Stochastic Renewal Process Models for Structural Reliability Analysis

by

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Abstract

Reliability analysis in structural engineering is utilized in the initial design phase and its application continues throughout the service life of a structural system in form of maintenance planning and optimization. Engineering structures are usually designed with extremely high reliability and with a long service life. However, deterioration with time and exposure to external hazards like earthquakes, strong winds etc., increase the structure's vulnerability to failure.

In structural reliability analysis, stochastic processes have been utilized to model time-dependent uncertain variations in environmental loads and structural resistance. The Homogeneous Poisson Process (HPP) is most commonly used as the driving process behind environmental hazards and shocks causing structural deterioration. The HPP model is justified on account of an asymptotic argument that exceedances of a process to a high threshold over a long lifetime converge to HPP model. This approach serves the purpose at the initial design stages. The combination of stochastic loads is an important part of design load estimation. Currently, solutions of the load combination problem are also based on HPP shock and pulse processes. The deterioration is typically modelled as a random variable problem, instead of a stochastic process. Among stochastic models of deterioration, the gamma process is popularly used. The reliability evaluation by combining a stochastic load process with a stochastic process of deterioration, such as gamma process, is a very challenging problem, and so its discussion is quite limited in the existing literature.

In case of reliability assessment of existing structures, such as nuclear power plants nearing the end of life, an indiscriminate use of HPP load models becomes questionable as asymptotic arguments may not be valid over a short remaining life. Thus, this thesis aims to generalize stochastic models used in the structural reliability analysis by considering more general models of environmental hazards based on the theory of the renewal process. These models include shock, pulse and alternating processes. The stochastic load combination problem is also solved in a more general setting by considering a renewal pulse process in combination with a Poisson shock process. The thesis presents a clear exposition of the stochastic load and strength combination problem. Several numerical algorithms have been developed to compute the stochastic reliability solution, and results have been compared with existing approximations. Naturally, existing approximations serve adequately in the routine design. However, in case of critical structures with high consequences to safety and reliability, the use of proposed methods would provide a more realistic assessment of structural reliability.

In summary, the results presented in this thesis contribute to the advancement in stochastic modeling of structural reliability analysis problems.

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Last but not least, I would like to thank my parents, Inolda and Erick, and my brother, Nolvienerick, for their love, support and encouragement throughout this journey far away from home.

To

My Family

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List of Acronyms

iid independent and identically distributed. 8, 14, 18, 112

CDF Cumulative Distribution Function. 5, 20–24, 49, 60, 86, 87, 91, 98, 101, 104, 106, 110

CFB Canadian Forces Base. 40

GEV Generalized Extreme Value. 64, 65, 67

GPD Generalized Pareto Distribution. 65–67

HPP Homogeneous Poisson Process. iv, 3, 5, 11, 14–16, 19, 31, 33, 39, 40, 59, 66, 67, 70, 74, 78, 97, 106, 113, 115, 118, 119

MoM Method of Moments. 53, 54, 56, 57

NHPP Nonhomogeneous Poisson Process. 83, 107, 112

PDF Probability Density Function. 7, 49, 92, 112

POE Probability of Exceedance. 17, 30, 33, 34, 39, 49, 70, 78

PPP Probability Paper Plot. 35, 40

TTF Time to Failure. 1

Nomenclature

$\mathbb{E}[X]$ Expectation of random variable X

$\mathbb{P}(X)$ Probability of event X

$F_T^{(i)}(t)$ i-fold convolution of $F_T(t)$ with itself

$f_{\mathbf{S}}(\mathbf{s})$ Joint probability density function of occurrence times

$F_T(t)$ Distribution of inter-arrival times

$F_X(x)$ Cumulative distribution function of a random variable X

$f_X(x)$ Probability density function of a random variable X

$M(t)$ Renewal function

$m(t)$ Renewal rate

$N(t)$ Counting process of events in $[0, t]$

$S_1, S_2, \ldots$ Sequence of random increasing values denoting epochs of occurrence

$T_1, T_2, \ldots$ Sequence of random values denoting subsequent inter-arrival times

$X(t)$ Value of stochastic process at time t

$X^*(t)$ Maximum value of stochastic process up to time t

Chapter 1

Introduction

1.1 Reliability Analysis

Reliability analysis supports decision making on the safeguarding of system performance from engineering design to the assessment of risk in existing engineering systems. Risk is the effect of adverse behaviour of a system, which can be quantified as the probability of failure and we have the following relation:

$$\text{Risk} = (\text{Probability of failure}) * \text{Consequences}.$$

Reliability, is related to the probability of failure as:

$$\text{Reliability} = 1 \text{ - } (\text{Probability of failure}).$$

This document concerns the development and explanation of models to compute the reliability of systems from a mechanistic point of view. This means setting up a mathematical formulation of the problem which includes the mechanistic components that can lead to failure, i.e. system strength and the loads impacting the system. It is easy to see that whenever there is no uncertainty in the strength of the system or the loads which the system is subject to, there is also no uncertainty in the behaviour of the system. The system will operate satisfactorily if the strength is larger than the impacting loads. However, usually the strength and load are uncertain and provided by a probability distribution as depicted in Figure 1.1.

Let the strength be denoted by the random variable X and the load by the random variable Y. Then the probability of failure is

$$\mathbb{P}(X \leq Y) = \mathbb{P}(X - Y \leq 0),$$

which clearly occurs only in the interference region given in Figure 1.1. This is still a problem which has no uncertainty in time, i.e. it is a time-invariant problem. In practice the strength and load can be uncertain in time and we are interested in estimating the Time to Failure (TTF) of the system. The TTF is the first time the magnitude of the load exceeds the strength of the system. In particular we are interested in non-repairable systems, which means the time to first failure is a critical event that should not occur before the system has completed its mission. Let the random variable T denote the unknown TTF. We are interested in computing:

$$\mathbb{P}(T > t), \qquad \text{for } \ t > 0,$$

which is the system reliability and denotes the probability of the TTF exceeding a certain time t.

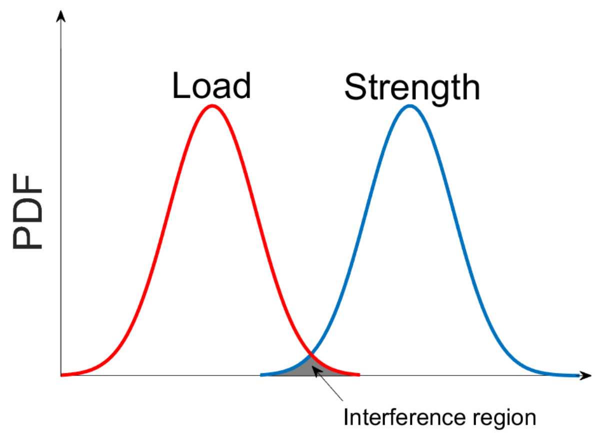


Figure 1.1: A basic reliability problem.

As mentioned, the strength and the loads can be uncertain in time and this has motivated the use of stochastic processes. This then leads to the following cases:

1. Load only is a stochastic process.
2. Strength only is a stochastic process.
3. Both load and strength are stochastic processes.

Case 1 is the subject of chapters 3 and 4 which besides reliability, leads to a different quantity of interest in safety assessment: the distribution of maximum load. Case 3 is the subject of chapter 5. Case 2 is a special and trivial variant of case 3.

1.2 Background and Motivation

Besides structural reliability analysis, also in equipment reliability probabilistic concepts and methods were developed. This means all components related to equipment capacity, applied loads and environmental factors are modeled with a random lifetime T. Therefore, reliability analysis of the entire non-repairable system can be formulated as a single lifetime distribution [10]. Non-repairable systems cannot be repaired after failure and usually have severe consequences. Examples are: collapsed buildings, bridges, dams or nuclear structures. Repairable systems however are assumed to be restored to an as good as new condition after failure. The lifetime T of the system is now interpreted as the time between each failure of the system. Because the system is renewed after each failure, stochastic renewal processes became a logical choice to model these problems and compute quantities such as: expected number of renewals in an interval, renewal rate and unavailability of the system.

Cornell [8] modeled seismic risk using the HPP and since then this model has been widely used in various studies on risk, reliability and life cycle cost analysis in civil engineering, [42]. As early as [62] and [61], Rosenblueth introduced the renewal process model for the estimation of expected loss caused by recurring hazards such as, earthquakes, winds, tsunamis etc.. In the most recent years, Rackwitz [57] has shown renewed interest in these models and they have been extended to combine with the Life Quality Index framework, [51]. Rackwitz and others,[33], [59], [17], have applied the renewal process model to analyze the effects of degradation and maintenance in life cycle analysis.

The HPP still remains overwhelmingly present in the literature since it leads to considerable analytical simplifications and avoids the complexities of stochastic theory, [57], [55]. In this document it becomes clear that the HPP performs well as an asymptotic solution but can become a crude approximation otherwise. Therefore, there is a need to develop more accurate methods that are robust under any condition, e.g. any time horizon, component capacity or inter-arrival times of hazards. This document intends to show that this can be achieved by utilizing renewal process theory.

Another quantity that plays a pivotal role in reliability analysis is the distribution of the maximum applied load. With a robust model this decision factor can also be calculated for non-HPP phenomena, [46]. Problems that involve a combination of two stochastic load processes, or a load and a degradation process pose many problems when analyzed in a general setting. In this document it is shown that this can be solved to some extent of generalization. A reliability problem involving a gamma degradation process and a marked HPP load process requires computing a complex stochastic integral of a gamma process [47]. Therefore, when stochastic processes are combined, special caution needs to be taken or some simplifying assumptions have to be included in the model.

As will be apparent in this document, the key issues in solving these reliability problems are the time component of the problems and possible correlations between loads. For case 3 mentioned above, information on the joint probability of occurrence of loads at different points in time is necessary. Solving a reliability problem with the joint probability of occurrence combined with possible correlation between load magnitudes can prove to be a daunting task. The assumption of independence between load magnitudes and their occurrence times following a Poisson process lead to significant reduction in the complexity of the problem. For case 1, because the strength does not change over time, one can argue that the process probabilistically starts all over again after the first occurrence of load. This naturally leads to the field of renewal process theory. Some basic properties of renewal processes, including the very well-known and widely used Poisson process are the subject of the following chapter.

1.3 Organization and Objectives

The main objectives of this document are to present accurate solutions to the following reliability analysis quantities:

- distribution of maximum of a load process
- distribution of maximum of combined load and shock process
- reliability of the interaction between gamma degradation and marked (non)-HPP load process

After getting familiarized with the general idea of reliability and safety analysis in Chapter 1, Chapter 2 introduces basic mathematical formulations and concepts. This chapter should also get the reader familiarized with the notations that are used throughout this document. In chapter 3, a reliability problem of a component with time-invariant strength subject to loads from a single source is treated. In this setting, renewal theory can be used with great effectiveness. From the formulations in this chapter the distribution of maximum of a load process is also derived. Chapter 4 builds on these achieved results and extends to the computation of the distribution of the maximum of two combined load processes. In Chapter 5 the component strength is time-dependent and a new approach is invoked to compute the reliability of the component. In this chapter it is shown that analytical results can be obtained even if both the strength and load process are stochastic in nature. In chapter 6 a summary of the results in this document are given, accompanied by suggestions for future research.

It is to be emphasized that when referring to load and strength, this means one load process and one strength process with the exception of chapter 4 which presents results for the combination of two load processes. An exact and relatively simple solution is difficult to analyze for the combination of three or more processes and simulations may be the only alternative.

Chapter 2

Stochastic Renewal Processes

In this chapter the terminology, concepts and notations of renewal theory used in this document are presented. In section 2.1 basic concepts of renewal theory are introduced while in section 2.2 terminology of renewal processes and renewal decomposition are presented. Section 2.3 explains more on the HPP. In section 2.4 it is shown why observing the maximum of the load process leads to both the quantities of maximum distribution and reliability of a component from the same model.

2.1 Renewal Theory

2.1.1 Renewal Process

This section presents the ordinary renewal process, starting with the basic concepts and ending with an example. Suppose we can model the lifetime of a component, which we denote as T, as a continuous random variable with Cumulative Distribution Function (CDF) given as

$$F_T(t) = \mathbb{P}(T \leq t), \quad x \geq 0 \tag{2.1}$$

and where $F_T(0) = 0$. Thus $F_T(t)$ is the probability of the component failing before time t.

A new component is put into service at $t = 0$ and survives a period of T_1. At time of failure it is replaced by e new component which survives a period of T_2. It is assumed that replacement time is negligible[1] so that the second component fails at time $(T_1 + T_2)$. When the second component fails it is also immediately replaced by a new one.

We let $T_n, n = 1, 2, \ldots$ be the length of the n^{th} survival period and S_n be the time of the n^{th} replacement. Hence,

$$S_n = \sum_{i=1}^{n} T_i, \quad n = 1, 2, \ldots,$$

where $S_0 = 0$. The point process $T_1, T_2, T_3, \ldots$ on the set of positive real values $\mathbb{R}^+$ represents the sequence of inter-arrival times. These inter-arrival times can equivalently be written as $T_n = S_n - S_{n-1}$ and are called *renewal intervals*.

The process

$$N(t) = \max\{n; S_n \leq t\}$$

represents the number of replacements on the interval $[0, t]$. The *counting process* $N = \{N(t); t \geq 0\}$ associated with the partial sums $S_i, i \geq 1$, is called the *ordinary renewal process*

[1] Since replacement time is negligible throughout this analysis, failure and replacement are interchangeable and will be used as equals.

(ORP) if $T_1, T_2, T_3, \ldots$ forms a sequence of non-negative iid random variables with distribution $F_T(t)$ (abstractly given in equation (2.1)).

The probability distribution $F_{S_i}(t)$ of the i-th renewal time S_i is the i-fold convolution $F_T^{(i)}(t)$ of $F_T(t)$ with itself. For the ordinary renewal process on $(0, t]$ we thus have

$$F_{S_i}(t) = \mathbb{P}(T_1 + \ldots + T_i \leq t) = F_T^{(i)}(t),$$

which can recursively be calculated as

$$\begin{aligned} F_T^{(i)}(t) &= F_T(x), \text{ for } i = 1 & (2.2) \\ F_T^{(i)}(t) &= \int_0^t F_T^{(i-1)}(t-y)\mathrm{d}F_T(y), \text{ for } i \geq 2, & (2.3) \end{aligned}$$

where $\mathrm{d}F_T(y) = f_T(y)\mathrm{d}y$ if the PDF of T exists.

2.1.2 Renewal Function

The expectation of the number of renewals up to time t, $M(t) = \mathbb{E}[N(t)]$, is called the *renewal function*. We notice that we can write the counting process $N(t)$ as a sum of indicator functions on the renewal times, i.e. counting all the renewal times on the interval $(0, t]$ as a unit occurrence and every renewal outside $(0, t]$ as a zero:

$$N(t) = \sum_{i=1}^{\infty} \mathbb{1}_{\{S_i \leq t\}}.$$

Taking expectations gives the renewal function[2]

$$M(t) = \mathbb{E}[N(t)] = \mathbb{E}\left[\sum_{i=1}^{\infty} \mathbb{1}_{\{S_i \leq t\}}\right] = \sum_{i=1}^{\infty} \mathbb{P}(S_i \leq t) = \sum_{i=1}^{\infty} F_T^{(i)}(t). \quad (2.4)$$

Because of the infinite sum in equation (2.4) it is impractical to compute. Instead we will rewrite the renewal equation in an integral form.

We can use equations (2.2)-(2.3) to write equation (2.4) as

$$\begin{aligned} M(t) &= \sum_{i=1}^{\infty} F_T^{(i)}(t) \\ &= F_T(t) + \int_0^t M(t-y)\mathrm{d}F_T(y). & (2.5) \end{aligned}$$

[2]Here we use that

$$\mathbb{E}\left[\mathbb{1}_{\{A\}}\right] = \mathbb{P}(A).$$

Intuitively, if Ω is the set of all renewals S_i, $i = 1, 2, \ldots$, and A is the set consisting of renewals only in the interval $(0, t]$, the indicator function assigns all S_i on $(0, t]$ as 1 and all others as 0. Thus, this gives exactly the amount of elements in A, which is the proportion of S_i, $i = 1, 2, \ldots$, that is in $(0, t]$. This is equal to the probability of S_i being in $(0, t]$.

We call the relation in equation (2.5) the *renewal integral equation*. A direct numerical approximation of this equation by a trapezoidal rule is a more simple, practical and sufficiently accurate method, [79] and [67, p. 311]. We notice that if $F_T(t)$ has Probability Density Function (PDF) given by $f_T(t)$, we can write equation (2.5) as

$$\begin{aligned} M(t) &= F_T(t) + \int_0^t M(t-y) f_T(y) \mathrm{d}y \\ &= F_T(t) + (M * f_T)(t). \end{aligned} \tag{2.6}$$

We notice that $M(t)$ is completely defined by $F_T(t)$. An important generalization is given in the following lemma.

Lemma 2.1. *Let $f_T(t)$ be a PDF with support $\mathbb{R}^+$ and let $b(t)$ be a given, integrable function that is bounded on finite intervals. Let $Z(t)$, $t \geq 0$, be defined by the integral equation*

$$Z(t) = b(t) + \int_0^t Z(t-y) f_T(y) \mathrm{d}y.$$

Then the solution of this equation is unique and bounded on finite intervals and is given as

$$Z(t) = \sum_{n=0}^{\infty} \left(F_T^{(n)} * b \right)(t) \tag{2.7}$$

or equivalently as

$$Z(t) = b(t) + \int_0^t b(t-y) m(y) \mathrm{d}y, \;\; t \geq 0, \tag{2.8}$$

where the renewal density $m(t)$ is the derivative of $M(t)$.

The proof of this lemma uses the fact that the process probabilistically starts over after each renewal and is outlined in [68].

The *renewal density*, $m(t)$, is defined as the expected amount of renewals per unit time:

$$m(t) = \frac{\mathrm{d}M(t)}{\mathrm{d}t} = f_T(t) + (m * f_T)(t). \tag{2.9}$$

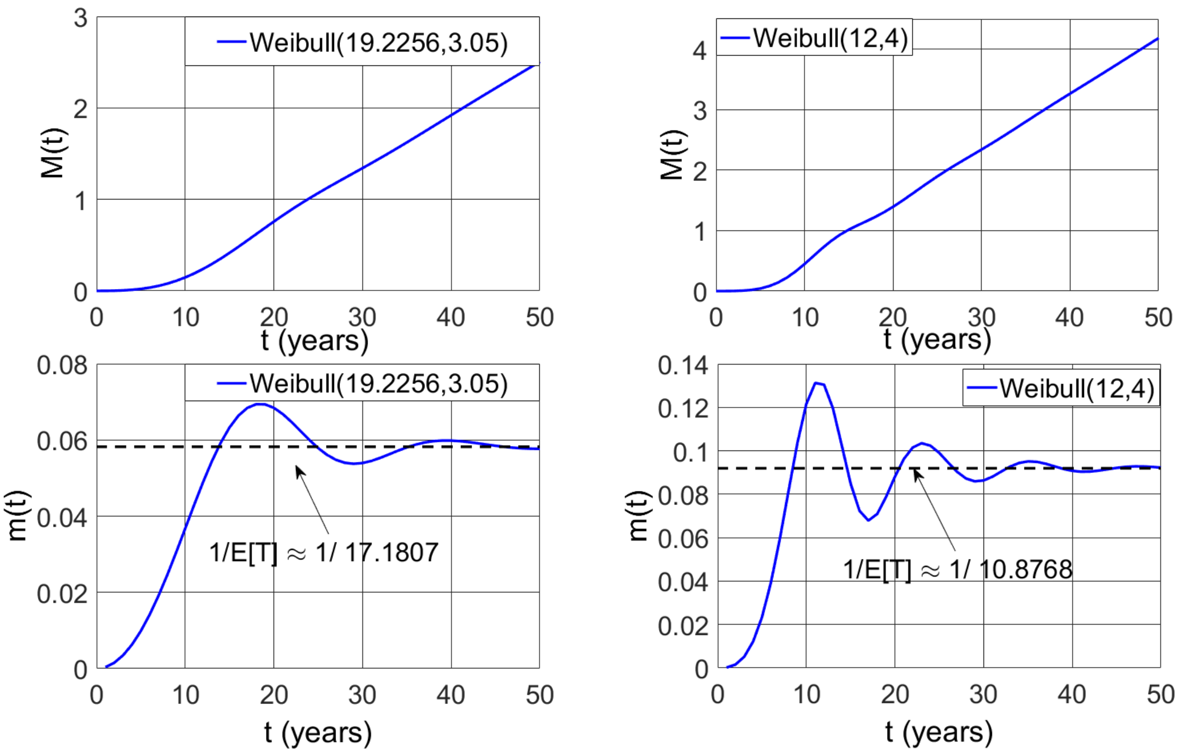
We can use the Erdös-Feller-Pollard [15] theorem to approximate $M(t)$:

$$\lim_{t\to\infty} \frac{M(t)}{t} = \lim_{t\to\infty} m(t) = \frac{1}{\mathbb{E}[T]}, \tag{2.10}$$

where $\mathbb{E}[T] = \int_0^\infty t \cdot f_T(t) \mathrm{d}t$ is the mean of T.

To help visualize and explain equations (2.6) and (2.9) we present two examples. In these examples the integrals in the equations are approximated by a trapezoidal rule with $\Delta t = 1$ (year) on a 50 year time horizon.

In Figure 2.1 the renewal function and the renewal density are depicted for a component lifetime given by a Weibull distribution with different parameters: in Figure 2.1 (a), $\alpha = 19.2256$ and shape parameter $\beta = 3.05$ while in Figure 2.1 (b), $\alpha = 12$ and shape parameter $\beta = 4$.



(a) $\alpha = 19.2256$ and $\beta = 3.05$.

(b) $\alpha = 12$ and $\beta = 4$.

Figure 2.1: Renewal function and renewal density for Weibull distributions.

2.2 Modelling Hazards

2.2.1 Terminology

Hazards are assumed to arrive at times $S_1, S_2, \ldots$ with independent and identically distributed (*iid*) inter-arrival times $T_1, T_2, \ldots$. So the partial sums of the sequence $T_1, T_2, \ldots$ is denoted as S_n and we have $S_n = S_{n-1} + T_n$ for $n \geq 1$ with the convention that $S_0 = 0$. At each arrival time S_n there is a corresponding mark/load of random magnitude X_n. Since inter-arrival times and load magnitudes are both non-negative, this model is based on *iid* random vectors $(T_1, X_1), (T_2, X_2), \ldots$ with components from distributions $T \sim F_T(t)$ and $X \sim F_X(x)$. Furthermore we assume $\mathbb{P}(T > 0) = 1$. The shock magnitude at time t can then be denoted as

$$X(t) = \sum_{n=1}^{\infty} X_n 1_{\{t=S_n\}}, \quad t \geq 0. \tag{2.11}$$

Figure 2.2 is a depiction of the marked renewal process which follows from the sequence $(T_1, X_1), (T_2, X_2), \ldots$. Let $N = \{N(t); t \geq 0\}$ be the counting process associated with the increasing sequence $0 < S_1 < S_2 < \cdots$. We are interested in calculating the probability distribution of the maximum load up to time t, where the maximum load is defined as

$$X^*(t) = \max\{X(s); s \leq t\} = \max(X_1, X_2, \ldots, X_{N(t)}).$$

2.2.2 Concept of Renewal Decomposition

Recall the shock load process given by equation (2.11) as

$$X(t) = \sum_{n=1}^{\infty} X_n 1_{\{t=S_n\}}, \quad t \geq 0. \tag{2.12}$$

Let $\widetilde{X}_n{=}X_{n+1}$, $\widetilde{T}_n{=}T_{n+1}$ and $\widetilde{S}_n{=}S_{n+1} - T_1$ for $n \geq 1$ with the convention $\widetilde{S}_0{=}S_1{=}T_1$. A schematic is given in Figure 2.3. Furthermore let $\widetilde{N}{=}\{\widetilde{N}(t-T_1); t \geq T_1\}$ be the counting process associated with the increasing sequence $T_1 < \widetilde{S}_1 < \widetilde{S}_2 < \cdots$, i.e.

$$\widetilde{N}(t) = \sum_{i=1}^{\infty} 1_{\{\widetilde{S}_i(t)\}}.$$

The renewal decomposition property means that

- The counting process $\widetilde{N}(t)$ has the same distribution as $N(t)$. It implies the expected value and other higher moments of the two processes are identical.
- The shifted process is independent of the time of shift, i.e. $\widetilde{N}(t)$ and T_1 are independent.

As shown by [52], a decomposition of the original counting process can be written as

$$N(t) = 1_{\{T_1 \leq t\}} + \widetilde{N}(t - T_1). \tag{2.13}$$

In Figure 2.4 (a) an example of a counting process $N(t)$ is depicted while in Figure 2.4 (b) a probabilistically equivalent process $\widetilde{N}(t-T_1)$ is depicted. Let $\widetilde{X}{=}\{\widetilde{X}(t); t \geq 0\}$ where

$$\widetilde{X}(t) = \sum_{n=1}^{\infty} \widetilde{X}_n 1_{\{t=\widetilde{S}_n\}}, \quad t \geq 0.$$

The processes X and $\widetilde{X}$ have the same distribution and the process $\widetilde{X}$ is independent of the vector of first occurrence (T_1, X_1). For $t > T_1$ it follows:

$$\begin{aligned}
\widetilde{X}(t-T_1) &= \sum_{n=1}^{\infty} X_{n+1} 1_{\{t=S_{n+1}-T_1\}} \\
&= \sum_{n=2}^{\infty} X_n 1_{\{t=S_n\}} \\
&= X(t) - X_1 1_{\{t=T_1\}}.
\end{aligned}$$

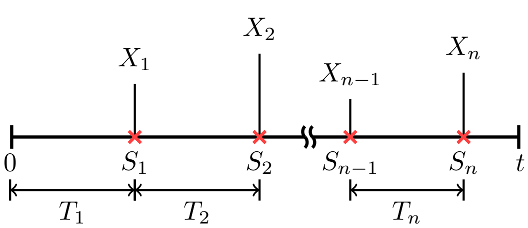
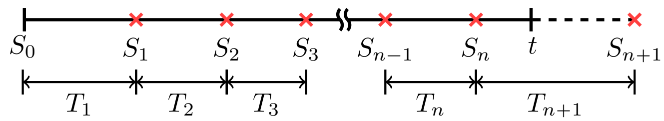
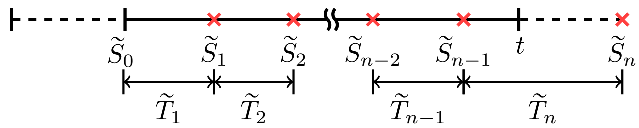


Figure 2.2: A marked renewal process.



(a) Renewal process



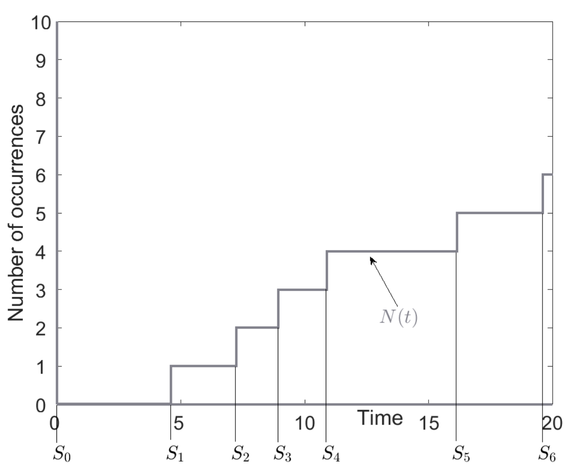
(b) Shifted renewal process

Figure 2.3: (a) A schematic of the renewal process, (b) an illustration of the renewal decomposition argument.

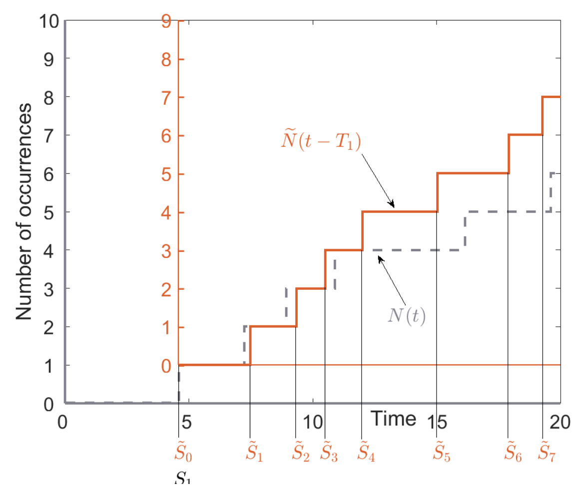
Taking $\widetilde{X}(t) = 0$ for $t < 0$ gives:

$$X(t) = X_1 1_{\{t=T_1\}} + \widetilde{X}(t - T_1), \;\; t \geq 0.$$

Hence, since the processes X and $\widetilde{X}$ have the same distribution, after the first occurrence (T_1, X_1) the shock load process probabilistically starts all over again.



(a) N.



(b) $\widetilde{N}$.

Figure 2.4: Counting processes.

2.3 Poisson Process

A special case of a renewal process is a Poisson process. The Poisson process is extensively used in reliability analysis due to its asymptotic behaviour and the simple, analytical solutions it provides. Let $N(t)$ be the integer valued random variable denoting the number of events in the interval $(0, t]$. Then, the HPP can be entirely derived from some intuitive postulates:

1. **Homogeneity** The statistical behaviour of the arrival time of an event in any time interval $[a + h, b + h]$, where $0 < a < b$ and $h > 0$, is independent of h. For our purposes this means for example: the number of events does not depend on any time interval but only on the length of the interval.

2. **Orderliness** For an interval of length h, the probability of having more than one event in the interval is very small compared to its length. Practically this implies the assumption of no simultaneous events.

3. **Independence** The number of events in disjoint intervals are statistically independent.

4. **Activity** The probability of at least one event occurring in a time period of length h (and from homogeneity independent of the time interval) is

   $$\mathbb{P}(N(h) \geq 1) = \lambda h + O(h), \;\; h \to 0, \;\; \lambda > 0.$$

   This is the same as saying

   $$\lim_{h \to 0} \frac{\mathbb{P}(N(h) \geq 1)}{h} = \lambda, \;\; \lambda > 0,$$

   so that there is a probability of an event occurring at every single time point.

The counting process $\{N(t), t \geq 0\}$ is then called a *homogenous Poisson process* with constant intensity parameter λ. From the above postulates it also follows that

$$\mathbb{P}(N(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}.$$

Thus $N(t)$ follows a Poisson distribution with parameter λt on an interval of length t. The proof of this formula is beyond the scope of this chapter and is omitted. The HPP with intensity λ is a renewal process with T following the exponential law, i.e. $F_T(t)$=$1 - e^{-\lambda t}$. The renewal function of an HPP is a linear function of time, $M(t)$=$\mathbb{E}[N(t)]$=λt, and the renewal rate is a constant, λ. It is worth noting that the Poisson process possesses the memoryless property in the sense that the probability of an event occurring in an interval does not depend on the past but only on the length of the interval. In certain formulations this is also the culprit of this process.

2.4 First Crossing Time of Maximum Process

In this section we show that the first passage time of a process $\{X(t), t \geq 0\}$ and its maximum process $\{X^*(t), t \geq 0\}$ are the same. Let $X{=}\{X(t); t \geq 0\}$ be the load process which a component is subject to. Let $\widetilde{T}$ be the random variable denoting the lifetime distribution of the component. We are interested in calculating the reliability of the component, i.e. $\mathbb{P}(\widetilde{T}{\leq}t){=}F_{\widetilde{T}}(t)$. This is a non-repairable component and in particular the component fails the first time $X(t)$ exceeds the fixed component capacity x. Thus, we are interested in the time

$$\widetilde{t} = \min\{t \geq 0; X(t) \geq x\}.$$

To analyze $\widetilde{t}$, and thus to compute $\mathbb{P}(\widetilde{T} \leq t)$, we instead use the maximum of the process $X(t)$, defined as:

$$X^*(t) = \max\{X(s); s \leq t\}. \tag{2.14}$$

This approach is easier since a characteristic of the maximum can easily be used[3].

Define t^* as the first time $X^*(t)$ is larger than the threshold x;

$$t^* = \min\{t \geq 0; X^*(t) \geq x\}.$$

It is left to be shown that calculating t^* is equivalent to calculating $\widetilde{t}$.

Lemma 2.2. *$\widetilde{t}$ is reached **if and only if** t^* is reached.*

Proof:
Assume $\widetilde{t}$ is reached. We have $X(t) \leq X^*(t)$ $\forall t$. For all $t \in [0, \widetilde{t})$ we have $X(t) < x$ and hence also $\max\limits_{0 \leq t < \widetilde{t}}\{X(t)\} = X^*(\widetilde{t}_-) < x$, where $\widetilde{t}_- = \widetilde{t} - \epsilon$ $\forall \epsilon > 0$.
Moreover, since $X(\widetilde{t}) \geq x$ and $X(t) < x$ for all $t \in [0, \widetilde{t})$ we have

$$X^*(\widetilde{t}) = \max_{0 \leq t \leq \widetilde{t}}\{X(t)\} = X(\widetilde{t}) \geq x.$$

But since $X^*(\widetilde{t}_-) < x$ and $X^*(\widetilde{t}) \geq x$, t^* is reached (at $\widetilde{t}$).

Assume t^* is reached. This implies $X^*(t^*) \geq x$ and $X^*(t^*_-) < x$ where $t^*_- = t^* - \epsilon$ $\forall \epsilon > 0$. Since $X^*(t)$ is a (positive) non-decreasing continuous function

$$x > X^*(t^*_-) = \max_{0 \leq t < t^*_-}\{X(t)\} \geq X(t), \quad \forall t \in [0, t^*),$$

[3] Let $X_1, \ldots, X_{N(t)}$ be the sequence of all shocks in the time interval $[0, t)$. It can be seen that

$$\mathbb{P}(\max\{X_1, \ldots, X_{N(t)}\} \leq x) = \mathbb{P}(X_1 \leq x, \ldots, X_{N(t)} \leq x).$$

Furthermore, if $X_1, \ldots, X_{N(t)}$ are *idd* and $X_1 \sim F_X(x)$:

$$\mathbb{P}(\max\{X_1, \ldots, X_{N(t)}\} \leq x) = F_X(x)^{N(t)}.$$

(implying that $\widetilde{t}$ has not yet been reached).

But since $X^*(t^*) = \max_{0 \le t \le t^*} \{X(t)\} \in \{X(t); t \in [0, t^*]\}$ and $X^*(t^*) \ge x$, we have that

$$X(t^*) = \max_{0 \le t \le t^*} \{X(t)\} = X^*(t^*) \ge x,$$

and hence $\widetilde{t}$ is reached (at t^*).

From the above it can be seen that the following always hold:

1. $\widetilde{t} = t^*$
2. $X(\widetilde{t}) = X^*(t^*)$.

Hence, $\min\{t \ge 0; X(t) \ge x\}$=$\min\{t \ge 0; X^*(t) \ge x\}$ and we can calculate the component reliability $\mathbb{P}(\widetilde{T} \le t)$ by observing the maximum process $X^*(t)$. We also notice that while computing the reliability of the component via the maximum of the load process, the distribution of the maximum load is also attained by the same formula $\mathbb{P}(X^*(t) \le x)$, i.e. when x is fixed we can calculate $\mathbb{P}(X^*(t) \le x; x)$=$\mathbb{P}(\widetilde{T} \le t)$ and for fixed t we can calculate $\mathbb{P}(X^*(t) \le x; t)$=$\mathbb{P}(X^* \le x)$.

Chapter 3

Distribution of Maximum Load: Single Process

3.1 Introduction

3.1.1 Background

Structures are designed to sustain the effect of many different types of loads generated by the surrounding environment. For example, wind, flood, snow and earthquakes generate load events that a structure must survive over its entire service life. Design codes specify provisions such that a structure can survive all extraordinary load events with a high degree of reliability, [43]. Recognizing the time-dependent and randomly fluctuating nature of such loads, the theory of stochastic processes became a natural candidate to model and analyze a wide variety of structural reliability problems.

The concept of extreme load distribution provided a convenient way to analyze structural reliability over a specified service life. The reason is that the extreme load distribution reduces the stochastic reliability problem to a simple time-invariant reliability analysis in the following manner. If $X_{max}(t)$ denotes the maximum of a stochastic load process over a time interval, $(0, t]$, and W is the time-invariant (i.e., no degradation) strength of a component, then the probability of the event, $\{X_{max}(t) > W\}$, represents the probability of failure over the entire service life of the component. This insight has made extreme value distribution an integral part of reliability-based design codes.

The mathematical developments in asymptotic theory of extreme value distribution further inspired this approach to reliability analysis. The asymptotic theory shows that there are well defined domains of attraction of extreme values generated by an *infinite* sequence of *iid* random variables. So the idea emerged that observations of maximum load values in a representative time period (like a year) are the only input data needed for modelling the distribution. The Gumbel distribution became a popular choice to the modelling of extremes of wind speed, flood and other similar environmental loads since the seminal work of [24]. Thus, in this framework the stochastic nature of the load process generating maximum values became irrelevant.

The interest in a more detailed modelling of load processes was renewed by the load combination problem. As a structure might be simultaneously subject to more than one load process, the interest in evaluating the maximum value of such combined loads became important. The short-term, impulsive loads are represented as a shock process and slowly-varying, sustained loads as a pulse process.

There are two primary approaches to analyze the maximum load. The first approach is based on the assumption that load occurrence process is an HPP, [76], for which an explicit solution is available. The second approach relies on the Rice formula for the mean crossing rate of a stochastic process exiting a domain, [38, 74]. The mean crossing rate is used in two ways: (1) the mean rate is directly used as an upper bound to the exceedance probability [37], and (2) the mean rate is taken as a parameter of the Poisson process under an (artificial) assumption that the nature of stochastic crossing is roughly like a Poisson process. With this assumption, the analytical solution of maximum load is continued to be used. A combination of such assumptions was used to analyze extremes generated by a renewal process, [3]. In case of an alternating (on and off) process, [11] used the Markov process model to analyze the problem. For a comprehensive account of stochastic modelling of maximum load, the readers are referred to a monograph by [77] and a more recent review by [16].

3.1.2 Basic Approaches

In the reliability literature, there are two basic approaches to derive the distribution of maximum value. The first approach is based on the asymptotic theory of extreme value distribution. The distribution of maximum value, M_n, of an *iid* sequence of random variables, $X_1, X_2, \ldots, X_n$, with a common distribution $F_X(x)$, can, in principle, be obtained as

$$\mathbb{P}(M_n \leq x) = \mathbb{P}(X_1 \leq x, X_2 \leq x, \ldots, X_n \leq x) = (F_X(x))^n. \tag{3.1}$$

As $n \to \infty$, there exists sequences of constants, $a_n > 0$ and b_n such that

$$\mathbb{P}\left(\frac{M_n - b_n}{a_n} \leq x\right) \to G(x), \qquad \text{as } n \to \infty, \tag{3.2}$$

where $G(x)$ is known as the asymptotic extreme value distribution that belongs to one of the following three types of the distributions: Gumbel, Fréchet and Reversed Weibull. More on extreme value distributions can be found in 3.C.4. The Gumbel distribution became a popular choice for modelling extremes of wind speed, flood and other similar environmental loads since the seminal work of [24].

The asymptotic distribution theory is used as the method of annual maxima. In this method, observations of the maximum value in a representative time period (e.g., one year) are collected for the parameter estimation of an appropriate asymptotic distribution, such as the Gumbel distribution. The method has been useful in modelling more frequent hazards (e.g., high wind and flood events) with long time series of data. This approach is conceptually simple, as it avoids a detailed modelling of stochasticity in the occurrence, intensity and duration of the hazard process.

The second approach to extreme value analysis relies on a more formal stochastic process model with well defined components (e.g., occurrence, intensity and duration). Since the seminal work of Cornell on seismic risk analysis [8], the HPP has become the most preferred model of hazards in structural reliability. The HPP model of flooding events, developed by Todorovic and Zelenhasic [69], had a similar influence in hydrology. The interest in stochastic process modelling was also driven by the load combination problem in which the maximum value generated by simultaneous occurrences of two (or more) hazards was investigated [76]. For a detailed account of stochastic modelling of various load processes, readers are referred to a monograph by [77] and a more recent review by [16]. It is worth remarking that the HPP model is almost exclusively used in the literature to model shock and pulse load processes. The analytical simplicity of this approach is certainly a contributing factor.

