Series-Parallel Posets and Polymorphisms

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Pure Mathematics

Waterloo, Ontario, Canada, 2018

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

We examine various aspects of the poset retraction problem for series-parallel posets. In particular we show that the poset retraction problem for series-parallel posets that are already solvable in polynomial time are actually also solvable in nondeterministic logarithmic space (assuming P $\neq$ NP). We do this by showing that these series-parallel posets when expanded by constants have bounded path duality. We also give a recipe for constructing members of this special class of series-parallel poset analogous to the construction of all series-parallel posets. Piecing together results from [5],[15],[14] and [12] one can deduce that if a relational structure expanded by constants has bounded path duality then it admits SD-$\vee$ operations. We directly prove the existence of SD-$\vee$ operations on members of this class by providing an algorithm which constructs them. Moreover, we obtain a polynomial upper bound to the length of the sequence of these operations. This also proves that for this class of series-parallel posets, having bounded path duality when expanded by constants is equivalent to admitting SD-$\vee$ operations. This equivalence is not yet known to be true for general relational structures; only the forward direction is proven. However the reverse direction is known to be true for structures that admit NU operations. Zádori has classified in [26] the class of series-parallel posets admitting an NU operation and has shown that every such poset actually admits a 5-ary NU operation. We give a recipe for constructing series-parallel posets of this class analogous to the one mentioned before. Then we show an alternative proof for Zádori's result.

Acknowledgements

I would like to thank Professor Ross Willard, whose extensive editing made this thesis possible.

Dedication

This is dedicated to my family and friends.

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Chapter 1

Introduction

In this thesis we will explore various aspects of the poset retraction problem on series-parallel posets. The poset retraction problem is a cousin of the constraint satisfaction problem (CSP) whose goal is to assign values to a set of variables such that a given set of constraints is satisfied. The computational complexity of such a problem refers to the amount of resources that is required to solve it. The resources that we are interested in are the amount of time and amount of memory space it requires to solve such a problem. A series-parallel poset is an example of a relational structure. It has been shown by Dalmau [5] that if a structure has bounded path duality then its CSP is in non-deterministic logarithmic space. As we will see later this means that the retraction problem will be in NL for any posets whose expansion by constants has bounded path duality. We will use these tools to establish a classification of those series-parallel posets whose retraction problem is in NL (assuming NL $\neq$ NP).

The existence of special operations on a relational structure gives us insight into the complexity of its CSP. We examine the existence of NU and SD-$\vee$ polymorphisms on series-parallel posets. A characterization of those series-parallel posets having an NU polymorphism has already been shown by Zádori [26]; what we present is merely an alternative proof. In the case when a structure is such that its expansion by constants has bounded path duality, it follows from known results that the poset has SD-$\vee$ polymorphisms, but no easy procedure for producing these polymorphisms is known. We provide an easy recipe for those posets that are series-parallel.

Series-parallel posets in general have a recipe for their construction. We will give similar recipes for all classes of series-parallel posets discussed in this paper.

The content of this thesis is as follows. In Chapter 2 we will introduce the background knowledge in universal algebra required for this thesis. Chapter 3 discusses Constraint satisfaction problems in detail as well as notable results from the research community. Starting from Chapter 4 we will focus on posets. We will describe what it means to be a series-parallel poset and the various properties such a poset may have. In Chapter 5 we will present our result that shows a certain class of series-parallel posets are such that their expansion by constants have bounded path duality. Next in Chapter 6 we will give our alternative proof to Zádori's result for NU polymorphisms mentioned before. Finally in Chapter 7 we will show our recipe for constructing SD-$\vee$ polymorphisms on those series-parallel posets mentioned in Chapter 5.

Chapter 2

Preliminaries

2.1 Language and Structures

The structures we will be working with in their most basic form are sets with operations and relations. In order to ensure we are not comparing apples to oranges we first have to introduce the notion of a language.

Definition 2.1.1. A (first-order) language $\mathcal{L}$ consists of a set $\mathcal{R}$ of *relation symbols* and a set $\mathcal{F}$ of *function symbols*, and to each element of $\mathcal{R}$ and $\mathcal{F}$ is assigned a natural number called the *arity* of the symbol. Let $\mathcal{R}_n$ ($\mathcal{F}_n$) denote the set of relation (function) symbols in $\mathcal{R}$ ($\mathcal{F}$) of arity n. If $\mathcal{R} = \emptyset$ then $\mathcal{L}$ is an *algebraic language*, and if $\mathcal{F} = \emptyset$ then $\mathcal{L}$ is a *relational language*.

The symbols in our languages will be used to identify operations and relations on our sets. When both $\mathcal{R}$ and $\mathcal{F}$ are finite we may express $\mathcal{L}$ as a set of its symbols. For example the language of abelian groups consist of no relation symbols, a 2-ary function symbol $+$, a 1-ary function symbol $-$ and a 0-ary function symbol 0. We will write it as $\{+, -, 0\}$. Similarly the language of rings will be $\{+, -, \times, 0, 1\}$. Both of these are examples of algebraic languages. As we will define later the language of partial orders will be a relational language.

Definition 2.1.2. Let $\mathcal{L}$ be a language. An *$\mathcal{L}$-structure* $\mathbb{M} = \langle U^{\mathbb{M}}, R^{\mathbb{M}}, F^{\mathbb{M}} \rangle$ consists of

- a non-empty universe $U^{\mathbb{M}}$;
- a set $R^{\mathbb{M}}$ of fundamental relations $r^{\mathbb{M}}$ indexed by $r \in \mathcal{R}$ where the arity of $r^{\mathbb{M}}$ is

equal to the arity of r; and,

- a set $F^{\mathbb{M}}$ of fundamental operations $f^{\mathbb{M}}$ indexed by $f \in \mathcal{F}$ where the arity of $f^{\mathbb{M}}$ is equal to the arity of f.

When $\mathcal{R} = \emptyset$ the $\mathcal{L}$-structure is called an *algebra*, and when $\mathcal{F} = \emptyset$ it is called a *relational structure*. If $R^{\mathbb{M}} = \{r_1^{\mathbb{M}}, r_2^{\mathbb{M}}, \ldots, r_n^{\mathbb{M}}\}$ and $F^{\mathbb{M}} = \{f_1^{\mathbb{M}}, f_2^{\mathbb{M}}, \ldots, f_m^{\mathbb{M}}\}$ contain only a small finite amount of elements, $\langle U^{\mathbb{M}}, R^{\mathbb{M}}, F^{\mathbb{M}}\rangle$ may be written as

$$\langle U^{\mathbb{M}}, r_1^{\mathbb{M}}, r_2^{\mathbb{M}}, \ldots, r_n^{\mathbb{M}}, f_1^{\mathbb{M}}, f_2^{\mathbb{M}}, \ldots, f_m^{\mathbb{M}}\rangle$$

instead. The superscript $\mathbb{M}$ may also be omitted when it is clear which $\mathcal{L}$-structure the object in question belongs to.

We will call the $r^{\mathbb{M}}$'s and $f^{\mathbb{M}}$'s the *interpretations* of the symbols of $\mathcal{L}$ in $\mathbb{M}$.

An abelian group $\mathbb{G}$ can be thought of as a structure of the language $\{+, -, 0\}$ with the obvious interpretations. Similarly a ring $\mathbb{R}$ will be a structure of the language $\{+, -, \times, 0, 1\}$. Both of these are examples of algebras. We will have plenty of examples of relational structures when we define partial orders.

Just as it is meaningless to compare a group with a ring, it only makes sense to compare structures when they are of the same language. Of course the group of integers can be expanded to the ring of integers by adding new interpretations for symbols that are in the language of rings but not in the language of groups. When we have a language that is the superset of another we can also expand the structures of the smaller language into a structure of the larger one by adding new interpretations for the additional symbols.

2.2 Substructures and Direct Products

There is a natural analogue of subgroups and subrings for $\mathcal{L}$-structures.

Definition 2.2.1. Let $\mathcal{L}$ be a language. Given two $\mathcal{L}$-structures $\mathbb{M} = \{U^{\mathbb{M}}, R^{\mathbb{M}}, F^{\mathbb{M}}\}$ and $\mathbb{N} = \{U^{\mathbb{N}}, R^{\mathbb{N}}, F^{\mathbb{N}}\}$, $\mathbb{N}$ is an $\mathcal{L}$-*substructure* of $\mathbb{M}$ if:

- $U^{\mathbb{N}} \subseteq U^{\mathbb{M}}$;
- for each $r \in \mathcal{R}$, $r^{\mathbb{M}} \cap (U^{\mathbb{N}})^n = r^{\mathbb{N}}$ (where n is the arity of r); and,
- for each $f \in \mathcal{R}$, $f^{\mathbb{M}}|_{(U^{\mathbb{N}})^n} = f^{\mathbb{N}}$ (where n is the arity of f).

$\mathbb{N}$ will be called a *subalgebra (substructure)* if $\mathbb{M}$ is an *algebra (relational structure)*.

It should be clear from the definition that substructures correspond to subgroups when we are working with abelian groups in their natural language. The same is true for rings.

Lemma 2.2.2. *Let $\mathcal{L}$ be a language. Let $\mathbb{M}$ be an $\mathcal{L}$-structure. Then a substructure $\mathbb{O}$ of a substructure $\mathbb{N}$ of $\mathbb{M}$ is a substructure of $\mathbb{M}$.*

Proof. First we have $U^{\mathbb{O}} \subseteq U^{\mathbb{N}} \subseteq U^{\mathbb{M}}$.

For each $r \in \mathcal{R}$ of arity n we have $r^{\mathbb{O}} = r^{\mathbb{N}} \cap (U^{\mathbb{O}})^n = r^{\mathbb{M}} \cap (U^{\mathbb{N}})^n \cap (U^{\mathbb{O}})^n$. Since $U^{\mathbb{O}} \subseteq U^{\mathbb{N}}$ this becomes $r^{\mathbb{O}} = r^{\mathbb{M}} \cap (U^{\mathbb{O}})^n$.

For each $f \in \mathcal{R}$ of arity n we have $f^{\mathbb{O}} = f^{\mathbb{N}}|_{(U^{\mathbb{O}})^n} = (f^{\mathbb{M}}|_{(U^{\mathbb{N}})^n})|_{(U^{\mathbb{O}})^n}$. Again because of $U^{\mathbb{O}} \subseteq U^{\mathbb{N}}$ This becomes $f^{\mathbb{O}} = f^{\mathbb{M}}|_{(U^{\mathbb{O}})^n}$.

By definition $\mathbb{O}$ is a substructure of $\mathbb{M}$. □

Definition 2.2.3. Let $\mathcal{L}$ be a language and $\mathbb{M}$ an $\mathcal{L}$-structure. We'll say $S \subseteq U^{\mathbb{M}}$ is a *subuniverse* of $\mathbb{M}$ if S is the universe of an $\mathcal{L}$-substructure of $\mathbb{M}$.

The intersection of subuniverses, if nonempty, is again a subuniverse. With this we can introduce the notion of a generated subalgebra.

Definition 2.2.4. Let $\mathbb{M}$ be an $\mathcal{L}$-structure and $\emptyset \neq S \subseteq U^{\mathbb{M}}$. We will define

$$\mathrm{Sg}(S) := \bigcap \{T : S \subseteq T \text{ and } T \text{ is a subuniverse of } \mathbb{M}\}.$$

We will call $\mathrm{Sg}(S)$ the *subuniverse generated by S*. We will call the $\mathcal{L}$-substructure with $\mathrm{Sg}(S)$ as its universe the *substructure generated by S*.

The substructure generated by S is said to be *finitely generated* if S is a finite set.

The notion of direct product also translates over to $\mathcal{L}$-structures.

Definition 2.2.5. Let $(\mathbb{M}_i)_{i \in I}$ be an indexed family of $\mathcal{L}$-structures. The *direct product* $\mathbb{M} = \prod_{i \in I} \mathbb{M}_i$ is an $\mathcal{L}$-structure with universe $\prod_{i \in I} U^{\mathbb{M}_i}$ and such that for $f \in \mathcal{F}_n$, $r \in \mathcal{R}_m$ and $a_1, \ldots, a_n, b_1, \ldots, b_m \in \prod_{i \in I} U^{\mathbb{M}_i}$,

$$f^{\mathbb{M}}(a_1, \ldots, a_n)(i) = f^{\mathbb{M}_i}(a_1(i), \ldots, a_n(i))$$

for every $i \in I$ and

$$(b_1, \dots, b_n) \in r^{\mathcal{M}} \iff \forall i \in I, (b_1(i), \dots, b_n(i)) \in r^{\mathcal{M}_i}.$$

When $I = n$ is a natural number and $\mathbb{M}_i = \mathbb{M}$ for all $i \in n$ we will write $\prod_{i \in I} \mathbb{M}_i$ as $\mathbb{M}^n$.

2.3 Homomorphisms

We will interact with the structures we have defined above by examining special operations that they admit. These operations in their most basic form are simply homomorphisms.

Definition 2.3.1. Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be $\mathcal{L}$-structures. A map $\phi : \mathbb{M}_1 \mapsto \mathbb{M}_2$ is called a *homomorphism* from $\mathbb{M}_1$ to $\mathbb{M}_2$ if for every $f \in \mathcal{F}_n$, $r \in \mathcal{R}_m$ and $a_1, \dots, a_n, b_1, \dots, b_m \in \mathbb{M}_1$ we have

$$\phi(f^{\mathbb{M}_1}(a_1, \dots, a_n)) = f^{\mathbb{M}_2}(\phi(a_1), \dots, \phi(a_n))$$

and

$$(b_1, \dots, b_2) \in r^{\mathbb{M}_1} \implies (\phi(b_1), \dots, \phi(b_m)) \in r^{\mathbb{M}_2}.$$

When $\mathbb{M}_1 = \mathbb{M}_2$ we will also call ϕ an *endomorphism.*

An endomorphism $r : \mathbb{M}_1 \mapsto \mathbb{M}_1$ is called a *retraction* if $r^2 = r \circ r = r$. In this case we say that the retraction r is *onto* $\mathbb{N}$ if $\mathbb{N}$ is the substructure of $\mathbb{M}_1$ whose universe is the range of r.

If ϕ has a homomorphic inverse we will call ϕ an *isomorphism.* In this case we say that $\mathbb{M}_1$ and $\mathbb{M}_2$ are *isomorphic.*

An isomorphism from $\mathbb{M}_1$ to itself is an *automorphism.*

Consider the Klein four-group of the language $\{+, -, 0\}$. It is an abelian group consisting of the tuples $\{(0,0), (0,1), (1,0), (1,1)\}$. Let r be a map on this group that does the following:

$$\begin{aligned}(0,0) &\mapsto (0,0)\\ (0,1) &\mapsto (0,0)\\ (1,0) &\mapsto (1,1)\\ (1,1) &\mapsto (1,1).\end{aligned}$$

This map is a homomorphism. Since it maps the Klein four-group back into itself it is also an endomorphism. Finally it fixes everything in its image, so it is a retraction as well.

Lemma 2.3.2. *The composition of homomorphisms is still a homomorphism.*

2.4 Algebras and Relational Structures

As I have mentioned before we will be studying partial orders which is a class of relational structures. So it may seem strange to also talk about algebras, but as we will see below they are actually two sides of the same coin.

Definition 2.4.1. Let $\mathbb{R}$ be a relational structure. For any natural number n, a homomorphism $\phi : \mathbb{R}^n \mapsto \mathbb{R}$ is said to be a *polymorphism* of $\mathbb{R}$. Let $\mathrm{Pol}(\mathbb{R})$ denote the set of all polymorphisms of $\mathbb{R}$.

We say that $\mathbb{R}$ *admits* an operation f if f is a polymorphism of $\mathbb{R}$.

The special operations we are interested in for our posets will all be polymorphisms.

Definition 2.4.2. Let $\mathcal{L}$ be an algebraic language. A *term of* $\mathcal{L}$ is a composition of function symbols of the language with variables representing projection maps of any arity.

Let $\mathbb{A}$ be an algebra of language $\mathcal{L}$. An operation on the set A is called a *term operation of* $\mathbb{A}$ if it can be obtained by composing fundamental operations of $\mathbb{A}$ with projection mappings (of any arity).

Each term of the algebraic language has a corresponding term operation in each algebra of that language which we will call its *interpretation*.

Definition 2.4.3. Let $\mathbb{A}$ be an algebra. An n-ary relation $r \subseteq A^n$ is said to be *preserved* by an m-ary fundamental operation f of $\mathbb{A}$ if for all $(a_{1,1}, \ldots, a_{1,n}), \ldots, (a_{m,1}, \ldots, a_{m,n})$ in A^n:

$$(a_{1,1}, \ldots, a_{1,n}), \ldots, (a_{m,1}, \ldots, a_{m,n}) \in r \implies (f(a_{1,1}, \ldots, a_{m,1}), \ldots, f(a_{1,n}, \ldots, a_{m,n})) \in r.$$

Let $\text{Inv}(\mathbb{A})$ denote the set of all relations on A that are preserved by all fundamental operations of $\mathbb{A}$.

Definition 2.4.4. Let $\mathbb{R}$ be a relational structure. The *algebra generated by* $\mathbb{R}$ is defined to be the pair $\langle R, \text{Pol}(\mathbb{R})\rangle$. The language of this new algebra will simply have a function symbol corresponding to each member of $\text{Pol}(\mathbb{R})$. Similarly the *induced relational structure by* an algebra $\mathbb{A}$ will be $\langle A, \text{Inv}(\mathbb{A})\rangle$.

Definition 2.4.5. We will refer to the equivalence relations in $\text{Inv}(\mathbb{A})$ as *congruences*. Let $\text{Con}(\mathbb{A})$ denote the set of all congruences on $\mathbb{A}$.

For those familiar with lattice theory, $\text{Con}(\mathbb{A})$ is an algebraic lattice under set containment.

2.5 Varieties

A variety is a standard way of collecting similar algebras. It is well known that each variety has a set of axioms that determine its membership. For our purposes we will use a different but equivalent definition.

Definition 2.5.1. Let $\mathcal{V}$ be a nonempty class of algebras of language $\mathcal{L}$. We'll call $\mathcal{V}$ a *variety* if it is closed under subalgebras, homomorphic images, and direct products.

It should be clear from their definition that the intersection of varieties is again a variety. The class of all algebras of a language $\mathcal{L}$ is a variety. Thus for every class K of algebras of the same language there exists a smallest variety containing K that is the intersection of all varieties containing K.

Definition 2.5.2. Let K be a class of algebras of some language $\mathcal{L}$. We'll denote the smallest variety (under containment) containing K as $V(K)$. If K has only a single member $\mathbb{A}$ then we may write $V(\mathbb{A})$ instead. We'll say a variety $\mathcal{V}$ is *finitely generated* if $\mathcal{V} = V(K)$ for some finite set K of finite algebras of the same language.

Definition 2.5.3. We'll say an algebra $\mathbb{A}$ is *locally finite* if every finitely generated subalgebra has a finite universe. A class K of algebras is *locally finite* if its members are locally finite.

2.6 Special Operations

Definition 2.6.1. Let f and g be n-ary operations on a set A. Let $x_1, \ldots, x_k$ be k distinct variables and $i_1, \ldots, i_n, j_1, \ldots, j_n$ be indices from the set $\{1, \ldots, k\}$.

The expression $f(x_{i_1}, \ldots, x_{i_n}) \approx g(x_{j_1}, \ldots, x_{j_n})$ denotes the claim that for all $a_1, \ldots, a_k \in A$, $f(a_{i_1}, \ldots, a_{i_n}) = g(a_{j_i}, \ldots, a_{j_n})$. This expression is called an *identity*.

We will be classifying the complexity of a given CSP through the existence of certain operations. Here we list the ones we will come across in this paper.

Definition 2.6.2. Let ϕ be an n-ary operation on a set A.

ϕ is said to be *idempotent* if it satisfies the following identity:

$$\phi\underbrace{(x, x, ..., x)}_{n \text{ times}} \approx x$$

(i.e. $\phi(a, a, \ldots, a) = a$ for every $a \in A$).

ϕ is said to be *near-unanimous (NU)* if it satisfies all of the following identities:

$$\phi(y, x, ..., x) \approx x$$

$$\phi(x, y, ..., x) \approx x$$

$$\vdots$$

$$\phi(x, x, ..., y) \approx x.$$

A ternary NU operation is called a *majority* operation.

ϕ is said to be *totally symmetric* if it satisfies the following identity for all sets of variables such that $\{x_1, \ldots, x_n\} = \{y_1, \ldots, y_n\}$:

$$\phi(x_1, \ldots, x_n) \approx \phi(y_1, \ldots, y_n).$$

When $n > 1$, ϕ is said to be *Taylor* if it is idempotent and for each $i \in \{1, \ldots, n\}$ satisfies an identity of the form

$$\phi(x_{i1}, \ldots, x_{in}) \approx \phi(y_{i1}, \ldots, y_{in})$$

where $x_{ij}, y_{ij} \in \{x, y\}$ and $x_{ii} \neq y_{ii}$.

Let us consider the set $A = \{0, 1\}$ and a 3-ary operation ϕ on A defined as follows:

$$\phi(1,1,1) = 1, \phi(0,0,0) = 0,$$
$$\phi(1,1,0) = 1, \phi(1,0,1) = 1, \phi(0,1,1) = 1,$$
$$\phi(0,0,1) = 0, \phi(0,1,0) = 0, \phi(1,0,0) = 0.$$

By inspection we can see that ϕ is idempotent and NU. Since it is 3-ary it is also a majority operation. However this is not a totally symmetric operation as we see that $\phi(1,1,0) \neq \phi(1,0,0)$. Finally it is a Taylor operation since it satisfies the following identities:

$$\phi(x,x,y) \approx \phi(y,x,x),$$
$$\phi(x,x,y) \approx \phi(x,y,x),$$
$$\phi(y,x,x) \approx \phi(x,x,y).$$

Definition 2.6.3. The sequence of 3-ary operations $d_0, d_1, \ldots, d_n$ is called *a sequence of Freese-McKenzie SD-$\vee$ operations* [12, Theorem 5.1] if they satisfy the following identities:

- $d_0(x,y,z) \approx x$ and $d_n(x,y,z) \approx z$;
- For each $0 \leq i < n$, at least two of the following identities must be true:
  1. $d_i(x,x,y) \approx d_{i+1}(x,x,y)$;
  2. $d_i(x,y,y) \approx d_{i+1}(x,y,y)$;
  3. $d_i(x,y,x) \approx d_{i+1}(x,y,x)$.

Continuing the example from before, let us define d_0 and d_2 to be the first and third ternary projection maps on A. Let $d_1 = \phi$. Then d_0, d_1, d_2 is a sequence of Freese-McKenzie SD-$\vee$ operations on A since they satisfy the following identities:

$$d_0(x,x,y) \approx d_1(x,x,y),$$
$$d_0(x,y,x) \approx d_1(x,y,x),$$
$$d_1(x,y,x) \approx d_2(x,y,x),$$
$$d_1(y,x,x) \approx d_2(y,x,x).$$

For each type of special operations that we have defined in definitions 2.6.2 and 2.6.3, a variety is said to have terms of these types if there exists terms of its language such that their interpretation in each algebra of the variety satisfies the corresponding identities.

Definition 2.6.4. An algebra $\mathbb{A}$ is an *idempotent* algebra if all of its fundamental operations are idempotent.

Proposition 2.6.5. *Let $\mathbb{A}$ be an algebra.*

1. *If $\mathbb{A}$ has an NU term operation, then $\mathbb{A}$ has Freese-McKenzie SD-$\vee$ term operations.*
2. *If $\mathbb{A}$ has Freese-McKenzie SD-$\vee$ term operations, then $\mathbb{A}$ has a Taylor term operation.*

Proof. (1) Let t be an n-ary NU term operation of $\mathbb{A}$. Define the 3-ary term operations $d_0, d_1, \ldots, d_{2n-4}$ by

$$d_{2i}(x, y, z) = t(\underbrace{x, \ldots, x}_{n-i-1 \text{ times}}, \underbrace{z, \ldots, z}_{i+1 \text{ times}})$$

and

$$d_{2i-1}(x, y, z) = t(\underbrace{x, \ldots, x}_{n-i-1 \text{ times}}, y, \underbrace{z, \ldots, z}_{i \text{ times}}).$$

Then $d_0, \ldots, d_{2n-4}$ is a sequence of Freese-McKenzie SD-$\vee$ term operations.

(2) Follows from [13, Lemma 9.4 and Theorem 9.6]. □

Corollary 2.6.6. *Let $\mathbb{R}$ be a relational structure.*

1. *If $\mathbb{R}$ has an NU polymorphism, then $\mathbb{R}$ has Freese-McKenzie SD-$\vee$ polymorphisms.*
2. *If $\mathbb{R}$ has Freese-McKenzie SD-$\vee$ polymorphisms, then $\mathbb{R}$ has a Taylor polymorphism.*

Proof. The term operations of the algebra generated by $\mathbb{R}$ are precisely the polymorphisms of $\mathbb{R}$. □

Chapter 3

Constraint Satisfaction Problem

3.1 Complexity Classes

The computational problems we will be discussing in this paper are all examples of decision problems. A decision problem will take in an input and output either 'yes' or 'no'. These problems are classified based on the complexity of the algorithms that exist to solve them, hence the name complexity classes.

The three main complexity classes we encounter are the classes P, NP, and NL. These are respectively the classes of decision problems which are solvable in polynomial time, in nondeterministic polynomial time, and in nondeterministic logarithmic-space.

What is known about these three classes is that NL $\subseteq$ P $\subseteq$ NP. Neither of the inclusions mentioned is known to be strict. It may just as well be that NL = NP. However it is the belief of most computer scientists that both of the inclusions are strict [24].

We can compare individual problems in NP using polynomial-time reductions or log-space reductions. These are algorithms that convert one problem into another. Log-space reductions are a subset of polynomial-time reductions. Later on we will use the fact that if a problem P is in NL and another problem Q has a log-space reduction to P, then Q is also in NL.

3.2 Definition

There are many ways to formulate the definition for a constraint satisfaction problem. For us we will view it as the homomorphism problem on relational structures.

Definition 3.2.1. Let $\mathbb{R}$ be a relational structure of some finite language $\mathcal{L}$. We will denote $\mathrm{CSP}(\mathbb{R})$ to be the set of all finite relational structures $\mathbb{R}'$ of the same language $\mathcal{L}$ such that there exists a homomorphism from $\mathbb{R}'$ to $\mathbb{R}$. We will denote $\neg\,\mathrm{CSP}(\mathbb{R})$ to be the class of all finite relational $\mathcal{L}$-structures not in $\mathrm{CSP}(\mathbb{R})$.

The CSP *of* $\mathbb{R}$ refers to the decision problem that takes as input a finite relational structure of the language $\mathcal{L}$ and decides whether it belongs to $\mathrm{CSP}(\mathbb{R})$. This is also called the *homomorphism problem of* $\mathbb{R}$ for obvious reasons.

Our definition of the CSP is different from how it is defined in general. The inputs to the CSP of $\mathbb{R}$ are more usually defined as triples $(V, R, \mathcal{C})$ where V is a finite set of variables, R is the domain of $\mathbb{R}$, and $\mathcal{C}$ is a set of *constaints*; each constraint consists of a basic relation from $\mathbb{R}$ and a tuple of variables from V of the same arity as the relation. The decision problem, given such an input, is to decide whether there exists an assignment of values from R to the variables in V such that all constraints in $\mathcal{C}$ are satisfied.

It is known that this version of $\mathrm{CSP}(\mathbb{R})$ is log-space equivalent to the homomorphism version as formulated above [4, Section 2.4].

Here in this paper we study a subproblem of the homomorphism problem called the retraction problem.

Definition 3.2.2. Let $\mathbb{R}$ be a finite relational structure of some finite language $\mathcal{L}$. We will denote $\mathrm{Ret}(\mathbb{R})$ to be the class of all finite relational structures $\mathbb{R}'$ of the same language $\mathcal{L}$ having $\mathbb{R}$ as an induced substructure such that there exists a retraction from $\mathbb{R}'$ onto $\mathbb{R}$.

The *retraction problem* of $\mathbb{R}$ refers to the decision problem that takes as input a finite relational structure of the language $\mathcal{L}$ having $\mathbb{R}$ as an induced substructure and decides whether it belongs to $\mathrm{Ret}(\mathbb{R})$.

It should be clear that $\mathrm{Ret}(\mathbb{R})$ is a subclass of $\mathrm{CSP}(\mathbb{R})$. Many of the results we will be using for our examination of the retraction problem are actually for the homomorphism problem. To bridge this gap we introduce the notion of expanding a structure by constants.

Definition 3.2.3. Let $\mathbb{M}$ be a structure of language $\mathcal{L}$ with universe M. Let $\mathcal{L}'$ be the expansion of $\mathcal{L}$ by adding to it a unary relational symbol U_a for each $a \in M$. Let $\exp(\mathbb{M})$ be the structure in language $\mathcal{L}'$ with universe M where every symbol in $\mathcal{L}$ is defined as it was in $\mathbb{M}$ and $U_a^{\exp(\mathbb{M})} = \{a\}$ for each $a \in M$.

It is known that for a finite relational structure $\mathbb{R}$ the retraction problem $\mathrm{Ret}(\mathbb{R})$ is equivalent to $\mathrm{CSP}(\exp(\mathbb{R}))$ under log-space reductions. So the pair $\mathrm{Ret}(\mathbb{R})$ and $\mathrm{CSP}(\exp(\mathbb{R}))$ are both in NL (non-deterministic logspace) or neither. Thus for us this means we may use the results that have been proven for $\mathrm{CSP}(\exp(\mathbb{R}))$.

3.3 Core Structures

Each structure $\exp(\mathbb{R})$ falls into the category of a special kind of structures called core structures.

Definition 3.3.1. We'll say two relational structures $\mathbb{R}$ and $\mathbb{R}'$ of some language $\mathcal{L}$ are *homomorphically equivalent* if there exist homomorphisms from $\mathbb{R}$ to $\mathbb{R}'$ and from $\mathbb{R}'$ to $\mathbb{R}$.

Definition 3.3.2. A finite relational structure $\mathbb{R}$ is called a *core* if all endomorphisms on $\mathbb{R}$ are automorphisms.

Cores can be thought of as minimal elements (size-wise) in its class of all homomorphically equivalent relational structures. For a finite relational structure there is always a homomorphically equivalent core structure, which is unique up to isomorphism.

Definition 3.3.3. We'll say $\mathbb{R}'$ is a *core* of a finite relational structure $\mathbb{R}$, if $\mathbb{R}'$ is minimal in size in the class of all relational structures homomorphically equivalent to $\mathbb{R}$.

From this we see that $\mathrm{CSP}(\mathbb{R}) = \mathrm{CSP}(\mathbb{R}')$. So typically one would only study the homomorphism problem for core structures.

Lemma 3.3.4. *If $\mathbb{R}$ is a finite relational structure then $\exp(\mathbb{R})$ is a core.*

We should note that although $\exp(\mathbb{R})$ is a core, it is not a core of $\mathbb{R}$. They are in different languages after all. However as we saw previously $\mathrm{CSP}(\exp(\mathbb{R}))$ is in NL if and only if $\mathrm{Ret}(\mathbb{R})$ is in NL. Furthermore if an operation is a polymorphism of $\exp(\mathbb{R})$ then it is an idempotent polymorphism of $\mathbb{R}$ and vice versa.

3.4 Obstructions and Pathwidth

One way to think about whether a structure belongs to CSP($\mathbb{R}$) is to look at all the problematic structural properties it may have. This leads to the study of obstruction sets.

Definition 3.4.1. Let $\mathbb{R}$ be a finite relational structure of a finite language $\mathcal{L}$. We call a set O of finite $\mathcal{L}$-structures an *obstruction set* for CSP($\mathbb{R}$) if for any finite $\mathcal{L}$-structure $\mathbb{R}'$

$$\mathbb{R}' \in \mathrm{CSP}(\mathbb{R}) \iff \forall \mathbb{O} \in O \text{ there does not exist a homomorphism from } \mathbb{O} \text{ to } \mathbb{R}'.$$

In general CSP($\mathbb{R}$) always has an obstruction set, namely the set $\neg\,\mathrm{CSP}(\mathbb{R})$. However this set is too big to be useful. We want our obstruction sets to be such that all members are relatively simple. The next definition explains what "simple" entails.

Definition 3.4.2. Let $\mathbb{O}$ be a finite relational structure of a finite language $\mathcal{L}$. A *path-decomposition* of $\mathbb{O}$ is a collection $S_1, \ldots, S_n$ of subsets of O such that:

1. for every $r \in \mathcal{R}$ and every $(a_1, \ldots, a_m) \in r^{\mathbb{O}}$, there exists $1 \leq i \leq n$ such that $\{a_1, \ldots, a_m\} \subseteq S_i$;
2. if $a \in S_i \cap S_j$, then $a \in S_l$ for all $i \leq l \leq j$.

The *width* of a path-decomposition is the pair of natural numbers

$$(\max\{|S_i \cap S_{i+1}| : 1 \leq i \leq n-1\}, \max\{|S_i| : 1 \leq i \leq n\}).$$

$\mathbb{O}$ is said to have *pathwidth* (j, k) if it has a path decomposition of width (j, k).

For example, let $\mathcal{L} = \{r, s\}$, where r and s are relation symbols of arity $2, 3$ respectively. Let $\mathbb{O}$ be a relational $\mathcal{L}$-structure of universe $O = \{a, b, c, e, d, f, g\}$ and relations

$$r^{\mathbb{O}} = \{(a, b), (a, d), (a, e), (a, f), (a, g), (b, c), (e, g), (f, g)\}$$

and $s^{\mathbb{O}} = \{(a, d, g)\}$. The figure below illustrates the relation $r^{\mathbb{O}}$ as a digraph.

Define the follow subsets:

$$S_1 := \{a, b, c\},$$
$$S_2 := \{a, e, g\},$$
$$S_3 := \{a, f, g\},$$
$$S_4 := \{a, d, g\}.$$

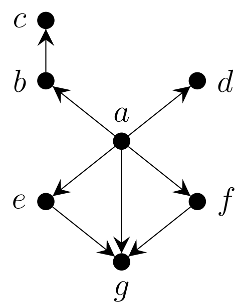

Figure 3.1

Then S_1, S_2, S_3, S_4 is a path-decomposition of $\mathbb{O}$ of width $(2, 3)$. It should be easy to see that $\mathbb{O}$ has pathwidth of at least $(2, 3)$.

3.5 Bounded Path Duality

Definition 3.5.1. A finite relational structure $\mathbb{R}$ in a finite language is said to have *bounded path duality* if for some $0 \leq j \leq k$, CSP($\mathbb{R}$) has an obstruction set whose every member has pathwidth at most (j, k) .

