

## Accepted Manuscript

Poissonian potential measures for Lévy risk models

David Landriault, Bin Li, Jeff T.Y. Wong, Di Xu

PII: S0167-6687(18)30052-0

DOI: <https://doi.org/10.1016/j.insmatheco.2018.07.004>

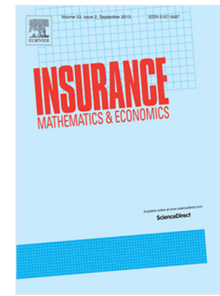
Reference: INSUMA 2483

To appear in: *Insurance: Mathematics and Economics*

Received date: February 2018

Revised date: June 2018

Accepted date: 25 July 2018



Please cite this article as: Landriault D., Li B., Wong J.T.Y., Xu D., Poissonian potential measures for Lévy risk models. *Insurance: Mathematics and Economics* (2018), <https://doi.org/10.1016/j.insmatheco.2018.07.004>

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.

# Poissonian Potential Measures for Lévy Risk Models

David Landriault\*    Bin Li†    Jeff T.Y. Wong‡    Di Xu§

June 5, 2018

## Abstract

This paper studies the potential (or resolvent) measures of spectrally negative Lévy processes killed on exiting (bounded or unbounded) intervals, when the underlying process is observed at the arrival epochs of an independent Poisson process. Explicit representations of these so-called Poissonian potential measures are established in terms of newly defined Poissonian scale functions. Moreover, Poissonian exit measures are explicitly solved by finding a direct relation with Poissonian potential measures. Our results generalize Albrecher et al. [4] in which Poissonian exit identities are solved. As an application of Poissonian potential measures, we extend the Gerber-Shiu analysis in Baurdoux et al. [7] to a (more general) Parisian risk model subject to Poissonian observations.

*Keywords:* Poissonian observations; Potential measures; Exit measures; Spectrally negative Lévy process; Parisian ruin problems

## 1 Introduction

In actuarial mathematics, the risk analysis of spectrally negative Lévy processes (SNLPs) has greatly benefited from the rich literature on fluctuation theory of Lévy processes. For instance, the analysis of exotic exit problems for SNLPs (which, among others, include the generalization of the classical time of ruin to more exotic ruin times) has been facilitated by the comprehensive body of literature on fluctuation identities of SNLPs. The Parisian ruin models of Dassios and Wu [13], Czarna and Palmowski [11], Loeffen et al. [23], Landriault et al. [20], Wong and Cheung [27], Baurdoux et al. [7], and Lkabous et al. [21], where the insurer is granted a grace period whenever the surplus is observed to be negative, are notable contributions on this exotic exit problem topic. We recall that Parisian ruin is deemed to occur at the end of the grace period if the surplus process fails to recover to level zero within the grace period. Interested readers are referred to Li et al. [25] for a more complete literature review on this research topic. Another class of risk processes which has drawn considerable interest in recent years is the drawdown risk models of, e.g., Zhang et al. [29], Landriault et al. [16][17][18], and Avram et al. [6], which use the drop of the insurance surplus from its maximum as a downside risk metric. In comparison to the traditional assessment of an insurer's solvency risk through a fixed level of capital adequacy,

---

\*Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, ON, N2L 3G1, Canada

†Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, ON, N2L 3G1, Canada

‡Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, ON, N2L 3G1, Canada

§Corresponding Author: Department of Finance, University of Nebraska Lincoln, 1400 R St, Lincoln, NE, USA

drawdown has the advantage of following more closely the dynamic growth of insurance surplus over time and hence, has the ability to provide timely warning to insurers on solvency matters. Applications of drawdown models in financial engineering can also be found in Zhang [28].

This paper considers another exotic risk model, namely the so-called *Poissonian observation model*, in which the underlying surplus process of an insurer is monitored discretely at the arrival epochs of an independent Poisson process. In insurance mathematics, the Poissonian observation model was first proposed by Albrecher et al. [1][2], and later generalized by e.g., Albrecher and Ivanovs [3] and Albrecher and Loutscham [5] to more general observation schemes (with surplus-dependent observation rates). Among its possible applications, the Poissonian observation scheme may be used to model the monitoring frequency by an exogenous regulatory authority of an insurer's surplus. The study of Poissonian observation models has been shown to be of interest on its own mathematical merits, and furthermore has helped to establish connections with other existing ruin-related problems in insurance mathematics (most notably, Parisian ruin problems with exponential clocks, see, e.g., Landriault [20]). In this regard, it should be mentioned that, for SNLPs with paths of unbounded variation, Parisian ruin and occupation time problems have typically relied on a spatial approximation technique to overcome difficulties arising from the standard renewal arguments (e.g., Loeffen et al. [23] and Landriault et al. [19]). In Li et al. [25], an alternative approach utilizing the Poissonian observation technique, henceforth referred as the *temporal approximation approach*, is proposed to study some Parisian ruin problems. The temporal approximation approach is shown to be well-suited to the analysis of these Parisian ruin problems, offering the added benefit of a unified treatment of SNLPs with bounded or unbounded variation paths. Note that Poissonian observation schemes have also been applied in queueing contexts (see, e.g., Bekker et al. [8] for more details).

In light of the aforementioned interest in Poissonian observation models, Albrecher et al. [4] established a complete set of exit probabilities for SNLPs. In this paper, we extend Albrecher et al. [4] by solving for the potential measures of SNLPs under Poissonian observations which we shall refer as *Poissonian potential measures* in what follows. Potential measures are known to play a fundamental role in the exit problems of SNLPs under the continuous-time observation scheme; see, for instance, Eq. (3.25) below, Pistorius [26], and Biffis and Kyprianou [9]. This will also be true in the Poissonian observation scheme framework. More precisely, simple relations between *Poissonian exit measures* and *Poissonian potential measures* will be given in Corollary 3.1. Another important contribution of this paper is the introduction of a new class of *Poissonian scale functions* which will allow to state the Poissonian potential and exit measures in the same form as their analogues in the continuous-time observation scheme framework. **Finally, it is worth noting that the observation time process can be generalized from Poisson to a renewal process for which the inter-observation times are assumed to be Erlang distributed or more generally have a rational Laplace transform (see, e.g., Albrecher et al.[1] and [2], and Zhang [30]). Since the proofs and results are considerably more complex in this context, we prefer to cover only the Poissonian observation process here.**

The rest of the paper is organized as follows: in Section 2, we review some preliminary results

on SNLPs. Section 3 contains our main results on Poissonian potential and exit measures. In Section 4, an application of the Poissonian potential measures is considered in a Parisian risk model under Poissonian observations. An explicit expression for a Gerber-Shiu type density at the Parisian ruin time is derived, generalizing its continuously-observed analogue in Baurdoux et al. [7]. All technical proofs are postponed to the Appendix.

## 2 Preliminaries on spectrally negative Lévy processes

In this section, we introduce some preliminary results on SNLPs including scale functions, exit identities, and potential measures. Interested readers are referred to Kyprianou [15] and Kuznetsov et al. [14] for more details. Throughout the paper, let  $X = \{X_t\}_{t \geq 0}$  be a SNLP defined on a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P} = \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$  satisfying the usual conditions of completeness and right continuity. We also adopt the convention that  $\mathbb{P}_x$  and  $\mathbb{E}_x$  are, respectively, the law and expectation when  $X_0 = x \in \mathbb{R}$  (with  $\mathbb{P} = \mathbb{P}_0$  and  $\mathbb{E} = \mathbb{E}_0$  for brevity).

The SNLP  $X$  can be fully characterized via its Laplace exponent  $\psi : [0, \infty) \rightarrow \mathbb{R}$  defined as

$$\psi(s) = \log \mathbb{E}[e^{sX_1}], \quad s \geq 0,$$

with

$$\psi(s) = \mu s + \frac{1}{2} \sigma^2 s^2 + \int_{(-\infty, 0)} (e^{sy} - 1 - sy1_{\{y > -1\}}) \Pi(dy).$$

To avoid triviality, we assume  $|X|$  is not a subordinator, i.e., almost surely non-decreasing sample paths. For any given  $q \geq 0$ , we write

$$\psi_q(s) = \psi(s) - q. \quad (2.1)$$

It is known that  $\psi$  is strictly convex with  $\psi(0) = 0$  and  $\psi(\infty) = \infty$ . Furthermore, we denote the largest solution of the equation  $\psi_q(s) = 0$  by  $\Phi_q$ .

### 2.1 Scale functions

Scale functions are known to play a fundamental role in the fluctuation theory of SNLPs. For any  $q \geq 0$ , the  $q$ -scale function  $W^{(q)} : \mathbb{R} \rightarrow [0, \infty)$  is continuous and (positively) supported on  $[0, \infty)$  (i.e.,  $W^{(q)}(x) = 0$  for all  $x < 0$ ), and constructed via its Laplace transform

$$\int_0^\infty e^{-sy} W^{(q)}(y) dy = \frac{1}{\psi_q(s)}, \quad s > \Phi_q. \quad (2.2)$$

The second  $q$ -scale function is defined by

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y) dy, \quad x \in \mathbb{R},$$

while the following generalized form is of particular interest in exit identities pertaining to  $X$ , namely, for  $\theta \geq 0$ ,

$$Z^{(q)}(x, \theta) = e^{\theta x} \left( 1 - \psi_q(\theta) \int_0^x e^{-\theta y} W^{(q)}(y) dy \right), \quad x \in \mathbb{R}. \quad (2.3)$$

It is immediate that  $Z^{(q)}(x, 0) = Z^{(q)}(x)$  and  $Z^{(q)}(x, \theta) = e^{\theta x}$  for  $x \leq 0$ . Also for  $\theta > \Phi_q$ , we can rewrite  $Z^{(q)}(x, \theta)$  as

$$Z^{(q)}(x, \theta) = \psi_q(\theta) \int_0^\infty e^{-\theta y} W^{(q)}(x+y) dy, \quad x \geq 0. \quad (2.4)$$

Moreover, for  $s, \theta > \Phi_q$ , the Laplace transform of  $Z^{(q)}(x, \theta)$  is given by

$$\int_0^\infty e^{-sx} Z^{(q)}(x, \theta) dx = \begin{cases} \frac{\psi(s) - \psi(\theta)}{\psi_q(s)(s-\theta)}, & \theta \neq s, \\ \frac{\psi'(\theta)}{\psi_q(\theta)}, & \theta = s, \end{cases} \quad (2.5)$$

where  $\psi'(\theta)$  is the derivative of  $\psi(\theta)$ .

Amongst the myriad of results on scale functions, we recall the following two identities from Loeffen et al. [24] which will be heavily relied upon in the later analysis. For any  $p, q, x \geq 0$  and  $p \neq q$ , we have

$$\int_0^x W^{(p)}(x-y) W^{(q)}(y) dy = \frac{W^{(p)}(x) - W^{(q)}(x)}{p-q}, \quad (2.6)$$

and

$$\int_0^x W^{(p)}(x-y) Z^{(q)}(y, \theta) dy = \frac{Z^{(p)}(x, \theta) - Z^{(q)}(x, \theta)}{p-q}. \quad (2.7)$$

## 2.2 Exit identities and potential measures

For any  $x \in \mathbb{R}$ , let

$$\tau_x^{+(-)} = \inf \{t \geq 0 : X_t > (<) x\},$$

where we adopt the convention that  $\inf \emptyset = \infty$ . The two-sided exit identities (2.8) and (2.9) are well known; see, e.g., Theorem 8.1 of Kyprianou [15] and Albrecher et al. [4].

Loeffen

**Lemma 2.1** For  $q \geq 0$  and  $x \in [0, a]$ ,

$$\mathbb{E}_x \left[ e^{-q\tau_a^+} 1_{\{\tau_a^+ < \tau_0^-\}} \right] = \frac{W^{(q)}(x)}{W^{(q)}(a)}, \quad (2.8)$$

and

$$\mathbb{E}_x \left[ e^{-q\tau_0^- + \theta X_{\tau_0^-}} 1_{\{\tau_0^- < \tau_a^+\}} \right] = Z^{(q)}(x, \theta) - \frac{W^{(q)}(x)}{W^{(q)}(a)} Z^{(q)}(a, \theta). \quad (2.9)$$

In particular, we also recall the following one-sided exit identities:

$$\mathbb{E} \left[ e^{-q\tau_a^+} 1_{\{\tau_a^+ < \infty\}} \right] = e^{-\Phi_q a}, \quad a > 0, \quad (2.10)$$

and

$$\mathbb{E}_x \left[ e^{-q\tau_0^- + \theta X_{\tau_0^-}} 1_{\{\tau_0^- < \infty\}} \right] = Z^{(q)}(x, \theta) - \frac{\psi_q(\theta)}{\theta - \Phi_q} W^{(q)}(x), \quad x \geq 0. \quad (2.11)$$

A comparison of (2.8) with (2.10), and (2.9) with (2.11) leads to the following limiting results related to scale functions:

$$\lim_{a \rightarrow \infty} \frac{W^{(q)}(a+x)}{W^{(q)}(a)} = e^{\Phi_q x}, \quad x \in \mathbb{R}, \quad (2.12)$$

and

$$\lim_{a \rightarrow \infty} \frac{Z^{(q)}(a, \theta)}{W^{(q)}(a)} = \frac{\psi_q(\theta)}{\theta - \Phi_q}. \quad (2.13)$$

Next we recall some results on potential measures for the SNLP  $X$  which are defined as follows:

$$\int_0^\infty e^{-qt} \mathbb{P}_x(X_t \in dy) dt = \theta^{(q)}(y - x) dy, \quad x, y \in \mathbb{R}, \quad (2.14)$$

$$\int_0^\infty e^{-qt} \mathbb{P}_x(X_t \in dy, t < \tau_0^+) dt = r_+^{(q)}(x, y) dy, \quad x, y \leq 0, \quad (2.15)$$

$$\int_0^\infty e^{-qt} \mathbb{P}_x(X_t \in dy, t < \tau_0^-) dt = r_-^{(q)}(x, y) dy, \quad x, y \geq 0, \quad (2.16)$$

$$\int_0^\infty e^{-qt} \mathbb{P}_x(X_t \in dy, t < \tau_0^- \wedge \tau_a^+) dt = u^{(q)}(x, y; a) dy, \quad x, y \in [0, a]. \quad (2.17)$$

A thorough derivation and discussion can be found in Chapter 8.4 of Kyprianou [15].

**Lemma 2.2** For  $q \geq 0$  and  $a > 0$ , the  $q$ -potential densities  $\theta^{(q)}$ ,  $r_+^{(q)}$ ,  $r_-^{(q)}$  and  $u^{(q)}$  are given by

$$\theta^{(q)}(y) = \Phi_q' e^{-\Phi_q y} - W^{(q)}(-y), \quad y \in \mathbb{R}, \quad (2.18)$$

$$r_+^{(q)}(x, y) = e^{\Phi_q x} W^{(q)}(-y) - W^{(q)}(x - y), \quad x, y \leq 0, \quad (2.19)$$

$$r_-^{(q)}(x, y) = e^{-\Phi_q y} W^{(q)}(x) - W^{(q)}(x - y), \quad x, y \geq 0, \quad (2.20)$$

$$u^{(q)}(x, y; a) = \frac{W^{(q)}(a - y)}{W^{(q)}(a)} W^{(q)}(x) - W^{(q)}(x - y), \quad x, y \in [0, a], \quad (2.21)$$

where we denote  $\Phi_q'$  as the derivative of  $\Phi_q$  with respect to  $q$ , which is known to satisfy  $\Phi_q' = 1/\psi'(\Phi_q)$ .

### 3 Main Results

This section culminates with our main results on Poissonian potential measures in Theorem 3.1. Subsequently, thanks to a simple relation between potential measures and exit measures under the Poissonian observation scheme, explicit formulas for (one-sided and two-sided) Poissonian exit measures will be given in Corollary 3.1.

Under the Poissonian observation scheme, let  $\{T_n\}_{n \in \mathbb{N}}$ , an increasing sequence of  $\mathbb{F}$ -stopping times, be the observation times which correspond to the arrival times of an independent Poisson process with intensity rate  $\lambda > 0$ . We note that the first observation occurs at time  $T_1$  (and not at time 0). Heuristically, when  $\lambda \rightarrow \infty$ , the Poissonian observation scheme reduces to the classical continuous observation scheme. The convergence of Poissonian potential measures to the “classical” potential measures will be shown in Proposition 3.1.