In cases where a load process can be modelled as a Gaussian process, a solution approach based on the Rice formula was developed [74]. In this approach, the mean crossing rate of a barrier by the process is computed and then used in the determination of the maximum load distribution in two approximate ways: (1) the mean crossing rate is directly used as an upper bound to the exceedance probability [37], and (2) relying on an asymptotic result that rare crossings converge to the Poisson process with its rate parameter equal to the mean crossing rate calculated by the Rice formula.

3.1.3 Limitations of Existing Literature

Although there is a fairly substantial body of engineering literature on stochastic modelling of maximum load, the following key limitations are worth noting:

- The literature is replete with approximate solutions based on the mean crossing rate and the Poisson crossing assumption. Although these assumptions can be justified in case of extremes of a continuous Gaussian stochastic process, their use for shock and pulse types load processes is unnecessary and conceptually incorrect.

- In the modelling of maximum wind speed distribution, [26] reported that a slow convergence of data to the asymptotic extreme value distribution introduces error in the parameter estimation and in some cases data appear to have converged to a wrong asymptotic form.

- The distribution of maximum load generated by the HPP process implies that the time between load occurrences is an exponential random variable. The exponential distribution may not be applicable all hazards faced by structures and system. In case of a non-exponential distribution of inter-occurrence time, the theory of renewal process must be used for which there is no explicit solution available. It is not expected that all hazards will comply with such assumptions of HPP model. It is therefore desirable to find a solution for the case of a renewal process with a non-exponential distribution of inter-arrival time. An accurate solution of this problem is lacking in the literature. An approximate approach to analyze this problem was presented by [3] who used the Rice formula and the Poisson crossing assumption.

  An asymptotic result about the Poisson nature of level crossings is applicable to a high barrier and a long interval of time. This approach becomes questionable in risk and reliability assessment over a short service life. For example, temporary structures are designed for a short service life and modest loading requirements. Another example comes from stress test of nuclear plant structures that are near the end of life.

- In addition to a shock and pulse type process, an alternating renewal process provides a more general approach to model an intermittent load process. This problem has received very limited attention in the literature. An alternating process with exponential on and off times was analyzed as a two phase Markov process by [11]. For a renewal type alternating process with non-exponential distributions of on and off time, no solution has been reported in the literature.

Therefore, there is a pressing need for developing an accurate, finite-time solution for the distribution of maximum value generated by a more general class of stochastic processes, i.e. the renewal process.

3.1.4 Contributions to Literature

In structural safety analysis the reliability and maximum impact of hazards are of fundamental importance for safe operation during the remaining life span of a structure. For the reliability computations it is well known that models based on a Poisson process are asymptotically justified and a Poisson model is appealing for its simple closed-form solution. However, if inter-occurrence times of hazards do not follow an exponential distribution or when solutions are needed for short life spans, these asymptotic solutions cannot be readily justified. This chapter presents a systematic development of a more general stochastic process model of reliability in which occurrences of a hazard and its impact are conceptually modelled as a marked renewal process from which exact solutions are obtained. The proposed model covers shock, pulse and alternating processes.

These same results can be used to compute the distribution of the maximum shock load on a structure. Since the solutions are obtained using the mechanistic model of inter-occurrence times and distribution of loads as inputs of the model, again relying on asymptotic behaviour of the maximum of long sequences approaching a Gumbel, Fréchet or reversed Weibull distribution is not needed as is often done in practice. Solutions can easily be approximated numerically using a trapezoidal scheme and in this chapter it is also shown, using an example, that closed-form solutions can be obtained via the theory of defective renewal equations. Other numerical examples showing the Probability of Exceedance (POE) of loads, highlight the differences between estimated solutions and their approximate limit solution as a Poisson model. From the practical example it can be seen that the true POE of extreme loads can be significantly underestimated using an approximate Poisson model.

3.1.5 Objectives and Organization

In this chapter the strength (or capacity) of a component is assumed to be time-invariant, i.e. the strength does not change over time. With this assumption the component reliability and distribution of the maximum load can be obtained through the theory of renewal processes. The loads are assumed to be from a single source and can be either: (1) a shock process having negligible duration, (2) a pulse process, where a load is sustained till the arrival of a new load, or (3) an alternating pulse process, which switches between an 'on' and 'off' state.

In section 3.2 the maximum distribution of the shock, pulse and alternating processes are derived as renewal integrals. In section 3.3 a numerical method and an analytical method to compute renewal integrals are discussed. In section 3.4 a numerical example is presented to explicitly show how to derive analytical solutions from a renewal equation while section 3.5 presents some practical numerical examples. The chapter concludes with an appendix where some mathematical proofs and further background material are provided.

3.2 Maximum of Single Load Process

3.2.1 Maximum of a Shock Load Process

A shock load process can be modelled as a marked renewal process, as shown in Figure 3.1. In this model, a random variable (or mark), Y, is attached to each occurrence of a shock to represent the load magnitude. A marked process is defined by a sequence of *iid* random vectors (T_i, Y_i), $i = 1, 2, \ldots$. The joint distribution of (T_i, Y_i) is independent of (T_j, Y_j) for $i \neq j$, [66, p. 321]. We want to calculate $\mathbb{P}(X^*(t) \leq x)$. Notice that since $\mathbb{P}(T > 0) = 1$ and there is no shock at time $t = 0$, there are $N(t)$ shocks in the interval $[0, t]$ and we define the maximum of the shock load process $X(t)$ on $[0, t]$ as

$$X^*(t) = \max_{0 \leq s \leq t} X(s) = \begin{cases} 0, & N(t) = 0, \\ \max\{Y_1, \ldots, Y_{N(t)}\}, & N(t) \geq 1. \end{cases} \tag{3.3}$$

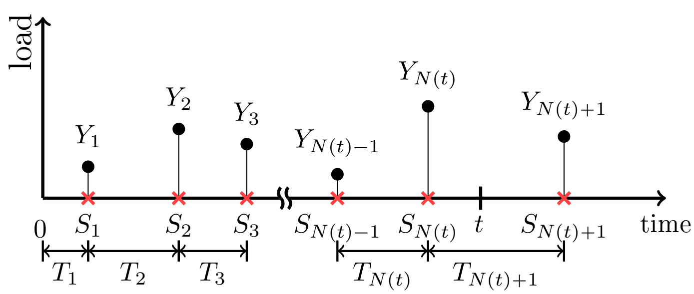


Figure 3.1: A schematic of a shock load process.

Unless mentioned explicitly, we assume loads and inter-arrival times are independent. We want to calculate the distribution of $X^*(t)$. The following general result can be derived:

Lemma 3.1. *Given the sequences $\{Y_n; n \geq 1\}$ and $\{T_n; n \geq 1\}$ are independent and let $\{Y_n; n \geq 1\}$ be iid with distribution $F_Y(y)$ where $F_Y(0) = 0$. Then,*

$$\mathbb{P}(X^*(t) \leq x) = \mathbb{E}\left[F_Y(x)^{N(t)}\right].$$

Proof:

$$\mathbb{P}(X^*(t) \leq x) = \mathbb{P}(\max\{Y_1, \ldots, Y_{N(t)}\} \leq x).$$

Now partition over the events $\{N(t) = n\}$ to get

$$\begin{aligned}
\mathbb{P}(\max\{Y_1, \ldots, Y_{N(t)}\} \leq x) &= \sum_{n=0}^{\infty} \mathbb{P}(\max\{Y_1, \ldots, Y_{N(t)}\} \leq x, N(t) = n) \\
&= \sum_{n=0}^{\infty} (F_Y(x))^n \, \mathbb{P}(N(t) = n) \qquad (3.4) \\
&= \mathbb{E}\left[F_Y(x)^{N(t)}\right]. \qquad (3.5)
\end{aligned}$$

3.2.1.1 Homogeneous Poisson Process

The distribution of maximum load for a general renewal shock process can be approximated by assuming an asymptotic limit of the process. This leads to the approximation of the distribution as an HPP with intensity λ. We notice that this is equivalent to approximating the inter-arrival times by an exponential distribution with parameter λ, i.e. $F_T(t) = 1 - e^{-\lambda t}$. The following well-known result can be derived from equation (3.4):

Lemma 3.2. *For the HPP with intensity λ we have:*

$$\mathbb{P}(X^*(t) \leq x) = e^{-\lambda t(1 - F_Y(x))}.$$

Proof: *Recall for an HPP we have: $\mathbb{P}(N(t) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$. Then*

$$\begin{aligned}
\mathbb{P}(X^*(t) \leq x) &= \sum_{n=0}^{\infty} (F_Y(x))^n \, \mathbb{P}(N(t) = n) \\
&= \sum_{n=0}^{\infty} (F_Y(x))^n \, \frac{(\lambda t)^n}{n!} e^{-\lambda t} \\
&= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(F_Y(x) \lambda t)^n}{n!} \\
&= e^{-\lambda t} e^{F_Y(x) \lambda t} \\
&= e^{-\lambda t(1 - F_Y(x))}. \qquad (3.6)
\end{aligned}$$

To use the HPP result as an approximate solution of a renewal shock process, the intensity parameter λ needs to be approximated. This can be done from elementary renewal theorem by observing that the asymptotic limit of the renewal rate is given by

$$\lambda_\infty = \lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{\mathbb{E}[T]}, \text{ with probability 1,} \qquad (3.7)$$

where $\mathbb{E}[T]$ is the expectation of the inter-arrival times of the shock process. Hence, as $t \to \infty$ the distribution of the maximum of a renewal shock process can be approximated by an HPP with parameter $\lambda = \lambda_\infty$.

3.2.1.2 General Renewal Process

When the inter-arrival times follow a general distribution $F_T(t)$, the distribution of $X^*(t)$ can be calculated by a renewal type equation given in the following lemma:

Lemma 3.3. *Let $\{T_n; n \geq 1\}$ be iid with distribution $F_T(t)$. Then we have:*

$$\mathbb{P}(X^*(t) \leq x) = \bar{F}_T(t) + F_Y(x) \int_0^t \mathbb{P}(X^*(t-s) \leq x)\mathrm{d}F_T(s). \tag{3.8}$$

Proof: *For the sake of notation, let $z(t){=}\mathbb{P}(X^*(t) \leq x){=}\mathbb{E}\left[F_Y(x)^{N(t)}\right]$ from equation (3.5). Now we condition on T_1, so fix $t > 0$ and use that the process probabilistically starts over again as explained in 2.2.2:*

$$\begin{aligned}
z(t) &= \mathbb{E}\left[F_Y(x)^{N(t)}; T_1 > t\right] + \mathbb{E}\left[F_Y(x)^{N(t)}; T_1 \leq t\right] \\
&= \mathbb{E}\left[F_Y(x)^0; T_1 > t\right] + \mathbb{E}\left[F_Y(x)^{(1+\tilde{N}(t-T_1))}; T_1 \leq t\right] \\
&= \int_t^\infty 1 \cdot \mathrm{d}F_T(s) + \int_0^t \mathbb{E}\left[F_Y(x)^{(1+\tilde{N}(t-s))}\right]\mathrm{d}F_T(s) \\
&= \left[1 - \int_0^t \mathrm{d}F_T(s)\right] + F_Y(x)\int_0^t \mathbb{E}\left[F_Y(x)^{\tilde{N}(t-s)}\right]\mathrm{d}F_T(s)
\end{aligned}$$

Now since $\tilde{N}(t-s) \sim N(t-s)$ we have $\mathbb{E}\left[F_Y(x)^{\tilde{N}(t-s)}\right] \sim \mathbb{E}\left[F_Y(x)^{N(t-s)}\right] = z(t-s)$.

Hence, $z(t) = \mathbb{P}(X^*(t) \leq x)$ satisfies the equation

$$z(t) = \bar{F}_T(t) + F_Y(x)\int_0^t z(t-s)\mathrm{d}F_T(s). \tag{3.9}$$

3.2.2 Maximum of a Pulse Load Process

An illustration of a pulse load process is given in Figure 3.2. We notice that mathematically speaking, the only difference between the pulse load process and the shock load process is the extra load at time $t{=}0$. This is because the sustained part of the load does not play a role in the failure of the process since the strength is time-invariant between load changes. The distribution of the maximum pulse load is given as $\mathbb{P}(X^*(t) \leq x) = \mathbb{P}(\max\{Y_1, \ldots, Y_{N(t)+1}\} \leq x)$ and we have the following result:

Lemma 3.4. *Given the sequences $\{Y_n; n \geq 1\}$ and $\{T_n; n \geq 1\}$ are independent and let $\{Y_n; n \geq 1\}$ be iid with distribution $F_Y(y)$ where $F_Y(0) = 0$. Then,*

$$\begin{aligned}
\mathbb{P}(X^*(t) \leq x) &= F_Y(x)\mathbb{E}\left[F_Y(x)^{N(t)}\right] \\
&= F_Y(x)\mathbb{P}^{shock}(X^*(t) \leq x),
\end{aligned}$$

where $\mathbb{P}^{shock}(X^*(t) \leq x)$ is the CDF of the shock load process.

Proof: *The result follows easily by following the same steps as in Lemma 3.1.*

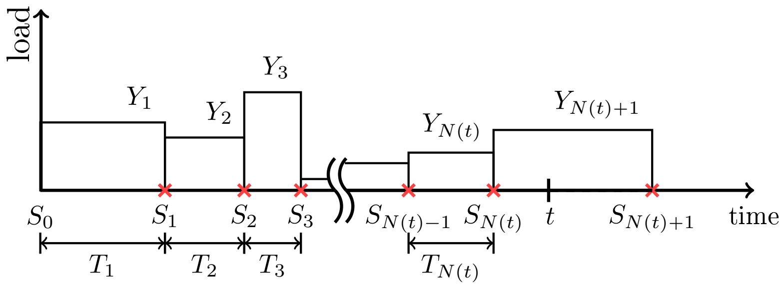


Figure 3.2: A schematic of a pulse load process.

3.2.2.1 Homogeneous Poisson Process

For an HPP we have the following result.

Lemma 3.5. *For the HPP with intensity λ we have:*

$$\mathbb{P}(X^*(t) \leq x) = F_Y(x)e^{-\lambda t(1-F_Y(x))}, \tag{3.10}$$

which follows easily from Lemmas 3.4 and 3.2.

We proceed by showing the results for a general renewal process. But since the HPP is a special case of a renewal process, with exponentially distributed inter-arrival times, the result in equation (3.10) can be derived using Lemma 3.6 for a general renewal process. This derivation is in the Appendix.

3.2.2.2 General Renewal Process

For a general renewal process we have the following result:

Lemma 3.6. *Let $\{T_n; n \geq 1\}$ be iid with distribution $F_T(t)$. Then we have:*

$$\mathbb{P}(X^*(t) \leq x) = \bar{F}_T(t)F_Y(x) + F_Y(x)\int_0^t \mathbb{P}(X^*(t-s) \leq x)\mathrm{d}F_T(s). \tag{3.11}$$

Proof: *Let $\mathbb{P}^{shock}(X^*(t) \leq x)$ denote the result in equation (3.8) for the CDF of the maximum shock load. From Lemma 3.4 we have the result*

$$F_Y(x)\mathbb{P}^{shock}(X^*(t) \leq x) = \bar{F}_T(t)F_Y(x) + (F_Y(x))^2 \int_0^t \mathbb{P}^{shock}(X^*(t-s) \leq x)\mathrm{d}F_T(s).$$

Now let $y(t){=}\mathbb{P}(X^*(t) \leq x){=}F_Y(x)\mathbb{P}^{shock}(X^*(t) \leq x)$, then $y(t)$ satisfies the renewal-type equation

$$y(t) = \bar{F}_T(t)F_Y(x) + F_Y(x)\int_0^t y(t-s)\mathrm{d}F_T(s). \tag{3.12}$$

Hence, to compute the CDF of the maximum pulse load process, equation (3.11) can be used or the result from the computation of the CDF of the maximum pulse load process can be directly multiplied by $F_Y(x)$.

Alternatively, the following result can also be used which is another renewal-type equation to compute the CDF of the maximum pulse load process. The detailed derivation can be found in the Appendix.

Lemma 3.7. *The CDF of the maximum pulse load process is given by*

$$\mathbb{P}(X^*(t) \leq x) = F_Y(x) - (1 - F_Y(x)) \sum_{n=1}^{\infty} F_Y(x)^n F_T^{(n)}(t), \tag{3.13}$$

where

$$y(t) = \sum_{n=1}^{\infty} F_Y(x)^n F_T^{(n)}(t)$$

satisfies the renewal-type equation

$$y(t) = F_T(t)F_Y(x) + F_Y(x) \int_0^t y(t-s)\mathrm{d}F_T(s).$$

Hence, equations (3.11) and (3.13) are equivalent.

3.2.3 Maximum of Alternating Pulse Process

Certain events exhibit behaviour more resembling an intermittent pulse process. Such an event could be a snow load process or a body of water temporarily occupying a space, e.g. a flood. In the model for an alternating process it is assumed that a load pulse of random magnitude Y is sustained for a random period W followed by a random period of inactivity Z. The alternating load process is therefore fully defined by an *iid* sequence of random vectors, $(Y_1, W_1, Z_1), (Y_2, W_2, Z_2), \ldots$. This is depicted in Figure 3.3.

The renewal cycle length is then $T_n = W_n + Z_n$, while in the n^{th} cycle the process is 'on' in the interval $[S_{n-1}, S_{n-1} + W_n)$ with magnitude Y_n. The maximum of the alternating pulse load process is defined as $\mathbb{P}(X^*(t) \leq x) = \mathbb{P}(\max\{Y_1, \ldots, Y_{N(t)+1}\} \leq x)$ which is similar to the pulse process since in the interval $(0, t]$ with $N(t)$ cycles there are $N(t)+1$ loads. The distribution of maximum load is given as

$$\begin{aligned} \mathbb{P}(X^*(t) \leq x) &= F_Y(x)\mathbb{E}\left[F_Y(x)^{N(t)}\right] \\ &= F_Y(x)\mathbb{P}^{shock}(X^*(t) \leq x), \end{aligned}$$

which is the same as the pulse process with the exception that the distribution of inter-arrival times is now given by the random variable $T{=}W{+}Z$ which is the sum of the random variables denoting 'on' and 'off' times, respectively.

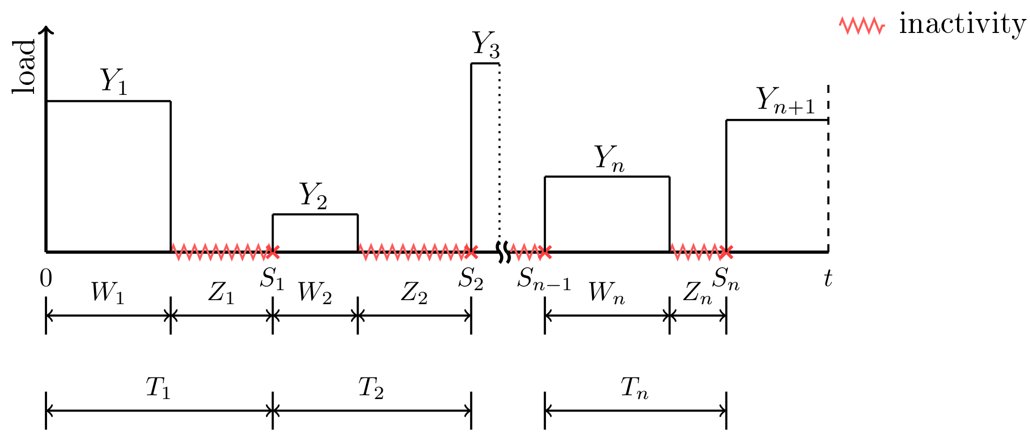


Figure 3.3: A schematic of an alternating load process.

3.3 Methods to Compute Renewal Integrals

3.3.1 Numerical Integration: Trapezoidal Rule

The method of numerical integration using the trapezoidal rule is demonstrated using a concrete example, equation (3.8) for the CDF of the maximum of a shock load process. Other renewal-type equations can be solved in a similar way. This solution is required to be computed recursively. Recall that we have the following equation:

$$\mathbb{P}(X^*(t) \le x) = \bar{F}_T(t) + F_Y(x) \int_0^t \mathbb{P}(X^*(t-s) \le x) \mathrm{d}F_T(s). \tag{3.14}$$

Assume $dF_T(s) = f_T(s)dt$ exists and divide time uniformly as $0 = s_0 < s_1 < \cdots < s_{N-1} < s_N = t$. Hence, the time domain is discretized into N equal parts of size $\frac{t-0}{N}$. For ease of notation let $h(t;x) = \mathbb{P}(X^*(t) \le x)$. Hence, we have

$$h(t;x) = \bar{F}_T(t) + F_Y(x) \int_0^t h(t-s;x) \mathrm{d}F_T(s). \tag{3.15}$$

Then, according to the trapezoidal rule, for fixed x, equation (3.15) can be approximated numerically as:

$$h(t;x) \approx \bar{F}_T(t) + F_X(x) \left[\frac{t}{2N} h(t;x) f_T(0) + \frac{t}{2N} 2 \sum_{i=1}^{N-1} h(t-s_i;x) f_T(s_i) + \frac{t}{2N} h(0;x) f_T(t) \right]$$

so that the reliability can be computed as

$$h(t;x) \approx \frac{1}{1 - \frac{t}{2N} F_X(x) f_T(0)} \left\{ \bar{F}_T(t) + \frac{t F_X(x)}{N} \left[\frac{1}{2} h(0;x) f_T(t) + \sum_{i=1}^{N-1} h(t-s_i;x) f_T(s_i) \right] \right\}, \tag{3.16}$$

where $h(0;x)=\bar{F}_T(t)=1$. Hence, $h(t;x)$ needs to be computed recursively as $h(0;x)$, $h(s_1;x)$, $h(s_2;x)$, ..., $h(t;x)$.

To compute the CDF of the maximum of the process for a fixed time horizon t, equation (3.16) can be used and repeated for values of x in the domain of interest to compute $h(x_1;t)$, $h(x_2;t)$, $h(x_3;t)$, ...

When computing $h(t,x)$ special caution has to be taken for approximation with the trapezoidal rule. For functions $f_T(s)$ that have a vertical asymptote, e.g. Weibull distribution with shape parameter less than one or very long-tailed distributions like the Lognormal distribution with high peaks, the solution may not converge even for large N.

3.3.2 Analytical Solution: Defective Renewal Equation

Using the method of defective renewal equation we try to rewrite the renewal equation into an analytical form. If this is possible, we get a much simpler solution, namely analytical form instead of (recursive) numerical approximation or Laplace transformation. Consider again the CDF of the maximum of the shock load process:

$$h(t) = \bar{F}_T(t) + F_Y(x)\int_0^t h(t-s)\mathrm{d}F_T(s). \tag{3.17}$$

This equation can be written in the form

$$h(t) = b(t) + \phi\int_0^t h(t-s)\mathrm{d}F_T(s), \;\; t \geq 0 \tag{3.18}$$

where $\phi=F_Y(x) \in (0,1)$ and equation (3.18) is called a *defective renewal equation*[1].

We can transform equation (3.18) into a proper renewal equation by finding a parameter γ such that

$$\mathrm{d}\widetilde{F}_T(t) = e^{\gamma t}\phi\mathrm{d}F_T(t) \tag{3.19}$$

is a probability distribution function, i.e

$$\int_0^\infty \mathrm{d}\widetilde{F}_T(t) = 1. \tag{3.20}$$

Now we can substitute $e^{-\gamma t}\mathrm{d}\widetilde{F}_T(t)=\phi\mathrm{d}F_T(t)$ into the defective renewal equation to obtain

$$h(t) = b(t) + \int_0^t h(t-s)e^{-\gamma s}\mathrm{d}\widetilde{F}_T(s), \;\; t \geq 0$$

[1]We notice that for $h(t) = b(t) + \phi\int_0^t h(t-s)\mathrm{d}F_T(s) = b(t) + \int_0^t h(t-s)\mathrm{d}G_T(s)$ where $G_T(s) = \phi F_T(s)$ it follows that $G_T(0) = 0$ and $G_T(\infty) < 1$ since $\phi \in (0,1)$. Hence, the improper renewal equation (3.18) is called *defective*. If $G_T(0) = 0$ and $G_T(\infty) > 1$ the improper renewal equation is called *excessive*.

and multiplying both sides by $e^{\gamma t}$ gives

$$e^{\gamma t}h(t) = e^{\gamma t}b(t) + \int_0^t h(t-s)e^{\gamma(t-s)}\mathrm{d}\widetilde{F}_T(s), \ \ t \geq 0.$$

Now let $\widetilde{h}(t){=}e^{\gamma t}h(t)$ and $\widetilde{b}(t){=}e^{\gamma t}b(t)$. Then we have transformed equation (3.18) into the following proper renewal equation:

$$\widetilde{h}(t) = \widetilde{b}(t) + \int_0^t \widetilde{h}(t-s)\mathrm{d}\widetilde{F}_T(s), \ \ t \geq 0. \tag{3.21}$$

The solution of this equation is given in the following lemma.

Lemma 3.8. *If $\widetilde{b}(t)$ is a given, integrable function that is bounded on finite intervals, then the solution of this equation is unique and bounded on finite intervals and is given as*

$$\widetilde{h}(t) = \sum_{n=0}^{\infty} \left(\widetilde{F}_T^{(n)} * \widetilde{b}\right)(t) \tag{3.22}$$

or equivalently as

$$\widetilde{h}(t) = \widetilde{b}(t) + \int_0^t \widetilde{b}(t-s)\mathrm{d}M(s), \ \ t \geq 0, \tag{3.23}$$

where

$$M(t) = F_T(t) + \int_0^t M(t-s)\mathrm{d}F_T(s), \ \ t \geq 0$$

is the renewal function associated with the distribution $F_T(t)$ of the inter-arrival times.

Now after finding $\widetilde{h}(t)$ the solution of equation (3.18) can be obtained by the transformation

$$h(t) = e^{-\gamma t}\widetilde{h}(t).$$

3.4 An Analytical Example: Erlang(2) Inter-Occurrence Times

3.4.1 Solution to Renewal Equation

We start with a general defective renewal equation

$$h(t) = b(t) + \theta \int_0^t h(t-s)dF_T(s), \ \ \theta \in (0,1). \tag{3.24}$$

We can transform equation (3.24) into a proper renewal equation by finding a parameter ϕ such that

$$\widetilde{F}_T(dt) = e^{\phi t}\theta F_T(dt) \tag{3.25}$$

is a PDF, i.e.: $\int_0^\infty \widetilde{F}_T(dt) = 1$. From this we can notice

$$1 = \int_0^\infty \widetilde{F}_T(dt) = \int_0^\infty e^{\phi t}\theta F_T(dt) \tag{3.26}$$

implies $\int_0^\infty e^{\phi t} F_T(dt) = \frac{1}{\theta}$. Now assume $T \sim Erlang(2) \equiv Gamma(2, \lambda)$, $\lambda > 0$, for the inter-occurrence times. Hence, we have

$$f_T(t) = \lambda^2 t e^{-\lambda t}, \qquad F_T(t) = 1 - (1 + \lambda t)e^{-\lambda t}, \;\; \lambda > 0. \tag{3.27}$$

From equation (3.26) we have:

$$\begin{aligned}
\theta\lambda^2 \int_0^\infty t e^{(\phi-\lambda)t} dt &= 1 \\
\theta\lambda^2 \left[t \frac{1}{\phi - \lambda} e^{(\phi-\lambda)t} \bigg|_0^\infty - \int_0^\infty \frac{1}{\phi - \lambda} e^{(\phi-\lambda)t} dt \right] &= 1 \\
-\frac{\theta\lambda^2}{\phi - \lambda} \left[\int_0^\infty e^{(\phi-\lambda)t} dt \right] &= 1 \\
\frac{\theta\lambda^2}{(\phi - \lambda)^2} &= 1,
\end{aligned}$$

where in the second step we used $\phi < \lambda$ for convergence. Solving for θ gives

$$\phi = \lambda(1 - \sqrt{\theta}). \tag{3.28}$$

Here we notice that since $\theta \in (0, 1)$, equation (3.28) indeed gives the relation $\phi < \lambda$. From equation (3.25) we can substitute $\theta F_T(dt) = e^{-\phi t}\widetilde{F}_T(dt)$ into equation (3.24), which leads to

$$h(t) = b(t) + \int_0^t h(t - s)e^{-\phi s} d\widetilde{F}_T(s).$$

Multiplying both sides by $e^{\phi t}$ gives the equivalent relation

$$e^{\phi t}h(t) = e^{\phi t}b(t) + \int_0^t h(t - s)e^{\phi(t-s)} d\widetilde{F}_T(s).$$

Now substituting $\widetilde{h}(t) = e^{\phi t}h(t)$ and $\widetilde{b}(t) = e^{\phi t}b(t)$ we get a proper renewal equation

$$\widetilde{h}(t) = \widetilde{b}(t) + \int_0^t \widetilde{h}(t - s) d\widetilde{F}_T(s). \tag{3.29}$$

From equation (3.25) we have

$$\begin{aligned}
\widetilde{F}_T(dt) &= e^{\phi t}\theta F_T(dt) \\
&= e^{\lambda(1-\sqrt{\theta})t}\theta\lambda^2 t e^{-\lambda t} dt \\
&= \theta\lambda^2 t e^{-\lambda\sqrt{\theta}t} dt \quad (\text{let } a = \lambda\sqrt{\theta}) \\
&= a^2 t e^{-at} dt,
\end{aligned}$$

hence, the new inter-occurrence times are also $Erlang(2)$ distributed with new parameter $a = \lambda\sqrt{\theta} > 0$. The solution to equation (3.29) is

$$\widetilde{h}(t) = \sum_{n=0}^{\infty} \left(\widetilde{F}_T^{(n)} * b^*\right)(t) \tag{3.30}$$

We notice that if $T \sim Erlang(2) \equiv Gamma(2, \lambda)$, $\lambda > 0$, then

$$\begin{aligned} F_T^{(n)}(dt) &= \frac{\lambda^{2n}}{\Gamma(2n)} t^{2n-1} e^{-\lambda t} dt \\ &= \lambda \frac{(\lambda t)^{2n-1}}{(2n-1)!} e^{\lambda t} dt. \end{aligned}$$

Now with[2] $F_T^0(t) = \delta_0(t)$ we have

$$\begin{aligned} \sum_{n=0}^{\infty} F_T^{(n)}(dt) &= \delta_0(dt) + \lambda\lambda t e^{-\lambda t} dt + \lambda \frac{(\lambda t)^3}{3!} e^{-\lambda t} dt + \dots \\ &= \delta_0(dt) + \lambda e^{-\lambda t} dt \left(\lambda t + \frac{(\lambda t)^3}{3!} + \dots\right) \\ &= \delta_0(dt) + \lambda e^{-\lambda t} dt \left(\frac{e^{\lambda t} - e^{-\lambda t}}{2}\right) \\ &= \delta_0(dt) + \frac{\lambda}{2}(1 - e^{-2\lambda t}) dt, \end{aligned}$$

where in the third inequality we used $\sum_{n>0,odd} \frac{x^n}{n!} = \frac{e^x - e^{-x}}{2}$. From equation (3.30) and noticing that $\widetilde{F}_T(dt) \equiv Erlang(2) \equiv Gamma(2, a)$ we have

$$\begin{aligned} \widetilde{h}(t) &= \sum_{n=0}^{\infty} \left(\widetilde{F}_T^{(n)} * b^*\right)(t) \\ &= \int_0^t \delta_0(s)\widetilde{b}(t-s)ds + \int_0^t \widetilde{b}(t-s)\frac{a}{2}(1 - e^{-2as})ds \\ &= \widetilde{b}(t) + \frac{a}{2}\int_0^t \widetilde{b}(t-s)(1 - e^{-2as})ds \\ &= e^{\lambda(1-\sqrt{\theta})t} b(t) + \frac{a}{2}\int_0^t e^{\lambda(1-\sqrt{\theta})(t-s)} b(t-s)(1 - e^{-2as})ds, \end{aligned} \tag{3.31}$$

where we recall $a = \lambda\sqrt{\theta}$. Now $b(t)$ and θ are still unknown and are determined by equation (3.24) and λ is a parameter of the inter-occurrence times.

<u>**Side Note**</u> We can also derive the solution using the formula given in equation (3.23), i.e.:

$$\widetilde{h}(t) = \widetilde{b}(t) + \int_0^t \widetilde{b}(t-s)dM(s), \tag{3.32}$$

[2] $\delta_0(t)$ equals 1 at $t = 0$ and is zero everywhere else.

where

$$M(t) = \frac{1}{2}at - \frac{1}{4}(1 - e^{-2at}), \tag{3.33}$$

which is the renewal function of the $Erlang(2)$ renewal process and is equal to equation (16) of the same manuscript with new parameter a. Substituting $\widetilde{b}(t)$ and $M(t)$ into equation (3.32) leads to the same result derived in equation (3.31).

3.4.2 Solution of Maximum Shock Load

The solution of the maximum shock load is given as

$$h(t) = \bar{F}_T(t) + F_X(x)\int_0^t h(t-s)dF_T(s), \tag{3.34}$$

where $h(t)$=$\mathbb{P}(X^*(t) \leq x)$. Now we have $b(t) = \bar{F}_T(t) = (1+\lambda t)e^{-\lambda t}$ and $\theta = F_X(x)$. For the sake of simplicity we will write the solution in equation (3.31) in terms of a. We have

$$\begin{aligned}
\widetilde{h}(t) &= e^{\lambda(1-\sqrt{\theta})t}b(t) + \frac{a}{2}\int_0^t e^{\lambda(1-\sqrt{\theta})(t-s)}b(t-s)(1-e^{-2as})ds \\
&= e^{\lambda(1-\sqrt{\theta})t}(1+\lambda t)e^{-\lambda t} + \frac{a}{2}\int_0^t e^{\lambda(1-\sqrt{\theta})(t-s)}(1+\lambda(t-s))e^{-\lambda(t-s)}(1-e^{-2as})ds \\
&= (1+\lambda t)e^{-at} + \frac{a}{2}\int_0^t e^{-a(t-s)}(1+\lambda(t-s))(1-e^{-2as})ds. \qquad (3.35)
\end{aligned}$$

Now we compute the integral on the right hand side:

$$\int_0^t e^{-a(t-s)}(1+\lambda(t-s))(1-e^{-2as})ds = \int_0^t e^{-a(t-s)} - e^{-a(t+s)} + e^{-a(t-s)}\lambda(t-s) - e^{-a(t+s)}\lambda(t-s)ds.$$

Calculating the four integrals separately gives:

$$\int_0^t e^{-a(t-s)}ds = \frac{1}{a}e^{-a(t-s)}\Big|_{s=0}^{t} = \frac{1}{a}\left(1 - e^{-at}\right). \tag{3.36}$$

$$-\int_0^t e^{-a(t+s)}ds = \frac{1}{a}e^{-a(t+s)}\Big|_{s=0}^{t} = \frac{1}{a}\left(e^{-2at} - e^{-at}\right). \tag{3.37}$$

$$
\begin{aligned}
\lambda\int_0^t e^{-a(t-s)}(t-s) &= \lambda\left(\left.\frac{(t-s)}{a}e^{-a(t-s)}\right|_{s=0}^{t} + \int_0^t \frac{1}{a}e^{-a(t-s)}ds\right)\\
&= \lambda\left(-\frac{t}{a}e^{-at} + \frac{1}{a^2}\left[e^{-a(t-s)}\right]_{s=0}^{t}\right)\\
&= \lambda\left(-\frac{t}{a}e^{-at} + \frac{1}{a^2}\left[1-e^{-at}\right]\right)\\
&= \frac{\lambda}{a}\left(\frac{1}{a} - te^{-at} - \frac{1}{a}e^{-at}\right). \qquad (3.38)\\
-\lambda\int_0^t e^{-a(t+s)}(t-s) &= -\lambda\left(\left.-\frac{(t-s)}{a}e^{-a(t+s)}\right|_{s=0}^{t} - \int_0^t \frac{1}{a}e^{-a(t+s)}ds\right)\\
&= -\lambda\left(\frac{t}{a}e^{-at} + \frac{1}{a^2}\left[e^{-a(t+s)}\right]_{s=0}^{t}\right)\\
&= -\lambda\left(\frac{t}{a}e^{-at} + \frac{1}{a^2}\left[e^{-2at}-e^{-at}\right]_{s=0}^{t}\right)\\
&= \frac{\lambda}{a}\left(\frac{1}{a}e^{-at} - te^{-at} - \frac{1}{a}e^{-2at}\right). \qquad (3.39)
\end{aligned}
$$

Now from equations (3.35), (3.36), (3.37), (3.38) and (3.39) we have

$$
\begin{aligned}
\widetilde{h}(t) &= (1+\lambda t)e^{-at} + \frac{a}{2}\left[\frac{1}{a}\left(1-e^{-at}\right) + \frac{1}{a}\left(e^{-2at}-e^{-at}\right) + \frac{\lambda}{a}\left(\frac{1}{a} - te^{-at} - \frac{1}{a}e^{-at}\right)\right.\\
&\quad\left. +\frac{\lambda}{a}\left(\frac{1}{a}e^{-at} - te^{-at} - \frac{1}{a}e^{-2at}\right)\right]\\
&= (1+\lambda t)e^{-at} + \frac{1}{2}\left[\left(1-e^{-at}\right) + \left(e^{-2at}-e^{-at}\right) + \lambda\left(\frac{1}{a} - te^{-at} - \frac{1}{a}e^{-at}\right)\right.\\
&\quad\left. +\lambda\left(\frac{1}{a}e^{-at} - te^{-at} - \frac{1}{a}e^{-2at}\right)\right].
\end{aligned}
$$

After simplification this leads to

$$
\widetilde{h}(t) = \frac{1}{2} + \frac{\lambda}{2a} + \frac{1}{2}e^{-2at} - \frac{\lambda}{2a}e^{-2at} \qquad (3.40)
$$

and substituting $a = \lambda\sqrt{\theta}$ gives

$$
\widetilde{h}(t) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{F_X(x)}} + \left(1 - \frac{1}{\sqrt{F_X(x)}}\right)e^{-2\lambda\sqrt{F_X(x)}t}\right). \qquad (3.41)
$$

Now the solution to (3.34) is $h(t) = e^{-\phi t}\widetilde{h}(t)$, hence we have:

$$h(t) = \frac{1}{2}e^{-\lambda\left(1-\sqrt{F_X(x)}\right)t}\left(1+\frac{1}{\sqrt{F_X(x)}}+\left(1-\frac{1}{\sqrt{F_X(x)}}\right)e^{-2\lambda\sqrt{F_X(x)}t}\right), \tag{3.42}$$

where $h(t) = \bar{F}_T(t)$ for $F_X(x)$=0.

