{-1}\| = \|UXV\|_s.$$

This proves the assertion. $\square$

Remark 2.9.6. We will later show (Chapter 7) that the converse of the above statement, namely that every unitarily invariant cross norm on $\mathcal{B}_{00}(\mathcal{H})$ is uniform (and thus symmetric).

Chapter 3

Characterization of operators in $\mathcal{AN}(\mathcal{H}, \mathcal{K})$

As pointed out in the introduction of this thesis, our purpose in this chapter is to lay bare the true nature of the spectral characterization theorem of absolutely norming operators in $\mathcal{B}(\mathcal{H}, \mathcal{K})$. This chapter is based on [PP17]. As promised, we begin by presenting a counterexample to [Ram14, Theorem 2.3].

Example 3.0.1. Consider the operator

$$T = \begin{bmatrix} \frac{1}{2} & & & & & \\ & 1 & & \text{\Large 0} & & \\ & & 1 & & & \\ & & & 1 & & \\ & \text{\Large 0} & & & \ddots & \\ & & & & & \ddots \end{bmatrix} \in \mathcal{B}(\ell^2).$$

That T is positive operator on a separable Hilbert space is obvious. T is not compact. The infimum of the eigenvalues of this operator, denoted $m(T)$ by Ramesh, is $1/2$. The operator $T - m(T)I = \text{diag}(0, 1/2, 1/2, ...)$ is not compact. Consequently, T is neither compact nor of the form $K + m(T)I$ for some positive compact operator K. Even more, there does not exist $\alpha \geq 0$ such that $T = K + \alpha I$ for some positive compact operator K. Thus, if [Ram14, Theorem 2.3] was correct, then T would not be absolutely norming.

However, we now prove that $T \in \mathcal{AN}(\ell^2)$. Suppose that $\mathcal{M}$ is an arbitrary nontrivial closed subspace of $\mathcal{H}$. If $\mathcal{M}$ is one dimensional, then $T|_{\mathcal{M}}$ attains its norm at any vector in $\mathcal{M}$ with unit norm.

If $\dim(\mathcal{M}) \geq 2$ and $\mathcal{M}$ contains two non-collinear vectors which are nonzero in the first entry, then there exists a linear combination of these two vectors with 0 in the first entry. Letting x_0 be the normalization of this vector, we get $1 = \|x_0\| = \|T(x_0)\| \leq \|T|_{\mathcal{M}}\| \leq \|T\| = 1$ and so we have equality throughout and T attains its norm on $\mathcal{M}$.

Finally, if $\dim(\mathcal{M}) \geq 2$ and it does not contain any two such vectors, then it either has a single such vector and its scalar multiples or no such vector. Since $\dim(\mathcal{M}) \geq 2$, $\mathcal{M}$ has at least one vector linearly independent from all vectors with non zero first entry and that vector must have 0 in its first entry. If we normalize this vector — we call this vector x_0 — we get $1 = \|x_0\| = \|T(x_0)\| \leq \|T|_{\mathcal{M}}\| \leq \|T\| = 1$ and hence T attains its norm on $\mathcal{M}$. This proves the assertion and serves to be a counterexample to the characterization Theorem 2.3 of [Ram14].

3.1 Properties of operators in $\mathcal{AN}(\mathcal{H}, \mathcal{K})$

Let us start with proving few fundamental results that we need for later purposes.

Proposition 3.1.1. *Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. If $\mathcal{H}$ is finite dimensional, then $T \in \mathcal{AN}(\mathcal{H}, \mathcal{K})$.*

Proof. Let $(\mathcal{H})_1 = \{x \in \mathcal{H} : \|x\| \leq 1\}$ be the closed unit ball of $\mathcal{H}$ and $\mathcal{M}$ be an arbitrarily chosen closed subspace of $\mathcal{H}$. Consider the function $f : (\mathcal{H})_1 \cap \mathcal{M} \longrightarrow [0, \infty)$ given by $f(x) = \|T|_{\mathcal{M}}(x)\|_{\mathcal{K}} = \|Tx\|_{\mathcal{K}}$. Since $\mathcal{H}$ is finite dimensional, $(\mathcal{H})_1$ is compact in norm topology. That $(\mathcal{H})_1 \cap \mathcal{M}$ is closed is a trivial observation. Further, $(\mathcal{H})_1 \cap \mathcal{M}$ is contained in the bounded set $(\mathcal{H})_1$ and is hence bounded. The Heine-Borel theorem guarantees the compactness of $(\mathcal{H})_1 \cap \mathcal{M}$. Also, f is continuous on $(\mathcal{H})_1 \cap \mathcal{M}$; since if (x_n) is a sequence in $(\mathcal{H})_1 \cap \mathcal{M}$ converging to x, then $T(x_n) \longrightarrow T(x)$ in the norm $\|\cdot\|_{\mathcal{K}}$, and so $f(x_n) = \|T(x_n)\|_{\mathcal{K}} \longrightarrow \|Tx\|_{\mathcal{K}} = f(x)$ since $\|\cdot\|_{\mathcal{K}}$ is a continuous function on $\mathcal{K}$. By extreme value theorem f has an absolute maximum on $(\mathcal{H})_1 \cap \mathcal{M}$. Thus there exists $x_0 \in (\mathcal{H})_1 \cap \mathcal{M}$ such that

$$\begin{aligned}
\|T|_{\mathcal{M}}(x_0)\|_{\mathcal{K}} &= f(x_0) \\
&= \max\{f(x) : x \in (\mathcal{H})_1 \cap \mathcal{M}\} \\
&= \sup\{f(x) : x \in (\mathcal{H})_1 \cap \mathcal{M}\} \\
&= \sup\{\|Tx\|_{\mathcal{K}} : x \in (\mathcal{H})_1 \cap \mathcal{M}\} \\
&= \sup\{\|T|_{\mathcal{M}}(x)\|_{\mathcal{K}} : x \in (\mathcal{H})_1\} \\
&= \sup\{\|T|_{\mathcal{M}}(x)\|_{\mathcal{K}} : \|x\|_{\mathcal{H}} \leq 1\} \\
&= \|T|_{\mathcal{M}}\|.
\end{aligned}$$

This proves that the operator $T|_{\mathcal{M}}$ attains its norm on the closed unit ball of $\mathcal{H}$. In order to show that it attains its norm on the unit sphere, notice that, $\|T|_{\mathcal{M}}(x_0)\|_{\mathcal{K}} \leq \|T|_{\mathcal{M}}\|\|x_0\|_{\mathcal{H}} \leq \|T|_{\mathcal{M}}\|$, and so $\|T|_{\mathcal{M}}\|\|x_0\|_{\mathcal{H}} = \|T|_{\mathcal{M}}\|$ which implies that $\|x_0\|_{\mathcal{H}} = 1$ and hence $T|_{\mathcal{M}} \in \mathcal{N}(\mathcal{H},\mathcal{K})$. Since $\mathcal{M}$ is arbitrary, we conclude that $T \in \mathcal{AN}(\mathcal{H},\mathcal{K})$. ☐

Notice that $\mathcal{K}$ need not be finite dimensional for $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ to qualify for an absolutely norming operator. In particular, if $\mathcal{H}$ is finite dimensional, then every operator $T \in \mathcal{B}(\mathcal{H})$ is absolutely norming, i.e., $\mathcal{AN}(\mathcal{H}) = \mathcal{B}(\mathcal{H})$. The above proposition, although simple in layout, yields an important result as its corollary.

Corollary 3.1.2. *Let $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$. If $\mathcal{H}$ is finite dimensional then $T \in \mathcal{N}(\mathcal{H},\mathcal{K})$.*

Proof. The result follows from the previous proposition when we replace the closed subspace $\mathcal{M}$ by the whole space $\mathcal{H}$. ☐

The key requirement in proofs is the compactness of $(\mathcal{H})_1 \cap \mathcal{M}$ (respectively, $(\mathcal{H})_1$) in the norm topology which is a consequence of the finite dimensionality of $\mathcal{H}$. This property is lost in infinite dimensional Hilbert spaces. However, if we assume the operator to be compact on $\mathcal{H}$, it gives us a similar tool to come up with the following proposition.

Proposition 3.1.3. *If $T \in \mathcal{B}_0(\mathcal{H},\mathcal{K})$, then $T \in \mathcal{AN}(\mathcal{H},\mathcal{K})$.*

Proof. If T is a compact operator from $\mathcal{H}$ to $\mathcal{K}$ then the restriction of T to any closed subspace $\mathcal{M}$ is a compact operator from $\mathcal{M}$ to $\mathcal{K}$. So it will be sufficient to prove that if T is a compact operator then $T \in \mathcal{N}(\mathcal{H},\mathcal{K})$.

Let $(\mathcal{H})_1 = \{x \in \mathcal{H} : \|x\| \leq 1\}$ be the closed unit ball of $\mathcal{H}$. Since T is a compact operator, $T((\mathcal{H})_1)$ is a compact subset of $\mathcal{K}$ in the norm topology [KR91, page 55]. Also, $\|\cdot\|_{\mathcal{K}} : T((\mathcal{H})_1) \longrightarrow [0,\infty)$ is a continuous function on $T((\mathcal{H})_1)$. Consequently we have $\sup\{\|Tx\|_{\mathcal{K}} : \|x\|_{\mathcal{H}} \leq 1\} = \max\{\|Tx\|_{\mathcal{K}} : \|x\|_{\mathcal{H}} \leq 1\}$. It therefore implies that there exists $x_0 \in (\mathcal{H})_1$ such that $\|T\| = \|Tx_0\|_{\mathcal{K}}$. This, together with $\|Tx_0\|_{\mathcal{K}} \leq \|T\|\|x_0\|_{\mathcal{H}} \leq \|T\|$, implies that $\|x_0\|_{\mathcal{H}} = 1$. This proves the proposition. ☐

Lemma 3.1.4. *Let $P \in \mathcal{B}(\mathcal{H})$ be a positive operator. If $x \in \mathcal{H}$ such that $\langle Px, x\rangle = 0$, then $Px = 0$.*

Proof. The assertion is true if $x = 0$. Assume that $x \neq 0$. We need to show that $\langle Px, x\rangle = 0$ implies $Px = 0$. Contrapositively, suppose that $Px \neq 0$. This means that $x \notin \ker(P) = \ker(P^*) = (\operatorname{ran}(P))^{\perp}$, which implies that x is not orthogonal to any non zero element of

$\operatorname{ran}(P)$, i.e., for every $z \in \operatorname{ran}(P) \setminus \{0\}$, we have $\langle x, z \rangle \neq 0$. In particular, $\langle Px, x \rangle \neq 0$. This proves the lemma.

Alternatively, $\langle Px, x \rangle = \left\langle P^{1/2}x, P^{1/2}x \right\rangle = \|P^{1/2}x\|^2 = 0$ implies that $P^{1/2}x = 0$ for every $x \in \mathcal{H}$, and hence $Px = P^{1/2}(P^{1/2}x) = 0$ for every $x \in \mathcal{H}$. $\square$

Proposition 3.1.5 ([CN12])**.** *Let $T \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator. Then $T \in \mathcal{N}(\mathcal{H})$ if and only if either $\|T\|$ or $-\|T\|$ is an eigenvalue of T.*

Proof. The backward implication is obvious. Indeed, if either $\|T\|$ or $-\|T\|$ is an eigenvalue of T and $x \in \mathcal{H}$ is a corresponding eigenvector of unit length, then $\|Tx\| = \|T\|$, and so T is norming.

For the forward implication we assume that there exists x_0 in the unit sphere of $\mathcal{H}$ such that $\|Tx_0\| = \|T\|$. Furthermore, let $\|T\| = \lambda$. We first prove that $(\lambda^2 I - T^2)x_0 = 0$. It is a trivial observation that $\lambda^2 I - T^2$ is a positive operator, since $(\lambda^2 I - T^2)^* = \lambda^2 I - T^* T^* = \lambda^2 I - T^2$ and for any $x \in \mathcal{H}$, we have $\langle (\lambda^2 I - T^2)x, x \rangle = \lambda^2 \|x\|^2 - \|Tx\|^2 \geqslant 0$. Also, we have

$$\left\langle (\lambda^2 I - T^2)x_0, x_0 \right\rangle = \lambda^2 \|x_0\|^2 - \|Tx_0\|^2 = \lambda^2 - \lambda^2 = 0,$$

and so, from the previous lemma, $(\lambda^2 I - T^2)x_0 = 0$. Since $(\lambda I - T)(\lambda I + T)x_0 = 0$, either $(\lambda I + T)x_o = 0$, in which case, $-\lambda$ is an eigenvalue of T or $y = (\lambda I + t)x_0 \neq 0$, in which case y is a non-zero vector in the kernel of $(\lambda I - T)$, in which case $+\lambda$ is an eigenvalue. $\square$

This result leads us to the following theorem.

Theorem 3.1.6. *Let $T \in \mathcal{B}(\mathcal{H})$ be a positive operator. Then $T \in \mathcal{N}(\mathcal{H})$ if and only if $\|T\|$ is an eigenvalue of T.*

Remark 3.1.7. It is desirable at this stage to make an important remark: an eigenvector of a positive operator T corresponding to the eigenvalue $\|T\|$ need not necessarily be an element in the unit sphere of $\mathcal{H}$ at which T attains its norm and more importantly, if T attains its norm at a point in the unit sphere of $\mathcal{H}$, that point need not necessarily be an eigenvector of T corresponding to the eigenvalue $\|T\|$.

The following theorem provides us with another interesting criterion for an operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ to be norming. This criterion is vitally important in establishing useful results in later sections.

Theorem 3.1.8. *Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then $T \in \mathcal{N}(\mathcal{H}, \mathcal{K})$ if and only if $T^*T \in \mathcal{N}(\mathcal{H})$.*

Proof. First assume that $T \in \mathcal{N}(\mathcal{H}, \mathcal{K})$. There exists x in the unit sphere of $\mathcal{H}$ such that $\|Tx\| = \|T\|$. Then

$$\|T^*T\| = \|T\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle \leq \|T^*Tx\| \leq \|T^*T\|,$$

and so we have equality throughout which implies that $T^*T \in \mathcal{N}(\mathcal{H})$.

Conversely, if $T^*T \in \mathcal{N}(\mathcal{H})$, then by Theorem 3.1.6 $\|T^*T\|$ is an eigenvalue of T^*T. Suppose $y \in \mathcal{H}$ is the corresponding eigenvector of unit length. Then $\|Ty\|^2 = \langle T^*Ty, y\rangle = \langle \|T^*T\|y, y\rangle = \|T\|^2$, and the result follows. □

Theorem 3.1.9. *If $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then the following statements are equivalent.*
(1) T *is norming.*
(2) T^* *is norming.*
(3) $\||T|\|$ *is an eigenvalue of* $|T|$.
(4) $\|T\|$ *is an eigenvalue of* $|T|$.
(5) $|T|$ *is norming.*
(6) $|T^*|$ *is norming.*
(7) $|T|^2$ *is norming.*
(8) $|T^*|^2$ *is norming.*
(9) $\|T\|$ *is an eigenvalue of* $|T^*|$.

Proof. The equivalence of (1) and (7) follows from Theorem 3.1.8 as does the equivalence of (5) and (7). Since $\||T|\| = \|T\|$, by Theorem 3.1.6, (5) is equivalent to (3) and (4).

Replacing T by T^* in these equivalences and using that $\|T\| = \|T^*\|$, shows the equivalence of (2), (6), (8) and (9).

All that remains is to show the equivalence of (1) and (2). Assume that T is norming. By equivalence of (1) and (4), $\|T\|$ is an eigenvalue of $|T|$. Let $z \in \mathcal{H}$ be an eigenvector of $|T|$ of unit norm corresponding to the eigenvalue $\|T\|$. Since $|T|(z) = \|T\|z$ we have $T^*Tz = |T|^2(z) = |T|(|T|(z)) = \|T\|^2 z$. Consequently, $\|T^*(Tz)\| = \|T\|\|T\|$, since $\|z_0\| = 1$. Notice that $T(\frac{z_0}{\|T\|})$ is in the unit sphere of $\mathcal{K}$ and hence $\|T^*(T\frac{z_0}{\|T\|})\| = \|T^*\|$ which means that $T^* \in \mathcal{N}(\mathcal{K}, \mathcal{H})$. This proves that (1) implies (2). The backward implication follows if we replace T by T^* in the proof and use $T^{**} = T$. This completes the proof. □

Remark 3.1.10. Later (see 3.5.2) we will give an example of an operator such that T is absolutely norming but T^* is not.

3.2 Necessary conditions for positive operators to belong to $\mathcal{AN}(\mathcal{H})$

The purpose of this section is to study the properties of positive absolutely norming operators. Let $\mathcal{AN}(\mathcal{H})^+$ denote the set of positive absolutely norming operators. We first briefly consider a general notion of summability in a Banach space (and thus in a Hilbert space).

Definition 3.2.1. Let $\{v_\alpha\}_{\alpha\in\Lambda}$ be a set of vectors in the Banach space X, where Λ is an index set. Let $\mathcal{F} = \{F \subseteq \Lambda : F \text{ is finite}\}$. If $\mathcal{F}$ is preordered by inclusion (that is define $F_1 \leq F_2$ for $F_1 \subseteq F_2$), then $\mathcal{F}$ is a directed set. For each $F \in \mathcal{F}$, let $h_F = \sum_{\alpha\in F} v_\alpha$. Since this is a finite sum, h_F is a well-defined element of X. If the net $(h_F)_{F\in\mathcal{F}}$ converges to some $h \in X$, then the sum $\sum_{\alpha\in\Lambda} v_\alpha$ is said to converge and we write $h = \sum_{\alpha\in\Lambda} v_\alpha$.

Theorem 3.2.2. *If $T \in \mathcal{AN}(\mathcal{H})^+$, then $\mathcal{H}$ has an orthonormal basis consisting of eigenvectors of T.*

Proof. Let $\mathcal{B} = \{v_\alpha : \alpha \in \Lambda\}$ be a maximal orthonormal set of eigenvectors of T. That $\mathcal{B}$ is non empty is a trivial observation; for T, being a positive absolutely norming operator, must have $\|T\|$ as one of its eigenvalues. Considering w to be a unit eigenvector corresponding to the eigenvalue $\|T\|$, we have $Tw = \|T\|w$ which implies that there exists $\theta \in [0, 2\pi)$ such that the unit vector $e^{i\theta}w \in \mathcal{B}$ serves as an eigenvector of T corresponding to the eigenvalue $\|T\|$.

To show that $\mathcal{H}$ has an orthonormal basis consisting entirely of eigenvectors of T we define $\mathcal{H}_0 := \text{clos}(\text{span}(\mathcal{B}))$ and show that $\mathcal{H}_0 = \mathcal{H}$. It suffices to show that $\mathcal{H}_0^\perp = \{0\}$; for then $\mathcal{H}_0 = \mathcal{H}_0^{\perp\perp} = \{0\}^\perp = \mathcal{H}$.

We first claim that $\mathcal{H}_0^\perp$ is an invariant subspace of $\mathcal{H}$ under T. To see this, let $\mathcal{F}$ denote the collection of finite subsets of Λ, that is, $\mathcal{F} = \{F \subseteq \Lambda : F \text{ is finite}\}$. If $v \in \mathcal{H}_0$, then by above definition we have

$$v = \sum_{\alpha\in\Lambda} \langle v, v_\alpha\rangle\, v_\alpha = \lim_{F\in\mathcal{F}} \sum_{\alpha\in F} \langle v, v_\alpha\rangle\, v_\alpha.$$

Since the above limit is norm limit and T is bounded (norm continuous), it follows that

$$\begin{aligned} Tv &= T\Big(\lim_{F\in\mathcal{F}}\sum_{\alpha\in F}\langle v, v_\alpha\rangle\, v_\alpha\Big) \\ &= \lim_{F\in\mathcal{F}} T\Big(\sum_{\alpha\in F}\langle v, v_\alpha\rangle\, v_\alpha\Big) \\ &= \lim_{F\in\mathcal{F}}\Big(\sum_{\alpha\in F}\langle v, v_\alpha\rangle\, Tv_\alpha\Big) \\ &= \lim_{F\in\mathcal{F}}\sum_{\alpha\in F}\langle v, v_\alpha\rangle\, \beta_\alpha v_\alpha \\ &= \sum_{\alpha\in\Lambda}\langle v, v_\alpha\rangle\, \beta_\alpha v_\alpha \in \mathcal{H}_0, \end{aligned}$$

considering $Tv_\alpha = \beta_\alpha v_\alpha$ where $\beta_\alpha \in \mathbb{C}$ for every $\alpha \in \Lambda$. This shows that $\mathcal{H}_0$ is an invariant subspace of $\mathcal{H}$ under T. Since $T = T^*$, we infer that $\mathcal{H}_0^\perp$ is also an invariant subspace of $\mathcal{H}$ under T.

We complete the proof by showing $\mathcal{H}_0^\perp = \{0\}$. Suppose, on the contrary, that $\mathcal{H}_0^\perp \neq \{0\}$, i.e. $\mathcal{H}_0^\perp$ is a non trivial closed subspace of $\mathcal{H}$. Since T is a positive absolutely norming operator, $T|_{\mathcal{H}_0^\perp} \in \mathcal{N}(\mathcal{H})$. Even more, $T|_{\mathcal{H}_0^\perp}$ is a positive operator on $\mathcal{H}_0^\perp$ which belongs to $\mathcal{N}(\mathcal{H})$ because $\mathcal{H}_0^\perp$ is invariant under T. Consequently, $\|T|_{\mathcal{H}_0^\perp}\|$ is an eigenvalue of $T|_{\mathcal{H}_0^\perp}$. Let z be a unit eigenvector of $T|_{\mathcal{H}_0^\perp}$ corresponding to the eigenvalue $\|T|_{\mathcal{H}_0^\perp}\|$. Clearly then $z \in \mathcal{H}_0^\perp$ such that $\|z\| = 1$ and $T|_{\mathcal{H}_0^\perp}(z) = \|T|_{\mathcal{H}_0^\perp}\|z$, which implies that $Tz = T|_{\mathcal{H}_0^\perp}(z) = \|T|_{\mathcal{H}_0^\perp}\|z$. But this means that $z \notin \mathcal{H}_0$ is an eigenvector of T which contradicts the maximality of the set $\mathcal{B} = \{v_\alpha : \alpha \in \Lambda\}$ of T and we conclude that $\mathcal{H}_0^\perp = \{0\}$. This completes the proof. $\square$

Corollary 3.2.3. *If $T \in \mathcal{AN}(\mathcal{H})^+$, then*

$$T = \sum_{\alpha\in\Lambda}\beta_\alpha v_\alpha \otimes v_\alpha,$$

where $\{v_\alpha : \alpha \in \Lambda\}$ is an orthonormal basis consisting entirely of eigenvectors of T and for every $\alpha \in \Lambda$, $Tv_\alpha = \beta_\alpha v_\alpha$ with $\beta_\alpha \geqslant 0$. Moreover, for every nonempty subset $\Gamma \subseteq \Lambda$ of Λ, we have $\sup\{\beta_\alpha : \alpha \in \Gamma\} = \max\{\beta_\alpha : \alpha \in \Gamma\}$.

Proof. That $T = \sum_{\alpha\in\Lambda}\beta_\alpha v_\alpha \otimes v_\alpha$ is obvious. Indeed, if we let $\mathcal{F}$ denote the collection of finite subsets of Λ, that is, $\mathcal{F} = \{F \subseteq \Lambda : F \text{ is finite}\}$ and let $z \in \mathcal{H}$, then we have

$$z = \sum_{\alpha\in\Lambda}\langle z, v_\alpha\rangle\, v_\alpha = \lim_{F\in\mathcal{F}}\sum_{\alpha\in F}\langle z, v_\alpha\rangle\, v_\alpha,$$

which yields

$$Tz = \lim_{F\in\mathcal{F}} \sum_{\alpha\in F} \langle z, v_\alpha\rangle Tv_\alpha = \sum_{\alpha\in\Lambda} \beta_\alpha \langle z, v_\alpha\rangle v_\alpha = \Big(\sum_{\alpha\in\Lambda} \beta_\alpha v_\alpha \otimes v_\alpha\Big) z,$$

and since $z \in \mathcal{H}$ is arbitrary, it follows that

$$T = \sum_{\alpha\in\Lambda} \beta_\alpha v_\alpha \otimes v_\alpha.$$

To prove the final claim we use the method of contradiction and assume, on the contrary, that $\sup\{\beta_\alpha : \alpha \in \Gamma\} \neq \max\{\beta_\alpha : \alpha \in \Gamma\}$ for some nonempty subset $\Gamma \subseteq \Lambda$, i.e., the supremum of the set $\{\beta_\alpha : \alpha \in \Gamma\}$ (say β) is not achieved. In that case, for any $x \in \mathcal{H}_\Gamma$ with $\|x\| = 1$, we have

$$\begin{aligned}
\|T|_{\mathcal{H}_\Gamma}(x)\|^2 &= \Big\|\sum_{\alpha\in\Gamma} \beta_\alpha \langle x, v_\alpha\rangle v_\alpha\Big\|^2 \\
&= \sum_{\alpha\in\Gamma} |\beta_\alpha|^2 |\langle x, v_\alpha\rangle|^2 \\
&< \sum_{\alpha\in\Gamma} \beta^2 |\langle x, v_\alpha\rangle|^2 \\
&= \beta^2 \sum_{\alpha\in\Gamma} |\langle x, v_\alpha\rangle|^2 \\
&= \beta^2 \|x\|^2 \\
&= \beta^2 = (\sup\{\beta_\alpha : \alpha \in \Gamma\})^2 = \|T|_{\mathcal{H}_\Gamma}\|^2.
\end{aligned}$$

This implies that $\|T|_{\mathcal{H}_\Gamma}(x)\| < \|T|_{\mathcal{H}_\Gamma}\|$ for every $x \in \mathcal{H}_\Gamma$ with $\|x\| = 1$ which contradicts the fact that T is absolutely norming. This proves the assertion. $\square$

The spectral conditions given in the above corollary do not characterize positive absolutely norming operators as the following example and result show.

Example 3.2.4. Let K_1, K_2 be positive compact operators that are not of finite rank on the complex Hilbert space ℓ^2, and $0 \leq a < b$. Consider the operator

$$T = \begin{bmatrix} aI + K_1 & 0 \\ 0 & bI + K_2 \end{bmatrix} \in \mathcal{B}(\ell^2 \oplus \ell^2).$$

Then the supremum of each subset of the spectrum is equal to the maximum of that subset since the spectrum of T consists of the closure of the union of two decreasing sequences, $\{a_n\} \cup \{b_n\}$ with $\lim_n a_n = a$ and $\lim_n b_n = b$. However, the spectrum of T has two limit points, and so by the following result $T \notin \mathcal{AN}(\ell^2 \oplus \ell^2)$. Thus, the spectral condition given by the above corollary does not characterize positive absolutely norming operators.

Proposition 3.2.5. *If $T \in \mathcal{AN}(\mathcal{H})^+$, then the spectrum $\sigma(T)$ of T has at most one limit point. Moreover, this unique limit point (if it exists) can only be the limit of a decreasing sequence in the spectrum.*

Proof. By the Corollary 3.2.3, we know that

$$T = \sum_{\alpha \in \Lambda} \beta_\alpha v_\alpha \otimes v_\alpha$$

where $\{v_\alpha : \alpha \in \Lambda\}$ is an orthonormal basis consisting entirely of eigenvectors of T and for every $\alpha \in \Lambda, Tv_\alpha = \beta_\alpha v_\alpha$ with $\beta_\alpha \geqslant 0$. All that remains is to show that the spectrum $\sigma(T)$, which is precisely the closure of $\{\beta_\alpha\}_{\alpha \in \Lambda}$, has at most one limit point and this unique limit point (if it exists) can only be the limit of a decreasing sequence in the spectrum.

First we show that whenever λ is a limit point of the spectrum $\sigma(T)$ of T, then there exists a decreasing sequence $(\lambda_n)_{n \in \mathbb{N}} \subseteq \{\beta_\alpha : \alpha \in \Lambda\}$ such that $\lambda_n \searrow \lambda$. To see this, it is sufficient to prove that there are at most only finitely many terms of the sequence of $(\lambda_n)_{n \in \mathbb{N}}$ that are strictly less than λ; for if there are infinitely many such terms, then there exists an increasing subsequence (λ_{n_k}) such that $\lambda_{n_k} \nearrow \lambda$ and for each $n_k \in \mathbb{N}$, $\lambda_{n_k} < \lambda$. But then if we define $\mathcal{M}_0 := \text{clos}[\text{span}\{v_{n_k}\}]$, where v_{n_k}'s are the eigenvectors corresponding to the eigenvalues λ_{n_k}, then it is a trivial observation that $\|T|_{\mathcal{M}_0}\| = \sup\{|\lambda_{n_k}|\} = \lambda$. However, for every $x = \sum_{n_k} \alpha_{n_k} v_{n_k} \in \mathcal{M}_0$ with $\sum_{n_k} |\alpha_{n_k}|^2 = 1$ so that $\|x\| = 1$, we have

$$\|T|_{\mathcal{M}_0}(x)\|^2 = \|\sum_{n_k} \alpha_{n_k} \lambda_{n_k} v_{n_k}\|^2 = \sum_{n_k} |\alpha_{n_k}|^2 |\lambda_{n_k}|^2 < \lambda^2 \sum_{n_k} |\alpha_{n_k}|^2 = \lambda^2$$

so that $\|T|_{\mathcal{M}_0}(x)\| < \lambda \leq \|T|_{\mathcal{M}_0}\|$. This contradicts the fact that $T \in \mathcal{AN}(\mathcal{H})^+$. This proves our first claim.

We next prove, by the method of contradiction, that the spectrum $\sigma(T)$ of T has at most one limit point. Suppose on the contrary that the spectrum $\sigma(T) = \text{clos}[\{\beta_\alpha\}_{\alpha \in \Lambda}]$ has two limit points $a < b$. By the discussion in the above paragraph, there exist decreasing sequences $(a_n)_{n \in \mathbb{N}} \subseteq \{\beta_\alpha\}_{\alpha \in \Lambda}$ and $(b_n)_{n \in \mathbb{N}} \subseteq \{\beta_\alpha\}_{\alpha \in \Lambda}$ such that $a_n \searrow a$ and $b_n \searrow b$. Let us rename and denote by $\{f_n\}$ and $\{g_n\}$ the eigenvectors corresponding to the eigenvalues

$\{a_n\}$ and $\{b_n\}$ respectively. Without any loss of generality we may assume that $a_1 < b$ so that $a_n < b_n$ for each $n \in \mathbb{N}$. (For if it happens otherwise then we can choose a natural number m such that $a_m < b$ and redefine the sequence $(a_n)_{n=m}^{\infty}$ by $(\tilde{a}_n)_{n=1}^{\infty}$.) Also note that $Tf_n = a_n f_n$ and $Tg_n = b_n g_n$ for each $n \in \mathbb{N}$. Define

$$\mathcal{M} := \operatorname{clos}\left[\operatorname{span}\left\{c_n f_n + \sqrt{1-c_n^2}\, g_n : n \in \mathbb{N}\right\}\right]$$

where $c_n^2 \in [0,1]$ are yet to be determined. Needless to say that $\mathcal{M}$ is a closed subspace of $\mathcal{H}$ and hence a Hilbert space in its own right. Moreover, it is a trivial observation that the set $\{e_n : n \in \mathbb{N}\}$, where $e_n := c_n f_n + \sqrt{1-c_n^2}\, g_n$ serves as an orthonormal basis of $\mathcal{M}$. Then we have,

$$\begin{aligned}
\|T|_{\mathcal{M}}\|^2 &= \sup\{\|Tx\|^2 : x \in \mathcal{M}, \|x\| = 1\} \\
&\geqslant \sup\{\|Te_n\|^2\} \\
&= \sup\{\|T(c_n f_n + \sqrt{1-c_n^2}\, g_n)\|^2 : n \in \mathbb{N}\} \\
&= \sup\{\|c_n a_n f_n + \sqrt{1-c_n^2}\, b_n g_n\|^2 : n \in \mathbb{N}\} \\
&= \sup\{c_n^2 a_n^2 + (1-c_n^2) b_n^2 : n \in \mathbb{N}\}.
\end{aligned}$$

At this point we define a sequence $(\gamma_n)_{n\in\mathbb{N}}$ by

$$\gamma_n := b + \frac{a_1 - b}{2n}; n \in \mathbb{N}.$$

Then, $(\gamma_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence such that for every $n \in \mathbb{N}$, $a_1^2 < \gamma_n^2 < b^2$ and $\lim_{n\to\infty} \gamma_n = \sup\{\gamma_n : n \in \mathbb{N}\} = b$. Notice that $c_n^2 a_n^2 + (1-c_n^2) b_n^2$ is a convex combination of a_n^2 and b_n^2, and hence it follows that $c_n^2 a_n^2 + (1-c_n^2) b_n^2 \in [a_n^2, b_n^2]$ for each $n \in \mathbb{N}$. In fact, by choosing the right value of $c_n^2 \in [0,1]$, $c_n^2 a_n^2 + (1-c_n^2) b_n^2$ can give any point in the interval $[a_n^2, b_n^2]$. Let us then choose a sequence $(c_n)_{n\in\mathbb{N}}$ such that $c_n^2 a_n^2 + (1-c_n^2) b_n^2 = \gamma_n^2$. With this chosen sequence the definition $\mathcal{M} := \operatorname{clos}\left[\operatorname{span}\left\{c_n f_n + \sqrt{1-c_n^2}\, g_n : n \in \mathbb{N}\right\}\right]$ now makes complete sense. Moreover, this also yields

$$\|T|_{\mathcal{M}}\|^2 \geqslant \sup\{c_n^2 a_n^2 + (1-c_n^2) b_n^2 : n \in \mathbb{N}\} = \sup\{\gamma_n^2 : n \in \mathbb{N}\} = b^2.$$

However, any $x \in \mathcal{M}$ with $\|x\| = 1$ can be written as

$$\sum_{n=1}^{\infty} \alpha_n (c_n f_n + \sqrt{1-c_n^2}\, g_n), \text{ with } \sum_{n=1}^{\infty} |\alpha_n|^2 = 1,$$

in which case,

$$\begin{aligned}
\|T|_{\mathcal{M}}(x)\|^2 &= \|Tx\|^2 \\
&= \left\| T\left(\sum_{n=1}^{\infty} \alpha_n (c_n f_n + \sqrt{1-c_n^2} g_n)\right)\right\|^2 \\
&= \left\| \sum_{n=1}^{\infty} \alpha_n T(c_n f_n + \sqrt{1-c_n^2} g_n)\right\|^2 \\
&= \left\| \sum_{n=1}^{\infty} \alpha_n (c_n a_n f_n + \sqrt{1-c_n^2} b_n g_n)\right\|^2 \\
&= \sum_{n=1}^{\infty} |\alpha_n|^2 (c_n^2 a_n^2 + (1-c_n^2) b_n^2) \\
&= \sum_{n=1}^{\infty} |\alpha_n|^2 \gamma_n^2 < \sum_{n=1}^{\infty} |\alpha_n|^2 b^2 = b^2.
\end{aligned}$$

This implies that for every element $x \in \mathcal{M}$ with $\|x\| = 1$, $\|T|_{\mathcal{M}}(x)\| < b \leq \|T|_{\mathcal{M}}\|$, which means that $T \notin \mathcal{AN}$. So we arrive at a contradiction. Hence, our hypothesis was wrong and we conclude that the spectrum of T can have at most one limit point. This completes the proof. □

We now use this as a tool to prove the following result.

Corollary 3.2.6. *If $T \in \mathcal{AN}(\mathcal{H})^+$, then the set $\{\beta_\alpha\}_{\alpha\in\Lambda}$ of distinct eigenvalues of T, that is, without counting multiplicities, is countable.*

Proof. This corollary is a direct consequence of the following fact: *if $E \subseteq \mathbb{R}$ is an uncountable subset, then E has at least two limit points.* Since the set $\{\beta_\alpha\}_{\alpha\in\Lambda}$ has at most one limit point, by the contrapositive of the above fact, it is countable. □

Corollary 3.2.7. *If $T \in \mathcal{AN}(\mathcal{H})^+$, then the set $\{\beta_\alpha\}_{\alpha\in\Lambda}$ of eigenvalues of T has at most one eigenvalue with infinite multiplicity.*

Proof. To show that this set has at most one eigenvalue with infinite multiplicity, we suppose that it has two distinct eigenvalues β_1 and β_2 with infinite multiplicity, and we deduce a contradiction from the supposition. Without loss of generality, we assume that $0 \leq \beta_1 < \beta_2$. Now let $(a_n)_{n\in\mathbb{N}} \subseteq \{\beta_\alpha\}_{\alpha\in\Lambda}$ and $(b_n)_{n\in\mathbb{N}} \subseteq \{\beta_\alpha\}_{\alpha\in\Lambda}$ be two sequences such that for every $n \in \mathbb{N}$, we have $a_n = \beta_1$ and $b_n = \beta_2$. Clearly then $a_n \longrightarrow \beta_1$ and $b_n \longrightarrow \beta_2$. Let us, like in the previous proof, rename and denote by $\{f_n\}$ and $\{g_n\}$ the eigenvectors corresponding to the eigenvalues $\{a_n\}$ and $\{b_n\}$ respectively where $Tf_n = a_n f_n = \beta_1 f_n$ and $Tg_n = b_n g_n = \beta_2 g_n$ for each $n \in \mathbb{N}$.

At this point we define a sequence $(\gamma_n)_{n\in\mathbb{N}}$ by

$$\gamma_n := \beta_2 + \frac{\beta_1 - \beta_2}{2n}; n \in \mathbb{N}.$$

That $(\gamma_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence with $\beta_1^2 < \gamma_n^2 < \beta_2^2$ for every $n \in \mathbb{N}$ such that $\lim_{n\to\infty} \gamma_n = \sup\{\gamma_n : n \in \mathbb{N}\} = \beta_2$ is obvious. Let $c_n^2 \in [0,1]$ be arbitrary, then since $c_n^2\beta_1^2 + (1-c_n^2)\beta_2^2$ is a convex linear combination of β_1^2 and β_2^2, it follows that for each $n \in \mathbb{N}$, we have $c_n^2\beta_1^2 + (1-c_n^2)\beta_2^2 \in [\beta_1^2, \beta_2^2]$. In fact, by choosing the right value of $c_n^2 \in [0,1]$, $c_n^2\beta_1^2 + (1-c_n^2)\beta_2^2$ gives any desired point in the interval $[\beta_1^2, \beta_2^2]$. This observation, together with the fact that $\beta_1^2 < \gamma_n^2 < \beta_2^2$ for every $n \in \mathbb{N}$, allows us to define the sequence $(c_n)_{n\in\mathbb{N}}$ concretely as follows: for each $n \in \mathbb{N}$, choose c_n so that $c_n^2\beta_1^2 + (1-c_n^2)\beta_2^2 = \gamma_n^2$. We will use this so defined sequence $(c_n)_{n\in\mathbb{N}}$ as a tool to define a closed subspace $\mathcal{M}$ of $\mathcal{H}$ by

$$\mathcal{M} := \operatorname{clos}\left[\operatorname{span}\left\{c_n f_n + \sqrt{1-c_n^2}\, g_n : n \in \mathbb{N}\right\}\right].$$

It is easy to see that the set $\{e_n : n \in \mathbb{N}\}$ serves as an orthonormal basis of $\mathcal{M}$, where $e_n := c_n f_n + \sqrt{1-c_n^2}\, g_n$. It now follows that

$$\begin{aligned}
\|T|_{\mathcal{M}}\|^2 &= \sup\{\|Tx\|^2 : x \in \mathcal{M}, \|x\| = 1\} \\
&\geq \sup\{\|Te_n\|^2\} \\
&= \sup\{\|T(c_n f_n + \sqrt{1-c_n^2}\, g_n)\|^2 : n \in \mathbb{N}\} \\
&= \sup\{\|c_n a_n f_n + \sqrt{1-c_n^2}\, b_n g_n\|^2 : n \in \mathbb{N}\} \\
&= \sup\{\|c_n \beta_1 f_n + \sqrt{1-c_n^2}\, \beta_2 g_n\|^2 : n \in \mathbb{N}\} \\
&= \sup\{c_n^2\beta_1^2 + (1-c_n^2)\beta_2^2 : n \in \mathbb{N}\} \\
&= \sup\{\gamma_n^2 : n \in \mathbb{N}\} \\
&= \beta_2^2.
\end{aligned}$$

However, any $x \in \mathcal{M}$ with $\|x\| = 1$ can be written as

$$\sum_{n=1}^{\infty} \alpha_n (c_n f_n + \sqrt{1-c_n^2}\, g_n) \text{ with } \sum_{n=1}^{\infty} |\alpha_n|^2 = 1.$$

In that case, we have

$$\begin{aligned}
\|T|_{\mathcal{M}}(x)\|^2 = \|Tx\|^2 &= \left\| T\left(\sum_{n=1}^{\infty} \alpha_n (c_n f_n + \sqrt{1-c_n^2} g_n) \right) \right\|^2 \\
&= \left\| \sum_{n=1}^{\infty} \alpha_n T(c_n f_n + \sqrt{1-c_n^2} g_n) \right\|^2 \\
&= \left\| \sum_{n=1}^{\infty} \alpha_n (c_n a_n f_n + \sqrt{1-c_n^2} b_n g_n) \right\|^2 \\
&= \left\| \sum_{n=1}^{\infty} \alpha_n (c_n \beta_1 f_n + \sqrt{1-c_n^2} \beta_2 g_n) \right\|^2 \\
&= \sum_{n=1}^{\infty} |\alpha_n|^2 (c_n^2 \beta_1^2 + (1-c_n^2)\beta_2^2) \\
&= \sum_{n=1}^{\infty} |\alpha_n|^2 \gamma_n^2 < \sum_{n=1}^{\infty} |\alpha_n|^2 \beta_2^2 = \beta_2^2.
\end{aligned}$$

This implies that for every element $x \in \mathcal{M}$ with $\|x\| = 1$, $\|T|_{\mathcal{M}}(x)\| < \beta_2 \leq \|T|_{\mathcal{M}}\|$ which means that $T \notin \mathcal{AN}(\mathcal{H})^+$. So we arrive at a contradiction. Hence, our hypothesis was wrong and we conclude that the spectrum of T can have at most one eigenvalue with infinite multiplicity. This completes the proof. $\square$

Corollary 3.2.8. *Let $T \in \mathcal{AN}(\mathcal{H})^+$. If the spectrum $\sigma(T) = clos\{\beta_\alpha : \alpha \in \Lambda\}$ of T has both a limit point β and an eigenvalue $\hat{\beta}$ with infinite multiplicity, then $\beta = \hat{\beta}$.*

Proof. To show that $\beta = \hat{\beta}$, we assume that $\beta \neq \hat{\beta}$, and we deduce a contradiction from the assumption. We first consider the case when $\beta < \hat{\beta}$. Because β is a limit point of the spectrum, we know that there exists a decreasing sequence $(a_n)_{n\in\mathbb{N}} \subseteq \{\beta_\alpha\}_{\alpha\in\Lambda}$ such that $a_n \searrow \beta$. Let $(b_n)_{n\in\mathbb{N}} \subseteq \{\beta_\alpha\}_{\alpha\in\Lambda}$ be the constant sequence whose each term is $\hat{\beta}$ so that $b_n \longrightarrow \hat{\beta}$. Without any loss of generality we may assume that $a_1 < \hat{\beta}$ so that $a_n < b_n$ for each $n \in \mathbb{N}$. Next we rename and denote by $\{f_n\}$ and $\{g_n\}$ the eigenvectors corresponding to the eigenvalues $\{a_n\}$ and $\{b_n\}$ respectively where $Tf_n = a_n f_n$ and $Tg_n = b_n g_n = \hat{\beta} g_n$ for each $n \in \mathbb{N}$.
As we did in the previous proof, we define a sequence $(\gamma_n)_n \in \mathbb{N}$ by

$$\gamma_n := \hat{\beta} + \frac{\beta - \hat{\beta}}{2n}; n \in \mathbb{N}.$$

Observe that $(\gamma_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence with $a_n^2 < \gamma_n^2 < \hat{\beta}^2$ for every $n \in \mathbb{N}$. It immediately follows then that $\lim_{n\to\infty} \gamma_n = \sup\{\gamma_n : n \in \mathbb{N}\} = \hat{\beta}$. Thereafter, for each $n \in \mathbb{N}$, we choose c_n so that $c_n^2 \in [0,1]$ and $c_n^2 a_n^2 + (1-c_n^2)\hat{\beta} = \gamma_n^2$. Finally, with the help of this sequence $(c_n)_{n\in\mathbb{N}}$ let us define a closed subspace $\mathcal{M}$ of $\mathcal{H}$ by

$$\mathcal{M} := \operatorname{clos}\left[\operatorname{span}\left\{c_n f_n + \sqrt{1-c_n^2}\, g_n : n \in \mathbb{N}\right\}\right].$$

We know that the set $\{e_n : n \in \mathbb{N}\}$, where $e_n := c_n f_n + \sqrt{1-c_n^2}\, g_n$, is an orthonormal basis of $\mathcal{M}$. It now follows, like the argument in the previous proof, that

$$\begin{aligned}
\|T|_{\mathcal{M}}\|^2 &= \sup\{\|Tx\|^2 : x \in \mathcal{M}, \|x\| = 1\} \geqslant \sup\{\|Te_n\|^2\} \\
&= \sup\{\|T(c_n f_n + \sqrt{1-c_n^2}\, g_n)\|^2 : n \in \mathbb{N}\} \\
&= \sup\{\|c_n a_n f_n + \sqrt{1-c_n^2}\, b_n g_n\|^2 : n \in \mathbb{N}\} \\
&= \sup\{\|c_n a_n f_n + \sqrt{1-c_n^2}\, \hat{\beta} g_n\|^2 : n \in \mathbb{N}\} \\
&= \sup\{c_n^2 a_n^2 + (1-c_n^2)\hat{\beta}^2 : n \in \mathbb{N}\} = \sup\{\gamma_n^2 : n \in \mathbb{N}\} = \hat{\beta}^2.
\end{aligned}$$

Since each $x \in \mathcal{M}$ with $\|x\| = 1$ can be written as

$$\sum_{n=1}^{\infty} \alpha_n (c_n f_n + \sqrt{1-c_n^2}\, g_n) \text{ with } \sum_{n=1}^{\infty} |\alpha_n|^2 = 1, \text{ we have}$$

$$\begin{aligned}
\|T|_{\mathcal{M}}(x)\|^2 = \|Tx\|^2 &= \left\| T\left(\sum_{n=1}^{\infty} \alpha_n (c_n f_n + \sqrt{1-c_n^2}\, g_n)\right)\right\|^2 \\
&= \left\| \sum_{n=1}^{\infty} \alpha_n T(c_n f_n + \sqrt{1-c_n^2}\, g_n)\right\|^2 \\
&= \left\| \sum_{n=1}^{\infty} \alpha_n (c_n a_n f_n + \sqrt{1-c_n^2}\, b_n g_n)\right\|^2 \\
&= \left\| \sum_{n=1}^{\infty} \alpha_n (c_n a_n f_n + \sqrt{1-c_n^2}\, \hat{\beta} g_n)\right\|^2 \\
&= \sum_{n=1}^{\infty} |\alpha_n|^2 (c_n^2 a_n^2 + (1-c_n^2)\hat{\beta}^2) \\
&= \sum_{n=1}^{\infty} |\alpha_n|^2 \gamma_n^2 < \sum_{n=1}^{\infty} |\alpha_n|^2 \hat{\beta}^2 = \hat{\beta}^2.
\end{aligned}$$

This implies that for every element $x \in \mathcal{M}$ with $\|x\| = 1$, $\|T|_{\mathcal{M}}(x)\| < \hat{\beta} \leq \|T|_{\mathcal{M}}\|$ which means that T is not absolutely norming. So we arrive at a contradiction. Hence, our hypothesis was wrong and we conclude that $\beta = \hat{\beta}$.

To prove the assertion for the case when $\hat{\beta} < \beta$, we follow the same line of argument. Let $(a_n)_{n\in\mathbb{N}} \subseteq \{\beta_\alpha\}_{\alpha\in\Lambda}$ be the decreasing sequence such that $a_n \searrow \beta$, $(b_n)_{n\in\mathbb{N}} \subseteq \{\beta_\alpha\}_{\alpha\in\Lambda}$ be the constant sequence whose each term is $\hat{\beta}$ so that $b_n \longrightarrow \hat{\beta}$, and rename and denote by $\{f_n\}$ and $\{g_n\}$ the eigenvectors corresponding to the eigenvalues $\{a_n\}$ and $\{b_n\}$ respectively where $Tf_n = a_n f_n$ and $Tg_n = b_n g_n = \hat{\beta} g_n$ for each $n \in \mathbb{N}$.
We define the sequence $(\gamma_n)_n \in \mathbb{N}$ a bit differently by

$$\gamma_n := \beta + \frac{\hat{\beta} - \beta}{2n}; n \in \mathbb{N}.$$

It is now a trivial observation that $(\gamma_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence with $\hat{\beta}^2 < \gamma_n^2 < a_n^2$ for every $n \in \mathbb{N}$. Consequently, $\lim_{n\to\infty} \gamma_n = \sup\{\gamma_n : n \in \mathbb{N}\} = \beta$.

Thereafter for each $n \in \mathbb{N}$, we choose c_n so that $c_n^2 \in [0, 1]$ and $c_n^2\hat{\beta}^2 + (1 - c_n^2)a_n^2 = \gamma_n^2$. Finally, with the help of this sequence $(c_n)_{n\in\mathbb{N}}$, we define a closed subspace $\hat{\mathcal{M}}$ of $\mathcal{H}$ by

$$\hat{\mathcal{M}} := \operatorname{clos}\left[\operatorname{span}\left\{c_n g_n + \sqrt{1 - c_n^2} f_n : n \in \mathbb{N}\right\}\right].$$

That the set $\{e_n : n \in \mathbb{N}\}$, where $e_n := c_n g_n + \sqrt{1 - c_n^2} g f_n$, is an orthonormal basis of $\hat{\mathcal{M}}$ can be easily verified. It now follows that

$$\begin{aligned}
\|T|_{\hat{\mathcal{M}}}\|^2 &= \sup\{\|Tx\|^2 : x \in \hat{\mathcal{M}}, \|x\| = 1\} \geqslant \sup\{\|Te_n\|^2\} \\
&= \sup\{\|T(c_n g_n + \sqrt{1 - c_n^2} f_n)\|^2 : n \in \mathbb{N}\} \\
&= \sup\{\|c_n b_n g_n + \sqrt{1 - c_n^2} a_n f_n\|^2 : n \in \mathbb{N}\} \\
&= \sup\{\|c_n \hat{\beta} g_n + \sqrt{1 - c_n^2} a_n f_n\|^2 : n \in \mathbb{N}\} \\
&= \sup\{c_n^2 \hat{\beta}^2 + (1 - c_n^2) a_n^2 : n \in \mathbb{N}\} = \sup\{\gamma_n^2 : n \in \mathbb{N}\} = \beta^2.
\end{aligned}$$

Since each $x \in \hat{\mathcal{M}}$ with $\|x\| = 1$ can be written as

$$\sum_{n=1}^{\infty} \alpha_n (c_n g_n + \sqrt{1 - c_n^2} f_n) \text{ with } \sum_{n=1}^{\infty} |\alpha_n|^2 = 1, \text{ we have}$$

$$
\begin{aligned}
\|T|_{\hat{\mathcal{M}}}(x)\|^2 = \|Tx\|^2 &= \left\| T\left(\sum_{n=1}^{\infty} \alpha_n (c_n g_n + \sqrt{1-c_n^2} f_n)\right)\right\|^2 \\
&= \left\| \sum_{n=1}^{\infty} \alpha_n T(c_n g_n + \sqrt{1-c_n^2} f_n)\right\|^2 \\
&= \left\| \sum_{n=1}^{\infty} \alpha_n (c_n \hat{\beta} g_n + \sqrt{1-c_n^2} a_n f_n)\right\|^2 \\
&= \sum_{n=1}^{\infty} |\alpha_n|^2 (c_n^2 \hat{\beta}^2 + (1-c_n^2) a_n^2) \\
&= \sum_{n=1}^{\infty} |\alpha_n|^2 \gamma_n^2 < \sum_{n=1}^{\infty} |\alpha_n|^2 \beta^2 = \beta^2.
\end{aligned}
$$

This implies that for every element $x \in \hat{\mathcal{M}}$ with $\|x\| = 1$, $\|T|_{\hat{\mathcal{M}}}(x)\| < \beta \leq \|T|_{\hat{\mathcal{M}}}\|$ which contradicts the fact that $T \in \mathcal{AN}(\mathcal{H})^+$. Thus, we conclude that $\beta = \hat{\beta}$. $\square$

We finish this section by stating the final proposition in its full strength.

Theorem 3.2.9. *If $T \in \mathcal{AN}(\mathcal{H})^+$, then*

$$T = \sum_{\alpha \in \Lambda} \beta_\alpha v_\alpha \otimes v_\alpha$$

where $\{v_\alpha : \alpha \in \Lambda\}$ is an orthonormal basis consisting entirely of eigenvectors of T and for every $\alpha \in \Lambda, Tv_\alpha = \beta_\alpha v_\alpha$ with $\beta_\alpha \geqslant 0$ such that

(i) for every nonempty subset $\Gamma \subseteq \Lambda$ of Λ, we have $\sup\{\beta_\alpha : \alpha \in \Gamma\} = max\{\beta_\alpha : \alpha \in \Gamma\}$;

(ii) the spectrum $\sigma(T) = \operatorname{clos}[\{\beta_\alpha : \alpha \in \Lambda\}]$ of T has at most one limit point. Moreover, this unique limit point (if it exists) can only be the limit of a decreasing sequence in the spectrum;

(iii) the set $\{\beta_\alpha\}_{\alpha \in \Lambda}$ of eigenvalues of T, without counting multiplicities, is countable and has at most one eigenvalue with infinite multiplicity;

(iv) if the spectrum $\sigma(T) = \operatorname{clos}[\{\beta_\alpha : \alpha \in \Lambda\}]$ of T has both, a limit point β and an eigenvalue $\hat{\beta}$ with infinite multiplicity, then $\beta = \hat{\beta}$.

3.3 Sufficient conditions for operators to belong to $\mathcal{AN}(\mathcal{H},\mathcal{K})$

We now discuss the sufficient conditions for an operator (not necessarily positive) to be absolutely norming.

Lemma 3.3.1. *For a closed linear subspace M of a complex Hilbert space $\mathcal{H}$ let P_M be the orthogonal projection of $\mathcal{H}$ onto M. An operator $T \in \mathcal{AN}(\mathcal{H},\mathcal{K})$ if and only if for every closed linear subspace M of $\mathcal{H}$, $TP_M \in \mathcal{N}(\mathcal{H},\mathcal{K})$.*

Proof. We first observe that for any given non trivial closed subspace M of $\mathcal{H}$, $\|TP_M\| = \|T|_M\|$; for

$$\begin{aligned}\|TP_M\|^2 &= \sup\{\|TP_M(x)\|^2 : \|x\| \leq 1\}\\ &= \sup\{\|Ty\|^2 : \|y\| \leq 1, y \in M\} = \|T|_M\|^2.\end{aligned}$$

We next assume that T is absolutely norming and prove the forward implication. Let M be an arbitrary non trivial closed subspace of $\mathcal{H}$. Clearly then there exists $x_0 \in M$ with $\|x_0\| = 1$ such that $\|T|_M\| = \|Tx_0\|$. It follows that there exists $x_0 \in \mathcal{H}$ such that $\|TP_M\| = \|T|_M\| = \|Tx_0\| = \|TP_M(x_0)\|$. Since M is arbitrary, it follows that $TP_M \in \mathcal{N}(\mathcal{H},\mathcal{K})$.