For any given level  $x \in \mathbb{R}$ , we define the *Poissonian exit times* by

$$T_x^{+(-), \lambda} = \inf \{T_i : X_{T_i} > (<) x\}.$$

In Albrecher et al. [4], the Laplace transform of Poissonian exit times and their corresponding overshoots/undershoots for SNLPs were studied. Their results are expressed in terms of the “classical” scale functions introduced in Section 2.1. To better formulate the results under Poissonian observations, we define the following *Poissonian scale functions*: for  $q \geq 0$  and  $\lambda > 0$ ,

$$W^{(q,\lambda)}(x) = (\Phi_{q+\lambda} - \Phi_q) \int_0^\infty e^{-\Phi_{q+\lambda}y} W^{(q)}(x+y) dy, \quad x \in \mathbb{R}, \quad (3.1)$$

and

$$Z^{(q,\lambda)}(x) = 1 + q \frac{\Phi_{q+\lambda}}{\Phi_{q+\lambda} - \Phi_q} \frac{\lambda}{q + \lambda} \int_0^x W^{(q,\lambda)}(y) dy, \quad x \in \mathbb{R}. \quad (3.2)$$

Using (2.4), we can also rewrite  $W^{(q,\lambda)}$  as

$$W^{(q,\lambda)}(x) = \frac{\Phi_{q+\lambda} - \Phi_q}{\lambda} Z^{(q)}(x, \Phi_{q+\lambda}), \quad x \in \mathbb{R}. \quad (3.3)$$

By applying the initial value theorem on (3.1), it is easily seen that the Poissonian scale functions converge to the classical scale functions as  $\lambda \rightarrow \infty$ , that is,

$$\lim_{\lambda \rightarrow \infty} W^{(q,\lambda)}(x) = W^{(q)}(x) \quad \text{and} \quad \lim_{\lambda \rightarrow \infty} Z^{(q,\lambda)}(x) = Z^{(q)}(x). \quad (3.4)$$

Note that, for the latter limit in (3.4), we can apply the dominated convergence theorem as, for fixed  $x \in \mathbb{R}$ , and  $\lambda$  large enough, we have

$$\frac{\Phi_{q+\lambda}}{\Phi_{q+\lambda} - \Phi_q} \frac{\lambda}{q + \lambda} W^{(q,\lambda)}(y) \leq 2W^{(q,\lambda)}(y) \leq 2W^{(q,\lambda)}(x) \leq 2W^{(q)}(x) + 1, \quad \text{for any } y \in [0, x].$$

In the following lemma, we make use of the Poissonian scale functions  $W^{(q,\lambda)}$  and  $Z^{(q,\lambda)}$  to re-state two Poissonian exit results in Albrecher et al. [4] in a form which is consistent with their continuously-observed analogues (2.8) and (2.9), respectively.

**Lemma 3.1** For  $q \geq 0$ , and  $x \in [0, a]$ ,

$$\mathbb{E}_x \left[ e^{-q\tau_a^+} \mathbf{1}_{\{\tau_a^+ < T_0^{-,\lambda}\}} \right] = \frac{W^{(q,\lambda)}(x)}{W^{(q,\lambda)}(a)}, \quad (3.5)$$

and

$$\mathbb{E}_x \left[ e^{-qT_0^{-,\lambda}} \mathbf{1}_{\{T_0^{-,\lambda} < \tau_a^+\}} \right] = Z^{(q,\lambda)}(x) - \frac{W^{(q,\lambda)}(x)}{W^{(q,\lambda)}(a)} Z^{(q,\lambda)}(a). \quad (3.6)$$

**Remark 3.1** Given their importance in Albrecher et al. [4] and the subsequent analysis, we limit the review of Albrecher et al. [4] to the exit results (3.5) and (3.6). We note that (3.5) was first proved by Albrecher and Ivanovs [3]. For both Eqs. (3.5) and (3.6), a spatial approximation argument is used to handle SNLPs with unbounded variation paths. Alternatively, simple conditioning arguments (coupled with the potential measure results in Lemma 2.2) can be called upon to derive these results in a more direct manner. As an illustrative example, we consider  $\mathbb{P}(\tau_a^+ < T_0^{-,\lambda})$ . The other cases can be similarly handled.

By conditioning on the first observation time  $T_1$  (which has the same distribution as an independent exponential random variable  $e_\lambda$  with mean  $1/\lambda$ ) and then using (2.15), we have

$$\begin{aligned}\mathbb{P}\left(\tau_a^+ < T_0^{-,\lambda}\right) &= \mathbb{P}\left(\tau_a^+ < e_\lambda\right) + \int_0^a \mathbb{P}\left(X_{e_\lambda} \in dy, e_\lambda < \tau_a^+\right) \mathbb{P}_y\left(\tau_a^+ < T_0^{-,\lambda}\right) \\ &= \mathbb{P}\left(\tau_a^+ < e_\lambda\right) + \int_0^a \lambda r_+^{(q,\lambda)}(-a, y-a) \mathbb{P}_y\left(\tau_a^+ < T_0^{-,\lambda}\right) dy.\end{aligned}\quad (3.7)$$

For  $x \in [0, a]$ , by conditioning on  $\tau_0^-$  and using (2.10), one finds that

$$\begin{aligned}\mathbb{P}_x\left(\tau_a^+ < T_0^{-,\lambda}\right) &= \mathbb{P}_x\left(\tau_a^+ < \tau_0^-\right) + \int_{-\infty}^0 \mathbb{P}_x\left(X_{\tau_0^-} \in dy, \tau_0^- < \tau_a^+\right) \mathbb{P}_y\left(\tau_0^+ < e_\lambda\right) \mathbb{P}\left(\tau_a^+ < T_0^{-,\lambda}\right) \\ &= \mathbb{P}_x\left(\tau_a^+ < \tau_0^-\right) + \mathbb{E}_x\left[e^{\Phi_\lambda X_{\tau_0^-}} 1_{\{\tau_0^- < \tau_a^+\}}\right] \mathbb{P}\left(\tau_a^+ < T_0^{-,\lambda}\right).\end{aligned}\quad (3.8)$$

Substituting (3.8) into (3.7) yields the desired renewal equation for  $\mathbb{P}\left(\tau_a^+ < T_0^{-,\lambda}\right)$ .

We now define the following set of Poissonian  $q$ -potential measures: for  $q \geq 0$  and  $a > 0$ ,

$$\begin{aligned}\int_0^\infty e^{-qt} \mathbb{P}_x\left(X_t \in dy, t < T_0^{+,\lambda} \wedge \tau_a^+\right) dt &= r_+^{(q,\lambda)}(x, y; a) dy, \quad x, y \leq a, \\ \int_0^\infty e^{-qt} \mathbb{P}_x\left(X_t \in dy, t < T_0^{-,\lambda} \wedge \tau_{-a}^-\right) dt &= r_-^{(q,\lambda)}(x, y; -a) dy, \quad x, y \geq -a, \\ \int_0^\infty e^{-qt} \mathbb{P}_x\left(X_t \in dy, t < T_0^{+,\lambda}\right) dt &= r_+^{(q,\lambda)}(x, y) dy, \quad x, y \in \mathbb{R}, \\ \int_0^\infty e^{-qt} \mathbb{P}_x\left(X_t \in dy, t < T_0^{-,\lambda}\right) dt &= r_-^{(q,\lambda)}(x, y) dy, \quad x, y \in \mathbb{R}, \\ \int_0^\infty e^{-qt} \mathbb{P}_x\left(X_t \in dy, t < T_0^{-,\lambda} \wedge \tau_a^+\right) dt &= u_{d:c}^{(q,\lambda)}(x, y; a) dy, \quad x, y \leq a, \\ \int_0^\infty e^{-qt} \mathbb{P}_x\left(X_t \in dy, t < \tau_0^- \wedge T_a^{+,\lambda}\right) dt &= u_{c:d}^{(q,\lambda)}(x, y; a) dy, \quad x \in [0, a], y \geq 0, \\ \int_0^\infty e^{-qt} \mathbb{P}_x\left(X_t \in dy, t < T_0^{-,\lambda} \wedge T_a^{+,\lambda}\right) dt &= u_{d:d}^{(q,\lambda)}(x, y; a) dy, \quad x \in [0, a], y \in \mathbb{R}.\end{aligned}$$

Among all of these Poissonian potential measures,  $r_-^{(q,\lambda)}(x, y; -a)$  and  $r_+^{(q,\lambda)}(x, y; a)$  are the two cornerstone quantities as the derivation of explicit expressions for all the other potential measures heavily relies on them. The Poissonian potential densities  $r_+^{(q,\lambda)}$ ,  $r_-^{(q,\lambda)}$ , and the triplet  $(u_{d:c}^{(q,\lambda)}, u_{c:d}^{(q,\lambda)}, u_{d:d}^{(q,\lambda)})$  are the Poissonian analogues to the classical potential densities  $r_+^{(q)}$ ,  $r_-^{(q)}$  and  $u^{(q)}$ , respectively. Note that the subscripts  $c$  and  $d$  are used to characterize the type of exit whether it is under continuous-time or discrete-time (Poissonian) observations, respectively.

Theorem 3.1 summarizes our main results on Poissonian potential measures for SNLPs. The proof of Theorem 3.1 is postponed to the Appendix. For  $q \geq 0$ ,  $\lambda > 0$  and  $x, y \in \mathbb{R}$ , we define an auxiliary function

$$A^{(q,\lambda)}(x, y) = W^{(q)}(x+y) + \lambda \int_0^y W^{(q)}(x+y-z) W^{(q+\lambda)}(z) dz, \quad (3.9)$$



which can also be rewritten as

$$A^{(q,\lambda)}(x, y) = W^{(q+\lambda)}(x+y) - \lambda \int_0^x W^{(q)}(z) W^{(q+\lambda)}(x+y-z) dz, \quad (3.10)$$

with the help of (2.6). Note that  $A^{(q,\lambda)}(x, y)$  is actually the same as  $g(q, \lambda, x, y)$  defined in Baurdoux et al. [7], and as  $\mathcal{W}_x^{(q,\lambda)}(x+y)$  defined in Loeffen et al. [24]. Moreover, it is seen from (3.9) and (3.10) that

$$A^{(q,\lambda)}(x, y) = W^{(q)}(x+y), \quad y \leq 0, \quad (3.11)$$

and

$$A^{(q,\lambda)}(x, y) = W^{(q+\lambda)}(x+y), \quad x \leq 0. \quad (3.12)$$

**Theorem 3.1** For  $q \geq 0$  and  $a > 0$ , the Poissonian  $q$ -potential densities are given by

$$r_+^{(q,\lambda)}(x, y; a) = \frac{A^{(q,\lambda)}(-y, a)}{Z^{(q+\lambda)}(a, \Phi_q)} Z^{(q+\lambda)}(x, \Phi_q) - A^{(q,\lambda)}(-y, x), \quad x, y \leq a, \quad (3.13)$$

$$r_-^{(q,\lambda)}(x, y; -a) = \frac{A^{(q,\lambda)}(x, a)}{Z^{(q+\lambda)}(a, \Phi_q)} Z^{(q+\lambda)}(-y, \Phi_q) - A^{(q,\lambda)}(x, -y), \quad x, y \geq -a, \quad (3.14)$$

$$r_+^{(q,\lambda)}(x, y) = W^{(q,\lambda)}(-y) Z^{(q+\lambda)}(x, \Phi_q) - A^{(q,\lambda)}(-y, x), \quad x, y \in \mathbb{R}, \quad (3.15)$$

$$r_-^{(q,\lambda)}(x, y) = W^{(q,\lambda)}(x) Z^{(q+\lambda)}(-y, \Phi_q) - A^{(q,\lambda)}(x, -y), \quad x, y \in \mathbb{R}, \quad (3.16)$$

$$u_{d;c}^{(q,\lambda)}(x, y; a) = \frac{A^{(q,\lambda)}(a, -y)}{W^{(q,\lambda)}(a)} W^{(q,\lambda)}(x) - A^{(q,\lambda)}(x, -y), \quad x, y \leq a, \quad (3.17)$$

$$u_{c;d}^{(q,\lambda)}(x, y; a) = \frac{W^{(q,\lambda)}(a-y)}{W^{(q,\lambda)}(a)} W^{(q)}(x) - W^{(q)}(x-y), \quad x \in [0, a], y \geq 0, \quad (3.18)$$

$$u_{d;d}^{(q,\lambda)}(x, y; a) = \frac{\int_a^\infty e^{-\Phi_{q+\lambda}z} A^{(q,\lambda)}(z, -y) dz}{\int_a^\infty e^{-\Phi_{q+\lambda}z} W^{(q,\lambda)}(z) dz} W^{(q,\lambda)}(x) - A^{(q,\lambda)}(x, -y), \quad x \in [0, a], y \in \mathbb{R}. \quad (3.19)$$

In fact, Eqs. (3.18) and (3.19) also hold for the case  $x < 0$ , and the proof in Appendix will concurrently handle these cases. More generally, one may further consider all these Poissonian potential measures for a general  $x \in \mathbb{R}$ . However, the corresponding expressions will become much more complicated and hence, we chose to limit the presentation to what is displayed in Theorem 3.1.

The following corollary confirms the convergence of Poissonian potential measures to the classical potential measures when the observation intensity rate  $\lambda$  goes to infinity. The proof is also postponed to the Appendix.

**Proposition 3.1** For  $q \geq 0$  and  $a > 0$ , we have

$$\lim_{\lambda \rightarrow \infty} r_+^{(q,\lambda)}(x, y) = r_+^{(q)}(x, y), \quad \text{for } x, y \leq 0, \quad (3.20)$$

$$\lim_{\lambda \rightarrow \infty} r_-^{(q,\lambda)}(x, y) = r_-^{(q)}(x, y), \quad \text{for } x, y \geq 0, \quad (3.21)$$

$$\lim_{\lambda \rightarrow \infty} u_{d:c}^{(q,\lambda)}(x, y; a) = u^{(q)}(x, y; a), \quad \text{for } x, y \in [0, a], \quad (3.22)$$

$$\lim_{\lambda \rightarrow \infty} u_{c:d}^{(q,\lambda)}(x, y; a) = u^{(q)}(x, y; a), \quad \text{for } x, y \in [0, a], \quad (3.23)$$

$$\lim_{\lambda \rightarrow \infty} u_{d:d}^{(q,\lambda)}(x, y; a) = u^{(q)}(x, y; a), \quad \text{for } x, y \in [0, a]. \quad (3.24)$$

For the rest of this section, we consider *Poissonian exit measures*, and simultaneously revisit some of the exit results given in Albrecher et al. [4]. First, we recall that under the continuous-time observation scheme, exit measures of SNLPs can be expressed as integrals of the Lévy measure and potential measures. For instance, for  $x \geq 0$  and  $y \leq 0$ ,

$$\mathbb{E}_x \left[ e^{-q\tau_0^-} 1_{\{\tau_0^- < \infty, X_{\tau_0^-} \in dy\}} \right] = \int_0^\infty e^{-qt} \mathbb{P}_x(t < \tau_0^-, X_t \in dz) \Pi(z - dy), \quad (3.25)$$

where  $\Pi$  is the Lévy measure of  $X$  on  $[0, \infty)$ . Interested readers are referred to Pistorius [26] or Chapter 8.4 of Kyprianou [15] for a detailed discussion, and Loeffen [22] for a general payoff function of the overshoot  $X_{\tau_0^-}$ .

Under the Poissonian observation scheme, the potential and exit measure relationship is even simpler. Again, we use the downward exiting as example. Since the probability that an observation is made within the infinitesimal time period  $(t, t + dt)$  is  $\lambda dt$  and by the independence of the observation process and  $X$ , the law of  $T_0^{-,\lambda}$  (which is an observation time) is

$$\mathbb{P}_x(T_0^{-,\lambda} \in dt) = \lambda \mathbb{P}_x(T_0^{-,\lambda} > t) dt.$$

More generally, for any  $x \geq 0$  and  $y \leq 0$ , a similar reasoning yields

$$\begin{aligned} \mathbb{E}_x \left[ e^{-qT_0^{-,\lambda}} 1_{\{T_0^{-,\lambda} < \infty, X_{T_0^{-,\lambda}} \in dy\}} \right] &= \int_0^\infty e^{-qt} \mathbb{P}_x(T_0^{-,\lambda} \in dt, X_t \in dy) \\ &= \lambda \int_0^\infty e^{-qt} \mathbb{P}_x(T_0^{-,\lambda} > t, X_t \in dy) dt \\ &= \lambda r_-^{(q,\lambda)}(x, y). \end{aligned}$$

Such duality further stresses the importance of Poissonian potential measures. By the same argument, we immediately have the following corollary on Poissonian exit measures.