In his 2005 paper Dalmau showed that having bounded path duality is sufficient for a CSP to be in the complexity class NL.

Proposition 3.5.2. *[5, Proposition 3] Let $\mathbb{R}$ be a finite relational structure. If $\mathbb{R}$ has bounded path duality then* CSP($\mathbb{R}$) *is in* NL *(nondeterministic log-space).*

This will allow us to show that Ret($\mathbb{R}$) is in NL by proving exp($\mathbb{R}$) has bounded path duality.

Another implication of having bounded path duality is the existence of Freese-McKenzie SD-$\vee$ operations. For the remainder of this section, we will fix a finite relational structure $\mathbb{R}$ such that exp($\mathbb{R}$) has bounded path duality. In each of the following theorems you may if you wish assume that $\mathbb{S}$ is exp($\mathbb{R}$). We will now state a series of results that when combined lead to the conclusion that $\mathbb{R}$ has Freese-McKenzie SD-$\vee$ operations. The first result in this series is another theorem in the aforementioned paper by Dalmau.

Theorem 3.5.3. *[5, Theorem 5] Let $\mathbb{S}$ be a finite relational structure. $\mathbb{S}$ has bounded path duality if and only if the class $\neg\,\mathrm{CSP}(\mathbb{S})$ is definable in linear Datalog.*

A class is definable in linear Datalog if and only if its members are distinguishable by a Datalog Program. We won't go into any detail on what exactly that means. The reader can simply think of it as a property on a class of relational structures. Larose and Tesson provide us with our next step.

Theorem 3.5.4. *[15, Theorem 4.2] Let $\mathbb{S}$ be a core relational structure and let $\mathbb{A}$ be the algebra it generates. If $\neg\,\mathrm{CSP}(\mathbb{S})$ is definable in linear Datalog then $V(\mathbb{A})$ omits types 1, 2, and 5.*

Omitting types is a powerful characteristic for finite algebras first described by Hobby and McKenzie in their book [13, Theorem 9.11]. They have shown that for a locally finite variety, omitting types 1, 2, and 5 is equivalent to its finite members being congruence join semi-distributive. For our purposes the reader does not need to know the exact definition of congruence join semi-distributivity. It is sufficient to know that it is some property of the set of congruences of an algebra. In 2001 Kearnes improved on this by extending it to all members of the variety (not just finite ones).

Theorem 3.5.5. *[14, Theorem 2.6] A locally finite variety V omits types 1, 2 and 5 if and only if every algebra in the variety is congruence join semi-distributive*

The variety generated by the algebra generated by $\exp(\mathbb{R})$ is finitely generated and hence is known to be locally finite. Thus by this theorem it is also congruence join semi-distributive, assuming $\exp(\mathbb{R})$ has bounded path duality. In a 2017 paper, Freese and McKenzie translate this to the existence of Freese-McKenzie SD-$\vee$ operations.

Theorem 3.5.6. *[12, Theorem 5.1] Let V be an idempotent variety. V is congruence join semi-distributive if and only if V has a sequence of Freese-McKenzie SD-$\vee$ term operations.*

Corollary 3.5.7. *Let V be an idempotent variety. If V is congruence join semi-distributive, then every member of V has a sequence of Freese-McKenzie SD-$\vee$ term operations.*

It is quite easy to see that the algebra generated by $\exp(\mathbb{R})$ is idempotent.

Lemma 3.5.8. *If $\mathbb{R}$ is a finite relational structure with universe R then every polymorphism in* $\mathrm{Pol}(\exp(\mathbb{R}))$ *is idempotent.*

Proof. Let $\phi \in \mathrm{Pol}(\exp(\mathbb{R}))$ be an n-ary polymorphism. For each $a \in R$ there exists the unary relation $U_a^{\exp(\mathbb{R})} = \{a\}$. ϕ must preserve this relation, so $\phi(\underbrace{a, \ldots, a}_{n \text{ times}}) \in \{a\}$. □

An idempotent algebra will generate an idempotent variety. Recall that $\mathbb{R}$ is a fixed finite relational structure. It follows from the above comment and Lemma 3.5.8 that the variety generated by the algebra generated by exp(R) is idempotent. It is easily shown that the term operations of this algebra are exactly its fundamental operations, which are the polymorphisms of $\exp(\mathbb{R})$. Hence if $\exp(\mathbb{R})$ has bounded path duality, then $\exp(\mathbb{R})$ admits a set of Freese-McKenzie SD-$\vee$ operations by Theorems 3.5.3 - 3.5.6 and Corollary 3.5.7. As $\mathbb{R}$ admits every operation admitted by $\exp(\mathbb{R})$, we have proved the following proposition.

Proposition 3.5.9. *Let $\mathbb{R}$ be a finite relational structure in a finite language $\mathcal{L}$. If* $\exp(\mathbb{R})$ *has bounded path duality then $\mathbb{R}$ admits a set of Freese-McKenzie SD-$\vee$ operations.*

The converse of this proposition is a long standing conjecture by Larose and Tesson stated in [15]. In this direction we know the statement is at least true under the stronger hypothesis that $\mathbb{R}$ admits a majority or an NU operation.

Theorem 3.5.10. *[6, Theorem 1] Let $\mathbb{R}$ be a finite relational structure in a finite language $\mathcal{L}$. If $\mathbb{R}$ admits a majority polymorphism then $\mathbb{R}$ has bounded path duality.*

Theorem 3.5.11. *[3, Theorem 7] Let $\mathbb{R}$ be a finite relational structure in a finite language $\mathcal{L}$. If $\mathbb{R}$ admits an NU polymorphism then $\mathbb{R}$ has bounded path duality.*

In the remainder of this paper we will examine the notions of bounded path duality, NU polymorphisms, and Freese-McKenzie operations for series-parallel posets.

Chapter 4

Posets

4.1 Partial Orders

Definition 4.1.1. A *partially ordered set (poset)* $\mathbb{P} = \langle P, \leq^{\mathbb{P}} \rangle$ is a relational structure of a language containing a single binary relation that satisfies the following three conditions for all a, b, and c in P:

- $a \leq^{\mathbb{P}} a$ (**reflexivity**);
- if $a \leq^{\mathbb{P}} b$ and $b \leq^{\mathbb{P}} a$, then $a = b$ (**antisymmetry**); and,
- if $a \leq^{\mathbb{P}} b$ and $b \leq^{\mathbb{P}} c$, then $a \leq^{\mathbb{P}} c$ (**transitivity**).

A *subposet* of $\mathbb{P}$ is a poset that is a substructure of $\mathbb{P}$.

A pair of elements a and b in $\mathbb{P}$ is said to be *comparable* if either $a \leq b$ or $a \geq b$.

A *partial order* is a binary relation that is reflexive, transitive and antisymmetric.

A *quasi order* is a binary relation that is reflexive and transitive.

We say a is above b to mean that $a \geq b$. Likewise we will say a is below b to mean $a \leq b$. In normal relation notation we would write $(a, b) \in \leq^{\mathbb{P}}$ to mean $a \leq^{\mathbb{P}} b$.

The integers equipped with their canonical ordering is a poset. A rooted tree is a poset if we declare x to be below y if and only if the unique path from x to the root passes through y. A powerset of any set is a poset with the containment relation as its order relation.

Definition 4.1.2. Let $\mathbb{P}$ be a poset. An element $a \in P$ is said to be *maximal* in $\mathbb{P}$ if for all $b \in P$, $b \geq a$ implies $a = b$. Likewise, an element is said to be *minimal* if for all $b \in P$, $b \leq a$ implies $a = b$.

An element a is said to be a *pinch point* of $\mathbb{P}$ if a is comparable to every element of $\mathbb{P}$.

A pair of elements $a < b \in P$ forms a *cover pair* if for all $c \in P$, $a \leq c \leq b$ implies $c = a$ or $c = b$. We write $a \prec b$ to signify this. The element a may be referred to as a *lower cover* of b and b an *upper cover* of a.

We will draw Hasse diagrams when we need to visualize certain posets. The elements of the poset will be represented by points, the order relation will be represented by lines connecting cover pairs such that the greater element is positioned higher than the lesser.

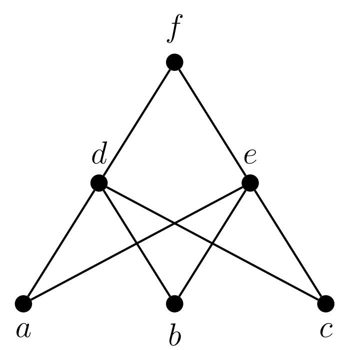

Figure 4.1: $\mathbb{P}$

Here we have the Hasse diagram for the poset $\mathbb{P}$ with universe $P = \{a, b, c, d, e, f\}$ and order relation

$$\leq^{\mathbb{P}} = \{(a,d),(a,e),(a,f),(b,d),(b,e),(b,f),(c,d),(c,e),(c,f),\\ (d,f),(e,f),(a,a),(b,b),(c,c),(d,d),(e,e),(f,f)\}.$$

Definition 4.1.3. Let $\mathbb{P}$ be a poset. An element $c \in P$ is said to be a *common upper bound* of $a, b \in P$ if $c \geq a$ and $c \geq b$. We call c a *common lower bound* of a and b if $c \leq a$ and $c \leq b$.

Definition 4.1.4. Let $\mathbb{P}$ be a poset. For a pair of elements $a, b \in P$ we define $\sup(a, b)$ to be the element in P (if it exists) such that $\sup(a, b) \geq a$, $\sup(a, b) \geq b$ and for all $c \in P$ such that $c \geq a$ and $c \geq b$ we have $c \geq \sup(a, b)$. We will say $\sup(a, b)$ is the supremum of a and b.

We define $\inf(a, b)$ to be the element in P (if it exists) such that $\inf(a, b) \leq a$, $\inf(a, b) \leq b$ and for all $c \in P$ such that $c \leq a$ and $c \leq b$ we have $c \leq \inf(a, b)$. We will say $\inf(a, b)$ is the infimum of a and b.

Let S be a subset of P. We define $u \in P$ to be an *upper bound* of S if $x \leq u$ for all $x \in S$. Similarly we define $l \in P$ to be a *lower bound* of S if $x \geq l$ for all $x \in S$.

We define $\sup(S)$ to be the unique upper bound of S (if it exists) such that $\sup(S) \leq u$ for all upper bounds u of S.

We define $\inf(S)$ to be the unique lower bound of S (if it exists) such that $\inf(S) \geq l$ for all lower bounds l of S.

In figure 4.1 we see that the supremum does not exist for any pair from $\{a, b, c\}$. However it does exist for the pair d, e, and any subset of P containing the two, as the element f.

Definition 4.1.5. Let $\mathbb{P}$ be a poset and $Q \subseteq P$. Then the poset $\mathbb{Q} = \langle Q, Q^2 \cap \leq^{\mathbb{P}} \rangle$ is the subposet of $\mathbb{P}$ induced by the set Q.

Since every subset of $\mathbb{P}$ induces a subposet of $\mathbb{P}$, for convenience we may refer to one as the other.

Lemma 4.1.6. *A subposet of a subposet of a poset is a subposet of that poset.*

Proof. This follow directly from the definition of subposets and Lemma 2.2.2. □

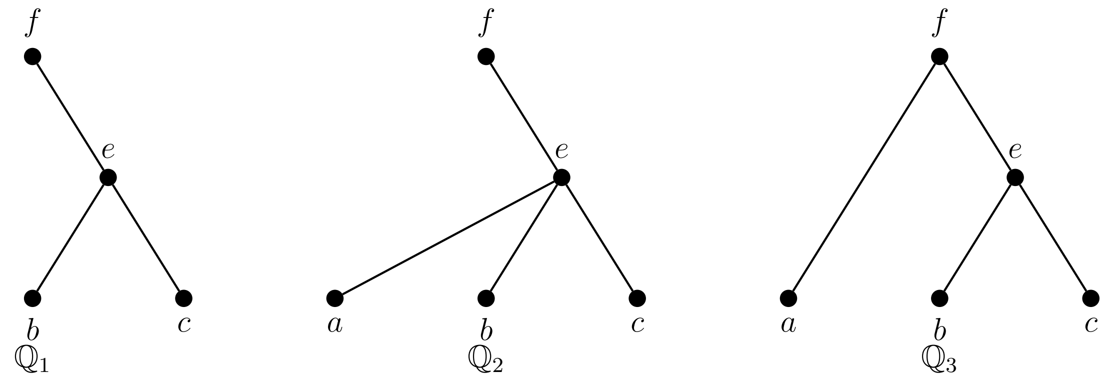

Figure 4.2

Here $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are both subposets of $\mathbb{P}$ from figure 4.1 where $\mathbb{Q}_3$ is not. $\mathbb{Q}_1$ is also a subposet of both $\mathbb{Q}_2$ and $\mathbb{Q}_3$.

Definition 4.1.7. Let $\mathbb{P}$ be a finite poset. We will denote PoRet($\mathbb{P}$) to be the to be the class of all finite posets $\mathbb{P}'$ having $\mathbb{P}$ as an induced subposet such that there exists a retraction from $\mathbb{P}'$ onto $\mathbb{P}$.

The *poset retraction problem* of $\mathbb{P}$ refers to the decision problem that take as input a finite poset containing $\mathbb{P}$ as an induced subposet and decides whether it belongs to PoRet($\mathbb{P}$).

The poset retraction problem is a more refined version of the retraction problem as we only take inputs that are already posets. Therefore Ret($\mathbb{P}$) belonging to NL implies PoRet($\mathbb{P}$) belongs to NL. PoRet($\mathbb{P}$) and Ret($\mathbb{P}$) are equivalent under many-one polynomial-time reductions, but not under log-space reductions. For example, if $\mathbb{P}'$ is the two-element poset $\{0, 1\}$ with $0 < 1$, then PoRet($\mathbb{P}'$) is in L (solvable in logarithmic space) because the answer is always yes. On the other hand, Ret($\mathbb{P}'$) is log-space equivalent to a well-known problem called ST-CON, which is known to be NL-complete under log-space reductions [21].

Definition 4.1.8. Let R be a binary relation on some set S. A finite sequence of elements $[a_1, a_2, \ldots, a_n]$ in S is called a *path* under R if for each consecutive pair a_i and a_{i+1} either (a_i, a_{i+1}) or (a_{i+1}, a_i) is in R. n will be the *length* of this path.

A *directed path* under R is a finite sequence of elements $[a_1, a_2, \ldots, a_n]$ in S such that for each consecutive pair a_i and a_{i+1} we have $(a_i, a_{i+1}) \in R$.

Let $b, c \in S$. We say that b and c are *connected* under R if there exists a path

$a_1, a_2, \ldots, a_n$ in R such that $a_1 = b$ and $a_n = c$.

A subset $S' \subseteq S$ is said to be *connected* under R if every pair of elements in S' is connected under R.

A *connected component* of S under R is a maximal connected subset of S under R.

When discussing connectivity in a poset we will drop the use of "under R" as it is clear that the binary relation we are referring to is the order relation.

Let us return to the example poset $\mathbb{P}$ from figure 4.1. Both $[a, d, f]$ and $[a, d, b, e, f]$ are paths from a to f where the former is a directed path and the latter is not. It should be clear that every pair of elements in $\mathbb{P}$ is connected and the only connected component of $\mathbb{P}$ is all of P.

Definition 4.1.9. An *antichain* is a poset such that any two distinct elements in the poset are incomparable. We may also refer to subsets of a poset whose induced subposet is an antichain as antichains.

We will denote the one element antichain **1** , two element antichain **2** and the n element antichain **n**.

The subsets $\{a, b, c\}$ and $\{d, e\}$ of the poset $\mathbb{P}$ from figure 4.1 are both antichains.

Definition 4.1.10. A $2n$-*crown* is any poset isomorphic to the poset with universe $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ such that $b_i < a_i$ and $b_{i+1} < a_i$ for each $1 \leq i \leq n$ where the indices are considered modulo n and all other pairs of elements are incomparable.

A $2n$-*fence* is the poset isomorphic to the $2n$-*crown* with exactly one of its comparabilities removed.

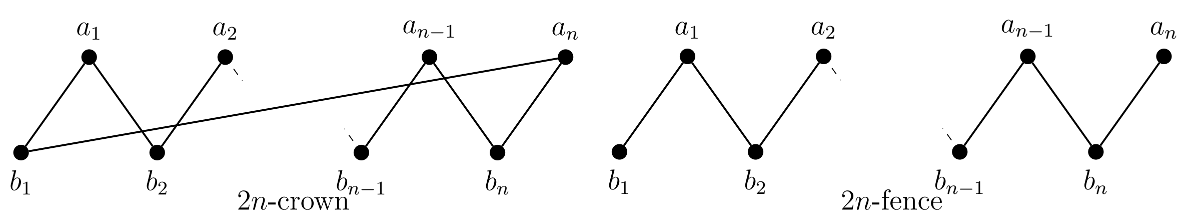

Figure 4.3

We will be mainly using the 4-crown and the 4-fence for the purpose of this paper. From now on we will call the 4-fence as the N-poset (since its Hasse diagram looks like the letter "N"). We will say a poset is N-*free* when it does not have the N-poset as a subposet.

4.2 Series-parallel Posets

Definition 4.2.1. Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be posets with universes P_1 and P_2. We will define their *linear sum* to be the poset $\mathbb{P}_1 + \mathbb{P}_2 := \langle P_1 \cup P_2, \leq^{\mathbb{P}_1} \cup \leq^{\mathbb{P}_2} \cup P_1 \times P_2 \rangle$. Likewise their *disjoint union* will be the poset $\mathbb{P}_1 \uplus \mathbb{P}_2 := \langle P_1 \cup P_2, \leq^{\mathbb{P}_1} \cup \leq^{\mathbb{P}_2} \rangle$

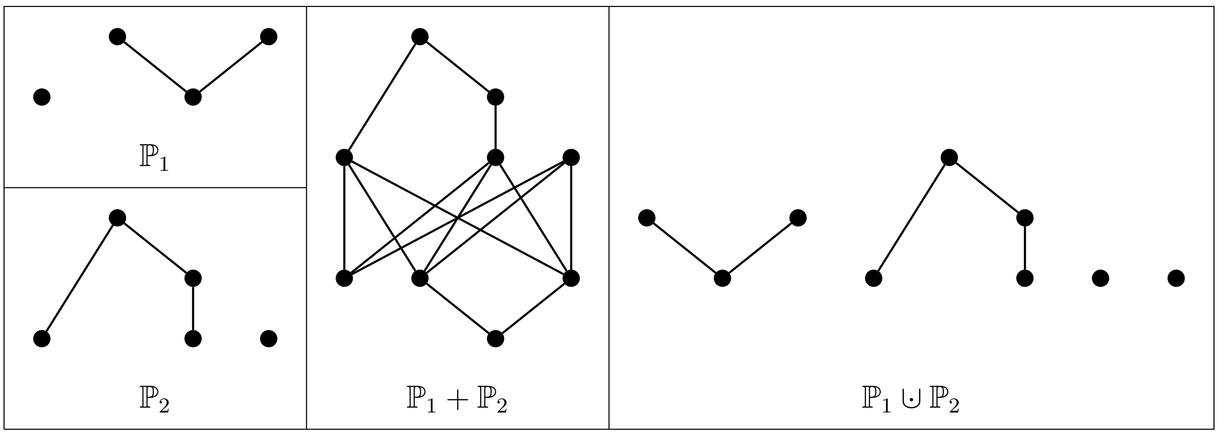

Figure 4.4

Each antichain $\mathbf{n}$ is equal to the disjoint union of n one element antichains $\biguplus_{i=1}^{n} \mathbf{1}$. A 4-crown can be thought of as $\mathbf{2} + \mathbf{2}$.

Definition 4.2.2. A poset will be called *series-parallel* if it can be constructed from $\mathbf{1}$ (the one element antichain) using only linear sum and disjoint union finitely many times.

Series-parallel posets are the poset version of series-parallel graphs or digraphs. Early works viewed them as analogues for electrical networks [9], whereas today they are used in many different computational problems such as job shop scheduling [18], machine learning of time series data [19], and transmission sequencing of multimedia data [1].

$\mathbf{2} + \mathbf{2}$ (4-crown) $\mathbf{2} + \mathbf{2} + \mathbf{1}$

Figure 4.5

Theorem 4.2.3. *[23] [25, Theorem 1] A finite poset is series-parallel if and only if it is N-free.*

Being N-free is a powerful characteristic that we will be utilizing whenever we prove something specific about series-parallel posets. Here it is in action as we prove several basic facts regarding series-parallel posets.

Lemma 4.2.4. *All subposets of a series-parallel poset are series-parallel.*

Proof. Let $\mathbb{P}$ be a series-parallel poset and $\mathbb{Q}$ a subposet of $\mathbb{P}$. Suppose $\mathbb{Q}$ admits the N-poset as an induced subposet. By lemma 4.1.6 this N-poset is also a subposet of $\mathbb{P}$, a contradiction to Theorem 4.2.3. Thus $\mathbb{Q}$ must be N-free. So by Theorem 4.2.3 again $\mathbb{Q}$ is series-parallel. □

Lemma 4.2.5. *Let $\mathbb{P}$ be a series-parallel poset. Every pair of connected elements in $\mathbb{P}$ has a common upper or lower bound.*

Proof. Let a, b be a pair of connected elements in $\mathbb{P}$. By definition we know there exists a path in $\leq^{\mathbb{P}}$ from a to b. Pick a path $x_1, \ldots, x_n$ of this kind that is minimal in length. If we have consecutive elements x_i, x_{i+1}, x_{i+2} on this path such that $x \leq^{\mathbb{P}} x_{i+1} \leq^{\mathbb{P}} x_{i+2}$ then a shorter path exists by removing x_{i+1}. Thus this minimal path must alternate between $\leq^{\mathbb{P}}$ and $\geq^{\mathbb{P}}$. Furthermore every non-consecutive pair of elements on the path must be incomparable or a shorter path will exist. If the length of this path is greater than or equal to 4 then we would have an induced N-subposet in $\mathbb{P}$, a contradiction since $\mathbb{P}$ is series-parallel. Thus the path is at most length 3. When the length is 1 we have $a = b$. When the length is 2 we have a comparable to b. When the length is 3 we get some c such that $a \geq c$ and $c \leq b$ or $a \leq c$ and $c \geq b$. In all three cases the lemma holds. □

Lemma 4.2.6. *Let $\mathbb{P}$ be a series-parallel poset. Suppose that for some incomparable $a, b \in P$, $\sup(a, b)$ exists. Then for all $c \geq a$, either $c \geq b$ or $c \leq \sup(a, b)$. Hence in either case, c is comparable to $\sup(a, b)$.*

Likewise, suppose that for some $a, b \in P$ $\inf(a, b)$ exists. Then for all $c \leq a$, either $c \leq b$ or $c \geq \inf(a, b)$. Hence in either case, c is comparable to $\inf(a, b)$.

Proof. Let a, b and c be stated as in the theorem. Then we must have one of the pairs $\{c, \sup(a, b)\}$ or $\{c, b\}$ be comparable or there would be an N-subposet for $\mathbb{P}$. Either one of outcomes satisfies the theorem. The second part of the theorem is proven similarly. □

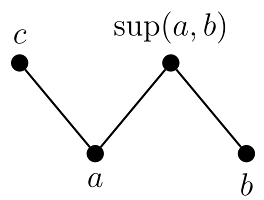

Figure 4.6

4.3 4-crown Condition

Definition 4.3.1. Let $\mathbb{P}$ be a series-parallel poset. We will say that $\mathbb{P}$ satisfies the *4-crown condition* if for every 4-crown $\{a_1, a_2, b_1, b_2\}$ in $\mathbb{P}$, where the a_i's are the lesser elements and b_i's the greater, at least one of the following conditions is true:

- there exists a midpoint $e \in \mathbb{P}$ such that $a_i < e < b_j$ for all $1 \leq i, j \leq 2$;
- $\inf(a_1, a_2)$ exists;
- $\sup(b_1, b_2)$ exists.

For example $\mathbf{2} + \mathbf{2} + \mathbf{1}$ satisfies the 4-crown condition where $\mathbf{2} + \mathbf{2}$ (the 4-crown) does not.

This condition was first introduced in a paper by Dalmau, Krokhin and Larose where they proved that it characterized all those series-parallel posets whose retraction problem is solvable in polynomial time (assuming P $\neq$ NP).

Theorem 4.3.2. *[7, Theorem 2] Let $\mathbb{P}$ be a connected series parallel poset. Then the following conditions are equivalent:*

1. *$\mathbb{P}$ satisfies the 4-crown condition;*
2. *$\mathbb{P}$ does not retract onto any of* $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{2}$, $\mathbf{2}+\mathbf{2}+\mathbf{2}+\mathbf{1}$, $\mathbf{2}+\mathbf{2}+\mathbf{2}+\mathbf{1}$, $\mathbf{2}+\mathbf{2}$, *and* $\mathbf{2}+\mathbf{2}+\mathbf{2}$*;*
3. *$\mathbb{P}$ admits a Taylor polymorphism;*
4. *$\mathbb{P}$ admits a TSI (totally symmetric idempotent) polymorphism of every arity $k \geq 2$;*

Furthermore, if $\mathbb{Q}$ is a series-parallel poset whose every connected component satisfies any of the conditions mentioned above then $\mathrm{PoRet}(\mathbb{Q})$ *is solvable in polynomial time. Otherwise it is* NP*-complete.*

As we have mentioned before in Chapter 3 we can expand on this result to show that if every connected component of $\mathbb{Q}$ satisfies any of 1-4 mentioned in the theorem, then $\mathrm{PoRet}(\mathbb{Q})$ is actually in NL by proving $\exp(\mathbb{Q})$ has bounded path duality. We will do so in the next chapter. For now we finish listing important properties of series-parallel posets.

Lemma 4.3.3. *Let $\mathbb{P}$ be a series-parallel poset satisfying the 4-crown condition. If there exists a retraction r of $\mathbb{P}$ then the subposet $\mathbb{Q}$ of $\mathbb{P}$ induced by the image of r will also be a series-parallel poset satisfying the 4-crown condition.*

Proof. By Lemma 4.2.4 we know that $\mathbb{Q}$ is also series-parallel. Let $\{a_1, a_2, b_1, b_2\}$ be a 4-crown in $\mathbb{Q}$ where the a_i's are the lesser elements and b_i's the greater. Since $\mathbb{Q}$ is a subposet of $\mathbb{P}$ this is also a 4-crown in $\mathbb{P}$. Because $\mathbb{P}$ satisfies the 4-crown condition we have three cases to consider.

The first case is when there exists a midpoint $e \in \mathbb{P}$ such that $a_i <^{\mathbb{P}} e <^{\mathbb{P}} b_j$ for all $1 \leq i, j \leq 2$. Since $\mathbb{Q}$ is the image of the retraction r the a_i's and b_j's are fixed by r. So we have $r(a_i) <^{\mathbb{Q}} r(e) <^{\mathbb{Q}} r(b_j) \implies a_i \leq^{\mathbb{Q}} r(e) \leq^{\mathbb{Q}} b_j$ for all $1 \leq i, j \leq 2$ (since retractions are homomorphisms which are order preserving). Since the a_i's and b_j's are both incomparable pairs we actually get strict inequalities in the implied inequality. Thus $r(e)$ is a midpoint of this 4-crown in $\mathbb{Q}$.

The second case is when $l = \inf(a_1, a_2)$ exists in $\mathbb{P}$. Just as above we have $r(l) <^{\mathbb{Q}} a_1, a_2$. Suppose we have some $c \in \mathbb{Q}$ such that $c <^{\mathbb{Q}} a_1, a_2$ also. Then as elements of $\mathbb{P}$ we have $c \leq^{\mathbb{P}} \inf(a_1, a_2) = l$. So $c = r(c) \leq^{\mathbb{Q}} r(l)$. Thus $r(l)$ is the infimum of a_1 and a_2 in $\mathbb{Q}$.

The third case is when $\sup(b_1, b_2)$ exists in $\mathbb{P}$. Using similar arguments as in the second case we can show that there also exists a supremum of b_1 and b_2 in $\mathbb{Q}$.

With all three cases considered we see that the 4-crown condition is satisfied for this particular 4-crown. As it was picked arbitrarily this shows that $\mathbb{Q}$ satisfies the 4-crown condition in general. □

Just as general series-parallel posets can be constructed using linear sum and disjoint union, those that satisfy the 4-crown condition have their own recipe.

Definition 4.3.4. Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be finite posets with universes P_1 and P_2. We call $\mathbb{P}_1 + \mathbb{P}_2$ the *restricted sum* and denote it as $\mathbb{P}_1 +^R \mathbb{P}_2$ when one of the following conditions is met:

- For every pair of maximal elements $a, b \in \mathbb{P}_1$, $\inf(a, b)$ exists;
- For every pair of minimal elements $a, b \in \mathbb{P}_2$, $\sup(a, b)$ exists.

Definition 4.3.5. Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be posets with universes P_1 and P_2. We call $\mathbb{P}_1 + \mathbb{P}_2$ the *connected sum* and denote it as $\mathbb{P}_1 +^C \mathbb{P}_2$ when both $\mathbb{P}_1$ and $\mathbb{P}_2$ are connected.

The restricted and connected sum are simply the linear sum with added conditions on the input that they take in.

Definition 4.3.6. Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be finite posets with universes P_1 and P_2. Assume that $\mathbb{P}_1$ has a unique maximal element and $\mathbb{P}_2$ a unique minimal element. We define $\mathbb{P}_1 ⧖ \mathbb{P}_2$ to be $\mathbb{P}_1 + \mathbb{P}_2'$, where $\mathbb{P}_2'$ is the subposet of $\mathbb{P}_2$ induced by all but its minimal element.

Although this is the proper definition of the $⧖$ operation, it helps notation wise to think of the unique minimum of $\mathbb{P}_2$ and the unique maximum of $\mathbb{P}_1$ as the same element. This way we have that $\mathbb{P}_1$ and $\mathbb{P}_2$ are induced subposets of $\mathbb{P}_1 ⧖ \mathbb{P}_2$ that share a common element.

Definition 4.3.7. Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be posets with universes P_1 and P_2. Assume that both $\mathbb{P}_1$ and $\mathbb{P}_2$ have a unique maximal element. We define $\mathbb{P}_1 \triangle \mathbb{P}_2$ to be $(\mathbb{P}_1' \cup \mathbb{P}_2') + \mathbf{1}$, where $\mathbb{P}_1'$ and $\mathbb{P}_2'$ is the subposet of $\mathbb{P}_1$ and $\mathbb{P}_2$ induced by all but its maximal element respectively.

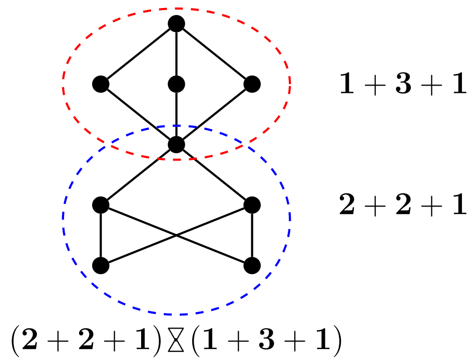

Figure 4.7

Definition 4.3.8. Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be posets with universes P_1 and P_2. Assume that both $\mathbb{P}_1$ and $\mathbb{P}_2$ have a unique minimal element. We define $\mathbb{P}_1 \triangledown \mathbb{P}_2$ to be $\mathbf{1} + (\mathbb{P}_1' \mathbin{\dot\cup} \mathbb{P}_2')$, where $\mathbb{P}_1'$ and $\mathbb{P}_2'$ is the subposet of $\mathbb{P}_1$ and $\mathbb{P}_2$ induced by all but its minimal element respectively.

As before it helps to think of $\mathbb{P}_1$ and $\mathbb{P}_2$ as induced subposets of $\mathbb{P}_1 \triangle \mathbb{P}_2$ and $\mathbb{P}_1 \triangledown \mathbb{P}_2$ that share a common maximal/minimal element.

Definition 4.3.9. Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be posets with universes P_1 and P_2. Assume that both $\mathbb{P}_1$ and $\mathbb{P}_2$ have a unique minimal element distinct from a unique maximal element. We define $\mathbb{P}_1 \lozenge \mathbb{P}_2$ to be $\mathbf{1} + (\mathbb{P}_1' \mathbin{\dot\cup} \mathbb{P}_2') + \mathbf{1}$, where $\mathbb{P}_1'$ and $\mathbb{P}_2'$ is the subposet of $\mathbb{P}_1$ and $\mathbb{P}_2$ induced by all but its minimal and maximal elements respectively.

We'll think of $\mathbb{P}_1$ and $\mathbb{P}_2$ as induced subposets of $\mathbb{P}_1 \lozenge \mathbb{P}_2$ that shares the maximal and minimal elements.

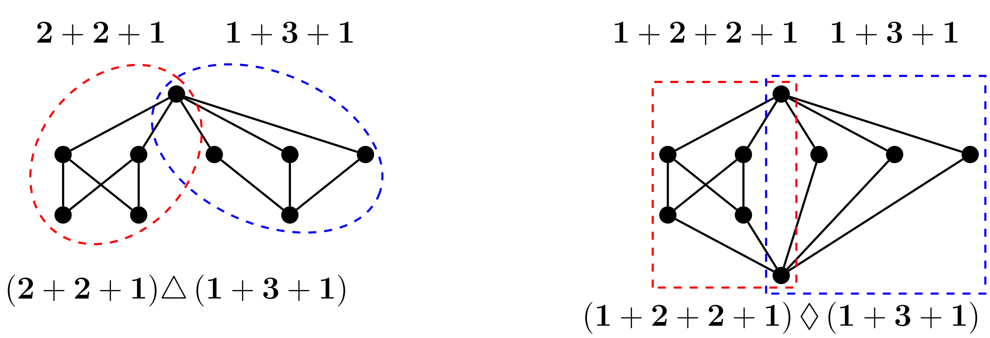

Figure 4.8

Definition 4.3.10. In the case of $\triangle$, $\triangledown$, $\lozenge$ and $\between$ operations the two input posets $\mathbb{P}_1$ and $\mathbb{P}_2$ when viewed as subposets of the output poset $\mathbb{P}$ will have some common elements (e.g. the maximal element in $\mathbb{P} = \mathbb{P}_1 \triangle \mathbb{P}_2$). We will refer to them as the *shared elements.*

We will show in the next two lemmas that all series-parallel posets satisfying the 4-crown condition can be constructed using some of the poset operations we have just defined. This result is inspired by discoveries of Larose and Willard that we will mention in chapter 6.