We complete the proof by showing that $T \in \mathcal{AN}(\mathcal{H},\mathcal{K})$ if $TP_M \in \mathcal{N}(\mathcal{H},\mathcal{K})$ for every non trivial closed subspace M of $\mathcal{H}$. Since TP_M is norming, there exists $x_M \in \mathcal{H}$(depending on M) with $\|x_M\| = 1$ and $\|TP_M\| = \|TP_M(x_M)\|$. This means that for every M, $\|T|_M\| = \|TP_M\| = \|TP_M(x_M)\| = \|T(P_M x_M)\|$ for some $P_M x_M \in M$ such that $\|P_M x_M\| \leq 1$. This shows that for every M, $T|_M$ attains its norm on the closed unit ball. To show that it attains its norm on the unit sphere, notice that

$$(\|T(P_M x_M)\| =)\|T|_M(P_M x_M)\| \leq \|T|_M\|\|(P_M x_M\| \leq \|T|_M\|.$$

But $\|T|_M(P_M x_M)\| = \|T|_M\|$. It follows then that $\|T|_M\|\|(P_M x_M\| = \|T|_M\|$ which in turn implies that $\|P_M x_M\| = 1$ and hence $T|_M$ attains its norm on the unit sphere. This completes the proof. $\square$

There is another important and useful criterion for an operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ to be absolutely norming which depends on the following facts: for a closed linear subspace $\mathcal{M}$ of a complex Hilbert space $\mathcal{H}$ let $V_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{H}$ be the inclusion map from $\mathcal{M}$ to $\mathcal{H}$ defined as $V_{\mathcal{M}}(x) = x$ for each $x \in \mathcal{M}$. It is then a trivial observation that the adjoint

$V_{\mathcal{M}}^* : \mathcal{H} \longrightarrow \mathcal{M}$ of $V_{\mathcal{M}}$ is the orthogonal projection of $\mathcal{H}$ on $\mathcal{M}$ (viewed as a map from $\mathcal{H}$ onto $\mathcal{M}$), that is, $V_{\mathcal{M}}^* : \mathcal{H} \longrightarrow \mathcal{M}$ such that

$$V_M^*(y) = \begin{cases} y & \text{if } y \in M \\ 0 & \text{if } y \in M^\perp \end{cases}.$$

The criterion referred to is the following: $T \in \mathcal{AN}(\mathcal{H}, \mathcal{K})$ if and only if for every closed linear subspace $\mathcal{M}$ of $\mathcal{H}$, $TV_{\mathcal{M}} \in \mathcal{N}(\mathcal{M}, \mathcal{K})$. To prove this assertion we first observe that for any given nontrivial closed subspace $\mathcal{M}$ of $\mathcal{H}$, $\|TV_{\mathcal{M}}\| = \|T|_{\mathcal{M}}\|$; for

$$\begin{aligned} \|TV_{\mathcal{M}}\|^2 &= \sup\{\|TV_{\mathcal{M}}(x)\|^2 : \|x\| \leq 1, x \in \mathcal{M}\} \\ &= \sup\{\|Tx\|^2 : \|x\| \leq 1, x \in \mathcal{M}\} = \|T|_{\mathcal{M}}\|^2. \end{aligned}$$

We next assume that $T \in \mathcal{AN}(\mathcal{H}, \mathcal{K})$ and prove the forward implication. Let $\mathcal{M}$ be an arbitrary nontrivial closed subspace of $\mathcal{H}$. Clearly then there exists $x_0 \in \mathcal{M}$ with $\|x_0\| = 1$ such that $\|T|_{\mathcal{M}}\| = \|Tx_0\|$. It follows then that there exists $x_0 \in \mathcal{H}$ such that $\|TV_{\mathcal{M}}\| = \|T|_{\mathcal{M}}\| = \|Tx_0\| = \|TV_{\mathcal{M}}(x_0)\|$. Since $\mathcal{M}$ is arbitrary, it follows that $TV_{\mathcal{M}} \in \mathcal{N}(\mathcal{M}, \mathcal{K})$. We complete the proof by showing that $T \in \mathcal{AN}(\mathcal{H}, \mathcal{K})$ if $TV_{\mathcal{M}} \in \mathcal{N}(\mathcal{M}, \mathcal{K})$ for every nontrivial closed subspace $\mathcal{M}$ of $\mathcal{H}$. Since $TV_{\mathcal{M}}$ is norming, there exists $x_{\mathcal{M}} \in \mathcal{H}$(depending on $\mathcal{M}$) with $\|x_{\mathcal{M}}\| = 1$ and $\|TV_{\mathcal{M}}\| = \|TV_{\mathcal{M}}(x_{\mathcal{M}})\|$. This means that for every $\mathcal{M}$, $\|T|_{\mathcal{M}}\| = \|TV_{\mathcal{M}}\| = \|TV_{\mathcal{M}}(x_{\mathcal{M}})\| = \|T(V_{\mathcal{M}}x_{\mathcal{M}})\| = \|Tx_{\mathcal{M}}\| = \|T|_{\mathcal{M}}(x_{\mathcal{M}})\|$ where $x_{\mathcal{M}} \in \mathcal{M}$ and $\|x_{\mathcal{M}}\| = 1$. This essentially guarantees that for every $\mathcal{M}$, $T|_{\mathcal{M}}$ attains its norm on unit sphere and is hence norming.

We can summarize the result of the above discussion in the following lemma.

Lemma 3.3.2. *For a closed linear subspace $\mathcal{M}$ of a complex Hilbert space $\mathcal{H}$ let $V_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{H}$ be the inclusion map from $\mathcal{M}$ to $\mathcal{H}$ defined as $V_{\mathcal{M}}(x) = x$ for each $x \in \mathcal{M}$. An operator $T \in \mathcal{AN}(\mathcal{H}, \mathcal{K})$ if and only if for every nontrivial closed linear subspace $\mathcal{M}$ of $\mathcal{H}$, $TV_{\mathcal{M}} \in \mathcal{N}(\mathcal{M}, \mathcal{K})$.*

The following application illustrates the power of this result.

Proposition 3.3.3. *If $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is an isometry, then $T \in \mathcal{AN}(\mathcal{H}, \mathcal{K})$.*

Proof. That an isometry is norming is obvious; for the operator norm of an isometry is 1 and it attains its norm on any vector of unit length. For a closed linear subspace $\mathcal{M}$ of the Hilbert space $\mathcal{H}$ let $V_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{H}$ be the inclusion map from $\mathcal{M}$ to $\mathcal{H}$ defined as $V_{\mathcal{M}}(x) = x$ for each $x \in \mathcal{M}$. To prove the assertion, it suffices to show that for every nonzero closed linear subspace $\mathcal{M}$, $TV_{\mathcal{M}}$ is norming. But $TV_{\mathcal{M}} \in \mathcal{B}(\mathcal{M}, \mathcal{K})$ is an isometry and hence attains its norm. $\square$

Lemma 3.3.4. *Let $T \in \mathcal{B}(\mathcal{H})$ be a diagonalizable operator on the complex Hilbert space $\mathcal{H}$, and $B = \{v_\alpha : \alpha \in \Lambda\}$ be an orthonormal basis of $\mathcal{H}$ corresponding to which T is diagonalizable. If T attains its norm on the unit sphere of $\mathcal{H}$, then it attains it norm on some $v_0 \in B$. Alternatively, if $T \in \mathcal{N}(\mathcal{H}, \mathcal{K})$, then there exists $v_0 \in B$ such that $\|T\| = \|Tv_0\|$.*

Proof. Let $\{\lambda_\alpha : \alpha \in \Lambda\}$ be the set of eigenvalues of T corresponding to the the eigenvectors $\{v_\alpha : \alpha \in \Lambda\}$. From [Hal82, Problem 61], we know that $\|T\| = \sup\{|\lambda_\alpha| : \alpha \in \Lambda\}$, so it suffices to prove that $\|T\| = \max\{|\lambda_\alpha| : \alpha \in \Lambda\}$; for then $\|T\| = |\lambda_0| = |\lambda_0|\|v_0\| = \|\lambda_0 v_0\| = \|Tv_0\|$ where $|\lambda_0| := \max\{|\lambda_\alpha| : \alpha \in \Lambda\}$ and v_0 is the corresponding eigenvector in B.

To this end, by the way of contradiction, we assume the negation of the above claim. It implies that for every $\alpha \in \Lambda$, we have $|\lambda_\alpha| < \|T\|$. However, for every $x \in \mathcal{H}$ with $\|x\| = 1$, we have $Tx = \sum_{\alpha\in\Lambda} \lambda_\alpha \langle x, v_\alpha \rangle v_\alpha$ so that

$$\begin{aligned}
\|Tx\|^2 &= \sum_{\alpha\in\Lambda} |\lambda_\alpha|^2 |\langle x, v_\alpha \rangle|^2 \\
&< \sum_{\alpha\in\Lambda} \|T\|^2 |\langle x, v_\alpha \rangle|^2 \\
&= \|T\|^2 \sum_{\alpha\in\Lambda} |\langle x, v_\alpha \rangle|^2 \\
&= \|T\|^2 \|x\|^2 \\
&= \|T\|^2;
\end{aligned}$$

which is a contradiction of the fact that $T \in \mathcal{N}(\mathcal{H})$. This proves the claim. □

Lemma 3.3.5. *Let $F \in \mathcal{B}(\mathcal{H})$ be a self-adjoint finite rank operator and $\alpha \geq 0$. Then $\alpha I + F \in \mathcal{N}(\mathcal{H})$.*

Proof. Let the range of F be k-dimensional. Since F is a self adjoint, there exists an orthonormal basis $B = \{v_\lambda : \lambda \in \Lambda\}$ of $\mathcal{H}$ corresponding to which the matrix $M_B(F)$ is a diagonal matrix with k nonzero real diagonal entries, say $\{\beta_1, \beta_2, ..., \beta_k\}$. Clearly then, $M_B(\alpha I + F)$ is also a diagonal matrix and

$$\begin{aligned}
\|\alpha I + F\| &= \sup\{|\alpha + \beta_1|, |\alpha + \beta_2|, ..., |\alpha + \beta_k|, \alpha\} \\
&= \max\{|\alpha + \beta_1|, |\alpha + \beta_2|, ..., |\alpha + \beta_k|, \alpha\}.
\end{aligned}$$

It is then a trivial observation that there exists $v_0 \in B$ such that $\|\alpha I + F\| = \|(\alpha I + F)v_0\|$. This proves that $\alpha I + F$ attains its norm on the unit sphere and hence is a norming operator. □

This lemma leads to the following proposition.

Proposition 3.3.6. *If $F \in \mathcal{B}(\mathcal{H})$ is a self-adjoint finite rank operator and $\alpha \geq 0$, then $\alpha I + F \in \mathcal{AN}(\mathcal{H})$.*

Proof. For a closed linear subspace $\mathcal{M}$ of the Hilbert space $\mathcal{H}$ let $V_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{H}$ be the inclusion map from $\mathcal{M}$ to $\mathcal{H}$ defined as $V_{\mathcal{M}}(x) = x$ for each $x \in \mathcal{M}$. Let us then define $T := \alpha I + F$ so that we have $T^* = \alpha I + F$ and $T^*T = (\alpha I + F)^2 = \alpha^2 I + 2\alpha F + F^2 = \beta I + \tilde{F}$ where $\beta = \alpha^2 \geq 0$ and $\tilde{F} = 2\alpha F + F^2$ is another self-adjoint finite rank operator. We observe that

$$\begin{aligned}
T \in \mathcal{AN}(\mathcal{H}) &\iff \text{for every closed subspace } \mathcal{M} \text{ of } \mathcal{H}, TV_{\mathcal{M}} \text{ is norming}\\
&\iff \text{for every closed subspace } \mathcal{M} \text{ of } \mathcal{H}, (TV_{\mathcal{M}})^*(TV_{\mathcal{M}}) \text{ is norming}\\
&\iff \text{for every closed subspace } \mathcal{M} \text{ of } \mathcal{H}, V_{\mathcal{M}}^*(T^*T)V_{\mathcal{M}} \text{ is norming}\\
&\iff \text{for every closed subspace } \mathcal{M} \text{ of } \mathcal{H}, V_{\mathcal{M}}^*(\beta I + \tilde{F})V_{\mathcal{M}} \text{ is norming}
\end{aligned}$$

So, it suffices to show that for every closed subspace $\mathcal{M}$ of $\mathcal{H}, V_{\mathcal{M}}^*(\beta I + \tilde{F})V_{\mathcal{M}}$ is norming. But $V_{\mathcal{M}}^*(\beta I + \tilde{F})V_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M}$ is an operator on $\mathcal{M}$ and

$$V_{\mathcal{M}}^*(\beta I + \tilde{F})V_{\mathcal{M}} = V_{\mathcal{M}}^*\beta I V_{\mathcal{M}} + V_{\mathcal{M}}^*\tilde{F}V_{\mathcal{M}} = \beta I_{\mathcal{M}} + \tilde{F}_{\mathcal{M}}$$

is the sum of a non negative scalar multiple of identity and a self-adjoint finite rank operator on a Hilbert space $\mathcal{M}$ which, by previous lemma, does attain its norm and thus proves our assertion. □

Lemma 3.3.7. *For any positive compact operator $K \in \mathcal{B}(\mathcal{H})$ and $\alpha \geq 0$, $\alpha I + K$ is norming.*

Proof. That K attains its norm is obvious, for K is compact. The positivity of K ascertains that there is an orthonormal basis $B = \{v_\lambda : \lambda \in \Lambda\}$ of $\mathcal{H}$, consisting entirely of eigenvectors of K, corresponding to which K is diagonalizable; this fact , together with the lemma 3.3.4 implies that there exists $v_0 \in B$ such that $\|K\| = \beta_0 = \max\ \{\beta_\lambda : \lambda \in \Lambda\} = \|Kv_0\|$, where $K(v_\lambda) = \beta_\lambda v_\lambda$ for each $\lambda \in \Lambda$. Since $\alpha \geq 0$, it readily follows that

$$\begin{aligned}
\|\alpha I + K\| &= \sup\{\alpha + \beta_\lambda : \lambda \in \Lambda\}\\
&= \alpha + \sup\{\beta_\lambda : \lambda \in \Lambda\}\\
&= \alpha + \max\ \{\beta_\lambda : \lambda \in \Lambda\}\\
&= \alpha + \beta_0 = \|(\alpha I + K)(v_0)\|.
\end{aligned}$$

$\alpha I + K$ therefore attains its norm on unit sphere for each $\alpha \geq 0$. □

This lemma is a special case of what the following proposition states.

Proposition 3.3.8. *For any positive compact operator $K \in \mathcal{B}(\mathcal{H})$ and $\alpha \geq 0$, $\alpha I + K$ is absolutely norming.*

Proof. Let us define $T := \alpha I + K$ so that we have $T^* = \alpha I + K$ and $T^*T = (\alpha I + K)^2 = \alpha^2 I + 2\alpha K + K^2 = \beta I + \tilde{K}$ where $\beta = \alpha^2 \geq 0$ and $\tilde{K} = 2\alpha K + K^2$ is another positive compact operator.

$$\begin{aligned}
T \in \mathcal{AN}(\mathcal{H}) &\iff \text{for every closed subspace } \mathcal{M} \text{ of } \mathcal{H}, TV_{\mathcal{M}} \text{ is norming} \\
&\iff \text{for every closed subspace } \mathcal{M} \text{ of } \mathcal{H}, (TV_{\mathcal{M}})^*(TV_{\mathcal{M}}) \text{ is norming} \\
&\iff \text{for every closed subspace } \mathcal{M} \text{ of } \mathcal{H}, V_{\mathcal{M}}^*(T^*T)V_{\mathcal{M}} \text{ is norming} \\
&\iff \text{for every closed subspace } \mathcal{M} \text{ of } \mathcal{H}, V_{\mathcal{M}}^*(\beta I + \tilde{K})V_{\mathcal{M}} \text{ is norming.}
\end{aligned}$$

So, it suffices to show that for every closed subspace $\mathcal{M}$ of $\mathcal{H}, V_{\mathcal{M}}^*(\beta I + \tilde{K})V_{\mathcal{M}}$ attains its norm. But $V_{\mathcal{M}}^*(\beta I + \tilde{K})V_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M}$ is an operator on $\mathcal{M}$ and

$$V_{\mathcal{M}}^*(\beta I + \tilde{K})V_{\mathcal{M}} = V_{\mathcal{M}}^*\beta I V_{\mathcal{M}} + V_{\mathcal{M}}^*\tilde{K}V_{\mathcal{M}} = \beta I_{\mathcal{M}} + \tilde{K}_{\mathcal{M}}$$

is the sum of a non negative scalar multiple of Identity and a positive compact operator on a Hilbert space $\mathcal{M}$ which, by the previous lemma, does attain its norm and hence the assertion is proved. □

Lemma 3.3.9. *Let $K \in \mathcal{B}(\mathcal{H})$ be a positive compact operator and $F \in \mathcal{B}(\mathcal{H})$ be a self-adjoint finite rank operator. Then $K + F$ can have at most finitely many negative eigenvalues.*

Proof. Since F is a self-adjoint finite rank operator, there is an orthonormal basis B of $\mathcal{H}$ consisting of eigenvectors of F corresponding to which it is diagonalizable. This allows us to write F as the difference of two positive finite rank operators, F_+ and F_- so that $F = F_+ - F_-$. Consider the set of all eigenvectors in B corresponding to which F_- has nonzero (positive) eigenvalues. Needless to say that they are finite in number. Define H_- to be the span of these eigenvectors. It is trivial to observe that H_- is a closed finite-dimensional subspace of $\mathcal{H}$ and $\mathcal{H} = H_- \oplus H_-^{\perp}$. We assume that the dimension of H_- is k, that is, $\dim H_- = k$.

We claim that the total number of negative eigenvalues of $K + F$ does not exceed k. To prove this claim, we first observe that $K + F$ can now be rewritten as $K + (F_+ - F_-) = (K + F_+) - F_- = \tilde{K} - F_-$ where $\tilde{K} = K + F_+$ is positive compact operator on $\mathcal{H}$. Also,

$\tilde{K} - F_{-}$ is a self-adjoint compact operator and thus there exists an orthonormal basis $\mathfrak{B}$ of $\mathcal{H}$ consisting entirely of eigenvectors of $\tilde{K} - F_{-}$ corresponding to which $\tilde{K} - F_{-}$ is diagonalizable. We next observe that

$$\text{for any } x \in H_{-}^{\perp}, \ \left\langle (\tilde{K} - F_{-})x, x \right\rangle \geq 0;$$

because $F_{-}(x) = 0$ for every $x \in H_{-}^{\perp}$ and $\left\langle \tilde{K}x, x \right\rangle \geq 0$ for each $x \in \mathcal{H}$ and hence for each $x \in H_{-}^{\perp}$. We are now ready to prove our claim. Consider the set of all orthonormal eigenvectors in $\mathfrak{B}$ corresponding to which $\tilde{K} - F_{-}$ has negative eigenvalues. By way of contradiction let us assume that the cardinality of this set is strictly bigger than k. We fix some $m > k$ and extract m eigenvectors from this set. Let the set of these extracted eigenvectors be $\{v_1, v_2, v_3, ..., v_m\}$ and the corresponding eigenvalues be $\{\lambda_1, \lambda_2, \lambda_3, ..., \lambda_m\}$. Since $m > k$, there exists $\alpha_1, \alpha_2, ..., \alpha_m$ not all zero such that $P_{H_{-}}(\sum_{i=1}^{m} \alpha_i v_i) = 0$. We then have

$$\begin{aligned}
\langle (\tilde{K} - F_{-}) \left(\sum_{i=1}^{m} \alpha_i v_i \right), \sum_{j=1}^{m} \alpha_j v_j \rangle &= \langle \sum_{i=1}^{m} \alpha_i \lambda_i v_i, \sum_{j=1}^{m} \alpha_j v_j \rangle \\
&= \sum_{i=1}^{m} |\alpha_i|^2 \lambda_i \\
&< 0.
\end{aligned}$$

But this contradicts the fact that $\sum_{i=1}^{m} \alpha_i v_i \in H_{-}^{\perp}$; for we established that for any $x \in H_{-}^{\perp}, \left\langle (\tilde{K} - F_{-})x, x \right\rangle \geq 0$. This proves our claim. ☐

This observation leads us directly to the following proposition.

Proposition 3.3.10. *Let $K \in \mathcal{B}(\mathcal{H})$ be a positive compact operator and $F \in \mathcal{B}(\mathcal{H})$ be a self-adjoint finite rank operator. Then for every $\alpha \geq 0$, $\alpha I + K + F \in \mathcal{N}(\mathcal{H})$.*

Proof. The assertion is trivial if $\alpha = 0$; for then $K + F$ is a compact operator which is norming. We assume that $\alpha > 0$. Notice that $K + F$ is a self-adjoint compact operator on $\mathcal{H}$ and thus there exists an orthonormal basis B of $\mathcal{H}$ consisting entirely of eigenvectors of $K + F$ corresponding to which it is diagonalizable. From the previous lemma, $K + F$ can have at most finitely many negative eigenvalues. Let $\{\lambda_1, \lambda_2, ..., \lambda_n\}$ be the set of all negative eigenvalues of $K + F$ with $\{v_1, v_2, ..., v_n\}$ as the corresponding eigenvectors in basis B; and let $\{\mu_\beta : \beta \in \Lambda\}$ be the set of all remaining nonnegative eigenvalues of $K + F$ with

$\{w_\beta : \beta \in \Lambda\}$ as the corresponding eigenvectors in B. We have $B := \{v_1, v_2, ..., v_n\} \cup \{w_\beta : \beta \in \Lambda\}$ and the matrix $M_B(K+F)$ of $K+F$ with respect to B is given by

$$K + F = \begin{bmatrix} \lambda_1 & & & \vdots & & & \\ & \ddots & & \vdots & & 0 & \\ & & \lambda_n & \vdots & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & \vdots & \ddots & & \\ & 0 & & \vdots & & \mu_\beta & \\ & & & \vdots & & & \ddots \end{bmatrix}.$$

Observing the fact that

$$\|K+F\| = \max\{\{|\lambda_i|\}_{i=1}^n \cup \{\mu_\beta\}_{\beta\in\Lambda}\}$$

we proceed to show that $\alpha I + K + F \in \mathcal{N}(\mathcal{H})$. To accomplish this we distinguish cases:
Case I. If $\mu_{\hat\beta} = \max\{\{|\lambda_i|\}_{i=1}^n \cup \{\mu_\beta\}_{\beta\in\Lambda}\}$ for some $\hat\beta \in \Lambda$.
Needless to say that $\|K+F\| = \mu_{\hat\beta} = \|(K+F)(w_{\hat\beta})\|$. Clearly then

$$\alpha + \mu_{\hat\beta} \geq \alpha + |\lambda_i| \geq |\alpha + \lambda_i| \text{ for each } i \in \{1, 2, ..., n\}, \text{ and}$$
$$\alpha + \mu_{\hat\beta} \geq \alpha + \mu_\beta \text{ for each } \beta \in \Lambda.$$

It is now easy to convince ourselves that if $w_{\hat\beta}$ be the eigenvector corresponding to the eigenvalue $\mu_{\hat\beta}$ then $\|\alpha I + K + F\| = \|\alpha + \mu_{\hat\beta}\| = \|(\alpha I + K + F)(w_{\hat\beta})\|$ which implies that $\alpha I + K + F$ achieves its norm at $w_{\hat\beta}$.
Case II. If $|\lambda_m| = \max\{|\lambda_i|\}_{i=1}^n \cup \{\mu_\beta\}_{\beta\in\Lambda}$ for some $m \in \{1, 2, ..., n\}$.
In this case it is important to observe that

$$\sup\{\mu_\beta : \beta \in \Lambda\} = \max\{\mu_\beta : \beta \in \Lambda\};$$

indeed the matrix $M_B(K+F)$ can be written as

$$
K+F=\begin{bmatrix}
\lambda_1 & & & \vdots & & & \\
 & \ddots & & \vdots & & 0 & \\
 & & \lambda_n & \vdots & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & & \vdots & & & \\
 & 0 & & \vdots & & 0 & \\
 & & & \vdots & & &
\end{bmatrix}
+
\begin{bmatrix}
 & & & \vdots & & & \\
 & 0 & & \vdots & & 0 & \\
 & & & \vdots & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & & \vdots & \ddots & & \\
 & 0 & & \vdots & & \mu_\beta & \\
 & & & \vdots & & & \ddots
\end{bmatrix},
$$

where the first matrix is compact. Consequently the second matrix is forced to be compact which implies that $\sup\{\mu_\beta : \beta \in \Lambda\} = \max\{\mu_\beta : \beta \in \Lambda\}$. Let $\max\{\mu_\beta : \beta \in \Lambda\} = \mu_{\tilde{\beta}}$ for some $\tilde{\beta} \in \Lambda$. It is then a trivial observation that $\sup\{|\alpha+\lambda_i|\}_{i=1}^n \cup \{\alpha+\mu_\beta\}_{\beta\in\Lambda} = \max\{\alpha+\mu_{\tilde{\beta}}, |\alpha+\lambda_1|, ..., |\alpha+\lambda_n|\}$ which ascertains that the operator $\alpha I+K+F$ is norming. We conclude the proof by a note that $\alpha I + K + F$ need not necessarily be positive for the proof to work. $\square$

This result is the key to the theorem that follows. The following result could be deduced from [CN12, Theorem 3.23] but there are some gaps in their proof of [CN12, Lemma 3.7] which is essential to their proof of [CN12, Theorem 3.23]; so we provide an independent proof.

Theorem 3.3.11. *Let $K \in \mathcal{B}(\mathcal{H})$ be a positive compact operator and $F \in \mathcal{B}(\mathcal{H})$ be a self-adjoint finite rank operator. Then for every $\alpha \geq 0$, $\alpha I + K + F \in \mathcal{AN}(\mathcal{H})$.*

Proof. Let $\mathcal{M}$ be an arbitrary nonempty closed linear subspace of the Hilbert space $\mathcal{H}$ and $V_\mathcal{M} : \mathcal{M} \longrightarrow \mathcal{H}$ be the inclusion map from $\mathcal{M}$ to $\mathcal{H}$ defined as $V_\mathcal{M}(x) = x$ for each $x \in \mathcal{M}$.

Let us then define $T := \alpha I + K + F$ so that we have $T^* = \alpha I + K + F$ and $T^*T = (\alpha I + K + F)^2 = (\alpha^2 I) + (2\alpha K + K^2) + (2\alpha F + FK + KF + F^2) = \beta I + \tilde{K} + \tilde{F}$ where $\beta = \alpha^2 \geq 0$, $\tilde{K} = 2\alpha K + K^2$ and $\tilde{F} = 2\alpha F + FK + KF + F^2$ are respectively positive compact and self-adjoint finite rank operators. Observe that

$$
\begin{aligned}
TV_\mathcal{M} \text{ is norming } &\iff (TV_\mathcal{M})^*(TV_\mathcal{M}) \text{ is norming} \\
&\iff V_\mathcal{M}^*(T^*T)V_\mathcal{M} \text{ is norming} \\
&\iff V_\mathcal{M}^*(\beta I + \tilde{K} + \tilde{F})V_\mathcal{M} \text{ is norming}\,.
\end{aligned}
$$

It suffices to show that $V_{\mathcal{M}}^*(\beta I + \tilde{K} + \tilde{F})V_{\mathcal{M}}$ is norming; for then, since $\mathcal{M}$ is arbitrary, it immediately follows from lemma 3.3.2 that T is an absolutely norming operator. To this end, notice that $V_{\mathcal{M}}^*(\beta I + \tilde{K} + \tilde{F})V_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M}$ is an operator on $\mathcal{M}$ and

$$V_{\mathcal{M}}^*(\beta I + \tilde{K} + \tilde{F})V_{\mathcal{M}} = V_{\mathcal{M}}^*\beta I V_{\mathcal{M}} + V_{\mathcal{M}}^*\tilde{K}V_{\mathcal{M}} + V_{\mathcal{M}}^*\tilde{F}V_{\mathcal{M}} = \beta I_{\mathcal{M}} + \tilde{K}_{\mathcal{M}} + \tilde{F}_{\mathcal{M}}$$

is the sum of a nonnegative scalar multiple of the identity, a positive compact operator and a self-adjoint finite rank operator on a Hilbert space $\mathcal{M}$ which, by the preceding proposition, attains its norm. This proves the assertion. $\square$

Remark 3.3.12. It is desirable at this stage to make an important remark: the sum of two absolutely norming operators need not necessarily be an absolutely norming operator. An example [CN12, Section 2, Page 182] appears in [CN12] which establishes that the sum of two norming operators need not necessarily be a norming operator. In what follows, we give an example of an operator $T \in \mathcal{H}$ which is absolutely norming but $2\mathrm{Re}(T)$ is not, which in turn implies that sum of two absolutely norming operators need not be absolutely norming.

Example 3.3.13. Let $\{e_i\}_{i\in\mathbb{N}}$ be the canonical orthonormal basis of the Hilbert space $\ell^2(\mathbb{N})$, $a \in (0,1]$, and $(a_i)_{i\in\mathbb{N}}, (b_i)_{i\in\mathbb{N}}$ be two sequences of real numbers such that

$$0 < a_1 < a_2 < ... < a, \quad a_i \nearrow a, \text{ and } a_i^2 + b_i^2 = 1.$$

Let $T \in \mathcal{B}(\ell^2(\mathbb{N}))$ defined as $Te_i = \lambda_i e_i$ for each $i \in \mathbb{N}$, where $\lambda_i = a_i + ib_i$. Then $T^*e_i = \overline{\lambda}_i e_i$. It is easy to observe that both T and T^* are isometries. Indeed, if $x \in \ell^2(\mathbb{N})$, then $x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$ which implies that

$$\begin{aligned}\|Tx\|^2 = \left\|\sum_{i=1}^{\infty} \langle x, e_i \rangle \lambda_i e_i\right\|^2 = \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 |\lambda_i|^2 = \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 = \|x\|^2 \\ = \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 = \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 |\overline{\lambda}_i|^2 = \left\|\sum_{i=1}^{\infty} \langle x, e_i \rangle \overline{\lambda}_i e_i\right\|^2 = \|T^*x\|^2.\end{aligned}$$

By Proposition 3.3.3, we infer that T and T^* are absolutely norming operators. We now show that $T + T^* \notin \mathcal{AN}(\ell^2(\mathbb{N}))$. Since $\mathcal{AN}(\ell^2(\mathbb{N})) \subseteq \mathcal{N}(\ell^2(\mathbb{N}))$, it suffices to show that $T+T^* \notin \mathcal{N}(\ell^2(\mathbb{N}))$. To this end, notice that $\|T+T^*\| \geqslant \sup\{\|Te_i\| : i \in \mathbb{N}\} = \sup\{|\lambda_i+\overline{\lambda}_i| :$

$i \in \mathbb{N}\} = \sup\{|2a_i| : i \in \mathbb{N}\} = 2a$. However, for every $x \in \ell^2(\mathbb{N})$ with $\|x\| = 1$, we have

$$\begin{aligned}\|(T+T^*)x\|^2 &= \left\|\sum_{i=1}^{\infty}(\lambda_i + \overline{\lambda}_i)\langle x, e_i\rangle e_i\right\|^2 = \sum_{i=1}^{\infty}|\lambda_i + \overline{\lambda}_i|^2|\langle x, e_i\rangle|^2 \\ &= \sum_{i=1}^{\infty}|2a_i|^2|\langle x, e_i\rangle|^2 < 4a^2\sum_{i=1}^{\infty}|\langle x, e_i\rangle|^2 = 4a^2.\end{aligned}$$

Consequently, for every $x \in \ell^2(\mathbb{N})$ of unit length $\|(T+T^*)x\| < 2a \leq \|T+T^*\|$ which implies that $T+T^*$ does not attain its norm.

3.4 Spectral characterization of positive operators in $\mathcal{AN}(\mathcal{H})$

The final theorem of the preceding section just established —— that for every $\alpha \geq 0, \alpha I + K + F \in \mathcal{AN}(\mathcal{H})$ where K and F are respectively positive compact and self-adjoint finite rank operators —— is the stronger version of the backward implication of our spectral theorem for positive absolutely norming operators. If the operator $\alpha I + K + F$ is also positive then the implication can be reversed and the two conditions are equivalent. This is what the next theorem states.

Theorem 3.4.1 (Spectral Theorem for Positive Absolutely Norming Operators). *Let $P \in \mathcal{B}(\mathcal{H})$ be a positive operator. Then $P \in \mathcal{AN}(\mathcal{H})$ if and only if P is of the form $P = \alpha I + K + F$, where $\alpha \geq 0, K$ is a positive compact operator and F is self-adjoint finite rank operator.*

Proof. It suffices to prove the forward implication. We assume that $P \in \mathcal{B}(\mathcal{H})$ is a positive absolutely norming operator. Theorem 3.2.9 asserts that there exists an orthonormal basis $B = \{v_\lambda : \lambda \in \Lambda\}$ consisting entirely of eigenvectors of P and for every $\lambda \in \Lambda, Tv_\lambda = \beta_\lambda v_\lambda$ with $\beta_\lambda \geq 0$. A moment's thought will convince the reader that there are four mutually exclusive and exhaustive set of possibilities for the spectrum $\sigma(P) = \text{clos}[\{\beta_\lambda : \lambda \in \Lambda\}]$ of P.

Case 1. $\sigma(P)$ has neither a limit point nor an eigenvalue with infinite multiplicity.
The index set Λ is then finite; for if it is not then the set $\{\beta_\lambda : \lambda \in \Lambda\}$ (counting multiplicities) of eigenvalues is also infinite. Since each eigenvalue in this set can have at most finite multiplicity, it is obvious then that the set $\{\beta_\lambda : \lambda \in \Lambda\}$ (without counting multiplicities)

of distinct eigenvalues of P is infinite. More interestingly, $\{\beta_\lambda : \lambda \in \Lambda\}$ is bounded above by the operator norm of P and below by 0. Since every infinite bounded subset of real numbers has a limit point, we arrive at a contradiction and hence Λ is finite. This forces the Hilbert space $\mathcal{H}$ to be finite dimensional. In that case P boils down to a positive (and hence self-adjoint) finite-rank operator and we can safely assume that $P = \alpha I + K + F$ with $\alpha = 0$, $K = 0$ and F the operator in question.

Case 2. $\sigma(P)$ has no limit point but has one eigenvalue with infinite multiplicity.
Let $\beta_0 \in \{\beta_\lambda : \lambda \in \Lambda\}$ be the eigenvalue with infinite multiplicity. Then the set $\Gamma := \Lambda \setminus \{\lambda \in \Lambda : \beta_\lambda = \beta_0\}$ is finite; for if it is not, then the set $\{\beta_\lambda : \lambda \in \Gamma\}$ (counting multiplicities) is also infinite which in turn implies that the set $\{\beta_\lambda : \lambda \in \Gamma\}$ (without counting multiplicities) is infinite because each eigenvalue in this set can have at most finite multiplicity. Since $\{\beta_\lambda : \lambda \in \Gamma\}$ is bounded and every infinite bounded subset of real numbers has a limit point, we arrive at a contradiction.This implies that Γ is finite. Observe that for an arbitrary $x \in \mathcal{H}$, we have $x = \sum_{\lambda\in\Lambda} \langle x, v_\lambda\rangle v_\lambda = \sum_{\lambda\in\Gamma} \langle x, v_\lambda\rangle v_\lambda + \sum_{\lambda\in\Lambda\setminus\Gamma} \langle x, v_\lambda\rangle v_\lambda$ so that for every $x \in \mathcal{H}$,

$$\begin{aligned}
Px &= \sum_{\lambda\in\Gamma} \langle x, v_\lambda\rangle P(v_\lambda) + \sum_{\lambda\in\Lambda\setminus\Gamma} \langle x, v_\lambda\rangle P(v_\lambda) \\
&= \sum_{\lambda\in\Gamma} \langle x, v_\lambda\rangle \beta_\lambda v_\lambda + \sum_{\lambda\in\Lambda\setminus\Gamma} \langle x, v_\lambda\rangle \beta_0 v_\lambda \\
&= \sum_{\lambda\in\Gamma} (\beta_\lambda - \beta_0) \langle x, v_\lambda\rangle v_\lambda + \beta_0 \sum_{\lambda\in\Lambda} \langle x, v_\lambda\rangle v_\lambda \\
&= \sum_{\lambda\in\Gamma} (\beta_\lambda - \beta_0)(v_\lambda \otimes v_\lambda)(x) + \beta_0 I x \\
&= \left(\sum_{\lambda\in\Gamma} (\beta_\lambda - \beta_0)(v_\lambda \otimes v_\lambda) + \beta_0 I \right)(x).
\end{aligned}$$

To conclude this case it suffices to observe that $\beta_0 \geq 0$ and $\sum_{\lambda\in\Gamma}(\beta_\lambda - \beta_0)(v_\lambda \otimes v_\lambda)$ is a self-adjoint finite-rank operator. It then readily follows that $P = \alpha I + K + F$, where $\alpha = \beta_0, K = 0$ and $F = \sum_{\lambda\in\Gamma}(\beta_\lambda - \beta_0)(v_\lambda \otimes v_\lambda)$.

Case 3. $\sigma(P)$ has no eigenvalue with infinite multiplicity but has a limit point.
The index set Λ is then countable; for if it is uncountable then the set $\{\beta_\lambda : \lambda \in \Lambda\}$ (counting multiplicities) is also uncountable thereby rendering the set $\{\beta_\lambda : \lambda \in \Lambda\}$ (without counting multiplicities) uncountable since each eigenvalue in this set has finite multiplicity. Then this uncountable set must have at least two limit points; and since this is impossible, we infer that Λ is countable and hence $\mathcal{H}$ is separable. Having shown that

Λ is countable, we can safely replace Λ by $\mathbb{N}$. This essentially redefines the spectrum $\sigma(P) = \text{clos}[\{\beta_n : n \in \mathbb{N}\}]$ of P.

Now let $\beta \in \sigma(P)$ be the unique limit point in the spectrum. We wish to reorder the elements of $\{\beta_n : n \in \mathbb{N}\}$ linearly in accordance with their size. To accomplish this, we first notice that there are at most only finitely many terms of the set $\{\beta_n : n \in \mathbb{N}\}$ that are strictly less than β—— represent this set of finite elements by $\{\beta_1, \beta_2, ..., \beta_k\}$ counting multiplicities. We next consider the set $\{\beta_n : \beta_n > \beta\}_{n\in\mathbb{N}}$ of all terms that are strictly bigger than β. We then inductively define a nonincreasing sequence $(\beta_{k+m})_{m\in\mathbb{N}}$ as

$$
\begin{aligned}
&\beta_{k+1} := \max\{\beta_n : \beta_n > \beta\}_{n\in\mathbb{N}},\\
&\beta_{k+2} := \max\{\beta_n : \beta_n > \beta\}_{n\in\mathbb{N}} \setminus \{\beta_{k+1}\},\\
&\vdots\\
&\beta_{k+m} := \max\{\beta_n : \beta_n > \beta\}_{n\in\mathbb{N}}, \setminus\{\beta_{k+1}, ..., \beta_{k+m-1}\},\\
&\vdots
\end{aligned}
$$

This decreasing sequence is bounded below by β, so it converges to β; for if it converges to any other point——which, in that case, happens to be a limit point of $\sigma(P)$—— then that contradicts the existence of only one limit point in the spectrum.
Before we go further, it is worth establishing that the set $\{\beta_n : \beta_n > \beta\}_{n\in\mathbb{N}}$ of all eigenvalues of P has been exhausted in the process of constructing the sequence $(\beta_{k+m})_{m\in\mathbb{N}}$, that is, each eigenvalue of P that is strictly bigger than β is a term of the sequence $(\beta_{k+m})_{m\in\mathbb{N}}$. This is rather a trivial observation if we show that whenever $\beta_n > \beta$ is an eigenvalue of P there exist only finitely many j's such that $\beta_j > \beta_n$. Now suppose, on the contrary, that there are infinitely many such j's. Then they form an infinite bounded set of real numbers with a limit point greater than or equal to β_n. But since $\beta_n > \beta$, it contradicts the fact that β is the unique limit point of the $\sigma(T)$.
This inductive method of constructing the decreasing sequence is exhaustive too and as an immediate consequence we re-order the eigenvalues of P:

$$\{\beta_n\}_{n=1}^{k} \cup \{\beta_n\}_{n=k+1}^{\infty}; \text{ where } \{\beta_n\}_{n=k+1}^{\infty} \text{ converges to } \beta.$$

Let us rename and denote by $\{v_n\}_{n=1}^{k}$ and $\{w_n\}_{n=k+1}^{\infty}$ the eigenvectors corresponding to the eigenvalues $\{\beta_n\}_{n=1}^{k}$ and $\{\beta_n\}_{n=k+1}^{\infty}$ respectively. Observe that for an arbitrary $x \in \mathcal{H}$, we have $x = \sum_{n=1}^{k} \langle x, v_n \rangle v_n + \sum_{n=k+1}^{\infty} \langle x, w_n \rangle w_n$ so that for every $x \in \mathcal{H}$,

$$\begin{aligned} Px &= \sum_{n=1}^{k} \langle x, v_n \rangle P(v_n) + \sum_{n=k+1}^{\infty} \langle x, w_n \rangle P(w_n) \\ &= \sum_{n=1}^{k} \langle x, v_n \rangle \beta_n v_n + \sum_{n=k+1}^{\infty} \langle x, w_n \rangle \beta_n w_n \\ &= \sum_{n=1}^{k} (\beta_n - \beta) \langle x, v_n \rangle v_n + \sum_{n=k+1}^{\infty} (\beta_n - \beta) \langle x, w_n \rangle w_n + \beta I x \\ &= \left(\sum_{n=1}^{k} (\beta_n - \beta)(v_n \otimes v_n) + \sum_{n=k+1}^{\infty} (\beta_n - \beta)(w_n \otimes w_n) + \beta I \right)(x). \end{aligned}$$

To conclude this case it suffices to observe that $\beta \geq 0$, $\sum_{n=1}^{k} (\beta_n - \beta)(v_n \otimes v_n)$ is a self-adjoint finite-rank operator, and $\sum_{n=k+1}^{\infty} (\beta_n - \beta)(w_n \otimes w_n)$ is a positive compact operator. It then readily follows that $P = \alpha I + K + F$, where $\alpha = \beta$, $K = \sum_{n=k+1}^{\infty} (\beta_n - \beta)(w_n \otimes w_n)$ and $F = \sum_{n=1}^{k} (\beta_n - \beta)(v_n \otimes v_n)$.

Case 4. $\sigma(P)$ has both a limit point and an eigenvalue with infinite multiplicity.
Let $\beta \in \{\beta_\lambda : \lambda \in \Lambda\}$ be the unique eigenvalue with infinite multiplicity which compels it to be the unique limit point of the spectrum $\sigma(P)$ of P. That the set $\Gamma := \Lambda \setminus \{\lambda : \beta_\lambda = \beta\}$ is countable is, at this stage, a trivial observation. This leaves us with$\{\beta_\lambda : \lambda \in \Lambda\} = \{\beta_\lambda : \lambda \in \Gamma\} \cup \{\beta\}$. Since $\{\beta_\lambda : \lambda \in \Gamma\}$ is countable, by the argument in the previous case, we can reorder the eigenvalues of this set in such a way that for some $k \in \mathbb{N}$,

$$\{\beta_\lambda : \lambda \in \Gamma\} = \{\beta_n\}_{n=1}^{k} \cup \{\beta_n\}_{n=k+1}^{\infty} \cup \{\beta\},$$

where, by the constructive method discussed previously, $\{\beta_n\}_{n=1}^{k}$ (counting multiplicities) is the set of all eigenvalues strictly less than β and $\{\beta_n\}_{n=k+1}^{\infty}$ is a nonincreasing sequence converging to β. We next rename and denote by $\{v_n\}_{n=1}^{k}$, $\{w_n\}_{n=k+1}^{\infty}$, and $\{v_\lambda\}_{\lambda \in \Lambda \setminus \Gamma}$ the eigenvectors corresponding to the eigenvalues $\{\beta_n\}_{n=1}^{k}$, $\{\beta_n\}_{n=k+1}^{\infty}$, and $\{\beta_\lambda\}_{\lambda \in \Lambda \setminus \Gamma}$ respectively. Observe that for an arbitrary $x \in \mathcal{H}$, we have $x = \sum_{n=1}^{k} \langle x, v_n \rangle v_n + \sum_{n=k+1}^{\infty} \langle x, w_n \rangle w_n +$

$\sum_{\lambda\in\Lambda\setminus\Gamma} \langle x, v_\lambda\rangle v_\lambda$. This yields that for every $x \in \mathcal{H}$

$$\begin{aligned}
Px &= \sum_{n=1}^{k} \langle x, v_n\rangle P(v_n) + \sum_{n=k+1}^{\infty} \langle x, w_n\rangle P(w_n) + \sum_{\lambda\in\Lambda\setminus\Gamma} \langle x, v_\lambda\rangle P(v_\lambda) \\
&= \sum_{n=1}^{k} \langle x, v_n\rangle \beta_n v_n + \sum_{n=k+1}^{\infty} \langle x, w_n\rangle \beta_n w_n + \sum_{\lambda\in\Lambda\setminus\Gamma} \langle x, v_\lambda\rangle \beta v_\lambda \\
&= \sum_{n=1}^{k} (\beta_n - \beta) \langle x, v_n\rangle v_n + \sum_{n=k+1}^{\infty} (\beta_n - \beta) \langle x, w_n\rangle w_n \\
&\qquad + \sum_{\lambda\in\Gamma} \langle x, v_\lambda\rangle \beta v_\lambda + \sum_{\lambda\in\Lambda\setminus\Gamma} \langle x, v_\lambda\rangle \beta v_\lambda \\
&= \left(\sum_{n=1}^{k} (\beta_n - \beta)(v_n \otimes v_n) + \sum_{n=k+1}^{\infty} (\beta_n - \beta)(w_n \otimes w_n) + \beta I \right)(x).
\end{aligned}$$

It then immediately follows that $P = \alpha I + K + F$, where $\alpha = \beta$, $K = \sum_{n=k+1}^{\infty} (\beta_n - \beta)(w_n \otimes w_n)$ and $F = \sum_{n=1}^{k} (\beta_n - \beta)(v_n \otimes v_n)$.

We complete the proof by observing that in all the four possibilities, we get the desired form. $\square$

Example 3.3.13 establishes the fact that the class of absolutely norming operators is not closed under addition. However, it is easy to see that it is closed under scalar multiplication, that is, if $T \in \mathcal{AN}(\mathcal{H}, \mathcal{K})$ and $\alpha \in \mathbb{C}$, then $\alpha T \in \mathcal{AN}(\mathcal{H}, \mathcal{K})$; for if $\mathcal{M}$ is an arbitrary non-trivial closed subspace of $\mathcal{H}$, then $\|\alpha T V_{\mathcal{M}}\| = |\alpha| \|T V_{\mathcal{M}}\| = |\alpha| \|T V_{\mathcal{M}}(x_0)\| = \|\alpha T V_{\mathcal{M}}(x_0)\|$, where $x_0 \in \mathcal{M}$, $\|x_0\| = 1$, and $\|T V_{\mathcal{M}}(x_0)\| = \|T V_{\mathcal{M}}\|$.

If we consider the class $\mathcal{AN}(\mathcal{H})^+$ of positive absolutely norming operators, what can be said about it in the similar vein? To answer this question, let $T_1, T_2 \in \mathcal{AN}(\mathcal{H})^+$. It is fairly obvious that $T_1 + T_2$ is positive. Moreover, by Theorem 3.4.1, $T_1 = \alpha_1 I + K_1 + F_1$ and $T_2 = \alpha_2 I + K_2 + F_2$ where $\alpha_1, \alpha_2 \geq 0; K_1, K_2$ are positive compact operators, and F_1, F_2 are self-adjoint finite-rank operators. Then $T_1 + T_2 = (\alpha_1 + \alpha_2)I + (K_1 + K_2) + (F_1 + F_2)$ and hence it is absolutely norming. Also, if $c \in \mathbb{R}, c \geq 0$, then $cT_1 \in \mathcal{AN}(\mathcal{H})^+$. Finally, if T and $-T$ are both in $\mathcal{AN}(\mathcal{H})^+$, then $\langle Tx, x\rangle \geq 0$ and $\langle Tx, x\rangle \leq 0$ which implies that $\langle Tx, x\rangle = 0$ for each $x \in \mathcal{H}$ and so $T = 0$. These observations, together with the fact that $\mathcal{B}(\mathcal{H})_{sa} := \{T \in \mathcal{B}(\mathcal{H}) : T = T^*\}$ is a real Banach space, implies that $\mathcal{AN}(\mathcal{H})^+$ is a cone in $\mathcal{B}(\mathcal{H})_{sa}$, which is proper in the sense that $\mathcal{AN}(\mathcal{H})^+ \cap (-\mathcal{AN}(\mathcal{H})^+) = \{0\}$.

We finish this section with the following example which shows that the set $\mathcal{AN}(\mathcal{H})$ of absolutely norming operators is not closed.

Example 3.4.2. Consider the sequence $(T_n)_{n\in\mathbb{N}}$ of operators in $\mathcal{B}(\ell^2)$ where T_n is defined by

$$T_n := \begin{bmatrix} \frac{1}{2} & & & & & & \\ & \frac{2}{3} & & & & \Large 0 & \\ & & \ddots & & & & \\ & & & 1-\frac{1}{n+1} & & & \\ & & & & 0 & & \\ & \Large 0 & & & & 0 & \\ & & & & & & \ddots \end{bmatrix},$$

for each $n \in \mathbb{N}$ and with respect to an orthonormal basis $B = \{e_i : i \in \mathbb{N}\}$ of ℓ^2. It is easy to see that for every $n \in \mathbb{N}$, the operator $T_n \in \mathcal{AN}(\ell^2)$; for T_n may be expressed as $T_n = I + F_n$ with $F_n = \operatorname{diag}(-\frac{1}{2}, -\frac{1}{3}, ..., -\frac{1}{n+1}, 0, 0, ...)$. We will show that $\lim_n T_n$ does not attain its norm. Observe that

$$T := \lim_n T_n = \begin{bmatrix} \frac{1}{2} & & & & & & \\ & \frac{2}{3} & & & & \Large 0 & \\ & & \frac{3}{4} & & & & \\ & & & \frac{4}{5} & & & \\ & & & & \ddots & & \\ & \Large 0 & & & & 1-\frac{1}{n+1} & \\ & & & & & & \ddots \end{bmatrix},$$

and the sequence of eigenvalues of T converge to 1. Consequently $\|T\| = 1$. Any arbitrarily chosen unit vector $x \in \ell^2$ can be represented as $x = \sum_i \langle x, e_i \rangle e_i$. It follows that

$$\begin{aligned}
\|Tx\|^2 &= \left\| \sum_i \langle x, e_i \rangle Te_i \right\|^2 \\
&= \left\| \sum_i \langle x, e_i \rangle (1 - \frac{1}{1+i}) e_i \right\|^2 \\
&= \sum_i |\langle x, e_i \rangle|^2 \left| (1 - \frac{1}{1+i}) \right|^2 \\
&< \sum_i |\langle x, e_i \rangle|^2 \\
&= \|x\|^2 = 1 = \|T\|^2.
\end{aligned}$$

Thus the operator $T \notin \mathcal{N}(\ell^2)$, and consequently the set $\mathcal{AN}(\ell^2)$ is not closed.

3.5 Spectral characterization of operators in $\mathcal{AN}(\mathcal{H}, \mathcal{K})$

We begin by proving the following lemma which is the key to the main theorem of this section.

Lemma 3.5.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces and let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then $T \in \mathcal{AN}(\mathcal{H}, \mathcal{K})$ if and only if $|T| \in \mathcal{AN}(\mathcal{H})$.*

Proof. Let $\mathcal{M}$ be an arbitrary nontrivial closed subspace of $\mathcal{H}$. For any $x \in \mathcal{M}$ notice that

$$\begin{aligned}
\|T|_{\mathcal{M}}(x)\| = \|Tx\| = \sqrt{\langle Tx, Tx \rangle} &= \sqrt{\langle T^*Tx, x \rangle} \\
&= \sqrt{\langle |T|^2 x, x \rangle} = \sqrt{\langle |T|x, |T|x \rangle} == \||T|(x)\| = \||T||_{\mathcal{M}}(x)\|,
\end{aligned}$$

which essentially guarantees that

$$\|T|_{\mathcal{M}}\| = \||T||_{\mathcal{M}}\|.$$

Since $\mathcal{M}$ is arbitrary, the assertion follows. □

Example 3.5.2. Let $V : \ell^2 \to \ell^2$ be an isometry onto a subspace $\mathcal{M}$ with infinite codimension. By Proposition 3.3.3, $V \in \mathcal{AN}(\ell^2)$. But $|V^*| = VV^* = P_{\mathcal{M}}$ is the orthogonal projection onto $\mathcal{M}$ and since $P_{\mathcal{M}}$ has two eigenvalues of infinite multiplicity, $P_{\mathcal{M}} \notin \mathcal{AN}(\ell^2)$ by Corollary 3.2.7. Thus, V is absolutely norming but V^* is not absolutely norming.

By the preceding lemma, the polar decomposition theorem and the spectral theorem for positive absolutely norming operators, we can safely consider the following theorem to be fully proved.

Theorem 3.5.3 (Spectral Theorem for Absolutely Norming Operators; Theorem 6.4 of [PP17])**.** *Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, and let $T = U|T|$ be its polar decomposition. Then $T \in \mathcal{AN}(\mathcal{H}, \mathcal{K})$ if and only if $|T|$ is of the form $|T| = \alpha I + K + F$, where $\alpha \geq 0, K$ is a positive compact operator and F is self-adjoint finite-rank operator.*

Chapter 4

Characterization of operators in $\mathcal{AN}_{[k]}(\mathcal{H},\mathcal{K})$

In the previous chapter we characterized the set $\mathcal{AN}(\mathcal{H},\mathcal{K})$ of absolutely norming operators in $\mathcal{B}(\mathcal{H},\mathcal{K})$ with respect to the usual operator norm. The spectral characterization theorem for absolutely norming operators (Theorem 3.5.3) indeed settles Chevreau's problem. But that is only the beginning of the story, as, like many problems in operator theory, the answer to the question is just a small step in the bigger picture. The spectral characterization of the set $\mathcal{AN}(\mathcal{H},\mathcal{K})$ opens up new territories to explore and serves to be the first hint to a more general situation. Suppose $\mathcal{B}(\mathcal{H},\mathcal{K})$ is equipped with a norm $|||\cdot|||$ equivalent to the usual operator norm and let $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$. What does it mean to say that T is norming or absolutely norming in this setting? What about characterizing these operators?

Our underlying purpose in this and the next two chapters, each of which is based on the paper[Pan17a] and its extended version [Pan16], is to first extend the concept of absolutely norming operators to several particular (symmetric) norms (that are equivalent to the operator norm) and then characterize these sets. In particular, we single out three norms on $\mathcal{B}(\mathcal{H},\mathcal{K})$: the "Ky Fan k-norm", "the weighted Ky Fan π, k-norm", and the "(p,k)-singular norm", and thereafter define and characterize the set of "absolutely norming" operators with respect to each of these three norms.

Our main business in this chapter is to give a detailed treatment of the theory of absolutely norming operators with respect to the Ky Fan k-norm and give a spectral characterization theorem for the set of such operators.

4.1 The sets $\mathcal{N}_{[k]}(\mathcal{H},\mathcal{K})$ and $\mathcal{AN}_{[k]}(\mathcal{H},\mathcal{K})$

Definition 4.1.1 (Ky Fan k-norm)**.** [Fan51] For a given natural number k, the Ky Fan k-norm $\|\cdot\|_{[k]}$ of an operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is defined to be the sum of the k largest singular values of T, that is,

$$\|T\|_{[k]} = \sum_{j=1}^{k} s_j(T).$$

Remark 4.1.2. It is not difficult to see that the Ky Fan k-norm is, in fact, a symmetric norm on $\mathcal{B}(\mathcal{H})$. Note that the smallest of Ky Fan norms, the Ky Fan 1-norm, is equal to the operator norm.

Definition 4.1.3. For any $k \in \mathbb{N}$, an operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is said to be $[k]$-*norming* if there are orthonormal elements $x_1, ..., x_k \in \mathcal{H}$ such that $\|T\|_{[k]} = \|Tx_1\| + ... + \|Tx_k\|$. If $\dim(\mathcal{H}) = r < k$, we define T to be $[k]$-*norming* if there exist orthonormal elements $x_1, ..., x_r \in \mathcal{H}$ such that $\|T\|_{[k]} = \|Tx_1\| + ... + \|Tx_r\|$. We let $\mathcal{N}_{[k]}(\mathcal{H},\mathcal{K})$ denote the set of $[k]$-norming operators in $\mathcal{B}(\mathcal{H},\mathcal{K})$.

A generalization of the above property leads to a new class of operators in $\mathcal{B}(\mathcal{H},\mathcal{K})$.