3.4.3 Computing the Renewal Integral

Equation (3.34) can also be solved recursively by writing the integral according to the trapezoidal rule.

$$h(t) \quad = \quad \bar{F}_T(t) + F_X(x)\int_0^t h(t-s)dF_T(s).$$

Assume $dF_T(s) = f_T(s)$ exists and divide time uniformly as $0 = s_1 < s_2 < \cdots < s_N < s_{N+1} = t$. Then we have:

$$h(t) \quad = \quad \bar{F}_T(t) + F_X(x)\left[\frac{t}{2N}h(t)f_T(0) + \frac{t}{2N}2\sum_{i=2}^{N}h(t-s_i)f_T(s_i) + \frac{t}{2N}h(0)f_T(t)\right]$$

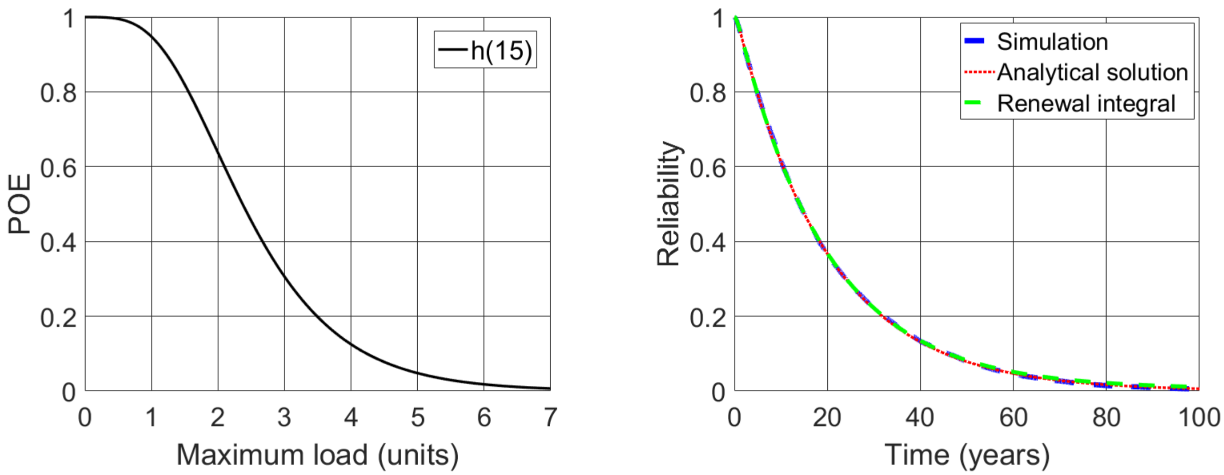
so that the reliability can be computed as

$$h(t) \quad = \quad \frac{1}{1-\frac{t}{2N}F_X(x)f_T(0)}\frac{t}{N}\left[\frac{1}{2}h(0)f_T(t) + \sum_{i=2}^{N}h(t-s_i)f_T(s_i)\right],$$

where $h(0) = \bar{F}_T(t) = 1$.

3.4.4 Numerical Example Shock Load

For the shock load magnitudes X: $X \sim Exp(1)$ with mean equal to 1 unit. For the inter-occurrence times T: $T \sim Erlang(2) \equiv Gamma(2, \lambda = 1)$. This distribution has mean equal to 2 years and standard deviation equal to $\sqrt{2}$ years. This translates to 90% of the waiting times for shocks between 0.36 and 4.74 years at an average of 2 years between shocks. Also, shocks are on average 1 unit while 90% of the shocks are smaller than the threshold which is 2.3 units. The POE is given in Figure 3.4 (a) for a time horizon of t=15 years while the reliability can be seen in Figure 3.4 (b) for a threshold $x = 2.3$ units, which is the 90-th percentile of X. The reliability is compared to a Monte Carlo simulation of 10000 runs and a numerical integration with Δt=1/30 years and Δx=1/100 units.



(a) POE. (b) Reliability.

Figure 3.4: Erlang(2) shock process.

3.4.5 Error Analysis Example Maximum Shock Load

The $Erlang(2)$ shock occurrences can be approximated by an HPP process where the inter-occurrence times are $Exp(1/mean(Erlang(2))$. We compare the relative error of the reliability $h(t;m)$ and its HPP approximation $h^{HPP}(t;m)$ at time $t = 14$ years for different values of the threshold x where capacity x is given as a percentile of $F_X(x)$:

$$error = \frac{h(14;x) - h^{HPP}(14;x)}{h(14;x)}. \tag{3.43}$$

In Figure 3.5 the errors are depicted for mean inter-occurrence time 2 and 10 years. It can be seen that for highly reliable components (relatively high thresholds) the process of shock occurrences is well approximated by an HPP process. Let δ be the difference in time between the solution h and its HPP approximation:

$$\delta = h(t;2.3) - h^{HPP}(t;2.3). \tag{3.44}$$

In Figure 3.6 we can see δ on a log-scale for mean inter-occurrence times 2 and 10 years.

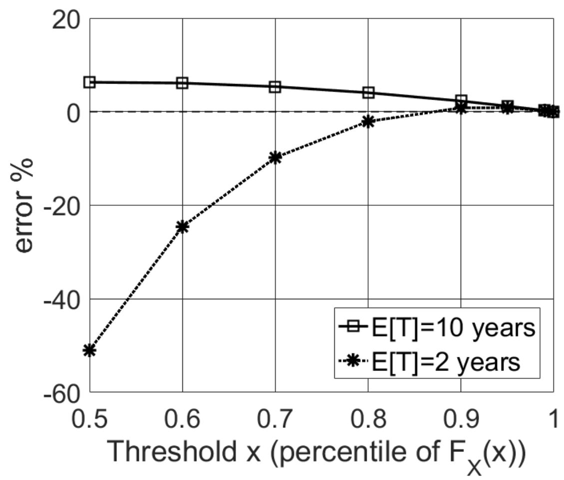


Figure 3.5: Comparison of error between a renewal load model and its HPP approximation as a function of threshold x.

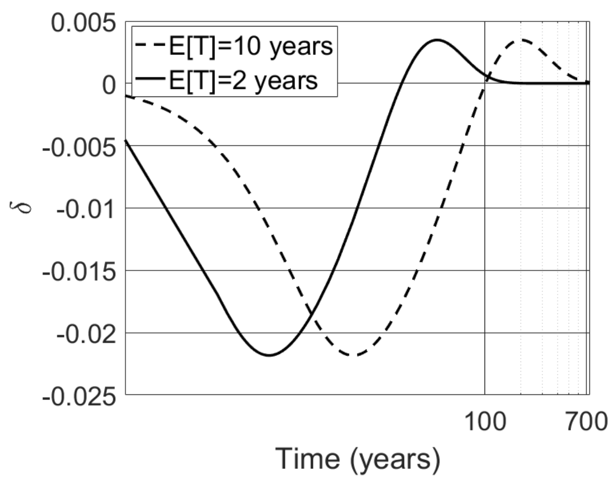


Figure 3.6: Comparison of error between a renewal load model and its HPP approximation as a function of time.

3.4.6 Solution and Numerical Example Maximum Pulse Load

We notice that for the distribution of maximum spike load we need to compute

$$\mathbb{P}^{spike}(X^*(t) \leq x) = \mathbb{P}\left(\max\left\{X_1, \ldots, X_{N(t)}\right\} \leq x\right), \tag{3.45}$$

which results in

$$\mathbb{P}^{spike}(X^*(t) \leq x) = \mathbb{E}\left[F_X(x)^{N(t)}\right]. \tag{3.46}$$

For the sustained load we have to compute

$$\mathbb{P}^{sustained}(X^*(t) \leq x) = \mathbb{P}\left(\max\left\{X_1, \ldots, X_{N(t)+1}\right\} \leq x\right), \tag{3.47}$$

which gives

$$\mathbb{P}^{sustained}(X^*(t) \leq x) = F_X(x)\mathbb{E}\left[F_X(x)^{N(t)}\right], \tag{3.48}$$

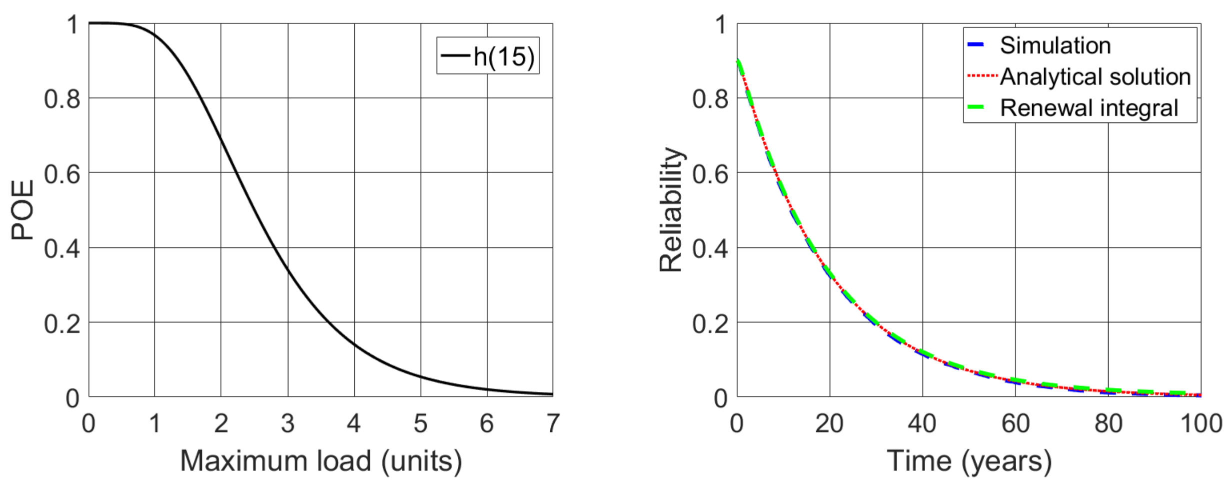
hence we have

$$\mathbb{P}^{sustained} = F_X(x)\mathbb{P}^{spike}. \tag{3.49}$$

From equation (3.42) the reliability $h(t)$ of the maximum sustained load process is

$$h(t) = \frac{1}{2}e^{-\lambda\left(1-\sqrt{F_X(x)}\right)t}\left(F_X(x) + \sqrt{F_X(x)} + \left(F_X(x) - \sqrt{F_X(x)}\right)e^{-2\lambda\sqrt{F_X(x)}t}\right). \tag{3.50}$$

Using the same example as in 3.4.4, the POE is depicted in Figure 3.7 (a) and the reliability in Figure 3.7 (b).



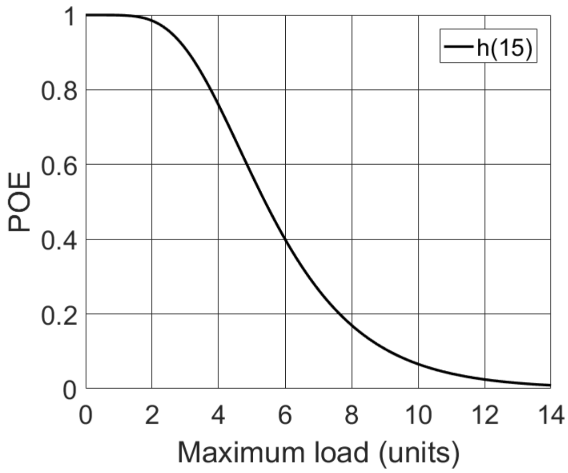
(a) POE. (b) Reliability.

Figure 3.7: Erlang(2) pulse process.

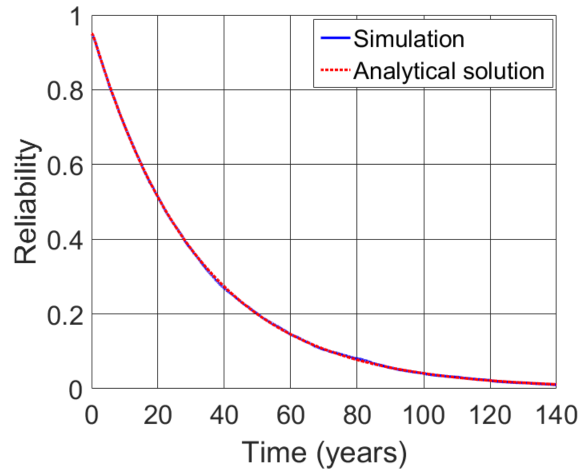
3.4.7 Numerical Example Maximum Alternating Process

In this section the pulse load process alternates between an 'on' and 'off' state. The on and off times each occur following the same exponential distribution (HPP) with parameter $\mu = 5/4$, i.e. we have a sequence of pairs $(V_1, X_1), (W_1, 0), (V_2, X_2), (W_2, 0), \ldots$ where V_i denotes the duration of an 'on' state and W_i denotes the duration of an 'off' state and $V_i, W_i \sim Exp(5/4)$ with mean 0.8 year. We notice that we always start in an 'on' state. The load magnitudes X_i are independent of V_i.

For this example we have $X \sim Exp(1/2)$, hence mean is 2 units. This alternating process has the same distribution as a (maximum) pulse load process where the inter-arrival times are $T \sim Erlang(2) \equiv Gamma(\cdot; 2, 4/5)$ with mean equal to 1.6 years and $X \sim Exp(1/2)$. The POE is given in Figure 3.8 (a) for t=15 years and the reliability can be seen in Figure 3.8 (b) for a threshold x=6 units which is the 95-th percentile of X. The reliability is compared to 10000 runs of a Monte Carlo simulation.



(a) Reliability.



(b) POE.

Figure 3.8: Erlang(2) alternating process.

3.5 Practical Numerical Examples

The data used in the Darlington example can be found in [23]. For the Trenton and Pearson examples the data can be found on the website of Environment and Climate Change Canada, [1].

3.5.1 Tornadoes near Darlington NGS, 1918-2003 (within 100,000 km^2)

3.5.1.1 Data analysis

The probabilistic safety assessment of a nuclear station requires the assessment of rare meteorological hazards, such as tornadoes, tropical (and extra-tropical) storms, exceptional wind gusts and freezing rain storm. This analysis is based on data about tornado occurrences in southern Ontario in a region of 100,000 km^2, which represents a circle of radius of about 180 km centred at the site of the Darlington (Nuclear Generating Station) NGS. This region includes a fair part of southern Ontario, parts of Western New York State, most of Lake Ontario, part of north-eastern Lake Erie and the southeast tip of Georgian Bay. The confirmed and probable numbers of tornado events that occurred from 1918 to 2003 are compiled in the report [23]. This data set includes the occurrence date and the intensity according to the Fujita scale as given in Table 3.1: F numbers 0 to 3. There were only two tornadoes of F category 4 measured ,which is insufficient for distribution fitting, and none in category 5. Any measurement within four days in the same category was considered a persisting storm and was consequently removed since we model the inter-arrival times of storms. Persisting storms which change category are kept as one measurement in each particular category, within four days. In Table 3.2 some statistics of the filtered data are summarised.

In Figures 3.9, 3.10, 3.11 and 3.12 the Exponential, Weibull and Lognormal distributions are fitted using Probability Paper Plot (PPP) to the inter-arrival times of tornadoes in each category 0, 1, 2 and 3 respectively.

In Tables 3.3, 3.4, 3.5 and 3.6 some statistics of the fitted distributions to the inter-arrival times are given for tornadoes in the category F_0, F_1, F_2 and F_3 respectively.

Table 3.1: **Tornadoes Darlington**: Fujita scale.

F Number	Wind speed (km/h)	Damage
0	64 - 116	Light
1	117 - 180	Moderate
2	181 - 252	Considerable
3	253 - 330	Severe
4	331 - 417	Devastating
5	418 - 509	Incredible

Table 3.2: **Tornadoes Darlington**: statistics inter-arrival times (in days) in each category.

	F_0	F_1	F_2	F_3
Amount data	110	67	36	6
Min	5	6	5	335
Max	3525	2172	5146	8832
Median	61.5	335	396	5107.5
Mean	275	430.42	858.17	4433
Variance	$2.396 \cdot 10^5$	$1.907 \cdot 10^5$	$1.205 \cdot 10^6$	$1.276 \cdot 10^7$
COV	0.562	0.986	0.782	1.241

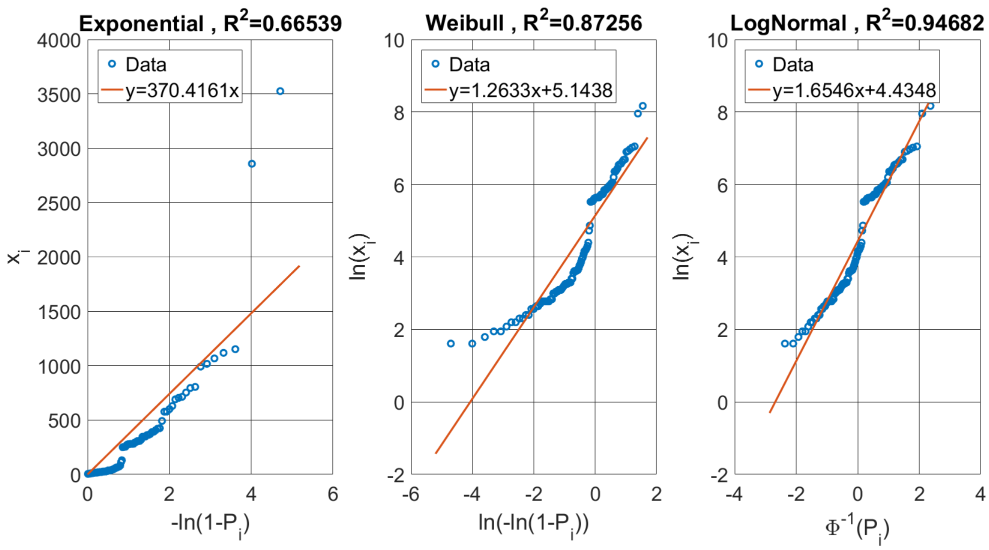


Figure 3.9: **Tornadoes Darlington**: Exponential, Weibull and Lognormal PPP for inter-arrival times in category F_0.

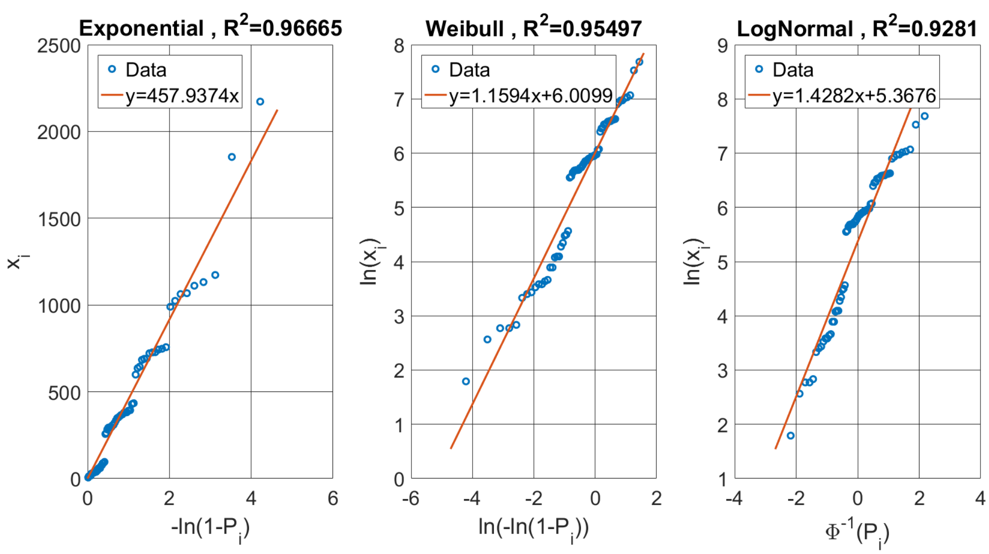


Figure 3.10: **Tornadoes Darlington**: Exponential, Weibull and Lognormal PPP for inter-arrival times in category F_1.

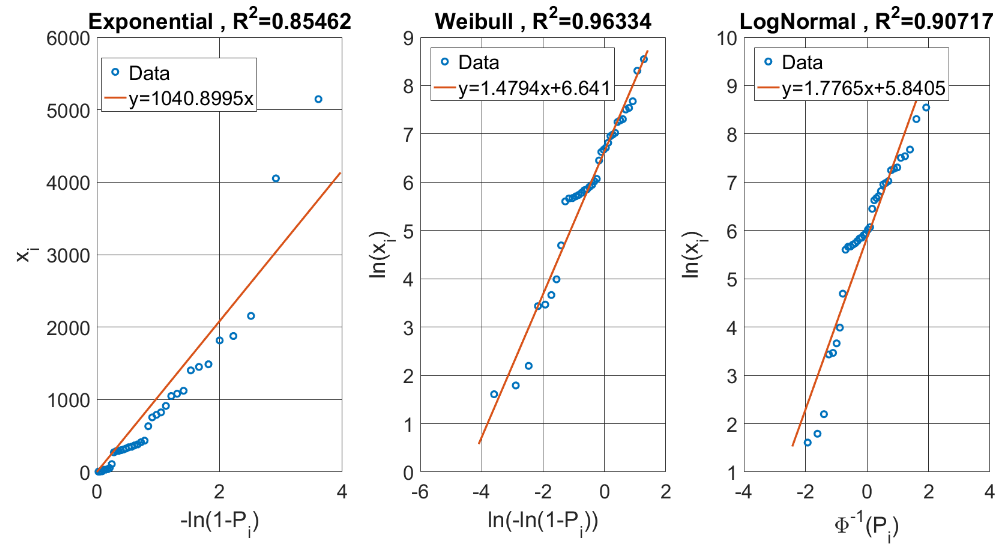


Figure 3.11: **Tornadoes Darlington**: Exponential, Weibull and Lognormal PPP for inter-arrival times in category F_2.

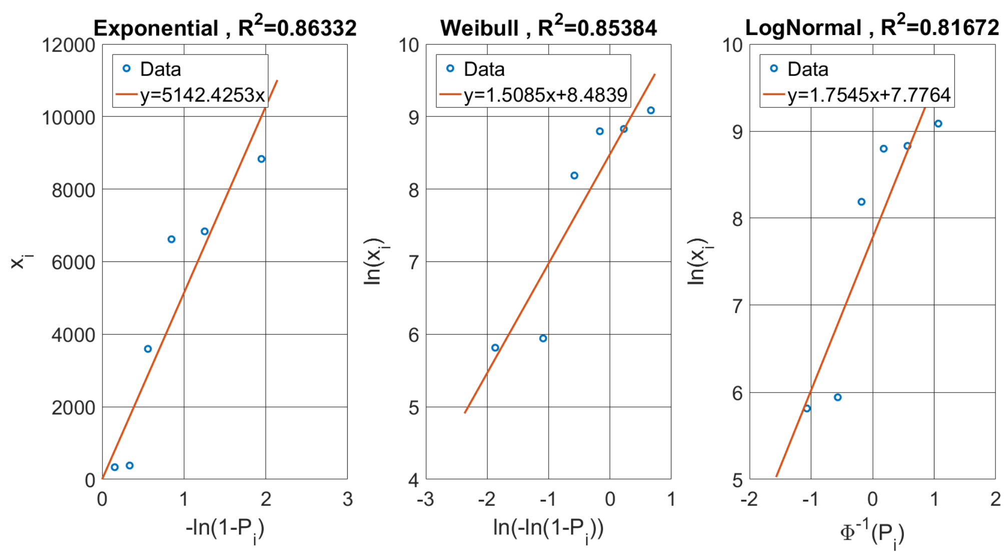


Figure 3.12: **Tornadoes Darlington**: Exponential, Weibull and Lognormal PPP for inter-arrival times in category F_3.

Table 3.3: **Tornadoes Darlington**: statistics fitted distributions for inter-arrival times in category F_0.

	Exponential	**Weibull**	**Lognormal**
Parameters	$\lambda = 1/370.4161$	$\alpha = 1/1.2633$, $\beta = \exp(5.1438)$	$\lambda = 4.4348$, $\zeta = 1.6546$
Mean (days)	370.4161	195.6516	331.5075
Variance	$0.14{\cdot}10^6$	$6.22{\cdot}10^4$	$1.59{\cdot}10^6$
COV	1	1.275	3.802

Table 3.4: **Tornadoes Darlington**: statistics fitted distributions for inter-arrival times in category F_1.

	Exponential	**Weibull**	**Lognormal**
Parameters	$\lambda = 1/457.9374$	$\alpha = 1/1.1594$, $\beta = \exp(6.0099)$	$\lambda = 5.3676$, $\zeta = 1.4282$
Mean (days)	457.9374	439.32	594.36
Variance	$0.21{\cdot}10^6$	$2.61{\cdot}10^5$	$2.36{\cdot}10^6$
COV	1	1.164	2.59

Table 3.5: **Tornadoes Darlington**: statistics fitted distributions for inter-arrival times in category F_2.

	Exponential	**Weibull**	**Lognormal**
Parameters	$\lambda = 1/1040.8995$	$\alpha = 1/1.4794$, $\beta = \exp(6.641)$	$\lambda = 5.8405$, $\zeta = 1.7765$
Mean (days)	1040.8995	1003.6	1666.5
Variance	$1.1{\cdot}10^6$	$2.3354{\cdot}10^6$	$62.42{\cdot}10^6$
COV	1	1.523	4.741

Table 3.6: **Tornadoes Darlington**: statistics fitted distributions for inter-arrival times in category F_3.

	Exponential	**Weibull**	**Lognormal**
Parameters	$\lambda = 1/5142.4253$	$\alpha = 1/1.5085$, $\beta = \exp(8.4839)$	$\lambda = 7.7764$, $\zeta = 1.7545$
Mean (days)	5142.4253	6467.7	1858.8
Variance	$26.4{\cdot}10^6$	$1.015{\cdot}10^8$	$25.8{\cdot}10^6$
COV	1	1.558	2.73

3.5.1.2 Maximum Distribution

In this section the maximum distribution is calculated for tornadoes of category F_2 for two different time horizons: 15 years and 50 years. The reason to focus on F_2 category is that tornadoes in lower categories (F_0 and F_1) do not have major impact on safety. Tornadoes exceeding F_2 category are not considered due to a small number of observations in the data set. The database does not contain any measurements of wind speeds generated by tornadoes. Therefore, the wind speed distribution is estimated in an approximate manner. The distribution of wind is chosen to be Weibull where the lower and upper bound of F_2 category tornadoes are fitted to the 2.5% and 97.5% percentiles respectively. The resulting distribution for the wind speeds is:

$$F_X(x) = 1 - e^{-\left(\frac{x}{231.0686}\right)^{15.0532}}, \;\; x \geq 0,$$

with mean 223.1595km/h. A depiction of this distribution is given in Figure 3.13.

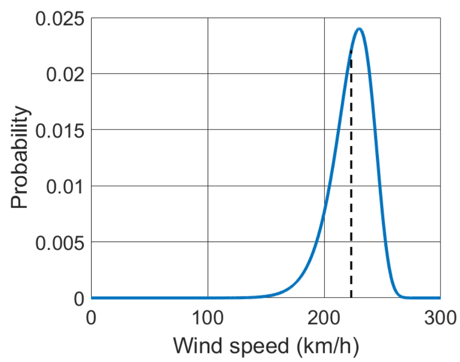
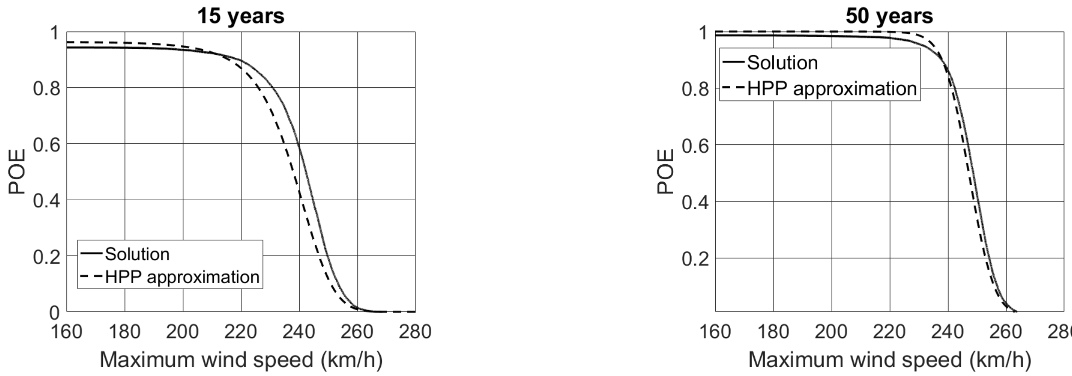


Figure 3.13: **Tornadoes Darlington**: Weibull distribution for wind speeds in category F_2.

The inter-arrival times are chosen to be the Lognormal fit for F_2 in Figure 3.11 and Table 3.5. The distribution of the maximum for a time horizon of 15 years is given in Figure 3.14 (a) while the distribution of the maximum for a time horizon of 50 years is given in Figure 3.14 (b). The HPP approximation of both distributions is also given. While the HPP approximation performs well as a limiting solution, in this case a longer time horizon, it can be seen that for both examples the HPP approximation is underestimating the POE around wind speeds of 240 km/h, making it a nonconservative estimate. This results in higher exposure to risks.

3.5.1.3 Explanation of Results

It is obvious from Figure 3.14 that the POE (at a given wind speed) is higher for longer time horizons since more occurrences are expected. It can also be seen that for longer time horizons there is less discrepancy between the proposed renewal solution and its HPP approximation. This is due to (3.7). From both Figures 3.14 (a) and (b) it can be seen that for high wind speeds, large x values, the discrepancy between the proposed model and its HPP approximation



(a) 15 years. (b) 50 years.

Figure 3.14: **Tornadoes Darlington**: POE.

vanishes. This can be explained from the fact that as $x \to \infty$ the probability of wind speeds exceeding x, approaches 0. This should hold for any time horizon and for any model. These results are consistent with the asymptotic behaviour of the HPP solution. The results in the following examples are qualitatively similar.

3.5.2 Wind Gusts at Trenton, Canadian Forces Base (Latitude 44°07'00.000" N, Longitude 77°32'00.000" W)

3.5.2.1 Data Analysis

The following analyses are of maximum wind gust measurements at Canadian Forces Base (CFB) Trenton, taken from Jan 1, 1955 up until the last occurrence prior to Dec 4, 2017. Persisting high measurements, within 4 days, were counted as one measurement. The analyses are based on wind gust measurements above 100km/h and above 110km/h. In both cases the inter-arrival times and wind gust speeds are taken into account. PPP of the Exponential, Weibull and Lognormal distributions are given and plots of the maximum distribution are given after choosing the best fits. For wind gusts above 120km/h only 12 measurements were observed, which is too few data points for analysis. In 3.5.2.4 the yearly maxima are analyzed where also the Gumbel distribution is fitted to the yearly maxima since this is a common methodology applied in practice.

Wind gusts above 100 km/h

The inter-arrival times were taken between Jan 1, 1955 and the last recorded wind gust above 100km/h, which was on November 20, 1993. In this period of 14202 days there were a total of 48 occurrences. In Table 3.7 some data statistics are given for the wind gust measurements. In Figures 3.15 and 3.16, respectively, the PPP of the Exponential, Weibull and Lognormal distributions are given for the inter-arrival times and wind gusts. In tables 3.8 and 3.9 some statistics of the fitted distributions are given for the inter-arrival times and wind gusts above 100km/h, respectively. In Figure 3.17 the distribution of the maximum wind gust speed is given on a time horizon of 15 years where the inter-arrival times and wind gusts are respectively Lognormal and Exponential with parameters given in tables 3.8 and 3.9.

Table 3.7: **CFB Trenton**: statistics wind gusts above 100 km/h.

	Inter-arrival times (days)	**Wind gusts (km/h)**
Amount data	47	48
Min	5	101
Max	2597	154
Median	134.5	111
Mean	295.875	115.0208
Variance	$2.7645 \cdot 10^5$	194.7442
COV	1.777	0.1213

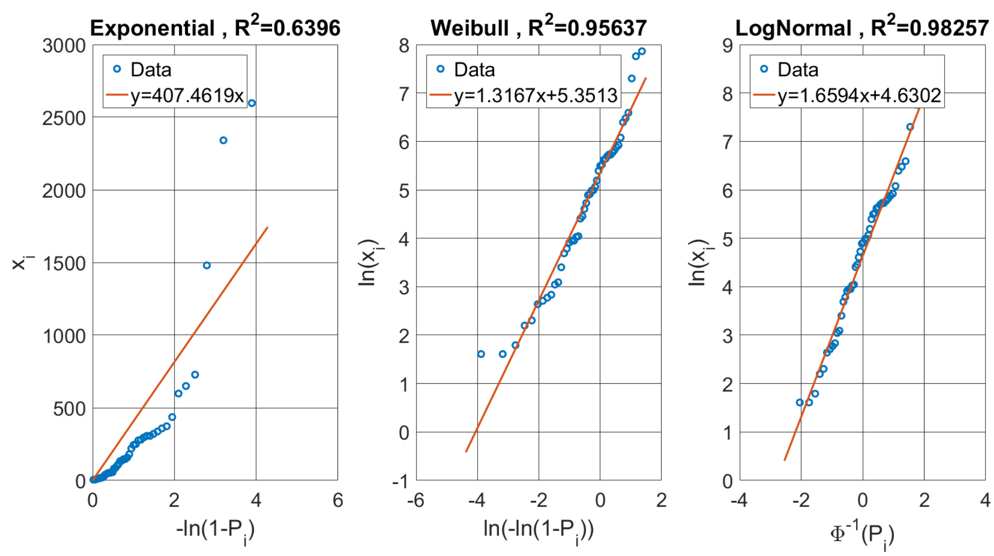


Figure 3.15: **CFB Trenton inter-arrival times**: wind speeds above 100km/h. Exponential, Weibull and Lognormal PPP.

Table 3.8: **CFB Trenton inter-arrival times**: statistics fitted distributions for inter-arrival times of wind speeds above 100km/h.

	Exponential	**Weibull**	**Lognormal**
Parameters	λ=1/407.4619	α=1/1.3167, β=exp(5.3513)	λ=4.6302, ζ=1.6594
Mean (days)	407.4619	248.5449	406.2786
Variance	$1.6603 \cdot 10^5$	$1.0996 \cdot 10^5$	$2.4264 \cdot 10^6$
COV	1	1.3342	3.8341

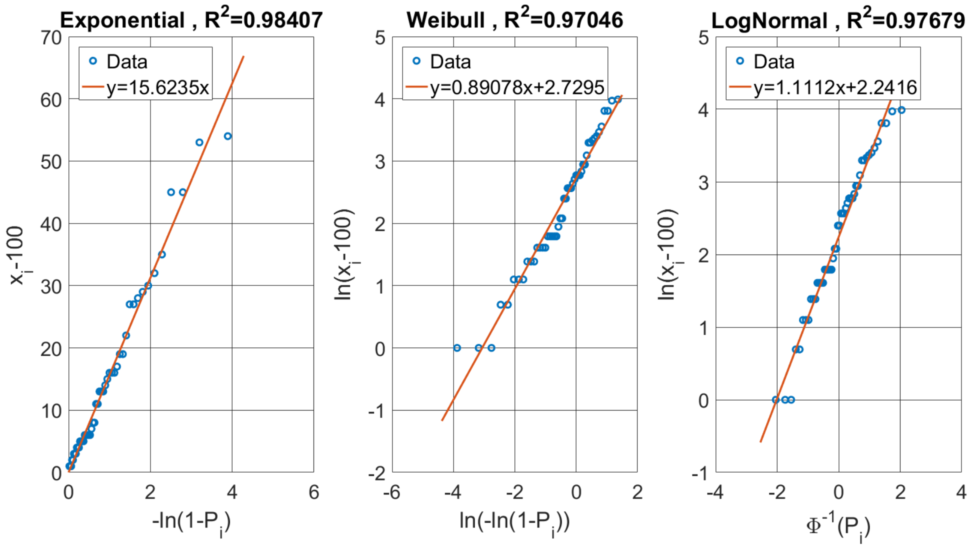


Figure 3.16: **CFB Trenton wind speeds above 100km/h**: Exponential, Weibull and Lognormal PPP.

Table 3.9: **CFB Trenton wind speeds above 100km/h**: statistics fitted distributions.

	Exponential	**Weibull**	**Lognormal**
Parameters	λ=1/15.6235	α=1/0.89078, β=exp(2.7295)	λ=2.2416, ζ=1.1112
Mean (km/h)	115.6235	114.6911	117.4436
Variance	244.0947	171.8754	741.6635
COV	0.1351	0.1143	0.2319

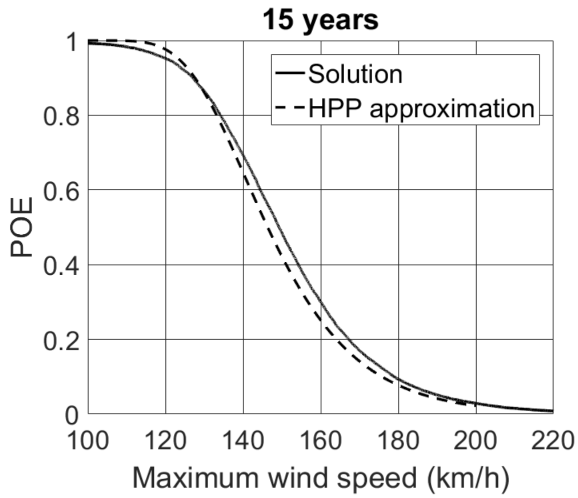


Figure 3.17: **CFB Trenton wind speeds above 100km/h**: POE wind speeds above 100km/h. Lognormal inter-arrival times and Exponential wind speeds as fitted in Tables 3.8 and 3.9.

Wind gusts above 110km/h

The same analysis is performed for wind gusts above 110km/h. The inter-arrival times were taken between Jan 1, 1955 and the last recorded wind gust above 110km/h, which was on November 12, 1992. In this period of 13829 days there were a total of 25 occurrences. In Table 3.10 some data statistics are given for the wind gust measurements. In Figures 3.18 and 3.19, respectively, the PPP of the Exponential, Weibull and Lognormal distributions are given for the inter-arrival times and wind gusts. In tables 3.11 and 3.12 some statistics of the fitted distributions are given for the inter-arrival times and wind gusts above 110km/h, respectively.

Table 3.10: **CFB Trenton**: statistics wind gusts above 110 km/h.

	Inter-arrival times (days)	**Wind gusts (km/h)**
Amount data	24	25
Min	5	111
Max	3169	154
Median	179.5	119
Mean	562.83	124.56
Variance	$8.4504 \cdot 10^5$	180.34
COV	1.6333	0.1078

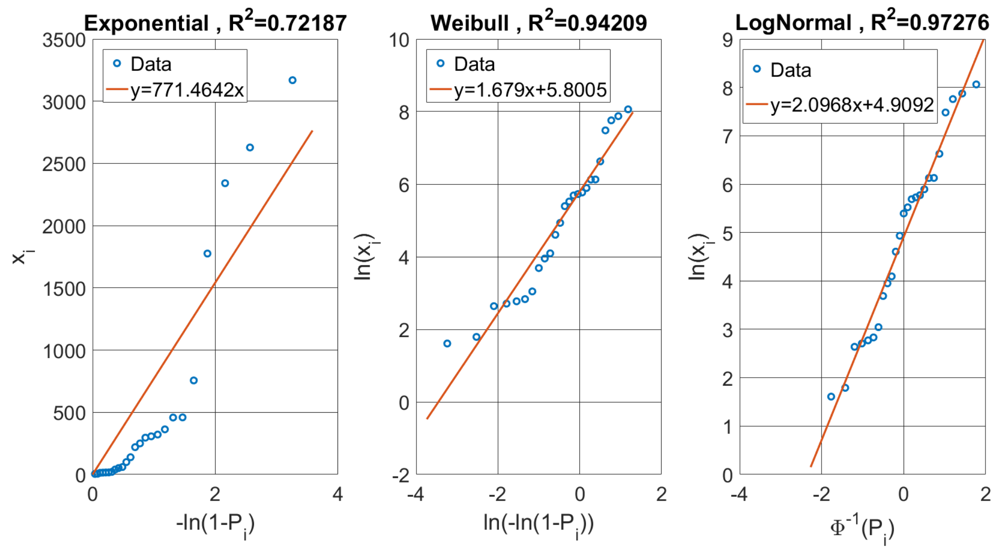


Figure 3.18: **CFB Trenton inter-arrival times**: wind speeds above 110km/h. Exponential, Weibull and Lognormal PPP.

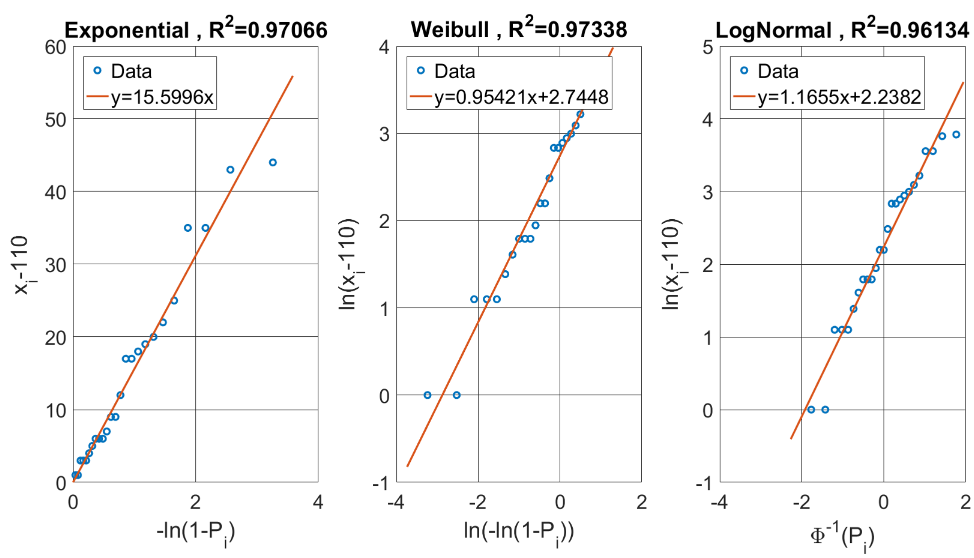


Figure 3.19: **CFB Trenton wind speeds above 110km/h**: Exponential, Weibull and Lognormal PPP.