Lemma 4.3.11. *Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be series-parallel posets. Then $\mathbb{P}_1 \uplus \mathbb{P}_2$, $\mathbb{P}_1 +^R \mathbb{P}_2$, $\mathbb{P}_1 +^C \mathbb{P}_2$, $\mathbb{P}_1 \between \mathbb{P}_2$, $\mathbb{P}_1 \triangle \mathbb{P}_2$, $\mathbb{P}_1 \triangledown \mathbb{P}_2$ and $\mathbb{P}_1 \lozenge \mathbb{P}_2$ (when the requirements of the operations are met) are still series-parallel posets. Moreover, if both $\mathbb{P}_1$ and $\mathbb{P}_2$ satisfy the 4-crown condition then so do their products under $\uplus$, $+^R$, $\between$, $\triangle$, $\triangledown$ and $\lozenge$.*

Proof. The fact that $\mathbb{P}_1 \uplus \mathbb{P}_2$, $\mathbb{P}_1 +^R \mathbb{P}_2$ and $\mathbb{P}_1 +^C \mathbb{P}_2$ are series-parallel posets should be clear from their definition.

To show that $\mathbb{P}_1 \between \mathbb{P}_2$ is series-parallel we will prove it is N-free. We know that $\mathbb{P}_1$ and $\mathbb{P}_2$ are both N-free. So if we assume for the sake of contradiction there exists an induced N-subposet in $\mathbb{P}_1 \between \mathbb{P}_2$ then the elements of this subposet must come from both $\mathbb{P}_1$ and $\mathbb{P}_2$. Since the upper two elements of the N-subposet are incomparable they must be elements of $\mathbb{P}_2$. Likewise the lower two elements must come from $\mathbb{P}_1$. By the definition of the $\between$ operation these four elements would form a 4-crown, a clear contradiction (see figure 4.9).

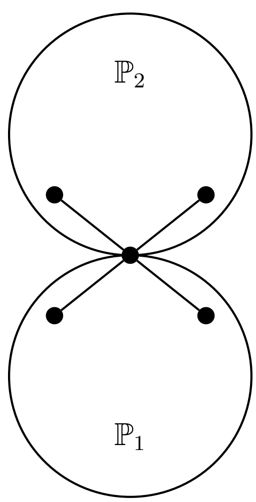

Figure 4.9: $\mathbb{P}_1 \between \mathbb{P}_2$

Suppose for a contradiction that $\mathbb{P}_1 \triangle \mathbb{P}_2$ contains an induced N-subposet. This immediately creates a contradiction unless the N-subposet isn't contained entirely inside either $\mathbb{P}_1'$ or $\mathbb{P}_2'$ (as described in the definition of $\triangle$). This would mean that one of the maximal elements in the subposet must be the unique element in $\mathbf{1}$. This is a contradiction since both maximal elements have to be incomparable with each other in $\mathbb{P}_1 \triangle \mathbb{P}_2$.

The case for $\triangledown$ is similar and therefore omitted.

Suppose for a contradiction that $\mathbb{P}_1 \lozenge \mathbb{P}_2$ contains an induced N-subposet. This again would immediately create a contradiction unless the N-subposet isn't contained entirely inside either $\mathbb{P}_1'$ or $\mathbb{P}_2'$ (as described in the definition of $\lozenge$). This would require at least one of the elements in the subposet to be one of the $\mathbf{1}$'s. Then this element would be comparable to all other three, a contradiction to the definition of an N-subposet.

Now assume both $\mathbb{P}_1$ and $\mathbb{P}_2$ satisfy the 4-crown condition. Let $\{a_1, a_2, b_1, b_2\}$ be a 4-crown in $\mathbb{P}_1 +^R \mathbb{P}_2$ where the a_i's are the lesser elements and b_i's the greater. As argued above if this 4-crown isn't fully contained in either $\mathbb{P}_1$ or $\mathbb{P}_2$ then we would have $a_1, a_2 \in \mathbb{P}_1$ and $b_1, b_2 \in \mathbb{P}_2$ (if it is then we are done). By the definition of $+^R$ let's assume without loss of generality that the inf of every pair of maximal elements in $\mathbb{P}_1$ exists. Pick a_1' and a_2' to be maximal elements of $\mathbb{P}_1$ greater than or equal to a_1 and a_2 respectively. Denote $m = \inf(a_1', a_2')$ (see figure 4.10). If $a_1 = a_1'$ and $a_2 = a_2'$ then this 4-crown satisfies the

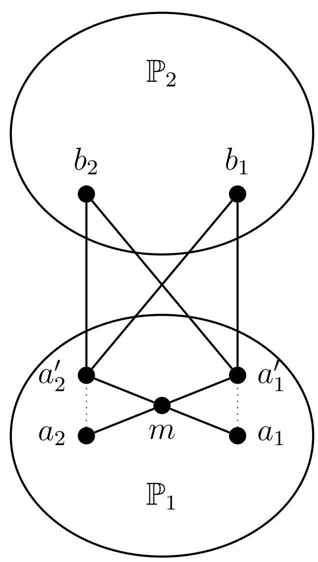

Figure 4.10: $\mathbb{P}_1 +^R \mathbb{P}_2$

4-crown condition. Suppose again without loss of generality that $a_1 \neq a_1'$. By Lemma 4.2.6 we have either $a_1 \leq a_2'$ or $a_1 \geq m$. In the first case a_2' would be the midpoint required by the 4-crown condition. In the second case we shift our focus onto a_2. If $a_2 = a_2'$ then

$m = \inf(a_1, a_2)$. If $a_2 \neq a_2'$ then by Lemma 4.2.6 again we have $a_2 \geq m$ or $a_2 \leq a_1$. In the first case we get $m = \inf(a_1, a_2)$ and in the second a_1 would be the midpoint required by the 4-crown condition. So we see that ultimately $\{a_1, a_2, b_1, b_2\}$ satisfies the condition one way or another.

Let $\{a_1, a_2, b_1, b_2\}$ be a 4-crown in $\mathbb{P}_1 ⧖ \mathbb{P}_2$ that is not fully contained in either $\mathbb{P}_1$ or $\mathbb{P}_2$ (setup as in the previous case). In this case the unique maximum element of $\mathbb{P}_1$ does not equal any of a_1, a_2, b_1 or b_2. It would also serve to satisfy the existence of a midpoint for the 4-crown.

The only way to have a 4-crown in $\mathbb{P}_1 \dot{\cup} \mathbb{P}_2$, $\mathbb{P}_1 \triangle \mathbb{P}_2$, $\mathbb{P}_1 \bigtriangledown \mathbb{P}_2$ or $\mathbb{P}_1 \Diamond \mathbb{P}_2$ is to have all four elements belonging to the same $\mathbb{P}_i$. Thus our assumptions provides the result. □

Lemma 4.3.12. *A poset $\mathbb{P}$ is series-parallel and satisfies the 4-crown condition if and only if it can be constructed from $\mathbf{1}$ using $+^R$ (restricted linear sum), $⧖$, $\triangle$, $\bigtriangledown$, $\Diamond$ and $\dot{\cup}$ (disjoint union) finitely many times.*

Proof. ($\Longleftarrow$) The $\mathbf{1}$ poset satisfies the 4-crown condition vacuously. It should be clear from the definition of disjoint union that it preserves the 4-crown condition. From Lemma 4.3.11 the result follows.

($\Longrightarrow$) Suppose now we have a series-parallel poset $\mathbb{P}$ satisfying the 4-crown condition. We will prove the desired result inductively on its size. When $|P| = 1$ we have $\mathbb{P} = \mathbf{1}$. Assume that $|P| > 1$ and all series-parallel posets satisfying the 4-crown condition with size strictly less than $\mathbb{P}$ are constructed from $\mathbf{1}$ using $+^R$, $⧖$, $\triangle$, $\bigtriangledown$, $\Diamond$ and $\dot{\cup}$ only finitely many times.

If $\mathbb{P}$ is not connected then let $P_1, \ldots, P_n$ ($n > 1$) be the connected components of $\mathbb{P}$. Let $\mathbb{P}_i$ be the induced subposet of P_i. Fix an arbitrary i and let $\{a_1, a_2, b_1, b_2\}$ be a 4-crown in $\mathbb{P}_i$ where the a_i's are the lesser elements and b_i's the greater. Since $\mathbb{P}$ satisfies the 4-crown condition there must be an element c in $\mathbb{P}$ witnessing that for $\{a_1, a_2, b_1, b_2\}$. By definition of the 4-crown condition c is connected to all of $\{a_1, a_2, b_1, b_2\}$. Thus $c \in P_i$ and we get that $\mathbb{P}_i$ satisfies the 4-crown condition also. Being a subposet of $\mathbb{P}$ also makes $\mathbb{P}_i$ N-free. Thus by our assumption it is constructed from $\mathbf{1}$ using $+^R$, $⧖$, $\triangle$, $\bigtriangledown$, $\Diamond$ and $\dot{\cup}$ finitely many times. Since $\mathbb{P} = \mathbb{P}_1 \dot{\cup} \ldots \dot{\cup} \mathbb{P}_n$ we have our desired result.

Now assume that $\mathbb{P}$ is connected. From the definition of series-parallel there exists series-parallel posets $\mathbb{P}_1$ and $\mathbb{P}_2$ such that $\mathbb{P} = \mathbb{P}_1 + \mathbb{P}_2$. First, assume that both of these subposets satisfies the 4-crown condition. Then by our assumption they are constructed from $\mathbf{1}$ using the above mentioned operations finitely many times. All that remains is to show that $\mathbb{P} = \mathbb{P}_1 +^R \mathbb{P}_2$ which amounts to show that one of the following conditions is satisfied:

- For every pair of maximal elements $a, b \in \mathbb{P}_1$ $\inf(a, b)$ exists;
- For every pair of minimal elements $a, b \in \mathbb{P}_2$ $\sup(a, b)$ exists.

Suppose that for some pair of maximal elements $a, b \in \mathbb{P}_1$ $\inf(a, b)$ does not exist in $\mathbb{P}_1$, then the pair cannot have an inf in $\mathbb{P}$ also. Since there does not exists an element between any maximal element of $\mathbb{P}_1$ and any minimal element of $\mathbb{P}_2$, every pair of minimal element of $\mathbb{P}_2$ will have a supremum in $\mathbb{P}$ by the 4-crown condition. But this is equivalent to having a sup in $\mathbb{P}_2$. Thus $\mathbb{P} = \mathbb{P}_1 +^R \mathbb{P}_2$ and we are done in the case when both $\mathbb{P}_1$ and $\mathbb{P}_2$ satisfies the 4-crown condition.

Next, suppose to the contrary that there exists $\{a_1, a_2, b_1, b_2\}$ a 4-crown in one of the $\mathbb{P}_i$'s where the a_i's are the lesser elements and b_i's the greater such that the 4-condition is not satisfied. We will only consider when this 4-crown belongs to $\mathbb{P}_1$ as the proof is similar in both cases. This set of four elements is also a 4-crown in $\mathbb{P}$ which satisfies the condition so there must be a c in $\mathbb{P}$ witnessing that. We cannot have c be the midpoint or the infimum of a_1 and a_2 since then c would be an element of $\mathbb{P}_1$ (which contradicts our assumption that this is a 4-crown in $\mathbb{P}_1$ such that the condition is not satisfied). Thus $c = \sup(b_1, b_2)$ and $c \in P_2$. Now $\mathbb{P} = \mathbb{P}_1 + \mathbb{P}_2$ so every element in P_2 is greater than b_1, b_2 and $\sup(b_1, b_2) = c$ in $\mathbb{P}$. This shows that c is the unique minimal element of $\mathbb{P}_2$ (see figure 4.11). If there are any other 4-crowns in $\mathbb{P}_1$ that also does not satisfy the condition in $\mathbb{P}_1$

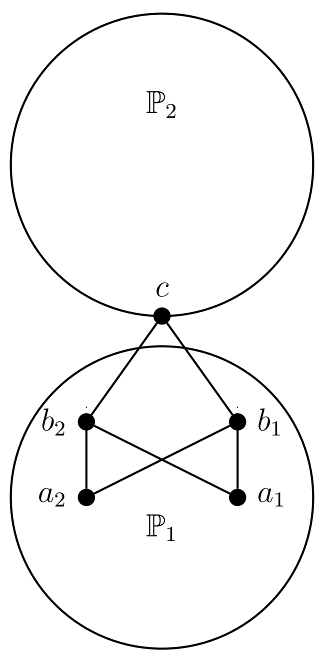

Figure 4.11

then c must also be supremum of its two greater elements in $\mathbb{P}$. Furthermore since the two greater elements of a 4-crown must be incomparable any 4-crown that belongs to $\mathbb{P}_1 + \mathbf{1}$ must also be in $\mathbb{P}_1$. Thus $\mathbb{P}_1 + \mathbf{1}$ is a series-parallel poset satisfying the 4-crown condition. As for $\mathbb{P}_2$ we know that c is its unique minimal element. This ensures that any element of $\mathbb{P}$ witnessing the 4-crown condition for any 4-crown in $\mathbb{P}_2$ must be in $\mathbb{P}_2$ already. Therefore $\mathbb{P}_2$ also satisfies the 4-crown condition.

If $|P_2| > 1$ then $|\mathbb{P}_1 + \mathbf{1}| < |P|$ and $\mathbb{P} = (\mathbb{P}_1 + \mathbf{1}) ⧖ \mathbb{P}_2$. By our inductive assumption the result follows. If $\mathbb{P}_2 = \mathbf{1}$ then we need to examine $\mathbb{P}_1$ further. If $\mathbb{P}_1$ is not connected then let $\mathbb{Q}_1, \ldots, \mathbb{Q}_n$ denote the subposets induced by the connected components of $\mathbb{P}$. In this case each $\mathbb{Q}_i + \mathbf{1}$ is a series-parallel poset satisfying the 4-crown condition with size less than $\mathbb{P}$. Then $\mathbb{P} = (\mathbb{Q}_1 + \mathbf{1}) \triangle (\mathbb{Q}_2 + \mathbf{1}) \triangle \ldots \triangle (\mathbb{Q}_n + \mathbf{1})$ and the result follows from our assumption.

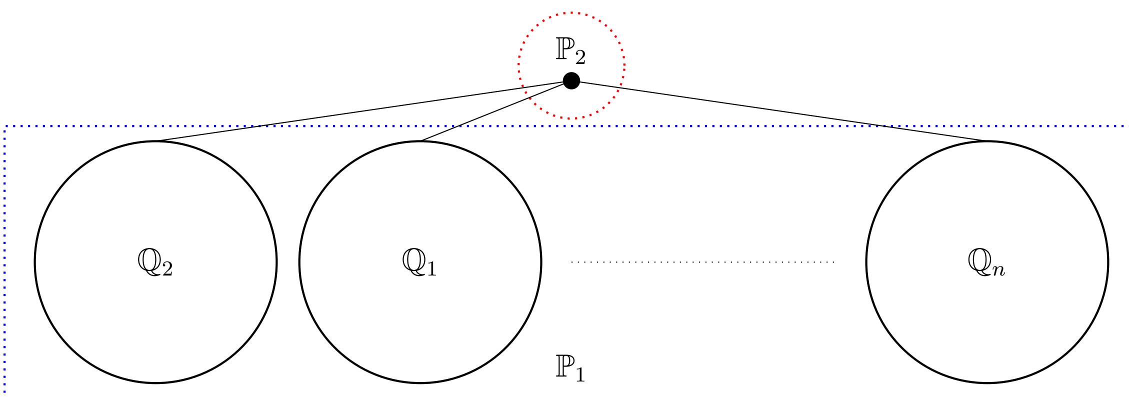

Figure 4.12

Now we arrive at the case where $\mathbb{P}_1$ is connected and $\mathbb{P}_2 = \mathbf{1}$. If $\mathbb{P}_1 = \mathbf{1}$ also then we are done. If not then since it is series-parallel $\mathbb{P}_1 = \mathbb{P}_3 + \mathbb{P}_4$ for some $\mathbb{P}_3$ and $\mathbb{P}_4$ also series parallel. Then $\mathbb{P} = \mathbb{P}_3 + (\mathbb{P}_4 + \mathbb{P}_2)$. If $|\mathbb{P}_3| > 1$ then since $|\mathbb{P}_4 + \mathbb{P}_2| > 1$ we may restart the argument from the beginning by replacing $\mathbb{P}_1$ with $\mathbb{P}_3$ and $\mathbb{P}_2$ with $\mathbb{P}_4 + \mathbb{P}_2$. This time the size of both posets are strictly greater than 1 and our proof will terminate on a prior case. If $|\mathbb{P}_3| = 1$ then we turn our attention to $\mathbb{P}_4$. If $|\mathbb{P}_4| = 1$ then $\mathbb{P} = \mathbf{1} +^R \mathbf{1} +^R \mathbf{1}$. If not then we have two cases depending on whether $\mathbb{P}_4$ is connected. If $\mathbb{P}_4$ is not connected then let $\mathbb{Q}_1, \ldots, \mathbb{Q}_n$ be the subposets induced by its connected components. These subposets are clearly series-parallel. Suppose there exists some 4-crown in some $\mathbb{Q}_i$ that does not satisfy the 4-crown condition. Since it is also a 4-crown of $\mathbb{P}$ there must be an element in $\mathbb{P}$ witnessing the condition. This element must either be the supremum of the two maximal elements or the infimum of the two minimal ones. The only elements in $\mathbb{P}$ that are comparable to anything in $\mathbb{Q}_i$ are the unique maximum and minimum of $\mathbb{P}$. Thus $\mathbf{1} + \mathbb{Q}_i + \mathbf{1}$

satisfies the 4-crown condition. Then $\mathbb{P} = (\mathbf{1} + \mathbb{Q}_1 + \mathbf{1}) \lozenge (\mathbf{1} + \mathbb{Q}_2 + \mathbf{1}) \lozenge \ldots \lozenge (\mathbf{1} + \mathbb{Q}_n + \mathbf{1})$.

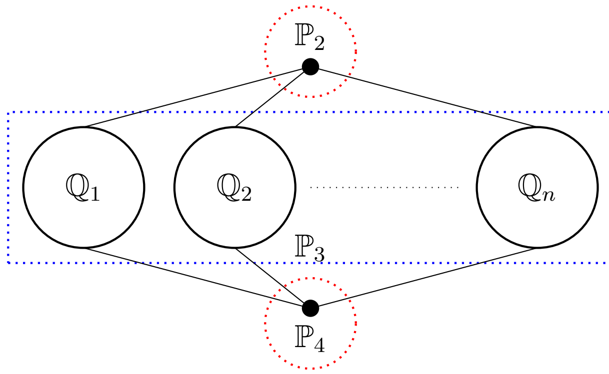

Figure 4.13

Now if $\mathbb{P}_4$ is connected, since it has size strictly bigger than one we can break it up further into $\mathbb{P}_4 = \mathbb{P}_5 + \mathbb{P}_6$. Then $\mathbb{P} = (\mathbf{1} + \mathbb{P}_5) + (\mathbb{P}_6 + \mathbf{1})$. With this we can return to the argument above for $\mathbb{P} = \mathbb{P}_1 + \mathbb{P}_2$ while guaranteeing both $\mathbb{P}_1$ and $\mathbb{P}_2$ have size strictly greater than one. □

Definition 4.3.13. Let $\mathbb{P}$ be a poset and let L and U be antichains in $\mathbb{P}$. Denote the subset $\mathbb{P}_{[L,U]}$ of $\mathbb{P}$ as

$$\mathbb{P}_{[L,U]} := \{p \in \mathbb{P} : l \leq p \leq u \text{ for all } l \in L \text{ and } u \in U\}.$$

Dalmau, Larose and Krokhin have shown in their paper the condition needed for any given poset to retract onto a series-parallel poset satisfying the 4-crown condition.

Theorem 4.3.14. *[7, Theorem 1] Let $\mathbb{P}$ be a connected series-parallel poset satisfying the 4-crown condition. Then a poset $\mathbb{Q}$ containing $\mathbb{P}$ as a subposet has a retraction onto $\mathbb{P}$ if and only if $\mathbb{Q}$ satisfies the following conditions:*

1. *For every pair (L, U) of antichains in $\mathbb{P}$, $\mathbb{Q}_{[L,U]}$ is empty whenever $\mathbb{P}_{[L,U]}$ is empty;*

2. *For every non-empty and non-connected subset of $\mathbb{P}$ of the form $\mathbb{P}_{[L',U']}$, and every pair of elements p_1, p_2 in different connected components of $\mathbb{P}_{[L',U']}$, there is no path*

in $\mathbb{Q}$ that connects p_1 and p_2 such that every element in this path belongs to some $\mathbb{Q}_{[L,U]}$ with $\mathbb{P}_{[L,U]} \subseteq \mathbb{P}_{[L',U']}$.

4.4 Pyramids

The restricted sum requires a special property from at least one of its inputs. We will study the consequences of having this property for a series-parallel poset.

Lemma 4.4.1. *Let $\mathbb{P}$ be a series parallel poset such that for every pair of minimal elements $a, b \in \mathbb{P}$ there exists $\sup(a, b)$. Let L be the set of minimal elements of $\mathbb{P}$. Then $\sup(S)$ exists for any $S \subseteq L$ and $\sup(L)$ is the minimal pinch point of $\mathbb{P}$.*

Proof. Fix $a \in S$. Let $U := \{\sup(a, b) : b \in S\}$. Suppose there exists x, y so that $\sup(a, x)$ and $\sup(a, y)$ are incomparable. This means that x is incomparable to $\sup(a, y)$. Then $\{x, \sup(a, x), a, \sup(a, y)\}$ would form an N-subposet in $\mathbb{P}$, a contradiction (see figure 4.14). Thus all pairs of elements from U are comparable. So there exist a maximum element

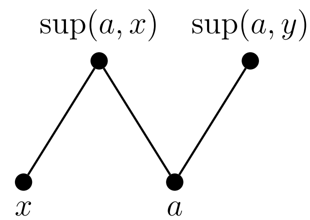

Figure 4.14

$\sup(a, b')$ for some $b' \in S$. By maximality $\sup(a, b')$ is above every element in S. Suppose there exists some $u \in P$ such that u is also above every element of S. By the definition of sup we must have $u \geq \sup(a, b')$. Therefore $\sup(a, b')$ is the $\sup(S)$ that we are looking for.

From the previous paragraph we know that $\sup(L) = \sup(a, b)$ for some $a, b \in L$. Let $c \in P$ and pick $c' \in L$ such that $c' \leq c$. Then by Lemma 4.2.6 c is comparable to $\sup(c', a)$. If $c \leq \sup(c', a)$ then $c \leq \sup(L)$. If $c \geq \sup(c', a)$ then $c \geq a$, so by Lemma 4.2.6 again c is comparable to $\sup(a, b) = \sup(L)$. This shows that $\sup(L)$ is a pinch point of $\mathbb{P}$. Any other pinch point in $\mathbb{P}$ would be above $\sup(L)$ due to its minimality (of being a supremum). □

Corollary 4.4.2. *Let $\mathbb{P}$ be a series-parallel poset such that for every pair of maximal* *elements $a, b \in \mathbb{P}$ there exists* $\inf(a, b)$. *Let U be the set of maximal elements of $\mathbb{P}$. Then* $\inf(S)$ *exists for any $S \subseteq L$ and* $\inf(L)$ *is the maximal pinch point of $\mathbb{P}$.*

Proof. The proof of this corollary is similar to the proof of the previous lemma and therefore is omitted. □

It will be helpful for us to find pinch points whenever possible as we can use them to partition a poset to subposets that connect together via $\bowtie$.

Lemma 4.4.3. *[7, Lemma 1] If $\mathbb{P}$ is a connected series-parallel poset satisfying the 4-crown condition then there exists a pinch point in $\mathbb{P}$.*

Proof. Since $\mathbb{P}$ is connected by Lemma 4.3.12 it is either the singleton poset **1** or constructed from $\mathbb{P}_1$ and $\mathbb{P}_2$ by using one of $+^R$, $\bowtie$, $\triangle$, $\bigtriangledown$ or $\Diamond$. When $\mathbb{P}$ is the singleton poset the only point in the poset will be the pinch point. When $\mathbb{P}$ is constructed using $+^R$ by Lemma 4.4.1 and Corollary 4.4.2 one of $\mathbb{P}_1$ or $\mathbb{P}_2$ will have a pinch point of their own, which also becomes a pinch point for $\mathbb{P}$. For the rest of the operations any shared element between $\mathbb{P}_1$ and $\mathbb{P}_2$ will be a pinch point in $\mathbb{P}$. □

Definition 4.4.4. Let $\mathbb{P}$ be a finite poset and L its set of minimal elements. We will call $\mathbb{P}$ a *pyramid* if $\sup(S)$ exists for every $S \subseteq L$ and $\sup(L)$ is the unique maximal element of $\mathbb{P}$.

Let $\mathbb{Q}$ be a finite poset and U its set of minimal elements. We will call $\mathbb{Q}$ a *reverse-pyramid* if $\inf(S)$ exists for every $S \subseteq U$ and $\sup(U)$ is the unique minimal element of $\mathbb{Q}$.

Suppose we are given a restricted sum of two series-parallel posets $\mathbb{P}_1 +^R \mathbb{P}_2$. If it is such that the first condition of the restricted sum is met (there exists a supremum for every pair of maximal elements of $\mathbb{P}_1$), then by corollary 4.4.2 the infimum of all maximal elements of $\mathbb{P}_1$ is a maximal pinch point. So the subposet $\mathbb{P}_3$ of all elements greater than or equal to this pinch point will form a reverse pyramid. Therefore we can write $\mathbb{P}_1 +^R \mathbb{P}_2 = \mathbb{P}_4 \bowtie (\mathbb{P}_3 +^R \mathbb{P}_2)$ for some $\mathbb{P}_4 \subseteq \mathbb{P}_1$.

Likewise if the second condition of the restricted sum is met then $\mathbb{P}_1 +^R \mathbb{P}_2 = (\mathbb{P}_1 +^R \mathbb{P}_3) \bowtie \mathbb{P}_2)$ for some $\mathbb{P}_3, \mathbb{P}_4 \subseteq \mathbb{P}_2$ where $\mathbb{P}_3$ is a pyramid.

Definition 4.4.5. Let $\mathbb{P}$ be a pyramid. Let L be the set of all minimal elements of $\mathbb{P}$. For every $a \in P$ define $L_a \subseteq L$ to be the set of minimal elements of $\mathbb{P}$ comparable to a. For each $S \subseteq P$ with $S \neq \emptyset$, we will define

$$l_S := \sup \left(\bigcup_{a \in S} L_a \right).$$

Lemma 4.4.6. *Let $\mathbb{P}$ be a series-parallel pyramid. The following are true:*

1. *For all $a \in P$ we have $l_{\{a\}} \leq a$;*
2. *For all $a \leq b$ in P we have $l_{\{a\}} \leq l_{\{b\}}$;*
3. *For all $S_1, S_2 \subseteq P$ we have $l_{S_1} \leq l_{S_2}$ if for all $a \in S_1$ there exists $b \in S_2$ such that $a \leq b$;*
4. *For all $a \in P$ and $S \subseteq P$, $l_S \leq a$ implies $l_S \leq l_{\{a\}}$;*
5. *For all $a, b \in P$, $l_{\{a\}} \leq b \leq a$ implies $l_{\{a\}} = l_{\{b\}}$;*
6. *For all $a, b \in P$, if a is comparable to b then a is comparable to $l_{\{b\}}$;*
7. *For all $S \subseteq P$ and $a \in P$, a is comparable to l_S if $a \leq b$ for some $b \in S$;*
8. *For all $S \subseteq P$ and $a \in S$, If $l_S \leq a$ then $l_{\{a\}} = l_S$;*
9. *For all $S \subseteq P$ we have $l_S = l_{\{l_S\}}$.*

Proof. Most of these facts are fairly simple to prove. We will only show the arguments for (6) and (7).

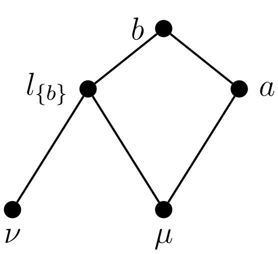

Figure 4.15

(6): If $a \geq b$ then by (2) $a \geq l_{\{b\}}$ follows. Assume $a < b$ and a is not comparable to $l_{\{b\}}$. This means that there exists a minimal element ν that is below $l_{\{b\}}$ but not a. Pick another minimal element μ below a. By definition we know $l_{\{b\}}$ is above μ. Then $\{a, l_{\{b\}}, \mu, \nu\}$ forms an N-subposet in $\mathbb{P}$, a contradiction (see figure 4.15).

(7): Let a and S be as described and assume $a \leq b$ for some $b \in S$. If a is not comparable to l_S then there must exists a minimal element ν that is below l_S but not a. Pick another minimal element μ below a. Since $a \leq b \in S$, l_S is also above μ. Then $\{a, l_S, \mu, \nu\}$ forms an N-subposet, a contradiction. □

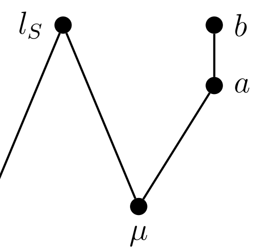

Figure 4.16

Chapter 5

Bounded Path Duality

In this chapter we will use bounded path duality to improve on Theorem 4.3.2 by showing the poset retraction problem for a series-parallel poset satisfying the 4-crown condition has complexity NL.

Theorem 5.0.1. *If $\mathbb{P}$ is a series-parallel poset satisfying the 4-crown condition, then* $\exp(\mathbb{P})$ *has bounded path duality; hence both* $\mathrm{Ret}(\mathbb{P})$ *and* $\mathrm{PoRet}(\mathbb{P})$ *are in* NL*.*

We will prove this theorem by showing that $\exp(\mathbb{P})$ has an obstruction set with bounded pathwidth. This means we will be working in the extended language containing the order relation symbol, which in this chapter we denote by R as well as a unary relation U_a for each element $a \in \mathbb{P}$. We will denote this extended language as $\mathcal{L}$.

5.1 The Obstruction Structures

The relational structures that will act as our obstructions are divided into three types. The first type is designed to weed out all structures that will violate the order relation of $\mathbb{P}$ should there exist a homomorphism into $\exp(\mathbb{P})$.

Definition 5.1.1. Let $a, b \in P$ and $1 \leq n \in \mathbb{N}$. Define $\mathbb{O}_{(n,a,b)}$ to be the $\mathcal{L}$-structure with universe

$$\{a_1, \dots, a_n\}$$

and relations:

$$R^{\mathbb{O}_{(n,a,b)}} := \{(a_1, a_2), \dots, (a_{n-1}, a_n)\},$$

$$U_a^{\mathbb{O}_{(n,a,b)}} := \{a_1\},$$

$$U_b^{\mathbb{O}_{(n,a,b)}} := \{a_n\},$$

and

$$U_c^{\mathbb{O}_{(n,a,b)}} := \emptyset$$

for all $c \neq a, b$ in P.

As we can see from the definition the homomorphic image of a_1 in a partially order set will be below that of a_n. So if $a, b \in P$ are such that $a \not\leq b$ then we certainly cannot have a homomorphism from $\mathbb{O}_{(n,a,b)}$ to $\exp(\mathbb{P})$.

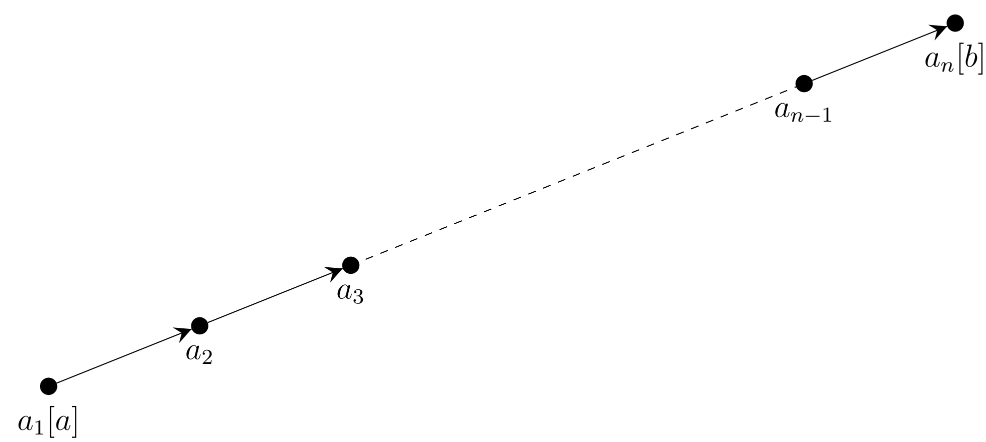

Figure 5.1: $\mathbb{O}_{(n,a,b)}$

Although we have drawn $\mathbb{O}_{(n,a,b)}$ here similar to a Hasse diagram, $R^{\mathbb{O}_{(n,a,b)}}$ is far from an actual order relation. However this illustration is helpful in visualizing the image of $\mathbb{O}_{(n,a,b)}$ under a homomorphism into a poset.

The remaining two types of obstructions corresponds to the two conditions in Theorem 4.3.14. They are designed to ensure both conditions are satisfied so that we can apply the theorem to get a homomorphism whenever none of the obstructions maps into $\exp(\mathbb{P})$.

Definition 5.1.2. Let $A, B \subseteq P$ and let f_A, f_B be mappings from A, B to $\mathbb{N}$ respectively. We define $\mathbb{X}_{(A,B,f_A,f_B)}$ to be the $\mathcal{L}$-structure with universe

$$\{x, x_i^a, x_j^b : a \in A, b \in B, 1 \leq i \leq f_A(a), 1 \leq j \leq f_B(b)\},$$

and relations:

$$R^{\mathbb{X}_{(A,B,f_A,f_B)}} := \{(x_1^a, x), (x, x_1^b) : a \in A, b \in B\} \cup \{(x_{i+1}^a, x_i^a), (x_j^b, x_{j+1}^b) : 1 \leq i < f_A(a), 1 \leq j < f_B(b)\},$$

$$U_a^{\mathbb{X}_{(A,B,f_A,f_B)}} := \{x^a_{f_A(a)}\}$$

for all $a \in A$,

$$U_b^{\mathbb{X}_{(A,B,f_A,f_B)}} := \{x^b_{f_B(b)}\}$$

for all $b \in B$, and

$$U_c^{\mathbb{X}_{(A,B,f_A,f_B)}} := \emptyset$$

for all $c \notin A \cup B$ in P.

We'll call x the *midpoint* of $\mathbb{X}_{(A,B,f_A,f_B)}$ for reference.