Definition 4.1.4. For any $k \in \mathbb{N}$, an operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is said to be *absolutely* $[k]$-*norming* if for every nontrivial closed subspace $\mathcal{M}$ of $\mathcal{H}$, $T|_{\mathcal{M}}$ is $[k]$-*norming*. We let $\mathcal{AN}_{[k]}(\mathcal{H},\mathcal{K})$ denote the set of absolutely $[k]$-norming operators in $\mathcal{B}(\mathcal{H},\mathcal{K})$.

Alternatively, an operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is said to be an absolutely $[k]$-norming operator if for every nontrivial closed subspace $\mathcal{M}$ of $\mathcal{H}$ with dimension k or more, there are orthonormal elements $x_1, ..., x_k \in \mathcal{M}$ such that $\|T|_{\mathcal{M}}\|_{[k]} = \|T|_{\mathcal{M}}x_1\| + ... + \|T|_{\mathcal{M}}x_k\|$. For a closed subspace $\mathcal{M}$ of $\mathcal{H}$ with $\dim(\mathcal{M}) = r < k$, the definition implies that T is absolutely $[k]$-norming if there exist orthonormal elements $x_1, ..., x_r \in \mathcal{M}$ such that $\|T|_{\mathcal{M}}\|_{[k]} = \|T|_{\mathcal{M}}x_1\| + ... + \|T|_{\mathcal{M}}x_r\|$. Needless to say, every absolutely $[k]$-norming operator is $[k]$-norming, that is, $\mathcal{AN}_{[k]}(\mathcal{H},\mathcal{K}) \subseteq \mathcal{N}_{[k]}(\mathcal{H},\mathcal{K})$.

Remark 4.1.5. Since, in the finite-dimensional setting, the geometric multiplicity of an eigenvalue of a diagonalizable operator is the same as its algebraic multiplicity and the singular values of an operator T are precisely the eigenvalues of the positive operator $|T|$, it immediately follows that *every operator on a finite-dimensional Hilbert space is* $[k]$-*norming for any* $k \in \mathbb{N}$. This is not true when the Hilbert space in question is not finite-dimensional (see Example 4.2.5).

There is an important and useful criterion for an operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ to be absolutely $[k]$-norming, which is stated in the following lemma.

Lemma 4.1.6. *For a closed linear subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ let $V_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{H}$ be the inclusion map from $\mathcal{M}$ to $\mathcal{H}$ defined as $V_{\mathcal{M}}(x) = x$ for each $x \in \mathcal{M}$ and let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. For any $k \in \mathbb{N}$, $T \in \mathcal{AN}_{[k]}(\mathcal{H}, \mathcal{K})$ if and only if for every nontrivial closed linear subspace $\mathcal{M}$ of $\mathcal{H}$, $TV_{\mathcal{M}} \in \mathcal{N}_{[k]}(\mathcal{M}, \mathcal{K})$.*

Proof. To prove this assertion we first observe that for any given nontrivial closed subspace $\mathcal{M}$ of $\mathcal{H}$, the maps $TV_{\mathcal{M}}$ and $T|_{\mathcal{M}}$ are identical and so are their singular values which implies $\|TV_{\mathcal{M}}\|_{[k]} = \|T|_{\mathcal{M}}\|_{[k]}$.

We next assume that $T \in \mathcal{AN}_{[k]}(\mathcal{H}, \mathcal{K})$ and prove the forward implication. Let $\mathcal{M}$ be an arbitrary but fixed nontrivial closed subspace of $\mathcal{H}$. Either $\dim(\mathcal{M}) = r < k$, in which case, there exist orthonormal elements $x_1, ..., x_r \in \mathcal{M}$ such that $\|T|_{\mathcal{M}}\|_{[k]} = \|T|_{\mathcal{M}}x_1\| + ... + \|T|_{\mathcal{M}}x_r\|$ which means that there exist orthonormal elements $x_1, ..., x_r \in \mathcal{M}$ such that $\|TV_{\mathcal{M}}\|_{[k]} = \|T|_{\mathcal{M}}\|_{[k]} = \|T|_{\mathcal{M}}x_1\| + ... + \|T|_{\mathcal{M}}x_r\| = \|TV_{\mathcal{M}}x_1\| + ... + \|TV_{\mathcal{M}}x_r\|$ proving that $TV_{\mathcal{M}} \in \mathcal{N}_{[k]}(\mathcal{M}, \mathcal{K})$, or $\dim(\mathcal{M}) \geq k$, in which case, there exist orthonormal elements $x_1, ..., x_k \in \mathcal{M}$ such that $\|T|_{\mathcal{M}}\|_{[k]} = \|T|_{\mathcal{M}}x_1\| + ... + \|T|_{\mathcal{M}}x_k\|$ which means that there exist orthonormal elements $x_1, ..., x_k \in \mathcal{M}$ such that $\|TV_{\mathcal{M}}\|_{[k]} = \|T|_{\mathcal{M}}\|_{[k]} = \|T|_{\mathcal{M}}x_1\| + ... + \|T|_{\mathcal{M}}x_k\| = \|TV_{\mathcal{M}}x_1\| + ... + \|TV_{\mathcal{M}}x_k\|$ proving that $TV_{\mathcal{M}} \in \mathcal{N}_{[k]}(\mathcal{M}, \mathcal{K})$. Since $\mathcal{M}$ is arbitrary, it follows that $TV_{\mathcal{M}} \in \mathcal{N}_{[k]}(\mathcal{M}, \mathcal{K})$ for every $\mathcal{M}$.

We complete the proof by showing that $T \in \mathcal{AN}_{[k]}(\mathcal{H}, \mathcal{K})$ if $TV_{\mathcal{M}} \in \mathcal{N}_{[k]}(\mathcal{M}, \mathcal{K})$ for every nontrivial closed subspace $\mathcal{M}$ of $\mathcal{H}$. We again fix $\mathcal{M}$ to be an arbitrary nontrivial closed subspace of $\mathcal{H}$. Since $TV_{\mathcal{M}} \in \mathcal{N}_{[k]}(\mathcal{M}, \mathcal{K})$, either $\dim(\mathcal{M}) = r < k$, in which case, there exist orthonormal elements $x_1, ..., x_r \in \mathcal{M}$ such that $\|TV_{\mathcal{M}}\|_{[k]} = \|TV_{\mathcal{M}}x_1\| + ... + \|TV_{\mathcal{M}}x_r\|$ which means that there exist orthonormal elements $x_1, ..., x_r \in \mathcal{M}$ such that $\|T|_{\mathcal{M}}\|_{[k]} = \|TV_{\mathcal{M}}\|_{[k]} = \|TV_{\mathcal{M}}x_1\| + ... + \|TV_{\mathcal{M}}x_r\| = \|T|_{\mathcal{M}}x_1\| + ... + \|T|_{\mathcal{M}}x_r\|$ proving that $T|_{\mathcal{M}} \in \mathcal{N}_{[k]}(\mathcal{M}, \mathcal{K})$, or $\dim(\mathcal{M}) \geq k$, in which case, there exist orthonormal elements $x_1, ..., x_k \in \mathcal{M}$ such that $\|TV_{\mathcal{M}}\|_{[k]} = \|TV_{\mathcal{M}}x_1\| + ... + \|TV_{\mathcal{M}}x_k\|$ which means that there exist orthonormal elements $x_1, ..., x_k \in \mathcal{M}$ such that $\|T|_{\mathcal{M}}\|_{[k]} = \|TV_{\mathcal{M}}\|_{[k]} = \|TV_{\mathcal{M}}x_1\| + ... + \|TV_{\mathcal{M}}x_k\| = \|T|_{\mathcal{M}}x_1\| + ... + \|T|_{\mathcal{M}}x_k\|$ proving that $T|_{\mathcal{M}} \in \mathcal{N}_{[k]}(\mathcal{M}, \mathcal{K})$. Because $\mathcal{M}$ is arbitrary, this essentially guarantees that $T \in \mathcal{AN}_{[k]}$. Since $k \in \mathbb{N}$ is arbitrary, the assertion holds for each $k \in \mathbb{N}$. $\square$

Proposition 4.1.7. *Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then for every $k \in \mathbb{N}$, $T \in \mathcal{AN}_{[k]}(\mathcal{H}, \mathcal{K})$ if and only if $|T| \in \mathcal{AN}_{[k]}(\mathcal{H})$.*

Proof. Let $\mathcal{M}$ be an arbitrary nontrivial closed subspace of $\mathcal{H}$ and let $V_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{H}$ be the inclusion map from $\mathcal{M}$ to $\mathcal{H}$ defined as $V_{\mathcal{M}}(x) = x$ for each $x \in \mathcal{M}$. We first show that

$|TV_{\mathcal{M}}| = |\,|T|V_{\mathcal{M}}\,|$. Indeed

$$|TV_{\mathcal{M}}|^2 = V_{\mathcal{M}}^* T^* T V_{\mathcal{M}} = V_{\mathcal{M}}^* |T|^2 V_{\mathcal{M}} = (V_{\mathcal{M}}^* |T|)(|T| V_{\mathcal{M}}) = (|T|V_{\mathcal{M}})^*(|T|V_{\mathcal{M}}) = |\,|T|V_{\mathcal{M}}\,|^2.$$

Consequently, for every j, $\lambda_j(|TV_{\mathcal{M}}|) = \lambda_j(|\,|T|V_{\mathcal{M}}\,|)$ and hence $s_j(TV_{\mathcal{M}}) = s_j(|T|V_{\mathcal{M}})$. This implies that for each $k \in \mathbb{N}$, we have

$$\|TV_{\mathcal{M}}\|_{[k]} = \|\,|T|V_{\mathcal{M}}\,\|_{[k]}.$$

That for each $x \in \mathcal{H}$, $\|TV_{\mathcal{M}}x\| = \|\,|T|V_{\mathcal{M}}x\,\|$ is a trivial observation. Since $\mathcal{M}$ is arbitrary, by Lemma 4.1.6 the assertion follows. □

4.2 Spectral characterization of positive operators in $\mathcal{AN}_{[k]}(\mathcal{H})$

The purpose of this section is to study the necessary and sufficient conditions for a positive operator on Hilbert space of arbitrary dimension to be absolutely $[k]$-norming for any $k \in \mathbb{N}$ and to characterize such operators.

Proposition 4.2.1. *Suppose $A \in \mathcal{B}(\mathcal{H})$ be a positive operator, $\mu = \sup\{\nu : \nu \in \sigma(A)\}$, and $\mu \notin \sigma_e(A)$, in which case, it is an eigenvalue of A with finite multiplicity, say m, so that for every $j \in \{1, ..., m\}, s_j(A) = \mu$. Then the following statements are equivalent.*

(1) $s_{m+1}(A)$ is an eigenvalue of A.

(2) $s_{m+1}(A)$ is an eigenvalue of $A - \mu P_{E_\mu}$, where P_{E_μ} is the orthogonal projection of $\mathcal{H}$ onto the eigenspace E_μ corresponding to the eigenvalue μ.

(3) $(A - \mu P_{E_\mu})|_{E_\mu^\perp} : E_\mu^\perp \longrightarrow E_\mu^\perp$ is norming, that is, $(A - \mu P_{E_\mu})|_{E_\mu^\perp} \in \mathcal{N}(E_\mu^\perp)$.

(4) $A|_{E_\mu^\perp} : E_\mu^\perp \longrightarrow E_\mu^\perp$ is norming, that is, $A|_{E_\mu^\perp} \in \mathcal{N}(E_\mu^\perp)$.

(5) $A \in \mathcal{N}_{[m+1]}(\mathcal{H})$.

Proof. (1) $\iff$ (2): The backward implication is trivial. For the forward implication, let $\lambda = s_{m+1}(A) := \sup\{\nu : \nu \in \sigma(A - \mu P_{E_\mu})\}$. Assume that λ is an eigenvalue of A. Then there exists some nonzero vector $x \in \mathcal{H}$ such that $Ax = \lambda x$. It suffices to prove that $x \perp E_\mu$, for then $(A - \mu P_{E_\mu})x = Ax = \lambda x$. But $A \geq 0$ and $\lambda \neq \mu$ which implies that $x \perp E_\mu$.

(2) $\iff$ (3): Since

$$A - \mu P_{E_\mu}(x) = \begin{cases} 0 & \text{if } x \in E_\mu \\ Ax & \text{if } x \in E_\mu^\perp, \end{cases}$$

$A-\mu P_{E_\mu}$ is a positive operator on $\mathcal{B}(\mathcal{H})$ and $E_\mu^\perp$ is a closed subspace of $\mathcal{H}$ which is invariant under $A-\mu P_{E_\mu}$ which implies that $(A-\mu P)_{E_\mu}|_{E_\mu^\perp} : E_\mu^\perp \longrightarrow E_\mu^\perp$, viewed as an operator on $E_\mu^\perp$, is positive and $\|(A-\mu P_{E_\mu})|_{E_\mu^\perp}\| = s_{m+1}(A)$. By [PP17, Theorem 2.3] we know that a positive operator T is norming if and only if $\|T\|$ is an eigenvalue of T. Thus $(A-\mu P_{E_\mu})|_{E_\mu^\perp}$ is norming if and only if $s_{m+1}(A)$ is an eigenvalue of $(A-\mu P_{E_\mu})|_{E_\mu^\perp}$ if and only if $s_{m+1}(A)$ is an eigenvalue of $A-\mu P_{E_\mu}$.

(3) $\iff$ (4): This equivalence follows trivially from the fact that the maps $(A-\mu P_{E_\mu})|_{E_\mu^\perp}$, and $A|_{E_\mu^\perp}$ are identical on $E_\mu^\perp$.

(3) $\iff$ (5): Notice that $A \in \mathcal{N}_{[m]}(\mathcal{H})$; for $\|A\|_{[m]} = m\mu$ and since the geometric multiplicity of μ is m, we can find a set $\{v_1, ..., v_m\}$ of m orthonormal vectors in $E_\mu \subseteq \mathcal{H}$ such that $\sum_{i=1}^m \|Av_i\| = m\mu = \|A\|_{[m]}$. Also, it is not very difficult to observe that if there exists any set $\{w_1, ..., w_m\}$ of m orthonormal vectors in $\mathcal{H}$ such that $\sum_{i=1}^m \|Aw_i\| = m\mu$, then this set has to be contained in E_μ. This observation implies that $A \in \mathcal{N}_{[m+1]}(\mathcal{H})$ if and only if there exists a unit vector $x \in E_\mu^\perp$ such that $\|Ax\| = s_{m+1}(A)$ which is possible if and only if $A-\mu P_{E_\mu}|_{E_\mu^\perp} : E_\mu^\perp \longrightarrow E_\mu^\perp$ is norming because $\|A-\mu P_{E_\mu}\| = \|(A-\mu P_{E_\mu})|_{E_\mu^\perp}\| = s_{m+1}(A)$. $\square$

Remark 4.2.2. The above proposition holds even if $\mu \in \sigma_e(A)$, μ is an accumulation point but not an eigenvalue; for we can consider it to be an eigenvalue with multiplicity 0. If $\mu \in \sigma_e(A)$ is an accumulation point as well as an eigenvalue with finite multiplicity, say m, then one can still prove (2) $\iff$ (3) $\iff$ (4) $\iff$ (5); the condition (1) no longer remains equivalent to other conditions.

Proposition 4.2.3. *If $A \in \mathcal{B}(\mathcal{H})$ is a positive operator and $s_{m+1}(A) \neq s_m(A)$ for some $m \in \mathbb{N}$, then $A \in \mathcal{N}_{[m]}(\mathcal{H})$. Moreover, in this case, $A \in \mathcal{N}_{[m+1]}(\mathcal{H})$ if and only if $s_{m+1}(A)$ is an eigenvalue of A.*

Proof. It is easy to see that for every $j \in \{1, ..., m\}, s_j(A) \notin \sigma_{\mathrm{e}}(A)$. Hence the set $\{s_1(A), ..., s_m(A)\}$ consists of eigenvalues (not necessarily distinct) of A, each having finite multiplicity. This guarantees the existence of an orthonormal set $\{v_1, ..., v_m\} \subseteq K \subseteq \mathcal{H}$ such that $Av_j = s_j(A)v_j$ which yields $\|A\|_{[m]} = \|Av_1\| + ... + \|Av_m\|$, where K is the joint span of the eigenspaces corresponding to the eigenvalues $\{s_1(A), ..., s_m(A)\}$, which implies that $A \in \mathcal{N}_{[m]}(\mathcal{H})$. Furthermore, we observe that if there exists any orthonormal set $\{w_1, ..., w_m\}$ of m vectors in $\mathcal{H}$ such that $\sum_{i=1}^m \|Aw_i\| = \sum_{j=1}^m s_j(A)$, then this set has to be contained in K. Note that $K^\perp$ is invariant under A and hence $A|_{K^\perp} : K^\perp \longrightarrow K^\perp$, viewed as an operator on $K^\perp$, is positive. It follows then that $A \in \mathcal{N}_{m+1}(\mathcal{H})$ if and only if there exists a unit vector $x \in K^\perp$ such that $\|Ax\| = s_{m+1}(A)$, which is possible if and only if $A|_{K^\perp} : K^\perp \longrightarrow K^\perp$, viewed as an operator on $K^\perp$, attains its norm, which in turn happens if and only if $s_{m+1}(A)$ is an eigenvalue of $A|_{K^\perp}$, since $\|A|_{K^\perp}\| = s_{m+1}(A)$. But $s_{m+1}(A) \neq s_m(A)$ implies that $s_{m+1}(A)$ is an eigenvalue of $A|_{K^\perp}$ if and only if $s_{m+1}(A)$ is an eigenvalue of A. This proves the assertion. $\square$

4.2.1 Necessary conditions for positive operators to belong to $\mathcal{AN}_{[k]}(\mathcal{H})$

The purpose of this subsection is to study the necessary conditions for a positive operator on complex Hilbert space of arbitrary dimension to be absolutely $[k]$-norming for any $k \in \mathbb{N}$.

Proposition 4.2.4. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator and $k \in \mathbb{N}$. If $A \in \mathcal{N}_{[k]}(\mathcal{H})$, then $s_1(A), ..., s_k(A)$ are eigenvalues of A.*

Proof. The proof is by contrapositive. Assuming that at least one of the elements from the set $\{s_1(A), ..., s_k(A)\}$ is not an eigenvalue of A, we show that $A \notin \mathcal{N}_{[k]}(\mathcal{H})$. Suppose that $s_1(A)$ is not an eigenvalue of A. Then it must be an accumulation point of the spectrum of A in which case none of the singular values of A is an eigenvalue of A and that $s_j(A) = s_1(A)$ for every $j \geq 2$. Since $s_1(A) = \|A\|$, it follows from [PP17, Theorem 2.3] that $A \notin \mathcal{N}(\mathcal{H})$ which means that for every $x \in \mathcal{H}, \|x\| = 1$, we have $\|Ax\| < \|A\| = s_1(A)$. Consequently, for every orthonormal set $\{x_1, ..., x_k\} \subseteq \mathcal{H}$ we have $\sum_{j=1}^k \|Ax_j\| < k\|A\| = \sum_{j=1}^k s_j(A)$ so that $A \notin \mathcal{N}_{[k]}(\mathcal{H})$.

Next suppose that $s_1(A)$ is an eigenvalue of A but $s_2(A)$ is not. Clearly then $s_1(A)$ is an eigenvalue with multiplicity 1, $s_2(A) \neq s_1(A)$ and $s_j(A) = s_2(A)$ for every $j \geq 3$ in which case Proposition 4.2.3 ascertains that $A \in \mathcal{N}(\mathcal{H})$ but $A \notin \mathcal{N}_{[2]}(\mathcal{H})$. This implies that there exists $y_1 \in \mathcal{H}$ with $\|y_1\| = 1$ such that $\|Ay_1\| = \|A\|$ and for every $y \in \text{span}\{y_1\}^\perp$ with $\|y\| = 1$ we have $\|Ay\| < s_2(A)$ which in turn implies that for every orthonormal set $\{y_2, ...y_k\} \subseteq \text{span}\{y_1\}^\perp$ we have $\sum_{j=2}^k \|Ay_j\| < (k-1)s_2(A) = \sum_{j=2}^k s_j(A)$. This yields $\sum_{j=1}^k \|Ay_j\| < \sum_{j=1}^k s_j(A)$ for every orthonormal set $\{y_1, ..., y_k\} \subseteq \mathcal{H}$ which implies that $A \notin \mathcal{N}_{[k]}(\mathcal{H})$.

If $s_1(A), s_2(A)$ are eigenvalues of A but $s_3(A)$ is not, then we have $s_3(A) \neq s_2(A)$ and $s_j(A) = s_3(A)$ for every $j \geq 4$ in which case Proposition 4.2.3 asserts that $A \in \mathcal{N}_{[2]}(\mathcal{H})$ but $A \notin \mathcal{N}_{[3]}(\mathcal{H})$. Consequently, there exists an orthonormal set $\{z_1, z_2\} \subseteq \mathcal{H}$ such that $\|Tz_1\|+$

$\|Tz_2\| = \|T\|_{[2]}$ and that for every unit vector $z \in \text{span}\{z_1, z_2\}^{\perp}$ we have $\|Tz\| < s_3(A)$ which in turn implies that for every orthonormal set $\{z_3, ...z_k\} \subseteq \text{span}\{z_1, z_2\}^{\perp}$ we have $\sum_{j=3}^{k} \|Az_j\| < (k-2)s_3(A) = \sum_{j=3}^{k} s_j(A)$. It then follows that $\sum_{j=1}^{k} \|Az_j\| < \sum_{j=1}^{k} s_j(A)$ for every orthonormal set $\{z_1, ..., z_k\} \subseteq \mathcal{H}$ which implies that $A \notin \mathcal{N}_{[k]}(\mathcal{H})$.

If we continue in this way, we can show at every step that $A \notin \mathcal{N}_{[k]}(\mathcal{H})$. We conclude the proof by discussing the final case when $s_1(A), ..., s_{k-1}(A)$ are all eigenvalues of A but $s_k(A)$ is not in which case $s_k(A) \neq s_{k-1}(A)$ and thus by Proposition 4.2.3, we infer that $A \notin \mathcal{N}_{[k]}(\mathcal{H})$. This exhausts all the possibilities and the assertion is thus proved contrapositively. $\square$

The converse of the above proposition is not necessarily true as the following example shows.

Example 4.2.5. Consider the operator

$$T = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & 0 & & \\ & & \frac{1}{2} & & & & \\ & & & \frac{2}{3} & & & \\ & & & & \ddots & & \\ & 0 & & & & 1-\frac{1}{n} & \\ & & & & & & \ddots \end{bmatrix} \in \mathcal{B}(\ell^2),$$

with respect to an orthonormal basis $B = \{v_i : i \in \mathbb{N}\}$. That T is positive diagonalizable operator with $\|T\| = 1$ is obvious. The spectrum $\sigma(T)$ of T is given by the set $\{1 - \frac{1}{n} : n \in \mathbb{N}, n > 1\} \cup \{1\}$ where $1 \in \sigma(T)$ is an accumulation point of the spectrum as well as an eigenvalue of T with multiplicity 2 and hence $s_j(T) = 1$ for each $j \in \mathbb{N}$. Notice that $\{v_1, v_2\} \subseteq B$ serves to be an orthonormal set such that $\|T\|_{[2]} = \|Tv_1\| + \|Tv_2\|$ which implies that $T \in \mathcal{N}_{[2]}(\ell^2)$. Also, if there exists an orthonormal set $\{w_1, w_2\} \subseteq \ell^2$ of two vectors such that $\|T\|_{[2]} = \|Tw_1\| + \|Tw_2\|$, then this set has to be contained in $\text{span}\{v_1, v_2\}$. T is, however, not [3]-norming. To show that there does not exist a unit vector $x \in \text{span}\{v_1, v_2\}^{\perp}$ such that $\|Tx\| = 1$, we consider the diagonal operator

$$A := T - P_{\text{span}\{v_1,v_2\}} = \begin{bmatrix} 0 & & & & & & \\ & 0 & & & \text{\huge 0} & & \\ & & \frac{1}{2} & & & & \\ & & & \frac{2}{3} & & & \\ & & & & \ddots & & \\ & \text{\huge 0} & & & & 1-\frac{1}{n} & \\ & & & & & & \ddots \end{bmatrix},$$

where $P_{\text{span}\{v_1,v_2\}}$ is the orthogonal projection of ℓ^2 onto the space $\text{span}\{v_1, v_2\}$. It is not very hard to see that there exists a unit vector $x \in \text{span}\{v_1, v_2\}^{\perp}$ with $\|Tx\| = 1$ if and only if $A|_{\text{span}\{v_1,v_2\}^{\perp}} : \text{span}\{v_1, v_2\}^{\perp} \longrightarrow \text{span}\{v_1, v_2\}^{\perp}$ achieves its norm on $\text{span}\{v_1, v_2\}^{\perp}$. Since $A|_{\text{span}\{v_1,v_2\}^{\perp}}$ is positive on $\text{span}\{v_1, v_2\}^{\perp}$, it follows that $A|_{\text{span}\{v_1,v_2\}^{\perp}}$ attains its norm on $\text{span}\{v_1, v_2\}^{\perp}$ if and only if $\|A|_{\text{span}\{v_1,v_2\}^{\perp}}\| = 1$ is an eigenvalue of $A|_{\text{span}\{v_1,v_2\}^{\perp}}$ which is indeed not the case.

Proposition 4.2.6. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator and $k \in \mathbb{N}$. If $s_1(A), ..., s_k(A)$ are mutually distinct eigenvalues of A, then there exists an orthonormal set $\{v_1, ..., v_k\} \subseteq \mathcal{H}$ such that $Av_j = s_j(A)v_j$ for every $j \in \{1, ..., k\}$. Thus $A \in \mathcal{N}_{[k]}(\mathcal{H})$.*

Proof. This is a direct consequence of the fact that the eigenvectors of a normal operator corresponding to distinct eigenvalues are mutually orthogonal. □

An immediate question that arises here is the following: suppose that $s_1(A), ..., s_k(A)$ are eigenvalues of the positive operator A with $s_1(A) = s_2(A) = ... = s_k(A)$. Is it possible for A to be in $\mathcal{N}_{[k]}(\mathcal{H})$, and if yes, then under what circumstances? The answer is affirmative and it happens if and only if the geometric multiplicity of the eigenvalue $s_1(A)$ is at least k.

Proposition 4.2.7. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator, $k \in \mathbb{N}$ and let $s_1(A), ..., s_k(A)$ be the first k singular values of A that are also the eigenvalues of A and are not necessarily distinct. Then either $s_1(A) = ... = s_k(A)$, in which case, $A \in \mathcal{N}_{[k]}(\mathcal{H})$ if and only if the multiplicity of $\alpha := s_1(A)$ is at least k; or there exists $t \in \{2, ..., k\}$ such that $s_{t-1}(A) \neq s_t(A) = s_{t+1}(A) = ... = s_k(A)$, in which case, $A \in \mathcal{N}_{[k]}(\mathcal{H})$ if and only if the multiplicity of $\beta := s_t(A)$ is at least $k - t + 1$.*

Proof. It suffices to establish the assertion of the first case; the second case follows similarly. We thus assume that $s_1(A) = ... = s_k(A)$ and prove that $A \in \mathcal{N}_{[k]}(\mathcal{H})$ if and only if the multiplicity of $\alpha := s_1(A)$ is at least k. The backward implication is trivial. To see the forward implication, let us assume contrapositively that the geometric multiplicity of α is strictly less that k, that is, the dimension of the eigenspace $E_\alpha = \ker(A - \alpha I)$ associated with the eigenvalue α is $m < k$. Then α has to be an accumulation point of the spectrum $\sigma(A)$ of A as well; for the number of times an eigenvalue with finite multiplicity appears in the sequence $(s_j(A))_{j\in\mathbb{N}}$ exceeds its multiplicity only when it is also an accumulation point of the spectrum. It is easy to see that $A \in \mathcal{N}_{[m]}(\mathcal{H})$ since there exists an orthonormal set $\{v_1, ..., v_m\} \subseteq E_\alpha$ such that $\|T\|_{[m]} = \|Tv_1\| + ... + \|Tv_m\|$. Even more, if there exists any orthonormal set $\{w_1, ..., w_m\} \subseteq \mathcal{H}$ such that $\|T\|_{[m]} = \|Tw_1\| + ... + \|Tw_m\|$, then this set has to be contained in E_α. We now show that $A \notin \mathcal{N}_{[k]}(\mathcal{H})$. Let P_{E_α} denote the orthogonal projection of $\mathcal{H}$ onto the eigenspace E_α. Now consider the positive operator $A - \alpha P_{E_\alpha}$ on $\mathcal{B}(\mathcal{H})$ and note that $E_\alpha^\perp$ is a closed subspace of $\mathcal{H}$ which is invariant under $A - \alpha P_{E_\alpha}$ which implies that $(A - \alpha P_{E_\alpha})|_{E_\alpha^\perp} : E_\alpha^\perp \longrightarrow E_\alpha^\perp$, viewed as an operator on $E_\alpha^\perp$, is positive and that $\|(A - \alpha P_{E_\alpha})|_{E_\alpha^\perp}\| = s_{m+1}(A) = \alpha$. It is easy to see that α is not an eigenvalue of the positive operator $(A - \alpha P_{E_\alpha})|_{E_\alpha^\perp}$ on $E_\alpha^\perp$. Consequently, this operator does not achieve its norm on $E_\alpha^\perp$ which means that for every $x \in E_\alpha^\perp$ with $\|x\| = 1$ we have $\|(A - \alpha P_{E_\alpha})|_{E_\alpha^\perp} x\| < s_{m+1}(A) = \alpha$. Thus for every orthonormal set $\{v_{m+1}, v_{m+2}, ..., v_k\} \subseteq E_\alpha^\perp$ we have $\|(A - \alpha P_{E_\alpha})|_{E_\alpha^\perp} v_j\| < s_j(A) = \alpha$, $m + 1 \leq j \leq k$ so that $\sum_{j=m+1}^k \|(A - \alpha P_{E_\alpha})|_{E_\alpha^\perp} v_j\| < \sum_{j=m+1}^k s_j(A)$. It now follows that for every orthonormal set $\{x_1, ..., x_k\} \subseteq \mathcal{H}, \sum j = 1^k \|Ax_j\| < \sum_{j=1}^k s_j(A) = \|A\|_{[k]}$ which implies that $A \notin \mathcal{N}_{[k]}(\mathcal{H})$. This proves the proposition. □

Theorem 4.2.8. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator and $k \in \mathbb{N}$. Then the following statements are equivalent.*

(1) $A \in \mathcal{N}_{[k]}(\mathcal{H})$.

(2) $s_1(A), ..., s_k(A)$ *are eigenvalues of* A *and there exists an orthonormal set* $\{v_1, ..., v_k\} \subseteq \mathcal{H}$ *such that* $Av_j = s_j(A)v_j$ *for every* $j \in \{1, ..., k\}$.

Proof. (1) follows from (2) trivially. Assume that $A \in \mathcal{N}_{[k]}(\mathcal{H})$. Since $A \geq 0$, by Proposition 4.2.4, $s_1(A), ..., s_k(A)$ are all eigenvalues of A. If $s_1(A), ..., s_k(A)$ are mutually distinct, then by Proposition 4.2.6 $A \in \mathcal{N}_{[k]}(\mathcal{H})$. However, if $s_1(A), ..., s_k(A)$ are not necessarily distinct then the Proposition 4.2.7 yields $A \in \mathcal{N}_{[k]}(\mathcal{H})$. This completes the proof. □

The following corollary is an immediate consequence of the above theorem.

Corollary 4.2.9. *Let $k \in \mathbb{N}$. If $A \in \mathcal{N}_{[k+1]}(\mathcal{H}, \mathcal{K})$ is positive, then $A \in \mathcal{N}_{[k]}(\mathcal{H}, \mathcal{K})$.*

Theorem 4.2.10. *Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $k \in \mathbb{N}$. Then the following statements are equivalent.*

(1) $T \in \mathcal{N}_{[k]}(\mathcal{H}, \mathcal{K})$.

(2) $|T| \in \mathcal{N}_{[k]}(\mathcal{H})$.

(3) $T^*T \in \mathcal{N}_{[k]}(\mathcal{H})$.

Proof. The equivalence of (1) and (2) follows from facts that for every $j, s_j(T) = \lambda_j(|T|) = s_j(|T|)$ and for every $x \in \mathcal{H}, \|Tx\| = \||T|x\|$. To prove the equivalence of (2) and (3), we first assume that $|T| \in \mathcal{N}_{[k]}(\mathcal{H})$. Since $|T|$ is positive, Theorem 4.2.8 guarantees that $s_1(|T|), ..., s_k(|T|)$ are eigenvalues of $|T|$ and there exists an orthonormal set $\{v_1, ..., v_k\} \subseteq \mathcal{H}$ such that $|T|v_j = s_j(|T|)v_j$ for every $j \in \{1, ..., k\}$. Consequently, for every $j \in \{1, ..., k\}$, we have $\||T|v_j\| = s_j(|T|)$. Using this we deduce that for every j,

$$\begin{aligned} s_j(T^*T) = s_j(|T|^2) = s_j^2(|T|) = \||T|v_j\|^2 = \langle |T|v_j, |T|v_j\rangle = \left\langle |T|^2 v_j, v_j\right\rangle \\ = \langle T^*Tv_j, v_j\rangle \leq \|T^*Tv_j\| = \||T|^2 v_j\| = \|s_j^2(|T|)v_j\| = s_j^2(|T|) = s_j(T^*T), \end{aligned}$$

and so we have equality throughout which implies that $s_j(T^*T) = \|T^*Tv_j\|$ for every $j \in \{1, ..., k\}$. This yields $\|T^*T\|_{[k]} = \sum_{j=1}^k s_j(T^*T) = \sum_{j=1}^k \|T^*Tv_j\|$ which implies that $T^*T \in \mathcal{N}_{[k]}(\mathcal{H})$.

Conversely, if $T^*T \in \mathcal{N}_{[k]}(\mathcal{H})$, then again by Theorem 4.2.8 $s_1(T^*T), ..., s_k(T^*T)$ are eigenvalues of T^*T and there exists an orthonormal set $\{w_1, ..., w_k\} \subseteq \mathcal{H}$ such that $T^*Tw_j = s_j(T^*T)w_j$ for every $j \in \{1, ..., k\}$. This gives

$$\begin{aligned} \||T|w_j\|^2 = \langle |T|w_j, |T|w_j\rangle = \langle T^*Tw_j, w_j\rangle = \langle s_j(T^*T)w_j, w_j\rangle \\ = s_j(T^*T)\langle w_j, w_j\rangle = s_j(T^*T) = s_j(|T|^2) = s_j^2(|T|), \end{aligned}$$

which in turn gives $\||T|w_j\| = s_j(|T|)$ for every $j \in \{1, ..., k\}$. Then $\||T|\|_{[k]} = \sum_{j=1}^k s_j(|T|) = \sum_{j=1}^k \||T|w_j\|$, and the result follows. □

Theorem 4.2.11. *Let $k \in \mathbb{N}$. Then $\mathcal{AN}_{[k+1]}(\mathcal{H}, \mathcal{K}) \subseteq \mathcal{AN}_{[k]}(\mathcal{H}, \mathcal{K})$.*

Proof. If $A \in \mathcal{AN}(\mathcal{H}, \mathcal{K})$, then Theorem 4.2.10 along with Corollary 4.2.9 implies that for every nontrivial closed subspace $\mathcal{M}$ of $\mathcal{H}$,

$$\begin{aligned} AV_{\mathcal{M}} \in \mathcal{N}_{[k+1]}(\mathcal{M}, \mathcal{K}) &\iff |AV_{\mathcal{M}}| \in \mathcal{N}_{[k+1]}(\mathcal{M}) \\ &\implies |AV_{\mathcal{M}}| \in \mathcal{N}_{[k]}(\mathcal{M}) \\ &\iff AV_{\mathcal{M}} \in \mathcal{N}_{[k]}(\mathcal{M}, \mathcal{K}). \end{aligned}$$

Since the above implications (and both way implications) hold for every $\mathcal{M}$, the assertion is proved. $\square$

Corollary 4.2.12. *Let $k \in \mathbb{N}$. Then every positive operator in $\mathcal{AN}_{[k+1]}(\mathcal{H})$ belongs to $\mathcal{AN}_{[k]}(\mathcal{H})$.*

Theorem 4.2.13. *Let A be a positive operator on $\mathcal{H}$, and $k \in \mathbb{N}$. If $A \in \mathcal{AN}_{[k]}(\mathcal{H})$, then A is of the form $A = \alpha I + K + F$, where $\alpha \geq 0, K$ is a positive compact operator and F is self-adjoint finite-rank operator.*

Proof. Since $A \in \mathcal{AN}_{[k]}(\mathcal{H})$, $A \in \mathcal{AN}(\mathcal{H})$. The forward implication of [PP17, Theorem 5.1], hence, implies the assertion. $\square$

We finish this subsection by proving a result which will be useful later in Chapter 8 for establishing the notion of absolutely norming operators in symmetrically-normed ideals.

Theorem 4.2.14. *For a closed linear subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ let $V_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{H}$ be the inclusion map from $\mathcal{M}$ to $\mathcal{H}$ defined as $V_{\mathcal{M}}(x) = x$ for each $x \in \mathcal{M}$, let $P_{\mathcal{M}} \in \mathcal{B}(\mathcal{H})$ be the orthogonal projection onto $\mathcal{M}$, and let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. For any $k \in \mathbb{N}$, the following statements are equivalent.*

1. *$T \in \mathcal{AN}_{[k]}(\mathcal{H}, \mathcal{K})$.*
2. *$TV_{\mathcal{M}} \in \mathcal{N}_{[k]}(\mathcal{M}, \mathcal{K})$ for every nontrivial closed linear subspace $\mathcal{M}$ of $\mathcal{H}$.*
3. *$TP_{\mathcal{M}} \in \mathcal{N}_{[k]}(\mathcal{H}, \mathcal{K})$ for every nontrivial closed linear subspace $\mathcal{M}$ of $\mathcal{H}$.*

Proof. The equivalence of (1) and (2) follows from Lemma 4.1.6. We will prove (1) $\iff$ (3). Fix $\mathcal{M}$ to be a nontrivial closed subspace of $\mathcal{H}$. A trivial verification shows that $\sigma(|T|_{\mathcal{M}}|) \setminus \{0\} = \sigma(|TP_{\mathcal{M}}|) \setminus \{0\}$ which implies that the singular values of $T|_{\mathcal{M}}$ and $TP_{\mathcal{M}}$ are identical, which gives $\|T|_{\mathcal{M}}\|_{[k]} = \|TP_{\mathcal{M}}\|_{[k]}$. Of course, $\| T|_{\mathcal{M}} x \| = \|TP_{\mathcal{M}}x\|$ for each $x \in \mathcal{M}$. This establishes the implication (1) $\implies$ (3). All that remains is to prove

(3) $\implies$ (1). Assume that $TP_{\mathcal{M}} \in \mathcal{N}_{[k]}(\mathcal{H}, \mathcal{K})$. Then by Theorem 4.2.10 $|TP_{\mathcal{M}}| \in \mathcal{N}_{[k]}(\mathcal{H})$. Theorem 4.2.8 guarantees the existence of an orthonormal set $\{x_1, ..., x_k\} \subseteq \mathcal{H}$ with $|TP_{\mathcal{M}}|x_j = s_j(|TP_{\mathcal{M}}|)x_j$ for every $j \in \{1, ..., k\}$ which implies that $|TP_{\mathcal{M}}|^2 x_j = s_j^2(|TP_{\mathcal{M}}|)x_j = s_j(|TP_{\mathcal{M}}|^2)x_j$ for every $j \in \{1, ..., k\}$. Without loss of generality we assume that for each $j, s_j(|TP_{\mathcal{M}}|^2) \neq 0$. Under this assumption it is obvious that $x_j \in \mathcal{M}$ for each j; for if x_j were not in $\mathcal{M}$, then it can't be an eigenvector of $|TP_{\mathcal{M}}|^2$ corresponding to the eigenvalue $s_j(|TP_{\mathcal{M}}|^2)$. It then follows immediately that

$$\begin{aligned}\|T|_{\mathcal{M}}\|_{[k]} = \|TP_{\mathcal{M}}\|_{[k]} = \|\,|TP_{\mathcal{M}}|\,\|_{[k]} &= \sum_{j=1}^{k} \|\,|TP_{\mathcal{M}}|x_j\| \\ &= \sum_{j=1}^{k} \|TP_{\mathcal{M}}x_j\| = \sum_{j=1}^{k} \|Tx_j\| = \sum_{j=1}^{k} \|\,T|_{\mathcal{M}}x_j\|,\end{aligned}$$

where $\{x_1, ..., x_k\}$ is an orthonormal set contained in $\mathcal{M}$. Using the fact $s_j(|TP_{\mathcal{M}}|) = s_j(TP_{\mathcal{M}})$ we conclude $T|_{\mathcal{M}} \in \mathcal{N}_{[k]}$. But $\mathcal{M}$ is arbitrary, so $T \in \mathcal{AN}_{[k]}(\mathcal{H}, \mathcal{K})$. Since $k \in \mathbb{N}$ is arbitrary, the assertion holds for every $k \in \mathbb{N}$. ☐

4.2.2 Sufficient conditions for operators to belong to $\mathcal{AN}_{[k]}(\mathcal{H}, \mathcal{K})$

In this subsection, we discuss the sufficient conditions for an operator (not necessarily positive) to be absolutely $[k]$-norming for every $k \in \mathbb{N}$. We begin with a relatively easy proposition, the proof of which is trivial and thus omitted, that gives a sufficient condition for a positive diagonalizable operator to be in $\mathcal{N}_{[k]}(\mathcal{H})$.

Proposition 4.2.15. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive diagonalizable operator on a Hilbert space $\mathcal{H}$, $B = \{v_\beta : \beta \in \Lambda\}$ be an orthonormal basis of $\mathcal{H}$ corresponding to which A is diagonalizable, and $k \in \mathbb{N}$. If there exists a subset $\{\beta_1, ..., \beta_k\} \subseteq \Lambda$ of cardinality k such that for every $j \in \{1, ..., k\}, A(v_{\beta_j}) = \lambda_j(A)v_{\beta_j}$, then $A \in \mathcal{N}_{[k]}(\mathcal{H})$. If $\dim(\mathcal{H}) = r < k$, then the existence of a subset $\{\beta_1, ..., \beta_r\} \subseteq \Lambda$ of cardinality r is required with the condition that for every $j \in \{1, ..., r\}, A(v_{\beta_j}) = \lambda_j(A)v_{\beta_j}$, for the operator A to be in $\mathcal{N}_{[k]}(\mathcal{H})$. Here $\lambda_j(A)$ is as introduced in the Definition 2.8.3.*

Proof. The existence of a subset $\{\beta_1, ..., \beta_k\} \subseteq \Lambda$ of cardinality k with the above mentioned property ascertains that there exist k orthonormal eigenvectors in B with $\lambda_j(A), j \in \{1, ..., k\}$ being their corresponding eigenvalues, which in turn implies that $A \in \mathcal{N}_k(\mathcal{H})$. ☐

Proposition 4.2.16. *If $T \in \mathcal{B}_0(\mathcal{H}, \mathcal{K})$, then $T \in \mathcal{AN}_{[k]}(\mathcal{H}, \mathcal{K})$ for every $k \in \mathbb{N}$.*

Proof. If T is a compact operator from $\mathcal{H}$ to $\mathcal{K}$ then the restriction of T to any closed subspace $\mathcal{M}$ is a compact operator from $\mathcal{M}$ to $\mathcal{K}$. So it will be sufficient to prove that if T is a compact operator then $T \in \mathcal{N}_{[k]}(\mathcal{H}, \mathcal{K})$ for each $k \in \mathbb{N}$.

The assertion is trivial if $\mathcal{H}$ is finite-dimensional; for then $|T| \in M_n(\mathbb{C})$ for some $n \in \mathbb{N}$ is a positive diagonal matrix. We thus assume $\mathcal{H}$ to be infinite-dimensional. Let us fix $k \in \mathbb{N}$. Since $|T|$ is positive compact operator the singular values of T are precisely the eigenvalues of $|T|$, the Courant Fisher theorem (**??**) guarantees the existence of an orthonormal set $\{v_1, ..., v_k\} \subseteq \mathcal{H}$ such that for every $j \in \{1, ..., k\}, |T|v_j = \lambda_j(|T|)v_j$ which implies that $\|T\|_{[k]} = \sum_{j=1}^{k} s_j(T) = \sum_{j=1}^{k} \lambda_j(|T|) = \sum_{j=1}^{k} \||T|v_j\| = \sum_{j=1}^{k} \|Tv_j\|$ so that $T \in \mathcal{N}_{[k]}(\mathcal{H}, \mathcal{K})$. Since $k \in \mathbb{N}$ is arbitrary, it follows that $T \in \mathcal{N}_{[k]}(\mathcal{H}, \mathcal{K})$ for every $k \in \mathbb{N}$. $\square$

Lemma 4.2.17. *If $F \in \mathcal{B}_{00}(\mathcal{H})$, $F = F^*$ and $\alpha \geq 0$, then $\alpha I + F \in \mathcal{N}_{[k]}(\mathcal{H})$ for every $k \in \mathbb{N}$.*

Proof. The assertion is trivial if $\alpha = 0$; for then F is a compact operator which belongs to $\mathcal{N}_{[k]}(\mathcal{H})$ for every $k \in \mathbb{N}$. We fix k and assume that $\alpha > 0$. There is no loss of generality in assuming that $\mathcal{H}$ is infinite-dimensional, for if it is not, then the operator is compact and hence belongs to $\mathcal{N}_{[k]}(\mathcal{H})$. Let the range of F be m-dimensional. It suffices to show that $|\alpha I + F| \in \mathcal{N}_{[k]}(\mathcal{H})$ operator.

Case I: If $k \leq m$. Since F is self-adjoint, there exists an orthonormal basis $B = \{v_\beta : \beta \in \Lambda\}$ of $\mathcal{H}$ corresponding to which the matrix $M_B(F)$ is a diagonal matrix with m nonzero real diagonal entries, say $\{\mu_1, \mu_2, ..., \mu_m\}$ which are not necessarily distinct. There is, then, a subset $\{\beta_1, ..., \beta_m\} \subseteq \Lambda$ of cardinality m such that for every $j \in \{1, ..., m\}$, we have $F(v_{\beta_j}) = \mu_j v_{\beta_j}$. Clearly then, $M_B(|\alpha I + F|)$ is also a diagonal matrix with respect to the basis B such that the spectrum $\sigma(|\alpha I + F|)$ of $|\alpha I + F|$ is given by $\sigma(|\alpha I + F|) = \sigma_p(|\alpha I + F|) = \{|\alpha + \mu_1|, ..., |\alpha + \mu_m|, \alpha\}$ where α is the only eigenvalue with infinite multiplicity. Let $i_1, ..., i_m$ be a permutation of the integers $1, ..., m$ which forces $|\alpha + \mu_{i_1}| \geq ... \geq |\alpha + \mu_{i_m}|$. Let us rename and denote by $v_{\beta_1}, ..., v_{\beta_m}$ the eigenvectors of $|\alpha I + F|$ corresponding to the eigenvalues $|\alpha + \mu_{i_1}|, ..., |\alpha + \mu_{i_m}|$ respectively. We can further rename and denote each $|\alpha + \mu_{i_j}|$ by $|\alpha + \mu_j|$ so that we have $|\alpha + \mu_1| \geq ... \geq |\alpha + \mu_m|$ and a subset $\{\beta_1, ..., \beta_m\} \subseteq \Lambda$ of cardinality m such that for every $j \in \{1, ..., m\}$, we have $|\alpha I + F|(v_{\beta_j}) = |\alpha + \mu_j| v_{\beta_j}$. Notice that $\sup\{|\alpha + \mu_1|, ..., |\alpha + \mu_m|, \alpha\} = \max\{|\alpha + \mu_1|, ..., |\alpha + \mu_m|, \alpha\} = \max\{|\alpha + \mu_1|, \alpha\}$. If $\max\{|\alpha + \mu_1|, ..., |\alpha + \mu_m|, \alpha\} = \alpha$, then we have $\lambda_j(|\alpha I + F|) = \alpha$ for every $j \in \{1, ..., k\}$, in which case, we can choose any k distinct eigenvectors from $B \setminus \{v_{\beta_1}, ..., v_{\beta_m}\}$, say $\{w_{\beta_1}, ..., w_{\beta_k}\}$, so that $\||\alpha I + F|\|_{[k]} = k\alpha = \||\alpha I + F|w_{\beta_1}\| + ... + \||\alpha I + F|w_{\beta_k}\|$ thereby implying that $|\alpha I + F| \in \mathcal{N}_{[k]}(\mathcal{H})$.

Otherwise, we have $|\alpha + \mu_1| = \max\{|\alpha + \mu_1|, ..., |\alpha + \mu_m|, \alpha\}$, so that $\lambda_1(|\alpha I + F|) = |\alpha + \mu_1|$. Further, if $\alpha = \max\{|\alpha + \mu_2|, ..., |\alpha + \mu_m|, \alpha\}$, then we have $\lambda_j(|\alpha I + F|) = \alpha$

for every $j \in \{2, ..., k\}$, in which case, we can choose the eigenvector v_{β_1} and any $k-1$ distinct eigenvectors from $B \setminus \{v_{\beta_2}, ..., v_{\beta_m}\}$, say $\{w_{\beta_2}, ..., w_{\beta_k}\}$ so that $\||\alpha I + F|\|_{[k]} = |\alpha + \mu_1| + (k-1)\alpha = \||\alpha I + F|v_{\beta_1}\| + |\alpha I + F|w_{\beta_2}\| + ... + \||\alpha I + F|w_{\beta_k}\|$ thereby implying that $|\alpha I + F| \in \mathcal{N}_{[k]}$; but if $\alpha \neq \max\{|\alpha + \mu_2|, ..., |\alpha + \mu_m|, \alpha\}$, then we have $|\alpha + \mu_2| = \max\{|\alpha + \mu_2|, ..., |\alpha + \mu_m|, \alpha\}$, so that $\lambda_1(|\alpha I + F|) = |\alpha + \mu_1|$, $\lambda_2(|\alpha I + F|) = |\alpha + \mu_2|$. Then, if $\alpha = \max\{|\alpha + \mu_3|, ..., |\alpha + \mu_m|, \alpha\}$, we get $\lambda_j(|\alpha I + F|) = \alpha$ for every $j \in \{3, ..., k\}$, in which case, we can choose the vectors v_{β_1}, v_{β_2} and any $k-2$ distinct eigenvectors from $B \setminus \{v_{\beta_3}, ..., v_{\beta_m}\}$, say $\{w_{\beta_3}, ..., w_{\beta_k}\}$ which yields $|\alpha I + F| \in \mathcal{N}_{[k]}(\mathcal{H})$. Carrying out this process of selecting appropriate eigenvectors from B depending upon the value $\lambda_j(|\alpha I + F|)$ takes, until we select k of those, establishes the fact that $|\alpha I + F| \in \mathcal{N}_{[k]}(\mathcal{H})$.

Case II: If $k \geq m$. The proof goes the same way except that now we terminate the process once we find a subset $\{\beta_1, ..., \beta_m\} \subseteq \Lambda$ of cardinality m such that for every $j \in \{1, ..., m\}$, we have $|\alpha I + F|(v_{\beta_j}) = |\alpha + \mu_j| v_{\beta_j}$; for $\lambda_j(|\alpha I + F|) = 0$ for $j > m$.

Since $k \in \mathbb{N}$ is arbitrary, it follows that $\alpha I + F \in \mathcal{N}_{[k]}(\mathcal{H})$ for every $k \in \mathbb{N}$. □

Proposition 4.2.18. *Let $K \in \mathcal{B}(\mathcal{H})$ be a positive compact operator, $F \in \mathcal{B}(\mathcal{H})$ be a self-adjoint finite-rank operator, and $\alpha \geq 0$. Then $\alpha I + K + F \in \mathcal{N}_{[k]}(\mathcal{H})$ for every $k \in \mathbb{N}$.*

Proof. The assertion is trivial if $\alpha = 0$; for then $K + F$ is a compact operator which sits in $\mathcal{N}_{[k]}(\mathcal{H})$ for every $k \in \mathbb{N}$. We fix k and assume that $\alpha > 0$. There is no loss of generality in assuming that $\mathcal{H}$ is infinite-dimensional, for if it is not, then the operator is compact and thus belongs to $\mathcal{N}_{[k]}(\mathcal{H})$. We can also assume, without loss of generality, that $\dim(\text{ran } K) > n$ for every $n \in \mathbb{N}$, for if K is a finite-rank operator then the operator is $[k]$-norming by the previous lemma. Due to the equivalence of (1) and (2) of Theorem 4.2.10, it suffices to show that $|\alpha I + K + F| \in \mathcal{N}_{[k]}(\mathcal{H})$.

Notice that $K + F$ is a self-adjoint compact operator on $\mathcal{H}$ and thus there exists an orthonormal basis B of $\mathcal{H}$ consisting entirely of eigenvectors of $K + F$ corresponding to which it is diagonalizable. From [PP17, Lemma 4.8], $K + F$ can have at most finitely many negative eigenvalues. Let $\{\nu_1, \nu_2, ..., \nu_m\}$ be the set of all negative eigenvalues of $K + F$ with $\{v_1, v_2, ..., v_m\}$ as the corresponding eigenvectors in basis B; and let $\{\mu_\beta : \beta \in \Lambda\}$ be the set of all remaining nonnegative eigenvalues of $K + F$ with $\{w_\beta : \beta \in \Lambda\}$ as the corresponding eigenvectors in B. We have $B := \{v_1, v_2, ..., v_n\} \cup \{w_\beta : \beta \in \Lambda\}$ and the matrix $M_B(K+F)$ of $K+F$ with respect to B is given by

$$K+F = \left[\begin{array}{ccccccc} \nu_1 & & & \vdots & & & \\ & \ddots & & \vdots & & \text{\huge 0} & \\ & & \nu_m & \vdots & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & \vdots & \ddots & & \\ & \text{\huge 0} & & \vdots & & \mu_\beta & \\ & & & \vdots & & & \ddots \end{array}\right]$$

Because $K+F$ is compact, the multiplicity of each nonzero eigenvalue is finite and there are at most countably many nonzero eigenvalues, counting multiplicities. In fact, we can safely assume that there are countably infinite nonzero eigenvalues (counting multiplicities) of $K+F$; for if there are only finitely many nonzero eigenvalues, then $K+F$ would be a self-adjoint finite-rank operator which, by Lemma 4.2.17, belongs to $\mathcal{N}_{[k]}(\mathcal{H})$. With this observation, the set $\Gamma := \Lambda \setminus \{\beta \in \Lambda : \mu_\beta = 0\}$ is countably infinite and can be safely replaced by $\mathbb{N}$. This essentially redefines the spectrum $\sigma(K+F) = \{\nu_n\}_{n=1}^m \cup \{\mu_n\}_{n=1}^\infty \cup \{0\}$ of $K+F$ and allows us to enumerate the positive eigenvalues $\{\mu_n\}_{n=1}^\infty$ in nonincreasing order $\mu_1 \geq \mu_2 \geq ...$ so that each eigenvalue appears as many times as is its multiplicity. This ensures that the set of all positive eigenvalues of $K+F$ has been exhausted in the process of constructing the sequence $\{\mu_n\}_{n\in\mathbb{N}}$. That the sequence $\{\mu_n\}$ converges to 0 is a trivial observation. So, 0 is an accumulation point of the spectrum. However, it can also be an eigenvalue with infinite multiplicity. At this point, we rename and denote by $\{v_n\}_{n=1}^m$, $\{w_n\}_{n=1}^\infty$, and $\{z_\beta\}_{\beta\in\Lambda\setminus\Gamma}$ the eigenvectors corresponding to the eigenvalues $\{\nu_n\}_{n=1}^m$, $\{\mu_n\}_{n=1}^\infty$, and $\{0\}$ respectively. With the reordering, we now have

$$B := \{v_n\}_{n=1}^m \cup \{w_n\}_{n=1}^\infty \cup \{z_\beta\}_{\beta\in\Lambda\setminus\Gamma},$$

and the matrix $M_B(K+F)$ of $K+F$ with respect to B is given by

$$K+F = \left[\begin{array}{ccccccc|c|ccc}
\nu_1 & & & \vdots & & & & \vdots & & & \\
& \ddots & & \vdots & & 0 & & \vdots & & & \\
& & \nu_m & \vdots & & & & \vdots & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & & 0 & \\
& & & \vdots & \ddots & & & \vdots & & & \\
& 0 & & \vdots & & \mu_n & & \vdots & & & \\
& & & \vdots & & & \ddots & \vdots & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
& & & & & & & \vdots & & & \\
& & & 0 & & & & \vdots & & 0 & \\
& & & & & & & \vdots & & &
\end{array}\right]$$

We now consider the operator $|\alpha I+K+F|$. With respect to the basis B, the matrix $M_B(|\alpha I+K+F|)$ of $|\alpha I+K+F|$ is given by

$$\left[\begin{array}{ccccccc|c|ccc}
|\alpha+\nu_1| & & & \vdots & & & & \vdots & & & \\
& \ddots & & \vdots & & 0 & & \vdots & & & \\
& & |\alpha+\nu_m| & \vdots & & & & \vdots & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & & 0 & \\
& & & \vdots & \ddots & & & \vdots & & & \\
& 0 & & \vdots & & \alpha+\mu_n & & \vdots & & & \\
& & & \vdots & & & \ddots & \vdots & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
& & & & & & & \vdots & \ddots & & \\
& & & 0 & & & & \vdots & & \alpha & \\
& & & & & & & \vdots & & & \ddots
\end{array}\right].$$

Observe that $\sigma_e(|\alpha I+K+F|)$ of $|\alpha I+K+F|$ is the singleton $\{\alpha\}$ and that for any given $k \in \mathbb{N}$, $\lambda_j(|\alpha I + K + F|) > \alpha$ for every $j \in \{1, ..., k\}$. In fact,

$$\lambda_j(|\alpha I + K + F|) \in \{|\alpha + \nu_n|\}_{n=1}^{m} \cup \{\alpha + \mu_n\}_{n=1}^{\infty} \text{ for every } j \in \{1, ..., k\}.$$

It then immediately follows that there exist k orthogonal eigenvectors in $\{v_n\}_{n=1}^{m} \cup \{w_n\}_{n=1}^{\infty}$ with $\lambda_j(|\alpha I + K + F|), j \in \{1, ..., k\}$ being their correspoding eigenvalues. This proves the assertion. Since $k \in \mathbb{N}$ is arbitrary in the above proof, the propostion holds for every $k \in \mathbb{N}$ and thus an operator of the form $\alpha I + K + F$ belongs to $\mathcal{N}_{[k]}(\mathcal{H})$ for every $k \in \mathbb{N}$. □

This result is the key to the following theorem.