**Corollary 3.1** For  $q \geq 0$  and  $a > 0$ ,

$$\begin{aligned} \mathbb{E}_x \left[ e^{-qT_0^{+, \lambda}} 1_{\{X_{T_0^{+, \lambda}} \in dy, T_0^{+, \lambda} < \tau_a^+\}} \right] &= \lambda r_+^{(q, \lambda)}(x, y; a) dy, \quad x \leq a, y \in [0, a], \\ \mathbb{E}_x \left[ e^{-qT_0^{-, \lambda}} 1_{\{X_{T_0^{-, \lambda}} \in dy, T_0^{-, \lambda} < \tau_{-a}^-\}} \right] &= \lambda r_-^{(q, \lambda)}(x, y; -a) dy, \quad x \geq -a, y \in [-a, 0], \\ \mathbb{E}_x \left[ e^{-qT_0^{+, \lambda}} 1_{\{X_{T_0^{+, \lambda}} \in dy, T_0^{+, \lambda} < \infty\}} \right] &= \lambda r_+^{(q, \lambda)}(x, y) dy, \quad x \leq 0, y \geq 0, \\ \mathbb{E}_x \left[ e^{-qT_0^{-, \lambda}} 1_{\{X_{T_0^{-, \lambda}} \in dy, T_0^{-, \lambda} < \infty\}} \right] &= \lambda r_-^{(q, \lambda)}(x, y) dy, \quad x \geq 0, y \leq 0, \end{aligned} \quad (3.26)$$

$$\mathbb{E}_x \left[ e^{-qT_0^{-, \lambda}} 1_{\{X_{T_0^{-, \lambda}} \in dy, T_0^{-, \lambda} < \tau_a^+\}} \right] = \lambda u_{d:c}^{(q, \lambda)}(x, y; a) dy, \quad x \in [0, a], y \leq 0, \quad (3.27)$$

$$\begin{aligned} \mathbb{E}_x \left[ e^{-qT_a^{+, \lambda}} 1_{\{X_{T_a^{+, \lambda}} \in dy, T_a^{+, \lambda} < \tau_0^-\}} \right] &= \lambda u_{c:d}^{(q, \lambda)}(x, y; a) dy, \quad x \in [0, a], y \geq a, \\ \mathbb{E}_x \left[ e^{-qT_a^{+, \lambda}} 1_{\{X_{T_a^{+, \lambda}} \in dy, T_a^{+, \lambda} < T_0^{-, \lambda}\}} \right] &= \lambda u_{d:d}^{(q, \lambda)}(x, y; a) dy, \quad x \in [0, a], y \geq a, \\ \mathbb{E}_x \left[ e^{-qT_0^{-, \lambda}} 1_{\{X_{T_0^{-, \lambda}} \in dy, T_0^{-, \lambda} < T_a^{+, \lambda}\}} \right] &= \lambda u_{d:d}^{(q, \lambda)}(x, y; a) dy, \quad x \in [0, a], y \leq 0. \end{aligned} \quad (3.28)$$

Corollary 3.1 generalizes Theorems 3.1 and 3.2 of Albrecher et al. [4] in which the joint Laplace transforms of the Poissonian exit times and the overshoots/undershoots are given.

To conclude this section, we provide another Poissonian exit measure, namely Eq. (3.29). Notice that the Poissonian exit measures (3.26), (3.27) and (3.29) are actually identical to Eqs. (1.12), (1.11), and (1.8), respectively, in Baurdoux et al. [7]. This is not surprising as the Parisian ruin time  $\tau_q$  in Baurdoux et al. [7] is well known to have the same distribution as  $T_0^{-, q}$  (defined in our paper). However, we point out that Baurdoux et al. [7] also relies on the spatial approximation argument to deal with the case of unbounded variation paths, while the present derivation relies more closely on the strength of the Poisson discretization technique to derive these results.

**Corollary 3.2** For  $q \geq 0$ ,  $a, b > 0$ ,  $x \in [-a, b]$  and  $y \in [-a, 0]$ , we have

$$\mathbb{E}_x \left[ e^{-qT_0^{-, \lambda}} 1_{\{X_{T_0^{-, \lambda}} \in dy, T_0^{-, \lambda} < \tau_{-a}^- \wedge \tau_b^+\}} \right] = \lambda \left( \frac{A^{(q, \lambda)}(x, a)}{A^{(q, \lambda)}(b, a)} A^{(q, \lambda)}(b, -y) - A^{(q, \lambda)}(x, -y) \right) dy. \quad (3.29)$$

The complete proof of the above corollary is again postponed to the Appendix for which the key step consists in proving that the following interesting identity holds:

$$\mathbb{E}_x \left[ e^{-q\tau_b^+} 1_{\{\tau_b^+ < T_0^{-, \lambda} \wedge \tau_{-a}^-\}} \right] = \frac{A^{(q, \lambda)}(x, a)}{A^{(q, \lambda)}(b, a)}, \quad x \in [-a, b].$$

## 4 Application: Parisian ruin with Poissonian observations

As an application of the Poissonian potential measures, we consider a generalization of the Parisian risk model in which the underlying SNLP  $X$  is subject to a Poissonian observation scheme with

intensity rate  $\lambda > 0$ . Our objective is to derive a Gerber-Shiu type density at the Poissonian Parisian ruin time which will generalize its continuously-observed analogue in Baurdoux et al. [7].

Under a Poissonian observation scheme, an excursion of  $X$  below level 0 starts whenever the SNLP  $X$  is observed below level 0 and ends whenever the SNLP  $X$  is subsequently observed above level 0. Recall  $\{T_n\}_{n \in \mathbb{N}}$  is the sequence of observation times which are the arrival epochs of an independent Poisson process with rate  $\lambda > 0$ . For  $n \in \mathbb{N}$ , we denote  $\xi_n$  the starting time of the  $n$ -th excursion below level 0, i.e.,

$$\begin{aligned}\xi_1 &= \inf \{T_i : X_{T_i} < 0\} \\ \xi_n &= \inf \{T_i : X_{T_i} < 0, X_{T_{i-1}} \geq 0 \text{ and } T_i > \xi_{n-1}\}, \text{ for } n \geq 2.\end{aligned}$$

Let  $\vartheta$  be the Markov shift operator acting as  $X_t \circ \vartheta_s = X_{t+s}$  for  $s, t \geq 0$ . The ending time of  $n$ -th excursion below level 0 is then given by  $T_0^{+, \lambda} \circ \vartheta_{\xi_n}$ . The excursion is deemed to have caused ruin if the length of the excursion exceeds an independent excursion-specific exponential time with mean  $1/q$ . Thus, the Parisian ruin time under the Poissonian observation is defined as

$$T^{\lambda, q} = \inf \left\{ \xi_n + e_q^{(n)} : T_0^{+, \lambda} \circ \vartheta_{\xi_n} - \xi_n > e_q^{(n)} \right\},$$

where  $e_q^{(n)}$  is an independent exponential clock with mean  $1/q$  for the  $n$ -th excursion below level 0, and each  $\xi_n + e_q^{(n)}$  is an  $\mathbb{F}$ -stopping time.

Our objective is to derive an explicit expression for the following Gerber-Shiu type density at the Parisian time:

$$\mathbb{E}_x \left[ e^{-sT^{\lambda, q}} 1_{\{X_{T^{\lambda, q}} \in dy, T^{\lambda, q} < T_{-a}^{-, \lambda} \wedge T_b^{+, \lambda}\}} \right], \quad x \in [-a, b], y \in \mathbb{R}. \quad (4.1)$$

For ease of notation, we define two auxiliary functions, for  $x \in [-a, b]$  and  $y \in \mathbb{R}$ ,

$$H_{a,b}^{(s, q, \lambda)}(x, y) = \int_0^a v^{(s, \lambda)}(x, -w; b) A^{(s+q, \lambda)}(a-w, y-a) dw, \quad (4.2)$$

and

$$Z_{a,b}^{(s, q, \lambda)}(x) = \int_0^a v^{(s, \lambda)}(x, -w; b) W^{(s+q, \lambda)}(a-w) dw, \quad (4.3)$$

where

$$v^{(s, \lambda)}(x, w; b) = \begin{cases} \delta_x(w), & x \in [-a, 0), \\ \lambda u_{d:d}^{(s, \lambda)}(x, w; b), & x \in [0, b], \end{cases}$$

and  $\delta_x(\cdot)$  is the Dirac delta function centered at  $x$ .

The proof of the following theorem is postponed to the Appendix.

**Theorem 4.1** *For  $x \in [-a, b]$  and  $y \in \mathbb{R}$ , the Gerber-Shiu density (4.1) satisfies*

$$\begin{aligned} & \frac{1}{qdy} \mathbb{E}_x \left[ e^{-sT^{\lambda, q}} 1_{\{X_{T^{\lambda, q}} \in dy, T^{\lambda, q} < T_{-a}^{-, \lambda} \wedge T_b^{+, \lambda}\}} \right] \\ &= \frac{\theta^{(s+q+\lambda)}(y) + A^{(s+q)}(a, -a-y) - \lambda \int_{-a}^b \theta^{(s+q+\lambda)}(z) H_{a,b}^{(s, q, \lambda)}(z, -y) dz}{W^{(s+q, \lambda)}(a) - \lambda \int_{-a}^b \theta^{(s+q+\lambda)}(z) Z_{a,b}^{(s, q, \lambda)}(z) dz} Z_{a,b}^{(s, q, \lambda)}(x) - H_{a,b}^{(s, q, \lambda)}(x, -y). \end{aligned}$$

On a side note, one expects the Gerber-Shiu density in Theorem 4.1 to reduce to the Gerber-Shiu density in Theorem 1.2 of Baurdoux et al. [7] (or equivalently Eq. 3.29) when the observation intensity rate  $\lambda$  goes to  $\infty$ . This result can be proven (see Appendix) when the SNLP  $X$  has bounded variation paths, namely for  $x \in [-a, b]$  and  $y \in [-a, 0]$ ,

$$\lim_{\lambda \rightarrow \infty} \frac{1}{qdy} \mathbb{E}_x \left[ e^{-sT^{\lambda,q}} 1_{\{X_{T^{\lambda,q}} \in dy, T^{\lambda,q} < T_{-a}^{-,\lambda} \wedge T_b^{+,\lambda}\}} \right] = \frac{A^{(s,q)}(x, a)}{A^{(s,q)}(b, a)} A^{(s,q)}(b, -y) - A^{(s,q)}(x, -y). \quad (4.4)$$

Unfortunately, there are non-trivial difficulties that arise in the case when  $X$  has **unbounded variation** paths which are related to the evaluation of the integrals  $\int_{-a}^b \theta^{(s+q+\lambda)}(z) H_{a,b}^{(s,q,\lambda)}(z, -y) dz$  and  $\int_{-a}^b \theta^{(s+q+\lambda)}(z) Z_{a,b}^{(s,q,\lambda)}(z) dz$  (unless  $a = \infty$ ). To complete this step, a non-trivial study of the two functions  $H_{a,b}^{(s,q,\lambda)}(x, y)$  and  $Z_{a,b}^{(s,q,\lambda)}(x)$  is necessary, as task which is left for future work.

We complement the above analysis with a numerical study of the Parisian ruin with Poissonian observations. More specifically, we consider the impact of the Poisson observation rate  $\lambda$  on some ruin-related quantities. Consider the classical compound Poisson model  $\{X_t\}_{t \geq 0}$  with

$$X_t = x + ct - \sum_{i=1}^{N_t} Y_i, \quad t \geq 0,$$

where  $c > 0$ ,  $\{Y_i\}_{i \in \mathbb{N}}$  is an iid sequence of exponential rv's with mean  $1/\alpha$ , and  $\{N_t\}_{t \geq 0}$  is a Poisson process with rate  $\eta > 0$ . Under this model, it is well known that

$$\psi(s) = cs - \eta + \frac{\eta\alpha}{s + \alpha}, \quad s \geq 0,$$

and

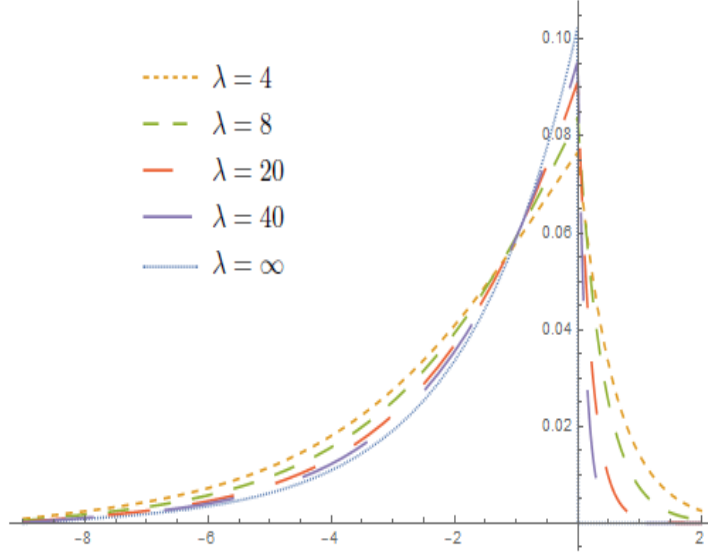
$$W^{(q)}(x) = \frac{e^{\Phi_q x}}{\psi'(e^{\Phi_q x})} + \frac{e^{-\zeta_q x}}{\psi'(e^{-\zeta_q x})}, \quad x \geq 0,$$

where the constants  $\Phi_q$  and  $-\zeta_q$  (with  $-\zeta_q < \Phi_q$ ) are roots to the equation  $\psi(s) = q$ , namely

$$\Phi_q = \frac{1}{2c} \left[ \sqrt{(c\alpha - q - \eta)^2 + 4cq\alpha} - (c\alpha - q - \eta) \right],$$

$$\zeta_q = \frac{1}{2c} \left[ \sqrt{(c\alpha - q - \eta)^2 + 4cq\alpha} + (c\alpha - q - \eta) \right].$$

For the subsequent numerical example, we choose the distributional parameters to be  $\alpha = 1$  and  $\eta = 5$ . Also, the Parisian ruin clock is assumed to make observations at rate  $q = 3$ . We focus on the computation of the Gerber-Shiu density (4.1) with  $s = 0$ ,  $x = 1$ ,  $a = 9$  and  $b = 2$ . In Figure 1, we plot the density of the deficit at ruin (i.e.  $X_{T^{\lambda,q}}$ ) for  $y \in (-9, 2)$  with different values of the observation rate  $\lambda$ . The cases  $\lambda = 4, 8, 20, 40$  are plotted using Theorem 4.1 whereas the case  $\lambda = \infty$  is plotted using Eq. 4.4 for  $y \leq 0$  and it remains at 0 for  $y > 0$ . It can be seen that as the observation rate  $\lambda$  increases, the Gerber-Shiu densities with Poissonian observations converge to that under a continuous observation scheme.

Figure 1: Density of  $X_{T^{\lambda,q}}$  with different observation rates

## 5 Appendix

### 5.1 Proof of Theorem 3.1

In the rest of the paper, we denote  $e_q$  and  $e'_\lambda$  as two exponential random variables with mean  $1/q$  and  $1/\lambda$ , respectively. We assume  $e_q$ ,  $e'_\lambda$ , and the underlying process  $X$  are mutually independent.

#### 5.1.1 Proof of Eq. (3.13)

For  $x, y \leq a$ , let

$$R_+^{(q,\lambda)}(x, dy; a) = \int_0^\infty e^{-qt} \mathbb{P}_x(X_t \in dy, t < T_0^{+, \lambda} \wedge \tau_a^+) dt = \frac{1}{q} \mathbb{P}_x(X_{e_q} \in dy, e_q < T_0^{+, \lambda} \wedge \tau_a^+).$$

We consider separately the cases where  $x < 0$  and  $x \in [0, a]$ .