Definition 5.1.3. Let $n \in \mathbb{N}$, $a, b \in P$, $\vec{p} \in (\{+,-\} \cup P)^{n-1}$ and $\vec{l} \in \{0,1\}^n$. Let X be a collection of n $\mathcal{L}$-structures of the form $\mathbb{X}_{(A,B,f_A,f_B)}$ for some $A, B \subseteq P$. Let p_i and l_j denote the i-th and j-th coordinate of $\vec{p}$ and $\vec{l}$ respectively. Label the members of X as $\mathbb{X}_1, \ldots, \mathbb{X}_n$. We define $\mathbb{Y}_{(n,a,b,\vec{p},\vec{l},X)}$ to be the $\mathcal{L}$-structure with universe

$$\dot{\bigcup_{i=1}^{n}} X_i,$$

where each X_i is the universe of $\mathbb{X}_i$. Let x_i denote the midpoint of $\mathbb{X}_i$ for $1 \leq i \leq n$ and relations:

$$R^{\mathbb{Y}_{(n,a,b,\vec{p},\vec{l},X)}} := \bigcup_{i=1}^{n} R^{\mathbb{X}_i} \cup \{(x_i, x_{i+1}) : \text{for every } p_i = +\} \cup \{(x_{i+1}, x_i) : \text{for every } p_i = -\},$$

$$U_a^{\mathbb{Y}_{(n,a,b,\vec{p},\vec{l},X)}} := \bigcup_{i=1}^{n} U_a^{\mathbb{X}_i} \cup \{x_1\} \cup \{x_i, x_{i+1} : \text{for every } p_i = a\},$$

$$U_b^{\mathbb{Y}_{(n,a,b,\vec{p},\vec{l},X)}} := \bigcup_{i=1}^{n} U_b^{\mathbb{X}_i} \cup \{x_n\} \cup \{x_i, x_{i+1} : \text{for every } p_i = b\},$$

and

$$U_c^{\mathbb{Y}_{(n,a,b,\vec{p},\vec{l},X)}} := \bigcup_{i=1}^{n} U_c^{\mathbb{X}_i} \cup \{x_i, x_{i+1} : \text{for every } p_i = c\}$$

for all other c in P.

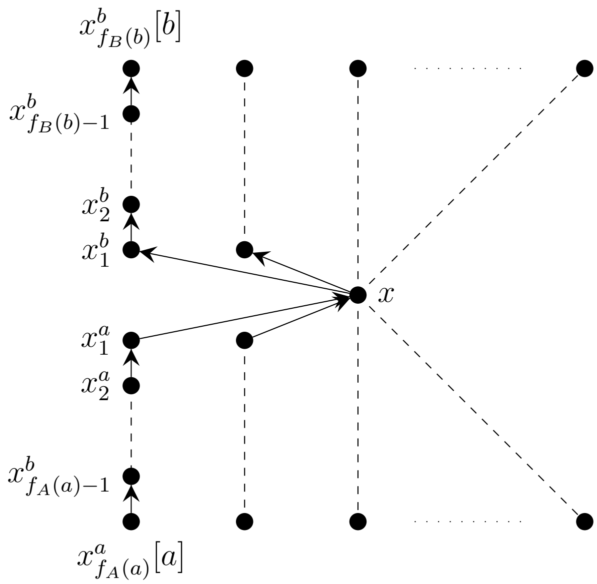

Figure 5.2: $\mathbb{X}_{(A,B,f_A,f_B)}$

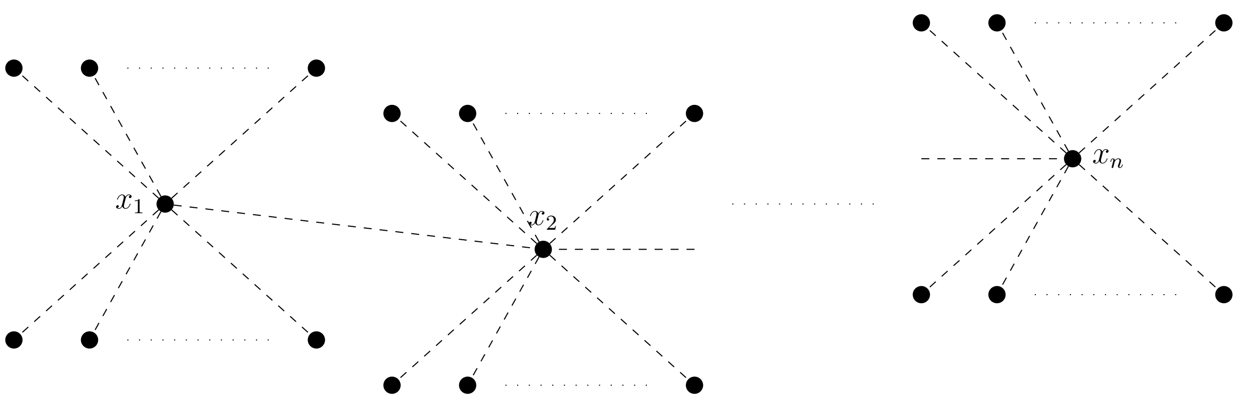

Figure 5.3: $\mathbb{Y}_{(n,a,b,\vec{p},\vec{l},X)}$

As stated before for $\mathbb{O}_{(n,a,b)}$, the relation R in $\mathbb{X}_{(A,B,f_A,f_B)}$ and $\mathbb{Y}_{(n,a,b,\vec{p},\vec{l},X)}$ are not order relations. The figures above merely illustrate what the homomorphic images of these structure would look like inside a poset. The structure of $\mathbb{Y}_{(n,a,b,\vec{p},\vec{l},X)}$ depends heavily on $\vec{p}$ so it might not look exactly as what is drawn.

5.2 The Connected Case

First we will prove that $\exp(\mathbb{P})$ has bounded path duality when $\mathbb{P}$ is connected.

Lemma 5.2.1. *Let $\mathbb{P}$ be a connected series-parallel poset satisfying the 4-crown condition. Then $\exp(\mathbb{P})$ has bounded path duality.*

Proof. Let $O_1 := \{\mathbb{O}_{(n,a,b)} : 1 \leq n \in \mathbb{N}, a \not\leq b\}$.

Let $O_2 := \{\mathbb{X}_{(L,U,f_L,f_U)} : L, U$ antichains in $\mathbb{P}$ such that $\mathbb{P}_{[L,U]} = \emptyset; f_L \in \mathbb{N}^L, f_U \in \mathbb{N}^U\}$. Note that in the definition of $\mathbb{X}_{(A,B,f_A,f_B)}$ A and B can be empty subsets, in which case the corresponding x_i^a's and x_j^b's don't exist. If A and B are both empty then it is simply the one element structure with all relations empty. Similarly L and U in the definition of O_2 can also be empty.

Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ denote the subsets used to denote $\mathbb{X}_1, \ldots, \mathbb{X}_n$ respectively. We'll say $(n, a, b, \vec{p}, \vec{l}, X)$ is primed when the following conditions are satisfied:

- There exists a $\mathbb{P}^X_{[A_0,B_0]}$ that is nonempty and disconnected;
- for every $1 \leq j \leq n$ we have $\mathbb{P}_{[A_j,B_j]} \subseteq \mathbb{P}^X_{[A_0,B_0]}$ when $l_j = 1$ or $A_j = B_j = \emptyset$ when $l_j = 0$;
- for every $1 \leq j < k \leq n$ such that l_j and l_k are consecutive non-zero coordinates of $\vec{l}$ the set of $\{p_j, p_{j+1} \ldots, p_{k-1}\}$ contains only one of $+$ or $-$;
- a, b belongs to different connected components of $\mathbb{P}^X_{[A_0,B_0]}$

Let $O_3 := \{\mathbb{Y}_{(n,a,b,\vec{p},\vec{l},X)} : (n, a, b, \vec{p}, \vec{l}, X)$ is primed$\}$.

Define $O := \bigcup_{i=1}^{3} O_i$ and we will now show that it is the obstruction set for $\exp(\mathbb{P})$. First we will prove that there does not exist a homomorphism from any $\mathcal{L}$-structure in O to $\exp(\mathbb{P})$. This in turn shows that $\mathbb{Q} \in \mathrm{CSP}(\exp(\mathbb{P}))$ implies $\forall \mathbb{O} \in O$ there does not exist a homomorphism from $\mathbb{O}$ to $\mathbb{Q}$.

Let $\mathbb{O}_{(n,a,b)}$ be as defined above for some $a \not\leq b$ in $\mathbb{P}$. Suppose there exists a homomorphism ϕ from $\mathbb{O}_{(n,a,b)}$ to $\exp(\mathbb{P})$. Then we would have $a = \phi(a_1) \leq \phi(a_2) \leq \cdots \leq \phi(a_n) = b$. A clear contradiction.

Let $\mathbb{X}_{(L,U,f_L,f_U)}$ be as defined above for some L, U antichains in $\mathbb{P}$ such that $\mathbb{P}_{[L,U]} = \emptyset$, $f_L \in \mathbb{N}^L$, and $f_U \in \mathbb{N}^U$. If there exists a homomorphism ϕ from $\mathbb{X}_{(L,U,f_L,f_U)}$ to $\exp(\mathbb{P})$ then $\phi(x)$ (the midpoint) must be an element from $\mathbb{P}_{[L,U]}$, establishing a contradiction.

Let $\mathbb{Y}_{(n,a,b,\vec{p},\vec{l},X)}$ be as defined above where $(n, a, b, \vec{p}, \vec{l}, X)$ is primed. Suppose there exists a homomorphism ϕ from $\mathbb{Y}_{(n,a,b,\vec{p},\vec{l},X)}$ to $\exp(\mathbb{P})$. For each $x_i \in \mathbb{Y}_{(n,a,b,\vec{p},\vec{l},X)}$ (the midpoints) where $l_i = 1$, $\phi(x_i)$ must be an element of $\mathbb{P}_{[A_i,B_i]} \subseteq \mathbb{P}^X_{[A_0,B_0]}$. Since $(n, a, b, \vec{p}, \vec{l}, X)$ is primed every $1 \leq j < k \leq n$ such that l_j and l_k are consecutive non-zero coordinates of $\vec{l}$ contains only one of $+$ or $-$. First assume it only contains $+$. This means that for every $j \leq i < k$ either $x_i < x_{i+1}$ or they share the same color. Then the images of these x_i's under ϕ will form a directed path between $\phi(x_j)$ and $\phi(x_k)$. Thus $\phi(x_j)$ and $\phi(x_k)$ must be comparable. Pick out all the x_i's such that $l_i = 1$. Their images under ϕ will then form a path from a to b in $\mathbb{P}^X_{[A_0,B_0]}$, a contradiction.

Now for the other direction we assume there exists an $\mathcal{L}$-structure $\mathbb{Q}$ such that there does not exists a homomorphism from any element of O to $\mathbb{Q}$. We will construct a series of homomorphisms starting from $\mathbb{Q}$ that eventually ends with a substructure of $\exp(\mathbb{P})$ as the final image.

For the first $\mathcal{L}$-structure we will add a new element χ_a to $\mathbb{Q}$ for each $a \in P$. Let $Q_1 := Q \cup \{\chi_a : a \in P\}$. Then define $\mathbb{Q}_1$ with universe Q_1 and relations

$$U_a^{\mathbb{Q}_1} := U_a^{\mathbb{Q}} \cup \{\chi_a\}$$

for each $a \in P$ and

$$R^{\mathbb{Q}_1} := R^{\mathbb{Q}}.$$

Clearly $\mathbb{Q}$ is an $\mathcal{L}$-substructure of $\mathbb{Q}_1$ so ϕ_1 defined to be the inclusion map of $\mathbb{Q}$ in $\mathbb{Q}_1$ will be a homomorphism. The important thing to note here is that $U_a^{\mathbb{Q}_1}$ for every $a \in P$ is nonempty.

The second $\mathcal{L}$-structure we will look at is such that its interpretation of R is a quasi-order. Define the color of an element α in an $\mathcal{L}$-structure to be a when α belongs to the unary relation labeled by U_a for some $a \in P$. If α does not belong to any such relation then we say that α is uncolored. Suppose there exists an element α of Q colored with both a and b for some $a \neq b \in P$. Since we cannot have both $a \leq b$ and $b \leq a$ in $\mathbb{P}$ there must be a homomorphism from one of $\mathbb{O}_{(1,a,b)}$ or $\mathbb{O}_{(1,b,a)}$ into $\mathbb{Q}$. This shows every element of $\mathbb{Q}_1$ have at most one color. Partition Q_1 according to the color of the elements, with each uncolored element in its own partition. Let $Q_2 = \{[\chi] : \chi \in Q\}$ be the set of partitions constructed this way. Q_2 will be the universe of the quasi-order $\mathbb{Q}_2$ we are constructing. Define $U_a^{\mathbb{Q}_2} := \{[\chi] : \chi \text{ is colored by } a\}$. It should be clear that each $U_a^{\mathbb{Q}_2}$ will contain exactly one element. Define $R^{\mathbb{Q}_2}$ to be the reflexive and transitive closure of set

$$\{([\alpha],[\beta]) : \text{ there exists } \alpha' \in [\alpha], \beta' \in [\beta] \text{ such that } (\alpha',\beta') \in R^{\mathbb{Q}_1} = R^{\mathbb{Q}}\}.$$

Then $\mathbb{Q}_2$ will be a quasi-order and $\phi_2 : \mathbb{Q}_1 \mapsto \mathbb{Q}_2$ defined by $\phi_2(\chi) = [\chi]$ for $\chi \in Q_2$ will be a homomorphism.

Before we move on it is helpful to note that if $a \not\leq b$ in $\mathbb{P}$ then $([\alpha],[\beta]) \notin R^{\mathbb{Q}_2}$ for a colored $[\alpha]$ and b colored $[\beta]$. Suppose for contradiction that this is not the case. Let $a, b, [\alpha]$ and $[\beta]$ be as mentioned above except that $([\alpha],[\beta]) \in R^{\mathbb{Q}_2}$. By definition of $R^{\mathbb{Q}_2}$ there exists $[\chi_1],[\chi_2],\ldots,[\chi_n]$ such that $[\chi_1] = [\alpha]$, $[\chi_n] = [\beta]$, and $\chi_i' \in [\chi_i]$ and $\chi_{i+1}' \in [\chi_{i+1}]$ such that $(\chi_i', \chi_{i+1}') \in R^{\mathbb{Q}}$ for $1 \leq i < n$. Let $a_1, a_2, \ldots, a_m$ be the colors that appear in the coloring of the $[\chi_i]$'s in order, where $a_1 = a$ and $a_m = b$. Because of $a \not\leq b$ we cannot have $a_1 \leq a_2 \leq \cdots \leq a_m$ in $\mathbb{P}$. Pick j_0 to be small index such that $a_{j_0} \not\leq a_{j_0+1}$. Let i_0 and i_1 denote the index of the corresponding $[\chi_i]$'s such that a_{j_0} is the color of $[\chi_{i_0}]$ and a_{j_0+1} is the color of $[\chi_{i_1}]$. This implies for all $i_0 < i < i_1$ $[\chi_i]$ is uncolored. By the definition of the partition on Q_1 each of these $[\chi_i]$'s contains only one element. So there exists a homomorphism from $\mathbb{O}_{((i_1-i_0),a_{j_0},a_{j_0+1})}$ to $\mathbb{Q}$, a contradiction.

Next we construct a partial order. Let Θ be the equivalence relation on Q_2 defined as follows:

$$\Theta := \{([\alpha],[\beta]) : ([\alpha],[\beta]),([\beta],[\alpha]) \in R^{\mathbb{Q}_2}\}.$$

We see that Θ is reflexive and transitive because $R^{\mathbb{Q}_2}$ is reflexive and transitive. Θ is also symmetric by definition. Let Q_3 be the set of Θ-equivalence classes on Q_2. We'll define an $\mathcal{L}$-structure $\mathbb{Q}_3$ with Q_3 as its universe.

$$U_a^{\mathbb{Q}_3} := \{[[\chi]] : [\chi] \text{ is colored by } a\}.$$

We will check that there does not exist two elements of Q_2 with different colors belonging to the same Θ-equivalence class (excluding uncolored elements). Suppose there exists $[\alpha],[\beta] \in Q_2$ colored by $a, b \in P$ respectively. Assuming $a \neq b$ it must be true that either $a \not\leq b$ or $b \not\leq a$ in $\mathbb{P}$. Without loss of generality we will assume that $a \not\leq b$. From what we have proven above this implies that $([\alpha],[\beta]) \notin R^{\mathbb{Q}_2}$, so $([\alpha],[\beta]) \notin \Theta$. This shows that each element of Q_3 belongs to at most one of the $U_a^{\mathbb{Q}_3}$'s. Since each $U_a^{\mathbb{Q}_2}$ contains exactly one element each $U_a^{\mathbb{Q}_3}$ contains exactly one element as well. Let

$$R^{\mathbb{Q}_3} := \{([[\alpha]],[[\beta]]) : ([\alpha],[\beta]) \in R^{\mathbb{Q}_2}\}.$$

$R^{\mathbb{Q}_3}$ inherits reflexivity and transitivity from $R^{\mathbb{Q}_2}$. If $([[\alpha]],[[\beta]])$ and $([[\beta]],[[\alpha]])$ are both in $R^{\mathbb{Q}_3}$ then $([\alpha],[\beta])$ and $([\beta],[\alpha])$ are both in $R^{\mathbb{Q}_2}$. By definition $[\alpha]$ and $[\beta]$ belongs to the same Θ-equivalence class. It is clear now that $\mathbb{Q}_3$ is a partial order and $\phi_3 : \mathbb{Q}_2 \mapsto \mathbb{Q}_3$ defined by $\phi_3([\chi]) := [[\chi]]$ is a homomorphism.

Before the next step we want to prove for $R^{\mathbb{Q}_3}$ the same thing we have proven for $R^{\mathbb{Q}_2}$. Let $a \not\leq b$ in $\mathbb{P}$ and suppose we have a colored $[[\alpha]]$ and b colored $[[\beta]]$ such that $([[\alpha]], [[\beta]]) \in R^{\mathbb{Q}_3}$. By definition there exists $[\alpha']$ and $[\beta']$ such that $[[\alpha']] = [[\alpha]]$, $[[\beta']] = [[\beta]]$, and $([\alpha'], [\beta']) \in R^{\mathbb{Q}_2}$. There must also exists $[\alpha'']$ and $[\beta'']$ in $\mathbb{Q}_2$ colored a and b respectively such that $[[\alpha'']] = [[\alpha]]$ and $[[\beta'']] = [[\beta]]$. Since $[\alpha']$ and $[\alpha'']$ belongs to the same Θ-equivalence class we have $([\alpha''], [\alpha']) \in R^{\mathbb{Q}_2}$, likewise $([\beta'], [\beta'']) \in R^{\mathbb{Q}_2}$. By transitivity we have $([\alpha''], [\beta'']) \in R^{\mathbb{Q}_2}$, which would imply the contradiction of $a \leq b$ in $\mathbb{P}$. Therefore if $a \not\leq b$ in $\mathbb{P}$ then $([[\alpha]], [[\beta]]) \notin R^{\mathbb{Q}_3}$ for a colored $[[\alpha]]$ and b colored $[[\beta]]$.

We see that the colored elements of $\mathbb{Q}_3$ form a one-to-one correspondence with the elements from $\mathbb{P}$. Let us refer to them according to their colors by denoting x_a to be the element of Q_3 colored by $a \in P$. Next we add to $\mathbb{Q}_3$ the order relation that exists on $\mathbb{P}$. Define $\mathbb{Q}_4$ with the same universe and U_a relations as $\mathbb{Q}_3$ and let

$$R^{\mathbb{Q}_4} := R^{\mathbb{Q}_3} \cup \{(x_a, x_b) : (a, b) \in \leq^{\mathbb{P}}\}.$$

Also let $\mathbb{Q}^*$ denote the poset $(Q_4, R^{\mathbb{Q}_4})$. From what we have shown for $R^{\mathbb{Q}_3}$ we get $(a, b) \in \leq^{\mathbb{P}}$ if and only if $(x_a, x_b) \in R^{\mathbb{Q}_4}$. So $\mathbb{P}$ is isomorphic to the subposet of $\mathbb{Q}^*$ induced by the colored elements of $\mathbb{Q}_4$. If there exists a retraction of $\mathbb{Q}^*$ onto this subposet then there would exists a homomorphism from $\mathbb{Q}_4$ to $\exp(\mathbb{P})$.

For this final step we will invoke Theorem 4.3.14. For the sake of convenience we will view $\mathbb{P}$ as a subposet of $\mathbb{Q}^*$. All that remains is to check that condition (1) and (2) are satisfied.

Let (L, U) be a pair of antichains in $\mathbb{P}$ such that $\mathbb{P}_{[L,U]}$is empty. Suppose there exists $x \in {\mathbb{Q}^*}_{[L,U]}$. Let L' be the set of maximal elements of $\mathbb{P}$ that is below x in $\mathbb{Q}^*$. Likewise let U' be the set of minimal elements in $\mathbb{P}$ above x. Note that for every element of L there exists one in L' that is above it. Similarly there is an element of U' below any element of U. Thus $\mathbb{P}_{[L',U']} \subseteq \mathbb{P}_{[L,U]} = \emptyset$ and $x \in {\mathbb{Q}^*}_{[L',U']}$. So without loss of generality let us assume $L = L'$ and $U = U'$.

Note that x must be an uncolored element of $\mathbb{Q}_4$. Since x is uncolored it is a Θ-equivalence class of uncolored elements from Q_2. The elements of Q_2 as we recall are partitions of Q_1 by colors. Being uncolored implies they each contain a single uncolored element of Q_1. The uncolored elements of Q_1 are all contained in Q. So we can fix an uncolored $\chi_0 \in Q$ such that $[[\chi_0]] = x$. Furthermore because these uncolored elements form partitions of Q_1 that are in the same Θ-class, any two χ_1 and χ_2 such that $[[\chi_1]] = [[\chi_2]] = x$ will be connected by a directed path in $R^{\mathbb{Q}}$. Enumerate the elements of L and U as $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_m\}$ respectively. For every $1 \leq i \leq n$ since $(a_i, x) \in R^{\mathbb{Q}_4}$ and

x is uncolored $(a_i, x) \in R^{\mathbb{Q}_3}$ as well. So there exists $\alpha_i \in Q_1$ such that $[[\alpha_i]] = a_i$ and $([\alpha_i], [\chi_i]) \in R^{\mathbb{Q}_2}$ for some $[[\chi_i]] = x$. So there exists a directed path in

$$\{([\alpha], [\beta]) : \text{ there exists } \alpha' \in [\alpha], \beta' \in [\beta] \text{ such that } (\alpha', \beta') \in R^{\mathbb{Q}_1} = R^{\mathbb{Q}}\}.$$

from $[\alpha_i]$ to $[\chi_i]$. Every element on this path except for $[\alpha_i]$ must be uncolored, otherwise it would contradict the maximality of the elements in L. Since these elements are uncolored they must be partitions of one element each. So there exists a directed path in $R^{\mathbb{Q}_1} = R^{\mathbb{Q}}$ from an element in $[\alpha_i]$ to χ_i. Without loss of generality let's assume that element is α_i. Then there exists a directed path in $R^{\mathbb{Q}}$ from α_i to χ_0. Let f_L be the function in $\mathbb{N}^L$ that maps each a_i to the length of such a directed path. Define f_U in $\mathbb{N}^U$ similarly for b_i's in U. Then there exists a homomorphism from $\mathbb{X}_{(L,U,f_L,f_U)}$ into $\mathbb{Q}$, a contradiction. This shows that condition (1) is satisfied.

Now for condition (2) assume for some non-empty and non-connected subset of $\mathbb{P}$ of the form $\mathbb{P}_{[L',U']}$, there exists a pair of elements p_1, p_2 in different connected components of $\mathbb{P}_{[L',U']}$, such that there exists a path in $\mathbb{Q}^*$ that connects p_1 and p_2 such that every element in this path belongs to some ${\mathbb{Q}^*}_{[L,U]}$ with $\mathbb{P}_{[L,U]} \subseteq \mathbb{P}_{[L',U']}$. Let's assume p_1 and p_2 are picked so that such a path has the minimal length (compared to other paths satisfying the same properties).

Since p_1 and p_2 are elements of $\mathbb{P}$ they are colored (by themselves). Suppose there exists another colored element in this path p_3. By assumption p_3 belongs to some ${\mathbb{Q}^*}_{[L,U]}$ with $\mathbb{P}_{[L,U]} \subseteq \mathbb{P}_{[L',U']}$. But since p_3 is colored it is an element of $\mathbb{P}$ as well. So $p_3 \in \mathbb{P}_{[L,U]} \subseteq \mathbb{P}_{[L',U']}$. If p_3 belongs to the same connected component as p_1 then we can remove all the elements of the aforementioned path between p_1 and p_2 up to but not including p_3 to get a shorter path satisfying the same properties. Similarly if p_3 belongs to the same connected component as p_2 we get a shorter path as well. Both of these cases are contradictions to the minimality of the path. The only option left is if p_3 belongs to a connected component of its own. In that case we can still chop off either the front or the back of the p_1 to p_2 path to get a shorter one. This shows p_1 and p_2 are the only colored elements of this path. Also note that this path is of length at least three since Otherwise they would be comparable in $\mathbb{Q}^*$. Comparability of elements of $\mathbb{P}$ in $\mathbb{Q}^*$ implies comparability in $\mathbb{P}$ (since $\mathbb{P}$ is a subposet of $\mathbb{Q}^*$).

So we see that in every consecutive pair of this path there is a uncolored element. Let x and y be a pair of consecutive elements on this path. Pick χ and ψ so that $x = [[\chi]]$ and $y = [[\psi]]$. We know one of (x, y) or (y, x) is in $R_4^{\mathbb{Q}}$. Since one of them is uncolored we have, as argued previously, a directed path between χ and ψ in $\mathbb{Q}$ such that the connections of this path comes from either $R^{\mathbb{Q}}$ or $(U_a^{\mathbb{Q}})^2$ for some $a \in P$ (each consecutive pair α, β on this path are either $R^{\mathbb{Q}}$ related or shares the same color).

Now back to the path between p_1 and p_2. For each element on this path pick a χ such that it equals $[[\chi]]$. From what we just discovered we know there exists a directed path between each such χ. Connected all of these χ's with these directed paths and number them accordingly by χ_i for $1 \leq i \leq n$ where n is the final length of this amalgamated path. Define $\vec{l} \in \{0,1\}^n$ so that l_i equals 1 when χ_i corresponds to an element on the original path and 0 otherwise. Define $\vec{p} \in (\{+,-\} \cup P)^{n-1}$ so that $p_i = +$ when we have (χ_i, χ_{i+1}) on the path, $p_i = -$ when it's (χ_{i+1}, χ_i) and $p_i = a \in P$ when χ_i and χ_{i+1} are both of color a (when χ_i and χ_{i+1} are comparable as well as sharing the same color either designation for p_i works).

When $l_i = 1$ $[[\chi_i]]$ belongs to some $\mathbb{Q}^*{}_{[L,U]}$ with $\mathbb{P}_{[L,U]} \subseteq \mathbb{P}_{[L',U']}$. Thus as shown in the previous case we can replace such an L and U with a set of maximal elements below and minimal elements above. Then with these new L and U there exists a homomorphism from $\mathbb{X}_{(L,U,f_L,f_U)}$ into $\mathbb{Q}$ where the midpoint will be mapped to χ_i. Let A_i's and B_i's denote the newly replaced L and U sets in order of the path. A_i and B_i will be the empty set when $l_i = 0$. Let X be the collection of $\mathbb{X}_{(A_i,B_i,f_{A_i},f_{B_i})}$. It should be clear that if we let $\mathbb{P}^X_{[A_0,B_0]} = \mathbb{P}_{[L',U']}$ then $(n, p_1, p_2, \vec{p}, \vec{l}, X)$ is primed. We just mentioned there is a homomorphism from each $\mathbb{X}_{(A_i,B_i,f_{A_i},f_{B_i})}$ into $\mathbb{Q}$. By its construction we then also have a homomorphism from $\mathbb{Y}_{(n,a,b,\vec{p},\vec{l},X)}$ into $\mathbb{Q}$ by mapping each x_i of $\mathbb{Y}_{(n,a,b,\vec{p},\vec{l},X)}$ onto χ_i. This is a contradiction, so condition (2) is also satisfied.

So O is an obstruction set for $\exp(\mathbb{P})$. We will now check that every element in O_1, O_2 and O_3 all have pathwidth at most $(2,3)$.

Let $\mathbb{O}_{(n,a,b)}$ be as defined above for some $n \in \mathbb{N}$ and $a, b \in P$. Then the sequence of subsets

$$\{a_1, a_2\}, \{a_2, a_3\}, \ldots \{a_{n-1}, a_n\}$$

is an $(1,2)$ path-decomposition of $\mathbb{O}_{(n,a,b)}$.

Let $\mathbb{X}_{(A,B,f_A,f_B)}$ be as defined above for some $A, B \in P$, $f_A \in \mathbb{N}^A$, and $f_B \in \mathbb{N}^B$. Then the sequence of subsets

$$\{x, x^a_1, x^a_2\}, \{x, x^a_2, x^a_3\}, \ldots, \{x, x^a_{f_A(a)-1}, x^a_{f_A(a)}\}$$

for each $a \in A$ and

$$\{x, x^b_1, x^b_2\}, \{x, x^b_2, x^b_3\}, \ldots, \{x, x^b_{f_B(b)-1}, x^b_{f_B(b)}\}$$

for each $b \in B$ is a $(2,3)$ path-decomposition of $\mathbb{X}_{(A,B,f_A,f_B)}$.

Let $\mathbb{Y}_{(n,a,b,\vec{p},\vec{l},X)}$ be as defined above for some $n \in \mathbb{N}$, $a, b \in P$, $\vec{p} \in \{+,-\}^{n-1}$, $\vec{l} \in \mathbb{N}^{n-1}$ and $X = \{\mathbb{X}_1, \ldots, \mathbb{X}_n\}$ some collection of $\mathcal{L}$-structures of the form $\mathbb{X}_{(A,B,f_A,f_B)}$. From above we know that for each $\mathbb{X}_i \in X$ there exists a $(2,3)$ path-decomposition. Denote this decomposition as $D_{\mathbb{X}_i}$. Then the sequence of subsets

$$D_{\mathbb{X}_1}, \{x_1, x_2\}, D_{\mathbb{X}_2}, \{x_2, x_3\}, \ldots, \{x_{n-1}, x_n\}, D_{\mathbb{X}_n}$$

is a $(2,3)$ path-decomposition of $\mathbb{Y}_{(n,a,b,X)}$. □

5.3 The Disconnected Case

For the disconnected case we can generalize the result to all posets. But first we need to define one more type of obstruction structure.

Definition 5.3.1. Let $a, b \in P$, $1 \leq n \in N$ and $\vec{p} \in \{+, -\}^n$. Denote the i-th coordinate of $\vec{p}$ as p_i. Define $\mathbb{C}_{(a,b,n,\vec{p})}$ to be the $\mathcal{L}$-structure with universe

$$\{x_0, \ldots, x_n\}$$

and relations:

$$R^{\mathbb{C}_{(a,b,n,\vec{p})}} := \{(x_{i-1}, x_i) : p_i = +\} \cup \{(x_i, x_{i-1}) : p_i = -\},$$

$$U_a^{\mathbb{C}_{(a,b,n,\vec{p})}} := \{x_0\},$$

$$U_b^{\mathbb{C}_{(a,b,n,\vec{p})}} := \{x_1\},$$

and

$$U_c^{\mathbb{C}_{(a,b,n,\vec{p})}} := \emptyset$$

for every other $c \in P$.

Lemma 5.3.2. *Let $\mathbb{P}$ be a finite poset such that for each of its connected components $\mathbb{Q}$, $\exp(\mathbb{Q})$ has bounded path duality. Then $\exp(\mathbb{P})$ has bounded path duality.*

Proof. There is a bit of logistics we have to work out before starting the proof of the theorem. Let $\mathbb{P}_1, \ldots, \mathbb{P}_n$ be the distinct subposets of $\mathbb{P}$ induced by its connected components. It is assumed that each $\exp(\mathbb{P}_i)$ has bounded path duality, which means that it has an obstruction set of finite pathwidth. However this is all working under the language of $\exp(\mathbb{P}_i)$ which does not contain as many constant relations (the U_a's) as $\mathcal{L}$ but is a subset of it. What we can do is to expand the language of the obstruction sets to $\mathcal{L}$ by adding in the missing relations so that each structure of each obstruction set becomes an $\mathcal{L}$-structure. All of the newly added relations will be empty.

Let O_1 be the union of obstruction sets of all of the $\exp(\mathbb{P}_i)$'s. This is not yet enough to get a obstruction set for the whole of $\exp(\mathbb{P})$.

Let O_2 be the set of all $\mathbb{C}_{(a,b,n,\vec{p})}$ such that a and b are disconnected in $\mathbb{P}$. We claim that $O := O_1 \cup O_2$ will be an obstruction set for $\exp(\mathbb{P})$.

First we will confirm that none of the $\mathcal{L}$-structures in O has a homomorphism to $\exp(\mathbb{P})$.

Let $\mathbb{O} \in O_1$. Suppose there exists a homomorphism h from $\mathbb{O}$ into $\exp(\mathbb{P})$. We know that $\mathbb{O}$ is the expansion of an obstruction set for some $\exp(\mathbb{P}_i)$. Thus the colored elements of $\mathbb{O}$ are all colored by elements of $\mathbb{P}_i$. So the image of each colored element of $\mathbb{O}$ under h must be in $\mathbb{P}_i$ (as a subset of $\mathbb{P}$). For every non-colored element of $\mathbb{O}$ if it is connected via $R^{\mathbb{O}}$ to some colored element then their image under h must still be connected via $R^{\mathbb{P}}$. Since $\mathbb{P}_i$ is a full connected component these non-colored elements are also mapped into $\mathbb{P}_i$ by h. Pick an arbitrary element a of $\mathbb{P}_i$. We can modify h so that all elements of $\mathbb{O}$ that are not connected (via $R^{\mathbb{O}}$) to a colored element is mapped to a. It is easy to see that this newly modified h is still a homomorphism. Furthermore this new h maps $\mathbb{O}$ in its entirety into $\mathbb{P}_i$. This translates to a homomorphism from $\mathbb{O}$ to $\exp(\mathbb{P}_i)$ in the original language of $\mathbb{O}$. A clear contradiction since $\mathbb{O}$ was assumed to be in the obstruction set of $\exp(\mathbb{P}_i)$.

