Quasi-Hopf Symmetry in Loop Quantum Gravity with Cosmological Constant and Spinfoams with Timelike Surfaces

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of contributions

The first section of chapter 5 contains the publication [1], of which I am the sole author.

Abstract

In this thesis we study two separate problems concerning improvements to the Loop quantum gravity and spinfoam approach to quantum gravity. In the first part we address the question about the origin of quantum group symmetries in Loop quantum gravity with non-vanishing cosmological constant Λ. Our focus is mainly the 3-dimensional Euclidean case with $\Lambda > 0$. We clarify, both at the classical and the quantum level, the quasi-Poisson and quasi-Hopf structures that arise in this case, respectively. This type of symmetry has, until recently, seen not much attention in the Loop quantum gravity literature, despite its importance for the approach. We explain the connection of our work with the Turaev-Viro state sum model, which relies heavily on the notion of twisting. To analyze our q - deformed model, for q being a root of unity, we construct for the first time certain gauge invariant geometric observables for the weak quasi-Hopf algebra $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$ with truncated coproduct, using so-called tensor operators. We show that these tensor operators satisfy the quasi-Hopf version of the Wigner-Eckart theorem and explicitly calculate the action of length- and angle- operators, which confirms the spherical curvature of our quantum geometry.

The second topic investigated in this thesis is the problem of timelike contributions for 4-dimensional Lorentzian spinfoam models, using the twistorial parametrization of Loop quantum gravity. We prove how the cotangent bundle $\mathrm{T}^*\mathrm{SU}(1,1)$ can be embedded into $\mathrm{T}^*\mathrm{SL}(2,\mathbb{C})$ via symplectic reduction by the simplicity constraints for a spacelike normal vector and an area matching constraint. This mathematical result is used to study timelike 2-surfaces in 4D Lorentzian gravity, both at the classical and quantum level. We investigate in particular the spectrum of the area operator for timelike faces and find that it is discrete. Furthermore, building on our results, we propose a new Lorentzian spinfoam model, which allows to include timelike contributions.

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Chapter 1

Introduction

Loop quantum gravity and spinfoam models [2, 3, 4] are two approaches to the problem of quantum gravity. This means that they try to find a consistent theory that unifies the principles of quantum mechanics with Einstein's theory of general relativity[1]. The problem of quantum gravity is itself almost as old as the two theoretical frameworks it aims to combine, since Einstein himself was already well aware of the need to modify his gravitational theory in the light of the new quantum physics [5].

General relativity and quantum physics have individually enjoyed unprecedented success in their respective domains of applicability, which was demonstrated yet again, very recently, by the experimental verification of gravitational waves [6], which would not have been feasible without the technological advances made possible by quantum physics.

Both theories have also fundamentally changed our view of nature and how it operates. General relativity has radically altered our understanding of space and time, teaching us that they are themselves relative notions that depend not only on the states of motion of observers, as in special relativity, but on the energy density and the matter distribution as well, which leads to well known effects like gravitational time dilation or black holes and singularities. Special relativity changed our understanding of simultaneity but, like most modern quantum physics, still uses a global background structure to measure distances and times (Minkowski space). General relativity, on the other hand, abandons the idea of a global notion of space and time altogether, replacing it with the local metric field, which

[1]Some might argue that Einstein's theory of gravity is itself not the correct theory of gravity and thus, should not be the basis for a quantum theory. However, it is still the best we have when it comes to matching our observed universe at large scales and in the absence of more compelling alternatives the most reasonable starting point.

eventually leads to the concepts of general covariance and background independence[2].

From the point of view of a relativist, the notion of background independence, or diffeomorphism invariance, which states that there are no preferred coordinates, or states of motion, in our description of physical phenomena, is the fundamental principle at the heart of general relativity. As such, it is considered to be most likely an important ingredient in any future quantum description of gravity and hence, a strong guiding principle in the search for such a theory. Loop quantum gravity and spinfoam models take this lesson seriously and provide a tentative model for a quantum theory of gravity that allows to implement diffeomorphism invariance.

Quantum physics has arguably changed our understanding of physics even more profoundly. However, it is much harder to pin down the precise physical principles at its core. Its lack of determinism, the pride of physicists since Newton's $F = ma$, is certainly one of its most significant features, but is it a fundamental principle of nature? Or is it an unfortunate property of our mathematical formalism? We prefer not to speculate and resort again to our agnostic point of view that without viable alternatives to standard quantum mechanics, these are the rules of the game. Which is not to diminish the astounding success that quantum mechanics has had as a predictive physical theory. Properties of quantum physics, that caused long theoretical and philosophical discussions, such as entanglement, superposition of states or quantum jumps, are an experimental fact of life that will not go away with other formulations or interpretations of the theory.

Thus, besides immense mathematical challenges, we see that the problem of quantum gravity is also a quest for the right physical principles underlying nature and our description of it and hence, it is at the center of research in fundamental physics. We will offer no solutions to these formidable questions in this thesis but follow loosely Wittgenstein's: "Whereof one cannot speak, thereof one must *calculate*"[3], an approach that worked surprisingly well in quantum mechanics.

In some sense, Loop quantum gravity follows the same philosophy, in that it does not attempt to radically change our current paradigms of physics, but tries to stick to the mathematical formalism of standard quantum theory complemented with the principle of background independence. Hence, the framework rests on well established physics and is a rather conservative approach that avoids introducing superfluous structures, like extra dimensions, or unobserved symmetries, such as supersymmetry, though both of these things

[2] The metric $g_{\mu\nu}(x)$ tells us locally how to measure distances and times but, as a tensor, transforms under general coordinate transformations and thus, is not a background independent quantity.

[3] Originally: "Wovon man nicht sprechen kann, darüber muss man schweigen", (Whereof one cannot speak, thereof one must be silent.).

can be incorporated, should experimental evidence be found in their favor.

The first and main topic of this thesis is concerned with a fairly precise mathematical problem, namely, to understand the origin of so-called quantum group symmetries in the framework of Loop quantum gravity with a non-zero cosmological constant Λ and the investigation of the resulting setting. Physically, this problem touches on another contentious issue in modern theoretical physics, the "Cosmological constant problem", which concerns the huge discrepancy between the observed value of $\Lambda_{obs} \approx 10^{-52} m^{-2}$, [7], and the calculated vacuum energy of standard quantum field theories, which is up to 120 orders of magnitude off from Λ_{obs}, [8]. We will not discuss the status or relevance of this problem further, since from a mathematical viewpoint the exact value is not really important to us and we focus on the fact that $\Lambda \neq 0$, where our main interest is the case of positive Λ in three Euclidean dimensions. For us, Λ is simply another parameter, whose qualitative effect on our quantum theory we would like to investigate. The system of interest to us is defined by the action

$$S_{GR}[g_{\mu\nu}] = \alpha \int_M \Omega_M \ (R[g_{\mu\nu}] - 2\,\Lambda)\,, \tag{1.1}$$

with $\Lambda \neq 0$, for M being a 3 - or 4 - dimensional (pseudo -) Riemannian manifold with volume element Ω_M. R denotes the Ricci curvature scalar and $\alpha = \frac{1}{16\pi G}$.

The treatment of Eq.(1.1) in the formalism of Loop quantum gravity for $\Lambda = 0$ is heavily based on the Lie group SU(2) and its representation theory. This follows from the fact that in three (Euclidean) dimensions Eq.(1.1) can be reformulated as a topological BF-theory for SU(2) and, similarly, using the so-called Ashtekar connection variables (A^i_b, E^a_j) in four dimensions (Euclidean and Lorentzian), we can give general relativity the structure of a SU(2) gauge theory. Finally, the fundamental configuration variables used in Loop quantum gravity and spinfoam models are not the connections A^i_b themselves, but their holonomies along certain paths.

The famous spin network states of Loop quantum gravity, which are graphs Γ[4], labeled in a gauge-invariant way by irreducible unitary representations $2j_i \in \mathbb{N}$ of SU(2), span the kinematical Hilbert space $\mathcal{H}^{LQG}_{\Gamma,kin}$ of the theory and correspond to the quantum states of spatial quantum geometry. These states are obtained via a generalized Fourier transform, where each link l is assigned a SU(2) group element, which is the holonomy of the (Ashtekar) connection and the spins j_i label the corresponding momentum modes of this transformation.

[4]By graphs Γ we mean a collection of L oriented, piecewise differentiable paths, or links, l, which (only) intersect at their end-points, called source $s(l)$ and target $t(l)$, to form N so-called nodes n, such that there are no loose ends.

The conjugate momentum to the holonomy $h_l[A]$ is the so-called flux $X_l \in \mathfrak{su}(2)$, which is obtained by an integration of the densitized triad E^a_j over a 2 - surface dual to the link l [5], [9, 10]. A detailed analysis shows that the variables $(h_l[A], X_l)$ satisfy the (canonical) Poisson brackets of the cotangent bundle $\mathrm{T}^*\mathrm{SU}(2)$, [2, 3], and hence, we see that the underlying (finite dimensional) phase space of a spin network state is given by L copies of the phase space $\mathrm{T}^*\mathrm{SU}(2)$. Thus, individual spin networks allow to study only a finite dimensional truncation of the infinitely many degrees of freedom of full general relativity (in four dimensions). The full field theoretic content of general relativity is formally restored by considering a limit of graph refinements and the full (kinematical) Hilbert space of Loop quantum gravity is defined as

$$\mathcal{H}^{LQG}_{kin} = \bigoplus_{\Gamma} \mathcal{H}^{LQG}_{\Gamma,kin} \,. \tag{1.2}$$

The information about the quantum geometry represented by this type of quantum theory can be extracted using so-called geometric operators, which are the quantization of classical expressions that measure, for example, lengths, areas, angles or volumes. This leads to one of the celebrated results of Loop quantum gravity, namely, that the area dual to a single link l, which carries the spin j_l, is quantized with a minimal area eigenvalue at the Planck scale as

$$\hat{A}\,|j_l\rangle = 8\pi\gamma\hbar G\,\sqrt{j_l(j_l+1)}\,. \tag{1.3}$$

The prefactor γ is the so-called Barbero-Immirzi parameter, which labels a 1 - parameter family of canonical transformations that are used in the derivation of the Ashtekar variables. Its physical meaning in the theory is not fully understood. Similarly, it is found that volume eigenvalues are discrete with a minimal value, [2, 3]. Note, that these are truly derived predictions for the kinematical eigenvalues of those operators, similar to discrete energy spectra in standard quantum mechanics, for example, and are not artifacts of the use of "discrete graphs", as it is sometimes criticized. The discreteness is, however, a result of the compact nature of the gauge group SU(2). This is a topic will will discuss in more detail in chapter 5.

Now, we have emphasized that those predictions hold on the kinematical Hilbert space Eq.(1.2), because the construction of the final physical Hilbert space $\mathcal{H}^{LQG}_{phys}$ has not yet been achieved in four spacetime dimensions. The problem is solved in three dimensions with $\Lambda = 0$, [11], but in the 4D case the implementation of the Hamiltonian constraint

[5] In the 3D case the link l in the spatial hypersurface is only dual to a 1-dimensional surface, i.e., an orthogonal link.

has not been solved in full generality[6]. However, this is rather an issue of mathematical complexity and certain quantization ambiguities, not of obvious or known fundamental obstructions and should be compared with the complexity of solving the Cauchy problem for the Einstein field equations [12].

These difficulties of implementing the "dynamics" in Loop quantum gravity have led to an investigation of the problem in the covariant picture, namely, to so-called spinfoam models. The current model for 4D spinfoam quantum gravity is the so-called EPRL-FK-KKL model named after the authors of [13, 14, 15, 16, 17].

The second main topic of this thesis is the investigation of a generalization of this particular spinfoam model, to allow for the inclusion of timelike contributions. In chapter 5 we will summarize [1] and propose a new spinfoam model based on our findings.

Loop quantum gravity with cosmological constant

The idea for q - deformed Loop quantum gravity, where the deformation corresponds to the replacement of the Lie group SU(2) by a so-called quantum group, or quasitriangular Hopf algebra, such as $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$, is not a new idea. In the process of this deformation one replaces not only the symmetry, but also the representations, which leads to q - deformed spin networks. Already in the early works [18, 19] the authors proposed such a deformation of the standard setting and since then, several investigations followed, for example, in four dimensional spinfoam models [20, 21, 22].

However, up until recently those q - deformed structures were usually introduced by hand into Loop quantum gravity or spinfoam models and the origin of this new symmetry was not really addressed. The motivation for this ad hoc introduction came from other approaches to 3-dimensional gravity or matter models, such as [23, 24, 25, 26], the combinatorial quantization of Chern-Simons theory [27, 28] or the so-called Turaev-Viro state sum model [29]. The Turaev-Viro model, in particular, was not intended at all to be relevant for physics, since it was constructed by mathematicians as a topological invariant of 3-manifolds. However, via an asymptotic analysis of this model it was later revealed that it could represent a discretized path integral for 3D Riemannian gravity with positive cosmological constant [30]. Similarly to the Ponzano-Regge model [31], which can be seen

[6] The kinematical Hilbert space of 4D Loop quantum gravity successfully implements the Gauss constraint, which is the generator of local SU(2) gauge transformations, and carries a representation of the 3-dimensional diffeomorphism group of the spatial hypersurface. The last missing constraint for the completion of the Dirac program is the Hamiltonian- or scalar- constraint, which generates time reparametrizations.

as the spinfoam model for $\Lambda = 0$, the Turaev-Viro model gives the discrete Regge action plus a cosmological constant term in the asymptotic limit.

The reason that the q - deformation is not directly apparent in Loop quantum gravity with $\Lambda \neq 0$ is the fact that Λ only appears as an extra term in the flatness (3D) or Hamiltonian (4D) constraint and hence, how this could lead to modifications of the kinematical Hilbert space was a longstanding puzzle. Recently there have been two approaches to tackle this interesting problem. In [32, 33, 34, 35, 36] the authors studied the 3D Riemannian case with $\Lambda > 0$, starting from the standard SU(2) setting, and showed that a q - deformation becomes necessary to avoid anomalies, when solving the Hamiltonian in the quantum theory. This can be interpreted as a first hint that the true symmetry of the problem is no longer given by SU(2).

The other approach, and the basis for our investigations in this thesis, was put forward in [37, 38, 39, 40, 41, 42] for the case of 3D Riemannian gravity with negative Λ. This series of works focuses more on the mathematical structures behind these deformed symmetries and shows how a non-zero Λ leads indeed to a q - deformation, already at the kinematical level. They showed furthermore, how indeed a Turaev-Viro model for a real deformation parameter q arises via a canonical quantization of the underlying gravitational theory.

Chapter 3 and 4 of this thesis will study the case of 3D Riemannian gravity with positive cosmological constant, along the lines laid out in [37, 38, 39, 40, 41, 42]. Classically, the deformation parameter q is related to the cosmological constant Λ and the spacetime signature $\sigma = \pm 1$ via

$$q = \exp\left(i\, \frac{\hbar G \sqrt{\sigma\Lambda}}{c}\right) \tag{1.4}$$

and we see that in our case q lies on the unit circle. In the quantum theory we will get a quantization condition and we consider q as a primitive l-th root of unity, i.e., $q = \exp(2\pi i/l)$ with $l \in \mathbb{Z}_{>2}$.

The deformation with q being a root of unity leads to many interesting mathematical structures, as we will see below. On the classical side we deal with so-called quasi-Poisson-manifolds and Lie groups [43, 44, 45, 46, 47, 48] and in the quantum theory this translates to so-called quasi-Hopf algebras [49, 50, 51, 52, 53] as the corresponding symmetries. One interesting feature for the q root of unity case are the non-standard fusion rules used in the Turaev-Viro model when building tensor products. For $q = \exp(2\pi i/l)$ we call the numbers (j_1, j_2, j_3) admissible when

$$(j_1, j_2, j_3) \in \left\{0, \frac{1}{2}, 1, \cdots, \frac{l-2}{2}\right\} \quad , \quad j_1 + j_2 + j_3 \in \mathbb{N}_0 \quad , \quad j_1 + j_2 + j_3 \leq l - 2 \quad , \tag{1.5}$$

$$j_1 \leq j_2 + j_3 \qquad , \qquad j_2 \leq j_3 + j_1 \qquad , \qquad j_3 \leq j_1 + j_2 \,. \tag{1.6}$$

We will try to present in this thesis the relationship between these fusion rules and 3D Riemannian gravity with positive cosmological constant. On the classical side we present a detailed investigation into the quasi-Poisson geometry of our system and how we can deform the standard phase space $\mathrm{T}^*\mathrm{SU}(2)$ into the double $D(\mathrm{SU}(2)) = \mathrm{SU}(2) \times \mathrm{SU}(2)$, which is only a quasi-phase space. In the quantum theory we find that the appropriate symmetry of the problem is given by a weak quasi-Hopf algebra $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$, for which we construct so-called tensor operators that allow us to probe the quantum geometry of our q - deformed spin networks. This allows us to confirm that for q root of unity these spin networks represent spherically curved quantum geometry. Our calculations for our geometrical observables, even though significantly more complex, match with their corresponding counterpart from the q real case from [37, 38, 39, 40, 41, 42].

Spinfoams with timelike contributions

In chapter 5 we use the so-called twistorial parametrization of Loop quantum gravity, [54, 55, 56], to investigate spacelike and timelike 2-surfaces on timelike hypersurfaces in four dimensions. We again begin with an investigation at the classical level and prove the symplectic reduction of $\mathrm{T}^*\mathrm{SL}(2,\mathbb{C})$ to $\mathrm{T}^*\mathrm{SU}(1,1)$ by the so-called simplicity constraints with a spacelike normal vector and an area matching constraint. This already indicates that in this setting we will deal with spin networks of the non-compact group $\mathrm{SU}(1,1)$. We present a quantum version of our classical model and find that all our area spectra are discrete.

Furthermore, we use our results to propose a modification of the EPRL spinfoam model, following ideas from [57, 58], which allows us to include timelike contributions in a 4D Lorentzian spinfoam model.

Chapter 2

Gravity in 3 and 4 dimensions with cosmological constant

In this chapter we briefly review how 3D gravity, with and without cosmological constant Λ, can be written as a BF-gauge theory for both Lorentzian and Riemannian signature σ. We consider the different symmetries of the system, which dependent on the sign of both Λ and σ. In section 2.1.1 and 2.1.2 we present certain coordinates that are adapted to the symmetries for non-vanishing cosmological constant. These new coordinates are the equivalent of certain adapted coordinates in the Chern-Simons formulation of 3D gravity, but adopted to our BF-theory formulation. For more details on 3D gravity in the Lorentzian signature we refer to [59]. In section 2.2 we present some preliminary results of 3D - inspired coordinate transformations in the 4D case. The reason for us to study these different coordinates - after all, general relativity famously does not care which coordinates we use - is the hope that these will lead to simplifications for the quantization and elucidate potential "deformed symmetries" when $\Lambda \neq 0$.

2.1 Euclidean and Lorentzian gravity in 3 dimensions

We start from the action Eq.(1.1) with a cosmological constant Λ, where we distinguish the Euclidean case ($\sigma = +1$) and the Lorentzian case ($\sigma = -1$),

$$S_{GR}[g_{\mu\nu}] = \alpha \int_M \Omega_M \, \left(R[g_{\mu\nu}] - 2\,\Lambda\right) = \alpha \int_M d^3x \, \sqrt{\sigma \det(g_{\mu\nu})} \, \left(R[g_{\mu\nu}] - 2\,\Lambda\right). \tag{2.1}$$

Our metric convention is $\eta = (\sigma, 1, 1)$. Furthermore, we fix $\varepsilon_{012} = 1 = \sigma\,\varepsilon^{012}$, where we use coordinates $(x^0, x^1, x^2) = (t, x, y)$ for $\sigma = -1$ or $(x^0, x^1, x^2) = (z, x, y)$ for $\sigma = 1$ and in both cases the coordinates (x, y) are associated with a spatial hypersurface. If we introduce the triad fields $e^I(x) = e^I_\mu(x)\mathrm{d}x^\mu$ such that

$$g_{\mu\nu} = \eta_{IJ} e^I_\mu e^J_\nu \qquad , \qquad e^I_\mu e^\mu_J = \delta^I_J \quad , \quad e^I_\mu e^\nu_I = \delta^\nu_\mu \,, \tag{2.2}$$

we can write the Ricci scalar as

$$R[g_{\mu\nu}] = R_{\mu\nu} g^{\mu\nu} = R^\tau{}_{\mu\tau\nu} g^{\mu\nu} = R^{\tau\nu}{}_{\tau\nu} = R^{IJ}{}_{\tau\nu} e^\tau_I e^\nu_J \,. \tag{2.3}$$

Furthermore, let us note that the volume element can be written in both signatures as

$$\Omega_M = d^3x \sqrt{\sigma \det(g_{\mu\nu})} = \frac{1}{6}\,\varepsilon_{IJK}\, e^I \wedge e^J \wedge e^K = e^0 \wedge e^1 \wedge e^3 = d^3x\, \det(e^I_\mu) \,. \tag{2.4}$$

If we define $R^{IJ} \equiv \frac{1}{2}\, R^{IJ}_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$, we can rewrite the actions as

$$S_{GR}[g_{\mu\nu}] = \alpha \int_M \varepsilon_{IJK} \left(e^I \wedge R^{JK} - \frac{\Lambda}{3}\, e^I \wedge e^J \wedge e^K \right) . \tag{2.5}$$

Variation with respect to the triad e^I gives, for both signatures, the following equations of motion

$$R^{JK} = \Lambda\, e^J \wedge e^K \,. \tag{2.6}$$

If we work in a first order formalism, where we consider the triads and the so-called spin connection ω^{IJ} as independent variables, we can express the curvature of the spin connection as

$$R^{IJ}[\omega] = \mathrm{d}\omega^{IJ} + \omega^I{}_K \wedge \omega^{KJ} \,. \tag{2.7}$$

Introducing the spin connection as an independent variable, however, leads to another equation of motion, the so-called torsion-less condition for the spin connection

$$T^I \equiv (D^\omega e)^I = \mathrm{d}e^I + \omega^I{}_J \wedge e^J = 0 \,. \tag{2.8}$$

We can use the Hodge operator $*$ to write

$$R^I[\omega] \equiv (*R[\omega])^I = \frac{1}{2}\,\varepsilon^I{}_{JK}\, R[\omega]^{JK} \qquad , \qquad R[\omega]^{JK} = \sigma\, \varepsilon^{JK}{}_I\, R^I[\omega] \,, \tag{2.9}$$

where we have

$$\varepsilon_{IJK}\varepsilon^{LMN} = \sigma\,\delta^{LMN}_{IJK} \quad , \quad \varepsilon_{IJK}\varepsilon^{ILM} = \sigma\,\delta^{LM}_{JK} = \sigma\left(\delta^L_J\delta^M_K - \delta^M_J\delta^L_K\right) , \tag{2.10}$$

$$\varepsilon_{IJK}\varepsilon^{IJL} = 2\sigma\,\delta^L_K \qquad , \qquad \varepsilon_{IJK}\varepsilon^{IJK} = 6\sigma\,. \tag{2.11}$$

Thus, we get for the 1-index curvature

$$R^I[\omega] = \mathrm{d}\omega^I - \frac{\sigma}{2}\,\varepsilon^I{}_{JK}\,\omega^J\wedge\omega^K , \tag{2.12}$$

where $\omega^I = \frac{1}{2}\,\varepsilon^I{}_{JK}\,\omega^{JK}$. In terms of R^I the equations of motion Eq.(2.6) become

$$R^I[\omega] = \frac{\Lambda}{2}\,\varepsilon^I{}_{JK}\,e^J\wedge e^K\,. \tag{2.13}$$

Now, if we want to identify the spin connection ω^I with a gauge theory connection A^I, note that we can write Eq.(2.12) as

$$\begin{aligned} R^I[\omega] &= \mathrm{d}\omega^I - \frac{\sigma}{2}\,\varepsilon^I{}_{JK}\,\omega^J\wedge\omega^K = -\sigma\left(\mathrm{d}(-\sigma\omega^I) + \frac{1}{2}\,\varepsilon^I{}_{JK}\,\omega^J\wedge\omega^K\right) \\ &= -\sigma\left(\mathrm{d}(-\sigma\omega^I) + \frac{1}{2}\,\varepsilon^I{}_{JK}\,(-\sigma\omega^J)\wedge(-\sigma\omega^K)\right) = -\sigma\,F[A]^I , \end{aligned} \tag{2.14}$$

where we have defined $A^I \equiv -\sigma\omega^I$ and the gauge theory curvature $F^I[A]$ is given via

$$F^I[A] = \mathrm{d}A^I + \frac{1}{2}\,\varepsilon^I{}_{JK}\,A^J\wedge A^K\,. \tag{2.15}$$

Using Eq.(2.9), we can write the action Eq.(2.5), in terms of the gauge curvature for the connection A^I as

$$S_{GR}[e,A] = \alpha\int_M K^\sigma_{IJ}\,e^I\wedge F^J[A] - \frac{\Lambda}{3}\,\varepsilon_{IJK}\,e^I\wedge e^J\wedge e^K \tag{2.16}$$

$$= S_{BF}[e,A] = \alpha\int_M K^\sigma_{IJ}\left(e^I\wedge F^J[A] + \frac{\sigma\,\Lambda}{12}\,e^I\wedge[e\wedge e]^J\right), \tag{2.17}$$

where we have introduced the Lie algebra valued commutator $[e\wedge e]^J$, which gives

$$-\frac{\Lambda}{3}\,\varepsilon_{IJK}\,e^I\wedge e^J\wedge e^K = \frac{\sigma\Lambda}{12}\,K^\sigma_{IJ}\,e^I\wedge[e\wedge e]^J , \tag{2.18}$$

and the σ - dependent Killing form $K^{\sigma}_{IJ} = (-2\sigma)\,\eta_{IJ} = (-2\sigma)(\sigma, 1, 1)$, which gives the Killing form of $\mathfrak{su}(2)$ for $\sigma = 1$ and the Killing form of $\mathfrak{so}(1, 2)$ for $\sigma = -1$. Thus, we see that general relativity in three spacetime dimensions, for both signatures, can be written as a (topological) BF-gauge theory. The only subtle point being that in general relativity we assume that the tetrad field e^I should be non-degenerate, a requirement that is usually dropped in gauge theory.

Now, in order to obtain the Poisson brackets and identify the conjugate coordinates, note that the only term in the action that contains a time derivative is $\mathrm{d}\omega$ in the curvature $R[\omega]$. If we consider the term $\varepsilon_{IJK}\, e^I \wedge R^{JK}[\omega]$ from the action Eq.(2.5) we can have a look at $\varepsilon_{IJK}\, e^I \wedge \mathrm{d}\omega^{JK}$, which gives

$$\begin{aligned}\varepsilon_{IJK}\, e^I \wedge \mathrm{d}\omega^{JK} &= \frac{1}{2}\,\varepsilon_{IJK}\, e^I_\mu (\mathrm{d}\omega^{JK})_{\nu\rho}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho = e^I_\mu (\mathrm{d}\omega_I)_{\nu\rho}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \\ &= \sigma\, e^I_\mu (\mathrm{d}\omega_I)_{\nu\rho}\, \varepsilon^{\mu\nu\rho}\, d^3x = \sigma\, e_{I,\mu} (\partial_\nu \omega^I_\rho - \partial_\rho \omega^I_\nu)\, \varepsilon^{\mu\nu\rho}\, d^3x \\ &= 2\sigma\, e_{I,\mu} (\varepsilon^{\mu t \rho}\, \partial_t \omega^I_\rho)\, d^3x \;+\; \text{(terms without time derivatives)} \\ &= -2\sigma\, e_{I,a} \varepsilon^{tab}\, \partial_t \omega^I_b\, dt\, d^2x \;+\; (\cdots) \\ &= \frac{1}{\alpha}\, E^b_I\, \partial_t \omega^I_b\, dt\, d^2x \;+\; (\cdots) \qquad , \qquad E^b_I \equiv -2\sigma\,\alpha\,\varepsilon^{tab}\, e_{I,a}\,, \end{aligned} \tag{2.19}$$

and thus we obtain the following Poisson brackets

$$\{\omega^I_a(x), E^b_J(y)\} = \delta^I_J \delta^b_a\, \delta^{(2)}(x - y)\,. \tag{2.20}$$

The full gravitational action Eq.(2.5) becomes

$$\begin{aligned} S_{GR}[\omega, E] = \int_{\mathcal{I}} dt \int_{\Sigma} d^2x \Bigg\{ & E^a_I \partial_t \omega^I_a \;+ (2\alpha)\, e_{I,t} \left(R^I_{ab}[\omega] - \frac{\Lambda}{4\alpha^2}\, \varepsilon^I{}_{JK}\, E^J_a E^K_b \right) \\ & + \; \omega_{I,t} \left(\partial_a E^{I,a} - \sigma\, \varepsilon^I{}_{JK}\, \omega^J_a E^{K,a} \right) \Bigg\}\,, \end{aligned} \tag{2.21}$$

where we clearly recognize the constraints from above and that $e_{I,t}$ and $\omega_{I,t}$ are the corresponding Lagrange multipliers. Note, that the (ab) for the flatness constraints corresponds to $(ab) = (xy)$. Performing a Legendre transformation we obtain the Hamiltonian

(constraint)

$$H(\omega, E) \equiv \int_\Sigma d^2x \;\; E^a_I \partial_t \omega^I_a - L(\omega, E) \qquad , \qquad S_{GR}[\omega, E] = \int_{\mathcal{I}} dt \; L(\omega, E)$$

$$= - \int_\Sigma d^2x \left\{ (2\alpha) \, e_{I,t} \, \left(R^I_{ab}[\omega] - \frac{\Lambda}{4\alpha^2} \, \varepsilon^I{}_{JK} \, E^J_a E^K_b \right) \; + \; \omega_{I,t} \left(\partial_a E^{I,a} - \sigma \, \varepsilon^I{}_{JK} \, \omega^J_a E^{K,a} \right) \right\}$$

$$= - \int_\Sigma d^2x \left\{ \, e_{I,t} \, \mathfrak{C}^I \; + \; \omega_{I,t} \, \mathfrak{T}^I \, \right\} \;\; \approx \;\; 0 \, , \tag{2.22}$$

where we have defined the curvature- and torsion-less constraints

$$\mathfrak{C}^I[\omega, E] \equiv (2\alpha) \left(R^I_{xy}[\omega] - \frac{\Lambda}{4\alpha^2} \, \varepsilon^I{}_{JK} \, E^J_x E^K_y \right) \;\; \approx \;\; 0 \, , \tag{2.23}$$

$$\mathfrak{T}^I[\omega, E] \equiv \left(\partial_a E^{I,a} - \sigma \, \varepsilon^I{}_{JK} \, \omega^J_a E^{K,a} \right) \;\; \approx \;\; 0 \, . \tag{2.24}$$

Furthermore, we can define the smeared constraints

$$\mathfrak{C}[N] \equiv \int_\Sigma d^2x \; N_I(x) \, \mathfrak{C}^I[\omega, E](x) \quad , \quad \mathfrak{T}[N] \equiv \int_\Sigma d^2x \; N_I(x) \, \mathfrak{T}^I[\omega, E](x) \tag{2.25}$$

and together with the Poisson brackets Eq.(2.20) one finds the following commutation relations

$$\{\mathfrak{T}[N], \mathfrak{T}[M]\} = (-\sigma) \, \mathfrak{T}[[N, M]] \, , \tag{2.26}$$

$$\{\mathfrak{T}[N], \mathfrak{C}[M]\} = (-2\alpha\sigma) \, \mathfrak{C}[[N, M]] \, , \tag{2.27}$$

$$\{\mathfrak{C}[N], \mathfrak{C}[M]\} = (-\Lambda) \, \mathfrak{T}[[N, M]] \, , \tag{2.28}$$

which shows that those constraints are all of first class and hence the reduced phase space has $6+6-2\times3-2\times3 = 0$ dynamical degrees of freedom. By $[N, M]$ we denote the $\mathfrak{su}(2)$ (or $\mathfrak{su}(1,1)$) commutator. From this we can clearly identity the following symmetry algebras for the different values of the signature and the sign of the cosmological constant. To better identify the symmetries in the Euclidean case, it is beneficial to use instead the variables $(-\omega, -E)$, which is a symmetry of the Poisson brackets Eq.(2.20). Under this transformations all minus signs in Eq.(2.26), Eq.(2.27) and Eq.(2.28) are absorbed into the transformed constraints.

Signature	$\Lambda = 0$	$\Lambda < 0$	$\Lambda > 0$
Lorentzian	$\mathfrak{iso}(1,2)$	$\mathfrak{so}(2,2) \cong \mathfrak{so}(1,2) \oplus \mathfrak{so}(1,2)$	$\mathfrak{so}(1,3)$
Euclidean	$\mathfrak{iso}(3)$	$\mathfrak{so}(1,3)$	$\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$

For the calculation of the Poisson brackets Eq.(2.26) - Eq.(2.28) we used

$$\frac{\delta \mathfrak{T}[N](x)}{\delta \omega_a^I(y)} = \sigma\, \varepsilon_{IJK}\, N^J(y) E^{K,a}(y) = \sigma\, [N, E^a]_I = \sigma\ \mathrm{ad}_N(E^a)_I\,, \tag{2.29}$$

$$\frac{\delta \mathfrak{T}[N](x)}{\delta E_I^a(y)} = -\left(\partial_a N^I(y) - \sigma\, \varepsilon^I{}_{JK}\, \omega_a^J(y) N^K(y)\right) = -(D_a^\omega N)^I\,, \tag{2.30}$$

$$\frac{\delta \mathfrak{C}[N](x)}{\delta \omega_a^I(y)} = (2\alpha\sigma)\, \varepsilon^{tab} \left(\partial_b N_I(y) - \sigma\, \varepsilon_{IJK}\, \omega_b^J(y) N^K(y)\right) = (2\alpha\sigma)\, \varepsilon^{tab} (D_b^\omega N)_I\,, \tag{2.31}$$

$$\frac{\delta \mathfrak{C}[N](x)}{\delta E_I^a(y)} = \frac{\Lambda}{2\alpha}\, \varepsilon_{tab}\, \varepsilon^I{}_{JK}\, N^J(y) E^{K,b}(y) = \frac{\Lambda}{2\alpha}\, \varepsilon_{tab}\, [N, E^b]^I = \frac{\Lambda}{2\alpha}\, \varepsilon_{tab}\ \mathrm{ad}_N(E^b)^I\,. \tag{2.32}$$

The action of the smeared constraints on the coordinates is calculated as follows

$$\{\mathfrak{T}[N], \omega^I(x)\} = (D^\omega N)^I(x) = \left(\partial_a N^I(x) - \sigma\, \varepsilon^I{}_{JK}\, \omega_a^J(x) N^K(x)\right) \mathrm{d}x^a\,, \tag{2.33}$$

$$\{\mathfrak{T}[N], E^I(x)\} = \sigma\, [N, E]^I(x) = \sigma\ \mathrm{ad}_N(E)^I(x)\,, \tag{2.34}$$

$$\{\mathfrak{T}[N], e^I(x)\} = \sigma\, [N, e\,]^I(x) = \sigma\ \mathrm{ad}_N(e)^I(x)\,, \tag{2.35}$$

$$\{\mathfrak{C}[N], \omega^I(x)\} = \Lambda\, [N, e\,]^I(x) = \Lambda\ \mathrm{ad}_N(e)^I(x) = \Lambda\ \mathrm{ad}_N \left(\frac{1}{2\alpha}\, \varepsilon_{tab} E^a \mathrm{d}x^b\right)^I (x)\,, \tag{2.36}$$

$$\{\mathfrak{C}[N], E^I(x)\} = (-2\alpha\sigma)\, \varepsilon^{ta}{}_b\, (D_a^\omega N^I(x))\, \mathrm{d}x^b\,, \tag{2.37}$$

$$\{\mathfrak{C}[N], e^I(x)\} = (D^\omega N(x))^I\,. \tag{2.38}$$

We see from these actions that the torsion-less constraint generates transformations that do not mix the coordinates, which is good. The flatness constraint, on the other hand, mixes the coordinates, which can cause problems for the discretization and quantization.

To be more precise, let us recall the strategy in the $\Lambda = 0$ case and why it fails for $\Lambda \neq 0$. To describe the physics of a system with constraints it is desirable to get an understanding of the reduced phase space and the induced Poisson structure thereon, which can be quite complicated. This holds similarly for the quantum theory, where one might want to quantize the reduced phase space directly, or, start from an auxiliary (pre-) quantization and then impose the constraints at the quantum level. In the Chern-Simons approach to the quantization of the above system[1] one pursues the first option, where the reduced phase space can be identified with the so-called moduli space of flat connections, cf. section 3.4. This means, one first solves the flatness constraint classically. The moduli space can then be quantized in a space of gauge-invariant functions of the (flat) connections ω, i.e., $f(\omega_{flat}) = f(\omega^G_{flat})$, where ω^G_{flat} is a gauge transformed flat connection. This takes care of the Gauss- or torsion constraint $\mathfrak{T}[N]$, and is possible, if the original Poisson structure is invariant under the gauge transformations, because then it pulls back onto the reduced phase space for gauge invariant functions. Now, it is crucial for this strategy to work, that we have a stable polarization under the gauge transformations, i.e., that after transforming ω_{flat} it is still just a function of ω_{flat}. We see from Eq.(2.37) that this is the reason why (normally) we do not work with a polarization of the conjugate variable E, since in that case the flatness constraint mixes the E- and the ω- sectors. Of course, the reduced phase space itself and the quantization of a system should not depend on the chosen polarization and one can still consider other options, which is for example investigated in the recent [60, 9]. The above described way, however, avoids that we have to study a complicated Dirac reduction of the original Poisson structure, which might be necessary, when we use unsuitable coordinates. The ω - polarization is also what is used in the Loop quantum gravity approach, however, we do not start with flat connections, but (try to) implement the flatness constraint as a quantum operator.

Now, this should already shed a light on the complications that arise when $\Lambda \neq 0$. We see from Eq.(2.36) that in this case the connection not only transforms non-trivially under the flatness constraint, but it mixes with the conjugate variable E. Hence, for $\Lambda \neq 0$ there is no hope of finding a simple quantization in terms of the ω- or E- variables, since neither of those polarizations would be stable under the symmetries of the system.

[1] Instead of the BF-theory formulation we can write 3D gravity, with and without Λ, also as a Chern-Simons gauge theory.

In the following two sections, however, we will show that we can use a suitable change of variables to circumvent these issues. These new coordinates, which also have been used in [35, 36], correspond to similar transformations used in the Chern-Simons formulation, [59], but adapted to the BF - formulation that we are using here.

2.1.1 Phase space coordinate transformations and momentum maps I

Consider the following coordinate transformation

$$\mathcal{X}_a^I \equiv \mu\,\omega_a^I + \nu\,e_a^I \qquad , \qquad \mathcal{Y}_a^I \equiv \mu\,\omega_a^I - \nu\,e_a^I\,, \tag{2.39}$$

with real (constant) functions μ and ν, i.e., they do not depend on ω or e, and inverse relations

$$\omega_a^I = \frac{1}{2\mu}\,(\mathcal{X}+\mathcal{Y})_a^I \qquad , \qquad e_a^I = \frac{1}{2\nu}\,(\mathcal{X}-\mathcal{Y})_a^I\,. \tag{2.40}$$

Note, from $E^a_I = 2\sigma\,\alpha\,\varepsilon^{tab}\,e_{I,b}$ we have $e_a^I = -\frac{1}{2\alpha}\,\varepsilon_{tab}E^{b,I}$. Using Eq.(2.20) we get the following new Poisson bracket relations

$$\{\mathcal{X}_a^I,\mathcal{X}_J^b\} = \frac{\mu\nu}{\alpha}\,\delta_J^I\,\varepsilon_{ta}{}^{b} \quad , \quad \{\mathcal{Y}_a^I,\mathcal{Y}_J^b\} = -\frac{\mu\nu}{\alpha}\,\delta_J^I\,\varepsilon_{ta}{}^{b} \quad , \quad \{\mathcal{X}_a^I,\mathcal{Y}_J^b\} = 0\,. \tag{2.41}$$

Or, more explicitly, the only non-vanishing brackets are given by the following 6 relations

$$\{\mathcal{X}_x^I,\mathcal{X}_I^y\} = \frac{\mu\nu}{\alpha} \qquad , \qquad \{\mathcal{Y}_x^I,\mathcal{Y}_I^y\} = -\frac{\mu\nu}{\alpha}\,. \tag{2.42}$$

This can also be concluded from the fact that one can rewrite the gravitational symplectic potential as follows

$$\int_{\mathcal{I}} dt \int_\Sigma d^2x\;\, E_I^a\partial_t\omega_a^I = \int_{\mathcal{I}} dt \int_\Sigma d^2x\;\, \frac{\alpha}{\mu\nu}\left(\mathcal{X}_{I,y}\partial_t\mathcal{X}_x^I \,+\, \mathcal{Y}_{I,x}\partial_t\mathcal{Y}_y^I\right) \quad + \quad \partial\mathcal{I}\,, \tag{2.43}$$

where $\partial\mathcal{I}$ denotes a boundary term of the time integration. Next, let us rewrite the constraints in this new set of variables and investigate the transformation behavior. The torsion-less constraint Eq.(2.24) can be rewritten as follows

$$\begin{aligned}\mathfrak{T}^I[\omega,E] &\equiv \left(\partial_a E^{I,a} - \sigma\,\varepsilon^I{}_{JK}\,\omega_a^J E^{K,a}\right)\\ &= \frac{\alpha}{\nu}\,\left(\tilde{R}^I_{xy}[\mathcal{X}] \,-\, \tilde{R}^I_{xy}[\mathcal{Y}]\right) \qquad , \qquad \tilde{R}^I_{xy}[\mathcal{X}] = R^I_{xy}[\mathcal{X}](\sigma\mapsto\sigma/\mu)\,.\end{aligned} \tag{2.44}$$

Furthermore, we find that the flatness constraint Eq.(2.23) can be rewritten as follows

$$\mathfrak{C}^I[\omega, E] \equiv (2\alpha) \left(R^I_{xy}[\omega] - \frac{\Lambda}{4\alpha^2}\, \varepsilon^I{}_{JK}\, E^J_x E^K_y \right)$$

$$= \frac{\alpha}{\mu} \, \left(\tilde{R}^I_{xy}[\mathcal{X}] \; + \; \tilde{R}^I_{xy}[\mathcal{Y}] \right) \qquad , \qquad \tilde{R}^I_{xy}[\mathcal{X}] = R^I_{xy}[\mathcal{X}](\sigma \mapsto \sigma/\mu) \tag{2.45}$$

$$= \alpha \, \left(R^I_{xy}[\mathcal{X}] \; + \; R^I_{xy}[\mathcal{Y}] \right) , \tag{2.46}$$

iff : $\nu^2 = \sigma\,\Lambda\,\mu^2$ holds. We set $\mu = 1$ in the last step. In the Euclidean case it is actually better to choose $\mu = -1$. Thus, we can summarize that we should choose $\mu = -\sigma$ and thus we get

$$\mathcal{X}^I_a \equiv -\sigma\,\omega^I_a \pm \sqrt{\sigma\Lambda}\, e^I_a \qquad , \qquad \mathcal{Y}^I_a \equiv -\sigma\,\omega^I_a \mp \sqrt{\sigma\Lambda}\, e^I_a\,. \tag{2.47}$$

Now, recall that we wanted to work with real coefficients μ and ν, in order to not complexify our system. This restricts the applicability of this coordinate transformations to the following cases. Eq.(2.47) shows how we get in the Lorentzian case for $\Lambda < 0$ a $\mathrm{SO}(1,2)\times\mathrm{SO}(1,2)$ gauge potential and in the Euclidean case for $\Lambda > 0$ a $\mathrm{SU}(2)\times\mathrm{SU}(2)$ gauge potential. We see furthermore, that for the Euclidean case we can directly get the change of variables $(\omega, E) \mapsto (-\omega, -E)$, which we have discussed before. Also, note that the constraints $\mathfrak{T}^I[\omega, E] \approx 0$ and $\mathfrak{C}^I[\omega, E] \approx 0$ imply $R^I_{xy}[\mathcal{X}] \approx 0$ and $R^I_{xy}[\mathcal{Y}] \approx 0$ and vice versa.

The full action, up to boundary terms, can be written in the new variables, when $\nu^2 = \sigma\,\Lambda\,\mu^2$ holds, as

$$S_{GR}[\mathcal{X},\mathcal{Y}] = \frac{\alpha}{\sqrt{\sigma\Lambda}} \int_{\mathcal{I}} dt \int_\Sigma d^2x \, \left(\mathcal{X}_{I,y}\partial_t \mathcal{X}^I_x \; + \; \mathcal{X}_{I,t}\, R^I_{xy}[\mathcal{X}] \right) \tag{2.48}$$

$$- \; \frac{\alpha}{\sqrt{\sigma\Lambda}} \int_{\mathcal{I}} dt \int_\Sigma d^2x \, \left(\mathcal{Y}_{I,y}\partial_t \mathcal{Y}^I_x \; + \; \mathcal{Y}_{I,t}\, R^I_{xy}[\mathcal{Y}] \right) ,$$

where the new Lagrange multipliers are given by

$$\mathcal{X}_{I,t} = \mu\,\omega_{I,t} + \nu\, e_{I,t} \qquad , \qquad \mathcal{Y}_{I,t} = \mu\,\omega_{I,t} - \nu\, e_{I,t}\,. \tag{2.49}$$

Thus, we see that the action separates into two independent (Poisson commuting) sectors. Another interesting observation about this change of variables is the fact that for gravity in the (ω, E) variables we have a clear distinction between the prefactor of the action, α, which depends only on the gravitational constant, and the cosmological constant. In the new variables, however, the overall prefactor depends on the gravitational constant as well as on the cosmological constant.

Now we consider again the smeared curvature constraints

$$\mathfrak{R}_{\mathcal{X}}[N] \equiv \int_\Sigma d^2x\, N_I(x)\, R^I_{xy}[\mathcal{X}] \qquad , \qquad \mathfrak{R}_{\mathcal{Y}}[N] \equiv \int_\Sigma d^2x\, N_I(x)\, R^I_{xy}[\mathcal{Y}]\,. \tag{2.50}$$

Since the $\mathcal{X}$ - sector Poisson commutes with the $\mathcal{Y}$ - sector, it is clear that we have

$$\{\mathfrak{R}_{\mathcal{X}}[N], \mathfrak{R}_{\mathcal{Y}}[M]\} = 0\,. \tag{2.51}$$

For the remaining brackets we find

$$\{\mathfrak{R}_{\mathcal{X}}[N], \mathfrak{R}_{\mathcal{X}}[M]\} \;=\; \mathfrak{R}_{\mathcal{X}}[[N,M]] \quad , \quad \{\mathfrak{R}_{\mathcal{Y}}[N], \mathfrak{R}_{\mathcal{Y}}[M]\} \;=\; -\,\mathfrak{R}_{\mathcal{Y}}[[N,M]]\,. \tag{2.52}$$

Note, that via a coordinate transformation of the $\mathcal{Y}$ - sector one could get rid of the minus sign and thus we see that in these variables we obtain two copies of $\mathfrak{su}(2)$ or $\mathfrak{so}(1,2)$, respectively. Taking the overall prefactor of the action into into account, does not change those brackets, which means in particular that the cosmological constant does not show up here. Finally, we consider again the action of those constraints on our new variables and we find

$$\{\mathfrak{R}_{\mathcal{X}}[N], \mathcal{X}^I_a(x)\} \;=\; \partial_a N^I(x) \,+\, \varepsilon^I{}_{JK}\, \mathcal{X}^J_a(x)\, N^K(x) \;=\; (D^{\mathcal{X}}_a N)^I(x)\,, \tag{2.53}$$

$$\{\mathfrak{R}_{\mathcal{X}}[N], \mathcal{Y}^I_a(x)\} \;=\; 0\,, \tag{2.54}$$

$$\{\mathfrak{R}_{\mathcal{Y}}[N], \mathcal{X}^I_a(x)\} \;=\; 0\,, \tag{2.55}$$

$$\{\mathfrak{R}_{\mathcal{Y}}[N], \mathcal{Y}^I_a(x)\} \;=\; -\,\partial_a N^I(x) \,-\, \varepsilon^I{}_{JK}\, \mathcal{Y}^J_a(x)\, N^K(x) \;=\; -\,(D^{\mathcal{Y}}_a N)^I(x)\,. \tag{2.56}$$

Changing coordinates again in the $\mathcal{Y}$ - sector allows to get rid of the overall minus sign in Eq.(2.56). More importantly, however, we see that we found a good set of variables, well adapted to the symmetries of the system, as generated by the first class constraints. We call these new constraints momentum maps, since they are in a form that generate nice transformations on our adapted new phase space coordinates. Comparing with our earlier analysis, we also see clearly that those constraints generate symmetries such that we stay in the same sector of variables, i.e., the $\mathcal{X}$ - sector and $\mathcal{Y}$ - sector are not mixed under the action of those new flatness constraints. We also learned, however, that this coordinate transformation only works if $\sigma\Lambda > 0$, if we want to avoid complexifying our system. In the next section we will find another set of coordinates that is well adapted to the remaining cases.

2.1.2 Phase space coordinate transformations and momentum maps II

We have learned before, by looking at the symmetry algebra generated by the constraints, that we are dealing with different symmetries, depending on the signature of spacetime and the sign of the cosmological constant. The coordinate transformation discussed in the previous section was appropriate for Lorentzian signature and negative Λ and corresponds to the isomorphism $\mathfrak{so}(2,2) \cong \mathfrak{so}(1,2) \oplus \mathfrak{so}(1,2)$, wheras for Euclidean signature with positive Λ we are dealing with $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Now, in this section we want to find a coordinate transformation that allows us to utilize the Iwasawa decomposition of $\mathfrak{so}(1,3)$ for Lorentzian signature with $\Lambda > 0$ and Euclidean signature with $\Lambda < 0$.

Let us next consider the following coordinate transformation, with constants κ and τ and some unspecified vector n^I,

$$\Omega_a^I \equiv \kappa\,\omega_a^I \,+\, \tau\,\varepsilon^I{}_{JK}\,n^J\,e_a^K \qquad , \qquad \Pi_a^I \equiv \tau\,E_a^I\,, \tag{2.57}$$

$$= \kappa\,\omega_a^I \,-\, \frac{\tau}{2\alpha}\,\varepsilon^I{}_{JK}\,n^J\,\varepsilon_{ta}{}^b\,E_b^K \tag{2.58}$$

where in the definition of Ω_a^I we expressed e_a^K again as a function of E_b^K via $e_a^K = -\frac{1}{2\alpha}\,\varepsilon_{tab}\,E^{K,b} = -\frac{1}{2\alpha}\,\varepsilon_{ta}{}^b\,E_b^K$. In these variables we have

$$\{\Omega_a^I(x), \Omega_b^J(y)\} \;=\; 0 \;=\; \{\Pi_a^I(x), \Pi_b^J(y)\}\,, \tag{2.59}$$

$$\{\Omega_a^I(x), \Pi_J^b(y)\} = \kappa\tau\,\{\omega_a^I(x), E_J^b(y)\} = \kappa\tau\,\delta_J^I\delta_a^b\delta^{(2)}(x-y) = \delta_J^I\delta_a^b\delta^{(2)}(x-y)\,. \tag{2.60}$$

We will see below, that we should choose $\kappa = \tau = -\sigma$, and thus we get the last equality for the last Poisson brackets. The inverse relations are given via

$$\omega_a^I = \frac{1}{\kappa}\left(\Omega_a^I \,+\, \frac{1}{2\alpha}\,\varepsilon^I{}_{JK}\,n^J\,\varepsilon_{ta}{}^b\,\Pi_b^K\right) \qquad , \qquad E_a^I = \frac{1}{\tau}\,\Pi_a^I\,. \tag{2.61}$$

Next, we want to investigate again the action and the constraints in those new variables. By plugging those new variables into our action Eq.(2.5), we find that we can cancel the cosmological constant term for a specific choice of the vector n^I. This condition is given by

$$n^2 = n_I n^I = \frac{\kappa^2}{\tau^2}(-\Lambda) = -\Lambda\,, \tag{2.62}$$

where we used again $\kappa = \tau = -\sigma$ in the last equality. From this condition we see that the coordinate transformation Eq.(2.57) is not meaningful for Euclidean signature and $\Lambda > 0$, because in this case we would need to complexify the vector n^I to satisfy condition Eq.(2.62). All other cases with $\Lambda \neq 0$ can be considered using those new variables. We can write the full action in the new variables, up to a boundary term, as

$$S_{GR}[e,\Omega] = (-2\alpha\sigma)\int_M e_I \wedge R^I[\Omega] \;+\; (n_I e^I)\wedge \Omega^J \wedge e_J\,, \tag{2.63}$$

where the curvature is given via $R^I[\Omega] = \mathrm{d}\Omega^I + \frac{1}{2}\,\varepsilon^I{}_{JK}\,\Omega^J\wedge\Omega^K$, for both signatures with $\kappa = \tau = -\sigma$. Now, if we define the Lie algebra valued wedge product for the Lie algebra $\mathfrak{an}(2)_n$ as follows

$$[e\wedge e]^I_{\mathfrak{an}(2)_n} = 2\,C_{AB}{}^I\,e^A\wedge e^B \qquad , \qquad C_{AB}{}^I = \xi\left(\delta^I_A\,n_B \;-\; \delta^I_B\,n_A\right)\,, \tag{2.64}$$

with some coefficient ξ, we see that we can write

$$(n_I e^I)\wedge\Omega^J\wedge e_J = -(n_A e^A)\wedge e^I \wedge \Omega_I = \frac{1}{4\xi}\,[e\wedge e]^I_{\mathfrak{an}(2)_n}\wedge\Omega_I \tag{2.65}$$

and thus we get

$$S_{GR}[e,\Omega] = (-2\alpha\sigma)\int_M e_I\wedge R^I[\Omega] \;+\; \frac{1}{4\xi}\,\Omega_I\wedge[e\wedge e]^I_{\mathfrak{an}(2)_n}\,. \tag{2.66}$$

From now on we will use $\xi = \sigma$ to match our conventions for the Lie algebra $\mathfrak{an}(2)_n$. In terms of those new coordinates (Π,Ω) we get for the full action

$$S_{GR}[\Pi,\Omega] = \int_{\mathcal{I}} dt \int_\Sigma d^2x\;\; \Pi^a_I \partial_t \Omega^I_a \;+\; \Omega_{I,t}\left(T^I_{xy}[\Omega,\Pi] \;-\; \frac{1}{4\alpha}\,\varepsilon_{tab}\,C_{AB}{}^I\,\Pi^{A,a}\,\Pi^{B,b}\right)$$
$$+\;(-2\alpha\sigma)\,e_{I,t}\left(F^I_{xy}[\Omega] \;-\; \frac{1}{2\alpha}\,C_A{}^I{}_B\,\Pi^A_a\,\Omega^{B,a}\right), \tag{2.67}$$

with curvature $F^I_{xy}[\Omega] = \partial_x\Omega^I_y - \partial_y\Omega^I_x + \varepsilon^I{}_{JK}\,\Omega^J_x\,\Omega^K_y$ and torsion $T^I_{xy}[\Omega,\Pi] = \partial_a\Pi^{I,a} + \varepsilon^I{}_{JK}\,\Omega^J_a\Pi^{K,a}$ and we also used

$$C_A{}^I{}_B = \eta^{IE}\eta_{BF}\,C_{AE}{}^F\,, \tag{2.68}$$

where η stands here both for the Euclidean and Lorentzian metric. Furthermore, we have : $\Pi_{I,t} \equiv \Pi^{xy}_I = (-2\alpha\sigma)\,e_{I,t}$. Thus, we get

$$S_{GR}[\Pi,\Omega] = \int_{\mathcal{I}} dt \int_\Sigma d^2x\;\Pi^a_I\partial_t\Omega^I_a \;+\; \Omega_{I,t}\,\mathcal{T}^I[\Omega,\Pi] \;+\; \Pi_{I,t}\,\mathcal{F}^I[\Omega,\Pi]\,, \tag{2.69}$$

with the new constraints

$$\mathcal{T}^I[\Omega,\Pi] \equiv \left(T^I_{xy}[\Omega,\Pi] \;-\; \frac{1}{4\alpha}\,\varepsilon_{tab}\,C_{AB}{}^I\,\Pi^{A,a}\,\Pi^{B,b}\right) \;\approx\; 0\,, \tag{2.70}$$

$$= \left(T^I_{xy}[\Omega,\Pi] \;-\; \frac{1}{2\alpha}\,C_{AB}{}^I\,\Pi^A_x\,\Pi^B_y\right)\,,$$

$$\mathcal{F}^I[\Omega,\Pi] \equiv \left(F^I_{xy}[\Omega] \;-\; \frac{1}{2\alpha}\,C_A{}^I{}_B\,\Pi^A_a\,\Omega^{B,a}\right) \;\approx\; 0 \tag{2.71}$$

and the Poisson structure

$$\{\Omega^I_a(x),\Pi^b_J(y)\} \;=\; \delta^I_J\delta^b_a\delta^{(2)}(x-y)\,. \tag{2.72}$$

The constraints Eq.(2.70) and Eq.(2.71) are still first class constraints and one can show that the following smeared constraints

$$\mathfrak{T}_q[N] \equiv \int_\Sigma d^2x\;N_I(x)\,\mathcal{T}^I[\Omega,\Pi](x) \quad , \quad \mathfrak{F}_q[N] \equiv (-2\alpha)\int_\Sigma d^2x\;N_I(x)\,\mathcal{F}^I[\Omega,\Pi](x)\,, \tag{2.73}$$

satisfy

$$\{\mathfrak{T}_q[N],\mathfrak{T}_q[M]\} = \mathfrak{T}_q[[N,M]]\,, \tag{2.74}$$

$$\{\mathfrak{T}_q[N],\mathfrak{F}_q[M]\} = \mathfrak{F}_q[[N,M]] + \mathfrak{T}_q[[N,M]_{\mathfrak{an}(2)_n}]\,, \tag{2.75}$$

$$\{\mathfrak{F}_q[N],\mathfrak{F}_q[M]\} = \mathfrak{F}_q[[N,M]_{\mathfrak{an}(2)_n}]\,. \tag{2.76}$$

We denote again by $[N,M]$ the $\mathfrak{su}(2)$ commutator and we see that $\mathfrak{T}_q[N]$ still generates a $\mathfrak{su}(2)$ subalgebra. $[N,M]^I_{\mathfrak{an}(2)_n} = C_{AB}{}^I\,N^AM^B$ is the $\mathfrak{an}(2)_n$ commutator. Hence, the (first class) constraints $\mathfrak{T}_q[N]$ and $\mathfrak{F}_q[N]$ again generate $\mathfrak{sl}(2,\mathbb{C})$, however, now corresponding to the Iwasawa decomposition. The action of the constraints on the phase space coordinates is more complicated than what we have found in section 2.1.1, but they show that those new coordinates transform correctly under the adjoint representation of $\mathfrak{sl}(2,\mathbb{C})$ on itself in the form given via the Iwasawa decomposition, cf. Eq.(3.49) and Eq.(3.50) below.

2.2 New coordinates for non-vanishing cosmological constant in 4 dimensions

Inspired by the coordinate transformations of the last section, we want to see how much of this game can be played in four spacetime dimensions. We start from the following action for the tetrad $e^I(x) = e^I_\mu(x)\,\mathrm{d}x^\mu$ and the spin connection $\omega^{IJ}_\mu(x) = -\omega^{JI}_\mu(x)$, with $\sigma = \pm 1$ and $\eta_{IJ} = (\sigma, +, +, +)$,

$$S[e,\omega] = \alpha \int_M \beta_1\,\varepsilon_{IJKL}\,e^I \wedge e^J \wedge R[\omega]^{KL} \;+\; \beta_2\,\varepsilon_{IJKL}\,e^I \wedge e^J \wedge e^K \wedge e^L \qquad (2.77)$$

$$+ \quad \beta_3\,e_I \wedge e_J \wedge R[\omega]^{IJ}\,.$$

This action corresponds to the Einstein-Cartan-Palatini-Holst action (with cosmological constant term) for the following values

$$\alpha = \frac{c^4}{16\pi G} \quad , \quad \beta_1 = \frac{1}{2} \quad , \quad \beta_2 = -\frac{\Lambda}{12} \quad , \quad \beta_3 = \frac{1}{\gamma}\,. \qquad (2.78)$$

The term proportional to β_3 is the topological Holst term (for zero torsion) and γ denotes the Barbero-Immirzi parameter. This action plays an important role in Loop quantum gravity, because adding the Holst term corresponds to the canonical transformation that gives rise to the so-called Ashtekar variables for real γ. Solving the torsion-less condition for the connection ω, which is one of the equations of motion of Eq.(2.77), gives back the Einstein-Hilbert action

$$S_{GR}[g_{\mu\nu}] = \alpha \int_M d^4x \sqrt{\sigma \det(g_{\mu\nu})}\,\left(R[g_{\mu\nu}] - 2\,\Lambda\right)\,, \qquad (2.79)$$

where the Holst term vanishes due to the Bianchi identities, when ω becomes the Levi-Civita connection. Let us consider first the action without Holst term,

$$S[e,\omega,\Lambda] \equiv \alpha \int_M \beta_1\,\varepsilon_{IJKL}\,e^I \wedge e^J \wedge R[\omega]^{KL} \;+\; \beta_2\,\varepsilon_{IJKL}\,e^I \wedge e^J \wedge e^K \wedge e^L\,. \qquad (2.80)$$

We define

$$\Omega^{IJ} \equiv \kappa_1\,\omega^{IJ} + \kappa_2\,\varepsilon^{IJ}{}_{KL}\,n^K e^L\,, \tag{2.81}$$

where again n^I is some vector. We get for the two terms of the curvature $R[\omega]^{IJ}$

$$\mathrm{d}\omega^{IJ} = \frac{1}{\kappa_1}\left(\mathrm{d}\Omega^{IJ} - \kappa_2\,\varepsilon^{IJ}{}_{KL}\,n^K \mathrm{d}e^L\right) \tag{2.82}$$

and

$$\omega^I{}_K \wedge \omega^{KJ} = \frac{1}{\kappa_1^2}\left(\Omega^I{}_K - \kappa_2\,\varepsilon^I{}_{KAB}\,n^A e^B\right) \wedge \left(\Omega^{KJ} - \kappa_2\,\varepsilon^{KJ}{}_{EF}\,n^E e^F\right)\,. \tag{2.83}$$

Note that Ω is still a proper affine connection, if ω was one, provided that the term we are adding is a tensor field. The first term of the action becomes

$$\begin{aligned}
\beta_1\,\varepsilon_{IJKL}\,e^I \wedge e^J \wedge R[\omega]^{KL} = \frac{\beta_1}{\kappa_1}\,\varepsilon_{IJKL}\,e^I \wedge e^J \wedge \left(\mathrm{d}\Omega^{KL} + \frac{1}{\kappa_1}\,\Omega^K{}_N \wedge \Omega^{NL}\right) \\
- \frac{\beta_1\kappa_2}{\kappa_1}\,\varepsilon_{IJKL}\,\varepsilon^{KL}{}_{AB}\,e^I \wedge e^J \wedge (n^A \mathrm{d}e^B) \\
- \frac{\beta_1\kappa_2}{\kappa_1^2}\,\varepsilon_{IJKL}\,e^I \wedge e^J \wedge \left(\Omega^K{}_N \wedge \varepsilon^{NL}{}_{CD}\,n^C e^D + \varepsilon^K{}_{NAB}\,n^A e^B \wedge \Omega^{NL}\right) \\
+ \frac{\beta_1\kappa_2^2}{\kappa_1^2}\,\varepsilon_{IJKL}\,\varepsilon^K{}_{NAB}\,\varepsilon^{NL}{}_{CD}\,e^I \wedge e^J \wedge \left(n^A n^C\,e^B \wedge e^D\right)\,.
\end{aligned} \tag{2.84}$$

The three terms in the second and third line can be combined and the last term is used to get rid of the cosmological constant term in Eq.(2.80) by choosing the vector n^I in a particular way, namely[2],

$$n^2 = n^A n_A = 2\sigma\,\frac{\beta_2}{\beta_1}\left(\frac{\kappa_1}{\kappa_2}\right)^2\,. \tag{2.86}$$

[2] One might wonder whether we would get a different result for the restriction on the norm of the vector n^I, if we chose instead of Eq.(2.81) the transformation

$$\tilde{\Omega}^{IJ} \equiv \kappa_1\,\omega^{IJ} + \kappa_2\,(n^I e^J - n^J e^I)\,, \tag{2.85}$$

i.e., without the Levi-Civita symbol, but one finds that one reproduces exactly the same restriction Eq.(2.86) again. Furthermore, as we will use below, we would still have to choose $n^I = n^0\delta^I_0$, when we want to work with the $\tilde{\Omega}$-connection, instead of the ω-connection and thus, overall, we do not get any benefits from working with Eq.(2.85) over Eq.(2.81).

We can write for the third line in Eq.(2.84)

$$
\begin{aligned}
&\frac{\beta_1\kappa_2}{\kappa_1^2}\,\varepsilon_{IJKL}\,e^I\wedge e^J\wedge\left(\Omega^K{}_N\wedge\varepsilon^{NL}{}_{CD}\,n^C e^D+\varepsilon^K{}_{NAB}\,n^A e^B\wedge\Omega^{NL}\right)\\
&=\;2\sigma\,\frac{\beta_1\kappa_2}{\kappa_1^2}\,\delta^{NCD}_{IJK}\,e^I\wedge e^J\wedge\Omega^K{}_N\wedge(n_C e_D)=\frac{\beta_1\kappa_2}{\kappa_1^2}\,\varepsilon_{IJAB}\,\varepsilon^{NCAB}\,e^I\wedge e^J\wedge\Omega^D{}_N\wedge(n_C e_D)\\
&=\;-\frac{\beta_1\kappa_2}{\kappa_1^2}\,\varepsilon_{IJKL}\,\varepsilon^{IJ}{}_{AB}\,e^K\wedge e^L\wedge(\Omega^A{}_C\wedge e^C)\,n^B\,. \qquad (2.87)
\end{aligned}
$$

Together with the particular value for n^2 from Eq.(2.86) we get for the action with the new connection

$$
\begin{aligned}
S[e,\Omega]=\alpha\int_M\frac{\beta_1}{\kappa_1}\,\varepsilon_{IJKL}\,e^I\wedge e^J\wedge\left(\mathrm{d}\Omega^{KL}+\frac{1}{\kappa_1}\,\Omega^K{}_N\wedge\Omega^{NL}\right) \qquad (2.88)\\
+\;\frac{\beta_1\kappa_2}{\kappa_1}\,\varepsilon_{IJKL}\,\varepsilon^{KL}{}_{AB}\,e^I\wedge e^J\wedge\left(\mathrm{d}e^A+\frac{1}{\kappa_1}\,\Omega^A{}_C\wedge e^C\right)n^B\,.
\end{aligned}
$$

We fix now the value $\kappa_1=1$ and get

$$
S[e,\Omega]=\alpha\int_M\beta_1\,\varepsilon_{IJKL}\,e^I\wedge e^J\wedge\left(R[\Omega]^{KL}\;+\;\kappa_2\,\varepsilon^{KL}{}_{AB}\,T^A[e,\Omega]\;n^B\right). \qquad (2.89)
$$

Also, let us note the following difference between the 3D and 4D case. In 3D we also get a term including de from our coordinate transformation Eq.(2.57), however, there it can be absorbed in a boundary term. In 4D this is not possible and the term with de survives. Thus, we have the torsion showing up. Now, if we add the Holst term again, we find that with the same coordinate transformation Eq.(2.81) we do not get a change for condition Eq.(2.86), i.e., we can absorb the cosmological constant term as before, and we get for the action with the new connection

$$S_\gamma[e,\Omega,\Lambda] = \alpha \int_M \frac{\beta_1}{\kappa_1}\,\varepsilon_{IJKL}\, e^I\wedge e^J\wedge \left(\mathrm{d}\Omega^{KL} + \frac{1}{\kappa_1}\,\Omega^K{}_N\wedge\Omega^{NL}\right) \tag{2.90}$$

$$+ \quad \frac{\beta_3}{\kappa_1}\, e_I\wedge e_J\wedge\left(\mathrm{d}\Omega^{IJ} + \frac{1}{\kappa_1}\,\Omega^I{}_K\wedge\Omega^{KJ}\right)$$

$$+ \quad \frac{\beta_1\kappa_2}{\kappa_1}\,\varepsilon_{IJKL}\,\varepsilon^{KL}{}_{AB}\, e^I\wedge e^J\wedge\left(\mathrm{d}e^A + \frac{1}{\kappa_1}\,\Omega^A{}_C\wedge e^C\right) n^B$$

$$+ \quad \frac{\beta_3\kappa_2}{\kappa_1}\,\varepsilon_{IJKL}\, e^I\wedge e^J\wedge\left(\mathrm{d}e^K - \frac{2}{\kappa_1}\,\Omega^K{}_N\wedge e^N\right) n^L\,.$$

Now, note that the second term including a "de", i.e.,

$$\frac{\alpha\beta_3\kappa_2}{\kappa_1}\,\varepsilon_{IJKL}\, e^I\wedge e^J\wedge \mathrm{d}e^K\, n^L\,, \tag{2.91}$$

can be written as a boundary term, i.e.,

$$\frac{\alpha\beta_3\kappa_2}{\kappa_1}\,\varepsilon_{IJKL}\, e^I\wedge e^J\wedge \mathrm{d}e^K\, n^L = \mathrm{d}\left(\frac{\alpha\beta_3\kappa_2}{3\kappa_1}\,\varepsilon_{IJKL}\, e^I\wedge e^J\wedge e^K\, n^L\right)\,. \tag{2.92}$$

This leaves us with the first term "de". Now, if we perform a the Hamiltonian analysis of this action we find that, due to the fact that the term de does not vanish, we still end up with the small ω as our canonical coordinate and not the capital Ω. However, in order to obtain the Ashtekar variables from Eq.(2.90) (for $\Lambda = 0$), we have to impose the so-called time gauge $e^I_\mu \mapsto e^0_a = 0$, where a is a spatial coordinate index. We find that we can work with the capital Ω connection as our canonical coordinate, and built the Ashtekar connection from Ω instead of ω, if we impose not just a restriction on the norm of the vector n^I, but also fix $n^I \propto \delta^I_0$, i.e.,

$$n^I = (n^0,0,0,0)^t \tag{2.93}$$

Then Eq.(2.86) becomes

$$n^2 = n^A n_A = \sigma(n^0)^2 = 2\sigma\,\frac{\beta_2}{\beta_1}\left(\frac{\kappa_1}{\kappa_2}\right)^2 = -\frac{\sigma\,\Lambda}{3} \quad\Rightarrow\quad n^0 = \sqrt{-\frac{\Lambda}{3}}\,, \tag{2.94}$$

where we have chosen the preferred values $\beta_1 = \frac{1}{2}$, $\beta_2 = -\frac{\Lambda}{12}$ and $(\kappa_1/\kappa_2)^2 = 1$ in the second-last equality. From this we see that we get a restriction on the coordinate transformation $\omega \to \Omega$, if we want to work with real variables, since it only works for negative Λ. In order to see whether this transformation is useful at all, we have to perform a similar analysis as before in the 3D case, which is still work in progress and more complicated because of the time gauge. The big difference with the 3D case, however, is that generally the constraints of general relativity in four dimensions do not form a nice Lie algebra, but a more complicated structure, with structure functions instead of structure constants. Our hope is that we get a similar "deformation" of the 4D constraints as in Eq.(2.70) and Eq.(2.71), which would transfer some information about the cosmological constant from the Hamiltonian constraint into the Gauss constraint and hence, indicate a deformation of the kinematics, especially, indicate a q - deformation of the spin networks in the 4D setting as well.

Chapter 3

Poisson-Lie groups and quasi-Poisson manifolds

Lie groups and the concept of phase space in classical mechanics are notions familiar to every physics student. In this chapter we will consider so-called Poisson-Lie groups, following mainly [61], which are a combination of those aforementioned structures, i.e., Lie groups that have a compatible Poisson structure. After a brief review of those topics we will then move on to a generalization of Poisson-Lie groups called quasi-Poisson-Lie groups, following [44, 62, 63, 64], where the "quasi-" denotes a specific relaxation of the Jacobi identity of the allowed Poisson structures.

Poisson-Lie groups can be seen as the underlying classical structure of certain types of quantum objects called quantum groups, which, in turn, can be understood as deformations of classical Lie group symmetries[1]. Similarly, quasi-Poisson-Lie groups, or their infinitesimal analogs, so-called quasi-Lie bialgebras, can be seen as the underlying classical structure of certain quasi-Hopf algebras.