For  $x < 0$ , conditioning on whether  $e_q$  or  $\tau_0^+$  happens first, one deduces that

$$\begin{aligned} R_+^{(q,\lambda)}(x, dy; a) &= \frac{1}{q} \mathbb{P}_x(X_{e_q} \in dy, e_q < \tau_0^+) + \mathbb{P}_x(\tau_0^+ < e_q) R_+^{(q,\lambda)}(0, dy; a) \\ &= r_+^{(q)}(x, y) dy + e^{\Phi_q x} R_+^{(q,\lambda)}(0, dy; a), \end{aligned} \quad (5.1)$$

where the last line holds due to (2.15) and (2.10).

For  $x \in [0, a]$ , comparing  $e_q$ ,  $\tau_a^+$ , and the first Poissonian observation time  $e'_\lambda$ , it follows that

$$\begin{aligned} R_+^{(q,\lambda)}(x, dy; a) &= \frac{1}{q} \mathbb{P}_x(X_{e_q} \in dy, e_q < e'_\lambda \wedge \tau_a^+) + \int_{-\infty}^0 \mathbb{P}_x(X_{e'_\lambda} \in dz, e'_\lambda < e_q \wedge \tau_a^+) R_+^{(q,\lambda)}(z, dy; a) \\ &= r_+^{(q+\lambda)}(x-a, y-a) dy + \int_{-\infty}^0 \lambda r_+^{(q+\lambda)}(x-a, z-a) dz R_+^{(q,\lambda)}(z, dy; a). \end{aligned} \quad (5.2)$$

Substituting (5.1) with  $x = z$  into (5.2) and using (2.19) yield

$$\begin{aligned}
 & R_+^{(q,\lambda)}(x, dy; a) \\
 &= r_+^{(q+\lambda)}(x-a, y-a) dy + \lambda \int_{-\infty}^0 r_+^{(q+\lambda)}(x-a, z-a) r_+^{(q)}(z, y) dz dy \\
 &+ \lambda \int_{-\infty}^0 r_+^{(q+\lambda)}(x-a, z-a) e^{\Phi_q z} dz R_+^{(q,\lambda)}(0, dy; a) \\
 &= r_+^{(q+\lambda)}(x-a, y-a) dy + \lambda \int_{-\infty}^0 r_+^{(q+\lambda)}(x-a, z-a) \left( e^{\Phi_q z} W^{(q)}(-y) - W^{(q)}(z-y) \right) dz dy \\
 &+ \lambda \int_{-\infty}^0 r_+^{(q+\lambda)}(x-a, z-a) e^{\Phi_q z} dz R_+^{(q,\lambda)}(0, dy; a). \tag{5.3}
 \end{aligned}$$

Letting  $x = 0$  in (5.3), we solve for  $R_+^{(q,\lambda)}(0, dy; a)$  and obtain

$$\begin{aligned}
 & R_+^{(q,\lambda)}(0, dy; a) / dy \\
 &= \frac{r_+^{(q+\lambda)}(-a, y-a) + W^{(q)}(-y) - \lambda \int_{-\infty}^0 r_+^{(q+\lambda)}(-a, z-a) W^{(q)}(z-y) dz}{1 - \lambda \int_{-\infty}^0 e^{\Phi_q z} r_+^{(q+\lambda)}(-a, z-a) dz} - W^{(q)}(-y). \tag{5.4}
 \end{aligned}$$

In what follows, we focus on specifying the two types of integrals in (5.3) and (5.4). On one hand, for  $x \leq a$ ,

$$\begin{aligned}
 & \lambda \int_{-\infty}^0 e^{\Phi_q z} r_+^{(q+\lambda)}(x-a, z-a) dz \\
 &= \int_{-\infty}^a e^{\Phi_q z} \mathbb{P}_x \left( X_{e'_\lambda} \in dz, e'_\lambda < e_q \wedge \tau_a^+ \right) dz - \lambda \int_0^a e^{\Phi_q z} r_+^{(q+\lambda)}(x-a, z-a) dz \\
 &= \int_{-\infty}^0 e^{\Phi_q(z+a)} \mathbb{P}_{x-a} \left( X_{e'_\lambda} \in dz, e'_\lambda < e_q \wedge \tau_0^+ \right) dz - \lambda \int_0^a e^{\Phi_q z} r_+^{(q+\lambda)}(x-a, z-a) dz \\
 &= e^{\Phi_q a} \mathbb{E}_{x-a} \left[ e^{-q e'_\lambda + \Phi_q X_{e'_\lambda}} 1_{\{e'_\lambda < \tau_0^+\}} \right] - \lambda \int_0^a e^{\Phi_q z} r_+^{(q+\lambda)}(x-a, z-a) dz.
 \end{aligned}$$

Furthermore, using Eq. (30) of Albrecher et al. [4], (2.19) and (2.3), one finds that

$$\begin{aligned}
 & \lambda \int_{-\infty}^0 e^{\Phi_q z} r_+^{(q+\lambda)}(x-a, z-a) dz \\
 &= e^{\Phi_q a} \left( e^{\Phi_q(x-a)} - e^{\Phi_{q+\lambda}(x-a)} \right) - e^{\Phi_{q+\lambda}(x-a)} \lambda \int_0^a e^{\Phi_q(a-z)} W^{(q+\lambda)}(z) dz + \lambda \int_0^x e^{\Phi_q(x-z)} W^{(q+\lambda)}(z) dz \\
 &= Z^{(q+\lambda)}(x, \Phi_q) - e^{\Phi_{q+\lambda}(x-a)} Z^{(q+\lambda)}(a, \Phi_q). \tag{5.5}
 \end{aligned}$$

On the other hand, for  $x \leq a$  and  $y < 0$ ,

$$\begin{aligned}
 & \lambda \int_{-\infty}^0 r_+^{(q+\lambda)}(x-a, z-a) W^{(q)}(z-y) dz \\
 &= \lambda \int_{-\infty}^0 \left[ e^{\Phi_{q+\lambda}(x-a)} W^{(q+\lambda)}(a-z) - W^{(q+\lambda)}(x-z) \right] W^{(q)}(z-y) dz \\
 &= e^{\Phi_{q+\lambda}(x-a)} \lambda \int_0^\infty W^{(q+\lambda)}(a+z) W^{(q)}(-y-z) dz - \lambda \int_0^\infty W^{(q+\lambda)}(x+z) W^{(q)}(-y-z) dz \\
 &= e^{\Phi_{q+\lambda}(x-a)} \lambda \int_a^{a-y} W^{(q+\lambda)}(z) W^{(q)}(a-y-z) dz - \lambda \int_x^{x-y} W^{(q+\lambda)}(z) W^{(q)}(x-y-z) dz \\
 &= e^{\Phi_{q+\lambda}(x-a)} \lambda \left[ \int_0^{a-y} W^{(q)}(a-y-z) W^{(q+\lambda)}(z) dz - \int_0^a W^{(q)}(a-y-z) W^{(q+\lambda)}(z) dz \right] \\
 &\quad - \lambda \left[ \int_0^{x-y} W^{(q)}(x-y-z) W^{(q+\lambda)}(z) dz - \int_0^x W^{(q)}(x-y-z) W^{(q+\lambda)}(z) dz \right] \\
 &= e^{\Phi_{q+\lambda}(x-a)} \left[ W^{(q+\lambda)}(a-y) - A^{(q,\lambda)}(-y, a) \right] - \left[ W^{(q+\lambda)}(x-y) - A^{(q,\lambda)}(-y, x) \right], \tag{5.6}
 \end{aligned}$$

where the last step is due to (2.6) and (3.9). Note that it is easily seen from (3.12) that the equality (5.6) also holds for  $y \geq 0$ .

Substituting (5.5) and (5.6) with  $x = 0$  into (5.4), and using (3.11), it is relatively easy to show that

$$R_+^{(q,\lambda)}(0, dy; a) / dy = \frac{A^{(q,\lambda)}(-y, a)}{Z^{(q+\lambda)}(a, \Phi_q)} - W^{(q)}(-y). \tag{5.7}$$

Lastly, substituting (2.19) and (5.7) into (5.1) yields, for  $x < 0$ ,

$$R_+^{(q,\lambda)}(x, dy; a) / dy = \frac{A^{(q,\lambda)}(-y, a)}{Z^{(q+\lambda)}(a, \Phi_q)} e^{\Phi_q x} - W^{(q)}(x-y).$$

Also, substituting (2.19), (5.5), (5.6), and (5.7) into (5.3) yields, for  $x \in [0, a]$ ,

$$R_+^{(q,\lambda)}(x, dy; a) / dy = \frac{A^{(q,\lambda)}(-y, a)}{Z^{(q+\lambda)}(a, \Phi_q)} Z^{(q+\lambda)}(x, \Phi_q) - A^{(q,\lambda)}(-y, x).$$

We complete the proof by unifying the above two expressions for  $x \leq a$ .

### 5.1.2 Proof of Eq. (3.14)

For  $x, y \geq -a$ , let

$$R_-^{(q,\lambda)}(x, dy; -a) = \int_0^\infty e^{-qt} \mathbb{P}_x \left( X_t \in dy, t < T_0^{-,\lambda} \wedge \tau_{-a}^- \right) dt = \frac{1}{q} \mathbb{P}_x \left( X_{e_q} \in dy, e_q < T_0^{-,\lambda} \wedge \tau_{-a}^- \right).$$

We consider separately the cases where  $y \in [-a, 0)$  and  $y \geq 0$ .

For  $y \in [-a, 0)$ , we shall have that  $\tau_0^- < e_q \wedge T_0^{-,\lambda}$  almost surely. Subsequently, at level  $X_{\tau_0^-}$ , we



know that the random time  $\tau_0^+ \wedge e_q$  should occur prior to the next observation time  $e'_\lambda$ . Therefore,

$$\begin{aligned} R_-^{(q,\lambda)}(x, dy; -a) &= \frac{1}{q} \int_{[-a,0]} \mathbb{P}_x \left( X_{\tau_0^-} \in dz, \tau_0^- < e_q \right) \mathbb{P}_z \left( X_{e_q} \in dy, e_q < T_0^{-,\lambda} \wedge \tau_{-a}^- \right) \\ &= \int_{[-a,0]} \mathbb{P}_x \left( X_{\tau_0^-} \in dz, \tau_0^- < e_q \right) \mathbb{P}_z \left( \tau_0^+ < e_q \wedge e'_\lambda \wedge \tau_{-a}^- \right) R_-^{(q,\lambda)}(0, dy; -a) \\ &\quad + \frac{1}{q} \int_{[-a,0]} \mathbb{P}_x \left( X_{\tau_0^-} \in dz, \tau_0^- < e_q \right) \mathbb{P}_z \left( X_{e_q} \in dy, e_q < \tau_0^+ \wedge e'_\lambda \wedge \tau_{-a}^- \right). \end{aligned}$$

Subsequently, using (2.8) and (2.17) leads to

$$\begin{aligned} R_-^{(q,\lambda)}(x, dy; -a) &= \int_{[-a,0]} \mathbb{P}_x \left( X_{\tau_0^-} \in dz, \tau_0^- < e_q \right) \left[ \frac{W^{(q+\lambda)}(z+a)}{W^{(q+\lambda)}(a)} R_-^{(q,\lambda)}(0, dy; -a) + u^{(q+\lambda)}(z+a, y+a; a) dy \right] \\ &= \mathbb{E}_x \left[ e^{-q\tau_0^-} \frac{W^{(q+\lambda)}(X_{\tau_0^-} + a)}{W^{(q+\lambda)}(a)} 1_{\{\tau_0^- < \infty\}} \right] R_-^{(q,\lambda)}(0, dy; -a) \\ &\quad + \mathbb{E}_x \left[ e^{-q\tau_0^-} u^{(q+\lambda)}(X_{\tau_0^-} + a, y+a; a) 1_{\{\tau_0^- < \infty\}} \right] dy, \end{aligned} \quad (5.8)$$

where the last line holds due to the fact that  $W^{(q+\lambda)}(x) = 0$  for any  $x < 0$ .

For  $y \geq 0$ , conditioning on whether  $\tau_0^-$  occurs before  $e_q$  (or not) leads to

$$\begin{aligned} R_-^{(q,\lambda)}(x, dy; -a) &= \frac{1}{q} \int_{[-a,0]} \mathbb{P}_x \left( X_{\tau_0^-} \in dz, \tau_0^- < e_q \right) \mathbb{P}_z \left( X_{e_q} \in dy, e_q < T_0^{-,\lambda} \wedge \tau_{-a}^- \right) \\ &\quad + \frac{1}{q} \mathbb{P}_x \left( X_{e_q} \in dy, e_q < \tau_0^- \right). \end{aligned} \quad (5.9)$$

Since  $z \leq 0$  and  $y \geq 0$ , by (2.8), we have

$$\begin{aligned} \frac{1}{q} \mathbb{P}_z \left( X_{e_q} \in dy, e_q < T_0^{-,\lambda} \wedge \tau_{-a}^- \right) &= \mathbb{P}_z \left( \tau_0^+ < e_q \wedge e'_\lambda \wedge \tau_{-a}^- \right) R_-^{(q,\lambda)}(0, dy; -a) \\ &= \frac{W^{(q+\lambda)}(z+a)}{W^{(q+\lambda)}(a)} R_-^{(q,\lambda)}(0, dy; -a). \end{aligned} \quad (5.10)$$

Substituting (5.10) into (5.9) and using (2.16) give

$$R_-^{(q,\lambda)}(x, dy; -a) = \mathbb{E}_x \left[ e^{-q\tau_0^-} \frac{W^{(q+\lambda)}(X_{\tau_0^-} + a)}{W^{(q+\lambda)}(a)} 1_{\{\tau_0^- < \infty\}} \right] R_-^{(q,\lambda)}(0, dy; -a) + r_-^{(q)}(x, y) dy. \quad (5.11)$$

We further note that (5.8) and (5.11) can be expressed in a unified manner as follows: for  $x, y \geq -a$ ,

$$\begin{aligned} R_-^{(q,\lambda)}(x, dy; -a) / dy &= \mathbb{E}_x \left[ e^{-q\tau_0^-} \frac{W^{(q+\lambda)}(X_{\tau_0^-} + a)}{W^{(q+\lambda)}(a)} 1_{\{\tau_0^- < \infty\}} \right] R_-^{(q,\lambda)}(0, dy; -a) / dy \\ &\quad + \mathbb{E}_x \left[ e^{-q\tau_0^-} u^{(q+\lambda)}(X_{\tau_0^-} + a, y+a; a) 1_{\{\tau_0^- < \infty\}} \right] 1_{\{-a \leq y < 0\}} + r_-^{(q)}(x, y) 1_{\{y \geq 0\}}. \end{aligned} \quad (5.12)$$

To solve for  $R_-^{(q,\lambda)}(0, dy; -a)$ , we condition on whether  $e_q$  arrives prior to the next observation time  $e'_\lambda$ . Using (2.14), we have

$$\begin{aligned} R_-^{(q,\lambda)}(0, dy; -a) &= \frac{1}{q} \mathbb{P}(X_{e_q} \in dy, e_q < e'_\lambda \wedge \tau_{-a}^-) + \int_0^\infty \mathbb{P}(X_{e'_\lambda} \in dz, e'_\lambda < e_q \wedge \tau_{-a}^-) R_-^{(q,\lambda)}(z, dy; -a) \\ &= r_-^{(q+\lambda)}(a, y+a) dy + \lambda \int_0^\infty r_-^{(q+\lambda)}(a, z+a) R_-^{(q,\lambda)}(z, dy; -a) dz. \end{aligned} \quad (5.13)$$