Table 3.11: **CFB Trenton inter-arrival times**: statistics fitted distributions for inter-arrival times of wind speeds above 110km/h.

	Exponential	**Weibull**	**Lognormal**
Parameters	λ=1/771.4642	α=1/1.679, β=exp(5.8005)	λ=4.9092, ζ=2.0968
Mean (days)	771.4642	502.0369	1221.2
Variance	$5.9516 \cdot 10^5$	$7.9344 \cdot 10^5$	$1.1959 \cdot 10^8$
COV	1	1.7743	8.9548

Table 3.12: **CFB Trenton wind speeds above 110km/h**: statistics fitted distributions.

	Exponential	**Weibull**	**Lognormal**
Parameters	λ=1/15.5996	α=1/0.95421, β=exp(2.7448)	λ=2.2382, ζ=1.1655
Mean (km/h)	125.5996	125.273	128.4933
Variance	243.3477	212.5231	988.38
COV	0.1242	0.1164	0.2447

3.5.2.2 Exponential Wind Speed

In Figure 3.20 the distribution of the maximum wind gust speed is given on a time horizon of 15 years where the inter-arrival times and wind gusts are respectively Lognormal and Exponential with parameters given in tables 3.11 and 3.12.

3.5.2.3 Weibull Wind Speed

In Figure 3.21 the distribution of the maximum wind gust speed is given on a time horizon of 15 years where the inter-arrival times and wind gusts are respectively Lognormal and Weibull with parameters given in tables 3.11 and 3.12.

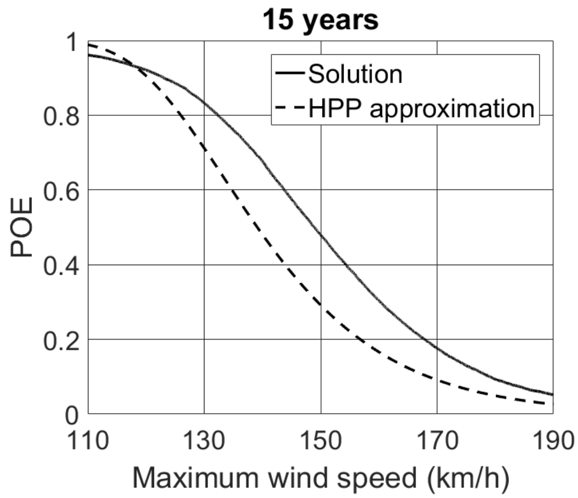


Figure 3.20: **CFB Trenton wind speeds above 110km/h**: POE wind speeds above 110km/h. Lognormal inter-arrival times and Exponential wind speeds as fitted in Tables 3.11 and 3.12.

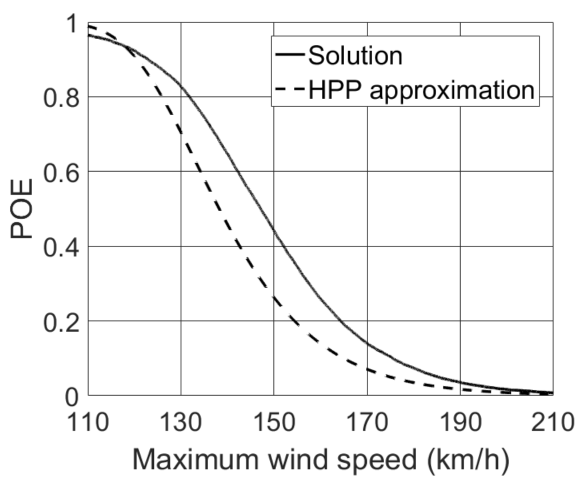


Figure 3.21: **CFB Trenton wind speeds above 110km/h**: POE wind speeds above 100km/h. Lognormal inter-arrival times and Weibull wind speeds as fitted in Tables 3.11 and 3.12.

3.5.2.4 Yearly Maximum: Data Analysis

In this analysis the maximum wind gust speed in each year is taken as the data set. In Table 3.13 statistics of the data are presented. In Figure 3.22 PPP's are given for the Exponential, Weibull, Lognormal and Gumbel distributions. Some statistics of the fitted distributions are given in Table 3.14.

Table 3.13: **CFB Trenton**: statistics wind gusts, yearly maximum .

	Wind gusts (km/h)
Amount data	63
Min	69
Max	154
Median	97
Mean	99.7619
Variance	362.8618
COV	0.1909

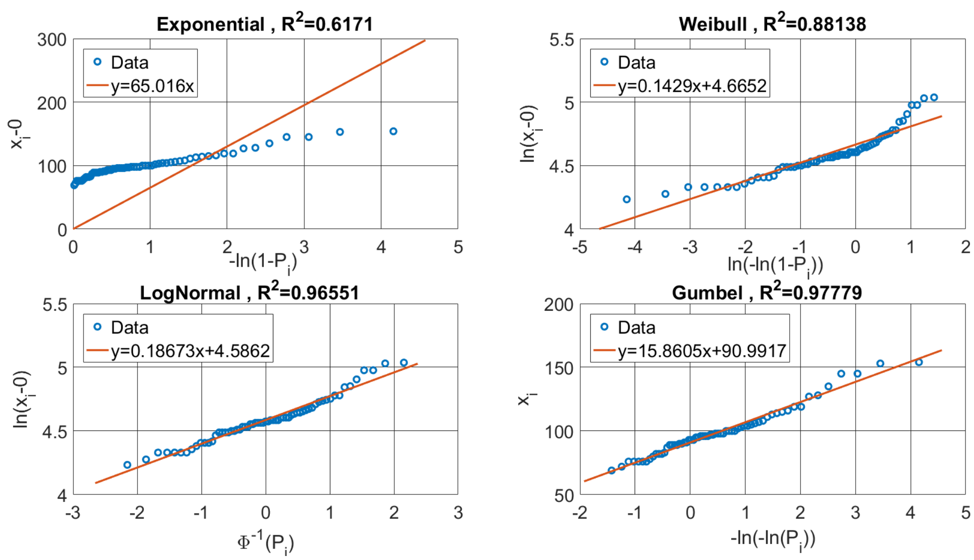


Figure 3.22: **CFB Trenton wind speeds above 110km/h**: Exponential, Weibull, Lognormal and Gumbel PPP.

Table 3.14: **CFB Trenton wind speeds above 110km/h**: statistics fitted distributions, yearly maximum.

	Exponential	**Weibull**	**Lognormal**	**Gumbel**
Parameters	λ=1/65.0160	α=1/0.1429, β=exp(4.6652)	λ=4.5862, ζ=0.18673	μ=90.9917, β=15.8605
Mean (km/h)	65.0160	99.3304	99.8454	100.1466
Variance	$4.2271 \cdot 10^3$	278.6991	353.7198	413.7903
COV	1	0.1681	0.1884	0.2031

3.5.2.5 Yearly Maximum: Distribution

In practice, often the maximum distribution is assumed to be a Gumbel distribution which has PDF and CDF given respectively as:

$$f_X(x) = \frac{1}{\beta} e^{-\left(\frac{x-\mu}{\beta} + e^{-\frac{x-\mu}{\beta}}\right)}, \quad F_X(x) = e^{-e^{-\frac{x-\mu}{\beta}}}, \quad \mu, x \in \mathbb{R}, \beta > 0.$$

Hence, if the distribution of the yearly maximum, $F_{X_{year\,max}}(x)$, is assumed to be a Gumbel distribution, we have:

$$F_{X_{year\,max}}(x) = e^{-e^{-\frac{x-\mu}{\beta}}}, \quad \mu, x \in \mathbb{R}, \beta > 0. \tag{3.51}$$

The distribution of the yearly maximum over a period of 15 years is then given as

$$\begin{aligned} G_X^{15}(x) = \left(F_{X_{year\,max}}(x)\right)^{15} &= \left(e^{-e^{-\frac{x-\mu}{\beta}}}\right)^{15} \\ &= e^{-e^{-\frac{x-(\mu+\beta\ln(15))}{\beta}}}. \end{aligned} \tag{3.52}$$

The POE of this maximum over 15 years, $G_X^{15}(x)$, is given in Figure 3.23, where the distribution parameters are given in Table 3.14. The plots in Figure 3.17 are also given for comparison to the results of the analysis of maximum wind speeds above 100km/h in a time horizon of 15 years. Similar plots are given in Figure 3.24 where $G_X^{15}(x)$ is compared to the results of the analysis of maximum wind speeds above 110km/h in a time horizon of 15 years.

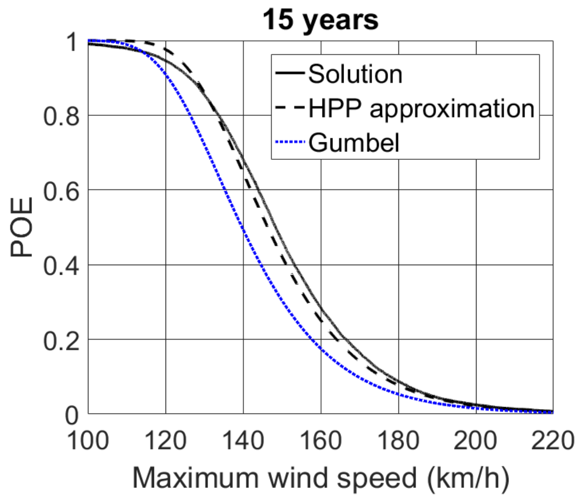


Figure 3.23: **CFB Trenton wind speeds above 100km/h**: POE of maximum Exponential gust over 15 years. Comparison between results in Figure 3.17 and Gumbel distribution as in Table 3.14.

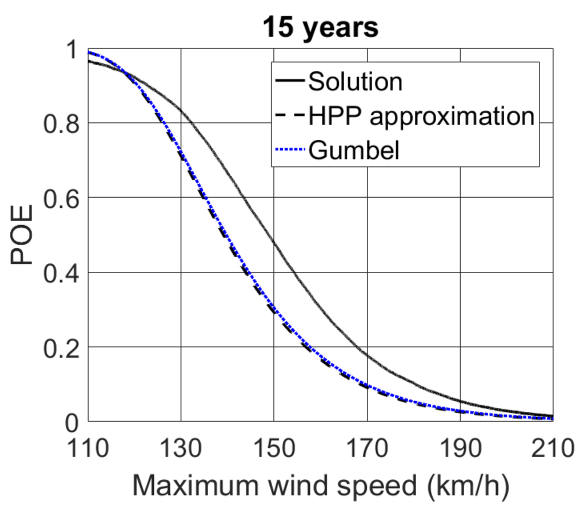


Figure 3.24: **CFB Trenton wind speeds above 110km/h**: POE of maximum Exponential gust over 15 years. Comparison between results in Figure 3.20 and Gumbel distribution as in Table 3.14.

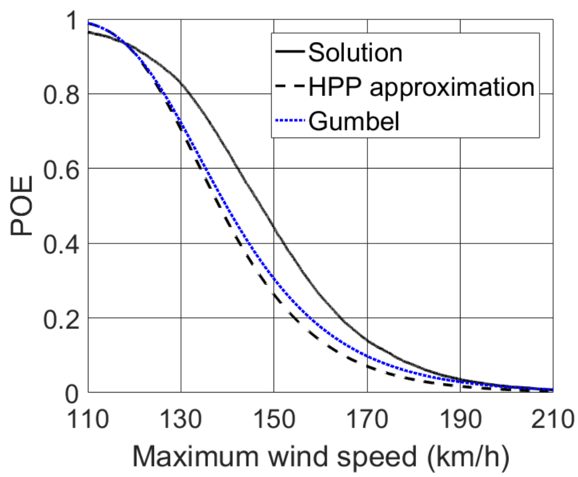
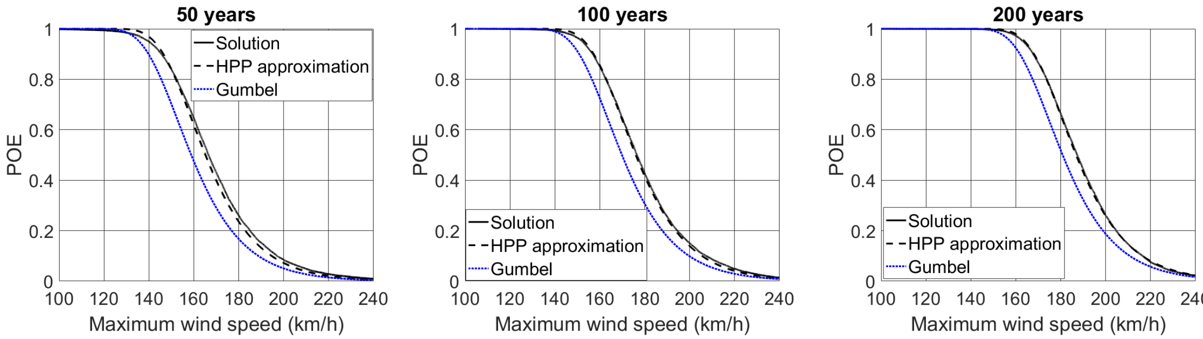


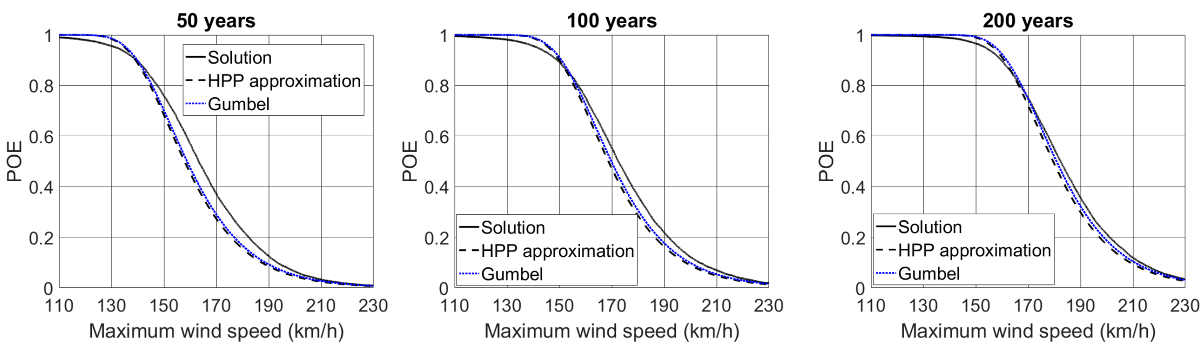
Figure 3.25: **CFB Trenton wind speeds above 110km/h**: POE of maximum Weibull gust over 15 years. Comparison between results in Figure 3.21 and Gumbel distribution as in Table 3.14.

For longer time horizons, 50, 100 and 200 years, $G_X^{50}(x)$, $G_X^{100}(x)$ and $G_X^{200}(x)$ are respectively given in Figure 3.26. These results are then compared to the renewal equation solution and its HPP approximation over the same time horizons for the parameters of wind gusts above 100km/h. In Figure 3.27 this is repeated for wind gusts above 110km/h.



(a) 50 years. (b) 100 years. (c) 200 years.

Figure 3.26: **CFB Trenton POE above 100km/h**: POE of maximum gust over 50, 100 and 200 years. Comparison between Lognormal inter-arrival times with Exponential wind gusts, its HPP approximation and Gumbel distribution.



(a) 50 years. (b) 100 years. (c) 200 years.

Figure 3.27: **CFB Trenton POE above 110km/h**: POE of maximum gust over 50, 100 and 200 years. Comparison between Lognormal inter-arrival times with Exponential wind gusts, its HPP approximation and Gumbel distribution.

3.5.2.6 Explanation of Results

Similar conclusions as in the previous example can be drawn from the results in Figures 3.26 and 3.27. In this particular example it can be seen that the Gumbel approximation is a better approximation for wind speeds above 110 km/h. This is a consequence of using different parameters because of the changed data subset.

3.5.3 Wind Gusts at Pearson International Airport

3.5.3.1 Data Analysis

The following analyses are of maximum wind gust measurements at Pearson Airport taken from Feb 1, 1955 to June 13, 2013. Persisting high measurements, within 4 days, were counted as one measurement. The analyses are based on the measurements of wind gusts above 100km/h and above 110km/h. In both cases the inter-arrival times and wind gust speeds are taken into account. Probability paper plots for the Exponential, Weibull and Lognormal distributions are given and a plot of the maximum distribution is given after choosing the best fits. For wind gusts above 120km/h only 4 measurements were observed and these are consequently not analyzed.

Wind gusts above 100km/h

The inter-arrival times were taken between Feb 1, 1955 and the last recorded wind gust above 100km/h, which was on September 22, 2010. In this period there were a total of 28 occurrences. In Table 3.15 some data statistics are given for the wind gust measurements. In Figures 3.28 and 3.29, respectively, the PPP of the Exponential, Weibull and Lognormal distributions are given for the inter-arrival times and wind gusts. In tables 3.16 and 3.17 some statistics of the fitted distributions are given for the inter-arrival times and wind gusts above 100km/h, respectively. Additionally, the Method of Moments (MoM) was used to fit a Lognormal distribution and its statistics are also given. In Figure 3.30 the distribution of the maximum wind gust speed is given on a time horizon of 15 years where the inter-arrival times and wind gusts are respectively Lognormal and Exponential with parameters given in tables 3.16 and 3.17.

Table 3.15: **Pearson airport**: statistics wind gusts above 100km/h.

	Inter-arrival times (days)	**Wind gusts (km/h)**
Amount data	27	28
Min	15	101
Max	4062	135
Median	343	107
Mean	741.9630	109.9643
Variance	$9.2116 \cdot 10^5$	71.4431
COV	1.2936	0.0769

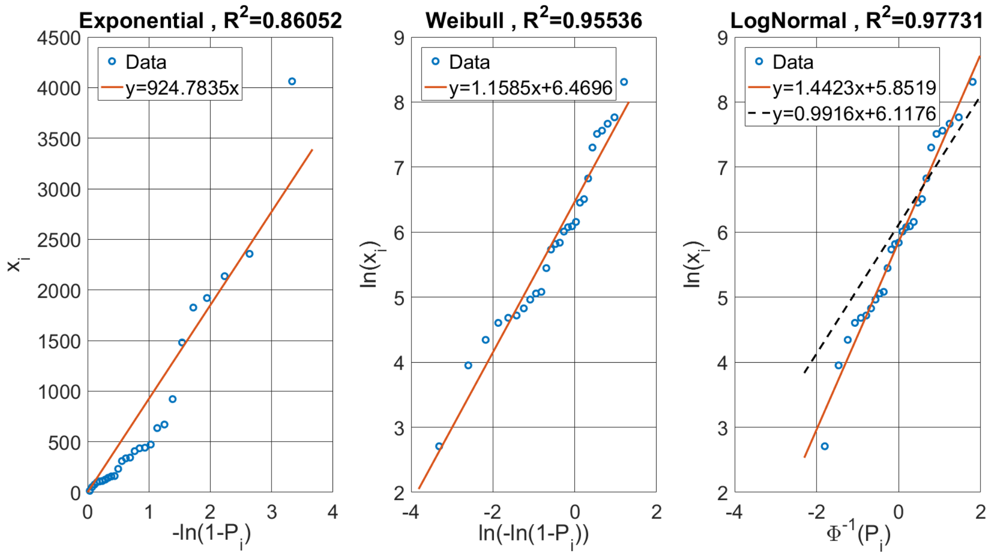


Figure 3.28: **Pearson airport inter-arrival times**: wind speeds above 100km/h. Exponential, Weibull and Lognormal PPP. Dashed line represents the MoM fit.

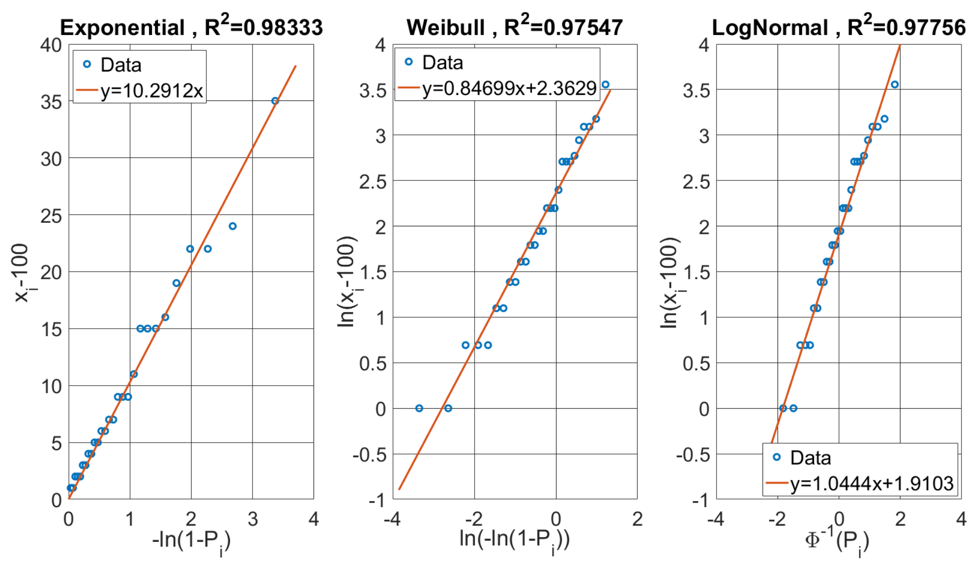


Figure 3.29: **Pearson airport wind speeds above 100km/h**: Exponential, Weibull and Lognormal PPP.

Table 3.16: **Pearson airport inter-arrival times**: statistics fitted distributions for inter-arrival times of wind speeds above 100km/h.

	Exponential	**Weibull**	**Lognormal**	**Lognormal**, (MoM) R^2=0.8404
Parameters	λ=1/924.7835	α=1/1.1585, β=exp(6.4696)	λ=5.8519, ζ=1.4423	λ=6.1176 ζ=0.9916
Mean (days)	924.7835	695.3994	984.4190	741.9630
Variance	$8.5522 \cdot 10^5$	$6.5358 \cdot 10^5$	$6.7905 \cdot 10^6$	$9.2116 \cdot 10^5$
COV	1	1.1626	2.6471	1.2936

Table 3.17: **Pearson airport wind speeds above 100km/h**: statistics fitted distributions.

	Exponential	**Weibull**	**Lognormal**
Parameters	λ=1/10.2912	α=1/0.84699, β=exp(2.3629)	λ=1.9103, ζ=1.0444
Mean (km/h)	110.2912	110.0339	111.6550
Variance	105.9097	72.7465	268.4954
COV	0.0933	0.0775	0.1468

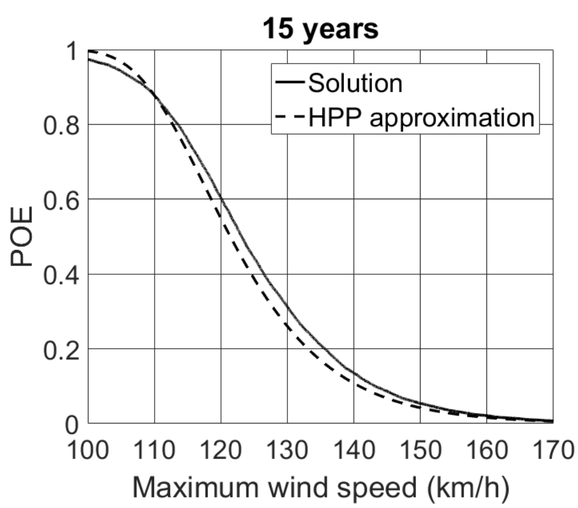


Figure 3.30: **Pearson airport wind speeds above 100km/h**: Lognormal inter-arrival times and Exponential wind speeds as fitted in Tables 3.16 and 3.17.

Wind gusts above 110km/h

The same analysis is performed for wind gusts above 110km/h. The inter-arrival times were taken between Feb 1, 1955 and the last recorded wind gust above 110km/h, which was on August 11, 2009. In this period there were a total of 10 occurrences. In Table 3.18 some data statistics are given for the wind gust measurements. In Figures 3.31 and 3.32, respectively, the PPP of the Exponential, Weibull and Lognormal distributions are given for the inter-arrival times and wind gusts. In tables 3.19 and 3.20 some statistics of the fitted distributions are given for the inter-arrival times and wind gusts above 110km/h, respectively. Additionally, the MoM was used to fit a Lognormal distribution and its statistics are also given. In Figure 3.33 the distribution of the maximum wind gust speed is given on a time horizon of 15 years where the inter-arrival times and wind gusts are respectively Lognormal and Exponential with parameters given in tables 3.19 and 3.20.

Table 3.18: **Pearson airport**: statistics wind gusts above 110km/h.

	Inter-arrival times (days)	**Wind gusts (km/h)**
Amount data	9	10
Min	108	111
Max	10977	135
Median	435	117.5
Mean	$2.1807 \cdot 10^3$	119.4
Variance	$1.3464 \cdot 10^7$	46.4889
COV	1.6827	0.0571

Table 3.19: **Pearson airport inter-arrival times**: statistics fitted distributions for inter-arrival times of wind speeds above 110km/h.

	Exponential	**Weibull**	**Lognormal**	**Lognormal**, (MoM) R^2=0.5802
Parameters	λ=1/3317.1494	α=1/1.626, β=exp(7.1595)	λ=6.3625, ζ=2.0501	λ=7.0158 ζ=1.1590
Mean (days)	3317.1494	1875.5	4741.2	$2.1807 \cdot 10^3$
Variance	$1.1003 \cdot 10^7$	$1.0228 \cdot 10^7$	$1.4812 \cdot 10^9$	$1.3464 \cdot 10^7$
COV	1	1.7052	8.1175	1.6827

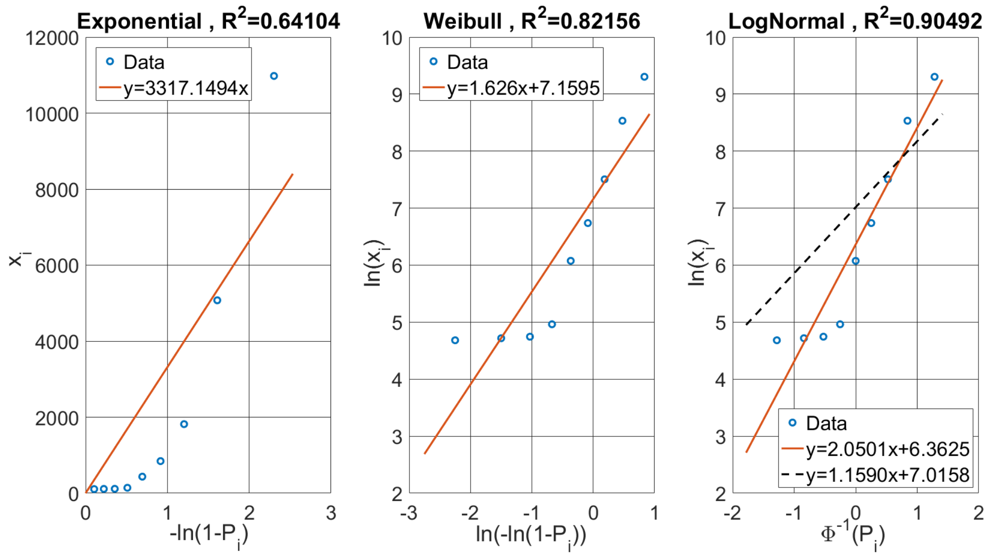


Figure 3.31: **Pearson airport inter-arrival times**: wind speeds above 110km/h. Exponential, Weibull and Lognormal PPP. Dashed line represents MoM fit.

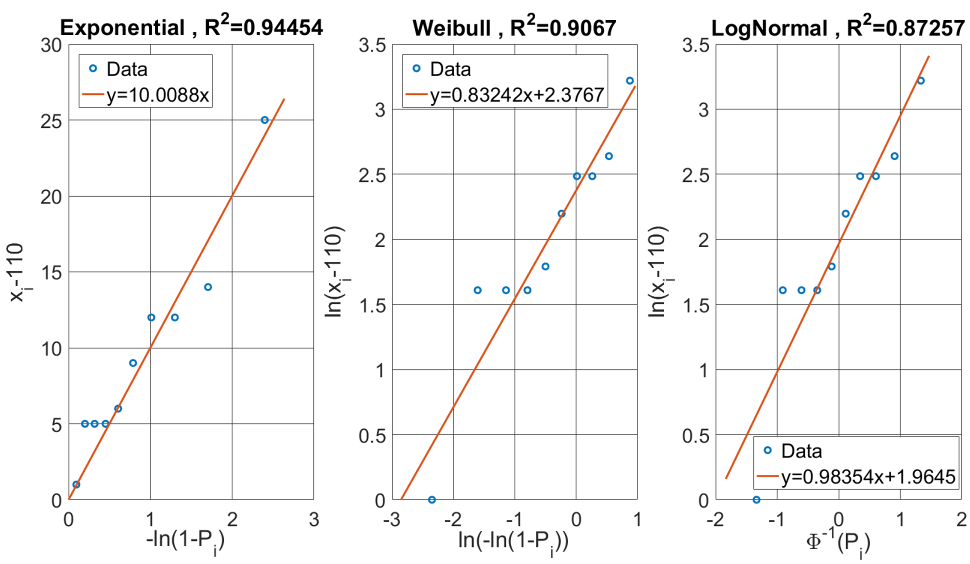


Figure 3.32: **Pearson airport wind speeds above 110km/h**: Exponential, Weibull and Lognormal PPP.

Table 3.20: **Pearson airport wind speeds above 110km/h**: statistics fitted distributions for wind speeds above 110km/h.

	Exponential	**Weibull**	**Lognormal**
Parameters	λ=1/10.0088	α=1/0.83242, β=exp(2.3767)	λ=1.9645, ζ=0.98354
Mean (km/h)	120.0088	120.1276	121.5673
Variance	100.1757	71.6878	218.2265
COV	0.0834	0.0705	0.1215

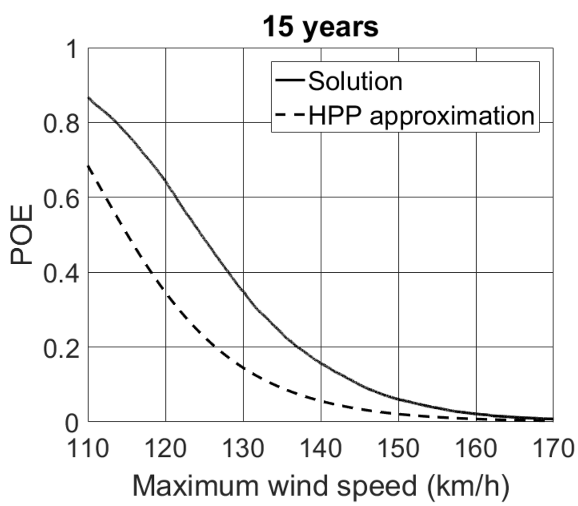


Figure 3.33: **Pearson airport wind speeds above 110km/h**: Lognormal inter-arrival times and Exponential wind speeds as fitted in Tables 3.19 and 3.20.

3.5.3.2 Explanation of Results

It is obvious from Figure 3.33 that the HPP approximation of the proposed model is not a good approximation in this example. The POE of wind speeds of 125 km/h is for example underestimated to less than half the value given by the model. This means, given the estimated parameters are correct and the inter-arrival times are indeed Lognormal distributed with given parameters, the HPP is not a good approximation of this problem. This shows the versatility of the proposed model since it is able to allow for inter-arrival times of any distribution.

3.6 Conclusions

In this chapter an accurate method to compute the maximum distribution of a stochastic hazard modeled as a shock, pulse and alternating process is presented. From practical numerical examples it is clearly shown that a simple solution given by the HPP approximation cannot be expected to perform well on short time horizons. The distribution of maximum given by a Gumbel distribution also exhibits similar behaviour. This chapter clearly shows the need for accurate models that do not use an HPP as underlying assumption or assume hazards follow the limiting behaviour of infinite sequences.

Appendix

3.A Proof of equation (3.10) via Renewal Process and Defective Renewal Equation.

The CDF of the maximum of a pulse load process with inter-arrival times given by a general distribution $F_T(t)$ is given by

$$\mathbb{P}(X^*(t) \leq x) = F_Y(x)\bar{F}_T(t) + F_Y(x)\int_0^t \mathbb{P}(X^*(t-s) \leq x)\mathrm{d}F_T(s). \tag{3.53}$$

This equation can be written in the form

$$a(t) = b(t) + \phi\int_0^t a(t-s)\mathrm{d}F_T(s), \;\; t \geq 0 \tag{3.54}$$

where ϕ=$F_Y(x) \in (0,1)$ and equation (3.54) is called a *defective renewal equation.*

We can transform equation (3.54) into a proper renewal equation by finding a parameter γ such that

$$\widetilde{F}_T(\mathrm{d}t) = e^{\gamma t}\phi F_T(\mathrm{d}t) \tag{3.55}$$

is a probability distribution function, i.e

$$\int_0^\infty \widetilde{F}_T(\mathrm{d}t) = 1. \tag{3.56}$$

Assume N=$\{N(t) = n\}$ is HPP with rate λ, hence, $\mathrm{d}F_T(t)$=$F_T(\mathrm{d}t)$=$\lambda e^{-\lambda t}\mathrm{d}t$. From equations (3.55) and (3.56) we see that

$$\begin{aligned}
\int_0^\infty e^{\gamma t}F_Y(x)\lambda e^{-\lambda t}\mathrm{d}t &= 1 \\
&\Downarrow \\
\lambda F_Y(x)\frac{1}{\gamma-\lambda}e^{-(\lambda-\gamma)t}\Big|_{t=0}^{\infty} &= 1 \\
&\Downarrow \\
\gamma &= \lambda(1-F_Y(x)),
\end{aligned}$$

where in the second equation we assume $\gamma < \lambda$ which holds in the last equation. So now we have transformed the improper renewal equation in (3.54) into a proper renewal equation

$$\widetilde{a}(t) = \widetilde{b}(t) + \int_0^t \widetilde{a}(t-s)\mathrm{d}\widetilde{F}_T(s), \;\; t \geq 0, \tag{3.57}$$

where $\widetilde{a}(t) = e^{\gamma t}a(t)$, $\widetilde{b}(t) = e^{\gamma t}b(t)$ and $\widetilde{F}_T(\mathrm{d}t) = e^{\gamma t}\phi F_T(\mathrm{d}t)$. The solution of

$$\widetilde{a} = \widetilde{b} + \widetilde{F}_T * \widetilde{a}$$

is given as

$$\widetilde{a}(t) = \sum_{n=0}^{\infty} \left(\widetilde{F}_T^{(n)} * \widetilde{b} \right)(t). \tag{3.58}$$

Now we have

$$\begin{aligned}
\widetilde{F}_T(\mathrm{d}t) &= e^{\gamma t}\phi F_T(\mathrm{d}t) \\
&= F_Y(x)e^{\lambda(1-F_Y(x))t}\lambda e^{-\lambda t}\mathrm{d}t \\
&= \lambda F_Y(x)e^{-\lambda F_Y(x)t}\mathrm{d}t,
\end{aligned}$$

hence $\widetilde{F}_T(\mathrm{d}t)$ is an exponential distribution with parameter $\lambda F_Y(x)$. We first notice that if F is the distribution of an exponential random variable with rate $\lambda > 0$, then

$$F_T^{(n)}(\mathrm{d}t) = \frac{(\lambda)^n}{\Gamma(n)} t^{n-1} e^{-\lambda t}\mathrm{d}t = \lambda \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t}\mathrm{d}t.$$

Now with

$$F_T^{(0)}(t) = \delta_0(t)$$

we have

$$\begin{aligned}
\sum_{n=0}^{\infty} F_T^{(n)}(\mathrm{d}t) &= \delta_0(dt) + \lambda e^{-\lambda t}\mathrm{d}t + \lambda\lambda t e^{-\lambda t}\mathrm{d}t + \lambda\frac{(\lambda t)^2}{2!}e^{-\lambda t}\mathrm{d}t + \ldots \\
&= \delta_0(dt) + \lambda e^{-\lambda t}\mathrm{d}t(1 + \lambda t + \frac{(\lambda t)^2}{2!} + \ldots) \\
&= \delta_0(dt) + \lambda \mathrm{d}t,
\end{aligned}$$

where $\delta_0(t)$ is the Dirac delta function which is equal to 1 at t and zero otherwise. Now from equation (3.58) it follows that

$$\widetilde{a}(t) = \sum_{n=0}^{\infty} \left(\widetilde{F}_T^{(n)} * \widetilde{b} \right)(t) = \widetilde{b}(t) + \lambda F_Y(x) \int_0^t \widetilde{b}(s)\mathrm{d}s. \tag{3.59}$$

From equation (3.53) we see that $b(t) = F_Y(x)\bar{F}_T(t)$ and hence $\widetilde{b}(t) = F_Y(x)\bar{F}_T(t)e^{\lambda(1-F_Y(x))t}$. Now substituting $\widetilde{b}(t)$ into equation (3.59) gives

$$\begin{aligned}
\widetilde{a}(t) &= F_Y(x)\left(\bar{F}_T(t)e^{\lambda(1-F_Y(x))t} + \lambda F_Y(x)\int_0^t \bar{F}_T(s)e^{\lambda(1-F_Y(x))s}\mathrm{d}s \right) \\
&= F_Y(x)\left(e^{-\lambda t}e^{\lambda(1-F_Y(x))t} + \lambda F_Y(x)\int_0^t e^{-\lambda s}e^{\lambda(1-F_Y(x))s}\mathrm{d}s \right) \\
&= F_Y(x)\left(e^{-\lambda t F_Y(x)} + \lambda F_Y(x)\int_0^t e^{-\lambda s F_Y(x)}\mathrm{d}s \right) \\
&= F_Y(x)\left(e^{-\lambda t F_Y(x)} - e^{-\lambda t F_Y(x)} + 1 \right) \\
&= F_Y(x).
\end{aligned}$$

Since $e^{\lambda(1-F_Y(x))t}a(t) = \widetilde{a}(t) = F_Y(x)$ it follows that

$$a(t) = F_Y(x)e^{-\lambda t(1-F_Y(x))}$$

which is the solution of equation (3.53) for an HPP. We can check this is the same solution we got in equation (3.10).

3.B Proof of Lemma 3.6.