The notion of Poisson-Lie groups was successfully applied to the study of 3-dimensional Euclidean gravity in [39, 40, 38], to solve the puzzle how a (negative) cosmological constant leads to a deformation of the classical symmetry to a quantum group symmetry in Loop quantum gravity. In those works the underlying symmetry of the system is given by $\mathrm{SL}(2,\mathbb{C})$, which can be seen almost as the standard example of (the double of) a Poisson-Lie group. Furthermore, when seen as a phase space, $\mathrm{SL}(2,\mathbb{C})$ can be considered in a precise sense as a deformation of $\mathrm{T}^*\mathrm{SU}(2)$, which is the fundamental phase space used

[1] The term "quantum group" denotes here a quasi-triangular (quasi-) Hopf algebra of the Drinfeld-Jimbo type $\mathcal{U}_q(\mathfrak{g})$. Otherwise, there might not be a connection with a standard Lie group symmetry.

in Loop quantum gravity for $\Lambda = 0$. When $\Lambda > 0$, as seen in chapter 2, the symmetry group of 3-dimensional Euclidean gravity is SO(4). We will describe how we learned that for an understanding of SO(4), along the lines of the the $SL(2, \mathbb{C})$ case, we have to use the generalized notion of quasi-Poisson-Lie group. We then investigate the relevant quasi-Poisson structure and calculate the deformed brackets of different (quasi-) phase space coordinates that represent the deformed analogs known from the $\Lambda = 0$ case in section 3.4. Most of those results contribute to or extend recent work in Loop quantum gravity, such as [47] and [48]. We close this chapter with some comments on the classical analog of the q root of unity fusion rules in section 3.4.3.

3.1 Poisson-Lie groups and Lie bialgebras

Most, if not all, (classical) physical systems can be given a description in terms of generalized coordinates q^i and their conjugate momenta p_i, satisfying the Poisson bracket relation $\{q^i, p_j\} = \delta^i_j$. The q^i are the coordinates of a smooth manifold Q, the configuration space, and the momenta are elements of the cotangent space T^*Q. Together with the Poisson brackets the cotangent bundle is the standard example of a classical phase space, $\mathcal{P} = (T^*Q, \{\cdot\,,\,\cdot\})$. This structure is captured in the general definition of a Poisson structure:

Definition (Poisson structure) : Let M be a smooth manifold M and denote the smooth functions on M by $\mathcal{C}^\infty(M)$. A Poisson structure on M is a $\mathbb{R}$ - bilinear map

$$\{\cdot\,,\,\cdot\}_M \,:\, \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)\,, \tag{3.1}$$

such that for all $f_1, f_2, f_3 \in \mathcal{C}^\infty(M)$ we have

$$\{f_1, f_2\}_M = -\{f_2, f_1\}_M\,, \tag{3.2}$$

$$\{f_1, \{f_2, f_3\}\}_M + \{f_3, \{f_1, f_2\}\}_M + \{f_2, \{f_3, f_1\}\}_M = 0\,, \tag{3.3}$$

$$\{f_1 f_2, f_3\}_M = f_1\{f_2, f_3\}_M + \{f_1, f_3\}_M f_2\,. \tag{3.4}$$

We consider only finite dimensional manifolds M in this thesis and if the manifold under consideration is obvious, we will drop the subscript M on the so-called *Poisson bracket* $\{\cdot\,,\,\cdot\}_M$. Eq.(3.2) tells us that the Poisson bracket is anti-symmetric under exchange of the two arguments and Eq.(3.3) is the so-called *Jacobi identity*. Eq.(3.4) is the so-called derivation property or *Leibniz identity*. A smooth manifold M together with a Poisson structure $\{\,\cdot\,,\,\cdot\,\}_M$ is called a *Poisson manifold*. Let us also note that the manifold M does not necessarily have to be even dimensional, whereas, if we use the term phase space, we mean that the underlying manifold has even dimensions and allows for a split into n coordinates q^i and n momenta p_i.

An equivalent way to characterize a Poisson structure on M is via the so-called **Poisson bivector** Π_M, which is determined via

$$\{f_1, f_2\}_M = \langle \Pi_M | df_1 \otimes df_2 \rangle \,, \tag{3.5}$$

where $\Pi_M \in \bigwedge^2 TM$ and $df_i(m) \in T^*_m M$. $\langle\,\cdot\,|\,\cdot\,\rangle$ denotes here the (linear extension of the) natural pairing between T^*M and TM. In coordinates m^i of M Eq.(3.5) can be written as

$$\{f_1, f_2\}_M(m) = \Pi_M^{ij}(m)\,\frac{\partial f_1(m)}{\partial m^i}\frac{\partial f_2(m)}{\partial m^j}\,, \tag{3.6}$$

where we used the Einstein summation convention, i.e., summation over repeated indices is understood, unless specified otherwise. The so-called *trivial Poisson structure* is given via $\Pi_M \equiv 0$. If the Poisson bivector Π_M is everywhere non-degenerate, i.e., $\Pi_M(df_1, df_2)(m) \neq 0$ for all $f_i \in \mathcal{C}^\infty(M)$ and $m \in M$, then it is called *symplectic*. This implies that it can be inverted to give an everywhere non-degenerate, closed 2-form $\omega = \Pi_M^{-1}$, where $\mathrm{d}\omega = 0$ is equivalent to the Jacobi-identity.

Now, let us consider maps between different Poisson manifolds.

Definition (Poisson map) : A smooth map $F : N \to M$ between two Poisson manifolds $(N, \{\,\cdot\,,\,\cdot\,\}_N)$ and $(M, \{\,\cdot\,,\,\cdot\,\}_M)$ is called a Poisson map if it satisfies

$$\{f_1, f_2\}_M \circ F = \{f_1 \circ F, f_2 \circ F\}_N \quad , \quad \forall f_1, f_2 \in \mathcal{C}^\infty(M)\,, \tag{3.7}$$

which implies that the Poisson bivectors Π_N and Π_M are related via $(dF)(\Pi_N) = (\Pi_M) \circ F$, where (dF) denotes the tangent map of F.

Given two Poisson manifolds $(M, \{\,\cdot\,,\,\cdot\,\}_M)$ and $(N, \{\,\cdot\,,\,\cdot\,\}_N)$ we can define a Poisson structure on the product manifold $M \times N$ as follows.

Definition (Product Poisson structure) : On the product manifold $M \times N$ we define the Product Poisson structure via

$$\{f_1, f_2\}_{M\times N}(m, n) = \{f_1(\,\cdot\,, n), f_2(\,\cdot\,, n)\}_M(m) + \{f_1(m,\,\cdot\,), f_2(m,\,\cdot\,)\}_N(n)\,, \tag{3.8}$$

for $f_1, f_2 \in \mathcal{C}^\infty(M \times N)$ and $m \in M$, $n \in N$.

A simple example for a Poisson manifold, used for instance in Newtonian point particle mechanics, is given when M is a (finite and $2n$ - dimensional) vector space $M = V \times V$, $\dim(V) = n$. Then we can express every $x \in V$ in terms of global coordinates $\{x^i\}$ and similarly every corresponding momentum as $p(x) = \{p_i(x)\} \in T^*_x V \cong V$. The functions $f \in \mathcal{C}^\infty(M)$ are just $f = f(x^i, p_j)$. The standard Poisson structure in this case is given by

$$\{f_1, f_2\}(x,p) = \frac{\partial f_1(x,p)}{\partial x^i}\frac{\partial f_2(x,p)}{\partial p_i} - \frac{\partial f_1(x,p)}{\partial p_i}\frac{\partial f_3(x,p)}{\partial x^i} \,. \tag{3.9}$$

More interesting examples occur when M is a non-trivial (e.g., curved) manifold, as we will see later on. For example, there is a canonical Poisson structure for the cotangent bundle T^*M of a (pseudo-) Riemannian manifold M. In this case the momentum space $T^*_m M$, however, is still a linear vector space. A special case of this type is the Poisson structure on coadjoint orbits in the dual of a Lie algebra $\mathfrak{g}^*$. The coadjoint orbits themselves are usually curved manifolds, unless the Lie algebra $\mathfrak{g}$ is abelian, and the dual $\mathfrak{g}^*$ is again a linear vector space. For functions of $\xi \in \mathfrak{g}^*$ we can write

$$\{f_1, f_2\}(\xi) = \langle [df_1, df_2](\xi) | \xi \rangle = \Pi^{ij}(\xi)\frac{\partial f_1(\xi)}{\partial \xi^i}\frac{\partial f_2(\xi)}{\partial \xi^j} = f^{ijk}\langle x_k | \xi \rangle \frac{\partial f_1(\xi)}{\partial \xi^i}\frac{\partial f_2(\xi)}{\partial \xi^j} \,, \tag{3.10}$$

where $[x_i, x_j] = f_{ij}{}^k\, x_k$ is the Lie algebra structure of $\mathfrak{g}$. Note, that $(\mathfrak{g}^*)^* \cong \mathfrak{g}$ and thus, we identified $df(\xi)_i \cong x_i$ and get $\langle x_k | \xi \rangle = \xi_k$. This is the famous *Kirillov-Kostant-Souriau symplectic structure* on the coadjoint orbits of a Lie group.

Now, unlike the previous examples, Poisson-Lie groups are concerned with the case where both the coordinate manifold, as well as the momentum manifold, carry curvature.

Definition (Poisson-Lie group) : A Poisson-Lie group is a Lie group G, which means in particular that it is a smooth manifold, together with a Poisson structure $\{\,\cdot\,,\,\cdot\,\}_G$ that is compatible with the group multiplication in the sense that $\mu : G \times G \to G$, $\mu(g_1, g_2) = g_1 g_2$ is a Poisson map, where $G \times G$ is given the product Poisson structure, as defined above.

The Lie algebra of a Poisson-Lie group is a so-called **Lie bialgebra**. Note, that in general the left- and right- translations L_g and R_g, as well as the inversion map $i : g \mapsto g^{-1}$, are in general not Poisson maps.

The compatibility requirement for a Poisson-Lie group means that the identity

$$(\{f_1, f_2\}_G \circ \mu)(g_1, g_2) = \{f_1, f_2\}_G(g_1 g_2) \stackrel{!}{=} \{f_1 \circ \mu, f_2 \circ \mu\}_{G\times G}(g_1, g_2) \\ = \{f_1 \circ R_{g_2}, f_2 \circ R_{g_2}\}(g_1) + \{f_1 \circ L_{g_1}, f_2 \circ L_{g_1}\}(g_2) \tag{3.11}$$

must hold, where $f_1, f_2 \in \mathcal{C}^\infty(G)$ and $g_1, g_2 \in G$. As a reminder, left-multiplication L_g and right-multiplication R_g are given by the following diffeomorphism $L_g, R_g : G \to G$ with $L_g(h) = gh$ and $R_g(h) = hg$. Note furthermore that L_g is a left-action, whereas R_g is a right-action, because $L_{(g_1 g_2)} = (L_{g_1} \circ L_{g_2})$ and $R_{(g_1 g_2)} = (R_{g_2} \circ R_{g_1})$.

In terms of the associated Poisson bivector Π_G the condition Eq.(3.11) can equivalently be written as

$$\Pi_G(g_1 g_2) = ((L_{g_1})_* \otimes (L_{g_1})_*)(\Pi_G)(g_2) + ((R_{g_2})_* \otimes (R_{g_2})_*)(\Pi_G)(g_1)\,, \tag{3.12}$$

where $(L_{g_1})_*$ and $(R_{g_2})_*$ denote the push-forward of the left- and right- multiplications, respectively. Note, that for any Poisson-Lie group we have $\{f_1, f_2\}_G(e) = 0$, or equivalently $\Pi_G(e) = 0$, which means that non-trivial Poisson-Lie structures are never symplectic.

An explicit example for the bivector of a connected semi-simple or compact Poisson-Lie group G can be given via

$$\Pi^\rho_G(g) \equiv (L_g)_*\rho - (R_g)_*\rho\,, \tag{3.13}$$

where $\rho \in \mathfrak{g} \wedge \mathfrak{g}$ and $\mathrm{ad}^{(3)}_X [\![\rho, \rho]\!] = 0$ for all $X \in \mathfrak{g}$. It is easy to show that this bivector indeed satisfies Eq.(3.12).

We saw in Eq.(3.11) that Poisson-Lie groups satisfy a compatibility requirement with respect to the group multiplication. Another condition, which could also be considered to be part of the general definition of a Poisson-Lie group, requires that any Lie group homomorphism $\Phi : G \to H$, i.e., homomorphism that satisfy $\Phi(g_1 g_2) = \Phi(g_1)\Phi(g_2) = h_1 h_2 \in H$, where $\Phi(g_i) = h_i$, are Poisson maps. This means that the Poisson structure must satisfy the compatibility condition

$$(\{f_1, f_2\}_H \circ \Phi)(g_1 g_2) = \{f_1, f_2\}_H(h_1 h_2) \stackrel{!}{=} \{f_1 \circ \Phi, f_2 \circ \Phi\}_G(g_1 g_2)\,. \tag{3.14}$$

If we study the action of a Poisson-Lie group G on some manifold M, $\varphi : G \times M \to M$, we might want to require the *equivariance* of the Poisson structure under the action. This means, we require that

$$(\{f_1, f_2\}_M \circ \varphi)(g, m) = \{f_1, f_2\}_M(m') \stackrel{!}{=} \{f_1 \circ \varphi, f_2 \circ \varphi\}_{G\times M}(g, m)\,, \tag{3.15}$$

where $\varphi(g,m) = m'$ and

$$\{f_1 \circ \varphi, f_2 \circ \varphi\}_{G\times M}(g,m) = \{(f_1 \circ \varphi)(\,\cdot\,,m),(f_2 \circ \varphi)(\,\cdot\,,m)\}_G(g) \\ + \{(f_1 \circ \varphi)(g,\,\cdot\,),(f_2 \circ \varphi)(g,\,\cdot\,)\}_M(m)\,. \tag{3.16}$$

This structure becomes especially interesting when $\{(f_1 \circ \varphi)(\,\cdot\,,m),(f_2 \circ \varphi)(\,\cdot\,,m)\}_G(g)$ does not vanish, which is the case for non-trivial Poisson-Lie groups, as they appear for example as semi-classical limits of certain Hopf algebras. Furthermore, it shows that if we have a certain action $\varphi : G \times M \to M$ and non-trivial Poisson brackets on the manifold M, we can possibly find non-vanishing brackets on the group G, such that we can still satisfy the equivariance condition.

As mentioned before, the infinitesimal version of a Poisson-Lie group is called a Lie bialgebra. We will now consider what are the extra structures on top of the Lie algebra structure, that make G into a Poisson Lie group.

Definition (Lie bialgebra) : A Lie bialgebra is a pair $(\mathfrak{g}, \delta_{\mathfrak{g}})$, with a Lie algebra $\mathfrak{g}$ and a skew-symmetric, linear 1-cocylce $\delta_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$, called the *cocommutator*. $\delta_{\mathfrak{g}}$ satisfies the cocycle condition

$$\begin{aligned} \delta_{\mathfrak{g}}([X,Y]) &= (\mathrm{ad}_X \otimes 1 + 1 \otimes \mathrm{ad}_X)\, \delta_{\mathfrak{g}}(Y) - (\mathrm{ad}_Y \otimes 1 + 1 \otimes \mathrm{ad}_Y)\, \delta_{\mathfrak{g}}(X) \\ &= (\mathrm{ad}_X^{(2)})\, \delta_{\mathfrak{g}}(Y) - (\mathrm{ad}_Y^{(2)})\, \delta_{\mathfrak{g}}(X) \end{aligned} \tag{3.17}$$

and defines a canonical Lie algebra structure on the dual Lie algebra $\mathfrak{g}^*$ via

$$[\xi_1, \xi_2]_{\mathfrak{g}^*} = (d\{f_1, f_2\})_e = \delta_{\mathfrak{g}}^*(\xi_1 \otimes \xi_2) \tag{3.18}$$

where $\xi_i \in \mathfrak{g}^*$ and $\xi_i = (df_i)_e$. We furthermore require that for Lie algebra homomorphisms $\varphi : \mathfrak{g} \to \mathfrak{h}$ we have $(\varphi \otimes \varphi) \circ \delta_{\mathfrak{g}} = \delta_{\mathfrak{h}} \circ \varphi$.

Eq.(3.18) can also be written also as

$$\langle X|[\xi_1, \xi_2]_{\mathfrak{g}^*}\rangle = \langle X|d\{f_1, f_2\}_e\rangle = \langle X|\delta_{\mathfrak{g}}^*(\xi_1 \otimes \xi_2)\rangle = \langle \delta(X)|\xi_1 \otimes \xi_2\rangle\,, \tag{3.19}$$

where $X \in \mathfrak{g}$ and $\langle\,\cdot\,|\,\cdot\,\rangle$ is the (linear extension of the) dual pairing $\langle\,\cdot\,|\,\cdot\,\rangle : \mathfrak{g} \times \mathfrak{g}^* \to \mathbb{R}$.

Note, that if $(\mathfrak{g},\delta_{\mathfrak{g}})$ is a Lie bialgebra then $(\mathfrak{g},k\,\delta_{\mathfrak{g}})$ is also a Lie bialgebra for any scalar k. In particular, we call $(\mathfrak{g},-\delta_{\mathfrak{g}})$ the *opposite Lie bialgebra*.

The relationship between the definition of a Lie bialgebra and a Poisson-Lie group, as introduced above, is given via a relation between the infinitesimal version of the Poisson bivector and the cocommutator as follows. First, we can rewrite the Poisson bracket in terms of the Poisson bivector as follows

$$
\begin{aligned}
\{f_1,f_2\}_G(g) &= \langle \Pi_G(g)|(\mathrm{d}f_1\otimes \mathrm{d}f_2)(g)\rangle \\
&= \langle (R_g\otimes R_g)_*(R_{g^{-1}}\otimes R_{g^{-1}})_*\Pi_G(g)|(\mathrm{d}f_1\otimes \mathrm{d}f_2)(g)\rangle \\
&= \langle (R_{g^{-1}}\otimes R_{g^{-1}})_*\Pi_G(g)|((R_g\otimes R_g)_*)^*(\mathrm{d}f_1\otimes \mathrm{d}f_2)(g)\rangle \\
&= \langle \Pi_G^R(g)|((R_g\otimes R_g)_*)^*(\mathrm{d}f_1\otimes \mathrm{d}f_2)(g)\rangle\,,
\end{aligned}
\tag{3.20}
$$

where we have defined $\Pi_G^R(g)\equiv(R_{g^{-1}}\otimes R_{g^{-1}})_*\Pi_G(g)$. Now, if we write $g=\exp(tX)$ and consider the derivative of Eq.(3.20) with respect to t at $t=0$, we find for the left-hand side

$$
\left.\frac{d}{dt}\right|_{t=0}\{f_1,f_2\}_G(\exp(tX))=\{f_1,f_2\}_G(e)+\mathrm{d}\{f_1,f_2\}_G(X)=\langle X|\mathrm{d}\{f_1,f_2\}_G(e)\rangle\,, \tag{3.21}
$$

where we used that $\{f_1,f_2\}_G(e)=0$, and for the right-hand side we get

$$
\begin{aligned}
&\langle \mathrm{d}\Pi_G^R(g)|((R_g\otimes R_g)_*)^*(\mathrm{d}f_1\otimes \mathrm{d}f_2)(g)\rangle_e+\langle \Pi_G^R(g)|\mathrm{d}[((R_g\otimes R_g)_*)^*(\mathrm{d}f_1\otimes \mathrm{d}f_2)(g)]\rangle_e \\
&=\langle \mathrm{d}\Pi_G^R(e)|(\mathrm{d}f_1\otimes \mathrm{d}f_2)(e)\rangle\,,
\end{aligned}
\tag{3.22}
$$

where we used that $\Pi_G^R(e)=0$ and $[((R_g\otimes R_g)_*)^*(\mathrm{d}f_1\otimes \mathrm{d}f_2)(g)]_e=(\mathrm{d}f_1\otimes \mathrm{d}f_2)(e)$. Thus, if we identify now

$$
\mathrm{d}\Pi_G^R(e)=\left.\frac{d}{dt}\right|_{t=0}\Pi_G^R(\exp(tX))\equiv\delta(X)\,, \tag{3.23}
$$

then we see that the infinitesimal version of the Poisson bracket relation Eq.(3.20) can be written as

$$
\langle X|\mathrm{d}\{f_1,f_2\}_G(e)\rangle=\langle \mathrm{d}\Pi_G^R(e)|(\mathrm{d}f_1\otimes \mathrm{d}f_2)(e)\rangle=\langle\delta(X)|\xi_1\otimes\xi_2\rangle\,, \tag{3.24}
$$

where we have identified $(\mathrm{d}f_i)(e) = \xi_i$ and thus reproduced Eq.(3.19). We see that Eq.(3.23) provides the mentioned relationship between the infinitesimal Poisson bivector of the group and the cocommutator of the Lie bialgebra. Note, that for the Poisson bivector we defined in Eq.(3.13) we get

$$\begin{aligned}\Pi_G^{\rho\, R}(g) &= (R_{g^{-1}})_* \Pi_G^{\rho}(g) = (R_{g^{-1}})_* \left((L_g)_*\rho - (R_g)_*\rho\right) \\ &= \left((R_{g^{-1}})(L_g)\right)_* \rho - \rho = \left(\mathrm{Ad}_g\right)_* \rho - \rho \,. \end{aligned} \tag{3.25}$$

Now, if we have a Lie bialgebra structure for a Lie algebra $\mathfrak{g}$, which naturally induces a (dual) Lie bialgebra structure on $\mathfrak{g}^*$, one can define a Lie bialgebra structure on $\mathfrak{g} \oplus \mathfrak{g}^*$ (direct sum of vector spaces) as follows.

Definition (Classical double) : The classical double $(\mathfrak{d}(\mathfrak{g}), \delta_{\mathfrak{d}})$ associated to a Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$ is the canonical Lie bialgebra defined on $\mathfrak{g} \oplus \mathfrak{g}^*$, where $\mathfrak{g}$ and $\mathfrak{g}^*$ are Lie subalgebras of $\mathfrak{g} \oplus \mathfrak{g}^*$, such that the inclusions $\mathfrak{g} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}^* \hookleftarrow (\mathfrak{g}^*)^{\mathrm{op}}$ are homomorphisms of Lie bialgebras.

We define a non-degenerate, symmetric bilinear form on $\mathfrak{g} \oplus \mathfrak{g}^*$ via

$$(X, \xi) = \langle X|\xi\rangle \quad , \quad (X, Y) = 0 = (\xi, \eta) \quad , \quad X, Y \in \mathfrak{g}\,,\ \xi, \eta \in \mathfrak{g}^*\,, \tag{3.26}$$

where $\langle X|\xi\rangle$ is the dual pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$[2]. The cocommutator of $\mathfrak{d}(\mathfrak{g})$ is defined with $r \in \mathfrak{g} \otimes \mathfrak{g}^*$ via

$$\delta_{\mathfrak{d}}(u) = (\mathrm{ad}_u \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_u)(r)\,, \tag{3.27}$$

where $u \in \mathfrak{d}(\mathfrak{g})$ and r corresponds to the identity map $\mathfrak{g} \to \mathfrak{g}$, i.e., $r = X_i \otimes \xi^i$.

Note that $\mathfrak{g} \oplus \mathfrak{g}^*$ is in general not a direct sum of Lie algebras, i.e., we do not require that $[\mathfrak{g}, \mathfrak{g}^*] = 0$. Furthermore, the inner product defined by Eq.(3.26) is invariant under the adjoint action of $\mathfrak{g} \oplus \mathfrak{g}^*$ on itself if and only if $(\mathfrak{g}, \delta_{\mathfrak{g}})$ is a Lie bialgebra. The classical double Lie bialgebra $\mathfrak{d}(\mathfrak{g})$ can be exponentiated to give a pair of dual Poisson-Lie groups (G, G^*), which is the so-called **classical double of Poisson-Lie groups**, i.e., $D(G) = (G, G^*)$ and $D(G)$ is by itself a Poisson-Lie group called the **Drinfeld double**.

Consider now a basis $\{X_i\}$ of $\mathfrak{g}$ and $\{\xi^i\}$ of $\mathfrak{g}^*$ with $\langle X_i|\xi^j\rangle = \delta_i^j$. We furthermore denote the Lie algebra structures as

$$[X_i, X_j] = f_{ij}{}^k\, X_k \qquad , \qquad [\xi^i, \xi^j] = c^{ij}{}_k\, \xi^k\,, \tag{3.28}$$

[2]This means, we get $((X, \xi), (Y, \eta)) = \langle X|\eta\rangle + \langle Y|\xi\rangle\,.$

which means that the cocommutator of $\mathfrak{g}$ is given by $\delta_{\mathfrak{g}}(X_i) = c_i^{\ jk} X_j \otimes X_k$. The Lie bracket between X_i and ξ^j can be calculating by noting that the invariance of the inner product[3] Eq.(3.26) implies for the $\mathfrak{g}$-component

$$([X_i, \xi^j])^k = ([X_i, \xi^j], \xi^k) = (X_i, [\xi^j, \xi^k]) = c^{jk}_{\ \ i} \tag{3.29}$$

and for the $\mathfrak{g}^*$ component

$$([X_i, \xi^j])_k = ([X_i, \xi^j], X_k) = (X_k, [X_i, \xi^j]) = ([X_k, X_i], \xi^j) = f_{ki}^{\ \ j} \tag{3.30}$$

and thus

$$[X_i, \xi^j] = c^{jk}_{\ \ i} X_k + f_{ki}^{\ \ j} \xi^k = c^{jk}_{\ \ i} X_k - f_{ik}^{\ \ j} \xi^k \,. \tag{3.31}$$

We mentioned already, that for a (finite dimensional) Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$ we get naturally a Lie bialgebra structure $\delta_{\mathfrak{g}^*}$ for the dual Lie algebra $\mathfrak{g}^*$ via duality

$$(X, [\xi, \eta]_{\mathfrak{g}^*}) = (\delta_{\mathfrak{g}}(X), \xi \otimes \eta) \qquad , \qquad ([X, Y]_{\mathfrak{g}}, \xi) = (X \otimes Y, \delta_{\mathfrak{g}^*}(\xi)) \tag{3.32}$$

for $X, Y \in \mathfrak{g}$ and $\xi, \eta \in \mathfrak{g}^*$ and where we used the scalar product from Eq.(3.26).

We saw that one of the conditions for a Lie bialgebra structure on $\mathfrak{g}$ was that $\delta_{\mathfrak{g}}$ needed to be a 1-cocycle. A special class of such maps are given when the 1-cocycle $\delta_{\mathfrak{g}}$ is in fact a co-boundary, i.e., there exists some $r \in \mathfrak{g} \otimes \mathfrak{g}$ such that $\delta_{\mathfrak{g}} = \partial r$, or, equivalently, such that $\delta_{\mathfrak{g}}(X) = \mathrm{ad}_X^{(2)}(r) = [X \otimes \mathrm{id} + \mathrm{id} \otimes X, r]$. Such an r defines a Lie bialgebra structure on $\mathfrak{g}$ if and only if its symmetric part $r + r^t$ and the element

$$[\![r, r]\!] \equiv [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \tag{3.33}$$

are both $\mathrm{ad}_{\mathfrak{g}}$ - invariant. Such Lie bialgebras are called **coboundary Lie bialgebras** and if r satisfies the **classical Yang-Baxter equation** $[\![r, r]\!] = 0$ we say r is a **classical** r **- matrix**. Note, that we already encountered an r - matrix in Eq.(3.13), where the element ρ is of this type.

With $r = \sum_i r_1^i \otimes r_2^i$ we denote $r_{12} = r \otimes \mathrm{id}$, $r_{23} = \mathrm{id} \otimes r$ and $r_{13} = \sum_i r_1^i \otimes \mathrm{id} \otimes r_2^i$.

The importance of the classical Yang-Baxter equation, or more generally the $\mathrm{ad}_{\mathfrak{g}}$ - invariance of $[\![r, r]\!]$, stems from that the fact that this guarantees that the dual Lie bracket

[3] A symmetric bilinear form B on a Lie algebra $\mathfrak{g}$ is called *invariant* if it satisfies $B(X, [Y, Z]) = B([X, Y], Z)$ for all $X, Y, Z \in \mathfrak{g}$.

$[\,\cdot\,,\,\cdot\,]_{\mathfrak{g}^*}$ satisfies the Jacobi identity. In fact, for any coboundary $\delta_{\mathfrak{g}}(r)$ with $\mathrm{ad}_{\mathfrak{g}}$ - invariant symmetric part we have

$$\mathrm{Jac}_\delta(X) + \mathrm{ad}_X^{(3)}([\![r,r]\!]) = 0\,, \tag{3.34}$$

where

$$\mathrm{Jac}_\delta(X) = \sum_{\text{cycl.}} (\delta \otimes \mathrm{id})\,\delta(X)\,. \tag{3.35}$$

With $\delta(X) = \sum_i X_1^i \otimes X_2^i$ and $\delta(X_1^i) = \sum_j X_{11}^{ij} \otimes X_{12}^{ij}$ we can thus write

$$\mathrm{Jac}_\delta(X) = \sum_{ij} \left(X_{11}^{ij} \otimes X_{12}^{ij} \otimes X_2^i + X_2^i \otimes X_{11}^{ij} \otimes X_{12}^{ij} + X_{12}^{ij} \otimes X_2^i \otimes X_{11}^{ij} \right)\,. \tag{3.36}$$

From Eq.(3.34) we see that for $\mathrm{ad}_{\mathfrak{g}}$ - invariant $[\![r,r]\!]$ we have $\mathrm{Jac}_\delta(X) = 0$ and thus, together with Eq.(3.19), that the Jacobi identity for $[\,\cdot\,,\,\cdot\,]_{\mathfrak{g}^*}$ is satisfied in that case.

If we decompose the r - matrix as $r = a + s$, where a is skew-symmetric and s is symmetric and $\mathrm{ad}_{\mathfrak{g}}$ - invariant, then we have $[\![a+s, a+s]\!] = [\![a,a]\!] + [\![s,s]\!]$. Furthermore, if $[\![r,r]\!] = \varphi \neq 0$, but $\mathrm{ad}_X^{(3)}([\![r,r]\!]) = 0$, then we call $[\![r,r]\!] = \varphi$ the **modified classical Yang-Baxter equation**.

Finally, if a Lie bialgebra is of the coboundary type and r satisfies the classical Yang-Baxter equation then we call the Lie bialgebra **quasi-triangular** and if r is a skew-symmetric solution of the classical Yang-Baxter equation then the corresponding Lie bialgebra is called **triangular**. In particular, one can show that the double Lie bialgebra introduced above is a quasi-triangular Lie bialgebra.

As an example consider the complex Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ with basis

$$X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad X^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.37}$$

and commutation relations $[X^+, X^-] = H$ and $[H, X^\pm] = \pm 2X^\pm$. Then one can classify all the (isomorphism classes of) Lie bialgebra structures on $\mathfrak{sl}(2,\mathbb{C})$ via the orbits of the adjoint action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathfrak{sl}(2,\mathbb{C})$, [61]. Note, that we can think of those r as elements in $\bigwedge^{(2)} \mathfrak{sl}(2,\mathbb{C})$, which, as a complex vector space, is generated by $\{X^+ \wedge X^-, X^+ \wedge H, X^- \wedge H\} \equiv \{b_1, b_2, b_3\}$. Thus, we can write $r = r^i b_i$ and one finds that the different orbits under the adjoint action give rise to the following classification

$$(i) \qquad \text{standard structure} \qquad r = \lambda\, X^+ \wedge X^- \quad , \quad \lambda \in \mathbb{C}_* \,, \tag{3.38}$$

$$(ii) \qquad \text{triangular structure} \qquad r = \frac{1}{2}\, H \wedge X^+ \quad , \tag{3.39}$$

$$(iii) \qquad \text{trivial structure} \qquad r = 0 \,. \tag{3.40}$$

These are all skew-symmetric solutions of the classical Yang-Baxter equation. With the inner product (Killing form) $\langle X|Y\rangle = \mathrm{Tr}(XY)$ on $\mathfrak{sl}(2,\mathbb{C})$ one finds that there is a unique, non-skew-symmetric solution of the classical Yang-Baxter equation, [61], and the corresponding r - matrix is given by

$$r = \frac{1}{4}\, H \otimes H + X^- \otimes X^+ \,, \tag{3.41}$$

which falls into class (i) above. In the fundamental representation it is given by

$$r = \frac{1}{4}\, H \otimes H + X^- \otimes X^+ = \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{3.42}$$

We will meet this r - matrix again in the next section, because it also corresponds to the r - matrix of $\mathfrak{su}(2)$ with the bilinear form $\langle X|Y\rangle_{\mathfrak{su}(2)} = \mathrm{Im}(\mathrm{Tr}(XY))$ and leads to the classical double $\mathfrak{d}(\mathfrak{su}(2)) \cong \mathfrak{sl}(2,\mathbb{C})$. The cocommutator on the basis elements, derived from this r - matrix, is given, as usual, via $\delta(X) = [X, r]$ and leads to

$$\delta(H) = 0 \quad , \quad \delta(X^\pm) = \frac{1}{2}\left(X^\pm \otimes H - H \otimes X^\pm\right) . \tag{3.43}$$

Going back now to the Poisson-Lie groups associated with a coboundary Lie bialgebra, with classical r - matrix $r = \sum_{ij} r^{ij} X_i \otimes X_j$, where $\{X_i\}$ constitutes a basis for $\mathfrak{g}$, we can define a Poisson-Lie group structure for the group G as follows

$$\{f_1, f_2\} = \sum_{ij} r^{ij} \left((X_i^L(f_1))(X_j^L(f_2)) - (X_i^R(f_1))(X_j^R(f_2)) \right) , \tag{3.44}$$

where $X_i^{L,R}$ are the left- and right- invariant vector fields associated with X_i. In fact, if we consider a matrix Lie group, i.e., $G = \mathrm{GL}(N)$ or a subgroup thereof, one finds for the coordinate functions $t_{ij} : G \to \mathbb{R}$, $t_{ij}(g) = g_{ij}$ with $X^L(t_{ij}) = (tX)_{ij}$ and $X^R(t_{ij}) = (Xt)_{ij}$ the explicit expression

$$\{t_{ij}, t_{kl}\} = \sum_{ab} \left(r^{aj,bl}\, t_{ia} t_{kb} - r^{ia,kb}\, t_{aj} t_{bl} \right). \tag{3.45}$$

We used that the basis of $\mathfrak{gl}(N)$ is given by X_{ij}, which are 1 in the (ij)-entry and otherwise zero. Thus, we have for the r-matrix $r = \sum_{ab,cd} r^{ab,cd} X_{ab} \otimes X_{cd}$. The brackets Eq.(3.45) can be summarized as

$$\{t \overset{\otimes}{,} t\} = [t \otimes t, r]\,, \tag{3.46}$$

where $\{t \overset{\otimes}{,} t\}$ denotes the matrix of all the Poisson brackets $\{t \overset{\otimes}{,} t\}_{ijkl} = \{t_{ij}, t_{kl}\}$ and thus, we see how the classical r - matrix enters in the definition of the Poisson-Lie structure on G.

3.2 Review of 3D Loop Quantum Gravity with $\Lambda < 0$ – $\mathrm{SL}(2, \mathbb{C})$ as a deformed phase space

In section 1 we saw how Loop quantum gravity is constructed from classical general relativity in the case of a vanishing cosmological constant $\Lambda = 0$ and how the choice of holonomy-flux variables assigns to each link of the graph $\Gamma \subset \Sigma$ the phase space $\mathrm{T}^*\mathrm{SU}(2) \cong \mathrm{SU}(2) \times \mathfrak{su}(2)^*$, together with its canonical Poisson structure, where the group element is identified with the holonomy of the Ashtekar connection along the link and the (dual) Lie algebra element is associated with the flux of the gravitational field through the surface dual to the link, [9, 10]. Furthermore, we discussed in chapter 2 the puzzle within Loop quantum gravity, how a non-zero cosmological constant $\Lambda \neq 0$ can lead to a deformation of the standard SU(2) symmetry to a quantum group symmetry. The reason for that puzzle being the fact that in a canonical analysis the cosmological constant only appears in the scalar-, or Hamiltonian constraint, which is usually associated with the dynamics of the theory - and thus should a priori not affect the kinematical structure, which is described in terms of the familiar SU(2) spin networks and the underlying phase space $\mathrm{T}^*\mathrm{SU}(2)$.

In this section we will review the case of Euclidean 3D gravity with $\Lambda < 0$ following [39, 41] and we will see how a consistent deformation of $\mathrm{T}^*\mathrm{SU}(2)$ leads to another phase space, namely $\mathrm{SL}(2, \mathbb{C})$, seen as the Heisenberg double of the Poisson-Lie group SU(2), which is different from the Drinfeld double we encountered above (and is itself not a Poisson-Lie group). This new phase space corresponds to the change of coordinates we used in section 2.1.2.

We will show now that there exists a Lie bialgebra structure on $\mathfrak{su}(2)$ such that the corresponding classical double $\mathfrak{d}(\mathfrak{su}(2))$ is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. The Lie algebra $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ is usually given by the following commutation relations

$$[J_i, J_j] = \varepsilon_{ij}{}^k J_k \quad , \quad [J_i, K_j] = \varepsilon_{ij}{}^k K_k \quad , \quad [K_i, K_j] = -\varepsilon_{ij}{}^k J_k \, , \tag{3.47}$$

where the J_i correspond to the rotation generators of the $\mathfrak{su}(2)$ subalgebra and the K_i are the so-called boost generators. If we compare these brackets with Eq.(3.28) and Eq.(3.31) we see that they do not yet have the structure of a classical double. The most obvious reason for this is the fact that the boost generators do not generate a Lie subalgebra. If we were to consider instead the Lie algebra $\mathfrak{iso}(3)$, given by

$$[J_i, J_j] = \varepsilon_{ij}{}^k J_k \quad , \quad [J_i, P_j] = \varepsilon_{ij}{}^k P_k \quad , \quad [P_i, P_j] = 0 \, , \tag{3.48}$$

we see that by comparing again with the brackets Eq.(3.28) and Eq.(3.31) that this has the structure of a classical double $\mathfrak{d}(\mathfrak{su}(2))$ with trivial cocommutator $\delta_{\mathfrak{su}(2)}(X) = 0$ and duality $\langle J_i | P_j \rangle = \delta_{ij}$. In fact, this is more than just a trivial example, because, as we will see below, this is the underlying Lie bialgebra structure from which one can reproduce the canonical Poisson structure on $\mathrm{ISO}(3) \cong \mathrm{SU}(2) \ltimes \mathbb{R}^3 \cong \mathrm{SU}(2) \ltimes \mathfrak{su}(2)^* \cong \mathrm{T}^*\mathrm{SU}(2)$ via the Heisenberg double construction.

Now, non-trivial Lie bialgebra structures on $\mathfrak{su}(2)$, generated by $\{\tau_i\}$, are characterized by non-trivial cocommutators such that $\delta(\tau_i) = c_i^{jk} \, \tau_j \otimes \tau_k = c^{jk}{}_i \, \tau_j \otimes \tau_k$ and the corresponding classical double has the brackets

$$[\tau_i, \tau_j] = \varepsilon_{ij}{}^k \, \tau_k \quad , \quad [\kappa^i, \kappa^j] = c^{ij}{}_k \, \kappa^k = c_k^{ij} \, \kappa^k \, , \tag{3.49}$$

where we denoted the basis of the dual Lie algebra $\mathfrak{su}(2)^*$, satisfying $\langle \tau_i | \kappa^j \rangle = \delta_i^j$, by $\{\kappa^i\}$, and

$$[\tau_i, \kappa^j] = c_i^{jk} \, \tau_k + \varepsilon_i{}^j{}_k \, \kappa^k \, . \tag{3.50}$$

Concerning the choice of duality, note that $\langle \tau_i | \kappa^j \rangle$ can not simply be given by the Killing form (trace form) of $\mathfrak{sl}(2,\mathbb{C})$ or its real part, because this would imply that $\kappa^i \propto \tau_i$, since $(-2)\,\mathrm{Tr}(\tau_i \tau_j) = (-2)\,\mathrm{Re}(\mathrm{Tr}(\tau_i \tau_j)) = \delta_{ij}$. We will see in section 3.3 that this is the case we have to study for positive cosmological constant $\Lambda > 0$. Now, since the trace form is the only non-trivial, non-degenerate ad - invariant bilinear form at our disposal, we are essentially left with only one remaining possibility for the scalar product, namely $\langle \tau_i | \kappa^j \rangle = \delta_i^j \propto \mathrm{Im}(\mathrm{Tr}(\tau_i \kappa^j))$. We will see below that this is indeed the correct scalar product. However, it is not enough to completely determine the explicit form the the dual basis $\{\kappa^i\}$.

To help us determine the $\{\kappa^i\}$ consider the **Iwasawa decomposition** of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$. The maximal compact Lie subalgebra of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ is given by $\mathfrak{su}(2)$, which can be seen from the Cartan decomposition $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}} = \mathfrak{su}(2) + i\,\mathfrak{su}(2)$. The root spaces of $\mathfrak{sl}(2,\mathbb{C})$ are spanned by $E = K_1 - J_2$ (positive root space) and $F = K_1 + J_2$ (negative root space) and the maximal abelian Lie subalgebra of $i\,\mathfrak{su}(2)$ is generated by K_3. Thus, the Iwasawa decomposition of the non-compact part of of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ is given by elements for the form $a = a^1 K_3 + a^2 E$ or $b = b^1 K_3 + b^2 F$, where a^1, b^1 are real numbers and a^2, b^2 are complex. Note, that we can write $-iE = J_1 + K_2$ and $iF = K_2 - J_1$. Thus, we get the 3-real-dimensional algebras

$$\mathfrak{an}(2)_+ \equiv \mathrm{gen}_{\mathbb{R}}(K_1 - J_2, K_2 + J_1, K_3) = \mathrm{gen}_{\mathbb{R}}(\kappa^1, \kappa^2, \kappa^3)\,, \tag{3.51}$$

$$\mathfrak{an}(2)_- \equiv \mathrm{gen}_{\mathbb{R}}(K_1 + J_2, K_2 - J_1, K_3) = \mathrm{gen}_{\mathbb{R}}(\lambda^1, \lambda^2, \lambda^3)\,. \tag{3.52}$$

If we exponentiate those Lie algebras we get the groups of upper- and lower triangular matrices, respectively,

$$e^{a\kappa^1 + b\kappa^2 + c\kappa^3} = \begin{pmatrix} e^{\frac{c}{2}} & \bar{z} \\ 0 & e^{-\frac{c}{2}} \end{pmatrix} \quad , \quad e^{a\lambda^1 + b\lambda^2 + c\lambda^3} = \begin{pmatrix} e^{\frac{c}{2}} & 0 \\ z & e^{-\frac{c}{2}} \end{pmatrix}\,, \tag{3.53}$$

with $z = 2\,\frac{(a+ib)}{c}\,\sinh\left(\frac{c}{2}\right)$. The commutation relations for $\mathfrak{an}(2)_+$ are given by

$$[\kappa^1, \kappa^2] = 0 \quad , \quad [\kappa^3, \kappa^1] = \kappa^1 \quad , \quad [\kappa^3, \kappa^2] = \kappa^2 \tag{3.54}$$

and similarly for $\mathfrak{an}(2)_-$

$$[\lambda^1, \lambda^2] = 0 \quad , \quad [\lambda^3, \lambda^1] = -\lambda^1 \quad , \quad [\lambda^3, \lambda^2] = -\lambda^2\,. \tag{3.55}$$

Those relations can be summarized as (with an extra minus sign for the structure constants of $\mathfrak{an}(2)_-$)

$$[\kappa^i, \kappa^j] = (\delta_3^i \delta_l^j - \delta_l^i \delta_3^j)\,\kappa^l = \varepsilon^{ija}\varepsilon_{3la}\,\kappa^l \equiv c^{ij}{}_l\,\kappa^l\,. \tag{3.56}$$

Now, first note that the algebras $\mathfrak{an}(2)_+$ and $\mathfrak{an}(2)_-$ are indeed subalgebras of $\mathfrak{sl}(2,\mathbb{C})_\mathbb{R}$. But are they really the dual of $\mathfrak{su}(2)$ that we are looking for? The answer is yes, since one finds that indeed

$$\langle \tau_i | \kappa^j \rangle \equiv (-2)\,\mathrm{Im}(\mathrm{Tr}(\tau_i \kappa^j)) = \delta_i^j = (-2)\,\mathrm{Im}(\mathrm{Tr}(\tau_i \lambda^j)) \equiv \langle \tau_i | \lambda^j \rangle \tag{3.57}$$

holds, which is the trace form we alluded to before. Furthermore, and most importantly, we can deduce from Eq.(3.56) the potential cocommutators for $\mathfrak{an}(2)_+$ and $\mathfrak{an}(2)_-$, which are given via

$$\delta_{\mathfrak{su}(2)}(\tau_i) \equiv \pm\nu\,(\delta_3^j \delta_i^k - \delta_i^j \delta_3^k)\,\tau_j \otimes \tau_k = \pm\nu\,(\tau_3 \wedge \tau_i)\,, \tag{3.58}$$

where we included the signs $\pm$ to distinguish $\mathfrak{an}(2)_+$ from $\mathfrak{an}(2)_-$, which is of course slightly redundant, since we also put a scalar multiplicative factor ν, which is possible, since we explained before that one can rescale the cocommutator[4] and still have the same Lie bialgebra structure. To reproduce Eq.(3.56) we set $\nu = 1$.

Now, so far we have used a change of basis of $\mathfrak{sl}(2,\mathbb{C})_\mathbb{R}$, using the Iwasawa decomposition, with the result that we obtained a split of $\mathfrak{sl}(2,\mathbb{C})_\mathbb{R}$ into two dual Lie subalgebras, $\mathfrak{su}(2)$ and $\mathfrak{su}(2)^* \cong \mathfrak{an}(2)_\pm$ and we deduced a potential Lie bialgebra structure given by the cocommutator Eq.(3.58). What remains to be shown is that the mixed Lie brackets $[\tau_i, \kappa^j]$ and $[\tau_i, \lambda^j]$ match the corresponding brackets of the classical double, i.e., Eq.(3.50), and that the potential cocommutator Eq.(3.58) is indeed a 1-cocycle. In fact, one can ask whether there is an associated classical r - matrix.

To answer the question about the mixed brackets, where we only consider the $\mathfrak{an}(2)_+$ case, one shows that

$$[\tau_1, \kappa^1] = -\tau_3 \quad , \qquad [\tau_1, \kappa^2] = \kappa^3 \quad , \qquad [\tau_1, \kappa^3] = \tau_1 - \kappa^2\,, \tag{3.59}$$

$$[\tau_2, \kappa^1] = -\kappa^3 \quad , \qquad [\tau_2, \kappa^2] = -\tau_3 \quad , \qquad [\tau_2, \kappa^3] = \tau_2 + \kappa^1\,, \tag{3.60}$$

$$[\tau_3, \kappa^1] = \kappa^2 \quad , \qquad [\tau_3, \kappa^2] = -\kappa_1 \quad , \qquad [\tau_3, \kappa^3] = 0\,, \tag{3.61}$$

which, indeed, can be summarized as

$$[\tau_i, \kappa^j] = c_i^{\;jk}\,\tau_k + \varepsilon_i^{\;j}{}_l\,\kappa^l = (\delta_3^j \delta_i^k - \delta_i^j \delta_3^k)\,\tau_k + \varepsilon_i^{\;j}{}_l\,\kappa^l\,. \tag{3.62}$$

[4] One could equivalently associate the rescaling ν with the scalar product.

Now, we know that the canonical r - matrix for a classical double is given by $r = X_i \otimes \xi^i$, or, in our case, $r = \tau_i \otimes \kappa^i$. If we consider this expression in the fundamental representation of $\mathfrak{sl}(2, \mathbb{C})$ we find

$$r = \tau_i \otimes \kappa^i \tag{3.63}$$

$$= \frac{1}{2i}\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \otimes \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \otimes \begin{pmatrix}0 & -i\\ 0 & 0\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes \begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix}\right)$$

$$= \frac{1}{4i}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 4 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \tag{3.64}$$

which, up to an overall factor of i, we recognize from Eq.(3.42) from the earlier example Eq.(3.37) - Eq.(3.43). This tells us directly that this r -matrix satisfied the classical Yang-Baxter equation, $[\![r, r]\!] = 0$ and hence that the corresponding $\delta_{\mathfrak{su}(2)}$ is indeed a proper cocommutator. Furthermore, this shows overall that the classical double, associated with the Lie bialgebra structure $(\mathfrak{su}(2), \delta_{\mathfrak{su}(2)})$, is indeed isomorphic to $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}}$, i.e., $(\mathfrak{d}(\mathfrak{su}(2)), \delta_{\mathfrak{d}}) \cong \mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}}$ and since the r - matrix in Eq.(3.64) is not anti-symmetric, we see that $(\mathfrak{su}(2), \delta_{\mathfrak{su}(2)})$ is a quasi-triangular Lie bialgebra.

The origin of the extra i - factor between Eq.(3.42) and Eq.(3.63) can be seen explicitly by writing

$$\tau_1 = \frac{1}{2i}(X^+ + X^-) \quad , \quad \tau_2 = \frac{1}{2}(X^- - X^+) \quad , \quad \tau_3 = \frac{1}{2i}\, H \tag{3.65}$$

and

$$\kappa^1 = X^+ \quad , \quad \kappa^2 = -i\, X^+ \quad , \quad \kappa^3 = \frac{1}{2}\, H\,, \tag{3.66}$$

which allows to show that the r - matrix of Eq.(3.42) and the r - matrix in Eq.(3.63) are related as

$$r = \frac{1}{4} H \otimes H + X^- \otimes X^+ = i\,(\tau_3 \otimes \kappa^3) + \, i\,(\tau_1 \otimes \kappa^1 + \tau_2 \otimes \kappa^2) = i\,(\tau_i \otimes \kappa^i)\,. \tag{3.67}$$

Note, that such factors of i are important when it comes to the quantization of such structures, as we will see in chapter 4. In the isomorphism we have described, i.e., $\mathfrak{d}(\mathfrak{su}(2)) \cong \mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}}$, the Lie bialgebra is an example of a so-called real-real form, [65], embedded in the larger complex Lie bialgebra $\mathfrak{d}(\mathfrak{sl}(2,\mathbb{C}))$. Being a real-real form means here that for both real Lie algebras, $\mathfrak{su}(2)$ as well as the dual Lie algebra $\mathfrak{su}(2)^*$, the structure constants are real. So-called half-real forms are given when the structure constants of $\mathfrak{g}$ are real and those of $\mathfrak{g}^*$ are purely imaginary, or vice versa. For example, we could consider $\kappa^j \mapsto \tilde{\kappa}^j \equiv i\,\kappa^j$ with $[\tilde{\kappa}^i, \tilde{\kappa}^j] = i\, c^{ij}{}_l\, \tilde{\kappa}^l$, which basically corresponds to an extra i - factor for the cocommutator Eq.(3.58). This also implies that the duality is now given by $\langle \tau_i | \tilde{\kappa}^j \rangle = \delta_i^j = 2\,\mathrm{Re}(\mathrm{Tr}(\tau_i \tilde{\kappa}^j))$ but, most importantly, we see that the corresponding r - matrix for this half-real form is given by $r = \tau_i \otimes \tilde{\kappa}^i = i\,(\tau_j \otimes \kappa^j)$ and thus, corresponds to the r - matrix in Eq.(3.42), but, of course, this transformation spoils the isomorphism $\mathfrak{d}(\mathfrak{su}(2)) \cong \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ and makes only sense within the larger, complex Lie bialgebra $\mathfrak{d}(\mathfrak{sl}(2,\mathbb{C}))$. Hence, the physically relevant cases, at least in the context of 3d gravity, are given by the real-real forms.

Next, let us consider the Poisson-Lie group structure on $\mathrm{SL}(2,\mathbb{C}) = \mathrm{SU}(2) \times \mathrm{AN}(2)_+$ induced from the classical r - matrix Eq.(3.64). Recall Eq.(3.46), which allows us to simply calculate the Poisson brackets for the coordinate functions t_{ij} for an element $g \in \mathrm{SL}(2,\mathbb{C})$ via

$$\{t_{ij} \overset{\otimes}{,} t_{kl}\}(g) = [t \otimes t, r]_{ijkl} \qquad , \qquad r = \alpha \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{3.68}$$

where we consider here a general $\alpha \in \mathbb{C}$, and get

$$\begin{aligned} \{t_{ij} \overset{\otimes}{,} t_{kl}\}(g) &= \begin{pmatrix} \{t_{11}, t_{11}\} & \{t_{11}, t_{12}\} & \{t_{12}, t_{11}\} & \{t_{12}, t_{12}\} \\ \{t_{11}, t_{21}\} & \{t_{11}, t_{22}\} & \{t_{12}, t_{21}\} & \{t_{12}, t_{22}\} \\ \{t_{21}, t_{11}\} & \{t_{21}, t_{12}\} & \{t_{22}, t_{11}\} & \{t_{22}, t_{12}\} \\ \{t_{21}, t_{21}\} & \{t_{21}, t_{22}\} & \{t_{22}, t_{21}\} & \{t_{22}, t_{22}\} \end{pmatrix} \\ &= 2\alpha \begin{pmatrix} 0 & t_{11}t_{12} & -t_{11}t_{12} & 0 \\ t_{11}t_{21} & 2\,t_{12}t_{21} & 0 & t_{12}t_{22} \\ -t_{11}t_{21} & 0 & -2\,t_{12}t_{21} & -t_{12}t_{22} \\ 0 & t_{21}t_{22} & -t_{21}t_{22} & 0 \end{pmatrix} . \end{aligned} \tag{3.69}$$

Using the Iwasawa decomposition of g into an $\mathrm{SU}(2)$ element u and a upper triangular matrix $l \in \mathrm{AN}(2)_+$ as $g = ul$ one can further specify their Poisson brackets by fixing

$$t = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = u = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \quad \text{or} \quad t = l = \begin{pmatrix} \lambda & \bar{z} \\ 0 & \lambda^{-1} \end{pmatrix} . \tag{3.70}$$

We will see in chapter 4 that the Poisson-Lie structure given in Eq.(3.69) is the semi-classical limit of a certain Hopf algebra, which is why this is the right quantum-symmetry associated with systems that have this type of classical symmetry.

Now, as mentioned before, Poisson-Lie groups G are Poisson manifolds whose Poisson bivector Π_G vanishes at the identity. Hence, those structures can not be considered as phase spaces. There is, however, a way to define a (at least close to the identity) non-degenerate Poisson structure using a classical r - matrix, the so-called **Heisenberg double**, which is associated with a Poisson-Lie group G. Similarly to the Drinfeld double, the Heisenberg double is defined on the manifold $G \times G^*$, where G^* denotes the dual Poisson-Lie group to G, but instead of defining a Poisson-Lie group structure on $G \times G^*$ the Heisenberg double realizes a phase space structure. The price to pay is, of course, that the group multiplication can no longer be a Poisson map for this new type of Poisson structure.

We saw in Eq.(3.46) how, given an r - matrix, we can define a Poisson-Lie group structure on $G \times G^*$ using the commutator between the tensor product of two matrices and the r - matrix. The Heisenberg double is defined similarly using instead the anti-commutator. Furthermore, we consider only the anti-symmetric part of the r - matrix, i.e., $\hat{r} \equiv (r_{12} - r_{21})/2$, to guarantee anti-symmetry of the Poisson structure. Similarly to Eq.(3.44) we can define the Poisson structure of the Heisenberg double also via

$$\{f_1, f_2\}_H(g) = \sum_{ij} \hat{r}^{ij} \left((X_i^L(f_1))(X_j^L(f_2)) + (X_i^R(f_1))(X_j^R(f_2)) \right)(g)\,. \tag{3.71}$$

Thus, if we write the r - matrix as $r = r^{ij}\, X_i \otimes X_j = X_i \otimes \xi^i$, i.e., $\xi^i = r^{ij} X_j$, we can summarize the Poisson bivectors of the Drinfeld $(-)$ and the Heisenberg $(+)$ double as

$$\Pi_\pm = \frac{1}{2}\left[\left(X_i^L \otimes \xi^{L\,i} - \xi^{L\,i} \otimes X_i^L \right) \pm \left(X_i^R \otimes \xi^{R\,i} - \xi^{R\,i} \otimes X_i^R \right) \right] \tag{3.72}$$

$$= \frac{1}{2}\left[r^{ij} \left(X_i^L \otimes X_j^L - X_j^L \otimes X_i^L \right) \pm r^{ij} \left(X_i^R \otimes X_j^R - X_j^R \otimes X_i^R \right) \right] \tag{3.73}$$

$$= \hat{r}^{ij} \left(X_i^L \otimes X_j^L \pm X_i^R \otimes X_j^R \right)\,. \tag{3.74}$$

In terms of the coordinate functions we can write

$$\{t_{ij} \overset{\otimes}{,} t_{kl}\}_\pm = [\, t \otimes t, r\,]_{\pm, ijkl}\,. \tag{3.75}$$

Working again with our r - matrix from Eq.(3.68), we find for the Heisenberg double the following brackets

$$\{t_{ij} \overset{\otimes}{,} t_{kl}\}_H(g) = 2\alpha \begin{pmatrix} 0 & t_{11}t_{12} & -t_{11}t_{12} & 0 \\ -t_{11}t_{21} & 0 & -2\,t_{11}t_{22} & -t_{12}t_{22} \\ t_{11}t_{21} & 2\,t_{11}t_{22} & 0 & t_{12}t_{22} \\ 0 & t_{21}t_{22} & -t_{21}t_{22} & 0 \end{pmatrix} . \tag{3.76}$$

For the SU(2) element u this implies the following non-zero brackets ($\alpha = \frac{1}{4i}$ for $r_+ = \tau_i \otimes \kappa^i$ and $r_- = \tau_i \otimes \lambda^i$)

$$\{a, b\} = 2\alpha\, ab \quad , \quad \{a, \bar{b}\} = -2\alpha\, a\bar{b} \quad , \quad \{\bar{a}, b\} = 2\alpha\, \bar{a}b \quad , \quad \{b, \bar{b}\} = 4\alpha\, |a|^2 \tag{3.77}$$

and for $l \in \mathrm{AN}(2)_+$ we get[5]

$$\{\lambda, \bar{z}\}_+ = 2\alpha\, \lambda\bar{z} \quad , \quad \{\lambda, z\}_+ = -2\alpha\, \lambda z \quad , \quad \{\bar{z}, z\}_+ = -2\alpha\,(\lambda^2 + \lambda^{-2})\,. \tag{3.78}$$

In order to calculate the last bracket $\{\bar{z}, z\}_+$ it is easiest to work directly with Eq.(3.71). The mixed brackets between the SU(2) and the $\mathrm{AN}(2)_+$ sector are calculated similarly, cf. [39, 66]. Note, that the anti-symmetric parts of the r - matrices $r_+ = \tau_i \otimes \kappa^i$ and $r_- = \tau_i \otimes \lambda^i$ are given by

$$\hat{r}_\pm = \pm\frac{i}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \pm(\tau_2 \wedge \tau_1) = \hat{r}^{ij}_\pm\, \tau_i \otimes \tau_j \quad \Rightarrow \quad \hat{r}^{12}_\pm = -\hat{r}^{21}_\pm = \mp 1\,. \tag{3.79}$$

For the calculation of $\{\bar{z}, z\}_+$ via Eq.(3.71) one uses

$$\begin{aligned} X^L_1(\bar{z})(g) &= \frac{d}{dt}\Big|_{t=0} \bar{z}(\exp(-t\tau_1).g) = \frac{d}{dt}\Big|_{t=0} t_{12}(\exp(-t\tau_1).g) \\ &= \frac{i}{2}\, g_{22} = \frac{i}{2}\, t_{22}(g) = \frac{i}{2}\,\lambda^{-1}(g)\,, \qquad X^L_1(z)(g) = -\frac{i}{2}\,\lambda^{-1}(g)\,, \end{aligned} \tag{3.80}$$

[5] From Eq.(3.76) we can only deduce the bracket involving $\bar{z}$. If we want to obtain for example $\{\lambda, z\}_+$ we can consider $\{\lambda, z\}_-$, associated with $\mathrm{AN}(2)_-$ by noting that in this case the matrix $l \in \mathrm{AN}(2)_-$ involves λ and z and that the Poisson structure $\{\,\cdot\,,\,\cdot\,\}_-$ is just minus the Poisson structure $\{\,\cdot\,,\,\cdot\,\}_+$ coming from $\mathrm{AN}(2)_+$. The functions on phase space, λ and z or $\bar{z}$, are of course the same, regardless of whether we use $\{\,\cdot\,,\,\cdot\,\}_-$ or $\{\,\cdot\,,\,\cdot\,\}_+$. The fact, that those two Poisson structures are just minus the other one, follows from the fact that the anti-symmetric parts of their two respective r - matrices are just minus the other one. Note, the SU(2) brackets in Eq.(3.77) also use the + - brackets, coming from $\mathrm{AN}(2)_+$.

$$X_2^L(z)(g) = \frac{d}{dt}\bigg|_{t=0} z(\exp(-t\tau_2).g) = \frac{d}{dt}\bigg|_{t=0} \overline{t_{12}(\exp(-t\tau_2).g)} \tag{3.81}$$

$$= \frac{1}{2}\, g_{22} = \frac{1}{2}\, t_{22}(g) = \frac{1}{2}\, \lambda^{-1}(g)\,, \qquad X_2^L(\bar{z})(g) = \frac{1}{2}\, \lambda^{-1}(g)\,,$$

$$X_1^R(\bar{z})(g) = -\, \frac{d}{dt}\bigg|_{t=0} \bar{z}(g.\exp(-t\tau_1)) = -\frac{i}{2}\, \lambda(g)\,, \qquad X_1^R(z)(g) = \frac{i}{2}\, \lambda(g)\,, \tag{3.82}$$

$$X_2^R(z)(g) = -\frac{1}{2}\, \lambda(g)\,, \qquad X_2^R(\bar{z})(g) = -\frac{1}{2}\, \lambda(g)\,. \tag{3.83}$$

Similarly, one finds with $g \in \mathrm{AN}(2)_+$

$$X_1^L(\lambda)(g) = 0 = X_2^L(\lambda)(g) \quad , \quad X_1^R(\lambda)(g) = -\frac{i}{2}\, \bar{z}(g) \quad , \quad X_2^R(\lambda)(g) = \frac{1}{2}\, \bar{z}(g)\,. \tag{3.84}$$

3.3 Quasi-Poisson manifolds and quasi-Lie bialgebras

In the following two sections we will consider the (classical) mathematical structures that are relevant for the study of Euclidean 3d gravity with a positive cosmological constant, along the lines of the previous section. As described in the introduction, this is the case underlying the Turaev-Viro state sum model, where the deformation parameter of the quantum symmetry algebra is a certain root of unity. Our aim in this section is to show how a positive cosmological constant leads to a deformation of the flat $\Lambda = 0$ phase space $\mathrm{T}^*\mathrm{SU}(2)$ to $\mathrm{SO}(4)$, instead of $\mathrm{SL}(2,\mathbb{C})$.

The first obstacle in understanding this problem was the statement that there is no classical r - matrix in our case of interest, [67, 68, 69, 70, 71]. This is problematic, because we saw in section 3.2 that the r - matrix plays a fundamental role in describing both the symmetries and the phase space structure. It took us a fairly long time to understand in detail how for the $\mathfrak{so}(4)$ case we can work without a cocommutator and hence, do not need an r - matrix for the same reasons as in the $\mathfrak{sl}(2,\mathbb{C})$ case. As we will see below, the important new structure for $\mathfrak{so}(4)$ is the so-called **coassociator**, which takes us into the realm of **quasi-Lie bialgebras**[6].

[6]Note, that the "quasi-" in quasi-Lie bialgebra or quasi-Poisson Lie group/manifold is unrelated with the "quasi-" in quasi-triangular r - matrix or quasi-triangular Lie bialgebra. This can become a bit confusing, when talking about quasi-triangular quasi-Hopf algebras, for example, but it is the common terminology in the literature.

Another problem we faced was related to the issue of quantization of these deformed symmetries and phase spaces. We will see in more detail in chapter 4 that the quasi-triangular Hopf algebra $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$, with its standard quantum R - matrix, results very clearly from a quantization of the symmetries associated with the deformed phase space $\mathrm{SL}(2,\mathbb{C})$. This quantum R - matrix is the quantum analog of the classical r - matrix that we encountered above. Since the Turaev-Viro model is constructed from certain representations of the Hopf algebra $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ at q root of unity and since it is also believed that the Turaev-Viro model describes Euclidean 3d gravity with $\Lambda > 0$, we see that there is a clear conflict with the previous statement about the non-existence of a classical r - matrix in the case of a positive cosmological constant.

A more practical challenge, that took a substantial amount of time, was to understand and apply the mathematical literature that was available, such as , [43, 44, 45, 63, 62, 64, 46], and which seemed to be relevant to our overall research question and make it accessible to the physics community.

However, we should mention that our first hint, that we have to deal with "quasi-" structures came actually from the quantum side, namely the works [50, 51, 52, 53], where it was first advocated that the real quantum symmetry for systems where the deformation parameter q is a root of unity, is actually given by certain quasi-Hopf algebras. The concept of a quasi-Hopf algebra was introduced by Drinfeld in [49], but later further generalizes in [50, 51, 52, 53] to be applicable to certain models in condensed matter physics. Another hint pointing towards the relevance of quasi-Poisson structures came from more recent work in Loop quantum gravity on classically deformed, or homogeneously curved, discrete geometries [47, 48].

Understanding all these intricate mathematical topics and successfully applying them to our system of interest is one of the major contributions of this thesis.

Before we give an overview of some of the relevant topics from quasi-Poisson geometry and things like quasi-Lie bialgebras, similar in style to section 3.1, we want to comment a bit more, why we have to deal with those "quasi-" structures when dealing with SO(4) and how we realized that there is no way around them.

We saw above that a classical r - matrix is a solution of the classical Yang-Baxter equation $[\![r,r]\!] = 0$ and that an r - matrix was used to define the cocommutator of a coboundary Lie bialgebra via $\delta(X) = [X,r]$. Furthermore, we discussed that the condition $[\![r,r]\!] = 0$ corresponds to the Jacobi identity for the dual Lie algebra. When considering $\mathfrak{sl}(2,\mathbb{C})$ as the classical (Drinfeld) double of $\mathfrak{su}(2)$, with respect to a certain non-trivial cocommutator, we also saw that the canonical r - matrix of the double is given via $r = X_i \otimes \xi^i$, where X_i and ξ^i are basis' of the Lie algebras $\mathfrak{su}(2)$ and its dual $\mathfrak{su}(2)^* \cong \mathfrak{an}(2,\mathbb{R})_{\pm}$.

Now, note that in our case of interest the Lie algebra $\mathfrak{so}(4)$ has a decomposition into $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, which is indeed a true direct sum of two Lie subalgebras. This is another way of saying that $\mathfrak{so}(4)$ is not a semi-simple Lie algebra, unlike $\mathfrak{su}(2)$ or $\mathfrak{sl}(2,\mathbb{C})$, which is, in fact, the root cause for most, if not all, the interesting new structures arising in this case. However, based on our knowledge about the Lie bialgebra setting and given the decomposition of $\mathfrak{so}(4)$ into $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, one might be tempted to interpret $\mathfrak{so}(4)$ also as some kind of classical double and $\mathfrak{su}(2)$ as being self-dual, i.e., $\mathfrak{su}(2)^* \cong \mathfrak{su}(2)$, because then we could again write $\mathfrak{so}(4) \cong \mathfrak{d}(\mathfrak{su}(2))$, similar to the $\mathfrak{sl}(2,\mathbb{C})$ case, but now meaning $\mathfrak{d}(\mathfrak{su}(2)) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)^* \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Giving in to this temptation, let us consider the corresponding r - matrix as $r = \tau_i \otimes \tau^i$, where of course τ^i is to denote the dual basis to τ_i, such that we have $\tau^i = \delta^{ij}\tau_j$, where τ_i is the standard basis of $\mathfrak{su}(2)$. Then one finds for

$$[\![r, r]\!] \equiv [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]\,, \tag{3.85}$$

given by

$$[r_{12}, r_{13}] = \sum_{ij} [r_i^{(1)}, r_j^{(1)}] \otimes r_i^{(2)} \otimes r_j^{(2)} = \sum_{ij} [\tau_i, \tau_j] \otimes \tau_i \otimes \tau_j = \sum_{ijk} \varepsilon_{ijk}\, \tau_k \otimes \tau_i \otimes \tau_j\,, \tag{3.86}$$

$$[r_{12}, r_{23}] = \sum_{ij} r_i^{(1)} \otimes [r_i^{(2)}, r_j^{(1)}] \otimes r_j^{(2)} = \sum_{ij} \tau_i \otimes [\tau_i, \tau_j] \otimes \tau_j = \sum_{ijk} \varepsilon_{ijk}\, \tau_i \otimes \tau_k \otimes \tau_j\,, \tag{3.87}$$

$$[r_{13}, r_{23}] = \sum_{ij} r_i^{(1)} \otimes r_j^{(1)} \otimes [r_i^{(2)}, r_j^{(2)}] = \sum_{ij} \tau_i \otimes \tau_j \otimes [\tau_i, \tau_j] = \sum_{ijk} \varepsilon_{ijk}\, \tau_i \otimes \tau_j \otimes \tau_k\,, \tag{3.88}$$

that

$$[\![r, r]\!] = \sum_{ijk} \varepsilon_{ijk}\, \tau_i \otimes \tau_j \otimes \tau_k = \tau_1 \wedge \tau_2 \wedge \tau_3 \neq 0\,. \tag{3.89}$$

A simple calculation shows furthermore that for any $X = X^i\tau_i \in \mathfrak{su}(2)$ we have indeed[7]

$$\mathrm{ad}_X^{(2)}(r) = 0 \qquad \text{and} \qquad \mathrm{ad}_X^{(3)}\left(\sum_{ijk} \varepsilon_{ijk}\, \tau_i \otimes \tau_j \otimes \tau_k\right) = 0\,, \tag{3.90}$$

which means that Eq.(3.89) is ad - invariant and thus, we actually have a (symmetric) solution of the modified Yang-Baxter equation, cf. our remark below Eq.(3.36), since

[7]Note, that if we would define our r - matrix as $r = \sum_i r^{ii}\, \tau_i \otimes \tau_i$ then one finds that in order to have $\mathrm{ad}_X^{(2)}(r) = 0$, we need $r^{11} = r^{22} = r^{33}$.

$r_{12} = r_{21}$. The problem with this type of solution, however, is that it is purely symmetric, i.e., the anti-symmetric part is zero, and thus, we have $\delta(X) = [X, r] = 0$ for all X and thus this type of r - matrix gives only rise to the trivial cocommutator. Explicitly, for $X = X^i \tau_i$ one finds

$$\delta(X) = [X, r] = [X, \tau_i] \otimes \tau^i + \tau_i \otimes [X, \tau^i] = X^i \, \varepsilon_{ijk} \, \left(\tau^j \otimes \tau^k + \tau^k \otimes \tau^j\right) = 0\,. \tag{3.91}$$

Without knowing it at the time, what we have encountered here is indeed (part of) the structure of a **quasi-Lie bialgebra**. The non-zero term on the right-hand side of Eq.(3.89) is the so-called ad-invariant **Cartan 3-tensor** or **coassociator**, usually denoted as φ. Our proposal for the r - matrix is usually written as t or Ω and called **Casimir element**. Of course, we can use an overall rescaling or normalization of t, i.e., $t = \alpha(\tau_i \otimes \tau^i)$, which allows us to write $[\![t, t]\!] = \varphi$ for different normalization of φ and vice versa.