Substituting (5.12) with  $x = z$  into (5.13), we then solve for  $R_-^{(q,\lambda)}(0, dy; -a)$  and obtain

$$\begin{aligned} &R_-^{(q,\lambda)}(0, dy; -a)/dy \\ &= \begin{cases} \frac{r_-^{(q+\lambda)}(a, y+a) + \lambda \int_0^\infty r_-^{(q+\lambda)}(a, z+a) \mathbb{E}_z \left[ e^{-q\tau_0^-} u^{(q+\lambda)}(X_{\tau_0^-} + a, y+a; a) 1_{\{\tau_0^- < \infty\}} \right] dz}{1 - \lambda \int_0^\infty r_-^{(q+\lambda)}(a, z+a) \mathbb{E}_z \left[ e^{-q\tau_0^-} \frac{W^{(q+\lambda)}(X_{\tau_0^-} + a)}{W^{(q+\lambda)}(a)} 1_{\{\tau_0^- < \infty\}} \right] dz}, & -a \leq y < 0, \\ \frac{r_-^{(q+\lambda)}(a, y+a) + \lambda \int_0^\infty r_-^{(q+\lambda)}(a, z+a) r_-^{(q)}(z, y) dz}{1 - \lambda \int_0^\infty r_-^{(q+\lambda)}(a, z+a) \mathbb{E}_z \left[ e^{-q\tau_0^-} \frac{W^{(q+\lambda)}(X_{\tau_0^-} + a)}{W^{(q+\lambda)}(a)} 1_{\{\tau_0^- < \infty\}} \right] dz}, & y \geq 0. \end{cases} \end{aligned} \quad (5.14)$$

With the help of (2.20) and (2.21), (5.14) can further simplified to

$$\begin{aligned} &\frac{1}{dy} R_-^{(q,\lambda)}(0, dy; -a) + W^{(q+\lambda)}(-y) \\ &= \begin{cases} \frac{e^{-\Phi q + \lambda y} W^{(q+\lambda)}(a) - \lambda W^{(q+\lambda)}(a) \int_0^\infty e^{-\Phi q + \lambda z} \mathbb{E}_z \left[ e^{-q\tau_0^-} W^{(q+\lambda)}(X_{\tau_0^-} - y) 1_{\{\tau_0^- < \infty\}} \right] dz}{e^{\Phi q + \lambda a} - \lambda \int_0^\infty e^{-\Phi q + \lambda z} \mathbb{E}_z \left[ e^{-q\tau_0^-} W^{(q+\lambda)}(X_{\tau_0^-} + a) 1_{\{\tau_0^- < \infty\}} \right] dz}, & -a \leq y < 0, \\ \frac{e^{-\Phi q + \lambda y} W^{(q+\lambda)}(a) + \lambda W^{(q+\lambda)}(a) \int_0^\infty e^{-\Phi q + \lambda z} r_-^{(q)}(z, y) dz}{e^{\Phi q + \lambda a} - \lambda \int_0^\infty e^{-\Phi q + \lambda z} \mathbb{E}_z \left[ e^{-q\tau_0^-} W^{(q+\lambda)}(X_{\tau_0^-} + a) 1_{\{\tau_0^- < \infty\}} \right] dz}, & y \geq 0. \end{cases} \end{aligned} \quad (5.15)$$

Next, we focus on simplifying (5.15). By the spatial homogeneity of  $X$  and the dominated convergence theorem, for any  $z > 0$  and  $y < 0$ ,

$$\begin{aligned} \mathbb{E}_z \left[ e^{-q\tau_0^-} W^{(q+\lambda)}(X_{\tau_0^-} - y) 1_{\{\tau_0^- < \infty\}} \right] &= \mathbb{E}_{z-y} \left[ e^{-q\tau_{-y}^-} W^{(q+\lambda)}(X_{\tau_{-y}^-}) 1_{\{\tau_{-y}^- < \infty\}} \right] \\ &= \lim_{b \rightarrow \infty} \mathbb{E}_{z-y} \left[ e^{-q\tau_{-y}^-} W^{(q+\lambda)}(X_{\tau_{-y}^-}) 1_{\{\tau_{-y}^- < \tau_b^+\}} \right]. \end{aligned} \quad (5.16)$$

Thanks to Lemma 2.2 of Loeffen et al. [24] or Theorem 2 of Loeffen [22], we deduce that

$$\begin{aligned}
& \mathbb{E}_{z-y} \left[ e^{-q\tau_{-y}^-} W^{(q+\lambda)} \left( X_{\tau_{-y}^-} \right) 1_{\{\tau_{-y}^- < \tau_b^+\}} \right] \\
&= W^{(q+\lambda)}(z-y) - \lambda \int_{-y}^{z-y} W^{(q)}(z-y-x) W^{(q+\lambda)}(x) dx \\
&\quad - \frac{W^{(q)}(z)}{W^{(q)}(b+y)} \left( W^{(q+\lambda)}(b) - \lambda \int_{-y}^b W^{(q)}(b-x) W^{(q+\lambda)}(x) dx \right) \\
&= W^{(q+\lambda)}(z-y) - \lambda \int_0^{z-y} W^{(q)}(z-y-x) W^{(q+\lambda)}(x) dx + \lambda \int_0^{-y} W^{(q)}(z-y-x) W^{(q+\lambda)}(x) dx \\
&\quad - \frac{W^{(q)}(z)}{W^{(q)}(b+y)} \left( W^{(q+\lambda)}(b) - \lambda \int_0^b W^{(q)}(b-x) W^{(q+\lambda)}(x) dx + \lambda \int_0^{-y} W^{(q)}(b-x) W^{(q+\lambda)}(x) dx \right) \\
&= W^{(q)}(z-y) + \lambda \int_0^{-y} W^{(q)}(z-y-x) W^{(q+\lambda)}(x) dx \\
&\quad - \frac{W^{(q)}(z)}{W^{(q)}(b+y)} \left( W^{(q)}(b) + \lambda \int_0^{-y} W^{(q)}(b-x) W^{(q+\lambda)}(x) dx \right), \tag{5.17}
\end{aligned}$$

where the last step is due to (2.6). We know from (2.12) that, for fixed  $y < 0$  and all  $b$  large enough,

$$\frac{W^{(q)}(b-x)}{W^{(q)}(b+y)} \leq \frac{W^{(q)}(b)}{W^{(q)}(b+y)} \leq e^{-\Phi_q y} + 1, \quad \text{for any } x \in [0-y].$$

Taking the limit  $b \rightarrow \infty$  in (5.17) and using (2.12), (2.3) and the dominated convergence theorem, Eq. (5.16) becomes

$$\mathbb{E}_z \left[ e^{-q\tau_0^-} W^{(q+\lambda)} \left( X_{\tau_0^-} - y \right) 1_{\{\tau_0^- < \infty\}} \right] = A^{(q,\lambda)}(z, -y) - W^{(q)}(z) Z^{(q+\lambda)}(-y, \Phi_q), \tag{5.18}$$

for any  $z > 0$  and  $y < 0$ . Substituting (5.18) into (5.15) yields

$$\begin{aligned}
& \frac{1}{dy} R_-^{(q,\lambda)}(0, dy; -a) + W^{(q+\lambda)}(-y) \\
&= \begin{cases} \frac{e^{-\Phi_q + \lambda y} W^{(q+\lambda)}(a) - \lambda W^{(q+\lambda)}(a) \int_0^\infty e^{-\Phi_q + \lambda z} [A^{(q,\lambda)}(z, -y) - W^{(q)}(z) Z^{(q+\lambda)}(-y, \Phi_q)] dz}{e^{\Phi_q + \lambda a} - \lambda \int_0^\infty e^{-\Phi_q + \lambda z} [A^{(q,\lambda)}(z, a) - W^{(q)}(z) Z^{(q+\lambda)}(a, \Phi_q)] dz}, & -a \leq y < 0, \\ \frac{e^{-\Phi_q + \lambda y} W^{(q+\lambda)}(a) + \lambda W^{(q+\lambda)}(a) \int_0^\infty e^{-\Phi_q + \lambda z} r_-^{(q)}(z, y) dz}{e^{\Phi_q + \lambda a} - \lambda \int_0^\infty e^{-\Phi_q + \lambda z} [A^{(q,\lambda)}(z, a) - W^{(q)}(z) Z^{(q+\lambda)}(a, \Phi_q)] dz}, & y \geq 0. \end{cases} \tag{5.19}
\end{aligned}$$

Note that by (3.9), (2.2), (2.4) and (2.20), we have

$$\int_0^\infty e^{-\Phi_q + \lambda z} A^{(q,\lambda)}(z, -y) dz = \frac{e^{-\Phi_q + \lambda y}}{\lambda}, \quad y < 0,$$

and

$$\int_0^\infty e^{-\Phi_q + \lambda z} r_-^{(q)}(z, y) dz = \frac{e^{-\Phi_q y} - e^{-\Phi_q + \lambda y}}{\lambda}, \quad y \geq 0.$$

Thus, (5.19) is further reduced to

$$R_-^{(q,\lambda)}(0, dy; -a) / dy = \frac{W^{(q+\lambda)}(a)}{Z^{(q+\lambda)}(a, \Phi_q)} Z^{(q+\lambda)}(-y, \Phi_q) - W^{(q+\lambda)}(-y). \tag{5.20}$$

Finally, substituting (5.20) into (5.12) and using (2.20), (2.21) and (5.18) yields (3.14).

### 5.1.3 Proof of Eqs. (3.15) and (3.16)

By (2.3) and (3.10), we have

$$\begin{aligned} \frac{A^{(q,\lambda)}(x, a)}{Z^{(q+\lambda)}(a, \Phi_q)} &= \frac{W^{(q+\lambda)}(x+a) - \lambda \int_0^x W^{(q)}(z) W^{(q+\lambda)}(x+a-z) dz}{Z^{(q+\lambda)}(a, \Phi_q)} \\ &= \frac{\frac{W^{(q+\lambda)}(x+a)}{W^{(q+\lambda)}(a)} - \lambda \int_0^x W^{(q)}(z) \frac{W^{(q+\lambda)}(x+a-z)}{W^{(q+\lambda)}(a)} dz}{\frac{Z^{(q+\lambda)}(a, \Phi_q)}{W^{(q+\lambda)}(a)}}. \end{aligned}$$

We know from (2.12) that, for fixed  $x \in \mathbb{R}$  and all  $a$  large enough,

$$\frac{W^{(q+\lambda)}(x+a-z)}{W^{(q+\lambda)}(a)} \leq \frac{W^{(q+\lambda)}(x+a)}{W^{(q+\lambda)}(a)} \leq e^{\Phi_{q+\lambda}x} + 1, \quad \text{for any } z \in [0, x].$$

By (2.12), (2.13), (3.3), and the dominated convergence theorem, it follows that

$$\lim_{a \rightarrow \infty} \frac{A^{(q,\lambda)}(x, a)}{Z^{(q+\lambda)}(a, \Phi_q)} = \frac{e^{\Phi_{q+\lambda}x} - \lambda \int_0^x W^{(q)}(z) e^{\Phi_{q+\lambda}(x-z)} dz}{\frac{\lambda}{\Phi_{q+\lambda} - \Phi_q}} = W^{(q,\lambda)}(x).$$

Therefore, it is straightforward to see from (3.13) and (3.14), that

$$\begin{aligned} r_+^{(q,\lambda)}(x, y) &= \lim_{a \rightarrow \infty} r_+^{(q,\lambda)}(x, y; a) \\ &= \lim_{a \rightarrow \infty} \frac{A^{(q,\lambda)}(-y, a)}{Z^{(q+\lambda)}(a, \Phi_q)} Z^{(q+\lambda)}(x, \Phi_q) - A^{(q,\lambda)}(-y, x) \\ &= W^{(q,\lambda)}(-y) Z^{(q+\lambda)}(x, \Phi_q) - A^{(q,\lambda)}(-y, x), \end{aligned}$$

and

$$\begin{aligned} r_-^{(q,\lambda)}(x, y) &= \lim_{a \rightarrow \infty} r_-^{(q,\lambda)}(x, y; -a) \\ &= \lim_{a \rightarrow \infty} \frac{A^{(q,\lambda)}(x, a)}{Z^{(q+\lambda)}(a, \Phi_q)} Z^{(q+\lambda)}(-y, \Phi_q) - A^{(q,\lambda)}(x, -y) \\ &= W^{(q,\lambda)}(x) Z^{(q+\lambda)}(-y, \Phi_q) - A^{(q,\lambda)}(x, -y). \end{aligned}$$

### 5.1.4 Proof of Eq. (3.17)

For  $x, y \leq a$ , due to the fact that  $\{t < \tau_a^+ \wedge T_0^{-,\lambda}\} = \{t < T_0^{-,\lambda}\} \setminus \{\tau_a^+ \leq t < T_0^{-,\lambda}\}$ , it is immediate from (3.5) that

$$u_{d;c}^{(q,\lambda)}(x, y; a) = r_-^{(q,\lambda)}(x, y) - \frac{W^{(q,\lambda)}(x)}{W^{(q,\lambda)}(a)} r_-^{(q,\lambda)}(a, y). \quad (5.21)$$

Substituting (3.16) into (5.21) yields (3.17).

### 5.1.5 Proof of Eq. (3.18)

For  $x \in [0, a]$  and  $y \geq 0$ , let

$$U_{c;d}^{(q,\lambda)}(x, dy; a) = \int_0^\infty e^{-qt} \mathbb{P}_x \left( X_t \in dy, t < \tau_0^- \wedge T_a^{+,\lambda} \right) dt = \frac{1}{q} \mathbb{P}_x \left( X_{e_q} \in dy, e_q < \tau_0^- \wedge T_a^{+,\lambda} \right).$$

Conditioning on whether or not  $\tau_a^+$  occurs prior to  $e_q$  and using (2.8) lead to

$$\begin{aligned} & U_{c:d}^{(q,\lambda)}(x, dy; a) \\ &= \frac{1}{q} \left\{ \mathbb{P}_x(X_{e_q} \in dy, e_q < \tau_0^- \wedge \tau_a^+) + \mathbb{P}_x(\tau_a^+ < e_q \wedge \tau_0^-) \mathbb{P}_a(X_{e_q} \in dy, e_q < \tau_0^- \wedge T_a^{+, \lambda}) \right\} \\ &= u^{(q)}(x, y; a) dy + \frac{W^{(q)}(x)}{W^{(q)}(a)} U_{c:d}^{(q,\lambda)}(a, dy; a), \end{aligned} \quad (5.22)$$

where we have extended the definition of  $u^{(q)}$  to  $u^{(q)}(x, y; a) = 0$  for  $x \in [0, a]$  and  $y > a$ .