Now pick a $\mathbb{C}_{(a,b,n,\vec{p})} \in O_2$ such that a and b are disconnected in $\mathbb{P}$. Suppose there exists a homomorphism h from $\mathbb{C}_{(a,b,n,\vec{p})}$ to $\exp(\mathbb{P})$. From construction we know that x_0 and x_n are $R^{\mathbb{C}_{(a,b,n,\vec{p})}}$ connected so their images under h must also be $R^{\mathbb{P}}$ connected. However since x_0 belongs to $U_a^{\mathbb{C}_{(a,b,n,\vec{p})}}$ and x_n to $U_b^{\mathbb{C}_{(a,b,n,\vec{p})}}$ this becomes a contradiction.

Now for the reverse direction we will show that for all $\mathcal{L}$-structure $\mathbb{Q}$ if there does not exists any homomorphism from elements of O to $\mathbb{Q}$ then there exists one for $\mathbb{Q}$ into $\exp(\mathbb{P})$.

We start by assuming there does not exists any homomorphism from elements of O into $\mathbb{Q}$. Let $Q_1, \ldots, Q_m$ be the $R^{\mathbb{Q}}$ connected components of $\mathbb{Q}$. Let a and b be a pair of $R^{\mathbb{P}}$ disconnected elements in P. Suppose we have some a and b colored element belonging to the same Q_j. By definition these exists a sequence of elements $x_0, \ldots, x_n$ connected by $R^{\mathbb{Q}}$ in $\mathbb{Q}$ where x_0 is a colored and x_n is b colored. Pick $\vec{p} \in \{+, -\}^n$ such that $p_i = +$ if $(x_{i-1}, x_i) \in R^{\mathbb{Q}}$ and $p_i = -$ if $(x_i, x_{i-1}) \in R^{\mathbb{Q}}$. Then there exists a homomorphism from $\mathbb{C}_{(a,b,n,\vec{p})} \in O_2$ to $\mathbb{Q}$, a contradiction. Thus we see that each Q_j contains only colored elements from one corresponding $\mathbb{P}_i$. We will construct a map from each Q_j to its corresponding $\mathbb{P}_i$.

I say 'construct' but a suitable map already exists. Each Q_j can be thought of as a substructure of $\mathbb{Q}$ (there are no functions to consider, only relations). Since there does not exist a homomorphism from any of the structures of O into $\mathbb{Q}$ there certainly does not exist one from the obstruction set of $\exp(\mathbb{P}_i)$ into Q_j. Thus by definition of obstruction sets there exists a homomorphism from Q_j as a structure into $\exp(\mathbb{P}_i)$.

Collect these maps and define one more from $\mathbb{Q}$ to $\mathbb{P}$ by applying each homomorphism onto its corresponding connected component Q_j. This final map will be our desired homomorphism.

This shows that O is indeed an obstruction set for $\exp(\mathbb{P})$. The structures in O_2 are all "paths" of $(1, 2)$ pathwidth. O_1 is a finite collection of structures each with a bounded pathwidth. Therefore O also has finite pathwidth. □

If a series-parallel poset $\mathbb{P}$ does not satisfy the 4-crown condition, then by Theorem 4.3.2 it will not have a Taylor polymorphism. Hence $\mathbb{P}$ does not admit Freese-McKenzie SD-$\vee$ polymorphisms by Corollary 2.6.6. Thus by Proposition 3.5.9, $\exp(\mathbb{P})$ does not have bounded path duality.

This observation combined with what we have just proven gives the following result.

Theorem 5.3.3. *Let $\mathbb{P}$ be a finite series-parallel poset. $\mathbb{P}$ satisfies the 4-crown condition if and only if* $\exp(\mathbb{P})$ *has bounded path duality.*

Proposition 3.5.9 shows that for a finite series-parallel poset $\mathbb{P}$ if $\exp(\mathbb{P})$ has bounded path duality then it will admit Freese-McKenzie SD-$\vee$ operations. Now we can show the converse as well.

Theorem 5.3.4. *Let $\mathbb{P}$ be a finite series-parallel poset. The following are equivalent:*

1. $\exp(\mathbb{P})$ *has bounded path duality;*
2. $\mathbb{P}$ *admits Freese-McKenzie SD-$\vee$ operations;*
3. $\mathbb{P}$ *admits a Taylor polymorphism.*

Proof. This follows directly from Proposition 3.5.9, Corollary 2.6.6, Theorem 4.3.2 and Theorem 5.3.3. □

Chapter 6

Majority and NU

6.1 Zádori's Result

Series-parallel posets admitting an NU polymorphism were first classified by Zádori in his 1993 paper.

Theorem 6.1.1. *[26, Theorem 2.3] [2, Corollary 5.1] If a series-parallel poset $\mathbb{P}$ retracts onto one of $\mathbf{1}+\mathbf{2}+\mathbf{2}, \mathbf{2}+\mathbf{2}+\mathbf{1}$ and $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{2}+\mathbf{1}$, then $\mathbb{P}$ does not admit an NU polymorphism.*

Theorem 6.1.2. *[26, Corollary 3.3] Every series parallel poset $\mathbb{P}$ that does not retract onto $\mathbf{2}+\mathbf{2}, \mathbf{1}+\mathbf{2}+\mathbf{2}, \mathbf{2}+\mathbf{2}+\mathbf{1}$ and $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{2}+\mathbf{1}$ admits a 5-ary NU polymorphism.*

If a poset retracts onto $\mathbf{2}+\mathbf{2}$ then it must not admit an NU polymorphism. If it did then certainly $\mathbf{2}+\mathbf{2}$ would have one as well. But $\mathbf{2}+\mathbf{2}$ does not satisfy the 4-crown condition, so by Theorem 4.3.2 it would not even have a Taylor polymorphism. Thus by Corollary 2.6.6, $\mathbf{2}+\mathbf{2}$ does not admit an NU polymorphism. With this we can deduce the following result by Zádori.

Theorem 6.1.3. *Let $\mathbb{P}$ be a series-parallel poset. The following are equivalent:*

1. *$\mathbb{P}$ admits an NU polymorphism.*

2. $\mathbb{P}$ *admits a* 5*-ary NU operation.*

3. $\mathbb{P}$ *does not retract onto* $\mathbf{2}+\mathbf{2}, \mathbf{1}+\mathbf{2}+\mathbf{2}, \mathbf{2}+\mathbf{2}+\mathbf{1}$ *or* $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{2}+\mathbf{1}$.

Zádori used this result to show that if the clone of a finite bounded series-parallel poset is finitely generated then it contains an NU operation. This was a question asked of general posets mentioned in [8], [20] and [22].

Recently an alternative classification was discovered by Larose and Willard which is the inspiration behind our classification of the 4-crown condition back in chapter 4. They've shown that similar to the fact that all series-parallel posets can be constructed from linear sum and disjoint union, $\{+^C, \triangle, \triangledown, \Diamond, \bowtie, \uplus\}$ produces all series-parallel posets with an NU polymorphism. We will show both of these classifications are equivalent to a third regarding the supremum and infimum of pairs of elements. We will then present an alternative proof of the existence of 5-ary NU polymorphisms on these posets.

6.2 Partial Lattices

Definition 6.2.1. Let $\mathbb{P}$ be a poset. We will say $\mathbb{P}$ is a *partial lattice* if for every pair of connected elements $a, b \in P$ at least one of $\sup(a, b)$ and $\inf(a, b)$ exists.

It's clear from the definition that series-parallel partial lattices satisfy the 4-crown condition.

Lemma 6.2.2. *Let* $\mathbb{P}$ *be a series-parallel partial lattice. If* r *is a retraction of* $\mathbb{P}$ *then the subposet* $\mathbb{Q}$ *of* $\mathbb{P}$ *induced by the image of* r *will also be a series-parallel partial lattice.*

Proof. This can be proved by repeating the same arguments from the second and third case of Lemma 4.3.3. □

We will show partial lattices are equivalent to Larose and Willard's classification in the next two lemmas.

Lemma 6.2.3. *Let* $\mathbb{P}_1$ *and* $\mathbb{P}_2$ *be series-parallel partial lattices. Then* $\mathbb{P}_1 \uplus \mathbb{P}_2$, $\mathbb{P}_1 +^C \mathbb{P}_2$, $\mathbb{P}_1 \triangle \mathbb{P}_2$, $\mathbb{P}_1 \triangledown \mathbb{P}_2$, $\mathbb{P}_1 \Diamond \mathbb{P}_2$ *and* $\mathbb{P}_1 \bowtie \mathbb{P}_2$*(when the requirements of the operations are met) are still series-parallel partial lattices.*

Proof. By Lemma 4.3.11 we know that the output of these operations will still be series-parallel.

Let $a, b \in \mathbb{P}_1 +^C \mathbb{P}_2$. If a, b both belong to the same $\mathbb{P}_i$ then either $\sup(a, b)$ or $\inf(a, b)$ would exist. If they belong to different $\mathbb{P}_i$'s then they are comparable so $\sup(a, b)$ and $\inf(a, b)$ exists anyways.

The case for ⧖ is proven in a similar manner.

Let $a, b \in \mathbb{P}_1 \triangle \mathbb{P}_2$. If a, b both belong to the same $\mathbb{P}'_i$ then $\sup(a, b)$ and $\inf(a, b)$ exists as in the previous case. Otherwise the unique element of $\mathbf{1}$ satisfy the conditions of $\sup(a, b)$.

The case for $\triangledown$ and $\lozenge$ is omitted due to similarity to the previous case.

The case for $\dot\cup$ is trivial. □

Lemma 6.2.4. *A poset $\mathbb{P}$ is a series-parallel partial lattice if and only if it can be constructed from $\mathbf{1}$ using only $+^C, \dot\cup, \triangle, \triangledown, \lozenge$ and ⧖ finitely many times.*

Proof. Since $\mathbf{1}$ satisfies the conditions of being a partial lattice, by the previous lemma any poset that can be constructed from $\mathbf{1}$ using only $+^C, \triangle, \triangledown, \lozenge$ and ⧖ finitely many times will be a series-parallel poset partial lattice.

Assume for contradiction that $\mathbb{P}$ is the minimal (size-wise) series-parallel partial lattice that cannot be constructed as described in the statement of the lemma. If $\mathbb{P}$ is disconnected then one of its connected components would contradict the minimality of $\mathbb{P}$. So $\mathbb{P}$ must be connected. Since $|\mathbb{P}| > 1$, we can pick $\mathbb{P}_1$ and $\mathbb{P}_2$ series parallel posets such that $\mathbb{P} = \mathbb{P}_1 + \mathbb{P}_2$. If $\mathbb{P}_1$ and $\mathbb{P}_2$ are both connected partial lattices then $\mathbb{P} = \mathbb{P}_1 +^C \mathbb{P}_2$. Since $\mathbb{P}_1$ and $\mathbb{P}_2$ have size strictly less than $\mathbb{P}$ by the minimality of $\mathbb{P}$ we have a contradiction. For the remainder of this argument, we assume that at least one of $\mathbb{P}_1$ and $\mathbb{P}_2$ is not a connected partial lattice.

Assume first that $\mathbb{P}_1$ is not a connected partial lattice. So either $\mathbb{P}_1$ is not a partial lattice, or $\mathbb{P}_1$ is disconnected. Pick a and b from different connected components of $\mathbb{P}_1$ if it is disconnected. Pick a and b that violate the definition of partial lattices if it is not a partial lattice. We know that $\inf(a, b)$ cannot exist in $\mathbb{P}$ since it would be contained in $\mathbb{P}_1$. So in both cases there exist a and b in $\mathbb{P}_1$ such that $\sup(a, b)$ exists in $\mathbb{P}$ but not in $\mathbb{P}_1$ (since $\mathbb{P}$ is connected).

The element $\sup(a, b)$ must be an element of $\mathbb{P}_2$. Since every element of $\mathbb{P}_2$ is above both a and b in $\mathbb{P}$ they must be above $\sup(a, b)$ as well. This shows that $\mathbb{P}_2$ is connected and possesses a unique minimal element. Let $\mathbb{P}'_1 := \mathbb{P}_1 + \mathbf{1}$; then $\mathbb{P} = \mathbb{P}'_1$ ⧖ $\mathbb{P}_2$. For every $a, b \in P'_1$ their supremum in $\mathbb{P}$ (if it exists) has to be below the maximal element of $\mathbb{P}'_1$. Thus $\mathbb{P}'_1$ is a partial lattice. If $\mathbb{P}'_1$ has size strictly less than $\mathbb{P}$, i.e. $|\mathbb{P}_2| > 1$, there would be a contradiction. A similar proof shows that if $\mathbb{P}_2$ is not a connected partial lattice and $|\mathbb{P}_1| > 1$ then we get a contradiction. This shows that if $\mathbb{P} = \mathbb{P}_1 + \mathbb{P}_2$ with $|\mathbb{P}_1| > 1$ and $|\mathbb{P}_2| > 1$ then we get a contradiction.

Next, assume that $\mathbb{P} = \mathbb{P}_1 + \mathbf{1}$ and $\mathbb{P}_1$ is not a connected partial lattice. Suppose $\mathbb{P}_1$ is connected. Since $|\mathbb{P}_1| > 1$, there exist $\mathbb{P}_3$ and $\mathbb{P}_4$ such that $\mathbb{P}_1 = \mathbb{P}_3 + \mathbb{P}_4$. If $\mathbb{P}_3 \neq \mathbf{1}$ then define $\mathbb{P}'_1 := \mathbb{P}_3$ and $\mathbb{P}'_2 := \mathbb{P}_4 + \mathbf{1}$. We would then have $\mathbb{P} = \mathbb{P}'_1 + \mathbb{P}'_2$ where both $\mathbb{P}'_1$ and $\mathbb{P}'_2$ have size greater than one. We can then use our previous argument and reach a contradiction. So it must be that $\mathbb{P}_3 = \mathbf{1}$, so $\mathbb{P} = \mathbf{1} + \mathbb{P}_4 + \mathbf{1}$. Now we have two subcases based on the connectivity of $\mathbb{P}_4$. If $\mathbb{P}_4$ is connected then $|\mathbb{P}_4| > 1$ (since otherwise $\mathbb{P} = \mathbf{1} + \mathbf{1} + \mathbf{1}$), so we can split it as we did $\mathbb{P}_1$ into $\mathbb{P}_4 = \mathbb{P}_5 + \mathbb{P}_6$. Define $\mathbb{P}'_1 := \mathbf{1} + \mathbb{P}_5$ and $\mathbb{P}'_2 := \mathbb{P}_6 + \mathbf{1}$ and we would reach a contradiction as before.

If $\mathbb{P}_4$ is disconnected then denote the connected components of $\mathbb{P}_4$ by $\mathbb{Q}_1, \ldots, \mathbb{Q}_n$. We get $(\mathbf{1} + \mathbb{Q}_1 + \mathbf{1}) \lozenge \ldots \lozenge (\mathbf{1} + \mathbb{Q}_n + \mathbf{1}) = \mathbb{P}$.

We will show that each $\mathbf{1} + \mathbb{Q}_i + \mathbf{1}$ for $i \in \{1, \ldots, n\}$ is a partial lattice. Fix an arbitrary $i_0 \in \{1, \ldots, n\}$. Let $a, b \in \mathbb{Q}_{i_0}$. We know that either $\inf(a, b)$ or $\sup(a, b)$ exists in $\mathbb{P}$ (since $\mathbb{P}$ is connected). If $\inf(a, b)$ exists then it must be the minimal element of $\mathbb{P}$ or contained in $\mathbb{P}_1$. It must also be connected to a and b so if it is not the minimal element then it is contained in $\mathbb{Q}_{i_0}$. If it is equal to the unique minimum then it exists in $\mathbf{1} + \mathbb{Q}_{i_0} + \mathbf{1}$ as its minimal element. If $\sup(a, b)$ exists then as we just argued it must be contained in $\mathbf{1} + \mathbb{Q}_{i_0} + \mathbf{1}$ as well. For any pair of elements in $\mathbf{1} + \mathbb{Q}_{i_0} + \mathbf{1}$ where one of the pair is the unique maximum or minimum then clearly their supremum or infimum exists. Therefore we see that $\mathbf{1} + \mathbb{Q}_{i_0} + \mathbf{1}$ is a partial lattice.

This would result in a contradiction as $(\mathbf{1} + \mathbb{Q}_1 + \mathbf{1}) \lozenge \ldots \lozenge (\mathbf{1} + \mathbb{Q}_n + \mathbf{1}) = \mathbb{P}$ where each $(\mathbf{1} + \mathbb{Q}_1 + \mathbf{1})$ has size strictly smaller than $\mathbb{P}$. Thus we see that no matter what, $\mathbb{P}_1$ being connected leads to a contradiction.

So $\mathbb{P}_1$ must be disconnected. Let $\mathbb{Q}_1, \ldots, \mathbb{Q}_n$ be the subposets of $\mathbb{P}_1$ induced by its connected components. Then $(\mathbb{Q}_1 + \mathbf{1}) \triangle \ldots \triangle (\mathbb{Q}_n + \mathbf{1}) = \mathbb{P}$.

Each $\mathbb{Q}_i + \mathbf{1}$ for $i \in \{1, \ldots, n\}$ is a partial lattice by arguments similar to what we have just done. For each $1 \leq i \leq n$, $\mathbb{Q}_i + \mathbf{1}$ has size strictly less than $\mathbb{P}$. The minimality of $\mathbb{P}$ would once again cause a contradiction.

The proof of when $\mathbb{P}_2$ is not being a connected partial lattice is similar to what we have just done. $\square$

6.3 Larose and Willard's Result

Since their result was never published we will show here the proof that Larose and Willard's classification is equivalent to Zádori's.

Lemma 6.3.1. *[16, Lemma 0.1] Let $\mathbb{P}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{S}$ be posets.*

1. *Let $n \geq 1$. If $\mathbb{P}$ has at least n connected components then it retracts onto $\boldsymbol{n}$ (the n element antichain).*
2. *If $\mathbb{P}$ retracts onto $\mathbb{R}$ and $\mathbb{Q}$ retracts onto $\mathbb{S}$ then $\mathbb{P} + \mathbb{Q}$ retracts onto $\mathbb{R} + \mathbb{S}$.*
3. *If $\mathbb{P}$ is connected and series-parallel with at least two minimal (maximal) elements then it retracts onto $\mathbf{2} + \mathbf{1}$ ($\mathbf{1} + \mathbf{2}$).*
4. *If $\mathbb{P}$ is connected and series-parallel with no pinch points then it retracts onto $\mathbf{2} + \mathbf{2} + \ldots + \mathbf{2}$ where there are at least two summands in the sum.*

Proof. (1): Pick n connected components of $\mathbb{P}$ and map each one onto a different element of $\mathbf{n}$. Map the rest onto a single element.

(2): Let r_1 be the retraction from $\mathbb{P}$ to $\mathbb{R}$ and r_2 the one from $\mathbb{Q}$ to $\mathbb{S}$. Define the retraction from $\mathbb{P} + \mathbb{Q}$ to $\mathbb{R} + \mathbb{S}$ as

$$r(x) := \begin{cases} r_1(x) & \text{if } x \in P \\ r_2(x) & \text{if } x \in Q. \end{cases}$$

(3): Let a and b be two distinct minimal (maximal) elements of $\mathbb{P}$. By Lemma 4.2.5 they have a common upper (lower bound) c. Define the retraction from $\mathbb{P}$ to $\mathbf{2} + \mathbf{1}$ ($\mathbf{1} + \mathbf{2}$) as

$$r(x) := \begin{cases} x & \text{if } x = a \text{ or } b \\ c & \text{else.} \end{cases}$$

(4): We will prove this by induction on the size of $\mathbb{P}$. The smallest connected series-parallel poset with no pinch point is the 4-crown. In that case the proof is trivial. Suppose all posets smaller than $\mathbb{P}$ satisfy (4).

Since $\mathbb{P}$ is connected and series-parallel it is a linear sum of smaller series-parallel posets $\mathbb{P}_1$ and $\mathbb{P}_2$. Let $i \in \{1, 2\}$ be arbitrary. Suppose $\mathbb{P}_i$ is connected. There does not exists a pinch point in $\mathbb{P}_i$ since $\mathbb{P}$ doesn't have one. By induction hypothesis $\mathbb{P}_i$ must retract onto $\mathbf{2} + \mathbf{2} + \cdots + \mathbf{2}$ where there are at least 2 summands in the sum. If $\mathbb{P}_i$ is disconnected then we can apply (1) to get a retraction onto $\mathbf{2}$. Now apply (2) to finish the proof. ∎

Definition 6.3.2. [16] Let $\mathcal{G}$ denote the set of series-parallel posets that do not retract onto a subposet of the form $\mathbf{2}+\mathbf{2}, \mathbf{1}+\mathbf{2}+\mathbf{2}, \mathbf{2}+\mathbf{2}+\mathbf{1}$ and $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{2}+\mathbf{1}$.

Lemma 6.3.3. *[16, Lemma 0.2] Let $\mathbb{P} \in \mathcal{G}$ be connected and of size at least 4. There exist connected series-parallel posets $\mathbb{P}_1$ and $\mathbb{P}_2$ each with at least 2 elements such that one of the following holds:*

1. $\mathbb{P} = \mathbb{P}_1 +^C \mathbb{P}_2$, *or*
2. $\mathbb{P} = \mathbb{P}_1 \triangle \mathbb{P}_2$, *or*
3. $\mathbb{P} = \mathbb{P}_1 \bigtriangledown \mathbb{P}_2$, *or*
4. $\mathbb{P} = \mathbb{P}_1 \Diamond \mathbb{P}_2$, *or*
5. $\mathbb{P} = \mathbb{P}_1 \between \mathbb{P}_2$.

Proof. If $\mathbb{P}$ contains no pinch points at all then by part (4) of Lemma 6.3.1 it retracts onto some $\mathbf{2}+\mathbf{2}+\cdots+\mathbf{2}$ where there are at least 2 summands in the sum. Thus by part (1) and (2) of the same lemma $\mathbb{P}$ retracts onto one of $\mathbf{2}+\mathbf{2}, \mathbf{1}+\mathbf{2}+\mathbf{2}$ or $\mathbf{2}+\mathbf{2}+\mathbf{1}$. This would cause a contradiction so $\mathbb{P}$ must have a pinch point.

If $\mathbb{P}$ contains a pinch point that is not a maximal or minimal element then (5) holds.

If $\mathbb{P}$ contains only pinch points that are maximal or minimal elements then $\mathbb{P} = \mathbb{Q}+\mathbf{1}$ or $\mathbf{1}+\mathbb{Q}$ for some series-parallel $\mathbb{Q}$ with at least three elements (it is a subposet of $\mathbb{P}$ so it is N-free). If $\mathbb{Q}$ is disconnected then (2) or (3) holds. If not then $\mathbb{Q} = \mathbb{Q}_1 + \mathbb{Q}_2$ where $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are also series-parallel. Without loss of generality let's assume $\mathbb{P} = \mathbb{Q}_1 + \mathbb{Q}_2 + \mathbf{1}$. By the first line of this paragraph $\mathbb{Q}_2$ must have at least two minimal elements and does not contain a pinch point. By part (3) of Lemma 6.3.1 $\mathbb{Q}_2 + \mathbf{1}$ has a retraction onto $\mathbf{2}+\mathbf{1}$. This means $\mathbb{Q}_1$ has to be connected or $\mathbb{P}$ will have a retraction onto $\mathbf{2}+\mathbf{2}+\mathbf{1}$. Thus either (1) holds or $\mathbb{Q}_1 = \mathbf{1}$. In the latter case $\mathbb{P} = \mathbf{1}+\mathbb{Q}_2+\mathbf{1}$. We know that $\mathbb{Q}_2$ has size at least two. If it is disconnected then (4) holds. If it is not then we can break it up into sums again. In which case $\mathbb{P} = (\mathbf{1}+\mathbb{Q}_3) +^C (\mathbb{Q}_4+\mathbf{1})$ and so (1) holds. □

Lemma 6.3.4. *[16, Lemma 0.3] Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be two connected series-parallel posets. Let $\star$ be an operation from the set $\{\triangle, \bigtriangledown, \Diamond, \between \cup\}$. $\mathbb{P}_1$ and $\mathbb{P}_2$ are both in $\mathcal{G}$ if and only if $\mathbb{P}_1 \star \mathbb{P}_2$ is in $\mathcal{G}$ (when the requirements of the operations are satisfied).*

Proof. Note that from definition the $\Diamond$ operation requires the two posets to have size strictly larger than 1. In the case of the rest of the operations when one of the $\mathbb{P}_i$'s is the singleton **1** their output will be the other $\mathbb{P}_i$. Thus we only need to prove the lemma for when $\mathbb{P}_1$ and $\mathbb{P}_2$ are of size strictly larger than one.

($\Longrightarrow$) First we look at the case where $\star = \triangle$. Suppose $\mathbb{P}_1 \triangle \mathbb{P}_2$ retracts onto a subposet $\mathbb{R}$ that is isomorphic to one of $\mathbf{2}+\mathbf{2}, \mathbf{1}+\mathbf{2}+\mathbf{2}, \mathbf{2}+\mathbf{2}+\mathbf{1}$ and $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{2}+\mathbf{1}$. The image of the unique maximum of $\mathbb{P}_1 \triangle \mathbb{P}_2$ should be a unique maximum as well. Thus $\mathbb{R}$ is isomorphic to either $\mathbf{2}+\mathbf{2}+\mathbf{1}$ or $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{2}+\mathbf{1}$. Let m and m' denote the two elements of $\mathbb{R}$ belonging to its minimal antichain. Without loss of generality we can assume m belongs to $\mathbb{P}_1$. Then every element of $\mathbb{R}$ comparable to m also belongs to $\mathbb{P}_1$. Since m and m' share a common upper bound that is not the unique maximum of $\mathbb{P}_1 \triangle \mathbb{P}_2$ m' must also be an element of $\mathbb{P}_1$. Therefore $\mathbb{R}$ is contained in $\mathbb{P}_1$ in its entirety. The retraction of $\mathbb{P}_1 \triangle \mathbb{P}_2$ into $\mathbb{R}$ when restricted to $\mathbb{P}_1$ will be a retraction of $\mathbb{P}_1$ into $\mathbb{R}$. This is a contradiction since we assumed $\mathbb{P}_1 \in \mathcal{G}$. Thus $\mathbb{P}_1 \triangle \mathbb{P}_2 \in \mathcal{G}$. Using similar arguments we can also show this for when $\star = \triangledown$ and $\Diamond$.

Now assume $\star = \between$. Let m be the element of $\mathbb{P}_1 \between \mathbb{P}_2$ that belongs to both $\mathbb{P}_i$'s. Suppose there exists a retraction of $\mathbb{P}_1 \between \mathbb{P}_2$ onto a subposet $\mathbb{R}$ defined as above. Due to the choices of what $\mathbb{R}$ can be, the image of m under this retraction will be either the unique maximum or minimum of $\mathbb{R}$. This means $\mathbb{R}$ is contained in one of the $\mathbb{P}_i$'s in its entirety. So we can finish by using similar arguments as before.

Finally assume $\star = \uplus$. Suppose $\mathbb{P}_1 \uplus \mathbb{P}_2$ can be retracted onto a subposet $\mathbb{R}$ as before. Then since $\mathbb{R}$ is a connected subposet it must be contained entirely in one of the $\mathbb{P}_i$'s. The rest of the proof follows as usual.

($\Longleftarrow$) When $\star = \uplus$ we can construct a retraction from $\mathbb{P}_1 \uplus \mathbb{P}_2$ into each $\mathbb{P}_i$ by mapping all of the other one onto a single point. For all other cases there exists a retraction from $\mathbb{P}_1 \star \mathbb{P}_2$ onto each $\mathbb{P}_i$'s if we map every element not belonging to said $\mathbb{P}_i$ onto the same shared element. Then if either of the $\mathbb{P}_i$'s admits a retraction onto one of $\mathbf{2}+\mathbf{2}, \mathbf{1}+\mathbf{2}+\mathbf{2}, \mathbf{2}+\mathbf{2}+\mathbf{1}$ and $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{2}+\mathbf{1}$ by composing the two retractions we get a contradiction. □

Theorem 6.3.5. *A finite poset $\mathbb{P}$ is in $\mathcal{G}$ if and only if it is a series-parallel partial lattice.*

Proof. ($\Longleftarrow$) It should be clear that $\mathbf{2}+\mathbf{2}, \mathbf{1}+\mathbf{2}+\mathbf{2}, \mathbf{2}+\mathbf{2}+\mathbf{1}$ and $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{2}+\mathbf{1}$ are not partial lattices. By Lemma 6.2.2 there cannot be a retraction of $\mathbb{P}$ onto any of them.

($\Longrightarrow$) Let $\mathbb{P}$ be the smallest (size-wise) $\mathbb{P}$ in $\mathcal{G}$ that is not a partial lattice. Since all series-parallel posets of size 3 or less are partial lattices we can assume $\mathbb{P}$ has size greater than or equal to 4.

If $\mathbb{P}$ is not connected then the subposets induced by its connected components will also be in $\mathcal{G}$ by Lemma 6.3.4. Due to the minimality of $\mathbb{P}$ these subposets will be partial lattices. Thus by Lemma 6.2.4 $\mathbb{P}$ is also a partial lattice. A contradiction. Thus $\mathbb{P}$ must be connected.

Applying Lemma 6.3.3 we find ourselves in one of five cases. In all but case (1) we can apply Lemma 6.3.4 to show the two smaller posets are in $\mathcal{G}$. Then by induction and Lemma 6.2.4 we are done.

Recall that in Lemma 6.3.3 case (1) occurs when all pinch points of $\mathbb{P}$, should they exist, are maximal or minimal elements. In this scenario we have $\mathbb{P} = \mathbb{P}_1 + \mathbb{P}_2$ such that $\mathbb{P}_1$ and $\mathbb{P}_2$ have at least 2 elements. Furthermore by the observation on the pinch points of $\mathbb{P}$ $\mathbb{P}_1$ must have at least two maximal elements. Likewise $\mathbb{P}_2$ possess at least two minimal elements. So by Lemma 6.3.1 part (3) $\mathbb{P}_1$ and $\mathbb{P}_2$ has a retraction onto $\mathbf{1}+\mathbf{2}$ and $\mathbf{2}+\mathbf{1}$ respectively.

If $\mathbb{P}_1$ does not have any pinch points at all then by Lemma 6.3.1 it has a retraction onto one of $\mathbf{2}+\mathbf{2}, \mathbf{1}+\mathbf{2}+\mathbf{2}$ or $\mathbf{2}+\mathbf{2}+\mathbf{1}$. In the first and third case $\mathbb{P}$ has a retraction onto $\mathbf{2}+\mathbf{2}+\mathbf{1}$. In the second case since $\mathbb{P}_2$ has a retraction onto $\mathbf{2}+\mathbf{1}$ it gives a retraction of $\mathbb{P}$ onto $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{2}+\mathbf{1}$. So all three cases leads to contradictions.

If $\mathbb{P}_1$ has a pinch point then it has to be the unique minimal element. So if it retracts onto one of $\mathbf{2}+\mathbf{2}, \mathbf{1}+\mathbf{2}+\mathbf{2}, \mathbf{2}+\mathbf{2}+\mathbf{1}$ and $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{2}+\mathbf{1}$ only $\mathbf{1}+\mathbf{2}+\mathbf{2}$ and $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{2}+\mathbf{1}$ makes sense. The first case is a contradiction as mentioned above. In the second case $\mathbb{P}$ has a retraction onto $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{2}+\mathbf{1}$ which is a contradiction.

Thus $\mathbb{P}_1$ must be in $\mathcal{G}$. The same thing can be proved for $\mathbb{P}_2$ as well.

Since $\mathbb{P}_1$ and $\mathbb{P}_2$ are have size strictly less than $\mathbb{P}$ by minimality they are partial lattices. So by Lemma 6.2.4 $\mathbb{P}$ is also a partial lattice. □

6.4 5-ary NU polymorphism

Finally we will present our alternative method of finding 5-ary NU polymorphisms on these posets.

Definition 6.4.1. Let S be a set. For a given n-tuple $(x_1, x_2, \ldots, x_n) \in S^n$ we will say that it is *diagonal* if $x_1 = x_2 = \cdots = x_n$. Denote the set of all diagonal n-tuples in S^n by $\Delta(S^n)$.

We will say that $(x_1, x_2, \ldots, x_n)$ is *NU (near unanimous)* if all but at most one of $x_1, x_2, \ldots, x_n$ are equal to each other. We'll call the ($(n-1)$ times) repeating element of an NU n-tuple the *majority element* and the element that is singled out will be the *renegade element* (should it exist).

Theorem 6.4.2. *Let $\mathbb{P}$ be a connected series-parallel partial lattice. Then $\mathbb{P}$ admits a 5-ary NU operation.*

Proof. Recall from the definition of polymorphisms we will need to show there exists a homomorphism from $\mathbb{P}^5$ to $\mathbb{P}$ satisfying the conditions of being an NU operation. We will do this by constructing an intermediate poset $\mathbb{Q}$ and homomorphisms from $\mathbb{P}^5$ to $\mathbb{Q}$ then from $\mathbb{Q}$ to $\mathbb{P}$.

For each $a \in P$ let U_a be the subset of P^5 that contains all NU 5-tuples that have a as their majority element. Let V_a be the convex closure of U_a (it contains all 5-tuples that are bounded above and below by elements of U_a). We will show that the V_a's are pairwise disjoint. Let a, b be distinct elements in P. Suppose we have some $(x_1, x_2, x_3, x_4, x_5)$ contained in both V_a and V_b. Then there exists some NU 5-tuple with a as their majority element above and below $(x_1, x_2, x_3, x_4, x_5)$. Each of these NU 5-tuples will have at least four coordinates equal to a. So at least three of the x_i's will be bounded above and below by a. Thus three coordinates of $(x_1, x_2, x_3, x_4, x_5)$ equal to a. But the same can be said for b. Since there aren't enough coordinates to fit in three a's and three b's this leads to a contradiction. So the V_a's are indeed mutually exclusive.

Now let's figure out what exactly is in each V_a. From above we know that every 5-tuple in it will be bounded above and below by some NU 5-tuple with majority a. This of course leads to three coordinates of every element in V_a to be a. Looking closer at the NU tuples that bound a particular element we consider their renegade elements. Suppose the upper and lower bound NU tuples have renegade elements in different coordinates. Then the renegade of the upper bound NU tuple must be above a. Similarly the renegade of the lower bound will be below a. So in this case the 5-tuples that are bounded by these NU tuples have all coordinates comparable to a with at most one coordinate above a and at most one coordinate below a. This is also true when one of the bounds is the a majority diagonal element. Now consider the case when both bounds have renegades in the same coordinate. In this case the bounded 5-tuple will also be an NU tuple. Thus we can conclude that within V_a we either have NU 5-tuples with majority a or 5-tuples with one coordinate above a, one below with the rest equaling a. This also means that the only way for a 5-tuple in V_a to have two coordinates the same is to have at least one more coordinate matching them.