One now might say, but what about cocommutators that do not come from an r - matrix? Coboundary Lie bialgebras are after all just a special class of Lie bialgebras. What about other cocycles δ? Remember, that the cocycle must be skew-symmetric, which means it is a map from $\mathfrak{su}(2)$ into $\bigwedge^{(2)} \mathfrak{su}(2) \subset \mathfrak{su}(2)^{\otimes 2}$, because otherwise the dual Lie bracket would not be anti-symmetric. For $\mathfrak{su}(2)$ we have $\bigwedge^{(2)} \mathfrak{su}(2) = \mathrm{gen}_{\mathbb{R}}(\tau_1 \wedge \tau_2, \tau_1 \wedge \tau_3, \tau_2 \wedge \tau_3)$ and $\dim\left(\bigwedge^{(3)} \mathfrak{su}(2)\right) = 1$, where the last relation tells us that the space of possible coassociators φ is only 1-dimensional. Thus, let us study now all the possible cocycles for $\mathfrak{su}(2)$, denoted by $\delta(\tau_i) = {c_i}^{jk} \, \tau_j \wedge \tau_k$, which must satisfy the cocycle condition, i.e, (remember ${c_i}^{jk} = -{c_i}^{kj}$)

$$\begin{aligned}\delta(\tau_1) = {c_1}^{ij} \tau_i \otimes \tau_j = \delta([\tau_2, \tau_3]) &\overset{!}{=} \mathrm{ad}^{(2)}_{\tau_2} \, \delta(\tau_3) - \mathrm{ad}^{(2)}_{\tau_3} \, \delta(\tau_2) \\ &= \mathrm{ad}^{(2)}_{\tau_2} ({c_3}^{ij} \tau_i \otimes \tau_j) - \mathrm{ad}^{(2)}_{\tau_3} ({c_2}^{ij} \tau_i \otimes \tau_j)\,,\end{aligned} \tag{3.92}$$

which leads to the condition

$${c_1}^{ij} \overset{!}{=} {c_3}^{mj} {\varepsilon_{2m}}^{i} + {c_3}^{im} {\varepsilon_{2m}}^{j} - {c_2}^{mj} {\varepsilon_{3m}}^{i} - {c_2}^{im} {\varepsilon_{3m}}^{j}\,, \tag{3.93}$$

from which we conclude for example ${c_1}^{12} = {c_3}^{32} = -{c_3}^{23}$, ${c_1}^{13} = {c_2}^{23}$ and ${c_1}^{23} = {c_3}^{12} - {c_2}^{13}$. Similarly, we can determine the other conditions from

$$\begin{aligned}\delta(\tau_2) = {c_2}^{ij} \tau_i \otimes \tau_j = \delta([\tau_3, \tau_1]) &\overset{!}{=} \mathrm{ad}^{(2)}_{\tau_3} \, \delta(\tau_1) - \mathrm{ad}^{(2)}_{\tau_1} \, \delta(\tau_3) \\ &= \mathrm{ad}^{(2)}_{\tau_3} ({c_1}^{ij} \tau_i \otimes \tau_j) - \mathrm{ad}^{(2)}_{\tau_1} ({c_3}^{ij} \tau_i \otimes \tau_j)\end{aligned} \tag{3.94}$$

and

$$\delta(\tau_3) = {c_3}^{ij}\tau_i \otimes \tau_j = \delta([\tau_1,\tau_2]) \stackrel{!}{=} \mathrm{ad}^{(2)}_{\tau_1}\,\delta(\tau_2) - \mathrm{ad}^{(2)}_{\tau_2}\,\delta(\tau_1) \tag{3.95}$$
$$= \mathrm{ad}^{(2)}_{\tau_1}({c_2}^{ij}\tau_i \otimes \tau_j) - \mathrm{ad}^{(2)}_{\tau_2}({c_1}^{ij}\tau_i \otimes \tau_j)\,,$$

which leads to the following solution matrix of allowed cocycles for $\mathfrak{su}(2)$

$$\delta \equiv \begin{pmatrix} {c_1}^{12} & {c_1}^{13} & {c_1}^{23} \\ {c_2}^{12} & {c_2}^{13} & {c_2}^{23} \\ {c_3}^{12} & {c_3}^{13} & {c_3}^{23} \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & 0 & \beta \\ 0 & \gamma & -\alpha \end{pmatrix} \quad , \quad \alpha,\beta,\gamma \in \mathbb{R}\,. \tag{3.96}$$

Thus, we see that we have a 3-parameter family[8] of Lie bialgebra structures on $\mathfrak{su}(2)$. We can also state already that, together with the one dimension of $\bigwedge^{(3)}\mathfrak{su}(2)$, which parametrizes the coassociators, we have a 4-parameter family of quasi-Lie bialgebra structures on $\mathfrak{su}(2)$. The cocycle corresponding to $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \oplus \mathfrak{an}(2,\mathbb{R})_\pm$ is given by $(\alpha,\beta,\gamma,\varphi) = (0,\mp,0,0)$, i.e., for the + - case we have

$$\delta(\tau_1) = \tau_3 \wedge \tau_1 \quad , \quad \delta(\tau_2) = \tau_3 \wedge \tau_2 \quad , \quad \delta(\tau_3) = 0 \qquad \text{or} \qquad \delta(\tau_i) = \tau_3 \wedge \tau_i\,. \tag{3.97}$$

Now, given all those possible cocycles, some of which might not correspond to coboundary cocycles, we ask, which of them allow to model a classical double for $\mathfrak{su}(2)$, such that the dual Lie algebra is given again by $\mathfrak{su}(2)$. This means, that the structure constants for $\mathfrak{su}(2)^*$, determined by the matrix elements of the cocycle ${c_i}^{jk}$, should be given by ${c_i}^{jk} = {\varepsilon_i}^{jk}$. However, one finds that

$$\varepsilon \equiv \begin{pmatrix} {\varepsilon_1}^{12} & {\varepsilon_1}^{13} & {\varepsilon_1}^{23} \\ {\varepsilon_2}^{12} & {\varepsilon_2}^{13} & {\varepsilon_2}^{23} \\ {\varepsilon_3}^{12} & {\varepsilon_3}^{13} & {\varepsilon_3}^{23} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{3.98}$$

and hence, we see by comparing with Eq.(3.96), that there are no cocycles $\delta(\alpha,\beta,\gamma)$, not even ones that are not of the coboundary type, that allow for a Lie bialgebra structure such that $\mathfrak{su}(2)^* \cong \mathfrak{su}(2)$. Furthermore, even if we could find (α,β,γ) such that $\mathfrak{su}(2)^* \cong \mathfrak{su}(2)$,

[8] If we remember our example below Eq.(3.37), it seems unlikely that $\mathfrak{su}(2)$ should have more (non-trivial) cocycle structures than its complexification $\mathfrak{sl}(2,\mathbb{C})$. The difference between our analysis here, and the one below Eq.(3.37), is that we did not sort the solutions into their isomorphism classes of cocycles. Thus, from the example on $\mathfrak{sl}(2,\mathbb{C})$ we know that our 3-parameter solution Eq.(3.96) includes isomorphic solutions.

this would not guarantee that the mixed Lie brackets between $(\tau_i, 0) \in \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $(0, \tau_i) \in \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ would give the right result, such that we really get $\mathfrak{d}(\mathfrak{su}(2)) \cong \mathfrak{so}(4)$. We will luckily see in this section that $\mathfrak{so}(4)$ can, however, be given the structure of a (canonical) quasi-Lie bialgebra instead, where the cocycle is actually trivial, i.e., $(\alpha, \beta, \gamma) = (0, 0, 0)$, and the dual and mixed brackets are entirely generated by the Cartan 3-tensor φ. This structure is the analog of the (classical) Drinfeld double to the quasi-Lie bialgebra setting.

Given the solution matrix Eq.(3.96) one could ask, what is the compatible r - matrix, i.e., is there an element $r = r(\alpha, \beta, \gamma)$, such that we can write the cocycle $\delta_{\alpha,\beta,\gamma}$ as $\delta_{\alpha,\beta,\gamma}(\tau_i) = [\tau_i, r(\alpha, \beta, \gamma)]$ and the answer to that question is given by

$$r = \{r^{ij}\} = \begin{pmatrix} \eta & \beta & -\alpha \\ -\beta & \eta & -\gamma \\ \alpha & \gamma & \eta \end{pmatrix} . \tag{3.99}$$

Any r - matrix of this form satisfies $\delta_{\alpha,\beta,\gamma}(\tau_i) = [\tau_i, r(\alpha, \beta, \gamma)]$. Most importantly, however, note that there is a fourth parameter η showing up, such that actually for all $r = r(\alpha, \beta, \gamma, \eta)$ we have $\delta_{\alpha,\beta,\gamma}(\tau_i) = [\tau_i, r(\alpha, \beta, \gamma, \eta)]$. By writing out the r - matrix as

$$r = \eta\,(\tau_1 \otimes \tau_1 + \tau_2 \otimes \tau_2 + \tau_3 \otimes \tau_3) \,+\, \beta(\tau_1 \wedge \tau_2) \,-\, \alpha(\tau_1 \wedge \tau_3) \,-\, \gamma(\tau_2 \wedge \tau_3)\,, \tag{3.100}$$

we see that η actually parametrizes the ad - invariant, symmetric part, or the Casimir element $t = \eta\,(\tau_i \otimes \tau^i)$.

We will now consider the analog notions of Lie bialgebras, their doubles and Poisson-Lie groups- and manifolds in the quasi-realm. We will also discuss the notion of twisting of those structures. From a mathematical perspective these quasi-structures were studied to generalize the notion of a **moment map**. In the standard setting a moment map, which is related to conserved quantities of systems with symmetries, is a map from a G - manifold M to the dual of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$, i.e., $J : M \to \mathfrak{g}^*$. Poisson-Lie groups are related to situations where the moment map J instead takes values in the dual Lie group G^* in a pair of dual Poisson-Lie groups (G, G^*). The notion of quasi-Poisson manifolds was investigated to understand situations where the moment map takes values in the group G itself, [43, 44, 45]. Examples for such situations are given, when we study gauge theories in holonomy- or group valued variables, such as lattice gauge theories or Chern-Simons theory and gravity in the combinatorial quantization- [27, 28] or Loop quantum gravity framework. In those cases it usually happens that the constraints of the system are expression in terms of group valued variables. An example is the flatness constraint in Loop quantum gravity, which is given by the requirement that the product of a set of holonomies around 2-surfaces is the group identity, which means, takes values in the Lie group.

The following notions were first introduced in [49]. We follow mostly [43, 63].

Definition (quasi-Lie bialgebra) : A quasi-Lie bialgebra is a triple $(\mathfrak{g}, \delta_{\mathfrak{g}}, \varphi_{\mathfrak{g}})$, where $\mathfrak{g}$ is a Lie algebra, $\delta_{\mathfrak{g}}$ denotes the cocommutator and the element $\varphi_{\mathfrak{g}} \in \wedge^3\mathfrak{g}$ is called the coassociator. As before, $\delta_{\mathfrak{g}}$ is a linear map $\delta_{\mathfrak{g}} : \mathfrak{g} \to \wedge^2\mathfrak{g}$, satisfying the cocycle condition. The coassociator $\varphi_{\mathfrak{g}} \in \wedge^3\mathfrak{g}$ is defined via the map of the same name $\varphi_{\mathfrak{g}} : \wedge^2\mathfrak{g}^* \to \mathfrak{g}$, given via $\varphi_{\mathfrak{g}}(\xi,\eta) = \mathrm{proj}_{\mathfrak{g}}[\xi,\eta]_{\mathfrak{d}}$, which denotes the projection of $[\xi,\eta]_{\mathfrak{d}}$ onto $\mathfrak{g}$, and has to satisfy the conditions

$$\mathrm{Alt}(\delta_{\mathfrak{g}} \otimes \mathrm{id})\, \delta_{\mathfrak{g}}(X) = \mathrm{ad}_X^{(3)}(\varphi_{\mathfrak{g}}) \quad , \quad \mathrm{Alt}(\delta_{\mathfrak{g}} \otimes \mathrm{id} \otimes \mathrm{id})(\varphi_{\mathfrak{g}}) = 0\,, \tag{3.101}$$

where $\mathrm{Alt}(x_1 \otimes \cdots \otimes x_n) = \sum_{\sigma\in\Sigma_n} \mathrm{sgn}(\sigma)\, x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$.

On $\mathfrak{d}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*$ we consider again the following scalar product, using the duality pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$, $\langle\,\cdot\,|\,\cdot\,\rangle : \mathfrak{g} \times \mathfrak{g}^* \to \mathbb{R}$,

$$((X,\xi),(Y,\eta))_{\mathfrak{d}} = \langle X|\eta\rangle + \langle Y|\xi\rangle \qquad , \qquad X, Y \in \mathfrak{g} \;,\; \xi,\eta \in \mathfrak{g}^*. \tag{3.102}$$

Then the Lie brackets for the **classical quasi-(Drinfeld) double** $\mathfrak{d}(\mathfrak{g})$ are given by[9]

$$[X,Y] = f(X,X) \quad , \quad [\xi,\eta] = \delta_{\mathfrak{g}}^*(\xi,\eta) + \varphi_{\mathfrak{g}}(\xi,\eta) \quad , \quad [X,\xi] = \mathrm{ad}_\xi^*\, X - \mathrm{ad}_X^*\, \xi\,, \tag{3.103}$$

where we denoted the Lie algebra structure on $\mathfrak{g}$ by $f : \wedge^2\mathfrak{g} \to \mathfrak{g}$. The classical quasi-(Drinfeld) double is itself again a quasi-Lie bialgebra. We clearly see that a quasi-Lie bialgebra becomes a Lie bialgebras for $\varphi_{\mathfrak{g}} = 0$. In order to express these brackets in a more familiar form, consider a basis of $\mathfrak{g}$ and $\mathfrak{g}^*$, i.e., $\{\tau_i\}$ and $\{\xi^i\}$, with $\langle \tau_i|\xi^j\rangle = \delta_i^j$. Then we write again

$$[\tau_i, \tau_j]_{\mathfrak{d}} = [\tau_i, \tau_j]_{\mathfrak{g}} = f_{ij}{}^k\, \tau_k \tag{3.104}$$

and with $\delta_{\mathfrak{g}}(\tau_i) = c_i{}^{jk}\, \tau_j \otimes \tau_k$ we still get from the relation $\langle \delta_{\mathfrak{g}}(X)|\xi\otimes\eta\rangle = \langle X|\delta_{\mathfrak{g}}^*(\xi,\eta)\rangle = \langle X|[\xi,\eta]_{\mathfrak{g}^*}\rangle$

$$[\xi^i,\xi^j]_{\mathfrak{d}} = \delta_{\mathfrak{g}}^*(\xi^i,\xi^j) + \varphi_{\mathfrak{g}}(\xi^i,\xi^j) = [\xi^i,\xi^j]_{\mathfrak{g}^*} + \varphi_{\mathfrak{g}}(\xi^i,\xi^j) = c_k{}^{ij}\, \xi^k + \varphi_{\mathfrak{g}}(\xi^i,\xi^j)\,. \tag{3.105}$$

[9]Note, that we have used an extra minus sign in the definition of the mixed brackets $[X,\xi]$, compared with [43], in order to match our definition used for the Lie bialgebras, i.e., Eq.(3.31).

Now, what is $\varphi_{\mathfrak{g}}(\xi^i,\xi^j)$? Per definition, we have $\varphi_{\mathfrak{g}}(\xi^i,\xi^j) \in \mathfrak{g}$ and thus we can write $\varphi_{\mathfrak{g}}(\xi^i,\xi^j) = \varphi_{\mathfrak{g}}(\xi^i,\xi^j)^k\,\tau_k$. We see that $\varphi_{\mathfrak{g}}(\xi^i,\xi^j)$ captures the $\mathfrak{g}$ - contribution of $[\xi^i,\xi^j]_{\mathfrak{d}}$. Thus, we write

$$[\xi^i,\xi^j]_{\mathfrak{d}} = c_k^{\ ij}\,\xi^k + \varphi^{ijk}\,\tau_k \tag{3.106}$$

and the relationship between the map $\varphi_{\mathfrak{g}} : \wedge^2\mathfrak{g}^* \to \mathfrak{g}$ and $\varphi_{\mathfrak{g}} \in \wedge^3\mathfrak{g}$ becomes clear. As an element of $\wedge^3\mathfrak{g}$ we can write $\varphi_{\mathfrak{g}} = \frac{1}{6}\varphi^{ijk}\,\tau_i\wedge\tau_j\wedge\tau_k$ and see that its components are also the components of $\varphi_{\mathfrak{g}}$ seen as a map from $\wedge^2\mathfrak{g}^*$ to $\mathfrak{g}$, i.e.

$$\begin{aligned}\varphi_{\mathfrak{g}}(\xi,\eta) \equiv \varphi_{\mathfrak{g}}(\xi,\eta,\,\cdot\,) &= \frac{1}{6}\,\varphi^{ijk}\,(\tau_i\wedge\tau_j\wedge\tau_k)(\xi,\eta,\,\cdot\,) = \varphi^{ijk}\,\tau_i(\xi)\tau_j(\eta)\,\tau_k \\ &= \varphi^{ijk}\,\xi_i\,\eta_j\,\tau_k\,,\end{aligned} \tag{3.107}$$

where the action of $\mathfrak{g}$ on $\mathfrak{g}^*$ is defined as usual via $X(\xi) \equiv \xi(X)$. For the mixed brackets we obtain the same result as in the Lie bialgebra case and we get the following brackets for the double $\mathfrak{d}(\mathfrak{g})$

$$[\tau_i,\tau_j]_{\mathfrak{d}} = f_{ij}^{\ \ k}\,\tau_k \quad , \quad [\xi^i,\xi^j]_{\mathfrak{d}} = c_k^{\ ij}\,\xi^k + \varphi^{ijk}\,\tau_k \quad , \quad [\tau_i,\xi^j]_{\mathfrak{d}} = c_i^{\ jk}\,\tau_k - f_{ik}^{\ \ j}\,\xi^k\,. \tag{3.108}$$

We just mention that quasi-Lie bialgebra structures for a Lie algebra $\mathfrak{g}$ are in one to one correspondence with so-called **Manin quasi-triples** $(\mathfrak{d},\mathfrak{g},\mathfrak{h})$, where $\mathfrak{d}$ denotes the double $\mathfrak{g}\oplus\mathfrak{h}$, $\mathfrak{g}$ is the Lie algebra and $\mathfrak{h}$ is the complement of $\mathfrak{g}$ in $\mathfrak{d}$, which is not necessarily a Lie subalgebra. Remember, that in the Lie bialgebra case $\mathfrak{g}^*$ was a Lie subalgebra. The so-called **standard Manin pair** for a Lie algebra $\mathfrak{g}$ is given by $\delta_{\mathfrak{g}} = 0$ and $\varphi_{\mathfrak{g}} = 0$ with brackets

$$[X,Y] = f(X,Y) \quad , \quad [\xi,\eta] = 0 \quad , \quad [X,\xi] = -\,\mathrm{ad}^*_X\,\xi\,. \tag{3.109}$$

In this case we have $\mathfrak{g}^* = \mathfrak{h}$. Furthermore, let us mention already that the classical quasi-double $\mathfrak{d}(\mathfrak{su}(2))$ with trivial cocycle $\delta_{\mathfrak{su}(2)} = 0$ and coassociator $\varphi^{ijk}_{\mathfrak{su}(2)} = \varepsilon^{ijk}$ is isomorphic to $\mathfrak{so}(4)$ and hence, is exactly the right analog of the classical double $\mathfrak{d}(\mathfrak{su}(2)) \cong \mathfrak{sl}(2,\mathbb{C})$ we were looking for to study the $\Lambda > 0$ case. We will discuss this in more detail below.

If the Lie algebra $\mathfrak{g}$ has an invariant, non-degenerate symmetric bilinear form, we can identify $\mathfrak{g}^*$ with $\mathfrak{g}$ and construct a quasi-Lie bialgebra on $\mathfrak{d} = \mathfrak{g}\oplus\mathfrak{g}$ using the scalar product

$$((X_1,X_2),(Y_1,Y_2))_{\mathfrak{g}\oplus\mathfrak{g}} = \langle X_1|Y_1\rangle - \langle X_2|Y_2\rangle\,. \tag{3.110}$$

This is the situation appropriate for the isomorphism we mentioned earlier, namely $\mathfrak{so}(4) \cong \mathfrak{su}(2)\oplus\mathfrak{su}(2)$. One embeds $\mathfrak{g}$ into $\mathfrak{d}$ by the diagonal embedding $\Delta : X \mapsto (X,X)$

and the isotropic complement is given by the anti-diagonal embedding, i.e., $\mathfrak{g}_- = \Delta_-(\mathfrak{g})$, with $\Delta_- : X \mapsto (X, -X)$. This gives in general a Manin quasi-triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}_-)$, since $\mathfrak{g}_-$ is not a Lie subalgebra. The cocommutator $\delta_\mathfrak{d}$ vanishes, because $[\mathfrak{g}_-, \mathfrak{g}_-] \subset \mathfrak{g}$. The coassociator in this case can be calculated via the anti-symmetric and ad - invariant map $\varphi_\mathfrak{d} : (X, Y, Z) \mapsto \frac{1}{4}\langle [X, Y] | Z\rangle$.

The **canonical r - matrix** for a quasi-Lie bialgebra takes the same form as before and we can write it in a basis as $r_\mathfrak{d} = X_i \otimes \xi^i$. The difference, of course, is that this canonical r - matrix is not a classical r - matrix in the sense that it generally does not satisfy the classical Yang-Baxter equation. In fact, with the right normalizations, this r - matrix satisfies $[\![r_\mathfrak{d}, r_\mathfrak{d}]\!] = \varphi_\mathfrak{d}$ and its symmetric, ad - invariant part is the Casimir element $t = \Omega = X_i \otimes X^i$.

Now, let us consider the important notion of **twisting**. This operation on the (quasi-) Lie bialgebra level has its analogs both at the group level, as well as in the quantum theory, where it was first studied. A twist can be seen as some kind of gauge transformation of the quasi-Lie bialgebra in the sense that all quasi-Lie bialgebras that correspond to the same Manin pair $(\mathfrak{d}, \mathfrak{g})$, are related by a twist transformation. A **twist** is simply given by an element $\chi \in \wedge^2\mathfrak{g}$ and is used to transform the cocommutator and the coassociator of a quasi-Lie bialgebra $(\mathfrak{g}, \delta_\mathfrak{g}, \varphi_\mathfrak{g})$ as follows

$$\delta_\mathfrak{g}^\chi(X) = \delta_\mathfrak{g}(X) + \mathrm{ad}_X^{(2)}(\chi) \qquad , \qquad \varphi_\mathfrak{g}^\chi = \varphi_\mathfrak{g} - \mathrm{Alt}(\delta_\mathfrak{g} \otimes \mathrm{id})(\chi) + [\![\chi, \chi]\!] \,. \tag{3.111}$$

The notion of twisting becomes especially relevant when there exist inequivalent isotropic complements to $\mathfrak{g}$ in $\mathfrak{d}$. From Eq.(3.111) we see that an ad - invariant twist χ leaves the cocommutator invariant. Furthermore, at the level of Lie bialgebras, i.e., $\varphi_\mathfrak{g} = 0$, we can consider so-called **Drinfeld twists**, $\chi_D \in \wedge^2\mathfrak{g}$, which are twists such that

$$\mathrm{Alt}(\delta_\mathfrak{g} \otimes \mathrm{id})(\chi_D) = [\![\chi_D, \chi_D]\!] \,. \tag{3.112}$$

These are the twists that leave the category of Lie bialgebras invariant, since they give $\varphi_\mathfrak{g}^{\chi_D} = \varphi_\mathfrak{g}$. However, Eq.(3.112) is quite a strong restriction on the allowed twists. Quasi-Lie bialgebras, and analogously quasi-Hopf algebras in the quantum theory, are precisely of interest because they allow for this larger class of twist transformations.

The canonical r - matrix of a quasi-Lie bialgebra transforms simply as $r^\chi = r + \chi$ under twisting. Furthermore, given a general twist $\chi = \chi^{ij}\, X_i \otimes X_j$ the general formula for the transformation of the dual basis is given by

$$\xi_\chi^i = \xi^i + \chi^{ij}\, X_j \,. \tag{3.113}$$

If we consider a general twist for $\mathfrak{g} = \mathfrak{su}(2)$, we can write with $\chi = \chi^{ij}\, \tau_i \otimes \tau_j \in \wedge^2 \mathfrak{su}(2)$,

$$\chi = \chi^1\, \tau_2 \wedge \tau_3 + \chi^2\, \tau_3 \wedge \tau_1 + \chi^3\, \tau_1 \wedge \tau_2 = \varepsilon_i{}^{jk}\, \chi^i\, \tau_j \otimes \tau_k \ , \quad \chi^{ij} = \varepsilon^{ij}{}_k\, \chi^k \tag{3.114}$$

and use this χ to study all the possible twist equivalent quasi-Lie bialgebras for $\mathfrak{su}(2)$. For example, we can apply the twist χ to $\mathfrak{d}(\mathfrak{su}(2)) \cong \mathfrak{iso}(3)$, with $\delta_{\mathfrak{iso}(3)} = 0 = \varphi_{\mathfrak{iso}(3)}$, using Eq.(3.111), to find that

$$\delta^\chi(Y) = \delta_{\mathfrak{iso}(3)}(Y) + \mathrm{ad}_Y^{(2)}(\chi) = [Y \otimes \mathrm{id} + \mathrm{id} \otimes Y, \chi] = Y^i \chi^j\, (\tau_i \wedge \tau_j) = Y \wedge (\chi^j \tau_j) \tag{3.115}$$

and a lengthy calculation gives

$$\varphi^\chi = \varphi_{\mathfrak{iso}(3)} - \mathrm{Alt}(\delta_{\mathfrak{iso}(3)} \otimes \mathrm{id})(\chi) + [\![\chi, \chi]\!] = [\![\chi, \chi]\!] = (\delta_{ij} \chi^i \chi^j)\, \tau_1 \wedge \tau_2 \wedge \tau_3 \,. \tag{3.116}$$

If we apply a second twist λ to δ^χ and φ^χ one finds

$$\delta^{\lambda\circ\chi}(Y) = \delta^\chi(Y) + \mathrm{ad}_Y^{(2)}(\lambda) = [Y \otimes \mathrm{id} + \mathrm{id} \otimes Y, \chi + \lambda] = Y \wedge ((\chi + \lambda)^j \tau_j) \tag{3.117}$$

and we see that in order to get back to $\delta_{\mathfrak{iso}(3)} = 0$, we must choose $\lambda = -\chi$. For the coassociator we find with $\varphi \equiv \tau_1 \wedge \tau_2 \wedge \tau_3$,

$$\begin{aligned} \varphi^{\lambda\circ\chi} &= \varphi^\chi - \mathrm{Alt}(\delta^\chi \otimes \mathrm{id})(\lambda) + [\![\lambda, \lambda]\!] \\ &= (\delta_{ij} \chi^i \chi^j)\, \varphi \ - \ \mathrm{Alt}(\delta^\chi \otimes \mathrm{id})(\lambda) \ + \ (\delta_{ij} \lambda^i \lambda^j)\, \varphi \\ &= (\delta_{ij} \chi^i \chi^j)\, \varphi \ + \ 2\, (\delta_{ij} \chi^i \lambda^j)\, \varphi \ + \ (\delta_{ij} \lambda^i \lambda^j)\, \varphi \,, \end{aligned} \tag{3.118}$$

where we used another lengthy calculation that shows that

$$\mathrm{Alt}(\delta^\chi \otimes \mathrm{id})(\lambda) = (-2)(\delta_{ij} \chi^i \lambda^j)\, \varphi \,. \tag{3.119}$$

From this we see that again for $\lambda = -\chi$, we do get back $\varphi_{\mathfrak{iso}(3)} = 0$, which is of course just a special case of the fact that χ and $-\chi$ are inverse twists.

Now, if we go back to our example $\mathfrak{d}(\mathfrak{su}(2)) \cong \mathfrak{su}(2) + \mathfrak{an}(2, \mathbb{R})_\pm \cong \mathfrak{sl}(2, \mathbb{C})$ from section 3.2, one can easily show that the cocommutator $\delta_{\mathfrak{sl}(2,\mathbb{C})}$ can be obtained via a simple twist from the "flat case" ($\Lambda = 0$), $\mathfrak{d}(\mathfrak{su}(2)) \cong \mathfrak{iso}(3)$. The relevant twist is given by the antisymmetric part of the standard r - matrix for $\mathfrak{d}(\mathfrak{su}(2)) \cong \mathfrak{sl}(2, \mathbb{C})$, i.e.,

$$\chi = \pm(\tau_2 \wedge \tau_1) = \hat{r}_\pm \,. \tag{3.120}$$

Plugging this twist into Eq.(3.111) we find with $\delta_{\mathfrak{iso}(3)} = 0 = \varphi_{\mathfrak{iso}(3)}$

$$\delta^\chi(X) = \mathrm{ad}_X^{(2)}(\chi) = [X, \hat{r}_\pm] = \delta_{\mathfrak{sl}(2,\mathbb{C})}(X) \quad , \quad \varphi^\chi = [\![\chi,\chi]\!] = (\delta_{ij}\chi^i\chi^j)\,\varphi \neq \varphi_{\mathfrak{sl}(2,\mathbb{C})}\,, \tag{3.121}$$

where we used that $\mathrm{Alt}(\delta_{\mathfrak{iso}(3)} \otimes \mathrm{id})(\chi) = 0$. Now, this shows clearly, that we can get $\delta_{\mathfrak{sl}(2,\mathbb{C})}(X)$, but there is no way, except for $\chi = 0$, to keep $\varphi_{\mathfrak{sl}(2,\mathbb{C})} = 0$. Hence, we find that indeed $\mathfrak{iso}(3)$ and $\mathfrak{sl}(2,\mathbb{C})$, as doubles of $\mathfrak{su}(2)$, are not twist equivalent. Similarly, Eq.(3.115) and Eq.(3.116) show clearly that for every twist χ with $\|\chi\|^2 = (\delta_{ij}\chi^i\chi^j) = 1$ we obtain $\varphi^\chi = \varphi = \tau_1 \wedge \tau_2 \wedge \tau_2$, which is the correct coassociator for $\mathfrak{d}(\mathfrak{su}(2)) \cong \mathfrak{so}(4)$, however, we can never get $\delta^\chi = 0$ for non-trivial χ. Hence, $\mathfrak{so}(4)$ is not twist equivalent to $\mathfrak{iso}(3)$. Similarly, one shows that it is not possible to find a twist between $\mathfrak{so}(4)$ and $\mathfrak{sl}(2,\mathbb{C})$. Finally, we want to mention that twist equivalence of quasi-Lie bialgebras, or their corresponding quasi-Hopf algebras, translates to an equivalence of their representation categories as monoidal- or quasi-tensor categories [61]. This we should keep in mind when studying representations of certain (quasi-) Hopf algebras at different values of the deformation parameter in chapter 4.

Next, we will consider the exponentiated version of the quasi-Lie bialgebras, which leads us to so-called quasi-Poisson Lie groups as symmetries of quasi-Poisson manifolds, where the quasi-Poisson structures are generalized in a precise sense, such that the Jacobi identity may be violated. We follow mainly [43, 44, 45].

Definition (quasi-Poisson structure) : A quasi-Poisson structure on a Lie group G is defined by a multiplicative bivector P_G and an element $\varphi \in \wedge^3\mathfrak{g}$ such that

$$\frac{1}{2}\,[P_G, P_G] = \varphi^R - \varphi^L \qquad \text{and} \qquad [P_G, \varphi^L] = 0\,, \tag{3.122}$$

where $[P_G, P_G]$ denotes the Schouten bracket of P_G and $\varphi^{L,R}$ are the left- and right-invariant multivector fields associated with φ.

Given the canonical r - matrix of a quasi-Lie bialgebra $\mathfrak{g}$, with double $\mathfrak{d}(\mathfrak{g})$, i.e., $r_\mathfrak{d} = X_i \otimes \xi^i \in \mathfrak{d} \otimes \mathfrak{d}$, one defines the associated quasi-Poisson bivector via

$$P_D \equiv r_\mathfrak{d}^L - r_\mathfrak{d}^R\,. \tag{3.123}$$

The notation D, or (D, G), denotes the group pair associated with the quasi-Lie bialgebra double $(\mathfrak{d}(\mathfrak{g}), \mathfrak{g}, \mathfrak{h})$. The bivector P_D vanishes at the identity, exactly like in the Poisson-Lie group case, i.e., $P_D(e) = 0$. Furthermore, one can show that the multiplication of the double D, i.e., $\mu_D : D\times D \to D$, is a quasi-Poisson map, i.e., $\{f_1\circ\mu_D, f_2\circ\mu_D\} = \{f_1, f_2\}\circ\mu_D$ and $\mu_{D*}(P_{D\times D}) = P_D$.

Definition (quasi-Poisson manifold) : A quasi-Poisson manifold is a G-manifold M with an invariant bivector field $P_M \in \mathcal{C}^\infty(M, \wedge^2 TM)$ such that $[P_M, P_M] = \phi_M$, where ϕ_M is the Cartan 3-form field on M. The quasi-Poisson bracket $\{f, g\} \equiv P_M(df, dg)$ does not satisfy the Jacobi identity in general, however, it holds that $\text{Jac}(f_1, f_2, f_3) = 2\phi_M(df_1, df_2, df_3)$. Furthermore, the G - action map $\Phi : G \times M \to M$, $(g, m) \mapsto g.m$ is a quasi-Poisson map for any quasi-Poisson manifold (M, P_M).

In section 3.4 we will consider two examples to better illustrate quasi-Poisson structures and quasi-Poisson manifolds, but first, we want to discuss the notion of **fusion**. The process of fusion is a new feature of Hamiltonian quasi-Poisson manifolds with a G - action[10], which is necessary because simply taking the direct product of two quasi-Poisson manifolds with the product Poisson structure does not satisfy the so-called moment map condition. The process of fusion guarantees that the product manifold is again a proper Hamiltonian quasi-Poisson manifold with the appropriate G - action. Fusion becomes necessary when we have a $G \times G$ - manifold M and we want to fuse the two factors to get a G - action on M.

Definition (Fusion) : Fusion defines a quasi-Poisson G - structure for the diagonal action of the quasi-Poisson $G \times G$ - manifold $(M, P_{M,G\times G})$ via the quasi-Poisson bivector

$$P_{\text{fus}} \equiv P_{M,G\times G} - \Psi_M \,, \tag{3.124}$$

where Ψ_M denotes the image of $\Psi = \frac{1}{2}\, e^1_a \wedge e^2_a \in \wedge^2(\mathfrak{g} \oplus \mathfrak{g})$ under the $G \times G$ - action map. Given the two components of the moment maps for the $G \times G$ - action, Φ_1 and Φ_2, one gets the moment map for the G - action via $\Phi = \Phi_1\Phi_2$.

If we have two quasi-Poisson G - manifolds (M_1, P_1) and (M_2, P_2), we can construct a quasi-Poisson G - manifold on $M_1 \times M_2$, with respect to the diagonal G - action and with bivector $(P_1 + P_2)_{\text{fus}}$. We denote this fusion product of quasi-Poisson manifolds by $M_1 \circledast M_2$.

In the next section we will briefly comment on the importance of the fusion product when studying the moduli space of flat connections on a Riemann surface $\Sigma_{g,n}$. However, since we will not directly use fusion ourselves in the following sections, we refer to [44, 45] for further information.

[10] We have not introduced the notion of quasi-Hamiltonian G - manifolds, which have associated with the group action a group valued moment map.

3.4 Loop Quantum Gravity with $\Lambda > 0 - \mathrm{SO}(4)$ as a deformed quasi-phase space

We are going to apply now what we have learned about quasi-Lie bialgebras to study the case of Euclidean 3d gravity with a positive cosmological constant, which has SO(4) symmetry. The Loop quantum gravity approach has certain similarities with the so-called combinatorial quantization of Chern-Simons theory in 3 dimensions [27, 28], but there are also differences. The same is true about the mathematical work in [45] and [46] on the classical aspects of the system. What they all have in common is the final object they try to describe, which is the so-called **moduli space of flat connections** on a 2-manifold Σ for structure group $G = \mathrm{SU}(2)$. From the gravity perspective and in physics language this object is called the physical phase space of the system, which means that all the constraints are dealt with. Another similarity of all three approaches is that their starting point is a finite dimensional system. It is known that the moduli space of flat connections for a 2-surface Σ is a finite dimensional manifold, but when starting from gravity, or Chern-Simons theory, the starting point is a field theory with infinitely many degrees of freedom. The way how the three approaches deal with the transition from the field theory picture to their finite dimensional picture is what distinguishes them. Whereas in Loop quantum gravity we consider graphs Γ embedded in Σ dual to a triangulation of Σ, the graphs used in the approaches [27, 28] and [45, 46] are related to a topological object, called the fundamental group of the 2-surface $\pi_1(\Sigma)$. Thus, the graphs in Loop quantum gravity can in general be bigger, or more refined than the graphs associated with $\pi_1(\Sigma)$. In some sense, the graphs coming from $\pi_1(\Sigma)$, for example from a so-called pants decomposition of Σ, are the minimal graphs that are necessary to capture the full topology of Σ[11]. This approach works in three dimensions, but has its limitations in higher dimensional situations. The Loop quantum gravity approach, however, is applicable to four dimensions (and higher) as well, since the graph Γ can still be associated with a triangulation (or more general discretization) of a 3-manifold, not just a 2-manifold. These larger graphs are then capable of measuring at least a finite number of the (local) degrees of freedom of the gravitational field. Another difference between the Loop quantum gravity approach and the other two is the order of reduction and quantization. In Loop quantum gravity we do not consider flat connections from the outset and then quantize the already reduced space of those flat connections, as is done in the other two approaches. Instead, we start with an auxiliary space of a priori general connections and impose gauge invariance and flatness in the quantum theory.

[11] For simplicity we have denoted our 2-manifold Σ. One can make explicit the number of holes (genus, g) and boundary components (punctures p) by writing $\Sigma_{g,p}$. This is also called a Riemann surface.

The work in [45, 46], using quasi-Poisson-Lie group structures, can be seen as a generalization of the Fock-Rosly description of the moduli space [72], which used Poisson-Lie group structures for semi-simple complex gauge groups, to the case of compact gauge groups, such as SU(2). The main result of [45, 46] is a (finite dimensional) description of the symplectic structure on the moduli spaces of flat connections on a Riemann surface $\mathcal{M}(\Sigma_{g,p})$ as a fusion product of g copies of the double $\mathbf{D}(G) = G \times G$, where G is a compact Lie group, and p conjugacy classes $\mathcal{C} \subset G = \mathrm{SU}(2)$. As mentioned before, it is here, where fusion is used in this approach[12]. If $\Sigma_{g,p}$ is a smooth 2-dimensional orientable manifold of genus g and p boundary components, the moduli space $\mathcal{M}(\Sigma_{g,p}) = \mathcal{A}_{flat}(\Sigma_{g,p})/G_{res}(\Sigma_{g,p})$ is isomorphic to

$$\mathrm{Hom}(\pi_1(\Sigma_{g,p}), G)/G = (\underbrace{\mathbf{D}(G) \circledast \cdots \circledast \mathbf{D}(G)}_{g} \circledast \underbrace{G \circledast \cdots \circledast G}_{p}) \,/\!\!/\, G\,, \tag{3.125}$$

where

$$\mathrm{Hom}(\pi_1(\Sigma_{g,p}), G) = \Big\{(a_1, \cdots, a_{2g}, b_1, \cdots, b_p) \in G^{2g+p} \;\; : \;\; \prod_j [a_j, a_{j+1}] \prod_k b_k = \mathrm{id}_G \Big\}\,. \tag{3.126}$$

We see, that in this description of $\mathcal{M}(\Sigma_{g,p})$ one uses the G - manifolds $\mathbf{D}(G)$ and $\mathcal{C} \subset G$. In the Loop quantum gravity approach, however, we will have to consider the $G \times G$ - manifold $D(G)$ to define our auxiliary Hilbert space. We will investigate this space in more detail in section 3.4.2. However, let us state here already their groups actions. $\mathbf{D}(\mathrm{SU}(2))$ has the following SU(2) - action

$$g \triangleright (h_1, h_2) = (gh_1g^{-1}, gh_2g^{-1}) \tag{3.127}$$

and $D(\mathrm{SU}(2))$ is defined as having the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ - action

$$(g_1, g_2) \triangleright (h_1, h_2) = (g_1h_1g_2^{-1}, g_2h_2g_1^{-1})\,. \tag{3.128}$$

In Loop quantum gravity we will consider a graph Γ, which is dual to some triangulation of the manifold Σ, and we want to attach to each link of Γ the double $D(\mathrm{SU}(2)) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ as a deformation of the flat- or $\Lambda = 0$ phase space $\mathrm{T}^*\mathrm{SU}(2)$. Unlike in the $\Lambda < 0$ case, where we could replace $\mathrm{T}^*\mathrm{SU}(2)$ by $\mathrm{SL}(2, \mathbb{C})$ as a phase space, $D(\mathrm{SU}(2)) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ turns out to be a quasi-Poisson manifold, i.e., not a proper phase space in

[12] In fact, fusion is used in two different instances. First, to obtain the G - manifold $\mathbf{D}(G)$ via fusion from the $G \times G$ - manifold $D(G)$ and second to glue the different $\mathbf{D}(G)$ together.

the standard sense. Also let us note already that we will not pay much attention to the fact that strictly we only have $\mathrm{SO}(4) \cong (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbb{Z}_2$ and not $\mathrm{SO}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$. This means, we consider $\mathrm{SO}(4)$ and $\mathrm{SU}(2) \times \mathrm{SU}(2)$ interchangeably.

Now, we want to investigate the quasi-Lie bialgebra structure of $\mathfrak{su}(2)$, which allows us to show that $\mathfrak{d}(\mathfrak{su}(2)) \cong \mathfrak{so}(4)$. Hence, let us first recall the real Lie algebra $\mathfrak{so}(4)$, usually given in the following form

$$[J_i, J_j] = \varepsilon_{ij}{}^k\, J_k \qquad , \qquad [J_i, K_j] = \varepsilon_{ij}{}^k\, K_k \qquad , \qquad [K_i, K_j] = \varepsilon_{ij}{}^k\, J_k\,, \tag{3.129}$$

where the J_i are the generators of rotations of a three dimensional subspace of $\mathbb{R}^4$ and the K_i, the so-called Euclidean boosts, are simply the remaining 3 rotations involving the "time - direction". Hence, we think in terms of coordinates of $\mathbb{R}^4$ given by $(0,1,2,3) = (t,x,y,z)$ and the metric is given by $\delta_{IJ} = (1,1,1,1)$. The main difference between $\mathfrak{so}(4)$ and $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$, besides the fact that $\mathfrak{so}(4)$ is compact and $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ is non-compact, is that $\mathfrak{so}(4)$ is not a semi-simple Lie algebra, which can be seen from the Lie algebra isomorphism $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. If we define

$$A_i \equiv \frac{1}{2}\,(J_i + K_i) \qquad , \qquad B_i \equiv \frac{1}{2}\,(J_i - K_i)\;, \tag{3.130}$$

with inverse $J_i = A_i + B_i$ and $K_i = A_i - B_i$, one can show that those new generators satisfy

$$[A_i, A_j] = \varepsilon_{ij}{}^k\, A_k \qquad , \qquad [A_i, B_j] = 0 \qquad , \qquad [B_i, B_j] = \varepsilon_{ij}{}^k\, B_k\,. \tag{3.131}$$

Hence, the A_i and B_i generate indeed two $\mathfrak{su}(2)$ - Lie-subalgebras, which commute with each other. Furthermore, one finds that the A_i and B_i are in fact the self-dual and anti-self dual components of $\mathfrak{so}(4)$, which can be seen by writing

$$M_{\mu\nu} = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & J_3 & -J_2 \\ -K_2 & -J_3 & 0 & J_1 \\ -K_3 & J_2 & -J_1 & 0 \end{pmatrix} \quad , \quad (*M)_{\mu\nu} = \begin{pmatrix} 0 & J_1 & J_2 & J_3 \\ -J_1 & 0 & K_3 & -K_2 \\ -J_2 & -K_3 & 0 & K_1 \\ -J_3 & K_2 & -K_1 & 0 \end{pmatrix}\,, \tag{3.132}$$

where we used the definition $(*M)_{\mu\nu} \equiv \frac{1}{2}\,\varepsilon_{\mu\nu}{}^{\rho\sigma} M_{\rho\sigma}$. Note, that in the Euclidean case we have $*^2 M = M$. The inverse relations are given, of course, by

$$J_i = (*M)_{0i} = \frac{1}{2}\,\varepsilon_{0i}{}^{jk} M_{jk} \qquad , \qquad K_i = M_{0i}\,. \tag{3.133}$$

The (anti-) self-dual components of M are given via $T^{\pm}_{\mu\nu} \equiv \alpha(M_{\mu\nu} \pm (*M)_{\mu\nu})$, since for all $\alpha \in \mathbb{R}_*$ we have $*T^{\pm} = \pm T^{\pm}$. We see that for $\alpha = \pm 1/2$ we reproduce the expressions for A_i and B_i in Eq.(3.130)[13].

The fact that $\mathfrak{so}(4)$ is really a direct sum of two Lie subalgebras can be seen as follows[14]. Recall that the Lie bracket on the direct sum of two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is given by $[(A_1, B_1), (A_2, B_2)]_{\mathfrak{g}_1\oplus\mathfrak{g}_2} = ([A_1, A_2]_{\mathfrak{g}_1}, [B_1, B_2]_{\mathfrak{g}_2})$. If we consider now the case $\mathfrak{g}_1 = \mathfrak{g}_2 = \mathfrak{su}(2)$, with generators A_i and B_i we have

$$[(A_i, B_a), (A_j, B_b)]_{\mathfrak{su}(2)\,\oplus\,\mathfrak{su}(2)} = ([A_i, A_j]_{\mathfrak{su}(2)}, [B_a, B_b]_{\mathfrak{su}(2)}) = ({\varepsilon_{ij}}^{k}\, A_k, {\varepsilon_{ab}}^{c}\, B_c)\,. \tag{3.134}$$

With the relations $J_i = A_i + B_i = (A_i, B_i) \in \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $K_i = A_i - B_i = (A_i, -B_i) \in \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ we can show that $\mathfrak{so}(4)$ is indeed isomorphic to the direct sum Lie algebra $(\mathfrak{su}(2) \oplus \mathfrak{su}(2), [\,\cdot\,,\,\cdot\,]_{\mathfrak{su}(2)\,\oplus\,\mathfrak{su}(2)})$, by writing

$$\begin{aligned}[][J_i, J_j] &= [(A_i, B_i), (A_j, B_j)]_{\mathfrak{su}(2)\,\oplus\,\mathfrak{su}(2)} = ([A_i, A_j]_{\mathfrak{su}(2)}, [B_i, B_j]_{\mathfrak{su}(2)}) \\ &= ({\varepsilon_{ij}}^{k}\, A_k, {\varepsilon_{ij}}^{k}\, B_k) = {\varepsilon_{ij}}^{k}\, (A_k, B_k) = {\varepsilon_{ij}}^{k}\, J_k\end{aligned} \tag{3.135}$$

and similarly

$$\begin{aligned}[][J_i, K_j] &= [(A_i, B_i), (A_j, -B_j)]_{\mathfrak{su}(2)\,\oplus\,\mathfrak{su}(2)} = ([A_i, A_j]_{\mathfrak{su}(2)}, -[B_i, B_j]_{\mathfrak{su}(2)}) \\ &= ({\varepsilon_{ij}}^{k}\, A_k, -{\varepsilon_{ij}}^{k}\, B_k) = {\varepsilon_{ij}}^{k}\, (A_k, -B_k) = {\varepsilon_{ij}}^{k}\, K_k\,,\end{aligned} \tag{3.136}$$

$$\begin{aligned}[][K_i, K_j] &= [(A_i, -B_i), (A_j, -B_j)]_{\mathfrak{su}(2)\,\oplus\,\mathfrak{su}(2)} = ([A_i, A_j]_{\mathfrak{su}(2)}, [B_i, B_j]_{\mathfrak{su}(2)}) \\ &= ({\varepsilon_{ij}}^{k}\, A_k, {\varepsilon_{ij}}^{k}\, B_k) = {\varepsilon_{ij}}^{k}\, (A_k, B_k) = {\varepsilon_{ij}}^{k}\, J_k\,.\end{aligned} \tag{3.137}$$

Now, if we recall Eq.(3.108), which gave the Lie brackets of the double of a quasi-Lie bialgebra, and compare those with the Lie brackets of $\mathfrak{so}(4)$ in Eq.(3.129) we see that

[13] Note, that there is a similar decomposition into (anti-) self-dual decomposition in the $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ case, which uses $A_i = (J_i + i\,K_i)/2$ and $B_i = (J_i - i\,K_i)/2$. However, this decomposition only works over $\mathbb{C}$, i.e., it only shows that $\mathfrak{sl}(2,\mathbb{C}) \hookrightarrow \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2)^{\mathbb{C}} \oplus \mathfrak{su}(2)^{\mathbb{C}}$ and not $\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. In the physics literature this necessary complexification is sometimes not pointed out carefully.

[14] Remember that the Iwasawa decomposition used for $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ is not a direct sum decomposition of Lie algebras, but only a direct sum decomposition of vector spaces. This has the effect that the mixed Lie brackets between rotations τ_i and the $\mathfrak{an}(2,\mathbb{R})_{\pm}$ parts κ^i or λ^i do not vanish.

we can identify $\mathfrak{so}(4)$ and $\mathfrak{d}(\mathfrak{su}(2))$ via $\tau_i = J_i$, which generate our $\mathfrak{g} = \mathfrak{su}(2)$, and the Euclidean boosts correspond to the dual, i.e., $\mathfrak{g}^* = \mathfrak{su}(2)^*$ with $\xi^i = \delta^{ij} K_j$ and $\langle \tau_i | \xi^j \rangle = \delta_i^j$. Then Eq.(3.129) becomes

$$[\tau_i, \tau_j] = \underbrace{\varepsilon_{ij}{}^k}_{= f_{ij}{}^k} \tau_k \quad , \quad [\tau_i, \xi^j] = \varepsilon_i{}^j{}_k \, \xi^k = - \underbrace{\varepsilon_{ik}{}^j}_{= f_{ik}{}^j} \xi^k \quad , \quad [\xi^i, \xi^j] = \underbrace{\varepsilon^{ijk}}_{= \varphi^{ijk}} \tau_k \, , \tag{3.138}$$

and by comparing again with Eq.(3.108) we can readily see that in our case the components of the cocycle are zero, i.e., $c_i{}^{jk} = 0 = c_k{}^{ij}$, but the components of the coassociator are non-trivial, i.e., $\varphi^{ijk} = \varepsilon^{ijk} \neq 0$. The Lie algebra structure of $\mathfrak{su}(2)$ is of course given by $f_{ij}{}^k = \varepsilon_{ij}{}^k$. This shows that the double of the quasi-Lie bialgebra $(\mathfrak{su}(2), \delta = 0, \varphi = \tau_1 \wedge \tau_2 \wedge \tau_3)$, which is the standard quasi-Lie bialgebra on $\mathfrak{su}(2)$, is indeed isomorphic to $\mathfrak{so}(4)$.

The description of $\mathfrak{so}(4)$ in terms of the generators $\tau_i = J_i$ and $\xi^i = \delta^{ij} K_j$ corresponds to the quasi-Lie bialgebra with double $\mathfrak{d}(\mathfrak{su}(2)) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)^*$, whereas the generators A_i and B_i correspond to the description using the bilinear form introduced in Eq.(3.110) and $\mathfrak{d}(\mathfrak{su}(2)) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Hence, we see that those two cases correspond merely to a different choice of basis and not really to different quasi-Lie bialgebra structures. This is possible because of the scalar product δ_{ij}, which is simply the (rescaled) Killing form on $\mathfrak{su}(2)$ and allows us to identify the K_i with the dual elements to J_i.

Similarly to our statement, that given a Lie bialgebra $(\mathfrak{g}, \delta_\mathfrak{g})$ we can rescale the cocycle with some number $k \in \mathbb{R}$ to get another Lie bialgebra $(\mathfrak{g}, k\, \delta_\mathfrak{g})$, one can make the statement that, given a quasi-Lie bialgebra $(\mathfrak{g}, \delta_\mathfrak{g}, \varphi_\mathfrak{g})$, we can rescale the coassociator with some number $k \in \mathbb{R}$ to get another quasi-Lie bialgebra $(\mathfrak{g}, \delta_\mathfrak{g}, k\, \varphi_\mathfrak{g})$, possibly up to some compatibility condition for non-trivial cocycles. Thus, we see clearly that the standard quasi-Lie bialgebra structure on $\mathfrak{su}(2)$, corresponding to $\mathfrak{d}(\mathfrak{su}(2)) \cong \mathfrak{so}(4)$, is a smooth deformation of the Lie bialgebra $(\mathfrak{su}(2), \delta = 0, \varphi = 0)$, corresponding to $\mathfrak{d}(\mathfrak{su}(2)) \cong \mathfrak{iso}(3)$, via $\varphi_k = k\,(\tau_1 \wedge \tau_2 \wedge \tau_3)$ with $k \in [0, 1]$.[15]

Now, before we consider the exponentiated version(s) of this quasi-Lie bialgebra double, once as a quasi-Poisson Lie group and once as a quasi-Poisson manifold, we will collect

[15] We are using the same notation $\mathfrak{d}(\mathfrak{g})$ for the classical double, regardless of whether we consider a Lie bialgebra structure on $\mathfrak{g}$ or a quasi-Lie bialgebra structure. This should hopefully not lead to confusions when we write $\mathfrak{d}(\mathfrak{su}(2)) \cong \mathfrak{iso}(3)$, $\mathfrak{d}(\mathfrak{su}(2)) \cong \mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{d}(\mathfrak{su}(2)) \cong \mathfrak{so}(4)$, which are obviously not all isomorphic to each other, but just denote each individually the isomorphisms between the standard Lie algebras and the corresponding classical doubles, depending on the appropriate cocycles and coassociators.

a few facts about the Lie group SO(4) and its parametrization in terms of two SU(2) elements.

In its defining representation an element G of the (real) Lie group of special orthogonal transformations of $\mathbb{R}^4$, commonly denoted as SO(4), is a real 4×4 matrix, such that $\det(G) = 1$ and $G^t = G^{-1}$. We use coordinates $(x^0, x^1, x^2, x^3) = (t, x, y, z)$ for $\mathbb{R}^4$ and call x^0 the time direction. The generators of this Lie group were denoted by J_i and K_i above. We can parametrize each G as $G = \exp(m^i J_i + n^i K_i)$, where again, in the defining representation, the J_i and K_i are 4-dimensional matrices. Using the decomposition in terms of the A_i and B_i basis, introduced above, every $G \in \mathrm{SO}(4)$ can be written as

$$G = \exp(m^i J_i + n^i K_i) = \exp(a^i A_i + b^i B_i) = \exp(a^i A_i)\exp(b^i B_i)\,, \tag{3.139}$$

because we have $[A_i, B_j] = 0$, but

$$G = \exp(m^i J_i + n^i K_i) \neq \exp(m^i J_i)\exp(n^i K_i)\,, \tag{3.140}$$

since $[J_i, K_j] \neq 0$. The relation between the components a^i, b^j and m^i, n^j is given via

$$\exp(a^i A_i) = \exp((m^i + n^i)A_i) \quad , \quad \exp(b^i B_i) = \exp((m^i - n^i)B_i)\,, \tag{3.141}$$

where we used that $J_i = A_i + B_i$ and $K_i = A_i - B_i$. Now, in order to understand the homomorphism between SO(4) and $\mathrm{SU}(2) \times \mathrm{SU}(2)$, note that there exists a similarity transformation U, such that we can always write

$$G = \exp(a^i A_i + b^i B_i) = U(\exp(a^i \tau_i) \otimes \exp(b^i \tau_i))\, U^{-1}\,, \tag{3.142}$$

where $\exp(a^i \tau_i)$ and $\exp(b^i \tau_i)$ are two SU(2) elements in terms of our standard basis τ_i of $\mathfrak{su}(2)$. Also note, that this similarity transformation does not depend on the coordinates a^i and b^i and is in fact just a particular constant matrix.

This means that to every $G \in \mathrm{SO}(4)$ we can associate two SU(2) elements and we usually write $G = (g_1, g_2)$, implying the use of the similarity transformation U. In particular, one finds that the elements with $\vec{a} = \vec{b}$, i.e., those G such that $G = (g, g)$, the so-called **diagonal elements**, correspond to the elements with $\vec{m} = \vec{a}$ and $\vec{n} = 0$. These are those rotations that leave the time direction invariant. On the other hand, elements with $\vec{b} = -\vec{a}$, i.e., G such that $G = (g, g^{-1})$, the so-called **anti-diagonal elements**, correspond to those transformations with $\vec{m} = 0$ and $\vec{n} = \vec{a}$. These are the Euclidean boosts that transform the time direction.

Thus, for a general element $G = (g_1, g_2)$, we write for a rotation of the 3d subspace $G_R = (g, g)$, and for a Euclidean boost $G_B = (h, h^{-1})$. This gives rise to an Iwasawa like decomposition, which means that for every $G = (g_1, g_2)$ there exist G_R and G_B such that we can write

$$\begin{aligned} G(\vec{a}, \vec{b}) &= (g_1, g_2) = (\exp(\vec{a} \cdot \vec{\tau}), \exp(\vec{b} \cdot \vec{\tau})) \\ &= (g, g)(h, h^{-1}) = (\exp(\vec{c} \cdot \vec{\tau}), \exp(\vec{c} \cdot \vec{\tau}))(\exp(\vec{d} \cdot \vec{\tau}), \exp(-\vec{d} \cdot \vec{\tau})) \\ &= G(\vec{c}, \vec{c})\, G(\vec{d}, -\vec{d}) = G_R\, G_B\,, \end{aligned} \tag{3.143}$$

where $g_1(\vec{a}) = g(\vec{c})\, h(\vec{d}) \in \mathrm{SU}(2)$ and $g_2(\vec{b}) = g(\vec{c})\, h(-\vec{d}) \in \mathrm{SU}(2)$. The vectors $\vec{a}$ and $\vec{b}$ can be explicitly related to the vectors $\vec{c}$ and $\vec{d}$ via the group multiplication law in SU(2), as we will use below. Analogously, on can of course also decompose G as $G = \tilde{G}_B\, \tilde{G}_R$.

Clarifying the interpretation of this decomposition is important for two reasons. First, if we want to think of the double $D(\mathrm{SU}(2))$ as a deformed version of the flat phase space $\mathrm{T}^*\mathrm{SU}(2)$, we would like to interpret one SU(2) element as the holonomy along the link such that it corresponds to the parallel transport map from the source node of the link to the target node. This is the same interpretation as in the flat case. The second SU(2) element is to be interpreted as the conjugate variable or the deformed flux. This is different from the flat case, where the flux is an element of $\mathfrak{su}(2)$. Hence, we would like to describe the holonomy in terms of elements of the form (g, g) and the fluxes as anti-diagonal elements (h, h^{-1}). We will consider the appropriate quasi-Poisson structure in the next section.

Secondly, if we want to think of the double $D(\mathrm{SU}(2))$ as the symmetry of the system in terms of quasi-Poisson Lie groups, for example, in a discretized setting, we do not have to consider the quasi-Poisson structures encountered in section 3.3, which make $D(\mathrm{SU}(2))$ into a quasi-phase space, but we have to consider the quasi-analog of the Drinfeld double. The Poisson brackets of this quasi-Drinfeld double will be much simpler than in the $\mathrm{SL}(2, \mathbb{C})$ case, because the canonical "r - matrix" t, associated with the standard quasi-Lie bialgebra structure of $\mathfrak{su}(2)$, is symmetric and leads to vanishing quasi-Poisson brackets Eq.(3.45).

Now, as in the $\Lambda = 0$ case, the picture we have in mind with the variables G_R and G_B from above is the following. We consider a graph Γ, whose links carry now the double $D(\mathrm{SU}(2))$ as our quasi-phase space, instead of the cotangent bundle $\mathrm{T}^*\mathrm{SU}(2)$ or $\mathrm{SL}(2, \mathbb{C})$. We think of the diagonal elements (g, g) as our holonomies along the link and the anti-diagonal elements (h, h^{-1}) as their conjugate (deformed) fluxes. Remember, that in their

4-dimensional representation the elements (g, g) correspond to rotations of the 3D subspace and the elements (h, h^{-1}) correspond to the (Euclidean) boosts. Hence, if we are interested in the "flat" limit, where the flux sector becomes undeformed, we should consider the logarithmic coordinates of the h in (h, h^{-1}), i.e., the vector $\vec{n}$ in $h^{\pm 1} = \exp(\pm n^i \tau_i)$. The $\mathrm{SU}(2) \times \mathrm{SU}(2)$ - symmetry of our system is discretized such that we have local SU(2) gauge transformations associated with the nodes of the graph and another set of SU(2) transformations (the deformed translations), associated with the faces bounded by the links of the graph. These symmetries are best understood in terms of the coordinates we have studied in section 2.1.1. In both cases we have group valued moment maps as the generators of our symmetries. At the nodes of the graph we require the fluxes to satisfy

$$\prod_{i=1}^{n} (G_B)_i = \mathrm{id}_{\mathrm{SO}(4)} \qquad \Leftrightarrow \qquad \prod_{i=1}^{n} h_i = \mathrm{id}_{\mathrm{SU}(2)} \,, \tag{3.144}$$

where n gives the number of links meeting at the node and $(G_B)_i = (h_i, h_i^{-1})$. Similarly, we impose the (deformed) flatness constraint at the faces via

$$\prod_{i=1}^{f} (G_R)_i = \mathrm{id}_{\mathrm{SO}(4)} \qquad \Leftrightarrow \qquad \prod_{i=1}^{f} g_i = \mathrm{id}_{\mathrm{SU}(2)} \,, \tag{3.145}$$

where we denote by f here the number of links bounding the face and $(G_R)_i = (g_i, g_i)$. We see that we have a nice duality between the two sectors in this case, cf. [48]. Furthermore, we will get back to study this setting for a 3-valent node in section 3.4.3, which will show nicely, how the q root of unity fusion rules in the quantum theory can be understood from these types of group valued Gauss constraints.

3.4.1 Quasi-Poisson brackets for the conjugacy classes of SU(2)

We will now consider the quasi-Poisson brackets for the conjugacy classes of SU(2) with the quasi-Poisson structure introduced in [44, 45] and which were studied in a Loop quantum gravity context already in [47]. Those variables are interpreted as the deformed fluxes and are associated with the links meeting at a node. We include our calculation here to show how to reproduce some of the results presented in [47]. We furthermore extend those results by calculating the brackets for the holonomy components, which also serves as a preparation for the calculations on the double $D(\mathrm{SU}(2))$ in section 3.4.2.

We start with the (left-) conjugation action of SU(2) on itself, i.e., SU(2) × SU(2) → SU(2), $(g,h) \mapsto ghg^{-1}$, and the corresponding bivector field

$$P_G \equiv \frac{\alpha}{2}(e_a^R \wedge e_a^L) \qquad , \qquad \alpha \in \mathbb{R}\,, \tag{3.146}$$

where the $e_a^{R,L}$ denote the right- and left invariant vector fields generated by $\tau_a \in \mathfrak{su}(2)$, respectively, and the τ_a are the standard $\mathfrak{su}(2)$ generators satisfying $[\tau_i, \tau_j] = \varepsilon_{ij}{}^k\,\tau_k$. The quasi-Poisson bracket for two functions $f_1, f_2 : \mathrm{SU}(2) \to \mathbb{C}$ is then given by

$$\{f_1, f_2\}(h) = P_G(df_1, df_2)(h) = \frac{\alpha}{2}(e_a^R \wedge e_a^L)(df_1, df_2)(h)\,. \tag{3.147}$$

Note, that the generating vector fields $\xi^\#(h)$ of the following left actions are related to the left- and right-invariant vector fields as follows

$$(g,h) \mapsto g.h\ , \quad \xi^\#(h) \equiv \left.\frac{d}{dt}\right|_{t=0}(\exp(-t\xi), h) = \left.\frac{d}{dt}\right|_{t=0}\exp(-t\xi).h = -\xi h = -\xi^R(h)\,, \tag{3.148}$$

$$(g,h) \mapsto h.g^{-1}, \quad \xi^\#(h) \equiv \left.\frac{d}{dt}\right|_{t=0}(\exp(-t\xi), h) = \left.\frac{d}{dt}\right|_{t=0} h.\exp(t\xi) = h\xi = \xi^L(h)\,. \tag{3.149}$$

Here we have $g = \exp(\xi)$, $\xi \in \mathfrak{su}(2)$ and $\xi^{L,R}$ denotes the left- and right invariant vector fields associated with $\xi \in \mathfrak{su}(2)$. In particular, we have

$$e_a^L(h) = h\tau_a \qquad , \qquad e_a^R(h) = \tau_a h\,. \tag{3.150}$$

Now, we are interested in calculating the quasi-Poisson brackets of the coordinate functions $f_{ij} : \mathrm{SU}(2) \to \mathbb{C}$, given via $f_{ij}(h) = h_{ij}$, which give the matrix-group elements and satisfy $\det(h) = 1$ and $\bar{f}_{22} = f_{11}$ and $\bar{f}_{21} = -f_{12}$, i.e.,

$$\begin{aligned}\{f_{ij}, f_{kl}\}(h) &= \frac{\alpha}{2}(e_a^R \wedge e_a^L)(df_{ij}, df_{kl})(h) \qquad (3.151)\\ &= \frac{\alpha}{2}\Big((e_a^R)(df_{ij})(e_a^L)(df_{kl}) - (e_a^R)(df_{kl})(e_a^L)(df_{ij})\Big)(h)\,.\end{aligned}$$

In this case we have

$$\begin{aligned}(e_a^R)(df_{ij})(h) = -\left.\frac{d}{dt}\right|_{t=0} f_{ij}(\exp(-t\tau_a).h) &= -\left.\frac{d}{dt}\right|_{t=0}(\exp(-t\tau_a))_{im}h_{mj}\\ &= (\tau_a)_{im}f_{mj}(h) \qquad (3.152)\end{aligned}$$

and similarly

$$(e_a^L)(df_{kl})(h) = \frac{d}{dt}\bigg|_{t=0} f_{kl}(h.\exp(t\tau_a)) = \frac{d}{dt}\bigg|_{t=0} h_{kn}(\exp(t\tau_a))_{nl}$$

$$= f_{kn}(h)(\tau_a)_{nl}\,. \tag{3.153}$$

We used that $f_{ij}(a.b) = f_{ik}(a)f_{kj}(b)$, where summation over k is understood. Thus, we find

$$\{f_{ij}, f_{kl}\}(h) = \frac{\alpha}{2}\Big((e_a^R)(df_{ij})(e_a^L)(df_{kl}) - (e_a^R)(df_{kl})(e_a^L)(df_{ij})\Big)(h)$$

$$= \frac{\alpha}{2}\left((\tau_a)_{im}f_{mj}(h)\, f_{kn}(h)(\tau_a)_{nl} - (\tau_a)_{km}f_{ml}(h)\, f_{in}(h)(\tau_a)_{nj}\right)\,. \tag{3.154}$$

Now, in order to simplify this expression, note that we can express the components of the $\mathfrak{su}(2)$ generators

$$\tau_1 = \frac{1}{2i}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad , \quad \tau_2 = \frac{1}{2i}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \quad , \quad \tau_3 = \frac{1}{2i}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{3.155}$$

as follows

$$(\tau_1)_{ij} = \frac{1}{2i}\,\varepsilon_{ij}\,(-1)^j \quad , \quad (\tau_2)_{ij} = -\frac{1}{2}\,\varepsilon_{ij} \quad , \quad (\tau_3)_{ij} = \frac{1}{2i}\,\delta_{ij}\,(-1)^{i+1} \tag{3.156}$$

and thus we get

$$(\tau_1)_{im}(\tau_1)_{nl} = \left(\frac{1}{2i}\right)^2 \varepsilon_{im}\,\varepsilon_{nl}\,(-1)^{m+l} = \frac{(-1)^{m+l+1}}{4}\,(\delta_{in}\delta_{ml} - \delta_{il}\delta_{mn})\,, \tag{3.157}$$

$$(\tau_1)_{km}(\tau_1)_{nj} = \left(\frac{1}{2i}\right)^2 \varepsilon_{km}\,\varepsilon_{nj}\,(-1)^{m+j} = \frac{(-1)^{m+j+1}}{4}\,(\delta_{kn}\delta_{mj} - \delta_{kj}\delta_{mn})\,, \tag{3.158}$$

$$(\tau_2)_{im}(\tau_2)_{nl} = \left(-\frac{1}{2}\right)^2 \varepsilon_{im}\,\varepsilon_{nl} = \frac{1}{4}\,(\delta_{in}\delta_{ml} - \delta_{il}\delta_{mn})\,, \tag{3.159}$$

$$(\tau_2)_{km}(\tau_2)_{nj} = \left(-\frac{1}{2}\right)^2 \varepsilon_{km}\,\varepsilon_{nj} = \frac{1}{4}\,(\delta_{kn}\delta_{mj} - \delta_{kj}\delta_{mn})\,, \tag{3.160}$$

$$(\tau_3)_{im}(\tau_3)_{nl} = \left(\frac{1}{2i}\right)^2 \delta_{im}\,\delta_{nl}\,(-1)^{m+l} = \frac{(-1)^{m+l+1}}{4}\,\delta_{im}\,\delta_{nl}\,, \tag{3.161}$$

$$(\tau_3)_{km}(\tau_3)_{nj} = \left(\frac{1}{2i}\right)^2 \delta_{km}\,\delta_{nj}\,(-1)^{m+j} = \frac{(-1)^{m+j+1}}{4}\,\delta_{km}\,\delta_{nj}\,. \tag{3.162}$$

This leads to

$$\{f_{ij}, f_{kl}\}(h) = \frac{\alpha}{8}\Big[f_{mj}(h)f_{kn}(h)\,\varepsilon_{im}\,\varepsilon_{nl}\,\Big(1-(-1)^{m+l}\Big) \tag{3.163}$$

$$-\; f_{ml}(h)f_{in}(h)\,\varepsilon_{km}\,\varepsilon_{nj}\,\Big(1-(-1)^{m+j}\Big) \;+\; f_{ij}(h)f_{kl}(h)\,\Big((-1)^{j+k}-(-1)^{i+l}\Big)\Big]\,,$$

which can be further simplified to give

$$\{f_{ij}, f_{kl}\}(h) = \frac{\alpha}{4}\left[f_{in}(h)f_{nl}(h)\,\delta_{jk} \;-\; f_{kn}(h)f_{nj}(h)\,\delta_{il}\right]. \tag{3.164}$$

For a single conjugacy class theses brackets are not yet what we are ultimately interested in. Hence, their calculation should be seen more of an exercise for the next section. However, they were an important step for us to understand the mathematical framework of quasi-Poisson geometry as presented in [44, 45, 47]. In the case for the double $D(\mathrm{SU}(2))$, however, the functions f_{ij} are interpreted as the components of a holonomy and the fact that the brackets Eq.(3.164) are not zero, is the result of our quasi-Poisson brackets, which corresponds to $\Lambda \neq 0$. Remember, that in the flat case with $\Lambda = 0$, where we have the standard Poisson structure of $\mathrm{T}^*\mathrm{SU}(2)$, the Poisson brackets of the components of the holonomy vanish. Now, a more interesting variable to look at is given by the "flux-" or "logarithmic-" coordinates a^i, because they directly correspond to the deformed version of the fluxes we know from standard Loop quantum gravity. This means, we are interested in calculating

$$\{a^i, a^j\}(h) = P_G(da^i, da^j)(h) = \frac{\alpha}{2}(e^R_a \wedge e^L_a)(da^i, da^j)(h)\,. \tag{3.165}$$

We use

$$e^R_a(da^i)(h) = -\left.\frac{d}{dt}\right|_{t=0} a^i(\exp(-t\tau_a).h) \quad , \qquad \exp(-t\tau_a).h \equiv \exp(a^k(t)\tau_k)$$

$$= -\left.\frac{d}{dt}\right|_{t=0} a^i(t) \tag{3.166}$$

and

$$e^L_a(da^j)(h) = \left.\frac{d}{dt}\right|_{t=0} a^j(h.\exp(t\tau_a)) \quad , \qquad h.\exp(t\tau_a) \equiv \exp(a^k(t)\tau_k)$$

$$= \left.\frac{d}{dt}\right|_{t=0} a^j(t) \tag{3.167}$$

where one should be careful, despite the same notation, not to confuse the $a^i(t)$ in Eq.(3.166), which corresponds to the right-invariant vector field, with the $a^j(t)$ in Eq.(3.167), which corresponds to the left-invariant vector field. We will see below how they are different, but we did not want to introduce too many different variables.