Theorem 4.2.19. *Let $K \in \mathcal{B}(\mathcal{H})$ be a positive compact operator, $F \in \mathcal{B}(\mathcal{H})$ be a self-adjoint finite-rank operator, and $\alpha \geq 0$. Then $\alpha I + K + F \in \mathcal{AN}_{[k]}(\mathcal{H})$ for every $k \in \mathbb{N}$.*

Proof. Let us define $T := \alpha I + K + F$ so that we have $|T| = |\alpha I + K + F|$ and $|T|^*|T| = |T|^2 = (\alpha I + K + F)^2 = (\alpha^2 I) + (2\alpha K + K^2) + (2\alpha F + FK + KF + F^2) = \beta I + \tilde{K} + \tilde{F}$ where $\beta = \alpha^2 \geq 0$, $\tilde{K} = 2\alpha K + K^2$ and $\tilde{F} = 2\alpha F + FK + KF + F^2$ are respectively positive compact and self-adjoint finite-rank operators. Further, let $\mathcal{M}$ be an arbitrary nonempty closed linear subspace of the Hilbert space $\mathcal{H}$ and $V_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{H}$ be the inclusion map from $\mathcal{M}$ to $\mathcal{H}$ defined as $V_{\mathcal{M}}(x) = x$ for each $x \in \mathcal{M}$. We fix $k \in \mathbb{N}$ and observe that

$$\begin{aligned} |T|V_{\mathcal{M}} \in \mathcal{N}_{[k]}(\mathcal{M}, \mathcal{H}) &\iff (|T|V_{\mathcal{M}})^*(|T|V_{\mathcal{M}}) \in \mathcal{N}_{[k]}(\mathcal{M}) \\ &\iff V_{\mathcal{M}}^*(|T|^*|T|)V_{\mathcal{M}} \in \mathcal{N}_{[k]}(\mathcal{M}) \\ &\iff V_{\mathcal{M}}^*(\beta I + \tilde{K} + \tilde{F})V_{\mathcal{M}} \in \mathcal{N}_{[k]}(\mathcal{M}). \end{aligned}$$

It suffices to show that $V_{\mathcal{M}}^*(\beta I + \tilde{K} + \tilde{F})V_{\mathcal{M}} \in \mathcal{N}_{[k]}(\mathcal{M})$; for then, since $\mathcal{M}$ is arbitrary, it immediately follows from lemma 4.1.6 that $|T| \in \mathcal{AN}_{[k]}(\mathcal{H})$ and so does T due to the equivalence of (1) and (2) of Theorem 4.2.10. To this end, notice that $V_{\mathcal{M}}^*(\beta I + \tilde{K} + \tilde{F})V_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M}$ is an operator on $\mathcal{M}$ and

$$V_{\mathcal{M}}^*(\beta I + \tilde{K} + \tilde{F})V_{\mathcal{M}} = V_{\mathcal{M}}^*\beta I V_{\mathcal{M}} + V_{\mathcal{M}}^*\tilde{K}V_{\mathcal{M}} + V_{\mathcal{M}}^*\tilde{F}V_{\mathcal{M}} = \beta I_{\mathcal{M}} + \tilde{K}_{\mathcal{M}} + \tilde{F}_{\mathcal{M}}$$

is the sum of a nonnegative scalar multiple of Identity, a positive compact operator and a self-adjoint finite-rank operator on a Hilbert space $\mathcal{M}$ which, by the preceding proposition, belongs to $\mathcal{N}_{[k]}(\mathcal{M})$. This proves the assertion. Moreover, since $k \in \mathbb{N}$ is arbitrary, the result holds for every $k \in \mathbb{N}$ and thus an operator of the above form belongs to $\mathcal{AN}_{[k]}(\mathcal{H})$ for every $k \in \mathbb{N}$. □

We are now ready to establish the spectral theorem for positive operators that belong to $\mathcal{AN}_{[k]}(\mathcal{H})$ for every $k \in \mathbb{N}$. Note that Theorem 4.2.19 we just proved —— that for every $\alpha \geq 0, \alpha I + K + F \in \mathcal{AN}_{[k]}(\mathcal{H})$ where K and F are respectively positive compact and self-adjoint finite-rank operators —— is the stronger version of the implication (3) $\implies$ (1) in the following spectral theorem for positive absolutely $[k]$-norming operators. If the operator $\alpha I + K + F$ is also positive then the implication can be reversed and the two conditions are equivalent. This is what the next theorem states.

Theorem 4.2.20 (Spectral Theorem for Positive Operators in $\mathcal{AN}_{[k]}(\mathcal{H})$)**.** *If P is a positive operator on $\mathcal{H}$, then the following statements are equivalent.*

(1) $P \in \mathcal{AN}_{[k]}(\mathcal{H})$ for every $k \in \mathbb{N}$.

(2) $P \in \mathcal{AN}_{[k]}(\mathcal{H})$ for some $k \in \mathbb{N}$.

(3) P is of the form $P = \alpha I + K + F$, where $\alpha \geq 0, K$ is a positive compact operator and F is self-adjoint finite-rank operator.

Proof. (1) implies (2) trivially. (2) implies (3) is due to Theorem 4.2.13. (1) follows from (3) due to Theorem 4.2.19. □

4.3 Spectral characterization of operators in $\mathcal{AN}_{[k]}(\mathcal{H}, \mathcal{K})$

In this section we extend the preceding theorem to bounded operators.

Theorem 4.3.1 (Spectral Theorem for Operators in $\mathcal{AN}_{[k]}(\mathcal{H}, \mathcal{K})$)**.** *Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and let $T = U|T|$ be its polar decomposition. Then the following statements are equivalent.*

1. *$T \in \mathcal{AN}_{[k]}(\mathcal{H}, \mathcal{K})$ for every $k \in \mathbb{N}$.*

2. *$T \in \mathcal{AN}_{[k]}(\mathcal{H}, \mathcal{K})$ for some $k \in \mathbb{N}$.*

3. *$|T|$ is of the form $|T| = \alpha I + K + F$, where $\alpha \geq 0, K$ is a positive compact operator and F is self-adjoint finite-rank operator.*

Proof. The proof follows from the Proposition 4.1.7, the polar decomposition theorem and the spectral characterization theorem of positive operators in $\mathcal{AN}_{[k]}(\mathcal{H})$. □

Chapter 5

Characterization of operators in $\mathcal{AN}_{[\pi,k]}(\mathcal{H},\mathcal{K})$

Let $(\pi_j)_{j\in\mathbb{N}}$ be a nonincreasing sequence of positive numbers with $\pi_1 = 1$ and let $k \in \mathbb{N}$. Let $\|\cdot\|_{[\pi,k]} : \mathcal{B}(\mathcal{H},\mathcal{K}) \to [0,\infty)$ be the function defined by

$$\|T\|_{[\pi,k]} = \sum_{j=1}^{k} \pi_j s_j(T) \qquad \text{for every } T \in \mathcal{B}(\mathcal{H},\mathcal{K}).$$

If we choose $(\pi_j)_{j\in\mathbb{N}}$ to be the constant sequence with each term equals to 1, then the function $\|\cdot\|_{[\pi,k]}$ is simply the Ky Fan k-norm $\|\cdot\|_{[k]}$. The above function is thus a generalization of the Ky Fan k-norm on $\mathcal{B}(\mathcal{H},\mathcal{K})$, and it can be shown, although not trivially, that it is also a norm on $\mathcal{B}(\mathcal{H},\mathcal{K})$; the proof relies heavily on the theory of symmetrically-normed ideals and uses properties of symmetric norming functions. For this reason we skip the proof and refer the reader to [GK69, Chapter 3, Lemma 15.1]. This norm, which we shall call the *weighted Ky Fan* π, k*-norm*, is defined to be the weighted sum of the k largest singular values of T, the weights being the first k terms of the sequence $(\pi_j)_{j\in\mathbb{N}}$. If, for instance, $\mathcal{H} = \mathcal{K}$, then the weighted Ky Fan π, k-norm on $\mathcal{B}(\mathcal{H})$ can easily be shown to be a symmetric norm.

This chapter is devoted to the study of the operators in $\mathcal{B}(\mathcal{H},\mathcal{K})$ that are absolutely norming with respect to the weighted Ky Fan π, k-norm. The central goal of this chapter is to present a spectral characterization theorem for the set of such operators.

5.1 The sets $\mathcal{N}_{[\pi,k]}(\mathcal{H},\mathcal{K})$ and $\mathcal{AN}_{[\pi,k]}(\mathcal{H},\mathcal{K})$

Definition 5.1.1. Let $(\pi_j)_{j\in\mathbb{N}}$ be a nonincreasing sequence of positive numbers with $\pi_1 = 1$ and let $k \in \mathbb{N}$. An operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is said to be $[\pi,k]$-*norming* if there are orthonormal elements $x_1, ..., x_k \in \mathcal{H}$ such that $\|T\|_{[\pi,k]} = \|Tx_1\| + \pi_2\|Tx_2\| + ... + \pi_k\|Tx_k\|$. If $\dim(\mathcal{H}) = r < k$, we define T to be $[\pi,k]$-*norming* if there exist orthonormal elements $x_1, ..., x_r \in \mathcal{H}$ such that $\|T\|_{[\pi,k]} = \|Tx_1\| + \pi_2\|Tx_2\| + ... + \pi_r\|Tx_r\|$. We let $\mathcal{N}_{[\pi,k]}(\mathcal{H},\mathcal{K})$ denote the set of $[\pi,k]$-norming operators in $\mathcal{B}(\mathcal{H},\mathcal{K})$.

Definition 5.1.2. Let $(\pi_j)_{j\in\mathbb{N}}$ be a nonincreasing sequence of positive numbers with $\pi_1 = 1$ and let $k \in \mathbb{N}$. An operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is said to be *absolutely* $[\pi,k]$-*norming* if for every nontrivial closed subspace $\mathcal{M}$ of $\mathcal{H}$, $T|_{\mathcal{M}}$ is $[\pi,k]$-norming. We let $\mathcal{AN}_{[\pi,k]}(\mathcal{H},\mathcal{K})$ denote the set of absolutely $[\pi,k]$-norming operators in $\mathcal{B}(\mathcal{H},\mathcal{K})$. Note that $\mathcal{AN}_{[\pi,k]}(\mathcal{H},\mathcal{K}) \subseteq \mathcal{N}_{[\pi,k]}(\mathcal{H},\mathcal{K})$.

Remark 5.1.3. We let Π denote the set of nonincreasing sequences of positive numbers with their first term equal to 1. Every operator on a finite-dimensional Hilbert space is $[\pi,k]$-norming for any $\pi \in \Pi$ and for any $k \in \mathbb{N}$. However, this is not true when the Hilbert space in question is not finite-dimensional. The operator in Example 4.2.5 is one such operator, that is, there exists $\tilde{\pi} = (1,1,1,1,...)$ such that $T \in \mathcal{N}_{[\tilde{\pi},2]}(\ell^2)$ but $T \notin \mathcal{N}_{[\tilde{\pi},3]}(\ell^2)$.

We now mention that Lemma 4.1.6 and Proposition 4.1.7 carries over word for word to operators in $\mathcal{AN}_{[\pi,k]}(\mathcal{H},\mathcal{K})$.

Lemma 5.1.4. *For a closed linear subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ let $V_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{H}$ be the inclusion map from $\mathcal{M}$ to $\mathcal{H}$ defined as $V_{\mathcal{M}}(x) = x$ for each $x \in \mathcal{M}$ and let $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$. For any sequence $\pi \in \Pi$ and for any $k \in \mathbb{N}$, $T \in \mathcal{AN}_{[\pi,k]}(\mathcal{H},\mathcal{K})$ if and only if for every nontrivial closed linear subspace $\mathcal{M}$ of $\mathcal{H}$, $TV_{\mathcal{M}} \in \mathcal{N}_{[\pi,k]}(\mathcal{H},\mathcal{K})$.*

Proof. From the proof of Lemma 4.1.6 we know that for any given nontrivial closed subspace $\mathcal{M}$ of $\mathcal{H}$, the maps $TV_{\mathcal{M}}$ and $T|_{\mathcal{M}}$ are identical and so are their singular values which implies $\|TV_{\mathcal{M}}\|_{[\pi,k]} = \|T|_{\mathcal{M}}\|_{[\pi,k]}$. We next assume that $T \in \mathcal{AN}_{[\pi,k]}(\mathcal{H},\mathcal{K})$ and prove the forward implication. Let $\mathcal{M}$ be an arbitrary but fixed nontrivial closed subspace of $\mathcal{H}$. Either $\dim(\mathcal{M}) = r < k$, in which case, there exist orthonormal elements $x_1, ..., x_r \in \mathcal{M}$ such that $\|T|_{\mathcal{M}}\|_{[\pi,k]} = \|T|_{\mathcal{M}}x_1\| + \pi_2\|T|_{\mathcal{M}}x_2\| + ... + \pi_r\|T|_{\mathcal{M}}x_r\|$ which means that there exist orthonormal elements $x_1, ..., x_r \in \mathcal{M}$ such that

$$\begin{aligned}\|TV_{\mathcal{M}}\|_{[\pi,k]} &= \|T|_{\mathcal{M}}\|_{[\pi,k]} \\ &= \|T|_{\mathcal{M}}x_1\| + \pi_2\|T|_{\mathcal{M}}x_2\| + ... + \pi_r\|T|_{\mathcal{M}}x_r\| \\ &= \|TV_{\mathcal{M}}x_1\| + \pi_2\|TV_{\mathcal{M}}x_2\| + ... + \pi_r\|TV_{\mathcal{M}}x_r\|,\end{aligned}$$

proving that $TV_{\mathcal{M}} \in \mathcal{N}_{[\pi,k]}(\mathcal{M},\mathcal{K})$, or $\dim(\mathcal{M}) \geq k$, in which case, there exist orthonormal elements $x_1, ..., x_k \in \mathcal{M}$ such that $\|T|_{\mathcal{M}}\|_{[\pi,k]} = \|T|_{\mathcal{M}}x_1\| + \pi_2\|T|_{\mathcal{M}}x_2\| ... + \pi_k\|T|_{\mathcal{M}}x_k\|$ which means that there exist orthonormal elements $x_1, ..., x_k \in \mathcal{M}$ such that

$$\begin{aligned}\|TV_{\mathcal{M}}\|_{[\pi,k]} &= \|T|_{\mathcal{M}}\|_{[\pi,k]} \\ &= \|T|_{\mathcal{M}}x_1\| + \pi_2\|T|_{\mathcal{M}}x_2\| + ... + \pi_k\|T|_{\mathcal{M}}x_k\| \\ &= \|TV_{\mathcal{M}}x_1\| + \pi_2\|TV_{\mathcal{M}}x_2\| + ... + \pi_k\|TV_{\mathcal{M}}x_k\|,\end{aligned}$$

proving that $TV_{\mathcal{M}} \in \mathcal{N}_{[\pi,k]}(\mathcal{M},\mathcal{K})$. Since $\mathcal{M}$ is arbitrary, it follows that $TV_{\mathcal{M}} \in \mathcal{N}_{[\pi,k]}(\mathcal{M},\mathcal{K})$ for every $\mathcal{M}$.

We complete the proof by showing that $T \in \mathcal{AN}_{[\pi,k]}(\mathcal{H},\mathcal{K})$ operator if for every nontrivial closed subspace $\mathcal{M}$ of $\mathcal{H}$, the operator $TV_{\mathcal{M}} \in \mathcal{N}_{[\pi,k]}(\mathcal{M},\mathcal{K})$. We again fix $\mathcal{M}$ to be an arbitrary nontrivial closed subspace of $\mathcal{H}$. Since $TV_{\mathcal{M}} \in \mathcal{N}_{[\pi,k]}(\mathcal{M},\mathcal{K})$, either $\dim(\mathcal{M}) = r < k$, in which case, there exist orthonormal elements $x_1, ..., x_r \in \mathcal{M}$ such that $\|TV_{\mathcal{M}}\|_{[\pi,k]} = \|TV_{\mathcal{M}}x_1\| + \pi_2\|TV_{\mathcal{M}}x_2\| + ... + \pi_r\|TV_{\mathcal{M}}x_r\|$ which means that there exist orthonormal elements $x_1, ..., x_r \in \mathcal{M}$ such that

$$\begin{aligned}\|T|_{\mathcal{M}}\|_{[\pi,k]} &= \|TV_{\mathcal{M}}\|_{[\pi,k]} \\ &= \|TV_{\mathcal{M}}x_1\| + \pi_2\|TV_{\mathcal{M}}x_2\| + ... + \pi_r\|TV_{\mathcal{M}}x_r\| \\ &= \|T|_{\mathcal{M}}x_1\| + \pi_2\|T|_{\mathcal{M}}x_1\| + ... + \pi_r\|T|_{\mathcal{M}}x_r\|,\end{aligned}$$

and hence $T|_{\mathcal{M}} \in \mathcal{N}_{[\pi,k]}(\mathcal{M},\mathcal{K})$, or $\dim(\mathcal{M}) \geq k$, in which case, there exist orthonormal elements $x_1, ..., x_k \in \mathcal{M}$ such that $\|TV_{\mathcal{M}}\|_{[\pi,k]} = \|TV_{\mathcal{M}}x_1\| + \pi_2\|TV_{\mathcal{M}}x_2\| + ... + \pi_k\|TV_{\mathcal{M}}x_k\|$ which means that there exist orthonormal elements $x_1, ..., x_k \in \mathcal{M}$ such that

$$\begin{aligned}\|T|_{\mathcal{M}}\|_{[\pi,k]} &= \|TV_{\mathcal{M}}\|_{[\pi,k]} \\ &= \|TV_{\mathcal{M}}x_1\| + \pi_2\|TV_{\mathcal{M}}x_2\| + ... + \pi_k\|TV_{\mathcal{M}}x_k\| \\ &= \|T|_{\mathcal{M}}x_1\| + \pi_2\|T|_{\mathcal{M}}x_2\| + ... + \pi_k\|T|_{\mathcal{M}}x_k\|.\end{aligned}$$

Because $\mathcal{M}$ is arbitrary, $T \in \mathcal{AN}_{[\pi,k]}$. It is worthwhile noticing that since $\pi \in \Pi$ and $k \in \mathbb{N}$ are arbitrary, the assertion holds for every sequence $\pi \in \Pi$ and for every $k \in \mathbb{N}$. $\square$

Proposition 5.1.5. *Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then for every $\pi \in \Pi$ and for every $k \in \mathbb{N}$, $T \in \mathcal{AN}_{[\pi,k]}(\mathcal{H}, \mathcal{K})$ if and only if $|T| \in \mathcal{AN}_{[\pi,k]}(\mathcal{H}, \mathcal{K})$.*

Proof. Let $\mathcal{M}$ be an arbitrary nontrivial closed subspace of $\mathcal{H}$ and let $V_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{H}$ be the inclusion map from $\mathcal{M}$ to $\mathcal{H}$ defined as $V_{\mathcal{M}}(x) = x$ for each $x \in \mathcal{M}$. From the proof of the Proposition 4.1.7, it is easy to see that $|TV_{\mathcal{M}}| = |\,|T|V_{\mathcal{M}}\,|$. Consequently, for every j, $\lambda_j(|TV_{\mathcal{M}}|) = \lambda_j(||T|V_{\mathcal{M}}|)$ and hence $s_j(TV_{\mathcal{M}}) = s_j(|T|V_{\mathcal{M}})$. This implies that for every $\pi \in \Pi$ and for each $k \in \mathbb{N}$, we have

$$\|TV_{\mathcal{M}}\|_{[\pi,k]} = \||T|V_{\mathcal{M}}\|_{[\pi,k]}.$$

Furthermore, for every $x \in \mathcal{H}$, we have $\|TV_{\mathcal{M}}x\| = \||T|V_{\mathcal{M}}x\|$. Since $\mathcal{M}$ is arbitrary, by Lemma 5.1.4 the assertion follows. $\square$

Remark 5.1.6. In what follows, we write $\mathcal{N}_{[\pi,k]}$ instead of $\mathcal{N}_{[\pi,k]}(\mathcal{H}, \mathcal{K})$ and $\mathcal{AN}_{[\pi,k]}$ in place of $\mathcal{AN}_{[\pi,k]}(\mathcal{H}, \mathcal{K})$ for brevity, as long as the domain and codomain spaces are obvious from the context. In a similar vein, we will denote by $\mathcal{N}$, $\mathcal{AN}$, $\mathcal{N}_{[k]}$ and $\mathcal{AN}_{[k]}$ respectively the sets $\mathcal{N}(\mathcal{H}, \mathcal{K})$, $\mathcal{AN}(\mathcal{H}, \mathcal{K})$, $\mathcal{N}_{[k]}(\mathcal{H}, \mathcal{K})$ and $\mathcal{AN}_{[k]}(\mathcal{H}, \mathcal{K})$ whenever no confusion can arise.

5.2 Spectral characterization of positive operators in $\mathcal{AN}_{[\pi,k]}(\mathcal{H})$

This section discusses the necessary and sufficient conditions for a positive operator on a Hilbert space to be absolutely $[\pi, k]$-norming for every $\pi \in \Pi$ and for every $k \in \mathbb{N}$. We first mention an easy proposition, the proof of which is left to the reader.

Proposition 5.2.1. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator. Then the following statements are equivalent.*

(1) $A \in \mathcal{N}(\mathcal{H})$.

(2) $A \in \mathcal{N}_{[\pi,1]}(\mathcal{H})$ *for some* $\pi \in \Pi$.

(3) $A \in \mathcal{N}_{[\pi,1]}(\mathcal{H})$ *for every* $\pi \in \Pi$.

The following result may be considered as an analogue of Proposition 4.2.3 and can be proved in much the same way.

Proposition 5.2.2. *Let $\pi \in \Pi$ and $A \in \mathcal{B}(\mathcal{H})$ be a positive operator. If $s_{m+1}(A) \neq s_m(A)$ for some $m \in \mathbb{N}$, then $A \in \mathcal{N}_{[\pi,m]}(\mathcal{H})$. Moreover, in this case, $A \in \mathcal{N}_{[\pi,m+1]}(\mathcal{H})$ if and only if $s_{m+1}(A)$ is an eigenvalue of A.*

Proof. It is easy to see that for every $j \in \{1, ..., m\}, s_j(A) \notin \sigma_e(A)$. Then the set $\{s_1(A), ..., s_m(A)\}$ consists of eigenvalues (not necessarily distinct) of A, each having finite multiplicity. This guarantees the existence of an orthonormal set $\{v_1, ..., v_m\} \subseteq K \subseteq \mathcal{H}$ such that $Av_j = s_j(A)v_j$ so that $\|Av_j\| = s_j(A)$ and thus $\|A\|_{[\pi,m]} = \sum_{j=1}^m \pi_j s_j(A) = \sum_{j=1}^m \pi_j \|Av_j\|$, where K is the closure of the joint span of the eigenspaces corresponding to the eigenvalues $\{s_1(A), ..., s_m(A)\}$, which implies that $A \in \mathcal{N}_{[\pi,m]}$. Furthermore, we observe that if there exists any orthonormal set $\{w_1, ..., w_m\}$ of m vectors in $\mathcal{H}$ such that $\sum_{i=1}^m \pi_i \|Aw_i\| = \sum_{j=1}^m \pi_j s_j(A)$, then this set has to be contained in K. Note that $K^\perp$ is invariant under A and hence $A|_{K^\perp} : K^\perp \to K^\perp$, viewed as an operator on $K^\perp$, is positive. Since $s_{m+1}(A) \neq s_m(A)$, it follows that $s_{m+1}(A)$ is an eigenvalue of A if and only if $s_{m+1}(A)$ is an eigenvalue of $A|_{K^\perp} : K^\perp \to K^\perp$ which is possible if and only if $A|_{K^\perp} : K^\perp \to K^\perp$, viewed as an operator on $K^\perp$, belongs to $\mathcal{N}$, that is, there is a unit vector $x \in K^\perp$ such that $\|Ax\| = s_{m+1}(A)$, which happens if and only if $\pi_{m+1}\|Ax\| = \pi_{m+1}s_{m+1}(A)$, or equivalently, $A \in \mathcal{N}_{[\pi,m+1]}$. This proves the assertion. □

5.2.1 Necessary conditions for positive operators to belong to $\mathcal{AN}_{[\pi,k]}(\mathcal{H})$

The last proposition in the previous section can be used to establish results analogous to Propositions 4.2.4, 4.2.6 and 4.2.7 (see 5.2.3, 5.2.4, and 5.2.5 respectively) for a given $\pi \in \Pi$ and a given $k \in \mathbb{N}$. The proofs for these are similar so we only prove one of these results and leave the rest for the reader.

Proposition 5.2.3. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator, $\pi \in \Pi$, and $k \in \mathbb{N}$. If $A \in \mathcal{N}_{[\pi,k]}(\mathcal{H})$, then $s_1(A), ..., s_k(A)$ are eigenvalues of A.*

Proof. The proof is by contrapositive. Assuming that at least one of the elements from the set $\{s_1(A), ..., s_k(A)\}$ is not an eigenvalue of A, we show that $A \notin \mathcal{N}_{[\pi,k]}$. Suppose that $s_1(A)$ is not an eigenvalue of A. Then it must be an accumulation point of the spectrum of A in which case none of the singular values of A is an eigenvalue of A and that $s_j(A) = s_1(A)$ for every $j \geq 2$. Since $s_1(A) = \|A\|$, it follows from [PP17, Theorem 2.3] that $A \notin \mathcal{N}$ which means that for every $x \in \mathcal{H}, \|x\| = 1$, we have $\|Ax\| < \|A\| = s_1(A) =$

$s_2(A) = \ldots = s_k(A)$. Consequently, for every orthonormal set $\{x_1, \ldots, x_k\} \subseteq \mathcal{H}$ we have $\sum_{j=1}^{k} \pi_j \|Ax_j\| < \sum_{j=1}^{k} \pi_j s_j(A)$ so that $A \notin \mathcal{N}_{[\pi,k]}$.

Next suppose that $s_1(A)$ is an eigenvalue of A but $s_2(A)$ is not. Clearly then $s_1(A)$ is an eigenvalue with multiplicity 1, $s_2(A) \neq s_1(A)$ and $s_j(A) = s_2(A)$ for every $j \geq 3$ in which case Proposition 5.2.2 ascertains that $A \in \mathcal{N}_{[\pi,1]}$ but $A \notin \mathcal{N}_{[\pi,2]}$. This implies that there exists $y_1 \in \mathcal{H}$ with $\|y_1\| = 1$ such that $\|Ay_1\| = \|A\|$ and for every $y \in \text{span}\{y_1\}^{\perp}$ with $\|y\| = 1$ we have $\|Ay\| < s_2(A)$ which in turn implies that for every orthonormal set $\{y_2, \ldots y_k\} \subseteq \text{span}\{y_1\}^{\perp}$ we have $\sum_{j=2}^{k} \pi_j \|Ay_j\| < \sum_{j=2}^{k} \pi_j s_j(A)$ so that $\sum_{j=1}^{k} \pi_j \|Ay_j\| < \sum_{j=1}^{k} \pi_j s_j(A)$ which implies that $A \notin \mathcal{N}_{[\pi,k]}$.

If $s_1(A), s_2(A)$ are eigenvalues of A but $s_3(A)$ is not, then we have $s_3(A) \neq s_2(A)$ and $s_j(A) = s_3(A)$ for every $j \geq 4$ in which case Proposition 5.2.2 asserts that $A \in \mathcal{N}_{[\pi,2]}$ but $A \notin \mathcal{N}_{[\pi,3]}$. Consequently, there exists an orthonormal set $\{z_1, z_2\} \subseteq \mathcal{H}$ such that $\|Tz_1\| + \pi_2 \|Tz_2\| = \|T\|_{[\pi,2]}$ and that for every unit vector $z \in \text{span}\{z_1, z_2\}^{\perp}$ we have $\|Tz\| < s_3(A)$ which in turn implies that for every orthonormal set $\{z_3, \ldots z_k\} \subseteq \text{span}\{z_1, z_2\}^{\perp}$ we have $\sum_{j=3}^{k} \pi_j \|Az_j\| < \sum_{j=3}^{k} \pi_j s_j(A)$. It then follows that $\sum_{j=1}^{k} \pi_j \|Az_j\| < \sum_{j=1}^{k} \pi_j s_j(A)$ for every orthonormal set $\{z_1, \ldots, z_k\} \subseteq \mathcal{H}$ which implies that $A \notin \mathcal{N}_{[\pi,k]}$.

If we continue in this way, we can show at every step that $A \notin \mathcal{N}_{[\pi,k]}$. We conclude the proof by discussing the final case when $s_1(A), \ldots, s_{k-1}(A)$ are all eigenvalues of A but $s_k(A)$ is not in which case $s_k(A) \neq s_{k-1}(A)$ and thus Proposition 5.2.2 again implies that $A \notin \mathcal{N}_{[\pi,k]}$. This exhausts all the possibilities and the assertion is thus proved contrapositively. $\square$

Proposition 5.2.4. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator, $\pi \in \Pi$, and $k \in \mathbb{N}$. If $s_1(A), \ldots, s_k(A)$ are mutually distinct eigenvalues of A, then there exists an orthonormal set $\{v_1, \ldots, v_k\} \subseteq \mathcal{H}$ such that $Av_j = s_j(A)v_j$ for every $j \in \{1, \ldots, k\}$. Thus $A \in \mathcal{N}_{[\pi,k]}$.*

Proposition 5.2.5. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator, $\pi \in \Pi$, $k \in \mathbb{N}$ and let $s_1(A), \ldots, s_k(A)$ be the first k singular values of A that are also the eigenvalues of A and are not necessarily distinct. Then either $s_1(A) = \ldots = s_k(A)$, in which case, $A \in \mathcal{N}_{[\pi,k]}$ if and only if the multiplicity of $\alpha := s_1(A)$ is at least k; or there exists $t \in \{2, \ldots, k\}$ such that $s_{t-1}(A) \neq s_t(A) = s_{t+1}(A) = \ldots = s_k(A)$, in which case, $A \in \mathcal{N}_{[\pi,k]}$ if and only if the multiplicity of $\beta := s_t(A)$ is at least $k - t + 1$.*

The above propositions leads us to prove the following result that adds another equivalent condition to the Theorem 4.2.8.

Theorem 5.2.6. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator, $\pi \in \Pi$, and $k \in \mathbb{N}$. Then the following statements are equivalent.*

1. $A \in \mathcal{N}_{[k]}(\mathcal{H})$.

2. $A \in \mathcal{N}_{[\pi,k]}(\mathcal{H})$.

3. $s_1(A), ..., s_k(A)$ *are eigenvalues of* A *and there exists an orthonormal set* $\{v_1, ..., v_k\} \subseteq \mathcal{H}$ *such that* $Av_j = s_j(A)v_j$ *for every* $j \in \{1, ..., k\}$.

Proof. (1) $\iff$ (3) has been established in Theorem 4.2.8 and (3) $\implies$ (2) is trivial. To establish (2) $\implies$ (3), note that by the Proposition 5.2.3, $s_1(A), ..., s_k(A)$ are all eigenvalues of A. If $s_1(A), ..., s_k(A)$ are mutually distinct eigenvalues, then by Proposition 5.2.4 there exists an orthonormal set $\{v_1, ..., v_k\} \subseteq \mathcal{H}$ such that $Av_j = s_j(A)v_j$ for every $j \in \{1, ..., k\}$. However, if $s_1(A), ..., s_k(A)$ are all eigenvalues but not necessarily distinct then also the existence of an orthonormal set $\{v_1, ..., v_k\} \subseteq \mathcal{H}$ with $Av_j = s_j(A)v_j$ for every $j \in \{1, ..., k\}$ is guaranteed by the Proposition 5.2.5. This completes the proof. □

The above theorem leads us immediately to the following rather obvious corollary.

Corollary 5.2.7. *Let* $A \in \mathcal{B}(\mathcal{H})$ *be a positive operator,* $\pi \in \Pi$, *and* $k \in \mathbb{N}$.

1. *If* $A \in \mathcal{N}_{[\pi,k+1]}(\mathcal{H})$, *then* $A \in \mathcal{N}_{[\pi,k]}(\mathcal{H})$.

2. *If* $A \in \mathcal{N}_{[\pi,k]}(\mathcal{H})$, *then* $A \in \mathcal{N}(\mathcal{H})$.

Theorem 4.2.10 extends word for word to the set $\mathcal{N}_{[\pi,k]}$ (see 5.2.8) and Theorem 4.2.11 alongwith the Corollary 4.2.12 extend to the set $\mathcal{AN}_{[\pi,k]}$ (see 5.2.9 and 5.2.10 respectively).

Theorem 5.2.8. *Let* $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $\pi \in \Pi$, *and* $k \in \mathbb{N}$. *Then the following statements are equivalent.*

1. $T \in \mathcal{N}_{[\pi,k]}(\mathcal{H}, \mathcal{K})$.

2. $|T| \in \mathcal{N}_{[\pi,k]}(\mathcal{H})$.

3. $T^*T \in \mathcal{N}_{[\pi,k]}(\mathcal{H})$.

Proof. It suffices to establish (1) $\iff$ (2); for then $|T|$ and T^*T are positive and since the sets $\mathcal{N}_{[k]}$ and $\mathcal{N}_{[\pi,k]}$ coincide for positive operators, Theorem 4.2.10 yields the equivalence of (2) and (3). But $s_j(T) = s_j(|T|)$ for every j and $\|Tx\| = \|\,|T|x\,\|$ for every $x \in \mathcal{H}$ which establishes the equivalence of (1) and (2). □

Theorem 5.2.9. *Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $\pi \in \Pi$, and $k \in \mathbb{N}$. If $T \in \mathcal{AN}_{[\pi,k+1]}(\mathcal{H}, \mathcal{K})$, then $T \in \mathcal{AN}_{[\pi,k]}(\mathcal{H}, \mathcal{K})$.*

The proof of the above theorem is similar to that of the Theorem 4.2.11 and hence omitted. It yields the following obvious corollary.

Corollary 5.2.10. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator, $\pi \in \Pi$, and $k \in \mathbb{N}$. If $A \in \mathcal{AN}_{[\pi,k+1]}(\mathcal{H})$, then $A \in \mathcal{AN}_{[\pi,k]}(\mathcal{H})$. In particular, if $A \in \mathcal{AN}_{[\pi,k]}(\mathcal{H})$, then $A \in \mathcal{AN}(\mathcal{H})$.*

The above corollary along with the forward implication of [PP17, Theorem 5.1] yields the following theorem.

Theorem 5.2.11. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator on $\mathcal{H}$, $\pi \in \Pi$, and $k \in \mathbb{N}$. If $A \in \mathcal{AN}_{[\pi,k]}(\mathcal{H})$, then A is of the form $A = \alpha I + K + F$, where $\alpha \geq 0, K$ is a positive compact operator and F is self-adjoint finite-rank operator.*

The following theorem is an analogue of Theorem 4.2.14 and its proof may be handled in much the same way. This result will not be needed until Chapter 8.

Theorem 5.2.12. *For a closed linear subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ let $V_{\mathcal{M}} : \mathcal{M} \to \mathcal{H}$ be the inclusion map from $\mathcal{M}$ to $\mathcal{H}$ defined as $V_{\mathcal{M}}(x) = x$ for each $x \in \mathcal{M}$, let $P_{\mathcal{M}} \in \mathcal{B}(\mathcal{H})$ be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$, and let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. For any sequence $\pi \in \Pi$ and for any $k \in \mathbb{N}$, the following statements are equivalent.*

1. *$T \in \mathcal{AN}_{[\pi,k]}(\mathcal{H}, \mathcal{K})$.*
2. *$TV_{\mathcal{M}} \in \mathcal{N}_{[\pi,k]}(\mathcal{M}, \mathcal{K})$ for every nontrivial closed linear subspace $\mathcal{M}$ of $\mathcal{H}$.*
3. *$TP_{\mathcal{M}} \in \mathcal{N}_{[\pi,k]}(\mathcal{H}, \mathcal{K})$ for every nontrivial closed linear subspace $\mathcal{M}$ of $\mathcal{H}$.*

5.2.2 Sufficient conditions for positive operators to belong to $\mathcal{AN}_{[\pi,k]}(\mathcal{H})$

We now mention the sufficient conditions for a positive operator to be absolutely $[\pi, k]$-norming for every $\pi \in \Pi$ and for every $k \in \mathbb{N}$. We begin by stating a proposition that gives a sufficient condition for a positive operator to be $[\pi, k]$-norming for every $\pi \in \Pi$ and for every $k \in \mathbb{N}$, the proof of which is easy to see.

Proposition 5.2.13. *Let $K \in \mathcal{B}(\mathcal{H})$ be a positive compact operator, $F \in \mathcal{B}(\mathcal{H})$ be a self-adjoint finite-rank operator, and $\alpha \geq 0$ such that $\alpha I + K + F \geq 0$. Then $\alpha I + K + F \in \mathcal{N}_{[\pi,k]}(\mathcal{H})$ for every $\pi \in \Pi$ and for every $k \in \mathbb{N}$.*

This proposition serves to be the key to the following theorem.

Theorem 5.2.14. *Let $K \in \mathcal{B}(\mathcal{H})$ be a positive compact operator, $F \in \mathcal{B}(\mathcal{H})$ be a self-adjoint finite-rank operator, and $\alpha \geq 0$ such that $\alpha I + K + F \geq 0$. Then $\alpha I + K + F \in \mathcal{AN}_{[\pi,k]}(\mathcal{H})$ for every $\pi \in \Pi$ and for every $k \in \mathbb{N}$.*

Proof. Let us fix $\pi \in \Pi$ and $k \in \mathbb{N}$, and let us define $T := \alpha I + K + F$. Due to Proposition 5.1.5, $T \in \mathcal{AN}_{[\pi,k]}$ if and only if $|T| \in \mathcal{AN}_{[\pi,k]}$, which due to Lemma 5.1.4, is possible if and only if for every nontrivial closed linear subspace $\mathcal{M}$ of $\mathcal{H}$, $|T|V_{\mathcal{M}} \in \mathcal{N}_{[\pi,k]}$, where $V_{\mathcal{M}} : \mathcal{M} \to \mathcal{H}$ is the inclusion map defined as $V_{\mathcal{M}}(x) = x$ for each $x \in \mathcal{M}$. We show the last of these equivalent statements.

Notice that $|T| = |\alpha I + K + F|$ and $|T|^*|T| = \beta I + \tilde{K} + \tilde{F}$ where $\beta = \alpha^2 \geq 0$, and, $\tilde{K} = 2\alpha K + K^2$ and $\tilde{F} = 2\alpha F + FK + KF + F^2$ are respectively positive compact and self-adjoint finite-rank operators. It is easy to see that $\beta I + \tilde{K} + \tilde{F} \geq 0$. Next we fix a closed linear subspace $\mathcal{M}$ of $\mathcal{H}$ and observe that

$$\begin{aligned} & |T|V_{\mathcal{M}} \in \mathcal{N}_{[\pi,k]} \\ \iff & (|T|V_{\mathcal{M}})^*(|T|V_{\mathcal{M}}) \in \mathcal{N}_{[\pi,k]} \\ \iff & V_{\mathcal{M}}^*(|T|^*|T|)V_{\mathcal{M}} \in \mathcal{N}_{[\pi,k]} \\ \iff & V_{\mathcal{M}}^*(\beta I + \tilde{K} + \tilde{F})V_{\mathcal{M}} \in \mathcal{N}_{[\pi,k]}, \end{aligned}$$

where the first equivalence is due to the Theorem 5.2.8. It suffices to show that $V_{\mathcal{M}}^*(\beta I + \tilde{K} + \tilde{F})V_{\mathcal{M}} \in \mathcal{N}_{[\pi,k]}$;for then, since $\mathcal{M}$ is arbitrary, the assertion immediately follows. To this end, notice that $V_{\mathcal{M}}^*(\beta I + \tilde{K} + \tilde{F})V_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is an operator on $\mathcal{M}$ and

$$V_{\mathcal{M}}^*(\beta I + \tilde{K} + \tilde{F})V_{\mathcal{M}} = V_{\mathcal{M}}^*\beta I V_{\mathcal{M}} + V_{\mathcal{M}}^*\tilde{K}V_{\mathcal{M}} + V_{\mathcal{M}}^*\tilde{F}V_{\mathcal{M}} = \beta I_{\mathcal{M}} + \tilde{K}_{\mathcal{M}} + \tilde{F}_{\mathcal{M}}$$

is the sum of a nonnegative scalar multiple of Identity, a positive compact operator and a self-adjoint finite-rank operator on the fixed Hilbert space $\mathcal{M}$ such that this sum is a positive operator on this Hilbert space $\mathcal{M}$ which, by the preceding proposition, belongs to $\mathcal{N}_{[\pi,k]}$. Moreover, since $\pi \in \Pi$ and $k \in \mathbb{N}$ are arbitrary, the result holds for every $\pi \in \Pi$ and for every $k \in \mathbb{N}$. This completes the proof. $\square$

As an immediate consequence of the Theorem 5.2.11 and Theorem 5.2.14, we get the following theorem which completely characterizes positive operators that are absolutely $[\pi, k]$-norming for any and every $\pi \in \Pi$ and $k \in \mathbb{N}$.

Theorem 5.2.15 (Spectral Theorem for Positive Operators in $\mathcal{AN}_{[\pi,k]}(\mathcal{H})$)**.** *Let P be a positive operator on $\mathcal{H}$. Then the following statements are equivalent.*

1. *$P \in \mathcal{AN}_{[\pi,k]}(\mathcal{H})$ for every $\pi \in \Pi$ and for every $k \in \mathbb{N}$.*
2. *$P \in \mathcal{AN}_{[\pi,k]}(\mathcal{H})$ for some $\pi \in \Pi$ and for some $k \in \mathbb{N}$.*
3. *P is of the form $P = \alpha I + K + F$, where $\alpha \geq 0, K$ is a positive compact operator and F is self-adjoint finite-rank operator.*

At this point, readers can move on to the result (see Theorem 5.3.1) in the next section which completes the proposed motive of characterizing operators on Hilbert spaces that attain their weighted Ky Fan π, k-norm on every closed subspace. However, it is perhaps worth a short digression to address the following question before closing this section: What can be said along the lines of Theorem 5.2.14 in the case of an operator in the same form of $\alpha I + K + F$ which is not necessarily positive? We still have our other hypotheses, that is, $K \in \mathcal{B}(\mathcal{H})$ is a positive compact operator, $F \in \mathcal{B}(\mathcal{H})$ is self-adjoint operator, and $\alpha \geq 0$. We address this question in the Proposition 5.2.18, the proof of which is left to the reader. The proof essentially requires the following lemma.

Lemma 5.2.16. *Let $K \in \mathcal{B}(\mathcal{H})$ be a positive compact operator, $F \in \mathcal{B}(\mathcal{H})$ be a self-adjoint finite-rank operator, and $\alpha \geq 0$. Then $\alpha I + K + F \in \mathcal{N}_{[\pi,k]}(\mathcal{H})$ for every $\pi \in \Pi$ and for every $k \in \mathbb{N}$.*

Proof. Fix $\pi \in \Pi$ and $k \in \mathbb{N}$. For any operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ we observe that

$$\begin{aligned} T \in \mathcal{N}_{[k]} &\iff |T| \in \mathcal{N}_{[k]} \\ &\iff |T| \in \mathcal{N}_{[\pi,k]} \\ &\iff T \in \mathcal{N}_{[\pi,k]}, \end{aligned}$$

where the first equivalence is due to the Theorem 4.2.10, the second equivalence is due to the Theorem 5.2.6 and the last equivalence is due to the Theorem 5.2.8. This observation when applied to the Proposition 4.2.18 proves that $\alpha I + K + F \in \mathcal{N}_{[\pi,k]}$. Since $\pi \in \Pi$ and $k \in \mathbb{N}$ are arbitrary, it follows that $\alpha I + K + F \in \mathcal{N}_{[\pi,k]}$ for every $\pi \in \Pi$ and for every $k \in \mathbb{N}$ thereby proving the assertion. ☐

Remark 5.2.17. The proof of the Lemma 5.2.16 uses a rather interesting result which deserves to be stated for its intrinsic interest. If $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $\pi \in \Pi$, and $k \in \mathbb{N}$, then

$$T \in \mathcal{N}_{[k]}(\mathcal{H}, \mathcal{K}) \iff T \in \mathcal{N}_{[\pi,k]}(\mathcal{H}, \mathcal{K}).$$

Proposition 5.2.18. *Let $K \in \mathcal{B}(\mathcal{H})$ be a positive compact operator, $F \in \mathcal{B}(\mathcal{H})$ be a self-adjoint finite-rank operator, and $\alpha \geq 0$. Then $\alpha I + K + F \in \mathcal{AN}_{[\pi,k]}(\mathcal{H})$ for every $\pi \in \Pi$ and for every $k \in \mathbb{N}$.*

Proof. This proof is very similar to the proof of the Theorem 5.2.14. As before, let us define $T := \alpha I + K + F$. We need to show that $T \in \mathcal{AN}_{[\pi,k]}$ for every $\pi \in \Pi$ and for every $k \in \mathbb{N}$. Let us fix $\pi \in \Pi$ and $k \in \mathbb{N}$. The Proposition 5.1.5, together with the Lemma 5.1.4 shows that it suffices to show that for every nontrivial closed linear subspace $\mathcal{M}$ of $\mathcal{H}$, $|T|V_{\mathcal{M}} \in \mathcal{N}_{[\pi,k]}$, where $V_{\mathcal{M}} : \mathcal{M} \to \mathcal{H}$ is the inclusion map as defined earlier. Next we fix a closed linear subspace $\mathcal{M}$ of $\mathcal{H}$ and observe that

$$\begin{aligned} &|T|V_{\mathcal{M}} \in \mathcal{N}_{[\pi,k]} \\ \iff &V_{\mathcal{M}}^*(\beta I + \tilde{K} + \tilde{F})V_{\mathcal{M}} \in \mathcal{N}_{[\pi,k]}, \end{aligned}$$

where $\beta I + \tilde{K} + \tilde{F} = |T|^*|T|$ with $\beta = \alpha^2 \geq 0$, $\tilde{K} = 2\alpha K + K^2$ and $\tilde{F} = 2\alpha F + FK + KF + F^2$. All that remains to be shown is that $V_{\mathcal{M}}^*(\beta I + \tilde{K} + \tilde{F})V_{\mathcal{M}} \in \mathcal{N}_{[\pi,k]}$; for then, since $\mathcal{M}$ is arbitrary, the assertion immediately follows. To this end, notice that $V_{\mathcal{M}}^*(\beta I + \tilde{K} + \tilde{F})V_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is an operator on $\mathcal{M}$ and $V_{\mathcal{M}}^*(\beta I + \tilde{K} + \tilde{F})V_{\mathcal{M}} = \beta I_{\mathcal{M}} + \tilde{K}_{\mathcal{M}} + \tilde{F}_{\mathcal{M}}$ is the sum of a nonnegative scalar multiple of Identity, a positive compact operator and a self-adjoint finite-rank operator on the fixed Hilbert space $\mathcal{M}$ which, by the preceding lemma, belongs to $\mathcal{N}_{[\pi,k]}$. Since $\pi \in \Pi$ and $k \in \mathbb{N}$ are arbitrary, the result holds for every $\pi \in \Pi$ and for every $k \in \mathbb{N}$ and thus an operator of the above form belongs to $\mathcal{AN}_{[\pi,k]}$ for every $\pi \in \Pi$ and for every $k \in \mathbb{N}$. This completes the proof. ☐

5.3 Spectral characterization of operators in $\mathcal{AN}_{[\pi,k]}(\mathcal{H}, \mathcal{K})$

By Proposition 5.1.5, the polar decomposition theorem (see Theorem 2.5.1) and the spectral theorem for positive operators in $\mathcal{AN}_{[\pi,k]}(\mathcal{H})$ (see Theorem 5.2.15), we can safely consider the following theorem to be fully proved.

Theorem 5.3.1 (Spectral Theorem for Operators in $\mathcal{AN}_{[\pi,k]}(\mathcal{H}, \mathcal{K})$)**.** *Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and let $T = U|T|$ be its polar decomposition. Then the following statements are equivalent.*

1. *$T \in \mathcal{AN}_{[\pi,k]}(\mathcal{H}, \mathcal{K})$ for every $\pi \in \Pi$ and for every $k \in \mathbb{N}$.*
2. *$T \in \mathcal{AN}_{[\pi,k]}(\mathcal{H}, \mathcal{K})$ for some $\pi \in \Pi$ and for some $k \in \mathbb{N}$.*
3. *$|T|$ is of the form $|T| = \alpha I + K + F$, where $\alpha \geq 0, K$ is a positive compact operator and F is self-adjoint finite-rank operator.*

Chapter 6

Characterization of operators in $\mathcal{AN}_{(p,k)}(\mathcal{H},\mathcal{K})$

In the preceding chapter we studied operators in $\mathcal{AN}_{[\pi,k]}(\mathcal{H},\mathcal{K})$ and obtained, in particular, a spectral characterization theorem (Theorem 5.3.1) for these operators. As we indicated, this result can be viewed as an appropriate generalization of the absolutely norming property from the set of Ky Fan k-norms to the set $\{\|\cdot\|_{[\pi,k]} : \pi \in \Pi, k \in \mathbb{N}\}$ of weighted Ky Fan π, k-norms. There is another set of norms on $\mathcal{B}(\mathcal{H},\mathcal{K})$ which are a generalization of the set of Ky Fan k-norms. In this chapter we introduce and study these norms and the operators which are absolutely norming with respect to these norms. Continuing in the same spirit, as that of the previous two chapters, we obtain a spectral characterization theorem for such operators.

6.1 The sets $\mathcal{N}_{(p,k)}(\mathcal{H},\mathcal{K})$ and $\mathcal{AN}_{(p,k)}(\mathcal{H},\mathcal{K})$

Govind S. Mudholkar and Marshall Freimer focussed on a particular class of norms in [MF85] — the vector p norm of the first k singular values — and found specific results about these norms. Nathaniel Johnston, in one of his blogs *Ky Fan Norms, Schatten Norms, and Everything in Between*, discusses these norms as the natural generalization of two well known families of norms, the Ky Fan norms and the Schatten norms. He coined the term "(p,k)-singular norm" for this class of norms.

Definition 6.1.1 ([MF85]). Let $p \in [1,\infty)$ and let $k \in \mathbb{N}$. The (p,k)-*singular norm* $\|\cdot\|_{(p,k)}$ of an operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is defined to be the vector p norm of the k largest singular values of T, that is,

$$\|T\|_{(p,k)} = \left(\sum_{j=1}^{k} s_j^p(T) \right)^{1/p}.$$

Remark 6.1.2. The (p,k)-singular norm on $\mathcal{B}(\mathcal{H},\mathcal{K})$ is, indeed, a norm. When $\mathcal{K} = \mathcal{H}$, it can be shown that this norm is symmetric. If we choose $p = 1$, then the $(1,k)$-singular norm $\|\cdot\|_{(1,k)}$ is simply the Ky Fan k-norm $\|\cdot\|_{[k]}$. If in addition, we also choose $k = 1$, we get the usual operator norm.

Definition 6.1.3. Let $p \in [1,\infty)$ and let $k \in \mathbb{N}$. An operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is said to be (p,k)-*norming* if there are orthonormal elements $x_1, ..., x_k \in \mathcal{H}$ such that

$$\|T\|_{(p,k)}^p = \sum_{j=1}^{k} \|Tx_j\|^p.$$

If $\dim(\mathcal{H}) = r < k$, we define T to be (p,k)-norming if there exist orthonormal elements $x_1, ..., x_r \in \mathcal{H}$ such that $\|T\|_{(p,k)}^p = \sum_{j=1}^{r} \|Tx_j\|^p$. We let $\mathcal{N}_{(p,k)}(\mathcal{H},\mathcal{K})$ denote the set of (p,k)-norming operators in $\mathcal{B}(\mathcal{H},\mathcal{K})$.

Definition 6.1.4. Let $p \in [1,\infty)$ and let $k \in \mathbb{N}$. An operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is said to be *absolutely* (p,k)-*norming* if for every nontrivial closed subspace $\mathcal{M}$ of $\mathcal{H}$, $T|_{\mathcal{M}}$ is (p,k)-norming. We let $\mathcal{AN}_{(p,k)}(\mathcal{H},\mathcal{K})$ denote the set of absolutely (p,k)-norming operators in $\mathcal{B}(\mathcal{H},\mathcal{K})$. Note that $\mathcal{AN}_{(p,k)}(\mathcal{H},\mathcal{K}) \subseteq \mathcal{N}_{(p,k)}(\mathcal{H},\mathcal{K})$.

Remark 6.1.5. Every operator on a finite-dimensional Hilbert space is (p,k)-norming for each $p \in [1,\infty)$ and for each $k \in \mathbb{N}$. However, this is not true when the Hilbert space in question is not finite-dimensional. The operator T in Example 4.2.5 is one such operator, as it can be easily shown that $T \notin \mathcal{N}_{(1,3)}(\ell^2)$.

The following lemma can be considered as an analogue of Lemma 4.1.6.