Let Θ be the equivalence relation on $\mathbb{P}^5$ with the V_a's as classes; those tuples that do not belong to any V_a will be in a class of their own. When a Θ-equivalence class equals to some V_a we will say it is an NU equivalence class. Let $Q := P^5/\Theta$ and define $\leq^{\mathbb{Q}}$ to be the transitive closure of

$$\{([(a_1,\ldots,a_5)]_\Theta,[(b_1,\ldots,b_5)]_\Theta) : ((a_1,\ldots,a_5),(b_1,\ldots,b_5)) \in \leq^{\mathbb{P}^5}\}.$$

The map ϕ_1 which sends every element of $\mathbb{P}$ to its corresponding Θ-equivalence class in $\mathbb{Q}$ is easily a relational structure homomorphism by inspection. What we need to make sure is that $\mathbb{Q}$ is still a poset with an induced subposet isomorphic to $\mathbb{P}$.

For convenience's sake we'll denote 5-tuples $(a_1,\ldots,a_5)$ by $\vec{a}$. $\leq^{\mathbb{Q}}$ is reflexive and transitive by definition. Suppose we have $[\vec{a}] \leq^{\mathbb{Q}} [\vec{b}]$ and $[\vec{b}] \leq^{\mathbb{Q}} [\vec{a}]$ in $\mathbb{Q}$ for some $\vec{a} \neq \vec{b}$. Then there exist $\vec{x_1},\vec{y_1},\ldots,\vec{x_n},\vec{y_n}$ in $\mathbb{P}^5$ such that $[\vec{a}] = [\vec{x_1}]$, $[\vec{b}] = [\vec{x_{i_0}}]$ for some $1 < i_0 \leq n$, $[\vec{x_i}] = [\vec{y_i}]$, and $\vec{y_i} \leq^{\mathbb{P}^5} \vec{x_{i+1}}$ modulo n for each $1 \leq i \leq n$. Suppose there exists some $1 < i \neq i_0$ such that $[\vec{x_i}]$ is not an NU equivalence class. Then $[\vec{x_i}] = \{\vec{x_i}\}$ so $\vec{y_{i-1}} \leq^{\mathbb{P}^5} \vec{x_i} = \vec{y_i} \leq^{\mathbb{P}^5} \vec{x_{i+1}}$ modulo n. This allows us to construct a shorter sequence by removing these equivalence classes. So without loss of generality let's assume the $[\vec{x_i}]$'s with $i \neq 1, i_0$ are NU equivalence classes.

Suppose $[\vec{a}]$ and $[\vec{b}]$ are also NU classes. Without loss of generality we can assume every $\vec{x_i}$ and $\vec{y_i}$ are also NU tuples since every element of an NU class is bounded above and below by some NU tuple. Let $z_i \in P$ be the majority element of each $\vec{x_i}$ and $\vec{y_i}$ (since they belong to the same Θ class they must have the same majority). $\vec{y_i} \leq^{\mathbb{P}^5} \vec{x_{i+1}}$ implies $z_i \leq^{\mathbb{P}} z_{i+1}$ modulo n for each $1 \leq i \leq n$. So we get $z_1 = z_2 = \cdots = z_n$. This means that $[\vec{a}] = [\vec{b}]$.

Suppose $[\vec{a}]$ is an NU class but $[\vec{b}]$ is not. Then $[\vec{b}]$ is bounded above and below by NU classes in this sequence. As argued above we can remove $[\vec{b}]$ from this sequence to get a new one satisfying all of the conditions and such that every class in it is an NU class. We can still apply the argument from the previous paragraph to show that all of the NU classes in this sequence are actually the same. Recall that we constructed these NU classes to be convex. Since $\vec{b}$ is bounded above and below by elements of the same NU class $[\vec{b}]$ has to equal said class, which is a contradiction.

Now suppose both $[\vec{a}]$ and $[\vec{b}]$ are not NU classes. If $n > 2$ then we can repeat the above argument once again to remove both $[\vec{a}]$ and $[\vec{b}]$ from the sequence to get a new sequence. This will imply $\vec{a}$ and $\vec{b}$ are bounded above and below by the elements of the same NU class, contradiction. If $n = 2$ then we have $\vec{a} \leq^{\mathbb{P}^5} \vec{b} \leq^{\mathbb{P}^5} \vec{a}$, another contradiction.

The antisymmetry of $\leq^{\mathbb{Q}}$ is verified.

<ins>Claim:</ins> Let $a, x_1, \ldots, x_5 \in \mathbb{P}$.

1. $[(a,a,a,a,a)] \leq^{\mathbb{Q}} [(x_1,\ldots,x_5)]$ if and only if at least four of the x_i's are above a.

2. $[(a,a,a,a,a)] \geq^{\mathbb{Q}} [(x_1,\ldots,x_5)]$ if and only if at least four of the x_i's are below a.

Proof: We will only show the proof for (1) as the proof for (2) is similar.

Suppose we have $[(a,a,a,a,a)] \leq^{\mathbb{Q}} [(x_1,\ldots,x_5)]$. Then we know there exists some $(a_1,\ldots,a_5) \in [(a,a,a,a,a)]$ such that $(a_1,\ldots,a_5) \leq^{\mathbb{P}^5} (x_1,\ldots,x_5)$. As we have noted before, at least four coordinates in $(a_1,\ldots,a_5)$ are greater than or equal to a. Thus at least four of the x_i's are above a.

Conversely suppose that at least four of the x_i's are above a. With out loss of generality let's assume they are $x_1,\ldots,x_4$. Then $(a,a,a,a,x_5) \leq^{\mathbb{P}^5} (x_1,\ldots,x_5)$. By definition of $\leq^{\mathbb{Q}}$ we have that $[(a,a,a,a,a)] \leq^{\mathbb{Q}} [(x_1,\ldots,x_5)]$. ■

Let $\mathbb{P}'$ denote the subposet of $\mathbb{Q}$ induced by the NU classes of diagonal elements. By the claim we get that $[(a,a,a,a,a)] \leq^{\mathbb{P}'} [(b,b,b,b,b)]$ if and only if $a \leq^{\mathbb{P}} b$. Thus $\mathbb{P}'$ is isomorphic $\mathbb{P}$ and we will view them as the same poset.

The map ϕ that sends each 5-tuple of $\mathbb{P}^5$ to its respective Θ-equivalence class is clearly a homomorphism.

Since we have $\mathbb{P}$ as a subposet of $\mathbb{Q}$ we may apply Theorem 4.3.14 to get a retraction, which is a homomorphism, if condition (1) and (2) (of the theorem) are satisfied.

Let L and U be anti-chains in $\mathbb{P}'$ such that $\mathbb{Q}_{[L,U]}$ is nonempty. We'll need to show that $\mathbb{P}'_{[L,U]}$ is nonempty as well . If there exists some $[\vec{x}] \in \mathbb{Q}_{[L,U]} \cap \mathbb{P}'$ then $\mathbb{P}'_{[L,U]}$ is nonempty. So let $[\vec{x}] \in \mathbb{Q}_{[L,U]} \backslash \mathbb{P}'$. Then $[\vec{x}] = \{\vec{x}\}$ and let us write $\vec{x} = (x_1,x_2,x_3,x_4,x_5)$. Define L' to be the set of all maximal elements below $[\vec{x}]$ in $\mathbb{P}'$ and U' the set of all minimal elements above. Then $[\vec{x}] \in \mathbb{Q}_{[L',U']} \subseteq \mathbb{Q}_{[L,U]}$ and $\mathbb{P}'_{[L',U']} \subseteq \mathbb{P}'_{[L,U]}$. Instead of equivalence classes of NU 5-tuples let us refer to the elements of L' and U' by their respective majority elements. For all $a \in L'$, if $a \leq^{\mathbb{Q}} [(x_1,x_2,x_3,x_4,x_5)]$ then by the claim above we get that a is below at least four out of five x_i's as elements of $\mathbb{P}$. We'll show there are at least three x_i's which sits above all (majority) elements of L'. Pick $a_1, a_2 \in L'$ should they exist that are incomparable to two different x_i's (if all $a \in L'$ are incomparable to the same x_i then they are all below the other four). Without loss of generality let's assume a_1 is incomparable to x_2 and a_2 is incomparable to x_1. This means that x_3 is above both a_1 and a_2. This would form two potential N-subposets in $\mathbb{P}$

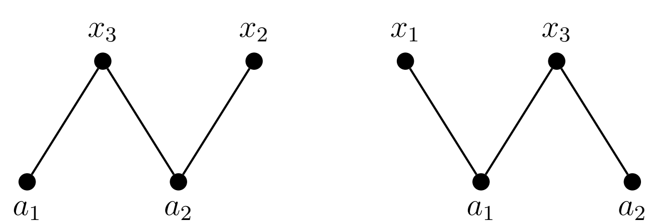

Figure 6.1

unless x_1 and x_2 are both below x_3. However, x_4 and x_5 are above a_1 and a_2 by the same argument. Following the same steps we get x_1 and x_2 below x_4 and x_5 as well.

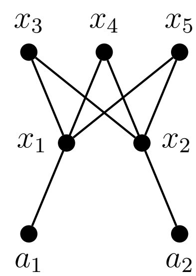

Figure 6.2

This shows that every $a \in L'$ can only be incomparable to one of x_1 or x_2 and is below all of x_3, x_4 and x_5. Similarly the same thing can be shown for U'. There are at least three x_i's which sit below every $b \in U'$. Putting these two statements together we get at least one x_i who is below all of U' and above all of L'. This means that $[(x_i, x_i, x_i, x_i, x_i)] \in \mathbb{P}'_{[L',U']} \subseteq \mathbb{P}'_{[L,U]}$. Therefore $\mathbb{P}'_{[L,U]}$ is nonempty if $\mathbb{Q}_{[L,U]}$ is nonempty.

Next suppose we are given L, U antichains in $\mathbb{P} = \mathbb{P}'$ such that $\mathbb{P}_{[L,U]}$ is nonempty. Since $\mathbb{P}$ is a connected partial lattice, given a pair of elements p_1 and p_2 in $\mathbb{P}_{[L,U]}$ they will either have a supremum or an infimum. Without loss of generality let's assume the former exists. Then by definition $\sup(p_1, p_2)$ is above all of L and below all of U. Thus $\sup(p_1, p_2)$

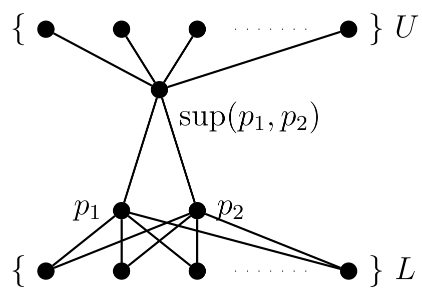

Figure 6.3

connects p_1 and p_2 with a path in $\mathbb{P}_{[L,U]}$. Similarly $\inf(p_1, p_2)$ connects p_1 and p_2 with a path in $\mathbb{P}_{[L,U]}$ if it exists. This shows that $\mathbb{P}_{[L,U]}$ is connected. Therefore condition (2) of theorem 4.3.14 is vacuously satisfied.

Now that we have a homomorphism from $\mathbb{P}^5$ to $\mathbb{Q}$ and from $\mathbb{Q}$ to $\mathbb{P}$, we can compose them to get a homomorphism from $\mathbb{P}^5$ to $\mathbb{P}$. The initial map ϕ from $\mathbb{P}^5$ to $\mathbb{Q}$ ensures that every NU 5-tuple will be mapped to its majority element in $\mathbb{P}$. The retraction from $\mathbb{Q}$ to $\mathbb{P}$ does not move any element of $\mathbb{P}$. Therefore the composition is a 5-ary NU operation, as advertised. ☐

Theorem 6.4.3. *Let $\mathbb{P}$ be a poset that is the disjoint union of $\mathbb{P}_1$ and $\mathbb{P}_2$, where both $\mathbb{P}_i$'s admit an n-ary NU operation. Then $\mathbb{P}$ admits an n-ary NU operation as well.*

Proof. For $i \in \{1, 2\}$ denote μ_i to be the n-ary NU operation on $\mathbb{P}_i$. Fix an element $a \in \mathbb{P}$ and define a function $\mu : \mathbb{P}^n \mapsto \mathbb{P}$ as follows:

$$\mu(x_1, \ldots, x_n) = \begin{cases} \mu_i(x_1, \ldots, x_n) & \text{if all of the } x_j\text{'s are in the same } P_i, \\ x_{j_0} & \text{if exactly } n-1 \text{ many } x_j\text{'s are in the same } P_i \text{ and } x_{j_0} \text{ is the left-most entry in } P_i, \\ a & \text{otherwise.} \end{cases}$$

It should be clear from its definition that μ is an n-ary NU operation. We just have to check that it is order preserving. Since elements from $\mathbb{P}_1$ and $\mathbb{P}_2$ are mutually disconnected, any pair of n-tuples $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)$ must have the same coordinates belonging to the same $\mathbb{P}_i$. Then it follows from the definition that μ is order preserving. ☐

Combining the previous two theorems we get the following result by Zádori.

Theorem 6.4.4. *[26, Corollary 3.3] Let $\mathbb{P}$ be a series-parallel partial lattice. Then $\mathbb{P}$ admits a 5-ary NU operation.*

Chapter 7

SD-join($\vee$) operations

In chapter 5 we saw that all series-parallel posets satisfying the 4-crown condition, when expanded with constants, have bounded path duality. From results in chapter 3 we know this means that each of these posets admits a series of Freese-McKenzie operations. We will use the classification of the 4-crown condition in chapter 4 to produce a recipe for constructing these Freese-McKenzie operations.

7.1 One-step Chain

When studying Freese-McKenzie operations on structures we discovered they closely resemble a sequence of retractions on the third power of that structure.

Definition 7.1.1. Let $\mathbb{R}$ be a relational structure and $\mathbb{R}'$ a substructure of $\mathbb{R}$. We will call a finite sequence of retractions $r_0, r_1, \ldots, r_n$ on $\mathbb{R}$ to be a *one-step chain* from $\mathbb{R}$ to $\mathbb{R}'$ if it satisfies the following conditions for all $0 \leq i < n$:

1. $r_{i+1}(\mathbb{R}) \subseteq r_i(\mathbb{R})$;
2. $|\{x \in r_i(\mathbb{R}) : r_{i+1}(x) \neq x\}| \leq 1$;
3. $r_0 = \mathrm{id}_{\mathbb{R}}$ and $r_n(\mathbb{R}) = \mathbb{R}'$.

Definition 7.1.2. Let S be a set. We will say that an operation $f : S^3 \mapsto S^3$ is *NU-preserving* if for all $f(x, y, z) = (x', y', z')$, we have:

- $x = y \implies x' = y'$,
- $x = z \implies x' = z'$ and
- $y = z \implies y' = z'$.

Lemma 7.1.3. *Let S be a set. If f and g are NU-preserving maps on S^3 then $f \circ g$ is also NU-preserving.*

As we will soon see having an NU-preserving one-step chain from the third power of a relational structure to the set of constant triples implies the existence of a set of Freese-McKenzie operations. The converse implication is not yet known, so it would seem this is a stronger condition.

7.2 One-step Chain on Substructure

The following lemmas help in the construction of one-step chains by allowing us to do so in segments.

Lemma 7.2.1. *Let $\mathbb{R}$ be a relational structure and $\mathbb{R}' \supseteq \mathbb{R}''$ two of its substructures. Suppose there exists a one-step chain from $\mathbb{R}$ to $\mathbb{R}'$ and another from $\mathbb{R}'$ to $\mathbb{R}''$. Then there exists a one-step chain from $\mathbb{R}$ to $\mathbb{R}''$.*

Proof. Let $r_0, \ldots, r_n$ be the one-step chain from $\mathbb{R}$ to $\mathbb{R}'$ and $s_0, \ldots, s_m$ be the one-step chain from $\mathbb{R}'$ to $\mathbb{R}''$. Define $r_{n+j} := s_j \circ r_n$ for $0 \leq j \leq m$. Note that we are not redefining r_n here since s_0 is the identity map on $\mathbb{R}'$, which happens to be the image of r_n. These are clearly retractions on $\mathbb{R}$ so we'll check that they form a one-step chain from $\mathbb{R}$ to $\mathbb{R}''$. Condition (3) is satisfied by construction. For $i < n$ condition (1) is satisfied from our assumption. Let $i = n + j$ for some $0 \leq j < m$ then

$$\begin{aligned} r_i(\mathbb{R}) &= s_j(r_n(\mathbb{R})) \\ &= s_j(\mathbb{R}') \\ &\supseteq s_{j+1}(\mathbb{R}') \\ &= s_{j+1}(r_n(\mathbb{R})) \\ &= r_{i+1}(\mathbb{R}). \end{aligned}$$

For condition (2) we only need to check it for $i = n + j$ for some $0 \le j < m$ as well. Let $a \in \{x \in r_i(\mathbb{R}) : r_{i+1}(x) \neq x\}$. Then

$$\begin{aligned} & r_i(a) = a \neq r_{i+1}(a) \\ \Longrightarrow\ & s_j(r_n(a)) = a \neq s_{j+1}(r_n(a)). \end{aligned}$$

However $a \in r_i(\mathbb{R}) \subseteq r_n(\mathbb{R})$ since $n \le i$. So we get $r_n(a) = a$ and

$$\begin{aligned} & s_j(a) = a \neq s_{j+1}(a) \\ \Longrightarrow\ & a \in \{x \in s_j(\mathbb{R}) : s_{j+1}(x) \neq x\}. \end{aligned}$$

Thus $|\{x \in r_i(\mathbb{R}) : r_{i+1}(x) \neq x\}| \le |\{x \in s_j(\mathbb{R}) : s_{j+1}(x) \neq x\}| \le 1$ as required. □

Lemma 7.2.2. *Let $\mathbb{R}$ be a relational structure and $\mathbb{R}' \supseteq \mathbb{R}''$ two of its substructures. Suppose there exists a one-step chain from $\mathbb{R}$ to $\mathbb{R}''$. Let r be a retraction on $\mathbb{R}$ whose image is $\mathbb{R}'$ such that for all $x \in R$ either $r(x) = x$ or $r(x) \in \mathbb{R}''$. Then there exists a one-step chain from $\mathbb{R}'$ to $\mathbb{R}''$.*

Proof. Let $r_0, \ldots, r_n$ be the one-step chain from $\mathbb{R}$ to $\mathbb{R}''$. We claim that $s_0, \ldots, s_n$ where $s_i := (r \circ r_i)|_{\mathbb{R}'}$ for $0 \le i \le n$ is a one-step chain from $\mathbb{R}'$ to $\mathbb{R}''$.

It should be clear from construction that the s_i's are homomorphisms from $\mathbb{R}'$ to $\mathbb{R}'$. Let $x \in \mathbb{R}'$. Then

$$s_i^2(x) = s_i(s_i(x)) = r(r_i(r(r_i(x)))).$$

We know that either $r(r_i(x)) = r_i(x)$ or $r(r_i(x)) \in \mathbb{R}''$. Both of these outcomes implies $r(r_i(x)) \in r_i(\mathbb{R})$. So $s_i^2(x) = r(r_i(r(r_i(x)))) = r(r(r_i(x)) = r(r_i(x)) = s_i(x)$. Therefore the s_i's are retractions.

Condition (3) is satisfied by construction.

Fix an $i \in \{0, \ldots, n-1\}$. Let $y \in s_{i+1}(\mathbb{R}')$ and choose $x \in \mathbb{R}'$ where $s_{i+1}(x) = y$. From our assumption r either fixes $r_{i+1}(x)$ or sends it into $\mathbb{R}''$. Once again this means that $y = s_{i+1}(x) = r(r_{i+1}(x)) \in r_{i+1}(\mathbb{R}) \subseteq r_i(\mathbb{R})$. Since $y \in r_i(\mathbb{R})$ we have $r_i(y) = y$ and $s_i(y) = r(r_i(y)) = r(y) = y$. This combined with $y \in \mathbb{R}'$ gives us $y \in s_i(\mathbb{R}')$. Therefore $s_{i+1}(\mathbb{R}') \subseteq s_i(\mathbb{R}')$ which satisfies condition (1).

Fix i as in the previous paragraph. Let $a \in \{x \in s_i(\mathbb{R}') : s_{i+1}(x) \neq x\}$. Then

$$\begin{aligned} & s_i(a) = a \neq s_{i+1}(a) \\ \Longrightarrow\ & r(r_i(a)) \neq r(r_{i+1}(a)) \\ \Longrightarrow\ & r_i(a) \neq r_{i+1}(a). \end{aligned}$$

Since r either fixes the output of r_i or sends it into $\mathbb{R}'' \subseteq r_i(\mathbb{R})$ we have

$$s_i(\mathbb{R}') = r(r_i(\mathbb{R}')) \subseteq r_i(\mathbb{R}') \subseteq r_i(\mathbb{R}).$$

Thus $a \in \{x \in r_i(\mathbb{R}) : r_{i+1}(x) \neq x\}$ and

$$|\{x \in s_i(\mathbb{R}') : s_{i+1}(x) \neq x\}| \leq |\{x \in r_i(\mathbb{R}) : r_{i+1}(x) \neq x\}| \leq 1.$$

Condition (2) is also satisfied. □

7.3 SD-join Operations

As promised we will show here that having an NU-preserving one-step chain implies the existence of Freese-McKenzie operations for general relational structures.

Lemma 7.3.1. *Let $\mathbb{R}$ be a relational structure. If there exists an NU-preserving one-step chain from $\mathbb{R}^3$ to (the substructure induced by) $\Delta(R^3)$ (the set of diagonal elements in R^3), then $\mathbb{R}$ admits a sequence of Freese-McKenzie SD-$\vee$ operations.*

Proof. Denote the first and third projection mapping on $\mathbb{R}^3$ as ϕ_1 and ϕ_3. Let $r_0, \ldots, r_n$ be the one-step chain described in the statement of the lemma. Define retractions $s_i := r_i \circ r_{i-1} \circ \cdots \circ r_0$ for $0 \leq i \leq n$. We'll check that $s_0, \ldots, s_n$ is also an NU-preserving one-step chain from $\mathbb{R}^3$ to $\Delta(R^3)$. This amounts to proving $s_i(\mathbb{R}^3) = r_i(\mathbb{R}^3)$ (composition of NU-preserving maps are NU-preserving), which we will do by induction. Starting with $s_0 = r_0$ as the base case we assume it to be true for some $i \in \{0, \ldots, n-1\}$. $s_{i+1}(\mathbb{R}^3) \subseteq r_{i+1}(\mathbb{R}^3)$ is clear by definition. Let $\vec{x} \in r_{i+1}(\mathbb{R}^3)$ (we'll use $\vec{\ }$ to denote 3-tuples). Then $\vec{x} \in r_i(\mathbb{R}^3) = s_i(\mathbb{R}^3)$ so there exists $\vec{y} \in \mathbb{R}^3$ such that $s_i(\vec{y}) = \vec{x}$. Therefore $\vec{x} = r_{i+1}(\vec{x}) = r_{i+1}(s_i(\vec{y})) = s_{i+1}(\vec{y}) \in s_{i+1}(\mathbb{R}^3)$ and we get $r_{i+1}(\mathbb{R}^3) \subseteq s_{i+1}(\mathbb{R}^3)$.

The Freese-McKenzie SD-$\vee$ operations $t_1, \ldots, t_{2n}$ will be defined as $t_i := \phi_1 \circ s_i$ and $t_{n+i} := \phi_3 \circ s_{n-i}$ for $0 \leq i \leq n$. Note that we are not defining t_n twice since $s_n(\mathbb{R}^3)$ contains only diagonal elements. Being the composition of two homomorphisms each t_j will be a homomorphism from $\mathbb{R}^3$ to $\mathbb{R}$. We just have to confirm that the sequence satisfies the Freese-McKenzie conditions.

Since s_0 is the identity mapping on $\mathbb{R}^3$, $t_0 = \phi_1$ and $t_{2n} = \phi_3$. This proves the first requirement of Freese-McKenzie SD-$\vee$ operations.

Fix an $i \in \{0, \ldots, n-1\}$. We'll first show that s_i and s_{i+1} satisfy two of the following conditions:

1. $s_i(x,x,y) = s_{i+1}(x,x,y)$ for all $x,y \in R$;
2. $s_i(x,y,y) = s_{i+1}(x,y,y)$ for all $x,y \in R$;
3. $s_i(x,y,x) = s_{i+1}(x,y,x)$ for all $x,y \in R$.

From construction we have $s_{i+1} = r_{i+1} \circ s_i$. Recall that r_{i+1} fixes all but at most one 3-tuple from $r_i(\mathbb{R}^3) = s_i(\mathbb{R}^3)$. Let (a,b,c) be the unique 3-tuple not fixed by r_{i+1} in $s_i(\mathbb{R}^3)$ should it exist. If it doesn't then $s_{i+1} = s_i$ and we are done. Then for all $(x,y,z) \in \mathbb{R}^3$ that is not an s_i preimage of (a,b,c)

$$s_i(x,y,z) = r_{i+1}(s_i(x,y,z)) = s_{i+1}(x,y,z).$$

If a 3-tuple of the form (u,u,v) is in the s_i preimage of (a,b,c) then since s_i is NU-preserving we must have $a = b$. If another 3-tuple of the form (s,t,t) or (t,s,t) is also in the s_i preimage of (a,b,c) then we would have $a = b = c$. This would mean that (a,b,c) is a diagonal element, which is a contradiction since it would be in the image of r_{i+1} and therefore fixed by it. Therefore conditions (2) and (3) from above are satisfied. Similarly if we began with a 3-tuple of the form (v,u,u) or (u,v,u) belonging to the s_i preimage of (a,b,c) then conditions (1) and (3) or (1) and (2) would be satisfied.

Since each pair of consecutive s_i's satisfies two out of the three conditions, composing them both with either ϕ_1 or ϕ_3 shows that each pair of consecutive t_j's for $0 \leq j < 2n$ satisfies two out of three conditions also. This proves the second requirement of Freese-McKenzie SD-$\vee$ operations. □

7.4 Constructing One-step Chains

Finally we begin the lengthy process of constructing NU-preserving one-step chains on the third power of series-parallel posets satisfying the 4-crown condition. The entire procedure is divided up into pieces to allow for easier digestion.

Recall that all series-parallel posets satisfying the 4-crown condition can be constructed from **1** using the operations $+^R$, $\triangle$, $\triangledown$, $\Diamond$, $⧖$ and $\uplus$ (Proposition 4.3.12). Our proof will be inductive along such a construction of the poset. For every induction step we will assume our poset $\mathbb{P}$ is constructed from $\mathbb{P}_1$ and $\mathbb{P}_2$ using one of the aforementioned operations where $\mathbb{P}_1$ and $\mathbb{P}_2$ each possess the desired one-step chain. Using the existing two NU-preserving one-step chains we will then create one for $\mathbb{P}$.

To start we will first construct one-step chains for $\mathbb{P}_1^3 \star \mathbb{P}_2^3$ where $\star \in \{+, \uplus, \triangle, \triangledown, ⧖\}$. Note that $\mathbb{P}_1^3 \star \mathbb{P}_2^3$ is almost never the entirety of $(\mathbb{P}_1 \star \mathbb{P}_2)^3$ but a subposet of it. It is still important as we will be using Lemma 7.2.1 to connect one-step chains from different subposets of $\mathbb{P}^3$. The $\Diamond$ operation is left out here as its case is not as straightforward and requires a different approach.

Lemma 7.4.1. *If $\mathbb{P}_1$ and $\mathbb{P}_2$ are series-parallel posets satisfying the 4-crown condition such that there exists an NU-preserving one-step chain from $\mathbb{P}_k^3$ to $\Delta(P_k^3)$ for each $k \in \{1,2\}$, then there exists an NU-preserving one-step chain from $\mathbb{P}_1^3 \triangle \mathbb{P}_2^3$ to $\Delta((P_1 \cup P_2)^3) = \Delta(P_1^3) \cup \Delta(P_2^3)$ (when requirements of $\triangle$ are satisfied).*

Proof. Denote each one-step chain from $\mathbb{P}_k^3$ to $\Delta(P_k^3)$ as $r_1^k, \dots, r_{n_k}^k$. We have noted after the definition of $\triangle$ that we'll view the universe of $\mathbb{P}_1^3 \triangle \mathbb{P}_2^3$ as $P_1^3 \cup P_2^3$. Since the requirements of $\triangle$ are assumed to be satisfied each $\mathbb{P}_k^3$ has a unique maximum. This implies each $\mathbb{P}_k$ has a unique maximum as well and its corresponding diagonal 3-tuple is the maximal of $\mathbb{P}_k^3$. Thus the unique maximum of $\mathbb{P}_1^3 \triangle \mathbb{P}_2^3$ is the diagonal 3-tuple that is the only 3-tuple in both $\mathbb{P}_k^3$'s.

For $i \in \{0, \dots, n_1\}$ define $r_i : \mathbb{P}_1^3 \triangle \mathbb{P}_2^3 \to \mathbb{P}_1^3 \triangle \mathbb{P}_2^3$ as follows:

$$r_i(x,y,z) = \begin{cases} r_i^1(x,y,z) & \text{if } (x,y,z) \in P_1^3 \\ (x,y,z) & \text{else.} \end{cases}$$

For $j \in \{0, \dots, n_2\}$ define $r_{n+j} : \mathbb{P}_1^3 \triangle \mathbb{P}_2^3 \to \mathbb{P}_1^3 \triangle \mathbb{P}_2^3$ as follows:

$$r_{n_1+j}(x,y,z) = \begin{cases} r_j^2(x,y,z) & \text{if } (x,y,z) \in P_2^3 \\ r_{n_1}(x,y,z) & \text{else.} \end{cases}$$

Note that r_{n_1} is not being defined twice since r_0^2 is the identity map on $\mathbb{P}_2^3$.

We will first check that these maps are order preserving. Let $(a,b,c) < (u,v,w)$ be two 3-tuples in $P_1^3 \cup P_2^3$. Since they are comparable they must both belong to some P_k^3. If it is P_1^3 then

$$r_i(a,b,c) = r_i^1(a,b,c) \leq r_i^1(u,v,w) = r_i(u,v,w).$$

Similarly if (a,b,c) and (u,v,w) both belong to P_2^3 then

$$r_i(a,b,c) = (a,b,c) < (u,v,w) = r_i(u,v,w).$$

For the very last equality of the above equation it is possible for (u,v,w) to belong to P_2^3 as well but this would only imply (u,v,w) is the unique maximal diagonal element which is fixed by every r_i^1. Thus each r_i is order preserving. Likewise so are the r_{n_1+j}'s.

Now we have to show that these maps are retractions. Let $(a,b,c) \in P_1^3$. Then since each r_i^1 is a retraction on $\mathbb{P}_i^3$ we have

$$r_i \circ r_i(a,b,c) = r_i \circ r_i^1(a,b,c) = r_i^1 \circ r_i^1(a,b,c) = r_i^1(a,b,c) = r_i(a,b,c).$$

If $(a,b,c) \in P_2^3 \backslash P_1^3$ then

$$r_i \circ r_i(a,b,c) = r_i \circ (a,b,c) = (a,b,c).$$

Thus each r_i is a retraction. Likewise so are the r_{n_1+j}'s.

These maps are NU-preserving since every r_i^1 and r_j^2 is NU-preserving. It should be clear from construction that for each $i \in \{0,\dots,n\}$ and $j \in \{0,\dots,m\}$ we have $r_i(\mathbb{P}_1^3 \triangle \mathbb{P}_2^3) = r_i^1(\mathbb{P}_1^3) \cup \mathbb{P}_2^3$ and $r_{n_1+j}(\mathbb{P}_1^3 \triangle \mathbb{P}_2^3) = r_{n_1}(\mathbb{P}_1^3) \cup r_j^2(\mathbb{P}_2^3)$. This makes it easy to see that all together $r_1,\dots,r_{n_1},\dots,r_{n_1+n_2}$ form a one-step chain from $\mathbb{P}_1^3 \triangle \mathbb{P}_2^3$ to $\Delta(P_1^3) \cup \Delta(P_2^3)$ as required. □

Lemma 7.4.2. *If $\mathbb{P}_1$ and $\mathbb{P}_2$ are series-parallel posets satisfying the 4-crown condition such that there exists an NU-preserving one-step chain from $\mathbb{P}_k^3$ to $\Delta(P_k^3)$ for each $k \in \{1,2\}$, then there exists an NU-preserving one-step chain from $\mathbb{P}_1^3 \triangledown \mathbb{P}_2^3$ to $\Delta((P_1 \cup P_2)^3) = \Delta(P_1^3) \cup \Delta(P_2^3)$ (when requirements of $\triangledown$ are satisfied).*

Proof. The proof of this lemma is similar to that of Lemma 7.4.1. □

Lemma 7.4.3. *If $\mathbb{P}_1$ and $\mathbb{P}_2$ are series-parallel posets satisfying the 4-crown condition such that there exists an NU-preserving one-step chain from $\mathbb{P}_k^3$ to $\Delta(P_k^3)$ for each $k \in \{1,2\}$, then there exists an NU-preserving one-step chain from $\mathbb{P}_1^3 ⧖ \mathbb{P}_2^3$ to $\Delta((P_1 \cup P_2)^3) = \Delta(P_1^3) \cup \Delta(P_2^3)$ (when requirements of ⧖ are satisfied).*

Proof. Denote each one-step chain from $\mathbb{P}_k^3$ to $\Delta(P_k^3)$ as $r_1^k,\dots,r_{n_k}^k$. We will view the universe of $\mathbb{P}_1^3 ⧖ \mathbb{P}_2^3$ as $P_1^3 \cup P_2^3$ as usual. Denote the unique diagonal 3-tuple that belongs to both $\mathbb{P}_1^3$ and $\mathbb{P}_2^3$ as (μ,μ,μ) (Revisit the first paragraph of the proof of Lemma 7.4.1 to see the existence and uniqueness of such a 3-tuple).