Now, in order to determine the t-dependent coordinates $a^i(t)$, we use the following general formula for the SU(2) multiplication of two group elements in terms of their logarithmic coordinates. We have in the physical convention

$$\exp(ia(\hat{n}\cdot\vec{\sigma})).\exp(ib(\hat{m}\cdot\vec{\sigma})) = \exp(ic(\hat{k}\cdot\vec{\sigma}))\,, \tag{3.168}$$

for unit vectors $(\hat{n}, \hat{m}, \hat{k})$ and the angles (a, b, c). We prefer the equivalent formula

$$\exp(\vec{n}\cdot\vec{\tau}).\exp(\vec{m}\cdot\vec{\tau}) = \exp(\vec{k}\cdot\vec{\tau})\,, \tag{3.169}$$

taking the appropriate rescaling into account. Now, one finds that

$$\cos(c) = \cos(a)\cos(b) - (\hat{n}\cdot\hat{m})\,\sin(a)\sin(b) \tag{3.170}$$

and

$$\hat{k} = \frac{1}{\sin(c)}\left(\hat{n}\,\sin(a)\cos(b)\,+\,\hat{m}\,\sin(b)\cos(a)\,-\,(\hat{n}\times\hat{m})\,\sin(a)\sin(b)\right)\,. \tag{3.171}$$

Now, if we first express $h = \exp(ib(\hat{m}\cdot\vec{\sigma}))$, then we get for the action of the right-invariant vector field

$$\begin{aligned}\exp(-t\tau_a).h = \exp(-t\tau_a).\exp(ib(\hat{m}\cdot\vec{\sigma})) &= \exp(ia(\hat{n}\cdot\vec{\sigma})).\exp(ib(\hat{m}\cdot\vec{\sigma}))\\ &= \exp(ic(\hat{k}\cdot\vec{\sigma})) = \exp(a^i(t)\tau_i)\end{aligned} \tag{3.172}$$

with

$$a = \frac{t}{2} \quad , \quad \hat{n}^i = \delta^i_a \quad , \quad a^i(t) = -2\,c(t)\,\hat{k}^i(t)\,. \tag{3.173}$$

Thus, we find, with $\hat{n}\cdot\hat{m} = \delta_{ij}\,\hat{n}^i\hat{m}^j = \hat{m}_a$,

$$c(t) = \arccos\left(\cos\left(\frac{t}{2}\right)\cos(b)\,-\,\hat{m}_a\,\sin\left(\frac{t}{2}\right)\sin(b)\right) \tag{3.174}$$

and

$$\hat{k}^i(t) = \frac{1}{\sin(c(t))} \left(\delta^i_a \sin\left(\frac{t}{2}\right) \cos(b) \, + \, \hat{m}^i \, \sin(b) \cos\left(\frac{t}{2}\right) \, - \, (\hat{n} \times \hat{m})^i \, \sin\left(\frac{t}{2}\right) \sin(b) \right) , \tag{3.175}$$

where $(\hat{n} \times \hat{m})^i = \varepsilon^i{}_{jk} \, \hat{n}^j \, \hat{m}^k = \varepsilon^i{}_{ak} \, \hat{m}^k$, and thus, finally,

$$a^i(t) = -2 \, c(t) \, \hat{k}^i(t) \tag{3.176}$$

$$= -2 \, \frac{\arccos\left(\cos\left(\frac{t}{2}\right) \cos(b) \, - \, \hat{m}_a \, \sin\left(\frac{t}{2}\right) \sin(b)\right)}{\sin\left(\arccos\left(\cos\left(\frac{t}{2}\right) \cos(b) \, - \, \hat{m}_a \, \sin\left(\frac{t}{2}\right) \sin(b)\right)\right)} \tag{3.177}$$

$$\times \, \left(\delta^i_a \sin\left(\frac{t}{2}\right) \cos(b) \, + \, \hat{m}^i \, \sin(b) \cos\left(\frac{t}{2}\right) \, - \, \varepsilon^i{}_{ak} \, \hat{m}^k \, \sin\left(\frac{t}{2}\right) \sin(b) \right) ,$$

where the denominator can be simplified, using $\sin(\arccos(x)) = \sqrt{1 - x^2}$. A lengthy but elementary calculation leads to

$$e^R_a(da^i)(h = \exp(ib(\hat{m} \cdot \vec{\sigma}))) = - \left.\frac{d}{dt}\right|_{t=0} a^i(t)$$

$$= \left(\left(1 \, - \, b \, \frac{\cos(b)}{\sin(b)} \right) \hat{m}_a \, \hat{m}^i \, + \, \delta^i_a \, b \, \frac{\cos(b)}{\sin(b)} \, - \, b \, \varepsilon^i{}_{ak} \, \hat{m}^k \right) . \tag{3.178}$$

Explicitly analyzing the action of the left-invariant vector field in an equivalent fashion reveals that the corresponding $a^i(t)$ can be obtained from Eq.(3.177) simply by switching the overall sign, i.e., $+2$ instead of -2, and mapping $\hat{m} \mapsto -\hat{m}$. The result is found to be

$$e^L_a(da^j)(h = \exp(ib(\hat{m} \cdot \vec{\sigma}))) = \left.\frac{d}{dt}\right|_{t=0} a^j(t)$$

$$= \left(\left(1 \, - \, b \, \frac{\cos(b)}{\sin(b)} \right) \hat{m}_a \, \hat{m}^j \, + \, \delta^j_a \, b \, \frac{\cos(b)}{\sin(b)} \, + \, b \, \varepsilon^j{}_{ak} \, \hat{m}^k \right) . \tag{3.179}$$

Another lengthy but elementary calculation gives, with $h = \exp(ib(\hat{m}\cdot\vec{\sigma}))$,

$$
\begin{aligned}
\{a^i, a^j\}(h) = P_G(da^i, da^j)(h) &= \frac{\alpha}{2}(e^R_a \wedge e^L_a)(da^i, da^j)(h) \\
&= \frac{\alpha}{2}\sum_{a=1}^{3}\left(e^R_a(da^i)\, e^L_a(da^j) \,-\, e^R_a(da^j)\, e^L_a(da^i)\right)(h) \\
&= -2\alpha\, b^2\, \frac{\cos(b)}{\sin(b)}\, \varepsilon^{ij}{}_k\, \hat{m}^k = \alpha\, b\, \frac{\cos(b)}{\sin(b)}\, \varepsilon^{ij}{}_k\, a^k(h)\,,
\end{aligned} \tag{3.180}
$$

where we used $a^k(h) = a^k(\exp(ib(\hat{m}\cdot\vec{\sigma}))) = -2b\,\hat{m}^k$ in the last equality. Now, if we express h as $h = \exp(a^k\tau_k)$, instead of b and $\hat{m}$, then we get $b = a/2$, where $a = \sqrt{a_k a^k}$, and with $\alpha = 1$ we reproduce

$$
\{a^i, a^j\}(h) = \frac{a}{2}\,\mathrm{ctg}\left(\frac{a}{2}\right)\, \varepsilon^{ij}{}_k\, a^k(h)\,, \tag{3.181}
$$

which is the result found in [47] and where $\mathrm{ctg}(x) = \frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)}$.

Note, that both brackets in Eq.(3.164) and Eq.(3.181) vanish for $h = e$. Hence, SU(2) with the quasi-Poisson structure Eq.(3.146) is not a quasi-phase space, i.e., not symplectic, which is also clear from the fact that the dimension of the manifold SU(2) is three. However, this does not imply that Eq.(3.146) gives the quasi-Poisson Lie structure for SU(2), which would be given via the canonical (quasi-) r - matrix and leads to vanishing brackets for the group elements. Nevertheless, SU(2) with Eq.(3.146) is still a quasi-Poisson manifold according to the definition from the previous section, following [44]. We will see shortly that the correct $\Lambda > 0$ analog of the phase spaces $\mathrm{T}^*\mathrm{SU}(2)$ and $\mathrm{SL}(2,\mathbb{C})$ as the Heisenberg double is indeed given by $D(\mathrm{SU}(2)) = \mathrm{SU}(2)\times\mathrm{SU}(2)$ with a quasi-analog of the Heisenberg Poisson structure and non-vanishing brackets at the identity.

3.4.2 Quasi-Poisson brackets for the double $D(\mathrm{SU}(2))$

Before we finally get to investigate the quasi-Poisson structure on the double $D(\mathrm{SU}(2)) = \mathrm{SU}(2)\times\mathrm{SU}(2)$ we want to state again the importance of $D(\mathrm{SU}(2))$ for the Loop quantum gravity approach. Recall, that $D(\mathrm{SU}(2))$ is defined as having a $\mathrm{SU}(2)\times\mathrm{SU}(2)$ - action, compared with the manifold $\mathbf{D}(\mathrm{SU}(2))$, which has only a SU(2) - action and is used in the combinatorial description of the moduli space of flat connections, as discussed above, cf. Eq.(3.127) to Eq.(3.128). Recall furthermore, that the SU(2) - action on $\mathbf{D}(\mathrm{SU}(2))$ is obtained via fusion from the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ - action on $D(\mathrm{SU}(2))$. The reason why we are interested in $D(\mathrm{SU}(2))$ over $\mathbf{D}(\mathrm{SU}(2))$ is the fact that in Loop quantum gravity the fundamental building blocks of our graph Γ are the individual links l, carrying the phase space $\mathrm{T}^*\mathrm{SU}(2)$ (for $\Lambda = 0$), with a group element, or holonomy $h_l[A]$ that transforms under gauge transformation as

$$h_l[A] \;\mapsto\; g_{s(l)}\, h_l[A]\, g_{t(l)}^{-1}\,, \tag{3.182}$$

which is a transformation under $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and not just SU(2). $g_{s(l)}$ and $g_{t(l)}$ denote here the gauge transformations associated with the source and target node of the link. Of course, if we close the link by identifying its source node $s(l)$ with its target node $t(l)$, i.e., creating a closed loop, then this reduces to an SU(2) - action, which is why in the combinatorial picture, where the graph is associated with the fundamental group $\pi_1(\Sigma)$ of the manifold, it is sufficient to work with $\mathbf{D}(\mathrm{SU}(2))$. In the end, the graphs that are considered in Loop quantum gravity are also closed graphs, which means, there are no loose ends, since this would render the corresponding (quantum) state gauge variant, but nevertheless, as basic building blocks, we are interested in open links.

In fact, if we were only interested in $\mathbf{D}(\mathrm{SU}(2))$, and if discovered earlier, we could have greatly benefited from the work [73], where the authors investigated some of the quasi-Poisson brackets of $\mathbf{D}(\mathrm{SU}(2))$ and punctures, carrying coadjoint orbits of SU(2) with the quasi-Poisson structure discussed in section 3.4.1, albeit they apply a somewhat different approach. The authors of [73] investigate a certain (quasi-) Poisson structure on $\Sigma_{g,p}$, which they show to be equivalent to the quasi-Poisson structure defined in [44]. After choosing an arbitrary base point for their graph they calculate the corresponding Poisson brackets for the generators of the fundamental group $\pi_1(\Sigma_{g,p})$. Those generators are valued in the group SU(2) and the corresponding components are denoted by z^u_{ij} for the puncture $u \in \{1, 2, \cdots, p\}$ and by q^u_{ij} and p^u_{ij} for the handles $u \in \{1, 2, \cdots, g\}$. They find for the generators on the same link, i.e. $u \in \{1, \cdots, p\}$,

$$\{z^u_{ij}, z^u_{kl}\} = -\delta_{kj} z^u_{ir} z^u_{rl} + z^u_{ks} z^u_{sj} \delta_{il} \tag{3.183}$$

and that the generators encircling different punctures, i.e., any $u, v \in \{1, \cdots, p\}$ with $u < v$, do not Poisson commute. Considering links associated with the same handle, i.e.,

$u \in \{1, \cdots, g\}$, they find

$$\{p^u_{ij}, p^u_{kl}\} = -\delta_{kj} p^u_{ir} p^u_{rl} + p^u_{ks} p^u_{sj} \delta_{il} \,,$$
$$\{q^u_{ij}, q^u_{kl}\} = \delta_{kj} q^u_{ir} q^u_{rl} - q^u_{ks} q^u_{sj} \delta_{il} \,,$$
$$\{p^u_{ij}, q^u_{kl}\} = \delta_{kj} p^u_{ir} q^u_{rl} + q^u_{ks} p^u_{sj} \delta_{il} - p^u_{kj} q^u_{il} + q^u_{kj} p^u_{il} \tag{3.184}$$

and for punctures and handles with $u \in \{1, \cdots, g\}$ and $v \in \{1, \cdots, p\}$ they get

$$\{p^u_{ij}, z^v_{kl}\} = \delta_{kj} p^u_{ir} z^v_{rl} + z^v_{ks} p^u_{sj} \delta_{il} - p^u_{kj} z^v_{il} - z^v_{kj} p^u_{il} \,, \tag{3.185}$$
$$\{q^u_{ij}, z^v_{kl}\} = \delta_{kj} q^u_{ir} z^v_{rl} + z^v_{ks} q^u_{sj} \delta_{il} - q^u_{kj} z^v_{il} - z^v_{kj} q^u_{il} \,. \tag{3.186}$$

Thus, we see that the brackets needed for the combinatorial description of the moduli space have been calculated. We will now investigate the case relevant to us and calculate the brackets coming from the quasi-Poisson structure for the double $D(\mathrm{SU}(2))$. We have to clarify whether they are the same as the brackets above or how they differ. But since $\mathbf{D}(\mathrm{SU}(2))$ is obtained from $D(\mathrm{SU}(2))$ via fusion, which can introduce non-trivial contributions to the Poisson structure, we do a priori not expect them to have the same quasi-Poisson brackets.

To be more explicit, recall that we consider $D(\mathrm{SU}(2))$ as the manifold $\mathrm{SU}(2) \times \mathrm{SU}(2)$ with action

$$(g_1, g_2) \triangleright (h_1, h_2) = (g_1 h_1 g_2^{-1}, g_2 h_2 g_1^{-1}) \,, \tag{3.187}$$

whereas $\mathbf{D}(\mathrm{SU}(2))$ denotes the manifold $\mathrm{SU}(2) \times \mathrm{SU}(2)$ with the (diagonal) action

$$g \triangleright (h_1, h_2) = (g h_1 g^{-1}, g h_2 g^{-1}) = (\mathrm{Ad}_g(h_1), \mathrm{Ad}_g(h_2)) \,. \tag{3.188}$$

The quasi-Poisson bivector for $D(\mathrm{SU}(2))$, according to [44, 45], is given by

$$P_{\mathrm{SU}(2) \times \mathrm{SU}(2)} = \frac{\alpha}{2} \left(e^{1,L}_a \wedge e^{2,R}_a + e^{1,R}_a \wedge e^{2,L}_a \right) \,, \tag{3.189}$$

where we will drop the subscript $\mathrm{SU}(2) \times \mathrm{SU}(2)$ in general. Let us first discuss how those vector fields in the definition of $P_{\mathrm{SU}(2) \times \mathrm{SU}(2)}$ actually act on functions. Because it took as a while to understand the exact meaning of $P_{\mathrm{SU}(2) \times \mathrm{SU}(2)}$ and to learn how to use it for explicit calculations. Note again that there is in general a difference between the fundamental vector field, associated with a particular action, and the left- or right-invariant vector fields, with respect to a certain group action. We can consider the action in Eq.(3.187) as an action of $(g_1, g_2) = g \in \mathrm{SO}(4)$ on $(h_1, h_2) = h \in \mathrm{SO}(4)$, i.e.,

$$\mu : \mathrm{SO}(4) \times \mathrm{SO}(4) \to \mathrm{SO}(4) \quad , \quad (g, h) \mapsto \mu(g, h) = (g_1 h_1 g_2^{-1}, g_2 h_2 g_1^{-1}) \in \mathrm{SO}(4) \,. \tag{3.190}$$

Note, however, that this action is neither the standard left-, right- or conjugation action of SO(4). The action defined via μ in Eq.(3.190) is not the conjugation action in SO(4) because the inverse factor acts with the two components exchanged, i.e., instead of Eq.(3.190) the conjugation action is given via $g.h.g^{-1} = (g_1 h_1 g_1^{-1}, g_2 h_2 g_2^{-1}) \neq \mu(g, h) = (g_1 h_1 g_2^{-1}, g_2 h_2 g_1^{-1})$.

If we take $\xi = (\xi_1, \xi_2) \in \mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ such that $g = \exp(\xi) = (g_1, g_2) = (\exp(\xi_1), \exp(\xi_2))$, then by the general definition of a fundamental vector field we get a vector field on SO(4) via

$$\xi_{\mathrm{SU}(2)\times\mathrm{SU}(2)}(h) = \left.\frac{d}{dt}\right|_{t=0} \mu(\exp(-t\xi), h) \tag{3.191}$$

$$= \left.\frac{d}{dt}\right|_{t=0} (\exp(-t\xi_1)\, h_1 \exp(t\xi_2), \exp(-t\xi_2)\, h_2 \exp(t\xi_1))$$

$$= (h_1\xi_2 - \xi_1 h_1, h_2\xi_1 - \xi_2 h_2) \,. \tag{3.192}$$

Next, we consider the two left-actions $\varphi_{L,R} : \mathrm{SO}(4) \times \mathrm{SO}(4) \to \mathrm{SO}(4)$

$$(g, h) \mapsto \varphi_L(g, h) = h.g^{-1} \qquad , \qquad (g, h) \mapsto \varphi_R(g, h) = g.h \,. \tag{3.193}$$

With $g = \exp(\xi)$ we find again a relation between the associated fundamental vector fields to the actions $\varphi_{L,R}$ and the left- and right-invariant vector fields $\xi^L(h)$ and $\xi^R(h)$ on SO(4)

$$\xi^L(h) = \left.\frac{d}{dt}\right|_{t=0} \varphi_L(\exp(-t\xi), h) = h.\xi = (h_1\xi_1, h_2\xi_2) \,, \tag{3.194}$$

$$-\,\xi^R(h) = \left.\frac{d}{dt}\right|_{t=0} \varphi_R(\exp(-t\xi), h) = -\xi.h = (-\xi_1 h_1, -\xi_2 h_2) \,. \tag{3.195}$$

Thus, note the difference between the fundamental vector field Eq.(3.192) associated with the action $\mu(g, h)$ and the left- and right- invariant vector fields in Eqs.(3.194, 3.195) associate with the actions $\varphi_{L,R}$.

Now, we consider $\xi_a^1 = (\tau_a, 0)$ and $\xi_a^2 = (0, \tau_a)$ to obtain

$$\xi_a^{1,L}(h) = \frac{d}{dt}\bigg|_{t=0} \varphi_L(\exp(-t\xi_a^1), h) = (h_1\tau_a, 0)\,, \tag{3.196}$$

$$\xi_a^{2,L}(h) = \frac{d}{dt}\bigg|_{t=0} \varphi_L(\exp(-t\xi_a^2), h) = (0, h_2\tau_a)\,, \tag{3.197}$$

$$\xi_a^{1,R}(h) = -\frac{d}{dt}\bigg|_{t=0} \varphi_R(\exp(-t\xi_a^1), h) = (\tau_a h_1, 0)\,, \tag{3.198}$$

$$\xi_a^{2,R}(h) = -\frac{d}{dt}\bigg|_{t=0} \varphi_R(\exp(-t\xi_a^2), h) = (0, \tau_a h_2)\,. \tag{3.199}$$

Now, one of the problems in understanding how to use the quasi-Poisson structure as defined by Eq.(3.189) was that we were not entirely sure whether the vector fields appearing in its definition correspond really just to the left- and right actions of SO(4) on itself, or whether we have to consider them as somehow explicitly related with the group action Eq.(3.187) and, if they really correspond simply to the left- and right actions, then what is the relevance of the explicit action Eq.(3.187)? What would happen if we would consider a different group action? Would Eq.(3.189) still give the correct quasi-Poisson structure on $D(\mathrm{SU}(2))$? The simple answer to these questions is yes, the vector fields in the definition of Eq.(3.189) are really just the left- and right actions. The connection with the group action Eq.(3.187), or any other action on $D(\mathrm{SU}(2))$, is made via the notion of "F - relation", where F denotes an equivariant map with respect to the group action under consideration[16]. This notion states that, if we have two (or more) G - manifolds M_i, i.e., there are actions φ_i of G on M_i, and a G - equivariant map $F_{ij} : M_i \to M_j$, then the (fundamental) vector fields ξ_{M_i} and ξ_{M_j} are F_{ij} - related, which means that the map F_{ij} induces a G - equivariant map $F_{ij,*} : TM_i \to TM_j$ that relates ξ_{M_i} to ξ_{M_j}.

Thus, we continue now with the Eqs.(3.196, 3.197, 3.198, 3.199) and calculate the quasi-Poisson brackets for the holonomy elements in different coordinates. First, we consider the "holonomy coordinates" $a_{ij}, b_{ij} : \mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathbb{C}$, defined via $a_{ij}(h) = a_{ij}(h_1, h_2) = (h_1)_{ij}$ and $b_{ij}(h) = b_{ij}(h_1, h_2) = (h_2)_{ij}$. We want to calculate the following quasi-Poisson brackets

$$\{a_{ij}, a_{kl}\}(h) \quad , \quad \{a_{ij}, b_{kl}\}(h) \quad , \quad \{b_{ij}, b_{kl}\}(h) \tag{3.200}$$

[16] We learned this concept from the lecture notes, http://www.math.toronto.edu/mein/teaching/LectureNotes/action.pdf, by one of the authors of [44, 45].

and write for example

$$\{a_{ij}, a_{kl}\}(h) = \frac{\alpha}{2}\left(e_a^{1,L} \wedge e_a^{2,R} + e_a^{1,R} \wedge e_a^{2,L}\right)(da_{ij}, da_{kl})(h)\,. \tag{3.201}$$

Since

$$a_{ij}(\varphi_L(g,h)) = a_{ij}(hg^{-1}) = a_{ij}((h_1,h_2)(g_1^{-1},g_2^{-1})) = (h_1 g_1^{-1})_{ij} = (h_1)_{im}(g_1^{-1})_{mj} \tag{3.202}$$

and

$$a_{kl}(\varphi_R(g,h)) = a_{kl}(gh) = a_{kl}((g_1,g_2)(h_1,h_2)) = (g_1 h_1)_{kl} = (g_1)_{kn}(h_1)_{nl}\,. \tag{3.203}$$

we get

$$(e_a^{1,L})(da_{ij})(h) = \left.\frac{d}{dt}\right|_{t=0} a_{ij}(\varphi_L(\exp(-t\xi_a^1),h)) = (h_1)_{im}(\tau_a)_{mj}\,, \tag{3.204}$$

$$(e_a^{2,R})(da_{kl})(h) = -\left.\frac{d}{dt}\right|_{t=0} a_{kl}(\varphi_R(\exp(-t\xi_a^2),h)) = 0\,, \tag{3.205}$$

$$(e_a^{1,R})(da_{ij})(h) = -\left.\frac{d}{dt}\right|_{t=0} a_{ij}(\varphi_R(\exp(-t\xi_a^1),h)) = (\tau_a)_{im}(h_1)_{mj}\,, \tag{3.206}$$

$$(e_a^{2,L})(da_{kl})(h) = \left.\frac{d}{dt}\right|_{t=0} a_{kl}(\varphi_L(\exp(-t\xi_a^2),h)) = 0\,, \tag{3.207}$$

which leads to

$$\{a_{ij}, a_{kl}\}(h) = \frac{\alpha}{2}\left(e_a^{1,L} \wedge e_a^{2,R} + e_a^{1,R} \wedge e_a^{2,L}\right)(da_{ij}, da_{kl})(h) = 0\,. \tag{3.208}$$

Similarly, due to

$$(e_a^{1,(L,R)})(db_{ij})(h) = \pm\left.\frac{d}{dt}\right|_{t=0} b_{ij}(\varphi_{L,R}(\exp(-t\xi_a^1),h)) = \pm\left.\frac{d}{dt}\right|_{t=0}(h_2)_{ij} = 0\,, \tag{3.209}$$

one finds

$$\{b_{ij}, b_{kl}\}(h) = 0\,. \tag{3.210}$$

Finally, together with

$$(e_a^{2,L})(db_{kl})(h) = (h_2)_{kn}(\tau_a)_{nl} \qquad , \qquad (e_a^{2,R})(db_{kl})(h) = (\tau_a)_{kn}(h_2)_{nl}\,, \tag{3.211}$$

we find

$$\begin{aligned}
\{a_{ij}, b_{kl}\}(h) &= \frac{\alpha}{2}\Big((e_a^{1,L})(da_{ij})\,(e_a^{2,R})(db_{kl}) \,+\, (e_a^{1,R})(da_{ij})\,(e_a^{2,L})(db_{kl})\Big)\,(h) \\
&= \frac{\alpha}{2}\,((h_1)_{im}(\tau_a)_{mj}\,(\tau_a)_{kn}(h_2)_{nl} \,+\, (\tau_a)_{im}(h_1)_{mj}\,(h_2)_{kn}(\tau_a)_{nl}) \\
&= \frac{\alpha}{2}\,(a(h)_{im}(\tau_a)_{mj}\,(\tau_a)_{kn}b(h)_{nl} \,+\, (\tau_a)_{im}a(h)_{mj}\,b(h)_{kn}(\tau_a)_{nl}) \qquad (3.212) \\
&= \frac{\alpha}{8}\Big(a(h)_{im}b(h)_{ml}\,\delta_{jk}\,((-1)^{m+j}-1) \,+\, b(h)_{kn}a(h)_{nj}\,\delta_{il}\,((-1)^{n+l}-1) \\
&\qquad -\, a(h)_{ij}b(h)_{kl}\,((-1)^{j+k}+(-1)^{i+l})\Big) \\
&= -\frac{\alpha}{4}\,(a(h)_{im}b(h)_{ml}\,\delta_{jk} \,+\, b(h)_{kn}a(h)_{nj}\,\delta_{il} \,-\, a(h)_{ij}b(h)_{kl})\,, \qquad (3.213)
\end{aligned}$$

where we have used again the Eqs.(3.157 - 3.162) in the fourth equality. For the simplification in the last equality we have used a computer.

Now, note that for $h = e$ we find with $a_{ij}(e) = \delta_{ij} = b_{ij}(e)$ that

$$\begin{aligned}
\{a_{ij}, b_{kl}\}(e) = \alpha\,(\tau_a)_{ij}\,(\tau_a)_{kl} &= -\frac{\alpha}{4}\,(\delta_{im}\,\delta_{ml}\,\delta_{jk} \,+\, \delta_{kn}\,\delta_{nj}\,\delta_{il} \,-\, \delta_{ij}\,\delta_{kl}) \\
&= -\frac{\alpha}{4}\,(2\,\delta_{il}\,\delta_{jk} \,-\, \delta_{ij}\,\delta_{kl})\,, \qquad (3.214)
\end{aligned}$$

which can also be written as

$$\{a \overset{\otimes}{,} b\}\,(e) = \begin{pmatrix} \{a_{11}, b_{11}\} & \{a_{11}, b_{12}\} & \{a_{12}, b_{11}\} & \{a_{12}, b_{12}\} \\ \{a_{11}, b_{21}\} & \{a_{11}, b_{22}\} & \{a_{12}, b_{21}\} & \{a_{12}, b_{22}\} \\ \{a_{21}, b_{11}\} & \{a_{21}, b_{12}\} & \{a_{22}, b_{11}\} & \{a_{22}, b_{12}\} \\ \{a_{21}, b_{21}\} & \{a_{21}, b_{22}\} & \{a_{22}, b_{21}\} & \{a_{22}, b_{22}\} \end{pmatrix}(e)$$

$$= -\frac{\alpha}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \alpha\,(\tau_a \otimes \tau_a) \neq 0 \tag{3.215}$$

and we see that indeed we have a proper quasi-Poisson structure on $D(\mathrm{SU}(2))$, which is non-degenerate at the identity. Note, however, that if we compare with Eq.(3.212), that we can not rewrite the whole bracket for $h \neq e$ using the Casimir element $t = (\tau_a \otimes \tau_a)$, but we find that, if we were to define the quasi-Poisson structure as,

$$\tilde{P}_{\mathrm{SU}(2)\times\mathrm{SU}(2)} = \frac{\alpha}{2}\left(e_a^{1,L} \wedge e_a^{2,L} + e_a^{1,R} \wedge e_a^{2,R}\right) \tag{3.216}$$

instead of Eq.(3.189), we still would get $\{a_{ij}, a_{kl}\}(h) = 0 = \{b_{ij}, b_{kl}\}(h)$, but for the mixed brackets we would find instead of Eq.(3.213)

$$\begin{aligned}
\{a_{ij}, b_{kl}\}(h) &= \frac{\alpha}{2}\left((e_a^{1,L})(da_{ij})\,(e_a^{2,L})(db_{kl}) \;+\; (e_a^{1,R})(da_{ij})\,(e_a^{2,R})(db_{kl})\right)(h) \\
&= \frac{\alpha}{2}\left((h_1)_{im}(\tau_a)_{mj}\,(h_2)_{kn}(\tau_a)_{nl} \;+\; (\tau_a)_{im}(h_1)_{mj}\,(\tau_a)_{kn}(h_2)_{nl}\right) \\
&= \frac{\alpha}{2}\left(a(h)_{im}(\tau_a)_{mj}\,b(h)_{kn}(\tau_a)_{nl} \;+\; (\tau_a)_{im}a(h)_{mj}\,(\tau_a)_{kn}b(h)_{nl}\right) \\
&= \frac{\alpha}{2}([a \otimes b, \tau_a \otimes \tau_a]_+)_{ij,kl}\,.
\end{aligned} \tag{3.217}$$

This also gives $\{a_{ij}, b_{kl}\}(e) = \alpha\,(\tau_a \otimes \tau_a)$ at $h = e$, but otherwise is different from Eq.(3.189). Given that we can express this bracket in terms of the Casimir element $t = (\tau_a \otimes \tau_a)$ it seems appropriate to take the quasi-Poisson structure Eq.(3.216) as the proper analog of the Heisenberg double to the quasi-setting, instead of Eq.(3.189). Indeed, this is the structure used in [48]. This way of expressing the quasi-Poisson structure on $D(\mathrm{SU}(2))$ is adapted to the (a, b) - coordinates and comes with the following group action

$$(\tilde{g}_1, \tilde{g}_2) \triangleright (h_1, h_2) = (\tilde{g}_2^{-1}, \tilde{g}_2^{-1})(h_1, h_2)(\tilde{g}_1, \tilde{g}_1)\,. \tag{3.218}$$

The action Eq.(3.187) and the quasi-Poisson bivector Eq.(3.189) correspond to the holonomy and flux variables (h, g), in the notation of [48], which is a different parametrization of $D(\mathrm{SU}(2))$. The h labels the diagonal elements $(h, h) \in \mathrm{SU}(2) \times \mathrm{SU}(2)$ and the g the anti-diagonal elements $(g, g^{-1}) \in \mathrm{SU}(2) \times \mathrm{SU}(2)$.

Now, we know that the a- and b- coordinates are not the coordinates we are mainly interested in, because they do not transform in a way that allows to interpret them as the deformation of the cotangent bundle $\mathrm{T}^*\mathrm{SU}(2)$. The reason is that both SU(2) elements transform under $\mathrm{SU}(2) \times \mathrm{SU}(2)$, however, we know that the flux sector (even the deformed one) ought to transform only under SU(2) only. The reason being, that in the discretized picture the fluxes are thought of as being attached to the nodes of the graph and hence the only transform under the gauge transformation associated with that node. The holonomy, on the other hand, connects two nodes of the graph and hence transforms under both gauge transformations, one at the source node and the other on the target node. However, we know how to transform the (a, b) - variables to achieve that. Namely, we are now looking at the quasi-Poisson brackets of the new variables (u, w) defined by $(u, w) = (a, ba)$, because they transform like

$$\mu(g,(u,w)) = \mu(g,(a,ba)) = (g_1 a g_2^{-1}, g_2 b g_1^{-1} g_1 a g_2^{-1}) = (g_1 u g_2^{-1}, g_2 w g_2^{-1}) \tag{3.219}$$

and thus we see that u transforms as an open holonomy and w as a flux at the target node Using instead the variable $w = ab$ for the flux, one finds that this flux transforms at the source node, i.e., $w \mapsto g_1 w g_1^{-1}$. We want to calculate first the brackets

$$\{u_{ij}, w_{kl}\}(h) \qquad , \qquad \{w_{ij}, w_{kl}\}(h)\,, \tag{3.220}$$

where it is clear from $\{a_{ij}, a_{kl}\} = 0$ that $\{u_{ij}, u_{kl}\}(h) = 0$. We also want to consider the coordinates w^i defined via $w = ba = \exp(w^i \tau_i)$ and then calculate

$$\{u_{ij}, w^k\}(h) \qquad , \qquad \{w^i, w^j\}(h)\,. \tag{3.221}$$

We use

$$(e_a^{2,R})(dw_{kl})(h) = (\tau_a)_{ks} w(h)_{sl}\,, \tag{3.222}$$

$$(e_a^{1,L})(dw_{kl})(h) = w(h)_{ks} (\tau_a)_{sl}\,, \tag{3.223}$$

$$\begin{aligned}(e_a^{2,L})(dw_{kl})(h) &= (h_2)_{ks}(\tau_a)_{st}(h_1)_{tl} = w(h)_{kr}(h_1^{-1}\tau_a h_1)_{rl} \\ &= w(h)_{kr}(\mathrm{Ad}_{u(h)^{-1}}(\tau_a))_{rl}\,,\end{aligned} \tag{3.224}$$

$$(e_a^{1,R})(dw_{kl})(h) = (h_2)_{ks}(\tau_a)_{st}(h_1)_{tl} = (e_a^{2,L})(dw_{kl})(h) \tag{3.225}$$

to find

$$\begin{aligned}\{u_{ij}, w_{kl}\}(h) &= \frac{\alpha}{2}\left((h_1)_{im}(\tau_a)_{mj}(\tau_a)_{ks}(h_2)_{sr}(h_1)_{rl} \;+\; (\tau_a)_{in}(h_1)_{nj}(h_2)_{ks}(\tau_a)_{st}(h_1)_{tl}\right) \\ &= \frac{\alpha}{2}\Big(u(h)_{im}(\tau_a)_{mj}(\tau_a)_{ks}w(h)_{sl} \;+\; (\tau_a)_{in}u(h)_{nj}(wu^{-1})(h)_{ks}(\tau_a)_{st}u(h)_{tl}\Big) \\ &= -\frac{\alpha}{4}\left(u(h)_{il}w(h)_{kj} \;-\; u(h)_{ij}w(h)_{kl} \;+\; u(h)_{ir}w(h)_{rl}\,\delta_{jk}\right)\,,\end{aligned} \tag{3.226}$$

where we have used that $h_2 = b(h) = (wu^{-1})(h) = w(h)u^{-1}(h) = h_2h_1h_1^{-1}$ and the last equality was again confirmed using a computer. Furthermore, we find

$$\{w_{ij}, w_{kl}\}(h) = \frac{\alpha}{2}\left(w(h)_{is}(\tau_a)_{sj}(\tau_a)_{kt}w(h)_{tl} \; - \; w(h)_{ks}(\tau_a)_{sl}(\tau_a)_{it}w(h)_{tj}\right) \tag{3.227}$$

which, as we know from above, equals

$$\{w_{ij}, w_{kl}\}(h) = -\frac{\alpha}{4}\left(w(h)_{in}w(h)_{nl}\,\delta_{jk} \; - \; w(h)_{kn}w(h)_{nj}\,\delta_{il}\right) . \tag{3.228}$$

This is a very nice results, because it shows us that the flux sector corresponds to the fluxes calculated in the conjugacy class case in section 3.4.1. Furthermore, we know directly that the brackets for w^i are given by

$$\{w^i, w^j\}(w) = -\frac{\alpha}{2}\,\frac{\tilde{w}}{2}\,\mathrm{ctg}\left(\frac{\tilde{w}}{2}\right)\;\varepsilon^{ij}{}_k\, w^k \,, \tag{3.229}$$

with $w = \exp(w^i\tau_i)$ and $\tilde{w} = \sqrt{w_i w^i}$. For the remaining bracket a lengthy calculation gives at $h = w = \exp(w^k\tau_k)$

$$\begin{aligned} \{u_{ij}, w^k\}(w) &= \frac{\alpha}{2}\Big((u\tau_a)_{ij}(e_a^{2,R})(dw^k) \; + \; (\tau_a u)_{ij}(e_a^{2,L})(dw^k)\Big)(w) \qquad (3.230) \\ &= \frac{\alpha}{2}\Big(u_{im}(\tau_a)_{mj}\Big(\kappa\,\hat{w}_a\hat{w}^k \; + \; \nu\,\delta_a^k \; + \; \varepsilon^k{}_{al}\,\lambda^l\Big) \; + \; (\tau_a)_{im}u_{mj}(e_a^{2,L})(dw^k)\Big) \,, \end{aligned}$$

with $\tilde{w} = \sqrt{w_l w^l}$ and

$$\kappa = 1 - \nu \quad , \quad \nu = \frac{\tilde{w}}{2}\,\mathrm{ctg}\left(\frac{\tilde{w}}{2}\right) \quad , \quad \lambda^l = \frac{w^l}{2} \,. \tag{3.231}$$

Note, that it is much harder to calculate the term $(e_a^{2,L})(dw^k)$ explicitly, compared with $(e_a^{2,R})(dw^k)$, which was already quite complicated. The reason is that we have to use the SU(2) group multiplication law twice for the logarithmic coordinates. While for

$$(e_a^{2,R})(dw^k)(h_1, h_2) = -\left.\frac{d}{dt}\right|_{t=0} w^k(\exp(-t\tau_a)\,h_2\,h_1) = -\left.\frac{d}{dt}\right|_{t=0} w^k(\exp(-t\tau_a)\,w(h_1, h_2)) \tag{3.232}$$

we can just use the result from the calculation for the conjugacy classes Eq.(3.178), for $(e_a^{2,L})(dw^k)$ we have to calculate

$$(e_a^{2,L})(dw^k)(h_1, h_2) = \frac{d}{dt}\bigg|_{t=0} w^k(h_2\, \exp(t\tau_a)\, h_1)\,, \tag{3.233}$$

which means we have to rewrite the product $h_2\, \exp(t\tau_a)\, h_1$ as $h_2\, \exp(t\tau_a)\, h_1 = \exp(w^a(t)\, \tau_a)$. We do not include this here explicitly, but just point out that one can alternatively relate $(e_a^{2,L})(dw^k)$ and $(e_a^{2,R})(dw^k)$ via

$$(e_a^{2,L}) = \mathrm{Ad}_{h_2}(e_a^{2,R})\,, \tag{3.234}$$

which can be used to determine the correct small curvature limit of those brackets. We know already that the flux brackets give back $\{w^i, w^j\} = \varepsilon^{ij}{}_k\, w^k$, with $\alpha = -2$, because for small $\tilde{w}$, or large radius R in $\frac{\tilde{w}}{R}$, we get

$$\nu = \frac{\tilde{w}}{2}\,\mathrm{ctg}\left(\frac{\tilde{w}}{2}\right) \approx 1 - \frac{\tilde{w}^2}{12} + \mathcal{O}(\tilde{w}^4) \qquad , \qquad \kappa = \mathcal{O}(\tilde{w}^2)\,. \tag{3.235}$$

We furthermore showed that $\{u_{ij}, u_{kl}\} = 0$ and with Eq.(3.235) and Eq.(3.234) we have

$$\begin{aligned}\{u_{ij}, w^k\}(w) &= \frac{\alpha}{2}\left(u_{im}(\tau_a)_{mj}\left(\delta_a^k + \varepsilon^k{}_{al}\,\lambda^l + \mathcal{O}(\tilde{w}^2)\right) + (\tau_a)_{im} u_{mj}\,(e_a^{2,L})(dw^k)(h)\right)\\ &= \frac{\alpha}{2}\left(u_{im}(\tau_a)_{mj}\,\delta_a^k\right) + \mathcal{O}(\tilde{w})\\ &= \frac{\alpha}{2}\,(u\tau_k)_{ij} + \mathcal{O}(\tilde{w})\end{aligned} \tag{3.236}$$

Hence, we see that we reproduce the correct brackets of $\mathrm{T}^*\mathrm{SU}(2)$ in the left-trivialization in the flat limit, i.e.

$$\{u_{ij}, u_{kl}\} = 0 \quad , \quad \{w^i, w^j\} = \varepsilon^{ij}{}_k\, w^k \quad , \quad \{u_{ij}, w^k\} = -(u\tau_k)_{ij}\,, \tag{3.237}$$

for $\alpha = -2$. Note, that in order to obtain the last bracket we have used Eq.(3.234) to argue that due to the adjoint action $\mathrm{Ad}_{h_2}(e_a^{2,R})$ the factor $(e_a^{2,L})(dw^k)$ is at least of linear order $\mathcal{O}(\tilde{w})$ and thus, in the flat limit, this term does not contribute to $\{u_{ij}, w^k\}(w)$. However, a more explicit calculation following Eq.(3.233) would be desirable. Similarly, one finds that one reproduces the correct brackets in the right trivialization of $\mathrm{T}^*\mathrm{SU}(2)$, i.e., $\{u_{ij}, w^k\} = -(\tau_k u)_{ij}$, when one chooses the flux to be given by $w(h) = h_1 h_2$. In this case, however, it is $(e_a^{2,L})(dw^k)$, which is easier to calculate and the factor $(e_a^{2,R})(dw^k)$ is non-trivial. Finally, let us note that one should alternatively investigate those brackets using the quasi-Poisson structure Eq.(3.216) and check, whether we get the correct flat limit. It is possible, however, that in this case one might have to define the correct flux variable via $w(h) = h_1^{-1} h_2$, which has to do with the different group action used with this Poisson structure and hence, different transformation behavior of the h_1 and h_2.

3.4.3 On the classical analog of q root of unity fusion rules

In this section we want to consider the node of a graph Γ, where three links meet and the corresponding flux holonomies satisfy the (curved) Gauss constraint. In section 1 we saw that in Loop quantum gravity the nodes of a graph Γ are decorated with so-called intertwiners, which are tensors that are invariant under the action of the gauge group. These intertwiners for 3-valent nodes have both, in the quantum theory and at the classical level, an interpretation as triangles[17]. Classically, the picture is given by a set of three (or n) arbitrary vectors $\vec{v}_i \in \mathbb{R}^3$, which can be interpreted as the gravitational flux. The vector $\vec{v}_i$ is assigned to the link i but together they are subject to the **Gauss constraint** at the node where all the links meet, i.e.,

$$\sum_{i=1}^{n} \vec{v}_i = 0\,, \tag{3.238}$$

which is the discretized version of the torsion-less condition. Due to the so-called **Minkowski theorem** one can then assign a unique convex polyhedron to a set of vectors $\{\vec{v}_i\}$ that satisfy Eq.(3.238). In the quantum theory, this constraint translates to the well-known Clebsch-Gordan restrictions on the allowed SU(2) representations j_i that meet at a node. When we work instead with q - deformed spin networks of $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ at q root of unity, as we intent to do in the next section, we get different **fusion rules**, i.e., Eq.(1.5) and Eq.(1.6), as we saw in in chapter 1. Now, the curved analog of Eq.(3.238) is given by the condition that the curved fluxes $g_i \in \mathrm{SU}(2)$ satisfy

$$\prod_{i=1}^{n} g_i = \mathrm{id}_{\mathrm{SU}(2)}\ . \tag{3.239}$$

This situation was already investigated in [47], where it was shown that there is a curved

[17]This correspondence between polyhedra and n - valent nodes holds also for $n > 3$, but we will not consider this here.

analog to the classical Minkowski theorem that allows to reconstruct a curved tetrahedron (for $n = 4$) from Eq.(3.239).

We want to review in this section some of the results from [74] and [75] on how the (quantum) fusion rules, which are known from the q root of unity case, can be understood already at the classical level from the (curved) Gauss constraint Eq.(3.239). The results presented here are mostly due to [74, 75] and we only give some more details on their calculations. The only difference between [74, 75] and our case is that again we are not tied to this picture of moduli spaces and flat connections, which is the setting in [74, 75], but rather consider general connections and a possibly larger graph Γ than the one associated with $\pi_1(\Sigma)$. Nevertheless, it was important to us to include these results, because they clarify the underlying deformed geometry of the q - deformed spin networks and neatly connect the topics presented in section 3.3 with our next chapter 4. Most importantly, however, they show that the correct quantization of the quasi-Poisson structures investigated before is given by the (quasi-) Hopf algebra $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{su}(2))$ at q root of unity. This is a very non-trivial statement and in some sense more important than in the $\Lambda < 0$ or q real case. For $\Lambda < 0$ the quantization of the symmetry of the deformed Poisson structure on $\mathrm{SL}(2,\mathbb{C})$ is done precisely via the quantum R - matrix, whose classical limit is the classical r - matrix of the classical double $\mathfrak{d}(\mathfrak{su}(2)) \cong \mathfrak{sl}(2,\mathbb{C})$. As we saw above, in the $\Lambda > 0$ case there is no such strict relation and thus it is a priori not certain that the correct quantization of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ should be related to $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ or $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{su}(2))$. In fact, in [74] the authors raise this very issue, namely, that "It would be interesting to see directly how quantization of such phase spaces provides invariant tensors of quantum groups $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ for q root of unity."

Now, let us consider a 3-valent node, where three holonomy fluxes $g_i \in \mathrm{SU}(2)$ meet, and assume that the three links all have the same orientation. We can write the holonomy g_i in terms of its logarithmic coordinates a^i as $g_i[A] = \exp(A)$, where $A = a^i\tau_i$, and then we have

$$\mathrm{Tr}\,(g_i[A]) = 2\cos\left(\frac{\sqrt{a\cdot a}}{2}\right) \quad , \quad a\cdot a = (a^1)^2 + (a^2)^2 + (a^3)^2\,. \tag{3.240}$$

The angles θ are obtained via

$$\theta \equiv \arccos(\mathrm{Tr}\,(g_i[A])\,/2) = \frac{\sqrt{a\cdot a}}{2} \quad , \quad \theta \in [0,\pi]\,. \tag{3.241}$$

Note, that $\frac{\sqrt{a\cdot a}}{2} \in \mathbb{R}$ is not restricted to those compact intervals, but due to the periodicity at the level of the holonomies we can not distinguish periodic connections.

Also note that we can think of $\tilde{\theta}^2 \equiv 4\theta^2 = a \cdot a$ as defining a 2-sphere in $\mathbb{R}^3$ with radius $\tilde{\theta}$. Now, if we were to uniformly rescale the coordinates as $a^i \to a^i_\kappa = \kappa\, a^i$ we get $A^\kappa = \kappa A$ and

$$\theta^\kappa \equiv \arccos(\mathrm{Tr}\,(g_i[A^\kappa])\,/2) = \kappa\,\frac{\sqrt{a \cdot a}}{2} = \frac{\sqrt{a_\kappa \cdot a_\kappa}}{2} = \kappa\,\theta \quad , \quad \theta^\kappa \in [0, \pi]\,, \tag{3.242}$$

from which we see that, no matter what our scale κ is, we always have $\theta^\kappa \in [0, \pi]$ because that is the image of the inverse cosine for all κ. However, we see that there is a linear relation between the angles at different scales, i.e., $\theta^\kappa = \kappa\,\theta$.

We consider now a 3-valent node with the constraint $g_1 g_2 g_3 = \mathrm{id}_{\mathrm{SU}(2)}$. Note that each g_i is conjugate to an element $\exp(2\theta_i \tau_3)$, which means that the angles $2\theta_i$ parametrize the conjugacy classes generated by the diagonal matrix τ_3. Note furthermore that Eq.(3.240) is invariant under conjugations of the holonomy, i.e.,

$$\mathrm{Tr}\,(g_i[A]) = 2\cos\left(\frac{\sqrt{a \cdot a}}{2}\right) = \mathrm{Tr}\left(m\, g_i[A]\, m^{-1}\right) \quad , \quad \forall\, m \in \mathrm{SU}(2) \tag{3.243}$$

and thus,

$$\mathrm{Tr}\,(g_i[A]) = 2\cos\left(\frac{\sqrt{a \cdot a}}{2}\right) = \mathrm{Tr}\left(m\, g_i[A]\, m^{-1}\right) = \mathrm{Tr}\,(\exp(2\theta_i\, \tau_3)) = 2\cos(\theta_i)\,. \tag{3.244}$$

Now we want to investigate which conditions on the angles $(\theta_1, \theta_2, \theta_3)$ guarantee that there is a solution to the constraint $g_1 g_2 g_3 = \mathrm{id}_{\mathrm{SU}(2)}$. Without loss of generality we can take g_1 to be already in diagonal form, because if we transform $g_1 \mapsto d_1 g_1 d_1^{-1} = \exp(2\theta_1 \tau_3)$, then we get for $g_1 g_2 g_3 = \mathrm{id}_{\mathrm{SU}(2)}$:

$$d_1 g_1 g_2 g_3 d_1^{-1} = d_1 g_1 d_1^{-1} d_1 g_2 d_1^{-1} d_1 g_3 d_1^{-1} = e^{2\theta_1 \tau_3} \tilde{g}_2 \tilde{g}_3 = \mathrm{id}_{\mathrm{SU}(2)}\,, \tag{3.245}$$

where $\tilde{g}_2$ and $\tilde{g}_3$ are, of course, still in the same conjugacy classes as g_2 and g_3, respectively. Then we are left with another $\exp(\alpha\tau_3)$ - freedom to transform $\tilde{g}_2$ and $\tilde{g}_3$, which allows us to take $\tilde{g}_2$ to be of the form $\tilde{g}_2 = \exp(2\theta_2(\sin(\beta)\tau_1 + \cos(\beta)\tau_3))$, for some β, and $\tilde{g}_3$ becomes $\breve{g}_3$, which is again still in the original conjugacy class as g_3. Thus, we have

$$e^{2\theta_1 \tau_3} e^{2\theta_2(\sin(\beta)\tau_1 + \cos(\beta)\tau_3)} = \breve{g}_3^{-1}\,. \tag{3.246}$$

Now we can conjugate the whole expression with the unique $n \in \mathrm{SU}(2)$, which takes $\check{g}_3^{-1}$ to $\exp(2\theta_3\,\tau_3)$, and then take the trace on both sides. This gives the following condition, where we used the cyclicity of the trace, i.e., $\mathrm{Tr}(n\,\mathcal{O}\,n^{-1}) = \mathrm{Tr}(\mathcal{O})$,

$$2\cos(\theta_1)\cos(\theta_2) - 2\sin(\theta_1)\sin(\theta_2)\cos(\beta) = 2\cos(\theta_3)\,. \tag{3.247}$$

For which β can this equation be satisfied? If we rewrite Eq.(3.247) as

$$\cos(\beta) = \pm 1 + \frac{\cos(\theta_1 \pm \theta_2) - \cos(\theta_3)}{\sin(\theta_1)\sin(\theta_2)}\,, \tag{3.248}$$

and note that $\cos(\beta)$ must be in the interval $[-1, 1]$, we conclude from

$$\cos(\beta) = 1 + \underbrace{\frac{\cos(\theta_1 + \theta_2) - \cos(\theta_3)}{\sin(\theta_1)\sin(\theta_2)}}_{\leq 0\ (!)} \qquad \Rightarrow \qquad \cos(\theta_1 + \theta_2) \leq \cos(\theta_3) \tag{3.249}$$

and from

$$\cos(\beta) = -1 + \underbrace{\frac{\cos(\theta_1 - \theta_2) - \cos(\theta_3)}{\sin(\theta_1)\sin(\theta_2)}}_{\geq 0\ (!)} \qquad \Rightarrow \qquad \cos(\theta_1 - \theta_2) \geq \cos(\theta_3)\,, \tag{3.250}$$

where those two conditions follow from the fact that $\sin(\theta_1)\sin(\theta_2) \in [0, 1]$ and $\cos(\theta_1 \pm \theta_2) \in [-1, 1]$ for $\theta_i \in [0, \pi]$. Hence, we get overall that Eq.(3.247) can be solved for $\cos(\beta)$ if and only if

$$\cos(\theta_1 + \theta_2) \leq \cos(\theta_3) \leq \cos(\theta_1 - \theta_2)\,. \tag{3.251}$$

This can also be expressed as

$$|\theta_1 - \theta_2| \leq \theta_3 \tag{3.252}$$

and

$$\theta_3 \leq \theta_1 + \theta_2 \quad \text{if} \quad |\theta_1 + \theta_2| \leq \pi \quad \text{or} \quad \theta_3 \leq 2\pi - (\theta_1 + \theta_2) \quad \text{if} \quad |\theta_1 + \theta_2| > \pi\,, \tag{3.253}$$

or, equivalently,

$$|\theta_1 - \theta_2| \leq \theta_3 \leq \min[\,\theta_1 + \theta_2\,,\, 2\pi - (\theta_1 + \theta_2)]\,. \tag{3.254}$$

If we consider now a constant R, which we call the radius, a constant $k = \pi R$, which is the circumference of a half-circle, and take instead of the angles θ_i their corresponding arc lengths, i.e., $l_i \equiv \frac{k\,\theta_i}{\pi} \in [0, k]$, then we can write Eq.(3.254) as

$$|l_1 - l_2| \leq l_3 \leq \min[\, l_1 + l_2\,,\, 2k - (l_1 + l_2)]\,. \tag{3.255}$$

Upon quantization of this system the constant k becomes discrete and by identifying $k = \frac{r-2}{2}$, where r is the root of q, i.e., $q = \exp(2\pi i/r)$, we see that we find that the arc lengths l_i correspond to the color representations j_i of $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2, \mathbb{C}))$ and we reproduce the correct fusion rules, i.e.,

$$|j_1 - j_2| \leq j_3 \leq \min[\, j_1 + j_2\,,\, r - 2 - (j_1 + j_2)]\,. \tag{3.256}$$

So, to summarize, this analysis shows that the quantum fusion rules for q a root of unity correspond exactly with the solution space of the classical deformed Gauss law $g_1 g_2 g_2 = \mathrm{id}_{\mathrm{SU}(2)}$, which is imposed at the nodes of a graph. The underlying reason for the necessary truncation of the tensor product in the root of unity case, i.e., that we have to take the minimum on the right-hand side in Eq.(3.256) and above, is the periodic nature of the flux holonomies $g_i \in \mathrm{SU}(2)$ when $\Lambda > 0$. For $\Lambda \leq 0$ the momentum space does not have such a periodicity and the fusion rules are the standard ones known from SU(2). This means, even in the deformed case with q real, for $\Lambda < 0$, the fusion rules are unchanged. This shows clearly that spin networks at q root of unity indeed correspond to the quasi-Poisson geometry which we have investigated in this chapter.

Chapter 4

Quantum theory

In the first part of this chapter we will review the mathematical background necessary to investigate the works [42, 40, 38] in the case when q is a root of unity. This means that we study the q - deformed Drinfeld - Jimbo universal enveloping algebra $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ and a certain finite dimensional subalgebra $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$. The representation category of these algebras, with a certain restriction on the allowed irreducible representations and a redefinition of their tensor products, provides the basis for the Turaev - Viro state sum model.

We will discuss the topic of star structures in section 4.1.1, which allow to define real forms of $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$, such as $\mathcal{U}_{\mathrm{q}}(\mathfrak{su}(2))$, when q is real. This is the quantum group used in [42, 40, 38] to construct the so-called Drinfeld double $\mathcal{D}(\mathcal{U}_{\mathrm{q}}(\mathfrak{su}(2)))$ as the symmetry of the classical deformed phase space $\mathrm{SL}(2,\mathbb{C})$ for $\Lambda < 0$.

After some progress adapting the results of [42, 40, 38] to the q root of unity case we learned from [50, 51] that the correct quantum symmetry for q - deformed models at q root of unity is actually a modified version of the standard Hopf algebra we used. This realization leads one into the realm of quasi-Hopf algebras, a fact that is not widely appreciated in the Loop quantum gravity literature on q - deformed models. Of course, given our current knowledge from section 3.3 on the quasi-Poisson structures that arise already at the classical level for $\Lambda > 0$, this comes as no surprise. Furthermore, it matches nicely with the recent works [47, 76, 48], where the authors introduced the quasi-Poisson language at the classical level into the Loop quantum gravity community. Our work presented here can be considered the quantum analog of [47, 76, 48] when $\Lambda > 0$.

However, learning about the underlying quasi-Hopf symmetry, when q is a root of unity, made it necessary to carefully adapt the work from [42, 40, 38] to this new setting and the calculations we will present in section 4.4 became significantly more complicated. We consider the generalized notion of tensor operators for quasitriangular Hopf algebras, which was introduced in [77] and used in [42, 40, 38], and adapt it to the quasi-Hopf setting for $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$[1]. We present concrete results for these quasi-Hopf tensor operators and, in particular, show that the Wigner-Eckart theorem holds for these tensor operators. This is an important result, which is used extensively in calculations to investigate the physics of spin networks. The tensor operators themselves are important, because they allow to generalize well know geometric observables from the $\Lambda = 0$ case to our q - deformed setting. The reason for the increased calculational complexity in our case, compared with [42, 40, 38] for q real, is that in the quasi-Hopf setting the quantum version of the coassociator φ, which we encountered in section 3.3, shows up in the representation of the braid group. This makes the calculations with tensor operators that act on tensor product spaces more cumbersome. Nevertheless, we successfully showed that we can correctly reproduce the correct eigenvalues for the length and angle operator in the q root of unity case. Those results prove that for q root of unity the corresponding spin network states indeed represent constantly curved spherical quantum geometry. Even though we did not yet fully reproduce the results from [42], where the authors derived the vertex amplitude of the Turaev-Viro model from a Hamiltonian constraint for q real, the fact, that we can match our calculations of geometric observables with the corresponding result of the q real case, hints that we have indeed the right formalism at hand, and that the Turaev-Viro model with q root of unity should follow as the covariant spinfoam model from the analog of the Hamiltonian constraint used in [42] at q root of unity.

The correct quantum symmetry in the Euclidean $\Lambda > 0$ case should then be given by a quasi-Hopf version of the Drinfeld double for the quasi-Hopf algebra $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{su}(2))$, [78]. This would generalize the statement, that the Drinfeld double is the symmetry of 3D quantum gravity (for the relevant Lie group, of course), both for $\Lambda = 0$ as well as $\Lambda \neq 0$, [67, 68, 79, 80], to inlcude a quasi-Drinfeld double.

4.1 The Hopf algebras $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$ for q root of unity

We will now review the necessary mathematical background necessary for the construction of the quantum theory corresponding to our classical model studied in section 3.4. The

[1] Tensor operators for quasi-Hopf algebra were considered in [50, 51] as well, but more from a quantum field theory like perspective as creation and annihilation operators.

main reference used here is [81]. For the quantum double we follow [65] and quasi-Hopf algebras are briefly discussed in [61].

The Hopf algebra $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$, which is defined for a general complex deformation parameter q, is a special case of the general Drinfeld-Jimbo Hopf algebra $\mathcal{U}_q(\mathfrak{g})$ that can be associated with a (complex) semi-simple Lie algebra $\mathfrak{g}$. As a reminder, a **bialgebra** $\mathcal{A}$ is an associative **algebra**, i.e., a vector space with a multiplication $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and a compatible **coalgebra** structure $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, called the **coproduct** together with maps $\eta : \mathbb{C} \to \mathcal{A}$, called the **unit**, and $\varepsilon : \mathcal{A} \to \mathbb{C}$, called the **counit**. The following condition of the coproduct is called coassociativity,

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \,, \tag{4.1}$$

and is satisfied for any Hopf algebra. A **Hopf algebra** $\mathcal{A}$ is a **bialgebra** with an **antipode** $S : \mathcal{A} \to \mathcal{A}$, which is a linear map such that $m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \varepsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta$. Below we will encounter so-called quasi-Hopf algebras, which violate the coassociativity condition in a certain way, which is the quantum analog of quasi-Lie bialgebras we have studied in section 3.3. If $\mathcal{A}$ is an algebra and a coalgebra then it is a bialgebra if and only if the following holds

$$\Delta(ab) = \Delta(a)\Delta(b) \qquad \varepsilon(ab) = \varepsilon(a)\varepsilon(b) \quad , \quad \Delta(1) = 1 \otimes 1 \quad , \quad \varepsilon(1) = 1 \,. \tag{4.2}$$

For the remaining compatibility conditions between multiplication, unit, comultiplication, counit and antipode for a Hopf algebra we refer to [81].

Now, the Hopf algebra $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ is a one-parameter deformation of the universal enveloping algebra of $\mathfrak{sl}(2,\mathbb{C})$ and it is defined in its Drinfeld-Jimbo form in terms of the generators E, F, G, G^{-1}, together with the unit 1, via

$$GG^{-1} = G^{-1}G = 1 \,, \quad GEG^{-1} = q^2 E \,, \quad GFG^{-1} = q^{-2} F \,, \quad [E,F] = \frac{G - G^{-1}}{q - q^{-1}} \,, \tag{4.3}$$

where $q \in \mathbb{C}_*$, such that $q^2 \neq 1$. A basis for this algebra is given by the set $\{E^a G^b F^c\}$ with $a, c \in \mathbb{N}_0$ and $b \in \mathbb{Z}$. The fact, that Eq.(4.3) corresponds to a deformation of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, given by the generators $\{E, F, H\}$, can be seen by writing q as $q = \exp(z)$, where z is a "small" (complex) number and taking $G = q^H = \exp(zH)$. Then we get from $GEG^{-1} = q^2 E$ that

$$\begin{aligned} GEG^{-1} = q^2 E \qquad &\Rightarrow \qquad (1 + zH)\, E = (1 + 2z)\, E\, (1 + zH) \;+\; \mathcal{O}(z^2) \\ &\Rightarrow \qquad z\, [H, E] = 2z\, E \;+\; \mathcal{O}(z^2) \end{aligned} \tag{4.4}$$

Hence, by dividing by z and taking the $q \to 1$, or $z \to 0$ limit, we get $[H, E] = 2E$. Similarly, one obtains $[H, F] = -2F$ from $GFG^{-1} = q^{-2}F$ and the last relation in Eq.(4.3) gives $[E, F] = H$ by noting that

$$[E, F] = \frac{G - G^{-1}}{q - q^{-1}} = \frac{e^{zH} - e^{-zH}}{e^{z} - e^{-z}} = \frac{\sinh(zH)}{\sinh(z)} = H \, + \, \mathcal{O}(z^2) \,. \tag{4.5}$$

The unique (up to Hopf algebra automorphisms) Hopf algebra structure for the algebra $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2, \mathbb{C}))$ is given by the following coproduct

$$\Delta(E) = E \otimes G + 1 \otimes E \,, \quad \Delta(F) = F \otimes 1 + G^{-1} \otimes F \,, \quad \Delta(G^{\pm 1}) = G^{\pm 1} \otimes G^{\pm 1} \,, \tag{4.6}$$

counit ε, satisfying $\varepsilon(E) = 0 = \varepsilon(F)$ and $\varepsilon(G^{\pm 1}) = 1$, and the antipode

$$S(E) = -EG^{-1} \;\; , \;\; S(F) = -GF \;\; , \;\; S(G^{\pm 1}) = G^{\mp 1} \,. \tag{4.7}$$

Let us point out that any Hopf algebra automorphism of the Hopf algebra $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2, \mathbb{C}))$ is of the form

$$\vartheta_\alpha(E) = \alpha \, E \quad , \quad \vartheta_\alpha(F) = \alpha^{-1} \, F \quad , \quad \vartheta_\alpha(G) = G \,, \tag{4.8}$$

for $\alpha \in \mathbb{C}_*$. Furthermore, for two complex numbers q_1 and q_2 the associated Hopf algebras $\mathcal{U}_{q_1}(\mathfrak{sl}(2, \mathbb{C}))$ and $\mathcal{U}_{q_2}(\mathfrak{sl}(2, \mathbb{C}))$ are isomorphic if and only if q_2 equals either $\{q_1, -q_1, q_1^{-1}, -q_1^{-1}\}$. The isomorphisms between $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$ and $\mathcal{U}_{q^{-1}}(\mathfrak{sl}(2, \mathbb{C}))$ are given via

$$\varphi_\alpha(E) = \alpha \, FG \,, \quad \varphi_\alpha(F) = -\alpha^{-1} \, q^2 \, EG^{-1} \,, \quad \varphi_\alpha(G^{\pm 1}) = G^{\pm 1} \tag{4.9}$$

and between $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$ and $\mathcal{U}_{-q}(\mathfrak{sl}(2, \mathbb{C}))$ via

$$\psi_\alpha(E) = \alpha \, E \,, \quad \psi_\alpha(F) = -\alpha^{-1} \, F \,, \quad \psi_\alpha(G^{\pm 1}) = G^{\pm 1} \tag{4.10}$$

for $\alpha \in \mathbb{C}_*$, respectively. We also point out that in the literature, for example in [82], $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2, \mathbb{C}))$ is sometimes defined by the relations, $q^4 \neq 0, 1$,

$$KK^{-1} = K^{-1}K = 1 \,, \; KEK^{-1} = qE \,, \; KFK^{-1} = q^{-1}F \,, \; [E, F] = \frac{K^2 - K^{-2}}{q - q^{-1}} \,, \tag{4.11}$$

with a more symmetric form of the coproduct, given by

$$\Delta(E) = E \otimes K + K^{-1} \otimes E \,, \quad \Delta(F) = F \otimes K + K^{-1} \otimes F \,, \quad \Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1} \tag{4.12}$$

and a different relation for the antipode. We denote this form as $\breve{\mathcal{U}}_q(\mathfrak{sl}(2,\mathbb{C}))$. Note that $\breve{\mathcal{U}}_q(\mathfrak{sl}(2,\mathbb{C}))$ is not isomorphic to our version $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$, however, $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ can be identified with a Hopf subalgebra of $\breve{\mathcal{U}}_q(\mathfrak{sl}(2,\mathbb{C}))$ via

$$E \mapsto EK \qquad , \qquad F \mapsto K^{-1}F \qquad , \qquad G \mapsto K^2 \,. \tag{4.13}$$

The version $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ with the non-symmetric coproduct of E and F, i.e., Eq.(4.6), which is the form that we will use, has the advantage that it gives rise to a factorisable quasi-triangular structure with non-degenerate quantum inverse Killing form when q is a root of unity, [65], which are properties we will explain below and which are necessary for certain applications in the quantum theory.

Now that we have seen for which values of q_1 and q_2 the Hopf algebras $\mathcal{U}_{q_1}(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathcal{U}_{q_2}(\mathfrak{sl}(2,\mathbb{C}))$ are isomorphic or automorphic, let us discuss what is special about the case when q is a root of unity, i.e., $q = \exp(2\pi i/l)$, for some $l \in \mathbb{N}_{>2}$. Thus, consider the following commutation relations

$$[E, F^m] = [m]_q \, F^{m-1} \, \frac{Gq^{1-m} - G^{-1}q^{-(1-m)}}{q - q^{-1}} \,, \tag{4.14}$$

$$[E^n, F] = [n]_q \, E^{n-1} \, \frac{Gq^{n-1} - G^{-1}q^{-(n-1)}}{q - q^{-1}} \,, \tag{4.15}$$

where we introduced the so-called q-numbers

$$[n]_q \equiv \frac{q^n - q^{-n}}{q - q^{-1}} \,. \tag{4.16}$$

When $q \in \mathbb{C}$ is not a root of unity the center of $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$, i.e., the set of all those elements in $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ that commute with all the other elements, is generated by the quantum Casimir element

$$C_q = EF + \frac{Gq^{-1} + G^{-1}q}{(q - q^{-1})^2} = FE + \frac{Gq + G^{-1}q^{-1}}{(q - q^{-1})^2} \,. \tag{4.17}$$

However, if q is given by $q = \exp(2\pi i/l)$ the center of $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ becomes much bigger, which follows from the commutation relations Eq.(4.14) and Eq.(4.15) and the fact that for q a l - th root of unity we have

$$[\,l\,]_q = \frac{q^l - q^{-l}}{q - q^{-1}} = \frac{e^{2\pi i} - e^{-2\pi i}}{e^{\frac{2\pi i}{l}} - e^{-\frac{2\pi i}{l}}} = 0 = \frac{e^{\pi i} - e^{-\pi i}}{e^{\frac{2\pi i}{l}} - e^{-\frac{2\pi i}{l}}} = \frac{q^{\frac{l}{2}} - q^{-\frac{l}{2}}}{q - q^{-1}} = [\,l/2\,]_q \,, \tag{4.18}$$

which implies $[E, F^l] = 0 = [E^l, F]$. Furthermore, we have

$$[G, E^m] = GE^m - E^mG = (q^{2m} - 1)\, E^mG \quad , \quad [G, F^m] = (q^{-2m} - 1)\, F^mG \, , \tag{4.19}$$

which implies $[G, E^l] = 0 = [G, F^l]$. Similarly,

$$[G^m, E] = G^mE - EG^m = (q^{2m} - 1)\, EG^m \quad , \quad [G^m, F] = (q^{-2m} - 1)\, FG^m \, , \tag{4.20}$$

which implies $[G^l, E] = 0 = [G^l, F]$. Note, that we can consider $E^{\frac{l}{2}}$ and $F^{\frac{l}{2}}$ only when $\frac{l}{2}$ is an integer. The previous three equations tell us that the elements $\{E^{l'}, F^{l'}, G^{\pm l'}\}$, together with the Casimir element Eq.(4.17), all belong to the center of $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ for $q = \exp(2\pi i/l)$ and where we defined l' to be equal to l, when l is odd and equal to $l/2$ when l is even. This means that $l' \in \mathbb{N}_{\geq 2}$ for $l \in \mathbb{N}_{>2}$ and we can distinguish three cases : (i) that l is odd, (ii) that l is even and l' is even and (iii) that l is even and l' is odd. This classification is relevant when studying representations at q root of unity.

After having learned that the elements $\{E^{l'}, F^{l'}, G^{\pm l'}\}$ belong to the center of $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ one can show that $\{E^{l'}, F^{l'}, G^{\pm l'} - 1\}$ generates a so-called two-sided Hopf ideal $\mathcal{J}$ and one can define the finite dimensional quotient Hopf algebra $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C})) \equiv \mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))\,/\mathcal{J}$. This means essentially that we set $E^{l'} = 0 = F^{l'}$ and $G^{\pm l'} = 1$, which implies that the basis for $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$ is given by

$$E^aG^bF^c \quad , \quad a, b, c \in \{0, 1, 2, 3, \cdots, l' - 1\} \quad , \quad \dim_{\mathbb{C}}(\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))) = (l')^3 \, . \tag{4.21}$$

Note, that now b is really non-negative, because in the quotient space we identify $G^{-n} = G^{l'-n}$ for $n < l'$. According to [65] the restriction to case (i), i.e., l must be an odd integer larger that two, has the advantage that the quasi-triangular Hopf algebra $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$ becomes factorizable, which means that the so-called quantum inverse Killing form $Q = R_{21}R$ is non-degenerate as a linear map $Q : \mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))^* \to \mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$. Hence, we restrict ourselves to this case.

The main reason for us to consider $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$ instead of $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ is the fact that for q root of unity only the former has a universal R - matrix, which is the quantum analog of the classical r - matrices we have encountered in section 3. Having such a universal R - matrix is necessary to construct a consistent model in the quantum theory. Note that in the deformed case when $q \neq 1$ the coproduct Eq.(4.6) is not cocommutative, which means that $\Delta^{cop} = \tau \circ \Delta \neq \Delta$, where τ denotes the permutation $\tau(a \otimes b) = b \otimes a$. However, there are special types of non-cocommutative Hopf algebras for which there exists a so-called

universal R **- matrix**, such that for all elements a of the Hopf algebra $\mathcal{A}$ the following holds

$$\Delta^{cop}(a) = R\Delta(a)R^{-1}\,. \tag{4.22}$$

Such a universal R - matrix is required to be an invertible element of $\mathcal{A} \otimes \mathcal{A}$ and to satisfy the **quantum Yang-Baxter equation**

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}\,. \tag{4.23}$$

We call a Hopf algebra **quasi-triangular** if its universal R - matrix satisfies

$$(\Delta \otimes \mathrm{id})R = R_{13}R_{23} \quad , \quad (\mathrm{id} \otimes \Delta)R = R_{13}R_{12}\,, \tag{4.24}$$

$$(\varepsilon \otimes \mathrm{id})R = (\mathrm{id} \otimes \varepsilon)R = 1\,, \tag{4.25}$$

$$(S \otimes \mathrm{id})R = R^{-1} \quad , \quad (\mathrm{id} \otimes S)R^{-1} = R \quad , \quad (S \otimes S)R = R\,. \tag{4.26}$$

The property of being quasi-triangular is physically motivated for example by the exchange symmetries of bosons and fermions when $q = 1$ or by the braiding properties of anyons for $q \neq 1$.

The quasi-triangular structure of the Hopf algebra $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$ at $q = \exp(2\pi i/l)$ for l odd is given by

$$R = \left(\frac{1}{l}\sum_{a,b=0}^{l-1} q^{-2ab}\, G^a \otimes G^b\right)\left(\sum_{k=0}^{l-1} \frac{(q-q^{-1})^k}{[k;q^{-2}]_q!}\, E^k \otimes F^k\right)\,, \tag{4.27}$$

with $[k;q^{-2}]_q! = [k;q^{-2}]_q[k-1;q^{-2}]_q \cdots [1;q^{-2}]_q$, $[k;q^{-2}]_q = \frac{1-q^{-2k}}{1-q^{-2}}$ and $[0;q^{-2}]_q! \equiv 1$, [65] [2].

4.1.1 Star structures and real forms

So far we have only considered deformed universal enveloping algebras $\mathcal{U}_{\mathrm{q}}(\mathfrak{g})$ for complex semi-simple Lie algebras $\mathfrak{g}$, where our main interest was on $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$. Similarly to the Lie algebra case, where one obtains real forms $\mathfrak{g}_{\mathbb{R}}$ of a complex Lie algebra from so-called

[2] Example 3.4.3, p.104/105 in [65]

star structures, we can define $*$ **- Hopf algebras**, which can then be thought of as $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$. In particular, we are interested in defining the compact real forms $\mathcal{U}_{\mathrm{q}}(\mathfrak{su}(2))$ and $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{su}(2))$ of $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$, respectively. The notion of star Hopf algebra is furthermore relevant when we want to consider their unitary representations.