First we notice that
$$N(t) \geq n \;\; \textit{iff} \;\; S_n \leq t$$
so $\mathbb{P}(N(t) \geq n)$=$F_T^{(n)}(t)$ where we use the notation $\mathbb{P}(S_n \leq t)$=$\mathbb{P}(T_1 + \ldots + T_n \leq t)$=$F_T^{(n)}(t)$ which is the convolution of n inter-arrival times. Next we use

$$\begin{aligned}
\mathbb{P}(N(t) = n) &= \mathbb{P}(N(t) \leq n) - \mathbb{P}(N(t) \leq n-1) \\
&= 1 - \mathbb{P}(N(t) > n) - (1 - \mathbb{P}(N(t) > n-1)) \\
&= 1 - \mathbb{P}(N(t) \geq n+1) - (1 - \mathbb{P}(N(t) \geq n)) \\
&= \mathbb{P}(N(t) \geq n) - \mathbb{P}(N(t) \geq n+1) \\
&= F_T^{(n)}(t) - F_T^{(n+1)}(t).
\end{aligned}$$

Now partition over the events $\{N(t) = n\}$:

$$\begin{aligned}
\mathbb{P}(X^*(t) \leq x) &= \mathbb{P}(\max\{Y_1\} \leq x, N(t) = 0) \\
&\quad + \mathbb{P}(\max\{Y_1, Y_2\} \leq x, N(t) = 1) \\
&\quad + \mathbb{P}(\max\{Y_1, Y_2, Y_3\} \leq x, N(t) = 2) \\
&\quad + \quad \vdots \\
&= F_Y(x) \left(F_T^{(0)}(t) - F_T^{(1)}(t) \right) \\
&\quad + F_Y(x)^2 \left(F_T^{(1)}(t) - F_T^{(2)}(t) \right) \\
&\quad + F_Y(x)^3 \left(F_T^{(2)}(t) - F_T^{(3)}(t) \right) \\
&\quad + \quad \vdots \\
&= \sum_{n=1}^{\infty} F_Y(x)^n \left(F_T^{(n-1)}(t) - F_T^{(n)}(t) \right) \\
&= \sum_{n=1}^{\infty} F_Y(x)^n F_T^{(n-1)}(t) - \sum_{n=1}^{\infty} F_Y(x)^n F_T^{(n)}(t) \\
&= F_Y(x) \sum_{n=1}^{\infty} F_Y(x)^{n-1} F_T^{(n-1)}(t) - \sum_{n=1}^{\infty} F_Y(x)^n F_T^{(n)}(t) \\
&= F_Y(x) \left(F_Y(x)^0 F_T^{(0)}(t) + \sum_{n=2}^{\infty} F_Y(x)^{n-1} F_T^{(n-1)}(t) \right) - \sum_{n=1}^{\infty} F_Y(x)^n F_T^{(n)}(t) \\
&= F_Y(x) \left(1 + \sum_{n=1}^{\infty} F_Y(x)^n F_T^{(n)}(t) \right) - \sum_{n=1}^{\infty} F_Y(x)^n F_T^{(n)}(t)
\end{aligned} \tag{3.60}$$

$$
\begin{aligned}
&= F_Y(x) + (F_Y(x)-1)\sum_{n=1}^{\infty} F_Y(x)^n F_T^{(n)}(t) \\
&= F_Y(x) - (1-F_Y(x))\sum_{n=1}^{\infty} F_Y(x)^n F_T^{(n)}(t). \qquad (3.61)
\end{aligned}
$$

Now let

$$y(t) = \sum_{n=1}^{\infty} F_Y(x)^n F_T^{(n)}(t)$$

be the summation given in equation (3.61). For $n>1$ we have

$$F_Y(x)^n F_T^{(n)}(t) = F_Y(x)^n \int_0^t F_T^{(n-1)}(t-s)\mathrm{d}F_T(s) = F_Y(x)\int_0^t F_Y(x)^{n-1}F_T^{(n-1)}(t-s)\mathrm{d}F_T(s).$$

Now take the summation over $n>1$:

$$
\begin{aligned}
\sum_{n=2}^{\infty} F_Y(x)^n F_T^{(n)}(t) &= \sum_{n=2}^{\infty} F_Y(x)\int_0^t F_Y(x)^{n-1}F_T^{(n-1)}(t-s)\mathrm{d}F_T(s) \\
&= F_Y(x)\int_0^t \sum_{n=2}^{\infty} F_Y(x)^{n-1}F_T^{(n-1)}(t-s)\mathrm{d}F_T(s) \\
&= F_Y(x)\int_0^t \sum_{n=1}^{\infty} F_Y(x)^{n}F_T^{(n)}(t-s)\mathrm{d}F_T(s).
\end{aligned}
$$

Now add $n=1$ to both sides to obtain:

$$
\begin{aligned}
y(t) &= F_Y(x)F_T(t) + \sum_{n=2}^{\infty} F_Y(x)^n F_T^{(n)}(t) \\
&= F_Y(x)F_T(t) + F_Y(x)\int_0^t \sum_{n=1}^{\infty} F_Y(x)^n F_T^{(n)}(t-s)\mathrm{d}F_T(s) \\
&= F_Y(x)F_T(t) + F_Y(x)\int_0^t y(t-s)\mathrm{d}F_T(s). \qquad \square
\end{aligned}
$$

3.C Generalized Extreme Value Distributions

3.C.1 Introduction

In modelling the distribution of the (annual) maximum one approach is to notice the following. Let $Y_1, Y_2, \ldots, Y_n$ be *iid* random variables with distribution $F_Y(y)$ and let $M_n = \max\{Y_1, Y_2, \ldots, Y_n\}$. Then

$$\mathbb{P}(M_n \leq y) = (F_Y(y))^n. \qquad (3.62)$$

Hence, the distribution of the maximum can be readily derived. However, since the distribution F is unknown and small errors in the estimation of F can lead to large errors in the estimation of $(F_Y(y))^n$ we can accept the fact that F is unknown and instead estimate $(F_Y(y))^n$.

3.C.2 Theoretical Results

From equation (3.62) first we notice that if we take any $y < y_+$ where y_+ is the smallest value such that $F(y_+)$=1, then $(F_Y(y))^n \to 0$ for $n \to \infty$. This implies that the distribution of M_n degenerates to a point mass at y_+. This is resolved by applying the normalization $(M_n - b_n)/a_n$. So we must first find sequences of constants $\{a_n > 0\}$ and $\{b_n\}$ such that

$$\mathbb{P}\left(\frac{M_n - b_n}{a_n} \leq y\right) = F(a_n x + b_n)^n \to G(y), \ \text{ as } n \to \infty.$$

If sequences of constants $\{a_n > 0\}$ and $\{b_n\}$ exist and G is a non-degenerate distribution, it is known that G is one of the following three types of distributions:

$$\text{Gumbel}: \quad G(y) = \exp\left(-\exp\left[-\left(\frac{y-b}{a}\right)\right]\right), \quad \forall y;$$

$$\text{Fréchet}: \quad G(y) = \begin{cases} 0, & y \leq 0, \\ \exp\left[-\left(\frac{y-b}{a}\right)^{-\alpha}\right], & y > 0; \end{cases}$$

$$\text{(reversed) Weibull}: \quad G(y) = \begin{cases} \exp\left(-\left[-\left(\frac{y-b}{a}\right)^{-\alpha}\right]\right), & y < 0, \\ 1, & y \geq 0, \end{cases}$$

where $a > 0$, b and $\alpha > 0$ for the Fréchet and Weibull distributions. It can be seen that these three distributions can be combined into one distribution called the Generalized Extreme Value (GEV) distribution:

$$G(y) = \exp\left(-\left[1 + \xi\left(\frac{y-\mu}{\sigma}\right)\right]^{-1/\xi}\right), \tag{3.63}$$

with $\{y : 1 + \xi(y - \mu)/\sigma > 0\}$, $-\infty < \mu < \infty$, $\sigma > 0$ and $-\infty < \xi < \infty$ where μ is called the location parameter, σ the scale parameter and ξ the shape parameter. The three distributions then belong to the three different cases: $\xi \to 0$ belongs to the Gumbel, $\xi > 0$ belongs to the Fréchet with $\alpha = 1/\xi$ and $\xi < 0$ belongs to the Weibull family of distributions with $\alpha = -1/\xi$.

The three distributions are distinguished by the tail behaviour of the original distribution F of the X_i:

- Gumbel, $\xi \approx 0$: exponential tail,
- Fréchet, $\xi > 0$: long/heavy tailed,
- Weibull, $\xi < 0$: short tailed. With finite end-point at $\mu - \xi/\sigma$.

3.C.3 The Generalized Pareto Distribution

In modelling extreme events one might be mostly interested in values above a certain threshold l_0. Hence, one might be interested in the conditional probability

$$\mathbb{P}(Y > l_0 + y | Y > l_0), \ \ y > 0. \tag{3.64}$$

This leads to the family of distributions called the Generalized Pareto Distribution (GPD) which is explained in the following lemma:

Lemma 3.9. *Let $Y_1, Y_2, \ldots, Y_n$ be iid random variables with distribution $F_Y(y)$ and M_n=$\max\{Y_1, Y_2, \ldots, Y_n\}$. Let Y be a random element of the Y_i's and assume $F_Y(y)$ satisfies*

$$\mathbb{P}\left(\frac{M_n - b_n}{a_n} \leq y\right) \to G(y),$$

as $n \to \infty$ where

$$G(y) = \exp\left(-\left[1 + \xi\left(\frac{y-\mu}{\sigma}\right)\right]^{-1/\xi}\right),$$

for ξ and $\mu, \sigma > 0$ as explained in section 3.C.2. Then, if a threshold l_0 is large enough the distribution of $Y - l_0$ under the condition $Y > l_0$, i.e. $\mathbb{P}(Y \leq l_0 + y | Y > l_0)$, can be approximated by the GPD

$$H(y) = 1 - \left(1 + \frac{\xi y}{\widetilde{\sigma}}\right)^{-1/\xi}, \; y > 0, \; 1 + \xi y/\widetilde{\sigma} > 0, \tag{3.65}$$

and $\widetilde{\sigma} = \sigma + \xi(l_0 - \mu)$.

Proof. An outline of this proof can also be found in [7] and [39].
Let Y follow the distribution $F_Y(y)$ and assume the maximum of the sequence $Y_1, \ldots, Y_n$ follows a GEV distribution. This means for large enough n we have

$$\left(F_Y(y)\right)^n \approx \exp\left(-\left[1 + \xi\left(\frac{y-\mu}{\sigma}\right)\right]^{-1/\xi}\right),$$

for ξ and $\mu, \sigma > 0$. Taking the logarithm leads to

$$n \ln\left(F_Y(y)\right) \approx -\left[1 + \xi\left(\frac{y-\mu}{\sigma}\right)\right]^{-1/\xi}. \tag{3.66}$$

But we are interested in maximum values and for large values y, a Taylor series expansion gives the approximation

$$\ln\left(F_Y(y)\right) \approx -\left(1 - F_Y(y)\right).$$

So for large values l_0, substituting this approximation into equation (3.66) gives the approximate relation

$$1 - F_Y(l_0) \approx \frac{1}{n}\left[1 + \xi\left(\frac{l_0 - \mu}{\sigma}\right)\right]^{-1/\xi}$$

and similarly for $l_0 + y$, given $y > 0$,

$$1 - F_Y(l_0 + y) \approx \frac{1}{n}\left[1 + \xi\left(\frac{l_0 + y - \mu}{\sigma}\right)\right]^{-1/\xi}.$$

Hence, the conditional probability in equation (3.64) gives

$$\begin{aligned}
\mathbb{P}(Y > l_0 + y | Y > l_0) &\approx \frac{1 - F_Y(l_0 + y)}{1 - F_Y(l_0)} \\
&= \frac{n^{-1}\left[1 + \xi(l_0 + y - \mu)/\sigma\right]^{-1/\xi}}{n^{-1}\left[1 + \xi(l_0 - \mu)/\sigma\right]^{-1/\xi}} \\
&= \left[\frac{\sigma + \xi(l_0 + y - \mu)}{\sigma + \xi(l_0 - \mu)}\right]^{-1/\xi} \\
&= \left[1 + \frac{\xi y}{\widetilde{\sigma}}\right]^{-1/\xi},
\end{aligned}$$

where $\widetilde{\sigma} = \sigma + \xi(l_0 - \mu)$. □

The Pareto distribution can also have varying tail behaviours:

- $\xi \approx 0$: exponential tail. When $\xi \to 0$ we get the limit $H(y) = 1 - \exp(-y/\sigma)$ which is an exponential distribution with intensity $1/\sigma$,
- $\xi > 0$: long/heavy tailed,
- $\xi < 0$: short tailed. With finite end-point at $-\sigma/\xi$.

3.C.4 Distribution of HPP Pareto

In this section we analyze the distribution of the annual maximum if loads arrive according to an HPP and the load magnitudes are given as a GPD.

Let the amount, N, of loads above a threshold l_0 in a year be an HPP with intensity λ. We notice that we can also take the load arrival process to be HPP with intensity $\hat{\lambda}$ and let the probability of exceeding the threshold l_0 be $p \in [0, 1]$. Then $\lambda = p\hat{\lambda}$. Let the load magnitudes $Y_1, Y_2, \ldots, Y_N$ exceeding the threshold l_0 be *iid* following a GPD and are furthermore independent of the HPP. Denote the maximum load in the year by, Y^*, i.e.:

$$Y^* = \max\{Y_1, Y_2, \ldots, Y_N\}, \quad N \geq 1,$$

and $Y^* = 0$ if $N = 0$. Assume $x > l_0 > 0$. Then, by first conditioning on the number of loads, the distribution of the annual maximum is given as

$$\begin{aligned}
\mathbb{P}(Y^* \leq x) &= \mathbb{P}(N = 0) + \sum_{n=1}^{\infty} \mathbb{P}(Y^* \leq x, N = n) \\
&= e^{-\lambda} + \sum_{n=1}^{\infty} \left[1 - \left(1 + \xi \frac{x - l_0}{\widetilde{\sigma}}\right)^{-1/\xi}\right]^n \frac{\lambda^n e^{-\lambda}}{n!}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}\left[1-\left(1+\xi\frac{x-l_0}{\widetilde{\sigma}}\right)^{-1/\xi}\right]^n \frac{\lambda^n e^{-\lambda}}{n!} \\
&= e^{-\lambda}\exp\left(\lambda\left[1-\left(1+\xi\frac{x-l_0}{\widetilde{\sigma}}\right)^{-1/\xi}\right]\right) \\
&= \exp\left(-\lambda\left(1+\xi\frac{x-l_0}{\widetilde{\sigma}}\right)^{-1/\xi}\right).
\end{aligned}
$$

This leads to the following lemma:

Lemma 3.10. *The distribution of the annual maximum of an HPP with GPD load magnitudes is GEV, i.e.:*

$$
\mathbb{P}(Y^* \leq x) = \exp\left(-\lambda\left(1+\xi\frac{x-l_0}{\widetilde{\sigma}}\right)^{-1/\xi}\right)
$$

is GEV with $\widetilde{\sigma}$=$\sigma+\xi(l_0-\mu)$ *and* λ=$\left(1+\xi\frac{l_0-\mu}{\sigma}\right)^{-1/\xi}$ *where the GPD parameter* $\widetilde{\sigma}$ *and HPP parameter* λ *are clearly functions of the threshold* l_0*.*

Proof. We have

$$
\mathbb{P}(Y^* \leq x) = \exp\left(-\lambda\left(1+\xi\frac{x-l_0}{\widetilde{\sigma}}\right)^{-1/\xi}\right),
$$

and let $\widetilde{\sigma}$=$\sigma+\xi(l_0-\mu)$ and λ=$\left(1+\xi\frac{l_0-\mu}{\sigma}\right)^{-1/\xi}$. Then,

$$
\begin{aligned}
\mathbb{P}(Y^* \leq x) &= \exp\left(-\left(1+\xi\frac{l_0-\mu}{\sigma}\right)^{-1/\xi}\left(1+\xi\frac{x-l_0}{\sigma+\xi(l_0-\mu)}\right)^{-1/\xi}\right) \\
&= \exp\left(-\left[\left(1+\xi\frac{l_0-\mu}{\sigma}\right)\left(1+\xi\frac{x-l_0}{\sigma+\xi(l_0-\mu)}\right)\right]^{-1/\xi}\right) \\
&= \exp\left(-\left[1+\xi\frac{x-l_0}{\sigma+\xi(l_0-\mu)}+\xi\frac{l_0-\mu}{\sigma}+\xi^2\frac{(l_0-\mu)(x-l_0)}{\sigma(\sigma+\xi(l_0-\mu))}\right]^{-1/\xi}\right) \\
&= \exp\left(-\left[1+\xi\frac{\sigma(x-l_0)+(l_0-\mu)(\sigma+\xi(l_0-\mu))+\xi(l_0-\mu)(x-l_0)}{\sigma(\sigma+\xi(l_0-\mu))}\right]^{-1/\xi}\right) \\
&= \exp\left(-\left[1+\xi\frac{\sigma(x-l_0)+\sigma(l_0-\mu)+\xi(l_0-\mu)\left[(l_0-\mu)+(x-l_0)\right]}{\sigma(\sigma+\xi(l_0-\mu))}\right]^{-1/\xi}\right) \\
&= \exp\left(-\left[1+\frac{\xi}{\sigma}\cdot\frac{\sigma(x-\mu)+\xi(l_0-\mu)(x-\mu)}{\sigma+\xi(l_0-\mu)}\right]^{-1/\xi}\right) \\
&= \exp\left(-\left[1+\frac{\xi(x-\mu)}{\sigma}\cdot\frac{\sigma+\xi(l_0-\mu)}{\sigma+\xi(l_0-\mu)}\right]^{-1/\xi}\right) \\
&= \exp\left(-\left[1+\frac{\xi(x-\mu)}{\sigma}\right]^{-1/\xi}\right).
\end{aligned}
$$

□

We notice that in this formulation for the annual maximum we assumed t=1 year. For the reliability of this load process we can set t as a variable with fixed threshold x. For the distribution of the maximum load in a time horizon $[0, t]$ years we can let x be a variable while fixing t. Both then lead to the formula

$$\mathbb{P}(Y^*(t) \leq x) = \exp\left(-\lambda t \left(1 + \xi \frac{x - l_0}{\widetilde{\sigma}} \right)^{-1/\xi} \right). \tag{3.67}$$

Chapter 4

Distribution of Maximum Load: Combined Processes

4.1 Introduction

4.1.1 Background

Structures are subject to loads from different sources and are built to withstand these during their intended lifespan. Examples of loads include: high winds, earthquakes, snow etc. The loads can generally be characterized as random occurrences in time that have effects (or impact) of a certain magnitude on a structure. However, more importantly, since the times of occurrence of these loads fluctuate randomly the total magnitude on a structure can be the combined load effect from different sources. This prompts the need to carefully model the superposition of loads in an understandable framework capturing their total effect during the (remaining) service life of a structure. In particular, we wish to capture the magnitude of the maximum combined impact of different loads by characterizing these loads as being random in their time of occurrence, magnitude and duration.

Generally, random loads can be characterized as being of short duration, such as earthquakes, which are modelled as stochastic shock processes or long-term slowly varying loads, such as dead loads or a body of water, which are modelled as stochastic pulse processes. The proposed model is confined to the superposition (sum) of a shock process and a pulse process but the random time and amount of load occurrences, random duration of occurrences and load magnitudes makes this a complex problem. Modelling the maximum of the combined process seemingly reduces the complexity of the time-variant problem to a time-invariant problem as follows. Suppose in the remaining life span $(0, t]$ of a structure, $Z^*(t)$ is the maximum of the sum process up to time t and z is the time-invariant capacity of the structure. Then $\mathbb{P}(Z^*(t) > z)$ indicates the probability of failure over the entire life span of the structure. However, in most available literature, still the load occurrences are often assumed to be generated by HPP's to further reduce the complexity of the problem.

4.1.2 Existing Literature and Limitations

Superposition of two Poisson pulse processes is the most dominant model used for load combination. The following types of work have been reported in the literature:

- Regarding the renewal pulse processes, there is not much work. [3] discussed this model assuming that the renewal process is stationary. Later, Rackwitz [56] analyzes the crossing rate of non-stationary renewal pulse processes. The solution approach is an approximation obtained by some modification of the solution for stationary renewal processes.

- Schrupp and Rackwitz [64] analyzed superposition of the Poisson cluster process. However, the derivation was based on assuming that resulting process is in equilibrium state.

  Winterstein [78] also analyzes the extreme value distribution of combined renewal pulse processes. This work also follows the crossing rate approach of [37] and the Rice formula.

- The combination of a Poisson pulse and Poisson shock process is given in [20] and [27]. This is how the literature reports this work.

- Combination of a "renewal" pulse process with a Poisson shock process is studied by Jacobs, [32]. This is the only formal study of the problem. This paper largely provides limiting approximation of the first passage distribution via the Laplace transform method. There is finite time solution of the problem.

  Our present work provides an explicit solution of this problem. A clear proof is presented to show that the combined process is a regenerative process. Jacobs assumed this property and derived his results.

- A model for alternating processes is studied by [11] as a Markov "on" and "off" process. The on and off times follow exponential distributions. The paper does not derive the extreme value distribution. Rather it goes on to derive the mean crossing rate of a sum process using the language of Rice and Leadbetter [39]. All these solutions are referred to as "bounds" on the POE.

- Rackwitz [31] studied this problem for Erlang distributed arrival times by keeping the Markov structure of the problem and deriving a matrix differential equation for the combination process to obtain the crossing rate. This solution is inspired by the Queueing theory.

- In [77] and [75], Wen provides a model for the superposition of two processes where parameters in the model can determine whether a process behaves as a pulse or a shock. The solution is a conservative approximation based on the rate of coincidence of the two processes. It is also used to compare the performance of the proposed model in this chapter.

- Other models have been considered in [58], [18], [13], [70], [2], [48], [53], [44], [45], [65], [30] and [29].

4.1.3 Contributions to Literature

As mentioned, in the literature there are solutions available for the load combination problem where the pulse and shock process are both HPP. This chapter relaxes the condition on the pulse process to have inter-arrival times following any distribution. At present, the work in this chapter provides the most precise formulation of the load superposition problem where the pulse process is a renewal process and the shock process is a Poisson process.

4.1.4 Objectives and Organization

In section 4.2 the load combination problem is explained and the solution is derived for the distribution of the maximum sum of loads. This section includes the details that justify the mathematical steps in the derivation which are often omitted in the literature. Moreover, a numerical example shows the accuracy of the model and its performance compared to a well-known existing method in the literature.

4.2 Combined Pulse and Shock Process

4.2.1 Pulse Process

4.2.1.1 Model

In this model the sustained load starts at time t=0 at load magnitude L_1 and lasts for time T_1. After T_1 the process repeats, creating a sequence $(L_1, T_1), (L_2, T_2), (L_3, T_3), \ldots$ of *iid* non-negative random vectors. The process of the sustained load can then be written as

$$X(t) = \sum_{n=1}^{\infty} L_n \mathbb{1}_{\{S_{n-1} \leq t < S_n\}}, \quad t \geq 0, \tag{4.1}$$

where

$$S_n = \sum_{i=1}^{n} T_i$$

and N=$\{N(t); t \geq 0\}$ is the counting process for the sequence $0 < S_1 < S_2 < \ldots$.

An illustration of the sustained load process, inter-arrival times and partial sums of the inter-arrival times is given in Figure 4.1.

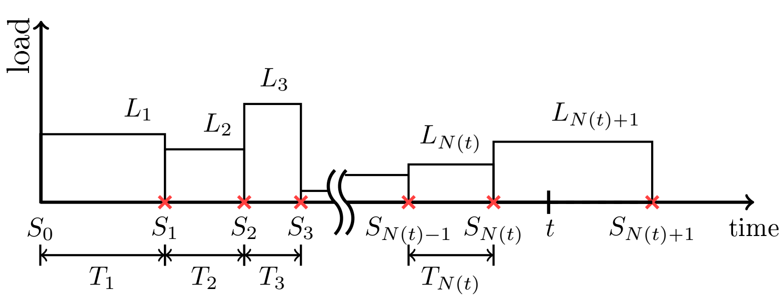


Figure 4.1: A schematic of a pulse load process.

4.2.1.2 Renewal Decomposition for Pulse Process

From the previous subsection, the sustained load process is given by equation (4.1). Now we look at the process starting after the first change in the sustained load size and argue that these two processes are equivalent in distribution.

Let $\widetilde{L}_n = L_{n+1}$, $\widetilde{T}_n = T_{n+1}$ and $\widetilde{S}_n = S_{n+1} - T_1$ be shifted sequences determining the shifted process

$$\widetilde{X}(t) = \sum_{n=1}^{\infty} \widetilde{L}_n \mathbb{1}_{\{\widetilde{S}_{n-1} \leq t < \widetilde{S}_n\}}, \quad t \geq 0.$$

Hence, $\{\widetilde{X}(t); t \geq 0\}$ is the shifted sustained load process starting after T_1. To show that $\{X(t); t \geq 0\}$ is a regenerative process it is to be shown that the processes X and $\widetilde{X}$ have the same distribution and the process $\widetilde{X}$ and the random vector (L_1, T_1) are independent. For $t > T_1$ we have

$$\begin{aligned}
\widetilde{X}(t - T_1) &= \sum_{n=1}^{\infty} L_{n+1} \mathbb{1}_{\{S_n - T_1 \leq t - T_1 < S_{n+1} - T_1\}} \\
&= \sum_{n=2}^{\infty} L_n \mathbb{1}_{\{S_{n-1} \leq t < S_n\}} \\
&= X(t) - L_1 \mathbb{1}_{\{t < T_1\}}.
\end{aligned}$$

Setting $\widetilde{X}(t) = 0$ for $t < 0$ leads to

$$X(t) = L_1 \mathbb{1}_{\{t < T_1\}} + \widetilde{X}(t - T_1), \quad t \geq 0. \tag{4.2}$$

4.2.2 Shock Process

4.2.2.1 Model

All shocks in this model are assumed to be of negligible duration. Hence, the shock process can be modelled as a point process. After a time period V_1 there is a shock of magnitude Y_1, subsequently after a time period V_2 there is a shock of magnitude Y_2 resulting in a sequence of *iid* non-negative random vectors $(Y_1, V_1), (Y_2, V_2), (Y_3, V_3), \ldots, (Y_{M(t)}, V_{M(t)})$ where the counting process $M = \{M(t); t \geq 0\}$ is associated with the increasing sequence $0 < U_1 < U_2 < \ldots$ given by

$$U_m = \sum_{i=1}^{m} V_i.$$

A depiction is given in Figure 4.2.

The process for the shock load can then be given as

$$Y(t) = \sum_{m=1}^{\infty} Y_m \mathbb{1}_{\{U_m = t\}}, \quad t \geq 0. \tag{4.3}$$

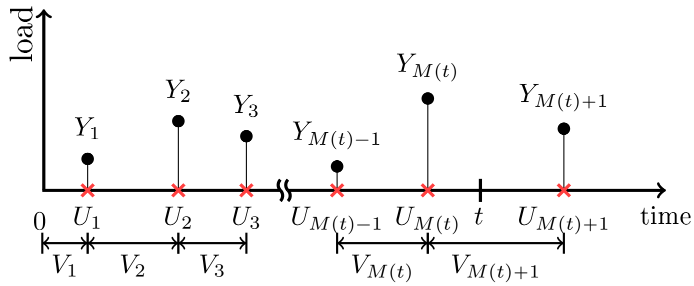


Figure 4.2: A schematic of a shock load process.

4.2.2.2 Renewal Decomposition for Combination Process

We want to write the reliability function of the component as a renewal function and the sustained load in the previous section has been written as a regenerative process after a time period T_1 at which the first sustained load, L_1 changes to L_2. This means that the shock load process needs to be analyzed as a regenerative process based on T_1 so that the sum of the processes can be a regenerative process with regeneration epoch T_1. At time T_1 there have been $M(T_1)$ occurrences of shock loads. This is depicted in Figure 4.3. The time between T_1

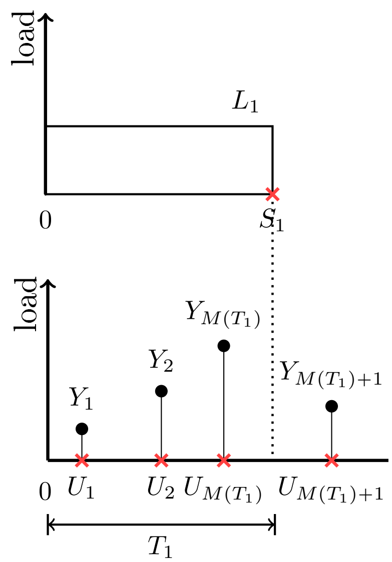


Figure 4.3: A schematic with regeneration epoch at T_1 for both processes.

and the next shock load equals $U_{M(T_1)+1} - T_1$. Define

$$\begin{aligned}
\widetilde{V}_1 &= U_{M(T_1)+1} - T_1, \;\; \widetilde{V}_j = V_{M(T_1)+j}, \quad j \geq 2 \\
\widetilde{Y}_m &= Y_{M(T_1)+m}, \quad m \geq 1.
\end{aligned}$$

First we notice that (V_1, Y_1) and $(\widetilde{V}_m, \widetilde{Y}_m)$ are independent. To show that the shock load process $\{Y(t); t \geq 0\}$ is a regenerative process with regeneration epoch T_1 it is left to be shown that the sequence $(\widetilde{V}_m, \widetilde{Y}_m)$ is *iid* and $(\widetilde{V}_m, \widetilde{Y}_m) \stackrel{d}{=} (V_m, Y_m)$. A necessary assumption for $(\widetilde{V}_1, \widetilde{Y}_1) \stackrel{d}{=} (V_1, Y_1)$ to hold, is that the counting process M is generated by an HPP. This assumption also renders (V_1, Y_1) and $(\widetilde{V}_m, \widetilde{Y}_m)$ independent. Now let μ be the intensity of the Poisson process M and we will show that $(\widetilde{V}_m, \widetilde{Y}_m) \stackrel{d}{=} (V_m, Y_m)$.

For the distribution of $(\widetilde{V}_m, \widetilde{Y}_m)$ we have

$$\begin{aligned}
&\mathbb{P}(\widetilde{V}_1 > v_1, \widetilde{Y}_1 \leq y_1, \ldots, \widetilde{V}_j > v_j, \widetilde{Y}_j \leq y_j) \\
&\quad = \int_0^\infty \mathbb{P}(U_{M(t_1)+1} - t_1 > v_1, Y_{M(t_1)+1} \leq y_1, \ldots, V_{M(t_1)+j} > v_j, Y_{M(t_1)+j} \leq y_j) dF_{T_1}(t_1).
\end{aligned}$$

Partitioning over the values of $M(t_1)$, the integrand becomes

$$\begin{aligned}
&\mathbb{P}(U_{M(t_1)+1} - t_1 > v_1, Y_{M(t_1)+1} \leq y_1, \ldots, V_{M(t_1)+j} > v_j, Y_{M(t_1)+j} \leq y_j) \\
&= \sum_{m=0}^{\infty} \mathbb{P}(U_{m+1} - t_1 > v_1, Y_{m+1} \leq y_1, \ldots, V_{m+j} > v_j, Y_{m+j} \leq y_j, M(t_1) = m) \\
&= F_Y(y_1) \cdots F_Y(y_j) \Big[\mathbb{P}(V_1 - t_1 > v_1, V_2 > v_2, \ldots, V_j > v_j) \\
&\quad + \sum_{m=1}^{\infty} \mathbb{P}(U_m + V_{m+1} - t_1 > v_1, V_{m+2} > v_2, \ldots, V_{m+j} > v_j, U_m \leq t_1 < U_m + V_{m+1}) \Big] \\
&= F_Y(y_1) \cdots F_Y(y_j) \Big[e^{-\mu(t_1 + v_1 + \ldots + v_j)} \\
&\quad + \sum_{m=1}^{\infty} \mathbb{P}(U_m + V_{m+1} - t_1 > v_1, V_{m+2} > v_2, \ldots, V_{m+j} > v_j, U_m \leq t_1 < U_m + V_{m+1}) \Big]. \qquad (4.4)
\end{aligned}$$

By independence, the summation term in equation (4.4) becomes

$$\begin{aligned}
\mathbb{P}(U_m + V_{m+1} - t_1 > v_1, V_2 > v_2, \ldots, V_{m+j} > v_j, U_m \leq t_1 < U_m + V_{m+1}) \\
= \int_0^{t_1} \mathbb{P}(V_{m+1} > t_1 + v_1 - u, V_{m+2} > v_2, \ldots, V_{m+j} > v_j) dF_{U_m}(u) \\
= \int_0^{t_1} e^{-\mu(t_1 + v_1 + \cdots + v_j - u)} dF_{U_m}(u).
\end{aligned}$$

This leads[1] to

$$\begin{aligned}
&\sum_{m=1}^{\infty} \mathbb{P}(U_m + V_{m+1} - t_1 > v_1, V_{m+2} > v_2, \dots, V_{m+j} > v_j) \\
&\qquad = \sum_{m=1}^{\infty} \int_0^{t_1} e^{-\mu(t_1+v_1+\cdots+v_j-u)} dF_{U_m}(u) \\
&\qquad = \int_0^{t_1} e^{-\mu(t_1+v_1+\cdots+v_j-u)} \mu du \\
&\qquad = e^{-\mu(t_1+v_1+\cdots+v_j)}[e^{\mu t_1} - 1].
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathbb{P}(U_{M(t_1)+1} - t_1 > v_1, Y_{M(t_1)+1} \le y_1, \dots, V_{M(t_1)+j} > v_j, Y_{M(t_1)+j} \le y_j) \\
&\qquad = F_Y(y_1) \cdots F_Y(y_j) e^{-\mu(v_1+\cdots+v_j)}
\end{aligned}$$

and it follows that

$$\begin{aligned}
\mathbb{P}(\widetilde{V}_1 > v_1, \widetilde{Y}_1 \le y_1, \dots, \widetilde{V}_j > v_j, \widetilde{Y}_j \le y_j) &= \int_0^{\infty} F_Y(y_1) \cdots F_Y(y_j) e^{-\mu(v_1+\cdots+v_j)} dF_{T_1}(t_1) \\
&= \mathbb{P}(V_1 > v_1, Y_1 \le y_1, \dots, V_j > v_j, Y_j \le y_j).
\end{aligned}$$

We conclude that $(\widetilde{V}_m, \widetilde{Y}_m) \stackrel{d}{=} (V_m, Y_m)$. □

Let the shifted process $\widetilde{Y} = \{Y(t); t \ge 0\}$ be the shock process determined by the sequence of random vectors $(\widetilde{V}_m, \widetilde{Y}_m)$, i.e.

$$\widetilde{Y}(t) = \sum_{n=1}^{\infty} \widetilde{Y}_m \mathbb{1}_{\{\widetilde{U}_m = t\}}, \quad t \ge 0,$$

where $\widetilde{U}_m = \widetilde{U}_{m-1} + \widetilde{V}_m, m \ge 1$ are the partial sums of the inter-occurrence times with the convention $\widetilde{U}_0 = 0$. The shock process Y and the shifted shock process $\widetilde{Y}$ have the same probability distribution. We note that

$$\widetilde{U}_m = \sum_{j=1}^{m} \widetilde{V}_j = U_{M(T_1)+1} - T_1 + \sum_{j=2}^{m} V_{M(T_1)+m} - T_1,$$

---

[1]

$$\begin{aligned}
M(u) = \sum_{m=1}^{\infty} \mathbb{1}_{\{U_m \le u\}} \quad &\Rightarrow \quad \mathbb{E}[M(u)] = \sum_{m=1}^{\infty} \mathbb{E}[\mathbb{1}_{\{U_m \le u\}}] \quad \Rightarrow \quad \mathbb{E}[M(u)] = \sum_{m=1}^{\infty} F_{U_m}(u) \\
&\Rightarrow \quad \mu u = \sum_{m=1}^{\infty} F_{U_m}(u) \quad \Rightarrow \quad \mu du = \sum_{m=1}^{\infty} dF_{U_m}(u).
\end{aligned}$$

so for $t > T_1$ and using equation (4.3) we get

$$\begin{aligned}
\widetilde{Y}(t-T_1) &= \sum_{m=M(T_1)+1}^{\infty} Y_m \mathbb{1}_{\{U_m=t\}} \\
&= \sum_{m=1}^{\infty} Y_m \mathbb{1}_{\{U_m=t\}} - \sum_{m=1}^{M(T_1)} Y_m \mathbb{1}_{\{U_m=t\}} \\
&= Y(t) - \sum_{m=1}^{M(T_1)} Y_m \mathbb{1}_{\{U_m=t\}}.
\end{aligned}$$

If we set $\widetilde{Y}(t) = 0$ for $t < 0$, we get

$$Y(t) = \sum_{m=1}^{M(T_1)} Y_m \mathbb{1}_{\{U_m=t\}} + \widetilde{Y}(t-T_1), \qquad t \geq 0. \tag{4.5}$$

4.2.3 Distribution of Maximum

Let

$$Z(t) = X(t) + Y(t), \qquad t \geq 0, \tag{4.6}$$

be the process defined by the sum of the two processes. We are interested in the reliability of the component, $\mathbb{P}(Z(t) \leq x)$. From equations (4.2) and (4.5) we see that the pulse process and the shock process can both be represented as regenerative processes with regeneration epoch T_1 denoting the first change in the pulse process. The decomposition of both processes for $t < T_1$ and $t \geq T_1$ will now be used to calculate the reliability function for the sum of process Z.

Recall that the combined load Z represents the sum of the pulse load and the shock load as in equation (4.6) and let Z^* represent the maximum of Z given as

$$Z^*(t) = \max\{Z(s); s \leq t\}. \tag{4.7}$$

From section 2.4 we can compute the reliability of Z by computing the reliability of Z^*.

It is first worth noting that $Z^*(t)$ is not equal to $X^*(t) + Y^*(t)$.

From equations (4.2) and (4.5) it follows that

$$\begin{aligned}
Z(t) &= X(t) + Y(t), \qquad t \geq 0 \\
&= L_1 \mathbb{1}_{\{t<T_1\}} + \widetilde{X}(t-T_1) + \sum_{m=1}^{M(T_1)} Y_m \mathbb{1}_{\{U_m=t\}} + \widetilde{Y}(t-T_1) \\
&= L_1 \mathbb{1}_{\{t<T_1\}} + \sum_{n=1}^{M(T_1)} Y_n \mathbb{1}_{\{U_n=t\}} + \widetilde{Z}(t-T_1),
\end{aligned} \tag{4.8}$$

where $\widetilde{Z}(t) = \widetilde{X}(t) + \widetilde{Y}(t)$, $t \geq 0$.

Now we use the results in equations (3.6) and (4.8) to calculate the reliability:

$$\mathbb{P}(Z^*(t) \leq x) = \mathbb{P}(Z^*(t) \leq x; T_1 > t) + \mathbb{P}(Z^*(t) \leq x; T_1 \leq t), \tag{4.9}$$

where

$$\begin{aligned}
\mathbb{P}(Z^*(t) \leq x; T_1 > t) &= \int_t^\infty \mathbb{P}(L_1 + \max\{Y_1, \ldots, Y_{M(t)}\} \leq x) dF_{T_1}(s) \\
&= \mathbb{P}(L_1 + \max\{Y_1, \ldots, Y_{M(t)}\} \leq x) \int_t^\infty dF_{T_1}(s) \\
&= \bar{F}_{T_1}(t) \mathbb{P}(L_1 + \max\{Y_1, \ldots, Y_{M(t)}\} \leq x) \\
&= \bar{F}_{T_1}(t) \int_0^x e^{-\mu t \bar{F}_Y(x-u)} dF_{L_1}(u)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}(Z^*(t) \leq x; T_1 \leq t) &= \mathbb{P}(L_1 + \max\{Y_1, \ldots, Y_{M(T_1)}\} \leq x, \bar{Z}^*(t - T_1) \leq x) \\
&= \int_0^t \mathbb{P}(L_1 + \max\{Y_1, \ldots, Y_{M(s)}\} \leq x, \bar{Z}^*(t-s) \leq x) dF_{T_1}(s) \\
&= \int_0^t \mathbb{P}(L_1 + \max\{Y_1, \ldots, Y_{M(s)}\} \leq x) \mathbb{P}(Z^*(t-s) \leq x) dF_{T_1}(s) \\
&= \int_0^t \left[\int_0^x e^{-\mu s \bar{F}_Y(x-u)} dF_{L_1}(u) \right] \mathbb{P}(Z^*(t-s) \leq x) dF_{T_1}(s),
\end{aligned}$$

where for the sake of notation we write $\bar{F}(\cdot) = 1 - F(\cdot)$.