To solve for  $U_{c:d}^{(q,\lambda)}(a, dy; a)$ , we condition on whether  $e_q$  occurs prior to the next observation time  $e'_\lambda$  and arrive at

$$\begin{aligned} U_{c:d}^{(q,\lambda)}(a, dy; a) &= \frac{1}{q} \mathbb{P}_a(X_{e_q} \in dy, e_q < \tau_0^- \wedge e'_\lambda) + \int_0^a \mathbb{P}_a(X_{e'_\lambda} \in dx, e'_\lambda < \tau_0^- \wedge e_q) U_{c:d}^{(q,\lambda)}(x, dy; a) \\ &= r_-^{(q+\lambda)}(a, y) dy + \lambda \int_0^a r_-^{(q+\lambda)}(a, x) U_{c:d}^{(q,\lambda)}(x, dy; a) dx. \end{aligned} \quad (5.23)$$

Substituting (5.22) into (5.23) gives

$$U_{c:d}^{(q,\lambda)}(a, dy; a) = \frac{\lambda \int_0^a r_-^{(q+\lambda)}(a, x) u^{(q)}(x, y; a) dx + r_-^{(q+\lambda)}(a, y)}{1 - \frac{\lambda}{W^{(q)}(a)} \int_0^a r_-^{(q+\lambda)}(a, x) W^{(q)}(x) dx} dy. \quad (5.24)$$

Next we simplify (5.24) by evaluating the two integral terms therein. By using (2.20), (2.3) and (2.6), we have

$$\begin{aligned} & \int_0^a r_-^{(q+\lambda)}(a, x) W^{(q)}(x-y) dx \\ &= W^{(q+\lambda)}(a) \int_0^a e^{-\Phi_{q+\lambda} x} W^{(q)}(x-y) dx - \int_0^a W^{(q+\lambda)}(a-x) W^{(q)}(x-y) dx \\ &= W^{(q+\lambda)}(a) \int_0^{a-y} e^{-\Phi_{q+\lambda}(z+y)} W^{(q)}(z) dz - \int_0^{a-y} W^{(q+\lambda)}(a-y-z) W^{(q)}(z) dx \\ &= \frac{1}{\lambda} W^{(q+\lambda)}(a) e^{-\Phi_{q+\lambda} y} \left[ 1 - e^{-\Phi_{q+\lambda}(a-y)} Z^{(q)}(a-y, \Phi_{q+\lambda}) \right] \\ &\quad - \frac{1}{\lambda} \left[ W^{(q+\lambda)}(a-y) - W^{(q)}(a-y) \right]. \end{aligned} \quad (5.25)$$

As for the other integral, using (2.21), (5.25) and (2.20) followed by simple algebraic manipulations, one finds that

$$\begin{aligned} & \lambda \int_0^a r_-^{(q+\lambda)}(a, x) u^{(q)}(x, y; a) dx \\ &= \lambda \int_0^a r_-^{(q+\lambda)}(a, x) \left[ \frac{W^{(q)}(x) W^{(q)}(a-y)}{W^{(q)}(a)} - W^{(q)}(x-y) \right] dx \\ &= \frac{W^{(q)}(a-y)}{W^{(q)}(a)} \left\{ W^{(q+\lambda)}(a) \left( 1 - e^{-\Phi_{q+\lambda} a} Z^{(q)}(a, \Phi_{q+\lambda}) \right) - \left( W^{(q+\lambda)}(a) - W^{(q)}(a) \right) \right\} \\ &\quad - \left\{ W^{(q+\lambda)}(a) e^{-\Phi_{q+\lambda} y} \left[ 1 - e^{-\Phi_{q+\lambda}(a-y)} Z^{(q)}(a-y, \Phi_{q+\lambda}) \right] - \left[ W^{(q+\lambda)}(a-y) - W^{(q)}(a-y) \right] \right\} \\ &= e^{-\Phi_{q+\lambda} a} W^{(q+\lambda)}(a) \left[ Z^{(q)}(a-y, \Phi_{q+\lambda}) - \frac{Z^{(q)}(a, \Phi_{q+\lambda})}{W^{(q)}(a)} W^{(q)}(a-y) \right] - r_-^{(q+\lambda)}(a, y). \end{aligned} \quad (5.26)$$

With the aid of (3.3), substituting (5.25) with  $y = 0$  and (5.26) into (5.24) yields

$$U_{c:d}^{(q,\lambda)}(a, dy; a) = \left\{ \frac{W^{(q,\lambda)}(a-y)}{W^{(q,\lambda)}(a)} W^{(q)}(a) - W^{(q)}(a-y) \right\} dy. \quad (5.27)$$

Finally, with the help of (2.21), (3.18) follows by substituting (5.27) into (5.22).

### 5.1.6 Proof of Eq. (3.19)

For  $x \leq a$  and  $y \in \mathbb{R}$ , let

$$U_{d:d}^{(q,\lambda)}(x, dy; a) = \int_0^\infty e^{-qt} \mathbb{P}_x \left( X_t \in dy, t < T_0^{-,\lambda} \wedge T_a^{+,\lambda} \right) = \frac{1}{q} \mathbb{P}_x \left( X_{e_q} \in dy, e_q < T_0^{-,\lambda} \wedge T_a^{+,\lambda} \right).$$

Conditioning on whether  $\tau_a^+$  occurs before  $e_q$  leads to

$$\begin{aligned} U_{d:d}^{(q,\lambda)}(x, dy; a) &= \frac{1}{q} \mathbb{P}_x \left( X_{e_q} \in dy, e_q < T_0^{-,\lambda} \wedge \tau_a^+ \right) \\ &\quad + \frac{1}{q} \mathbb{P}_x \left( \tau_a^+ < e_q \wedge T_0^{-,\lambda} \right) \mathbb{P}_a \left( X_{e_q} \in dy, e_q < T_0^{-,\lambda} \wedge T_a^{+,\lambda} \right) \\ &= u_{d:c}^{(q,\lambda)}(x, y; a) dy + \frac{W^{(q,\lambda)}(x)}{W^{(q,\lambda)}(a)} U_{d:d}^{(q,\lambda)}(a, dy; a), \end{aligned} \quad (5.28)$$

where the last step is due to (3.5) and the definition of  $u_{d:c}^{(q,\lambda)}$  was extended to  $u_{d:c}^{(q,\lambda)}(x, y; a) = 0$  for  $y > a$  and  $x \leq a$ .

To solve for  $U_{d:d}^{(q,\lambda)}(a, dy; a)$ , we consider whether  $e_q$  occurs before the next observation time  $e'_\lambda$  and obtain

$$\begin{aligned} U_{d:d}^{(q,\lambda)}(a, dy; a) &= \frac{1}{q} \mathbb{P}_a \left( X_{e_q} \in dy, e_q < e'_\lambda \right) + \frac{1}{q} \int_0^a \mathbb{P}_a \left( X_{e'_\lambda} \in dx, e'_\lambda < e_q \right) q U_{d:d}^{(q,\lambda)}(x, dy; a) \\ &= \theta^{(q+\lambda)}(y-a) dy + \lambda \int_0^a \theta^{(q+\lambda)}(x-a) U_{d:d}^{(q,\lambda)}(x, dy; a) dx. \end{aligned} \quad (5.29)$$

Substituting (5.28) into (5.29) and using (3.17) give

$$\begin{aligned} &U_{d:d}^{(q,\lambda)}(a, dy; a) \\ &= \frac{\theta^{(q+\lambda)}(y-a) + \lambda \int_0^a \theta^{(q+\lambda)}(x-a) u_{d:c}^{(q,\lambda)}(x, y; a) dx}{1 - \frac{\lambda}{W^{(q,\lambda)}(a)} \int_0^a \theta^{(q+\lambda)}(x-a) W^{(q,\lambda)}(x) dx} dy \\ &= \frac{\theta^{(q+\lambda)}(y-a) + A^{(q,\lambda)}(a, -y) - \lambda \int_0^a \theta^{(q+\lambda)}(x-a) A^{(q,\lambda)}(x, -y) dx}{1 - \frac{\lambda}{W^{(q,\lambda)}(a)} \int_0^a \theta^{(q+\lambda)}(x-a) W^{(q,\lambda)}(x) dx} - A^{(q,\lambda)}(a, -y). \end{aligned} \quad (5.30)$$

Next, we simplify the expression of  $U_{d:d}^{(q,\lambda)}(a, dy; a)$  in (5.30). Using (3.9), one obtains

$$\begin{aligned} \int_0^a W^{(q+\lambda)}(a-x) W^{(q)}(x-y) dx &= \int_0^a W^{(q)}(-y+a-x) W^{(q+\lambda)}(x) dx \\ &= \frac{A^{(q,\lambda)}(-y, a) - W^{(q)}(a-y)}{\lambda}, \end{aligned}$$

which implies that

$$\begin{aligned}
 & \lambda \int_0^a W^{(q+\lambda)}(a-x) A^{(q,\lambda)}(x, -y) dx \\
 &= A^{(q,\lambda)}(-y, a) - W^{(q)}(a-y) + \lambda^2 \int_0^a \int_0^{-y} W^{(q)}(x-y-z) W^{(q+\lambda)}(z) W^{(q+\lambda)}(a-x) dz dx \\
 &= \lambda \int_0^{-y} W^{(q+\lambda)}(-y-z) A^{(q,\lambda)}(z, a) dz + A^{(q,\lambda)}(-y, a) - A^{(q,\lambda)}(a, -y). \tag{5.31}
 \end{aligned}$$

By (2.7), it can be seen that

$$\int_0^a W^{(q+\lambda)}(a-x) W^{(q,\lambda)}(x) dx = \frac{\Phi_{q+\lambda} - \Phi_q}{\lambda^2} e^{\Phi_{q+\lambda} a} - \frac{W^{(q,\lambda)}(a)}{\lambda}. \tag{5.32}$$

Invoking (2.18), (5.31), (5.32) and also (3.10) for the term  $A^{(q,\lambda)}(-y, a)$ , one can rewrite (5.30) as

$$\begin{aligned}
 & u_{d;d}^{(q,\lambda)}(a, y; a) \\
 &= \frac{\Phi'_{q+\lambda} [e^{-\Phi_{q+\lambda} y} - \lambda \int_0^a e^{-\Phi_{q+\lambda} x} A^{(q,\lambda)}(x, -y) dx]}{\frac{\Phi_{q+\lambda} - \Phi_q}{\lambda} - \lambda \Phi'_{q+\lambda} \int_0^a e^{-\Phi_{q+\lambda} x} W^{(q,\lambda)}(x) dx} W^{(q,\lambda)}(a) - A^{(q,\lambda)}(a, -y) \\
 &+ \frac{\lambda e^{-\Phi_{q+\lambda} a} \int_0^{-y} [W^{(q+\lambda)}(-y-z) A^{(q,\lambda)}(z, a) - W^{(q+\lambda)}(a-y-z) W^{(q)}(z)] dz}{\frac{\Phi_{q+\lambda} - \Phi_q}{\lambda} - \lambda \Phi'_{q+\lambda} \int_0^a e^{-\Phi_{q+\lambda} x} W^{(q,\lambda)}(x) dx} W^{(q,\lambda)}(a). \tag{5.33}
 \end{aligned}$$

Furthermore, by (3.9), (2.4), (2.7), (3.3) and (2.5), it can be shown that

$$\int_0^\infty e^{-\Phi_{q+\lambda} x} A^{(q,\lambda)}(x, -y) dx = \frac{e^{-\Phi_{q+\lambda} y}}{\lambda},$$

and

$$\int_0^\infty e^{-\Phi_{q+\lambda} x} W^{(q,\lambda)}(x) dx = \frac{\Phi_{q+\lambda} - \Phi_q}{\lambda^2 \Phi'_{q+\lambda}}.$$

Using the above two relations, (5.33) can be rewritten as

$$\begin{aligned}
 u_{d;d}^{(q,\lambda)}(a, y; a) &= \frac{\int_a^\infty e^{-\Phi_{q+\lambda} x} A^{(q,\lambda)}(x, -y) dx}{\int_a^\infty e^{-\Phi_{q+\lambda} x} W^{(q,\lambda)}(x) dx} W^{(q,\lambda)}(a) - A^{(q,\lambda)}(a, -y) \\
 &+ \frac{\int_0^{-y} [W^{(q+\lambda)}(-y-z) A^{(q,\lambda)}(z, a) - W^{(q+\lambda)}(a-y-z) W^{(q)}(z)] dz}{\Phi'_{q+\lambda} \int_a^\infty e^{\Phi_{q+\lambda}(a-x)} W^{(q,\lambda)}(x) dx} W^{(q,\lambda)}(a). \tag{5.34}
 \end{aligned}$$

Substituting (5.34) into (5.28) leads to

$$\begin{aligned}
 u_{d;d}^{(q,\lambda)}(x, y; a) &= \frac{\int_a^\infty e^{-\Phi_{q+\lambda} z} A^{(q,\lambda)}(z, -y) dz}{\int_a^\infty e^{-\Phi_{q+\lambda} z} W^{(q,\lambda)}(z) dz} W^{(q,\lambda)}(x) - A^{(q,\lambda)}(x, -y) \\
 &+ \frac{\int_0^{-y} [W^{(q+\lambda)}(-y-z) A^{(q,\lambda)}(z, a) - W^{(q+\lambda)}(a-y-z) W^{(q)}(z)] dz}{\Phi'_{q+\lambda} \int_a^\infty e^{\Phi_{q+\lambda}(a-z)} W^{(q,\lambda)}(z) dz} W^{(q,\lambda)}(x).
 \end{aligned}$$

In light of (3.19), it remains to show that, for any  $y \in \mathbb{R}$  and  $a > 0$ ,

$$\int_0^{-y} W^{(q+\lambda)}(-y-z) A^{(q,\lambda)}(z, a) dz = \int_0^{-y} W^{(q+\lambda)}(a-y-z) W^{(q)}(z) dz. \quad (5.35)$$

It suffices to prove (5.35) for the case when  $y < 0$  because (5.35) clearly holds for  $y \geq 0$ . For large enough  $s > 0$ , it follows that from (3.9), (2.2) and (2.4) that

$$\begin{aligned} & \int_0^\infty e^{-sx} \int_0^x W^{(q+\lambda)}(x-z) A^{(q,\lambda)}(z, a) dz dx \\ &= \int_0^\infty e^{-sx} W^{(q+\lambda)}(x) dx \int_0^\infty e^{-sz} A^{(q,\lambda)}(z, a) dz \\ &= \frac{Z^{(q+\lambda)}(a, s)}{\psi_{q+\lambda}(s)} \frac{1}{\psi_q(s)} \\ &= \int_0^\infty e^{-sx} W^{(q+\lambda)}(a+x) dx \cdot \int_0^\infty e^{-sz} W^{(q)}(z) dz \\ &= \int_0^\infty e^{-sx} \int_0^x W^{(q+\lambda)}(a+x-z) W^{(q)}(z) dz dx. \end{aligned} \quad (5.36)$$

Taking Laplace inversion to (5.36) yields, for  $x \geq 0$ ,

$$\int_0^x W^{(q+\lambda)}(x-z) A^{(q,\lambda)}(z, a) dz = \int_0^x W^{(q+\lambda)}(a+x-z) W^{(q)}(z) dz. \quad (5.37)$$

This completes the proof of (5.35) by letting  $x = -y > 0$  in (5.37).

## 5.2 Proof of Proposition 3.1

Relations (3.20) and (3.23) are immediate from (3.4). In addition, relations (3.21) and (3.22) are direct consequences of (3.11), (3.4), and the fact that  $Z^{(q)}(x, \theta) = e^{\theta x}$  for  $x \leq 0$ . We are only left to prove (3.24).

For  $x, y \in [0, a]$ , by (3.19) and (3.9),

$$u_{d:d}^{(q,\lambda)}(x, y; a) = \frac{\int_a^\infty e^{-\Phi_{q+\lambda} z} W^{(q)}(z-y) dz}{\int_a^\infty e^{-\Phi_{q+\lambda} z} W^{(q,\lambda)}(z) dz} W^{(q,\lambda)}(x) - W^{(q)}(x-y).$$

Note that by (3.1), it follows that

$$\frac{\int_a^\infty e^{-\Phi_{q+\lambda} z} W^{(q)}(z-y) dz}{\int_a^\infty e^{-\Phi_{q+\lambda} z} W^{(q,\lambda)}(z) dz} = \frac{W^{(q,\lambda)}(a-y)}{(\Phi_{q+\lambda} - \Phi_q) \int_0^\infty e^{-\Phi_{q+\lambda} z} W^{(q,\lambda)}(z+a) dz}. \quad (5.38)$$

From (3.4) and (5.38), it remains to show that

$$\lim_{\lambda \rightarrow \infty} (\Phi_{q+\lambda} - \Phi_q) \int_0^\infty e^{-\Phi_{q+\lambda} z} W^{(q,\lambda)}(z+a) dz = W^{(q)}(a). \quad (5.39)$$



For any fixed  $\varepsilon > 0$ , by (3.1), we have

$$\begin{aligned}
 & (\Phi_{q+\lambda} - \Phi_q) \int_{\varepsilon}^{\infty} e^{-\Phi_{q+\lambda}z} W^{(q,\lambda)}(z+a) dz \\
 &= (\Phi_{q+\lambda} - \Phi_q)^2 \int_{\varepsilon}^{\infty} \int_0^{\infty} e^{-\Phi_{q+\lambda}(z+y)} W^{(q)}(z+y+a) dy dz \\
 &= (\Phi_{q+\lambda} - \Phi_q)^2 \int_{\varepsilon}^{\infty} \int_z^{\infty} e^{-\Phi_{q+\lambda}x} W^{(q)}(x+a) dx dz \\
 &= (\Phi_{q+\lambda} - \Phi_q)^2 \int_{\varepsilon}^{\infty} (x-\varepsilon) e^{-\Phi_{q+\lambda}x} W^{(q)}(x+a) dx.
 \end{aligned} \tag{5.40}$$