A Hopf algebra $\mathcal{A}$ is called a $*$ - Hopf algebra if it is a $*$ - bialgebra. This means that there is an algebra involution $* : \mathcal{A} \to \mathcal{A}$, $*(\alpha v + \beta w) = \bar{\alpha}(*v) + \bar{\beta}(*w)$, $*(*v) = v$ for $\alpha, \beta \in \mathbb{C}$ and $v, w \in \mathcal{A}$ and $*(\overline{vw}) = (*w)(*v)$. Furthermore, we must have $\Delta(*v) = *\Delta(v) = (* \otimes *)\Delta(v)$ and $\varepsilon(*v) = \overline{\varepsilon(v)}$ in order to have a $*$ - coalgebra.

The Hopf algebra $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ has several real forms in accordance with the given definition, [81]. However, only when q is real there exists a real form that corresponds to $\mathcal{U}_{\mathrm{q}}(\mathfrak{su}(2))$. It is given by

$$* \, E = FG \quad , \quad * \, F = G^{-1}E \quad , \quad * \, G^{\pm 1} = G^{\pm 1} \, . \tag{4.28}$$

When q is a root of unity, or more generally, $|q| = 1$, then there exists a real form denoted by $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$, which is defined via

$$* \, E = -E \quad , \quad * \, F = -F \quad , \quad * \, G^{\pm 1} = G^{\pm 1} \, . \tag{4.29}$$

Those real forms strictly satisfy $\Delta(*v) = *\Delta(v)$, which guarantees that the tensor product of two $*$ - representations is again a $*$ - representation. Hence, the expression $\mathcal{U}_{\mathrm{q}}(\mathfrak{su}(2))$ technically makes only sense for a real deformation parameter. However, one can define a modified, or flipped star structure, [50, 51, 65], which allows to obtain compact real forms $\mathcal{U}_{\mathrm{q}}(\mathfrak{su}(2))$ and $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{su}(2))$ from $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$ when q is a root of unity. The difference here is that the star operator, when acting on tensor products, uses an extra permutation as follows

$$\Delta(\star v) = (\star \otimes \star)\Delta^{cop}(v) = (\star \otimes \star)\, \tau \circ \Delta(v) \neq \star\Delta(v) \, . \tag{4.30}$$

Recall that, as long as $q \neq 1$, we have $\tau \circ \Delta(v) \neq \Delta(v)$. With this extra permutation the star structure for $\mathcal{U}_{\mathrm{q}}(\mathfrak{su}(2))$ and $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{su}(2))$ at q root of unity is given by

$$\star \, E = F \quad , \quad \star \, F = E \quad , \quad \star \, G^{\pm 1} = G^{\mp 1} \, . \tag{4.31}$$

In [83] it is argued that this procedure of defining the compact real form for q root of unity is not the right way because this extra permutation does not guarantee that the tensor product of $*$ - representations is again a $*$ - representation, which they claim is better achieved by defining a different scalar product on the representation space. However, in our opinion, defining such a new scalar product, which essentially corresponds to a Euclidean scalar product in a complex vector space, is also not a natural choice and gives a first hint that the quantization of $\mathfrak{su}(2)$ for $\Lambda > 0$ and q root of unity in terms of $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ is maybe not the most rigorous thing to do.

Based on our analysis on quasi-Lie bialgebras in section 3.3 we saw that the standard quasi-Lie bialgebra for $\mathfrak{su}(2)$ is actually given by a trivial cocycle $\delta = 0$. Up to first order in $\hbar$ those cocycles are the semi-classical limit of the difference between Δ and Δ^{cop} via

$$\delta(v) = \frac{\Delta(v) - \Delta^{cop}(v)}{\hbar} \mod \hbar\,. \tag{4.32}$$

Note that the non-deformed coproduct of the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$ is given by $\tilde{\Delta}(v) = v \otimes 1 + 1 \otimes v$ and satisfies $\tilde{\Delta}^{cop}(v) = \tilde{\Delta}(v)$ and hence gives $\delta = 0$. It is possible, that the proper quantization of the quasi-Lie bialgebra $\mathfrak{su}(2)$ for $\Lambda > 0$ and q root of unity is actually such that the coproduct is not deformed. The reason why we mention this idea here, however, is that if we work with this undeformed coproduct then the standard star structure Eq.(4.31) that gives $\mathcal{U}(\mathfrak{su}(2))$ from $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$ also works for q root of unity and we do not need this odd permutation. Instead, we have properly $\tilde{\Delta}(\star v) = \star\tilde{\Delta}(v)$. That Eq.(4.31) is the right star structure that gives $\mathcal{U}(\mathfrak{su}(2))$ from $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$ at q root of unity can be seen by noting that we can write

$$E = i\,\tau_1 + \tau_2 \quad , \quad F = i\,\tau_1 - \tau_2 \quad , \quad H = -2i\,\tau_3\,, \tag{4.33}$$

which is compatible with $\star E = F$, $\star F = E$ and $\star H = H$ or $\star G^{\pm 1} = G^{\mp 1}$, when $G = q^H$ and q a phase or root of unity. Of course we have $\star\,\tau_i = -\tau_i$. We close by noting that in this possible alternative quantization the undeformed coproduct is given via

$$\tilde{\Delta}(E) = E \otimes 1 + 1 \otimes E \quad , \quad \tilde{\Delta}(F) = F \otimes 1 + 1 \otimes F \quad , \quad \tilde{\Delta}(G) = G \otimes G \tag{4.34}$$

and the universal R - matrix would be given by $R = \exp(\hbar\, t)$, where t is the Casimir element $t = \tau_a \otimes \tau^a$. In this form this coproduct is coassociative. However, below we will see that for q root of unity we will have to work with certain truncated tensor products, which can be seen as coproducts that do not satisfy $\Delta(1) = 1 \otimes 1$ but are projectors, such that $\Delta(1) = P \neq 1 \otimes 1$. Such truncated coproducts are not coassociative and hence, $\tilde{\Delta}$ from Eq.(4.34) together with a projector is possibly the right quantization of the standard quasi-Lie bialgebra of $(\mathfrak{su}(2), \delta = 0, \varphi \neq 0)$. The projector P is related to the coassociator φ and gives the truncated 6j - symbol at q root of unity.

4.1.2 Representations of $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathcal{U}_{\mathrm{q}}^{\mathbf{res}}(\mathfrak{sl}(2,\mathbb{C}))$

In this section we want to collect some facts about the representation theory of $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$ at q root of unity. When q is not a root of unity the representation theory of $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$, or $\mathcal{U}_{\mathrm{q}}(\mathfrak{su}(2))$ for q real, is very similar to the case of $\mathfrak{sl}(2,\mathbb{C})$, or $\mathfrak{su}(2)$, respectively. These irreducible representations are labeled by a discrete number $\omega \in \{-1, 1\}$ and a non-negative half integer j, which relates to the q - deformed Casimir Eq.(4.17). They act in a complex vector space V^j of dimension $(2j+1)$ with basis vectors $|j,m\rangle$, $m \in \{-j, -j+1, \cdots, j\}$, that diagonalize C_q and G and the tensor product decomposition works exactly as in the $\mathfrak{sl}(2,\mathbb{C})$ or $\mathfrak{su}(2)$ case. We can write

$$\pi_\omega^{(j)}(E)|j,m\rangle = \mathcal{C}(j,m)\,|j,m+1\rangle \quad , \quad \pi_\omega^{(j)}(F)|j,m\rangle = \omega\,\mathcal{C}(j,-m)\,|j,m-1\rangle\,, \tag{4.35}$$

$$\pi_\omega^{(j)}(G)|j,m\rangle = \omega\, q^{2m}\,|j,m\rangle\,, \tag{4.36}$$

where $\mathcal{C}(j,m) = \sqrt{[j-m]_q[j+m+1]_q}$. Note, that it is not a typo that there is no ω - prefactor for the E - action, as can be seen from looking at $[E,F]$. We also briefly mention that the irreducible representations of $\breve{\mathcal{U}}_q(\mathfrak{sl}(2,\mathbb{C}))$ are also labeled by the spin j, but they have instead $\omega \in \{\pm 1, \pm i\}$ and

$$\pi_\omega^{(j)}(E)|j,m\rangle = \omega\,\mathcal{C}(j,m)\,|j,m+1\rangle \quad , \quad \pi_\omega^{(j)}(F)|j,m\rangle = \omega\,\mathcal{C}(j,-m)\,|j,m-1\rangle\,, \tag{4.37}$$

$$\pi_\omega^{(j)}(K)|j,m\rangle = \omega\, q^{m}\,|j,m\rangle\,, \tag{4.38}$$

where one should note that here the E - action has the ω - prefactor. The fact that $\breve{\mathcal{U}}_q(\mathfrak{sl}(2,\mathbb{C}))$ has four 1-dimensional irreducible representations, whereas $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ has just two, shows again that those two Hopf algebras are indeed not isomorphic. Now, the reason why we mention the representations of both $\breve{\mathcal{U}}_q(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ is that in [81] the Clebsch-Gordan coefficients, which depend on the coproduct, are determined for $\breve{\mathcal{U}}_q(\mathfrak{sl}(2,\mathbb{C}))$. However, we prefer the version $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ and this raised the question how, if at all, the Clebsch-Gordan coefficients for those two versions relate to each other and furthermore, which are the right Clebsch-Gordan coefficients for the Turaev-Viro model? The short answer is that one can use the $\breve{\mathcal{U}}_q(\mathfrak{sl}(2,\mathbb{C}))$ Clebsch-Gordan coefficients as the $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ Clebsch-Gordan coefficients by (slightly) modifying the representation Eq.(4.35). This modification was already used in [20], however, without giving any explanation why. The reason for this modification is that it gives the same recursion relations for the Clebsch-Gordan coefficients, for both the coproduct of $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ with the modified representation, cf. Eq.(4.43) and Eq.(4.44), and the coproduct of $\breve{\mathcal{U}}_q(\mathfrak{sl}(2,\mathbb{C}))$ with the representation Eq.(4.37) and Eq.(4.38).

Now, when q is a root of unity the representation theory becomes somewhat more complicated, because there occur so-called indecomposable irreducible representations, which also make the tensor product decomposition more complicated. Furthermore, as discussed above already, the issue of the choice of correct star structure and scalar product for V^j makes the notion of unitarity of representations more complicated as well. These things are more or less related to the fact that for q root of unity the center of $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ becomes larger. The representations of $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ at root of unity are thus distinguished by the action of the central elements $E^{l'}$ and $F^{l'}$, where, as a reminder, we have $q = \exp(2\pi i/l)$ and $l' = l$ for l odd and $l' = l/2$ for l even. If $E^{l'}$ and $F^{l'}$ act as zero, i.e., $E^{l'} = 0 = F^{l'}$, then we call the representation **nilpotent**. If $E^{l'} \neq 0 \neq F^{l'}$ the representation is called **cyclic** and in the remaining cases **semi-cyclic**. For the first case one can show that the representation is irreducible if and only if $2j < l'$, i.e., if $\dim V^j \leq l'$. For more details in the general representation theory for $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ at q root of unity we refer to [81]. The main interest for us is $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$. One finds that the irreducible representations for $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$ can only be of the nilpotent type and are given by Eq.(4.35) and Eq.(4.36) as follows

$$(i) \qquad \pi_1^{(j)} \quad , \quad 0 \leq 2j+1 \leq l \quad , \quad l \text{ odd}\,, \tag{4.39}$$

$$(ii) \qquad \pi_{\pm 1}^{(j)} \quad , \quad 0 \leq 2j+1 < l' \quad , \quad 2j \quad \text{even, if} \quad l, l' \text{ even}\,, \tag{4.40}$$

$$(iii) \qquad \pi_1^{(j)} \quad , \quad 0 \leq 2j+1 < l' \quad , \quad 2j \quad \text{even, if} \quad l \text{ even} \ , \ l' \text{ odd}\,, \tag{4.41}$$

$$\pi_{-1}^{(j)} \quad , \quad 0 \leq 2j+1 \leq l' \quad , \quad 2j \quad \text{odd, if} \quad l \text{ even} \ , \ l' \text{ odd}\,. \tag{4.42}$$

Thus, we see that for l odd there is, up to equivalence, only one type of irreducible nilpotent representation $\pi_1^{(j)}$, which have to satisfy $0 \leq j \leq \frac{l-1}{2}$. They are given by Eq.(4.35) and Eq.(4.36) with $\omega = 1$ via

$$\pi_1^{(j)}(E)|j,m\rangle = q^{m+\frac{1}{2}}\,\mathcal{C}(j,m)\,|j,m+1\rangle \quad , \quad \pi_1^{(j)}(G^{\pm 1})|j,m\rangle = q^{\pm 2m}\,|j,m\rangle\,, \tag{4.43}$$

$$\pi_1^{(j)}(F)|j,m\rangle = q^{-m+\frac{1}{2}}\,\mathcal{C}(j,-m)\,|j,m-1\rangle\,, \tag{4.44}$$

where again $\mathcal{C}(j,m) = \sqrt{[j-m]_q[j+m+1]_q}$. Furthermore, we already introduced the modification mentioned before, namely, the q - prefactors for the action of E and F. These prefactors are a special choice of the more general modification that allows to define with $a, b \in \mathbb{N}_0$

$$\tilde{\pi}_1^{(j)}(E)|j,m\rangle = q^{am+b}\,\mathcal{C}(j,m)\,|j,m+1\rangle\,, \tag{4.45}$$

$$\tilde{\pi}_1^{(j)}(F)|j,m\rangle = q^{-am-a-b}\,\mathcal{C}(j,-m)\,|j,m-1\rangle\,, \tag{4.46}$$

and still have a representation for $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$. Those prefactors guarantee that one obtains the same recursion relations for the Clebsch-Gordan coefficients as in the case of the symmetric coproduct of $\breve{\mathcal{U}}_q(\mathfrak{sl}(2,\mathbb{C}))$. As an example, we can consider the $j = 1/2$ representation of Eq.(4.43) and Eq.(4.44), which gives

$$\pi^{(\frac{1}{2})}(G) = \begin{pmatrix} \frac{1}{q} & 0 \\ 0 & q \end{pmatrix} \quad , \quad \pi^{(\frac{1}{2})}(E) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \quad \pi^{(\frac{1}{2})}(F) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{4.47}$$

Using this we can give an explicit expression of the universal R-matrix Eq.(4.27) in the $(\frac{1}{2}, \frac{1}{2})$ - tensor representation. One finds

$$(R)^{(\frac{1}{2},\frac{1}{2})} = q^{-\frac{1}{2}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & (q-q^{-1}) & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \tag{4.48}$$

If we use the approximation $q^{\frac{1}{2}} = e^{\frac{\pi i}{l}} \approx 1 + \frac{\pi i}{l}$ we find

$$(R)^{(\frac{1}{2},\frac{1}{2})} \approx \mathbb{I}_4 + \frac{\pi i}{l} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbb{I}_4 - \frac{4\pi}{l}\, r\,, \tag{4.49}$$

where we recognize as a first order approximation the classical r - matrix of the Lie bialgebra of $\mathfrak{su}(2)$ corresponding to the double $\mathfrak{sl}(2,\mathbb{C})$, i.e., Eq.(3.64).

Before we consider the Clebsch-Gordan problem in the next section we recall the notion of a **quantum trace**, which is defined for any irreducible representation of $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$ and for any linear map $f : V^j \to V^j$ via

$$_q\mathrm{Tr}(f) \equiv \mathrm{Tr}(G \circ f) = \sum_{m=-j}^{j} \langle j,m|\,\pi^{(j)}(G \circ f)\,|j,m\rangle\,. \tag{4.50}$$

Using this trace we can define the quantum dimension of the j - representation using $f = \mathrm{id}_{V^j}$ and obtain

$$ {}_q\mathrm{dim}(V^j) \equiv {}_q\mathrm{Tr}(\mathrm{id}_{V^j}) = \sum_{m=-j}^{j} q^{2m} = [2j+1]_q \, . \tag{4.51} $$

One observes that for q root of unity, with l odd, all $0 \leq j < \frac{l-1}{2}$ have non-zero quantum dimension, $[2j+1]_q \neq 0$, whereas for $j = \frac{l-1}{2}$, which is the largest allowed irreducible representation for $q = \exp(2\pi i/l)$, one has ${}_q\mathrm{dim}(V^j) = [\,l\,]_q = 0$. Note, that for l odd we can not have $2j+1 = \frac{l}{2}$. Irreducible representations with ${}_q\mathrm{dim}(V^j) = 0$ are called **indecomposable**, which means that they have invariant sub spaces, yet, they can not be decomposed in terms of irreducibles. Such representations are also not unitarizable and lead to problems in the decomposition of tensor products of irreducible representations, as we will see in the next section. The main issue with those indecomposable representations, however, is that the category of irreducible representations of $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$ does not give a monoidal category, which is needed for applications to physics. We will review in the next section how these problems are dealt with by a certain re-definition of the tensor product of representations and a restriction to so-called **admissible representations** $0 \leq j \leq \frac{l-2}{2}$. The final object will then be a rigid, semisimple, monoidal or fusion category [61], which is the object used by physicists in conformal field theory or condensed matter systems.

If we consider again the representation Eq.(4.43) and Eq.(4.44) in a vector spaces V^j, one can ask whether there is a scalar product that respects our star structure Eq.(4.31) and thus would be a $\star$ - representation. If we consider V^j to be a complex vector space with its standard inner product $\langle x|y\rangle_{\mathbb{C}} = \bar{x}_i y^i$ then the adjoint operator $\pi^{(j)}(\xi)^\dagger$ is given by the conjugate transpose. One finds that we have for all $\xi \in \mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$

$$ \pi^{(j)}(\xi)^\dagger = \bar{\pi}^{(j)}(\xi)^t = \pi^{(j)}(\star\xi) \tag{4.52} $$

only when $0 \leq j \leq \frac{l-1}{4}$. This results from the fact that with this restriction there can be no negative numbers under the square roots in Eq.(4.43) and Eq.(4.44). On the other hand, if $\frac{l}{4} < j \leq \frac{l-2}{2}$ we have $\pi^{(j)}(\xi)^\dagger = \bar{\pi}^{(j)}(\xi)^t = -\pi^{(j)}(\star\xi)$ for all $\xi \in \mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$. If we consider V^j instead to be a real vector space with inner product $\langle x|y\rangle_{\mathbb{R}} = x_i y^i$, as advocated in [83], then the adjoint operator $\pi^{(j)}(\xi)^\dagger$ is given by the transpose. In this case we have for example

$$ (\pi^{(j)}(F)^\dagger)_{mn} = \pi^{(j)}(F)_{nm} = q^{-(2n+1)}\,\pi^{(j)}(\star F)_{mn} = q^{-(2n+1)}\,\pi^{(j)}(E)_{mn} \tag{4.53} $$

and see that due to our special q-prefactors in Eq.(4.43) and Eq.(4.44) we do not have a $\star$ - representation with the real scalar product. If we cancel those prefactors then the real scalar product would give a $\star$ - representation for all admissible $j \in \{0, \frac{1}{2}, \cdots, \frac{l-2}{2}\}$, but we would have to alter the Clebsch-Gordan coefficients. Both options are satisfactory and allow us to consider the representation Eq.(4.43) and Eq.(4.44) as a $\star$ - representation of the algebra $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{su}(2))$. In [83] the authors chose to work with the real scalar product but they where also working with the symmetric version of the Hopf algebra, i.e., $\breve{\mathcal{U}}_q(\mathfrak{sl}(2,\mathbb{C}))$, and different conventions for q - numbers. We see that these details do have a significant influence. One of their arguments for the real scalar product was that it guarantees that the decoupled states $|J, M\rangle$ in $V^{j_1}\bar{\otimes}V^{j_2}$ are orthonormal only with the real scalar product. However, we point out that with the correct definition of hermitian adjoint those decoupled states $|J, M\rangle$ are orthonormal for the hermitian scalar product as well. Thus, we see that we can either consider V^j with complex scalar product and the restriction $0 \leq j \leq \frac{l-1}{4}$, which also should be implemented in the truncation of the tensor product via

$$V^{j_1}\underline{\otimes}V^{j_2} \cong \bigoplus_{J=|j_1-j_2|}^{\min(j_1+j_2,\frac{l-1}{4})} V^J\,, \tag{4.54}$$

or consider V^j with real scalar product and redefinition of Clebsch-Gordan coefficients and cancellation of the q-prefactors in Eq.(4.43) and Eq.(4.44). However, note that this is not the truncation used in the Turaev-Viro model and hence, corresponds to $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$ instead of $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{su}(2))$.

4.1.3 Clebsch-Gordan decomposition, truncated tensor products and 6j-symbols

As mentioned before, at q root of unity the decomposition of tensor product representations of irreducible factors is not always decomposable into irreducible ones. The occurrence of indecomposable representations, characterized by their quantum dimension being zero, makes it necessary to modify the tensor product of representations if one wants to obtain a modular or fusion category. In the undeformed case for SU(2), as well as in the deformed case when q is real, we have

$$V^{j_1} \otimes V^{j_2} \cong \bigoplus_{J=|j_1-j_2|}^{j_1+j_2} V^J\,, \tag{4.55}$$

where all the V^J on the right-hand side are irreducible. We say that the decomposition is semi-simple. When q is a root of unity, however, there appear V^J with quantum dimension zero in the tensor product decomposition even when j_1 and j_2 are physical. As stated before, V^j has quantum trace zero for $j = \frac{l-1}{2}$. Hence, the **admissible representations** of [29], or so-called **color representations** of [84], are given by $j_c \in \{0, \frac{1}{2}, 1, \cdots, \frac{l-2}{2}\}$. Thus, if $0 \leq j_1 + j_2 \leq \frac{l-2}{2}$ the formula Eq.(4.55) can be used unaltered for $j_1, j_2 \leq \frac{l-2}{2}$. However, when $j_1 + j_2$ becomes larger than $\frac{l-2}{2}$ the tensor product has to be redefined such that no quantum trace zero representations occur. One defines in this case [20, 61]

$$V^{j_1} \bar{\otimes} V^{j_2} \cong \bigoplus_{J=|j_1-j_2|}^{l-2-j_1-j_2} V^J \,. \tag{4.56}$$

We can combine Eq.(4.55) and Eq.(4.56) by writing[3]

$$V^{j_1} \bar{\otimes} V^{j_2} \cong \bigoplus_{J=|j_1-j_2|}^{\min(j_1+j_2, l-2-j_1-j_2)} V^J \,. \tag{4.57}$$

An equivalent way of avoiding the issue of indecomposable representations, is to consider a priori only triples of so-called admissible representations. In the Turaev-Viro model [29] one calls a set of three **color representations** (j_1, j_2, j_3), associated with the three edges of triangles of the triangulation, **admissible** when the following conditions are satisfied

1. $j_1 + j_2 + j_3 \in \mathbb{N}_0$.
2. $j_1 + j_2 + j_3 \leq l - 2$.
3. $j_1 \leq j_2 + j_3$ and $j_3 \leq j_1 + j_2$ and $j_2 \leq j_1 + j_3$.

This inherently discards the problems associated with the tensor product decomposition by saying that we only label the triangulation of the 3-manifold M such that there are no indecomposable contributions. Either way, what one ends up with is the physical fusion

[3] This tensor product Eq.(4.57) is associative with a non-trivial associativity isomorphism. It is easy to show that the restriction to color representations $j \leq \frac{l-2}{2}$ is necessary, because otherwise such a truncated tensor product $\tilde{\otimes}$, with irreducible states $j \leq \frac{l-1}{2}$, would not be associative. For example, for $l = 3$ one finds that

$$\left(\frac{1}{2} \tilde{\otimes} \frac{1}{2}\right) \tilde{\otimes} 1 = 0 \oplus 2 \cdot 1 \neq 0 \oplus 1 = \frac{1}{2} \tilde{\otimes} \left(\frac{1}{2} \tilde{\otimes} 1\right) .$$

category of representations at q root of unity, which means that one can essentially calculate as one knows from the $\mathfrak{su}(2)$ or q real case, as long as one keeps the truncation of the tensor product in mind. However, one thing that is somewhat hidden from this procedure, as was first pointed out in [50, 51], is the fact that these truncated tensor products should be interpreted in the framework of so-called weak quasi-Hopf algebras. We will discuss this issue in more detail in section 4.2. Now, in this setting of truncated tensor product we can define the q - deformed Clebsch-Gordan coefficients of the decomposition of $V^{j_1}\bar{\otimes}V^{j_2}$ via

$$|J,M\rangle = \sum_{m_1,m_2} {}_qC\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M\end{pmatrix} |j_1,m_1\rangle\bar{\otimes}|j_2,m_2\rangle \,. \tag{4.58}$$

The coefficients of the inverse direction are given by the same coefficients and we have

$$\sum_{m_1,m_2} {}_qC\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M\end{pmatrix} {}_qC\begin{pmatrix} j_1 & j_2 & K \\ m_1 & m_2 & N\end{pmatrix} = \delta_{JK}\delta_{MN} \,, \tag{4.59}$$

$$\sum_{\substack{J,M \\ \text{adm.}}} {}_qC\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M\end{pmatrix} {}_qC\begin{pmatrix} j_1 & j_2 & J \\ n_1 & n_2 & M\end{pmatrix} = \Pi^{(j_1,j_2)}_{(m_1,m_2,n_1,n_2)} \overset{*}{\neq} \delta_{m_1n_1}\delta_{m_2n_2} \,, \tag{4.60}$$

where $\Pi^{(j_1,j_2)}_{(m_1,m_2,n_1,n_2)}$ denotes a non-trivial projector when $j_1 + j_2 \geq \frac{l-1}{2}$, otherwise we get the identity. Note that for q root of unity the Clebsch-Gordan coefficients can not be chosen to be real, as is possible in the $\mathfrak{su}(2)$ or q real case. Furthermore, note in Eq.(4.59) there is no complex conjugation of the coefficients. The Clebsch-Gordan coefficients can be explicitly calculated giving [85, 81]

$$\begin{aligned} {}_qC\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M\end{pmatrix} &= \delta_{m_1+m_2,M} \frac{(-1)^{j_1-m_1}\, q^{\beta_1} \sqrt{[2J+1]_q\,\beta_2}\,[j_1+j_2-J]_q!}{\beta_3\,[j_1+j_2+J+1]_q!\,[j_1+m_1]_q!\,[j_2+m_2]_q!} \\ &\quad \times \sum_s \frac{(-1)^s\, q^{s(J+M+1)}\,[j_1+m_1+s]_q!\,[j_2+J-m_1-s]_q!}{[s]_q!\,[J-M-s]_q!\,[j_1-m_1-s]_q!\,[j_2-J+m_1+s]_q!}\,, \end{aligned} \tag{4.61}$$

with

$$\beta_1 = \frac{1}{2}\left(j_2(j_2+1) - j_1(j_1+1) - J(J+1) + 2m_1(M+1)\right)\,, \tag{4.62}$$

$$\beta_2 = [j_1+m_1]_q!\,[j_1-m_1]_q!\,[j_2+m_2]_q!\,[j_2-m_2]_q!\,[J+M]_q!\,[J-M]_q!\,, \tag{4.63}$$

$$\beta_3 = \left(\frac{[j_1+j_2-J]_q!\,[j_1-j_2+J]_q!\,[j_2-j_1+J]_q!}{[j_1+j_2+J+1]_q!}\right)^{\frac{1}{2}}\,. \tag{4.64}$$

The non-vanishing coefficients up to spin-1 (for $r > 3$) are given by

$$ {}_qC\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix} = 1 = {}_qC\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ -\frac{1}{2} & -\frac{1}{2} & -1 \end{pmatrix}, \tag{4.65} $$

$$ {}_qC\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = \frac{q^{\frac{1}{2}}}{\sqrt{[2]_q}} = {}_qC\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}, \tag{4.66} $$

$$ {}_qC\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} = \frac{q^{-\frac{1}{2}}}{\sqrt{[2]_q}} = -{}_qC\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}. \tag{4.67} $$

We can furthermore use the Clebsch-Gordan coefficients to give an expression for the R - matrix Eq.(4.27) via

$$ (R^{(j_1,j_2)})_{(m_1,n_1),(m_2,n_2)} \equiv \langle j_1, m_1| \bar{\otimes} \langle j_2, m_2|\; R\; |j_1, n_1\rangle \bar{\otimes} |j_2, n_2\rangle $$

$$ = \sum_{\substack{J,M \\ \text{adm.}}} (-1)^{j_1+j_2-J} q^{J(J+1)-j_1(j_1+1)-j_2(j_2+1)}\, {}_qC\begin{pmatrix} j_2 & j_1 & J \\ m_2 & m_1 & M \end{pmatrix} {}_qC\begin{pmatrix} j_1 & j_2 & J \\ n_1 & n_2 & M \end{pmatrix} \tag{4.68} $$

$$ = \sum_{\substack{J,M \\ \text{adm.}}} q^{J(J+1)-j_1(j_1+1)-j_2(j_2+1)}\, {}_q\overline{C}\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M \end{pmatrix} {}_qC\begin{pmatrix} j_1 & j_2 & J \\ n_1 & n_2 & M \end{pmatrix} \tag{4.69} $$

and similarly we can write for $R_{21} \equiv \tau \circ R_{12} = \tau \circ R$

$$ (R_{21}^{(j_1,j_2)})_{(m_1,n_1),(m_2,n_2)} = (R_{12}^{(j_2,j_1)})_{(m_2,n_2),(m_1,n_1)} $$

$$ = \sum_{\substack{J,M \\ \text{adm.}}} (-1)^{j_1+j_2-J} q^{J(J+1)-j_1(j_1+1)-j_2(j_2+1)}\, {}_qC\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M \end{pmatrix} {}_qC\begin{pmatrix} j_2 & j_1 & J \\ n_2 & n_1 & M \end{pmatrix} \tag{4.70} $$

$$ = \sum_{\substack{J,M \\ \text{adm.}}} q^{J(J+1)-j_1(j_1+1)-j_2(j_2+1)}\, {}_q\overline{C}\begin{pmatrix} j_2 & j_1 & J \\ m_2 & m_1 & M \end{pmatrix} {}_qC\begin{pmatrix} j_2 & j_1 & J \\ n_2 & n_1 & M \end{pmatrix}. \tag{4.71} $$

We can also consider the inverse R - matrix via

$$ ((R^{-1})^{(j_1,j_2)})_{(m_1,n_1),(m_2,n_2)} \equiv \langle j_1, m_1| \bar{\otimes} \langle j_2, m_2|\; (R^{-1})\; |j_1, n_1\rangle \bar{\otimes} |j_2, n_2\rangle $$

$$ = \sum_{\substack{J,M \\ \text{adm.}}} (-1)^{J-j_1-j_2} q^{-J(J+1)+j_1(j_1+1)+j_2(j_2+1)}\, {}_qC\begin{pmatrix} j_2 & j_1 & J \\ n_2 & n_1 & M \end{pmatrix} {}_qC\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M \end{pmatrix}. \tag{4.72} $$

Now, we want to consider the quantum Racah coefficients and the quantum 6j-symbol, which provide the isomorphism between $(V^{j_1}\bar{\otimes}V^{j_2})\bar{\otimes}V^{j_3}$ and $V^{j_1}\bar{\otimes}(V^{j_2}\bar{\otimes}V^{j_3})$. Using the Clebsch-Gordan maps one decomposes the tensor product spaces as indicated by the brackets and defines the Racah coefficients as the unitary matrix between the representations of the vectors

$$|(j_1,j_2);(j_{12},j_3);J,M\rangle \equiv \tag{4.73}$$

$$\sum_{m_1,m_2}\sum_{m_{12},m_3} {}_qC\begin{pmatrix} j_1 & j_2 & j_{12}\\ m_1 & m_2 & m_{12}\end{pmatrix} {}_qC\begin{pmatrix} j_{12} & j_3 & J\\ m_{12} & m_3 & M\end{pmatrix} \left(|j_1,m_1\rangle\bar{\otimes}|j_2,m_2\rangle\right)\bar{\otimes}|j_3,m_3\rangle\,,$$

$$|(j_2,j_3);(j_1,j_{23});J,M\rangle \equiv \tag{4.74}$$

$$\sum_{m_2,m_3}\sum_{m_1,m_{23}} {}_qC\begin{pmatrix} j_1 & j_{23} & J\\ m_1 & m_{23} & M\end{pmatrix} {}_qC\begin{pmatrix} j_2 & j_3 & j_{23}\\ m_2 & m_3 & m_{23}\end{pmatrix} |j_1,m_1\rangle\bar{\otimes}\left(|j_2,m_2\rangle\bar{\otimes}|j_3,m_3\rangle\right)\,,$$

via

$$|(j_1,j_2);(j_{12},j_3);J,M\rangle = \sum_{j_{23}} {}_qR(j_1,j_2,j_3;j_{12},j_{23};J)\;|(j_2,j_3);(j_1,j_{23});J,M\rangle\,. \tag{4.75}$$

where all sums only add admissible contributions. The Racah coefficients are given explicitly by

$${}_qR(j_1,j_2,j_3;j_{12},j_{23};J) = \tag{4.76}$$

$$\sum_{m_1,m_2,m_3} {}_qC\begin{pmatrix} j_1 & j_2 & j_{12}\\ m_1 & m_2 & m_{12}\end{pmatrix} {}_qC\begin{pmatrix} j_{12} & j_3 & J\\ m_{12} & m_3 & M\end{pmatrix} {}_qC\begin{pmatrix} j_1 & j_{23} & J\\ m_1 & m_{23} & M\end{pmatrix} {}_qC\begin{pmatrix} j_2 & j_3 & j_{23}\\ m_2 & m_3 & m_{23}\end{pmatrix}\,.$$

We can define the q-6j-symbols in terms of the Racah coefficients as

$$\begin{Bmatrix} j_1 & j_2 & j_{12}\\ j_3 & J & j_{23}\end{Bmatrix}_q = \frac{(-1)^{j_1+j_2+j_3+J}}{\sqrt{[2j_{12}+1]_q[2j_{23}+1]_q}}\;{}_qR(j_1,j_2,j_3;j_{12},j_{23};J)\,. \tag{4.77}$$

The Biedenharn-Elliott or pentagon identity for the Racah coefficients is given by

$$\sum_{j_{23}} {}_qR(j_1,j_2,j_3;j_{12},j_{23};j_{123})\,{}_qR(j_1,j_{23},j_4;j_{123},j_{234};J)\,{}_qR(j_2,j_3,j_4;j_{23},j_{34};j_{234}) \tag{4.78}$$

$$= {}_qR(j_{12},j_3,j_4;j_{123},j_{34};J)\,{}_qR(j_1,j_2,j_{34};j_{12},j_{234};J)$$

and holds in a similar form for the 6j-symbol [81].

4.2 $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$ as quasi-Hopf algebras

We have introduced $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$ above as Hopf algebras, which implies in particular that their coproduct Δ is coassociative. This means that we have for all Hopf algebra elements ξ

$$(\Delta \otimes \mathrm{id})\Delta(\xi) = (\mathrm{id} \otimes \Delta)\Delta(\xi)\,. \tag{4.79}$$

It was first noted in [50, 51] that for q root of unity this relation is actually violated, though indirectly, as we will see. Hence, the notion of Hopf algebras is not sufficient to properly describe symmetries of this type. The authors of [50, 51] realized that the truncation of the tensor product of the physical representations should be reflected in a modification of the underlying coproduct, which led them to advocate that the true symmetry for q root of unity is actually a quasi-Hopf algebra. In hindsight, after having studied the classical version in more detail, i.e., $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and its quasi-Poisson geometry, this comes as no surprise. However, since we started our investigations in the quantum theory this realization was not trivial and required several adjustments in our calculations, as we will see below.

We saw in section 4.1.3 that in order to obtain a physically sensible tensor product of representations, we had to use a truncation of the irreducible representations to the physical states with $0 \leq j \leq \frac{l-2}{2}$ as well as a truncation of the representations appearing in the decomposition of the usual tensor product when $j_1 + j_2 > \frac{l-2}{2}$, to get rid of unwanted contributions with vanishing quantum dimension. Now, note that the canonical way of defining the tensor product of representation of a Hopf-algebra, however, is given by the formula

$$(\pi^{(j_1)} \otimes \pi^{(j_2)})(\xi) \equiv (\pi^{(j_1)} \otimes \pi^{(j_2)})(\Delta(\xi)) = \sum_p \pi^{(j_1)}(\xi_1^p) \otimes \pi^{(j_2)}(\xi_2^p)\,, \tag{4.80}$$

where $\Delta(\xi) = \sum_p \xi_1^p \otimes \xi_2^p$, and is thus connected to the coproduct of the Hopf algebra. The choice of admissible labellings in the Turaev-Viro model or in rational conformal field theories can be seen as an ad hoc modification of the tensor product of representations that is put in by hand to obtain the desired fusion category. However, as argued in [50, 51] it is preferable and possible, to explain this modified tensor product of physical representations by working with certain (weak) quasi-Hopf algebras and a modification of the coproduct, where indeed the tensor product of representations corresponds to the underlying coproduct as in Eq.(4.80). However, in this framework one finds that the modified coproduct becomes non-coassociative but they also show that it is only quasi-coassociative, which means that there exists a so-called coassociator or Drinfeld associator $\Phi \in \mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))^{\otimes 3}$ such that

$$(\Delta \otimes \mathrm{id})\Delta(\xi) = \Phi\,(\mathrm{id} \otimes \Delta)\Delta(\xi)\,\Phi^{-1}\,. \tag{4.81}$$

Thus, instead of working with the Hopf algebra $(\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C})),\Delta)$ and modifying the tensor product by hand, to give a nice fusion category, they propose that the correct underlying symmetry of those fusion categories arising from quantum groups at q root of unity is given by the (weak) quasi-Hopf algebra $(\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C})),\Delta_{new})$. It is important to note that, per construction, the final fusion category of physical representations is not changed, only the underlying symmetry is corrected. In fact, the components of the non-trivial coassociator φ are exactly given by the restricted 6j-symbol (or Racah coefficients, depending on conventions) at q root of unity as we know them from the Turaev-Viro model. Note furthermore that, even though we have a mild violation of coassociativity for quasi-Hopf algebras, the resulting categories of their representations still form monoidal or tensor categories where $(V^{j_1} \otimes V^{j_2}) \otimes V^{j_3} \cong V^{j_1} \otimes (V^{j_2} \otimes V^{j_3})$ holds. Before we continue, let us give the following definitions.

Definition (quasi-Hopf algebra) : A quasi-Hopf algebra $(\mathcal{A},\Delta,\Phi,\varepsilon,S)$ is a quasi-bialgebra $\mathcal{A}$ with an antipode S, which means, there exists a **coassociator** $\Phi = \Phi_{123} \in \mathcal{A}^{\otimes 3}$ such that, for all $\xi \in \mathcal{A}$,

$$(\Delta \otimes \mathrm{id})\Delta(\xi) = \Phi.(\mathrm{id} \otimes \Delta)\Delta(\xi).\Phi^{-1}\,. \tag{4.82}$$

Furthermore, we must have

$$(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi).(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi) = (\Phi \otimes \mathrm{id}).(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi).(1 \otimes \Phi)\,, \tag{4.83}$$

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta \qquad , \qquad (\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(\Phi) = 1\,. \tag{4.84}$$

For the compatibility relations with the antipode we refer to [61]. Quasi-Hopf algebras are interesting objects, because they are in some sense the most general algebras such that their category of representations form a monoidal- or tensor- category, which means that there is a canonical isomorphism between the tensor product spaces $(j_1 \otimes j_2) \otimes j_3$ and $j_1 \otimes (j_2 \otimes j_3)$ given by the action of the coassociator Φ. Furthermore, recall the notion of twisting we mentioned in section 3. This notion has its analog in the quasi-Hopf algebra setting and quasi-Hopf algebras are interesting because they are still quasi-Hopf algebras after twisting, whereas standard Hopf algebras are not “invariant” under twists in general.

Now, similarly to the Hopf algebra case we define

Definition (quasitriangular quasi-Hopf algebra) : A quasitriangular quasi-Hopf algebra $\mathcal{A}$ is a quasi-Hopf algebra with a universal R - matrix, $R \in \mathcal{A}^{\otimes 2}$, such that, for all $\xi \in \mathcal{A}$,

$$\Delta^{cop}(\xi) = R.\Delta(\xi).R^{-1}\,, \tag{4.85}$$

$$(\Delta \otimes \mathrm{id})(R) = \Phi_{231}^{-1}\, R_{13}\, \Phi_{132}\, R_{23}\, \Phi_{123}^{-1}\,, \tag{4.86}$$

$$(\mathrm{id} \otimes \Delta)(R) = \Phi_{312}\, R_{13}\, \Phi_{213}^{-1}\, R_{12}\, \Phi_{123} \tag{4.87}$$

holds.

Quasitriangular quasi-Hopf algebras are defined such that the **quasi-quantum Yang-Baxter equation**

$$R_{12}\, \Phi_{231}^{-1}\, R_{13}\, \Phi_{132}\, R_{23}\, \Phi_{123}^{-1} = \Phi_{132}^{-1}\, R_{23}\, \Phi_{312}\, R_{13}\, \Phi_{213}^{-1}\, R_{12} \tag{4.88}$$

is satisfied. This relation, as in the Hopf algebra case, guarantees that the representation category of the algebra carries a braid group representation. Now, at the quasi-Hopf algebra level we can **twist** the coproduct, coassociator and R - matrix by an (invertible) element $\mathcal{F} \in \mathcal{A}^{\otimes 2}$ as follows

$$\Delta^{\mathcal{F}}(\xi) = \mathcal{F}.\Delta(\xi).\mathcal{F}^{-1}\,, \tag{4.89}$$

$$\Phi^{\mathcal{F}} = \mathcal{F}.(\Delta \otimes \mathrm{id})(\mathcal{F}).\Phi.(\mathrm{id} \otimes \Delta)(\mathcal{F})^{-1}.\mathcal{F}_{23}^{-1}\,, \tag{4.90}$$

$$R^{\mathcal{F}} = \mathcal{F}_{21}.R.\mathcal{F}^{-1} \tag{4.91}$$

and the resulting algebra $(\mathcal{A}, \varepsilon, \Delta^{\mathcal{F}}, \Phi^{\mathcal{F}}, R^{\mathcal{F}})$ is again a quasi-Hopf algebra. The notion of **weak quasi-Hopf algebras**, as introduced in [50, 51], weakens the invertibility requirement for R and Φ and is necessary to deal with coproducts that are obtained via truncated tensor product. Also, it should be clear that the definition of a (quasitriangular) quasi-Hopf algebra encompasses the notion of a (quasitriangular) Hopf algebra, which is obtained for trivial coassociator $\Phi = 1 \otimes 1 \otimes 1$. In this case the quasi-quantum Yang-Baxter equation Eq.(4.88) reduces to the quantum Yang-Baxter equation Eq.(4.23).

If we think of our quasi-Hopf algebra $\mathcal{A}$ as the deformation of the universal enveloping algebra of some Lie algebra $\mathfrak{g}$ we write $\mathcal{A}_h(\mathfrak{g}) = \mathcal{U}_h(\mathfrak{g})$. We denote the deformation parameter here as h instead of q. The deformed object $\mathcal{U}_h(\mathfrak{g})$ is a deformation of $\mathcal{U}(\mathfrak{g})$ if

$\mathcal{U}_h(\mathfrak{g}) \,/\, h\, \mathcal{U}_h(\mathfrak{g}) \cong \mathcal{U}(\mathfrak{g})$. Note that the coproduct of the (undeformed) UEA $\mathcal{U}(\mathfrak{g})$ is given by $\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi$, which satisfies $\Delta = \Delta^{cop}$. We say that a quasi-Hopf quantum universal enveloping algebra $\mathcal{U}_h(\mathfrak{g})$ is a quasi-Hopf deformation of $\mathcal{U}(\mathfrak{g})$ when

$$\sum_{\sigma \in \Sigma_3} (\Phi_h)_{\sigma(1)\sigma(2)\sigma(3)} = 1 \otimes 1 \otimes 1 \mod h^2 \tag{4.92}$$

and, in the quasitriangular case, $R_h = 1 \otimes 1 \mod h$, [61].

The relation of a (quasi-) Hopf algebra with the corresponding classical structures of a (quasi-) Lie bialgebra, such as cocycle δ, coassociator φ and classical r - matrix, are obtained from the quantum analogs via

$$\delta(x) = \frac{\Delta_h(\xi) - \Delta_h^{cop}(\xi)}{h} \mod h \;, \;\; x = \xi \bmod h \,, \tag{4.93}$$

$$\varphi = -\frac{1}{h^2} \sum_{\sigma \in \Sigma_3} \mathrm{sgn}(\sigma)\, (\Phi_h)_{\sigma(1)\sigma(2)\sigma(3)} \mod h \,, \tag{4.94}$$

$$R_h = 1 \otimes 1 \,+\, h\, r \,+\, \mathcal{O}(h^2)\,. \tag{4.95}$$

Now, if $\mathcal{U}_h(\mathfrak{g})$ is a quasitriangular quasi-Hopf QUE algebra with universal R - matrix R_h one can obtain the symmetric $\mathfrak{g}$ - invariant Casimir element t, which we encountered in section 3.3, via

$$t = \frac{R_h^{21} R_h^{12} - 1 \otimes 1}{h} \mod h \,. \tag{4.96}$$

Furthermore, one can always find a twist[4] such that $R_h = 1 \otimes 1 \,+\, h\, t \,+\, \mathcal{O}(h^2)$ and $\Phi_h = 1 \otimes 1 \otimes 1 \pmod{h^2}$, [61]. Then we call the Lie algebra $\mathfrak{g}$ together with the trivial cocycle $\delta = 0$ and coassociator

$$\varphi = -\frac{1}{4}\, [t_{12}, t_{23}] \tag{4.97}$$

[4]Note, that the Lie algebra $\mathfrak{g}$ is still assumed to be complex and simple. Thus, this statement from [61] does not contradict our findings in section 3.3 about the impossibility to twist between different twist equivalence classes at the level of real Lie algebras. For example, the three (real) Lie algebras $\mathfrak{sl}(2,\mathbb{C})$, $\mathfrak{iso}(3)$ and $\mathfrak{so}(4)$, seen as the classical (quasi-) doubles of $\mathfrak{su}(2)$, are truly in different twist equivalence classes. If, however, we embed those Lie algebras in $\mathfrak{so}(4,\mathbb{C})$ and allow complex twists, then it is possible to twist between the different classes and hence, get to different classical limits. See, for example, appendix A.2.

the **classical limit** of the **quasitriangular quasi-Hopf QUE algebra** $\mathcal{U}_h(\mathfrak{g})$. Also note that, if $\mathcal{U}_h(\mathfrak{g})$ happens to be a quasitriangular Hopf algebra as well, which means, that $\mathcal{U}_h(\mathfrak{g})$ is actually just obtained by some twist from a Hopf algebra, then it can have a classical limit $(\mathfrak{g}, \delta_{\mathfrak{g}}, \varphi_{\mathfrak{g}} = 0)$, which is different from the one just mentioned. Thus, in the second case the classical limit is a Lie bialgebra, whereas in the first case the classical limit is a quasi-Lie bialgebra. To be clear, we emphasize that the possibility of those two distinct limits is only possible at the level of complex algebras and complex twists.

Now, following [50, 51, 52] we can define a modified coproduct Δ and a modified R - matrix R via

$$\Delta(\xi) \equiv P\Delta_q(\xi) \qquad , \qquad R \equiv R_q P \qquad , \qquad \Delta^{cop}(\xi) R = R\Delta(\xi)\,, \tag{4.98}$$

where P denotes some projector that takes the truncation of the tensor product into account. Δ_q and R_q denote here the old structures Eq.(4.6) and Eq.(4.27). Furthermore, note the way we have written the last relation, avoiding the use of the inverse R - matrix R^{-1}. This is intentional, to denote that for weak quasi-Hopf algebras we only require there to exists weak inverse elements for R and Φ.

With this new coproduct we have now the canonical relation connecting the coproduct of the underlying algebra and the tensor product representation, i.e.,

$$\begin{aligned}(\pi^{(j_1)} \otimes \pi^{(j_2)})(\xi) &= (\pi^{(j_1)} \otimes \pi^{(j_2)})(\Delta(\xi)) = (\pi^{(j_1)} \otimes \pi^{(j_2)})(P\Delta_q(\xi)) \\ &= (\pi^{(j_1)} \bar{\otimes}\, \pi^{(j_2)})(\Delta_q(\xi))\,,\end{aligned} \tag{4.99}$$

where we used again the bar over the tensor product to denote the truncated tensor product introduced in Eq.(4.56) in the last step. The action of this projector is essentially to truncate the coupled spin $J = J(j_1, j_2)$, which normally ranges from $|j_1 - j_2|$ to $j_1 + j_2$, to its admissible values, if $j_1 + j_2 > \frac{l-2}{2}$, in the sum

$$(\pi^{(j_1)} \otimes \pi^{(j_2)})(\xi) = \sum_{\substack{J(j_1,j_2),\\ \text{adm}}} \sum_{M,N=-J}^{J} {}_qC \begin{pmatrix} j_1 & j_2 & J \\ \cdot & \cdot & M \end{pmatrix} \pi^{(J)}(\xi)_{MN}\, {}_qC \begin{pmatrix} j_1 & j_2 & J \\ \cdot & \cdot & N \end{pmatrix} . \tag{4.100}$$

We point out that[5] our definition of the projector P (potentially) differs from the expression in [50, 51, 52], in that we do not use a complex conjugation for neither of the

[5] In [50, 51, 52] the first Clebsch-Gordan coefficient in their relation, corresponding to our Eq.(4.100),

Clebsch-Gordan coefficients appearing in Eq.(4.100). In fact, our projector corresponds to the Π of Eq.(4.60). Note that with this new coproduct and R-matrix we have $\Delta(1) = P \neq 1 \otimes 1$, when acting on $j_1 \bar{\otimes} j_2$, if $j_1 + j_2 > \frac{l-2}{2}$, which is another feature of weak quasi-Hopf algebras. Furthermore, with quasi-inverse R^{-1} we have $RR^{-1} = \Delta^{cop}(1) = \tau \circ P$ and $R^{-1}R = \Delta(1) = P$.

Now, we can see the non-coassociativity of the new coproduct by considering

$$
\begin{aligned}
& & (\Delta \otimes \mathrm{id})(\Delta(\xi)) &= (P\Delta_q \otimes \mathrm{id})(P\Delta_q(\xi)) \\
(P\Delta_q(\xi)) = (\Delta_q(\xi))P & \quad\Rightarrow & &= (P \otimes \mathrm{id})(\Delta_q \otimes \mathrm{id})(\Delta_q(\xi))P \\
\text{co-associativity of } \Delta_q & \quad\Rightarrow & &= (P \otimes \mathrm{id})(\mathrm{id} \otimes \Delta_q)(\Delta_q(\xi))P \\
(P\Delta_q(\xi)) = (\Delta_q(\xi))P & \quad\Rightarrow & &= (P \otimes \mathrm{id})(\mathrm{id} \otimes \Delta_q)(P\Delta_q(\xi)) \\
& & &= (P \otimes \mathrm{id})(\mathrm{id} \otimes \Delta_q)(\Delta(\xi))\,,
\end{aligned} \tag{4.101}
$$

where we used twice that the projector is a central element and thus commutes with any element, [50, 51]. From Eq.(4.101) we get by left-multiplication with $(\mathrm{id} \otimes P)$

$$
\begin{aligned}
(\mathrm{id} \otimes P)(\Delta \otimes \mathrm{id})(\Delta(\xi)) &= (\mathrm{id} \otimes P)(P \otimes \mathrm{id})(\mathrm{id} \otimes \Delta_q)(\Delta(\xi)) \\
&= (P \otimes \mathrm{id})(\mathrm{id} \otimes P)(\mathrm{id} \otimes \Delta_q)(\Delta(\xi)) \\
&= (P \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)(\Delta(\xi))
\end{aligned} \tag{4.102}
$$

and, again due to the centrality of P, we have

$$
(\mathrm{id} \otimes P)(\Delta \otimes \mathrm{id})(\Delta(\xi)) = (\Delta \otimes \mathrm{id})(\Delta(\xi))(\mathrm{id} \otimes P)\,, \tag{4.103}
$$

which shows that, unless $P = \mathrm{id} \otimes \mathrm{id}$, we have a non-coassociative co-product Δ.

carries an asterisk to denote the dual Clebsch-Gordan coefficient. It is, however, not clarified whether their asterisk is to include complex conjugation or not. It makes a significant difference, whether we take as the dual coefficient the complex conjugate of the original one or not. In our expression Eq.(4.100) the projector is only non-trivial if $j_1 + j_2 > \frac{l-2}{2}$. If one defines the projector such that one of the Clebsch-Gordan coefficients uses complex conjugation, then the projector is non-trivial for all values of j_1 and j_2, independent of l. In the latter case, we would, furthermore, run into problems defining tensor operators that satisfy the Wigner-Eckart theorem.

4.3 Tensor operators and Wigner-Eckart theorem for $\mathcal{U}_q^{\mathbf{res}}(\mathfrak{sl}(2,\mathbb{C}))$

When we deal with physical systems with symmetries, we are most interested in operators that transform well under the symmetries at hand. Such operators are called tensor operators and are well known from non-relativistic quantum mechanics and relativistic quantum field theory. In our case of interest, however, the symmetries are not given by some Lie group but by generalized symmetries given in terms of a weak quasi-Hopf algebra, which makes the notion of tensor operators more involved. Tensor operators for quasitriangular Hopf algebras have been defined in [77] and where already successfully applied in [38, 39, 40, 42] in the study of 3D gravity with a negative cosmological constant, which corresponds, in the quantum theory, to the quasitriangular Hopf algebra $\mathcal{U}_q(\mathfrak{su}(2))$ for q real.

In this section we give a brief review of the construction of such tensor operators and show that it readily extends to the case when the symmetry is given by a weak quasitriangular quasi-Hopf algebra as in [50, 51, 52]. We also consider here the so-called Jordan-Schwinger representation in the q-deformed case, which we will need for the construction of spinor operators. Furthermore, we show how to define tensor operators and tensor products of tensor operators, using a representation of the braid group in the quasi-Hopf setting. This allows us to construct the q-analogs of geometrical operators known from standard Loop Quantum Gravity with $\Lambda = 0$. The results of the calculations are in the end similar to the q real case and could have been obtained via an analytic continuation. However, the technical details along the way are more involved, due to the truncated tensor product and the non-coassociative features.

4.3.1 q-boson algebras and Jordan-Schwinger representations

In this section we review a realization of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathcal{U}_q^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$ in terms of a pair of q-boson operators, i.e., the q-analog of the so-called Jordan-Schwinger representation [82, 81]. This will be necessary for the construction of our spinor operators in the next section. The q-deformed version of the harmonic oscillator algebra or q-boson algebra can be given by a set of operators $\{a, \bar{a}, L_a, L_a^{-1}\}$ satisfying the following relations

$$\bar{a}a - qa\bar{a} = L_a^{-1} \quad , \quad L_a a L_a^{-1} = qa \quad , \quad L_a \bar{a} L_a^{-1} = q^{-1}\bar{a} \tag{4.104}$$

and of course $L_a L_a^{-1} = L_a^{-1} L_a = 1$. Furthermore, it holds that

$$L_a = \bar{a}a - q^{-1} a\bar{a} \,. \tag{4.105}$$

One can show that

$$a\bar{a} = \frac{L_a - L_a^{-1}}{q - q^{-1}} \overset{*}{=} [N_a]_q \quad , \quad \bar{a}a = \frac{qL_a - q^{-1}L_a^{-1}}{q - q^{-1}} \overset{*}{=} [N_a + 1]_q \tag{4.106}$$

where the equalities $\overset{*}{=}$ hold for q real and $L_a = q^{N_a}$. For the root of unity case $q^l = 1$, one finds again that the central elements are given by a q-Casimir and the elements $\{a^{l'}, \bar{a}^{l'}, L_a^{l'}, L_a^{-l'}\}$, where again $l' = l$ for l odd and $l' = l/2$ for l even. Thus, one can again define a finite dimensional quotient Hopf algebra using the ideal generated by $a^{l'} = 0 = \bar{a}^{l'}$ and $L_a^{\pm l'} - 1 = 0$. Using now two mutually commuting copies of this q-boson algebras $\{a, \bar{a}, L_a, L_a^{-1}\}$ and $\{b, \bar{b}, L_b, L_b^{-1}\}$, we can give the Jordan-Schwinger representation of $\mathcal{U}_{\rm q}(\mathfrak{sl}(2,\mathbb{C}))$ or $\mathcal{U}_{\rm q}^{\rm res}(\mathfrak{sl}(2,\mathbb{C}))$ via

$$E = a\bar{b} \quad , \quad F = b\bar{a} \quad , \quad G = L_a L_b^{-1} \quad , \quad G^{-1} = L_b L_a^{-1} \,. \tag{4.107}$$

It is easily verified that these expressions indeed satisfy Eq.(4.3).

4.3.2 Representations of the q-boson algebras

As for $\mathcal{U}_{\rm q}(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathcal{U}_{\rm q}^{\rm res}(\mathfrak{sl}(2,\mathbb{C}))$, there are several inequivalent representations for the q - boson algebra, depending on the value of q and, especially for q root of unity, whether we want to consider cyclic-, semi-cyclic or non-cyclic representations. One such representation of the non-cyclic type, given in [82], is the following finite dimensional Fock space representation with

$$L_a^{\pm 1} |n\rangle = q^{\pm n} |n\rangle \quad , \quad a\,|n\rangle = \sqrt{|[n+1]_q|}\,|n+1\rangle \quad , \quad \bar{a}\,|n\rangle = \sqrt{|[n]_q|}\,|n-1\rangle \tag{4.108}$$

and $\bar{a}\,|0\rangle = 0\,|0\rangle = 0$, for $n \in \{0, 1, 2, \cdots, l-1\}$, l odd. Note the absolute value under the square brackets, which is necessary to guarantee a real action. In this representation, however, a and $\bar{a}$ are not the hermitian conjugates of each other. Furthermore, we have for all $n \in \{0, 1, \cdots, l-1\}$, with l odd,

$$a^l\,|n\rangle = 0 \quad , \quad \bar{a}^l\,|n\rangle = 0 \quad , \quad L_a^{\pm l}\,|n\rangle = |n\rangle \tag{4.109}$$

and thus, a representation of the restricted, or truncated q - boson algebra. For a classification of representations of the q - boson algebra for q root of unity we refer to [81]. Another form of the representation, which does not need this absolute value under the square bracket, is given via

$$T_\omega(L_a^{\pm 1})\,|\omega+m\rangle = q^{\pm(\omega+m)}\,|\omega+m\rangle \qquad , \qquad T_\omega(a)\,|\omega+m\rangle = |\omega+m+1\rangle\,, \tag{4.110}$$

$$T_\omega(\bar{a})\,|\omega+m\rangle = q^{-\omega}[m]_q\,|\omega+m-1\rangle\,, \tag{4.111}$$

which is a representation with lowest and highest weights, $\omega \in \mathbb{C}$ and with $|\omega-1\rangle = |\omega+l\rangle = 0$. Note, that Eq.(4.108) can be obtained from Eq.(4.110) and Eq.(4.111) by setting $\omega = 0$ and a rescaling of the basis vectors $|m\rangle \mapsto |m\rangle/\sqrt{|[m]_q!|}$.

Using now two q-boson algebras with the representation Eq.(4.108), we can consider their tensor product representation and starting from the vacuum $|0,0\rangle$ we can obtain the representation of $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ introduced in Eq.(4.43) and Eq.(4.44), without the extra q-factors, by defining the states

$$|j,m\rangle \equiv |j+m,j-m\rangle = \frac{a^{j+m}\,b^{j-m}}{\sqrt{[j+m]_q!}\sqrt{[j-m]_q!}}\,|0,0\rangle\,. \tag{4.112}$$

We did not write any absolute values in the square brackets here. Compare also with our statements below Eq.(4.52), where we showed that one has to restrict $j \leq \frac{l-1}{4}$ to guarantee positivity under the square root. In order to restore the extra q-factors in Eq.(4.43) and Eq.(4.44) we can define instead of Eq.(4.108) the following action

$$a\,|n\rangle = q^{\frac{n}{2}+\beta}\sqrt{|[n+1]_q|}\,|n+1\rangle \quad , \quad \bar{a}\,|n\rangle = q^{-\frac{n}{2}+\frac{1}{2}-\beta}\sqrt{|[n]_q|}\,|n-1\rangle\,, \tag{4.113}$$

for any parameter β and we still have a representation of our q-boson algebra and Eq.(4.112) holds unchanged. The most symmetric case is achieved for $\beta = \frac{1}{4}$, which also respects the $*$-structure $*a = \bar{a}$ and $*(L_a^{\pm 1}) = L_a^{\mp 1}$, which is the star structure compatible with Eq.(4.31). With this representation Eq.(4.107) and Eq.(4.112) are still valid and we obtain Eq.(4.43) and Eq.(4.44) with our choice of q-factors.

4.3.3 Tensor operators and Wigner-Eckart theorem

We first review the definition of tensor operators as presented in [77] for Hopf algebras and will see that it holds similarly for weak quasi-Hopf algebras with some new features showing up. If we consider H to be a (weak quasi-) Hopf algebra and consider two H-modules V and W then there are two (equivalent) ways to define tensor operators. Consider two linear maps

$$t \,:\, V \longrightarrow L(W,W) \qquad , \qquad \tilde{t} \,:\, V \otimes W \longrightarrow W \,, \tag{4.114}$$

where $L(W,W)$ is the algebra of all linear maps from W into W. $\tilde{t}$ is defined via $\tilde{t}(v \otimes w) = t(v) \triangleright w$. Now, according to [77], t is an H-module homomorphism iff $\tilde{t}$ is an H-module isomorphism. If those (equivalent) statements are satisfied, t is called a **tensor operator**. Now, a linear map $f \,:\, V \to W$ is called a homomorphism (intertwiner) of the H-modules V and W if we have, for all $h \in H$,

$$f \pi_V(h) = \pi_W(h) f \,, \tag{4.115}$$

where

$$\pi_V \,:\, H \longrightarrow L(V,V) \tag{4.116}$$

denotes a representation of H. This means for our definition of a tensor operator that t and $\tilde{t}$ must satisfy

$$t\, \pi_V(h) = \pi_{L(W,W)}(h)\, t \qquad , \qquad \tilde{t}\, \pi_{V\otimes W}(h) = \pi_W(h)\, \tilde{t} \tag{4.117}$$

for all $h \in H$. The representations of H in $L(W,W)$ and in $V \otimes W$ are given by

$$\pi_{L(W,W)}(h) \triangleright f = \sum_i \pi_W(h_i^1)\, f\, \pi_W(S(h_i^2)) \quad , \quad \pi_{V\otimes W}(h) = \sum_i \pi_V(h_i^1) \otimes \pi_W(h_i^2) \tag{4.118}$$

for $f \in L(W,W)$ and $\Delta(h) = \sum_i h_i^1 \otimes h_i^2$. Note, that the first action is basically a representation of the definition of the adjoint action for Hopf algebras $\mathrm{ad}_h(f) = \sum_i h_i^1 f S(h_i^2)$. Consider now the modules $V^j = \{|jm\rangle\}$ of $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$. In this notation we get from the condition for t in Eq.(4.117)

$$t\, \pi_V(h) = \pi_{L(W,W)}(h)\, t = \sum_i \pi_W(h_i^1)\, t\, \pi_W(S(h_i^2)) \tag{4.119}$$

and with the definition $t\,|jm\rangle \equiv t_m^j$ we get

$$\sum_{m=-j}^{j} t\,|jm\rangle\langle jm|\pi^j(h)|jn\rangle = \sum_i \pi_W(h_i^1)\, t(|jn\rangle)\, \pi_W(S(h_i^2))\,, \tag{4.120}$$

$$\sum_{m=-j}^{j} t^j_m \, \pi^j(h)_{mn} = \sum_i \pi_W(h^1_i) \, t^j_n \, \pi_W(S(h^2_i)) \,. \tag{4.121}$$

From the condition for $\tilde{t}$ in Eq.(4.117) we get

$$\tilde{t} \; \sum_i \pi_V(h^1_i) \otimes \pi_W(h^2_i) = \pi_W(h) \, \tilde{t} \tag{4.122}$$

and in a basis $|IM\rangle \otimes |JN\rangle$ of the tensor product of $V = V^I$ and $W = V^J$ this gives

$$\tilde{t} \sum_{Q,R} |IQ\rangle \otimes |JR\rangle \sum_i \langle IQ|\pi^I(h^1_i)|IM\rangle \otimes \langle JR|\pi^J(h^2_i)|JN\rangle = \pi^J(h) \, \tilde{t}(|IM\rangle \otimes |JN\rangle) \,, \tag{4.123}$$

and

$$\sum_{Q,R} \sum_i (t^I_Q \triangleright |JR\rangle) \, \pi^I(h^1_i)_{QM} \, \pi^J(h^2_i)_{RN} = \pi^J(h) \, (t^I_M \triangleright |JN\rangle) \,. \tag{4.124}$$

Multiplying by $\langle JT|$ gives

$$\sum_{Q,R} \sum_i (t^I_Q)_{TR} \, \pi^I(h^1_i)_{QM} \, \pi^J(h^2_i)_{RN} = \sum_{U=-J}^{J} \pi^J(h)_{TU} \, (t^I_M)_{UN} \,. \tag{4.125}$$

We see that this is the notion of tensor operator used in [50, 51, 52].

Now, for the case of q being a root of unity $q^l = 1$, remember that in this case we have a truncation of the tensor product, such that $J \in \{|j_1 - j_2|, \cdots, \min(j_1 + j_2, l - 2 - j_1 - j_2)\}$. How is this truncation implemented in the definition of the tensor operators? Observe that in general we can use the Clebsch-Gordan coefficients to rewrite the tensor product representation, i.e.,

$$\pi_{V \otimes W}(h) \equiv \pi_{V \otimes W}(\Delta h) = \sum_i \pi_V(h^1_i) \otimes \pi_W(h^2_i) \,, \tag{4.126}$$

as

$$\pi_{V^{j_1} \otimes V^{j_2}}(h) \equiv \sum_{J(j_1,j_2)} \sum_{M,N=-J}^{J} {}_qC \begin{pmatrix} j_1 & j_2 & J \\ \cdot & \cdot & M \end{pmatrix} \pi^{(J)}(h)_{MN} \, {}_qC \begin{pmatrix} j_1 & j_2 & J \\ \cdot & \cdot & N \end{pmatrix} . \tag{4.127}$$

Thus, the information about the truncation of the coproduct is hiding in the Clebsch-Gordan coefficients in the above formula by summing only over admissible spins $J(j_1, j_2)$.

This implies for the definition of weak quasi-Hopf tensor operators the condition, i.e., the modified or truncated version of Eq.(4.125),

$$\sum_{F=-J}^{J} (t^I_M)_{FN}\, \pi^J(h)_{EF} = \tag{4.128}$$

$$\sum_{\substack{K(I,J),\\ \text{adm}}} \sum_{A=-I}^{I} \sum_{B=-J}^{J} \sum_{C=-K}^{K} \sum_{D=-K}^{K} (t^I_A)_{EB}\ {}_qC \begin{pmatrix} I & J & K \\ A & B & C \end{pmatrix} \pi^K(h)_{CD}\ \ {}_qC \begin{pmatrix} I & J & K \\ M & N & D \end{pmatrix} .$$

Note that the Clebsch-Gordan coefficients vanish unless $A + B = C$ and $M + N = D$. Thus, we can write

$$\sum_{F=-J}^{J} (t^I_M)_{FN}\, \pi^J(h)_{EF} = \tag{4.129}$$

$$\sum_{\substack{K(I,J),\\ \text{adm}}} \sum_{A=-I}^{I} \sum_{B=-J}^{J} (t^I_A)_{EB}\ {}_qC \begin{pmatrix} I & J & K \\ A & B & A+B \end{pmatrix} \pi^K(h)_{A+B,M+N}\ \ {}_qC \begin{pmatrix} I & J & K \\ M & N & M+N \end{pmatrix} .$$

The defining condition for weak quasi-Hopf tensor operators Eq.(4.128) shows now that the **Wigner-Eckart theorem**, which states that the matrix elements of tensor operators can be expressed in terms of Clebsch-Gordan coefficients and a reduced matrix element, holds in the weak quasi-Hopf case as well, since we can solve Eq.(4.128) with

$$\langle j_1 m_1 |\, t^I_M\, | j_2 m_2 \rangle = N(j_1, j_2, I)\ {}_qC \begin{pmatrix} I & j_2 & j_1 \\ M & m_2 & m_1 \end{pmatrix} , \tag{4.130}$$

for some reduced matrix elements $N(j_1, j_2, I)$. Note, that if we had been using the truncation of the coproduct using one complex conjugated Clebsch-Gordan coefficients, we could not use the orthogonality of the Clebsch-Gordan coefficients and thus the fate of the Wigner-Eckart theorem in that case seems not clear.

An interesting feature of those weak quasi-Hopf tensor operators is that they have a cut-off, exactly like the physical representations of $\mathcal{U}^{\text{res}}_{\text{q}}(\mathfrak{sl}(2,\mathbb{C}))$, i.e., the rank I of a tensor operator t^I can be at most $\frac{l-2}{2}$ for $q^l = 1$, with l odd. For example, with $l = 3$ we can only have scalar and spinor operators, but no vector operators or higher. This behavior is implemented in Eq.(4.130), because if any of the spins j_1, j_2, I is bigger than $\frac{l-2}{2}$, then the tensor operator vanishes. When we want to calculate explicit expressions for those truncated tensor operators we can still use the (equivalent) first condition in Eq.(4.118) using the adjoint action, as long as we make sure that we consider tensor operators of admissible rank.