Hence,

$$\begin{aligned}
&\mathbb{P}(Z^*(t) \leq x) = \\
&\bar{F}_{T_1}(t) \int_0^x e^{-\mu t \bar{F}_Y(x-u)} dF_{L_1}(u) + \int_0^t \left[\int_0^x e^{-\mu s \bar{F}_Y(x-u)} dF_{L_1}(u) \right] \mathbb{P}(Z^*(t-s) \leq x) dF_{T_1}(s).
\end{aligned} \tag{4.10}$$

4.2.4 Numerical Integration

Using the trapezoidal rule, $\mathbb{P}(Z^*(t) \leq x)$ can recursively be computed as follows:

$$\begin{aligned}
\mathbb{P}(Z^*(t) \leq x) &= \bar{F}_{T_1}(t) \int_0^x e^{-\mu \bar{F}_Y(x-u) t} dF_{L_1}(u) \\
&\quad + \frac{t}{2N} f_{T_1}(0) \mathbb{P}(Z^*(t) \leq x) \int_0^x dF_{L_1}(u) \\
&\quad + \sum_{i=2}^{N} 2 \frac{t}{2N} f_{T_1}(s_i) \mathbb{P}(Z^*(t - s_i) \leq x) \int_0^x e^{-\mu \bar{F}_Y(x-u) s_i} dF_{L_1}(u) \\
&\quad + \frac{t}{2N} f_{T_1}(t) F_{L_1}(x) \int_0^x e^{-\mu \bar{F}_Y(x-u) t} dF_{L_1}(u).
\end{aligned}$$

$$\begin{aligned}\mathbb{P}(Z^*(t) \leq x) &= \frac{1}{1-\frac{t}{2N}F_{L_1}(x)}\left\{\left[\bar{F}_{T_1}(t) + \frac{t}{2N}f_{T_1}(t)F_L(x)\right]\int_0^x e^{-\mu\bar{F}_Y(x-u)t}dF_{L_1}(u)\right. \\ &\left. + \sum_{i=2}^{N}\frac{t}{N}f_{T_1}(s_i)\mathbb{P}(Z^*(t-s_i) \leq x)\int_0^x e^{-\mu\bar{F}_Y(x-u)s_i}dF_{L_1}(u)\right\}. \end{aligned} \tag{4.11}$$

4.2.5 Numerical Example

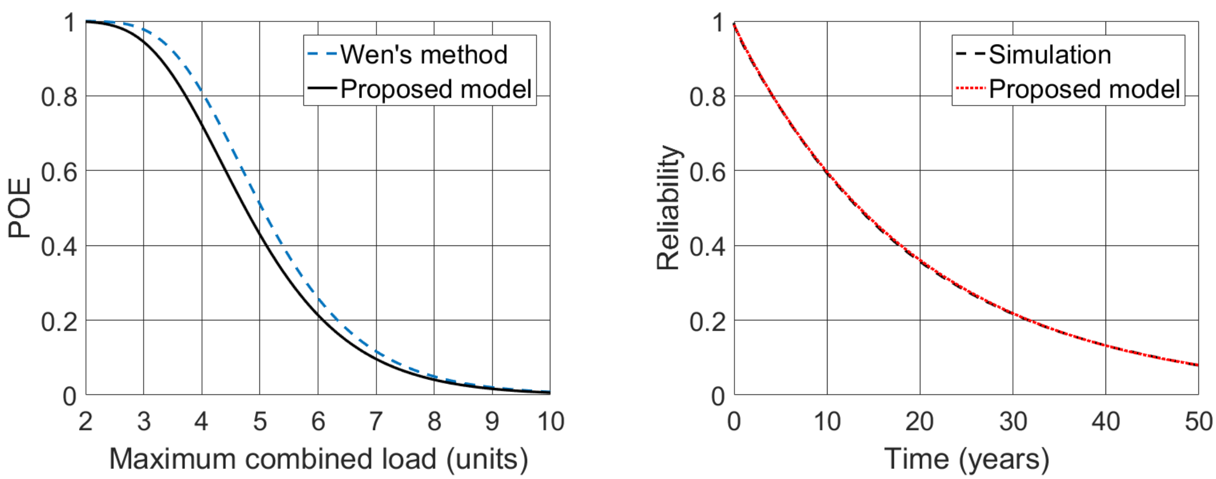
As an example equation (4.11) is implemented with Δt=1/30 (years) and Δx=1/100 using the following parameters:

- the pulse load changes and shock occurrences are Poisson processes with an average of one occurrence/year, i.e. $F_T(t) \sim Exp(1)$ and μ=1
- $L \sim Exp(1)$
- $Y \sim Exp(1)$
- for the POE, t=15 years
- for the reliability, failure threshold $x = 4.61$ units

The POE is given in Figure 4.4 (a) and compared to the solution obtained by Wen's method, [77]. Some more details about Wen's method and the formula used are given in the Appendix. It can be confirmed that Wen's method is a conservative approach and its errors can be quite large. In particular this would lead to excess and unnecessary costs in design and safety measures. The reliability is given in Figure 4.4 (b) and compared to 100000 runs of a Monte Carlo simulation.

4.3 Conclusions

In this chapter an accurate model to compute the distribution of maximum of a combined process is carefully derived. The combined process consists of a pulse load and an HPP shock load process. By comparing the results to an existing model it is shown that the proposed approach leads to better decision quantities. The proposed model is also more versatile than many models in the literature since it allows for the use of a non-HPP approach for the pulse load process.



(a) POE.

(b) Reliability.

Figure 4.4: Combined loads.

Appendix

4.A Load Coincidence Method

In this method proposed by Wen [77], the pulse process $\{X(t), t \geq 0\}$ and shock process $\{Y(t), t \geq 0\}$ are assumed to be mutually independent homogeneous Poisson processes with rate λ and μ, respectively. Define $X^*(t)$=max$\{X(s), 0 \leq s \leq t\}$, $Y^*(t)$=max$\{Y(s), 0 \leq s \leq t\}$ and $C^*(t)$=max$\{C_{XY}(s), 0 \leq s \leq t\}$ where $C_{XY}(t)$ is the coincidence process, i.e. when the two processes overlap. It can be noticed that if the pulse process is always in an 'on' state, the shock process never occurs alone. Therefore, the distribution of the maximum combined process $Z^*(t) = \max\{X(s) + Y(s), 0 \leq s \leq t\}$ is approximated as

$$\begin{aligned} \mathbb{P}(Z^*(t) < z) &= \mathbb{P}((X^*(t) < z) \cap (C^*(t) < z)) \\ &\approx \mathbb{P}(X^*(t) < z)\mathbb{P}(C^*(t) < z). \end{aligned} \tag{4.12}$$

Notice that $X^*(t)$ and $C^*(t)$ are positively correlated since $X(t)$ and $C_{XY}(t)$ are dependent. Therefore, equation (4.12) is an underestimation. This means the probability of exceedance, i.e. $\mathbb{P}(Z^*(t) > z)$, gives an approximation on the conservative side. First recall from equation (3.10) the following:

$$\mathbb{P}(X^*(t) \leq z) = F_X(z)e^{-\lambda t(1-F_X(z))}, \tag{4.13}$$

where $F_X(x)$ is the distribution of the load magnitudes. Since the coincidence process exists if and only if the shock process is 'on', the coincidence process is also a shock process with rate equal to μ. However, the distribution of the coincidence load magnitudes is now given by the distribution of the sum $\mathbb{P}(X + Y \leq x) = F_{XY}(x)$. Hence, from equation (3.6) it follows that

$$\mathbb{P}(C^*(t) < z) = e^{-\mu t(1-F_{XY}(z))}. \tag{4.14}$$

It now follows that

$$\mathbb{P}(Z^*(t) < z) \approx F_X(z)e^{-t[\lambda(1-F_X(z))+\mu(1-F_{XY}(z))]}. \tag{4.15}$$

Chapter 5

Reliability of Degrading Structures

5.1 Introduction

5.1.1 Background and Existing Literature

In a highly reliable environment such as a nuclear power plant, reliability assessment based on measurements/data analysis is challenging due to the lack of data from equivalent sources. Moreover, an approximation of the reliability function for a component, equal to for example the flexible Weibull distribution, is only justifiable for a large population of identical components. Therefore, in this paper effects on a structure/component are mechanistically modeled leading to a mathematical formulation to compute the reliability of an individual (and unique) component.

The deterioration in structures or engineering components can be modeled as the interaction between two different stochastic processes: a process of continuous deterioration (aging) and the process describing the external loads (stress) on the component. A depiction of this interaction can be seen in Figure 5.1. A component loses strength (or capacity) over time due

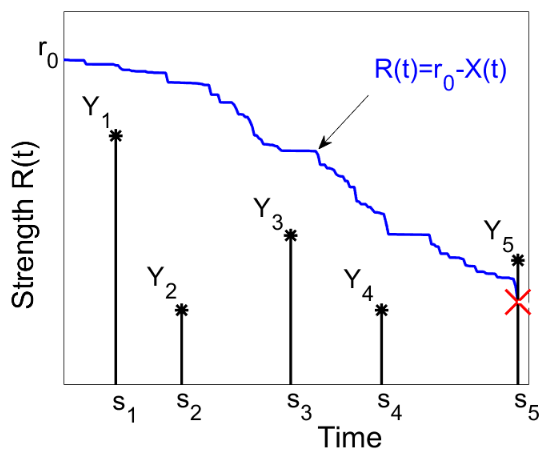


Figure 5.1: Schematic of stochastic process of deterioration $R(t)$ and stochastic loads Y_i. At time s_5 a failure occurs since $Y_5 > R(s_5)$.

to *progressive deterioration*. This type of deterioration stands for (continuous) time-dependent degradation of the system caused by for example environmental stressors, corrosion, creep, extreme pressure or temperature, components exposed to chemicals etc. It can immediately be remarked that as long as there is no sporadic regeneration or intervention, the component strength is a non-increasing process. Hence, the continuous deterioration in the component is a non-decreasing process. The gamma process is such a process and has been widely used to model deterioration in civil engineering, e.g. coastal safety [73], [72] and structural reliability [71], [54], [22], [6], [4], [5] and [36]. The second type of effect on a component is commonly modeled is sudden and short-lived high levels of *stress*, *loads* or *shocks* caused by for example earthquakes and other rare events. These effects are usually assumed to be point processes and thus having a negligible time duration and also with a non-lasting effect. These external loads occur randomly in time and are usually described by a Poisson process with stochastic load magnitudes, although diffusion stresses with continuous intensity have been discussed in [19]. Iervolino et al. [28] discuss a reliability model with a gamma process of stochastic degradation and cumulative stress. In the computations however, the amount of stresses is taken as the expectation of the amount of stresses to simplify calculations and since this is a rough assumption it may not always be favourable. Sánchez-Silva et al. [63] present a model to compute the instantaneous rate of intervention (heuristically the hazard rate) of a component. The total deterioration in the component is a sum of the aging process and the Poisson process where aging occurs at fixed time points. In Riascos-Ochoa et al. [60] the reliability of a component can be computed when the total deterioration is again a sum of aging and external loads where now this sum is a Lévy process. Mori and Ellingwood [49], [14] present a model where the stress process is an HPP resembling external incidental loads affecting an aging component. However, in this model the aging process is deterministic. The model described in this chapter is an extension of the Mori and Ellingwood [49], [14], models in the sense that the deterministic aging is extended to stochastic aging. Since the stress process in this model is treated as external loads separate from the aging process, a model treating the stress process as part of the aging process can be viewed as a special case of the model in this chapter. This model has been presented in [72].

5.1.2 Contributions to Literature

As mentioned, the model in this chapter has been presented in [72]. However, without the proper explanation of the mathematical intricacies that justify the results. These details are included in this chapter. In addition, this chapter includes a special case that greatly simplifies the results in [72]. As an extension to the model, in this chapter the rate of occurrence of the shock process is assumed to be time-dependent to establish a more general model.

5.1.3 Objectives and Organization

In section 5.2 a basic two shock challenge is presented to illustrate all the different components that play a role in this problem. This section highlights the challenges that should be overcome in order to solve this reliability problem. Section 5.3 briefly summarizes the model in [49], [14] which assumes deterioration occurs as a deterministic function of time. In section 5.4 the gamma process is presented. In section 5.5 the gamma process is introduced in the model as the deterioration process as is done in [72] and the section serves as a practical numerical example to illustrate computations. In section 5.6 the reliability function that needs to be computed is further analyzed to give insight into its computation method. Section 5.7 shows the method used to arrive at analytical solutions instead of resorting to numerical integration. In section 5.8 it is shown how analytical solutions are derived for Uniformly distributed shocks. In section 5.9 the model is further generalized to accommodate for shocks that arrive according to a Nonhomogeneous Poisson Process (NHPP). The chapter concludes with an appendix with further results and some mathematical proofs.

5.2 Basic Challenge

This section discusses a basic challenge in reliability modelling where computation of the reliability is presented for a component that is put into service at time t=0 and is subject to two shocks.

In Figure 5.2 a deterministic model is depicted. In this model there is no uncertainty regarding the deterioration process, the occurrence times of stresses and the stress magnitudes. Assume the following:

- $R(t) = r_0 - at$ is the strength process where $r_0 > 0, a > 0$ and $t > 0$
- Stress sizes $y > 0$ are stresses exceeding a threshold[1] $l_0 \geq 0$
- $\delta > 0$ is the time interval between stress occurrences.

This situation is depicted in Figure 5.2.

We notice that in a completely deterministic model there is no uncertainty and hence there is no reliability computation, so from now on we will add uncertainty to the model. This means uncertainty in the degradation process and/or uncertainty in the stress process. We want to compute the reliability of a component surviving two shocks of magnitudes Y_1 and Y_2 occurring at fixed times s_1 and s_2. The shock magnitudes are moreover assumed to be dependent and given by the joint distribution $F_{Y_1,Y_2}(y_1, y_2)$. In the remainder of this chapter we will illustrate the effect of uncertainty in reliability computation with a few examples given three different types of deterioration processes: no deterioration, deterministic deterioration and stochastic deterioration.

5.2.1 No Deterioration

First we assume there is no deterioration. Hence, the component strength $R(t)$ is a constant r_0>0. This strength can either be identified with certainty or is uncertain. Hence, if the component strength is constant over time it can be a known value r_0 or a random variable R_0 with

[1]The threshold l_0 is justified for a system that cannot fail in the absence of stresses larger than l_0. In the case any stress size has an impact on the probability of failure of the system, the threshold l_0 equals 0.

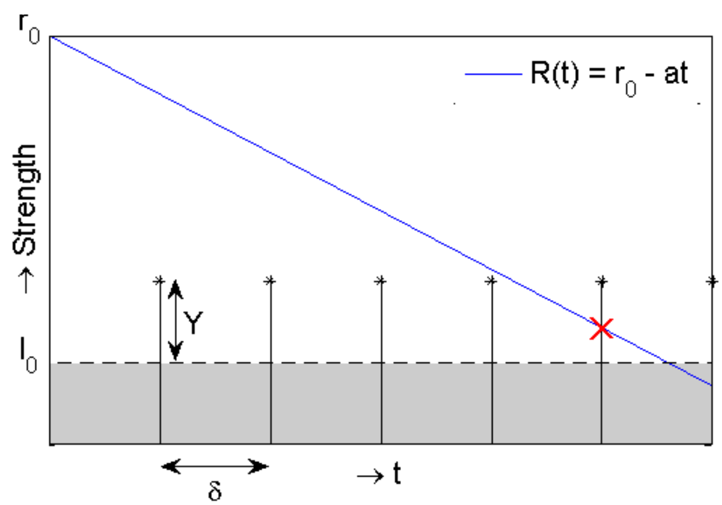


Figure 5.2: A deterministic model where all variables are known.

distribution $F_{R_0}(r_0)$.

Component strength known

Let the component strength be known and equal to r_0. This situation is given in Figure 5.3. The probability that the component survives both shocks is given by

$$\mathbb{P}(T > s_2) = \mathbb{P}(Y_1 < r_0, Y_2 < r_0) = F_{Y_1,Y_2}(r_0, r_0).$$

Component strength uncertain

Let the component strength R_0 be a random variable. This situation is given in Figure 5.4. As the previous case, given $R_0 = r_0$ the probability of surviving both shocks is given by

$$\begin{aligned}\mathbb{P}(T > s_2 | R_0 = r_0) &= \mathbb{P}(Y_1 < r_0, Y_2 < r_0) \\ &= F_{Y_1,Y_2}(r_0, r_0).\end{aligned}$$

Hence, unconditioning on the value r_0 leads to:

$$\begin{aligned}\mathbb{P}(T > s_2) &= \int_0^\infty \mathbb{P}(T > s_2 | R_0 = r_0) \mathrm{d}F_{R_0}(r_0) \\ &= \int_0^\infty F_{Y_1,Y_2}(r_0, r_0) \mathrm{d}F_{R_0}(r_0).\end{aligned}$$

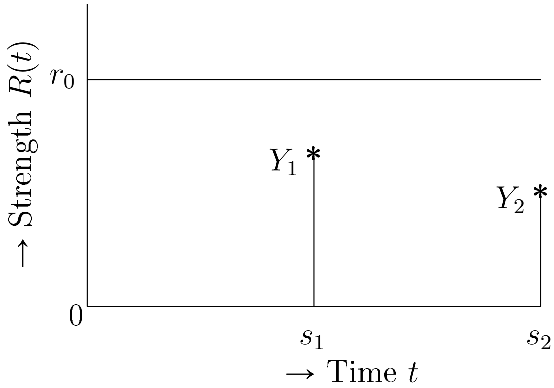


Figure 5.3: **No deterioration**: component strength known.

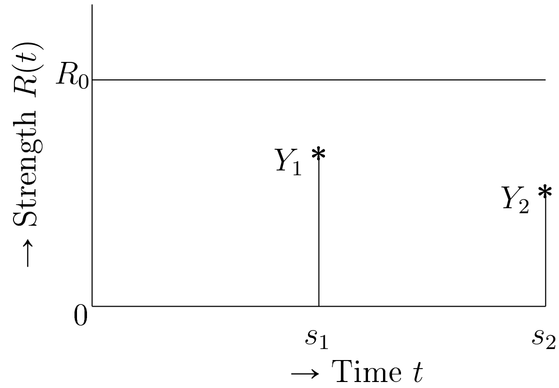


Figure 5.4: **No deterioration**: component strength uncertain.

5.2.2 Deterministic Deterioration

In this section we assume the component deteriorates over time according to a specified function. We write the component strength as

$$R(t) = R_0 \cdot g(t), \quad t \geq 0,$$

where $R_0>0$ is a constant and $g(t)\geq 0$ is a non-increasing function. In Figure 5.5 this situation is depicted. For the sake of simplicity, in the following we assume $g(0)=1$ so that $R(0)=R_0$ and $g(t)$ can be interpreted as the loss of initial capacity over time.

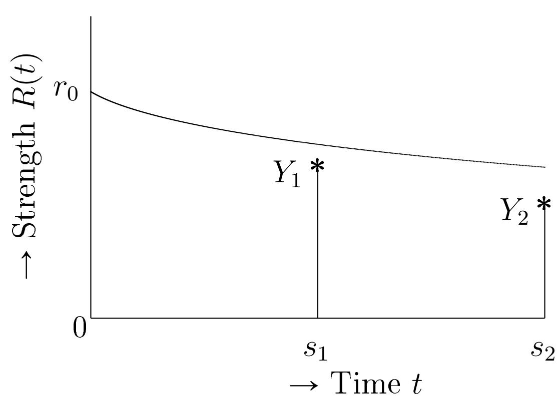


Figure 5.5: **Deterministic deterioration**: schematic of component with non-increasing strength $R(t)$ affected by two loads.

Initial component strength known

If the component strength is known to be $R_0=r_0$, the deterioration function contains no uncertainty and is given by $R(t)=r_0 \cdot g(t)$. This is shown in Figure 5.6. The reliability of the component is given as

$$\mathbb{P}(T > s_2) = \mathbb{P}(Y_1 < r_0 \cdot g(s_1), Y_2 < r_0 \cdot g(s_2)) = F_{Y_1,Y_2}(r_0 \cdot g(s_1), r_0 \cdot g(s_2)).$$

Initial component strength uncertain

In the case the initial component strength R_0 is a random variable following a distribution $F_{R_0}(r_0)$, Figure 5.7, the reliability becomes

$$\mathbb{P}(T > s_2) \quad = \quad \int_0^\infty F_{Y_1,Y_2}(r_0 \cdot g(s_1), r_0 \cdot g(s_2)) \mathrm{d}F_{R_0}(r_0).$$

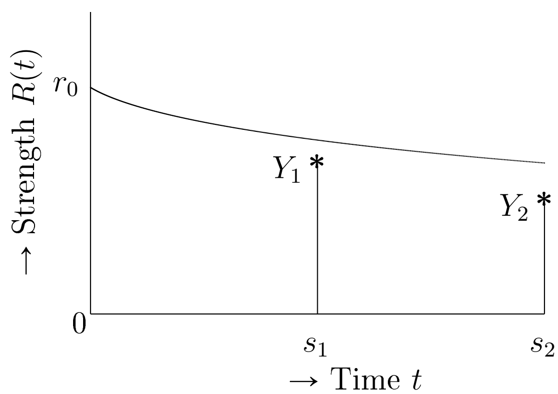


Figure 5.6: **Deterministic deterioration**: initial component strength known.

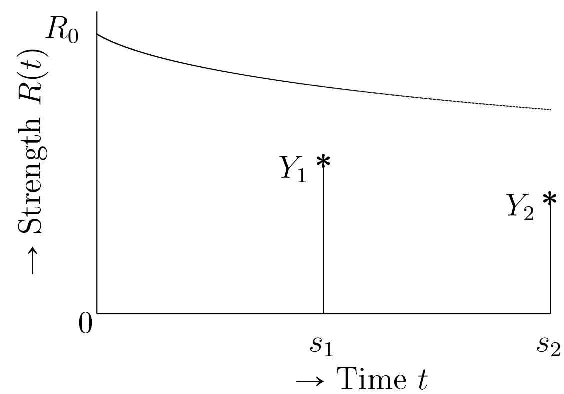


Figure 5.7: **Deterministic deterioration**: initial component strength uncertain.

A Numerical Example

The computation of the reliability with the assumption of deterministic deterioration is illustrated for an exponential deterioration function $g(t)$. The joint distribution for the shock magnitudes is a bivariate Exponential distribution, i.e. the margins are Exponential, proposed by Gumbel [25].

The joint CDF of the load magnitudes (Y_1, Y_2) is given by

$$F(y_1, y_2; \delta) = 1 - e^{-y_1} - e^{-y_2} + e^{-y_1 - y_2 - \delta y_1 y_2}, \;\; y_1 \geq 0, y_2 \geq 0, 0 \leq \delta \leq 1. \tag{5.1}$$

The correlation ρ between Y_1 and Y_2 is determined by the parameter δ. Y_1 and Y_2 are independent when δ=0. Furthermore, the correlation is non-positive and we have $\rho \in [-0.40365, 0]$. It is to be noticed that natural hazards can have a positive or negative effect on other hazards, or even trigger other hazards [21]:

- **positive**: surface subsidence can increase the risk and impact of flooding
- **negative**: heavy rainfall can decrease the risk and impact of wildfires
- **trigger**: an earthquake can directly cause a tsunami or a landslide.

The joint density function of (5.1) is given by

$$f(y_1, y_2; \delta) = e^{-y_1-y_2-\delta y_1 y_2}\left[(1+\delta y_1)(1+\delta y_2)-\delta\right].$$

For this distribution it follows that:

$$F(y_1, \infty; \cdot) = F_1(y_1) = 1 - e^{-y_1} \quad \text{and} \quad F(\infty, y_2; \cdot) = F_2(y_2) = 1 - e^{-y_2}.$$

Moreover,

$$\int_{-\infty}^{\infty} f(y_1, y_2; \cdot)dy_2 = f_1(y_1) = \frac{\mathrm{d}}{\mathrm{d}y_1}F_1(y_1) = e^{-y_1}$$

and

$$\int_{-\infty}^{\infty} f(y_1, y_2; \cdot)dy_1 = f_2(y_2) = \frac{\mathrm{d}}{\mathrm{d}y_2}F_2(y_2) = e^{-y_2},$$

hence the marginal density functions are the exponential distribution with parameter equal to 1.

The probability of survival, $\mathbb{P}(T > s_2)$, for this component is given as

$$\begin{aligned} \mathbb{P}(T > s_2) &= \mathbb{P}(Y_1 < R(s_1), Y_2 < R(s_2)) \\ &= F(R(s_1), R(s_2); \cdot), \end{aligned} \tag{5.2}$$

where F denotes the bivariate CDF of the load magnitudes with corresponding parameter value.

If $0 < S_1 < S_2 \leq t$ are now *random* times of occurrence and the strength process $R(t)$ is considered deterministic, equation (5.2) becomes:

$$\mathbb{P}(T > t) = \int_{s_1}^{t}\int_{0}^{t} F(r_0 \cdot g(s_1), r_0 \cdot g(s_2)) f_{\boldsymbol{S}}(s_1, s_2)\mathrm{d}s_1\mathrm{d}s_2, \tag{5.3}$$

where $f_{\boldsymbol{S}}(s_1, s_2)$ is the joint PDF of random occurrence times $\boldsymbol{S}$=$\{S_1, S_2\}$. Assume the PDF of the occurrence times is given by

$$f_{\boldsymbol{S}}(s_1, s_2) = \frac{1}{t^2}, \;\; t > 0.$$

In the following chapter we will see that this is assumption equivalent to the case of the two shocks being occurrences of a Poisson process on $(0, t]$. Equation (5.3) then becomes

$$\mathbb{P}(T > t) = \frac{1}{t^2}\left(\int_{0}^{t}\int_{0}^{t} F(r_0 \cdot g(s_1), r_0 \cdot g(s_2))\mathrm{d}s_1\mathrm{d}s_2\right). \tag{5.4}$$

If $g(t) = e^{-t}$, the degradation function is given as

$$r(t) = r_0 e^{-t}.$$

The survival probability is then

$$\begin{aligned}
\mathbb{P}(T > t) &= \frac{1}{t^2}\left(\int_0^t \int_0^t F(r_0 \cdot g(s_1), r_0 \cdot g(s_2)) \mathrm{d}s_1 \mathrm{d}s_2\right) \\
&= \frac{1}{t^2}\int_0^t \int_0^t F\left(r_0 e^{-t}, r_0 e^{-t}\right) \mathrm{d}s_1 \mathrm{d}s_2.
\end{aligned}$$

This probability is given in Figure 5.8 for different parameter values.

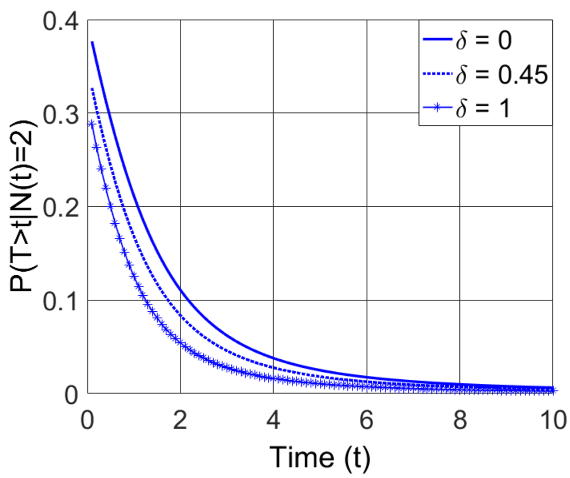


Figure 5.8: $\mathbb{P}(T > t | N(t) = 2)$ where $r_0 = 2$ and $F_Y \sim Exp(1)$.

5.2.3 Stochastic Deterioration

In this section the deterioration is stochastic and given as

$$R(t) = R_0 - X(t), \quad t \geq 0,$$

where $R_0 > 0$ is a constant and $X(t)$ is a non-decreasing stochastic process with $X(0) = 0$.

Initial component strength known

If the initial component strength is known to be $R_0 = r_0$, the deterioration process is given by $R(t) = r_0 - X(t)$ and the reliability of the component is

$$\begin{aligned}\mathbb{P}(T > s_2) &= \mathbb{P}(Y_1 < R(s_1), Y_2 < R(s_2)) = \mathbb{P}(Y_1 < r_0 - X(s_1), Y_2 < r_0 - X(s_2)) \\ &= \int_0^\infty \int_0^\infty \mathbb{P}(Y_1 < r_0 - x_1, Y_2 < r_0 - x_2) f_{X_1(s_1),X_2(s_2)}(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2 \\ &= \int_0^\infty \int_0^\infty F_{Y_1,Y_2}(r_0 - x_1, r_0 - x_2) f_{X_1(s_1),X_2(s_2)}(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2.\end{aligned}$$

Initial component strength uncertain

If the component strength is given by $R(t) = R_0 - X(t)$, where R_0 is unknown, the reliability is

$$\begin{aligned}\mathbb{P}(T > s_2) &= \mathbb{P}(Y_1 < R(s_1), Y_2 < R(s_2)) = \mathbb{P}(Y_1 < R_0 - X(s_1), Y_2 < R_0 - X(s_2)) \\ &= \int_0^\infty \int_0^\infty \mathbb{P}(Y_1 < R_0 - x_1, Y_2 < R_0 - x_2) f_{X_1(s_1),X_2(s_2)}(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2 \\ &= \int_0^\infty \left(\int_0^\infty \int_0^\infty F_{Y_1,Y_2}(r_0 - x_1, r_0 - x_2) f_{X_1(s_1),X_2(s_2)}(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2 \right) \mathrm{d}F_{R_0}(r_0).\end{aligned}$$

5.2.4 Randomness in Occurrence Times and Number of Shocks

Now we keep the last assumption of stochastic deterioration and notice that besides the reasonable assumption of uncertainty in the deterioration process, generally the times of occurrence of the two shocks are also uncertain. The times of occurrence are now given by the (joint) density $f_{S_1,S_2}(s_1, s_2)$. The reliability given two shocks of uncertain magnitudes occurring at random times affecting a component with randomly decreasing strength can now be written as

$$\begin{aligned}\mathbb{P}(T > s_2) = & \int_0^\infty \left(\int_0^\infty \int_0^\infty \left(\int_0^{s_2} \int_{s_1}^{s_2} F_{Y_1,Y_2}\left(r_0 - X(s_1), r_0 - X(s_2)\right) f_{S_1,S_2}(s_1, s_2) \mathrm{d}s_2 \mathrm{d}s_1 \right) \right. \\ & \left. \times f_{X_1(s_1),X_2(s_2)}(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2 \right) \mathrm{d}F_{R_0}(r_0).\end{aligned}$$

It is also obvious that the number of shocks is not known a priori so the randomness in the number of shocks needs to be taken into account. Assuming there are three shocks instead of two occurring at times $0 < s_1 < s_2 < s_3$, the reliability function becomes

$$\begin{aligned}\mathbb{P}(T > s_3) = & \int_0^\infty \left(\int_0^\infty \int_0^\infty \int_0^\infty \left(\int_0^{s_3} \int_{s_1}^{s_3} \int_{s_2}^{s_3} F_{Y_1,Y_2,Y_3}\left(r_0 - X(s_1), r_0 - X(s_2), r_0 - X(s_3)\right) \right. \right. \\ & \left. \left. \times f_{S_1,S_2,S_3}(s_1, s_2, s_3) \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1 \right) f_{X_1(s_1),X_2(s_2),X(s_3)}(x_1, x_2, x_3) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \right) \mathrm{d}F_{R_0}(r_0).\end{aligned}$$

In general, for a total of n shocks occurring at times $0 < s_1 < s_2 < \ldots < s_n$, the reliability becomes

$$\mathbb{P}(T > s_n) = \int_0^\infty \left(\underbrace{\int_0^\infty \cdots \int_0^\infty}_{n-fold} \left(\underbrace{\int_0^{s_n} \cdots \int_{s_{n-1}}^{s_n}}_{n-fold} F_{Y_1,\ldots,Y_n}\left(r_0 - X(s_1), \ldots, r_0 - X(s_n)\right) \right.\right.$$
$$\left.\left. \times f_{S_1,\ldots,S_n}(s_1, \ldots, s_n) \mathrm{d}s_n \cdots \mathrm{d}s_1 \right) f_{X(s_1)}(x_1) \cdots f_{X(s_n)}(x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n \right) \mathrm{d}F_{R_0}(r_0).$$

Let $0 < s_1 < s_2 < s_3 < \ldots \leq t$ denote the time occurrences of shocks on the interval $(0, t]$. Unconditioning on the amount of shocks the reliability function becomes

$$\begin{aligned}
\mathbb{P}(T > t) &= \sum_{i=0}^{\infty} \mathbb{P}(T > t | i \text{ shocks}) \cdot \mathbb{P}(i \text{ shocks}) \\
&= 1 \cdot \mathbb{P}(0 \text{ shocks}) + \sum_{i=1}^{\infty} \mathbb{P}(T > t | i \text{ shocks}) \cdot \mathbb{P}(i \text{ shocks}) \\
&= \mathbb{P}(0 \text{ shocks}) + \sum_{i=1}^{\infty} \int_0^\infty \left(\underbrace{\int_0^\infty \cdots \int_0^\infty}_{i-fold} \left(\underbrace{\int_0^t \cdots \int_{s_{i-1}}^t}_{i-fold} F_{Y(s_1),\ldots,Y(s_i)}\left(r_0 - X(s_1), \ldots, r_0 - X(s_i)\right) \right.\right. \\
&\quad \left.\left. \times f_{S_1,\ldots,S_i}(s_1, \ldots, s_i) \mathrm{d}s_i \cdots \mathrm{d}s_1 \right) f_{X(s_1),\ldots,X(s_i)}(x_1, \ldots, x_i) \mathrm{d}x_1 \cdots \mathrm{d}x_i \right) \mathrm{d}F_{R_0}(r_0) \cdot \mathbb{P}(i \text{ shocks}).
\end{aligned}$$

Up to this point we have seen illustrations of the complexities involved in the calculation of the reliability of a component subject to shocks. It should now be clear that the intricacies in the model arise from the uncertainties in the time component of the model. Three key elements need to be identified in order to compute the reliability in this model formulation:

1. the (stochastic) process of deterioration: $\{X(t), t \geq 0\}$
2. joint distribution of occurrence times of shocks: $f_{S_1,\ldots,S_i}(s_1, \ldots, s_i)$
3. joint distribution of shock magnitudes: $F_{Y_1}, \ldots, F_{Y_i}(y_1, \ldots, y_i)$.

In addition, if the initial component strength R_0 is also uncertain, the distribution $F_{R_0}(r_0)$ needs to be provided. From these elements, it is not straightforward to identify a joint distribution for the occurrence times of shocks while this quantity also gives rise to an infinite amount of integrals in the computation of the reliability. To circumvent this difficulty the Poisson process is used. If moreover the shock magnitudes are *iid*, the computational difficulties can be significantly reduced. All of this is illustrated in the proceeding sections.

5.3 Deterministic Strength and Stochastic Shocks

So far we have presented reliability problems where the occurrence times of shocks have been fixed or given by a probability density $f_{\mathbf{S}}(\mathbf{s})$. Now we explain why the Poisson process is a convenient assumption for the shock process.

We model the (deterministic) component strength according to

$$R(t) = r_0 \cdot g(t),$$

where r_0 is a constant, $g(t)$ is a non-increasing degradation function and $R(t)$ is the strength at time t. This is depicted in Figure 5.9.

We assume $g(t)$ does not account for shocks, hence we model the strength deterioration as a consequence of environmental factors. This also implies that $g(t)$ does not include fatigue, since the time-dependent strength degradation due to shocks would need to be included [50].

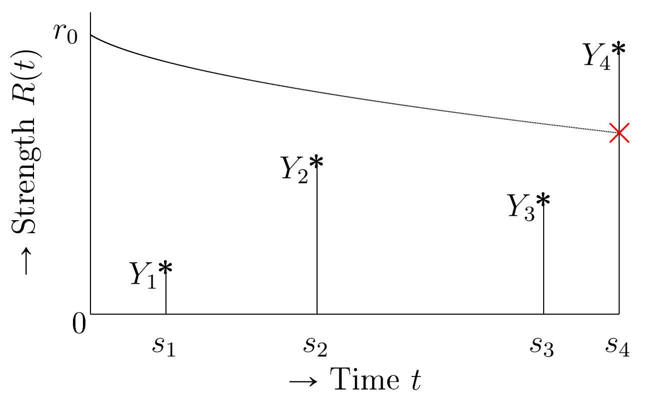


Figure 5.9: A component with deterministic deterioration affected by independent random shocks. The Time To Failure (TTF) is s_4.

In case of k stresses of sizes $Y_1, \dots, Y_k$ occurring at deterministic times $s_i, i = 1, \dots, k$, the probability of the component surviving after t is

$$\begin{aligned} \mathbb{P}(T > t | N(t) = k) &= \mathbb{P}(Y_1 < r_0 \cdot g(s_1), Y_2 < r_0 \cdot g(s_2), \dots, Y_k < r_0 \cdot g(s_k) > 0 | N(t) = k) \\ &= \left(\prod_{i=1}^{k} F_Y(r_0 \cdot g(s_i)) \right), \end{aligned}$$

where F_Y is the CDF of the independent and identically distributed load sizes.

When the k loads occur at random times $\mathbf{S} = \{S_1, S_2, \ldots, S_k\}$ according to their joint probability density function $f_{\mathbf{S}}(\mathbf{s})$, the probability of the component surviving after t becomes

$$\mathbb{P}(T > t|N(t) = k) = \underbrace{\int_{t_{k-1}}^{t} \cdots \int_{0}^{t}}_{k-fold} \left(\prod_{i=1}^{k} F_Y(r_0 \cdot g(s_i)) \right) f_{\mathbf{S}}(\mathbf{s}) \mathrm{d}\mathbf{s}. \tag{5.5}$$

In 5.A is it explained why given k load occurrences of a Poisson process on $(0, t]$, the occurrence times $\mathbf{S} = \{S_1, S_2, \ldots, S_k\}$ are the k order statistics of a random variable $\mathbf{U} = \{U_1, U_2, \ldots, U_k\}$ with a uniform distribution on $(0, t]$ and where $U_1, U_2, \ldots, U_k$ are statistically independent. This means we can assume the occurrence times are uniformly distributed on $(0, t]$. Hence, the joint PDF of $\mathbf{S}$ can then be written as

$$f_{\mathbf{S}}(\mathbf{s}) \equiv f_{\mathbf{U}}(\mathbf{u}) = \left(\frac{1}{t} \right)^k.$$

Moreover, since the load magnitudes are statistically independent and identically distributed (identical to a uniform distribution), we can write

$$\prod_{i=1}^{k} F_Y\left(r_0 \cdot g(S_i)\right) = \prod_{j=1}^{k} F_Y\left(r_0 \cdot g(U_j)\right).$$

Now we can write equation (5.5) expressed in $\mathbf{U}$ as

$$\begin{aligned} \mathbb{P}(T > t|N(t) = k) &= \underbrace{\int_{0}^{t} \cdots \int_{0}^{t}}_{k-fold} \left(\prod_{i=1}^{k} F_Y(r_0 \cdot g_{u_j}) \right) f_{\mathbf{U}}(\mathbf{u}) \mathrm{d}\mathbf{u} \\ &= \left(\int_0^t F_Y(r_0 \cdot g(u_j)) \frac{1}{t} \mathrm{d}u \right)^k \\ &= \left(\int_0^t F_Y(r_0 \cdot g(t)) \frac{1}{t} \mathrm{d}t \right)^k. \end{aligned} \tag{5.6}$$

We now relax the conditional probability on the number of loads on the interval $(0, t]$ to get $\mathbb{P}(T > t)$ by using the fact that the occurrence times are modeled as a Poisson process with intensity λ:

$$\begin{aligned} \mathbb{P}(T > t) &= \sum_{k=0}^{\infty} \mathbb{P}(T > t|N(t) = k) \mathbb{P}(N(t) = k) \\ &= \sum_{k=0}^{\infty} \left(\int_0^t F_Y(r_0 \cdot g(t)) \frac{1}{t} \mathrm{d}t \right)^k \cdot \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\ &= e^{-\lambda t \left(1 - \frac{1}{t} \int_0^t F_Y(r_0 \cdot g(t)) \mathrm{d}t \right)}. \end{aligned} \tag{5.7}$$

This model has been presented in [49] and [14]. We proceed by examining $\mathbb{P}(T > t)$ for a few choices of the degradation function $g(t)$.

5.3.1 Linear Deterioration

Let the degradation function $g(t)$ be given as

$$g(t) = 1 - \frac{1}{at_H}t,$$

where $a > 0$ is a constant and t_H is the time horizon (usually chosen as 15 years). The time-dependent component strength is then given by the function

$$R(t) = r_0 \left(1 - \frac{1}{at_H}t\right).$$

Hence, a regulates where $R(t)$ crosses the x-axis, e.g. at $t = at_H$. This means that the component survives at most a period $[0, at_H)$ when it fails (naturally) due to environmental factors.

Equation (5.7) then becomes:

$$\mathbb{P}(T > t) \quad = \quad e^{-\lambda t\left(1-\frac{1}{t}\int_0^t F_Y\left(r_0\cdot\left(1-\frac{1}{at_H}u\right)\right)\mathrm{d}u\right)}. \tag{5.8}$$

The survival probability $\mathbb{P}(T > t)$ given in equation (5.8) is computed numerically using the trapezoidal rule. The results are given in Figure 5.10 for different values of a where t=15 (years), $\lambda = 2, r_0 = 20$ and $F_Y \sim Ga(\cdot, 15, 1)$.