Observe that for any fixed  $x \geq \varepsilon$ , the function  $\beta \mapsto \beta^2 e^{-\beta x}$  is monotone decreasing in  $\beta$  for any  $\beta \geq \frac{x}{\varepsilon}$ . By (2.1), we deduce that for any  $x \geq \varepsilon$ , the function  $\lambda \mapsto \Phi_{q+\lambda}^2 e^{-\Phi_{q+\lambda}x}$  is monotone decreasing in  $\lambda$  for any  $\lambda \geq \psi(\frac{x}{\varepsilon}) - q$ . By (5.40) and the monotone convergence theorem, we deduce that

$$\begin{aligned}
 0 &\leq \limsup_{\lambda \rightarrow \infty} (\Phi_{q+\lambda} - \Phi_q) \int_{\varepsilon}^{\infty} e^{-\Phi_{q+\lambda}z} W^{(q,\lambda)}(z+a) dz \\
 &= \limsup_{\lambda \rightarrow \infty} (\Phi_{q+\lambda} - \Phi_q)^2 \int_{\varepsilon}^{\infty} (x-\varepsilon) e^{-\Phi_{q+\lambda}x} W^{(q)}(x+a) dx \\
 &\leq \lim_{\lambda \rightarrow \infty} \Phi_{q+\lambda}^2 \int_{\varepsilon}^{\infty} (x-\varepsilon) e^{-\Phi_{q+\lambda}x} W^{(q)}(x+a) dx \\
 &= \int_{\varepsilon}^{\infty} (x-\varepsilon) W^{(q)}(x+a) \lim_{\lambda \rightarrow \infty} \Phi_{q+\lambda}^2 e^{-\Phi_{q+\lambda}x} dx \\
 &= 0.
 \end{aligned} \tag{5.41}$$

On the other hand, thanks to the monotonicity of  $W^{(q,\lambda)}$ , we have

$$\begin{aligned}
 (\Phi_{q+\lambda} - \Phi_q) \int_0^{\varepsilon} e^{-\Phi_{q+\lambda}z} W^{(q,\lambda)}(z+a) dz &\geq \frac{(\Phi_{q+\lambda} - \Phi_q)(1 - e^{-\Phi_{q+\lambda}\varepsilon})}{\Phi_{q+\lambda}} W^{(q,\lambda)}(a), \\
 (\Phi_{q+\lambda} - \Phi_q) \int_0^{\varepsilon} e^{-\Phi_{q+\lambda}z} W^{(q,\lambda)}(z+a) dz &\leq \frac{(\Phi_{q+\lambda} - \Phi_q)(1 - e^{-\Phi_{q+\lambda}\varepsilon})}{\Phi_{q+\lambda}} W^{(q,\lambda)}(a+\varepsilon).
 \end{aligned}$$

It follows that

$$\liminf_{\lambda \rightarrow \infty} (\Phi_{q+\lambda} - \Phi_q) \int_0^{\varepsilon} e^{-\Phi_{q+\lambda}z} W^{(q,\lambda)}(z+a) dz \geq W^{(q)}(a), \tag{5.42}$$

$$\limsup_{\lambda \rightarrow \infty} (\Phi_{q+\lambda} - \Phi_q) \int_0^{\varepsilon} e^{-\Phi_{q+\lambda}z} W^{(q,\lambda)}(z+a) dz \leq W^{(q)}(a+\varepsilon). \tag{5.43}$$

From the arbitrariness of  $\varepsilon$ , we conclude from (5.41)–(5.43) that (5.39) holds.

### 5.3 Proof of Corollary 3.2

For  $x \in [-a, b]$  and  $y \in [-a, 0]$ , we have

$$\begin{aligned}
& \mathbb{E}_x \left[ e^{-qT_0^-, \lambda} 1_{\{X_{T_0^-, \lambda} \in dy, T_0^-, \lambda < \tau_{-a}^- \wedge \tau_b^+\}} \right] \\
&= \mathbb{E}_x \left[ e^{-qT_0^-, \lambda} 1_{\{X_{T_0^-, \lambda} \in dy, T_0^-, \lambda < \tau_{-a}^-\}} \right] - \mathbb{E}_x \left[ e^{-q\tau_b^+} 1_{\{\tau_b^+ < T_0^-, \lambda \wedge \tau_{-a}^-\}} \right] \mathbb{E}_b \left[ e^{-qT_0^-, \lambda} 1_{\{X_{T_0^-, \lambda} \in dy, T_0^-, \lambda < \tau_{-a}^-\}} \right] \\
&= \lambda r_-^{(q, \lambda)}(x, y; -a) dy - \mathbb{E}_x \left[ e^{-q\tau_b^+} 1_{\{\tau_b^+ < T_0^-, \lambda \wedge \tau_{-a}^-\}} \right] \lambda r_-^{(q, \lambda)}(b, y; -a) dy. \tag{5.44}
\end{aligned}$$

In what follows, we focus on characterizing  $\mathbb{E}_x \left[ e^{-q\tau_b^+} 1_{\{\tau_b^+ < T_0^-, \lambda \wedge \tau_{-a}^-\}} \right]$ .

Conditioning on whether  $\tau_b^+$  or  $\tau_0^-$  occurs first, by (2.8) and (5.17), it follows that

$$\begin{aligned}
& \mathbb{E}_x \left[ e^{-q\tau_b^+} 1_{\{\tau_b^+ < T_0^-, \lambda \wedge \tau_{-a}^-\}} \right] \\
&= \int_{-a}^0 \mathbb{E}_x \left[ e^{-q\tau_0^-} 1_{\{X_{\tau_0^-} \in dz, \tau_0^- < \tau_b^+\}} \right] \mathbb{E}_z \left[ e^{-q\tau_0^+} 1_{\{\tau_0^+ < e'_\lambda \wedge \tau_{-a}^-\}} \right] \mathbb{E} \left[ e^{-q\tau_b^+} 1_{\{\tau_b^+ < T_0^-, \lambda \wedge \tau_{-a}^-\}} \right] \\
&+ \mathbb{E}_x \left[ e^{-q\tau_b^+} 1_{\{\tau_b^+ < \tau_0^-\}} \right] \\
&= \frac{W^{(q)}(x)}{W^{(q)}(b)} + \frac{1}{W^{(q+\lambda)}(a)} \mathbb{E}_x \left[ e^{-q\tau_0^-} W^{(q+\lambda)}(a + X_{\tau_0^-}) 1_{\{\tau_0^- < \tau_b^+\}} \right] \mathbb{E} \left[ e^{-q\tau_b^+} 1_{\{\tau_b^+ < T_0^-, \lambda \wedge \tau_{-a}^-\}} \right] \\
&= \frac{W^{(q)}(x)}{W^{(q)}(b)} + \left( \frac{A^{(q, \lambda)}(x, a)}{W^{(q+\lambda)}(a)} - \frac{W^{(q)}(x) A^{(q, \lambda)}(b, a)}{W^{(q)}(b) W^{(q+\lambda)}(a)} \right) \mathbb{E} \left[ e^{-q\tau_b^+} 1_{\{\tau_b^+ < T_0^-, \lambda \wedge \tau_{-a}^-\}} \right]. \tag{5.45}
\end{aligned}$$

Note that (5.45) holds for  $x \in [-a, b]$ . To evaluate  $\mathbb{E} \left[ e^{-q\tau_b^+} 1_{\{\tau_b^+ < T_0^-, \lambda \wedge \tau_{-a}^-\}} \right]$ , we condition on whether  $e'_\lambda$  or  $\tau_b^+$  occurs first and obtain

$$\begin{aligned}
& \mathbb{E} \left[ e^{-q\tau_b^+} 1_{\{\tau_b^+ < T_0^-, \lambda \wedge \tau_{-a}^-\}} \right] \\
&= \mathbb{E} \left[ e^{-q\tau_b^+} 1_{\{\tau_b^+ < e'_\lambda \wedge \tau_{-a}^-\}} \right] + \int_0^b \mathbb{E} \left[ e^{-qe'_\lambda} 1_{\{X_{e'_\lambda} \in dz, e'_\lambda < \tau_{-a}^- \wedge \tau_b^+\}} \right] \mathbb{E}_z \left[ e^{-q\tau_b^+} 1_{\{\tau_b^+ < T_0^-, \lambda \wedge \tau_{-a}^-\}} \right] \\
&= \frac{W^{(q+\lambda)}(a)}{W^{(q+\lambda)}(a+b)} + \lambda \int_0^b u^{(q+\lambda)}(a, z+a; b+a) \mathbb{E}_z \left[ e^{-q\tau_b^+} 1_{\{\tau_b^+ < T_0^-, \lambda \wedge \tau_{-a}^-\}} \right] dz. \tag{5.46}
\end{aligned}$$

Substituting (5.45) with  $x = z$  into (5.46) and using (2.21), we have

$$\begin{aligned}
& \mathbb{E} \left[ e^{-q\tau_b^+} 1_{\{\tau_b^+ < T_0^-, \lambda \wedge \tau_{-a}^-\}} \right] \\
&= \frac{\frac{W^{(q+\lambda)}(a)}{W^{(q+\lambda)}(a+b)} + \lambda \int_0^b u^{(q+\lambda)}(a, z+a; a+b) \frac{W^{(q)}(z)}{W^{(q)}(b)} dz}{1 - \lambda \int_0^b u^{(q+\lambda)}(a, z+a; a+b) \left[ \frac{A^{(q, \lambda)}(z, a)}{W^{(q+\lambda)}(a)} - \frac{W^{(q)}(z) A^{(q, \lambda)}(b, a)}{W^{(q)}(b) W^{(q+\lambda)}(a)} \right] dz} \\
&= \frac{\frac{W^{(q+\lambda)}(a)}{W^{(q+\lambda)}(a+b)} + \frac{\lambda W^{(q+\lambda)}(a)}{W^{(q+\lambda)}(a+b) W^{(q)}(b)} \int_0^b W^{(q+\lambda)}(b-z) W^{(q)}(z) dz}{1 - \frac{\lambda}{W^{(q+\lambda)}(a+b)} \int_0^b W^{(q+\lambda)}(b-z) \left( A^{(q, \lambda)}(z, a) - \frac{W^{(q)}(z) A^{(q, \lambda)}(b, a)}{W^{(q)}(b)} \right) dz}. \tag{5.47}
\end{aligned}$$

From (5.37) and (2.6), one easily finds that

$$\int_0^b W^{(q+\lambda)}(b-z) A^{(q,\lambda)}(z, a) dz = \int_0^b W^{(q+\lambda)}(a+b-z) W^{(q)}(z) dz,$$

and

$$\int_0^b W^{(q+\lambda)}(b-z) W^{(q)}(z) dz = \frac{W^{(q+\lambda)}(b) - W^{(q)}(b)}{\lambda}.$$

Further substituting the above two equalities into (5.47), and using (3.10) lead to

$$\mathbb{E} \left[ e^{-q\tau_b^+} \mathbf{1}_{\{\tau_b^+ < T_0^{-,\lambda} \wedge \tau_{-a}^-\}} \right] = \frac{W^{(q+\lambda)}(a)}{A^{(q,\lambda)}(b, a)}.$$

Hence, (5.45) reduces to

$$\mathbb{E}_x \left[ e^{-q\tau_b^+} \mathbf{1}_{\{\tau_b^+ < T_0^{-,\lambda} \wedge \tau_{-a}^-\}} \right] = \frac{A^{(q,\lambda)}(x, a)}{A^{(q,\lambda)}(b, a)}. \quad (5.48)$$

Lastly, by substituting (5.48) into (5.44) and using (3.14), the proof is complete.

#### 5.4 Proof of Theorem 4.1

For  $x \in [-a, 0)$  and  $y \in \mathbb{R}$ , we separately consider the contributions to (4.1) by the following two possible events:  $\{e_q < T_{-a}^{-,\lambda} \wedge \tau_0^+\}$  and  $\{\tau_0^+ < T_{-a}^{-,\lambda} \wedge e_q\}$ . It follows that

$$\begin{aligned} & \mathbb{E}_x \left[ e^{-sT^{\lambda,q}} \mathbf{1}_{\{X_{T^{\lambda,q}} \in dy, T^{\lambda,q} < T_{-a}^{-,\lambda} \wedge T_b^{+,\lambda}\}} \right] \\ &= \mathbb{E}_x \left[ e^{-se_q} \mathbf{1}_{\{X_{e_q} \in dy, e_q < T_{-a}^{-,\lambda} \wedge \tau_0^+\}} \right] + \mathbb{E}_x \left[ e^{-s\tau_0^+} \mathbf{1}_{\{\tau_0^+ < e_q \wedge T_{-a}^{-,\lambda}\}} \right] \mathbb{E}_{0^-} \left[ e^{-sT^{\lambda,q}} \mathbf{1}_{\{X_{T^{\lambda,q}} \in dy, T^{\lambda,q} < T_{-a}^{-,\lambda} \wedge T_b^{+,\lambda}\}} \right] \\ &= qu_{d:c}^{(s+q,\lambda)}(x+a, y+a; a) dy \mathbf{1}_{\{y < 0\}} + \frac{W^{(s+q,\lambda)}(x+a)}{W^{(s+q,\lambda)}(a)} \mathbb{E}_{0^-} \left[ e^{-sT^{\lambda,q}} \mathbf{1}_{\{X_{T^{\lambda,q}} \in dy, T^{\lambda,q} < T_{-a}^{-,\lambda} \wedge T_b^{+,\lambda}\}} \right], \end{aligned} \quad (5.49)$$

where  $0^-$  means the surplus is at level 0 and the Parisian clock is on. For  $x \in [0, b]$  and  $y \in \mathbb{R}$ , we shall have  $T_0^{-,\lambda} \leq T^{\lambda,q}$  almost surely. Hence, by conditioning on  $T_0^{-,\lambda}$ , one finds that

$$\begin{aligned} & \mathbb{E}_x \left[ e^{-sT^{\lambda,q}} \mathbf{1}_{\{X_{T^{\lambda,q}} \in dy, T^{\lambda,q} < T_{-a}^{-,\lambda} \wedge T_b^{+,\lambda}\}} \right] \\ &= \int_{-a}^0 \mathbb{E}_x \left[ e^{-sT_0^{-,\lambda}} \mathbf{1}_{\{X_{T_0^{-,\lambda}} \in dw, T_0^{-,\lambda} < T_b^{+,\lambda}\}} \right] \mathbb{E}_w \left[ e^{-sT^{\lambda,q}} \mathbf{1}_{\{X_{T^{\lambda,q}} \in dy, T^{\lambda,q} < T_{-a}^{-,\lambda} \wedge T_b^{+,\lambda}\}} \right] \\ &= \lambda \int_{-a}^0 u_{d:d}^{(s,\lambda)}(x, w; b) \mathbb{E}_w \left[ e^{-sT^{\lambda,q}} \mathbf{1}_{\{X_{T^{\lambda,q}} \in dy, T^{\lambda,q} < T_{-a}^{-,\lambda} \wedge T_b^{+,\lambda}\}} \right] dw. \end{aligned} \quad (5.50)$$

Substituting (5.49) into (5.50) leads to

$$\begin{aligned} & \mathbb{E}_x \left[ e^{-sT^{\lambda,q}} \mathbf{1}_{\{X_{T^{\lambda,q}} \in dy, T^{\lambda,q} < T_{-a}^{-,\lambda} \wedge T_b^{+,\lambda}\}} \right] \\ &= q\lambda \int_{-a}^0 u_{d:d}^{(s,\lambda)}(x, w; b) u_{d:c}^{(s+q,\lambda)}(w+a, y+a; a) dw dy \mathbf{1}_{\{y < 0\}} \\ &+ \lambda \int_{-a}^0 u_{d:d}^{(s,\lambda)}(x, w; b) \frac{W^{(s+q,\lambda)}(w+a)}{W^{(s+q,\lambda)}(a)} dw \mathbb{E}_{0^-} \left[ e^{-sT^{\lambda,q}} \mathbf{1}_{\{X_{T^{\lambda,q}} \in dy, T^{\lambda,q} < T_{-a}^{-,\lambda} \wedge T_b^{+,\lambda}\}} \right]. \end{aligned} \quad (5.51)$$

Let

$$v^{(s,\lambda)}(x, w; b) = \begin{cases} \delta_x(w), & x \in [-a, 0), \\ \lambda u_{d:d}^{(s,\lambda)}(x, w; b), & x \in [0, b]. \end{cases}$$