4.3.4 Spinor operators

We will consider now explicitly the case of spin-$\frac{1}{2}$ or spinor operators. They are particularly useful since they allow to construct all higher order tensor operators using the Clebsch-Gordan coefficients. Furthermore, they are useful in Loop quantum gravity to build so called spinor-networks that appear in the twisted geometries parametrization [41]. Using the representation Eq.(4.43) and Eq.(4.44) the first relation in Eq.(4.121) for $j = 1/2$ becomes for the generators of $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$, i.e., for $h = \{G, E, F\}$,

$$\mathrm{ad}_G \triangleright t^{\frac{1}{2}}_m = G t^{\frac{1}{2}}_m G^{-1} = q^{2m} t^{\frac{1}{2}}_m = \pi^{(\frac{1}{2})}(G) \triangleright t^{\frac{1}{2}}_m \,, \tag{4.131}$$

$$\mathrm{ad}_E \triangleright t^{\frac{1}{2}}_m = E t^{\frac{1}{2}}_m G^{-1} - t^{\frac{1}{2}}_m E G^{-1} = q^{m+\frac{1}{2}} \sqrt{[1/2-m]_q [3/2+m]_q}\; t^{\frac{1}{2}}_{m+1} = \pi^{(\frac{1}{2})}(E) \triangleright t^{\frac{1}{2}}_m \,, \tag{4.132}$$

$$\mathrm{ad}_F \triangleright t^{\frac{1}{2}}_m = F t^{\frac{1}{2}}_m - G^{-1} t^{\frac{1}{2}}_m G F = q^{-m+\frac{1}{2}} \sqrt{[1/2+m]_q [3/2-m]_q}\; t^{\frac{1}{2}}_{m-1} = \pi^{(\frac{1}{2})}(F) \triangleright t^{\frac{1}{2}}_m \,. \tag{4.133}$$

Now, if we define

$$t^{\frac{1}{2}} = \begin{pmatrix} t^{\frac{1}{2}}_{-\frac{1}{2}} \\ t^{\frac{1}{2}}_{\frac{1}{2}} \end{pmatrix} \equiv \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix} \,, \tag{4.134}$$

we obtain, together with Eq.(4.107) the following conditions

$$L_a L_b^{-1} \mathcal{A} L_b L_a^{-1} = q^{-1} \mathcal{A} \quad , \quad L_a L_b^{-1} \mathcal{B} L_b L_a^{-1} = q \mathcal{B} \,, \tag{4.135}$$

$$a\bar{b} \mathcal{A} L_b L_a^{-1} - \mathcal{A} a\bar{b} L_b L_a^{-1} = \mathcal{B} \quad , \quad a\bar{b} \mathcal{B} L_b L_a^{-1} - \mathcal{B} a\bar{b} L_b L_a^{-1} = 0 \,, \tag{4.136}$$

$$b\bar{a} \mathcal{A} - L_b L_a^{-1} \mathcal{A} L_a L_b^{-1} b\bar{a} = 0 \quad , \quad b\bar{a} \mathcal{B} - L_b L_a^{-1} \mathcal{B} L_a L_b^{-1} b\bar{a} = \mathcal{A} \,, \tag{4.137}$$

from which one finds that

$$\mathcal{A}_{\alpha,\beta} = \alpha \bar{a} L_b^{-1} + \beta b L_a \quad , \quad \mathcal{B}_{\alpha,\beta} = -q\alpha \bar{b} + \beta a \tag{4.138}$$

satisfy Eq.(4.135) - Eq.(4.137) for all $(\alpha, \beta) \in \mathbb{C}^2$. Hence, we get the 2-parameter family of spin-$\frac{1}{2}$ tensor operators for $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$

$$t^{\frac{1}{2}}_{\alpha,\beta} = \begin{pmatrix} \mathcal{A}_{\alpha,\beta} \\ \mathcal{B}_{\alpha,\beta} \end{pmatrix} = \begin{pmatrix} \alpha \bar{a} L_b^{-1} + \beta b L_a \\ -q\alpha\bar{b} + \beta a \end{pmatrix} . \tag{4.139}$$

One can take the following two spinors both as basis vectors for a spin-$\frac{1}{2}$ module (note that we redefine $\mathcal{A}$ and $\mathcal{B}$)

$$t^{\frac{1}{2}}_{\alpha} = t^{\frac{1}{2}}_{\alpha,0} = \alpha \begin{pmatrix} \bar{a} L_b^{-1} \\ -q\bar{b} \end{pmatrix} \equiv \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix} \quad , \quad t^{\frac{1}{2}}_{\beta} = t^{\frac{1}{2}}_{0,\beta} = \beta \begin{pmatrix} b L_a \\ a \end{pmatrix} \equiv \begin{pmatrix} \mathcal{C} \\ \mathcal{D} \end{pmatrix} . \tag{4.140}$$

4.3.5 Vector operators

Similarly, we consider now the vector operators with spin $j = 1$. In this case the condition Eq.(4.121) gives together with the representation Eq.(4.43) and Eq.(4.44) the following equations

$$\mathrm{ad}_G \triangleright t^1_m = G t^1_m G^{-1} = q^{2m} t^1_m = \pi^{(1)}(G) \triangleright t^1_m \, , \tag{4.141}$$

$$\mathrm{ad}_E \triangleright t^1_m = E t^1_m G^{-1} - t^1_m E G^{-1} = q^{m+\frac{1}{2}} \sqrt{[1-m]_q [2+m]_q} \, t^1_{m+1} = \pi^{(1)}(E) \triangleright t^1_m \, , \tag{4.142}$$

$$\mathrm{ad}_F \triangleright t^1_m = F t^1_m - G^{-1} t^1_m G F = q^{-m+\frac{1}{2}} \sqrt{[1+m]_q [2-m]_q} \, t^1_{m-1} = \pi^{(1)}(F) \triangleright t^1_m \, . \tag{4.143}$$

For $m = 1$ we get

$$G t^1_1 G^{-1} = q^2 t^1_1 \, , \tag{4.144}$$

$$E t^1_1 G^{-1} - t^1_1 E G^{-1} = 0 \, , \tag{4.145}$$

$$F t^1_1 - G^{-1} t^1_1 G F = q^{-\frac{1}{2}} \sqrt{[2]_q} \, t^1_0 \, . \tag{4.146}$$

The equations Eq.(4.144) and Eq.(4.145) are solved by choosing $t^1_{1,\alpha} = \alpha E$, for all $\alpha \in \mathbb{C}_*$. For $m = -1$ we get

$$G t^1_{-1} G^{-1} = q^{-2} t^1_{-1} \, , \tag{4.147}$$

$$E t^1_{-1} G^{-1} - t^1_{-1} E G^{-1} = q^{-\frac{1}{2}} \sqrt{[2]_q} \, t^1_0 \, , \tag{4.148}$$

$$F t^1_{-1} - G^{-1} t^1_{-1} G F = 0 \tag{4.149}$$

and using now $t^1_{1,\alpha} = \alpha E$ in Eq.(4.146) we find that we have to define t^1_0 as follows

$$t^1_{0,\alpha} = \frac{\alpha q^{\frac{1}{2}}}{\sqrt{[2]_q}}(FE - q^{-2}EF)\,. \tag{4.150}$$

This satisfies the equations for $m = 0$, which are given by

$$Gt^1_0G^{-1} = t^1_0\,, \tag{4.151}$$

$$Et^1_0G^{-1} - t^1_0EG^{-1} = q^{\frac{1}{2}}\sqrt{[2]_q}\,t^1_1\,, \tag{4.152}$$

$$Ft^1_0 - G^{-1}t^1_0GF = q^{\frac{1}{2}}\sqrt{[2]_q}\,t^1_{-1} \tag{4.153}$$

and from Eq.(4.153) we learn that we should define

$$t^1_{-1,\alpha} = -\alpha GF \tag{4.154}$$

in order to satisfy the last remaining equation. Hence, we find the following (one-parameter family of) tensor operators

$$t^1_\alpha = \begin{pmatrix} t^1_{-1,\alpha} \\ t^1_{0,\alpha} \\ t^1_{1,\alpha} \end{pmatrix} = \begin{pmatrix} -\alpha GF \\ \frac{\alpha q^{\frac{1}{2}}}{\sqrt{[2]_q}}(FE - q^{-2}EF) \\ \alpha E \end{pmatrix} \quad \xrightarrow{q\to 1} \quad t^1_\alpha = (-\alpha)\begin{pmatrix} F \\ \frac{1}{\sqrt{2}}H \\ -E \end{pmatrix}\,, \tag{4.155}$$

which reproduces the correct vector operator for the algebra $\mathfrak{sl}(2,\mathbb{C})$ in the spherical basis in the $q \to 1$ limit. Now, if we consider the following commutator relations

$$[(t^1_\alpha)_{-1}, (t^1_\alpha)_0] = -\alpha\sqrt{q}\sqrt{[2]_q}\,G\,(t^1_\alpha)_{-1}\,, \tag{4.156}$$

$$[(t^1_\alpha)_{-1}, (t^1_\alpha)_1] = -\alpha\frac{\sqrt{[2]_q}}{\sqrt{q}}\,G\,(t^1_\alpha)_0\,, \tag{4.157}$$

$$[(t^1_\alpha)_0, (t^1_\alpha)_1] = -\alpha q^{-\frac{3}{2}}\sqrt{[2]_q}\,G\,(t^1_\alpha)_1 \tag{4.158}$$

and, furthermore, fix α to be

$$\alpha = -\frac{\sqrt{q}}{\sqrt{[2]_q}}\,, \tag{4.159}$$

then we get

$$[(t^1)_0, (t^1)_{-1}] = -qG(t^1)_{-1} = -q^{-1}(t^1)_{-1}G\,, \tag{4.160}$$

$$[(t^1)_{-1}, (t^1)_1] = G(t^1)_0 = (t^1)_0 G\,, \tag{4.161}$$

$$[(t^1)_0, (t^1)_1] = q^{-1}G(t^1)_1 = q(t^1)_1 G\,. \tag{4.162}$$

With the identification $G = q^{J_z}$ and $K_m = (t^1)_m$, $m \in \{-1, 0, 1\}$ these are exactly the same commutation relations found in [86] in the q real case and with $q^{\frac{1}{2}}$ instead of our convention working with q.

As mentioned before, we can now use our spinor operators and the Clebsch-Gordan coefficients Eq.(4.65) - Eq.(4.67) to construct all (admissible) higher order tensor operators. As an example, we see that we can construct the spin-1 tensor operators Eq.(4.155), for $\alpha = -q$, from the spin-$\frac{1}{2}$ operators Eq.(4.140) via

$$t^1_{-1} = \sum_{m_1,m_2} {}_qC \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ m_1 & m_2 & -1 \end{pmatrix} t^{\frac{1}{2}}_{m_2} \tilde{t}^{\frac{1}{2}}_{m_2} = t^{\frac{1}{2}}_{-\frac{1}{2}} \tilde{t}^{\frac{1}{2}}_{-\frac{1}{2}} = \bar{a} L_b^{-1} b L_a = qGF\,, \tag{4.163}$$

$$t^1_{1} = \sum_{m_1,m_2} {}_qC \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ m_1 & m_2 & 1 \end{pmatrix} t^{\frac{1}{2}}_{m_2} \tilde{t}^{\frac{1}{2}}_{m_2} = t^{\frac{1}{2}}_{\frac{1}{2}} \tilde{t}^{\frac{1}{2}}_{\frac{1}{2}} = -q\bar{b}a = -qE\,, \tag{4.164}$$

$$t^1_{0} = \sum_{m_1,m_2} {}_qC \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ m_1 & m_2 & 0 \end{pmatrix} t^{\frac{1}{2}}_{m_2} \tilde{t}^{\frac{1}{2}}_{m_2} = {}_qC \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} t^{\frac{1}{2}}_{-\frac{1}{2}} \tilde{t}^{\frac{1}{2}}_{\frac{1}{2}} + {}_qC \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} t^{\frac{1}{2}}_{\frac{1}{2}} \tilde{t}^{\frac{1}{2}}_{-\frac{1}{2}}$$

$$= -q\, \frac{q^{\frac{1}{2}}}{\sqrt{[2]_q}} \left(FE - q^{-2}EF \right)\,, \tag{4.165}$$

where $t^{\frac{1}{2}}$ corresponds to $t^{\frac{1}{2}}_{\alpha}$ and $\tilde{t}^{\frac{1}{2}}$ corresponds to $t^{\frac{1}{2}}_{\beta}$.

4.4 Geometrical observables

In this section we will construct certain gauge-invariant operators that allow us to probe the quantum geometry of our q-deformed spin networks. In particular, we are interested in the q-deformed version of the so-called length operator and the angle operator. Our fundamental building block, or 'atom of space', is given by a three valent node, which is labeled by a so-called intertwiner state. The length operator acts on the three individual legs and the angle operator acts on pairs of legs. We will find that the eigenvalues of these quantum operators indeed correspond to the classical expressions that we know from spherical triangles. From this we can conclude that q-deformed spin networks at q root of unity really represent quantized spherical geometry, or quantum geometry with a positive cosmological constant.

As we saw in section 4.3 we can use the spinor operators and the q-Clebsch-Gordan coefficients to build tensor operators for Hilbert spaces associated with a single link of the spin network. However, when we want to consider tensor operators acting on tensor product states, i.e., states which are not just supported on a single link, we have to be more careful, because of the non-trivial braiding properties of $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$. The difference, compared with the flat $\mathfrak{su}(2)$ case, is the non-cocommutativity and, compared with the q real case, additionally, the non-coassociativity, which means that we need the quasitriangular structure as well as the coassociator to define tensor products of tensor operators. In order to construct proper tensor operators that transform well even for tensor product states, we will make use of the fact that the representation category of quasitriangular (quasi-) Hopf algebras carry a representation of the braid group.

Recall that the **braid group $\mathcal{B}_n$**, acting on n-strands, is generated by the elements b_i and b_i^{-1} with $i \in \{1, 2, \cdots, n-1\}$ subject to the so-called **Artin relations**

$$b_i b_j = b_j b_i \quad \text{if} \quad |i-j| > 1\,, \tag{4.166}$$

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \quad \text{otherwise}\,, \tag{4.167}$$

$$b_i b_i^{-1} = \mathrm{id}_{\mathcal{B}} = b_i^{-1} b_i\,. \tag{4.168}$$

With the following notation

$$\Delta^n = (\mathrm{id}^{n-1} \otimes \Delta) \cdots (\mathrm{id} \otimes \Delta)\Delta \qquad \text{for} \qquad n \geq 2\,, \tag{4.169}$$

$$\Delta^1 = \Delta \quad , \quad \Delta^0 = \mathrm{id} \quad , \quad \Delta^{-1} = \varepsilon \tag{4.170}$$

and with $T^+ \equiv \tau \circ R$ and $T^- \equiv R^{-1} \circ \tau$, where τ is the permutation operator, one can define a representation of the braid group $\mathcal{B}_n$ acting on the weak quasi-Hopf algebra $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))^{\otimes n}$ (or its representations) via, [52],

$$b_k^{\pm} \equiv \Delta^{n-1}(1)(\mathrm{id}^{n-k+1} \otimes \Delta^{k-2})(1^{n-k-1} \otimes \Phi\,(T^{\pm} \otimes 1)\,\Phi^{-1}) \;\in\; \mathcal{B}_n\,, \tag{4.171}$$

where $\mathrm{id}_{\mathcal{B}} = \Delta^{n-1}(1)$. Since we are mainly interested in trivalent graphs, i.e., $n = 3$, let us write those braid operators explicitly. If we consider an arbitrary state on a 3-valent node we can write

$$|\Psi\rangle = \sum_{m_1 m_2 m_3} \Psi_{m_1 m_2 m_3} \, |j_1, m_1\rangle \bar{\otimes} (|j_2, m_2\rangle \bar{\otimes} |j_3, m_3\rangle) \,, \tag{4.172}$$

where we had to make a choice for the order of the tensor products, i.e., whether we consider $(V^{j_1} \otimes V^{j_2}) \otimes V^{j_3}$ or $V^{j_1} \otimes (V^{j_2} \otimes V^{j_3})$. In order to braid the last two factors we need the elements $b_1^{\pm}$, which leave the first factor unchanged. This braid is simply given by

$$b_1^{\pm} = 1 \otimes T^{\pm} \tag{4.173}$$

and since we do not change the order of the brackets, we do not encounter the coassociator yet. For the braiding involving the first factor, however, we need to take care of the brackets first using the coassociator Φ. We get

$$b_2^{\pm} = \Phi \, (T^{\pm} \otimes 1) \, \Phi^{-1} \,. \tag{4.174}$$

Now, before we consider the length operator we have to consider how to build tensor products of tensor operators in such a way that we still have another tensor operator at the end. The problem in the non-cocommutative or non-coassociative case is that, starting with any tensor operator t, we can not simply let it act in an arbitrary way on a tensor product space. While it can be shown that ${}^{(1)}t \equiv t \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}$ is still a good tensor operator, any other operator, where t is not in the first factor, is in general not a good tensor operator. This problem can be solved, however, using the braiding Eq.(4.171). If we consider the simplest case of two factors ($n = 2$), then we can construct the tensor operator corresponding to t, acting on the second leg as follows

$${}^{(2)}t \equiv b_1^{-1} (t \otimes \mathrm{id}) \, b_1 = R^{-1} \circ \tau \, (t \otimes \mathrm{id}) \, \tau \circ R = R^{-1} (\mathrm{id} \otimes t) \, R \,. \tag{4.175}$$

We have seen before already that in this case the coassociator does not show up yet, but only for higher order product. This procedure continues and one can use the braid group to move the tensor operator t to any position. Furthermore, this is also the correct procedure to define the tensor product of tensor operators that are acting not only on the first factor of the product state [77]. This means that, for example, the tensor product of t_1 and t_2 has to be defined via

$$t_1 \otimes t_2 \equiv (t_1 \otimes \mathrm{id}) \, b_1^{-1} (t_2 \otimes \mathrm{id}) \, b_1 \,. \tag{4.176}$$

4.4.1 Length operator

In analogy with standard Loop quantum gravity for $\Lambda = 0$, we will now use our vector operators t^1 to define different geometrical operators. For example, the q-deformed length (squared) operator, acting on the i-th leg, is defined as

$$ {}^q\vec{L}^2{}_i \equiv {}^{(i)}t \cdot {}^{(i)}t = \sum_{m_1,m_2} {}_qC \begin{pmatrix} 1 & 1 & 0 \\ m_1 & m_2 & 0 \end{pmatrix} {}^{(i)}t^1{}_{m_1} \; {}^{(i)}t^1{}_{m_2} \, . \tag{4.177} $$

The contraction of the components of the vector operator with the Clebsch-Gordan coefficient in this particular form guarantees that the resulting object transforms as a scalar, and thus, is invariant under gauge-transformations. Note, that in four dimensions this operator would measure the area (squared) dual to a link in the 3-dimensional hypersurface. For a 3-valent node with state $|1\rangle \otimes (|2\rangle \otimes |3\rangle)$ we can consider for example

$$ {}^{(1)}t^1{}_m \equiv (t^1_m \otimes \mathrm{id} \otimes \mathrm{id}) \, , \tag{4.178} $$

$$ {}^{(2)}t^1{}_m \equiv b_2^{-1} \, (t^1_m \otimes \mathrm{id} \otimes \mathrm{id}) \, b_2 \tag{4.179} $$

$$ {}^{(3)}t^1{}_m \equiv b_1^{-1} b_2^{-1} \, (t^1_m \otimes \mathrm{id} \otimes \mathrm{id}) \, b_2 \, b_1 \, . \tag{4.180} $$

For the length operators one finds that the braiding matrices in between the two vector operators cancel, e.g. for

$$ \begin{aligned} {}^q\vec{L}^2{}_3 \equiv {}^{(3)}t \cdot {}^{(3)}t &= \sum_{m_1,m_2} {}_qC \begin{pmatrix} 1 & 1 & 0 \\ m_1 & m_2 & 0 \end{pmatrix} {}^{(3)}t^1{}_{m_1} \; {}^{(3)}t^1{}_{m_2} \\ &= \sum_{m_1,m_2} {}_qC \begin{pmatrix} 1 & 1 & 0 \\ m_1 & m_2 & 0 \end{pmatrix} b_1^{-1} b_2^{-1} \, (t^1_{m_1} \otimes \mathrm{id} \otimes \mathrm{id}) \, b_2 \, b_1 b_1^{-1} b_2^{-1} \, (t^1_{m_2} \otimes \mathrm{id} \otimes \mathrm{id}) \, b_2 \, b_1 \\ &= \sum_{m_1,m_2} {}_qC \begin{pmatrix} 1 & 1 & 0 \\ m_1 & m_2 & 0 \end{pmatrix} b_1^{-1} b_2^{-1} \, (t^1_{m_1} t^1_{m_2} \otimes \mathrm{id} \otimes \mathrm{id}) \, b_2 \, b_1 \, . \end{aligned} \tag{4.181} $$

Furthermore, note that we can actually move the braiding as follows, and thus can relate the different length operators with each other, i.e.,

$$ \begin{aligned} {}^q\vec{L}^2{}_3 &= b_1^{-1} b_2^{-1} \left(\sum_{m_1,m_2} {}_qC \begin{pmatrix} 1 & 1 & 0 \\ m_1 & m_2 & 0 \end{pmatrix} (t^1_{m_1} t^1_{m_2} \otimes \mathrm{id} \otimes \mathrm{id}) \right) b_2 \, b_1 \\ &= b_1^{-1} b_2^{-1} \left({}^q\vec{L}^2{}_1 \right) b_2 \, b_1 = b_1^{-1} \left({}^q\vec{L}^2{}_2 \right) \, b_1 \, . \end{aligned} \tag{4.182} $$

Now, let us start with the simplest case. That is, we consider the length operator ${}^{q}\vec{L}^{2}{}_{1}$ acting on the first outwardly oriented leg of a 3-valent intertwiner $|i_{123}\rangle$. In this case the orientation of the last two legs has no influence and it would not change the result if we where to consider, e.g., $|i_{12^*3}\rangle$, $|i_{123^*}\rangle$ or $|i_{12^*3^*}\rangle$, where legs with an inward orientation are labeled by a dual vector j^*, which is, obviously, just a convention. However, it tells us already that outwardly oriented edges transform under the irreducible representation and the inwardly oriented edges under the corresponding dual action.

As a reminder, we consider spin networks, i.e., states that are gauge invariant with respect to the Gauss constraint (the generator of local $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$ transformations). This means, that we have to consider states that transform like scalars. Since the scalar representation of $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$ is given by the counit ε we know that a scalar object $\mathcal{O}$ must transform like

$$\xi \triangleright \mathcal{O} = \varepsilon(\xi)\, \mathcal{O}\,. \tag{4.183}$$

This means in particular that any n-valent intertwiner, independent of the orientation of any of its legs, must satisfy

$$E \triangleright |i_{12\cdots n}\rangle \stackrel{!}{=} \varepsilon(E)\, |i_{12\cdots n}\rangle = 0 = \varepsilon(F)\, |i_{12\cdots n}\rangle \stackrel{!}{=} F \triangleright |i_{12\cdots n}\rangle\,, \tag{4.184}$$

$$G^{\pm 1} \triangleright |i_{12\cdots n}\rangle \stackrel{!}{=} \varepsilon(G^{\pm 1})\, |i_{12\cdots n}\rangle = |i_{12\cdots n}\rangle\,. \tag{4.185}$$

If we consider an arbitrary 3-valent state with three outwardly oriented legs, the action of $\mathcal{U}_{\mathrm{q}}^{\mathrm{res}}(\mathfrak{sl}(2,\mathbb{C}))$ on it is given by

$$\xi \triangleright |1(23)\rangle = \tag{4.186}$$

$$\sum_{m_1,m_2,m_3} f^{j_1 j_2 j_3}_{m_1 m_2 m_3}\, (\pi^{(j_1)} \otimes (\pi^{(j_2)} \otimes \pi^{(j_3)}))(\mathrm{id} \otimes \Delta)\Delta(\xi)\, |j_1, m_1\rangle \bar{\otimes} \left(|j_2, m_2\rangle \bar{\otimes} |j_3, m_3\rangle\right)\,.$$

If we consider states with reversed orientation, for instance for the third leg, than the action is given by

$$\xi \triangleright |1(23^*)\rangle = \tag{4.187}$$

$$\sum_{m_1,m_2,m_3} \tilde{f}^{j_1 j_2 j_3}_{m_1 m_2 m_3}\, (\pi^{(j_1)} \otimes (\pi^{(j_2)} \otimes \pi^{(j_3),*}))(\mathrm{id} \otimes \Delta)\Delta(\xi)\, |j_1, m_1\rangle \bar{\otimes} \left(|j_2, m_2\rangle \bar{\otimes} \langle j_3, m_3|\right)\,.$$

Note the change of $\pi^{(j_3)}(\xi^{(3)})$ to the dual representation $\pi^{(j_3)*}(\xi^{(3)}) = \pi^{(j_3)}(S(\xi^{(3)}))^t$. One can use these equations, together with Eq.(4.184) and Eq.(4.185) to determine the coefficients $f^{j_1 j_2 j_3}_{m_1 m_2 m_3}$ and $\tilde{f}^{j_1 j_2 j_3}_{m_1 m_2 m_3}$ in Eq.(4.186) and Eq.(4.187).

A different way to determine the intertwiner states is to map a state in the triple tensor product onto its spin-0 irreducible subspace with $\langle 0,0|$, where $\langle 0,0|$ corresponds to the dual as defined by the standard scalar product such that $\langle J,M|K,N\rangle = \delta_{JK}\delta_{MN}$. This means for the standard state $|1(23)\rangle$ that we have to consider

$$\begin{aligned}
\langle 0,0|j_1,m_1\rangle\bar{\otimes}\left(|j_2,m_2\rangle\bar{\otimes}|j_3,m_3\rangle\right) &= \sum_{\substack{J,M\\ \text{adm.}}} {}_qC\begin{pmatrix} j_2 & j_3 & J\\ m_2 & m_3 & M\end{pmatrix}\langle 0,0|j_1,m_1\rangle\bar{\otimes}|J,M\rangle\\
&= \sum_{\substack{J,M\\ \text{adm.}}}\sum_{\substack{K,N\\ \text{adm.}}} {}_qC\begin{pmatrix} j_2 & j_3 & J\\ m_2 & m_3 & M\end{pmatrix} {}_qC\begin{pmatrix} j_1 & J & K\\ m_1 & M & N\end{pmatrix}\langle 0,0|K,N\rangle\\
&= \sum_{\substack{J,M\\ \text{adm.}}} {}_qC\begin{pmatrix} j_2 & j_3 & J\\ m_2 & m_3 & M\end{pmatrix} {}_qC\begin{pmatrix} j_1 & J & 0\\ m_1 & M & 0\end{pmatrix}\\
&= \sum_{\substack{J,M\\ \text{adm.}}} {}_qC\begin{pmatrix} j_2 & j_3 & J\\ m_2 & m_3 & M\end{pmatrix}\delta_{j_1,J}\delta_{m_1,-M}\,\frac{(-1)^{j_1-m_1}q^{m_1}}{\sqrt{[2j_1+1]_q}}\\
&= {}_qC\begin{pmatrix} j_2 & j_3 & j_1\\ m_2 & m_3 & -m_1\end{pmatrix}\frac{(-1)^{j_1-m_1}q^{m_1}}{\sqrt{[2j_1+1]_q}}\,.
\end{aligned} \tag{4.188}$$

For $\langle 0,0|\left(|j_1,m_1\rangle\bar{\otimes}|j_2,m_2\rangle\right)\bar{\otimes}|j_3,m_3\rangle$ one gets similarly

$$\langle 0,0|\left(|j_1,m_1\rangle\bar{\otimes}|j_2,m_2\rangle\right)\bar{\otimes}|j_3,m_3\rangle = {}_qC\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & -m_3\end{pmatrix}\frac{(-1)^{j_3+m_3}q^{-m_3}}{\sqrt{[2j_3+1]_q}}\,. \tag{4.189}$$

Now, if we want to consider states which have inwardly oriented legs, e.g., $|1(23^*)\rangle$, we have to consider $\langle 0,0|j_1,m_1\rangle\bar{\otimes}\left(|j_2,m_2\rangle\bar{\otimes}\langle j_3,m_3|\right)$. In order to get the correct dual $\langle J,M|$ from $|J,M\rangle$ we need an invariant bilinear form $\mathcal{B}$ on V^J. Then we can define $\langle J,M| \equiv \mathcal{B}(|J,M\rangle,\cdot)$. In fact, the Clebsch-Gordan coefficients allow us to construct such an invariant bilinear form explicitly. Note that the Clebsch-Gordan coefficients of the decomposition $|j_1,m_1\rangle\bar{\otimes}|j_2,m_2\rangle \to |J,M\rangle$ are the components of the canonical isomorphism given by [87]

$$\mathrm{Hom}(V^{j_1}\otimes V^{j_2}, V^J) \cong \mathrm{Hom}(V^{j_1}, V^J\otimes (V^{j_2})^*)\,. \tag{4.190}$$

This isomorphism allows us to give an invariant map between V^{j_1} and $(V^{j_2})^*$ by setting

$J = 0$ and using $\mathrm{Hom}(V^{j_1} \otimes V^{j_2}, \mathbb{C}) \cong \mathrm{Hom}(V^{j_1}, (V^{j_2})^*)$. One can write this explicitly as

$$\langle j_2, m_2| = \sum_{m_1} {}_qC \begin{pmatrix} j_1 & j_2 & 0 \\ m_1 & m_2 & 0 \end{pmatrix} |j_1, m_1\rangle = \delta_{j_1,j_2} \frac{(-1)^{j_1+m_2}\, q^{-m_2}}{\sqrt{[2j_1+1]_q}} |j_1, -m_2\rangle$$
$$= \frac{(-1)^{j_2+m_2}\, q^{-m_2}}{\sqrt{[2j_2+1]_q}} |j_2, -m_2\rangle\,. \tag{4.191}$$

In the opposite direction one can use similarly $\mathrm{Hom}(V^J, V^{j_1} \otimes V^{j_2}) \cong \mathrm{Hom}((V^{j_1})^* \otimes V^J, V^{j_2})$, which gives with $J = 0$ that $\mathrm{Hom}(\mathbb{C}, V^{j_1} \otimes V^{j_2}) \cong \mathrm{Hom}((V^{j_1})^*, V^{j_2})$. Thus, we can define for example

$$|j_2, m_2\rangle = \sum_{m_1} {}_qC \begin{pmatrix} j_1 & j_2 & 0 \\ m_1 & m_2 & 0 \end{pmatrix} \langle j_1, m_1|\,. \tag{4.192}$$

Note, however, that the maps Eq.(4.191) and Eq.(4.192) need to be properly normalized to be inverse with respect to each other[6]. With the following normalizations those two maps become inverses of each other,

$$\langle j_2, m_2| = (-1)^{-j_2}\sqrt{[2j_2+1]_q}\, \sum_{m_1} {}_qC \begin{pmatrix} j_1 & j_2 & 0 \\ m_1 & m_2 & 0 \end{pmatrix} |j_1, m_1\rangle = (-1)^{m_2} q^{-m_2}\, |j_2, -m_2\rangle \tag{4.193}$$

and

$$|j_2, m_2\rangle = (-1)^{-j_2}\sqrt{[2j_2+1]_q}\, \sum_{m_1} {}_qC \begin{pmatrix} j_1 & j_2 & 0 \\ m_1 & m_2 & 0 \end{pmatrix} \langle j_1, m_1| = (-1)^{m_2} q^{-m_2}\, \langle j_2, -m_2|\,. \tag{4.194}$$

[6]This results from the fact that for $J = 0$ we did not write explicitly the functions coming from $V^0 = \mathbb{C}$, which is not the identity function, in general. We also point out that the dual basis $\langle j, m|$ of the dual space $(V^j)^*$, constructed using the Clebsch-Gordan coefficients, despite the notation, is not the same dual basis that is obtained from the scalar product on V^j via $\langle j, m| \equiv \langle |j, m\rangle | \cdot \rangle_{V^j}$.

Using these expressions we can write for $|1(23^*)\rangle$

$$
\begin{aligned}
\langle 0,0|j_1,m_1\rangle \bar{\otimes} \left(|j_2,m_2\rangle \bar{\otimes} \langle j_3,m_3|\right) &= \langle 0,0|j_1,m_1\rangle \bar{\otimes} \Big(|j_2,m_2\rangle \bar{\otimes} (-1)^{m_3}\, q^{-m_3}\, |j_3,-m_3\rangle\Big) \\
&= {}_qC \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & -m_3 & -m_1 \end{pmatrix} (-1)^{m_3}\, q^{-m_3} \frac{(-1)^{j_1-m_1} q^{m_1}}{\sqrt{[2j_1+1]_q}} \\
&= {}_qC \begin{pmatrix} j_3 & j_2 & j_1 \\ m_3 & -m_2 & m_1 \end{pmatrix} \frac{(-1)^{j_1-m_1+m_3}\, q^{m_1-m_3}}{\sqrt{[2j_1+1]_q}} \\
&= {}_qC \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \frac{(-1)^{j_3}}{\sqrt{[2j_3+1]_q}}\,. \qquad (4.195)
\end{aligned}
$$

Note, that for a 3-valent node all these spin-0 states are equivalent, because in this case the dimension of the invariant subspace of any type of triple tensor product space, independent of the orientations involved, is always of dimension one. One can show for example explicitly, using the symmetries of the Clebsch-Gordan coefficients, that Eq.(4.188) and Eq.(4.189) are the same.

Now we will calculate ${}^q\vec{L}^2{}_1 \triangleright |i_{1(23)}\rangle$ and ${}^q\vec{L}^2{}_1 \triangleright |i_{1^*(23)}\rangle$. In those cases we will not need the braiding yet. The details of the calculations can be found in appendix B.1. By the definition of the length operator Eq.(4.177) we get

$$
\begin{aligned}
{}^q\vec{L}^2{}_1 \triangleright |i_{1(23)}\rangle &= \sum_{k_1,k_2} {}_qC \begin{pmatrix} 1 & 1 & 0 \\ k_1 & k_2 & 0 \end{pmatrix} {}^{(1)}t^1{}_{k_1}\; {}^{(1)}t^1{}_{k_2} \triangleright |i_{1(23)}\rangle \\
&= \sum_{k_1,k_2} \sum_{m_1,m_2,m_3} {}_qC \begin{pmatrix} 1 & 1 & 0 \\ k_1 & k_2 & 0 \end{pmatrix} \frac{(-1)^{j_3+m_3} q^{-m_3}}{\sqrt{[2j_3+1]_q}}\, {}_qC \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} \\
&\quad \times \quad t^1{}_{k_1} t^1{}_{k_2}\, |j_1,m_1\rangle \left(|j_2,m_2\rangle |j_3,m_3\rangle\right)\,. \qquad (4.196)
\end{aligned}
$$

Using now the Wigner-Eckart theorem and certain symmetries of the Clebsch-Gordan coefficients we can write

$$t^1{}_{k_1} t^1{}_{k_2} \, |j_1, m_1\rangle = \sum_{n_1,n_2=-j_1}^{j_1} N(j_1,j_1,1)^2 {}_qC \begin{pmatrix} 1 & j_1 & j_1 \\ k_1 & n_2 & n_1 \end{pmatrix} {}_qC \begin{pmatrix} 1 & j_1 & j_1 \\ k_2 & m_1 & n_2 \end{pmatrix} \, |j_1, n_1\rangle$$

$$= N(j_1,j_1,1)^2 \sum_{n_1,n_2=-j_1}^{j_1} (-1)^{-k_1} q^{-k_1} {}_qC \begin{pmatrix} 1 & j_1 & j_1 \\ k_1 & n_2 & n_1 \end{pmatrix} {}_qC \begin{pmatrix} 1 & j_1 & j_1 \\ k_1 & n_2 & m_1 \end{pmatrix} \, |j_1, n_1\rangle \, . \quad (4.197)$$

Combining Eq.(4.196) and Eq.(4.197) we finally find

$$ {}^q\vec{L}^2{}_1 \triangleright |i_{1(23)}\rangle = \frac{N(j_1,j_1,1)^2}{\sqrt{[3]_q}} \sum_{k_1} \sum_{m_1,m_2,m_3} \sum_{n_1,n_2} (-1)^{1-2k_1} \frac{(-1)^{j_3+m_3} q^{-m_3}}{\sqrt{[2j_3+1]_q}} {}_qC \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}$$

$$\times \quad {}_qC \begin{pmatrix} 1 & j_1 & j_1 \\ k_1 & n_2 & n_1 \end{pmatrix} {}_qC \begin{pmatrix} 1 & j_1 & j_1 \\ k_1 & n_2 & m_1 \end{pmatrix} \, |j_1, n_1\rangle \left(|j_2, m_2\rangle |j_3, m_3\rangle \right)$$

$$= -\frac{N(j_1,j_1,1)^2}{\sqrt{[3]_q}} \sum_{m_1,m_2,m_3} \frac{(-1)^{j_3+m_3} q^{-m_3}}{\sqrt{[2j_3+1]_q}} {}_qC \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} \, |j_1, m_1\rangle \left(|j_2, m_2\rangle |j_3, m_3\rangle \right)$$

$$= -\frac{N(j_1,j_1,1)^2}{\sqrt{[3]_q}} \, |i_{1(23)}\rangle \, . \quad (4.198)$$

In order to calculate now ${}^q\vec{L}^2{}_1 \triangleright |i_{1^*(23)}\rangle$ with the reversed orientation of the first link, we have to make sure to use the right duality. The details are found again in appendix B.1. We get

$$ {}^q\vec{L}^2{}_1 \triangleright |i_{1^*(23)}\rangle = \sum_{k_1,k_2} {}_qC \begin{pmatrix} 1 & 1 & 0 \\ k_1 & k_2 & 0 \end{pmatrix} {}^{(1)}t^1{}_{k_1} \; {}^{(1)}t^1{}_{k_2} \triangleright |i_{1^*(23)}\rangle$$

$$= \sum_{k_1,k_2} \sum_{m_1,m_2,m_3} {}_qC \begin{pmatrix} 1 & 1 & 0 \\ k_1 & k_2 & 0 \end{pmatrix} \frac{(-1)^{j_1+3m_1} q^{-3m_1}}{\sqrt{[2j_1+1]_q}} {}_qC \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix}$$

$$\times \quad t^1{}_{k_1} t^1{}_{k_2} \, |j_1, -m_1\rangle \left(|j_2, m_2\rangle |j_3, m_3\rangle \right) \quad (4.199)$$

and after a similar calculation as before we obtain

$$ {}^q\vec{L}^2{}_1 \triangleright |i_{1^*(23)}\rangle = -\frac{N(j_1,j_1,1)^2}{\sqrt{[3]_q}} \, |i_{1^*(23)}\rangle \, . \quad (4.200)$$

Thus, we see that with the properly normalized duality map Eq.(4.193) and Eq.(4.194) we get the same result, regardless of whether we take the first leg inwardly or outwardly oriented. This is of course the result we would expect, however, it depends on the correct choice of the dual vectors.

If we want to calculate the length operator acting on the two remaining legs, we have to use the braiding, as should be clear from the expressions Eq.(4.179) and Eq.(4.180). Since we just saw that the change of orientation does not change the result of the length operator we will consider only outwardly oriented legs now. Schematically, the braiding action of the first (inverse) braid operator is given by

$$b_1^{-1} \triangleright |a\rangle \otimes (|b\rangle \otimes |c\rangle) = (\mathrm{id} \otimes T^-) \triangleright |a\rangle \otimes (|b\rangle \otimes |c\rangle) = |a\rangle \otimes (|c\rangle \otimes |b\rangle) \tag{4.201}$$

and for the second (inverse) braid operator we have

$$\begin{aligned} b_2^{-1} \triangleright |a\rangle \otimes (|b\rangle \otimes |c\rangle) &= \Phi\,(T^- \otimes \mathrm{id})\,\Phi^{-1} \triangleright |a\rangle \otimes (|b\rangle \otimes |c\rangle) \\ &= \Phi\,(T^- \otimes \mathrm{id}) \triangleright (|a\rangle \otimes |b\rangle) \otimes |c\rangle \\ &= \Phi \triangleright (|b\rangle \otimes |a\rangle) \otimes |c\rangle \\ &= |b\rangle \otimes (|a\rangle \otimes |c\rangle) \ . \end{aligned} \tag{4.202}$$

Combining those two we get

$$b_2^{-1}\, b_1^{-1} \triangleright |a\rangle \otimes (|b\rangle \otimes |c\rangle) = b_2^{-1} \triangleright |a\rangle \otimes (|c\rangle \otimes |b\rangle) = |c\rangle \otimes (|a\rangle \otimes |b\rangle) \ . \tag{4.203}$$

Thus, we find with Eq.(4.182) and Eq.(4.200) that, not surprisingly,

$${}^q\vec{L}^2{}_2 \triangleright |i_{1(23)}\rangle = -\frac{(N(j_2, j_2, 1))^2}{\sqrt{[3]_q}}\,|i_{1(23)}\rangle = {}^q\vec{L}^2{}_2 \triangleright |i_{1(2^*3)}\rangle \,, \tag{4.204}$$

$${}^q\vec{L}^2{}_3 \triangleright |i_{1(23)}\rangle = -\frac{(N(j_3, j_3, 1))^2}{\sqrt{[3]_q}}\,|i_{1(23)}\rangle = {}^q\vec{L}^2{}_3 \triangleright |i_{1(23^*)}\rangle \,. \tag{4.205}$$

Now, what are those reduced matrix elements $N(j, j, 1)$? They can be determined by using the Wigner-Eckart theorem and the explicit expression for the corresponding

Clebsch-Gordan coefficient, calculated from Eq.(4.61), on the one hand and the action of the components of our vector operator Eq.(4.155) on the other hand in the equation

$$\langle j_1, m_1 | t^1_M | j_2, m_2 \rangle = N(j_1, j_2, 1)\ {}_qC \begin{pmatrix} 1 & j_2 & j_1 \\ M & m_2 & M + m_2 \end{pmatrix} , \tag{4.206}$$

where in the vector case we actually have $j_1 = j_2$. We then find the following

$$N(j, j, 1) = \alpha\, q^{-\frac{1}{2}} \sqrt{\frac{[2j]_q [2j+2]_q}{[2]_q}} = -\frac{\sqrt{[2j]_q [2j+2]_q}}{[2]_q} , \tag{4.207}$$

where we used the specific value for α found in Eq.(4.159) in the second equality. If we rescale now our length operator by the overall $-\sqrt{[3]_q}$ - factor, we finally find

$${}^q\vec{L}^2{}_1 \triangleright |i_{1(23)}\rangle = N(j_1, j_1, 1)^2\, |i_{1(23)}\rangle = \frac{[2j_1]_q [2j_1 + 2]_q}{[2]_q^2}\, |i_{1(23)}\rangle . \tag{4.208}$$

and equivalently for the other two legs and the opposite orientations. We see that in the limit $q \to 1$ this gives the result we know from standard Loop quantum gravity, i.e., $\vec{L}_1^2 = j_1(j_1 + 1)$. Furthermore, let us mention again that this result holds regardless of whether we consider 3D or 4D Loop quantum gravity, i.e., if we were to consider a 4-valent intertwiner and associate the four spin labels of the links with the dual areas of a tetrahedron, we would still obtain the above result for this quantum operator.

4.4.2 Angle operator

In this section we will calculate the action of the so-called angle operator on a 3-valent intertwiner. This operator acts on two legs of the node and will require the full braiding machinery. The q-deformed angle operator, acting on the legs i and j, is defined in terms of two vector operators as

$${}^qW_{ij} \equiv {}^{(i)}t \cdot {}^{(j)}t = \sum_{m_1, m_2} {}_qC \begin{pmatrix} 1 & 1 & 0 \\ m_1 & m_2 & 0 \end{pmatrix} {}^{(i)}t^1{}_{m_1}\, {}^{(j)}t^1{}_{m_2} , \tag{4.209}$$

where the tensor operators ${}^{(i)}t^1$ for the leg i are defined again via Eq.(4.178) - Eq.(4.180). By the contraction of the vector operators with this particular Clebsch-Gordan coefficient we are certain to have a gauge-invariant operator at hand.

We want to consider specifically ${}^qW_{12} \triangleright |i_{1(23)}\rangle$, because, for example, for ${}^qW_{12} \triangleright |i_{(12)3}\rangle$ we would not need the coassociator in the braiding and the calculation is the same as in the Hopf case.

Now, we will see that in the case of a 3-valent node, decorated with an intertwiner state, there are some simplifications concerning the braiding, following essentially from the fact that the invariant subspace is at most 1-dimensional. Furthermore, we will see that it is very beneficial for the calculation to work in the coupled basis. We denote the basis vectors for $(V^{j_1}\bar{\otimes}V^{j_2})\,\bar{\otimes}V^{j_3}$ and $V^{j_1}\bar{\otimes}\,(V^{j_2}\bar{\otimes}V^{j_3})$, respectively, by

$$e_M^{j_1j_2,j_{12},j_3,J} = \sum_{m_i,m_{12}} {}_qC\begin{pmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12}\end{pmatrix} {}_qC\begin{pmatrix} j_{12} & j_3 & J \\ m_{12} & m_3 & M\end{pmatrix} |j_1,m_1\rangle|j_2,m_2\rangle|j_3,m_3\rangle \tag{4.210}$$

and

$$e_M^{j_1,j_2j_3,j_{23},J} = \sum_{m_i,m_{12}} {}_qC\begin{pmatrix} j_1 & j_{23} & J \\ m_1 & m_{23} & M\end{pmatrix} {}_qC\begin{pmatrix} j_2 & j_3 & j_{23} \\ m_2 & m_3 & m_{23}\end{pmatrix} |j_1,m_1\rangle|j_2,m_2\rangle|j_3,m_3\rangle\,. \tag{4.211}$$

Because of the truncation of the tensor product at q root of unity J takes values only in the range of admissible spins. As we have discussed before, those two basis vectors are connected by the q-Racah coefficients via

$$e_M^{j_1j_2,j_{12},j_3,J} = \sum_{j_{23}} {}_qR(j_1,j_2,j_3;j_{12},j_{23};J)\; e_M^{j_1,j_2j_3,j_{23},J}\,, \tag{4.212}$$

$$e_M^{j_1,j_2j_3,j_{23},J} = \sum_{j_{12}} {}_qR(j_1,j_2,j_3;j_{12},j_{23};J)\; e_M^{j_1j_2,j_{12},j_3,J}\,. \tag{4.213}$$

For the gauge invariant space of intertwiners we have to consider $J=0$. In this case, however, one finds that the q-Racah coefficients are non-zero only when $j_{23}=j_1$ and $j_{12}=j_3$, which is clear because we must have $|j_1-j_{23}|=J=0$ and $|j_{12}-j_3|=J=0$. In fact, we have

$${}_qR(j_1,j_2,j_3;j_{12},j_{23};0) = \delta_{j_1,j_{23}}\,\delta_{j_{12},j_3} \tag{4.214}$$

and thus

$$e_0^{j_1j_2,j_{12},j_3,0} = \sum_{j_{23}} {}_qR(j_1,j_2,j_3;j_{12},j_{23};0)\; e_0^{j_1,j_2j_3,j_{23},0} = \delta_{j_{12},j_3}\; e_0^{j_1,j_2j_3,j_1,0}\,, \tag{4.215}$$

$$e_0^{j_1,j_2j_3,j_{23},0} = \sum_{j_{12}} {}_qR(j_1,j_2,j_3;j_{12},j_{23};0)\; e_0^{j_1j_2,j_{12},j_3,0} = \delta_{j_1,j_{23}}\; e_0^{j_1j_2,j_3,j_3,0}\,. \tag{4.216}$$

Hence, we see that on the gauge invariant level for a 3-valent node the choice of brackets for the triple tensor product does not matter and we have $e_0^{j_1 j_2, j_3, j_3, 0} = e_0^{j_1 j_2 j_3, j_1, 0}$. This simplifies in particular the action of the braiding, because now we have

$$\Phi^{-1} \triangleright e_0^{j_1, j_2 j_3, j_1, 0} = e_0^{j_1 j_2, j_3, j_3, 0} \quad , \quad \Phi \triangleright e_0^{j_1 j_2, j_3, j_3, 0} = e_0^{j_1, j_2 j_3, j_1, 0} . \tag{4.217}$$

Note, however, that this does not imply that we can forget about the coassociator when calculating with general tensor operators. The reason is that intermediate states, after acting with a tensor operator for example, are not necessarily invariant $J = 0$ states and thus the braiding becomes more complicated, as we will see below.

Let us first consider

$$^{(2)}t^1{}_{k_2} \triangleright |i_{1(23)}\rangle = b_2^{-1} \, {}^{(1)}t^1{}_{k_2} \, b_2 \triangleright |i_{1(23)}\rangle = b_2^{-1} \, (t^1{}_{k_2} \otimes \mathrm{id} \otimes \mathrm{id}) \, b_2 \triangleright |i_{1(23)}\rangle . \tag{4.218}$$

We have

$$b_2 \triangleright e_M^{j_1, j_2 j_3, j_{23}, J} = \sum_{j_{12}, j_{13}} (-1)^{j_1 + j_2 - j_{12}} \, q^{c_{j_{12}} - c_{j_1} - c_{j_2}} \, {}_qR(j_1, j_2, j_3; j_{12}, j_{23}; J) \tag{4.219}$$

$$\times \quad {}_qR(j_2, j_1, j_3; j_{12}, j_{13}; J) \, e_M^{j_2, j_1 j_3, j_{13}, J} ,$$

which gives for $J = 0 = M$, together with

$${}_qR(j_1, j_2, j_3; j_{12}, j_{23}; 0) = 1 = {}_qR(j_2, j_1, j_3; j_{12}, j_{13}; 0) , \tag{4.220}$$

the following

$$b_2 \triangleright |i_{1(23)}\rangle = (-1)^{j_1 + j_2 - j_3} \, q^{c_{j_3} - c_{j_1} - c_{j_2}} \, |i_{2(13)}\rangle . \tag{4.221}$$

Next, one finds, using the Wigner-Eckart theorem,

$$^{(1)}t^1{}_{k_2} \triangleright |i_{2(13)}\rangle = -\frac{N(j_2, j_2, 1)}{\sqrt{[3]_q}} \, e_{k_2}^{j_2, j_1 j_3, j_2, 1} . \tag{4.222}$$

Now, calculating $b_2^{-1} \triangleright e_{k_2}^{j_2, j_1 j_3, j_2, 1}$, we see that the braiding is highly non-trivial, even though we started from a simple intertwiner state, because

$$^{(2)}t^1{}_{k_2} \triangleright |i_{1(23)}\rangle = -\frac{N(j_2, j_2, 1)}{\sqrt{[3]_q}} \sum_{j_{21}, j_{23}} (-1)^{j_{21} - j_3} q^{c_{j_3} - c_{j_{21}}} \, {}_qR(j_2, j_1, j_3; j_{21}, j_2; 1) \tag{4.223}$$

$$\times \quad {}_qR(j_1, j_2, j_3; j_{21}, j_{23}; 1) \, e_{k_2}^{j_1, j_2 j_3, j_{23}, 1} .$$

Lastly, we have to calculate ${}^{(1)}t^1{}_{k_1} \triangleright e_{k_2}^{j_1,j_2 j_3,j_{23},1}$ and collecting everything together, using Eq.(4.209), gives after a lengthy calculation

$$
\begin{aligned}
{}^{q}W_{12} \triangleright |i_{1(23)}\rangle = &- \frac{N(j_1,j_1,1)N(j_2,j_2,1)}{\sqrt{[3]_q}} \left([2j_1+1]_q[2j_2+1]_q\right)^{\frac{1}{2}} \\
&\times \quad (-1)^{j_1+j_2+j_3}\, q^{-2} \begin{Bmatrix} j_1 & j_3 & j_2 \\ j_2 & 1 & j_1 \end{Bmatrix}_q |i_{1(23)}\rangle \\
= &\frac{N(j_1,j_1,1)N(j_2,j_2,1)}{\sqrt{[3]_q}}\, q^{-2}\, {}_qR(j_1,j_3,j_2;j_2,j_1;1)\, |i_{1(23)}\rangle\,.
\end{aligned} \tag{4.224}
$$

This result is exactly the same as in the q real case, however, completely calculated in the weak quasi-Hopf setting for q root of unity.

We can rewrite the 6j-symbol above using

$$
\begin{Bmatrix} j_1 & j_3 & j_2 \\ j_2 & 1 & j_1 \end{Bmatrix}_q = \begin{Bmatrix} j_2 & 1 & j_2 \\ j_1 & j_3 & j_1 \end{Bmatrix}_q = \begin{Bmatrix} j_2 & j_2 & 1 \\ j_1 & j_1 & j_3 \end{Bmatrix}_q = \begin{Bmatrix} j_1 & j_1 & 1 \\ j_2 & j_2 & j_3 \end{Bmatrix}_q\,. \tag{4.225}
$$

From [81] we have the following explicit expression for the quantum 6j-symbol[7]

$$
\begin{aligned}
\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}_q = &\frac{(-1)^a\, \Delta(j_1,j_2,j_{12})\Delta(j_1,j_{23},j)\Delta(j_2,j_3,j_{23})\Delta(j_{12},j_3,j)}{[j+j_3-j_{12}]_q![j_1-j_2+j_{12}]_q![j_2-j_1+j_{12}]_q![j_1-j_{23}+j]_q!} \\
\times \quad &\frac{[j_{12}+j_3+j+1]_q![j_1+j+j_{23}+1]_q!}{[j_2-j_3+j_{23}]_q![j_3-j_2+j_{23}]_q!} \sum_s \frac{(-1)^s\,[2j_2-s]_q![j_1-j_2+j_{12}+s]_q!}{[s]_q![j_1+j_2-j_{12}-s]_q!} \\
\times \quad &\frac{[j_3+j_{23}-j_2+s]_q!}{[j_2+j_3-j_{23}-s]_q![j_{23}+j_{12}-j_2-j+s]_q![j_{23}+j_{12}-j_2+j+s+1]_q!}\,,
\end{aligned} \tag{4.226}
$$

where the sum over s goes over all integer values such that the q-factorials are non-negative, $a = j - j_1 + j_2 + j_3$ and

$$
\Delta(j_1,j_2,j_{12}) = \left(\frac{[j_1+j_2-j_3]_q[j_1-j_2+j_3]_q[j_2-j_1+j_3]_q}{[j_1+j_2+j_3+1]_q}\right)^{\frac{1}{2}}\,. \tag{4.227}
$$

[7] Instead of $a = j - j_1 + j_2 + j_3$, which is used in [81], we consider a different phase, namely $a = j + j_1 + j_2 + j_3$, which matches the definition of the quantum 6j-symbol used in [85].

Using the symmetries Eq.(4.225) and Eq.(4.226) gives, with our changed phase factor, i.e., $a = j + j_1 + j_2 + j_3$ instead of $a = j - j_1 + j_2 + j_3$,

$$\begin{Bmatrix} j_1 & j_3 & j_2 \\ j_2 & 1 & j_1 \end{Bmatrix}_q = \begin{Bmatrix} j_1 & 1 & j_1 \\ j_2 & j_3 & j_2 \end{Bmatrix}_q \tag{4.228}$$

$$= (-1)^{j_1+j_2+j_3} \frac{[2j_1]_q[2j_2]_q - [2]_q[j_1+j_2-j_3]_q[j_1+j_2+j_3+1]_q}{\sqrt{[2j_1]_q[2j_1+1]_q[2j_1+2]_q[2j_2]_q[2j_2+1]_q[2j_2+2]_q}} .$$

Thus, after rescaling again by the overall factor $-\sqrt{[3]_q}$, we get

$$-\sqrt{[3]_q}\ {}^qW_{12} \triangleright |i_{1(23)}\rangle = N(j_1,j_1,1)\, N(j_2,j_2,1)\, ([2j_1+1]_q[2j_2+1]_q)^{\frac{1}{2}} \tag{4.229}$$

$$\times \quad (-1)^{j_1+j_2+j_3}\, q^{-2} \begin{Bmatrix} j_1 & j_3 & j_2 \\ j_2 & 1 & j_1 \end{Bmatrix}_q |i_{1(23)}\rangle$$

$$= N(j_1,j_1,1)\, N(j_2,j_2,1)\, ([2j_1+1]_q[2j_2+1]_q)^{\frac{1}{2}}\ q^{-2}$$

$$\times \quad \frac{[2j_1]_q[2j_2]_q - [2]_q[j_1+j_2-j_3]_q[j_1+j_2+j_3+1]_q}{\sqrt{[2j_1]_q[2j_1+1]_q[2j_1+2]_q[2j_2]_q[2j_2+1]_q[2j_2+2]_q}}\ |i_{1(23)}\rangle$$

$$= q^{-2}\ \frac{[2j_1]_q[2j_2]_q - [2]_q[j_1+j_2-j_3]_q[j_1+j_2+j_3+1]_q}{[2]_q^2}\ |i_{1(23)}\rangle \tag{4.230}$$

where we used Eq.(4.207), with the specific value for α from Eq.(4.159), in the last step. In the limit $q \to 1$ we find that Eq.(4.230) becomes

$$-\sqrt{[3]_q}\ {}^qW_{12} \triangleright |i_{1(23)}\rangle \overset{q\to 1}{=} -\frac{1}{2}\left(j_1(j_1+1) + j_2(j_2+1) - j_3(j_3+1)\right). \tag{4.231}$$

If we consider the $q \to 1$ limit of the 6j-symbol directly, we get

$$\begin{Bmatrix} j_1 & 1 & j_1 \\ j_2 & j_3 & j_2 \end{Bmatrix}_q \overset{q\to 1}{=} 2\,(-1)^{j_1+j_2+j_3} \frac{j_3(j_3+1) - j_1(j_1+1) - j_2(j_2+1)}{\sqrt{(2j_1)(2j_1+1)(2j_1+2)(2j_2)(2j_2+1)(2j_2+2)}} \tag{4.232}$$

and we obtain the following limit

$$-([2j_1+1]_q[2j_2+1]_q)^{\frac{1}{2}}\,(-1)^{j_1+j_2+j_3}\,q^{-2}\,\begin{Bmatrix} j_1 & 1 & j_1 \\ j_2 & j_3 & j_2 \end{Bmatrix}_q$$

$$\overset{q\to 1}{=} \quad \frac{j_1(j_1+1)+j_2(j_2+1)-j_3(j_3+1)}{2\sqrt{j_1(j_1+1)\,j_2(j_2+1)}}\,. \tag{4.233}$$

As mentioned before, in Loop quantum gravity with $\Lambda = 0$, which corresponds to the limit $q \to 1$, we can identify the length of a link with $L_i = \sqrt{j_i(j_i+1)}$. Hence, Eq.(4.233) should be compared with the classical (flat) law of cosines

$$\cos(\theta_c) = \frac{a^2+b^2-c^2}{2ab}\,, \tag{4.234}$$

where θ_i is the angle opposite the link i. Thus, we see that, up to the reduced matrix elements $N(j_1, j_1, 1)$ and $N(j_2, j_2, 1)$, in the $q \to 1$ limit the operator $\sqrt{[3]_q}\;{}^qW_{12}$, note the dropped minus sign, reproduces the flat law of cosines Eq.(4.234).

Now, recall the classical (spherical) law of cosines for a triangle on the 2-sphere

$$\cos(\theta_c) = \frac{\cos(c)-\cos(a)\cos(b)}{\sin(a)\sin(b)}\,, \tag{4.235}$$

which gives back Eq.(4.234) for short arc lengths a, b, c, relative to the radius or the sphere, via $\sin(a)\sin(b) \approx ab$ and $\cos(c) \approx 1 - c^2/2$, etc. We will see now that we do not have to consider the $q \to 1$ limit of our angle operator, to relate it with this classical geometry. With $q = e^{\frac{2\pi i}{l}}$ we can write

$$[2j_1]_q[2j_2]_q - [2]_q[j_1+j_2-j_3]_q[j_1+j_2+j_3+1]_q = \tag{4.236}$$

$$= \frac{(q^{2j_1}-q^{-2j_1})(q^{2j_2}-q^{-2j_2}) - [2]_q(q^{j_1+j_2-j_3}-q^{-j_1-j_2+j_3})(q^{j_1+j_2+j_3+1}-q^{-j_1-j_2-j_3-1})}{(q-q^{-1})^2}$$

$$= 4\,(q-q^{-1})^{-2}\left[\cos\left(\frac{2\pi}{l}\right)\cos\left(\frac{2\pi}{l}(2j_3+1)\right) - \cos\left(\frac{2\pi}{l}(2j_1+1)\right)\cos\left(\frac{2\pi}{l}(2j_2+1)\right)\right]$$

and we get

$$\sqrt{[3]_q}\ {}^{q}W_{12} \triangleright |i_{1(23)}\rangle = N(j_1,j_1,1)\,N(j_2,j_2,1)\ q^{-2}$$
$$\times \quad \frac{\left[\cos\left(\frac{2\pi}{l}\right)\cos\left(\frac{2\pi}{l}(2j_3+1)\right) - \cos\left(\frac{2\pi}{l}(2j_1+1)\right)\cos\left(\frac{2\pi}{l}(2j_2+1)\right)\right]}{\sqrt{\left[\sin\left(\frac{2\pi}{l}\right)^2 - \sin\left(\frac{2\pi}{l}(2j_1+1)\right)^2\right]\left[\sin\left(\frac{2\pi}{l}\right)^2 - \sin\left(\frac{2\pi}{l}(2j_2+1)\right)^2\right]}}\ |i_{1(23)}\rangle$$

If we consider a limit such that $l \gg 1$ and $j_i \gg 1$, but $\frac{j_i}{l} \sim 1$, we have $\cos(2\pi/l) \approx 1$ and $\sin(2\pi/l) \approx 0$, and thus we get

$$\sqrt{3}\ {}^{q}W_{12} \triangleright |i_{1(23)}\rangle \approx N(j_1,j_1,1)\,N(j_2,j_2,1) \tag{4.237}$$
$$\times \quad \frac{\left[\cos\left(\frac{2\pi}{l}(2j_3+1)\right) - \cos\left(\frac{2\pi}{l}(2j_1+1)\right)\cos\left(\frac{2\pi}{l}(2j_2+1)\right)\right]}{\sin\left(\frac{2\pi}{l}(2j_1+1)\right)\sin\left(\frac{2\pi}{l}(2j_2+1)\right)}\ |i_{1(23)}\rangle\,,$$

from which we see that we reproduce the spherical cosine law if we identify the arc lengths of the triangle with $\frac{2\pi}{l}(2j_i+1)$. The mismatch between the arc length for the angle operator and the result for the eigenvalue of the length operator Eq.(4.208) was already noticed in the hyperbolic case in [40, 38]. However, with $q = e^{\frac{2\pi i}{l}}$ we can write the length eigenvalue from Eq.(4.208) as

$$\sqrt{\frac{[2j]_q[2j+2]_q}{[2]_q^2}} = \sqrt{\frac{\cos\left(\frac{4\pi}{l}(2j+1)\right) - \cos\left(\frac{4\pi}{l}\right)}{\cos\left(\frac{8\pi}{l}\right) - 1}} \approx \frac{1}{2}\sqrt{(2j+1)^2 - 1}\ +\ \mathcal{O}(l^{-2}) \tag{4.238}$$

and we see that, at least for large j, we have

$$\sqrt{\frac{[2j]_q[2j+2]_q}{[2]_q^2}} \approx \left(j + \frac{1}{2}\right)\ +\ \mathcal{O}(j^{-1})\ +\ \mathcal{O}(l^{-2})\,, \tag{4.239}$$

which resembles the values for the arc length in the angle eigenvalue, i.e., $\frac{2\pi}{l}(2j_i+1) = \frac{4\pi}{l}\left(j_i + \frac{1}{2}\right)$. Note, that for both Eq.(4.237) and Eq.(4.239) we considered the limits $l \gg 1$ and $j_i \gg 1$, but kept $\frac{j_i}{l} \sim 1$ fixed.

Chapter 5

Timelike twisted geometries and Lorentzian spinfoam models

In this chapter we summarize and extend our own work [1], which is concerned with Loop quantum gravity and spinfoam models in 4D Lorentzian spacetime and a certain gauge choice, that leads to $\mathrm{SU}(1,1)$ as the (small) gauge group, instead of the more common $\mathrm{SU}(2)$, associated with the standard (real) Ashtekar variables. Section 5.1 presents almost one to one the results of [1] and in section 5.2 we present some unpublished work on Lorentzian 4D spinfoam models.

In [1] we used the so-called twistorial parametrization of Loop quantum gravity and investigated the consequences of choosing a spacelike normal vector in the so-called linear simplicity constraints[1]. In the (current) standard formulation of 4D Loop quantum gravity and the so-called EPRL-FK-KKL spinfoam model[2], whose amplitudes are based on $\mathrm{SU}(2)$ boundary states, we only have spacelike building blocks in the bulk of spacetime. In our approach, using a spacelike normal vector, instead of a timelike normal vector, in the linear simplicity constraints allows us to distinguish spacelike from timelike 2-surfaces. We proposed in [1] a quantum theory that includes both spatial and temporal building blocks and hence, in our opinion, a more complete picture of quantum spacetime. At the classical

[1]In current 4D spinfoam models gravity is formulated as a topological BF - theory, which gives general relativity upon imposition of those simplicity constraints. They are called that, because they impose that the B - field of BF - theory be simple, which means, that it can be expressed as a wedge product of two vectors $B = e_1 \wedge e_2$.

[2]The Engle-Pereira-Rovelli-Livine-Freidel-Krasnov-Kaminski-Kisielowski-Lewandowski (EPRL-FK-KKL) spinfoam model is named after the authors of [13, 14, 15, 16, 17].

level, we show how we can describe $\mathrm{T}^*\mathrm{SU}(1,1)$ as a symplectic quotient of 2-twistor space $\mathbb{T}^2$ by area matching and simplicity constraints. This provides us with the underlying classical phase space for $\mathrm{SU}(1,1)$ spin networks describing timelike boundaries and their extension into the bulk. Applying a Dirac quantization, we show that the reduced Hilbert space is spanned by $\mathrm{SU}(1,1)$ spin networks and hence is able to give a quantum description of both spacelike and timelike faces. We discuss in particular the spectrum of the area operator and argue that for spacelike and timelike 2-surfaces it is discrete.

5.1 Timelike twisted geometries

Spinfoam models aim to give a covariant description of Loop quantum gravity (LQG), which is a canonical quantization of standard Einstein gravity in so-called connection variables [2, 3]. The current EPRL-FK-KKL spinfoam model solved several issues of its predecessors [88, 89, 90], such as having the correct SU(2) boundary states to match the states of LQG and having a good semiclassical limit [91, 92, 93]. There are, however, further questions that are worth investigating. The main motivation of [1] was related to the problem of timelike boundaries and the occurrence of non-spacelike building blocks in the bulk of spinfoam models, which, in turn, relates to the study of timelike boundaries as motivated by the so-called general boundary formulation (GBF) [94, 95, 96]. The absence of such non-spatial contributions in the current spinfoam models was also discussed in [97]. Within the GBF we are led to the possibility of timelike boundaries and their corresponding amplitudes in Lorentzian spinfoam models, a question which has some history in the field [98, 99, 100].

Currently, the new spinfoam model is constructed in such a way that all its building blocks, even in the bulk, are strictly spacelike, which follows from the imposition of the linear simplicity constraints using a timelike normal vector N^I. This is necessary for achieving the matching of the spinfoam boundary states with the kinematical SU(2) spin network states of LQG. From a covariant standpoint, however, it is not clear why we should make such a restriction in the bulk. Based on this reasoning, a generalization of the new spinfoam model that uses both timelike as well as spacelike normal vectors N^I for the linear simplicity constraints was proposed in [57, 58, 101]. Their derivation is based on the Freidel-Krasnov model [16] and uses coherent states techniques to implement the simplicity constraints in the quantum theory.

The main objective of [1] was, whether we can give a twistorial description of the Conrady-Hnybida model [57, 58, 101] with the hope that this would allow for an asymptotic analysis of such generalized spinfoam models with timelike components. The use of the twistorial parametrization of LQG [102, 103, 54, 55, 104] has in the past proven very useful for the investigation of the covariance properties of LQG [105, 106] and the underlying phase space geometry. It has already been used in [107] to investigate the possibility of a null normal vector N^I in the simplicity constraints and the subsequent quantization of null hypersurfaces with spacelike 2-surfaces. Similar to [107] we use these techniques here to consider timelike hypersurfaces with spacelike and timelike 2-surfaces, where we are mainly interested in the (quantum) description of timelike 2-surfaces. We find for example, similarly to the results obtained in [108] and [109] in a slightly different model, that the area spectrum of the timelike faces might be independent from the Barbero-Immirzi parameter.

One crucial question that has often been discussed in the literature on Lorentzian spinfoam models is whether the (kinematical) spectra of geometrical operators are (all) discrete or continuous [110, 111, 112]. In 2+1 spacetime dimensions the situation is clear; see, for example, [113] or the recent work [114, 115, 116], where one obtains continuous spectra for timelike 2-surfaces, because in that case the representations are labeled by a continuous parameter, which is a result of the non-compactness of the underlying gauge group. In 3+1 dimensions, however, the simplicity constraints can lead to relations between continuous and discrete representation labels, which amounts to the possibility that continuous spectra can become discrete. We will show that, indeed, also timelike faces can have discrete spectra when the simplicity constraints are imposed. This, however, requires a more detailed analysis than in the standard case with timelike N^I.