For $i \in \{0,\dots,n_1\}$ define $r_i : \mathbb{P}_1^3 ⧖ \mathbb{P}_2^3 \to \mathbb{P}_1^3 ⧖ \mathbb{P}_2^3$ as follows:

$$r_i(x,y,z) = \begin{cases} r_i^1(x,y,z) & \text{if } (x,y,z) \in P_1^3 \\ (x,y,z) & \text{else.} \end{cases}$$

For $j \in \{0, \dots, n_2\}$ define $r_{n+j} : \mathbb{P}_1^3 \mathbin{\rotatebox[origin=c]{90}{$\bowtie$}} \mathbb{P}_2^3 \to \mathbb{P}_1^3 \mathbin{\rotatebox[origin=c]{90}{$\bowtie$}} \mathbb{P}_2^3$ as follows:

$$r_{n_1+j}(x, y, z) = \begin{cases} r_j^2(x, y, z) & \text{if } (x, y, z) \in P_2^3 \\ r_{n_1}(x, y, z) & \text{else.} \end{cases}$$

Note that r_{n_1} is not being defined twice since r_0^2 is the identity map on $\mathbb{P}_2^3$. This construction is similar to what we have done in Lemma 7.4.1. As such, the proof that this is an NU-preserving one step chain from $\mathbb{P}_1^3 \mathbin{\rotatebox[origin=c]{90}{$\bowtie$}} \mathbb{P}_2^3$ to $\Delta(P_1^3) \cup \Delta(P_2^3)$ is also similar. The only noteworthy part unique to this case is to show order preservation when we have $(a, b, c) < (u, v, w)$ where (a, b, c) and (u, v, w) belong to different P_k^3's. In this case $(a, b, c) \in P_1^3$ and $(u, v, w) \in P_2^3 \backslash P_1^3$. Since every r_i^1 is a retraction on $\mathbb{P}_1^3$ and every 3-tuple in $\mathbb{P}_1^3$ is below every 3-tuple from $\mathbb{P}_2^3$ we get

$$r_i(a, b, c) = r_i^1(a, b, c) \leq (u, v, w) = r_i(u, v, w).$$

Thus each r_i is a order preserving. Then using similar arguments we can show each r_{n_1+j} is also order preserving. □

Lemma 7.4.4. *If $\mathbb{P}_1$ and $\mathbb{P}_2$ are series-parallel posets satisfying the 4-crown condition such that there exists an NU-preserving one-step chain from $\mathbb{P}_k^3$ to $\Delta(P_k^3)$ for each $k \in \{1, 2\}$, then there exists an NU-preserving one-step chain from $\mathbb{P}_1^3 + \mathbb{P}_2^3$ to $\Delta((P_1 \cup P_2)^3) = \Delta(P_1^3) \cup \Delta(P_2^3)$.*

Proof. The proof of this lemma is similar to that of Lemma 7.4.3. □

Lemma 7.4.5. *If $\mathbb{P}_1, \dots, \mathbb{P}_n$ are series-parallel posets satisfying the 4-crown condition such that there exists an NU-preserving one-step chain from $\mathbb{P}_k^3$ to $\Delta(P_k^3)$ for each $k \in \{1, \dots, n\}$, then there exists an NU-preserving one-step chain from $\mathbb{P}_1^3 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} \mathbb{P}_n^3$ to $\Delta((P_1 \cup \cdots \cup P_n)^3) = \Delta(P_1^3) \cup \cdots \cup \Delta(P_n^3)$.*

Proof. Denote each one-step chain from $\mathbb{P}_k^3$ to $\Delta(P_k^3)$ as $r_1^k, \dots, r_{n_k}^k$. Let $n_0 = 0$.

For each $k \in \{0, \dots, n\}$ and each $i_k \in \{0, \dots, n_k\}$ define:

$$r_{\left(\sum_{l=0}^{k-1} n_l + i_k\right)}(x, y, z) = \begin{cases} r_{i_k}^k(x, y, z) & \text{if } (x, y, z) \in P_k^3 \\ r_{\left(\sum_{l=0}^{k-1} n_l\right)}(x, y, z) & \text{else.} \end{cases}$$

It is fairly easy to check that this is the desired one-step chain. □

Now we turn to the task of constructing retractions from $(\mathbb{P}_1 \star \mathbb{P}_2)^3$ to $\mathbb{P}_1^3 \star \mathbb{P}_2^3$. Due to the lemmas we have proven above, for all cases except where $\star = \Diamond$ we only need to build an NU-preserving one-step chain to a proper subset of $\mathbb{P}_1^3 \star \mathbb{P}_2^3$ then apply Lemma 7.2.2.

We want to construct our retractions so that each consecutive one differs from the prior at exactly one input. The map in the following definition allows us to do just that. Furthermore if we are careful in selecting the new output we can easily ensure the new map is also a retraction.

Definition 7.4.6. Let S be a set and $a, b \in S$. Define the map $I_{a \to b} : S \mapsto S$ as follows:

$$I_{a\to b}(x) = \begin{cases} b & \text{if } x = a \\ x & \text{otherwise.} \end{cases}$$

Lemma 7.4.7. *[10, Lemma 6.5] Let $\mathbb{P}$ be a poset and $r : \mathbb{P} \mapsto \mathbb{P}$ a retraction on $\mathbb{P}$. Suppose there exist $a, b \in r(\mathbb{P})$ such that b is either the unique upper cover or the unique lower cover of a in $r(\mathbb{P})$. Then $r' := I_{a\to b} \circ r$ is a retraction on $\mathbb{P}$. Moreover we have $r'(\mathbb{P}) \subseteq r(\mathbb{P})$ and $|\{x \in r(\mathbb{R}) : r'(x) \neq x\}| \leq 1$.*

Proof. We'll assume b is the unique upper cover of a since the proof for the other case is analogous.

First we'll check that r' is a homomorphism (order preserving map). Let $x < y$ be elements of $\mathbb{P}$. If neither $r(x)$ nor $r(y)$ equals a then $r'(x) = r(x) \leq r(y) = r'(x)$ since r is a homomorphism. If $r(x) = a$ then since $r(x) \leq r(y)$ we have $b \leq r(y)$ as well. So $r'(x) = b \leq r(y) = r'(y)$. If $r(y) = a$ then

$$r'(x) = r(x) \leq r(y) = a \leq b = r'(y).$$

Thus r' is a homomorphism.

$r'(\mathbb{P}) \subseteq r(\mathbb{P})$ and $|\{x \in r(\mathbb{R}) : r'(x) \neq x\}| \leq 1$ should be clear since $a, b \in r(\mathbb{P})$.

Let $x = r'(y) \in r'(\mathbb{P}) \subseteq r(\mathbb{P})$. By definition of r', $x \neq a$. Thus

$$r'(x) = I_{a\to b}(r(x)) = r(x) = x.$$

Thus r' is a retraction. □

Each of the following lemmas will be the proof of one of the inductive steps, $\Diamond$ included.

Lemma 7.4.8. *Let $\mathbb{P}$ be a series-parallel poset satisfying the 4-crown condition. Suppose $\mathbb{P} = \mathbb{P}_1 \triangle \mathbb{P}_2$ (or $\mathbb{P}_1 \triangledown \mathbb{P}_2$) for some series-parallel posets $\mathbb{P}_1$ and $\mathbb{P}_2$ satisfying the 4-crown condition and there exist NU-preserving one-step chains from $\mathbb{P}_1^3$ to $\Delta(P_1^3)$ and $\mathbb{P}_2^3$ to $\Delta(P_2^3)$. Then there exists an NU-preserving one-step chain from $\mathbb{P}^3$ to $\Delta(P^3)$.*

Proof. We will only show the proof for when $\mathbb{P} = \mathbb{P}_1 \triangle \mathbb{P}_2$ as the case for $\triangledown$ is similar.

Let $\mu \in P$ denote the unique maximal element that is also the maximum of $\mathbb{P}_1$ and $\mathbb{P}_2$ as induced subposets. We will construct NU-preserving one-step chains from subposets to subposets then link them together using Lemma 7.2.1.

Let T_1 be the set of 3-tuples in P^3 where exactly two of their coordinates are equal to μ (e.g. (μ, μ, x) for some $x \in P$, $x \neq \mu$). Let $S_1 := P^3 \backslash T_1$ and denote $\mathbb{S}_1$ as its induced subposet. Our first one-step chain will be from $\mathbb{P}^3$ to $\mathbb{S}_1$.

Let r_0 be the identity map on $\mathbb{P}^3$. Assume there exists $r_0, \dots, r_i$ an NU-preserving one-step chain from $\mathbb{P}^3$ to $r_i(\mathbb{P}^3) \supseteq \mathbb{S}_1$. If $r_i(\mathbb{P}^3) = \mathbb{S}_1$ then we have the chain we wanted, so let's assume not. Pick (a, b, c) to be maximal in $r_i(\mathbb{P}^3) \backslash \mathbb{S}_1$. By definition of $\mathbb{S}_1$, (a, b, c) must have exactly two coordinates equal to μ. Without loss of generality let's assume $(a, b, c) = (\mu, \mu, c)$. We will show that (μ, μ, μ) is the unique upper cover of (μ, μ, c) in $r_i(\mathbb{P}^3)$. (μ, μ, μ) has μ in all three coordinates so it does not belong to T_1. Thus it is in $S_1 \subset r_i(\mathbb{P}^3)$. Let $(u, v, w) \in r_i(\mathbb{P}^3)$ be such that $(u, v, w) > (\mu, \mu, \mu)$. Since μ is the maximum element of $\mathbb{P}$ we must have $u = \mu = v$. Then by the maximality of $(a, b, c) = (\mu, \mu, c)$ we have $w = \mu$ also. Then by Lemma 7.4.7

$$r_{i+1} := I_{(\mu,\mu,c)\to(\mu,\mu,\mu)} \circ r_i$$

is a retraction on $\mathbb{P}^3$ such that $r_{i+1}(\mathbb{P}^3) \subseteq r_i(\mathbb{P}^3)$ and $|\{x \in r_i(\mathbb{P}^3) : r_{i+1}(x) \neq x\}| \leq 1$. $I_{(\mu,\mu,c)\to(\mu,\mu,\mu)}$ is clearly NU-preserving which makes r_{i+1} NU-preserving also. Thus $r_0, \dots, r_{i+1}$ is an NU-preserving one-step chain.

Since the image of each newly constructed retraction is strictly decreasing in size, this process will provide us with the one-step chain we want after finitely many steps.

Now let T_2 be the set of 3-tuples in P^3 where exactly one of their coordinates equals to μ. Let $S_2 := S_1 \backslash T_2$ and denote its induced subposet as $\mathbb{S}_2$. We will construct an NU-preserving one-step chain from $\mathbb{S}_1$ to $\mathbb{S}_2$.

Let r_0 be the identity map on $\mathbb{S}_1$. Assume there exists $r_0, \dots, r_i$ an NU-preserving one-step chain from $\mathbb{S}_1$ to $r_i(\mathbb{S}_1) \supseteq \mathbb{S}_2$. If $r_i(\mathbb{S}_1) = \mathbb{S}_2$ then the construction is finished so let's assume not. Pick a maximal (a, b, c) in $r_i(\mathbb{S}_1) \backslash \mathbb{S}_2$. By definition exactly one of a, b or c equals to μ. Without loss of generality let's assume it is a. We will show that (μ, μ, μ) is the unique upper cover of (a, b, c) in $r_i(\mathbb{S}_1)$. From above we know that it belongs to S_1. Since it clearly does not belong to T_2 it must be in $S_2 \subseteq r_i(\mathbb{S}_1)$. Suppose there exists in

$r_i(\mathbb{S}_1)$ some $(u,v,w) > (a,b,c)$. Since $a = \mu$ we must have $u = \mu$ also. By the maximality of (a,b,c), (u,v,w) must have more than one coordinate equal to μ. The only 3-tuple in $\mathbb{S}_1$ with such description is the diagonal (μ,μ,μ). Thus we see that (μ,μ,μ) is not only the unique upper cover of (a,b,c) it is also the only element greater than it in $r_i(\mathbb{S}_1)$. By Lemma 7.4.7

$$r_{i+1} := I_{(\mu,b,c)\to(\mu,\mu,\mu)} \circ r_i$$

is a retraction on $\mathbb{S}_1$ such that $r_{i+1}(\mathbb{S}_1) \subseteq r_i(\mathbb{S}_1)$ and $|\{x \in r_i(\mathbb{S}_1) : r_{i+1}(x) \neq x\}| \leq 1$. Just as before this retraction is clearly NU-preserving so we have an NU-preserving one-step chain $r_0, \ldots, r_{i+1}$.

Repeating this process as needed, we will get a chain from $\mathbb{S}_1$ to $\mathbb{S}_2$ within a finite amount of steps.

The next subposet will be $\mathbb{S}_3$ which is induced by $S_3 := (P_1^3 \cup P_2^3) \cap S_2$.

Let r_0 be the identity map on $\mathbb{S}_2$. Assume there exists $r_0, \ldots, r_i$ an NU-preserving one-step chain from $\mathbb{S}_2$ to $r_i(\mathbb{S}_2) \supseteq \mathbb{S}_3$. If $r_i(\mathbb{S}_2) = \mathbb{S}_3$ then we are done let's assume not. Pick (a,b,c) to be maximal in $r_i(\mathbb{S}_2)\backslash\mathbb{S}_3$. This means $(a,b,c) \notin P_1^3 \cup P_2^3$ so it must have one coordinate in each of $P_1\backslash\{\mu\}$ and $P_2\backslash\{\mu\}$. Without loss of generality let's assume $a \in P_1\backslash\{\mu\}$ and $c \in P_2\backslash\{\mu\}$. We will show that (μ,μ,μ) is the unique upper cover of (a,b,c) in $r_i(\mathbb{S}_2)$. From above we have $(\mu,\mu,\mu) \in S_2$, and since it is also in $P_1^3 \cup P_2^3$ it belongs to $S_3 \subseteq r_i(\mathbb{S}_2)$ as well. Now suppose there exists in $r_i(\mathbb{S}_2)$ some $(u,v,w) > (a,b,c)$. By the maximality of (a,b,c) one of the P_i's must contain all of u, v and w. But $a \in P_1$ and $c \in P_2$ implies $u \in P_1$ and $w \in P_2$. So one of u or w must equal μ. However the only 3-tuple in S_2 that has μ as any of its coordinates is the diagonal (μ,μ,μ). Thus (μ,μ,μ) is the unique upper cover of (a,b,c) in $r_i(\mathbb{S}_2)$. By Lemma 7.4.7

$$r_{i+1} := I_{(a,b,c)\to(\mu,\mu,\mu)} \circ r_i$$

is a retraction on $\mathbb{S}_2$ such that $r_{i+1}(\mathbb{S}_2) \subseteq r_i(\mathbb{S}_2)$ and $|\{x \in r_i(\mathbb{S}_2) : r_{i+1}(x) \neq x\}| \leq 1$. This is clearly NU-preserving so we get an NU-preserving one-step chain $r_0, \ldots, r_{i+1}$.

After finitely many steps we get an NU-preserving one-step chain from $\mathbb{S}_2$ to $\mathbb{S}_3$.

□

Lemma 7.4.9. *Let $\mathbb{P}$ be a series-parallel poset satisfying the 4-crown condition. Suppose $\mathbb{P} = \mathbb{P}_1 \between \mathbb{P}_2$ for some series-parallel posets $\mathbb{P}_1$ and $\mathbb{P}_2$ satisfying the 4-crown condition and there exist NU-preserving one-step chains from $\mathbb{P}_1^3$ to $\Delta(P_1^3)$ and $\mathbb{P}_2^3$ to $\Delta(P_2^3)$. Then there exists an NU-preserving one-step chain from $\mathbb{P}^3$ to $\Delta(P^3)$.*

Proof. Let $\mu \in P$ denote the element that is both the unique maximum of $\mathbb{P}_1$ as well as the unique minimum of $\mathbb{P}_2$. As always we will construct the one-step chain by going from subposets to subposets.

Recall that $P = P_1 \cup P_2'$ where $P_2' = P_2 \setminus \{\mu\}$. Let T_1 be the set of 3-tuples in P^3 such that exactly one of its coordinates is in $P_1 \backslash \{\mu\}$ and let $S_1 := P^3 \backslash T_1$. Denote $\mathbb{S}_1$ to be the subposet of $\mathbb{P}^3$ induced by S_1. We will first construct an NU-preserving one-step chain from $\mathbb{P}^3$ to $\mathbb{S}_1$.

Denote r_0 to be the identity map on $\mathbb{P}^3$. Recursively let us assume there exists $r_0, \ldots, r_i$ an NU-preserving one-step chain from $\mathbb{P}^3$ to $r_i(\mathbb{P}^3)$ such that $\mathbb{S}_1 \subseteq r_i(\mathbb{P}^3)$. If $\mathbb{S}_1 = r_i(\mathbb{P}^3)$ then we have what we wanted. If not then let (a, b, c) be a maximal 3-tuple of $r_i(\mathbb{P}^3) \backslash \mathbb{S}_1$. By definition it must be true that exactly one of a, b or c is in $P_1 \backslash \{\mu\}$. Without loss of generality let us assume $c \in P_1 \backslash \{\mu\}$. Suppose there exists $(u, v, w) \in r_i(\mathbb{P}^3)$ such that $(a, b, c) < (u, v, w)$. Since $a, b \in P_2' \cup \{\mu\}$ then $u, v \in P_2' \cup \{\mu\}$ as well. By maximality of (a, b, c), w must be in $P_2' \cup \{\mu\}$ or more importantly $w \geq \mu$. Thus $(u, v, w) \geq (a, b, \mu)$. Clearly (a, b, μ) is not an element of T_1 so it must be in S_1 in the image of r_i. Combining with the fact that $c \neq \mu$ shows (a, b, μ) is the unique upper cover of (a, b, c) in $r_i(\mathbb{P}^3)$. By Lemma 7.4.7

$$r_{i+1} := I_{(a,b,c) \to (a,b,\mu)} \circ r_i$$

is a retraction on $\mathbb{P}^3$ such that $r_{i+1}(\mathbb{P}^3) \subseteq r_i(\mathbb{P}^3)$ and $|\{x \in r_i(\mathbb{P}^3) : r_{i+1}(x) \neq x\}| \leq 1$. Due to assumption on c the only way (a, b, c) can be an NU 3-tuple is if $a = b$. In this case $r_{i+1}(a, a, c) = (a, a, \mu)$. Since r_{i+1} behaves the same way on every other input as r_i and r_i is NU-preserving, r_{i+1} must also be NU-preserving. Thus we get an NU-preserving one-step chain from $\mathbb{P}^3$ to $r_{i+1}(\mathbb{P}^3)$ in $r_0, \ldots, r_{i+1}$.

Since series-parallel posets are finite and the image of each new retraction is strictly decreasing this process will yield a one-step chain from $\mathbb{P}$ to $\mathbb{S}_1$ in finitely many steps.

For the next step define T_2 to be the set of all 3-tuples in S_1 such that exactly one of its coordinates belongs to $P_2 \backslash \{\mu\}$. Let $S_2 := S_1 \backslash T_2$ and denote $\mathbb{S}_2$ as the induced subposet. We now construct an NU-preserving one-step chain from $\mathbb{S}_1$ to $\mathbb{S}_2$.

As before denote r_0 to be the identity map on $\mathbb{S}_1$ and assume there exists $r_0, \ldots, r_i$ an NU-preserving one-step chain from $\mathbb{S}_1$ to $r_i(\mathbb{S}_1)$ such that $\mathbb{S}_2 \subseteq r_i(\mathbb{S}_1)$. If $r_i(\mathbb{S}_1) = \mathbb{S}_2$ then we have the one-step chain that we wanted. Assume that is not the case and let (a, b, c) be a minimal element in $r_i(\mathbb{S}_1) \backslash \mathbb{S}_2$. By arguments similar to the previous step we have that (a, b, μ) is the unique lower cover of (a, b, c) in $r_i(\mathbb{S}_1)$. Thus by Lemma 7.4.7 we get

$$r_{i+1} := I_{(a,b,c) \to (a,b,\mu)} \circ r_i$$

as a retraction on $\mathbb{S}_1$ such that $r_{i+1}(\mathbb{S}_1) \subseteq r_i(\mathbb{S}_1)$ and $|\{x \in r_i(\mathbb{S}_1) : r_{i+1}(x) \neq x\}| \leq 1$. The fact that r_{i+1} is NU-preserving can be checked just as before. Thus we get an NU-preserving one-step chain from $\mathbb{S}_1$ to $r_{i+1}(\mathbb{S}_1)$. Once again the image of the final retraction will be $\mathbb{S}_2$ after finitely many steps.

Combining the two one-step chains together we get a single chain from $\mathbb{P}^3$ to $\mathbb{S}_2 \subseteq \mathbb{P}_1^3 \between \mathbb{P}_2^3$. Let r denote the final retraction on this chain whose image is exactly $\mathbb{S}_2$.

Note that $S_2 \subseteq P_1^3 \cup P_2^3$ and $\mathbb{S}_2$ is an induced subposet of $\mathbb{P}_1^3 \between \mathbb{P}_2^3$. Let (a,b,c) be an element in $\mathbb{P}_1^3 \between \mathbb{P}_2^3$ but not of $\mathbb{S}_2$. Then either $(a,b,c) \in \mathbb{P}_1^3$ or $(a,b,c) \in \mathbb{P}_2^3$. In the first case since $(a,b,c) \notin \mathbb{S}_2$ it must be in T_1 (no element of P_1 can be in $P_2 \backslash \{\mu\}$). Thus exactly one of a, b or c is in $P_1 \backslash \{\mu\}$. This means the other two must be equal to μ. It doesn't matter which is which since it would all imply $r(a,b,c) = (\mu,\mu,\mu) \in \Delta(P^3)$. The same is true for the second case. Since r fixes all of $\mathbb{S}_2$ we have shown that r either fixes a 3-tuple in $P_1^3 \cup P_2^3$ or maps it into $\Delta(P^3)$. Now we can using Lemma 7.4.3 and 7.2.2 to get an NU-preserving one-step chain from $\mathbb{S}_2$ to $\Delta(P^3)$. □

Lemma 7.4.10. *Let $\mathbb{P}$ be a series-parallel poset satisfying the 4-crown condition. Suppose $\mathbb{P} = \mathbb{P}_1 \lozenge \mathbb{P}_2$ for some series-parallel posets $\mathbb{P}_1$ and $\mathbb{P}_2$ satisfying the 4-crown condition and there exist NU-preserving one-step chains from $\mathbb{P}_1^3$ to $\Delta(P_1^3)$ and $\mathbb{P}_2^3$ to $\Delta(P_2^3)$. Then there exists an NU-preserving one-step chain from $\mathbb{P}^3$ to $\Delta(P^3)$.*

Proof. Denote the NU-preserving one-step chains from $\mathbb{P}_1^3$ to $\Delta(P_1^3)$ and $\mathbb{P}_2^3$ to $\Delta(P_2^3)$ as $r_1^1, \ldots, r_n^1$ and $r_1^2, \ldots, r_m^2$ respectively. Denote the unique maximal and minimal elements of $\mathbb{P}$ as μ_1 and μ_0 respectively. We will be constructing the one-step chain from subsets to subsets in $\mathbb{P}^3$ then connecting them using Lemma 7.2.1.

Let T_1 be the set of 3-tuples in $\mathbb{P}^3$ that have μ_1 as at least one and at most two of its coordinates. Define $S_1 := P^3 \backslash T_1$ and let $\mathbb{S}_1$ be its induced subposet. We will construct an NU-preserving one-step chain from $\mathbb{P}^3$ to $\mathbb{S}_1$.

Let r_0 be the identity map on $\mathbb{P}^3$. Let's assume recursively there already exists $r_0, \ldots, r_i$ an NU-preserving one-step chain from $\mathbb{P}^3$ to $r_i(\mathbb{P}^3)$ such that $\mathbb{S}_1 \subseteq r_i(\mathbb{P}^3)$. If $\mathbb{S}_1 = r_i(\mathbb{P}^3)$ then we have the chain we needed. If not then pick (a,b,c) to be maximal in $r_i(\mathbb{P}^3) \backslash \mathbb{S}_1$. We will now show that (μ_1,μ_1,μ_1) is the unique upper cover of (a,b,c) in $r_i(\mathbb{P}^3)$. Since (μ_1,μ_1,μ_1) is diagonal it is clearly in $r_i(\mathbb{P}^3)$. Suppose there exists some other $(u,v,w) > (a,b,c)$ in $r_i(\mathbb{P}^3)$. By definition of $\mathbb{S}_1$ we see that (a,b,c) must be in T_1. So at least one of a, b or c equals to μ_1. This means that at least one of u, v or w must also be μ_1. By maximality of (a,b,c) we must have $(u,v,w) = (\mu_1,\mu_1,\mu_1)$. Thus (μ_1,μ_1,μ_1) is not only the unique upper cover of (a,b,c) but also the only element above it in $r_i(\mathbb{P}^3)$. So by Lemma 7.4.7

$$r_{i+1} := I_{(a,b,c)\to(\mu_1,\mu_1,\mu_1)} \circ r_i$$

is a retraction on $\mathbb{P}^3$ such that $r_{i+1}(\mathbb{P}^3) \subseteq r_i(\mathbb{P}^3)$ and $|\{x \in r_i(\mathbb{P}^3) : r_{i+1}(x) \neq x\}| \leq 1$. Since (μ_1, μ_1, μ_1) is diagonal, r_{i+1} is clearly NU-preserving. Thus $r_0, \ldots, r_{i+1}$ is an NU-preserving one-step chain.

Continue this process until the image of the final retraction is exactly $\mathbb{S}_1$. We see that the final retraction maps any 3-tuples in $\mathbb{P}^3$ that has μ_1 in one of its coordinates onto (μ_1, μ_1, μ_1).

Now let T_2 be the set of 3-tuples in $\mathbb{P}^3$ that have μ_0 as at least one and at most two of its coordinates. Define $S_2 := S_1 \backslash T_2$ and let $\mathbb{S}_2$ be its induced subposet. We will construct an NU-preserving one-step chain from $\mathbb{S}_1$ to $\mathbb{S}_2$.

Let r_0 be the identity map on $\mathbb{S}_1$. Let's assume recursively there already exists $r_0, \ldots, r_i$ an NU-preserving one-step chain from $\mathbb{S}_1$ to $r_i(\mathbb{S}_1)$ such that $\mathbb{S}_2 \subseteq r_i(\mathbb{S}_1)$. If $\mathbb{S}_2 = r_i(\mathbb{S}_1)$ then we have the chain and we are done. If not then pick (a, b, c) to be minimal in $r_i(\mathbb{S}_1) \backslash \mathbb{S}_2$. Using similar arguments to the previous construction we get that (μ_0, μ_0, μ_0) is the unique lower cover of (a, b, c) in $r_i(\mathbb{S}_1)$. So by Lemma 7.4.7

$$r_{i+1} := I_{(a,b,c) \to (\mu_0, \mu_0, \mu_0)} \circ r_i$$

will be the next retraction on the list.

Repeat this step as needed until we get an NU-preserving one-step chain from $\mathbb{S}_1$ to $\mathbb{S}_2$. Note that the final retraction will map all 3-tuples in S_1 that have μ_0 as one of its coordinates onto (μ_0, μ_0, μ_0).

Next let T_3 be the set of 3-tuples in S_2 that do not belong to $P_1^3 \cup P_2^3$. Let $S_3 := S_2 \backslash T_3$ and denote $\mathbb{S}_3$ to be its induced subposet.

Let r_0 be the identity map on $\mathbb{S}_2$. Suppose there exists $r_0, \ldots, r_i$ an NU-preserving one-step chain from $\mathbb{S}_2$ to $r_i(\mathbb{S}_2) \supseteq \mathbb{S}_3$. If $r_i(\mathbb{S}_2) = \mathbb{S}_3$ then we are done. If not then pick (a, b, c) to be maximal in $r_i(\mathbb{S}_2) \backslash \mathbb{S}_3$. We will show that (μ_1, μ_1, μ_1) is its unique upper cover in $r_i(\mathbb{S}_2)$. Suppose there exists some $(u, v, w) > (a, b, c)$ in $r_i(\mathbb{S}_2)$. Then by the maximality of (a, b, c), (u, v, w) must belong to $P_1^3 \cup P_2^3$. However since (a, b, c) is in T_3 it must have a coordinate exclusively in each of the P_i's. Since (u, v, w) has all three coordinates in the same P_i at least one of them must be μ_1. But the only 3-tuple in S_2 with μ_1 as one of its coordinates is the diagonal (μ_1, μ_1, μ_1). Thus (μ_1, μ_1, μ_1) is not only the unique upper cover of (a, b, c) but the only one above it in $r_i(\mathbb{S}_2)$. By Lemma 7.4.7

$$r_{i+1} := I_{(a,b,c) \to (\mu_1, \mu_1, \mu_1)} \circ r_i$$

is a retraction on $\mathbb{S}_2$ such that $r_{i+1}(\mathbb{S}_2) \subseteq r_i(\mathbb{S}_2)$ and $|\{x \in r_i(\mathbb{S}_2) : r_{i+1}(x) \neq x\}| \leq 1$. Since (μ_1, μ_1, μ_1) is diagonal r_{i+1} is clearly NU-preserving. Thus $r_0, \ldots, r_{i+1}$ is an NU-preserving one-step chain.

After finitely many iterations we will get an NU-preserving one step chain from $\mathbb{S}_2$ to $\mathbb{S}_3$. Note that $\mathbb{S}_3$ is fully contained in $\mathbb{P}^3_1 \lozenge \mathbb{P}^3_2$. Combine the three chains we have constructed so far using Lemma 7.2.1. Let r be the final retraction of the new chain that goes from $\mathbb{P}^3$ to $\mathbb{S}_3$.

The case for $\lozenge$ differs from the other operations in that we cannot use Lemma 7.2.2 to finish proof as we did in Lemma 7.4.8 and 7.4.9. This is due to the fact that we do not have the analogue of Lemma 7.4.1 and 7.4.3 for $\lozenge$. However the idea from Lemma 7.2.2 still works.

Finally we want to construct an NU-preserving one-step chain from $\mathbb{S}_3$ to $\Delta(P^3)$. Recall that we had assumed the existence of one-step chains $r^1_1, \dots, r^1_n$ from $\mathbb{P}^3_1$ to $\Delta(P^3_1)$ and $r^2_1, \dots, r^2_m$ from $\mathbb{P}^3_2$ to $\Delta(P^3_2)$. For $i \in \{0, \dots, n\}$ define $r_i : \mathbb{S}_3 \to \mathbb{S}_3$ as follows:

$$r_i(x, y, z) = \begin{cases} r \circ r^1_i(x, y, z) & \text{if } (x, y, z) \in P^3_1 \\ (x, y, z) & \text{else.} \end{cases}$$

For $j \in \{0, \dots, m\}$ define $r_{n+j} : \mathbb{S}_3 \to \mathbb{S}_3$ as follows:

$$r_{n+j}(x, y, z) = \begin{cases} r \circ r^2_j(x, y, z) & \text{if } (x, y, z) \in P^3_2 \\ r_n(x, y, z) & \text{else.} \end{cases}$$

Note that r_n is not being defined twice since r^2_0 is the identity map on $\mathbb{P}^3_2$. It should be clear that each of these maps are NU-preserving. We will check that $r_0, \dots, r_{n+m}$ is a one-step chain. First we need to confirm they are all order preserving. For $i \in \{0, \dots, n\}$ let's consider a pair of 3-tuples in $\mathbb{S}_3$ $(u, v, w) \geq (a, b, c)$. Suppose both of these are in P^3_1. By its construction r either fixes an element of $\mathbb{P}^3$ or maps it onto a diagonal (either (μ_0, μ_0, μ_0) or (μ_1, μ_1, μ_1)). Thus r maps P^3_i back into itself for each $i \in \{1, 2\}$. Since r^1_i and r are both order preserving we must have $r_i(u, v, w) \geq r_i(a, b, c)$. Suppose one of the 3-tuples is in P^3_1 while the other one is not. Then the only way they are comparable is if the one that is in P^3_1 contains only μ_0 and μ_1 as it's coordinates. The only 3-tuples in S_3 fitting this criteria are the diagonal tuples (μ_0, μ_0, μ_0) and (μ_1, μ_1, μ_1). Both of these are fixed by r and r^1_i. So either way we get $r_i(u, v, w) \geq r_i(a, b, c)$ again. The final consideration is where both 3-tuples are not in P^3_1. Then $r_i(u, v, w) \geq r_i(a, b, c)$ should be clear from the construction of r_i. The r_{n+j}'s for $j \in \{0, \dots, m\}$ are order preserving by similar arguments.

To see that these are retractions we will also only show the arguments for the r_i's. Let (a, b, c) be in S_3. If $(a, b, c) \notin P^3_1$ then we have trivially $r_i \circ r_i(a, b, c) = (a, b, c) = r_i(a, b, c)$. Suppose $(a, b, c) \in P^3_1$. Then $r_i(a, b, c) = r(r^1_i(a, b, c))$. We know that r either fixes $r^1_i(a, b, c)$ or maps it onto a diagonal tuple (x, x, x). If it is the latter then

$$r_i(r_i(a, b, c)) = r(r^1_i(r(r^1_i(a, b, c)))) = r(r^1_i(x, x, x)) = (x, x, x) = r_i(a, b, c).$$

If r fixes $r_i^1(a,b,c)$ then

$$r_i(r_i(a,b,c)) = r(r_i^1(r(r_i^1(a,b,c)))) = r(r_i^1(r_i^1(a,b,c))) = r(r_i^1(a,b,c)) = r_i(a,b,c).$$

Therefore each r_i is a retraction.

The image of each r_i equals to $r \circ r_i^1(S_3 \cap P_1^3) \cup (S_3 \backslash P_1^3)$ while the image of each r_{n+j} is $r \circ r_j^2(S_3 \cap P_2^3) \cup r_n(S_3 \backslash P_2^3)$. From this it is easy to see that $r_0, \ldots, r_n, \ldots, r_{n+m}$ is a one-step chain from $\mathbb{S}_3$ to $\Delta(P^3)$.

Connect all of the one-step chains we have constructed using Lemma 7.2.1 and we will have an NU-preserving one-step chain from $\mathbb{P}^3$ to $\Delta(P^3)$. □

The case for the restricted sum will be the most complicated out of all. Take a deep breath before you begin and remember to have fun!