We note that (5.49) and (5.51) can be expressed in a unified way as follows: for any  $x \in [-a, b]$  and  $y \in \mathbb{R}$ ,

$$\begin{aligned} & \mathbb{E}_x \left[ e^{-sT^{\lambda,q}} \mathbf{1}_{\{X_{T^{\lambda,q}} \in dy, T^{\lambda,q} < T_{-a}^{-,\lambda} \wedge T_b^{+,\lambda}\}} \right] \\ &= q \int_{-a}^0 v^{(s,\lambda)}(x, w; b) u_{d:c}^{(s+q,\lambda)}(w+a, y+a; a) dw dy \mathbf{1}_{\{y < 0\}} \\ &+ \int_{-a}^0 v^{(s,\lambda)}(x, w; b) \frac{W^{(s+q,\lambda)}(w+a)}{W^{(s+q,\lambda)}(a)} dw \mathbb{E}_{0-} \left[ e^{-sT^{\lambda,q}} \mathbf{1}_{\{X_{T^{\lambda,q}} \in dy, T^{\lambda,q} < T_{-a}^{-,\lambda} \wedge T_b^{+,\lambda}\}} \right]. \end{aligned} \quad (5.52)$$

Next, we focus on characterizing  $\mathbb{E}_{0-} \left[ e^{-sT^{\lambda,q}} \mathbf{1}_{\{X_{T^{\lambda,q}} \in dy, T^{\lambda,q} < T_{-a}^{-,\lambda} \wedge T_b^{+,\lambda}\}} \right]$ . Conditioning on whether  $e'_\lambda$  or  $e_q$  occurs first and using (5.52), one obtains

$$\begin{aligned} & \mathbb{E}_{0-} \left[ e^{-sT^{\lambda,q}} \mathbf{1}_{\{X_{T^{\lambda,q}} \in dy, T^{\lambda,q} < T_{-a}^{-,\lambda} \wedge T_b^{+,\lambda}\}} \right] \\ &= \mathbb{E}_{0-} \left[ e^{-se_q} \mathbf{1}_{\{X_{e_q} \in dy, e_q < e'_\lambda\}} \right] + \int_{-a}^b \mathbb{E}_{0-} \left[ e^{-se'_\lambda} \mathbf{1}_{\{X_{e'_\lambda} \in dz, e'_\lambda < e_q\}} \right] \mathbb{E}_x \left[ e^{-sT^{\lambda,q}} \mathbf{1}_{\{X_{T^{\lambda,q}} \in dy, T^{\lambda,q} < T_{-a}^{-,\lambda} \wedge T_b^{+,\lambda}\}} \right] \\ &= q\theta^{(s+q+\lambda)}(y)dy + \lambda q dy \int_{-a}^b \int_{-a}^0 \theta^{(s+q+\lambda)}(x) v^{(s,\lambda)}(x, w; b) u_{d:c}^{(s+q,\lambda)}(w+a, y+a; a) dw dz \\ &+ \lambda \int_{-a}^b \int_{-a}^0 \theta^{(s+q+\lambda)}(z) v^{(s,\lambda)}(z, w; b) \frac{W^{(s+q,\lambda)}(w+a)}{W^{(s+q,\lambda)}(a)} dw dz \mathbb{E}_{0-} \left[ e^{-sT^{\lambda,q}} \mathbf{1}_{\{X_{T^{\lambda,q}} \in dy, T^{\lambda,q} < T_{-a}^{-,\lambda} \wedge T_b^{+,\lambda}\}} \right]. \end{aligned} \quad (5.53)$$

Thus, it is direct from (5.53) that

$$\begin{aligned} & \frac{1}{qdy} \mathbb{E}_{0-} \left[ e^{-sT^{\lambda,q}} \mathbf{1}_{\{X_{T^{\lambda,q}} \in dy, T^{\lambda,q} < T_{-a}^{-,\lambda} \wedge T_b^{+,\lambda}\}} \right] \\ &= \frac{\theta^{(s+q+\lambda)}(y) + \lambda \int_{-a}^b \int_{-a}^0 \theta^{(s+q+\lambda)}(z) v^{(s,\lambda)}(z, w; b) u_{d:c}^{(s+q,\lambda)}(w+a, y+a; a) dw dz}{1 - \lambda \int_{-a}^b \int_{-a}^0 \theta^{(s+q+\lambda)}(z) v^{(s,\lambda)}(z, w; b) \frac{W^{(s+q,\lambda)}(w+a)}{W^{(s+q,\lambda)}(a)} dw dz} \\ &= \frac{\theta^{(s+q+\lambda)}(y) + A^{(s+q,\lambda)}(a, -a-y) - \lambda \int_{-a}^b \theta^{(s+q+\lambda)}(z) H_{a,b}^{(s,q,\lambda)}(z, -y) dz}{1 - \lambda \int_{-a}^b \theta^{(s+q+\lambda)}(z) \frac{Z_{a,b}^{(s,q,\lambda)}(z)}{W^{(s+q,\lambda)}(a)} dz} - A^{(s+q,\lambda)}(a, -a-y), \end{aligned} \quad (5.54)$$

where the last step is due to the definitions of  $u_{d:c}$ ,  $H_{a,b}^{(s,q,\lambda)}$ , and  $Z_{a,b}^{(s,q,\lambda)}$ , in (3.17), (4.2), and (4.3), respectively. Finally, the substitution of (5.54) into (5.51) completes the proof.

## 5.5 Proof of Equation (4.4)

By (3.11), we have

$$\begin{aligned} & \frac{1}{q} \mathbb{E}_x \left[ e^{-sT^{\lambda,q}} 1_{\{X_{T^{\lambda,q}} \in dy, T^{\lambda,q} < T_{-a}^{-,\lambda} \wedge T_b^{+,\lambda}\}} \right] \\ &= \frac{\theta^{(s+q+\lambda)}(y) + W^{(s+q)}(-y) - \lambda \int_{-a}^b \theta^{(s+q+\lambda)}(z) H_{a,b}^{(s,q,\lambda)}(z, -y) dz}{W^{(s+q,\lambda)}(a) - \lambda \int_{-a}^b \theta^{(s+q+\lambda)}(z) Z_{a,b}^{(s,q,\lambda)}(z) dz} Z_{a,b}^{(s,q,\lambda)}(x) - H_{a,b}^{(s,q,\lambda)}(x, -y), \end{aligned} \quad (5.55)$$

where  $H_{a,b}^{(s,q,\lambda)}(x, -y) = \int_0^a v^{(s,\lambda)}(x, -w; b) W^{(s+q)}(-y-w) dw$ . By (3.28), we know that  $v^{(s,\lambda)}(x, -w; b) dw$  converges (weakly) to  $\mathbb{E}_x \left[ e^{-q\tau_0^-} 1_{\{-X_{\tau_0^-} \in dw, \tau_0^- < \tau_b^+\}} \right]$  as  $\lambda \rightarrow \infty$ . By the boundedness and continuity of  $W^{(s+q)}(-y-w)$  for  $w \in [0, a]$ , it follows that

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} H_{a,b}^{(s,q,\lambda)}(x, -y) &= \int_0^a \mathbb{E}_x \left[ e^{-q\tau_0^-} 1_{\{-X_{\tau_0^-} \in dw, \tau_0^- < \tau_b^+\}} \right] W^{(s+q)}(-y-w) dw \\ &= \mathbb{E}_x \left[ e^{-q\tau_0^-} W^{(s+q)}(-y + X_{\tau_0^-}) 1_{\{\tau_0^- < \tau_b^+\}} \right] \\ &= A^{(s,q)}(x, -y) - \frac{W^{(s)}(x)}{W^{(s)}(b)} A^{(s,q)}(b, -y), \end{aligned} \quad (5.56)$$

where the last line is due to Lemma 2.2 of Loeffen et al. [24]. By the same argument, we have

$$\lim_{\lambda \rightarrow \infty} Z_{a,b}^{(s,q,\lambda)}(x) = \lim_{\lambda \rightarrow \infty} H_{a,b}^{(s,q,\lambda)}(x, a) = A^{(s,q)}(x, a) - \frac{W^{(s)}(x)}{W^{(s)}(b)} A^{(s,q)}(b, a). \quad (5.57)$$

From (2.14), we deduce that  $\lambda\theta^{(\lambda)}(z)$  converges (weakly) to  $\delta_0(z)$  when  $\lambda \rightarrow \infty$ . By the boundedness and continuity of  $H_{a,b}^{(s,q,\lambda)}(z, -y)$  and  $Z_{a,b}^{(s,q,\lambda)}(z)$  for  $z \in [-a, b]$ , with the application of (5.56), (5.57) and (3.12), the limit of (5.55) is given by

$$\begin{aligned} & \lim_{\lambda \rightarrow \infty} \frac{1}{q} \mathbb{E}_x \left[ e^{-sT^{\lambda,q}} 1_{\{X_{T^{\lambda,q}} \in dy, T^{\lambda,q} < T_{-a}^{-,\lambda} \wedge T_b^{+,\lambda}\}} \right] \\ &= \frac{W^{(s+q)}(-y) - A^{(s,q)}(0, -y) + \frac{W^{(s)}(0+)}{W^{(s)}(b)} A^{(s,q)}(b, -y)}{W^{(s+q)}(a) - A^{(s,q)}(0, a) + \frac{W^{(s)}(0+)}{W^{(s)}(b)} A^{(s,q)}(b, a)} \left( A^{(s,q)}(x, a) - \frac{W^{(s)}(x)}{W^{(s)}(b)} A^{(s,q)}(b, a) \right) \\ &\quad - A^{(s,q)}(x, -y) + \frac{W^{(s)}(x)}{W^{(s)}(b)} A^{(s,q)}(b, -y) \\ &= \frac{A^{(s,q)}(b, -y)}{A^{(s,q)}(b, a)} \left( A^{(s,q)}(x, a) - \frac{W^{(s)}(x)}{W^{(s)}(b)} A^{(s,q)}(b, a) \right) - A^{(s,q)}(x, -y) + \frac{W^{(s)}(x)}{W^{(s)}(b)} A^{(s,q)}(b, -y) \\ &= \frac{A^{(s,q)}(b, -y)}{A^{(s,q)}(b, a)} A^{(s,q)}(x, a) - A^{(s,q)}(x, -y), \end{aligned}$$

where we have used the fact that  $W^{(s)}(0+) \neq 0$  when  $X$  has bounded variation paths (e.g., Lemma 3.1 of Kuznetsov et al. [14]). This completes the proof.

## 6 Acknowledgement

The authors would like to thank two anonymous referees for helpful comments and suggestions. Support from grants from the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged by David Landriault and Bin Li (grant numbers 341316 and 05828, respectively). Support from the Canada Research Chair Program is gratefully acknowledged by David Landriault. Support from the James C. Hickman Scholar program and Educational Institution Grant of the Society of Actuaries is gratefully acknowledged by Jeff T.Y. Wong and Di Xu, respectively.

## References

- [1] Albrecher, H.; Cheung, E.C.K.; Thonhauser, S. (2011). Randomized observation periods for the compound Poisson risk model: dividends. *ASTIN Bulletin*, 41(2), 645-672.
- [2] Albrecher, H.; Cheung, E.C.K.; Thonhauser, S. (2013). Randomized observation periods for the compound Poisson risk model: The discounted penalty function. *Scandinavian Actuarial Journal* 2013(6), 424-452.
- [3] Albrecher, H.; Ivanovs, J. (2013). A risk model with an observer in a Markov environment. *Risks*, 1(3), 148-161.
- [4] Albrecher, H.; Ivanovs, J.; Zhou, X. (2016). Exit identities for Lévy processes observed at Poisson arrival times. *Bernoulli* 22(3), 1364-1382.
- [5] Albrecher, H.; Lautscham, V. (2013). From ruin to bankruptcy for compound Poisson surplus processes. *ASTIN Bulletin*, 43(2), 213-243.
- [6] Avram, F.; Vu, N.L.; Zhou, X. (2017). On taxed spectrally negative Lévy processes with draw-down stopping. *Insurance: Mathematics and Economics*, 76, 69-74.
- [7] Baurdoux, E.J.; Pardo, J.C.; Perez, J.L.; Renaud, J.-F. (2016). Gerber-Shiu distribution at Parisian ruin for Lévy insurance risk processes. *Journal of Applied Probability*, 53(2), 572-584
- [8] Bekker, R.; Boxma, O.J.; Resing, J.A.C. (2009). Lévy processes with adaptable exponent. *Advances in Applied Probability*, 41(1), 177-205.
- [9] Biffis, E.; Kyprianou, A.E. (2010). A note on scale functions and the time value of ruin for Lévy insurance risk processes. *Insurance: Mathematics and Economics*, 46(1), 85-91
- [10] Choi, M.C.H.; Cheung, E.C.K. (2014). On the expected discounted dividends in the Cramér-Lundberg risk model with more frequent ruin monitoring than dividend decisions. *Insurance: Mathematics and Economics*, 59, 121-132..
- [11] Czarna, I.; Palmowski, Z. (2011). Ruin probability with Parisian delay for a spectrally negative Lévy risk process. *Journal of Applied Probability*, 48(4), 984-1002.

- [12] Cheung, E.C.K.; Wong, J.T.Y. (2017). On the dual risk model with Parisian implementation delays in dividend payments. *European Journal of Operational Research*, 257(1), 159-173.
- [13] Dassios, A.; Wu, S. (2008). Parisian ruin with exponential claims. Technical report, working paper L.S.E.
- [14] Kuznetsov, A.; Kyprianou, A.E.; Rivero, V. (2012). The theory of scale functions for spectrally negative Lévy processes. *Lévy Matters II*: 97-186. Springer Berlin Heidelberg.
- [15] Kyprianou, A.E. (2014). Fluctuations of Lévy processes with applications. *Introductory Lectures (second ed.) Universitext*. Springer, Heidelberg.
- [16] Landriault, D.; Li, B.; Li, S. (2015). Analysis of a drawdown-based regime-switching Lévy insurance model. *Insurance: Mathematics and Economics*, 60, 98-107.
- [17] Landriault, D.; Li, B.; Zhang, H. (2017). On magnitude, asymptotics and duration of drawdowns for Lévy models. *Bernoulli* 23(1), 432-458.
- [18] Landriault, D.; Li, B.; Li, S. Expected utility of the drawdown-based regime-switching risk model with state-dependent termination. *Insurance: Mathematics and Economics*, forthcoming.
- [19] Landriault, D.; Renaud, J.-F.; Zhou, X. (2011). Occupation times of spectrally negative Lévy processes with applications. *Stochastic Processes and their Applications*, 121(11), 2629-2641.
- [20] Landriault, D.; Renaud, J.-F.; Zhou, X. (2014). An insurance risk model with Parisian implementation delays. *Methodology and Computing in Applied Probability*, 16(3), 583-607.
- [21] Lkabous, M.A.; Czarna, I.; Renaud, J.-F. (2017). Parisian ruin for a refracted Lévy process. *Insurance: Mathematics and Economics*, 74, 153-163.
- [22] Loeffen, R. (2014). On obtaining simple identities for overshoots of spectrally negative Lévy processes. arXiv preprint arXiv:1410.5341.
- [23] Loeffen, R.; Czarna, I.; Palmowski, Z. (2013). Parisian ruin probability for spectrally negative Lévy processes. *Bernoulli* 19(2), 599-609.
- [24] Loeffen, R.; Renaud, J.-F.; Zhou, X. (2014). Occupation times of intervals until first passage times for spectrally negative Lévy processes. *Stochastic Processes and their Applications*, 124(3), 1408-1435.
- [25] Li, B.; Willmot, G.E.; Wong, J.T.Y. (2017). A temporal approach to the Parisian risk model. *Journal of Applied Probability*, forthcoming.
- [26] Pistorius, M.R. (2005). A potential-theoretical review of some exit problems of spectrally negative Lévy processes. *Séminaire de Probabilités XXXVIII*, 1857, 30-41.

- [27] Wong, J.T.Y.; Cheung, E.C.K. (2015). On the time value of Parisian ruin in (dual) renewal risk processes with exponential jumps. *Insurance: Mathematics and Economics*, 65, 280-290.
- [28] Zhang, H. (2017). *Stochastic Drawdowns (Modern Trends in Financial Engineering)*. World Scientific.
- [29] Zhang, H.; Leung, T.; Hadjiladis, O. (2013). Stochastic modeling and fair valuation of draw-down insurance. *Insurance: Mathematics and Economics*, 53(3), 840-850.
- [30] Zhang, Z.M. (2014). On a risk model with randomized dividend-decision times. *Journal of Industrial and Management Optimization*, 10(4): 1041-1058.