Lemma 6.1.6. *For a closed linear subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ let $V_{\mathcal{M}} : \mathcal{M} \to \mathcal{H}$ be the inclusion map from $\mathcal{M}$ to $\mathcal{H}$ defined as $V_{\mathcal{M}}(x) = x$ for each $x \in \mathcal{M}$ and let $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$. For any real number $p \in [1,\infty)$ and for any $k \in \mathbb{N}$, $T \in \mathcal{AN}_{(p,k)}(\mathcal{H},\mathcal{K})$ if and only if for every nontrivial closed linear subspace $\mathcal{M}$ of $\mathcal{H}$, $TV_{\mathcal{M}} \in \mathcal{N}_{(p,k)}(\mathcal{H},\mathcal{K})$.*

Proof. To prove this assertion we first observe that for any given nontrivial closed subspace $\mathcal{M}$ of $\mathcal{H}$, the maps $TV_{\mathcal{M}}$ and $T|_{\mathcal{M}}$ are identical and so are their singular values which implies $\|TV_{\mathcal{M}}\|_{(p,k)} = \|T|_{\mathcal{M}}\|_{(p,k)}$. We next assume that $T \in \mathcal{AN}_{(p,k)}(\mathcal{H},\mathcal{K})$ and prove the forward implication. Let $\mathcal{M}$ be an arbitrary but fixed nontrivial closed subspace of $\mathcal{H}$. Either $\dim(\mathcal{M}) = r < k$, in which case, there exist orthonormal elements $x_1, ..., x_r \in \mathcal{M}$ such that $\|T|_{\mathcal{M}}\|^p_{(p,k)} = \sum_{j=1}^r \|T|_{\mathcal{M}} x_j\|^p$ which means that there exist orthonormal elements $x_1, ..., x_r \in \mathcal{M}$ such that $\|TV_{\mathcal{M}}\|^p_{(p,k)} = \|T|_{\mathcal{M}}\|^p_{(p,k)} = \sum_{j=1}^r \|T|_{\mathcal{M}} x_j\|^p = \sum_{j=1}^r \|TV_{\mathcal{M}} x_j\|^p$ proving that $TV_{\mathcal{M}} \in \mathcal{N}_{(p,k)}(\mathcal{M},\mathcal{K})$, or $\dim(\mathcal{M}) \geq k$, in which case, there exist orthonormal elements $x_1, ..., x_k \in \mathcal{M}$ such that $\|T|_{\mathcal{M}}\|^p_{(p,k)} = \sum_{j=1}^k \|T|_{\mathcal{M}} x_j\|^p$ which means that there exist orthonormal elements $x_1, ..., x_k \in \mathcal{M}$ such that $\|TV_{\mathcal{M}}\|^p_{(p,k)} = \|T|_{\mathcal{M}}\|^p_{(p,k)} = \sum_{j=1}^k \|T|_{\mathcal{M}} x_j\|^p = \sum_{j=1}^k \|TV_{\mathcal{M}} x_j\|^p$ proving that $TV_{\mathcal{M}} \in \mathcal{N}_{(p,k)}(\mathcal{M},\mathcal{K})$. Since $\mathcal{M}$ is arbitrary, it follows that $TV_{\mathcal{M}} \in \mathcal{N}_{(p,k)}(\mathcal{M},\mathcal{K})$.

We complete the proof by showing that T is an $\mathcal{AN}_{(p,k)}(\mathcal{H},\mathcal{K})$ operator if $TV_{\mathcal{M}} \in \mathcal{N}_{(p,k)}(\mathcal{M},\mathcal{K})$ for every nontrivial closed subspace $\mathcal{M}$ of $\mathcal{H}$. We again fix $\mathcal{M}$ to be an arbitrary nontrivial closed subspace of $\mathcal{H}$. Since $TV_{\mathcal{M}} \in \mathcal{N}_{(p,k)}(\mathcal{M},\mathcal{K})$, either $\dim(\mathcal{M}) = r < k$, in which case, there exist orthonormal elements $x_1, ..., x_r \in \mathcal{M}$ such that $\|TV_{\mathcal{M}}\|^p_{(p,k)} = \sum_{j=1}^r \|TV_{\mathcal{M}} x_j\|^p$ which means that there exist orthonormal elements $x_1, ..., x_r \in \mathcal{M}$ such that $\|T|_{\mathcal{M}}\|^p_{(p,k)} = \|TV_{\mathcal{M}}\|^p_{(p,k)} = \sum_{j=1}^r \|TV_{\mathcal{M}} x_j\|^p = \sum_{j=1}^r \|T|_{\mathcal{M}} x_j\|^p$ proving that $T|_{\mathcal{M}} \in \mathcal{N}_{(p,k)}(\mathcal{M},\mathcal{K})$, or $\dim(\mathcal{M}) \geq k$, in which case, there exist orthonormal elements $x_1, ..., x_k \in \mathcal{M}$ such that $\|TV_{\mathcal{M}}\|^p_{(p,k)} = \sum_{j=1}^k \|TV_{\mathcal{M}} x_j\|^p$ which means that there exist orthonormal elements $x_1, ..., x_k \in \mathcal{M}$ such that $\|T|_{\mathcal{M}}\|^p_{(p,k)} = \|TV_{\mathcal{M}}\|^p_{(p,k)} = \sum_{j=1}^k \|TV_{\mathcal{M}} x_j\|^p = \sum_{j=1}^k \|T|_{\mathcal{M}} x_j\|^p$ proving that $T|_{\mathcal{M}} \in \mathcal{N}_{(p,k)}(\mathcal{M},\mathcal{K})$. Because $\mathcal{M}$ is arbitrary, this essentially guarantees that $T \in \mathcal{AN}_{(p,k)}(\mathcal{H},\mathcal{K})$. It is worthwhile noticing that since $p \in [1,\infty)$ and $k \in \mathbb{N}$ are arbitrary, the assertion holds for every $p \in [1,\infty)$ and for every $k \in \mathbb{N}$. $\square$

Without going into details, we mention that Proposition 4.1.7 carries over word for word to operators in $\mathcal{AN}_{(p,k)}(\mathcal{H},\mathcal{K})$.

Proposition 6.1.7. *Let $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$. Then for every $p \in [1,\infty)$ and for every $k \in \mathbb{N}$, $T \in \mathcal{AN}_{(p,k)}(\mathcal{H},\mathcal{K})$ if and only if $|T| \in \mathcal{AN}_{(p,k)}(\mathcal{H},\mathcal{K})$.*

Remark 6.1.8. By abuse of notation, we continue to write $\mathcal{N}_{(p,k)}$ and $\mathcal{AN}_{(p,k)}$ respectively for the sets $\mathcal{N}_{(p,k)}(\mathcal{H},\mathcal{K})$ and $\mathcal{AN}_{(p,k)}(\mathcal{H},\mathcal{K})$ whenever the domain and codomain spaces are obvious from the context.

6.2 Spectral characterization of positive operators in $\mathcal{AN}_{(p,k)}(\mathcal{H})$

This section discusses the necessary and sufficient conditions for a positive operator on Hilbert space of arbitrary dimension to be absolutely (p,k)-norming for every $p \in [1,\infty)$ and for every $k \in \mathbb{N}$. We state the following proposition that adds few equivalent conditions to the Theorem 5.2.1

Proposition 6.2.1. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator. Then the following statements are equivalent.*

1. *$A \in \mathcal{N}(\mathcal{H})$.*
2. *$A \in \mathcal{N}_{(p,1)}(\mathcal{H})$ for some $p \in [1,\infty)$.*
3. *$A \in \mathcal{N}_{(p,1)}(\mathcal{H})$ for every $p \in [1,\infty)$.*

The following proposition is analogous to Proposition 4.2.3 and can be proved in much the same way.

Proposition 6.2.2. *Let $p \in [1,\infty)$ and $A \in \mathcal{B}(\mathcal{H})$ be a positive operator. If $s_{m+1}(A) \neq s_m(A)$ for some $m \in \mathbb{N}$, then $A \in \mathcal{N}_{(p,m)}(\mathcal{H})$. Moreover, in this case, $A \in \mathcal{N}_{(p,m+1)}(\mathcal{H})$ if and only if $s_{m+1}(A)$ is an eigenvalue of A.*

Proof. From the proof of the Proposition 5.2.2, we deduce that there exists an orthonormal set $\{v_1,...,v_m\} \subseteq K \subseteq \mathcal{H}$ such that $Av_j = s_j(A)v_j$ so that $\|Av_j\|^p = s_j^p(A)$ and thus $\|A\|^p_{(p,m)} = \sum_{j=1}^m s_j^p(A) = \sum_{j=1}^m \|Av_j\|^p$, where K is the joint span of the eigenspaces corresponding to the eigenvalues $\{s_1(A),...,s_m(A)\}$, which implies that $A \in \mathcal{N}_{(p,m)}$. Furthermore, we observe that if there exists any orthonormal set $\{w_1,...,w_m\}$ of m vectors in $\mathcal{H}$ such that $\sum_{i=1}^m \|Aw_i\|^p = \sum_{j=1}^m s_j^p(A)$, then this set has to be contained in K. Note that $K^\perp$ is invariant under A and hence $A|_{K^\perp} : K^\perp \to K^\perp$, viewed as an operator on $K^\perp$, is positive. Since $s_{m+1}(A) \neq s_m(A)$, it follows that $s_{m+1}(A)$ is an eigenvalue of A if and only if $s_{m+1}(A)$ is an eigenvalue of $A|_{K^\perp} : K^\perp \to K^\perp$ which is possible if and only if $A|_{K^\perp} : K^\perp \to K^\perp$, viewed as an operator on $K^\perp$, belongs to $\mathcal{N}$, that is, there is a unit vector $x \in K^\perp$ such that $\|Ax\| = s_{m+1}(A)$, which in turn happens if and only if $\|Ax\|^p = s_{m+1}^p(A)$ or equivalently, $A \in \mathcal{N}_{(p,m+1)}$. This proves the assertion. $\square$

6.2.1 Necessary conditions for positive operators to belong to $\mathcal{AN}_{(p,k)}(\mathcal{H})$

Using the above proposition, it is not too hard to establish results analogous to Propositions 4.2.4, 4.2.6 and 4.2.7 (see 6.2.3, 6.2.4, and 6.2.5 respectively) for a given $p \in [1, \infty)$ and a given $k \in \mathbb{N}$.

Proposition 6.2.3. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator, $p \in [1, \infty)$, and $k \in \mathbb{N}$. If $A \in \mathcal{N}_{(p,k)}(\mathcal{H})$, then $s_1(A), ..., s_k(A)$ are eigenvalues of A.*

The proof of the above proposition is along the lines of the proof of Proposition 4.2.4 or Propositoin 5.2.3. The following two propositions are also not too difficult to see and hence we omit their proofs.

Proposition 6.2.4. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator, $p \in [1, \infty)$, and $k \in \mathbb{N}$. If $s_1(A), ..., s_k(A)$ are mutually distinct eigenvalues of A, then there exists an orthonormal set $\{v_1, ..., v_k\} \subseteq \mathcal{H}$ such that $Av_j = s_j(A)v_j$ for every $j \in \{1, ..., k\}$. Thus $A \in \mathcal{N}_{(p,k)}(\mathcal{H})$.*

Proposition 6.2.5. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator, $p \in [1, \infty)$, $k \in \mathbb{N}$ and let $s_1(A), ..., s_k(A)$ be the first k singular values of A that are also the eigenvalues of A and are not necessarily distinct. Then either $s_1(A) = ... = s_k(A)$, in which case, $A \in \mathcal{N}_{(p,k)}(\mathcal{H})$ if and only if the multiplicity of $\alpha := s_1(A)$ is at least k; or there exists $t \in \{2, ..., k\}$ such that $s_{t-1}(A) \neq s_t(A) = s_{t+1}(A) = ... = s_k(A)$, in which case, $A \in \mathcal{N}_{(p,k)}(\mathcal{H})$ if and only if the multiplicity of $\beta := s_t(A)$ is at least $k - t + 1$.*

These propositions yield the following result that adds yet another equivalent condition to the Theorem 5.2.6.

Theorem 6.2.6. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator, $p \in [1, \infty)$, $\pi \in \Pi$, and $k \in \mathbb{N}$. Then the following statements are equivalent.*

1. $A \in \mathcal{N}_{[k]}(\mathcal{H})$.
2. $A \in \mathcal{N}_{(p,k)}(\mathcal{H})$.
3. *$s_1(A), ..., s_k(A)$ are eigenvalues of A and there exists an orthonormal set $\{v_1, ..., v_k\} \subseteq \mathcal{H}$ such that $Av_j = s_j(A)v_j$ for every $j \in \{1, ..., k\}$.*
4. $A \in \mathcal{N}_{[\pi,k]}(\mathcal{H})$.

Proof. (1) $\iff$ (3) $\iff$ (4) has been established in Theorem 5.2.6 and (3) $\implies$ (2) is trivial. All that remains to show is (2) $\implies$ (3). By the Proposition 6.2.3, $s_1(A), ..., s_k(A)$ are all eigenvalues of A. If $s_1(A), ..., s_k(A)$ are mutually distinct eigenvalues, then by Proposition 6.2.4 there exists an orthonormal set $\{v_1, ..., v_k\} \subseteq \mathcal{H}$ such that $Av_j = s_j(A)v_j$ for every $j \in \{1, ..., k\}$. However, if $s_1(A), ..., s_k(A)$ are all eigenvalues but not necessarily distinct then also the existence of an orthonormal set $\{v_1, ..., v_k\} \subseteq \mathcal{H}$ with $Av_j = s_j(A)v_j$ for every $j \in \{1, ..., k\}$ is guaranteed by the Proposition 6.2.5. This completes the proof. $\square$

The following corollary is easy to deduce from the above theorem.

Corollary 6.2.7. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator, $p \in [1, \infty)$, and let $k \in \mathbb{N}$.*

1. *If $A \in \mathcal{N}_{(p,k+1)}(\mathcal{H})$, then $A \in \mathcal{N}_{(p,k)}(\mathcal{H})$.*
2. *If $A \in \mathcal{N}_{(p,k)}(\mathcal{H})$, then $A \in \mathcal{N}(\mathcal{H})$.*

Theorem 4.2.10 extends word for word to the set $\mathcal{N}_{(p,k)}$ (see 6.2.8) and Theorem 4.2.11 alongwith the Corollary 4.2.12 extend to the set $\mathcal{AN}_{(p,k)}$ (see 6.2.9 and 6.2.10 respectively).

Theorem 6.2.8. *Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $p \in [1, \infty)$, and $k \in \mathbb{N}$. Then the following statements are equivalent.*

1. *$T \in \mathcal{N}_{(p,k)}(\mathcal{H}, \mathcal{K})$.*
2. *$|T| \in \mathcal{N}_{(p,k)}(\mathcal{H})$.*
3. *$T^*T \in \mathcal{N}_{(p,k)}(\mathcal{H})$.*

Proof. It suffices to establish (1) $\iff$ (2); for then $|T|$ and T^*T being positive and the sets $\mathcal{N}_{[k]}$ and $\mathcal{N}_{(p,k)}$ being identical for positive operators, Theorem 4.2.10 yields the equivalence of (2) and (3). But $s_j(T) = s_j(|T|)$ for every j and $\|Tx\| = \|\, |T|x \,\|$ for every $x \in \mathcal{H}$ which implies that $\sum_{j=1}^{k} \|Tx_j\|^p = \sum_{j=1}^{k} s_j^p(T) \iff \sum_{j=1}^{k} \|\, |T|x_j \,\|^p = \sum_{j=1}^{k} s_j^p(|T|)$. This establishes the equivalence of (1) and (2). $\square$

Theorem 6.2.9. *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $p \in [1, \infty)$, and $k \in \mathbb{N}$. If $A \in \mathcal{AN}_{(p,k+1)}(\mathcal{H}, \mathcal{K})$, then $A \in \mathcal{AN}_{(p,k)}(\mathcal{H}, \mathcal{K})$.*

The proof of the above theorem is similar to that of the Theorem 4.2.11 and the following corollary is easy to deduce from it.

Corollary 6.2.10. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator, $p \in [1,\infty)$, and $k \in \mathbb{N}$. If $A \in \mathcal{AN}_{(p,k+1)}(\mathcal{H})$, then $A \in \mathcal{AN}_{(p,k)}(\mathcal{H})$. In particular, if $A \in \mathcal{AN}_{(p,k)}(\mathcal{H})$, then $A \in \mathcal{AN}(\mathcal{H})$.*

The above corollary alongwith the forward implication of [PP17, Theorem 5.1] yields the main result of this subsection as the following theorem.

Theorem 6.2.11. *Let A be a positive operator on $\mathcal{H}$, $p \in [1,\infty)$, and $k \in \mathbb{N}$. If $A \in \mathcal{AN}_{(p,k)}(\mathcal{H})$, then A is of the form $A = \alpha I + K + F$, where $\alpha \geq 0, K$ is a positive compact operator and F is self-adjoint finite-rank operator.*

Theorem 4.2.14 extends word for word to the family $\mathcal{AN}_{(p,k)}(\mathcal{H},\mathcal{K})$ and its proof is similar. This result will be needed in Chapter 8.

Theorem 6.2.12. *For a closed linear subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ let $V_{\mathcal{M}} : \mathcal{M} \to \mathcal{H}$ be the inclusion map from $\mathcal{M}$ to $\mathcal{H}$ defined as $V_{\mathcal{M}}(x) = x$ for each $x \in \mathcal{M}$, let $P_{\mathcal{M}} \in \mathcal{B}(\mathcal{H})$ be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$, and let $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$. For any real number $p \in [1,\infty)$ and for any $k \in \mathbb{N}$, the following statements are equivalent.*

1. *$T \in \mathcal{AN}_{(p,k)}(\mathcal{H},\mathcal{K})$.*
2. *$TV_{\mathcal{M}} \in \mathcal{N}_{(p,k)}(\mathcal{M},\mathcal{K})$ for every nontrivial closed linear subspace $\mathcal{M}$ of $\mathcal{H}$.*
3. *$TP_{\mathcal{M}} \in \mathcal{N}_{(p,k)}(\mathcal{H},\mathcal{K})$ for every nontrivial closed linear subspace $\mathcal{M}$ of $\mathcal{H}$.*

6.2.2 Sufficient conditions for positive operators to belong to $\mathcal{AN}_{(p,k)}(\mathcal{H})$

We now discuss the sufficient conditions for a positive operator to be absolutely (p,k)-norming for every $p \in [1,\infty)$ and for every $k \in \mathbb{N}$. We begin by stating a proposition that gives a sufficient condition for a positive operator to be (p,k)-norming for every $p \in [1,\infty)$ and for every $k \in \mathbb{N}$.

Proposition 6.2.13. *Let $K \in \mathcal{B}(\mathcal{H})$ be a positive compact operator, $F \in \mathcal{B}(\mathcal{H})$ be a self-adjoint finite-rank operator, and $\alpha \geq 0$ such that $\alpha I + K + F \geq 0$. Then $\alpha I + K + F \in \mathcal{N}_{(p,k)}(\mathcal{H})$ for every $p \in [1,\infty)$ and for every $k \in \mathbb{N}$.*

The above result follows immediately from the Proposition 4.2.18 and Theorem 6.2.6. In fact, this proposition is a special case of the following theorem.

Theorem 6.2.14. *Let $K \in \mathcal{B}(\mathcal{H})$ be a positive compact operator, $F \in \mathcal{B}(\mathcal{H})$ be a self-adjoint finite-rank operator, and $\alpha \geq 0$ such that $\alpha I + K + F \geq 0$. Then $\alpha I + K + F \in \mathcal{AN}_{(p,k)}(\mathcal{H})$ for every $p \in [1, \infty)$ and for every $k \in \mathbb{N}$.*

Proof. Let us fix $p \in [1, \infty)$ and $k \in \mathbb{N}$, and let us define $T := \alpha I + K + F$. Due to Proposition 6.1.7, $T \in \mathcal{AN}_{(p,k)}$ if and only if $|T| \in \mathcal{AN}_{(p,k)}$, which due to Lemma 6.1.6, is possible if and only if for every nontrivial closed linear subspace $\mathcal{M}$ of $\mathcal{H}$, $|T|V_{\mathcal{M}} \in \mathcal{N}_{(p,k)}$ for every $p \in [1, \infty)$ and for every $k \in \mathbb{N}$, where $V_{\mathcal{M}} : \mathcal{M} \to \mathcal{H}$ is the inclusion map as defined earlier. We show the last of these equivalent statements.

Notice that $|T| = |\alpha I + K + F|$ and $|T|^*|T| = \beta I + \tilde{K} + \tilde{F}$ where $\beta = \alpha^2 \geq 0$, and, $\tilde{K} = 2\alpha K + K^2$ and $\tilde{F} = 2\alpha F + FK + KF + F^2$ are respectively positive compact and self-adjoint finite-rank operators. It is easy to see that $\beta I + \tilde{K} + \tilde{F} \geq 0$. Next we fix a closed linear subspace $\mathcal{M}$ of $\mathcal{H}$ and observe that

$$\begin{aligned} & |T|V_{\mathcal{M}} \in \mathcal{N}_{(p,k)} \\ \Longleftrightarrow\ & (|T|V_{\mathcal{M}})^*(|T|V_{\mathcal{M}}) \in \mathcal{N}_{(p,k)} \\ \Longleftrightarrow\ & V_{\mathcal{M}}^*(|T|^*|T|)V_{\mathcal{M}} \in \mathcal{N}_{(p,k)} \\ \Longleftrightarrow\ & V_{\mathcal{M}}^*(\beta I + \tilde{K} + \tilde{F})V_{\mathcal{M}} \in \mathcal{N}_{(p,k)}, \end{aligned}$$

where the first equivalence is due to the Theorem 6.2.8. It suffices to show that $V_{\mathcal{M}}^*(\beta I + \tilde{K} + \tilde{F})V_{\mathcal{M}} \in \mathcal{N}_{(p,k)}$;for then, since $\mathcal{M}$ is arbitrary, the assertion immediately follows. To this end, notice that $V_{\mathcal{M}}^*(\beta I + \tilde{K} + \tilde{F})V_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is an operator on $\mathcal{M}$ and

$$V_{\mathcal{M}}^*(\beta I + \tilde{K} + \tilde{F})V_{\mathcal{M}} = V_{\mathcal{M}}^*\beta I V_{\mathcal{M}} + V_{\mathcal{M}}^*\tilde{K}V_{\mathcal{M}} + V_{\mathcal{M}}^*\tilde{F}V_{\mathcal{M}} = \beta I_{\mathcal{M}} + \tilde{K}_{\mathcal{M}} + \tilde{F}_{\mathcal{M}}$$

is the sum of a nonnegative scalar multiple of Identity, a positive compact operator and a self-adjoint finite-rank operator on the fixed Hilbert space $\mathcal{M}$ such that this sum is a positive operator on this Hilbert space $\mathcal{M}$ which, by the preceding proposition, belongs to $\mathcal{N}_{(p,k)}$. Since $p \in [1, \infty)$ and $k \in \mathbb{N}$ are arbitrary, it follows that an operator of the above form belongs to $\mathcal{AN}_{(p,k)}$ for every $p \in [1, \infty)$ and for every $k \in \mathbb{N}$. This completes the proof. $\square$

As an immediate consequence of the Theorem 6.2.11 and Theorem 6.2.14, we get the following theorem which completely characterizes positive operators that are absolutely (p, k)-norming for any and every $p \in [1, \infty)$ and $k \in \mathbb{N}$.

Theorem 6.2.15 (Spectral Theorem for Positive Operators in $\mathcal{AN}_{(p,k)}(\mathcal{H})$)**.** *Let P be a positive operator on $\mathcal{H}$. Then the following statements are equivalent.*

1. $P \in \mathcal{AN}_{(p,k)}(\mathcal{H})$ *for every* $p \in [1, \infty)$ *and for every* $k \in \mathbb{N}$.
2. $P \in \mathcal{AN}_{(p,k)}(\mathcal{H})$ *for some* $p \in [1, \infty)$ *and for some* $k \in \mathbb{N}$.
3. P *is of the form* $P = \alpha I + K + F$, *where* $\alpha \geq 0, K$ *is a positive compact operator and* F *is self-adjoint finite-rank operator.*

We have all that is required to move on to the next section and establish the result (see Theorem 6.3.1) which characterizes bounded operators that attain their (p, k)-singular norm on every closed subspace. Lemma 5.2.16 and Proposition 5.2.18 carry over word for word to the operators in $\mathcal{N}_{(p,k)}$ and operators in $\mathcal{AN}_{(p,k)}$ respectively (see 6.2.16 and 6.2.17). The Proposition 6.2.17 addresses the question of whether an operator of the form $\alpha I + K + F$, which is not necessarily positive, belongs to $\mathcal{AN}_{(p,k)}$.

Lemma 6.2.16. *Let* $K \in \mathcal{B}(\mathcal{H})$ *be a positive compact operator,* $F \in \mathcal{B}(\mathcal{H})$ *be a self-adjoint finite-rank operator, and* $\alpha \geq 0$. *Then* $\alpha I + K + F \in \mathcal{N}_{(p,k)}(\mathcal{H})$ *for every* $p \in [1, \infty)$ *and for every* $k \in \mathbb{N}$.

Proof. Fix $p \in [1, \infty)$ and $k \in \mathbb{N}$. For any bounded operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ we observe that

$$\begin{aligned} T \in \mathcal{N}_{[k]} &\iff |T| \in \mathcal{N}_{[k]} \\ &\iff |T| \in \mathcal{N}_{(p,k)} \\ &\iff T \in \mathcal{N}_{(p,k)}, \end{aligned}$$

where the first equivalence is due to the Theorem 4.2.10, the second equivalence is due to the Theorem 6.2.6 and the last equivalence is due to the Theorem 6.2.8. This observation when applied to the Proposition 4.2.18 proves that $\alpha I + K + F \in \mathcal{N}_{(p,k)}$. Since $p \in [1, \infty)$ and $k \in \mathbb{N}$ are arbitrary, it follows that $\alpha I + K + F \in \mathcal{N}_{(p,k)}$ for every $p \in [1, \infty)$ and for every $k \in \mathbb{N}$ which proves the assertion. □

Proposition 6.2.17. *Let* $K \in \mathcal{B}(\mathcal{H})$ *be a positive compact operator,* $F \in \mathcal{B}(\mathcal{H})$ *be a self-adjoint finite-rank operator, and* $\alpha \geq 0$. *Then* $\alpha I + K + F \in \mathcal{AN}_{(p,k)}(\mathcal{H})$ *for every* $p \in [1, \infty)$ *and for every* $k \in \mathbb{N}$.

Proof. This proof is very similar to the proof of the Theorem 6.2.14. As before, let us define $T := \alpha I + K + F$. We need to show that $T \in \mathcal{AN}_{(p,k)}$ for every $p \in [1, \infty)$ and for every $k \in \mathbb{N}$. Let us fix $p \in [1, \infty)$ and $k \in \mathbb{N}$. The Proposition 6.1.7, together with the Lemma 6.1.6 shows that it suffices to show that for every nontrivial closed linear subspace $\mathcal{M}$ of $\mathcal{H}$, $|T|V_{\mathcal{M}} \in \mathcal{N}_{(p,k)}$ for every $p \in [1, \infty)$ and for every $k \in \mathbb{N}$, where $V_{\mathcal{M}} : \mathcal{M} \to \mathcal{H}$

is the inclusion map as defined earlier. Next we fix a closed linear subspace $\mathcal{M}$ of $\mathcal{H}$ and observe that

$$|T|V_{\mathcal{M}} \in \mathcal{N}_{(p,k)} \iff V_{\mathcal{M}}^*(\beta I + \tilde{K} + \tilde{F})V_{\mathcal{M}} \in \mathcal{N}_{(p,k)},$$

where $\beta I+\tilde{K}+\tilde{F} = |T|^*|T|$ with $\beta = \alpha^2 \geq 0$, $\tilde{K} = 2\alpha K+K^2$ and $\tilde{F} = 2\alpha F+FK+KF+F^2$. All that remains to be shown is that $V_{\mathcal{M}}^*(\beta I + \tilde{K} + \tilde{F})V_{\mathcal{M}} \in \mathcal{N}_{(p,k)}$. To this end, notice that $V_{\mathcal{M}}^*(\beta I + \tilde{K} + \tilde{F})V_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is an operator on $\mathcal{M}$ and $V_{\mathcal{M}}^*(\beta I + \tilde{K} + \tilde{F})V_{\mathcal{M}} = \beta I_{\mathcal{M}} + \tilde{K}_{\mathcal{M}} + \tilde{F}_{\mathcal{M}}$ is the sum of a nonnegative scalar multiple of Identity, a positive compact operator and a self-adjoint finite-rank operator on the fixed Hilbert space $\mathcal{M}$ which, by the preceding lemma, belongs to $\mathcal{N}_{(p,k)}$. Finally, since $p \in [1, \infty)$ and $k \in \mathbb{N}$ are arbitrary, the result holds for every $p \in [1, \infty)$ and for every $k \in \mathbb{N}$ and thus an operator of the above form belongs to $\mathcal{AN}_{(p,k)}$ for every $p \in [1, \infty)$ and for every $k \in \mathbb{N}$. This completes the proof. $\square$

Remark 6.2.18. The proof of the Lemma 6.2.16 uses an interesting result which deserves to be stated for its intrinsic interest. If $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $p \in [1, \infty)$, and $k \in \mathbb{N}$, then $T \in \mathcal{N}_{[k]}(\mathcal{H}, \mathcal{K}) \iff T \in \mathcal{N}_{(p,k)}(\mathcal{H}, \mathcal{K})$. This result, together with the result stated in Remark 5.2.17 yields the following:

Suppose $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $\pi \in \Pi$, $p \in [1, \infty)$, and $k \in \mathbb{N}$. Then the following statements are equivalent.

1. $T \in \mathcal{N}_{[k]}(\mathcal{H}, \mathcal{K})$.
2. $T \in \mathcal{N}_{(p,k)}(\mathcal{H}, \mathcal{K})$.
3. $T \in \mathcal{N}_{[\pi,k]}(\mathcal{H}, \mathcal{K})$.

6.3 Spectral characterization of operators in $\mathcal{AN}_{(p,k)}(\mathcal{H}, \mathcal{K})$

By Proposition 6.1.7, the polar decomposition theorem (see Theorem 2.5.1) and the spectral theorem for positive operators in $\mathcal{AN}_{(p,k)}(\mathcal{H})$ (see Theorem 6.2.15), we can safely consider the following theorem to be fully proved.

Theorem 6.3.1 (Spectral Theorem for Operators in $\mathcal{AN}_{(p,k)}(\mathcal{H}, \mathcal{K})$)**.** *Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and let $T = U|T|$ be its polar decomposition. Then the following statements are equivalent.*

1. *$T \in \mathcal{AN}_{(p,k)}(\mathcal{H}, \mathcal{K})$ for every $p \in [1, \infty)$ and for every $k \in \mathbb{N}$.*

2. $T \in \mathcal{AN}_{(p,k)}(\mathcal{H}, \mathcal{K})$ *for some* $p \in [1, \infty)$ *and for some* $k \in \mathbb{N}$*.*

3. $|T|$ *is of the form* $|T| = \alpha I + K + F$*, where* $\alpha \geq 0, K$ *is a positive compact operator and* F *is self-adjoint finite-rank operator.*

6.4 A survey of the situation

What have we achieved so far? The spectral characterization theorem for absolutely norming operators (Theorem 3.5.3) at the end of Chapter 3 settles Chevreau's problem of characterizing the set $\mathcal{AN}(\mathcal{H}, \mathcal{K})$ of absolutely norming operators in $\mathcal{B}(\mathcal{H}, \mathcal{K})$ with respect to the usual operator norm. Thereafter, in Chapters 4, 5, and 6 we embarked on unraveling the analogous notion in a slightly more general case where $\mathcal{B}(\mathcal{H}, \mathcal{K})$ is equipped with a (symmetric) norm $|||\cdot|||$ (in particular, the Ky Fan k-norm, the weighted Ky Fan π, k-norm, and the (p, k)-singular norm) that is equivalent to the usual operator norm, and we not only defined the notion of norming and absolutely norming operators in each of these contexts, but also completely characterized the set of absolutely norming operators with respect to each of these three families of norms. Unfortunately, (or fortunately?) these characterization theorems imply that the set of absolutely norming operators in all the three cases are identical! However, the question which motivated this work is far from being solved — we are yet to establish the definition of "norming" and "absolutely norming" operators with respect to an arbitrary symmetric norm on $\mathcal{B}(\mathcal{H})$. What shall be our next path of exploration?

All the spectral characterization theorems we established in Chapters $3 - 6$ exhibit a common phenomenon: if an operator in $\mathcal{B}(\mathcal{H}, \mathcal{K})$ belongs to one of the families $\mathcal{AN}_{[k]}(\mathcal{H}, \mathcal{K})$, $\mathcal{AN}_{[\pi,k]}(\mathcal{H}, \mathcal{K})$, or $\mathcal{AN}_{(p,k)}(\mathcal{H}, \mathcal{K})$, it belongs to all of them and its absolute value is of the form $\alpha I + K + F$, where $\alpha \geq 0, K$ is a positive compact operator and F is a self-adjoint finite-rank operator. Conversely, any operator in $\mathcal{B}(\mathcal{H}, \mathcal{K})$ with its absolute value of the form $\alpha I + K + F$ is absolutely norming with respect to each of the norms we discussed. As a corollary of these results, we have that every positive operator of the form $\alpha I + K + F$ belongs to each of the families $\mathcal{AN}_{[k]}(\mathcal{H}), \mathcal{AN}_{[\pi,k]}(\mathcal{H})$ and $\mathcal{AN}_{(p,k)}(\mathcal{H})$. So, it might appear at this stage that with respect to *every* symmetric norm $\|\cdot\|_s$ on $\mathcal{B}(\mathcal{H})$, the positive operators on $\mathcal{B}(\mathcal{H})$, that are of the above form, are "absolutely s-norming".

If this were true, if the end result of the analysis of absolutely norming operators with respect to various symmetric norms were that they are all of the same form, then this theory would be relatively straightforward. But this is not the case, for we prove the existence of a symmetric norm $\|\cdot\|_{\Phi_\pi^*}$ on $\mathcal{B}(\ell^2)$ with respect to which the identity operator does not attain its norm; see Chapter 8, Theorem 8.3.1. This is the path that we explore in in Chapters 8 and 9. In order to discover this not-so-usual symmetric norm on $\mathcal{B}(\mathcal{H})$, we need to put down some definitions and collect some facts that we will be using extensively in the sequel. This is precisely the goal of the next chapter; and the last two chapters elaborately study the theory of "symmetric norming" and "absolutely symmetric norming" operators.

Chapter 7

Preliminaries II: Symmetrically-normed ideals

Having given a fairly comprehensive treatment of the theory of s-numbers in the second chapter of this thesis, we turn our attention to the theory of symmetrically-normed ideals ("norm ideals" in the older literature; see, for instance, [Sch50, Sch60, Mit64, Hol73, Hol74, Hol75]). The central purpose of this chapter is to develop in a systematic manner the main elementary facts about symmetrically-normed ideals of the algebra $\mathcal{B}(\mathcal{H})$ of operators on a Hilbert space, with most of the attention centered around symmetrically-normed ideals of the algebra of compact operators on a (separable infinite-dimensional) Hilbert space. These facts are important for their own sake, and also for the sake of motivation they provide, as an essential prerequisite, for our later work in Chapters 8 and 9.

This chapter draws heavily from, and hence summarizes, the third chapter of the monograph by Gohberg and Kreĭn [GK69], and it includes the basic content of the elegant monograph by Schatten [Sch60]. Dictated by the intention of exposing the main ideas and their interconnections in a minimal amount of time, a number of topics are discussed only briefly. It has been necessary, for the same reason, to omit a number of proofs and to give only sketches of others.

7.1 Ideals of operators

This section contains an account of those basic facts related to the ideals of the algebra $\mathcal{B}(\mathcal{H})$ of operators on $\mathcal{H}$ that are needed, later in this chapter, in the study of symmetrically-normed ideals. As mentioned in Chapter 2, the term "ideal" will always mean a "two-sided" ideal. An ideal in a given algebra will be termed *nontrivial* if it is not the *zero ideal* $\{0\}$ and it does not embrace the whole algebra; it is defined to be *proper* if it is properly contained in the algebra. (The zero ideal is, of course, proper as per our definition.) Our terminologies differ from that of Schatten's monograph [Sch60], and Gohberg and Kreĭn's text [GK69]; Schatten uses the terms "nontrivial" and "proper" interchangeably for the ideals that are not trivial in the sense of our definition; Gohberg and Kreĭn assume, in their definition of an ideal, that it is neither the zero ideal nor the full algebra.

By definition, an ideal $\mathfrak{I}$ of an algebra $\mathfrak{A}$ is *minimal* if it is a nonzero ideal which does not properly contain any other nonzero ideal, i.e. if (a) $\mathfrak{I} \neq \{0\}$, and (b) for every ideal $\mathfrak{H} \subseteq \mathfrak{A}$, such that $\{0\} \subseteq \mathfrak{H} \subseteq \mathfrak{I}$, either $\mathfrak{H} = \{0\}$ or $\mathfrak{H} = \mathfrak{I}$. Also recall that an ideal $\mathfrak{M}$ of an algebra $\mathfrak{A}$ is said to be a *maximal* if it is a proper ideal that is not properly contained in any other proper ideal, i.e. if (a) $\mathfrak{M} \neq \mathfrak{A}$, and (b) for every ideal $\mathfrak{N} \subseteq \mathfrak{A}$, such that $\mathfrak{M} \subseteq \mathfrak{N} \subseteq \mathfrak{A}$, either $\mathfrak{N} = \mathfrak{M}$ or $\mathfrak{N} = \mathfrak{A}$.

Some algebras have a multitude of nontrivial ideals, while others have none at all. It is of great interest indeed that both $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_0(\mathcal{H})$ possess nontrivial ideals if and only if $\mathcal{H}$ is infinite-dimensional. Needless to say, a nonzero ideal in the algebra of linear transformations on a finite-dimensional Hilbert space, necessarily coincides with the whole algebra. However, the situation is strikingly different when $\mathcal{H}$ is an infinite-dimensional; for the set $\mathcal{B}_{00}(\mathcal{H})$ of finite rank operators on $\mathcal{H}$ is always a nontrivial ideal of $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_0(\mathcal{H})$ both. Even more, $\mathcal{B}_{00}(\mathcal{H})$ is the minimal ideal of the algebra $\mathcal{B}(\mathcal{H})$ in this case. Also, in the case when $\mathcal{H}$ is infinite-dimensional, one can always construct a maximal nontrivial ideal in $\mathcal{B}(\mathcal{H})$. The following lemma and corollary provides the required machinery for the desired construction .

Lemma 7.1.1 ([Sch60])**.** *Let $A \in \mathcal{B}(\mathcal{H})$ such that* $\operatorname{ran} A$ *contains a closed subspace $\mathcal{M}$ of the same dimension as $\mathcal{H}$. Then there exist two partial isometries U and V such that VAU has an inverse in $\mathcal{B}(\mathcal{H})$.*

Corollary 7.1.2. *No proper ideal $\mathfrak{I}$ of $\mathcal{B}(\mathcal{H})$ has an operator whose range contains a subspace of the same dimension as $\mathcal{H}$.*

Proof. Suppose, on the contrary, that there exists $A \in \mathfrak{I}$ such that $\operatorname{ran} A$ contains a subspace of the same dimension as $\mathcal{H}$. Then, by the previous lemma, there are partial isometries U and V such that VAU has an inverse. But since $VAU \in \mathfrak{I}$, its inverse too belongs to $\mathfrak{I}$. Consequently, $\mathfrak{I}$ contains the identity and thus coincides with $\mathcal{B}(\mathcal{H})$. $\square$

Now we are prepared to propose the construction of a maximal nontrivial ideal of $\mathcal{B}(\mathcal{H})$ in the case when $\mathcal{H}$ is infinite-dimensional.

Theorem 7.1.3 ([Sch60]). *The set $\mathfrak{S}$ of operators on an infinite-dimensional Hilbert space $\mathcal{H}$, none of which contains in its range a closed subspace of the same dimension as $\mathcal{H}$, is a maximal nontrivial ideal in $\mathcal{B}(\mathcal{H})$.*

Proof. That $\mathfrak{S}$ form a nontrivial ideal is not difficult to verify. Rest of the proof follows from the previous corollary. □

We conclude this section with the following interesting result due to Calkin [Cal41, page 841], though we give no proof.

Theorem 7.1.4. *Let $\mathcal{H}$ be a separable Hilbert space. For any ideal $\mathfrak{I} \subseteq \mathcal{B}(\mathcal{H})$ we have either $\mathfrak{I} = \mathcal{B}(\mathcal{H})$ or $\mathfrak{I} \subseteq \mathcal{B}_0(\mathcal{H})$.*

7.2 Symmetric norms revisited

We discussed symmetric norms in Section 2.9 of Chapter 2. This section picks up from where we left, and lists further properties of symmetric norms. First, a brief recapitulation. Recall that a norm $\|\cdot\|_s$ on an ideal $\mathfrak{I}$ of $\mathcal{B}(\mathcal{H})$ is said to be symmetric if it is a uniform crossnorm. Explicitly, a symmetric norm on $\mathfrak{I} \subseteq \mathcal{B}(\mathcal{H})$ is a function $\|.\|_s : \mathfrak{I} \to [0, \infty)$ which satisfies the following conditions:

(a) $\|X\|_s$ is a norm;

(b) $\|X\|_s = \|X\|$ for every rank one operator $X \in \mathfrak{I}$ (crossnorm property); and

(c) $\|AXB\|_s \leq \|A\|\|X\|_s\|B\|$ for every $X \in \mathfrak{I}$ and for every pair A, B of operators in $\mathcal{B}(\mathcal{H})$ (uniformity).

Before we can proceed we need one more technical result by Douglas, often called Douglas's Range Inclusion Theorem.

Theorem 7.2.1 ([Dou66]). *If $A, B \in \mathcal{B}(\mathcal{H})$, then the following statements are equivalent.*

(a) $\operatorname{ran}(A) \subseteq \operatorname{ran}(B)$.

(b) There is a positive constant c such that $AA^* \leq c^2 BB^*$.

(c) There is an operator $C \in \mathcal{B}(\mathcal{H})$ such that $A = BC$.

If $\mathfrak{I}$ is an ideal in $\mathcal{B}(\mathcal{H})$, and if either of the operators $T, |T|, T^*, |T^*|$ is in $\mathfrak{I}$, then the Polar Decomposition theorem implies that the remaining three are also in $\mathfrak{I}$. In particular, $\mathfrak{I}$ is self-adjoint. If, in addition, $\|.\|_s$ is a symmetric norm on $\mathfrak{I}$, then

$$\|T\|_s = \|T^*\|_s = \| \, |T| \, \|_s = \| \, |T^*| \, \|_s.$$

To prove this, assume that $T = U|T|$ is the polar decomposition of the operator T. This yields

$$\|T\|_s = \| \, U|T| \, \|_s = \|U|T|I\|_s \leq \|U\| \| \, |T| \, \|_s \|I\| = \| \, |T| \, \|_s.$$

Since the polar decomposition of T also asserts that $|T| = U^*T$, it follows that $\||T|\|_s \leq \|T\|_s$ (same argument), which implies that $\|T\|_s = \||T|\|_s$. Apply the same technique on the polar decomposition of T^* to obtain $\|T^*\|_s = \||T|\|_s$. Conclude the proof by showing that $\|T^*\|_s = \||T|\|_s$; use the equation $T^* = |T|U^*$ obtained by taking adjoint of the polar decomposition of T.

It is immediately clear that if $\mathfrak{I}$ is an ideal in $\mathcal{B}(\mathcal{H})$, and if is not contained in the compacts, then it must contains $\mathcal{B}_0(\mathcal{H})$. If $\mathfrak{I}$ contains $\mathcal{B}_0(\mathcal{H})$, then every compact operator is in $\mathfrak{I}$. Given an ideal $\mathfrak{I}$ in $\mathcal{B}_0(\mathcal{H})$, it is not always easy to determine whether or not a particular compact operator belongs to $\mathfrak{I}$, especially when the ideal $\mathfrak{I}$ is described vaguely. The following results provide a solution to this problem with an elegant criterion which is often easy to check.

Proposition 7.2.2 ([GK69]). *Let $\mathfrak{I}$ be an ideal of the algebra $\mathcal{B}_0(\mathcal{H})$ of compact operators that is equipped with a symmetric norm $\|.\|_s$ and let $A \in \mathfrak{I}$. If $B \in \mathcal{B}_0(\mathcal{H})$ such that, for some positive constant c, $s_j(B) \leq c s_j(A)$ for each $j \in \mathbb{N}$, then $B \in \mathfrak{I}$ and $\|B\|_s \leq c\|A\|_s$.*

Corollary 7.2.3. *If $\mathfrak{I}$ is an ideal of $\mathcal{B}_0(\mathcal{H})$, then every symmetric norm $\|.\|_s$ on $\mathfrak{I}$ satisfies the following inequalities:*

(1) $s_1(X) \leq \|X\|_s$ for every $X \in \mathfrak{I}$; and

(2) $\|X\|_s \leq \sum_j s_j(X)$ for every finite rank operator $X \in \mathfrak{I}$.

7.3 Symmetrically-normed ideals

We now return to the central purpose of this chapter, namely, the study of *symmetrically-normed ideals* of the algebra of compact operators on a separable infinite-dimensional Hilbert space (the only kind to be considered in this chapter from now on).

Definition 7.3.1 (Symmetrically-normed ideals)**.** An ideal $\mathfrak{S}$ of the algebra $\mathcal{B}(\mathcal{H})$ is said to be a *symmetrically-normed ideal* (abbreviated an *s.n.ideal*) of $\mathcal{B}(\mathcal{H})$ if on it there is defined a symmetric norm $\|.\|_{\mathfrak{S}}$ which makes $\mathfrak{S}$ a Banach space, i.e. $\mathfrak{S}$ is complete in the metric given by this norm.

Definition 7.3.2. We say that two s.n.ideals $\mathfrak{S}_I$ and $\mathfrak{S}_{II}$ in $\mathcal{B}(\mathcal{H})$ *coincide elementwise* if $\mathfrak{S}_I$ and $\mathfrak{S}_{II}$ consist of the same elements.

Theorem 7.3.3. *If $(\mathfrak{S}_I, \|\cdot\|_{\mathfrak{S}_I})$ and $(\mathfrak{S}_{II}, \|\cdot\|_{\mathfrak{S}_{II}})$ are two s.n.ideals in $\mathcal{B}(\mathcal{H})$ that coincide elementwise, then their norms are topologically equivalent.*

Proof. Denote by $\mathfrak{S}$ the set of elements of the s.n.ideal $\mathfrak{S}_I$ (or of the s.n.ideal $\mathfrak{S}_{II}$ which is identical to $\mathfrak{S}_I$ as a set) and set

$$\|X\|_{\mathfrak{S}} = \max\{\|X\|_{\mathfrak{S}_I}, \|X\|_{\mathfrak{S}_{II}}\} \text{ for every } X \in \mathfrak{S}.$$

We first show that $(\mathfrak{S}, \|\cdot\|_{\mathfrak{S}})$ is a Banach space. This is, of course, a normed linear space. To show the completeness in the metric given by the norm $\|\cdot\|_{\mathfrak{S}}$, let us assume that $(T_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathfrak{S}, \|\cdot\|_{\mathfrak{S}})$. Obviously then $(T_n)_n$ is Cauchy in $(\mathfrak{S}_I, \|\cdot\|_{\mathfrak{S}_I})$ and $(\mathfrak{S}_{II}, \|\cdot\|_{\mathfrak{S}_{II}})$, and since these two spaces are complete, the sequence $(T_n)_n$ converges to say R and S, respectively, in $(\mathfrak{S}_I, \|\cdot\|_{\mathfrak{S}_I})$ and $(\mathfrak{S}_{II}, \|\cdot\|_{\mathfrak{S}_{II}})$. Since the first singular value of an operator is equal to its usual operator norm, it follows from the first inference of Corollary 7.2.3 that

$$\|T_n - R\| \leq \|T_n - R\|_{\mathfrak{S}_I} \to 0 \text{ and}$$
$$\|T_n - S\| \leq \|T_n - S\|_{\mathfrak{S}_{II}} \to 0,$$

where $\|\cdot\|$ is the usual operator norm. This shows that $(T_n)_n$ converges to R and S both in the operator norm, which implies that $R = S$. All that remains to show is $\lim_{n\to\infty} \|T_n - R\|_{\mathfrak{S}} = 0$ which follows from the following:

$$\|T_n - R\|_{\mathfrak{S}} \leq \|T_n - R\|_{\mathfrak{S}_I} + \|T_n - R\|_{\mathfrak{S}_{II}} \to 0.$$

This implies that the the sequence $(T_n)_n \to R$, and hence $(\mathfrak{S}, \|\cdot\|_{\mathfrak{S}})$ is complete.

It is then a routine to show the norms $\|\cdot\|_{\mathfrak{S}_I}$ and $\|\cdot\|_{\mathfrak{S}_{II}}$ are toplogically equivalent. The identity maps from $(\mathfrak{S}, \|\cdot\|_{\mathfrak{S}})$ to each of the s.n.ideals $(\mathfrak{S}_I, \|\cdot\|_{\mathfrak{S}_I})$ and $(\mathfrak{S}_{II}, \|\cdot\|_{\mathfrak{S}_{II}})$ is then shown to be bicontinuous, which implies that each of the norms $\|\cdot\|_{\mathfrak{S}_I}$ and $\|\cdot\|_{\mathfrak{S}_{II}}$ is (topologically) equivalent to the norm $\|\cdot\|_{\mathfrak{S}}$, and therefore, the norms $\|X\|_{\mathfrak{S}_I}$ and $\|X\|_{\mathfrak{S}_{II}}$ are equivalent. This proves the assertion. (Notice that the proof does not make use of the separability of the Hilbert space, and hence this result is valid for arbitrary Hilbert spaces.) $\square$

Remark 7.3.4. The above result makes a clever use of the fact that every symmetric norm is always greater than or equal to the usual operator norm, and hence, this result is not true in the setting of a general Banach space.

Let $(\mathfrak{S}, \|\cdot\|)$ be an s.n.ideal in $\mathcal{B}_0(\mathcal{H})$ and let $A \in \mathfrak{S}$. If $B \in \mathcal{B}_0(\mathcal{H})$, and if $s_j(B) = s_j(A)$ for each $j \in \mathbb{N}$, then Theorem 7.2.2 implies that $B \in \mathfrak{S}$ and $\|B\|_s = \|A\|_s$. In other words, for every X in an s.n.ideal $(\mathfrak{S}, \|\cdot\|) \subseteq \mathcal{B}_0(\mathcal{H})$, the value $\|X\|_s$ depends entirely on the sequence $s(X) := (s_n(X))_{n\in\mathbb{N}}$ of s-numbers of X. Consequently, $\|X\|_s$ can be viewed as the value of some function Φ at $s(X)$, i.e.,

$$\|X\|_s = \Phi(s(X)).$$

Too little is known about Φ as of now, which, however, does not bar us from listing few of its obvious properties: it is a real-valued function; its domain is the set $\{s(X) : X \in \mathfrak{S}\}$ of sequences of s-numbers of operators in $\mathfrak{S}$, which is often referred to as the *characteristic set* of the s.n.ideal $\mathfrak{S}$; and that Φ is well-defined on the characteristic set of $\mathfrak{S}$. Indeed, if $X, Y \in \mathfrak{S}, X \neq Y$ and $s(X) = s(Y) =: (\alpha_j)_{j\in\mathbb{N}}$, then on one hand we have $\Phi((\alpha_j)_j) = \|X\|_s$, while on the other hand we have $\Phi((\alpha_j)_j) = \|Y\|_s$. Since $s(X) = s(Y)$, it follows from the above discussion that $\|X\|_s = \|Y\|_s$, and therefore, the well-definedness of Φ is established.

A useful observation: consider the ideal $\mathcal{B}_{00}(\mathcal{H})$ of finite rank operators on $\mathcal{H}$. This is not an s.n.ideal; for no symmetric norm on $\mathcal{B}_{00}(\mathcal{H})$ makes it a Banach space. However, the notion of the characteristic set makes sense for every ideal (not just for s.n.ideals) and it is perhaps worth a short short pause to derive/examine the characteristic set of $\mathcal{B}_{00}(\mathcal{H})$. A moment's thought will convince the reader that the characteristic set of $\mathcal{B}_{00}(\mathcal{H})$ is the set of nonincreasing sequences of nonnegative numbers with a finite number of nonzero terms.

What more can we say about Φ and what significance does it hold? As is evident that every s.n.ideal $(\mathfrak{S}, \|\cdot\|_s)$ gives rise to a function

$$\Phi : \{s(X) : X \in \mathfrak{S}\} \to [0, \infty),$$

and that Φ intrinsically depends on the s.n.ideal $\mathfrak{S}$, it would be natural to ask whether an s.n.ideal (in $\mathcal{B}_0(\mathcal{H})$) can be constructed from such a function? Any attempt towards answering this question requires putting precisely the definition of such functions in place, independent of any s.n.ideal whatsoever. Once defined, these functions (known as "symmetric norming functions") naturally generate s.n. ideals, as we shall see later. In the next section, we study these symmetric norming functions extensively.

7.4 Symmetric norming functions

Definition 7.4.1 (Symmetric norming function)**.** Let $c_{00}(\mathbb{N})$ (or simply c_{00}) be the subspace of c_0 consisting of sequences with a finite number of nonzero terms. A *norming function* Φ on c_{00} is a real valued function which satisfies the following properties:

(i) $\Phi(\xi) \geq 0$ for every $\xi := (\xi_j)_{j\in\mathbb{N}} \in c_{00}$;

(ii) $\Phi(\xi) = 0 \iff \xi = 0$;

(iii) $\Phi(\alpha\xi) = |\alpha|\Phi(\xi)$ for every $\xi \in c_{00}$ and for every scalar $\alpha \in \mathbb{R}$; and

(iv) $\Phi(\xi + \psi) \leq \Phi(\xi) + \Phi(\psi)$ for every pair ξ, ψ of sequences in c_{00}.

We call $\Phi : c_{00} \to [0, \infty)$ a *symmetric norming function* (or simply an *s.n.function*) if in addition to the properties (i)-(iv), it also satisfies the following property:

(v) $\Phi((\xi_1, \xi_2, ..., \xi_n, 0, 0, ...)) = \Phi((|\xi_{j_1}|, |\xi_{j_2}|, ..., |\xi_{j_n}|, 0, 0, ...))$ for every $\xi \in c_{00}$ and for every $n \in \mathbb{N}$, where $j_1, j_2, ..., j_n$ is any permutation of the integers $1, 2, ..., n$.

To simplify the formula, it is always assumed that an s.n.function Φ also satisfies the following condition:

(vi) $\Phi((1, 0, 0, ...)) = 1$.

We conclude this section by adding few symbols to our existing list of notations. As mentioned earlier, we write c_{00} for the subspace of c_0 consisting of sequences with a finite number of nonzero terms. By c_{00}^+ we denote the positive cone of c_{00} and we use $c_{00}^* \subseteq c_{00}$ to denote the cone of nonincreasing nonnegative sequences from c_{00}.

7.4.1 Properties of symmetric norming functions

Proposition 7.4.2. *Let $\xi = (\xi_j)_{j\in\mathbb{N}}, \eta = (\eta_j)_{j\in\mathbb{N}} \in c_{00}$ and Φ be an arbitrary s.n function on c_{00}. If $|\xi_j| \leq |\eta_j|$ for every $j \in \mathbb{N}$, then $\Phi(\xi) \leq \Phi(\eta)$.*

Proof. Let us assume, without loss of generality, that ξ_j's and η_j's are nonnegative. Needless to say, it suffices to prove the assertion for the following case: $\xi_j = \eta_j$ for $j \neq k$, and $\xi_k = \eta_k$, where k is some positive integer. Let us denote by α the ratio $\frac{\xi_k}{\eta_k}$. Then

$$\Phi(\xi) = \Phi(\xi' + \xi'') \leq \Phi(\xi') + \Phi(\xi''), \tag{7.4.1}$$

where

$$\xi_j' = \frac{1+\alpha}{2}\xi_j, \qquad \xi_j'' = \frac{1-\alpha}{2}\xi_k, \qquad (j \neq k),$$

and

$$\xi_k' = \frac{1+\alpha}{2}\eta_k, \qquad \xi_k'' = -\frac{1-\alpha}{2}\eta_k.$$

Since $\Phi(\xi') = \frac{1+\alpha}{2}\Phi(\eta)$, and since

$$\Phi(\xi'') = \frac{1-\alpha}{2}\Phi\left((\xi_1, \xi_2, ..., -\eta_k, ...)\right) = \frac{1-\alpha}{2}\Phi(\eta),$$

the desired inequality follows from (7.4.1). $\square$

From a given vector $\eta = (\eta_1, ..., \eta_n) \in \mathbb{R}^n$, one can obtain $n!$ vectors by permuting its coordinates, and, from each of these $n!$ vectors, 2^n vectors can be generated by multiplying its coordinates either by $+1$ or by -1. This gives a set of $2^n n!$ vectors in $\mathbb{R}^n$ generated by $\eta \in \mathbb{R}^n$. By $\{\eta^{(\nu)} : 1 \leq \nu \leq 2^n n!\}$ let us denote the set of $2^n n!$ vectors in $\mathbb{R}^n$ generated by $\eta \in \mathbb{R}^n$ in the above sense.