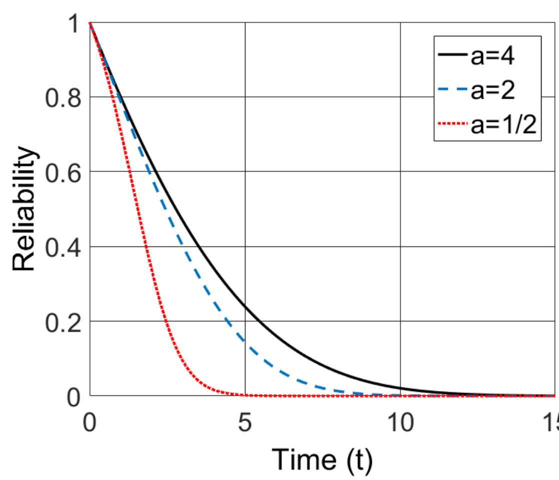


Figure 5.10: Reliability in case of linear deterioration.

5.3.2 Exponential Deterioration

We now let the degradation function $g(t)$ be given as

$$g(t) = e^{-at}$$

where the constant $a > 0$. The time-dependent component strength is then given as

$$R(t) = r_0 \cdot e^{-at},$$

so that a regulates the exponential decay of the component strength.

We notice that in this case the component implicitly can never directly fail as a consequence of loss of strength but failure occurs only due to loads incurred.

The survival probability in equation (5.7) then becomes:

$$\mathbb{P}(T > t) \quad = \quad e^{-\lambda t\left(1-\frac{1}{t}\int_0^t F_Y\left(r_0 \cdot e^{-at}\right)\mathrm{d}t\right)}. \tag{5.9}$$

Equation (5.9) is computed numerically using the trapezoidal rule. The results are given in Figure 5.11 for different values of a and parameter values $\lambda = 2, r_0 = 20$ and $F_Y \sim Ga(\cdot, 15, 1)$.

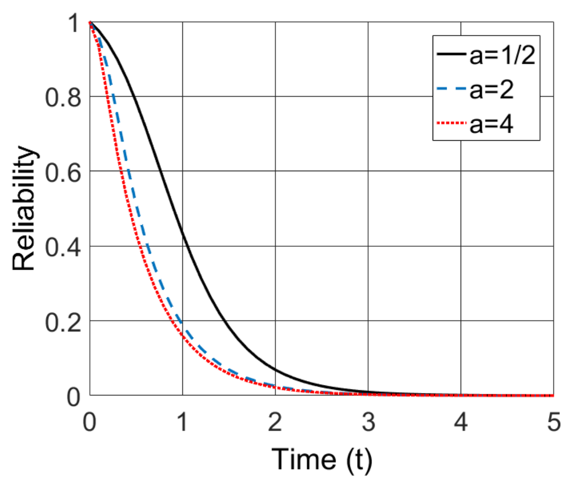


Figure 5.11: Reliability in case of exponential deterioration.

In the next section the process of deterioration is assumed to be, more realistically, a stochastic process.

5.4 Gamma Process of Deterioration

5.4.1 Gamma Process

The gamma process is widely used in the modelling of deterioration since it is a monotonically non-increasing process starting at zero. This means it allows for the use of the reasonable assumption that a system starts in its original (perfect) condition and is able to deteriorate continuously over time without sporadic regeneration. We first start with stating the gamma distribution and continue with the definition of the gamma process.

A random variable X has a gamma distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$ if its PDF is given as

$$Ga(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{\{x \geq 0\}},$$

where the indicator function $\mathbb{1}_{x\geq 0}$ is 1 if $x \geq 0$ and zero otherwise. This means that the gamma distribution only exists for non-negative values x.

Furthermore, the expectation and variance of X are given by

$$\mathbb{E}[X] = \frac{\alpha}{\beta}, \ \ Var(X) = \frac{\alpha}{\beta^2}.$$

Now let $\alpha(t) : [0, \infty) \to \mathbb{R}$ be a right-continuous, non-decreasing function with $\alpha(0) = 0$. The *gamma process* with shape function $\alpha(t)$ and scale parameter β is a continuous-time process $\{X(t)\}_{t\geq 0}$ with the characteristics:

1. $X(0) = 0$ with probability one,
2. $X(t) - X(s) \sim Ga(x; \alpha(t) - \alpha(s), \beta) \ \forall \ t > s \geq 0$,
3. $X(t)$ has independent increments.

Following this definition, the process $X(t)$ has PDF given by

$$f_{X(t)}(x) = Ga(x; \alpha(t), \beta), \ \ \forall t,$$

with expectation and variance given by

$$\mathbb{E}[X(t)] = \frac{\alpha(t)}{\beta}, \ \ Var(X(t)) = \frac{\alpha(t)}{\beta^2}.$$

In the case this shape function is linear in time, i.e. $\alpha(t) = \alpha t$, $t \geq 0$, the gamma process is called a *stationary gamma process*. This implies that the independent increments now have a gamma distribution $X_t - X_s \sim Ga(x; \alpha(t-s), \beta)$, for $t > s \geq 0$. The distribution of the increments are thus now solely dependent on the length of the interval, $t - s$.

5.4.2 Gamma sampling

The gamma process is a jump process with infinitely many jumps in each finite interval. It can be simulated in a straightforward manner by simulating increments according to the gamma process characteristics given in the previous subsection. This means, partition the time interval and simulate increments from a gamma distribution with shape parameter conforming the length of the interval and non-changing scale parameter. Since this approach entails the simulation of increments, the value of the process in each time point t is the cumulative sum of the previous increments up to and including t.

However, we can also use the *gamma-bridge sampling* method [12]. Gamma-bridge sampling uses the following relation between a gamma process and the beta distribution [9]: Let $\{X(t)\}_{t\geq 0}$ be a stationary gamma process with parameters αt and β. For time $\tau > 0$, the conditional probability distribution $f_{[X(\tau/2)/X(\tau)]|X(\tau)}$ of $X(\tau/2)/X(\tau)$ given $X(\tau)$, is a symmetric beta distribution with parameter $\alpha\tau/2$, i.e.

$$f_{[X(\tau/2)/X(\tau)]|X(\tau)} = Be(x; \alpha\tau/2, \alpha\tau/2).$$

To simulate a sample path for the process $X(t)$ on the interval $[0, \tau]$ we can use the following:

1. Simulate a value for $X(\tau)$ from a $Ga(x; \alpha\tau, \beta)$ distribution. *Now $X(\tau)$ is known.*
2. Simulate the ratio $\frac{X(\frac{\tau}{2})}{X(\tau)}$ from $Be(x, \frac{\tau}{2}, \frac{\tau}{2})$. *Now $X(\frac{\tau}{2})$ is known.*
3. Simulate the ratios $\frac{X(\frac{\tau}{4})}{X(\frac{\tau}{2})}$ and $\frac{X(\frac{3\tau}{4})-X(\frac{\tau}{2})}{X(\tau)-X(\frac{\tau}{2})}$ both from $Be(x, \frac{\tau}{4}, \frac{\tau}{4})$. *Now $X(\frac{\tau}{4})$ and $X(\frac{3\tau}{4})$ are known.*
4. Simulate the ratios $\frac{X(\frac{\tau}{8})}{X(\frac{\tau}{4})}$, $\frac{X(\frac{3\tau}{8})-X(\frac{\tau}{4})}{X(\frac{\tau}{2})-X(\frac{\tau}{4})}$, $\frac{X(\frac{5\tau}{8})-X(\frac{\tau}{2})}{X(\frac{3\tau}{4})-X(\frac{\tau}{2})}$ and $\frac{X(\frac{7\tau}{8})-X(\frac{3\tau}{4})}{X(\tau)-X(\frac{3\tau}{4})}$, all from $Be(x, \frac{\tau}{8}, \frac{\tau}{8})$. *Now $X(\frac{\tau}{8})$, $X(\frac{3\tau}{8})$, $X(\frac{5\tau}{8})$ and $X(\frac{7\tau}{8})$ are known.*
5. etc.

5.5 Stochastic Strength and HPP Stress

At this point we have presented a stochastic process for deterioration, the gamma process, and a stochastic process for shocks. In this section an example is used to derive the reliability of a component using these stochastic processes. The illustrated example is based on water waves exceeding a dike with declining height. Hence, the system strength is dike height and the shocks are waves. This example has been presented in [72].

Strength
Let the initial dike height be r_0. The degradation process $X = \{X(t), t \geq 0\}$ on the interval $[0, t]$ is modelled as a stationary gamma process described in section 5.4. The strength (or height) of the dike can be written as

$$R(t) = r_0 - X(t), \quad t \geq 0.$$

Stress
Denote the stress magnitude (wave height) at time t as $Y(t)$. We assume the number of stress events (waves) are described by an HPP with intensity λ on the time interval $(0, t]$ and we consider events that are larger than a certain threshold l_0 (minimum wave height).

In Figure 5.12 an example of the dike height decline process is given with wave occurrences exceeding l_0. In the depiction an example of a failure is indicated at time s_6 where $Y_6 > R(s_6) - l_0$.

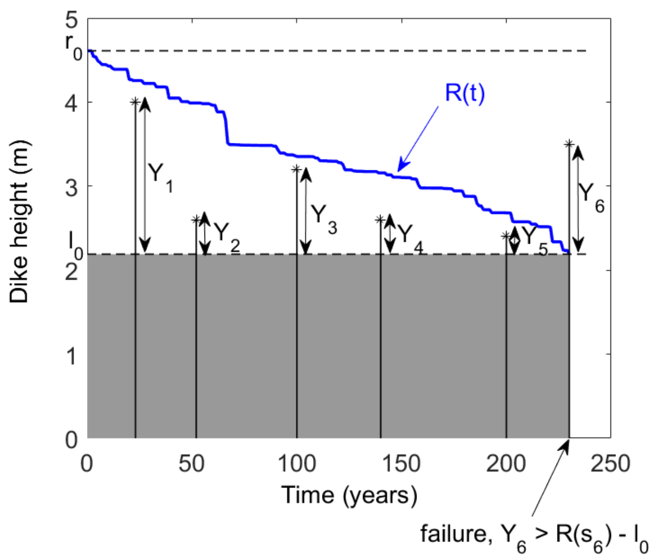


Figure 5.12: Schematic of stochastic process of deterioration and HPP stress. A failure indicates the time point where $Y_6 > R(s_6) - l_0$.

Reliability
A failure is the event of a wave magnitude being larger than the remaining dike height. Hence, the dike cannot fail due to the degradation process itself.

Let $Y_i, i = 1, 2, \ldots$ be wave occurrences exceeding l_0 at time points s_i. Denote the dike height at time point $s_j, j = 1, 2, \ldots$ as $R(s_j)$ and similarly the degradation process at time point $s_j, j = 1, 2, \ldots$ is $X(s_j)$. In this notation the reliability (or survival probability) on $[0, t]$, $\mathbb{P}(t)$, given the occurrence of n stress events on the interval $[0, t]$ is

$$\mathbb{P}(T > t) = (Y_1 < R(s_1) - l_0, Y_2 < R(s_2) - l_0, \ldots, Y_n < R(s_n) - l_0).$$

If the sequence $Y_i, i = 1, 2, \ldots, n$, consists of *iid* random variables with CDF $F_Y(y)$ the reliability can be written as

$$\begin{aligned}
\mathbb{P}(T > t|N(t) = n) &= \mathbb{P}(Y_1 < R(s_1) - l_0, Y_2 < R(s_2) - l_0, \ldots, Y_n < R(s_n) - l_0) \\
&= \mathbb{P}(Y_1 < r_0 - l_0 - X(s_1), \ldots, Y_n < r_0 - l_0 - X(s_n)) \\
&= \mathbb{P}(Y_1 < r_0 - l_0 - X(s_1)) \cdots \mathbb{P}(Y_n < r_0 - l_0 - X(s_n)) \qquad (5.10) \\
&= \prod_{i=1}^{n} F_Y(r_0 - l_0 - X(s_i)) \qquad (5.11)
\end{aligned}$$

and $s_1 < s_2 < \cdots < s_n$.

The wave occurrences follow a Poisson process and hence are random in time. Let us write the joint PDF of the n random occurrence times $\mathbf{S} = \{S_1, S_2, \ldots, S_n\}$ as $f_{\mathbf{S}}(\mathbf{s})$. Writing the reliability function (5.11) as a time-dependent reliability function then gives

$$\begin{aligned}
\mathbb{P}(T > t|N(t) = n) &= \int_0^t \int_{s_1}^t \cdots \int_{s_{n-1}}^t \left(\prod_{j=1}^{n} F_Y(r_0 - l_0 - X(s_j)) \right) f_{S_1,\ldots,S_n}(s_1, \ldots, s_n)\mathrm{d}s_n \cdots \mathrm{d}s_1 \\
&= \underbrace{\int_0^t \cdots \int_{s_{n-1}}^t}_{n-\text{fold}} \left(\prod_{j=1}^{n} F_Y(r_0 - l_0 - X(s_j)) \right) f_{\mathbf{S}}(\mathbf{s})\mathrm{d}\mathbf{s}. \qquad (5.12)
\end{aligned}$$

But according to [34] and in section 5.A we have seen that given n occurrences of a Poisson process on $[0, t]$ their random occurrence times $\mathbf{S}$ are the n order statistics of uniformly distributed and independent random variables $\mathbf{U} = \{U_1, U_2, \ldots, U_n\}$ on the same interval $[0, t]$. Thus we can write

$$f_{\mathbf{S}}(\mathbf{s}) \equiv f_{\mathbf{U}}(\mathbf{u}) = \left(\frac{1}{t}\right)^n.$$

Moreover from equation (5.11) we can write

$$\prod_{j=1}^{n} F_Y(r_0 - l_0 - X(S_j)) = \prod_{k=1}^{n} F_Y(r_0 - l_0 - X(U_k)).$$

Hence, we can write the time-dependent reliability in (5.12) as

$$\begin{aligned}
\mathbb{P}(T > t|N(t) = n) &= \underbrace{\int_0^t \cdots \int_0^t}_{n-\text{fold}} \left(\prod_{k=1}^{n} F_Y(r_0 - l_0 - X(u_k)) \right) f_{\mathbf{U}}(\mathbf{u})\mathrm{d}\mathbf{u} \qquad (5.13) \\
&= \underbrace{\int_0^t \cdots \int_0^t}_{n-\text{fold}} \left(\prod_{k=1}^{n} F_Y(r_0 - l_0 - X(u_k)) \right) \left(\frac{1}{t}\right)^n \mathrm{d}\mathbf{u} \\
&= \left(\int_0^t \frac{F_Y(r_0 - X(u_1))}{t} \mathrm{d}u_1 \right) \cdots \left(\int_0^t \frac{F_Y(r_0 - l_0 - X(u_n))}{t} \mathrm{d}u_n \right) \\
&= \left(\int_0^t \frac{F_Y(r_0 - l_0 - X(u))}{t} \mathrm{d}u \right)^n. \qquad (5.14)
\end{aligned}$$

We notice that equation (5.14) resembles n time points where the probability $F_Y(\cdot)$ needs to be computed, see e.g. equation (5.10). We can write the PDF of the n occurrences of the random variables $\mathbf{X} = \{X(s_1), X(s_2), \ldots, X(s_n)\}$
as $f_{X(s_1),X(s_2),\ldots,X(s_n)}(x(s_1), x(s_2), \ldots, x(s_n))$. Incorporating the randomness of the process $\mathbf{X}$ into equation (5.14) gives

$$\begin{aligned}
\mathbb{P}(T > t|N(t) = n) &= \int_0^\infty \cdots \int_0^\infty \left(\int_0^t \frac{F_Y(r_0 - l_0 - X(u))}{t} \mathrm{d}u \right)^n f_{X(s_1),\ldots,X(s_n)}(x(s_1), \ldots, x(s_n)) \\
&\quad \times \mathrm{d}x(s_1) \cdots \mathrm{d}x(s_n) \\
&= \mathbb{E}_X \left[\left(\int_0^t \frac{F_Y(r_0 - l_0 - X(u))}{t} \mathrm{d}u \right)^n \right],
\end{aligned}$$

where $\mathbb{E}_X$ indicates the expectation taken w.r.t. the random variables
$\mathbf{X} = \{X(s_1), X(s_2), \ldots, X(s_n)\}$. Using the law of total probability we can remove the condition on the amount of occurrences by noticing that the stress occurrences are modelled as a Poisson process:

$$\begin{aligned}
\mathbb{P}(T > t) &= \sum_{n=0}^\infty \mathbb{P}(T > t|N(t) = n)\mathbb{P}(N(t) = n) \qquad (5.15) \\
&= \sum_{n=0}^\infty \mathbb{E}_X \left[\left(\int_0^t \frac{F_Y(r_0 - l_0 - X(u))}{t} \mathrm{d}u \right)^n \right] e^{-\lambda t} \frac{(\lambda t)^n}{n!} \\
&= \sum_{n=0}^\infty \mathbb{E}_X \left[\left(\int_0^t \frac{F_Y(r_0 - l_0 - X(u))}{t} \mathrm{d}u \right)^n e^{-\lambda t} \frac{(\lambda t)^n}{n!} \right] \\
&= e^{-\lambda t} \sum_{n=1}^\infty \mathbb{E}_X \left[\left(\lambda \int_0^t F_Y(r_0 - l_0 - X(u))\mathrm{d}u \right)^n \frac{1}{n!} \right] \\
&= e^{-\lambda t} \mathbb{E}_X \left[\sum_{n=1}^\infty \left(\lambda \int_0^t F_Y(r_0 - l_0 - X(u))\mathrm{d}u \right)^n \frac{1}{n!} \right] \\
&= e^{-\lambda t} \mathbb{E}_X \left[e^{\lambda \int_0^t F_Y(r_0 - l_0 - X(u))\mathrm{d}u} \right] \\
&= \mathbb{E}_X \left[e^{-\lambda \left(t - \int_0^t F_Y(r_0 - l_0 - X(u))\mathrm{d}u \right)} \right] \\
&= \mathbb{E}_X \left[e^{-\lambda \left(\int_0^t 1 - F_Y(r_0 - l_0 - X(u))\mathrm{d}u \right)} \right]. \qquad (5.16)
\end{aligned}$$

In the appendix it is shown that the expectation in equation (5.16) is indeed a probability. We can calculate the formula in equation (5.16) in two steps: first write the integral in the expectation as a Riemann sum (*) and second, perform a Monte Carlo simulation to determine the integrand paths (**) and finally compute the Riemann sum.

Explicitly this can be written as

$$\begin{aligned}
\mathbb{P}(T>t) &= \mathbb{E}_X\left[\exp\left(-\lambda\int_0^t(1-F_Y(r_0-l_0-X(u))\mathrm{d}u\right)\right] \\
&\overset{*}{=} \mathbb{E}_X\left[\exp\left(-\lambda\sum_{i=1}^{k}(1-F_Y(r_0-l_0-X(s_i))(s_i-s_{i-1})\right)\right] \\
&\overset{**}{=} \lim_{k\to\infty}\underbrace{\int_0^\infty\cdots\int_0^\infty}_{k-\text{fold}}\exp\left(-\lambda\sum_{i=1}^{k}(1-F_Y(r_0-l_0-x(s_i))(s_i-s_{i-1})\right) \\
&\quad\times f_{X(s_1),\ldots,X(s_k)}(x(s_1),\ldots,x(s_k))\mathrm{d}x(s_1)\cdots\mathrm{d}x(s_k),
\end{aligned}$$

where the time horizon is uniformly partitioned in k steps, i.e. $s_i=(i/k)t, i=0,\ldots,k$.

As done in [72] we use the parameters in Table 5.1 to evaluate the reliability $\mathbb{P}(T>t)$. The survival probability $\mathbb{P}(T>t)$ is depicted in Figure 5.13.

Parameter	**value**
(Monte Carlo runs)	(10000)
T (time horizon)	400 (years)
$s_i - s_{i-1}$ (time grid steps for all i)	2 (years)
r_0 (initial dike hight)	4.61 (m)
l_0 (load/wave threshold)	2.19 (m)
λ (wave frequency)	0.5 (waves/year)
σ (Pareto waves/loads parameter)	0.3245
c (Pareto waves/loads parameter)	0.05465
v (Gamma strength deterioration process shape parameter)	0.1111
u (Gamma strength deterioration process scale parameter)	15.8730

Table 5.1: Parameters used to compute reliability in Figure 5.13.

5.6 Kac Functional Equation

5.6.1 Understanding the Expectation

We want to compute the expectation:

$$G(t)=\mathbb{E}_X\left[e^{-\lambda\left(\int_0^t 1-F_Y(r_0-X(u))\mathrm{d}u\right)}\right]. \tag{5.17}$$

We first notice that the expectation needs to be computed w.r.t. the process X=$\{X(u), 0\ \leq u\ \leq t\}$, i.e. the gamma process on the time horizon $[0,t]$. So $X(u)$ is not a

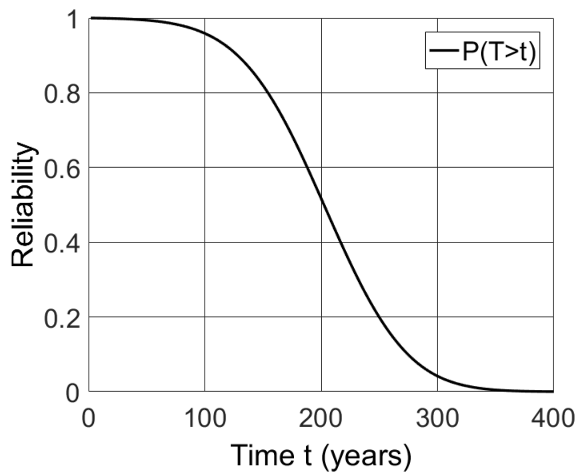


Figure 5.13: Reliability and CDF.

random variable but a random path. This random path is in the integration

$$\int_0^t 1 - F_Y(r_0 - X(u))\mathrm{d}u. \tag{5.18}$$

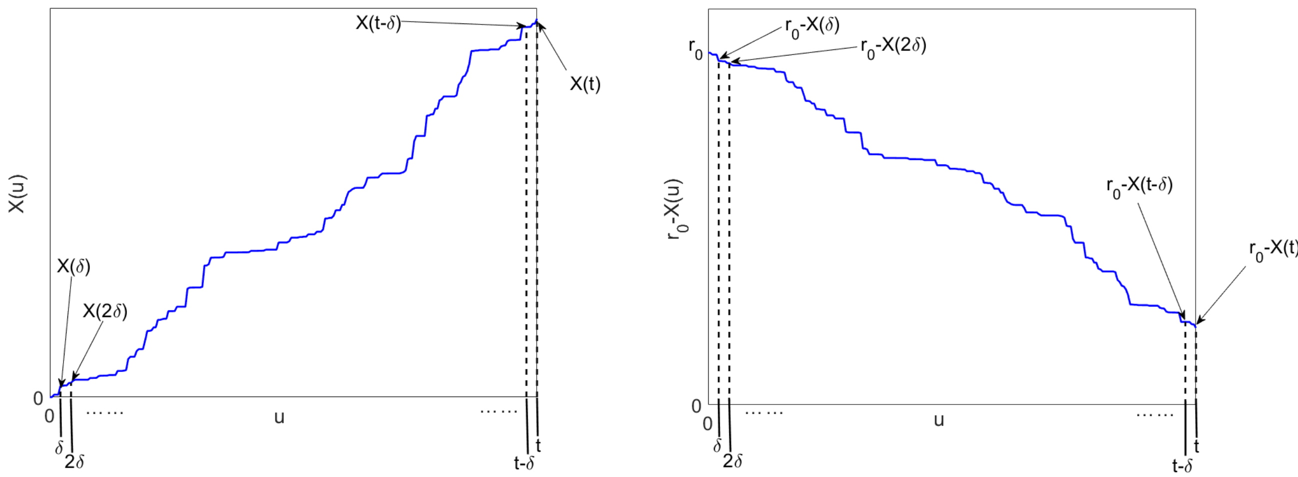
We also notice that the random process is a gamma process starting at $X(0){=}0$ and is defined for an infinite number of points on the interval $[0, t]$. However for the sake of explanation, and as will also be apparent later for computation, we will look at the process every δ, so we consider the sequence $\mathbf{X}{=}X(0), X(\delta), X(2\delta), \ldots, X(t-\delta), X(t)$. This stochastic process $\mathbf{X}$ is a non-decreasing function of time. Consequently the stochastic process $r_0 - \mathbf{X}$, which is the argument of the CDF of Y, is a non-increasing function of time. Graphical representations of these two processes are given in Figures 5.14(a) and 5.14(b) respectively.

Since $r_0 - \mathbf{X}$ is non-increasing $F_Y(r_0 - \mathbf{X})$ is also a non-increasing function of time and the integrand $1 - F_Y(r_0 - \mathbf{X})$ is a non-decreasing function of time. The integral in equation (5.18) is then the area under the function $1 - F_Y(r_0 - \mathbf{X})$. These can be seen in Figures 5.15(a) and 5.15(b).

The steps that rest to do are multiplying this area by $-\lambda$ and taking the exponential. Then we have *one* value of

$$e^{-\lambda\left(\int_0^t 1 - F_Y(r_0 - X(u))\mathrm{d}u\right)}. \tag{5.19}$$

From equation (5.17) we see that we need to compute the expected value of the quantity in equation (5.19), w.r.t. the paths generated by $\mathbf{X}$. To get this expected value we generate multiple paths to compute equation (5.19) and take the mean of these values as the expectation in equation (5.17).



(a) Process $\mathbf{X}$.

(b) Process $r_0 - \mathbf{X}$.

Figure 5.14: Graphical representations of $\mathbf{X}$ and $r_0 - \mathbf{X}$.

5.6.2 Evaluating the Expectation

From the previous section we have established that we need to do the following:

1. Generate a path of the process $X = \{X(u), 0 \leq u \leq t\}$.
2. Compute the finite value $e^{-\lambda\left(\int_0^t 1 - F_Y(r_0 - X(u))\mathrm{d}u\right)}$ in equation (5.19).
3. Repeat steps 1 and 2 n times, and take the mean of these n values.

Step 1

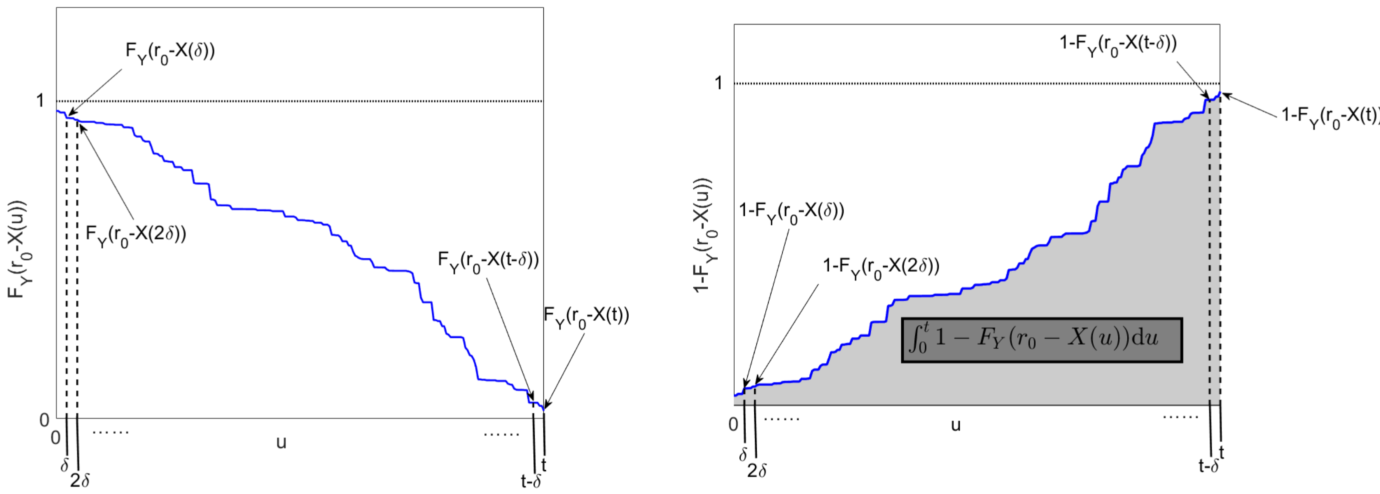
Generate a path $0, x(\delta), x(2\delta), \ldots, x(t-\delta), x(t)$ where the time horizon $[0, t]$ is uniformly partitioned in N steps of length δ, i.e. $0, \delta, 2\delta, \ldots, t-\delta, t = N\delta$.

Step 2

Approximate the integral $\int_0^t 1 - F_Y(r_0 - X(u))\mathrm{d}u$ as a Riemann-Stieltjes sum:

$$\begin{aligned}\int_0^t 1 - F_Y(r_0 - X(u))\mathrm{d}u &\approx \sum_{i=1}^{N} \left(1 - F_Y(r_0 - x(i\delta))\right)(i\delta - (i-1)\delta) \\ &= \delta \sum_{i=1}^{N} \left(1 - F_Y(r_0 - x(i\delta))\right).\end{aligned}$$

This can be seen in Figure 5.16. This sum is then multiplied by $-\lambda$ and subsequently the exponential is taken.



(a) Process $F_Y(r_0 - \mathbf{X})$.

(b) Process $1 - F_Y(r_0 - \mathbf{X})$.

Figure 5.15: Graphical representations of $F_Y(r_0 - \mathbf{X})$ and $1 - F_Y(r_0 - \mathbf{X})$.

Step 3

Repeat step 1 and 2 n times. Sum these n obtained values of $\left[e^{-\lambda\delta\sum_{i=1}^{N}(1-F_Y(r_0-x(i\delta)))}\right]$ and divide by n.

Remark 5.6.1. *Notice that we have just evaluated the expectation* $\mathbb{E}_X\left[e^{-\lambda\int_0^t 1-F_Y(r_0-X(u))\mathrm{d}u)}\right]$ *for* t*. Hence we have evaluated* $G(t) = \mathbb{P}(T \geq t)$ *which is the probability of surviving at least* t*. To compute the distribution* $G(t)$ *on the entire time horizon we can simply take step 2 in the computation cumulatively, i.e. take the cumulative sum of* $\delta\sum_{i=1}^{N}(1 - F_Y(r_0 - x(i\delta)))$*, multiply by* $-\lambda$ *and take the exponential of this cumulative sum. Now perform step 3 and take the mean of the cumulative sum to get* $G(\delta), G(2\delta), \ldots, G(t-\delta), G(t)$*. Notice that* $G(0) = 1$*.*

5.6.3 Numerical Example

In this numerical example we take:

- λ=1
- δ=1 year and t=100 years. So N=100
- r_0=30
- Y follows a Weibull distribution with mean 11.6 and COV 0.36
- X is a stationary gamma process with parameters α=2 and β=1/10

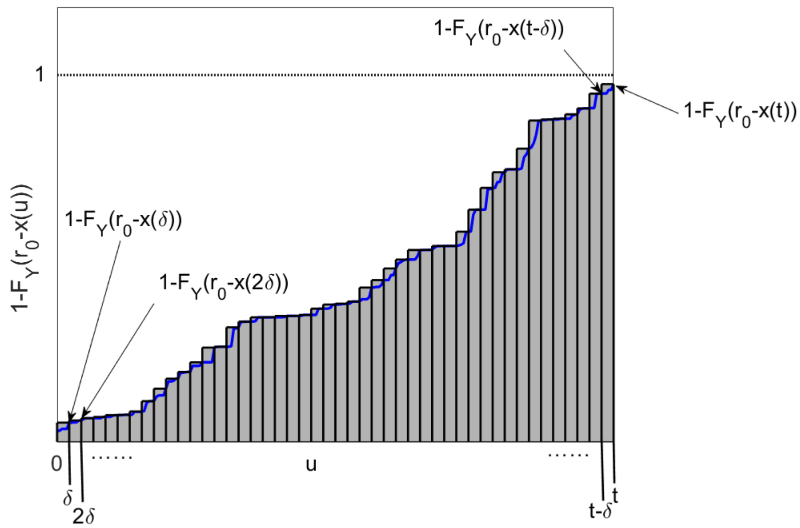


Figure 5.16: Riemann-Stieltjes sum.

The reliability curve, $G(t) = \mathbb{P}(t > T)$, can be seen in Figure 5.17 together with the CDF $1 - G(t)$.

5.6.4 Solving the Kac Functional Analytically

5.6.4.1 General Idea

We notice that the reliability $G(t)$ can be written as

$$G(t) = \mathbb{E}\left[e^{H_{X,Y}(t;\lambda,r_0)}\right],$$

where $H_{X,Y}(t;\lambda,r_0) = H(t) = -\lambda\int_0^t 1 - F_Y(r_0 - X(u))\mathrm{d}u$ is a stochastic integral. So for each fixed t, $H(t)$ is a random variable following a distribution $\Lambda(h_1(t),\ldots,h_n(t))$ where $h_1(t),\ldots,h_n(t)$ are its distribution parameters. To understand this visually, Figures 5.18 and 5.19 give histograms at different values of t for the previous examples $Y \sim Weibull$ and $Y \sim Uniform$ respectively.

We can calculate the moment-generating function of a random variable H as

$$M_H(z) = \mathbb{E}\left[e^{zH}\right], \ \ z \in \mathbb{R}.$$

We also notice that

$$G(t) = \mathbb{E}\left[e^{H(t)}\right], \ \ t \in \mathbb{R}^+.$$

Hence, we want to calculate the moment-generating function at

$$G(t) = M_H(1,t) = \mathbb{E}\left[e^{H(t)}\right], \ \ t \in \mathbb{R}^+.$$

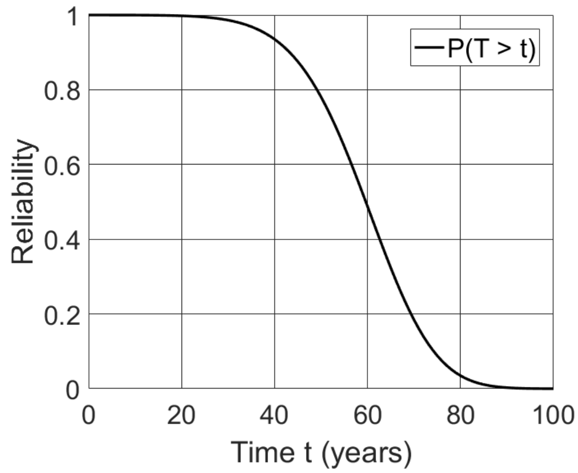


Figure 5.17: Reliability and CDF.

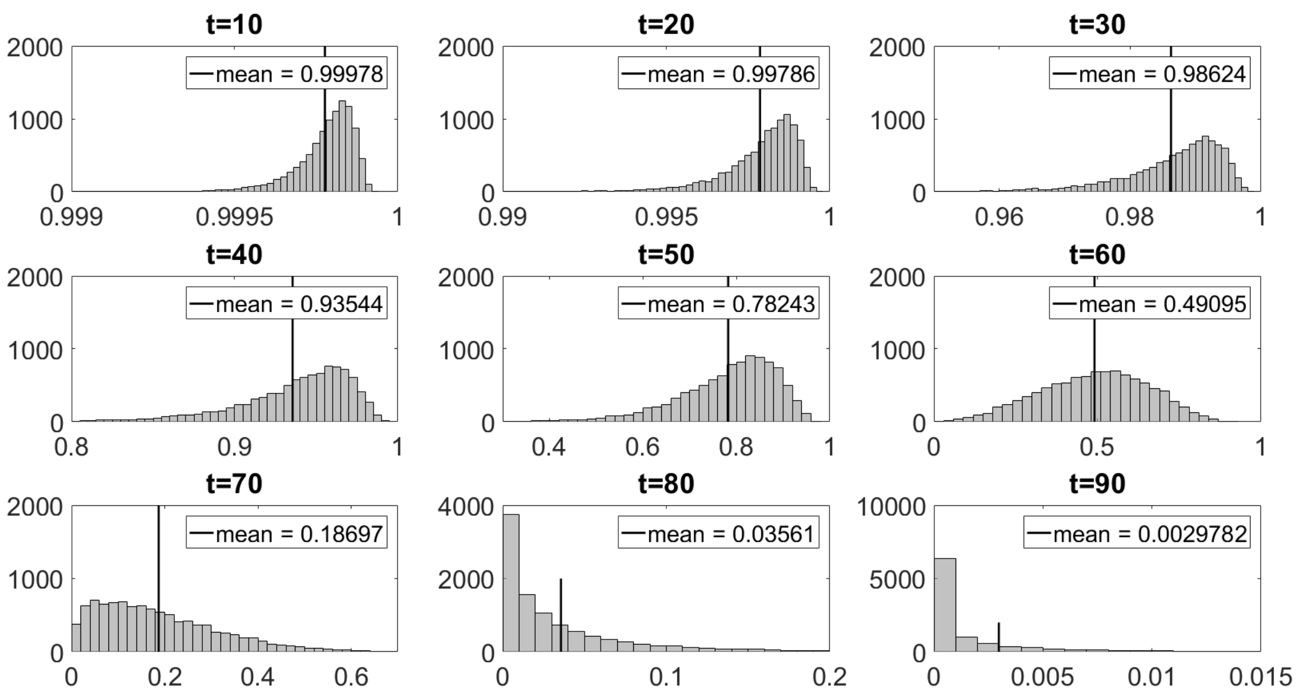


Figure 5.18: Distribution of $e^{-\lambda\int_0^t 1-F_Y(r_0-X(u))\mathrm{d}u)}$ for different values of t, $Y \sim Weibull$.

In order to calculate this the distribution of $H(t)$ must be known for each t. Hence, we need to derive the distributions $\Lambda(h_1(t),\ldots,h_n(t))$.