5.1.1 Twistors and Spinors in LQG and Spinfoams

In the current spinfoam models, the starting point is the quantization of BF-theory, on which one imposes the simplicity constraints, which reduce BF-theory to general relativity, in the quantum theory. The BF-action relates to the BF-action with a Holst term and the Barbero-Immirzi parameter $\gamma \in \mathbb{R}_*$ through the so-called Immirzi shift and is given by

$$S_{\mathrm{BF}}[B,A] = \int_M \mathrm{Tr}\left(B \wedge F[A]\right) = \int_M \mathrm{Tr}\left(*\Sigma \wedge F[A] - \frac{1}{\gamma}\Sigma \wedge F[A]\right). \tag{5.1}$$

The B- and Σ- bivector fields take values in $\mathfrak{sl}(2,\mathbb{C})$, and $F[A]$ is the curvature of a $\mathfrak{sl}(2,\mathbb{C})$-valued spacetime connection A. The trace is taken with respect to the $\mathfrak{sl}(2,\mathbb{C})$ Cartan metric. The Immirzi shift amounts to a change of basis for $\mathfrak{sl}(2,\mathbb{C})$ in a way that leaves the equations of motion unaltered but changes the symplectic structure by introducing γ. BF-theory is a topological theory and hence has only global degrees of freedom. By requiring that the Σ field should be simple, i.e., $\Sigma^{IJ} = e^I \wedge e^J$, one obtains gravity (in the Einstein-Cartan form and up to a prefactor $1/16\pi G$) with a Holst term [117], i.e.,

$$S_{\text{Holst}}[e, A] = \int_M \text{Tr}\left(*e \wedge e \wedge F[A] - \frac{1}{\gamma}\, e \wedge e \wedge F[A] \right). \tag{5.2}$$

In their linear form, those **simplicity constraints** are given by

$$N_I \Sigma^{IJ} = 0\,, \tag{5.3}$$

for some auxiliary normal vector N^I. Those constraints lead to two solutions, namely, $\Sigma^{IJ} = \pm e^I \wedge e^J$, where the sign relates to the orientation of the underlying frame field. Using now a discretization of the spacetime manifold M and a smearing of the continuous variables gives us a 2-complex decorated with $\mathrm{T}^*\mathrm{SL}(2,\mathbb{C})$ on each one-dimensional edge e of the dual 2-complex. The group element g corresponds to the holonomy of the connection A along e and can be used to measure the curvature associated with faces f bounded by the edges e_i. The Lie algebra element corresponds to the smeared B field over some 2-surface dual to f. We can now consider a three-dimensional intersection between this discrete structure and some hypersurface of spacetime. This leads us to some abstract, oriented graph Γ with N nodes n and L links l. Induced from the 2-complex, $\mathrm{T}^*\mathrm{SL}(2,\mathbb{C})$ is again associated with the links l. One reason for the name twisted geometries is the fact that $\mathrm{T}^*\mathrm{SL}(2,\mathbb{C})$ can be embedded in 2-twistor space as a symplectic quotient with respect to the so-called area matching constraint. Hence, we consider on each link a set of two twistors $(Z, W) \in \mathbb{T}^2 \cong \mathbb{C}^8$, where the first twistor is associated with the source node of the link and the second one is associated with the target node. Each twistor by itself is composed of two spinors $Z^\alpha = (\omega^A, i\bar{\pi}_{\bar{B}})$ and $W^\alpha = (\lambda^A, i\bar{\sigma}_{\bar{B}})$, where $\omega, \lambda \in \mathbb{C}^2$ transforms under the $(\frac{1}{2}, 0)$ (left-handed) and $\bar{\pi}, \bar{\sigma} \in (\bar{\mathbb{C}}^2)^*$ transforms under the $(0, \frac{1}{2})$ (right-handed) representation of $\mathrm{SL}(2,\mathbb{C})$. The adjoint twistors are given via $\bar{Z}_\alpha = (-i\pi_A, \bar{\omega}^{\bar{B}})$ such that the twistor norm is given by $\frac{1}{2}\bar{Z}_\alpha Z^\alpha = \mathrm{Im}(\pi\omega)$. We use the convention $\epsilon^{01} = \epsilon_{01} = 1$, $\epsilon^{AB} = -\epsilon^{BA}$ for the two-dimensional ϵ tensor, which allows us to move spinor indices as

$$\omega^A = \epsilon^{AB}\omega_B \qquad , \qquad \omega_A = \epsilon_{BA}\omega^B \tag{5.4}$$

and analogously for the complex conjugate sector. The 2-twistor space $\mathbb{T}^2$ comes equipped with a natural Poisson structure which is $\mathrm{SL}(2,\mathbb{C})$ invariant and is given by the 2-form [118, 119, 120][3]

$$\Omega = i\,\mathrm{d}Z^{\alpha} \wedge \mathrm{d}\bar{Z}_{\alpha} + i\,\mathrm{d}W^{\alpha} \wedge \mathrm{d}\overline{W}_{\alpha}\,. \tag{5.5}$$

In terms of the spinors Eq.(5.5) gives

$$i\,\mathrm{d}Z^{\alpha} \wedge \mathrm{d}\bar{Z}_{\alpha} = \mathrm{d}\omega^{A} \wedge \mathrm{d}\pi_{A} + \mathrm{d}\bar{\omega}^{\bar{B}} \wedge \mathrm{d}\bar{\pi}_{\bar{B}}\,, \tag{5.6}$$

$$i\,\mathrm{d}W^{\alpha} \wedge \mathrm{d}\overline{W}_{\alpha} = \mathrm{d}\lambda^{A} \wedge \mathrm{d}\sigma_{A} + \mathrm{d}\bar{\lambda}^{\bar{B}} \wedge \mathrm{d}\bar{\sigma}_{\bar{B}}\,, \tag{5.7}$$

which gives rise to the Poisson brackets

$$\{\pi_A, \omega^B\} = \delta_A^B = \{\sigma_A, \lambda^B\} \qquad , \qquad \{\bar{\pi}_{\bar{A}}, \bar{\omega}^{\bar{B}}\} = \delta_{\bar{A}}^{\bar{B}} = \{\bar{\sigma}_{\bar{A}}, \bar{\lambda}^{\bar{B}}\} \tag{5.8}$$

and all others vanishing. Thus, $\mathbb{T}^2$ together with the above brackets constitutes a Poisson manifold. For two functions f, g on $\mathbb{T}^2$ we calculate their Poisson bracket via

$$\begin{aligned} \{f, g\} = {} & \frac{\partial f}{\partial \pi_A}\frac{\partial g}{\partial \omega^A} - \frac{\partial f}{\partial \omega^A}\frac{\partial g}{\partial \pi_A} + \frac{\partial f}{\partial \sigma_A}\frac{\partial g}{\partial \lambda^A} - \frac{\partial f}{\partial \lambda^A}\frac{\partial g}{\partial \sigma_A} \\ & + \frac{\partial f}{\partial \bar{\pi}_{\bar{A}}}\frac{\partial g}{\partial \bar{\omega}^{\bar{A}}} - \frac{\partial f}{\partial \bar{\omega}^{\bar{A}}}\frac{\partial g}{\partial \bar{\pi}_{\bar{A}}} + \frac{\partial f}{\partial \bar{\sigma}_{\bar{A}}}\frac{\partial g}{\partial \bar{\lambda}^{\bar{A}}} - \frac{\partial f}{\partial \bar{\lambda}^{\bar{A}}}\frac{\partial g}{\partial \bar{\sigma}_{\bar{A}}}\,. \end{aligned} \tag{5.9}$$

The **area matching constraint**

$$C = \pi\omega - \lambda\sigma = 0 \tag{5.10}$$

is a first-class constraint and defines the embedding $\mathbb{T}^2_* \,/\!\!/\, C = \mathrm{T}^*\mathrm{SL}(2, \mathbb{C})$, [54, 55]. We assume throughout that $\pi\omega = \epsilon_{AB}\pi^A\omega^B = -\omega\pi \neq 0$ or $\sigma\lambda \neq 0$. Hence, we consider $\mathbb{T}^2_*$ where we remove the null configurations $\pi\omega = 0$ or $\sigma\lambda = 0$. One finds that the holonomy g and the fluxes Π of the gauge-invariant phase space $\mathrm{T}^*\mathrm{SL}(2, \mathbb{C})$ are parametrized in terms of the spinors via

$$g^A{}_B = \frac{\lambda^A\pi_B + \sigma^A\omega_B}{\sqrt{\pi\omega}\sqrt{\lambda\sigma}}\,, \tag{5.11}$$

[3] Following the conventions of the original twisted geometries literature [102, 103, 54, 55, 107], we remove the i appearing in the original spinorial Poisson brackets by parametrizing the twistors Z and W with an extra i in front of $\bar{\pi}_{\bar{B}}$ and $\bar{\sigma}_{\bar{B}}$. As in [107], we furthermore use the Poisson structure as defined by Eq.(5.5) and not with a relative minus sign. This leads to the symmetric Poisson brackets as shown in Eq.(5.8).

which satisfies $\det g = 1$ and $\{C, g^A{}_B\} = 0$, and

$$\Pi^{AB} = \frac{1}{4}\left(\pi^A\omega^B + \omega^A\pi^B\right) = \frac{1}{2}\pi^{(A}\omega^{B)} \quad , \quad \tilde{\Pi}^{AB} = \frac{1}{4}\left(\sigma^A\lambda^B + \lambda^A\sigma^B\right) = \frac{1}{2}\sigma^{(A}\lambda^{B)} . \tag{5.12}$$

Furthermore, one can show that

$$\{g^A{}_B, g^C{}_D\} = \frac{2C}{(\pi\omega)^2(\lambda\sigma)^2}\left[(\lambda\sigma)\epsilon^{AC}\Pi_{BD} - (\pi\omega)\epsilon_{BD}\tilde{\Pi}^{AC}\right] \tag{5.13}$$

and hence on the constraint surface $C = 0$, we get $\{g^A{}_B, g^C{}_D\} \approx 0$. The group element g defines a linear map from Z to W:

$$g^A{}_B\omega^B = \sqrt{\frac{\pi\omega}{\lambda\sigma}}\,\lambda^A \approx \lambda^A \qquad , \qquad g^A{}_B\pi^B = -\sqrt{\frac{\pi\omega}{\lambda\sigma}}\,\sigma^A \approx -\sigma^A . \tag{5.14}$$

A real bivector B^{IJ} can be decomposed into a self-dual and an anti-self-dual part which, in spinorial variables, takes the following form:

$$B^{A\bar{B}C\bar{D}} = \Pi^{AC}\bar{\epsilon}^{\bar{B}\bar{D}} + \bar{\Pi}^{\bar{B}\bar{D}}\epsilon^{AC} . \tag{5.15}$$

Using

$$\{C, \omega^A\} = \omega^A \quad , \quad \{C, \pi^A\} = -\pi^A \quad , \quad \{C, \lambda^A\} = \lambda^A \quad , \quad \{C, \sigma^A\} = -\sigma^A \tag{5.16}$$

we show that $g, \Pi, \tilde{\Pi}$ are invariant under the flow of C. The fluxes transform like $\tilde{\Pi} \approx -g\Pi g^{-1}$ on the constraint surface $C = 0$, and they furthermore satisfy two copies of the $\mathfrak{sl}(2,\mathbb{C})$ algebra,

$$\{\Pi^{AB}, \Pi^{CD}\} = \frac{1}{4}\left(\Pi^{AC}\epsilon^{BD} + \Pi^{AD}\epsilon^{BC} + \Pi^{BC}\epsilon^{AD} + \Pi^{BD}\epsilon^{AC}\right) \tag{5.17}$$

and similarly for the tilded fluxes, and we have $\{\Pi^{AB}, \tilde{\Pi}^{CD}\} = 0$. Thus, the variables g and Π suffice to fully parametrize $\mathrm{T}^*\mathrm{SL}(2,\mathbb{C})$, and $\tilde{\Pi}$ is obtained from g and Π via $\tilde{\Pi} \approx -g\Pi g^{-1}$. We can now employ the following isomorphism between $\mathfrak{sl}(2,\mathbb{C})$ and $\mathbb{C}^3$ to rewrite the fluxes in terms of their rotation and boost generators according to

$$\Pi^A{}_B = \Pi^i(\tau_i)^A{}_B = \left(L^i + i\,K^i\right)(\tau_i)^A{}_B , \tag{5.18}$$

with $i \in \{1,2,3\}$ and where the τ_i are related to the Pauli matrices via $\tau_i = \frac{1}{2i}\sigma_i$. They satisfy $[\tau_i, \tau_j] = \varepsilon_{ij}{}^{k}\,\tau_k$, and we use them to calculate the components $\Pi^i \in \mathbb{C}$ via

$$\Pi^i = -2\,\mathrm{Tr}(\Pi\,\tau_i) = -2\,\Pi^{A}{}_{B}(\tau_i)^{B}{}_{A}\,, \tag{5.19}$$

which gives

$$\Pi^1 = i\,(\Pi^{00} - \Pi^{11}) \quad , \quad \Pi^2 = -(\Pi^{00} + \Pi^{11}) \quad , \quad \Pi^3 = -2i\,\Pi^{01}\,. \tag{5.20}$$

Together with Eq.(5.17), this leads to $\{\Pi^i, \Pi^j\} = \varepsilon^{ij}{}_{k}\Pi^k$. Hence, on $C = 0$, we reproduce the Poisson structure of $\mathrm{T}^*\mathrm{SL}(2,\mathbb{C})$ given by

$$\{\Pi^i, g^{A}{}_{B}\} = g^{A}{}_{C}(\tau_i)^{C}{}_{B} \quad , \quad \{\tilde{\Pi}^i, g^{A}{}_{B}\} = -(\tau_i)^{A}{}_{C}g^{C}{}_{B} \quad , \quad \{g^{A}{}_{B}, g^{C}{}_{D}\} \approx 0\,. \tag{5.21}$$

5.1.2 Twistorial description of timelike hypersurfaces

We will use the twistorial parametrization reviewed above to investigate the reduction of $\mathbb{T}^2_*$ by the linear simplicity constraints and the area matching constraint. But first, let us consider the bivector field $B \in \bigwedge^2 \mathbb{R}^{1,3} \otimes \mathfrak{sl}(2,\mathbb{C})$. In $\mathrm{SL}(2,\mathbb{C})$ BF theory the B field is valued in $\mathfrak{sl}(2,\mathbb{C})$ and hence can be expanded in terms of a $\mathfrak{sl}(2,\mathbb{C})$ basis. This means that we can express B^{IJ} with the $\mathfrak{sl}(2,\mathbb{C})$ generators L^i and K^i as

$$B = \{B^{IJ}\} = \begin{pmatrix} 0 & K^1 & K^2 & K^3 \\ -K^1 & 0 & L^3 & -L^2 \\ -K^2 & -L^3 & 0 & L^1 \\ -K^3 & L^2 & -L^1 & 0 \end{pmatrix} \tag{5.22}$$

or, equivalently,

$$K^i = -K_i = B^{0i} \qquad , \qquad L^i = L_i = (*B)^{0i} = \frac{1}{2}\varepsilon^{0i}{}_{jk}B^{jk}\,, \tag{5.23}$$

where we used the Hodge star operator $*$, which satisfies $*^2 = -1$ in four dimensions with Lorentzian signature $(-,+,+,+)$. This gives furthermore

$$\{*B^{IJ}\} = \begin{pmatrix} 0 & L^1 & L^2 & L^3 \\ -L^1 & 0 & -K^3 & K^2 \\ -L^2 & K^3 & 0 & -K^1 \\ -L^3 & -K^2 & K^1 & 0 \end{pmatrix}\,. \tag{5.24}$$

The two $\mathfrak{sl}(2,\mathbb{C})$-invariant Casimirs $C_1 = \vec{L}^2 - \vec{K}^2$ and $C_2 = -2\,\vec{L}\cdot\vec{K}$ are obtained from $B^2 = \frac{1}{2}B_{IJ}B^{IJ} = -\vec{K}^2 + \vec{L}^2$ and $C_2 = \frac{1}{2}(*B)_{IJ}B^{IJ} = -2\left(L^1K^1 + L^2K^2 + L^3K^2\right) = 2K_iL^i = -2L_iK^i$. Note that for the Lorentzian signature we have $(*B)^2 = -B^2$. Not surprisingly, this already shows the possibility of nondefinite bivectors in the case of a spacelike normal vector in the linear simplicity constraints. For the standard time gauge, where $N^I = (1,0,0,0)^t$, we have $B^{0i} = 0$ and hence see that B is projected onto a Euclidean subspace with $(+,+,+)$ signature where we are only left with $B^2 > 0$ (we exclude the degenerate case of null bivectors in our considerations). If we choose the spacelike vector $N^I = (0,0,0,1)^t$, we deal with a subspace of signature $(+,-,-)$ and hence have, even after using the simplicity constraints, the possibility of bivectors with positive or negative areas. Let us also point out that in four spacetime dimensions every bivector can be written as the sum of two simple bivectors [121].

5.1.3 Phase space structure and timelike simplicity constraints

Using the Immirzi shift and identifying B^{IJ} with the $\mathfrak{sl}(2,\mathbb{C})$ generators as in Eq.(5.22) and Eq.(5.24), the linear simplicity constraints for spacelike normal $N^I = (0,0,0,1)^t$, i.e., $\Sigma^{3i} = 0$, become

$$L^3 = -\frac{1}{\gamma}K^3 \quad , \quad K^1 = \frac{1}{\gamma}L^1 \quad , \quad K^2 = \frac{1}{\gamma}L^2\,. \tag{5.25}$$

Using these constraints, we can already see that the $\mathfrak{sl}(2,\mathbb{C})$ Casimirs C_1 and C_2 reduce to

$$C_1 \longrightarrow (1-\gamma^2)\,Q_{\mathfrak{su}(1,1)}\,, \quad C_2 \longrightarrow 2\gamma\,Q_{\mathfrak{su}(1,1)}\,, \tag{5.26}$$

where the $\mathfrak{su}(1,1)$ Casimir is given by $Q_{\mathfrak{su}(1,1)} = (L^3)^2 - (K^1)^2 - (K^2)^2$. Following the procedure laid out in [55, 104, 107] we aim now for a decomposition of the constraints $\Sigma^{3i} = 0$ in their spinorial parametrization into a Lorentz-invariant part and a second part, specified by the little group of N^I. This has the advantage that the nature of those constraints becomes more transparent, which simplifies the phase space analysis as well as the quantization. We begin by rewriting B^{IJ} in spinorial variables. The simplicity constraints become

$$n_{A\bar{B}}\Sigma^{A\bar{B}C\bar{D}} = 0 \tag{5.27}$$

with

$$n_{A\bar{B}} = \epsilon_{CA}\epsilon_{\bar{D}\bar{B}}n^{C\bar{D}} = \frac{i}{\sqrt{2}}(\sigma_I)_{A\bar{B}}N^I \quad , \quad \Sigma^{A\bar{B}C\bar{D}} = -\frac{1}{2}(\sigma_I)^{A\bar{B}}(\sigma_J)^{C\bar{D}}\Sigma^{IJ}\,. \tag{5.28}$$

We use the following basis for the isomorphism between 4-vectors and anti-Hermitian matrices [note the extra factor of i in Eq. (5.28)]:

$$(\sigma_0)^{A\bar{B}} = (\sigma_0)_{A\bar{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad (\sigma_1)^{A\bar{B}} = -(\sigma_1)_{A\bar{B}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \tag{5.29}$$

$$(\sigma_2)^{A\bar{B}} = (\sigma_2)_{A\bar{B}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad (\sigma_3)^{A\bar{B}} = -(\sigma_3)_{A\bar{B}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{5.30}$$

Next, write B in terms of its self-dual and anti-self-dual components Π and $\bar{\Pi}$ as

$$B^{A\bar{B}C\bar{D}} = B_1^{AC}\epsilon^{\bar{B}\bar{D}} + \bar{B}_2^{\bar{B}\bar{D}}\epsilon^{AC} , \tag{5.31}$$

where

$$B_1^{AC} = -\frac{1}{2}B^{A\bar{B}C}{}_{\bar{B}} = B_1^{CA} \qquad , \qquad \bar{B}_2^{\bar{B}\bar{D}} = -\frac{1}{2}B^{A\bar{B}}{}_A{}^{\bar{D}} = \bar{B}_2^{\bar{D}\bar{B}} . \tag{5.32}$$

Note that for real bivectors we have $B_2 = \bar{B}_1$; otherwise the self-dual and anti-self-dual parts are not complex conjugates of each other. Including the Immirzi shift, we have

$$B = B_1\bar{\epsilon} + \bar{B}_1\epsilon = \Pi\bar{\epsilon} + \bar{\Pi}\epsilon = (i\Sigma_1 - \frac{1}{\gamma}\Sigma_1)\bar{\epsilon} + (-i\bar{\Sigma}_1 - \frac{1}{\gamma}\bar{\Sigma}_1)\epsilon \tag{5.33}$$

and hence

$$B_1 = \Pi = (i - \frac{1}{\gamma})\Sigma_1 \qquad , \qquad \Sigma_1 = -\frac{i\gamma}{\gamma + i}\Pi . \tag{5.34}$$

The difference in decomposing B or $*B$ into self-dual and anti-self-dual components is an extra i factor for the self-dual part and a $-i$ factor for the anti-self-dual part. This will be relevant for the distinction of spacelike and timelike 2-surfaces. Hence, we get for the linear simplicity constraints from Eq.(5.27)

$$n_{A\bar{B}}\left(-\frac{i\gamma}{\gamma + i}\,\Pi^{AC}\epsilon^{\bar{B}\bar{D}} + \frac{i\gamma}{\gamma - i}\,\bar{\Pi}^{\bar{B}\bar{D}}\epsilon^{AC}\right) = 0 \tag{5.35}$$

and the dual constraint $N_I(*\Sigma)^{IJ} = 0$ gives

$$n_{A\bar{B}}\left(\frac{\gamma}{\gamma + i}\,\Pi^{AC}\epsilon^{\bar{B}\bar{D}} + \frac{\gamma}{\gamma - i}\,\bar{\Pi}^{\bar{B}\bar{D}}\epsilon^{AC}\right) = 0 . \tag{5.36}$$

This distinction is important for the following reason. In order to split Eq.(5.35) according to the decomposition used in [55], [104], and [107] into a Lorentz-invariant part and the part invariant under the little group, we use two linearly independent null vectors (one real and one complex), which are furthermore orthogonal to each other (there is nothing that forces us to use the same procedure, except its success in the timelike and null cases, and thus we prefer to stay as close as possible). Now, even though we are using the spacelike normal vector $N^I = (0,0,0,1)^t$, which projects onto a pseudo-Riemannian subspace and hence allows for bivectors with non-definite norm, decomposing the simplicity constraint with respect to those null vectors always leads to subspaces where the bivectors have a definite norm. However, since we have seen that under the Hodge dual the bivector norm changes its sign, we can use this to distinguish the simplicity constraints for spacelike from those for timelike 2-surfaces. This essentially corresponds to the necessity of choosing another auxiliary vector U^I to distinguish those two cases in the Conrady-Hnybida construction [57, 58].

To be more explicit, we know that for a timelike normal vector N^I the solutions to the simplicity constraints lead to positive definite bivectors because they lie in a subspace with Euclidean signature. Hence, we can conclude from $N_I \Sigma^{IJ} = 0$ that $\Sigma = \pm e_1 \wedge e_2$ with $\Sigma^2 > 0$ and hence $(*\Sigma)^2 < 0$ and, vice versa, we can conclude from $N_I(*\Sigma)^{IJ} = 0$ that $*\Sigma = \pm \tilde{e}_1 \wedge \tilde{e}_2$ with $(*\Sigma)^2 > 0$ and hence $(\Sigma)^2 < 0$. Now, for a spacelike normal N^I, we still obtain from $N_I \Sigma^{IJ} = 0$ that $\Sigma = \pm e_1 \wedge e_2$ but now this does not imply $\Sigma^2 > 0$ any longer (because we are in a space with Lorentzian signature). The question arises as to how we should distinguish whether Σ is spacelike or timelike. Note that *a priori* it should be possible to obtain spacelike as well as timelike solutions from one constraint, i.e., either $N_I \Sigma^{IJ} = 0$ or $N_I(*\Sigma)^{IJ} = 0$. However, for now, we will investigate the reduction of $\mathrm{T}^*\mathrm{SL}(2,\mathbb{C})$ by both constraints Eq.(5.35) and Eq.(5.36) and discuss the results further in Sec. 5.1.11.

Following again [55], [104], and [107], we decompose Eq.(5.35) and Eq.(5.36) by projecting them onto the two null vectors $\frac{i}{\sqrt{2}}\omega_C \bar{\omega}_{\bar{D}}$ (real) and $\frac{i}{\sqrt{2}} n_{C\bar{E}} \bar{\omega}^{\bar{E}} \bar{\omega}_{\bar{D}}$ (complex). Contracting Eq.(5.35) with $\frac{i}{\sqrt{2}}\omega_C \bar{\omega}_{\bar{D}}$ gives us

$$\frac{\pi\omega}{\gamma + i} - \frac{\bar{\pi}\bar{\omega}}{\gamma - i} = 0 \qquad \text{or} \qquad F_1 \equiv \mathrm{Re}(\pi\omega) - \gamma\, \mathrm{Im}(\pi\omega) = 0\,, \tag{5.37}$$

where we exclude cases where $\|\omega\|^2 = -(\sigma_3)_{A\bar{B}}\omega^A \bar{\omega}^{\bar{B}} = |\omega^0|^2 - |\omega^1|^2 = 0$. This is the Lorentz-invariant constraint that one obtains for the time gauge, and hence it makes sense to associate it with spacelike bivectors. The contraction of Eq.(5.35) with $\frac{i}{\sqrt{2}} n_{C\bar{E}} \bar{\omega}^{\bar{E}} \bar{\omega}_{\bar{D}}$

and assuming that $\|\omega\|^2 \neq 0$ gives similarly the following complex constraint, which, due to the presence of the normal, is only invariant under the little group, which is in this case SU(1, 1):

$$F_2 = G_2 \equiv n^{A\dot{B}} \pi_A \bar{\omega}_{\dot{B}} = 0\,. \tag{5.38}$$

Applying the same procedure to Eq.(5.36) only changes the Lorentz invariant constraint, and Eq.(5.38) is valid for both cases. Hence, we have for the dual case the constraints Eq.(5.38) together with

$$\frac{\pi\omega}{\gamma+i} + \frac{\bar{\pi}\bar{\omega}}{\gamma-i} = 0 \qquad \text{or} \qquad G_1 \equiv \mathrm{Re}(\pi\omega) + \frac{1}{\gamma}\,\mathrm{Im}(\pi\omega) = 0\,, \tag{5.39}$$

as an equivalent set of constraints replacing Eq.(5.36). Since they are dual to the first set, we interpret them as the ones corresponding to the timelike case. A more direct way to see that this is the correct way to associate the (F_1, F_2) with spacelike bivectors and (G_1, G_2) with timelike bivectors is to consider the area form

$$\begin{aligned}\mathcal{A}^2 &= \frac{1}{2}\left(\Sigma_1\bar{\epsilon} + \bar{\Sigma}_1\epsilon\right)\left(\Sigma_1\bar{\epsilon} + \bar{\Sigma}_1\epsilon\right) = \left(-\frac{i\gamma}{\gamma+i}\right)^2 \Pi_{AC}\Pi^{AC} + \left(\frac{i\gamma}{\gamma-i}\right)^2 \bar{\Pi}_{\bar{B}\bar{D}}\bar{\Pi}^{\bar{B}\bar{D}} \\ &= \frac{\gamma^2}{8}\left(\frac{(\pi\omega)^2}{(\gamma+i)^2} + \frac{(\bar{\pi}\bar{\omega})^2}{(\gamma-i)^2}\right) = \frac{\gamma^2}{4}\,\mathrm{Re}\left(\frac{(\pi\omega)^2}{(\gamma+i)^2}\right)\,.\end{aligned} \tag{5.40}$$

One finds that the solutions of the simplicity constraint $F_1 = 0$, which are given by $\pi\omega = (\gamma + i)\,\mathcal{J}$, with $\mathcal{J} \in \mathbb{R}_*$, lead to a positive area,

$$\mathcal{A}^2\Big|_{F_1=0} = \frac{\gamma^2}{4}\mathcal{J}^2 > 0\,, \tag{5.41}$$

whereas the solutions of $G_1 = 0$, which are given by $\pi\omega = i(\gamma+i)\,\mathcal{K}$, with $\mathcal{K} \in \mathbb{R}_*$, lead to a negative area,

$$\mathcal{A}^2\Big|_{G_1=0} = -\frac{\gamma^2}{4}\mathcal{K}^2 < 0\,. \tag{5.42}$$

Note, that in both cases the area (squared) depends quadratically on γ. Since we only used F_1 in Eq.(5.41) and G_1 in Eq.(5.42), it is clear that this statement is independent of the other constraints F_2 and G_2. Furthermore, this suggests that also in the quantum theory the area spectra of spacelike and timelike areas should depend on γ.

Spacelike faces

We consider in this subsection the classical analysis of the constraints F_1, F_2 together with the area matching constraint C from Eq.(5.10) and investigate the symplectic reduction $\mathbb{T}_* /\!\!/ F_1 /\!\!/ F_2$. We will also use the following version of F_1:

$$\mathring{F}_1 \equiv (\gamma - i)(\pi\omega) - (\gamma + i)(\bar{\pi}\bar{\omega}) = 0\,. \tag{5.43}$$

We first look for the classical solutions to the constraints F_1 and F_2. From twistor theory and the solutions of the simplicity constraints in the standard time gauge case, we know that the spinors are linearly dependent, and hence we are working with simple twistors, which are determined by a single spinor. This motivates to make the ansatz

$$\pi_A = -\xi\,(\sigma_3)_{A\bar{B}}\bar{\omega}^{\bar{B}} \quad , \quad \xi \in \mathbb{C}_* \tag{5.44}$$

and one finds that this indeed solves $G_2 = F_2 = 0$ for all $\xi \in \mathbb{C}_*$. Plugging our ansatz into $F_1 = 0$, we find with $\xi = r_\xi \exp(i\varphi_\xi)$

$$F_1 = \|\omega\|^2\, r_\xi \left[\cos(\varphi_\xi) - \gamma\sin(\varphi_\xi)\right] \stackrel{!}{=} 0\,, \tag{5.45}$$

where we have defined $\|\omega\|^2 = -(\sigma_3)_{A\bar{B}}\omega^A\bar{\omega}^{\bar{B}} = |\omega^0|^2 - |\omega^1|^2$. Hence, we get

$$\varphi_\xi = \varphi(\gamma) = \mathrm{arccot}(\gamma) = \arctan\left(\frac{1}{\gamma}\right)\,. \tag{5.46}$$

We see that we can solve $F_1 = 0 = F_2$ by choosing

$$\pi_A = -r_\xi\, e^{i\varphi(\gamma)}\,(\sigma_3)_{A\bar{B}}\bar{\omega}^{\bar{B}} \quad , \quad r_\xi \in \mathbb{R}_* \tag{5.47}$$

and that (r_ξ, ω^A) span our five-dimensional solution space within $\mathbb{T}$, which has eight real dimensions. We have the system of constraints

$$\{\mathring{F}_1, F_2\} = -2\gamma F_2 \approx 0 \quad , \quad \{\mathring{F}_1, \bar{F}_2\} = 2\gamma\bar{F}_2 \approx 0 \quad , \quad \{F_2, \bar{F}_2\} = -i\,\mathrm{Im}(\pi\omega)\,, \tag{5.48}$$

and together with the area matching constraint, we have $\{\mathring{F}_1, C\} = 0 = \{\mathring{F}_1, \bar{C}\}$ and

$$\{F_2, C\} = -\{F_2, \bar{C}\} = F_2 \approx 0 \qquad , \qquad \{\bar{F}_2, C\} = -\{\bar{F}_2, \bar{C}\} = -\bar{F}_2 \approx 0\,. \tag{5.49}$$

Hence, we see that F_1 and C are of first class and F_2 is of second class. On the fundamental spinors, $\mathring{F}_1$ generates the following transformations:

$$\{\mathring{F}_1, \omega^A\} = (\gamma - i)\,\omega^A \qquad , \qquad \{\mathring{F}_1, \pi^A\} = -(\gamma - i)\,\pi^A\,, \tag{5.50}$$

$$\{\mathring{F}_1, \bar{\omega}^{\bar{A}}\} = -(\gamma + i)\,\bar{\omega}^{\bar{A}} \qquad , \qquad \{\mathring{F}_1, \bar{\pi}^{\bar{A}}\} = (\gamma + i)\,\bar{\pi}^{\bar{A}}\,. \tag{5.51}$$

Since F_1 is a first-class constraint, it generates gauge transformations, and we are interested in the gauge-invariant four-dimensional solution space. Consider the following bracket, with $\|\omega\|^2 = -(\sigma_3)_{A\bar{B}}\omega^A\bar{\omega}^{\bar{B}}$, for which we have

$$\{\mathring{F}_1, \|\omega\|^\alpha\} = -i\alpha\,\|\omega\|^\alpha\,. \tag{5.52}$$

Can we find an expression of r_ξ in terms of ω^A, in order to parametrize the reduced phase space? Note that

$$\{\mathring{F}_1, \pi\omega\} = 0\,. \tag{5.53}$$

If we use the solution Eq.(5.47) and assume that r_ξ is a function of ω^A, we find with

$$\pi\omega = r_\xi(\omega^A)\,e^{i\varphi(\gamma)}\,\|\omega\|^2 \tag{5.54}$$

and Eq.(5.53) that $r_\xi(\omega^A)$ must satisfy

$$\{\mathring{F}_1, r_\xi(\omega^A)\} \stackrel{!}{=} 2i\,r_\xi(\omega^A)\,. \tag{5.55}$$

From this, we conclude that

$$r_\xi(\omega^A) = \frac{N}{\|\omega\|^2} \tag{5.56}$$

for some arbitrary numerical prefactor $N \in \mathbb{R}_*$. Hence, the four-dimensional reduced phase space (the symplectic quotient $\mathbb{T} /\!\!/ F_1 /\!\!/ F_2$) can be parametrized by a single spinor. However, we know from Eq.(5.50) that ω^A itself is not a gauge-invariant variable and hence not a good coordinate on the reduced phase space. Before we get to this point, let us choose N such that

$$\pi\omega = (\gamma + i)\,\mathcal{J} \tag{5.57}$$

for some $\mathcal{J} \in \mathbb{R}_*$. This is achieved for

$$N = (\gamma + i)\,\mathcal{J}\,e^{-i\varphi(\gamma)} = \sqrt{1+\gamma^2}\,\mathcal{J}\,, \tag{5.58}$$

where we used that

$$e^{i\varphi(\gamma)} = \cos(\operatorname{arccot}(\gamma)) + i\sin(\operatorname{arccot}(\gamma)) = \sqrt{\frac{\gamma+i}{\gamma-i}} \tag{5.59}$$

and hence we get

$$\pi_A = -(\gamma+i)\,\mathcal{J}\,\frac{(\sigma_3)_{A\bar{B}}\bar{\omega}^{\bar{B}}}{\|\omega\|^2}\,. \tag{5.60}$$

On the non-gauge-invariant solution space of F_1 and F_2, the variable $\mathcal{J}$ is given by

$$\mathcal{J} = \frac{\|\omega\|^2\, r_\xi}{\sqrt{1+\gamma^2}} \qquad \Rightarrow \qquad \{\mathring{F}_1, \mathcal{J}\} = 0\,. \tag{5.61}$$

Now, let us find the spinor that parametrizes the reduced phase space. Making the ansatz

$$z^A(\omega^B) = \sqrt{M}\,\frac{\omega^A}{\|\omega\|^\tau}\,, \tag{5.62}$$

for some number M, and requiring that $\{\mathring{F}_1, z^A\} = 0$, gives

$$\{\mathring{F}_1, z^A\} = z^A\,[\gamma - i + i\tau] \stackrel{!}{=} 0 \qquad \Leftrightarrow \qquad \tau = i\gamma + 1\,. \tag{5.63}$$

Furthermore, we have

$$\|z\|^2 = -(\sigma_3)_{A\bar{B}} z^A \bar{z}^{\bar{B}} = M \tag{5.64}$$

and we will choose $M = 2\mathcal{J}$. Note that $\mathcal{J}$ can be positive or negative, and if we wish to emphasize this point, we write $\varepsilon\mathcal{J}$ where we consider $\mathcal{J} > 0$ and $\varepsilon \in \{\pm 1\}$.

Timelike faces

We consider now the symplectic reduction of $\mathbb{T}_*$ by the dual simplicity constraints Eq.(5.36). We will use again the following expression for G_1:

$$\mathring{G}_1 \equiv (\gamma - i)(\pi\omega) + (\gamma + i)(\bar{\pi}\bar{\omega}) = 0\,. \tag{5.65}$$

To obtain the classical solutions of G_1 and G_2, we use now the ansatz

$$\pi_A = -i\,\zeta\,(\sigma_3)_{A\bar{B}}\bar{\omega}^{\bar{B}} \quad , \quad \zeta \in \mathbb{C}_*\,, \tag{5.66}$$

where we use the extra i factor compared with the spacelike case and find that this solves $G_2 = 0$ for all $\zeta \in \mathbb{C}_*$. To solve $G_1 = 0$, we find that $\zeta = r_\zeta \exp(i\varphi_\zeta)$ has to satisfy

$$G_1 = \frac{1}{\gamma} \|\omega\|^2 r_\zeta \left[\cos(\varphi_\zeta) - \gamma \sin(\varphi_\zeta)\right] \stackrel{!}{=} 0\,, \tag{5.67}$$

from which we get

$$\varphi_\zeta = \varphi_\xi = \varphi(\gamma) = \operatorname{arccot}(\gamma) = \arctan\left(\frac{1}{\gamma}\right)\,. \tag{5.68}$$

The fact that we obtain the same dependence of the phase and the Barbero-Immirzi parameter in the standard case Eq.(5.46) as well as the dual case Eq.(5.68) is a result of our i factor, which we used in Eq.(5.66). Thus, we see that we can solve $G_1 = 0 = G_2$ by choosing

$$\pi_A = -i\, r_\zeta\, e^{i\varphi(\gamma)}\, (\sigma_3)_{A\bar{B}} \bar{\omega}^{\bar{B}} \quad , \quad r_\zeta \in \mathbb{R}_*, \tag{5.69}$$

and again (r_ζ, ω^A) can be seen to span our five-dimensional solution space. The same procedure as in the spacelike case leads us the the gauge-invariant spinor variables. We have the relations between the simplicity constraints,

$$\{\mathring{G}_1, G_2\} = 2i\, G_2 \approx 0\,, \quad \{\mathring{G}_1, \bar{G}_2\} = -2i\, \bar{G}_2 \approx 0\,, \quad \{G_2, \bar{G}_2\} = -i\, \mathrm{Im}(\pi\omega) \tag{5.70}$$

and together with the area matching constraint we have

$$\{\mathring{G}_1, C\} = 0 = \{\mathring{G}_1, \bar{C}\}\,. \tag{5.71}$$

Because $G_2 = F_2$, the brackets with C and $\bar{C}$ are equivalently given by Eq.(5.49). $\mathring{G}_1$ acts with an extra minus sign on the complex conjugated spinors

$$\{\mathring{G}_1, \omega^A\} = (\gamma - i)\, \omega^A \quad , \quad \{\mathring{G}_1, \pi^A\} = -(\gamma - i)\, \pi^A\,, \tag{5.72}$$

$$\{\mathring{G}_1, \bar{\omega}^{\bar{A}}\} = (\gamma + i)\, \bar{\omega}^{\bar{A}} \quad , \quad \{\mathring{G}_1, \bar{\pi}^{\bar{A}}\} = -(\gamma + i)\, \bar{\pi}^{\bar{A}}\,. \tag{5.73}$$

Hence, we find that the constraint structure is the same as in the spacelike case with G_1 and C being of first class and F_2 being a complex second-class constraint. We consider again

$$\{\mathring{G}_1, \|\omega\|^\alpha\} = \alpha\gamma\, \|\omega\|^\alpha \tag{5.74}$$

and ask whether we can find an expression of r_ζ in terms of ω^A. Using the the solution Eq.(5.69) and the assumption that we can express r_ζ as a function of ω^A, we find with

$$\pi\omega = i\, r_\zeta(\omega^A)\, e^{i\varphi(\gamma)}\, \|\omega\|^2 \tag{5.75}$$

and $\{\mathring{G}_1, \pi\omega\} = 0$ that $r_\zeta(\omega^A)$ must satisfy

$$\{\mathring{G}_1, r_\zeta(\omega^A)\} \overset{!}{=} -2\gamma\, r_\zeta(\omega^A) \qquad \Rightarrow \qquad r_\zeta(\omega^A) = \frac{\kappa}{\|\omega\|^2}\,, \tag{5.76}$$

for some arbitrary numerical prefactor $\kappa \in \mathbb{R}_*$. Now, we want to choose κ such that

$$\pi\omega = i(\gamma + i)\,\mathcal{K} \tag{5.77}$$

for some $\mathcal{K} \in \mathbb{R}_*$ which is achieved for

$$\kappa = (\gamma + i)\,\mathcal{K}\, e^{-i\varphi(\gamma)} = \sqrt{1+\gamma^2}\,\mathcal{K} \tag{5.78}$$

and hence we get

$$\pi_A = -i\,(\gamma + i)\,\mathcal{K}\,\frac{(\sigma_3)_{A\bar{B}}\bar{\omega}^{\bar{B}}}{\|\omega\|^2}\,. \tag{5.79}$$

On the non-gauge-invariant solution space of G_1 and G_2, the variable $\mathcal{K}$ is given by

$$\mathcal{K} = \frac{\|\omega\|^2\, r_\zeta}{\sqrt{1+\gamma^2}} = -\frac{i\,\|\omega\|^2\, r_\xi}{\sqrt{1+\gamma^2}} = -i\mathcal{J} \tag{5.80}$$

and hence $\{\mathring{G}_1, \mathcal{K}\} = 0$. The spinor that parametrizes the reduced phase space is again found by making the ansatz

$$y^A(\omega^B) = \sqrt{M}\,\frac{\omega^A}{\|\omega\|^\tau}\,, \tag{5.81}$$

for some complex number M, and further requiring that $\{\mathring{G}_1, y^A\} = 0$ holds, which gives

$$\{\mathring{G}_1, y^A\} = y^A\,[-\gamma\tau + \gamma - i] \overset{!}{=} 0 \qquad \Leftrightarrow \qquad \tau = 1 - \frac{i}{\gamma}\,. \tag{5.82}$$

Hence,

$$y^A(\omega^B) = \sqrt{M}\,\frac{\omega^A}{\|\omega\|^{1-i/\gamma}} \qquad , \qquad \|y\|^2 = -(\sigma_3)_{A\bar{B}}\, y^A \bar{y}^{\bar{B}} = M\,. \tag{5.83}$$

Note that we choose the normalization of y^A such that $M = 2\gamma\mathcal{K}$, which is motivated by the simple form the Dirac bracket attains on the reduced phase space[4]. Note furthermore that in the standard timelike case one restricts $\mathcal{J}$ to be strictly positive, because in that case $\|z\|^2 = |z^0|^2 + |z^1|^2 \geq 0$ and $\mathcal{J} = 0$ is ruled out since we assumed throughout that $\pi\omega \neq 0$. This restriction was used to get rid of a $\mathbb{Z}_2$ symmetry of the reduction of $\mathbb{T}^2_*$ to $\mathrm{T}^*\mathrm{SL}(2,\mathbb{C})$ and we have the same symmetry present. In our case, however, the norm of z^A and y^A is not positive definite. Hence, if we want to focus on this nondefiniteness we can write $\varepsilon\mathcal{J}$ and $\varepsilon\mathcal{K}$, where $\varepsilon \in \{\pm 1\}$.

Now, we want to calculate the Dirac bracket of the reduced spinor with its complex conjugate. We need the Dirac bracket on the reduced space to take care of the second-class constraints $F_2 = G_2$ and $\bar{F}_2 = \bar{G}_2$. We use

$$z^A = \sqrt{2\mathcal{J}}\frac{\omega^A}{\|\omega\|^{i\gamma+1}} = \sqrt{\frac{2\pi\omega}{(\gamma+i)}}\frac{\omega^A}{\|\omega\|^{i\gamma+1}} \tag{5.84}$$

and

$$\underset{\sim}{z}^A = \sqrt{2\underset{\sim}{\mathcal{J}}}\frac{\lambda^A}{\|\lambda\|^{i\gamma+1}} = \sqrt{\frac{2\sigma\lambda}{(\gamma+i)}}\frac{\lambda^A}{\|\lambda\|^{i\gamma+1}} \tag{5.85}$$

as coordinates on the reduced space $\mathbb{T}^2_* /\!\!/ F \cong \mathbb{C}^2 \times \mathbb{C}^2$, where $F = \{F_1, F_2, \underset{\sim}{F}_1, \underset{\sim}{F}_2\}$, and

$$y^A = \sqrt{2\gamma\mathcal{K}}\frac{\omega^A}{\|\omega\|^{1-i/\gamma}} = \sqrt{\frac{2\gamma\pi\omega}{(i\gamma-1)}}\frac{\omega^A}{\|\omega\|^{1-i/\gamma}} \tag{5.86}$$

and

$$\underset{\sim}{y}^A = \sqrt{2\gamma\underset{\sim}{\mathcal{K}}}\frac{\lambda^A}{\|\lambda\|^{1-i/\gamma}} = \sqrt{\frac{2\gamma\sigma\lambda}{(i\gamma-1)}}\frac{\lambda^A}{\|\lambda\|^{1-i/\gamma}} \tag{5.87}$$

as coordinates on the reduced space $\mathbb{T}^2_* /\!\!/ G \cong \mathbb{C}^2 \times \mathbb{C}^2$, where $G = \{G_1, G_2, \underset{\sim}{G}_1, \underset{\sim}{G}_2\}$. Let us already note that the system of constraints F or G together with the area matching constraint is reducible, which means that after imposing $F = 0$ or $G = 0$ part of C is already satisfied. Hence, the final step of the reduction is only with a reduced area matching constraint. Now, we calculate the Dirac bracket on $\mathbb{C}^2 \times \mathbb{C}^2$ via

$$\{z^A, \bar{z}^{\bar{B}}\}_{\mathrm{D}} = \{z^A, \bar{z}^{\bar{B}}\} - \{z^A, F_2\}\, M^{-1}_{12}\, \{\bar{F}_2, \bar{z}^{\bar{B}}\} - \{z^A, \bar{F}_2\}\, M^{-1}_{21}\, \{F_2, \bar{z}^{\bar{B}}\}\,. \tag{5.88}$$

[4]This is a possible choice we can make. However, as we will discuss in Sec. 5.1.11, whether it is worth keeping track of the fate of the Barbero-Immirzi parameter γ. Also cf. footnote 4.

Together with

$$M = \begin{pmatrix} \{F_2, F_2\} & \{F_2, \bar{F}_2\} \\ \{\bar{F}_2, F_2\} & \{\bar{F}_2, \bar{F}_2\} \end{pmatrix} = i\,\mathrm{Im}(\pi\omega) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \Rightarrow \quad M^{-1} = \frac{i}{\mathrm{Im}(\pi\omega)} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{5.89}$$

we find

$$\{z^A, \bar{z}^{\bar{B}}\} = \frac{i}{2\mathcal{J}}\, z^A \bar{z}^{\bar{B}} \quad , \quad \{z^A, F_2\} \approx -\frac{n^{A\bar{B}} \bar{z}_{\bar{B}}}{\|\omega\|^{2i\gamma}} \quad , \quad \{\bar{F}_2, \bar{z}^{\bar{B}}\} \approx \frac{\bar{n}^{\bar{B}C} z_C}{\|\omega\|^{-2i\gamma}}\,, \tag{5.90}$$

where the last two exressions hold weakly on F_2 and $\bar{F}_2$, respectively. Furthermore, with

$$\{z^A, \bar{F}_2\} = \{F_2, \bar{z}^{\bar{B}}\} = 0 \tag{5.91}$$

we finally obtain

$$\{z^A, \bar{z}^{\bar{B}}\}_{\mathrm{D}} \approx i(\sigma_3)^{A\bar{B}} \approx \{\underset{\sim}{z}^A, \bar{\underset{\sim}{z}}^{\bar{B}}\}_{\mathrm{D}}\,, \tag{5.92}$$

where we used

$$n^{A\bar{B}} = \frac{i}{\sqrt{2}}(\sigma_3)^{A\bar{B}}\,. \tag{5.93}$$

Similarly, we find for the dual case with

$$\{y^A, \bar{y}^{\bar{B}}\} = \frac{i}{2\gamma\mathcal{K}}\, y^A \bar{y}^{\bar{B}} \quad , \quad \{y^A, G_2\} \approx -\frac{n^{A\bar{B}} \bar{y}_{\bar{B}}}{\|\omega\|^{-\frac{2i}{\gamma}}} \quad , \quad \{\bar{G}_2, \bar{y}^{\bar{B}}\} \approx \frac{\bar{n}^{\bar{B}C} y_C}{\|\omega\|^{\frac{2i}{\gamma}}}\,, \tag{5.94}$$

where again the last two relations hold weakly on G_2 and $\bar{G}_2$, respectively. And with

$$\{y^A, \bar{G}_2\} = \{G_2, \bar{y}^{\bar{B}}\} = 0\,, \tag{5.95}$$

we get[5]

$$\{y^A, \bar{y}^{\bar{B}}\}_{\mathrm{D}} \approx i(\sigma_3)^{A\bar{B}} \approx \{\underset{\sim}{y}^A, \bar{\underset{\sim}{y}}^{\bar{B}}\}_{\mathrm{D}}\,. \tag{5.96}$$

In the standard case, using the time gauge, one obtains for the Dirac brackets of the reduced spinors the harmonic oscillator brackets where $(\sigma_3)^{A\bar{B}}$ is replaced by $(\sigma_0)^{A\bar{B}} = \delta^{A\bar{B}}$. In our case, instead, we find that we have an additional relative minus sign between

[5]If we would have not put the extra γ in the normalization of the reduced spinor in Eq.(5.86) and Eq.(5.87), these two Dirac brackets would be given by $\{y^A, \bar{y}^{\bar{B}}\}_{\mathrm{D}} \approx \frac{i}{\gamma}(\sigma_3)^{A\bar{B}} \approx \{\underset{\sim}{y}^A, \bar{\underset{\sim}{y}}^{\bar{B}}\}_{\mathrm{D}}$.

brackets for the spinor components, which reflects the Lorentzian structure underlying our reduction. Furthermore, let us point out that those reduced brackets can be obtained equivalently as the Kirillov-Kostant-Souriau brackets [87] on the coadjoint orbits of $\mathrm{SU}(1,1)$ for a timelike representative. We will further discuss this point in Sec. 5.1.4. Before that, however, we will consider again the second-class constraints $F_2 = G_2$ and show that it can be exchanged for an equivalent real first-class constraint, the so-called master constraint, which will be important for the quantum theory, where it is easier to impose the first-class constraints strongly than properly taking care of the second class constraints. We follow again the procedure known from the standard time-gauge case, where the first-class master constraint is defined via (equivalently for G_2)

$$\mathbf{M} \equiv \bar{F}_2 F_2 = 0\,. \tag{5.97}$$

We can now rewrite $\mathbf{M}$ in terms of quantities that simplify the identification of the solution space to $\mathbf{M} = 0$ in the quantum theory. This is achieved by the fact that we can rewrite it in terms of one of the $\mathfrak{sl}(2,\mathbb{C})$ Casimirs and the $\mathfrak{su}(1,1)$ Casimir plus an extra term, and for all of those, we know the spectrum on the noncanoncial basis of $\mathrm{SL}(2,\mathbb{C})$, which diagonalizes not $\mathrm{SU}(2)$ but $\mathrm{SU}(1,1)$. We follow [104] closely and adapt it to the timelike case. We have

$$\begin{aligned}\mathbf{M} = \bar{F}_2 F_2 &= \bar{n}^{\dot{A}B} n^{C\dot{D}} \bar{\pi}_{\dot{A}} \omega_B \pi_C \bar{\omega}_{\dot{D}} \\ &= \bar{n}^{\dot{A}B} n^{C\dot{D}} \left(\omega_{(B}\pi_{C)} + \omega_{[B}\pi_{C]}\right)\left(\bar{\pi}_{(\dot{A}}\bar{\omega}_{\dot{D})} + \bar{\pi}_{[\dot{A}}\bar{\omega}_{\dot{D}]}\right)\,,\end{aligned} \tag{5.98}$$

where we used that $\omega_B \pi_C = \left(\omega_{(B}\pi_{C)} + \omega_{[B}\pi_{C]}\right)$. We obtain

$$\begin{aligned}\mathbf{M} =& \bar{n}^{\dot{A}B} n^{C\dot{D}} \left(2\,\Pi_{BC} + (\omega\pi)\,\epsilon_{BC}\right)\left(2\,\bar{\Pi}_{\dot{A}\dot{D}} + (\bar{\pi}\bar{\omega})\,\epsilon_{\dot{A}\dot{D}}\right) \\ =& \bar{n}^{\dot{A}B} n^{C\dot{D}} \Big(4\,\Pi_{BC}\bar{\Pi}_{\dot{A}\dot{D}} + 2\,(\bar{\pi}\bar{\omega})\,\Pi_{BC}\,\epsilon_{\dot{A}\dot{D}} \\ &+ \quad 2\,(\omega\pi)\,\bar{\Pi}_{\dot{A}\dot{D}}\,\epsilon_{BC} - |\pi\omega|^2 \epsilon_{BC}\epsilon_{\dot{A}\dot{D}}\Big)\,.\end{aligned} \tag{5.99}$$

Together with $N^I = (0,0,0,1)$ and $n^{A\dot{B}} = \frac{i}{\sqrt{2}}(\sigma_I)^{A\dot{B}} N^I = \frac{i}{\sqrt{2}}\,\mathrm{diag}(1,-1)$, one can now show explicitly that

$$\mathbf{M} = 4\,\bar{n}^{\dot{A}B} n^{C\dot{D}}\,\Pi_{BC}\bar{\Pi}_{\dot{A}\dot{D}} - |\pi\omega|^2 \bar{n}^{\dot{A}B} n^{C\dot{D}}\,\epsilon_{BC}\epsilon_{\dot{A}\dot{D}} = 4\,\bar{n}^{\dot{A}B} n^{C\dot{D}}\,\Pi_{BC}\bar{\Pi}_{\dot{A}\dot{D}} + |\pi\omega|^2\,. \tag{5.100}$$

For the first term in Eq.(5.100), we get

$$4\,\bar{n}^{\dot{A}B} n^{C\dot{D}}\,\Pi_{BC}\bar{\Pi}_{\dot{A}\dot{D}} = 2\,|\Pi_{00}|^2 - 4\,|\Pi_{01}|^2 + 2\,|\Pi_{11}|^2\,. \tag{5.101}$$

Let us now rewrite the fluxes in terms of their rotation and boost generators using Eq.(5.18) and Eq.(5.19), which gives us

$$|\Pi_{00}|^2 = |\Pi_{11}|^2 = \frac{1}{4}\left(|\Pi^1|^2 + |\Pi^2|^2\right) = \frac{1}{4}\left((L^1)^2 + (L^2)^2 + (K^1)^2 + (K^2)^2\right) \tag{5.102}$$

and

$$|\Pi_{01}|^2 = \frac{1}{4}|\Pi^3|^2 = \frac{1}{4}\left((L^3)^2 + (K^3)^2\right)\,. \tag{5.103}$$

Hence, we finally get for Eq.(5.101)

$$4\,\bar{n}^{\dot{A}B} n^{C\dot{D}}\,\Pi_{BC}\bar{\Pi}_{\dot{A}\dot{D}} = \left[(\vec{L})^2 - (\vec{K})^2 - 2\left((L^3)^2 - (K^1)^2 - (K^2)^2\right)\right]\,. \tag{5.104}$$

Now, we note that $(\vec{L})^2 - (\vec{K})^2$ is the quadratic $\mathfrak{sl}(2,\mathbb{C})$ Casimir and furthermore $Q_{\mathfrak{su}(1,1)} = (L^3)^2 - (K^1)^2 - (K^2)^2$ is the Casimir of $\mathfrak{su}(1,1)$, and we get for the master constraint for a spacelike normal N^I

$$\mathbf{M} = \left(C_{\mathrm{SL}(2,\mathbb{C})} - 2\,Q_{\mathfrak{su}(1,1)}\right) + |\pi\omega|^2\,. \tag{5.105}$$

Recall that for the case of timelike normal vector we obtain the $\mathfrak{su}(2)$ Casimir instead of $Q_{\mathfrak{su}(1,1)}$, but otherwise it looks exactly the same. Finding the complete solution space in the quantum theory, however, is more involved than in the standard case.

5.1.4 Reduction by the area matching constraint

As we have mentioned before, the system of all constraints is reducible. On $\mathbb{T}^2_* /\!\!/ F$ or $\mathbb{T}^2_* /\!\!/ G$ part of the area matching constraint C is already satisfied. One finds that the reduced area matching constraint is given by

$$C_{\mathrm{red}} = \left\|z\right\|^2 + \left\|\underset{\sim}{z}\right\|^2 = 0 \tag{5.106}$$

or in the dual case by

$$D_{\mathrm{red}} = \left\|y\right\|^2 + \left\|\underset{\sim}{y}\right\|^2 = 0\,. \tag{5.107}$$

Note that this constraint has nontrivial solutions, since the "norm" of the spinors $\|z\|^2$, etc., is not positive definite in our case. We will see that these constraints will be solved by $\mathcal{J} = -\underset{\sim}{\mathcal{J}}$ and $\mathcal{K} = -\underset{\sim}{\mathcal{K}}$. We will use $\mathcal{J}, \mathcal{K}, \underset{\sim}{\mathcal{J}}, \underset{\sim}{\mathcal{K}} > 0$ and solve the constraints by using opposite ε's. Equivalently, we could have chosen the normalization of the tilded sector to be $M = -2\underset{\sim}{\mathcal{J}}$ to obtain a reduced area matching with a minus sign, which was used in [102, 103, 54]. However, the important point is the gauge transformations that are generated by C_{red} and D_{red}, and those are not affected by this sign. The origin of this minus sign can be traced back to our choice to have the standard Poisson structure on $\mathbb{T}^2$ and not the sign-flipped one used, for example, in [102, 103, 54, 55, 56].

We are now interested in the reductions $(\mathbb{C}^2 \times \mathbb{C}^2) \mathbin{/\mkern-6mu/} C_{\text{red}}$ and $(\mathbb{C}^2 \times \mathbb{C}^2) \mathbin{/\mkern-6mu/} D_{\text{red}}$ and whether we end up with $\mathrm{T}^*\mathrm{SU}(1,1)$ in both cases. Remember that from now on we are using the Dirac bracket on the reduced phase space. We have

$$\{C_{\text{red}}, z^A\} = -iz^A \,,\ \{C_{\text{red}}, \underset{\sim}{z}^A\} = -i\underset{\sim}{z}^A \,,\ \{C_{\text{red}}, \bar{z}^{\bar{A}}\} = i\bar{z}^{\bar{A}} \,,\ \{C_{\text{red}}, \bar{\underset{\sim}{z}}^{\bar{A}}\} = i\bar{\underset{\sim}{z}}^{\bar{A}} \tag{5.108}$$

and similarly

$$\{D_{\text{red}}, y^A\} = -iy^A \,,\ \{D_{\text{red}}, \underset{\sim}{y}^A\} = -i\underset{\sim}{y}^A \,,\ \{D_{\text{red}}, \bar{y}^{\bar{A}}\} = i\bar{y}^{\bar{A}} \,,\ \{D_{\text{red}}, \bar{\underset{\sim}{y}}^{\bar{A}}\} = i\bar{\underset{\sim}{y}}^{\bar{A}} \,. \tag{5.109}$$

Inspired by the holonomy and the fluxes constructed in Sec. 5.1.1 we find that we can analogously parametrize the gauge-invariant reduced phase space $(\mathbb{C}^2 \times \mathbb{C}^2) \mathbin{/\mkern-6mu/} C_{\text{red}}$ with the holonomy

$$h^A{}_B = \frac{\underset{\sim}{z}^A (\sigma_3)_{B\bar{C}} \bar{z}^{\bar{C}} + (\sigma_3)^{A\bar{C}} \bar{\underset{\sim}{z}}_{\bar{C}} z_B}{\|z\| \|\underset{\sim}{z}\|} \tag{5.110}$$

and similarly for $(\mathbb{C}^2 \times \mathbb{C}^2) \mathbin{/\mkern-6mu/} D_{\text{red}}$,

$$h^A{}_B = \frac{\underset{\sim}{y}^A (\sigma_3)_{B\bar{C}} \bar{y}^{\bar{C}} + (\sigma_3)^{A\bar{C}} \bar{\underset{\sim}{y}}_{\bar{C}} y_B}{\|y\| \|\underset{\sim}{y}\|} \,. \tag{5.111}$$

They are both of the form

$$h = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \tag{5.112}$$

and Eq.(5.110) gives

$$a = \frac{(z^1 \bar{\underset{\sim}{z}}^{\bar{1}} - \bar{z}^{\bar{0}} \underset{\sim}{z}^0)}{\|z\| \|\underset{\sim}{z}\|} \qquad , \qquad b = \frac{(\underset{\sim}{z}^0 \bar{z}^{\bar{1}} - \bar{\underset{\sim}{z}}^{\bar{1}} z^0)}{\|z\| \|\underset{\sim}{z}\|} \tag{5.113}$$

and similarly for Eq.(5.111). One finds that both satisfy $\det h = 1$. Hence, we see that we obtain SU(1, 1) on the reduced phase space. Furthermore, on C_{red}

$$\underset{\sim}{z}^A \approx h^A{}_B z^B \qquad , \qquad \underset{\sim}{y}^A \approx h^A{}_B y^B \tag{5.114}$$

and one shows explicitly that, using the Dirac bracket, we have

$$\{C_{\text{red}}, h^A{}_B\} = 0 \qquad , \qquad \{h^A{}_B, h^C{}_D\} \approx 0\,. \tag{5.115}$$

The fluxes Π^{BD} from Eq.(5.12) become

$$\pi^{BD} = \frac{(\gamma + i)}{8}\left[(\sigma_3)^{B\bar{C}}\bar{z}_{\bar{C}}z^D + (\sigma_3)^{D\bar{C}}\bar{z}_{\bar{C}}z^B\right]\,, \tag{5.116}$$

which gives

$$\pi = -\frac{(\gamma + i)}{8}\begin{pmatrix} 2z^0\bar{z}^{\bar{1}} & (|z^0|^2 + |z^1|^2) \\ (|z^0|^2 + |z^1|^2) & 2\bar{z}^{\bar{0}}z^1 \end{pmatrix}\,. \tag{5.117}$$

They satisfy, of course, $\{C_{\text{red}}, \pi^{BD}\} = 0$. We can now expand π in terms of a $\mathfrak{su}(1,1)$ basis, i.e., $\pi = \pi^i\tau_i$. With

$$(\tau_1)^A{}_B = \frac{1}{2i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad (\tau_2)^A{}_B = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad (\tau_3)^A{}_B = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{5.118}$$

and by a rescaling with a factor $-2i/(\gamma + i)$, we get

$$\pi^1 = \frac{1}{2}(|z^0|^2 + |z^1|^2) \quad , \quad \pi^2 = \operatorname{Im}(\bar{z}^{\bar{0}}z^1) \quad , \quad \pi^3 = -\operatorname{Re}(\bar{z}^{\bar{0}}z^1)\,. \tag{5.119}$$

They satisfy

$$\{\pi^1, \pi^3\} = \pi^2 \quad , \quad \{\pi^1, \pi^2\} = -\pi^3 \quad , \quad \{\pi^3, \pi^2\} = -\pi^1 \tag{5.120}$$

and hence, we see that we get indeed a $\mathfrak{su}(1,1)$ algebra where $(\pi^1, \pi^2, \pi^3) \cong (J_3, K_2, K_1)$. Thus, we finally obtain $T^*SU(1,1)$ via a symplectic reduction of $\mathbb{T}^2_*$ by the simplicity constraints and the area-matching constraint. This holds in both cases of constraints (F, C) and (G, C). In terms of the reduced spinors, one finds that with Eq.(5.119) the $\mathfrak{su}(1,1)$ Casimir operator is given by

$$Q_{\mathfrak{su}(1,1)} = (\pi^1)^2 - (\pi^2)^2 - (\pi^3)^2 = \frac{1}{4}(|z^0|^2 - |z^1|^2) = \frac{1}{4}\|z\|^2\,. \tag{5.121}$$

Now, as we have mentioned before, let us show that the Poisson structure we have obtained via reduction from $\mathbb{T}^2_*$ by the simplicity and area matching constraint is exactly the canonical symplectic structure (Kirillov-Kostant-Souriau symplectic structure [87]) on the coadjoint orbits of SU(1,1). If we take an element $g \in \mathrm{SU}(1,1)$ with

$$g = \begin{pmatrix} z^0 & z^1 \\ \bar{z}^{\bar{1}} & \bar{z}^{\bar{0}} \end{pmatrix} \qquad , \qquad |z^0|^2 - |z^1|^2 = 1 \,. \tag{5.122}$$

(note that the components of g are not to be confused with our reduced spinor components), we can consider the right invariant 1-forms $\theta = dg \cdot g^{-1}$, and together with $\det(g) = 1$, we have

$$\theta = \begin{pmatrix} \bar{z}^{\bar{0}} dz^0 - \bar{z}^{\bar{1}} dz^1 & z^0 dz^1 - z^1 dz^0 \\ \bar{z}^{\bar{0}} d\bar{z}^{\bar{1}} - \bar{z}^{\bar{1}} d\bar{z}^{\bar{0}} & \bar{z}^{\bar{1}} dz^1 - \bar{z}^{\bar{0}} dz^0 \end{pmatrix} \,. \tag{5.123}$$

Using the basis Eq.(5.118), we can expand $\theta = a\tau_1 + b\tau_2 + c\tau_3$ with

$$a = 2i(\bar{z}^{\bar{0}} dz^0 - \bar{z}^{\bar{1}} dz^1) \quad , \quad b = 2\,\mathrm{Re}(z^0 dz^1 - z^1 dz^0) \quad , \quad c = -2\,\mathrm{Im}(z^0 dz^1 - z^1 dz^0) \,. \tag{5.124}$$

The coefficients b and c are obviously real. To show that a is real as well, use again $\det(g) = 1$. To obtain the symplectic structure on the different coadjoint orbits we have to consider certain representatives of those orbits, for example, $f_1 = (s, 0, 0)$, $f_2 = (0, s, 0)$, or $f_3 = (0, 0, s)$. We get, for example,

$$\theta_{f_1} = 2is(\bar{z}^{\bar{0}} dz^0 - \bar{z}^{\bar{1}} dz^1) \,, \tag{5.125}$$

which leads to

$$\omega_1 = -d\theta_{f_1} = 2is(dz^0 \wedge d\bar{z}^{\bar{0}} - dz^1 \wedge d\bar{z}^{\bar{1}}) \,. \tag{5.126}$$

This symplectic 2-forms induces the following Poisson bracket for functions f, g on the coadjoint orbit of f_1,

$$\{f, g\}_1 = 2is \left(\frac{\partial f}{\partial z^0} \frac{\partial g}{\partial \bar{z}^{\bar{0}}} - \frac{\partial f}{\partial \bar{z}^{\bar{0}}} \frac{\partial g}{\partial z^0} - \frac{\partial f}{\partial z^1} \frac{\partial g}{\partial \bar{z}^{\bar{1}}} + \frac{\partial f}{\partial \bar{z}^{\bar{1}}} \frac{\partial g}{\partial z^1} \right) \,. \tag{5.127}$$

Hence, for $s = \frac{1}{2}$, we get the Poisson structure

$$\{z^A, \bar{z}^{\bar{B}}\}_1 = i(\sigma_3)^{A\bar{B}} \,, \tag{5.128}$$

which is exactly Eq.(5.92). Note, that we can choose different values for s, even negative ones. Using the coadjoint representation of a $g \in \mathrm{SU}(1,1)$, we can build a representation of $\mathfrak{su}(1,1)$ using those spinors and the Poisson brackets. Consider, for example,

$$J_3 \equiv |z^0|^2 + |z^1|^2 \quad , \quad K_1 \equiv 2\,\mathrm{Im}(\bar{z}^{\bar{0}} z^1) \quad , \quad K_2 \equiv 2\,\mathrm{Re}(\bar{z}^{\bar{0}} z^1)\,. \tag{5.129}$$

Together with the Poisson bracket Eq.(5.128), one shows that this gives indeed a (vector) representation of $\mathfrak{su}(1,1)$ with

$$\{J_3, K_1\}_1 = 2K_2 \quad , \quad \{J_3, K_2\}_1 = -2K_1 \quad , \quad \{K_1, K_2\}_1 = -2J_3\,. \tag{5.130}$$

Using the other coadjoint orbits f_2 or f_3, one can similarly construct different representations of $\mathfrak{su}(1,1)$.

We will now consider the quantum theory of the presented model. For the sections on the actual "twisted geometries parametrization" and the "closure constraint", which we have omitted here, we refer the reader to [1].

5.1.5 Quantization and timelike spin networks

Our starting point for the quantization, following [55], [104], and [107], are quantum twistor networks, which are graphs labeled with 2-twistor space $\mathbb{T}^2_*$ on each link. This space $\mathbb{T}_*$, one for each half-link, can easily be quantized by promoting the spinorial components of the twistors to operators and their Poisson brackets to the corresponding commutators in a Schrödinger representation. This will provide us with our unconstrained Hilbert space on which we then impose the quantized simplicity constraints, (reduced) area matching constraint, and closure constraints (in this order). For each link, we consider the auxiliary Hilbert space of homogeneous functions of degree (a, b). Hence, we consider $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ such that $\forall \lambda \in \mathbb{C}_*$,

$$f(\lambda \omega^A) = \lambda^a \bar{\lambda}^b f(\omega^A)\,. \tag{5.131}$$

These functions are essentially functions on $\mathbb{CP}^1$. To deal with single valued functions, we have to require that $a - b$ must be an integer. Note, furthermore, that these functions are not assumed to be holomorphic or antiholomorphic, since they are general polynomials in the spinor components as well as their complex conjugates. In certain cases, however, they can be reduced to give holomorphic representations. Together with

$$(g \triangleright f)(\omega^A) = f(g^{-1} \triangleright \omega^A)\,, \tag{5.132}$$

this provides, for certain values of the numbers (a, b), a unitary and irreducible representation for SL$(2,\mathbb{C})$ [122]. The SL$(2,\mathbb{C})$-invariant measure on this space of functions is given by

$$d\Omega(\omega^A) = \frac{i}{2}(\omega^0 d\omega^1 - \omega^1 d\omega^0) \wedge (\bar{\omega}^{\bar{0}} d\bar{\omega}^{\bar{1}} - \bar{\omega}^{\bar{1}} d\bar{\omega}^{\bar{0}})\,. \tag{5.133}$$

Under rescaling, it transforms as $d\Omega(\lambda\omega^A) = |\lambda|^4\, d\Omega(\omega^A)$ so that the SL$(2,\mathbb{C})$ and scaling-invariant scalar product is given by

$$\langle f_1|f_2\rangle = \frac{i}{2}\int_{\mathbb{CP}^1} d\Omega(\omega^A)\, \bar{f}_1(\omega^A) f_2(\omega^A)\,. \tag{5.134}$$

This representation belongs to the principal series of SL$(2,\mathbb{C})$. With $n \in \mathbb{Z}/2$ and $p \in \mathbb{R}$, it is unitary, and we denote the corresponding Hilbert space of those functions by $\mathcal{H}^{(n,p)}$. The numbers (a, b) and (n, p) are related by

$$a = -n - 1 + ip \qquad \text{and} \qquad b = n - 1 + ip\,. \tag{5.135}$$

Since the representations (n, p) and $(-n, -p)$ are unitarily equivalent, we restrict those labels to be $n \in \mathbb{N}_0/2$ and $p \in \mathbb{R}$. The labels (n, p) are related to the eigenvalues of the $\mathfrak{sl}(2,\mathbb{C})$ Casimirs $C_1 = \vec{L}^2 - \vec{K}^2$ and $C_2 = -2\vec{L}\cdot\vec{K}$ as follows:

$$\hat{C}_1 \triangleright f^{(n,p)} = (n^2 - p^2 - 1) f^{(n,p)}\,, \tag{5.136}$$

$$\hat{C}_2 \triangleright f^{(n,p)} = -2np\, f^{(n,p)}\,. \tag{5.137}$$

If we consider the half-link phase space $\mathbb{T}_*$ with $Z^\alpha = (\omega^A, i\bar{\pi}_{\bar{B}})$ and $\pi\omega = \epsilon_{AB}\pi^A\omega^B \neq 0$, the Poisson structure of which is given by

$$\{\pi_A, \omega^B\} = \delta_A^B \qquad , \qquad \{\bar{\pi}_{\bar{A}}, \bar{\omega}^{\bar{B}}\} = \delta_{\bar{A}}^{\bar{B}}\,, \tag{5.138}$$

and similarly for $W^\alpha = (\lambda^A, i\bar{\sigma}_{\bar{B}})$ with $\sigma\lambda \neq 0$, we use for the commutators

$$[\hat{\pi}_A, \hat{\omega}^B] = -i\hbar\,\delta_A^B \qquad , \qquad [\hat{\bar{\pi}}_{\bar{A}}, \hat{\bar{\omega}}^{\bar{B}}] = -i\hbar\,\delta_{\bar{A}}^{\bar{B}} \tag{5.139}$$

the following Schrödinger representation:

$$\hat{\omega}^B f(\omega^A) = \omega^B f(\omega^A) \qquad , \qquad \hat{\pi}_B f(\omega^A) = -i\hbar\frac{\partial}{\partial\omega^B} f(\omega^A)\,. \tag{5.140}$$

The homogeneous functions are furthermore interesting because they diagonalize the Euler dilatation operator $\omega^A \partial_A$,

$$\omega^A \frac{\partial}{\partial \omega^A} f^{(a,b)}(\omega^A) = a\, f^{(a,b)}(\omega^A) \qquad , \qquad \bar{\omega}^{\bar{A}} \frac{\partial}{\partial \bar{\omega}^{\bar{A}}} f^{(a,b)}(\omega^A) = b\, f^{(a,b)}(\omega^A)\,, \tag{5.141}$$

which holds for all homogeneous functions. The Hilbert space for each single link is now given by the homogeneous functions of the form

$$f^{(a,b)}(\omega^A, \lambda^B) \equiv f^{(a_s,b_s)}(\omega^A) \otimes f^{(a_t,b_t)}(\lambda^A)\,, \tag{5.142}$$

where the subscripts s and t stand for the source and target half-links. It is easy to see that these are now homogeneous functions of degree $(a,b) = (a_s + a_t, b_s + b_t)$. Recall that the complex area matching constraint Eq.(5.10) was given by $C = \pi\omega - \lambda\sigma = 0$. We can use Eq.(5.141) to impose $\hat{C} = 0$ as follows. We can write $\pi\omega = \pi_A \omega^A = \frac{1}{2}(\pi\omega + \pi\omega) = \frac{1}{2}(\pi\omega - \omega\pi)$. This gives us a normal ordering for $\widehat{\pi\omega}$,

$$\widehat{\pi\omega} = \frac{\hbar}{2i}\left[\frac{\partial}{\partial\omega^A}\omega^A - \omega_A \frac{\partial}{\partial\omega_A}\right] = \frac{\hbar}{2i}\left[\omega^A \frac{\partial}{\partial\omega^A} + \frac{\partial}{\partial\omega^A}\omega^A\right], \tag{5.143}$$

where we have used that switching the position of spinorial indices gives a minus sign in the second equality. Analogously one obtains for the complex conjugate contribution

$$\widehat{\bar{\pi}\bar{\omega}} = \frac{\hbar}{2i}\left[\bar{\omega}^{\bar{A}} \frac{\partial}{\partial\bar{\omega}^{\bar{A}}} + \frac{\partial}{\partial\bar{\omega}^{\bar{A}}}\bar{\omega}^{\bar{A}}\right] \tag{5.144}$$

and the corresponding expressions in terms of (σ, λ) variables. Using now the commutation relations and Eq.(5.141), we can show that for a homogeneous function with degree (a,b) we have

$$\begin{aligned}\widehat{\pi\omega}\, f^{(a,b)} &= \frac{\hbar}{2i}\left[\omega^A \frac{\partial}{\partial\omega^A} + \frac{\partial}{\partial\omega^A}\omega^A\right] f^{(a,b)} = \frac{\hbar}{2i}\left[\omega^A \frac{\partial}{\partial\omega^A} + 2 + \omega^A \frac{\partial}{\partial\omega^A}\right] f^{(a,b)} \\ &= \frac{\hbar}{i}\,[a+1]\, f^{(a,b)}\end{aligned} \tag{5.145}$$

and similarly $\widehat{\bar{\pi}\bar{\omega}}\, f^{(a,b)} = \frac{\hbar}{i}\,[b+1]\, f^{(a,b)}$. The action of the area-matching constraint becomes

$$\hat{C} \triangleright f^{(a,b)}(\omega^A, \lambda^B) = \hat{C} \triangleright \left(f^{(a_s,b_s)}(\omega^A) \otimes f^{(a_t,b_t)}(\lambda^A)\right) \tag{5.146}$$
$$= \left(\hat{C} \otimes 1 + 1 \otimes \hat{C}\right) \left(f^{(a_s,b_s)}(\omega^A) \otimes f^{(a_t,b_t)}(\lambda^A)\right)$$
$$= \left(\widehat{\pi\omega} \triangleright f^{(a_s,b_s)}(\omega^A)\right) \otimes f^{(a_t,b_t)}(\lambda^A) - f^{(a_s,b_s)}(\omega^A) \otimes \left(\widehat{\lambda\sigma} \triangleright f^{(a_t,b_t)}(\lambda^A)\right)$$
$$= \frac{\hbar}{i}\left[a_s + a_t + 2\right] \left(f^{(a_s,b_s)}(\omega^A) \otimes f^{(a_t,b_t)}(\lambda^A)\right)$$

and analogously the complex conjugate area-matching constraint gives

$$\hat{\bar{C}} \triangleright f^{(a,b)}(\omega^A, \lambda^B) = \frac{\hbar}{i}\left[b_s + b_t + 2\right] f^{(a,b)}(\omega^A, \lambda^B)\,. \tag{5.147}$$

Using Eq.(5.135), one finds that $a_s + a_t + 2 = -(n_s + n_t) + i(p_s + p_t)$, and hence both constraints are solved by $n_t = -n_s$ and $p_t = -p_s$. Since we want to work with $n_i \in \frac{\mathbb{N}_0}{2}$, we have to consider on the source link states with (n_s, p_s) and on the target link states with $(-n_s, -p_s)$, which are states from two different (but unitarily equivalent) Hilbert spaces.

Before we investigate the imposition of the simplicity constraints in the next sections, we recall that the so-called canonical basis for $\mathcal{H}^{(n,p)}$, which stems from an induced representation using the SU(2) subgroup of $\mathrm{SL}(2,\mathbb{C})$, is used in the quantization of the EPRL model using the time gauge. This is possible because we can further diagonalize $\vec{L}^2$ and L^3 besides the two $\mathfrak{sl}(2,\mathbb{C})$ Casimirs, which gives the states $|(n,p); j, m\rangle$, where $j \in \mathbb{N}_0/2$ denotes the spin and $m \in \{-j, -j+1, \cdots, j\}$ denotes its magnetic number. In particular, this leads to a decomposition of $\mathcal{H}^{(n,p)}$ as

$$\mathcal{H}^{(n,p)} \simeq \bigoplus_{n \leq j} \mathcal{H}^{(j)}\,, \tag{5.148}$$

where $\mathcal{H}^{(j)}$ denotes the standard $(2j+1)$-dimensional unitary and irreducible representation space of SU(2). Since the stabilizing subgroup for our spacelike normal vector $N^I = (0,0,0,1)^t$ is given by SU(1,1), it is more suitable to employ a decomposition in terms of a SU(1,1) basis. This was also used in [57] and [58]. For that reason, we briefly review some representation theory of SU(1,1) in the following section.