Markus showed in [Mar62] that if $\xi = (\xi)_{j=1}^n, \eta = (\eta_j)_{j=1}^n \in \mathbb{R}^n$ satisfy the following conditions:

(1) $\xi_1 \geq \xi_2 \geq ... \geq \xi_n \geq 0$, $\eta_1 \geq \eta_2 \geq ... \geq \eta_n$; and

(2) $\sum_{j=1}^k \xi_j \leq \sum_{j=1}^k \eta_j$ for every $k \in \{1, ..., n\}$,

then the vector ξ admits the following representation:

$$\xi = \sum_{\nu=1}^{2^n n!} t_\nu \eta^{(\nu)},$$

where $\eta^{(\nu)}$'s ($\nu = 1, 2, ..., 2^n n!$) are the vectors in $\mathbb{R}^n$ generated from $n \in \mathbb{R}^n$ in the sense described above, and t_ν's are nonnegative numbers such that $\sum_{\nu=1}^{2^n n!} t_\nu = 1$, i.e., ξ belongs to the convex hull of the set $\{\eta^{(\nu)} : 1 \leq \nu \leq 2^n n!\}$ of n-dimensional vectors. (This result was also proved by Mitjagin in [Mit64]; this is the proof presented in [GK69]. We choose to skip the proof this result.)

Let us change the setting of our discussion from $\mathbb{R}^n$ to c_{00} with $\xi = (\xi_j)_{j\in\mathbb{N}}, \eta = (\eta_j)_{j\in\mathbb{N}} \in c_{00}$, such that

(1) $\xi_1 \geq \xi_2 \geq ... \geq 0$, and $\eta_1 \geq \eta_2 \geq ... \geq 0$; and

(2) $\sum_{j=1}^k \xi_j \leq \sum_{j=1}^k \eta_j$ for every $k \in \mathbb{N}$.

Suppose $n \in \mathbb{N}$ is the index beyond which all the coordinates of vectors ξ and η equal zero. By abuse of the notation let us continue to write $\eta^{(\nu)}$ for vectors from c_{00} such that their first n coordinates are obtained by permuting the first n coordinates of η and by multiplying them by ± 1. Then by Markus's result discussed above

$$\xi = \sum_{\nu=1}^{2^n n!} t_\nu \eta^{(\nu)}, \text{ and } \sum_{\nu=1}^{2^n n!} t_\nu = 1,$$

with $t_\nu \geq 0$ ($1 \leq \nu \leq 2^n n!$). If $\Phi : c_{00} \to [0, \infty)$ is any s.n.function, then

$$\Phi(\xi) \leq \sum_{\nu=1}^{2^n n!} t_\nu \Phi(\eta^{(\nu)}) \leq \sum_{\nu=1}^{2^n n!} t_\nu \Phi(\eta) = \Phi(\eta) \sum_{\nu=1}^{2^n n!} t_\nu = \Phi(\eta).$$

We have thus proved the following proposition.

Proposition 7.4.3 ([Fan51])**.** *Let* $\xi = (\xi_j)_{j\in\mathbb{N}}, \eta = (\eta_j)_{j\in\mathbb{N}} \in c_{00}$. *If*

(1) $\xi_1 \geq \xi_2 \geq ... \geq \xi_n \geq ... \geq 0$, *and* $\eta_1 \geq \eta_2 \geq ... \geq \eta_n \geq ... \geq 0$; *and*

(2) $\sum_{i=1}^k \xi_i \leq \sum_{j=1}^k \psi_j$ *for every* $k \in \mathbb{N}$,

then for every s.n. function Φ *we have*

$$\Phi(\xi) \leq \Phi(\psi).$$

7.4.2 An alternative definition of symmetric norming functions

The discussion at the end of Section 7.3 suggests why c_{00} is chosen to be the default domain of an arbitrary s.n.function; for our main goal is to construct an s.n.ideal $(\mathfrak{S}, \|\cdot\|_{\mathfrak{S}})$ corresponding to a given s.n.function Φ in such a way that the symmetric norm $\|\cdot\|_{\mathfrak{S}}$ of each operator X in the s.n.ideal $\mathfrak{S}$ agrees with the value of the s.n.function Φ at $(s(X))$, (i.e., $\|X\|_{\mathfrak{S}} = \Phi(s(X))$ for every $X \in \mathfrak{S}$). This requires the domain of the s.n.function to contain the characteristic set $\{s(X) : X \in \mathfrak{S}\}$ of the s.n.ideal $\mathfrak{S}$. Since every ideal in $\mathcal{B}(\mathcal{H})$ contains the ideal $\mathcal{B}_{00}(\mathcal{H})$ of finite rank operators, and since the characteristic set of $\mathcal{B}_{00}(\mathcal{H})$ precisely consists of nonincreasing sequences of nonnegative numbers with a finite number of nonzero terms (this is the sequence set which we denoted by c_{00}^*), the default domain of an arbitrary s.n.function should and must contain the set c_{00}^*. Since c_{00} is the closest vector space to c_{00}^*, it seems natural to define an s.n.function on c_{00}. Here is, however, an interesting question that this discussion leads us to: what could have gone wrong if we would have chosen to have the set c_{00}^* as the default domain of an arbitrary s.n.function instead of c_{00}? Is it feasible to describe a typical s.n.function entirely through its action on the cone c_{00}^*? The answer to the second question is a resounding yes (which of course answers the first question). In what follows, we discuss this aspect.

To every vector $\xi = (\xi_j)_j \in c_{00}$, we associate the unique vector $\xi^* = (\xi_j^*)_j \in c_{00}^*$, where $\xi_j^* = |\xi_{n_j}|$ for every $j \in \mathbb{N}$ and $n_1, n_2, ..., n_j, ...$ is a permutation of the positive integers such that the sequence $(|\xi_{n_j}|)_j$ is nonincreasing. It is known that for any s.n.function Φ defined on c_{00}, we have

$$\Phi(\xi) = \Phi(\xi^*) \text{ for every } \xi \in c_{00}. \tag{7.4.2}$$

Next suppose that $\Phi : c_{00}^* \to [0, \infty)$ is a function defined on the cone c_{00}^* and extend it to c_{00} via the above formula $\Phi(\xi) = \Phi(\xi^*)$ for every $\xi \in c_{00}$. When can this extension be an s.n.function in the sense of the Definition 7.4.1? The next result answers this; it states the necessary and sufficient conditions for a function defined on c_{00}^* to be an s.n. function defined on c_{00} by replacing conditions (i)-(vi) of Definition 7.4.1 with equivalent conditions for vectors from c_{00}^*.

Proposition 7.4.4. *Let $\Phi : c_{00}^* \to [0, \infty)$ be a function, defined on c_{00}^*. The relation*

$$\Phi(\xi) = \Phi(\xi^*) \textit{ for every } \xi \in c_{00},$$

defines an s.n.function on c_{00} if and only if the following conditions are satisfied:

(a) $\Phi(\xi) \geq 0$ *for every* $\xi := (\xi_j)_j \in c_{00}^*$*;*

(b) $\Phi(\xi) = 0 \iff \xi = 0$*;*

(c) $\Phi(\alpha\xi) = \alpha\Phi(\xi)$ *for every* $\xi \in c_{00}^*$ *and for every nonnegative scalar* $\alpha \in \mathbb{R}^+$*;*

(d) $\Phi(\xi + \psi) \leq \Phi(\xi) + \Phi(\psi)$ *for every pair* ξ, ψ *of sequences in* c_{00}^**;*

(e) If $\xi, \psi \in c_{00}^*$ *and* $\sum_{i=1}^{n} \xi_i \leq \sum_{j=1}^{n} \psi_j$ *for every* $n \in \mathbb{N}$*, then* $\Phi(\xi) \leq \Phi(\psi)$*; and*

(f) $\Phi((1, 0, 0, ...)) = 1$.

Proof. Assume that Φ is an s.n.function defined on c_{00}. Then Φ obviously satisfies properties (a)-(d) and the property (f). From Proposition7.4.3, it can be immediately deduced that Φ satisfies the property (e).

Conversely, let the function Φ, defined on c_{00}^*, have the properties (a)-(f). Then the corresponding function Φ (by abuse of notation), defined on c_{00} by the formula (7.4.2) obviously has all the properties (i)-(vi), except possibly the property (iv) in Definition 7.4.1. To see that Φ (defined on c_{00}^*) has the property (iv), we let ξ and η be arbitrary vectors from c_{00} and set $\zeta = \xi + \eta$. Then, it is easy to see that

$$\sum_{j=1}^{n} \zeta_j^* \leq \sum_{j=1}^{n} (\xi_j^* + \eta_j^*), \qquad (n \in \mathbb{N}).$$

Consequently, using the property (e), we obtain

$$\Phi(\xi + \eta) = \Phi(\zeta) = \Phi(\zeta^*) \leq \Phi(\xi^* + \eta^*).$$

On the other hand, the property (d) yields

$$\Phi(\xi^* + \eta^*) \leq \Phi(\xi^*) + \Phi(\eta^*) = \Phi(\xi) + \Phi(\eta).$$

This gives $\Phi(\xi + \eta) \leq \Phi(\xi) + \Phi(\eta)$. Since $\xi, \eta \in c_{00}$ are arbitrary, the result follows. □

Remark 7.4.5. The above proposition serves to be an alternative definition for s.n.functions on c_{00}^*. Due to this result, the requirement of proving a certain property for any s.n.function defined on c_{00} is shifted to the requirement of proving it on c_{00}^*, assuming that the function satisfies the conditions (a)-(f) in the above result.

Example 7.4.6 (Minimal s.n.function)**.** Consider the function $\Phi_\infty : c_{00}^* \to [0, \infty)$ defined by

$$\Phi_\infty(\xi) = \xi_1 \text{ for every } \xi = (\xi_j)_j \in c_{00}^*.$$

This is an s.n.function and is called the *minimal* s.n.function. It can also be realized as an s.n.function defined on c_{00} via the formula

$$\Phi_\infty(\xi) = |\xi_1| \text{ for every } \xi = (\xi_j)_j \in c_{00}.$$

Example 7.4.7 (Maximal s.n.function)**.** Consider the function $\Phi_1 : c_{00}^* \to [0, \infty)$ defined by

$$\Phi_1(\xi) = \sum_j \xi_j \text{ for every } \xi = (\xi_j)_j \in c_{00}^*.$$

This is also an s.n.function and is called the *maximal* s.n.function. Moreover, it can be viewed as an s.n.function defined on c_{00}, in which case, we have

$$\Phi_1(\xi) = \sum_j |\xi_j| \text{ for every } \xi = (\xi_j)_j \in c_{00}.$$

Definition 7.4.8. Let Φ and Ψ be two s.n.functions defined on c_{00}^* (or c_{00}). We say that $\Phi \leq \Psi$ if for every every $\xi \in c_{00}^*$ (or c_{00}), we have $\Phi(\xi) \leq \Psi(\xi)$.

The next proposition justifies the name "minimal" and "maximal" given to the symmetric norming functions Φ_∞ and Φ_1 respectively.

Proposition 7.4.9. *Let Φ be any s.n.function defined on c_{00}^* (or c_{00}). Then*

$$\Phi_\infty \leq \Phi \leq \Phi_1.$$

Proof. Let $\xi = (\xi_j)_j \in c_{00}^*$ be arbitrary. Then there exists $n \in \mathbb{N}$ such that $\xi = (\xi_1, \xi_2, ..., \xi_n, 0, 0, ...)$. Consider the following vectors in c_{00}^*:

$$\xi_{\max} = (\xi_1, 0, 0, ...), \text{ and } \xi_{\text{sum}} = \left(\sum_{j=1}^n \xi_j, 0, 0, ... \right),$$

and observe that $\Phi(\xi_{\max}) = \xi_1$ and $\Phi(\xi_{\text{sum}}) = \sum_{j=1}^n \xi_j$. Then the pair $\xi_{\max}, \xi$ of vectors in c_{00}^* satisfies the hypothesis of Proposition 7.4.2 and hence $\Phi(\xi_{\max}) \leq \Phi(\xi)$, which implies $\xi_1 \leq \Phi(\xi)$. Similarly, since the pair ξ, ξ_{sum} of vectors in c_{00}^* satisfies the hypothesis of Proposition 7.4.2 and hence $\Phi(\xi) \leq \Phi(\xi_{\text{sum}}) = \sum_{j=1}^n \xi_j$. Since $\xi \in c_{00}^*$ is arbitrary, it follows that

$$\Phi_\infty(\xi) \leq \Phi(\xi) \leq \Phi_1(\xi), \qquad (\xi \in c_{00}).$$

□

Proposition 7.4.10. *Every symmetric norming function is continuous on c_{00}.*

Proof. The assertion follows from the inequality below.

$$|\Phi(\xi) - \Phi(\eta)| \leq \Phi(\xi - \eta) \leq \sum_j |\xi_j - \eta_j|.$$

$\square$

Remark 7.4.11. In the above proposition, we actually proved that every symmetric norming function is Lipschitz continuous, Lipschitz constant being 1, which is a much more stronger condition.

7.4.3 Equivalence of symmetric norming functions

Definition 7.4.12 (Equivalence of s.n.functions)**.** Two s.n.functions Φ and Ψ are said to be equivalent if

$$\sup_{\xi \in c_{00}} \frac{\Phi(\xi)}{\Psi(\xi)} < \infty \quad \text{and} \quad \sup_{\xi \in c_{00}} \frac{\Psi(\xi)}{\Phi(\xi)} < \infty.$$

The above definition implies that an s.n.function Φ is equivalent to the minimal s.n.function Φ_∞ if and only if

$$\sup \left\{ \frac{\Phi(\xi)}{\xi_1} : \xi \in c_{00}^* \right\} < \infty \quad \text{and} \quad \sup \left\{ \frac{\xi_1}{\Phi(\xi)} : \xi \in c_{00}^* \right\} < \infty.$$

Since $\xi_1 \leq \Phi(\xi)$, it suffices to show the first condition in order to show that Φ is equivalent to the minimal s.n.function.

Proposition 7.4.13. *For any s.n.function Φ, we have*

$$\sup_{\xi \in c_{00}^*} \frac{\Phi(\xi)}{\xi_1} = \sup_{n \in \mathbb{N}} \Phi(\underbrace{1, 1, ..., 1}_{n}, 0, 0, ...).$$

In particular, Φ is equivalent to the minimal s.n.function Φ_∞ if and only if

$$\sup_n \Phi(\underbrace{1, 1, ..., 1}_{n}, 0, 0, ...) < \infty.$$

Proof. Observe that

$$\frac{\Phi(\xi)}{\xi_1} = \Phi\left(1, \frac{\xi_2}{\xi_1}, \frac{\xi_3}{\xi_1}, ...\right), \qquad \text{where } \frac{\xi_2}{\xi_1} \leq 1, \frac{\xi_3}{\xi_1} \leq 1, \ldots.$$

Consequently, we have

$$\left\{\frac{\Phi(\xi)}{\xi_1} : \xi \in c_{00}^*\right\} = \{\Phi(\xi) : \xi \in c_{00}^* \text{ and } \xi_1 = 1\}, \text{ and hence}$$
$$\sup\left\{\frac{\Phi(\xi)}{\xi_1} : \xi \in c_{00}^*\right\} = \sup\{\Phi(\xi) : \xi \in c_{00}^* \text{ and } \xi_1 = 1\}.$$

Clearly then, for all $n \in \mathbb{N}$,

$$\Phi(\underbrace{1, ..., 1}_{n}, 0, 0, ...) \leq \sup\{\Phi(\xi) : \xi \in c_{00}^* \text{ and } \xi_1 = 1\},$$

which implies

$$\begin{aligned}\sup_n \Phi(\underbrace{1, ..., 1}_{n}, 0, 0, ...) &\leq \sup\{\Phi(\xi) : \xi \in c_{00}^* \text{ and } \xi_1 = 1\} \\ &= \sup_{\xi \in c_{00}^*} \frac{\Phi(\xi)}{\xi_1}.\end{aligned}$$

For the reverse inequality we let $\xi = (\xi_1, ..., \xi_n, 0, 0, ...)$ be an arbitrary but fixed $\xi \in c_{00}^*$. Then

$$\frac{\Phi(\xi)}{\xi_1} = \Phi\left(1, \frac{\xi_2}{\xi_1}, ..., \frac{\xi_n}{\xi_1}, 0, ...\right) \leq \Phi\left(\underbrace{1, ..., 1}_{n}, 0, 0, ...\right) \leq \sup_n \Phi\left(\underbrace{1, ..., 1}_{n}, 0, 0, ...\right).$$

Notice that this inequality is now independent of n. This implies,

$$\sup_{\xi \in c_{00}^*} \frac{\Phi(\xi)}{\xi_1} \leq \sup_n \Phi\left(\underbrace{1, ..., 1}_{n}, 0, 0, ...\right),$$

$\square$

The next result presents a necessary and sufficient condition for an s.n.function to be equivalent to the maximal s.n.function. We skip the proof of this one though.

Proposition 7.4.14. *For any s.n. function Φ, we have*

$$\sup_{\xi \in c_{00}^*} \frac{\sum_j \xi_j}{\Phi(\xi)} = \sup_n \frac{n}{\Phi(\underbrace{1, 1, ..., 1}_{n}, 0, 0, ...)}.$$

In particular, Φ is equivalent to the maximal s.n. function Φ_1 if and only if

$$\sup_n \frac{n}{\Phi(\underbrace{1,1,...,1}_{n},0,0,...)} < \infty.$$

7.4.4 The s.n.functions and the unitarily invariant crossnorms

In this subsection we illustrate the relationship between the class of symmetric norming functions on c_{00}^* and the unitarily invariant crossnorms on $\mathcal{B}_{00}(\mathcal{H})$.

Theorem 7.4.15. *Let $\|\cdot\|_{\mathfrak{S}}$ be a unitarily invariant crossnorm on the ideal $\mathcal{B}_{00}(\mathcal{H})$. Then the equation*

$$\Phi(s(A)) := \|A\|_{\mathfrak{S}}, \text{ for every } s(A) \in \{s(X) : X \in \mathcal{B}_{00}(\mathcal{H})\} \tag{7.4.3}$$

defines an s.n.function Φ on c_{00}^*. Conversely, if Φ is an s.n.function defined on c_{00}^*, then the equation

$$\|A\|_{\mathfrak{S}} := \Phi(s(A)), \text{ for every } A \in \mathcal{B}_{00}(\mathcal{H}) \tag{7.4.4}$$

defines a unitarily invariant crossnorm on $\mathcal{B}_{00}(\mathcal{H})$.

Thus there is a one-to-one correspondence between the unitarily invariant crossnorms on $\mathcal{B}_{00}(\mathcal{H})$ and the class of symmetric norming functions on c_{00}^*.

Proof. Let $\|\cdot\|_{\mathfrak{S}}$ be a unitarily invariant crossnorm on the ideal $\mathcal{B}_{00}(\mathcal{H})$. Fix an orthonormal basis $B = \{\phi_i : i \in I\}$ of $\mathcal{H}$ (of arbitrary dimension), and let $\xi = (\xi_j)_{j\in\mathbb{N}} = (\xi_1, \xi_2, ..., \xi_n, 0, 0, ...) \in c_{00}^*$, where $n \in \mathbb{N}$ is the index beyond which all the coordinates of vector ξ equal zero. Define a diagonal operator D_ξ on $\mathcal{H}$ by $D_\xi = \sum_{i\in I} \xi_i \phi_i \otimes \phi_i$. That is, the matrix of D_ξ with respect to the orthonormal basis B is given by

$$D_\xi = \begin{bmatrix} \xi_1 & & & & & \\ & \ddots & & & \text{\Large 0} & \\ & & \xi_n & & & \\ & & & 0 & & \\ & \text{\Large 0} & & & 0 & \\ & & & & & \ddots \end{bmatrix}.$$

Set $\Phi(\xi) = \left\|\sum_{i\in I} \xi_i \phi_i \otimes \phi_i\right\|_{\mathfrak{S}}$ for every $\xi \in c_{00}^*$. From the properties of unitarily invariant crossnorms on $\mathcal{B}_{00}(\mathcal{H})$, it can be easily verified that Φ is an s.n.function on c_{00}^*.

Conversely, if Φ is some s.n.function defined on c_{00}^*, then Equation (7.4.4) defines a functional which obviously satisfies nonnegativity ($\|A\|_{\mathfrak{S}} \geq 0$), nondegeneracy ($\|A\|_{\mathfrak{S}} = 0 \iff A = 0$), and multiplicativity ($\|\lambda A\|_{\mathfrak{S}} = |\lambda|\|A\|_{\mathfrak{S}}$), for each $A \in \mathcal{B}_{00}(\mathcal{H})$.

To see that it satisfies the triangle inequality, let $A, B \in \mathcal{B}_{00}(\mathcal{H})$. Then by Lemma 2.7.3 we have

$$\sum_{j=1}^{n} s_j(A+B) \leq \sum_{j=1}^{n} s_j(A) + \sum_{j=1}^{n} s_j(B) = \sum_{j=1}^{n} (s_j(A) + s_j(B)).$$

Notice that $s(A+B) = (s_j(A+B))_j$ and $s(A)+s(B) = (s_j(A)+s_j(B))_j$, so that, by using the property (e) of s.n.function in Proposition 7.4.4, we get $\Phi(s(A+B)) \leq \Phi(s(A)+s(B))$. Moreover, $\Phi(s(A) + s(B)) \leq \Phi(s(A)) + \Phi(s(B))$. Consequently,

$$\|A+B\|_{\mathfrak{S}} = \Phi(s(A+B)) \leq \Phi(s(A)) + \Phi(s(B)) = \|A\|_{\mathfrak{S}} + \|B\|_{\mathfrak{S}},$$

which proves the triangle inequality.

For $\|.\|_{\mathfrak{S}}$ to be unitarily invariant, notice that for any pair U, V of unitary operators $s_j(UAV) = s_j(A)$ for every $j \in \mathbb{N}$ and hence $\|UAV\|_{\mathfrak{S}} = \Phi(s(UAV)) = \Phi(s(A)) = \|A\|_{\mathfrak{S}}$.

Finally, for rank one operator A, we have $s_1(A) = \|A\|$ and $s_{j+1}(A) = 0$ for every $j \in \mathbb{N}$. Thus,

$$\|A\|_{\mathfrak{S}} = \Phi(s(A)) = \Phi\left((\|A\|, 0, 0, ...)\right) = \|A\|\Phi((1, 0, 0, ...)) = \|A\|.$$

This proves the theorem and establishes the one-to-one correspondence between the unitarily invariant crossnorms on $\mathcal{B}_{00}(\mathcal{H})$ and the class of s.n.functions on c_{00}^*. $\square$

Recall that in Chapter 2 we proved that every symmetric norm is unitarily invariant, and we promised to prove its converse; but on $\mathcal{B}_{00}(\mathcal{H})$. We are now in a position of proving this much anticipated result—every unitarily invariant crossnorm on $\mathcal{B}_{00}(\mathcal{H})$ is a symmetric norm.

Corollary 7.4.16. *Every unitarily invariant crossnorm on $\mathcal{B}_{00}(\mathcal{H})$ is uniform, and hence a symmetric norm.*

Proof. Let $\|.\|_{\mathfrak{S}}$ be a unitarily invariant crossnorm on $\mathcal{B}_{00}(\mathcal{H})$ and let Φ be the s.n.function on c_{00}^* generated by this norm. We only need to prove the uniformity, i.e., $\|AXB\|_{\mathfrak{S}} \leq \|A\|\|X\|_{\mathfrak{S}}\|B\|$ for every $X \in \mathcal{B}_{00}(\mathcal{H})$ and for every pair A, B of operators in $\mathcal{B}(\mathcal{H})$.

Observe that for any operators $A, B \in \mathcal{B}(\mathcal{H})$ we have $s_j(AXB) \le \|A\|\|B\|s_j(X)$ ($X \in \mathcal{B}_{00}(\mathcal{H}); j \in \mathbb{N}$). This implies that

$$\Phi(s(AXB)) \le \Phi(\|A\|\|B\|s(X)) = \|A\|\|B\|\Phi(s(X)),$$

which yields $\|AXB\|_{\mathfrak{S}} \le \|A\|\|B\|\|X\|_{\mathfrak{S}}$. This proves the assertion. □

Remark 7.4.17. Theorem 7.4.15 together with Corollary 7.4.16 guarantees the existence of a one-to-one correspondence between s.n.functions on c_{00}^* and symmetric norms on $\mathcal{B}_{00}(\mathcal{H})$.

7.5 Symmetrically-normed ideals generated by a symmetric norming function

Having defined s.n.functions, we shall now address our earlier question about how to construct an s.n.ideal from a given s.n.function. We do this in two steps; first we extend the domain of the s.n.function and then we associate a set of operators to this s.n.function.

7.5.1 Step I: Extending the default domain to the natural domain

Suppose $\Phi : c_{00} \to [0, \infty)$ be an arbitrary s.n.function. The default domain of Φ, as that of every s.n.function, is c_{00}. We wish to extend the domain of Φ. Needless to say, the desired extended domain of Φ must be contained in c_0 and should contain c_{00}. To this end, let $\xi = (\xi_j)_{j\in\mathbb{N}} \in c_0$. We define $\xi^{(n)} = (\xi_1, \xi_2, ..., \xi_n, 0, 0, ...)$ for every $n \in \mathbb{N}$. Since $\xi^{(n)} \in c_{00}$ for every $n \in \mathbb{N}$, $\Phi(\xi^{(n)})$ makes sense. It is a trivial observation that for every natural number n, $\Phi(\xi^{(n)}) \le \Phi(\xi^{(n+1)})$ so that the sequence $(\Phi(\xi^{(n)}))_{n\in\mathbb{N}}$ is nondecreasing and we have

$$\sup_n \Phi(\xi^{(n)}) = \lim_{n\to\infty} \Phi(\xi^{(n)}).$$

There is no reason why $\lim_{n\to\infty} \Phi(\xi^{(n)})$ should be finite as it depends on the s.n.function Φ. So, if the limit exists and is finite, then we include the sequence ξ in the (extended) domain of Φ and define

$$\Phi(\xi) := \sup_n \Phi(\xi^{(n)}) = \lim_{n\to\infty} \Phi(\xi^{(n)}).$$

We thus define c_Φ to be the set of all sequences $\xi \in c_0$ for which $\sup_n \Phi(\xi^{(n)}) < \infty$, or $\lim_{n\to\infty} \Phi(\xi^{(n)}) < \infty$. Notice that $c_{00} \subseteq c_\Phi \subseteq c_0$. From the definition of the set c_Φ and the properties of an s.n.function Φ, it can be shown that the set c_Φ is a real linear subspace of c_0, that is,

(a) if $\xi, \eta \in c_\Phi$, then so does $\xi + \eta$; and

(b) if $\alpha \in \mathbb{R}$ and $\xi \in c_\Phi$, then $\alpha\xi \in c_\Phi$.

Lemma 7.5.1. *If $\xi \in c_0$ is a sequence, then $\xi \in c_\Phi \iff \xi^* \in c_\Phi$, and in that case $\Phi(\xi) = \Phi(\xi^*)$.*

Proof. First the backward implication; let us assume that $\xi^* \in c_\Phi$. Indeed, for any $n \in \mathbb{N}$ we have

$$\begin{aligned}\Phi(\xi^{(n)}) &= \Phi(\xi_1, ..., \xi_n, 0, 0, ...)\\ &= \Phi(|\xi_1|, ..., |\xi_n|, 0, 0, ...)\\ &\leq \Phi(\xi_1^*, ..., \xi_n^*, 0, 0, ...)\\ &= \Phi(\xi^{*(n)}),\end{aligned}$$

where the inequality results from Proposition 7.4.3 and consequently, as n approaches infinity, we have $\Phi(\xi) \leq \Phi(\xi^*) < \infty$ thereby ascertaining that $\xi \in c_\Phi$.

Next, we suppose that $\xi \in c_\Phi$ and show that $\xi^* \in c_\Phi$. As before, we fix $n \in \mathbb{N}$ and notice that the truncated sequence $\xi^* = (\xi_1^*, ..., \xi_n^*, 0, 0, ...) \in c_{00}^*$. Then there exist $k_j \in \mathbb{N}, 1 \leq j \leq n$ such that $\xi_1^* = |\xi_{k_1}|, \xi_2^* = |\xi_{k_2}|, ..., \xi_n^* = |\xi_{k_n}|$ where k_j's is a permutation of positive integers such that the sequence $(|\xi_{k_j}|)_{k_j \in \mathbb{N}}$ is nonincreasong. Let $k = \max\{k_1, ..., k_n\}$. Since $k_1, ..., k_n$ are all distinct, we have $k \geq n$. Consider the sequence $\xi^{(k)} \in c_{00}$. This sequence has at most k nonzero terms, n of which are $\pm\xi_1^*, ..., \pm\xi_n^*$ in some order. Since $\Phi(\xi^{(k)})$ does not depend upon the sign and permutation of the terms of the sequence $\xi^{(k)}$, we have $\Phi(\xi^{(k)}) = \Phi(\xi_1^*, ..., \xi_n^*, ..., \xi_k, 0, 0, ...)$, but from Proposition 7.4.2 we have $\Phi(\xi_1^*, ..., \xi_n^*, ..., \xi_k, 0, 0, ...) \geq \Phi(\xi^{*(n)})$ and thus $\Phi(\xi^{(k)}) \geq \Phi(\xi^{*(n)})$. Taking supremum over $k \in \mathbb{N}$ and using the fact that $\xi \in c_\Phi$ we have

$$\Phi(\xi^{*(n)}) \leq \sup_k\{\Phi(\xi^{(k)})\} = \Phi(\xi) < \infty.$$

Now taking the supremum over n, we have $\sup\{\Phi(\xi^{*(n)})\} \leq \Phi(\xi) < \infty$, which guarantees that $\xi^* \in c_\Phi$ and hence $\Phi(\xi^*) = \sup\{\Phi(\xi^{*(n)})\} < \infty$. From the proof of both implications, we have the inequalities $\Phi(\xi) \leq \Phi(\xi^*)$ and $\Phi(\xi^*) \leq \Phi(\xi)$ which proves that $\Phi(\xi) = \Phi(\xi^*)$. $\square$

We use this lemma to prove the next proposition that illustrates an important property of the linear subspace c_Φ, which is generally referred to as the dominance property of c_Φ.

Proposition 7.5.2 (Dominance Property of c_Φ). *Suppose $\xi = (\xi_j)_{j\in\mathbb{N}} \in c_\Phi$. If a sequence $\eta = (\eta_j)_{j\in\mathbb{N}} \in c_0$ satisfies the condition*

$$\sum_{j=1}^{n} \eta_j^* \le \sum_{j=1}^{n} \xi_j^* \text{ for every } n \in \mathbb{N},$$

then $\eta = (\eta_j)_{j\in\mathbb{N}} \in c_\Phi$. Moreover, $\Phi(\eta) \le \Phi(\xi)$.

Proof. Fix $k \in \mathbb{N}$. Consider the truncated sequence $\eta^{(k)} = (\eta_1, ..., \eta_k, 0, 0, ...) \in c_{00}$. Since $\eta \in c_0$, η^* makes sense and $\eta^{*(k)} = (\eta_1^*, ..., \eta_k^*, 0, 0, ...)$. Notice that for every $1 \le \ell \le k$ we have

$$\sum_{j=1}^{\ell} |\eta_j| \le \sum_{j=1}^{\ell} \eta_j^*, \text{ and according to the hypothesis } \sum_{j=1}^{\ell} \eta_j^* \le \sum_{j=1}^{\ell} \xi_j^*.$$

These two inequalities yield

$$\sum_{j=1}^{\ell} |\eta_j| \le \sum_{j=1}^{\ell} \xi_j^* \text{ for every } 1 \le \ell \le k,$$

which, by Proposition 7.4.3, yields $\Phi(|\eta_1|, ..., |\eta_k|, 0, 0, ...) \le \Phi(\xi^{*(k)})$. However, because

$$\Phi(\eta^{(k)}) = \Phi(\eta_1, ..., \eta_k, 0, 0, ...) = \Phi(|\eta_1|, ..., |\eta_k|, 0, 0, ...),$$

it follows that $\Phi(\eta^{(k)}) \le \Phi(\xi^{*(k)})$. Since k is arbitrarily fixed, we get

$$\lim_{k\to\infty} \Phi(\eta^{(k)}) \le \lim_{k\to\infty} \Phi(\xi^{*(k)}) < \infty,$$

where the last inequality follows from the previous lemma and the hypothesis $\xi \in c_0$. This proves that $\eta \in c_\Phi$ and that $\Phi(\eta) \le \Phi(\xi)$. □

It is then a routine matter to verify that the s.n.function Φ preserves its properties described in the Definition 7.4.1, or equivalently, in the Proposition 7.4.4, in the extended domain c_Φ. The space c_Φ is conventionally known as the *natural domain* of the s.n.function Φ.

7.5.2 Step II: Associating an s.n.ideal to the natural domain

Recall that we have arbitrarily fixed the s.n.function Φ at the beginning of our discussion. We want to associate a set of operators to the s.n.function Φ in such a way that this set of operators forms a symmetrically-normed ideal. Since every ideal of $\mathcal{B}(\mathcal{H})$ is contained in $\mathcal{B}_0(\mathcal{H})$, our task boils down to determine which operators from the ideal $\mathcal{B}_0(\mathcal{H})$ should form the set we are looking for. Thus, to the s.n.function Φ, we associate the set $\mathfrak{S}_\Phi$ of all operators $X \in \mathcal{B}_0(\mathcal{H})$ for which $s(X) = (s_j(X))_{j\in\mathbb{N}} \in c_\Phi$. Next we define a norm $\|\cdot\|_\Phi$ on $\mathfrak{S}_\Phi$ by the formula $\|X\|_\Phi := \Phi(s(X))$ for every $X \in \mathfrak{S}_\Phi$ It is not hard to verify that the norm $\|\cdot\|_\Phi$ is symmetric. The following proposition states an obvious criterion for an operator $X \in \mathcal{B}_0(\mathcal{H})$ to be in $\mathfrak{S}_\Phi$.

Proposition 7.5.3. *Let Φ be an s.n.function and $X \in \mathcal{B}_0(\mathcal{H})$. Then $X \in \mathfrak{S}_\Phi$ if and only if $\sup_n \|X_n\|_\Phi < \infty$, where X_n is the n-th partial Schmidt series of the operator X. Moreover, in that case, $\|X\|_\Phi$ is given by $\|X\|_\Phi = \lim_{n\to\infty} \|X_n\|_\Phi = \Phi(s(X))$.*

We complete the construction by showing that the set $\mathfrak{S}_\Phi$ is an s.n.ideal.

Theorem 7.5.4. *Let Φ be an s.n.function. Then the set $(\mathfrak{S}_\Phi, \|\cdot\|_\Phi)$ is an s.n.ideal with $\|X\|_\Phi = \|X\|_{\mathfrak{S}_\Phi} = \Phi(s(X))$ for every $X \in \mathfrak{S}_\Phi$.*

Proof. First we show that $A_1 + A_2 \in \mathfrak{S}_\Phi$ whenever $A_1, A_2 \in \mathfrak{S}_\Phi$. Suppose $A_1, A_2 \in \mathfrak{S}_\Phi$, then $s(A_1), s(A_2) \in c_\Phi$ and hence $s(A_1) + s(A_2) \in c_\Phi$. Since

$$\sum_{j=1}^{n} s_j(A_1 + A_2) \le \sum_{j=1}^{n} s_j(A_1) + \sum_{j=1}^{n} s_j(A_2) = \sum_{j=1}^{n} s_j(A_1) + s_j(A_2), \qquad \text{for each } n \in \mathbb{N},$$

it immediately follows from the dominance property of c_Φ in Proposition 7.5.2 that $s(A_1 + A_2) \in c_\Phi$; for $s(A_1) + s(A_2) \in c_\Phi$. This shows that $A_1 + A_2 \in \mathfrak{S}_\Phi$. Even more, $\Phi(s(A_1 + A_2)) \le \Phi(s(A_1)) + \Phi(s(A_2))$ which means $\|A_1 + A_2\|_\Phi \le \|A_1\|_\Phi + \|A_2\|_\Phi$, and the triangle inequality is proved to hold for $\|.\|_\Phi$.

Next, it is easy to see that if $A \in \mathfrak{S}_\Phi$, then $\lambda A \in \mathfrak{S}_\Phi$ for any complex number λ, and that $\|\lambda A\|_\Phi = |\lambda|\|A\|_\Phi$.

We next proceed to show that $\mathfrak{S}_\Phi$ is complete. Let $(A_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of operators in $(\mathfrak{S}_\Phi, \|.\|_\Phi)$. Then

$$\|A_m - A_n\| = s_1(A_m - A_n) \le \|A_m - A_n\|_\Phi,$$

and so the sequence $(A_n)_{n\in\mathbb{N}}$ of operators is Cauchy with respect to the operator norm $\|.\|$. Then there exists $A \in \mathcal{B}_0(\mathcal{H})$ such that $\lim_{n\to\infty} \|A_n - A\| = 0$. It is then not too hard to convince oneself that for each $j \in \mathbb{N}$, we have $\lim_{n\to\infty} s_j(A_n) = s_j(A)$. Notice that

$$\Phi\left(s_1(A_r), s_2(A_r), ..., s_n(A_r), 0, 0, ...\right) \leq \sup_{p\in\mathbb{N}} \|A_p\|_\Phi \ (<\infty),$$

for every $r \in \mathbb{N}$. Since Φ is continuous, as $r \to \infty$ it follows that

$$\Phi(s_1(A), s_2(A), ..., s_n(A), 0, 0, ...) \leq \sup_{p\in\mathbb{N}} \|A_p\|_\Phi \ (<\infty).$$

Also note that the above inequality is true for every $n \in \mathbb{N}$. This simply means, in our earlier notation, that $\Phi\left(s_j^{(n)}(A)\right) \leq \sup_{p\in\mathbb{N}} \|A_p\|_\Phi < \infty$ for every $n \in \mathbb{N}$, which implies that

$$\|A\|_\Phi = \Phi(s(A)) = \lim_{n\to\infty} \Phi\left(s_j^{(n)}(A)\right) \leq \sup_{p\in\mathbb{N}} \|A_p\|_\Phi < \infty.$$

This shows that $A \in \mathfrak{S}_\Phi$. We still need to show that the sequence $(A_n)_{n\in\mathbb{N}}$ of operators converge to A in the norm $\|.\|_\Phi$ of the space $\mathfrak{S}_\Phi$. Since $(A_n)_{n\in\mathbb{N}}$ is Cauchy in $(\mathfrak{S}_\Phi, \|.\|_\Phi)$, letting $\epsilon > 0$ we obtain $N \in \mathbb{N}$ such that for all natural numbers p, q, the operators A_p and A_q satisfy $\|A_p - A_q\|_\Phi < \epsilon$. This means

$$\Phi\left(s_1(A_p - A_q), s_2(A_p - A_q), ..., s_n(A_p - A_q), 0, 0, ...\right) < \epsilon \qquad (n \in \mathbb{N}; p, q > \mathbb{N}). \tag{7.5.1}$$

Recalling that $\lim_{q\to\infty} s_j(A_p - A_q) = s_j(A_p - A)$ for every $j \in \mathbb{N}$, and letting $q \to \infty$ in (7.5.1) we deduce

$$\Phi\left(s_1(A_p - A), s_2(A_p - A), ..., s_n(A_p - A), 0, 0, ...\right) \leq \epsilon \qquad (n \in \mathbb{N}); p > N).$$

It follows that $\|A_p - A\|_\Phi \leq \epsilon$, $(p \geq N)$, and hence $(\mathfrak{S}_\Phi, \|.\|_\Phi)$ is complete.

Finally we show that $\mathfrak{S}_\Phi$ is an ideal and the norm $\|.\|_\Phi$ is uniform. We know that for each $j \in \mathbb{N}$, $s_j(BAC) \leq \|B\|\|C\|s_j(A)$ for each $B, C \in \mathcal{B}(\mathcal{H})$ and for every $A \in \mathfrak{S}_\Phi$. This essentially shows two things: first, via the dominance property of c_Φ in Proposition 7.5.2, $BAC \in \mathfrak{S}_\Phi$; and second, $\|BAC\|_\Phi \leq \|B\|\|C\|\|A\|_\Phi$ for every $B, C \in \mathcal{B}(\mathcal{H})$ and for every $A \in \mathfrak{S}_\Phi$, which means that $\|.\|_\Phi$ is uniform. This completes the proof. □

Having constructed the s.n.ideal $\mathfrak{S}_\Phi$ from an s.n.function Φ, it is perhaps worth noticing how the dominance property of the natural domain c_Φ discussed in the Propostion 7.5.2 gets transferred to the symmetric norm $\|\cdot\|_\Phi$ on the s.n.ideal $\mathfrak{S}_\Phi$. We can thus consider the following proposition proved.

Proposition 7.5.5 (Dominance Property)**.** *Let Φ be an s.n.function, let $(\mathfrak{S}_\Phi, \|\cdot\|_\Phi)$ be the s.n.ideal generated by Φ and let $A \in \mathfrak{S}_\Phi$. If an operator $B \in \mathcal{B}_0(\mathcal{H})$ satisfies the condition $\sum_{j=1}^n s_j(B) \leq \sum_{j=1}^n s_j(A)$ for every $n \in \mathbb{N}$, then $B \in \mathfrak{S}_\Phi$ and $\|B\|_\Phi \leq \|A\|_\Phi$.*

We end this section with the following result which supplements Theorem 7.5.4.

Proposition 7.5.6. *The s.n.ideals $\mathfrak{S}_{\Phi_1}$ and $\mathfrak{S}_{\Phi_2}$ coincide elementwise if and only if the s.n.functions Φ_1 and Φ_2 are equivalent.*

Proof. If the s.n.functions are equivalent, then the assertion follows trivially. If the s.n.ideals coincide elementwise, then Theorem 7.3.3 yields the result. $\square$

7.6 Two examples of extremal s.n.ideals

In the next couple of examples, we have constructed s.n.ideals corresponding to the minimal and the maximal s.n.functions.

Example 7.6.1 ($\mathfrak{S}_{\Phi_\infty}$)**.** Given the minimal s.n.function Φ_∞, we want to realize the s.n.ideal $\mathfrak{S}_{\Phi_\infty}$ generated by it. That $\mathfrak{S}_{\Phi_\infty} \subseteq \mathcal{B}_0(\mathcal{H})$ is a trivial observation. We claim that $\mathcal{B}_0(\mathcal{H}) \subseteq \mathfrak{S}_{\Phi_\infty}$. Suppose $X \in \mathcal{B}_0(\mathcal{H})$. Then, the sequence $\xi = (\xi_j)$ defined as $\xi := s(X) = (s_j(X)) \in c_0$. Notice that $\xi^{(n)} = (s_1(X), s_2(X), ..., s_n(X), 0, 0, ...)$ which implies that $\Phi_\infty(\xi^{(n)}) = s_1(X) = \|X\|$ for every $n \in \mathbb{N}$, where $\|\cdot\|$ is the operator norm. Consequently, $\lim_{n\to\infty} \Phi_\infty(\xi^{(n)}) = s_1(X) = \|X\| < \infty$. This means that the sequence of singular values of X is in the natural domain c_{Φ_∞} of Φ_∞, that is, $s(X) \in c_{\Phi_\infty}$ which in turn yields $X \in \mathfrak{S}_{\Phi_\infty}$. Since the opertaor X with which we started the discussion is arbitrary, our claim follows and

$$\mathfrak{S}_{\Phi_\infty} = \mathcal{B}_0(\mathcal{H}).$$

We conclude this example by remarking that the natural domain c_{Φ_∞} of Φ_∞ is

$$c_{\Phi_\infty} = \left\{ \xi \in c_0 : \sup_{j\in\mathbb{N}} |\xi_j| < \infty \right\} = c_0.$$

The forward containment is obvious from definition. To show the backward containment, suppose that $\xi \in c_0$. It then suffices to show that $\lim_{n\to\infty} \Phi_\infty(\xi^{(n)}) < \infty$. However, for every $n \in \mathbb{N}$, we have $\Phi_\infty(\xi^{(n)}) = \Phi_\infty(\xi_1, ..., \xi_n, 0, 0, ...) = \max\{|\xi_j| : 1 \leq n\}$ which implies that $\lim_{n\to\infty} \Phi_\infty(\xi^{(n)}) = \sup\{|\xi_j| : j \in \mathbb{N}\}$ which is finite since every convergent sequence is bounded. Thus, $\lim_{n\to\infty} \Phi_\infty(\xi^{(n)}) < \infty$ so that $\xi \in c_{\Phi_\infty}$ and hence $c_0 \subseteq c_{\Phi_\infty}$.

Remark 7.6.2. Let Φ be an arbitrary s.n.function. The s.n.ideals $\mathfrak{S}_\Phi$ and $\mathcal{B}_0(\mathcal{H})$ coincide elementwise if and only if Φ is equivalent to the minimal s.n.function Φ_∞. This result follows immediately from Theorem 7.3.3 and Example 7.6.1.

Example 7.6.3 ($\mathfrak{S}_{\Phi_1}$)**.** Given the maximal s.n.function Φ_1, we want to realize the s.n.ideal $\mathfrak{S}_{\Phi_1}$ generated by it.

First we observe that if $\xi = (\xi_j) \in c_0$, then $\Phi_1(\xi^{(n)}) = \Phi_1(\xi_1, ..., \xi_n, 0, 0, ...) = \sum_{j=1}^n |\xi_j|$ so that $\xi \in c_{\Phi_1}$ if and only if $\lim_{n\to\infty} \Phi_1(\xi^{(n)}) = \lim_{n\to\infty} \sum_{j=1}^n |\xi_j| < \infty$, that is, $\lim_{n\to\infty} s_n < \infty$, where $(s_n)_{n\in\mathbb{N}}$ is the sequence of the partial sums given by $s_n = \sum_{j=1}^n |\xi_j|$, which is possible if and only if the series $\sum_{j=1}^\infty |\xi_j|$ converges absolutely, that is, $\sum_{j=1}^\infty |\xi_j| < \infty$. Thus, the natural domain c_{Φ_1} of Φ_1 is

$$c_{\Phi_1} = \{\xi \in c_0 : \sum_{j=1}^{\infty} |\xi_j| < \infty\}.$$

Next, we claim that

$$\mathfrak{S}_{\Phi_1} = \mathcal{B}_1(\mathcal{H}),$$

which follows from the fact that a compact operator T is trace class if and only if the sum $\sum_j s_j(T)$ of all singular values of T is finite, which in turn is possible if and only if $\Phi_1(s(T)) = \Phi_1(s_j(T)) < \infty$, that is, $T \in \mathfrak{S}_{\Phi_1}$.

Remark 7.6.4. Let Φ be an arbitrary s.n.function. The s.n.ideals $\mathfrak{S}_\Phi$ and $\mathcal{B}_1(\mathcal{H})$ coincide elementwise if and only if Φ is equivalent to the maximal s.n.function Φ_1. This result also follows from Theorem 7.3.3 and Example 7.6.3.

7.7 A criterion for an operator to belong to s.n.ideal $\mathfrak{S}_\Phi$

In this section, we present a criterion for a given operator to belong to a space $\mathfrak{S}_\Phi$, which is used, in particular, to construct what is called the "dual space" of $\mathfrak{S}_\Phi$. For the proof of this criterion, we refer the reader to [GK69, Chapter 3, Section 5]. We, however, list the following lemma (without proof) which is essential for the proof of this criterion.

Recall that if $\mathcal{H}, \mathcal{K}$ are Hilbert spaces, then the weak operator topology (or WOT) on $\mathcal{B}(\mathcal{H}, \mathcal{K})$ is the weak topology on $\mathcal{B}(\mathcal{H}, \mathcal{K})$ induced by the functionals, $T \mapsto \langle Tx, y\rangle$ for all $x \in \mathcal{H}, y \in \mathcal{K}$. Thus, a sequence $T_n \to T_n$ in WOT if and only if $\langle T_n x, y\rangle \to \langle Tx, y\rangle$ for all

$x \in \mathcal{H}, y \in \mathcal{K}$. If $\mathcal{X}, \mathcal{Y}$ are normed spaces, then the strong operator topology (or SOT) on $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ is the topology on $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ induced by the family of seminorms $T \mapsto \|Tx\|$, for all $x \in X$. Thus a sequence $T_n \to T$ strongly, or in SOT, if and only if $T_n(x) \to T(x)$ in the norm topology of Y, for all $x \in X$.

Lemma 7.7.1. *Suppose that $A \in \mathcal{B}(\mathcal{H})$ is the weak limit of a sequence $(A_n)_{n\in\mathbb{N}}$ of operators in $\mathcal{B}_0(\mathcal{H})$, i.e.,*

$$\lim_{n\to\infty} \langle A_n x, y\rangle = \langle Ax, y\rangle, \qquad \text{for every } x, y \in \mathcal{H}.$$

If $\lim_{j\to\infty} \sup_m s_j(A_m) = 0$, then $A \in \mathcal{B}_0(\mathcal{H})$ and

$$\sum_{j=1}^{n} s_j(A) \leq \sum_{j=1}^{n} \sup_m s_j(A_m), \quad (n = 1, 2, \ldots).$$

Let Φ be an arbitrary s.n.function. From Theorem 7.3.3 and Example 7.6.1, we know that the s.n.ideals $\mathfrak{S}_\Phi$ and $\mathcal{B}_0(\mathcal{H})$ coincide elementwise if and only if Φ is equivalent to the minimal s.n.function Φ_∞.

Theorem 7.7.2. *Let Φ be an arbitrary s.n.function not equivalent to the minimal s.n.function Φ_∞ (i.e., $\mathfrak{S}_\Phi$ and $\mathcal{B}_0(\mathcal{H})$ do not coincide elementwise). If an operator $A \in \mathcal{B}(\mathcal{H})$ is the weak limit of a sequence of operators $(A_n)_{n\in\mathbb{N}}$ from $\mathfrak{S}_\Phi$ (i.e., $A_n \to A$ in WOT) and if $\sup_n \|A_n\|_\Phi < \infty$, then $A \in \mathfrak{S}_\Phi$ and $\|A\|_\Phi \leq \sup_n \|A_n\|_\Phi$.*

We now present the required criterion.

Theorem 7.7.3. *Let $\mathfrak{S}_\Phi$ be an s.n.ideal not coinciding elementwise with $\mathcal{B}_0(\mathcal{H})$. Further, let $(P_n)_{n\in\mathbb{N}}$ be a monotonically increasing sequence of finite dimensional orthogonal projections which converges strongly to the identity operator, that is, $\lim_{n\to\infty} \|P_n x - x\| = 0$, for every $x \in \mathcal{H}$. Then $A \in \mathfrak{S}_\Phi$ if and only if $\sup_n \|P_n A P_n\|_\Phi < \infty$.*

Remark 7.7.4. For an s.n.ideal $\mathfrak{S}_\Phi$ which coincides elementwise with $\mathcal{B}_0(\mathcal{H})$ the above theorem does not hold, since the condition $\sup_n \|P_n A P_n\|_\Phi < \infty$ is, in this case, fulfilled for all bounded operators.

7.8 Separable symmetrically-normed ideals

Recall that if $n \in \mathbb{Z}^+$, we use $\mathcal{B}_{00}^n(\mathcal{H})$ to denote the set of finite rank operators on $\mathcal{H}$ with rank less than or equal to n. The following lemma sets the stage for this section.

Lemma 7.8.1. *Let Φ be an s.n.function and let $A \in \mathfrak{S}_\Phi$. Then*

$$\min\{\|A-K\|_\Phi : K \in \mathcal{B}_{00}^n(\mathcal{H})\} = \|A-A_n\|_\Phi = \Phi((s_{n+1}(A), s_{n+2}(A), ...)); \qquad (7.8.1)$$

$$\textit{and} \;\; \inf\{\|A-K\|_\Phi : K \in \mathcal{B}_{00}^n(\mathcal{H})\} = \lim_{n\to\infty} \Phi((s_{n+1}(A), s_{n+2}(A), ...)); \qquad (7.8.2)$$

where A_n $(n = 1, 2, ...)$ is the n-th partial Schmidt series of the operator A.

Proof. From Corollary 2.6.2 we have

$$s_j(A-K) \geq s_{n+j}(A) \qquad (K \in \mathcal{B}_{00}^n(\mathcal{H}); j, n = 1, 2, ...).$$

Since $s_{n+j}(A) = s_j(A-A_n)$, it follows that $s_j(A-K) \geq s_j(A-A_n)$ for every $j \in \mathbb{N}$. Consequently, we have $\Phi(s(A-K)) \geq \Phi(s(A-A_n))$, which gives

$$\|A-K\|_\Phi \geq \|A-A_n\|_\Phi.$$

Taking minimum over the set of all finite rank operators with rank at most n, we get

$$\min\{\|A-K\|_\Phi : K \in \mathcal{B}_{00}^n(\mathcal{H})\} \geq \|A-A_n\|_\Phi.$$

Also, we have

$$\|A-A_n\|_\Phi \geq \min\{\|A-K\|_\Phi : K \in \mathcal{B}_{00}^n(\mathcal{H})\}.$$

Consequently, we get our first assertion,

$$\min\{\|A-K\|_\Phi : K \in \mathcal{B}_{00}^n(\mathcal{H})\} = \|A-A_n\|_\Phi = \Phi((s_{n+1}(A), s_{n+2}(A), ...)).$$

For the second assertion notice that

$$\mathcal{B}_{00}(\mathcal{H}) = \cup_n \mathcal{B}_{00}^n(\mathcal{H}).$$

The result follows immediately from the first one. □

As mentioned earlier, we are interested in minimal symmetrically-normed ideals in $\mathcal{B}_0(\mathcal{H})$. However, there exist symmetrically-normed ideals $\mathfrak{S}_\Phi$ in which the set $\mathcal{B}_{00}(\mathcal{H})$ of finite rank operators are not dense (of course, with respect to the respective Φ-norm). This circumstance suggests the necessity of introducing the subspace $\mathfrak{S}_\Phi^{(0)}$, the closure of $\mathcal{B}_{00}(\mathcal{H})$ in the norm of $\mathfrak{S}_\Phi$. The following result explicitly describes those operators that belong to this subspace.

Proposition 7.8.2. *Let Φ be an s.n.function, $\mathfrak{S}_\Phi$ be an s.n.ideal generated by the s.n.function Φ, and $\mathfrak{S}_\Phi^{(0)}$ be the closure of $\mathcal{B}_{00}(\mathcal{H})$ in the norm of $\mathfrak{S}_\Phi$. Then $\mathfrak{S}_\Phi^{(0)}$ is a subspace consisting of all operators $A \in \mathfrak{S}_\Phi$ for which*

$$\lim_{n\to\infty} \Phi((s_{n+1}(A), s_{n+2}(A), ...)) = 0; \;\; \textit{or}$$

$$\lim_{n,p\to\infty} \Phi((s_{n+1}(A), s_{n+2}(A), ..., s_{n+p}(A), 0, 0, ...)) = 0$$

Proof. Since $\mathfrak{S}_\Phi^{(0)}$ is the closure of $\mathcal{B}_{00}(\mathcal{H})$ in the norm of $\mathfrak{S}_\Phi$, we have $A \in \mathfrak{S}_\Phi^{(0)}$ if and only if there exists $(K_n)_n \subseteq \mathcal{B}_{00}(\mathcal{H})$ such that $\|K_n - A\|_\Phi \to 0$. Now

$$\begin{aligned}
&\inf_{K\in\mathcal{B}_{00}(\mathcal{H})} \|A-K\|_\Phi = \lim_{n\to\infty} (s_{n+1}(A), ...) = 0 \\
\iff & \;\forall\, \epsilon > 0, \exists\, K_\epsilon \in \mathcal{B}_{00}(\mathcal{H}) \text{ such that } 0 < \|A - K_\epsilon\|_\Phi < \epsilon \\
\iff & \;\forall\, n \in \mathbb{N}, \exists\, K_n \in \mathcal{B}_{00}(\mathcal{H}) \text{ such that } 0 \leq \|A - K_n\|_\Phi < \frac{1}{n} \\
\iff & \;\exists \text{ a sequence } \{K_n\} \subseteq \mathcal{B}_{00}(\mathcal{H}) \text{ such that } \|A - K_n\|_\Phi \to 0.
\end{aligned}$$

This proves the result. □

Remark 7.8.3. Of course, $\mathfrak{S}_\Phi^{(0)}$ consists of all $A \in \mathfrak{S}_\Phi$ such that

$$\lim_{n\to\infty} \|A - A_n\|_\Phi = \lim_{n\to\infty} \Phi((s_{n+1}(A), s_{n+2}(A), ...)) = 0.$$

Definition 7.8.4 (Mononormalizing s.n.function)**.** A symmetric norming function Φ is said to be *mononormalizing* if the condition

$$\lim_{n\to\infty} \Phi((s_{n+1}(A), s_{n+2}(A), ...)) = 0,$$

or the equivalent condition

$$\lim_{n,p\to\infty} \Phi((s_{n+1}(A), s_{n+2}(A), ..., s_{n+p}(A), 0, 0, ...)) = 0$$

is fulfilled for every $\xi \in c_\Phi$. Every s.n.function which is not mononormalizing is referred to as *binormalizing*.