Generally this is not an easy task. Since integration involves summation, and in this case

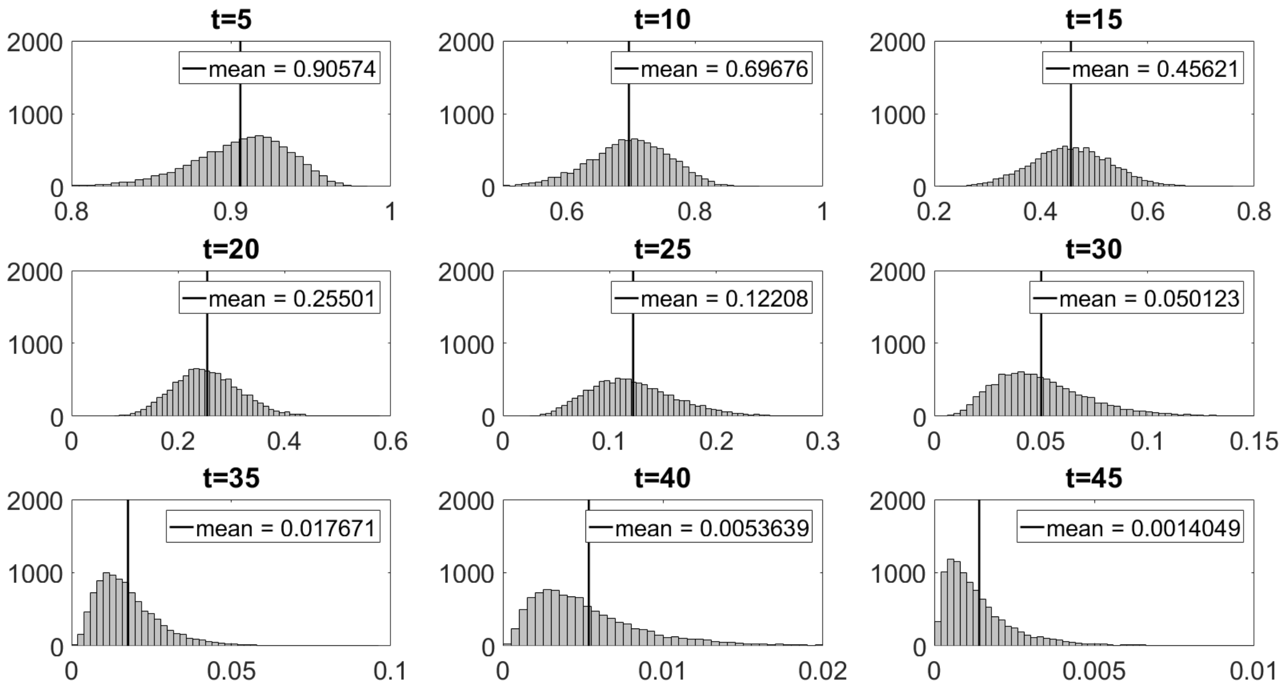
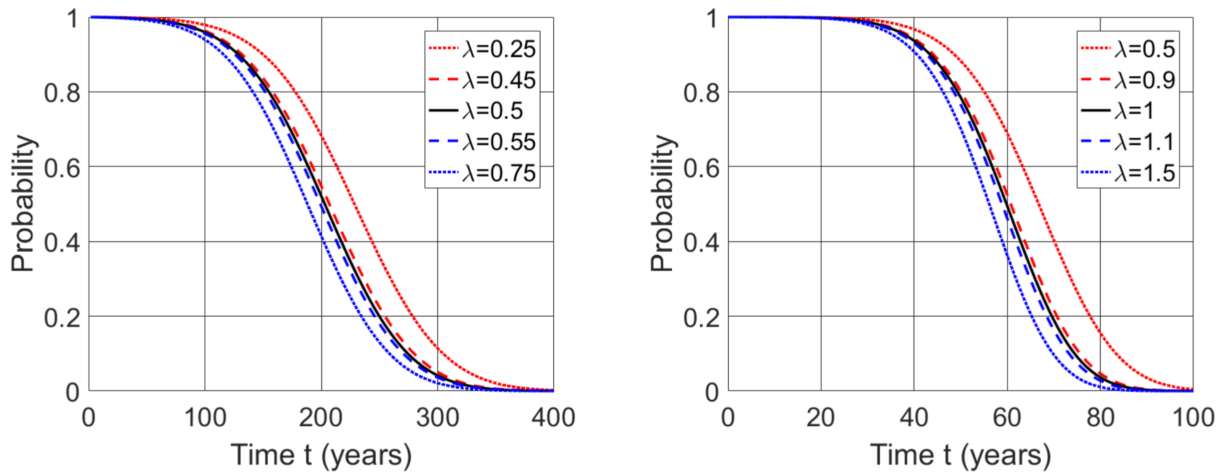


Figure 5.19: Distribution of $e^{-\lambda \int_0^t 1-F_Y(r_0-X(u))\mathrm{d}u)}$ for different values of t, $Y \sim Uniform$.

summation needs to be done after the CDF transformation of gamma increments, it can be particularly difficult or impossible to separate these increments. It is however crucial to separate the increments to use their *iid* property in the expectation.

5.6.5 Sensitivity Analysis

In Figure 5.20 the numerical examples in Figures 5.13 and 5.17 are presented again with different values for the parameter λ. In Figure 5.20 (a) the estimated intensity of the HPP load process is λ=0.5 occurrences/year. Assuming this is the actual load intensity value we can analyze the effect of epistemic uncertainty on λ. If λ is estimated within 10% error of the actual value, i.e. $\lambda \in [0.45, 0.55]$, the solution is still relatively close to the actual reliability. In the case $\lambda \in [0.25, 0.75]$, i.e. maximum 50% off the actual value, errors due to parameter uncertainty in the load intensity estimation become more apparent. It can also be seen that an underestimation of λ leads to bigger errors in the model when compared to overestimation. Similar conclusions can be drawn from Figure 5.20 (b).



(a) Dike example. (b) Example 5.6.3.

Figure 5.20: Sensitivity to λ.

5.7 Solution for a Poisson Process with Stochastic Rate Function

Let T be the time to system failure and we assume for some parameter γ that $\mathbb{P}(T \leq \delta + u | T > u, X(u) = x) = \delta\gamma x + o(\delta)$, $\forall u$, where $\{X(u); u \geq 0\}$ is a stochastic process with $X(0) = 0$. So γx is interpreted as the rate of occurrence of failure when the system level is x as in [41], [40]. Since we are interested in time to system failure, T is the epoch of the first occurrence. Assume the rate function $\{\gamma X(u); u \geq 0\}$ is of a NHPP $\{N(t); t \geq 0\}$. The reliability of the system is then

$$\mathbb{P}(T > t) = \mathbb{P}(N(t) = 0) = \mathbb{E}\left[\frac{\left(\int_0^t \gamma X(u)\mathrm{d}u\right)^0}{0!} e^{-\int_0^t \gamma X(u)\mathrm{d}u}\right] = \mathbb{E}\left[e^{-\int_0^t \gamma X(u)\mathrm{d}u}\right], t \geq 0. \quad (5.20)$$

Hence, the deterioration problem with stochastic loads can be equivalently interpreted as a problem of first occurrence of a Poisson process with stochastic rate. Since we are interested in the gamma process, let us assume the rate function $\{\gamma X(u); u \geq 0\}$ is a stationary gamma process as defined in 5.4.1. To compute equation (5.20) we discretize the integral $\int_0^t X(u)\mathrm{d}u$.

Lemma 5.1. *Let $\int_0^t X(u)\mathrm{d}u, t \geq 0$, where $\{X(u); u \geq 0\}$ is a stationary gamma process, be approximated as a Riemann-Stieltjes sum. We have the "Lebesgue sum":*

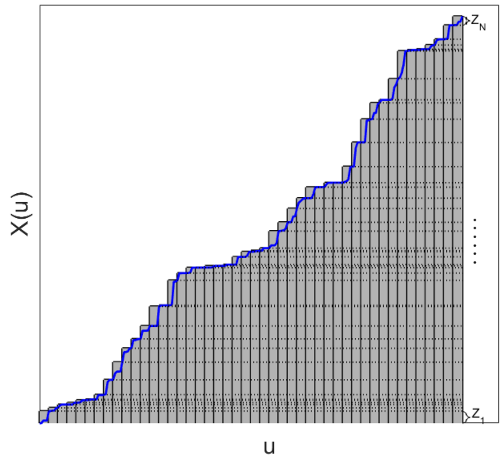
$$\int_0^t X(u)\mathrm{d}u = \lim_{\Delta t \to 0} \sum_{k=1}^{N} Z_k(t - (k-1)\Delta t), \quad (5.21)$$

where $X_u = X(u)$, $Z_{k+1} = (X_{(k+1)\Delta t} - X_{k\Delta t}) \sim Ga(z; \Delta t \cdot \alpha, \beta)$ for all $k = 0, 1, \ldots, N-1$ and $N = \frac{t}{\Delta t}$ is the number of partitions in $[0, t]$.

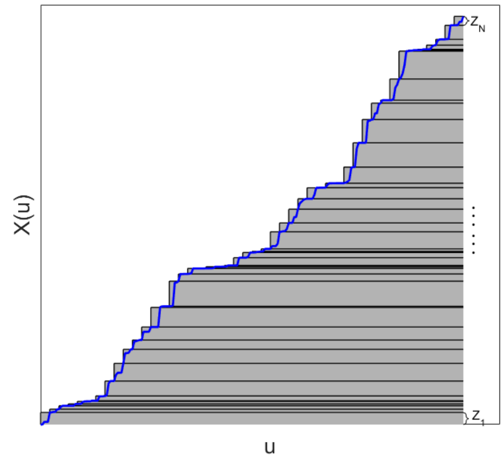
Proof: *First we partition the integration interval uniformly as* $0 = t_0 < t_1 < \ldots < t_N = t$, *where* $t_{k+1} - t_k = (k+1)\Delta t - k\Delta t = \Delta t$ *for* $k = 0, 1, \ldots, N-1$. *Notice that for the sake of simplicity we have* $N = \left\lfloor \frac{t}{\Delta t} \right\rfloor = \frac{t}{\Delta t}$. *Next we approximate the integral as the Riemann-Stieltjes sum:*

$$
\begin{aligned}
\int_0^t X(u)\mathrm{d}u &\approx \sum_{k=1}^{N} X_{k\Delta t}(t_{k\Delta t} - t_{(k-1)\Delta t}) \qquad (5.22)\\
&= \sum_{k=1}^{N} X_{k\Delta t} t_{k\Delta t} - \sum_{k=1}^{N} X_{k\Delta t} t_{(k-1)\Delta t}\\
&= \sum_{k=1}^{N} X_{k\Delta t} t_{k\Delta t} - \sum_{k=0}^{N-1} X_{k\Delta t} t_{(k-1)\Delta t}\\
&= \sum_{k=0}^{N} X_{k\Delta t} t_{k\Delta t} - \sum_{k=0}^{N-1} X_{k\Delta t} t_{(k-1)\Delta t}\\
&= tX_t - \sum_{k=0}^{N-1} (X_{(k+1)\Delta t} - X_{k\Delta t}) t_{k\Delta t}\\
&= t\sum_{k=0}^{N-1} (X_{(k+1)\Delta t} - X_{k\Delta t}) - \sum_{k=0}^{N-1} (X_{(k+1)\Delta t} - X_{k\Delta t}) t_{k\Delta t}\\
&= \sum_{k=0}^{N-1} (X_{(k+1)\Delta t} - X_{k\Delta t})(t - t_{k\Delta t}).
\end{aligned}
$$

Now taking the limit as $\Delta t \to 0$ *of the right-hand side of equation* (5.22) *gives the result.*



(a) Process **X**.



(b) Process **X**.

Figure 5.21: Graphical representations of equation (5.21), changing the integral from a partition on the time scale to a partition on the increments.

Next we notice that $Z_1, Z_2, \ldots, Z_M$ are *iid* and we compute the system reliability

$$
\begin{aligned}
\mathbb{P}(T > t) &= \mathbb{E}\left[e^{-\int_0^t \gamma X(u)\mathrm{d}u}\right] \\
&= \lim_{\Delta t \to 0} \mathbb{E}\left[e^{-\gamma \sum_{k=1}^{\frac{t}{\Delta t}} Z_k (t-(k-1)\Delta t)}\right] \\
&= \lim_{\Delta t \to 0} \prod_{k=1}^{\frac{t}{\Delta t}} \mathbb{E}\left[e^{-\gamma Z_k (t-(k-1)\Delta t)}\right] \\
&= \lim_{\Delta t \to 0} \prod_{k=1}^{\frac{t}{\Delta t}} M_{Z_1}(-\gamma(t-(k-1)\Delta t)) \\
&= \lim_{\Delta t \to 0} \prod_{k=1}^{\frac{t}{\Delta t}} \left(\frac{1}{1+\gamma\beta(t-(k-1)\Delta t)}\right)^{\alpha \Delta t} \\
&= \lim_{\Delta t \to 0} \prod_{k=0}^{\frac{t}{\Delta t}} \left(\frac{1}{1+\gamma\beta k \Delta t)}\right)^{\alpha \Delta t} \\
&= \lim_{\Delta t \to 0} \left(\frac{1}{(\gamma\beta\Delta t)^{\frac{t}{\Delta t}} Pochhammer(\frac{\gamma\beta\Delta t+1}{\gamma\beta\Delta t}, \frac{t}{\Delta t})}\right)^{\alpha\Delta t} \\
&= \lim_{\Delta t \to 0} \left(\frac{\Gamma\left(\frac{\gamma\beta\Delta t+1}{\gamma\beta\Delta t}\right)}{(\gamma\beta\Delta t)^{\frac{t}{\Delta t}} \Gamma\left(\frac{\gamma\beta\Delta t+1}{\gamma\beta\Delta t} + \frac{t}{\Delta t}\right)}\right)^{\alpha\Delta t}, \qquad (5.23)
\end{aligned}
$$

where $M_{Z_1}(s)$ is the moment-generating function of Z_1 evaluated at $s \in \mathbb{R}$. Hence, the system reliability can be computed analytically with equation (5.23) by choosing a small value for $\Delta t = \frac{t}{N}$. Since the gamma function can cause overflow when implementing in for example Matlab, for computational reasons it is better to take the logarithm of equation (5.23) and use readily available functions to compute the logarithm of the gamma function (`\gammaln`):

$$\ln(\mathbb{P}(T > t)) = \alpha\Delta t \left\{ \ln\left[\Gamma\left(\frac{\gamma\beta\Delta t + 1}{\gamma\beta\Delta t}\right)\right] - \frac{t}{\Delta t}\ln(\gamma\beta\Delta t) - \ln\left[\Gamma\left(\frac{\gamma\beta\Delta t + 1}{\gamma\beta\Delta t} + \frac{t}{\Delta t}\right)\right]\right\}. \tag{5.24}$$

Then the system reliability can be computed as

$$\mathbb{P}(T > t) = e^{\ln(\mathbb{P}(T>t))}. \tag{5.25}$$

5.8 An Analytical Solution

We now want to solve the system reliability given in equation (5.17), we recall:

$$\mathbb{P}(T > t) = \mathbb{E}_X\left[e^{-\lambda\int_0^t 1-F_Y(r_0-X(u))\mathrm{d}u)}\right]. \tag{5.26}$$

First we notice that this system reliability can be seen in the same light as the problem in section 5.7. Let T be the time to system failure for a system with non-decreasing stochastic degradation given as $\{1 - F_Y(r_0 - X(u)); u \geq 0\}$ where $\{X(u); u \geq 0\}$ is the gamma process and F_Y is the CDF of the random *iid* loads $Y_1, Y_2, \ldots$. Assume for some parameter $\lambda \in \mathbb{R}^+$ that $\mathbb{P}(T \leq \delta + u | T > u, 1 - F_Y(r_0 - X(u)) = x) = \delta\lambda x + o(\delta)$, $\forall u$. So λx is interpreted as the rate of occurrence of failure when the system level of degradation is x. As the system degradation increases the probability of failure increases. Since we are interested in time to system failure, T is the epoch of the first occurrence. Assume the stochastic rate function $\{\lambda(1 - F_Y(r_0 - X(u))); u \geq 0\}$ is of an NHPP $\{N(t); t \geq 0\}$. The reliability of the system is then given in equation (5.26). Notice that since $X(0) = 0$, at $t = 0$ the system already has amount of degradation $1 - F_Y(r_0)$.

We assume the loads follow a Uniform distribution $U(a, b)$, $0 \leq a < b$, with CDF given as:

$$F_Y(y) = \begin{cases} 0, & \text{for } y < a, \\ \frac{y-a}{b-a}, & \text{for } y \in [a, b], \\ 1, & \text{for } y > b. \end{cases}$$

To compute the integral in equation (5.26) we partition the integration interval uniformly as $0 = t_0 < t_1 < \ldots < t_N = t$, where $t_{k+1} - t_k = (k+1)\Delta t - k\Delta t = \Delta t$ for $k = 0, 1, \ldots, N-1$. With the notation $X_u = X(u)$ we notice that $Z_{k+1} = (X_{(k+1)\Delta t} - X_{k\Delta t}) \sim Ga(z; \Delta t \cdot \alpha, \beta)$ for all $k = 0, 1, \ldots, N-1$ and $N = \frac{t}{\Delta t}$ is the number of partitions in $[0, t]$. Now we compute the system reliability:

$$\begin{aligned}
\mathbb{P}(T>t) &= \mathbb{E}\left[e^{-\lambda\int_0^t 1-F_Y(r_0-X(u))\mathrm{d}u}\right] \\
&= e^{-\lambda t}\lim_{\Delta t\to 0}\mathbb{E}\left[e^{\lambda\sum_{k=1}^{\frac{t}{\Delta t}}\frac{r_0-X_{k\Delta t}-a}{b-a}\Delta t}\right] \\
&= e^{-\lambda t}e^{\lambda t\frac{r_0-a}{b-a}}\lim_{\Delta t\to 0}\mathbb{E}\left[e^{-\frac{\lambda}{b-a}\sum_{k=1}^{\frac{t}{\Delta t}}(t-(k-1)\Delta t)Z_k}\right] \\
&= e^{-\lambda t\frac{b-r_0}{b-a}}\lim_{\Delta t\to 0}\prod_{k=1}^{\frac{t}{\Delta t}}M_Z\left(-\frac{\lambda\Delta t}{b-a}\left[\frac{t}{\Delta t}-(k-1)\right]\right) \\
&= e^{-\lambda t\frac{b-r_0}{b-a}}\lim_{\Delta t\to 0}\prod_{k=1}^{\frac{t}{\Delta t}}\left(\frac{1}{1+\frac{\lambda\beta\Delta t}{b-a}\left[\frac{t}{\Delta t}-(k-1)\right]}\right)^{\alpha\Delta t} \qquad (5.27)\\
&= e^{-\lambda t\frac{b-r_0}{b-a}}\lim_{\Delta t\to 0}\prod_{k=1}^{\frac{t}{\Delta t}}\left(\frac{1}{1+\frac{\lambda\beta\Delta t}{b-a}k}\right)^{\alpha\Delta t} \\
&= e^{-\lambda t\frac{b-r_0}{b-a}}\lim_{\Delta t\to 0}\left(\frac{1}{\left(\frac{\lambda\beta\Delta t}{b-a}\right)^{\frac{t}{\Delta t}}\mathit{Pochhammer}\left(1+\left(\frac{\lambda\beta\Delta t}{b-a}\right)^{-1},\frac{t}{\Delta t}\right)}\right)^{\alpha\Delta t} \\
&= e^{-\lambda t\frac{b-r_0}{b-a}}\lim_{\Delta t\to 0}\left(\frac{\Gamma\left(1+\left(\frac{\lambda\beta\Delta t}{b-a}\right)^{-1}\right)}{\left(\frac{\lambda\beta\Delta t}{b-a}\right)^{\frac{t}{\Delta t}}\Gamma\left(1+\left(\frac{\lambda\beta\Delta t}{b-a}\right)^{-1}+\frac{t}{\Delta t}\right)}\right)^{\alpha\Delta t}, \qquad (5.28)
\end{aligned}$$

where in the third equality Lemma 5.1 is used and equation (5.28) is subject to the condition $\{r_0-X(u);u\geq 0\}\in[a,b]$.

Remark 5.8.1. *Notice that*

$$\frac{r_0-X_{k\Delta t}-a}{b-a}=\begin{cases}0, & \text{for } r_0-X_{k\Delta t}<a,\\ \frac{r_0-X_{k\Delta t}-a}{b-a}, & \text{for } r_0-X_{k\Delta t}\in[a,b],\\ 1, & \text{for } r_0-X_{k\Delta t}>b,\end{cases}$$

for all $k=1,\ldots,\frac{t}{\Delta t}$. Hence, since $\{X(u);u\geq 0\}$ is a non-decreasing function, for any $k^*\leq\frac{t}{\Delta t}$ where $r_0-X_{(k^*-1)\Delta t}>a$ and $r_0-X_{k^*\Delta t}<a$ we have $\frac{r_0-X_{k\Delta t}-a}{b-a}=0$ for all $k=k^*,\ldots,\frac{t}{\Delta t}$. Then we have

$$\mathbb{P}(T>t)=e^{-\lambda t}\lim_{\Delta t\to 0}\mathbb{E}\left[e^{\lambda\sum_{k=1}^{k^*-1}\frac{r_0-X_{k\Delta t}-a}{b-a}\Delta t}\right].$$

While equation (5.28) is easy to compute by simply choosing a small value for Δt, a completely analytical formula does exist for Uniformly distributed loads and is given in the following lemma which has derivation given in the Appendix.

Lemma 5.2. *Let the loads be HPP with intensity λ where the load magnitudes are uniformly distributed random variables $Y\sim U(a,b)$. Furthermore, let the stationary gamma degradation* *process $X = \{r_0 - X(t), t \geq 0\}$ with parameters α and β as in 5.4.1 be of slow decrease such that $r_0 - X$ is always above the threshold l_0. Then the system reliability is given as*

$$\mathbb{P}(T > t) = \exp\left(-\lambda t \frac{b - r_0}{b - a}\right) \exp\left(-\frac{\alpha}{\theta}\left[(1 + \theta t)\left(\ln(1 + \theta t)\right) - \theta t\right]\right), \tag{5.29}$$

where $\theta = \lambda\beta/(b - a)$.

Hence, equations (5.28) and (5.29) are equivalent. A proof of 5.2 is given in 5.C.

5.9 Stochastic Strength and NHPP Stress

The formulation in section 5.5 can be extended to the case when the occurrence of stress has intensity which is time-dependent. This generalization of the intensity incorporates the possibility to model problems where climate change leads to changing behaviour in stress occurrences. Let $\lambda(t) : \mathbb{R} \to \mathbb{R}$ represent the rate of stress occurrence of an NHPP and

$$\Lambda(t) = \int_0^t \lambda(u)\mathrm{d}u.$$

Then, from [35], the joint PDF of the occurrence times $S_1, S_2, \ldots, S_n$ given $N(t) = n$ is

$$f_{S_1,\ldots,S_n|N(t)=n}(s_1, \ldots, s_n|n) = \frac{n! \prod_{i=1}^n \lambda(s_i)}{(\Lambda(t))^n},$$

where $0 < s_1 < \cdots < s_n \leq t$. So conditioned on $N(t) = n$, the occurrence times $S_1, S_2, \ldots, S_n$ have the same distribution as the order statistics of n *iid* random variables $U_1, U_2, \ldots, U_n$ with density $f_U(u)$ where

$$f_U(u) = \frac{\lambda(u)}{\Lambda(t)}, \quad u \in (0, t].$$

Hence, from equations (5.13) and (5.15) the reliability is

$$\begin{aligned}
\mathbb{P}(T > t) &= \sum_{n=0}^{\infty} \mathbb{P}(t|N(t) = n)\mathbb{P}(N(t) = n) \qquad (5.30)\\
&= \sum_{n=0}^{\infty} \mathbb{E}_X\left[\left(\int_0^t F_Y(r_0 - l_0 - X(u))\frac{\lambda(u)}{\Lambda(t)}\mathrm{d}u\right)^n\right] e^{-\Lambda(t)}\frac{(\Lambda(t))^n}{n!}\\
&= e^{-\Lambda(t)} \cdot \mathbb{E}_X\left[\sum_{n=0}^{\infty}\left(\int_0^t F_Y(r_0 - l_0 - X(u))\lambda(u)\mathrm{d}u\right)^n \frac{1}{n!}\right]\\
&= e^{-\Lambda(t)} \cdot \mathbb{E}_X\left[e^{\int_0^t F_Y(r_0 - l_0 - X(u))\lambda(u)\mathrm{d}u}\right]\\
&= \mathbb{E}_X\left[e^{-\int_0^t \lambda(u)(1 - F_Y(r_0 - l_0 - X(u)))\mathrm{d}u}\right]. \qquad (5.31)
\end{aligned}$$

Thus equation (5.16) is the special case, when $\lambda(t)=\lambda>0$ is a constant, of the more general result in equation (5.31). The result in equation (5.31) is to the best of knowledge not known to have appeared in the literature.

By setting the intensity equal to a constant λ and the process $X(t)=0$ so there is no deterioration, we notice that from (5.31) the same result is obtained as the HPP solution in equation (3.6) where the threshold is $r_0 - l_0 = x$.

5.10 Conclusions

This chapter presents an accurate reliability model for components deteriorating according to a gamma process where shocks are (N)HPP. The model requires simulation of deterioration paths in order to arrive at a solution. However, since this method does not require simulation of the load process, it is more efficient than a full scale Monte Carlo implementation. Moreover, by exploiting the solution formula, it is shown that completely analytical solutions are possible in certain cases.

Appendix

5.A Poisson Processes and Order Statistics

Homogeneous Poisson Process

In this section it is explained why conditioned on a number of occurrences from an HPP in a time horizon $(0, t]$, the epochs of these occurrences are uniformly distributed in a sense. This leads to the substitution of the joint PDF $f_{\mathbf{S}}(\mathbf{s})$ in equation (5.12) by the joint PDF

$$f_{\mathbf{S}}(\mathbf{s}) \equiv f_{\mathbf{U}}(\mathbf{u}) = \left(\frac{1}{t}\right)^n.$$

We begin with a proof of the assertion and continue with an explanation of why the assertion is true by introducing the equivalent result for the uniform distribution.

Theorem 5.1. *On a time horizon $(0, t]$, let there be n occurrences of a Poisson process with rate $\lambda>0$ and let $S_1, S_2, \ldots, S_n$ be the occurrence epochs. The random variables $S_1, S_2, \ldots, S_n$ have joint PDF*

$$f_{S_1,\ldots,S_n|N(t)=n}(s_1, \ldots, s_n) = \frac{n!}{t^n}, \quad \text{for } 0 < s_1 < \ldots < s_n \le t. \tag{5.32}$$

Proof. Each occurrence time S_i is in an interval $s_i < S_i \le s_i + \Delta s_i$ for $i = 1, \ldots, n$ and $N(t) = n$. So we need to compute the probability of having exactly one occurrence in each of the disjoint intervals $s_i < S_i \le s_i + \Delta s_i$, for $i = 1, \ldots, n$:

$$\begin{aligned}
&\mathbb{P}\left(N((s_1, s_1 + \Delta s_1]) = 1, \ldots, N((s_n, s_n + \Delta s_n]) = 1\right) \\
&\qquad = \frac{\lambda(\Delta s_1) e^{-\lambda(\Delta s_1)}}{1!} \cdots \frac{\lambda(\Delta s_n) e^{-\lambda(\Delta s_n)}}{1!} \\
&\qquad = \lambda(\Delta s_1) \cdots \lambda(\Delta s_n) e^{-\lambda(\Delta s_1 + \ldots + \Delta s_n)} \\
&\qquad = \lambda(\Delta s_1) \cdots \lambda(\Delta s_n) \left(1 + o(\max\{\Delta s_i\})\right).
\end{aligned}$$

At the same time there are no occurrences in the disjoint intervals $(0, s_1], (s_1 + \Delta s_1, s_2], \ldots, (s_{n-1} + \Delta s_{n-1}, s_n], (s_n + \Delta s_n, t]$. The probability of this is given by

$$\begin{aligned}
&\mathbb{P}\left(N((0, s_1]) = 0, \ldots, N((s_n + \Delta s_n, t]) = 0\right) \\
&\qquad = e^{-\lambda \Delta s_1} e^{-\lambda(s_2 - s_1 - \Delta s_1)} \cdots e^{-\lambda(s_n - s_{n-1} - \Delta s_{n-1})} e^{-\lambda(t - s_n - \Delta s_n)} \\
&\qquad = e^{-\lambda t} e^{\lambda(\Delta s_1 + \ldots + \Delta s_n)} \\
&\qquad = e^{-\lambda t} \left(1 + o(\max\{\Delta s_i\})\right).
\end{aligned}$$

Hence, the probability of having exactly one occurrence at each occurrence time is

$$
\begin{aligned}
&f_{S_1,\ldots,S_n|N(t)=n}\Delta s_1 \cdots \Delta s_n \\
&\quad = \mathbb{P}(s_1 < S_1 \le s_1 + \Delta s_1, \ldots, s_n < S_n \le s_n + \Delta s_n | N(t) = n) + o(\Delta s_1, \ldots, \Delta s_n) \\
&\quad = \frac{\mathbb{P}(s_i < S_i \le s_i + \Delta s_i, i = 1, \ldots, n, N(t) = n)}{\mathbb{P}(N(t) = n)} + o(\Delta s_1, \ldots, \Delta s_n) \\
&\quad = \frac{e^{-\lambda t}\lambda(\Delta s_1) \cdots \lambda(\Delta s_n)}{e^{-\lambda t}(\lambda t)^n/n!}\left(1 + o(\max\{\Delta s_i\})\right) \\
&\quad = \frac{n!}{t^n}(\Delta s_1) \cdots (\Delta s_n)\left(1 + o(\max\{\Delta s_i\})\right), \qquad (5.33)
\end{aligned}
$$

where in the first equality the probability

$$\mathbb{P}(s_i < S_i \le s_i + \Delta s_i) = f_{S_i}(s_i)\Delta s_i + o(\Delta s_i), \ \ \textit{for} \ \ \Delta s_i \downarrow 0,$$

is a more accurate mathematical formulation of

$$\mathbb{P}(x < X \le x + dx) = F(x + dx) - F(x) = dF(x) = f(x)dx.$$

Dividing both sides of equation (5.35) by $\Delta s_1 \cdots \Delta s_n$ and letting $\Delta s_1 \to 0, \ldots, \Delta s_n \to 0$ results in (5.32). □

Before we continue with the results for a uniformly distributed random variable we notice that the Poisson process naturally orders the occurrences on the time interval such that we always have occurrence times ordered as $S_1 < S_2 < \ldots < S_n$.

Uniform Distribution and Order Statistics

For the uniformly distributed random variables we set up the following scenario. Suppose there are n time epochs $U_1, U_2, \ldots, U_n$ on the time horizon $(0, t]$. Each of these n epochs have been occupied by randomly selecting the values $U_1, U_2, \ldots$ and lastly U_n from a uniform distribution on $(0, t]$ independent from previously chosen epochs. Now let $S_1 \le S_2 \le \ldots \le S_n$ denote these epochs arranged in increasing order instead of the order of selection. The joint PDF of $S_1, S_2, \ldots, S_n$ is given by

$$f_{S_1,\ldots,S_n|N(t)=n}(s_1, \ldots, s_n) = \frac{n!}{t^n}, \ \ \textit{for} \ \ 0 < s_1 < \ldots < s_n \le t, \qquad (5.34)$$

which is the same result obtained in the HPP case. To understand this result we first notice that when rearranging the random variables $U_1, U_2, \ldots, U_n$ into increasing order $S_1 \le S_2 \le \ldots \le S_n$

there are $n!$ possibilities of rearrangement. Thus,

$$\begin{aligned}
f_{S_1,\ldots,S_n}(s_1,\ldots,s_n)\Delta s_1\cdots\Delta s_n &= \mathbb{P}(s_1 < S_1 \le s_1+\Delta s_1,\ldots,s_n < S_n \le s_n+\Delta s_n) \\
&= \mathbb{P}(s_1 < U_1 \le s_1+\Delta s_1,\ldots,s_n < U_n \le s_n+\Delta s_n) \\
&\quad \vdots \\
&\quad +\mathbb{P}(s_1 < U_n \le s_1+\Delta s_1,\ldots,s_n < U_1 \le s_n+\Delta s_n) \\
&= n!\frac{\Delta s_1}{t}\cdots\frac{\Delta s_n}{t} \\
&= \frac{n!}{t^n}\Delta s_1\cdots\Delta s_n, \qquad (5.35)
\end{aligned}$$

where in the second equality each term represents one permutation of the rearrangements. Now dividing both sides of equation (5.35) by $\Delta s_1\cdots\Delta s_n$ and letting $\Delta s_1\to 0,\ldots,\Delta s_n\to 0$ results in equation (5.34).

Hence, conditioned on the number of occurrences from an HPP in an interval $(0,t]$, the epochs in this interval can instead be assumed to be occurrences from a uniform distribution.

5.B Kac Functional as a Distribution

In this section we show that the expectation in equation (5.16) is a reliability. We recall

$$\mathbb{P}(T>t)=\mathbb{E}_X\left[e^{-\lambda\int_0^t 1-F_Y(r_0-l_0-X(u))\mathrm{d}u}\right].$$

It suffices to show the following:

1. $\mathbb{P}(T>0)=1$.
2. $\lim_{t\to\infty}\mathbb{P}(T>t)=0$.
3. $\mathbb{P}(T>t)$ is a non-increasing function, i.e. $\mathbb{P}(T>t+\Delta t)\le\mathbb{P}(T>t)$ for $\Delta t>0$.
4. $\mathbb{P}(T>t)$ is right-continuous, i.e. $\lim_{\Delta t\downarrow 0}\mathbb{P}(T>t+\Delta t)=\mathbb{P}(T>t)$ for $\Delta t>0$.

Proof. 1. $\mathbb{P}(T>0)=\mathbb{E}_X\left[e^{-\lambda\cdot 0}\right]=1$.

2. We first notice that $e^a>0$, $\forall a$. Hence, $\mathbb{P}(T>t)\ge 0$, $\forall t$. Now since $X(0)=0$ and $\{X(t),t\ge 0\}$ is non-decreasing, $\max_{0\le u\le t}\{F_Y(r_0-l_0-X(u))\}=F_Y(r_0-l_0)$. Hence, $\min_{0\le u\le t}\{1-F_Y(r_0-l_0-X(u))\}=1-F_Y(r_0-l_0)$. So we have

$$\begin{aligned}
0\le\lim_{t\to\infty}\mathbb{P}(T>t) &= \lim_{t\to\infty}\mathbb{E}_X\left[e^{-\lambda\int_0^t 1-F_Y(r_0-l_0-X(u))\mathrm{d}u}\right] \\
&\le \lim_{t\to\infty}\mathbb{E}_X\left[e^{-\lambda\int_0^t 1-F_Y(r_0-l_0)\mathrm{d}u}\right] \\
&= \lim_{t\to\infty}e^{-\lambda t(1-F_Y(r_0-l_0))} \\
&= 0,
\end{aligned}$$

since $F_Y(r_0 - l_0) \in (0,1)$.

3.

$$
\begin{aligned}
\mathbb{P}(T > t + \Delta t) &= \mathbb{E}_X\left[e^{-\lambda \int_0^{t+\Delta t} 1 - F_Y(r_0 - l_0 - X(u))\mathrm{d}u}\right] \\
&= \mathbb{E}_X\left[e^{-\lambda \int_0^{t} 1 - F_Y(r_0 - l_0 - X(u))\mathrm{d}u} \cdot e^{-\lambda \int_t^{t+\Delta t} 1 - F_Y(r_0 - l_0 - X(u))\mathrm{d}u}\right] \\
&\leq \mathbb{E}_X\left[e^{-\lambda \int_0^{t} 1 - F_Y(r_0 - l_0 - X(u))\mathrm{d}u} \cdot e^{-\lambda \int_t^{t+\Delta t} 1 - F_Y(r_0 - l_0)\mathrm{d}u}\right] \\
&= \mathbb{E}_X\left[e^{-\lambda \int_0^{t} 1 - F_Y(r_0 - l_0 - X(u))\mathrm{d}u} \cdot e^{-\lambda \Delta t(1 - F_Y(r_0 - l_0))}\right] \\
&= e^{-\lambda \Delta t(1 - F_Y(r_0 - l_0))} \cdot \mathbb{E}_X\left[e^{-\lambda \int_0^{t} 1 - F_Y(r_0 - l_0 - X(u))\mathrm{d}u}\right] \\
&\leq \mathbb{E}_X\left[e^{-\lambda \int_0^{t} 1 - F_Y(r_0 - l_0 - X(u))\mathrm{d}u}\right] \\
&= \mathbb{P}(T > t),
\end{aligned}
$$

where the last inequality holds since $e^{-\lambda \Delta t(1 - F_Y(r_0 - l_0))} \in (0,1)$.

4.

$$
\begin{aligned}
\lim_{\Delta t \downarrow 0} \mathbb{P}(T > t + \Delta t) &= \lim_{\Delta t \downarrow 0} \mathbb{E}_X\left[e^{-\lambda \int_0^{t+\Delta t} 1 - F_Y(r_0 - l_0 - X(u))\mathrm{d}u}\right] \\
&= \mathbb{E}_X\left[e^{-\lambda \int_0^{t} 1 - F_Y(r_0 - l_0 - X(u))\mathrm{d}u} \cdot \lim_{\Delta t \downarrow 0} e^{-\lambda \int_t^{t+\Delta t} 1 - F_Y(r_0 - l_0 - X(u))\mathrm{d}u}\right] \\
&= \mathbb{E}_X\left[e^{-\lambda \int_0^{t} 1 - F_Y(r_0 - l_0 - X(u))\mathrm{d}u} \cdot 1\right] \\
&= \mathbb{P}(T > t).
\end{aligned}
$$

□

5.C An Analytical Solution

Proof of Lemma 5.2 We begin we equation (5.27) and notice the following:

$$
\begin{aligned}
\mathbb{P}(T > t) &= e^{-\lambda t \frac{b - r_0}{b - a}} \lim_{\Delta t \to 0} \prod_{k=1}^{\frac{t}{\Delta t}} \left(\frac{1}{1 + \frac{\lambda \beta}{b - a}\left[t - \Delta t(k-1)\right]}\right)^{\alpha \Delta t} \\
&= e^{-\lambda t \frac{b - r_0}{b - a}} e^{\lim_{\Delta t \to 0} \sum_{k=1}^{\frac{t}{\Delta t}} \ln\left(\frac{1}{1 + \theta(t - \Delta t(k-1))}\right)^{\alpha \Delta t}} \\
&= e^{-\lambda t \frac{b - r_0}{b - a}} e^{\lim_{\Delta t \to 0} -\alpha \Delta t \ln(1 + \theta t) - \alpha \Delta t \ln(1 + \theta(t - \Delta t)) - \ldots - \alpha \Delta t \ln(1 + \theta \Delta t)} \\
&= e^{-\lambda t \frac{b - r_0}{b - a}} e^{-\alpha \int_0^t \ln(1 + \theta(t - u))\mathrm{d}u} \\
&= e^{-\lambda t \frac{b - r_0}{b - a}} e^{-\frac{\alpha}{\theta}\left[(1 + \theta t)(\ln(1 + \theta t)) - \theta t\right]},
\end{aligned}
$$

where $\theta = \lambda \beta/(b - a)$.

Chapter 6

Summary and Future Research

6.1 Summary

The distribution of maximum load generated by stochastic and recurring hazards is of primary importance in structural reliability analysis. In the current literature, this distribution is estimated by either relying on the asymptotic extreme value theory or assuming that occurrences of a hazard follow the homogeneous Poisson process. However, assumptions underlying these approaches become questionable when maximum load distribution is required for a short service life, such as in reliability assessment of temporary structures and ageing infrastructure systems nearing the end of life (e.g., old nuclear power plants). This paper aims to fill this gap in the literature by presenting more general and accurate solutions for the probability distribution of the maximum load generated by stochastic hazards which can be modelled as a shock, pulse and alternating renewal process. This work is a considerable advancement of the state of the art in probabilistic analysis of maximum value distribution.

In the case a component is subject to different types of loads at the same time, a model is presented to compute the component reliability and the distribution of the maximum sum of these loads. The model includes two load processes: a pulse process and an HPP shock process. The model is a more general form of solutions in the literature that assume both the pulse and shock process are HPP. In this model the pulse process is allowed to have inter-arrival times following any distribution.

For a component exhibiting gamma deterioration, the reliability can be computed if it is subject to shocks arriving according to an (N)HPP. The solution requires simulation of the deterioration process but avoids simulation of the shock process which would be necessary in a full scale Monte Carlo. In the current setting of computation of reliability for degrading systems, it has been shown that if the shock process is a marked HPP with Uniformly distributed magnitudes, the problem can be solved completely analytically.

6.2 Future Research

In order to compute the component reliability using the single load process model, the component necessarily has no deterioration in time. This can be a rather crude assumption in reliability modelling, especially if longer time horizons are to be considered. It would be an addition to current knowledge to derive solutions in the case of deteriorating components and non Poisson loads. It is however believed that this cannot be done in the current setting of renewal processes since a moving threshold, (component strength), renders the process incapable of having a regeneration point. A workaround is needed to use the same method or a new approach must be invoked to solve this problem.

In the load combination model, as a limitation again of the renewal process approach, the shock process cannot be non HPP. This, to guarantee a regeneration point. A model allowing for both the pulse and shock process to have inter-arrival times following an arbitrary distribution would be a contribution to the literature. Another contribution is to derive solutions, reliability and distribution of maximum sum of loads, for two sources of pulse loads. This proves to be a challenge to solve even if both processes are assumed to be HPP. So far only approximate solutions and solutions in the sense of crossing rates exist. It is advised that future work on these models consider the combination of two or more pulse processes. For more than two load processes, solutions may be difficult to analyze due to analytical complexities. For these load combination problems, it would especially be a challenge to find an accurate model which includes non-exponential inter-arrival times for all the load processes.

For the model with gamma deterioration, it is not clear how the model should be extended to the case when inter-arrival times follow an arbitrary distribution and should be challenging work for the future. It is also not clear how to arrive at a solution if the shock process is replaced by a pulse process, alternating process or sum of processes. To the best of knowledge, these extensions are not known to exist and should pose considerable challenge as future work to arrive at practical, accurate and comprehensive solutions. It has also been shown that analytical solutions exist by exploiting the proposed solution for specifically uniform distributed shock magnitudes. In the future it can be researched if similar analytical results can be obtained for shock magnitudes following different distributions.

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