Lemma 7.4.11. *Let $\mathbb{P}$ be a series-parallel poset satisfying the 4-crown condition. Suppose $\mathbb{P} = \mathbb{P}_1 +^R \mathbb{P}_2$ for some series-parallel posets $\mathbb{P}_1$ and $\mathbb{P}_2$ satisfying the 4-crown condition and there exist NU-preserving one-step chains from $\mathbb{P}_1^3$ to $\Delta(P_1^3)$ and $\mathbb{P}_2^3$ to $\Delta(P_2^3)$. Then there exists an NU-preserving one-step chain from $\mathbb{P}^3$ to $\Delta(P^3)$.*

Proof. By the definition of $+^R$ it must be true that either all pairs of maximal elements of $\mathbb{P}_1$ have an infimum or all pairs of minimal elements of $\mathbb{P}_2$ have a supremum. The proof will be similar in both cases so let us assume without loss of generality the latter. Then by Lemma 4.4.1 there exists a minimum pinch point of $\mathbb{P}_2$ that is the supremum of all minimal elements. Let $\mathbb{P}_3$ be the subposet of $\mathbb{P}_2$ induced by all elements below this minimum pinch point and $\mathbb{P}_4$ be induced by all those above it. Then we can break up $\mathbb{P}_2$ such that $\mathbb{P}_2 = \mathbb{P}_3 \mathbin{⧖} \mathbb{P}_4$. Clearly $\mathbb{P}_3$ and $\mathbb{P}_4$ are series-parallel posets satisfying the 4-crown condition. By restricting the NU-preserving one-step chain of $\mathbb{P}_2$ we get NU-preserving one-step chains on both of these subposets. Thus if we can construct an NU-preserving one-step chain for $\mathbb{P}_1 +^R \mathbb{P}_3$ then by Lemma 7.4.9 above we get one for $(\mathbb{P}_1 +^R \mathbb{P}_3) \mathbin{⧖} \mathbb{P}_4 = \mathbb{P}_1 +^R \mathbb{P}_2$. So let us assume $\mathbb{P}_2 = \mathbb{P}_3$ so that the minimum pinch point of $\mathbb{P}_2$, the supremum of all minimal elements, is its unique maximum. By this assumption $\mathbb{P}_2$ will be a pyramid so we can denote L as its set of minimal elements and use the facts from Lemma 4.4.6.

The 3-tuples in $\mathbb{P}^3$ fall into four categories. The first two are the 3-tuples that belong to P_1^3 and P_2^3 respectively. What is left are the 3-tuples that have coordinates in both P_1 and P_2. We will denote

$$P_{1,1,2} := (P_1 \times P_1 \times P_2) \cup (P_1 \times P_2 \times P_1) \cup (P_2 \times P_1 \times P_1)$$

and

$$P_{1,2,2} := (P_1 \times P_2 \times P_2) \cup (P_2 \times P_1 \times P_2) \cup (P_2 \times P_2 \times P_1).$$

Then $P^3 = P_1^3 \cup P_2^3 \cup P_{1,1,2} \cup P_{1,2,2}$. We will be using this fact to divide our construction into stages.

Let $Q := \{a \in P_2 : a = l_{\{a\}}\}$ and

$$P_{1,1,Q} := (P_1 \times P_1 \times Q) \cup (P_1 \times Q \times P_1) \cup (Q \times P_1 \times P_1).$$

Clearly Q is a subset of P_2 so $P_{1,1,Q}$ is a subset of $P_{1,1,2}$. Define $S_1 := P_1^3 \cup P_2^3 \cup P_{1,1,Q} \cup P_{1,2,2}$ and let $\mathbb{S}_1$ denote the subposet induced by S_1. Our first step is to construct an NU-preserving one-step chain from $\mathbb{P}^3$ to $\mathbb{S}_1$.

Let r_0 be the identity map on $\mathbb{P}^3$. Recursively let us assume there exists $r_0, \ldots, r_i$ an NU-preserving one-step chain from $\mathbb{P}^3$ to $r_i(\mathbb{P}^3)$ such that $\mathbb{S}_1 \subseteq r_i(\mathbb{P}^3)$. If $\mathbb{S}_1 = r_i(\mathbb{P}^3)$ then we have the one-step chain we wanted. Assume not, and let (a, b, c) be a minimal 3-tuple in $r_i(\mathbb{P}^3)\backslash\mathbb{S}_1$. Then $(a, b, c) \in P_{1,1,2}\backslash P_{1,1,Q}$ and by definition exactly one of a, b or c belongs to $P_2\backslash Q$ while the rest are in P_1. Without loss of generality let us assume $c \in P_2\backslash Q$. By Lemma 4.4.6(1) we have $l_x \leq x$ for all $x \in P_2$. So it must be that $c > l_{\{c\}}$.

We will show that $(a, b, l_{\{c\}})$ is the unique lower cover of (a, b, c) in $r_i(\mathbb{P}^3)$. By Lemma 4.4.6(9) $(a, b, l_{\{c\}})$ is in S_1 which is in the image of r_i. Suppose there exists $(u, v, w) \in r_i(\mathbb{P}^3)$ such that $(u, v, w) < (a, b, c)$. By minimality of (a, b, c), (u, v, w) must be in $\mathbb{S}_1$. Since $u < a$ and $v < b$, u and v are elements of P_1. If w comes from P_1 also then $(u, v, w) < (a, b, l_{\{c\}})$. If $w \in P_2$ then it must be true that $w = l_{\{w\}}$. So Lemma 4.4.6(2) tells us that $w \leq l_{\{c\}}$ and $(u, v, w) \leq (a, b, l_{\{c\}})$.

Using Lemma 7.4.7 we can define the retraction

$$r_{i+1} := I_{(a,b,c)\to(a,b,l_{\{c\}})} \circ r_i$$

where $r_{i+1}(\mathbb{P}^3) \subseteq r_i(\mathbb{P}^3)$ and $|\{x \in r_i(\mathbb{P}^3) : r_{i+1}(x) \neq x\}| \leq 1$. If (a, b, c) is an NU 3-tuple then it must be that $a = b$. Thus $I_{(a,b,c)\to(a,b,l_{\{c\}})}$ is NU-preserving which implies r_{i+1} is also NU-preserving. Therefore $r_0, \ldots, r_{i+1}$ is an NU-preserving one-step chain.

The image of each retraction is constructed to be strictly less than the previous one. After finitely many iterations we will have an NU-preserving one-step chain from $\mathbb{P}^3$ to $\mathbb{S}_1$.

For the second step we will let $R := \{(a, b) \in P_2^2 : a, b \leq l_{\{a,b\}}\}$ and

$$P_{1,R} := (P_1 \times R) \cup \{(a, b, c) : b \in P_1 \text{ and } (a, c) \in R\} \cup (R \times P_1).$$

Let $S_2 := P_1^3 \cup P_2^3 \cup P_{1,1,Q} \cup P_{1,R}$ and denote its induced subposet as $\mathbb{S}_2$. We'll now construct an NU-preserving one-step chain from $\mathbb{S}_1$ to $\mathbb{S}_2$.

As before we start with r_0 being the identity map on $\mathbb{S}_1$ and assume the existence of $r_0, \ldots, r_i$ an NU-preserving one-step chain from $\mathbb{S}_1$ to $r_i(\mathbb{S}_1) \supseteq \mathbb{S}_2$. If $r_i(\mathbb{S}_1) = \mathbb{S}_2$ then we have what we wanted so assume otherwise. Let (a,b,c) be minimal in $r_i(\mathbb{S}_1)\backslash\mathbb{S}_2$. Then $(a,b,c) \in P_{1,2,2}\backslash P_{1,R}$ and by definition exactly one of a,b or c is in P_1, we'll assume it is a since the construction will be similar in every case. Then at least one of b,c is strictly greater than $l_{\{b,c\}}$. Our construction of r_{i+1} will depend on how many that is.

Suppose only one of b or c is strictly greater than $l_{\{b,c\}}$. Without loss of generality let it be c. Then $b \leq l_{\{b,c\}} = l_{\{c\}} < c$ by Lemma 4.4.6(3,4). We'll show that $(a,b,l_{\{c\}})$ is the unique lower cover of (a,b,c) in $r_i(\mathbb{S}_1)$. Again we can use Lemma 4.4.6(9) to show that $(a,b,l_{\{c\}})$ indeed belongs to $r_i(\mathbb{S}_1)$. Suppose there exists $(u,v,w) \in r_i(\mathbb{S}_1)$ such that $(u,v,w) < (a,b,c)$. We would only need to show $w \leq l_{\{c\}}$. If $w \in P_1$ then it is obviously true. If $w \in P_2$ then we will have to take a look at u and v. Now u must be in P_1 since a is. If v is also in P_1 then according to the observation at the end of the previous step we have $w = l_{\{w\}} \leq l_{\{c\}} < c$ by Lemma 4.4.6(2). If v is in P_2 then by the minimality of (a,b,c) it must be that $v,w \leq l_{\{u,v,w\}\cap P_2} \leq l_{\{a,b,c\}\cap P_2} = l_{\{c\}}$ (using Lemma 4.4.6(2) again). So $(a,b,l_{\{c\}})$ is indeed the unique lower cover of (a,b,c) in $r_i(\mathbb{S}_1)$. Applying Lemma 7.4.7 we get the retraction

$$r_{i+1} := I_{(a,b,c)\to(a,b,l_{\{c\}})} \circ r_i$$

where $r_{i+1}(\mathbb{S}_1) \subseteq r_i(\mathbb{S}_1)$ and $|\{x \in r_i(\mathbb{S}_1) : r_{i+1}(x) \neq x\}| \leq 1$. This time it is not possible for (a,b,c) to be an NU 3-tuple so it's easy to see that $I_{(a,b,c)\to(a,b,l_{\{c\}})}$ and therefore r_{i+1} is NU-preserving.

Now suppose both b and c are strictly greater than $l_{\{b,c\}}$. Then by Lemma 4.4.6(3,4) we have $l_{\{b,c\}} = l_{\{b\}} = l_{\{c\}}$. We will show that $(a, l_{\{b,c\}}, l_{\{b,c\}})$ is the unique lower cover of (a,b,c) in $r_i(\mathbb{S}_1)$. By Lemma 4.4.6(9) $(a, l_{\{b,c\}}, l_{\{b,c\}})$ belongs to $r_i(\mathbb{S}_1)$. Let $(u,v,w) \in r_i(\mathbb{S}_1)$ such that $(u,v,w) < (a,b,c)$. Once again we have $u \in P_1$ because of a. If both v and w are also in P_1 then we get $(u,v,w) \leq (a, l_{\{b,c\}}, l_{\{b,c\}})$ as needed. Assume at least W is in P_2. If $v \in P_1$ then by observation at the end of step one again we get $w = l_{\{w\}} \leq l_{\{c\}} = l_{\{b,c\}}$ (by Lemma 4.4.6(2)) so $(u,v,w) \leq (a, l_{\{b,c\}}, l_{\{b,c\}})$. If $v \in P_2$ then by the minimality of (a,b,c) $v,w \leq l_{\{u,v,w\}\cap P_2} \leq l_{\{a,b,c\}\cap P_2} = l_{\{b,c\}}$. This shows that $(a, l_{\{b,c\}}, l_{\{b,c\}})$ is the unique lower cover of (a,b,c) in $r_i(\mathbb{S}_1)$. Using Lemma 7.4.7 we can define the retraction

$$r_{i+1} := I_{(a,b,c)\to(a,l_{\{b,c\}},l_{\{b,c\}})} \circ r_i$$

where $r_{i+1}(\mathbb{S}_1) \subseteq r_i(\mathbb{S}_1)$ and $|\{x \in r_i(\mathbb{S}_1) : r_{i+1}(x) \neq x\}| \leq 1$. The only way for (a,b,c) to be an NU 3-tuple is if $b = c$. Thus $I_{(a,b,c)\to(a,l_{\{b,c\}},l_{\{b,c\}})}$ and r_{i+1} are both NU-preserving.

We see that either way we get $r_0, \ldots, r_{i+1}$ as an NU-preserving one-step chain. Since the image of each retraction is strictly decreasing this process will yield the desired chain from $\mathbb{S}_1$ to $\mathbb{S}_2$ after finitely many steps.

For the third step define $T := \{(a,b,c) \in P_2^3 : l_{\{a,b,c\}} \leq a,b,c\}$. Denote $S_3 := P_1^3 \cup T$ and let $\mathbb{S}_3$ be the induced subposet. We will construct an NU-preserving one-step chain from $\mathbb{S}_2$ to $\mathbb{S}_3$.

As usual we start with r_0 being the identity map on $\mathbb{S}_2$ and assume there exists an NU-preserving one-step chain $r_0, \ldots, r_i$ from $\mathbb{S}_2$ to $r_i(\mathbb{S}_2) \supseteq \mathbb{S}_3$. If $\mathbb{S}_3 = r_i(\mathbb{S}_2)$ then we have the desired chain so let's assume not. Let (a, b, c) be maximal in $r_i(\mathbb{S}_2) \backslash \mathbb{S}_3$. For the sake of convenience denote $l' := l_{\{a,b,c\}}$. By Lemma 4.4.6(7) a, b and c are all comparable to l'. We will show that $(\max(a, l'), \max(b, l'), \max(c, l'))$ is the unique upper cover of (a, b, c) in $r_i(\mathbb{S}_2)$. Using Lemma 4.4.6(3,4) (if $a \geq l'$ then $l_{\{a\}} = l'$ so $L_a = L_{l'}$) and then Lemma 4.4.6(9) we get

$$\begin{aligned} l_{\{\max(a,l'),\max(b,l'),\max(c,l')\}} &= \sup(L_{\max(a,l')} \cup L_{\max(b,l')} \cup L_{\max(c,l')}) \\ &= \sup(L_{l'} \cup L_{l'} \cup L_{l'}) \\ &= \sup(L_{l'}) \\ &= l_{l'} \\ &= l' \end{aligned}$$

so $(\max(a, l'), \max(b, l'), \max(c, l'))$ is an element of $\mathbb{S}_3 \subseteq r_i(\mathbb{S}_2)$. Suppose there exists $(u, v, w) \in r_i(\mathbb{S}_2)$ such that $(u, v, w) > (a, b, c)$. Let $l'' := l_{\{u,v,w\}}$. By maximality of (a, b, c) we have $(u, v, w) \geq (l'', l'', l'') \geq (l', l', l')$ by Lemma 4.4.6(3). Thus $(u, v, w) \geq (\max(a, l'), \max(b, l'), \max(c, l'))$ as needed.

Using Lemma 7.4.7 we can define the retraction

$$r_{i+1} := I_{(a,b,c)\to(\max(a,l'),\max(b,l'),\max(c,l'))} \circ r_i$$

where $r_{i+1}(\mathbb{S}_2) \subseteq r_i(\mathbb{S}_2)$ and $|\{x \in r_i(\mathbb{S}_2) : r_{i+1}(x) \neq x\}| \leq 1$. If any pair of the three elements a, b and c are equal to each other then the corresponding pair from $\max(a, l'), \max(b, l')$ and $\max(c, l')$ are also equal. Thus $I_{(a,b,c)\to(\max(a,l'),\max(b,l'),\max(c,l'))}$ and r_{i+1} are NU-preserving maps. So $r_0, \ldots, r_{i+1}$ is an NU-preserving one-step chain from $\mathbb{S}_2$ to $r_{i+1}(\mathbb{S}_2)$.

After repeating finitely many times we get an NU-preserving one-step chain from $\mathbb{S}_2$ to $\mathbb{S}_3$. Combining the chains we have constructed from the first three steps using Lemma 7.2.1 we get one from $\mathbb{P}^3$ to $\mathbb{S}_3$. Denote the final retraction of this new chain as r.

Before we define our final one-step chain we need to turn our attention to a particular subposet of $\mathbb{P}_1^3 + \mathbb{P}_2^3$. Define

$$\Lambda := \{l_{\{a,b,c\}} : (a, b, c) \in P_2^3\}.$$

Now Λ is a subposet of $\mathbb{P}_2$ and the unique maximum of $\mathbb{P}_2$ is also the unique maximum of Λ. We will show that for each $l \in \Lambda$ either l is the unique maximal element of Λ or there exists a unique $l' \in \Lambda$ such that $l \prec l'$ in $(\Lambda, \leq^\Lambda)$. Clearly if l is not the unique maximal element then there exists an upper cover of it. Suppose l' and l'' are both upper covers of l. Now l' and l'' are both suprema of sets of minimal elements of $\mathbb{P}_2$. If they are incomparable then there exists some a minimal in $\mathbb{P}_2$ such that $a < l'$ but $a \not< l''$. This implies $a \not< l$ also which means $\{a, l'', l, l'\}$ form an induced N-subposet of $\mathbb{P}_2$. This is a contradiction since $\mathbb{P}_2$ is series-parallel. So l' and l'' must be comparable so they are equal.

Let $U := \{(a, b, c) \in P_2^3 : \exists l \prec l' \text{ in } \Lambda \text{ with } l \leq a, b, c \leq l'\}$ and $S_4 = P_1^3 \cup U$. Recall that T contains all $(a, b, c) \in P_2^3$ such that $l_{\{a,b,c\}} \leq a, b, c$. Lemma 4.4.6(4,8) tells us that if $l' \succ l_{\{a,b,c\}}$ then $(a, a, a) < (l', l', l')$. Thus $T \subseteq U$ and $S_3 \subseteq S_4$. Let $\mathbb{S}_4$ be the subposet induced by S_4.

Notice that U is the union of intervals in P_2^3 with diagonal elements as end points. Any endomorphism of $\mathbb{P}^3$ that fixes the diagonal elements must map $\mathbb{S}_4$ back into itself. In particular r is such an endomorphism. Thus $r|_{\mathbb{S}_4}$ is well defined. Since $\mathbb{S}_3 \subseteq \mathbb{S}_4$ $r|_{\mathbb{S}_4}$ is actually a retraction of $\mathbb{S}_4$ with $\mathbb{S}_3$ as its image. Let $(a, b, c) \in \mathbb{S}_4 \backslash \mathbb{S}_3 = U \backslash T$. We want to find its image under $r|_{\mathbb{S}_4}$.

By the proof of Lemma 7.2.1 and the construction in the first three steps, it follows that

$$r = \underbrace{I''_k \circ \cdots \circ I''_2 \circ I''_1}_{\text{3rd step}} \circ \underbrace{I'_m \circ \cdots \circ I'_2 \circ I'_1}_{\text{2nd step}} \circ \underbrace{I_n \circ \cdots \circ I_2 \circ I_1}_{\text{1st step}}$$

where each I_r, I'_s, I''_t is a function of the form $I_{(x,y,z)\to(x',y',z')}$. Within each of the three steps these functions are considered with a different subset of $\mathbb{P}^3$ as their domain. Note that each I_r and I'_s from the first two steps fixes all 3-tuples from P_2^3. Every 3-tuple in T is fixed by every I''_t in the third step. As for those in in $P_2^3 \backslash T$ each I''_t either fixes it or maps it into T. Hence for $(a, b, c) \in \mathbb{S}_4 \backslash \mathbb{S}_3 \subseteq P_2^3 \backslash T$ there is a corresponding

$$I''_{t_0} = I_{(a,b,c)\to(\max(a,l_{\{a,b,c\}}),\max(b,l_{\{a,b,c\}}),\max(c,l_{\{a,b,c\}}))}$$

such that $r(a, b, c) = (\max(a, l_{\{a,b,c\}}), \max(b, l_{\{a,b,c\}}), \max(c, l_{\{a,b,c\}}))$.

Since it is in $\mathbb{S}_4$ (a, b, c) must be between some (l, l, l) and (l', l', l') where $l \prec l'$ in Λ. Being in $P_2^3 \backslash T$ means that $(a, b, c) \not\geq (l_{\{a,b,c\}}, l_{\{a,b,c\}}, l_{\{a,b,c\}})$ so at least one of a, b or c is less than $l_{\{a,b,c\}}$ by Lemma 4.4.6(7). Assume $a < l_{\{a,b,c\}}$. Since $l \leq a < l_{\{a,b,c\}}$ and $l \prec l'$ then $l' \leq l_{\{a,b,c\}}$. By Lemma 4.4.6(3,9) we have $l' = l_{\{a,b,c\}}$. Thus

$$r(a, b, c) = (\max(a, l_{\{a,b,c\}}), \max(b, l_{\{a,b,c\}}), \max(c, l_{\{a,b,c\}})) = (l_{\{a,b,c\}}, l_{\{a,b,c\}}, l_{\{a,b,c\}}).$$

So we see that $r|_{\mathbb{S}_4}$ maps $\mathbb{S}_4 \backslash \mathbb{S}_3$ into $\Delta(P^3)$ and fixes $\mathbb{S}_3$.

By Lemma 7.2.2 all we need is an NU-preserving one-step chain from $\mathbb{S}_4$ to $\Delta(P^3)$ to get one from $\mathbb{S}_3$ to $\Delta(P^3)$. As mentioned three paragraphs back if we have a retraction on any subposet of $\mathbb{P}^3$ containing $\mathbb{S}_4$ as an induced subposet, and if the retraction fixes all of the diagonal elements of $\mathbb{P}^3$, then restricting it onto $\mathbb{S}_4$ produces a retraction on $\mathbb{S}_4$. It is clear that $S_4 \subseteq P_1^3 \cup P_2^3$ so $\mathbb{S}_4$ is an induced subposet of $\mathbb{P}_1^3 + \mathbb{P}_2^3$. Thus instead of from $\mathbb{S}_4$

to $\Delta(P^3)$ all we need is an NU-preserving one-step chain starting from $\mathbb{P}_1^3 + \mathbb{P}_2^3$. Such a chain exists by Lemma 7.4.4. □

Lemma 7.4.12. *Let $\mathbb{P}$ be a series-parallel poset satisfying the 4-crown condition. Suppose $\mathbb{P} = \mathbb{P}_1 \cup \mathbb{P}_2$ for some series-parallel posets $\mathbb{P}_1$ and $\mathbb{P}_2$ satisfying the 4-crown condition and there exist NU-preserving one-step chains from $\mathbb{P}_1^3$ to $\Delta(P_1^3)$ and $\mathbb{P}_2^3$ to $\Delta(P_2^3)$. Then there exists an NU-preserving one-step chain from $\mathbb{P}^3$ to $\Delta(P^3)$.*

Proof. If $\mathbb{P}_1$ and $\mathbb{P}_2$ are not connected posets then we can split them up by their connected components where each component is an induced subposet satisfying the conditions we just mentioned. So without loss of generality let's assume $\mathbb{P} = \mathbb{P}_1 \cup \mathbb{P}_2 \cup \ldots \cup \mathbb{P}_n$ where each $\mathbb{P}_i$ for $i \in \{1, \ldots, n\}$ is connected and possessing the required properties.

Note that $\mathbb{P}^3$ will also be separated into connected components. There will be a copy of $\mathbb{P}_i^3$ for each $i \in \{1, \ldots, n\}$. Label the other connected components as $C_1, C_2, \ldots, C_k$. We will call these mixed components, as the 3-tuples they contain cannot have all three elements belonging to the same P_i. We will first build an NU-preserving one-step chain from $\mathbb{P}^3$ to $\mathbb{P}_1^3 \cup \ldots \cup \mathbb{P}_n^3$. To do so we will first need to construct such a chain for each connected component C_i to a single point.

The process for each C_i will be the same. Fix an arbitrary C_i. Note that $C_i = P_j \times P_k \times P_l$ for some $j, k, l \in \{1, \ldots, n\}$. Since each $\mathbb{P}_i$ is a connected series parallel poset satisfying the 4-crown condition by Lemma 4.4.3 they must all have a pinch point. Let m_j, m_k and m_l denote a pinch point from $\mathbb{P}_j, \mathbb{P}_k$ and $\mathbb{P}_l$ respectively.

For all $(x, y, z) \in C_i$, each coordinate of the 3-tuple will be comparable to their respective pinch points. Let T_1 be the set of all 3-tuples in C_i such that exactly one of its coordinates is strictly less than its respective pinch point. Define $S_1 := C_i \backslash T_1$ and let $\mathbb{S}_1$ denote the induced subposet. We will build a chain of one-step NU-preserving retraction from $\mathbb{P}_j \times \mathbb{P}_k \times \mathbb{P}_l$ to $\mathbb{S}_1$.

Define r_0 to be the identity map on $\mathbb{P}_j \times \mathbb{P}_k \times \mathbb{P}_l$. Assume $r_0, \ldots, r_i$ is an NU-preserving one-step chain from $\mathbb{P}_j \times \mathbb{P}_k \times \mathbb{P}_l$ to $r_i(\mathbb{P}_j \times \mathbb{P}_k \times \mathbb{P}_l) \supseteq \mathbb{S}_1$. If $r_i(\mathbb{P}_j \times \mathbb{P}_k \times \mathbb{P}_l) = \mathbb{S}_1$ then we are done. Otherwise pick $(a, b, c) \in r_i(\mathbb{P}_j \times \mathbb{P}_k \times \mathbb{P}_l)$ maximal with respect to $(a, b, c) \notin S_1$. By definition one of a, b or c is strictly less than its respective pinch point. Without loss of generality assume it is c. We will show that (a, b, m_l) is the unique upper cover of (a, b, c) in $r_i(\mathbb{P}_j \times \mathbb{P}_k \times \mathbb{P}_l)$. Clearly (a, b, m_l) is an element of $r_i(\mathbb{P}_j \times \mathbb{P}_k \times \mathbb{P}_l)$. Suppose there exists some $(u, v, w) > (a, b, c)$ in $r_i(\mathbb{P}_j \times \mathbb{P}_k \times \mathbb{P}_l)$. By maximality of (a, b, c) w must be greater than or equal to its comparable pinch point, which in this case is m_l. Thus $(u, v, w) \geq (a, b, m_l)$ as desired. By Lemma 7.4.7 we may define

$$r_{i+1} := I_{(a,b,c)\to(a,b,m_l)} \circ r_i$$

where $r_{i+1}(\mathbb{P}_j \times \mathbb{P}_k \times \mathbb{P}_l) \subseteq r_i(\mathbb{P}_j \times \mathbb{P}_k \times \mathbb{P}_l)$ and $|\{x \in r_i(\mathbb{P}_j \times \mathbb{P}_k \times \mathbb{P}_l) : r_{i+1}(x) \neq x\}| \leq 1$. This map is clearly NU-preserving. After finitely many steps this process will yield a one-step chain from $\mathbb{P}_j \times \mathbb{P}_k \times \mathbb{P}_l$ to $\mathbb{S}_1$.

Let T_1 be the set of all 3-tuples in S_1 such that exactly one of its coordinates is strictly greater than its respective pinch point. Denote $S_2 := S_1 \backslash T_2$ and let $\mathbb{S}_2$ be its induced subposet.

Let r_0 be the identity map on $\mathbb{S}_1$ and assume $r_0, \ldots, r_i$ is an NU-preserving one-step chain from $\mathbb{S}_1$ to $r_i(\mathbb{S}_1) \supseteq \mathbb{S}_2$. If $r_i(\mathbb{S}_1) = \mathbb{S}_2$ then we are done so suppose not. Then pick (a, b, c) minimal in $r_i(\mathbb{S}_1) \backslash \mathbb{S}_2$. By definition of T_2 one of a, b or c will be strictly greater than its respective pinch point. We will assume it is c. Then (a, b, m_l) will be the unique lower cover of (a, b, c) in $r_i(\mathbb{S}_1)$ and the next retraction will be

$$r_{i+1} := I_{(a,b,c) \to (a,b,m_l)} \circ r_i.$$

Repeat this process until we get a one-step chain from $\mathbb{S}_1$ to $\mathbb{S}_2$.

Now $\mathbb{S}_2$ will be a subposet of C_i where every 3-tuple is comparable to (m_j, m_k, m_l). So we can define a one-step chain from $\mathbb{S}_2$ to the singleton subposet containing only (m_j, m_k, m_l) by picking a maximal below or a minimal above for each retraction and map it onto (m_j, m_k, m_l).

Fix some $a \in \mathbb{P}$. We can construct an NU-preserving one-step chain from $\mathbb{P}^3$ to $\mathbb{P}_1^3 \uplus \ldots \uplus \mathbb{P}_n^3$ by reducing the C_i's one at a time onto a point then mapping each of these points one at a time onto (a, a, a).

Finally to go from $\mathbb{P}_1^3 \uplus \ldots \uplus \mathbb{P}_n^3$ to $\Delta(P^3)$ we simply apply Lemma 7.4.5. □

Theorem 7.4.13. *Let $\mathbb{P}$ be a series-parallel poset satisfying the 4-crown condition. Then there exists an NU-preserving one-step chain from $\mathbb{P}^3$ to $\Delta(P^3)$.*

Proof. By Lemma 4.3.12 we know that such a $\mathbb{P}$ can be constructed from $\mathbf{1}$ using $\uplus$, $\triangle$, $\bigtriangledown$, $\Diamond$, ⧖ and $+^R$ finitely many times. We will prove the stated result by induction on the construction of $\mathbb{P}$. The bases case of when $\mathbb{P} = \mathbf{1}$ is trivial. The inductive step is split up into lemmas 7.4.10, 7.4.8, 7.4.9, 7.4.11 and 7.4.12. □

Corollary 7.4.14. *If $\mathbb{P}$ is a series-parallel poset and $\mathbb{P}$ admits a sequence of Freese-McKenzie SD-$\vee$ operations, then $\mathbb{P}$ admits a sequence of Freese-McKenzie SD-$\vee$ operations of length $2(n^3 - n) - 1$ where n is the size of $\mathbb{P}$.*

7.5 Example

Here we will show an example of a poset that does not have SD-$\vee$ operations. It was mentioned in a paper by Larose and Zádori also as an example for having TSI operations but no semilattice operation [17].

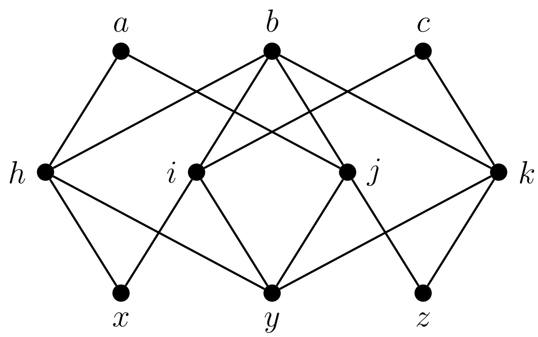

Figure 7.1: $\mathbb{A}$

Let $\mathbb{A}$ denote the 10-element poset pictured in Figure 7.1.

Lemma 7.5.1. *The poset $\mathbb{A}$ does not admit Freese-McKenzie SD-$\vee$ polymorphisms.*

Proof. Suppose for the sake of contradiction that there does exists a sequence $\{r_0, \ldots, r_n\}$ of Freese-McKenzie SD-$\vee$ polymorphisms. Choose one of minimum length.

By definition we know that r_0 must be the first projection on $\mathbb{A}^3$. By minimality of the chain r_1 has to differ from r_0 at some $(\alpha, \beta, \gamma) \in \mathbb{A}^3$; we will examine such a triple to find our contradiction.

First let us focus on a special case where all three coordinates in the triple (α, β, γ) are maximal in $\mathbb{A}$.

Since r_1 disagrees with r_0 on (α, β, γ), we have $r_1(\alpha, \beta, \gamma) = \theta$ for some $\theta \neq \alpha$. Since α, β and γ are all maximal, we have (α, β, γ) greater than all of $(x, x, x), (y, y, y)$ and (z, z, z). The output of the diagonal elements are the same for every r_i. So $r_1(x, x, x) = x$, $r_1(y, y, y) = y$ and $r_1(z, z, z) = z$. Therefore it must be that $\theta \geq x, y, z$. This limits the possible candidates for θ to a maximal element of $\mathbb{A}$ aside from α.

Next we see that $(\alpha, \beta, \gamma) \geq (\alpha, y, y)$. Since each r_i preserves the order relation and α, θ are incomparable, (α, y, y) is another triple that r_0 and r_1 disagree on (in other words $r_1(\alpha, y, y) \neq \alpha$). So condition (2) in the definition of Freese-McKenzie SD-$\vee$ operations is violated for r_0 and r_1. Another violation would cause the desired contradiction.

(α, y, γ) is a triple that is less than both (α, β, γ) and (α, α, γ). Since condition (1) in the definition of a Freese-McKenzie SD-$\vee$ operations must be satisfied between r_0 and r_1, we have $r_1(\alpha, \alpha, \gamma) = \alpha$. Thus in order to preserve the order relation r_1 has to map (α, y, γ) to some ψ where ψ is less than both α and θ. This means that ψ cannot be a maximal element in $\mathbb{A}$, and so due to the unique structure of $\mathbb{A}$ there exists some δ minimal in $\mathbb{A}$ that is incomparable with ψ. Now (δ, y, δ) is less than (α, y, γ), so their images under r_i must follow suit. Due to the incomparability of δ and ψ, (δ, y, δ) cannot be sent to δ by r_1. This means that r_0 and r_1 violates condition (3) in the definition of a Freese-McKenzie SD-$\vee$ operations. Combined with what we have already shown, a contradiction appears.

Similar proof will also show a contradiction for when α, β and γ are all minimal elements.

Going back to our original assumption, we will now find contradictions by showing that if r_1 does not agree with r_0 on some arbitrary (α, β, γ) then there exists a triple of the special cases above that r_1 does not agree with r_0 on.

Let (α, β, γ) be an arbitrary triple in $\mathbb{A}^3$ such that $r_1(\alpha, \beta, \gamma) \neq \alpha$. Then $r_1(\alpha, \beta, \gamma) = \theta$ for some $\theta \neq \alpha$ in $\mathbb{A}$. We will split the rest of the proof into cases.

First assume that α, θ are both maximal in $\mathbb{A}$. Since they are not equal they have to be incomparable. Let β' and γ' be maximal elements greater than β and γ respectively. Then r_0 and r_1 does not agree on $(\alpha, \beta', \gamma')$.

Next assume that α is maximal but θ is not. Then there exists δ minimal such that δ is incomparable to θ. Let β' and γ' be minimal elements less than β and γ respectively. Then r_0 and r_1 does not agree on $(\delta, \beta', \gamma')$.

Now assume that α is not maximal or minimal in $\mathbb{A}$. If θ is maximal then there exists δ also maximal such that $\delta \geq \alpha$. Let β' and γ' be maximal elements greater than β and γ respectively. Then r_0 and r_1 does not agree on $(\delta, \beta', \gamma')$. Similar proof applies when θ is minimal.

Finally assume that both α and θ are not maximal or minimal. There must exists δ minimal or maximal that is comparable to α but not δ. Our usual method applies either way.

The cases where α is minimal is similar to when it is maximal. This concludes our proof. $\square$

Chapter 8

Concluding Remarks

As stated before we do not know if having NU-preserving one-step chains is equivalent to admitting SD-$\vee$ operations for relational structures. In this paper only the forward direction is proved. Additional work is needed to prove converse direction or find a counter example.

We have also shown that for a series-parallel poset $\mathbb{P}$, $\exp(\mathbb{P})$ has bounded path duality if and only if it admits Freese-McKenzie SD-$\vee$ operations. In general the reverse direction remains an open question. A sensible next step would be to check if this statement holds for posets of dimension 2, or any other class of posets containing series-parallel posets.

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