Proposition 7.8.5. *Let Φ be an s.n.function, $\mathfrak{S}_\Phi$ be an s.n.ideal generated by the s.n.function Φ, and $\mathfrak{S}_\Phi^{(0)}$ be the closure of $\mathcal{B}_{00}(\mathcal{H})$ in the norm of $\mathfrak{S}_\Phi$. Then the following statements are equivalent:*

(1) *The spaces* $\mathfrak{S}_\Phi$ *and* $\mathfrak{S}_\Phi^{(0)}$ *coincide.*

(2) Φ *is a mononormalizing function.*

Proof. Suppose Φ is mononormalizing. We want to show that $\mathfrak{S}_\Phi = \mathfrak{S}_\Phi^{(0)}$. We know, by definition of $\mathfrak{S}_\Phi^{(0)}$, that $\mathfrak{S}_\Phi^{(0)} \subseteq \mathfrak{S}_\Phi$. To show the reverse containment, let $A \in \mathfrak{S}_\Phi$. This implies $(s(A)) = (s_1(A), s_2(A), ...) \in c_\Phi$, which further implies $\lim_{n\to\infty} \Phi\left((s_{n+1}(A), s_{n+2}(A), ...)\right) = 0$. Thus $A \in \mathfrak{S}_\Phi^{(0)}$.

Conversely, if $\mathfrak{S}_\Phi$ and $\mathfrak{S}_\Phi^{(0)}$ coincide then for all $A \in \mathfrak{S}_\Phi$, $\Phi((s_{n+1}(A), ...)) \to 0$ as $n \to \infty$, which implies that for all $\xi \in \mathfrak{S}_\Phi$, $\Phi((\xi_{n+1}, \xi_{n+2,...})) \to 0$ as $n \to \infty$. Thus Φ is mononormalizing function. $\square$

Proposition 7.8.6. *The minimal and the maximal s.n.functions are mononormalizing.*

Proof. Straightforward. $\square$

Proposition 7.8.7. *Let* Φ *be a mononormalizing s.n.function and* $\mathfrak{S}_\Phi$ *be an s.n.ideal generated by* Φ*. For any operator* $A \in \mathfrak{S}_\Phi$*, the Schimdt series of* A *converges to* A *in the norm* $\|\cdot\|_\Phi$*.*

Proof. The proof is a series of if and only if statements. We have,

$$\begin{aligned} & \Phi \text{ is mononormalizing} \\ \iff & \lim_{n\to\infty} \Phi\left((\xi_{n+1}, \xi_{n+2}, ...)\right) = 0, \qquad \forall\, \xi \in c_\Phi \\ \iff & \lim_{n\to\infty} \Phi\left((s(A - A_n))\right) = 0, \qquad \forall\, A \in \mathfrak{S}_\Phi \\ \iff & \lim_{n\to\infty} \|A - A_n\|_\Phi = 0, \qquad \forall\, A \in \mathfrak{S}_\Phi. \end{aligned}$$

This implies the assertion. $\square$

Theorem 7.8.8. *The subspace* $\mathfrak{S}_\Phi^{(0)}$ *is a separable s.n.ideal of the algebra* $\mathcal{B}(\mathcal{H})$*.*

Proof. Let $A \in \mathfrak{S}_\Phi^{(0)}$ and $B, C \in \mathcal{B}(\mathcal{H})$. Let A_n be the n-th partial Schmidt series of A. Then BA_nC is finite rank operator, i.e., $BA_nC \in \mathcal{B}_{00}(\mathcal{H})$. This implies $BA_nC \in \mathfrak{S}_\Phi^{(0)} = \text{clos}_{\|.\|_\Phi}[\mathcal{B}_{00}(\mathcal{H})]$. Hence $\|BA_nC - BAC\|_\Phi \leq \|B\|\|C\|\|A_n - A\|_\Phi \to 0$, which implies $\lim_{n\to\infty} \|BA_nC - BAC\|_\Phi = 0$. Hence $BAC \in \mathfrak{S}_\Phi^{(0)}$.

Since $\mathcal{H}$ is separable, there exists a countable set $\mathcal{N} = \{v_i\}$ of vectors, dense in $\mathcal{H}$. Define $\mathcal{M}$ to be the set of all finite dimensional operators of the form $K = \sum_j v_j \otimes u_j = \sum_j \langle ., u_j \rangle v_j$

(finite sum), where $u_j, v_j \in \mathcal{N}$. Clearly, $\mathcal{M}$ is a countable set. Also it is easy to see that $\mathcal{M}$ is dense in $\mathcal{B}_{00}(\mathcal{H})$. Since $\mathcal{M}$ is countable and dense in $\mathcal{B}_{00}(\mathcal{H})$ with respect to the norm $\|.\|_\Phi$, and $\mathfrak{S}_\Phi^{(0)}$ is the closure of $\mathcal{B}_{00}(\mathcal{H})$ with respect to the norm $\|.\|_\Phi$, it follows that $\mathfrak{S}_\Phi^{(0)}$ is separable. $\square$

The following theorem proves the converse.

Theorem 7.8.9 ([GK69]). *Every separable s.n.ideal coincides with some ideal $\mathfrak{S}_\Phi^{(0)}$.*

Corollary 7.8.10. *An s.n.ideal $\mathfrak{S}_\Phi^{(0)}$ is nonseparable if and only if the s.n.function Φ is binormalizing, i.e., $\mathfrak{S}_\Phi \neq \mathfrak{S}_\Phi^{(0)}$.*

Theorem 7.8.11 ([GK69]). *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of self-adjoint operators from $\mathcal{B}(\mathcal{H})$ which converges strongly to some operator $X \in \mathcal{B}(\mathcal{H})$. If $\mathfrak{S}$ is a separable s.n.ideal and $A \in \mathfrak{S}$, then the sequences $(X_nA)_{n\in\mathbb{N}}, (AX_n)_{n\in\mathbb{N}}, (X_nAX_n)_{n\in\mathbb{N}}$ converge in the norm of the ideal $\mathfrak{S}$ to the operators XA, AX, and XAX, respectively.*

7.9 The symmetrically-normed ideals $\mathfrak{S}_{\Phi_p}$

Let $1 \leq p \leq \infty$ and consider the function $\Phi_p(\xi) = \left(\sum_j \xi_j^p\right)^{\frac{1}{p}}$ for every $\xi \in c_{00}^*$. It is not too difficult to see that for every $p \in [1,\infty]$, Φ_p is an s.n.function, and that its natural domain c_{Φ_p} coincides with ℓ^p, i.e., it consists of all sequences $\xi = (\xi_j)_{j\in\mathbb{N}} \in c_0$ such that $\sum_j |\xi_j|^p < \infty$. Furthermore, it is obvious that for any sequence $\xi = (\xi_j)_{j\in\mathbb{N}} \in \ell^p$, we have

$$\lim_{n\to\infty}\left(\sum_{j=n+1}^{\infty} |\xi_j|^p\right)^{\frac{1}{p}} = 0,$$

and consequently, the s.n.function Φ_p is mononormalizing for $1 \leq p < \infty$. For $p = \infty$, the s.n.function Φ_p reduces to the minimal s.n.function Φ_∞, which is known to be mononormalizing.

Theorem 7.9.1. *Let $1 \leq p \leq \infty$. Then the s.n.ideal $\mathfrak{S}_{\Phi_p}$ generated by the s.n.function Φ_p consists of compact operators A for which $\sum_{j=1}^{\infty} s_j^p(A) < \infty$, with the norm defined by*

$$\|A\|_{\Phi_p} = \left(\sum_{j=1}^{\infty} s_j^p(A)\right)^{\frac{1}{p}}.$$

Also, for every $1 \leq p \leq \infty$, the s.n.ideal $\mathfrak{S}_{\Phi_p}$ is separable. Furthermore, the set $\mathcal{B}_{00}(\mathcal{H})$ of a finite rank operators is dense in $\mathfrak{S}_{\Phi_p}$ with respect to the norm $\|.\|_{\Phi_p}$, and

$$\min_{K \in \mathcal{B}_{00}^n(\mathcal{H})} \|A - K\|_{\Phi_p} = \left(\sum_{j=n+1}^{\infty} s_j^p(A) \right)^{\frac{1}{p}}, \quad (A \in \mathfrak{S}_{\Phi_p}; n \in \mathbb{N}).$$

7.10 Functions adjoint to s.n.functions

This section is devoted to the study of real valued functions that are "adjoint" to an s.n.function; the end goal is to establish that the adjoint of an s.n.function is itself an s.n.function, and that the adjoint of the adjoint is the function itself.

Definition 7.10.1. Let Φ be an s.n.function defined on c_{00}^*. Then the function given by the formula

$$\Phi^*(\eta) = \sup \left\{ \frac{\sum_j \eta_j \xi_j}{\Phi(\xi)} : \xi \in c_{00}^*, \xi \neq 0 \right\} \qquad (\eta \in c_{00}^*) \tag{7.10.1}$$

is defined to be the *adjoint of the function* Φ.

Other useful formulae for Φ^* are

$$\Phi^*(\eta) = \sup \left\{ \sum_j \eta_j \xi_j : \xi \in c_{00}^*, \Phi(\xi) = 1 \right\} \qquad (\eta \in c_{00}^*) \tag{7.10.2}$$

$$\Phi^*(\eta) = \sup \left\{ \sum_j \eta_j \xi_j : \xi \in c_{00}^*, \Phi(\xi) \leq 1 \right\} \qquad (\eta \in c_{00}^*) \tag{7.10.3}$$

— these numbers turn out to be all the same. In fact, the supremum in the above formulae is attained. To see this consider the formula given in (7.10.3). Notice that $\eta \in c_{00}^*$ is fixed. Let k be the largest index corresponding to a nonzero component of the vector η. Then it is easy to see that

$$\sup \left\{ \sum_j \eta_j \xi_j : \xi \in c_{00}^*, \Phi(\xi) \leq 1 \right\} = \max \left\{ \sum_{j=1}^{k} \eta_j \xi_j : \xi \in c_{00}^*, \Phi(\xi) \leq 1, \xi_j = 0 \, \forall j \geq k \right\}.$$

Thus for each η, $\Phi^*(\eta)$ is attained and hence we have

$$\Phi^*(\eta) = \max \left\{ \frac{\sum_j \eta_j \xi_j}{\Phi(\xi)} : \xi \in c_{00}^*, \xi \neq 0 \right\} \qquad (\eta \in c_{00}^*). \tag{7.10.4}$$

Remark 7.10.2. Notice that for any pair η, ξ of vectors in c_{00}^*,

$$\sum_j \eta_j \xi_j \leq \Phi(\xi)\Phi^*(\eta) \tag{7.10.5}$$

follows directly from the definition of Φ^*.

Theorem 7.10.3 ([GK69]). *A function Φ^* (defined on c_{00}^*) which is adjoint of some s.n.function Φ is itself an s.n.function. The adjoint of the function Φ^* is the function Φ.*

Remark 7.10.4. Let $\Phi(\xi)$ $(\xi \in c_{00}^*)$ be some s.n.function. If the function $\psi(\xi)$ $(\xi \in c_{00}^*)$ satisfies the condition

$$\sum_j \xi_j \eta_j \leq \Phi(\xi)\Psi(\eta) \qquad (\xi, \eta \in c_{00}^*) \tag{7.10.6}$$

and for any vector ξ (or, η) from c_{00}^*, one can find a vectore η (or, ξ) for which the equality holds in (7.10.6), i.e.,

$$\sum_j \xi_j \eta_j \leq \Phi(\xi)\Psi(\eta), \tag{7.10.7}$$

then we have

$$\Phi^*(\eta) = \Psi(\eta) \qquad (\eta \in c_{00}^*).$$

Example 7.10.5. Let us consider the s.n.function

$$\Phi_p(\xi) = \left(\sum_j \xi_j^p\right)^{\frac{1}{p}}, \quad (1 \leq p \leq \infty; \xi \in c_{00}^*).$$

By Hölder's inequality, we have

$$\sum_j \eta_j \xi_j \leq \Phi_p(\xi)\Phi_q(\eta), \quad \text{for every } \xi, \eta \in c_{00}^*, \tag{7.10.8}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Now, for any fixed $\xi \in c_{00}^*$, equality holds in Inequality (7.10.8) if and only if $\eta_j = c\xi_j^{p-1}$ $(j = 1, 2, \ldots)$, where c is an arbitrary positive constant; and for any fixed $\eta \in c_{00}^*$, equality holds in Inequality (7.10.8) if and only if $\xi_j = d\eta_j^{q-1}$ $(j = 1, 2, \ldots)$ for some positive constant d. Thus,

$$\Phi_p^*(\eta) = \Phi_q(\eta), \quad \left(\eta \in c_{00}^*; \frac{1}{p} + \frac{1}{q} = 1\right).$$

7.11 Symmetrically-normed ideals and their dual spaces

Definition 7.11.1. If X is a normed linear space, then we define X^* to be $\mathcal{B}(X, \mathbb{C})$. Then X^* is said to be the *conjugate* or *dual space* of X, and an element of X^* is often called a *bounded linear functional* on X.

Of course for $\varphi \in X^*$, we have $\|\varphi\| = \sup\{|\varphi(x)| : x \in X, \|x\| \leq 1\}$, and that X^* is a Banach space. We recall few notations and fundamental facts before proceeding further.

For any set I (equipped with the discrete topology so that any function on I is continuous), we use $\ell^\infty(I)$ to denote the collection of all bounded complex functions $f : I \to \mathbb{C}$ with $\|f\| = \sup\{\|f(i)\| : i \in I\}$. It is a well known fact that $\ell^\infty(I)$ is a Banach space with respect to this norm. Further, we reserve $c_0(I)$ to denote the set of all functions $f : I \to \mathbb{C}$ in $\ell^\infty(I)$ such that for every $\epsilon > 0$, the set $\{i \in I : |f(i)| \geq \epsilon\}$ is finite. Again, $c_0(I)$ is a closed subspace of $\ell^\infty(I)$ and hence a Banach space. If $I = \mathbb{N}$, the usual notation for these spaces are simply ℓ^∞ and c_0; ℓ^∞ consists of all bounded sequences of scalars (complex numbers) and c_0 consists of all sequences of scalars that converge to zero. In addition, let $\ell^1(\mathbb{N})$ (or, simply ℓ^1) denote the collection of all complex functions φ on $\mathbb{N}$ such that $\sum_{n=1}^\infty |\varphi(n)| < \infty$, with the norm defined by $\|\varphi\|_1 = \sum_{n=1}^\infty |\varphi(n)|$. Alternatively, ℓ^1 consists of all sequences $x = (x_n)_{n\in\mathbb{N}}$ of scalars (complex numbers) such that $\sum_{n=1}^\infty |x_n| < \infty$, with the norm defined by $\|x\|_1 = \sum_{n=1}^\infty |x_n|$. It is again a fundamental fact that ℓ^1 is a Banach space with respect to this norm. In general, consider a real number p with the property that $1 \leq p < \infty$ and let $\ell^p(\mathbb{N})$ (or, simply ℓ^p) denote the set of all complex valued functions φ on $\mathbb{N}$ such that $\sum_{n=1}^\infty |\varphi(n)|^p < \infty$, with the norm defined by $\|\varphi\|_p = \left(\sum_{n=1}^\infty |\varphi(n)|^p\right)^{1/p}$. In other words, for $1 \leq p < \infty$, the space ℓ^p consists of all sequences $x = (x_n)_{n\in\mathbb{N}}$ of complex numbers such that $\sum_{n=1}^\infty |x_n|^p < \infty$, with the norm defined by $\|x\|_p = \left(\sum_{n=1}^\infty |x_n|^p\right)^{1/p}$. It is yet another well known fact in functional analysis that for each real number p satisfying $1 \leq p < \infty$, the space ℓ^p is a Banach space with respect to the vector p norm $\|\cdot\|_p$.

We next recall the following elementary definition; it tells us when to think of two Banach spaces to be the "same".

Definition 7.11.2. Two Banach spaces X and Y are considered to be (copies of) the same space if there exists an isometric isomorphism of X onto Y, and we write $X \cong Y$ or $Y \cong X$ isometrically.

For the convenience of the reader, let us briefly illustrate the problem of identifying dual spaces. The dual of c_0 is (isometrically isomorphic to) ℓ^1, and the dual of ℓ^1 is ℓ^∞, that is,

$$(c_0)^* \cong \ell^1 \text{ isometrically, and } \left(\ell^1\right)^* \cong \ell^\infty \text{ isometrically.}$$

In addition, for each $p \in (1, \infty)$, we have

$$(\ell^p)^* \cong \ell^q, \text{ where } \frac{1}{p} + \frac{1}{q} = 1.$$

How about unraveling the one-to-one correspondence between these spaces? As for example, the assertion $(\ell^1)^* \cong \ell^\infty$ isometrically, amounts to convey the following details.

Proposition 7.11.3. *For every bounded linear functional $f \in (\ell^1)^*$, there exists a sequence $\beta = (\beta_j)_{j \in \mathbb{N}} \in \ell^\infty$ such that the general form of f is given by the formula*

$$f(\xi) = \sum_{j=1}^{\infty} \xi_j \beta_j, \text{ for every } \xi = (\xi_j) \in \ell^1,$$

and $\|f\|_1 = \sup\{|f(\xi)| : \xi \in \ell^1,\ \|\xi\|_1 \le 1\} = \|\beta\|$.

The above proposition is, in fact, a special case of the following result. Following the same analogy, the roles of the spaces c_0, ℓ^1, and ℓ^∞ are played, respectively, by the spaces $\mathcal{B}_0(\mathcal{H})$, $\mathcal{B}_1(\mathcal{H})$, and $\mathcal{B}(\mathcal{H})$. In fact, to these fundamental results, there correspond, in a certain sense, analogous results for the dual of separable s.n.ideal of $\mathfrak{S}_\Phi$.

Theorem 7.11.4. *To every bounded linear functional f on the space $\mathcal{B}_1(\mathcal{H})$ of trace class operators, we associate an element $A \in \mathcal{B}(\mathcal{H})$ such that $f(X) = \operatorname{Tr}(XA)$, for every $X \in \mathcal{B}_1(\mathcal{H})$, and*

$$\|f\| = \sup\{|\operatorname{Tr}(XA)| : X \in \mathcal{B}_1(\mathcal{H}),\ \|X\|_1 \le 1\} = \|A\|.$$

Thus the dual of $\mathcal{B}_1(\mathcal{H})$ is isometrically isomorphic to $\mathcal{B}(\mathcal{H})$, that is, $(\mathcal{B}_1(\mathcal{H}))^* \cong \mathcal{B}(\mathcal{H})$ isometrically.

Theorem 7.11.5. *Let Φ be an arbitrary s.n.function not equivalent to the maximal s.n.function Φ_1. Then for each bounded linear functional f on the separable space $\mathfrak{S}_\Phi^{(0)}$, there exists an operator $A \in \mathfrak{S}_{\Phi^*}$ such that the general form of f is given by the formula $f(X) = \operatorname{Tr}(XA)$ for every $X \in \mathfrak{S}_\Phi^{(0)}$, and*

$$\|f\| = \sup\{|\operatorname{Tr}(XA)| : X \in \mathfrak{S}_\Phi^{(0)}, \|X\|_\Phi \le 1\} = \|A\|_{\Phi^*}.$$

Thus, the dual of separable space $\mathfrak{S}_\Phi^{(0)}$ is isometrically isomorphic to $\mathfrak{S}_{\Phi^*}$, that is, ${\mathfrak{S}_\Phi^{(0)}}^* \cong \mathfrak{S}_{\Phi^*}$ isometrically. In particular, if both functions Φ and Φ^* are mononormalizing, the space $\mathfrak{S}_\Phi$ is reflexive.

For $1 < p \leq \infty$, it can be verified that the mononormalizing function

$$\Phi_p(\xi) = \left(\sum_j \xi_j^p\right)^{\frac{1}{p}}, \quad (\xi \in c_{\Phi_p}),$$

is not equivalent to the maximal s.n.function Φ_1. Since $\Phi_p^* = \Phi_q$, where $\frac{1}{p} + \frac{1}{q} = 1$, we can safely consider the following theorem fully proved.

Theorem 7.11.6. *Let $1 < p \leq \infty$ and let $\mathfrak{S}_\Phi^{(0)}$ be the s.n.ideal generated by the mononormalizing s.n.function Φ_p. Then for each bounded linear functional f on the (separable) space $\mathfrak{S}_{\Phi_p}$, there exists an operator $A \in \mathfrak{S}_{\Phi_q}$ (where $\frac{1}{p} + \frac{1}{q} = 1$) such that the general form of f is given by the formula $f(X) = \operatorname{Tr}(XA)$ for every $X \in \mathfrak{S}_{\Phi_p}$, and*

$$\|f\| = \sup\{|\operatorname{Tr}(XA)| : X \in \mathfrak{S}_{\Phi_p}, \|X\|_{\Phi_p} \leq 1\} = \|A\|_{\Phi_q}.$$

Thus the dual $\mathfrak{S}_{\Phi_p}^*$ of $\mathfrak{S}_{\Phi_p}$ is isometrically isomorphic to $\mathfrak{S}_{\Phi_q}$, that is, $\left(\mathfrak{S}_{\Phi_p}\right)^* = \left(\mathfrak{S}_{\Phi_q}\right)$ isometrically. Consequently, for each $p \in (1, \infty)$, the s.n.ideal $\mathfrak{S}_{\Phi_p}$ is reflexive.

In Section 2.8 of Chapter 2 the notion of s-numbers of an arbitrary operator on $\mathcal{H}$ was introduced and studied. Using the definition and properties of the s-numbers of an arbitrary operator, one can generalize Theorem 7.11.4 to any separable s.n.ideal $\mathfrak{S}_\Phi^{(0)}$ generated by an arbitrary s.n.function Φ equivalent to the maximal s.n.function Φ_1. The formulation of this theorem would also require the notion of a symmetric norm on $\mathcal{B}(\mathcal{H})$ — a notion which we have elaborately studied in Chapter 2, Section 2.9 and Chapter 7, Section **??**.

We give a brief description of the construction of a symmetric norm on $\mathcal{B}(\mathcal{H})$ from a given s.n.function that is equivalent to the minimal s.n.function. Let Φ be an arbitrary s.n.function equivalent to the minimal one. Then the formula

$$\|A\|_\Phi = \lim_{n\to\infty} \Phi\left((s_1(A), s_2(A), ..., s_n(A), 0, 0, ...)\right), \text{ for every } A \in \mathcal{B}(\mathcal{H}), \tag{7.11.1}$$

defines a symmetric norm on $\mathcal{B}(\mathcal{H})$. Let us examine why the limit of the right hand side exists and finite. Since Φ is equivalent to the minimal s.n.function Φ_∞, we have

$$\lim_{n\to\infty} \Phi((\underbrace{1, 1, ..., 1}_{n}, 0, 0, ...)) < \infty \quad \text{or} \quad \sup_n \Phi((\underbrace{1, 1, ..., 1}_{n}, 0, 0, ...)) < \infty.$$

So for any $A \in \mathcal{B}(\mathcal{H})$, we have

$$\begin{aligned}
\|A\|_\Phi &= \sup_n \Phi\left((s_1(A), s_2(A), ..., s_n(A), 0, 0, ...)\right) \\
&= s_1(A) \sup_n \Phi\left(\left(1, \frac{s_2(A)}{s_1(A)}, ..., \frac{s_n(A)}{s_1(A)}, 0, 0, ...\right)\right) \\
&\leq \|A\| \sup_n \Phi((\underbrace{1, 1, ..., 1}_{n}, 0, 0, ...)) < \infty.
\end{aligned}$$

It is then a routine matter to verify that the norm $\|.\|_\Phi$ on $\mathcal{B}(\mathcal{H})$ is indeed a symmetric norm. We let $(\mathcal{B}(\mathcal{H}), \|.\|_\Phi)$ denote the algebra of operators on $\mathcal{H}$ equipped with the symmetric norm $\|.\|_\Phi$.

Theorem 7.11.7. *Let Φ be an arbitrary s.n.function equivalent to the maximal s.n.function Φ_1. Then for each bounded linear functional f on the separable space $\mathfrak{S}_\Phi$, there exists an operator $A \in \mathcal{B}(\mathcal{H})$ such that the general form of f is given by the formula $f(X) = \mathrm{Tr}(XA)$ for every $X \in \mathfrak{S}_\Phi$, and*

$$\|f\| = \sup\{|\mathrm{Tr}(XA)| : X \in \mathfrak{S}_\Phi, \|X\| \leq 1\} = \|A\|_{\Phi^*}.$$

Thus, the dual space $(\mathfrak{S}_\Phi)^*$ is isometrically isomorphic to $(\mathcal{B}(\mathcal{H}), \|.\|_{\Phi^*})$, that is, $(\mathfrak{S}_\Phi)^* \cong (\mathcal{B}(\mathcal{H}), \|.\|_{\Phi^*})$ isometrically.

Chapter 8

Counterexamples: Norms that are not attained by the identity

The setting for our discussion is a separable infinite-dimensional Hilbert space $\mathcal{H}$. This chapter, using the theory of symmetrically-normed ideals, primarily aims at extending the concept of "norming" and "absolutely norming" operators from the usual operator norm to arbitrary symmetric norms that are equivalent to the operator norm. That accomplished, the final goal (and one of the most important goals) of this chapter is to construct the promised symmetric norm on $\mathcal{B}(\mathcal{H})$ with respect to which even the identity operator does not attain its norm. In fact, we present a family of symmetric norms on $\mathcal{B}(\mathcal{H})$ with respect to each of which the identity operator is rendered nonnorming. We begin by recalling few background results for this discussion.

8.1 Background and motivation

Henceforth, we assume $\mathcal{H}$ to be a separable Hilbert space.

In [GK69, Chapter 3, Section 14] there are examples of s.n. ideals in which the set $\mathcal{B}_{00}(\mathcal{H})$ of finite rank operators is not dense. This circumstance suggests the necessity of introducing the subspace $\mathfrak{S}_\Phi^{(0)}$, the norm closure of the set $\mathcal{B}_{00}(\mathcal{H})$ in the norm of $\mathfrak{S}_\Phi$, that is,

$$\mathfrak{S}_\Phi^{(0)} := \text{clos}_{\|\cdot\|_\Phi}[\mathcal{B}_{00}(\mathcal{H})].$$

In our exposition we will need the following elementary piece of folklore from [GK69], which we have already seen in the previous chapter.

Proposition 8.1.1. [GK69, Chapter 3, Theorems 12.2 and 12.4] *Let Φ be an arbitrary s.n. function.*

1. *If Φ is not equivalent to the maximal s.n. function, then the general form of a continuous linear functional f on the separable space $\mathfrak{S}_\Phi^{(0)}$ is given by $f(X) = \mathrm{Tr}(AX)$ for some $A \in \mathfrak{S}_{\Phi^*}$ and*

$$\|f\| := \sup\{|\operatorname{Tr}(AX)| : X \in \mathfrak{S}_\Phi^{(0)}, \|X\|_\Phi \le 1\} = \|A\|_{\Phi^*}.$$

   Thus, the dual space of $\mathfrak{S}_\Phi^{(0)}$ is isometrically isomorphic to $\mathfrak{S}_{\Phi^*}$, that is, $\mathfrak{S}_\Phi^{(0)^*} \cong \mathfrak{S}_{\Phi^*}$. In particular, if both functions Φ and Φ^* are mononormalizing, the space $\mathfrak{S}_\Phi$ is reflexive.

2. *If Φ is equivalent to the maximal s.n. function, then the general form of a continuous linear functional f on the separable space $\mathfrak{S}_\Phi$ is given by $f(X) = \mathrm{Tr}(AX)$ for some $A \in \mathcal{B}(\mathcal{H})$ and*

$$\|f\| := \sup\{|\operatorname{Tr}(AX)| : X \in \mathfrak{S}_\Phi, \|X\|_\Phi \le 1\} = \|A\|_{\Phi^*}.$$

   Thus, the dual space $\mathfrak{S}_\Phi^*$ is isometrically isomorphic to $(\mathcal{B}(\mathcal{H}), \|\cdot\|_{\Phi^*})$, that is, $\mathfrak{S}_\Phi^* \cong (\mathcal{B}(\mathcal{H}), \|\cdot\|_{\Phi^*})$.

Let Φ be an arbitrary s.n.function equivalent to the maximal one. The remaining part of this section is intended to provide the motivation to establish the notion of "Φ^*-norming" operators on the s.n.ideal $(\mathcal{B}(\mathcal{H}), \|\cdot\|_{\Phi^*})$ which agrees with the Definitions 4.1.3, 5.1.1, and 6.1.3, and in essentially the same spirit, generalizes the concept. In what follows, we meet this purpose by establishing a sequence of results which provide us with the machinery required to convert the concept of norming operators in the language of s.n.ideals.

8.1.1 Norming operators

Theorem 8.1.2. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $T \in \mathcal{N}(\mathcal{H})$ in the sense of the Definition 1.1.1 if and only if there exists an operator $K \in \mathcal{B}_1(\mathcal{H})$ with $\|K\|_1 = 1$ such that $|\operatorname{Tr}(TK)| = \|T\|$.*

Proof. We first assume that $T \in \mathcal{N}(\mathcal{H})$. Then there exists x in the unit sphere of $\mathcal{H}$ such that $\|Tx\| = \|T\|$. Let

$$y = \frac{Tx}{\|Tx\|},$$

and define a rank one operator $K_0 := x \otimes y \in \mathcal{B}_1(\mathcal{H})$. Notice that $\|K_0\|_1 = 1$ and

$$|\operatorname{Tr}(TK_0)| = |\operatorname{Tr}(T(x \otimes y))| = |\operatorname{Tr}(Tx \otimes y)| = |\langle Tx, y\rangle| = \|Tx\| = \|T\|,$$

which proves the forward implication. To see the backward implication, we assume that there exists an operator $K \in \mathcal{B}_1(\mathcal{H})$ such that $\|K\|_1 = 1$ and $|\operatorname{Tr}(TK)| = \|T\|$. Since $\mathcal{B}_1(\mathcal{H}) \subseteq \mathcal{B}_0(\mathcal{H})$, the Schmidt expansion allows us to write $K = \sum_j s_j(K)(x_j \otimes y_j)$, where $\{x_j\}$ is an orthonormal basis of clos[ran K] and $\{y_j\}$ is an orthonormal basis of clos[ran $|K|$]. (We would like to bring to the reader's attention the convention we have been following since Chapter 2: unless otherwise stated, it is assumed that in every expression of the form $\sum_j \lambda_j e_j \otimes f_j$ in this thesis, $\{e_j\}_j$ and $\{f_j\}_j$ are orthonormal sets of vectors and all scalars λ_j are nonzero.) We now have

$$\begin{aligned}
\|T\| &= |\operatorname{Tr}(TK)| \\
&= \left|\operatorname{Tr}\left(T\left(\sum_j s_j(K)(x_j \otimes y_j)\right)\right)\right| \\
&= \left|\sum_j s_j(K)\operatorname{Tr}(Tx_j \otimes y_j)\right| \\
&= \left|\sum_j s_j(K)\,\langle Tx_j, y_j\rangle\right| \\
&\leq \sum_j s_j(K)|\langle Tx_j, y_j\rangle| \\
&\leq \sum_j s_j(K)\|T\| \\
&= \|T\|\|K\|_1 \\
&= \|T\|,
\end{aligned}$$

which forces all inequalities to be equalities, so

$$\sum_j s_j(K)\|Tx_j\| = \sum_j s_j(K)\|T\|,$$

which implies that $\sum_j s_j(K)(\|T\| - \|Tx_j\|) = 0$. Notice that, for every j, $s_j(K) > 0$ and $\|T\| - \|Tx_j\| \geq 0$. Thus for every j we have $\|T\| = \|Tx_j\|$ which implies that $T \in \mathcal{N}$. This completes the proof. $\square$

8.1.2 [2]-Norming operators

Lemma 8.1.3. *If* Φ *and* Ψ *are s.n.functions defined as*

$$\Phi(\eta) = \max\left\{\eta_1, \frac{\sum_j \eta_j}{2}\right\};$$
$$\Psi(\xi) = \xi_1 + \xi_2;$$

with $\eta = (\eta_i)_{i\in\mathbb{N}}$, *and* $\xi = (\xi_j)_{j\in\mathbb{N}} \in c_{00}^*$, *then* Φ *and* Ψ *are mutually adjoint, that is,*

$$\Phi^* = \Psi \text{ and } \Psi^* = \Phi.$$

Proof. Recall that the adjoint Ψ^* of the s.n.function Ψ is given by

$$\Psi^*(\eta) = \max\left\{\sum_j \eta_j\xi_j : \xi \in c_{00}^*, \Psi(\xi) = 1\right\}, \text{ for every } \eta \in c_{00}^*,$$

which can be rewritten as

$$\Psi^*(\eta) = \max\left\{\sum_j \eta_j\xi_j : \xi \in c_{00}^*, \xi_1 + \xi_2 = 1\right\}.$$

Since $\xi \in c_{00}^*$, it is a nonincreasing sequence which allows us to infer that

$$\begin{aligned}
\Psi^*(\eta) &= \max\left\{\eta_1\xi_1 + \left(\sum_{j\neq 1}\eta_j\right)\xi_2 : \xi_1 \geq \xi_2, \xi_1+\xi_2 = 1\right\}\\
&= \max\left\{\left(\eta_1 - \left(\sum_{j\neq 1}\eta_j\right)\right)\xi_1 + \left(\sum_{j\neq 1}\eta_j\right) : \xi_1 \in \left[\frac{1}{2}, 1\right]\right\}\\
&= \begin{cases} \eta_1 & \text{if } \eta_1 \geq \sum_{j\neq 1}\eta_j \\ \frac{\sum_j \eta_j}{2} & \text{if } \eta_1 \leq \sum_{j\neq 1}\eta_j \end{cases}\\
&= \max\left\{\eta_1, \frac{\sum_j \eta_j}{2}\right\},
\end{aligned}$$

which is indeed equal to $\Phi(\eta)$. Here the penultimate equality is a direct consequence of the fact that a function of the form $ax + b, x \in [\frac{1}{2}, 1]$ achieves its maximum at $x = 1$ if $a > 0$, and at $x = \frac{1}{2}$ if $a < 0$. The final equality arrives from the following simple calculation:

$$\eta_1 \geq \sum_{j\neq 1}\eta_j \iff \eta_1 \geq \frac{\sum_j \eta_j}{2} \text{ and } \eta_1 \leq \sum_{j\neq 1}\eta_j \iff \eta_1 \leq \frac{\sum_j \eta_j}{2}.$$

This completes the proof. $\square$

Remark 8.1.4. It is easy to see that Ψ is equivalent to the minimal s.n.function and that it corresponds to the Ky Fan 2-norm on $\mathcal{B}(\mathcal{H})$. Notice that

$$\sup_n \left\{ \frac{n}{\Psi^*(\underbrace{1, 1, ..., 1}_{n}, 0, 0, ...)} \right\} = \sup_n \left\{ \frac{n}{\max\{1, \frac{n}{2}\}} \right\} = 2 < \infty,$$

which implies that $\Phi = \Psi^*$ is equivalent to the maximal s.n.function Φ_1. Consequently, the dual $\mathfrak{S}_\Phi^*$ of the s.n.ideal $\mathfrak{S}_\Phi$ is isometrically isomorphic to the space $(\mathcal{B}(\mathcal{H}), \|\cdot\|_{\Phi^*})$, that is, $\mathfrak{S}_\Phi^* \cong (\mathcal{B}(\mathcal{H}), \|\cdot\|_{\Phi^*})$. Moreover, the s.n.ideal $\mathfrak{S}_\Phi$ generated by Φ and the ideal $\mathcal{B}_1(\mathcal{H})$ of trace class operators coincide elementwise. Clearly, Φ and Φ^* are s.n.functions considered on their natural domain instead of merely c_{00}^*.

Theorem 8.1.5. *Let $T \in \mathcal{B}(\mathcal{H}), \Phi$ be an s.n.function equivalent to the maximal s.n.function defined by*

$$\Phi(\eta) = \max\left\{\eta_1, \frac{\sum_j \eta_j}{2}\right\},$$

where $\eta = (\eta_i)_{i\in\mathbb{N}} \in c_\Phi$, and let Φ^* be its dual norm so that $\mathfrak{S}_\Phi^* \cong (\mathcal{B}(\mathcal{H}), \|\cdot\|_{\Phi^*})$ isometrically with $\|T\|_{\Phi^*} = \|T\|_{[2]}$ for every $T \in \mathcal{B}(\mathcal{H})$. Then $T \in \mathcal{N}_{[2]}(\mathcal{H})$ in the sense of the Definition 4.1.3 if and only if there exists an operator $K \in \mathfrak{S}_\Phi = \mathcal{B}_1(\mathcal{H})$ with $\|K\|_\Phi = 1$ such that $|\operatorname{Tr}(TK)| = \|T\|_{\Phi^*}$.

Proof. First we assume that $T \in \mathcal{N}_{[2]}$. There exist $x_1, x_2 \in \mathcal{H}$ with $\|x_1\| = \|x_2\| = 1$ and $x_1 \perp x_2$ such that $\|T\|_{\Phi^*} = \|T\|_{[2]} = \|Tx_1\| + \|Tx_2\|$. Let

$$y_1 = \frac{Tx_1}{\|Tx_1\|},\ y_2 = \frac{Tx_2}{\|Tx_2\|},$$

and define $K := \sum_{j=1}^2 x_j \otimes y_j$. That $K \in \mathcal{B}_1(\mathcal{H})$ and $s_1(K) = s_2(K) = 1$ is obvious, so

$\|K\|_\Phi = 1$. Then

$$\begin{aligned}
|\operatorname{Tr}(TK)| &= \left|\operatorname{Tr}\left(T\left(\sum_{j=1}^{2} x_j \otimes y_j\right)\right)\right| \\
&= |\sum_{j=1}^{2} \langle Tx_j, y_j\rangle| \\
&= \left|\sum_{j=1}^{2} \left\langle Tx_j, \frac{Tx_j}{\|Tx_j\|}\right\rangle\right| \\
&= \sum_{j=1}^{2} \|Tx_j\| = \|T\|_{\Phi^*}.
\end{aligned}$$

This finishes the proof of the forward implication. To see the backward implication, we assume that there exists an operator $K \in \mathcal{B}_1(\mathcal{H})$ with $\|K\|_\Phi = 1$ such that $|\operatorname{Tr}(TK)| = \|T\|_{\Phi^*}$. Let $\alpha := \|T\|_{\Phi^*} = \|T\|_{[2]} = s_1(T) + s_2(T)$. Consequently,

$$\begin{aligned}
\alpha &= \|T\|_{\Phi^*} \\
&= |\operatorname{Tr}(TK)| \\
&= \left|\operatorname{Tr}\left(T\left(\sum_j s_j(K)(x_j \otimes y_j)\right)\right)\right| \\
&= \left|\sum_j s_j(K)\langle Tx_j, y_j\rangle\right| \\
&\leq \sum_j s_j(K)|\langle Tx_j, y_j\rangle| \\
&= \left\langle \begin{bmatrix} s_1(K) \\ \vdots \\ s_j(K) \\ \vdots \end{bmatrix}, \begin{bmatrix} |\langle Tx_1, y_1\rangle| \\ \vdots \\ |\langle Tx_j, y_j\rangle| \\ \vdots \end{bmatrix} \right\rangle \\
&\leq \Phi((s_j(K))_j)\Phi^*((|\langle Tx_j, y_j\rangle|)_j) \\
&= \Phi^*((|\langle Tx_j, y_j\rangle|)_j) \\
&\leq \|T\|_{\Phi^*} = \alpha.
\end{aligned}$$

This forces $\Phi^*((|\langle Tx_j, y_j\rangle|)_j) = \alpha$. That is, $\|(|\langle Tx_j, y_j\rangle|)_j\|_{\Phi^*} = \alpha$. This observation along with the fact that the sequence $(s_j(K))_j$ is nonincreasing implies that the sequence $(|\langle Tx_j, y_j\rangle|)_j$ is also nonincreasing, that is, $|\langle Tx_1, y_1\rangle| \geq |\langle Tx_2, y_2\rangle| \geq \ldots \geq |\langle Tx_j, y_j\rangle| \geq \ldots$; for if it is not, then there exists $\ell \in \mathbb{N}$ such that $|\langle Tx_\ell, y_\ell\rangle| < |\langle Tx_{\ell+1}, y_{\ell+1}\rangle|$ which yields

$$\begin{aligned}
\alpha &= \left\langle \begin{bmatrix} \vdots \\ s_\ell(K) \\ s_{\ell+1}(K) \\ \vdots \end{bmatrix}, \begin{bmatrix} \vdots \\ |\langle Tx_\ell, y_\ell\rangle| \\ |\langle Tx_{\ell+1}, y_{\ell+1}\rangle| \\ \vdots \end{bmatrix} \right\rangle \\
&< \left\langle \begin{bmatrix} \vdots \\ s_\ell(K) \\ s_{\ell+1}(K) \\ \vdots \end{bmatrix}, \begin{bmatrix} \vdots \\ |\langle Tx_{\ell+1}, y_{\ell+1}\rangle| \\ |\langle Tx_\ell, y_\ell\rangle| \\ \vdots \end{bmatrix} \right\rangle \\
&\leq \Phi\left(\begin{bmatrix} \vdots \\ s_\ell(K) \\ s_{\ell+1}(K) \\ \vdots \end{bmatrix}\right) \Phi^*\left(\begin{bmatrix} \vdots \\ |\langle Tx_{\ell+1}, y_{\ell+1}\rangle| \\ |\langle Tx_\ell, y_\ell\rangle| \\ \vdots \end{bmatrix}\right) \\
&= \Phi\left(\begin{bmatrix} \vdots \\ s_\ell(K) \\ s_{\ell+1}(K) \\ \vdots \end{bmatrix}\right) \Phi^*\left(\begin{bmatrix} \vdots \\ |\langle Tx_\ell, y_\ell\rangle| \\ |\langle Tx_{\ell+1}, y_{\ell+1}\rangle| \\ \vdots \end{bmatrix}\right) = \alpha,
\end{aligned}$$

which is indeed a contradiction. Consequently,

$$\begin{aligned}
\alpha &= \|(|\langle Tx_j, y_j\rangle|)_j\|_{\Phi^*} \\
&= |\langle Tx_1, y_1\rangle| + |\langle Tx_2, y_2\rangle| \\
&\leq \|Tx_1\| + \|Tx_2\| \\
&\leq s_1(T) + s_2(T) = \alpha,
\end{aligned}$$

which forces $\|Tx_1\| + \|Tx_2\| = s_1(T) + s_2(T)$ thereby establishing that $T \in \mathcal{N}_{[2]}$. This completes the proof. $\square$

8.1.3 $[\pi, 2]$-Norming operators

Lemma 8.1.6. *Given $\pi = (\pi_j)_{j\in\mathbb{N}} \in \Pi$, if Φ and Ψ are s.n.functions defined as*

$$\Phi(\eta) = \max\left\{\eta_1, \frac{\sum_j \eta_j}{1+\pi_2}\right\};$$
$$\Psi(\xi) = \xi_1 + \pi_2\xi_2;$$

where $\eta = (\eta_i)_{i\in\mathbb{N}}$, and $\xi = (\xi_j)_{j\in\mathbb{N}} \in c_{00}^*$, then Φ and Ψ are mutually adjoint, that is, $\Phi^* = \Psi$ and $\Psi^* = \Phi$. Moreover, ϕ is equivalent to the maximal s.n.function and Ψ to the minimal.

Proof. Without much hassle it can be shown that

$$\Psi^*(\eta) = \max\left\{\eta_1 t + \frac{(\sum_{j\neq 1}\eta_j)(1-t)}{\pi_2} : t \in \left[\frac{1}{1+\pi_2}, 1\right]\right\}.$$

But

$$\begin{aligned}\eta_1 t + \frac{(\sum_{j\neq 1}\eta_j)(1-t)}{\pi_2} &= \left(\eta_1 - \left(\frac{\sum_{j\neq 1}\eta_j}{\pi_2}\right)\right)t + \left(\frac{\sum_{j\neq 1}\eta_j}{\pi_2}\right)\\ &= \begin{cases}\eta_1 & \text{when } t = 1\\ \frac{\sum_j \eta_j}{1+\pi_2} & \text{when } t = \frac{1}{1+\pi_2},\end{cases}\end{aligned}$$

which implies that

$$\Phi(\eta) = \Psi^*(\eta) = \max\left\{\eta_1, \frac{\sum_j \eta_j}{1+\pi_2}\right\}.$$

The final part of the assertion is trivial. □

Using this result we establish the following theorem.

Theorem 8.1.7. *Given $\pi = (\pi_j)_{j\in\mathbb{N}} \in \Pi$, let $T \in \mathcal{B}(\mathcal{H})$ and Φ be an s.n.function equivalent to the maximal s.n.function defined by*

$$\Phi(\eta) = \max\left\{\eta_1, \frac{\sum_j \eta_j}{1+\pi_2}\right\},$$

where $\eta = (\eta_i)_{i\in\mathbb{N}} \in c_\Phi$, and let Φ^* be its dual norm so that $\mathfrak{S}_\Phi^* \cong (\mathcal{B}(\mathcal{H}), \|\cdot\|_{\Phi^*})$ isometrically with $\|T\|_{\Phi^*} = \|T\|_{[\pi,2]}$ for every $T \in \mathcal{B}(\mathcal{H})$. Then $T \in \mathcal{N}_{[\pi,2]}$ in the sense of the Definition 5.1.1 if and only if there exists an operator $K \in \mathfrak{S}_\Phi = \mathcal{B}_1(\mathcal{H})$ with $\|K\|_\Phi = 1$ such that $|\operatorname{Tr}(TK)| = \|T\|_{\Phi^*}$.

Proof. Let $\{x_1, x_2\} \in \mathcal{H}$ be an orthonormal set such that $\|T\|_{\Phi^*} = \|T\|_{[\pi,2]} = \|Tx_1\| + \pi_2\|Tx_2\|$ and let

$$y_1 = \frac{Tx_1}{\|Tx_1\|},\ y_2 = \frac{Tx_2}{\|Tx_2\|}.$$

Define $K := (x_1 \otimes y_1) + \pi_2(x_2 \otimes y_2)$. Clearly, $K \in \mathcal{B}_1(\mathcal{H})$ and $s_1(K) = 1$, $s_2(K) = \pi_2$ with $\|K\|_\Phi = 1$. Then

$$|\operatorname{Tr}(TK)| = \left|\left\langle Tx_1, \frac{Tx_1}{\|Tx_1\|}\right\rangle + \pi_2\left\langle Tx_2, \frac{Tx_2}{\|Tx_2\|}\right\rangle\right| = \|Tx_1\| + \pi_2\|Tx_2\| = \|T\|_{\Phi^*},$$

proves the forward implication. Next we assume that there exists an operator $K \in \mathcal{B}_1(\mathcal{H})$ with $\|K\|_\Phi = 1$ such that $|\operatorname{Tr}(TK)| = \|T\|_{\Phi^*}$. Let

$$\alpha := \|T\|_{\Phi^*} = \|T\|_{[\pi,2]} = s_1(T) + \pi_2 s_2(T).$$

By slightly tweaking the proof of Theorem 8.1.5, we infer that $\|(|\langle Tx_j, y_j\rangle|)_j\|_{\Phi^*} = \alpha$, and that the sequence $(|\langle Tx_j, y_j\rangle|)_j$ is nonincreasing. This yields

$$\begin{aligned}\alpha &= \|(|\langle Tx_j, y_j\rangle|)_j\|_{\Phi^*}\\ &= |\langle Tx_1, y_1\rangle| + \pi_2|\langle Tx_2, y_2\rangle|\\ &\le \|Tx_1\| + \pi_2\|Tx_2\|\\ &\le s_1(T) + \pi_2 s_2(T) = \alpha,\end{aligned}$$

which forces $\|Tx_1\| + \pi_2\|Tx_2\| = s_1(T) + \pi_2 s_2(T)$ thereby establishing that $T \in \mathcal{N}_{[\pi,2]}$. This completes the proof. $\square$

8.2 Symmetric norming and absolutely symmetric norming operators

Given an arbitrary s.n.function Φ that is equivalent to the maximal s.n.function, we are now ready to establish the definition of operators in $\mathcal{B}(\mathcal{H})$ that attain their Φ^*-norm.

Definition 8.2.1. Let Φ be an s.n.function equivalent to the maximal s.n.function. An operator $T \in (\mathcal{B}(\mathcal{H}), \|\cdot\|_{\Phi^*})$ is said to be Φ^*-*norming* or *symmetric norming with respect to the symmetric norm* $\|\cdot\|_{\Phi^*}$ if there exists an operator $K \in \mathfrak{S}_\Phi = \mathcal{B}_1(\mathcal{H})$ with $\|K\|_\Phi = 1$ such that $|\operatorname{Tr}(TK)| = \|T\|_{\Phi^*}$. We let $\mathcal{N}_{\Phi^*}(\mathcal{H})$ denote the set of Φ^*-norming operators in $\mathcal{B}(\mathcal{H})$.

Remark 8.2.2. It is worth mentioning that the above definition is motivated by the results we proved in the preceding section, namely, Theorems 8.1.2, 8.1.5 and 8.1.7.

The following proposition is a trivial observation and its prinicipal significance lies in the fact that it can be taken as a new equivalent definition of Φ^*-norming operators in $\mathcal{B}(\mathcal{H})$.

Proposition 8.2.3. *Let Φ be an s.n.function equivalent to the maximal s.n.function and let Φ^* be its dual norm so that $\mathfrak{S}_\Phi^* \cong (\mathcal{B}(\mathcal{H}), \|\cdot\|_{\Phi^*})$ isometrically. If $T \in \mathcal{B}(\mathcal{H})$ is identified with $f_T \in \mathfrak{S}_\Phi^*$, then the following statements are equivalent.*

1. *$T \in \mathcal{N}_{\Phi^*}(\mathcal{H})$.*
2. *f_T attains its norm.*

We let $\mathcal{N}(\mathfrak{S}_\Phi, \mathbb{C})$ denote the set of functionals on $\mathfrak{S}_\Phi$ that attain their norm.

Notice that Theorem 4.2.14 is a reformulation of the definition of an absolutely $[k]$-norming operators in $\mathcal{B}(\mathcal{H}, \mathcal{K})$ by identifying $T|_\mathcal{M} \in \mathcal{B}(\mathcal{M}, \mathcal{K})$ with $TP_\mathcal{M} \in \mathcal{B}(\mathcal{H}, \mathcal{K})$; and so are Theorems 5.2.12 and 6.2.12 for absolutely $[\pi, k]$-norming and absolutely (p, k)-norming operators, respectively. These reformulations motivate the following definition.

Definition 8.2.4. Let Φ be an s.n.function equivalent to the maximal s.n.function. An operator $T \in (\mathcal{B}(\mathcal{H}), \|\cdot\|_{\Phi^*})$ is said to be *absolutely Φ^*-norming* or *absolutely symmetric norming with respect to the symmetric norm* $\|\cdot\|_{\Phi^*}$ if for every nontrivial closed subspace $\mathcal{M}$ of $\mathcal{H}$, $TP_\mathcal{M} \in \mathcal{B}(\mathcal{H})$ is Φ^*-norming. We let $\mathcal{AN}_{\Phi^*}(\mathcal{H})$ denote the set of absolutely Φ^*-norming operators in $\mathcal{B}(\mathcal{H})$.

Example 8.2.5. For any $\pi \in \Pi$ and for any $k \in \mathbb{N}$, choose Φ to be the s.n.function such that $\Phi^* = \|\cdot\|_{[\pi,k]}$. Then $T \in (\mathcal{B}(\mathcal{H}), \|\cdot\|_{\Phi^*})$ belongs to $\mathcal{AN}_{\Phi^*}(\mathcal{H})$ if and only if $|T|$ is of the form $|T| = \alpha I + F + K$, where $\alpha \geq 0, K$ is a positive compact operator and F is a self-adjoint finite rank operator. Also, given $p \in [1, \infty)$ and $k \in \mathbb{N}$, choose Ψ to be the s.n.function such that $\Psi^* = \|\cdot\|_{(p,k)}$. Then $T \in \mathcal{AN}_{\Psi^*}(\mathcal{H})$ if and only if $|T|$ is of the form $|T| = \alpha I + F + K$, where $\alpha \geq 0, K$ is a positive compact operator and F is a self-adjoint finite rank operator.

With the above definitions established to guide the way, we prove and collect certain fundamental results concerning symmetric norming and absolutely symmetric norming operators in $\mathcal{B}(\mathcal{H})$. Our first goal is to prove Theorem 8.2.7, which can be thought of as an analogue of Proposition 4.2.16 except that we are in the setting of $(\mathcal{B}(\mathcal{H}), \|\cdot\|_{\Phi^*})$ instead of $(\mathcal{B}(\mathcal{H}, \mathcal{K}), \|\cdot\|)$ with Φ being an s.n.function equivalent to the maximal s.n.function. We need the following lemma in order to prove this theorem.

Lemma 8.2.6. *Let Φ be an arbitrary s.n.function equivalent to the maximal s.n.function. Then $(\mathcal{B}_0(\mathcal{H}), \|\cdot\|_{\Phi^*})^{**} \cong (\mathcal{B}(\mathcal{H}), \|\cdot\|_{\Phi^*})$ isometrically.*

Proof. Since Φ is equivalent to the maximal s.n.function Φ_1, the first part of the Proposition 8.1.1 guarantees that $(\mathfrak{S}_\Phi, \|\cdot\|_\Phi)^* \cong (\mathcal{B}(\mathcal{H}), \|\cdot\|_{\Phi^*})$ and $(\mathfrak{S}_{\Phi^*}, \|\cdot\|_{\Phi^*})^* \cong (\mathfrak{S}_\Phi, \|\cdot\|_\Phi)$ isometrically. Moreover, $\mathfrak{S}_\Phi$ and $\mathcal{B}_1(\mathcal{H})$ coincide elementwise and so does $\mathfrak{S}_{\Phi^*}$ and $\mathcal{B}_0(\mathcal{H})$. Consequently, $(\mathcal{B}_1(\mathcal{H}), \|\cdot\|_\Phi)^* \cong (\mathcal{B}(\mathcal{H}), \|\cdot\|_{\Phi^*})$ and $(\mathcal{B}_0(\mathcal{H}), \|\cdot\|_{\Phi^*})^* \cong (\mathcal{B}_1(\mathcal{H}), \|\cdot\|_\Phi)$ isometrically, which yields $(\mathcal{B}_0(\mathcal{H}), \|\cdot\|_{\Phi^*})^{**} \cong (\mathcal{B}(\mathcal{H}), \|\cdot\|_{\Phi^*})$ isometrically. This completes the proof. $\square$

Theorem 8.2.7 ([Pan17a])**.** *Let Φ be an arbitrary s.n.function equivalent to the maximal s.n.function. If T is a compact operator, then $T \in \mathcal{AN}_{\Phi^*}(\mathcal{H})$, that is,*

$$\mathcal{B}_0(\mathcal{H}) \subseteq \mathcal{AN}_{\Phi^*}(\mathcal{H}).$$

Proof. If $T \in \mathcal{B}_0(\mathcal{H})$, then $TP_\mathcal{M} \in \mathcal{B}_0(\mathcal{H})$ for any closed subspace $\mathcal{M}$ of $\mathcal{H}$. So it suffices to show that $T \in \mathcal{N}_{\Phi^*}(\mathcal{H})$. Since $(\mathcal{B}_0(\mathcal{H}), \|\cdot\|_{\Phi^*})$ and $(\mathcal{B}_0(\mathcal{H}), \|\cdot\|_{\Phi^*})^{**}$ are Banach spaces, the Banach space theory guarantees the existence of the canonical map

$$\wedge : (\mathcal{B}_0(\mathcal{H}), \|\cdot\|_{\Phi^*}) \to (\mathcal{B}_0(\mathcal{H}), \|\cdot\|_{\Phi^*})^{**} \text{ given by } T \mapsto \hat{T},$$

and $\|\hat{T}\| = \max\{|\hat{T}(\varphi)| : \varphi \in (\mathcal{B}_0(\mathcal{H}), \|\cdot\|_{\Phi^*})^*, \|\varphi\| = 1\}$, so that there exists $\varphi_0 \in (\mathcal{B}_0(\mathcal{H}), \|\cdot\|_{\Phi^*})^*$ with $\|\varphi_0\| = 1$ such that $\|\hat{T}\| = |\hat{T}(\varphi_0)| = |\varphi_0(T)|$. Corresponding to this φ_0 there exists a unique $A_0 \in (\mathcal{B}_1(\mathcal{H}), \|\cdot\|_\Phi)$ so that $\varphi_0(X) = \operatorname{Tr}(A_0 X)$ for every $X \in (\mathcal{B}_0(\mathcal{H}), \|\cdot\|_{\Phi^*})$ with $\|\varphi_0\| = \|A_0\|_\Phi$. Since the diagram below commutes

$$\begin{tikzcd}
(\mathcal{B}_0(\mathcal{H}), \|\cdot\|_{\Phi^*}) \arrow[r, "\wedge"] \arrow[dr, hook, "i"'] & (\mathcal{B}_0(\mathcal{H}), \|\cdot\|_{\Phi^*})^{**} \arrow[d, "f"] \\
& (\mathcal{B}(\mathcal{H}), \|\cdot\|_{\Phi^*})
\end{tikzcd}$$

where $\wedge$ is the canonical map from the space $(\mathcal{B}_0(\mathcal{H}), \|\cdot\|_{\Phi^*})$ to its double dual, f is the isometric isomorphism resulting from Lemma 8.2.6, and i is the inclusion map. The operator in $(\mathcal{B}(\mathcal{H}), \|\cdot\|_{\Phi^*})$ associated with $\hat{T} \in (\mathcal{B}_0(\mathcal{H}), \|\cdot\|_{\Phi^*})^{**}$ is the operator T itself. So, $\|T\|_{\Phi^*} = \|\hat{T}\|$ which implies that there exists $A_0 \in (\mathcal{B}_1(\mathcal{H}), \|\cdot\|_\Phi)$ such that $\|\hat{T}\| = |\operatorname{Tr}(A_0 T)|$ with $\|A_0\|_\Phi = 1$. This proves that $T \in \mathcal{N}_{\Phi^*}(\mathcal{H})$. $\square$

Following our notation (and of course the well-e1stablished precedent), we continue to use $s_j(T)$ to denote the j-th singular number of $T \in \mathcal{B}(\mathcal{H})$. The following proposition from [Pan17b] allows us to concentrate on the positive operators that are symmetrically norming.

Proposition 8.2.8. *Let Φ be an s.n.function equivalent to the maximal s.n.function. Then $T \in \mathcal{N}_{\Phi^*}(\mathcal{H})$ if and only if $|T| \in \mathcal{N}_{\Phi^*}(\mathcal{H})$.*

Proof. We first assume that $T \in \mathcal{N}_{\Phi^*}(\mathcal{H})$ and observe that $\|T\|_{\Phi^*} = \| \, |T| \, \|_{\Phi^*}$ since for each j, $s_j(T) = s_j(|T|)$. Then there exists $K \in \mathcal{B}_1(\mathcal{H})$ with $\|K\|_\Phi = 1$ such that $\|T\|_{\Phi^*} = |\operatorname{Tr}(TK)|$. If $T = U|T|$ is the polar decomposition of T, then

$$\| \, |T| \, \|_{\Phi^*} = \|T\|_{\Phi^*} = |\operatorname{Tr}(TK)| = |\operatorname{Tr}(U|T|K)| = |\operatorname{Tr}(|T|KU)|,$$

where $KU \in \mathcal{B}_1(\mathcal{H})$ with $\|KU\|_\Phi = \|IKU\|_\Phi \leq \|I\|\|K\|_\Phi\|U\| = \|K\|_\Phi = 1$. In fact, $\|KU\|_\Phi = 1$; for if not, then the operator $S := KU/\|KU\|_\Phi \in \mathcal{B}_1(\mathcal{H})$ satisfies $\|S\|_\Phi = 1$ and yields

$$|\operatorname{Tr}(|T|S)| = \left|\operatorname{Tr}\left(\frac{|T|KU}{\|KU\|_\Phi}\right)\right| = \frac{1}{\|KU\|_\Phi}|\operatorname{Tr}(|T|KU)| > |\operatorname{Tr}(|T|KU)| = \| \, |T| \, \|_{\Phi^*},$$

which contradicts the fact that the supremum of the set

$$\{|\operatorname{Tr}(|T|X)| : X \in \mathcal{B}_1(\mathcal{H}), \|X\|_\Phi \leq 1\}$$

is attained at KU. This shows that $|T| \in \mathcal{N}_{\Phi^*}(\mathcal{H})$.

Conversely, if $|T| \in \mathcal{N}_{\Phi^*}(\mathcal{H})$, then by replacing T with $|T|$ in the above argument using $|T| = U^*T$, we can prove the existence of $\hat{K} \in \mathcal{B}_1(\mathcal{H})$ with $\|\hat{K}\|_\Phi = 1$ such that $\|T\|_{\Phi^*} = |\operatorname{Tr}(T\hat{K}U^*)|$ where $\hat{K}U^* \in \mathcal{B}_1(\mathcal{H})$ with $\|\hat{K}U*\|_\Phi \leq 1$. It can then be shown that $\|\hat{K}U^*\|_\Phi = 1$ and the result follows. $\square$

In the section that follows we introduce a certain family of symmetric norms on $\mathcal{B}(\mathcal{H})$ which is going to be the subject of a detailed study in the next chapter. Before we can proceed we need one more result concerning the computation of the symmetric norm of an operator.

Proposition 8.2.9 ([Pan17b])**.** *Let Φ be an s.n.function equivalent to the maximal s.n.function and let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\|T\|_{\Phi^*} = \sup\left\{\sum_j s_j(T)s_j(K) : K \in \mathcal{B}_1(\mathcal{H}), K = \operatorname{diag}\{s_j(K)\}_j, \|K\|_\Phi = 1\right\}$$

Proof. Since Φ is equivalent to the maximal s.n.function, we know that $\mathfrak{S}_\Phi^* \cong (\mathcal{B}(\mathcal{H}), \|\cdot\|_{\Phi^*})$ isometrically and by Definition 8.2.1 the $\|\cdot\|_{\Phi_\pi^*}$ norm for any operator $T \in \mathcal{B}(\mathcal{H})$ is given by $\|T\|_{\Phi^*} = \sup\{|\operatorname{Tr}(TK)| : K \in \mathfrak{S}_\Phi,\ \|K\|_\Phi = 1\}$. But the ideal $\mathcal{B}_1(\mathcal{H})$ and $\mathfrak{S}_{\Phi_\pi}$ coincide elementwise and hence $\|T\|_{\Phi^*} = \sup\{|\operatorname{Tr}(TK)| : K \in \mathcal{B}_1(\mathcal{H}),\ \|K\|_\Phi = 1\}$.

First we set $\alpha := \sup\{|\operatorname{Tr}(TK)| : K \in \mathcal{B}_1(\mathcal{H}),\ \|K\|_\Phi = 1\}$ and $\beta := \sup\{\sum_j s_j(T)s_j(K) : K \in \mathcal{B}_1(\mathcal{H}),\ \|K\|_\Phi = 1\}$, and thereafter we claim that $\alpha = \beta$. That $\alpha \le \beta$ is a trivial observation since $|\operatorname{Tr}(TK)| \le \sum_j s_j(TK) \le \sum_j s_j(T)s_j(K)$. To see $\beta \le \alpha$, let us choose an operator $K \in \mathcal{B}_1(\mathcal{H})$ with $\|K\|_\Phi = 1$. An easy computation yields

$$\begin{aligned}
\sum_j s_j(T)s_j(K) &= \left\langle \begin{bmatrix} s_1(T) \\ \vdots \\ s_j(T) \\ \vdots \end{bmatrix}, \begin{bmatrix} s_1(K) \\ \vdots \\ s_j(K) \\ \vdots \end{bmatrix} \right\rangle \\
&\le \Phi^* \left(\begin{bmatrix} s_1(T) \\ \vdots \\ s_j(T) \\ \vdots \end{bmatrix} \right) \Phi \left(\begin{bmatrix} s_1(K) \\ \vdots \\ s_j(K) \\ \vdots \end{bmatrix} \right) \\
&= \Phi^* \left(\begin{bmatrix} s_1(T) \\ \vdots \\ s_j(T) \\ \vdots \end{bmatrix} \right) = \|T\|_{\Phi^*} \\
&= \sup\{|\operatorname{Tr}(TK)| : K \in \mathcal{B}_1(\mathcal{H}),\ \|K\|_\Phi = 1\} = \alpha.
\end{aligned}$$

It then follows that $\beta \le \alpha$ and this proves our first claim.

We next let $\gamma := \sup\{\sum_j s_j(T)s_j(K) : K \in \mathcal{B}_1(\mathcal{H}),\ K = \operatorname{diag}\{s_j(K)\},\ \|K\|_\Phi = 1\}$ and prove that $\gamma = \beta$. That $\gamma \le \beta$ is obvious. To prove $\beta \le \gamma$, we choose an operator $K \in \mathcal{B}_1(\mathcal{H})$ with $\|K\|_\Phi = 1$ and define

$$\tilde{K} := \begin{pmatrix} s_1(K) & & & & \\ & s_2(K) & & \text{\Large 0} & \\ & & \ddots & & \\ & \text{\Large 0} & & s_j(K) & \\ & & & & \ddots \end{pmatrix}.$$

Notice that for every j, we have $s_j(\tilde{K}) = s_j(K)$ which implies that $\|\tilde{K}\|_\Phi = \|K\|_\Phi = 1$. Even more, $\tilde{K} \in \mathcal{B}_1(\mathcal{H})$ and hence $\sum_j s_j(T)s_j(K) = \sum_j s_j(T)s_j(\tilde{K})$. But since

$$\sum_j s_j(T)s_j(\tilde{K}) \leq \sup\left\{\sum_j s_j(T)s_j(K) : K \in \mathcal{B}_1(\mathcal{H}),\ K = \operatorname{diag}\{s_j(K)\},\ \|K\|_\Phi = 1\right\},$$

it follows that $\beta \leq \gamma$ which establishes our second claim. From the above two observations we conclude that $\alpha = \gamma$, and consequently the assertion is proved. $\square$

8.3 Norm(s) that are not attained by the identity

Theorem 8.3.1 ([Pan17a]). *There exists a symmetric norm $\|\cdot\|_{\Phi_\pi^*}$ on $\mathcal{B}(\ell^2)$ such that $I \notin \mathcal{N}_{\Phi_\pi^*}(\ell^2)$.*

Proof. Let $\{e_i\}_{i\in\mathbb{N}}$ be the canonical orthonormal basis of the concrete infinite-dimensional separable Hilbert space ℓ^2, and let $\pi = (\pi_n)_{n\in\mathbb{N}}$ be a strictly decreasing convergent sequence of positive numbers with $\pi_1 = 1$ such that $\lim_n \pi_n > 0$. Let us define a symmetrically norming function Φ_π by

$$\Phi_\pi(\xi_1, \xi_2, \ldots) := \sum_j \pi_j \xi_j.$$

Notice that for every $n \in \mathbb{N}$, we have

$$\frac{n}{\Phi_\pi(\underbrace{1, \ldots, 1}_{n \text{ times}}, 0, 0, \ldots)} = \frac{n}{1 + \pi_2 + \ldots + \pi_n} < \frac{1}{\lim_n \pi_n},$$

which implies

$$\sup_n \left\{\frac{n}{\Phi_\pi(\underbrace{1, \ldots, 1}_{n \text{ times}}, 0, 0, \ldots)}\right\} \leq \sup_n \left\{\frac{1}{\lim_n \pi_n}\right\} < \infty.$$

Φ_π is thus equivalent to the maximal symmetric norming function Φ_1 due to Proposition 7.4.14. The dual $\mathfrak{S}_{\Phi_\pi}^*$ of the symmetrically normed ideal $\mathfrak{S}_{\Phi_\pi}$ is thus isometrically isomorphic to $(\mathcal{B}(\ell^2), \|\cdot\|_{\Phi_\pi^*})$, that is, $\mathfrak{S}_{\Phi_\pi}^* \cong (\mathcal{B}(\ell^2), \|\cdot\|_{\Phi_\pi^*})$ isometrically, and the $\|\cdot\|_{\Phi_\pi^*}$ norm for any operator $T \in \mathcal{B}(\ell^2)$ is given by $\|T\|_{\Phi_\pi^*} = \sup\{|\operatorname{Tr}(TK)| : K \in \mathfrak{S}_{\Phi_\pi},\ \|K\|_{\Phi_\pi} = 1\}$. By Proposition 8.2.9, this reduces to

$$\|T\|_{\Phi_\pi^*} = \sup\left\{\sum_j s_j(T)s_j(K) : K \in \mathcal{B}_1(\ell^2),\ K = \operatorname{diag}\{s_j(K)\}_j,\ \|K\|_{\Phi_\pi} = 1\right\}.$$

We will show that I does not attain its Φ_π^*-norm in $\mathcal{B}(\ell^2)$. To show this, we assume that $I \in \mathcal{N}_{\Phi_\pi^*}(\ell^2)$, and we deduce a contradiction from this assumption. Accordingly, assume that $I \in \mathcal{N}_{\Phi_\pi^*}$, then the supremum,

$$\sup\left\{\sum_j s_j(K) : K \in \mathcal{B}_1(\ell^2),\ K = \operatorname{diag}\{s_1(K), s_2(K), ...\},\ \|K\|_{\Phi_\pi} = 1\right\},$$

is attained, that is, there exists $K_0 = \operatorname{diag}\{s_1(K_0), s_2(K_0), ...\} \in \mathcal{B}_1(\ell^2)$ with $\sum_j \pi_j s_j(K_0) = 1$ such that $\|I\|_{\Phi_\pi^*} = |\operatorname{Tr}(K_0)| = \sum_j s_j(K_0)$. Since $K_0 \in \mathcal{B}_1(\ell^2) \subseteq \mathcal{B}_0(\ell^2)$, we have $\lim_{j\to\infty} s_j(K_0) = 0$. This forces the existence of a natural number M such that $s_M(K_0) > s_{M+1}(K_0)$. All that remains is to show the existence of an operator $\tilde{K} \in \mathcal{B}_1(\ell^2)$, $\|\tilde{K}\|_{\Phi_\pi} = 1$ of the form $\tilde{K} = \operatorname{diag}\{s_1(\tilde{K}), s_2(\tilde{K}), ...\}$ such that $\sum_i s_i(\tilde{K}) > \sum_j s_j(K_0)$.

If we define

$$\lambda := \frac{\sum_{j=M}^{M+1} \pi_j s_j(K_0)}{\sum_{j=M}^{M+1} \pi_j} = \frac{\pi_M s_M(K_0) + \pi_{M+1} s_{M+1}(K_0)}{\pi_M + \pi_{M+1}},$$

and let $\tilde{K}$ be the diagonal operator defined by

$$\tilde{K} := \begin{bmatrix} s_1(K_0) & & & & & & \\ & \ddots & & & & & \\ & & s_{M-1}(K_0) & & & & \\ & & & \lambda & & & \\ & & & & \lambda & & \\ & & & & & s_{M+1}(K_0) & \\ & & & & & & \ddots \end{bmatrix},$$

then we have

$$\begin{aligned}
\sum_j \pi_j s_j(\tilde{K}) &= \sum_{j=1}^{M-1} \pi_j s_j(K_0) + (\pi_M + \pi_{M+1})\lambda + \sum_{j>M+1} \pi_j s_j(K_0) \\
&= \sum_{j=1}^{M-1} \pi_j s_j(K_0) + (\pi_M + \pi_{M+1}) \frac{\sum_{j=M}^{M+1} \pi_j s_j(K_0)}{\sum_{j=M}^{M+1} \pi_j} + \sum_{j>M+1} \pi_j s_j(K_0) \\
&= \sum_{j=1}^{M-1} \pi_j s_j(K_0) + \sum_{j=M}^{M+1} \pi_j s_j(K_0) + \sum_{j>M+1} \pi_j s_j(K_0) \\
&= \sum_j \pi_j s_j(K_0).
\end{aligned}$$

It then follows that $\|\tilde{K}\|_{\Phi_\pi} = \|K_0\|_{\Phi_\pi} = 1$, and that $\tilde{K} \in \mathcal{B}_1(\ell^2)$. Even more, $\tilde{K}$ is of the form $\tilde{K} = \operatorname{diag}\{s_1(\tilde{K}), s_2(\tilde{K}), ...\}$. However, because

$$2\lambda = 2\left(\frac{\pi_M s_M(K_0) + \pi_{M+1} s_{M+1}(K_0)}{\pi_M + \pi_{M+1}}\right) > s_M(K_0) + s_{M+1}(K_0),$$

it immediately follows that

$$\begin{aligned}
\sum_j s_j(\tilde{K}) &= \sum_{j=1}^{M-1} s_j(K_0) + 2\lambda + \sum_{j>M+1} \pi_j s_j(K_0) \\
&> \sum_{j=1}^{M-1} s_j(K_0) + (s_M(K_0) + s_{M+1}(K_0)) + \sum_{j>M+1} \pi_j s_j(K_0) \\
&= \sum_j s_j(K_0),
\end{aligned}$$

which contradicts the assumption that $\sum_j s_j(K_0)$ $(= |\operatorname{Tr}(K_0)|)$ is the supremum of the set

$$\left\{\sum_j s_j(K) : K \in \mathcal{B}_1(\ell^2),\ K = \operatorname{diag}\{s_1(K), s_2(K), ...\},\ \|K\|_{\Phi_\pi} = 1\right\}.$$

Since the operator K_0 with which we began our discussion is arbitrary, it follows that for any given operator in $\mathcal{B}_1(\ell^2)$(or $\mathfrak{S}_{\Phi_\pi}$) with unit norm, one can find another operator in $\mathfrak{S}_{\Phi_\pi}$ with unit norm with trace of larger magnitude and hence the supremum of the above set can never be attained. This shows that the identity operator does not attain its norm. $\square$

The proof of this theorem is constructive and illustrates an elegant technique of producing symmetric norms on $\mathcal{B}(\mathcal{H})$ with respect to which the identity operator does not attain its norm. In particular, the proof demonstrates a family of s.n.functions — s.n.functions affiliated to strictly decreasing weights — which naturally generate such symmetric norms. This is the family of s.n.functions which we promised to introduce at the beginning of this chapter. Even more, this family of s.n.functions lies at the very foundation of the results we prove in the next chapter. In what follows we formally introduce this family of s.n.functions.

Let $\widehat{\Pi}$ denote the set of all strictly decreasing convergent sequences of positive numbers with their first term equal to 1 and positive limit, that is,

$$\widehat{\Pi} = \{\pi := (\pi_n)_{n\in\mathbb{N}} : \pi_1 = 1, \lim_n \pi_n > 0, \text{ and } \pi_k > \pi_{k+1} \text{ for each } k \in \mathbb{N}\}.$$

(Recall that we have used Π, in Chapter 5, to denote the set of all nonincreasing sequences of positive numbers with their first term equal to 1. Let us, by abuse of notation, continue using Π to denote the set of all nonincreasing sequences of positive numbers with their first term equal to 1 and positive limit. Hence, in accordance with this notation, we have $\widehat{\Pi} \subseteq \Pi$.) For each $\pi \in \widehat{\Pi}$, let Φ_π denote the symmetric norming function defined by $\Phi_\pi(\xi_1, \xi_2, ...) = \sum_j \pi_j \xi_j$ and observe that Φ_π is equivalent to the maximal s.n.function Φ_1. Theorem 8.3.1 essentially proves that $I \notin \mathcal{N}_{\Phi^*_\pi}(\ell^2)$ for every Φ_π from the family $\{\Phi_\pi : \pi \in \widehat{\Pi}\}$ of s.n.functions. Thus, instead of merely one symmetric norm, we have a family of symmetric norms on $\mathcal{B}(\mathcal{H})$ with respect to each of which the identity operator does not attain its norm.

8.4 A few more norms that are not attained by the identity

It is perhaps worth a short digression to construct more symmetric norms on $\mathcal{B}(\mathcal{H})$ with respect to which the identity operator does not attain its norm. Observe that the proof of Theorem 8.3.1, in its construction of symmetric norms, assumes that $\pi \in \widehat{\Pi}$ and is thus a strictly decreasing sequence. However, choosing a sequence $\pi \in \Pi$ which is not necessarily strictly decreasing and considering the s.n.function defined by it also generates a symmetric norm with respect to which the identity operator does not attain its norm, as long as there exists a natural number N such that $\pi_N > \pi_{N+1}$. Equivalently, if the sequence $\pi \in \Pi$ is not the constant sequence $\mathbf{1} = (1, 1, ...)$, then there exists a natural number N such that $\pi_N > \pi_{N+1}$, and it can be shown that the s.n.function defined by such a sequence also generates a symmetric norm on $\mathcal{B}(\ell^2)$ with respect to which the identity operator does not attain its norm. We conclude this chapter by proving the following result which, in effect, extends the preceding theorem to every symmetric norm Φ_π in the family $\{\Phi_\pi : \pi \in \Pi\}$ of s.n.functions except when $\pi \in \Pi$ is the constant sequence $\mathbf{1} = (1, 1, ...)$; earlier the result was shown to hold for the family $\{\Phi_\pi : \pi \in \widehat{\Pi}\}$ of s.n.functions.

Theorem 8.4.1 ([Pan17b])**.** *Let Π be the set of all nonincreasing convergent sequence of positive numbers with their first term equal to 1 and positive limit, that is,*

$$\Pi = \{\pi := (\pi_n)_{n\in\mathbb{N}} : \pi_1 = 1, \lim_n \pi_n > 0, \text{ and } \pi_k \geq \pi_{k+1} \text{ for each } k \in \mathbb{N}\},$$

and consider the subset $\Pi \setminus \{\mathbf{1}\}$ of Π consisting of all nonincreasing convergent sequence of positive numbers with their first term equal to 1 and positive limit except the constant sequence $\mathbf{1}$. For each $\pi \in \Pi \setminus \{\mathbf{1}\}$, let Φ_π denote the symmetric norming function defined by $\Phi_\pi(\xi_1, \xi_2, ...) = \sum_j \pi_j \xi_j$. Then

1. *Φ_π is equivalent to the maximal s.n.function Φ_1; and*
2. *for every $\pi \in \Pi \setminus \{\mathbf{1}\}$, $I \notin \mathcal{N}_{\Phi^*}(\ell^2)$.*

Alternatively, $I \notin \mathcal{N}_{\Phi^*}(\ell^2)$ for every Φ_π from the family $\{\Phi_\pi : \pi \in \Pi\setminus\{\mathbf{1}\}\}$ of s.n.functions.

Proof. That each s.n.function from the family $\{\Phi_\pi : \pi \in \Pi \setminus \{\mathbf{1}\}\}$ of s.n.functions is equivalent to the maximal s.n.function Φ_1 is, now, a trivial observation.

The proof of the second claim is almost along the lines of the proof of Theorem 8.3.1. Let $\pi = (\pi_n)_{n\in\mathbb{N}} \in \Pi\setminus\{\mathbf{1}\}$ and let Φ_π be the s.n.function generated by π, that is, $\Phi_\pi(\xi_1, \xi_2, ...) = \sum_j \pi_j \xi_j$. Now, contrapositively assume that $I \in \mathcal{N}_{\Phi_\pi^*}$, then the supremum,

$$\sup\left\{\sum_j s_j(K) : K \in \mathcal{B}_1(\ell^2),\ K = \operatorname{diag}\{s_1(K), s_2(K), ...\},\ \|K\|_{\Phi_\pi} = 1\right\},$$

is attained, that is, there exists $K_0 = \operatorname{diag}\{s_1(K_0), s_2(K_0), ...\} \in \mathcal{B}_1(\ell^2)$ with $\sum_j \pi_j s_j(K_0) = 1$ such that $\|I\|_{\Phi_\pi^*} = \sum_j s_j(K_0)$. We will prove the existence of an operator $\tilde{K} \in \mathcal{B}_1(\ell^2)$, $\|\tilde{K}\|_{\Phi_\pi} = 1$ of the form $\tilde{K} = \operatorname{diag}\{s_1(\tilde{K}), s_2(\tilde{K}), ...\}$ such that $\sum_i s_i(\tilde{K}) > \sum_j s_j(K_0)$. To this end, since $\pi \in \Pi \setminus \{\mathbf{1}\}$, there exists a natural number M such that $\pi_M > \pi_{M+1}$. Set

$$\lambda = \frac{\pi_M}{\pi_{M+1}},$$

and choose $\epsilon > 0$ such that

$$s_M(K_0) - \epsilon = s_{M+1}(K_0) + \lambda\epsilon.$$

(Of course there is only one such ϵ.) Now define $\tilde{K}$ to be the diagonal operator given by

$$\tilde{K} := \begin{bmatrix} s_1(K_0) & & & & & & \\ & \ddots & & & & & \\ & & s_{M-1}(K_0) & & & & \\ & & & s_M(K_0) - \epsilon & & & \\ & & & & s_{M+1}(K_0) + \lambda\epsilon & & \\ & & & & & s_{M+1}(K_0) & \\ & & & & & & \ddots \end{bmatrix}.$$

Before proceeding further, notice that the singular numbers of the above defined opearator $\tilde{K}$ are precisely the diagonal elements, and that the equation preceding the definition of $\tilde{K}$ guarantees that these s-numbers are arranged in a nonincreasing manner on the diagonal. Next we observe that

$$-\pi_M\epsilon + \pi_{M+1}\lambda\epsilon = -\pi_M\epsilon + \pi_{M+1}\frac{\pi_M}{\pi_{M+1}}\epsilon = -\pi_M\epsilon + -\pi_M\epsilon = 0,$$

and hence

$$\pi_M(s_M(K_0) - \epsilon) + \pi_{M+1}(s_{M+1}(K_0) + \lambda\epsilon) = \pi_M s_M(K_0) + \pi_{M+1}s_{M+1}(K_0).$$

This yields

$$\sum_j \pi_j s_j(\tilde{K}) = \sum_j \pi_j s_j(K_0), \text{ and hence } \|\tilde{K}\|_{\Phi_\pi} = \|K_0\|_{\Phi_\pi} = 1.$$

Consequently, we have $\tilde{K} \in \mathcal{B}_1(\ell^2)$. Moreover, $\tilde{K}$ is of the form $\tilde{K} = \mathrm{diag}\{s_1(\tilde{K}), s_2(\tilde{K}), ...\}$.

However, since $\lambda > 1$ and $\epsilon > 0$, we have

$$(s_M(K_0) - \epsilon) + (s_{M+1}(K_0) + \lambda\epsilon) = s_M(K_0) + s_{M+1}(K_0) + (\lambda - 1)\epsilon > s_M(K_0) + s_{M+1}(K_0),$$

which allows us to infer that

$$\begin{aligned}\sum_j s_j(\tilde{K}) &= \sum_{j=1}^{M-1} s_j(K_0) + (s_M(K_0) - \epsilon) + (s_{M+1}(K_0) + \lambda\epsilon) + \sum_{j>M+1} \pi_j s_j(K_0) \\ &> \sum_{j=1}^{M-1} s_j(K_0) + (s_M(K_0) + s_{M+1}(K_0)) + \sum_{j>M+1} \pi_j s_j(K_0) \\ &= \sum_j s_j(K_0),\end{aligned}$$

which contradicts the assumption that $\sum_j s_j(K_0)$ $(= |\operatorname{Tr}(K_0)|)$ is the supremum of the set

$$\left\{ \sum_j s_j(K) : K \in \mathcal{B}_1(\ell^2),\ K = \operatorname{diag}\{s_1(K), s_2(K), ...\},\ \|K\|_{\Phi_\pi} = 1 \right\}.$$

Since the operator K_0 with which we began our discussion is arbitrary, it follows that for any given operator in $\mathcal{B}_1(\ell^2)$ with unit norm, one can find another operator in $\mathcal{B}_1(\ell^2)$ with unit norm with trace of larger magnitude and hence the supremum of the above set can never be attained. This shows that the identity operator does not attain its norm. $\square$

Chapter 9

Universally symmetric norming operators are compact

Again the setting for our discussion is a separable infinite-dimensional Hilbert space. In this chapter, we introduce and study the notion of "universally symmetric norming operators" (see Definition 9.2.1) and "universally absolutely symmetric norming operators" (see Definition 9.2.2). These refer to the operators that are, respectively, norming and absolutely norming, with respect to every symmetric norm. The goal of this chapter is to characterize such operators.

The main result of this chapter is Theorem 9.2.6 which states that an operator in $\mathcal{B}(\mathcal{H})$ is universally symmetric norming if and only if it is universally absolutely symmetric norming, which is true if and only if it is compact. We hence establish a characterization theorem for such operators on $\mathcal{B}(\mathcal{H})$. In particular, this result provides an alternative characterization theorem for compact operators on a separable Hilbert space.

One of the most non-intuitive and important results that motivated this work is that there exist symmetric norms on $\mathcal{B}(\mathcal{H})$ with respect to which even the identity operator does not attain its norm.

9.1 Symmetric norming operators affiliated to strictly decreasing weights

In this section we return to the study of the family of symmetric norms on $\mathcal{B}(\mathcal{H})$ generated by the duals (or adjoints) of s.n.functions from the family $\{\Phi_\pi : \pi \in \widehat{\Pi}\}$ we encountered in the preceding chapter, and establish a characterization theorem for operators in $\mathcal{B}(\mathcal{H})$ that are symmetric norming with respect to *every* such symmetric norm. (Recall that for each $\pi \in \widehat{\Pi}$, Φ_π denotes the s.n. function defined by $\Phi_\pi(\xi_1, \xi_2, ...) = \sum_j \pi_j \xi_j$, and that Φ_π is equivalent to the maximal s.n.function Φ_1.) This section also studies the operators in $\mathcal{B}(\mathcal{H})$ that are absolutely symmetric norming with respect to *every* symmetric norm in the family and presents a characterization theorem for those as well. It turns out that an operator is symmetric norming with respect to every symmetric norm in the family if and only if it is absolutely symmetric norming with respect to every symmetric norm in the family. This "characterization theorem" is the main theorem of this section.

We know that, in general, $\mathcal{N}_{\Phi^*}(\mathcal{H}) \nsubseteq \mathcal{B}_0(\mathcal{H})$ for an arbitrary s.n.function Φ equivalent to the maximal s.n.function Φ_1. However, it is of interest to know whether $\mathcal{N}_{\Phi_\pi^*}(\mathcal{H}) \subseteq \mathcal{B}_0(\mathcal{H})$ if Φ_π belongs to the family $\{\Phi_\pi : \pi \in \widehat{\Pi}\}$ of s.n.functions; for if the answer to this question is affirmative, then Theorem 8.2.7 would yield $\mathcal{N}_{\Phi_\pi^*}(\mathcal{H}) = \mathcal{B}_0(\mathcal{H})$ and would thus characterize the Φ_π^*-norming operators in $\mathcal{B}(\mathcal{H})$. By Proposition 8.2.8 it suffices to know whether $\mathcal{N}_{\Phi_\pi^*}(\mathcal{H}) \cap \mathcal{B}(\mathcal{H})_+ \subseteq \mathcal{B}_0(\mathcal{H})$ where $\mathcal{B}(\mathcal{H})_+ = \{T \in \mathcal{B}(\mathcal{H}) : T \geq 0\}$. The following lemma and example prove the existence of $\pi \in \widehat{\Pi}$ such that $\mathcal{N}_{\Phi_\pi^*}(\mathcal{H}) \nsubseteq \mathcal{B}_0(\mathcal{H})$.

Lemma 9.1.1 ([GK69]). *If $\Phi_\pi \in \{\Phi_\pi : \pi \in \widehat{\Pi}\}$, then its adjoint Φ_π^* is given by*

$$\Phi_\pi^*(\xi) = \sup_n \left\{ \frac{\sum_{j=1}^n \xi_j}{\sum_{j=1}^n \pi_j} \right\} \quad \textit{for every } \xi = (\xi_i)_{i\in\mathbb{N}} \in c_{00}^*.$$

Moreover, the s.n.function Φ_π^* is equivalent to the minimal s.n.function.

For the proof of the above lemma we refer the reader to [GK69, Chapter 3, Lemma 15.1, Page 147]; readers can also see Pages 148-149, and the paragraph preceding Theorem 15.2 of the monograph [GK69].

Example 9.1.2. Consider the positive diagonal operator

$$P = \begin{bmatrix} 2 & & & & & & & \\ & 1+\frac{1}{2} & & & & \text{\Large 0} & & \\ & & 1+\frac{1}{3} & & & & & \\ & & & 1+\frac{1}{4} & & & & \\ & & & & \ddots & & & \\ & \text{\Large 0} & & & & 1+\frac{1}{n} & & \\ & & & & & & \ddots \end{bmatrix} \in \mathcal{B}(\ell^2),$$

with respect to an orthonormal basis $B = \{v_i : i \in \mathbb{N}\}$. Let $\pi = (\pi_n)_{n\in\mathbb{N}}$ be a sequence of real numbers defined by $\pi_n := \frac{1}{2} + \frac{1-1/2}{n} = \frac{n+1}{2n}$. That $\pi \in \widehat{\Pi}$ is obvious and hence $\Phi_\pi \in \{\Phi_\pi : \pi \in \widehat{\Pi}\}$. Consequently, Φ_π is equivalent to the maximal s.n.function Φ_1 and the dual space $\mathfrak{S}^*_{\Phi_\pi}$ of the s.n.ideal $\mathfrak{S}_{\Phi_\pi}$ is isometrically isomorphic to $(\mathcal{B}(\ell^2), \|\cdot\|_{\Phi^*_\pi})$, that is, $\mathfrak{S}_{\Phi_\pi} \cong (\mathcal{B}(\ell^2), \|\cdot\|_{\Phi^*_\pi})$ isometrically. An easy computation yields

$$\|P\|_{\Phi^*_\pi} = \sup_n \left\{ \frac{\sum_{j=1}^n s_j(P)}{\sum_{j=1}^n \pi_j} \right\} = \sup_n \left\{ \frac{n + (1 + 1/2 + ... + 1/n)}{\frac{1}{2}(n + (1 + 1/2 + ... + 1/n))} \right\} = 2.$$

If we define K to be the diagonal operator given by

$$K = \begin{pmatrix} 1 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix} \in \mathcal{B}_1(\ell^2) = \mathfrak{S}_{\Phi_\pi},$$

then we have $\|K\|_{\Phi_\pi} = \sum_j \pi_j s_j(K) = 1$ and $|\operatorname{Tr}(PK)| = |\operatorname{Tr}(\operatorname{diag}\{2, 0, 0, ...\})| = 2 = \|P\|_{\Phi^*_\pi}$ which implies that $P \in \mathcal{N}_{\Phi^*_\pi}(\mathcal{H})$. However, $P \notin \mathcal{B}_0(\ell^2)$. This proves the existence of an s.n.function $\Phi_\pi \in \{\Phi_\pi : \pi \in \widehat{\Pi}\}$ equivalent to the maximal s.n.function such that $\mathcal{N}_{\Phi^*_\pi}(\mathcal{H}) \nsubseteq \mathcal{B}_0(\mathcal{H})$.

The above example establishes the fact that even for the family of s.n.functions given by $\{\Phi_\pi : \pi \in \widehat{\Pi}\}$, it is too much to ask for the set $\mathcal{N}_{\Phi^*_\pi}(\mathcal{H})$ to be contained in the compacts for a given Φ_π from the family. So let us be more modest and ask whether $P \in \mathcal{B}(\mathcal{H})$ is compact whenever $P \in \mathcal{N}_{\Phi^*_\pi}(\mathcal{H}) \cap \mathcal{B}(\mathcal{H})_+$ for every $\Phi_\pi \in \{\Phi_\pi : \pi \in \widehat{\Pi}\}$. The answer to this question is a resounding yes as is stated in the Theorem 9.1.6. However, before we prove this theorem rigorously, let us pause to find an s.n.function Φ_π from the family $\{\Phi_\pi : \pi \in \widehat{\Pi}\}$ of s.n.functions such that the positive noncompact operator P of Example 9.1.2 does not belong to $\mathcal{N}_{\Phi^*_\pi}(\mathcal{H})$. The example which follows illustrates this and hence agrees with the Theorem 9.1.6.

Example 9.1.3. Let $\pi = (\pi_n)_{n\in\mathbb{N}}$ be a sequence defined by

$$\pi_n := \frac{1}{3} + \frac{1 - 1/3}{n} = \frac{n+2}{3n}.$$

Then $\pi \in \widehat{\Pi}$, $\Phi_\pi \in \{\Phi_\pi : \pi \in \widehat{\Pi}\}$ and $\mathfrak{S}_{\Phi_\pi} \cong (\mathcal{B}(\ell^2), \|\cdot\|_{\Phi^*_\pi})$ isometrically. We consider the operator P of Example 9.1.2 and prove that $P \notin \mathcal{N}_{\Phi^*_\pi}(\ell^2)$. To show this, we assume that

$P \in \mathcal{N}_{\Phi_\pi^*}(\ell^2)$, that is, the supremum,

$$\sup \left\{ \sum_j s_j(P) s_j(K) : K \in \mathcal{B}_1(\ell^2), K = \operatorname{diag}\{s_1(K), s_2(K), ...\}, \|K\|_{\Phi_\pi} = 1 \right\},$$

is attained, and we deduce a contradiction from this assumption. So there exists $K = \operatorname{diag}\{s_1(K), s_2(K), ...\} \in \mathcal{B}_1(\ell^2)$ with $\|K\|_{\Phi_\pi} = \sum_j \pi_j s_j(K) = 1$ such that $\|P\|_{\Phi_\pi^*} = |\operatorname{Tr}(PK)| = \sum_j s_j(P) s_j(K)$. Since $K \in \mathcal{B}_1(\mathcal{H}) \subseteq \mathcal{B}_0(\mathcal{H})$, we have $\lim_{j\to\infty} s_j(K) = 0$. This forces the existence of a natural number M such that $s_M(K) > s_{M+1}(K)$. All that remains is to show the existence of an operator $\tilde{K} \in \mathcal{B}_1(\mathcal{H})$, $\|\tilde{K}\|_{\Phi_\pi} = 1$ of the form $\tilde{K} = \operatorname{diag}\{s_1(\tilde{K}), s_2(\tilde{K}), ...\}$ such that $\sum_j s_j(P) s_j(\tilde{K}) > \sum_j s_j(P) s_j(K)$.

If we define

$$\lambda := \frac{\sum_{j=M}^{M+1} \pi_j s_j(K)}{\sum_{j=M}^{M+1} \pi_j} = \frac{\pi_M s_M(K) + \pi_{M+1} s_{M+1}(K)}{\pi_M + \pi_{M+1}}$$

and let $\tilde{K}$ be the diagonal operator defined by

$$\tilde{K} := \begin{pmatrix} s_1(K) & & & & & & \\ & \ddots & & & & & \\ & & s_{M-1}(K) & & & & \\ & & & \lambda & & & \\ & & & & \lambda & & \\ & & & & & s_{M+1}(K) & \\ & & & & & & \ddots \end{pmatrix},$$

then it can be easily verified that $\|\tilde{K}\|_{\Phi_\pi} = \|K\|_{\Phi_\pi} = 1$ so that $\tilde{K} \in \mathcal{B}_1(\ell^2)$ and is of the form $\tilde{K} = \operatorname{diag}\{s_1(\tilde{K}), s_2(\tilde{K}), ...\}$. We now prove that $\tilde{K}$ is the required candidate. It is not too hard to see that

$$\frac{\pi_M}{\pi_{M+1}} > \frac{s_M(P)}{s_{M+1}(P)},$$

which yields,

$$\pi_M s_{M+1}(P)(s_M(K) - s_{M+1}(K)) > \pi_{M+1} s_M(P)(s_M(K) - s_{M+1}(K)).$$

Simplification and rearrangement of terms in the above inequality gives

$$(s_M(P) + s_{M+1}(P)) \left(\frac{\pi_M s_M(K) + \pi_{M+1} s_{M+1}(K)}{\pi_M + \pi_{M+1}} \right) > s_M(P) s_M(K) + s_{M+1}(P) s_{M+1}(K).$$

But the left hand side of the above inequation is actually

$$s_M(P)s_M(\tilde{K}) + s_{M+1}(P)s_{M+1}(\tilde{K}),$$

which implies that

$$s_M(P)s_M(\tilde{K}) + s_{M+1}(P)s_{M+1}(\tilde{K}) > s_M(P)s_M(K) + s_{M+1}(P)s_{M+1}(K).$$

It then immediately follows that $\sum_j s_j(P)s_j(\tilde{K}) > \sum_j s_j(P)s_j(K)$ which contradicts the assumption that $\sum_j s_j(P)s_j(K)$ is the supremum of the set

$$\left\{\sum_j s_j(P)s_j(K) : K \in \mathcal{B}_1(\ell^2), K = \operatorname{diag}\{s_1(K), s_2(K), ...\}, \|K\|_{\Phi_\pi} = 1\right\}$$

and this is precisely the assertion of our claim.

The working rule of the above example is illuminating. The sequence $\pi = (\pi_n)_{n\in\mathbb{N}} \in \widehat{\Pi}$ has been cleverly chosen to construct the example. The significance of choosing this sequence lies in the fact that it guarantees the existence of a natural number M so that $s_M(K) > s_{M+1}(K)$ as well as $\frac{\pi_M}{\pi_{M+1}} > \frac{s_M(P)}{s_{M+1}(P)}$. We use this example as a tool to prove the following proposition.

Proposition 9.1.4. *Let $P \in \mathcal{B}(\mathcal{H})$ be a positive operator. If $\pi \in \widehat{\Pi}$ such that*

$$\frac{\pi_n}{\pi_{n+1}} > \frac{s_n(P)}{s_{n+1}(P)} \text{ for every } n \in \mathbb{N},$$

then $P \notin \mathcal{N}_{\Phi_\pi^*}(\mathcal{H})$.

Proof. To show that $P \notin \mathcal{N}_{\Phi_\pi^*}(\mathcal{H})$, we assume that $P \in \mathcal{N}_{\Phi_\pi^*}(\mathcal{H})$, and we deduce a contradiction from this assumption. If $P \in \mathcal{N}_{\Phi_\pi^*}(\mathcal{H})$, then there exists $K = \operatorname{diag}(s_1(K), s_2(K), ...) \in \mathcal{B}_1(\mathcal{H})$ with $\|K\|_{\Phi_\pi} = \sum_j \pi_j s_j(K) = 1$ such that $\|P\|_{\Phi_\pi^*} = |\operatorname{Tr}(PK)| = \sum_j s_j(P)s_j(K)$. Since $K \in \mathcal{B}_1(\mathcal{H}) \subseteq \mathcal{B}_0(\mathcal{H})$, we have $\lim_{j\to\infty} s_j(K) = 0$. This forces the existence of a natural number M such that $s_M(K) > s_{M+1}(K)$. We complete the proof by establishing the existence of an operator $\tilde{K} \in \mathcal{B}_1(\mathcal{H})$, $\|\tilde{K}\|_{\Phi_\pi} = 1$ of the form $\tilde{K} = \operatorname{diag}\{s_1(\tilde{K}), s_2(\tilde{K}), ...\}$ such that $\sum_j s_j(P)s_j(\tilde{K}) > \sum_j s_j(P)s_j(K)$.

To this end we define

$$\lambda := \frac{\sum_{j=M}^{M+1} \pi_j s_j(K)}{\sum_{j=M}^{M+1} \pi_j},$$

and let

$$\tilde{K} := \begin{pmatrix} s_1(K) & & & & & & \\ & \ddots & & & & & \\ & & s_{M-1}(K) & & & & \\ & & & \lambda & & & \\ & & & & \lambda & & \\ & & & & & s_{M+1}(K) & \\ & & & & & & \ddots \end{pmatrix}.$$

It can be verified that $\|\tilde{K}\|_{\Phi_\pi} = \|K\|_{\Phi_\pi} = 1$ so that $\tilde{K} \in \mathcal{B}_1(\ell^2)$ and is of the form $\tilde{K} = \operatorname{diag}\{s_j(\tilde{K})\}$. However, since

$$\frac{\pi_n}{\pi_{n+1}} > \frac{s_n(P)}{s_{n+1}(P)} \text{ for every } n \in \mathbb{N},$$

it follows that

$$\frac{\pi_M}{\pi_{M+1}} > \frac{s_M(P)}{s_{M+1}(P)},$$

and thus we have,

$$s_M(P)s_M(\tilde{K}) + s_{M+1}(P)s_{M+1}(\tilde{K}) > s_M(P)s_M(K) + s_{M+1}(P)s_{M+1}(K),$$

which yields

$$\sum_j s_j(P)s_j(\tilde{K}) > \sum_j s_j(P)s_j(K) = \|P\|_{\Phi_\pi^*},$$

which contradicts the assumption that $\sum_j s_j(P)s_j(K)$ is the supremum of the set

$$\left\{\sum_j s_j(P)s_j(K) : K \in \mathcal{B}_1(\ell^2), K = \operatorname{diag}\{s_1(K), s_2(K), ...\}, \|K\|_{\Phi_\pi} = 1\right\}.$$

This proves our assertion. $\square$

Theorem 9.1.5 ([Pan17b]). *Let $P \in \mathcal{B}(\mathcal{H})$ be a positive operator and $\lim_{j\to\infty} s_j(P) \neq 0$, that is, P is not compact. Then there exists $\pi \in \widehat{\Pi}$ such that*

$$\frac{\pi_n}{\pi_{n+1}} > \frac{s_n(P)}{s_{n+1}(P)} \text{ for every } n \in \mathbb{N}.$$

Alternatively, if $P \in \mathcal{B}(\mathcal{H})$ is positive noncompact operator then there exists $\pi \in \widehat{\Pi}$ such that $P \notin \mathcal{N}_{\Phi_\pi^*}(\mathcal{H})$.

Proof. Since $P \geq 0$ and $\lim_{j\to\infty} s_j(P) \neq 0$, there exists $s > 0$ such that $\lim_{j\to\infty} s_j(P) = s$. If we take $\alpha_n := \frac{1}{e^{1/n^2}}$ for all $n \in \mathbb{N}$ and define a sequence $\pi = (\pi_n)_{n\in\mathbb{N}}$ recursively by

$$\pi_1 = 1 \ \text{ and } \ \frac{\pi_{n+1}}{\pi_n} := \alpha_n \frac{s_{n+1}(P)}{s_n(P)} \ \text{ for all } n \in \mathbb{N},$$

we have $\alpha_n < 1$ for all $n \in \mathbb{N}$. Then the fact that $s_n(P)$ is a nonincreasing sequence implies that $\frac{\pi_{n+1}}{\pi_n} < \frac{s_{n+1}(P)}{s_n(P)}$ for all $n \in \mathbb{N}$. Therefore, $\frac{\pi_n}{\pi_{n+1}} > \frac{s_n(P)}{s_{n+1}(P)}$ for every $n \in \mathbb{N}$. All that remains is to show that $\pi \in \widehat{\Pi}$. That $\pi_1 = 1$ and $(\pi_n)_{n\in\mathbb{N}}$ is a strictly decreasing sequence of positive real numbers are trivial observations. We complete the proof by showing that $\lim_{n\to\infty} \pi_n > 0$. An easy calculation shows that

$$\pi_{n+1} = \left(\prod_{m=1}^{n} \alpha_m\right) \frac{s_{n+1(P)}}{s_1(P)} \ \text{ for each } \ n \in \mathbb{N}.$$

Let $x_n = (\prod_{m=1}^{n} \alpha_m)$ for every $n \in \mathbb{N}$ and observe that

$$\pi_{n+1} = (x_n)\left(\frac{s_{n+1(P)}}{s_1(P)}\right),$$

which yields

$$\lim_{n\to\infty} (\pi_{n+1}) = \frac{1}{s_1(P)} \lim_{n\to\infty} x_n \lim_{n\to\infty} s_{n+1}(P).$$

This observation, together with the facts that $s_1(P) > 0$ and $\lim_{n\to\infty} s_{n+1}(P) = s > 0$ allows us to infer that $\lim_{n\to\infty}(\pi_{n+1}) > 0$ if and only if $\lim_{n\to\infty} x_n > 0$. But

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{e^{\sum_{m=1}^{n} 1/m^2}} = \frac{1}{e^{\pi/6}} > 0,$$

and we conclude that $\lim_{n\to\infty}(\pi_n) > 0$. This completes the proof.

$\square$

We are now in a position to prove a key result — a characterization theorem for positive operators in $\{\mathcal{N}_{\Phi_\pi^*}(\mathcal{H}) : \pi \in \widehat{\Pi}\}$ — which answers the question we asked in the paragraph preceding the Example 9.1.3. Moreover, this result is a special case of a more general result that will be presented in the next section (see Theorem 9.2.4).

Theorem 9.1.6. *Let P be a positive operator on $\mathcal{H}$. Then the following statements are equivalent.*

1. $P \in \mathcal{B}_0(\mathcal{H})$.

2. $P \in \mathcal{AN}_{\Phi_\pi^*}(\mathcal{H})$ *for every* $\pi \in \widehat{\Pi}$.

3. $P \in \mathcal{N}_{\Phi_\pi^*}(\mathcal{H})$ *for every* $\pi \in \widehat{\Pi}$.

Proof. (1) implies (2) follows from Theorem 8.2.7. (2) implies (3) is obvious. (3) implies (1) is a direct consequence of the Theorem 9.1.5. □

We conclude this section by proving the following result that extends the above theorem to bounded operators in $\mathcal{B}(\mathcal{H})$, the above theorem required the operator to be positive. This is the main theorem of this section.

Theorem 9.1.7 ([Pan17b]). *If* $T \in \mathcal{B}(\mathcal{H})$, *then the following statements are equivalent.*

1. $T \in \mathcal{B}_0(\mathcal{H})$.

2. $T \in \mathcal{AN}_{\Phi_\pi^*}(\mathcal{H})$ *for every* $\pi \in \widehat{\Pi}$.

3. $T \in \mathcal{N}_{\Phi_\pi^*}(\mathcal{H})$ *for every* $\pi \in \widehat{\Pi}$.

Proof. (2) implies (3) is obvious, as is (1) implies (2) from the Theorem 8.2.7. The Proposition 8.2.8 along with the Theorem 9.1.5 proves (3) implies (1). □

The above result, although very important, is transitory. We will see a much more general result than this — the characterization theorem for universally symmetric norming operators (see Theorem 9.2.6).

9.2 Characterization of universally symmetric norming operators

In the preceding section we considered a certain family $\{\Phi_\pi : \pi \in \widehat{\Pi}\}$ of s.n.functions and a family of symmetric norms on $\mathcal{B}(\mathcal{H})$ generated by the dual of these, and we studied the symmetric norming operators and absolutely symmetric norming operators with respect to each of these symmetric norms. The fact that each member of the family $\{\Phi_\pi : \pi \in \widehat{\Pi}\}$ is equivalent to the maximal s.n.function Φ_1 suggests the possibility of extending the Theorem 9.1.7 to a larger family of s.n.functions. With this in mind, our attention is drawn to the family of all s.n.functions that are equivalent to the maximal s.n.function, that is, the family $\{\Phi : \Phi \text{ is equivalent to } \Phi_1\}$ of s.n.functions. This larger family of s.n. functions provides us with the leading idea on which we develop the notions of "universally symmetric norming operators" and "universally absolutely symmetric norming operators" on a separable Hilbert space. The study of these operators are taken up in this section.

We begin by defining the relevant classes of operators.

Definition 9.2.1. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *universally symmetric norming* if $T \in \mathcal{N}_{\Phi^*}(\mathcal{H})$ for every s.n.function Φ equivalent to the maximal s.n.function Φ_1. Alternatively, an operator $T \in (\mathcal{B}(\mathcal{H}))$ is said to be universally symmetric norming if $T \in \mathcal{N}_{\Phi^*}(\mathcal{H})$ for every Φ from the family $\{\Phi : \Phi \text{ is equivalent to } \Phi_1\}$ of s.n.functions.

Definition 9.2.2. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *universally absolutely symmetric norming* if $T \in \mathcal{AN}_{\Phi^*}(\mathcal{H})$ for every s.n.function Φ equivalent to the maximal s.n.function Φ_1.

Remark 9.2.3. Since every symmetric norm on $\mathcal{B}(\mathcal{H})$ is topologically equivalent to the usual operator norm, it follows that $T \in \mathcal{B}(\mathcal{H})$ is universally symmetric norming (respectively universally absolutely symmetric norming) if and only if T is symmetric norming (respectively absolutely symmetric norming) with respect to every symmetric norm on $\mathcal{B}(\mathcal{H})$. Another important observation worth mentioning here is that every universally absolutely symmetric norming operator is universally symmetric norming.

The following theorem gives a useful characterization of positive universally symmetric norming operators in $\mathcal{B}(\mathcal{H})$.

Theorem 9.2.4. *Let P be a positive operator on $\mathcal{H}$ and let Φ_1 denote the maximal s.n.function. Then the following statements are equivalent.*

1. *$P \in \mathcal{B}_0(\mathcal{H})$.*
2. *P is universally absolutely symmetric norming, that is, $P \in \mathcal{AN}_{\Phi^*}(\mathcal{H})$ for every s.n.function Φ equivalent to Φ_1.*
3. *P is universally symmetric norming, that is, $P \in \mathcal{N}_{\Phi^*}(\mathcal{H})$ for every s.n.function Φ equivalent to Φ_1.*

Proof. The implication (1) $\implies$ (2) is an immediate consuequence of Theorem 8.2.7 and (2) $\implies$ (3) is straightforward. To prove (3) $\implies$ (1), assume that the positive operator P is universally symmetrically norming on $\mathcal{H}$. Then the statement (3) of Theorem 9.1.6 holds which implies that P is compact and the proof is complete. $\square$

We next establish the following result which allows us to extend the above theorem to operators that are not necessarily positive.

Proposition 9.2.5. *An operator $T \in \mathcal{B}(\mathcal{H})$ is universally symmetric norming if and only if $|T|$ is universally symmetrically norming.*

Proof. This follows immediately from the Proposition 8.2.8. $\square$

We are now prepared to extend the Theorem 9.2.4 for an arbitrary operator on a separable Hilbert space.

Theorem 9.2.6 ([Pan17b])**.** *Let $T \in \mathcal{B}(\mathcal{H})$ and let Φ_1 denote the maximal s.n.function. Then the following statements are equivalent.*

1. *$T \in \mathcal{B}_0(\mathcal{H})$.*
2. *T is universally absolutely symmetric norming, that is, $T \in \mathcal{AN}_{\Phi^*}(\mathcal{H})$ for every s.n.function Φ equivalent to Φ_1.*
3. *T is universally symmetric norming, that is, $T \in \mathcal{N}_{\Phi^*}(\mathcal{H})$ for every s.n.function Φ equivalent to Φ_1.*

Proof. Theorem 9.2.4 and the preceding proposition yield this result. $\square$

Remark 9.2.7. The preceding result, in effect, states that an operator in $\mathcal{B}(\mathcal{H})$ is universally symmetric norming if and only if it is universally absolutely symmetric norming, which holds if and only if it is compact. As an immediate consequence, this result provides an alternative characterization of compact operators on $\mathcal{H}$.

It is worth noticing that Theorem 9.1.5 essentially states that given any positive noncompact operator on a (infinite-dimensional separable) Hilbert space $\mathcal{H}$, there exists a symmetric norm on $\mathcal{B}(\mathcal{H})$ with respect to which the operator does not attain its norm. The following corollary extends Theorem 9.1.5 to any noncompact operator.

Corollary 9.2.8. *If $T \in \mathcal{B}(\mathcal{H})$ is a noncompact operator then there exists $\pi \in \widehat{\Pi}$ such that $T \notin \mathcal{N}_{\Phi_\pi^*}(\mathcal{H})$*

Proof. Contrapositively, if for every $\pi \in \widehat{\Pi}$ the operator $T \in \mathcal{N}_{\Phi_\pi^*}(\mathcal{H})$, then by the preceding theorem T must be a compact operator. $\square$

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