5.1.6 Representations of $\mathrm{SU}(1,1)$

The $\mathrm{SL}(2,\mathbb{C})$ representations from above provide representations for the subgroup $\mathrm{SU}(1,1)$ as well. They are, however, not irreducible. But similarly to Eq.(5.148), they can be decomposed into $\mathrm{SU}(1,1)$ irreducible representations. To fix our conventions, we consider here the unitary and irreducible representations of $\mathrm{SU}(1,1)$ belonging to the principal series. The early works on the representation theory of the three-dimensional Lorentz group are [123] or the book [124]. The Plancherel decomposition was investigated, for example, in [125], and for a newer account, see [122]. The Clebsch-Gordan problem for $\mathrm{SU}(1,1)$ was investigated in [126, 127, 128, 129]. Note, that in this work we have so far used the mathematical convention for the rotation and boost generators, i.e., $L_i^\dagger = -L_i$ and $K_i^\dagger = K_i$. In [57] and [58] or [114, 115, 116], for example, the authors use the physical convention where the Hermiticity property is reversed. This will not be an obstacle in what follows, since the simplicity constraints are invariant under this choice. This can easily be seen from Eq.(5.18), where one can simply define the Π^i with an additional factor of $\pm i$ and this would not change the form of the master constraint, as can be seen from Eq.(5.102) to Eq.(5.104). For the covariant simplicity constraints F_1 and G_1, this convention is irrelevant as well, since for them we do not use the generators L_i and K_i explicitly. Now, with this in mind, we can consider the physical convention, where L_3 is Hermitian and hence can be diagonalized with a real eigenvalue. Furthermore, we look for states that diagonalize the $\mathfrak{su}(1,1)$ Casimir $Q_{\mathfrak{su}(1,1)} = (L^3)^2 - (K^1)^2 - (K^2)^2$. We denote those eigenstates of the two $\mathfrak{sl}(2,\mathbb{C})$ Casimirs C_1 and C_2 as well as $Q_{\mathfrak{su}(1,1)}$ and L^3 by $f_{j,m}^{(n,p)} = |(n,p);j,m\rangle \in \mathcal{H}^{(n,p)}$. The eigenvalues of the $\mathfrak{sl}(2,\mathbb{C})$ Casimirs are given by Eq.(5.136) and Eq.(5.137), and we have furthermore

$$Q_{\mathfrak{su}(1,1)} \triangleright f_{j,m}^{(n,p)} = \pm j(j+1)\, f_{j,m}^{(n,p)}\,, \tag{5.149}$$

$$L^3 \triangleright f_{j,m}^{(n,p)} = m\, f_{j,m}^{(n,p)}\,. \tag{5.150}$$

The action of $Q_{\mathfrak{su}(1,1)}$ with a plus is the convention as used, for example, in [57] and [58], whereas in [116] the authors use the additional minus sign in front of $j(j+1)$. We will see that this sign plays a role for our final result. We will find that the solutions to the master constraint with the discrete states on both half-links do not give us the full reduced Hilbert space necessary to decompose all functions on $\mathrm{SU}(1,1)$ in a spin network basis[6]. Hence, we are eventually forced to work with the convention from [116], i.e., with

[6]Rather, one would obtain only the discrete states with integer spin and the continuous states with even parity.

eigenvalues $-j(j+1)$. Furthermore, let us point out that if we compare our approach with the coherent state approach used in [57] and [58], where it was stated that it is necessary to diagonalize a noncompact generator K^1 or K^2 instead of L^3, in order to be able to describe timelike faces, we do not find this to be necessary, which makes our considerations more comprehensible.

For SU(1,1), we have the following unitary irreducible representations (that appear in the Plancherel decomposition), which are all infinite dimensional, since SU(1,1) is noncompact. First, we have the discrete series $\mathcal{D}_k^{\pm}$ where $j=-k$ with $k\in\frac{\mathbb{N}}{2}$. For $\mathcal{D}_k^{+}$, we have $m\in\{k,k+1,k+2,\cdots\}$, and for $\mathcal{D}_k^{-}$, we have $m\in\{-k,-k-1,-k-2,\cdots\}$. The state with $j=-1/2$ is somewhat special in that it is not normalizable and hence does not appear in the Plancherel decomposition. We see that using the plus convention in Eq.(5.149) and if we do not consider the state with $j=-1/2$ then we have for all other possible values of j in the discrete series

$$Q^d_{\mathrm{SU}(1,1)}\in\{0,\frac{3}{4},2,\frac{15}{4},\cdots\}\geq 0\,. \tag{5.151}$$

Second, we have the continuous series $\mathcal{C}_s^{\varepsilon}$ with $j=-\frac{1}{2}+is$ and $\varepsilon\in\{0,\frac{1}{2}\}$. For $\varepsilon=0$ (even functions), we have $s\geq 0$ and $m\in\{0,\pm1,\pm2,\cdots\}$, and for $\varepsilon=\frac{1}{2}$ (odd functions), we have $s>0$ and $m\in\{\pm\frac{1}{2},\pm\frac{3}{2},\pm\frac{5}{2},\cdots\}$. Hence, using again the plus convention in Eq.(5.149), we have for all states from $\mathcal{C}_s^{\varepsilon}$

$$Q^c_{\mathrm{SU}(1,1)}=j(j+1)=-s^2-\frac{1}{4}<0\,. \tag{5.152}$$

In what follows, we will first use this convention and only later change to the opposite case. We explicitly include the full analysis in order to pinpoint exactly where the problem with this convention lies. We just mention that the analog of Eq.(5.148) reads in this noncanonical basis [122, 101]

$$\mathcal{H}^{(n,p)}\simeq\left(\bigoplus_{k>1/2}^{n}\mathcal{D}_k^{+}\oplus\int_0^{\infty\oplus}ds\,\mathcal{C}_s^{\varepsilon}\right)\oplus\left(\bigoplus_{k>1/2}^{n}\mathcal{D}_k^{-}\oplus\int_0^{\infty\oplus}ds\,\mathcal{C}_s^{\varepsilon}\right)\,, \tag{5.153}$$

where the sum over the discrete states ranges over values for which $k-n$ is an integer and similarly ε is determined by the condition that $\varepsilon-n$ is an integer. The Clebsch-Gordan decomposition for the coupling of those representations is given by [126, 127, 128, 129]

$$\mathcal{D}_{k_1}^{\pm}\otimes\mathcal{D}_{k_2}^{\pm}=\bigoplus_{K=k_1+k_2}^{\infty}\mathcal{D}_K^{\pm}\,, \tag{5.154}$$

and

$$\mathcal{D}^{\pm}_{k_1} \otimes \mathcal{D}^{\mp}_{k_2} = \bigoplus_{K=K_{\min}}^{k_1-k_2} \mathcal{D}^{\pm}_{K} \oplus \bigoplus_{K=K_{\min}}^{k_2-k_1} \mathcal{D}^{\mp}_{K} \oplus \int_0^{\infty\oplus} \mathcal{C}^{\varepsilon}_{s}\, ds\,, \tag{5.155}$$

where $K_{\min} = 1$ and $\varepsilon = 0$ if $k_1 + k_2$ is an integer and $K_{\min} = \frac{3}{2}$ and $\varepsilon = \frac{1}{2}$ otherwise. Furthermore, note that the discrete contributions vanish when the upper limits $k_1 - k_2$ or $k_2 - k_1$, respectively, are smaller than 1; i.e, we must have $k_1 - k_2 \geq 1$ for the first sum and $k_2 - k_1 \geq 1$ for the second. The coupling of two continuous states gives

$$\mathcal{C}^{\varepsilon_1}_{s_1} \otimes \mathcal{C}^{\varepsilon_2}_{s_2} = \bigoplus_{K=K_{\min}}^{\infty} \mathcal{D}^{+}_{K} \oplus \bigoplus_{K=K_{\min}}^{\infty} \mathcal{D}^{-}_{K} \oplus 2\int_0^{\infty\oplus} \mathcal{C}^{\varepsilon}_{s}\, ds\,, \tag{5.156}$$

where $K_{\min} = 1$ and $\varepsilon = 0$ if $\varepsilon_1 + \varepsilon_2 \in \mathbb{Z}$ and $K_{\min} = \frac{3}{2}$ and $\varepsilon = \frac{1}{2}$ otherwise. The coupling of discrete states $k \in \frac{\mathbb{N}}{2}$ with continuous states $\varepsilon \in \{0, \frac{1}{2}\}$ and $0 < s < \infty$ gives

$$\mathcal{D}^{\pm}_{k} \otimes \mathcal{C}^{\varepsilon}_{s} = \bigoplus_{K=K_{\min}}^{\infty} \mathcal{D}^{\pm}_{K} \oplus \int_0^{\infty\oplus} \mathcal{C}^{\varepsilon'}_{s'}\, ds'\,, \tag{5.157}$$

where $K_{\min} = 1$ and $\varepsilon' = 0$ if $k + \varepsilon$ is an integer and $K_{\min} = \frac{3}{2}$ and $\varepsilon' = \frac{1}{2}$ otherwise. The Clebsch-Gordan coefficients for $\mathrm{SU}(1,1)$ can be defined, and explicit formulas for their calculation can be found in [130]. However, due to the noncompactness of $\mathrm{SU}(1,1)$ and the different representation series, their explicit calculation is more complicated than in the $\mathrm{SU}(2)$ case.

5.1.7 Spacelike faces

We consider now the imposition of the quantized simplicity constraints in the quantum theory. For the Lorentz-invariant part Eq.(5.43), we use Eq.(5.145) to obtain

$$\begin{aligned}\hat{\mathring{F}}_1\, f^{(a,b)} &= \left[(\gamma - i)\,\widehat{\pi\omega} - (\gamma + i)\,\widehat{\bar{\pi}\bar{\omega}}\right] f^{(a,b)} = \frac{\hbar}{i}\left[(\gamma - i)\,[a+1] - (\gamma + i)\,[b+1]\right] f^{(a,b)} \\ &= \frac{\hbar}{i}\left[\gamma[a-b] - i[a+b+2]\right] f^{(a,b)}\,. \end{aligned}\tag{5.158}$$

In terms of the labels (n, p), we have $a - b = -2n$ and $a + b + 2 = 2ip$, and thus we get

$$\hat{\mathring{F}}_1\, f^{(a,b)} = \frac{\hbar}{i}\left[-2\gamma n + 2p\right] f^{(a,b)} \overset{!}{=} 0 \qquad \Leftrightarrow \qquad p = \gamma n\,, \tag{5.159}$$

which is the well-known result from the EPRL model. Note that this provides a new way of describing spacelike faces in a nonstandard gauge and hence is interesting by itself. However, it is important to remember that our solution states $f_{j,m}^{(n,\gamma n)}$ (the master constraint not yet imposed) are not to be confused with the states one obtains with the standard time gauge. Those states are also denoted in the same way [or as $|(n,\gamma n);j,m\rangle$] but are very different states, because they diagonalize $\vec{L}^2$ and not $Q_{\mathfrak{su}(1,1)}$. How to connect those states (when $j=-k$ for the discrete series) can be found in [57] and [58] or [122].

5.1.8 Timelike faces

For the dual constraint $\mathring{G}_1$, one obtains now similarly

$$\hat{\mathring{G}}_1\, f^{(a,b)} = \left[(\gamma-i)\,\widehat{\pi\omega} + (\gamma+i)\,\widehat{\bar{\pi}\bar{\omega}}\right] f^{(a,b)} = \frac{\hbar}{i}\left[(\gamma-i)\,[a+1] + (\gamma+i)\,[b+1]\right] f^{(a,b)}$$
$$= \frac{\hbar}{i}\left[\gamma[a+b+2] - i[a-b]\right] f^{(a,b)} \tag{5.160}$$

and again in terms of the labels (n,p), we have $a+b+2=2ip$ and $a-b=-2n$, and thus we get

$$\hat{\mathring{G}}_1\, f^{(a,b)} = 2\hbar\,[\gamma p + n]\, f^{(a,b)} \stackrel{!}{=} 0 \qquad \Leftrightarrow \qquad p = -\frac{n}{\gamma}\,. \tag{5.161}$$

This result was also found in [57] and [58], and we will see in section 5.1.11 that those states indeed can be associated to timelike faces. This is one of the main results of our research. It not only confirms the solution found in [57] and [58] but, in fact, provides or more rigorous derivation, since it does not resort to some sort of large spin argument, which is typical for the coherent state approach to the imposition of the simplicity constraints. However, we will also see that we do not necessarily need those dual solutions in order to obtain timelike area spectra on the reduced Hilbert space. We will see that we can stay within solutions with $n=\gamma p$ and still obtain faces with negative area eigenvalues on the reduced Hilbert space.

5.1.9 Master constraint

Compared with the solutions to the covariant simplicity constraints F_1 and G_1, the more interesting part follows now when we study the master constraint Eq.(5.105) and how to solve it in the quantum theory,

$$\mathbf{M} = \Big(C_{\mathrm{SL}(2,\mathbb{C})} - 2\,Q_{\mathfrak{su}(1,1)}\Big) + |\pi\omega|^2\,. \tag{5.162}$$

Since we have already expressed this constraint in terms of the Casimirs, we only have to find a proper quantization of the last term. One finds [55] that the quantization of $|\pi\omega|^2$ should be given by

$$\hat{\omega}^A\hat{\pi}_A\hat{\bar{\pi}}_{\dot{B}}\hat{\bar{\omega}}^{\dot{B}} = -\omega^A\frac{\partial}{\partial\omega^A}\frac{\partial}{\partial\bar{\omega}^{\dot{B}}}\,\bar{\omega}^{\dot{B}}\,. \tag{5.163}$$

Acting with Eq.(5.163) on a state, we get

$$-\,\omega^A\frac{\partial}{\partial\omega^A}\frac{\partial}{\partial\bar{\omega}^{\dot{B}}}\,\bar{\omega}^{\dot{B}}\,f^{(a,b)} = -\omega^A\frac{\partial}{\partial\omega^A}\left(\bar{\omega}^{\dot{B}}\frac{\partial}{\partial\bar{\omega}^{\dot{B}}}+2\right)f^{(a,b)} = -a(b+2)\,f^{(a,b)}\,, \tag{5.164}$$

where $-a(b+2)$ gives $(n^2+2n+1+p^2)$ when we use states in (n,p) with non-negative n. If we use states from $(-n,-p)$, with $n\in\frac{\mathbb{N}_0}{2}$, then this gives $-a(b+2) = (n^2-2n+1+p^2)$. This distinction is important given our knowledge about the solutions of the area matching constraint Eq.(5.146). Now, what is the action of those two Casimirs on a general state $f^{(n,p)}$? The $\mathfrak{sl}(2,\mathbb{C})$ Casmir $C_{\mathrm{SL}(2,\mathbb{C})} = C_2$ was given in Eq.(5.136) and gives

$$(\vec{L}^2 - \vec{K}^2)\,f^{(n,p)} = (n^2-1-p^2)\,f^{(n,p)}\,, \tag{5.165}$$

which, as we have already pointed out, is not sensitive to the change between (n,p) and $(-n,-p)$, and the $\mathfrak{su}(1,1)$ Casimir $Q_{\mathfrak{su}(1,1)}$ gives with the plus convention

$$((L^3)^2-(K^1)^2-(K^2)^2)\,f^{(n,p)}_{j,m} = j(j+1)\,f^{(n,p)}_{j,m}\,. \tag{5.166}$$

One can show that this operator is also invariant with respect to the change between (n,p) and $(-n,-p)$. Hence, we finally obtain

$$\widehat{\mathbf{M}}\,f^{(n,p)}_{j,m} = \left[2n(n+1)-2j(j+1)\right]\,f^{(n,p)}_{j,m} \overset{!}{=} 0 \tag{5.167}$$

and

$$\widehat{\mathbf{M}}\,f^{(-n,-p)}_{j,m} = \left[2n(n-1)-2j(j+1)\right]\,f^{(-n,-p)}_{j,m} \overset{!}{=} 0\,. \tag{5.168}$$

In the standard time gauge, where the states $f^{(n,p)}_{j,m}$ diagonalize the $\mathfrak{su}(2)$ Casimir $\vec{L}^2$, the master constraint is solved by $n=j$. The solution with $n=-(j+1)$ does not occur in the decomposition Eq.(5.148). Even if we use that the representations (n,p) and $(-n,-p)$ are unitarily equivalent, one finds that with $n=-n=j+1$ we have $j=n-1<n$, which again does not occur in the decomposition Eq.(5.148), and hence $n=j$ is the only available solution. Now, in contrast to the SU(2) case, the spectrum of $Q_{\mathfrak{su}(1,1)}$ is determined by the four series $\mathcal{D}_k^{\pm}$ and $\mathcal{C}_s^{\varepsilon}$. Can the master constraint Eq.(5.167) and Eq.(5.168) be solved with any of these states? Recall that for the principal series of the unitary irreducible representations of $\mathrm{SL}(2,\mathbb{C})$ the parameter n is an integer or half-integer. A *priori* we can assume positive and negative values alike. But for $n(n\pm1)$, there is a minimum value given by $-1/4$ for $n=-1/2$ or $n=1/2$. Otherwise, we have $n(n\pm1)\geq 0$ for all other n. Now, if we consider first the states of the two continuous series $\mathcal{C}_s^{\varepsilon}$ (with $\varepsilon\in\{0,\frac{1}{2}\}$), we see that Eq.(5.167) and Eq.(5.168) with the plus convention for the $\mathfrak{su}(1,1)$ Casimir $Q_{\mathfrak{su}(1,1)}$ lead to

$$\left[n(n\pm1)+\frac{1}{4}+s^2\right]\stackrel{!}{=}0 \tag{5.169}$$

for both ε. It is clear that for most n there is no solution to this condition. The only possible singular solution occurs for $n=\pm\frac{1}{2}$ and $\varepsilon=0$, which is, however, of no relevance to us, since we consider $n\geq0$ [even though we can solve Eq.(5.168) with $n=\frac{1}{2}$, this state will later be ruled out when solving the reduced area matching constraint]. Hence, for real $s\in\mathbb{R}_{\geq0}$ we see that the master constraint cannot be solved by the states of the continuous series and the plus convention for $Q_{\mathfrak{su}(1,1)}$. Note that this analysis transfers exactly to the other half-link in the (λ,σ) variables.

Now, for the states of the discrete series $\mathcal{D}_k^{\pm}$, we obtain for Eq.(5.167) with $j=-k$

$$[n(n+1)-k(k-1)]\stackrel{!}{=}0 \tag{5.170}$$

and see that the master constraint can be satisfied by the solutions

$$k=n+1 \qquad , \qquad k=-n\,. \tag{5.171}$$

However, since we have $k\in\frac{\mathbb{N}}{2}$ and $n\in\frac{\mathbb{N}_0}{2}$, the second solution is not admissible. The first solution restricts furthermore the occurrence of the non-normalizable state $k=\frac{1}{2}$. For state with $(-n,-p)$, Eq.(5.168) gives with $j=-k$

$$[n(n-1)-k(k-1)]\stackrel{!}{=}0\,, \tag{5.172}$$

and we see that this is satisfied by the solutions

$$k=n \qquad , \qquad k=-n+1\,. \tag{5.173}$$

Again, the second solution is not compatible with our range of parameter values. Using then the first solution in Eq.(5.171), we see that all the discrete states in $\mathcal{D}_k^{\pm}$ with $k \in \{1, \frac{3}{2}, 2, \cdots\}$ and $n \in \frac{\mathbb{N}_0}{2}$ solve the master constraint Eq.(5.167). For the first solution of Eq.(5.173), we see that $k, n \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \cdots\}$ solves the master constraint Eq.(5.168). However, we will see in the next section why it is preferable to change from the plus convention for $Q_{\mathfrak{su}(1,1)}$ to the minus convention and to solve the master constraint using the continuous states instead.

5.1.10 Reduced area matching constraint

Now, we will consider the full reduced Hilbert space by imposing the reduced area matching constraint on the states that solve the simplicity constraints on the two half-links. From Eq.(5.146) and Eq.(5.147), we learned that the area matching constraint imposes the conditions $n_t = -n_s$ and $p_t = -p_s$ on the tensor product states

$$f_{\text{left}}^{(n_s,p_s)} \otimes f_{\text{right}}^{(n_t,p_t)} \,. \tag{5.174}$$

However, since we prefer to work with non-negative values for the n_i labels we choose from the beginning states of the form

$$f_{\text{left}}^{(n_s,p_s)} \otimes f_{\text{right}}^{(-n_t,-p_t)} \,, \tag{5.175}$$

which leads to the area matching condition $n_t = n_s \in \frac{\mathbb{N}_0}{2}$ and $p_t = p_s$. Since we already know from the simplicity constraints that $p_s = \gamma n_s$ or $p_s = -\frac{n_s}{\gamma}$ and similarly for the target half-link [which are not sensitive to a change between (n, p) and $(-n, -p)$], we see that the area matching constraint reduces to only one condition, namely, $n_t = n_s$. After imposing the master constraint on both half-links, we are left with the following possibilities on which we can impose the reduced area matching. First, we consider the case with $-j_s = k_s = n_s + 1$ and $-j_t = k_t = n_t$. Solving the reduced area matching

$$\hat{C}_{\text{red}} \triangleright \left(f_{n_s+1,m_s}^{(n_s,p_s(n_s)),\pm} \otimes f_{n_t,m_t}^{(-n_t,-p_t(n_t)),\pm} \right) \stackrel{!}{=} 0 \tag{5.176}$$

leads to $n_t = n_s$ and hence both n_i must be $n_i \in \frac{\mathbb{N}}{2}$. It furthermore implies $k_s = k_t + 1$ and hence $k_s \in \{\frac{3}{2}, 2, \frac{5}{2}, \cdots\}$ and $k_t \in \{\frac{1}{2}, 1, \frac{3}{2}, \cdots\}$. From this, we obtain $K = k_s + k_t = 2n_s + 1$. Using now the decomposition Eq.(5.154), we find that we can obtain all the (integer) discrete states $\mathcal{D}_K^\pm$ with $K \geq 2$ as solutions to Eq.(5.176) from states satisfying the simplicity constraints. Explicitly, we have

$$f_{n_s+1,m_s}^{(n_s,p_s(n_s)),\pm} \otimes f_{n_s,m_t}^{(-n_s,-p_t(-n_s)),\pm} = \bigoplus_{K=2n_s+1}^{\infty} \mathcal{D}_K^\pm \,. \tag{5.177}$$

Changing the order of the two states in the tensor product gives the same result. Now, let us consider the action of the reduced area matching operator on discrete states with opposite signs. Hence,

$$\hat{C}_{\text{red}} \triangleright \left(f_{n_s+1,m_s}^{(n_s,p_s(n_s)),\pm} \otimes f_{n_t,m_t}^{(-n_t,-p_t(n_t)),\mp} \right) \stackrel{!}{=} 0 \,. \tag{5.178}$$

Using again the solution $n_t = n_s$, we find that $k_s + k_t = 2n_s + 1 \in \mathbb{Z}$, and hence for the decomposition Eq.(5.155), we get $K_{\text{min}} = 1$ and $\varepsilon = 0$. Furthermore, we have $k_s - k_t = 1$ and $k_t - k_s = -1$, and hence one finds that those states that satisfy the simplicity constraints and the reduced area matching are given by

$$f_{n_s+1,m_s}^{(n_s,p_s(n_s)),\pm} \otimes f_{n_s,m_t}^{(-n_s,-p_t(n_s)),\mp} = \mathcal{D}_1^\pm \, \oplus \int_0^{\infty \oplus} \mathcal{C}_s^0 \, ds \,. \tag{5.179}$$

Hence, we see that we do not obtain all the states we need to span $\mathrm{SU}(1,1)$ spin networks, i.e., all the states that appear in the harmonic analysis of functions on $\mathrm{SU}(1,1)$. We only obtain the discrete states $\mathcal{D}_K^\pm$ with $K \in \mathbb{N}$ and are missing all the half-integral values $K \in \frac{\mathbb{N}}{2}$. Similarly, we only obtain the even continuous states $\mathcal{C}_s^0$, but we are missing the odd states with $\varepsilon = \frac{1}{2}$. This is a result of the reduced area matching constraint, which does not allow for tensor-product states that have integer labels on the left factor and half-integer labels on the right factor (or vice versa). Hence, in the decomposition, only states with integer labels and/or states with $\varepsilon = 0$ appear. However, this problem can be solved as follows. The requirement that we need all unitary irreducible Plancherel representations of $\mathrm{SU}(1,1)$ forces us to choose the minus convention in Eq.(5.149). This gives for the master constraint now the conditions

$$\widehat{\mathbf{M}} \, f_{j,m}^{(\pm n,\pm p)} = [2n(n \pm 1) + 2j(j+1)] \; f_{j,m}^{(\pm n,\pm p)} \stackrel{!}{=} 0 \,, \tag{5.180}$$

which can now not be satisfied by the states of the discrete series anymore but by the states of the continuous series. For the states $f_{s,m}^{(\pm n,\pm p)}$, one obtains the solution

$$s^\pm(n) = \frac{\sqrt{(2n \pm 1)^2 - 2}}{2} \,. \tag{5.181}$$

For the states $f_{s,m}^{(n,p)}$, this is strictly positive for $n \in \frac{\mathbb{N}}{2}$, hence $n = 0$ is ruled out, and for the states $f_{s,m}^{(-n,-p)}$, we have to restrict n such that $n \in \{\frac{3}{2}, 2, \frac{5}{2}, \cdots\}$. The reason why we can now use those states to obtain the full reduced Hilbert space is that neither the simplicity constraints nor the reduced area matching constraint restricts the labels ε_s and ε_t, which, according to Eq.(5.156), determine which states appear in the decomposition, i.e., $K_{\min}$ and ε are now determined by $\varepsilon_s + \varepsilon_t$, which can now be freely chosen to be integral or half-integral. Explicitly, we find that the simplicity and reduced area matching constraints are now solved by the states

$$\Psi^{n_s,\varepsilon_s,\varepsilon_t}_{m_s,m_t} \equiv f^{(n_s,p_s(n_s)),\varepsilon_s}_{s_1^+(n_s),m_s} \otimes f^{(-n_s,-p_t(n_s)),\varepsilon_t}_{s_2^-(n_s),m_t}\,, \tag{5.182}$$

where now $n_s \geq \frac{3}{2}$. Again, we can now freely choose whether $\varepsilon_s + \varepsilon_t$ is integral, which gives from Eq.(5.156) the states

$$\bigoplus_{K=1}^{\infty} \mathcal{D}^+_K \oplus \bigoplus_{K=1}^{\infty} \mathcal{D}^-_K \oplus 2\int_0^{\infty\oplus} \mathcal{C}^0_s\, ds\,, \tag{5.183}$$

or whether $\varepsilon_s + \varepsilon_t$ is half-integral, which gives the states

$$\bigoplus_{K=\frac{3}{2}}^{\infty} \mathcal{D}^+_K \oplus \bigoplus_{K=\frac{3}{2}}^{\infty} \mathcal{D}^-_K \oplus 2\int_0^{\infty\oplus} \mathcal{C}^{\frac{1}{2}}_s\, ds \tag{5.184}$$

and thus we see that we obtain all the discrete states with $K \in \{1, \frac{3}{2}, 2, \frac{5}{2}, \cdots\}$ as well as all the continuous states spanning our reduced Hilbert space. Note that, due to the integral over the continuous parameter s in both decompositions Eq.(5.183) and Eq.(5.184), we obtain all continuous states for arbitrary $s \in \mathbb{R}_{\geq 0}$ in the coupled basis and not just those that satisfy the discreteness constraint Eq.(5.181). This can be seen explicitly by considering the Clebsch-Gordan coefficients of the above decompositions. Even when both states in the decoupled basis satisfy the condition Eq.(5.181), one obtains nonzero Clebsch-Gordan coefficients for general $s \in \mathbb{R}_{\geq 0}$ in the coupled basis. This means in particular that the reduced Hilbert space includes indeed all the necessary SU(1,1) Plancherel representations that are necessary to expand states in the holonomy representation, i.e., certain $\mathbb{C}$-valued functions on SU(1,1), in terms of a spin network basis. Thus, this gives perfect agreement of our reduced Hilbert space and the quantization of 3D Lorentzian gravity [113, 116]. Note, that for such spin network states we can obtain links that are labeled by arbitrary continuous states with $s \in \mathbb{R}_{\geq 0}$. On the level of the coupled basis of the reduced Hilbert space, one then finds that the area associated with such links can be continuous, again in agreement with the 3D Lorentzian case. However, those states are not physical, in the sense that they do not satisfy simplicity constraints and area matching, i.e., they are not of the form Eq.(5.182). If we consider a general $\mathrm{SU}(1,1)$ spin network state, which is labeled by continuous s values, we know from the inverse decompositions of Eq.(5.183) and Eq.(5.184) how to embed those states into our solution space of simplicity and area matching constraints via Eq.(5.204) and Eq.(5.205). This is basically the Lorentzian version of the Livine-Dupuis map known from the standard EPRL model and shows nicely how to embed the three-dimensional Lorentzian Ponzano-Regge model into our four-dimensional setting. This gives, furthermore, an explicitly mechanism that shows how we can have continuous eigenvalues on the 3D level, but when we embed those states into the solution space of simplicity and area matching constraint those eigenvalues become strictly discrete. Note that one does not need this decomposition explicitly to calculate, for example, the area operator eigenvalues of the state Eq.(5.182) as we will see in the next section. We consider it another important result of our work that we obtain a reduced Hilbert space with enough states such that one obtains a valid $\mathrm{SU}(1,1)$ spin network decomposition. Compared with the standard time gauge case, where one solves both simplicity constraints on each half-link and obtains already all the necessary $\mathrm{SU}(2)$ states on each half-link (which are then glued using the area matching), it was necessary in our case to understand that, even though we just obtain a subclass of representations per half-link as solutions to the simplicity constraints, all the required $\mathrm{SU}(1,1)$ states arise after the decomposition of the tensor product states and imposition of the reduced area matching.

5.1.11 Area spectra

In Lorentzian spinfoam models [98, 99, 100, 57, 58] and LQG, there are two major issues concerning the spectra of geometrical operators and the area operator in particular. The first is about the question of whether those operators have discrete or continuous spectra [110, 111, 112], and the second concerns the appearance of the Barbero-Immirzi parameter [109]. The first problem can, at least in four dimensions, be further separated into whether we are talking about spectra on the kinematical level or at the level of the physical Hilbert space; see, for example, [131, 132].

In LQG, the area operator is essentially given by (the square root of) the $\mathfrak{su}(2)$ Casimir since the (densitized) flux operators satisfy a $\mathfrak{su}(2)$ algebra and thus the quantization of the classical expression for the area (squared) leads explicitly to $\vec{L}^2$ (with a γ-dependent prefactor), [2, 3]. This leads then to the discrete spectra for the area (on the kinematical Hilbert space). However, there have been other proposals for the area operator within covariant formulations of LQG [111, 100] that potentially lead to continuous and γ-independent area spectra. That there are cases when the Barbero-Immirzi parameter disappears from the area spectra was also observed in [109] and is a result we will discuss in this section using our twistorial description. Our definition of the area operator was given in Eq.(5.40) by the Plebanski 2-form Σ, and we consider

$$\hat{\mathcal{A}}^2 \equiv \frac{1}{2}\,\hat{\Sigma}_{IJ}\hat{\Sigma}^{IJ} = \frac{\gamma^2}{4}\ \mathrm{Re}\left(\frac{(\pi\omega)^2}{(\gamma+i)^2}\right) . \tag{5.185}$$

Using the vector representation in terms of rotation and boost generators allows us to understand its reduction classically as follows. Recall that we have associated the $\mathfrak{sl}(2,\mathbb{C})$ generators with B^{IJ} as in Eq.(5.22). Furthermore, we have

$$\Sigma = -\frac{\gamma^2}{1+\gamma^2}\left(* + \frac{1}{\gamma}\right) B\,, \tag{5.186}$$

which, together with Eq.(5.185), gives

$$\begin{aligned}\mathcal{A}^2 &= \frac{\gamma^4}{2(1+\gamma^2)^2}\left((*B)+\frac{B}{\gamma}\right)_{IJ}\left((*B)+\frac{B}{\gamma}\right)^{IJ} \\ &= \frac{\gamma^4}{(1+\gamma^2)^2}\left(\left(\frac{1}{\gamma^2}-1\right)(\vec{L}^2-\vec{K}^2)+\frac{1}{\gamma}(*B)_{IJ}B^{IJ}\right) .\end{aligned} \tag{5.187}$$

Using that

$$(*B)_{IJ}B^{IJ} = -4\left(L^1K^1+L^2K^2+L^3K^3\right) \tag{5.188}$$

we get with the simplicity constraints $\Sigma^{3i}=0$, i.e.,

$$K^3 = -\gamma L^3 \quad , \quad L^1 = \gamma K^1 \quad , \quad L^2 = \gamma K^2\,, \tag{5.189}$$

that

$$\vec{L}^2 - \vec{K}^2 = (1-\gamma^2)\, Q_{\mathfrak{su}(1,1)}\,, \tag{5.190}$$

$$(*B)_{IJ}B^{IJ} = 4\gamma\, Q_{\mathfrak{su}(1,1)}\,, \tag{5.191}$$

which finally leads to

$$\mathcal{A}^2 = \gamma^2\, Q_{\mathfrak{su}(1,1)}\,. \tag{5.192}$$

Now, if we use the dual simplicity constraints $(*\Sigma)^{3i} = 0$, or

$$K^3 = \frac{1}{\gamma} L^3 \quad , \quad L^1 = -\frac{1}{\gamma} K^1 \quad , \quad L^2 = -\frac{1}{\gamma} K^2 \,, \tag{5.193}$$

we obtain instead

$$\vec{L}^2 - \vec{K}^2 = \left(1 - \frac{1}{\gamma^2}\right) Q_{\mathfrak{su}(1,1)} \,, \tag{5.194}$$

$$(*B)_{IJ} B^{IJ} = -\frac{4}{\gamma} Q_{\mathfrak{su}(1,1)} \tag{5.195}$$

and hence

$$\mathcal{A}^2 = -Q_{\mathfrak{su}(1,1)} \,. \tag{5.196}$$

This already indicates that the Barbero-Immirzi parameter γ seems to disappear in the spectrum for states that solve the dual simplicity constraints $(*\Sigma)^{3i} = 0$, similarly to the results found in [109][7]. Now, let us consider the quantized area operator in the twistorial parametrization. Using the action of $\widehat{\pi\omega}$ and $\widehat{\bar{\pi}\bar{\omega}}$ on the homogeneous functions $f^{(a,b)} \in \mathcal{H}^{(a,b)}$

$$\widehat{\pi\omega} \triangleright f^{(a,b)} = -i\hbar \, [a+1] \, f^{(a,b)} \qquad , \qquad \widehat{\bar{\pi}\bar{\omega}} \triangleright f^{(a,b)} = -i\hbar \, [b+1] \, f^{(a,b)} \,, \tag{5.197}$$

we obtain with Eq.(5.185) and Eq.(5.135) that

$$\begin{aligned}
\hat{\mathcal{A}}^2 \triangleright f^{(a,b)} &= \frac{\gamma^2}{8} \left(\frac{\widehat{\pi\omega}\,\widehat{\pi\omega}}{(\gamma+i)^2} + \frac{\widehat{\bar{\pi}\bar{\omega}}\,\widehat{\bar{\pi}\bar{\omega}}}{(\gamma-i)^2} \right) \triangleright f^{(a,b)} \\
&= -\frac{\hbar^2}{8} \frac{\gamma^2}{(\gamma^2+1)^2} \Big[(\gamma^2-1)(a^2+b^2+2a+2b+2) - 2i\gamma(a^2-b^2+2a-2b) \Big] \, f^{(a,b)} \\
&= -\frac{\hbar^2}{4} \frac{\gamma^2}{(\gamma^2+1)^2} \Big[(\gamma^2-1)(n^2-p^2) - 4\gamma n p \Big] \, f^{(a,b)} \,.
\end{aligned} \tag{5.198}$$

[7] However, note that the same reasoning works for the SU(2) case, where we can equally consider $\Sigma^{0i} = 0$ or the dual $(*\Sigma)^{0i} = 0$ but with a timelike normal vector N^I, and we still obtain that in the first case we have a γ dependence, i.e., $\mathcal{A}^2_{\mathrm{SU}(2)} = \gamma^2 \, \vec{L}^2$, and in the other case, we have a sign flip, and γ disappears, i.e., $\mathcal{A}^2_{\mathrm{SU}(2)} = -\vec{L}^2$.

Now, if we consider the solutions to the simplicity constraints, $p = \gamma n$ for $F_1 = 0$ and $p = -n/\gamma$ for $G_1 = 0$, we obtain

$$\hat{\mathcal{A}}^2 \triangleright f^{(n,\gamma n)} = \frac{\hbar^2}{4}\gamma^2 n^2 \, f^{(n,\gamma n)} \tag{5.199}$$

and

$$\hat{\mathcal{A}}^2 \triangleright f^{(n,-n/\gamma)} = -\frac{\hbar^2}{4} n^2 \, f^{(n,-n/\gamma)} \tag{5.200}$$

respectively. First, note that we find that, indeed, the area eigenvalues switch sign between the two branches with $p = \gamma n$ and $p = -n/\gamma$, respectively. Hence, our identification of the constraints (F_1, F_2) with the spacelike case and the constraints (G_1, G_2) with the timelike case seems justified. Furthermore, we again confirm that the area spectrum for timelike faces seems to not depend on γ. Second, note the different nature between Eq.(5.199) and Eq.(5.200) on the one hand and Eq.(5.192) and Eq.(5.196) on the other. For the calculation in Eq.(5.199) and Eq.(5.200), we have used the covariant version of the area operator Eq.(5.185) and then imposed the solutions of the simplicity constraints on the area eigenvalues, which leads us, in the spacelike case as well as in the timelike case, to discrete area eigenvalues, which is in contrast to the statement often made in the literature, e.g, [109, 113, 106, 131], that in Lorentzian models we have necessarily continuous spectra, due to the noncompactness of the gauge group. In the formulas leading to Eq.(5.192) and Eq.(5.196), on the other hand, we have first reduced the operator by the simplicity constraints. If we use now for the (reduced) area operators Eq.(5.192) and Eq.(5.196) instead, we first would notice that this operator does not act on the covariant labels (a, b) but on the $\mathrm{SU}(1,1)$ labels $j(k)$ and $j(s)$. In this situation, one might wonder whether we actually recover continuous spectra for the continuous states with $j(s)$ and $Q^c_{\mathrm{SU}(1,1)} = -j(j+1) = \frac{1}{4} + s^2$, which is related to our discussion about whether we have all the continuous states available in the reduced Hilbert space or just a discrete subset. We will see now that both ways, reducing the eigenvalues of the covariant area operator or first reducing the area operator, are consistent and lead in both cases to a discrete area eigenvalue spectrum for those states that solve the area matching and simplicity constraints.

Consider first a state of the form Eq.(5.182) with $p_s = \gamma n_s = p_t$. Then, $Q_{\mathfrak{su}(1,1)}$ acts as

$$
\begin{aligned}
&Q_{\mathfrak{su}(1,1)} \triangleright \left(f^{(n_s,\gamma n_s),\varepsilon_s}_{s_1^+(n_s),m_s} \otimes f^{(-n_s,-\gamma n_s),\varepsilon_t}_{s_2^-(n_s),m_t} \right) \\
&= \left(Q_{\mathfrak{su}(1,1)} \triangleright f^{(n_s,\gamma n_s),\varepsilon_s}_{s_1^+(n_s),m_s} \right) \otimes f^{(-n_s,-\gamma n_s),\varepsilon_t}_{s_2^-(n_s),m_t} + f^{(n_s,\gamma n_s),\varepsilon_s}_{s_1^+(n_s),m_s} \otimes \left(Q_{\mathfrak{su}(1,1)} \triangleright f^{(-n_s,-\gamma n_s),\varepsilon_t}_{s_2^-(n_s),m_t} \right) \\
&= \left(\frac{1}{4} + (s_1^+(n_s))^2 + \frac{1}{4} + (s_2^-(n_s))^2 \right) \times f^{(n_s,\gamma n_s),\varepsilon_s}_{s_1^+(n_s),m_s} \otimes f^{(-n_s,-\gamma n_s),\varepsilon_t}_{s_2^-(n_s),m_t} \\
&= 2n_s^2 \; f^{(n_s,\gamma n_s),\varepsilon_s}_{s_1^+(n_s),m_s} \otimes f^{(-n_s,-\gamma n_s),\varepsilon_t}_{s_2^-(n_s),m_t}
\end{aligned}
\tag{5.201}
$$

and hence with $\hat{\mathcal{A}}^2 = \gamma^2 \, Q_{\mathfrak{su}(1,1)}$, we get

$$
\hat{\mathcal{A}}^2 \triangleright \left(f^{(n_s,\gamma n_s),\varepsilon_s}_{s_1^+(n_s),m_s} \otimes f^{(-n_s,-\gamma n_s),\varepsilon_t}_{s_2^-(n_s),m_t} \right) = 2\,\gamma^2 n_s^2 \; \left(f^{(n_s,\gamma n_s),\varepsilon_s}_{s_1^+(n_s),m_s} \otimes f^{(-n_s,-\gamma n_s),\varepsilon_t}_{s_2^-(n_s),m_t} \right) . \tag{5.202}
$$

Comparing this with Eq.(5.199), where the missing factor of $\hbar^2$ is included in $Q_{\mathfrak{su}(1,1)}$ and up to an irrelevant factor of $\frac{1}{8}$, we showed the consistency between the two ways of obtaining the area eigenvalues. If we consider now similarly the dual case with $p_s = -\frac{n_s}{\gamma} = p_t$, we have $\hat{\mathcal{A}}^2 = -Q_{\mathfrak{su}(1,1)}$, cf. Eq.(5.196), and we obtain instead

$$
\begin{aligned}
\hat{\mathcal{A}}^2 \triangleright \left(f^{(n_s,-\frac{n_s}{\gamma}),\varepsilon_s}_{s_1^+(n_s),m_s} \otimes f^{(-n_s,\frac{n_s}{\gamma}),\varepsilon_t}_{s_2^-(n_s),m_t} \right) = \\
- 2n_s^2 \; \left(f^{(n_s,-\frac{n_s}{\gamma}),\varepsilon_s}_{s_1^+(n_s),m_s} \otimes f^{(-n_s,\frac{n_s}{\gamma}),\varepsilon_t}_{s_2^-(n_s),m_t} \right) .
\end{aligned}
\tag{5.203}
$$

This matches the result of Eq.(5.200) and γ seems to not appear. Note that, due to the area matching constraint, we must have $p_t = p_s$. Hence, if we were to consider the coupling of states with $p_s = \gamma n_s$ and $p_t = -\frac{n_s}{\gamma}$, or vice versa, the condition $p_t = p_s$ leads to the requirement that γ must be imaginary, i.e., $\gamma = \pm i$, which might be related to the self-dual Ashtekar variables that have recently been investigated in [133, 134, 135]. It is tempting to interpret this in some way as a coupling of a spacelike state on one side of the link with a timelike state on the other side. However, throughout this work, we have assumed real γ, and hence considering complex γ is merely a speculation at this level. Furthermore, it is important to note that taking γ to be complex would take us out of the unitary representations of $\mathrm{SL}(2,\mathbb{C})$.

If we want to avoid using the dual constraints $p = -\frac{n}{\gamma}$, because spacelike states as well as timelike states should be included already in just the case with $p = \gamma n$, we can consider the explicit decomposition of the solution state Eq.(5.182) into its irreducible components following Eq.(5.183) and Eq.(5.184). Acting with $Q_{\mathfrak{su}(1,1)}$ onto those irreducible states will give positive as well as negative eigenvalues of the continuous series and the discrete series, respectively. Hence, in this picture, the timelike states are associated with the discrete series states, which are composed as the tensor product of two continuous states. In the reversed direction, imagine we have a spin network decorated with SU(1,1) representations $j(k)$ or $j(s)$; then, we can think of a generalized Livine-Dupuis map[8], which maps the states of the SU(1,1) spin network into the solution states of the area matching and simplicity constraint as

$$|j(k), m\rangle \mapsto \sum_{m_s, m_t} C(n_s) f^{(n_s, p_s(n_s)), \varepsilon_s}_{s_1^+(n_s), m_s} \otimes f^{(-n_s, -p_t(n_s)), \varepsilon_t}_{s_2^-(n_s), m_t} \,, \tag{5.204}$$

or for the continuous states with $j(s)$ as

$$|j(s), m\rangle \mapsto \sum_{m_s, m_t} \tilde{C}(n_s) f^{(n_s, p_s(n_s)), \varepsilon_s}_{s_1^+(n_s), m_s} \otimes f^{(-n_s, -p_t(n_s)), \varepsilon_t}_{s_2^-(n_s), m_t} \,, \tag{5.205}$$

where $C(n_s)$ and $\tilde{C}(n_s)$ depend besides n_s on k or s and on m_s, m_t and denote the Clebsch-Gordan coefficients corresponding to the inverse of the decompositions in Eq.(5.183) and Eq.(5.184).

Finally, let us comment again on the fate of the Barbero-Immirzi parameter. We point out that we discuss here only the appearance of γ in the eigenvalues of the area operator for timelike faces and not whether the physical Hilbert space will be γ dependent or not. From Eq.(5.196) and Eq.(5.200), with the solution of the dual simplicity constraints $(*\Sigma)^{3i} = 0$, i.e., $p = -n/\gamma$, we confirmed the statement that was made in [108] and [109] that the spectrum of timelike faces does not depend on γ. However, there is a possibility that γ might actually reappear as follows. Note that when we introduce dimensionful constants the area operator $\sqrt{\hat{\mathcal{A}}^2}$ has eigenvalues proportional to the Planck length [2]; i.e., for the standard SU(2) case, we have

$$\hat{\mathcal{A}} \triangleright |j\rangle = 8\pi\gamma l_P^2 \sqrt{j(j+1)}\, |j\rangle \,, \tag{5.206}$$

with $l_P^2 = \hbar G/c^3$, and hence we see that it depends on the gravitational constant G. This certainly holds true for the spacelike faces and the space-gauge simplicity constraints

[8] In the SU(2) case, the Livine-Dupuis map embeds the SU(2) representations into the subspace of the canonical basis that satisfies the simplicity constraints as $|j, m\rangle \hookrightarrow |(j, \gamma j), j, m\rangle$.

(F_1, F_2). If we consider now the area spectrum of timelike faces, we would assume that it is proportional to either $t_P l_P$ or t_P^2, where t_P is the Planck time with $t_P^2 = l_P^2/c^2$. In either case, we again find that the spectrum is proportional to G. However, if we go back to the original Holst action we started with in Eq.(5.2) and note that there is a prefactor of $1/(16\pi G)$, then we notice that the dual simplicity constraints $(*\Sigma)^{3i} = 0$, i.e., (G_1, G_2), lead to Einstein-Cartan gravity with the dual Barbero-Immirzi parameter $\tilde{\gamma} = -1/\gamma$ and a scaled gravitational constant $\tilde{G} = G\gamma$. Now, in this situation, it appears as if γ does not appear in the area operator, but, in fact, if we consider the proportionality with $\tilde{G} = G\gamma$, we see that it still appears via the rescaling of G. Following this reasoning would imply that all our area spectra are linearly dependent on γ as in the standard SU(2) case.

5.2 A new Spinfoam model

As mentioned before, our main objective in [1] was to give a twistorial description of the Conrady-Hnybida spinfoam model [57, 58], in order to facilitate, or simplify, an asymptotic analysis. However, as we saw in the previous section, we find a different set of solutions to the simplicity constraints for timelike faces from solving the master constraint, cf. Eq.(5.181). In general, the structure of our solution states is different from the formulation using coherent states used in [57, 58], since we are using those tensor product states Eq.(5.182). Even though we do not consider an asymptotic analysis of the Conrady-Hnybida model, we can still use our results from [1] to construct a new 4D Lorentzian spinfoam model that is equivalent in spirit to [57, 58], but slightly differs in the details of the construction. Recall that the relation found in [57, 58], corresponding to Eq.(5.181), is given by

$$s(n_1) = \frac{1}{2}\sqrt{\frac{n_1^2}{\gamma^2} - 1} = \frac{1}{2\gamma}\sqrt{n_1^2 - \gamma^2}\,. \tag{5.207}$$

Our solution to the master constraint, Eq.(5.181), can be expressed in a similar form as

$$s^{\pm}(n_2) = \frac{\sqrt{(2n_2 \pm 1)^2 - 2}}{2} = \frac{1}{2\gamma}\sqrt{(2\gamma n_2)^2\left(1 \pm \frac{1}{n_2}\right) - \gamma^2}\,, \tag{5.208}$$

from which we see that for large n_2, and with the identification $n_1 = 2\gamma n_2$, the two expressions Eq.(5.207) and Eq.(5.208) actually become the same. This is no surprise and is a phenomenon already well known in the standard SU(2) setting where for large spins the EPRL - model and the FK - model also give the same solution states. However, note that $n_1 = 2\gamma n_2$ is a somewhat strange relation, because the labels n_i are integer numbers and the real Barbero-Immirzi parameter γ is usually taken as a non-zero real number.

Since our work on this topic there has been increased interest in those types of generalized EPRL - models. For example, there was a first investigation into the asymptotic analysis of the Conrady-Hnybida spinfoam model in [136] and the recent [137], investigates the underlying phase space of $\mathrm{SU}(1,1)$ spin networks, similar to our work in section 5.1.

The overall question, when thinking about including timelike contributions into a 4D Lorentzian spinfoam model, is whether the current EPRL-FK-KKL spinfoam model [13, 14, 15, 16, 17] is our final spinfoam model, or, equivalently, whether this model is our final non-perturbative definition of the dynamics of 4D Lorentzian quantum gravity. Based on several arguments in the literature, it is probably safe to say, that there is room for improvement. For example, it was argued in [138, 139] that the EPRL-model does not define a so-called rigging map, which means that in its current form the model is not a proper projector onto the physical Hilbert space of Loop quantum gravity. In other works, such as [140, 141], it is argued that one should modify the vertex amplitude of the model such that the asymptotic limit takes a certain preferred form.

What we are proposing in this section, similar in philosophy to [57, 58], is not likely to address any of the concerns raised in either [138, 139] or [140, 141]. However, what we want to do is to cure the shortcoming (if it is one) of the Lorentzian EPRL-model, that it does not allow to include timelike contributions in the spacetime triangulation. This is in stark contrast to the Causal Dynamical Triangulations approach to quantum gravity [142], where a clear distinction is made into spacelike and timelike parts of the triangulation of spacetime. In fact, the separation into spacelike and timelike contributions is crucial for the causality of the approach. Furthermore, in the 3D Lorentzian Ponzano-Regge model, cf. [113] or [114, 115, 116], one finds that timelike building blocks do contribute to the path integral and are necessary to provide the correct quantization of the system. The solution to the quantized flatness constraint $F = 0$ on the spatial 2-surface in 3D Lorentzian gravity leads to the $\mathrm{SU}(1,1)$ 6j-symbol as the vertex amplitude of the Lorentzian Ponzano-Regge spinfoam model. This vertex amplitude is a function of spacelike and timelike contributions. Even if we were to restrict the allowed $\mathrm{SU}(1,1)$ representations on the spatial slice to be only of the continuous series, having positive area with $\hat{A}^2 \equiv Q_{\mathfrak{su}(1,1)}$, the solutions to $\hat{F}|\Psi\rangle = 0$ would generate timelike contributions, i.e., states of the discrete series with negative area.

Similarly, in its covariant form, i.e., without using a 3+1 split of the spacetime manifold, the action of general relativity in four dimensions is a function of the 4D Ricci tensor $R_{\mu\nu}$ and in order to capture the full geometry of the manifold we transport vectors along little parallelograms to measure the curvature. To get the full information about the spacetime geometry we have to measure the curvature not only along spacelike parallelograms but also along timelike parallelograms. Of course, with a $3+1$ split on can obtain the full 4D curvature from the spatial 3D curvature, when we have all the necessary information about the extrinsic curvature. This is the idea followed in a Hamiltonian approach. But a spinfoam model should be closer to the covariant picture and hence be able to make a statement about curvature along timelike contributions. A simple example that the curvature in the timelike direction is very important is simple cosmology. In flat $k=0$ Friedmann-Lemaitre-Robertson-Walker cosmology the spatial slice is flat, but there is a nontrivial total 4D curvature, which comes from the curvature along the timelike direction.

These are sufficient reasons to be convinced that in a 4D Lorentzian spinfoam model we also should a able to include timelike contributions. For a final judgment, however, we would need a full asymptotic analysis.

5.2.1 Spinfoam quantization of BF-theory and EPRL model

We will now briefly collect a few facts about the spinfoam quantization of $G = \mathrm{SL}(2,\mathbb{C})$ BF-theory, which is a topological theory and is the basis for both the Lorentzian EPRL-FK-KKL spinfoam model, as well as the Conrady-Hnybida generalization thereof. The BF-action for a 4D manifold M is given by

$$S_{\mathrm{BF}}[B,A] = \int_M \mathrm{Tr}\left(B \wedge F[A]\right), \tag{5.209}$$

where $B^{IJ} = -B^{JI}$ is a $\mathfrak{g}$-valued 2-form and $F[A]$ is the curvature of a G - connection A. This theory defines a diffeomorphism invariant gauge theory, since no metric is used. The equations of motion are given by $F = D_A B = 0$, which are solved by flat connections A, [3]. The trace is defined by the Cartan-Killing form of the gauge group. If the B - field is constrained to satisfy $B = *(e \wedge e)$, where e denotes the tetrad fields, then Eq.(5.209) reduces to the first order Einstein-Cartan action and the B - field is called simple, hence the name **simplicity constraints**. These constraints break much of the gauge invariance of topological BF-theory and unlock the metric degrees of freedom of general relativity.

The path integral of BF-theory is

$$Z_{\mathrm{BF}} = \int [dA][dB]\, e^{i \int_M \mathrm{Tr}(B \wedge F[A])} \tag{5.210}$$

and by integrating out the B-field one obtains

$$Z_{\rm BF} \propto \int [dA]\, \delta(F[A])\,. \tag{5.211}$$

Hence, we integrate over flat connections A. Using a spinfoam quantization for this model we discretize the manifold M, which gives a triangulation Δ and a dual 2-complex $\Delta^* = \{v, e, f\}$ with vertices v, edges e and faces f and instead of the smooth variables B and A we use the smeared, or integrated variables B_f and $g_e[A]$

$$B_f^{IJ} = \int_f B^{IJ} \,\in\, \mathfrak{g} \qquad , \qquad g_e[A] = P\,\exp\left(\int_e A\right) \,\in\, G\,. \tag{5.212}$$

We see that the B - field, as a 2-form, is naturally integrated over a 2-surface f and $g_e[A]$ is the holonomy of the connection A along an edge e. The holonomies $g_e[A]$ around a face f are used to construct a 'face holonomy' G_f

$$G_f = g_{e_1} g_{e_2} \cdots g_{e_n} = P \exp\left(\oint_{\partial f} A\right)\,, \tag{5.213}$$

which measures the curvature of A associated with the face f. In terms of the B_f and G_f variables the action Eq.(5.209) on a triangulation Δ becomes

$$S^{\Delta}_{\rm BF}[B_f, G_f[A]] = \sum_{f\in\Delta^*} {\rm Tr}\,(B_f\, G_f)\,, \tag{5.214}$$

which leads to the following discretized analog of Eq.(5.210)

$$Z_{\rm BF}(\Delta) = \int \prod_{e\in\Delta^*} [dG_f] \prod_{f\in\Delta^*} [dB_f]\, e^{i\,{\rm Tr}\left(B_f\, G_f\right)} = \int \prod_{e\in\Delta^*} [dG_f] \prod_{f\in\Delta^*} \delta_G\,(G_f)\,. \tag{5.215}$$

In this expression we have replaced the integration over the connection A by an Haar measure over the group G on each edge e. In the second step we have integrated out the B-field again and replace the exponential with the delta-distribution of the Lie group G. Using now the Peter-Weyl theorem we can write the delta-distribution in terms of the irreducible representations ρ of G, which gives

$$Z_{\rm BF}(\Delta) = \int \prod_{e\in\Delta^*} [dG_f] \prod_{f\in\Delta^*} \sum_{\rho} d_\rho\, {\rm Tr}_\rho(G_f)\,. \tag{5.216}$$

Explicitly, for $G_f \in \mathrm{SL}(2,\mathbb{C})$ we have

$$\delta_G\left(G_f\right) = \sum_{n=0}^{\infty} \int_0^{\infty} dp\,(n^2+p^2)\,\mathrm{Tr}\left(\mathcal{D}^{((n,p))}(G_f)\right), \tag{5.217}$$

in terms of the unitary irreducible representations $(n,p) \in \mathbb{N}_0/2 \times \mathbb{R}$ of $\mathrm{SL}(2,\mathbb{C})$, which defines our spinfoam model for BF-theory.

A spinfoam model for quantum gravity, which describes not a topological theory but gravity, has to incorporate a quantum version of the simplicity constraints into this state sum Eq.(5.216). For example, the EPRL partition function is given by

$$Z_{\mathrm{EPRL}}(\Delta) = \int \prod_{e\in\Delta^*} [dG_f] \prod_{f\in\Delta^*} \delta_{\mathrm{EPRL}}\left(G_f\right), \tag{5.218}$$

where the solutions to the simplicity constraints, $p = \gamma n$ and $n = j$, lead to

$$\delta_{\mathrm{EPRL}}\left(G_f\right) = \sum_{j_f=0}^{\infty} (1+\gamma^2)\, j_f^2\ \mathrm{Tr}\left(\mathcal{D}^{((j_f,\gamma j_f))}(G_f)\right). \tag{5.219}$$

Now, instead of using the face holonomy G_f it is possible to rewrite the partition function Eq.(5.218) in terms of a so-called vertex amplitude A_v, which is associated with the (internal) vertices v. This amplitude is obtained by a reparametrization of the holonomies g_{e_i}, which can be split into half-edge holonomies $g_{e_i} = g_{v_{i+1},e_i}\, g_{e_i,v_i}$, and is then a function of the new holonomies $h_f^{v_i} \equiv g_{e_i,v_i}\, g_{v_i,e_{i-1}}$, [14, 13], and we can write

$$Z_{\mathrm{EPRL}} = \int dh_f^v \prod_f \delta(h_f) \prod_v A_v(h_f^v) \qquad \text{with} \qquad h_f \equiv \prod_{v\subset\partial f} h_f^v \tag{5.220}$$

and the vertex amplitude

$$A_v(h_f^v) = \int dg_{ve} \prod_{f(v)} \delta_{\mathrm{EPRL}}(h_f^v g_{e'v} g_{ve}). \tag{5.221}$$

5.2.2 New spinfoam model with timelike contributions

It should be clear that the correct treatment of the simplicity constraints is very important in order to obtain the correct model for quantum gravity. The idea behind the Conrady-Hnybida model [57, 58] and the model presented here is that the solutions of the linear

EPRL simplicity constraints miss the sector of the B - field configurations that correspond to timelike 2-surfaces and hence, do not represent the full dynamics of general relativity. When we impose the simplicity constraints in a path integral, where we denoted them by $\delta(C(B))$, and consider only the linear simplicity constraints $N_I B^{IJ} = 0$, for some auxiliary timelike vector N^I, then it seems that we only integrate over B - field configurations that correspond to spacelike bivectors B. As an analogy one can think of this problem in terms of the following relation

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_0^i)}{|g'(x_0^i)|}\,, \tag{5.222}$$

for some real function $g : \mathbb{R} \to \mathbb{R}$, where the sum is over all the zeros of the function, i.e., all x_0 such that $g(x_0) = 0$. Similarly, in order to integrate over all the curvature components in the path integral, not just the spacelike ones, we should not only integrate over spacelike B - fields, but also over timelike B - fields, which leads to the idea that we should write for the simplicity constraints $\delta(C(B))$

$$\delta(C(B)) = \delta(N_t \cdot B) + \delta(N_z \cdot B)\,, \tag{5.223}$$

where $N_t \cdot B = 0$ denotes the standard linear simplicity constraints with some timelike vector N_t and the new contribution $N_z \cdot B = 0$ uses as spacelike vector N_z, such that the solutions are timelike 2-surfaces. In fact, we even get a third term in Eq.(5.223), because for spacelike normal vector N_z the solution B - field can be spacelike and timelike (even null) and the spacelike solutions are not gauge equivalent to the spacelike solutions of $N_t \cdot B = 0$.

Now, without going into any detail about the Conrady-Hnybida spinfoam model and their derivation, we just mention that they obtain the following solutions for the quantum states for the timelike sector. Instead of $p = \gamma n$ they find $p = -n/\gamma$ and Eq.(5.207), i.e.,

$$s = \frac{1}{2}\sqrt{\frac{n^2}{\gamma^2} - 1} \qquad \text{with} \qquad j = -\frac{1}{2} + i\,s\,, \tag{5.224}$$

Based on their solutions for the simplicity constraints they propose the following generalization of the EPRL partition function, where one sums over both normals N_1^I and N_2^I and thus over spacelike and timelike 2-surfaces. They implement those constraints on the BF-theory spinfoam model via certain projectors, such that the BF-quantum states are mapped into the solution spaces of the simplicity constraints. The Conrady-Hnybida partition function is then defined as

$$Z_{\text{CH}} = \int dh_f^v \sum_{n_f \in \mathbb{N}_0} \sum_{N_e} \sum_{\zeta_{ef} = \pm 1} \prod_f (1 + \gamma^{2\zeta_{ef}})\, n_f^2 \prod_v A_v((p_f, n_f), \zeta_{ef}, N_e, h_f^v))\,, \tag{5.225}$$

where the label $\zeta_{ef} = \pm 1$ distinguishes spacelike and timelike 2-surfaces for spacelike normal N_e and the vertex amplitude is given via

$$A_v = \int dg_{ve} \prod_{f(v)} \langle \psi^{vf}_{j,\vec{N}} | \, \mathcal{D}^{(\rho,n)} \left(g_{ve} g^{-1}_{ve'} \right) \, | \psi^{vf}_{j,\vec{N}} \rangle \, . \tag{5.226}$$

In this expression the vertex amplitude is evaluated for coherent states $|\psi^{vf}_{j,\vec{N}}\rangle$, which are labeled by a normal vector $\vec{N}$ which is either $\vec{N} \in S^2$ for SU(2) states or $\vec{N} \in \mathbb{H}_\pm, \mathbb{H}_{sp}$ for states in $\mathcal{D}^\pm_j, \mathcal{C}^\epsilon_s$, respectively. Also, note that $h^v_f = g_{ve} g^{-1}_{ve'}$, which is just a change of variables.

Now, in the same spirit as above we start from $\mathrm{SL}(2,\mathbb{C})$ BF-theory as a spinfoam model, Eq.(5.216) and Eq.(5.217), and impose the simplicity constraints, which we found in section 5.1. Together with Eq.(5.223) this leads us to the same expression for our partition function, i.e.,

$$Z = \int dh^v_f \sum_{n_f \in \mathbb{N}_0} \sum_{N_e} \sum_{\zeta_{ef} = \pm 1} \prod_f (1 + \gamma^{2\zeta_{ef}}) \, n_f^2 \prod_v A_v((p_f, n_f), \zeta_{ef}, N_e, h^v_f)) \, , \tag{5.227}$$

because we derived the same results for the relation between p and n in section 5.1, i.e., $p = \gamma n$ for spacelike faces and $p = -n/\gamma$ for timelike faces[9]. The difference between our formulation and the Conrady-Hnybida approach becomes apparent only in the definition of the vertex amplitude. First, we found a different solution for the simplicity constraints for timelike faces, i.e., Eq.(5.208) with

$$s^\pm(n) = \frac{\sqrt{(2n \pm 1)^2 - 2}}{2} \, , \tag{5.228}$$

instead of Eq.(5.207) and secondly, we do not use coherent states. In fact, our physical states Eq.(5.182), which solve the simplicity and area matching constraints, are already in a factorized- or tensor product form, and we can easily define our new vertex amplitude $\mathcal{A}_v$ via

$$A_v \equiv \int dg_{ve} \prod_{f(v)} \left\langle f^{(n_s, p_s(n_s)), \varepsilon_s}_{s_1^+(n_s), m_s} \,\middle|\, g_{ve'} g^{-1}_{ve} \,\middle|\, f^{(n_s, p_s(n_s)), \varepsilon_t}_{s_1^-(n_s), m_t} \right\rangle \tag{5.229}$$

[9] In fact, our derivation of those relations is stronger, since in the twistorial formulation we do not need a large spin argument, which is necessary in the coherent state approach.

If we want to consider a spacetime manifold M with boundaries, for example in a Hamiltonian formulation, we can investigate in this formalism spacelike as well as timelike hypersurfaces. In order to embed the corresponding boundary states $|j,m\rangle$, $|k,m\rangle$ or $|s,m\rangle$, we will use the standard Dupuis-Livine map for SU(2) or the generalized versions Eq.(5.204) and Eq.(5.205) for the timelike case with $\mathrm{SU}(1,1)$.

In general, we should investigate this model in more detail to evaluate whether we gain any actual physical benefits over the standard EPRL model. The fact, that we can consider now those timelike contributions is certainly a satisfactory achievement in itself. For example, we can interpret the discrete eigenvalues of the area operator for timelike 2-surfaces as a measurement of time. Hence, we can say that in our Lorentzian spinfoam model we have shown that time intervals are quantized in this approach. Concerning the long debate in the spinfoam community, whether for Lorentzian models we have discrete or continuous spectra, we have shown that in our approach, due to the simplicity constraints, in 4D we do not encounter continuous spectra for our area operator. From a covariant standpoint, this is a much more consistent situation than having one type discrete and the other continuous.

One should, of course, attempt an the asymptotic analysis of this model. First steps into this direction for the Conrady-Hnybida model have been taken in [136] already. Recall, that one of our motivations for this project was the hope that in the twistorial parametrization the asymptotic analysis might become easier, compared with the coherent states used in [57, 58]. However, due to the continuous series representation of $\mathrm{SU}(1,1)$ showing up, our model might not provide a real simplification.

One could also try to further investigate questions about causality in these Lorentzian spinfoam models, [143]. Recall, that the occurrence of the cosine of the Regge action in the asymptotic analysis, instead of the exponential, is usually attributed to the claim that spinfoam models sum over both orientations of the spacetime manifold, which leads to the two branches. This problem could actually be studied in the 3D Lorentzian Ponzano-Regge model [113]. What happens to the (asymptotics of the) vertex amplitude if we restrict to only $\mathcal{D}_k^+$ or only $\mathcal{D}_k^-$ representations on the spatial slice?

Finally, one might wonder whether we can study our model numerically along the lines of the recent [144, 145]? However, again due to the occurrence of the continuous series states, it is not clear to us at the moment, whether this can be achieved.

Chapter 6

Conclusions

We have addressed in this thesis two important research questions within the Loop quantum gravity approach to the problem of quantum gravity. The results from [37, 38, 39, 40, 41, 42] for $\Lambda < 0$, together with our work presented in the chapters 3 and 4 for $\Lambda > 0$, provide, for the first time, a very detailed and rigorous bottom-up explanation of how a non-vanishing cosmological constant results in the q - deformation of standard Loop quantum gravity.

In chapter 3 we hope to have given a somewhat pedagogical overview of the mathematical structures behind the deformed (quasi-) phase spaces $\mathrm{SL}(2,\mathbb{C})$ and $D(\mathrm{SU}(2))$, such as (quasi-) Poisson manifolds and (quasi-) Poisson-Lie groups and their Lie algebras. Extracting these things from the mathematical literature [43, 44, 45, 46] and applying them to our particular example, including the novel calculations of section 3.4.2, was a considerable challenge and we hope, might make it more accessible to a wider physical audience.

The results of the calculations for our geometrical observables in section 4.4 confirm nicely the spherical nature of q - deformed spin networks at q root of unity. However, working explicitly with the projected tensor products and the coassociator in the braid representation makes these calculations very complex. The fact, that we (structurally) reproduce the same eigenvalues for our operators as in the q real case, makes us wonder, whether for future investigations, if we are just interested in certain eigenvalues for example, the explicit use of the weak- and quasi- structures are really necessary. What we have shown with our explicit calculations, though, is that doing calculations for q real or a general complex q, and then evaluating the result at q root of unity, seems to lead to the correct result. Working with such an "analytic continuation" has been the strategy in most of the considerations on q - deformed models at q root of unity in the literature. The drawback with this approach, however, is that one does not see the interesting, and technically necessary, quasi-structures hiding underneath. Now, if we lend some credibility to these ideas, which are indeed supported by our calculations, it is clear that the results from [42] also imply that we should correctly reproduce the Turaev-Viro model, now for q root of unity instead of q real, from the same Hamiltonian constraint and the same solution strategy. One important ingredient in the derivation of the Turaev-Viro vertex amplitude in [42], and similarly in the $\Lambda = 0$ case in [37], is the so-called Biedenharn-Elliott- or pentagon-identity. We know that this identity holds in the q root of unity case as well for the truncated 6j-symbol. Hence, we have no doubt, that the Turaev-Viro model follows from our quantum theory.

Building on our tensor operators there is a vast amount of possible follow-up work we can do, partially inspired by results from the q real case, as well as $\Lambda = 0$. For example, we can investigate a q - deformation of the so-called $U(N)$ formalism and could consider q - deformed twisted geometries for q root of unity. This is ongoing work, which has not been included in this thesis.

The results presented in chapter 5 about timelike 2-surfaces and 4D Lorentzian spinfoam models with timelike contributions answers a longstanding question within this field, namely, whether the area spectrum of either spacelike or timelike areas should be continuous in Lorentzian spacetime. Our answer is clearly that both areas have a discrete spectrum, which follows at once from the simplicity constraints. As in the standard EPRL- versus the FK- (coherent state) approach, we reproduce the solutions for the simplicity constraints found be Conrady and Hnybida in [57, 58] in a more rigorous way (recall, that their solutions and ours match for large spins), because thanks to our formulation we do not need to make a large spin argument, which is used in the coherent state approach.

Due to this fact, and our solution states Eq.(5.182) being structurally very different, compared with the coherent states of the approach in [57, 58], we defined a new 4D Lorentzian spinfoam model via the vertex amplitude in Eq.(5.229). The most relevant investigation concerning this model, in our opinion, is certainly its asymptotic analysis, which requires a more detailed understanding of the explicit structure of the solution states.

APPENDICES

Appendix A

Additional topics for chapter 3

A.1 On the classical double $\mathfrak{d}(\mathfrak{sl}(2,\mathbb{C}))$

In this appendix we want to mention another approach we were pursuing for a while, which is in fact related to our example below Eq.(3.37). This approach was particularly inspired by the problem of quantization. Because even if we understood the quasi-geometric structure of SO(4) or SU(2) × SU(2), it is a priori not clear whether we can simply quantize them in terms of the standard Hopf algebra $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$ or its quantum double, which was the quantum object we encountered in the quantization of $\mathrm{SL}(2,\mathbb{C})$.

We saw in the example Eq.(3.37) that there is a Lie bialgebra structure for the complex semi-simple Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, which has the same classical r - matrix as $\mathfrak{su}(2)$ for the double $\mathfrak{d}(\mathfrak{su}(2)) \cong \mathfrak{sl}(2,\mathbb{C})$, i.e., Eq.(3.42). This implies that the quantization of those two objects has the same quantum R - matrix, at least up to first order in an $\hbar$ - expansion, and thus, both could be related to the same quantum object $\mathcal{U}_{\mathrm{q}}(\mathfrak{sl}(2,\mathbb{C}))$. Furthermore, since both situations are described by the same r - matrix and since $\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2)^{\mathbb{C}}$, i.e., one can characterize $\mathfrak{sl}(2,\mathbb{C})$ as a complex Lie algebra with the $\mathfrak{su}(2)$ basis $\{\tau_i\}$, one finds that indeed the classical double of the Lie bialgebra $\mathfrak{sl}(2,\mathbb{C})$ is given by $\mathfrak{d}(\mathfrak{sl}(2,\mathbb{C})) \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})^*$, where the dual algebra $\mathfrak{sl}(2,\mathbb{C})^*$ is again given by $\mathfrak{an}(2,\mathbb{C})_{\pm}$, but now as a complex Lie algebra. Now, knowing that $\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ we can consider the following (Lie algebra) isomorphisms

$$\begin{aligned}\mathfrak{so}(4,\mathbb{C}) &\cong \mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C})^{\mathbb{C}} \\ &\cong \mathfrak{sl}(2,\mathbb{C})\oplus i\,\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{d}(\mathfrak{su}(2))\oplus i\,\mathfrak{d}(\mathfrak{su}(2)) \\ &\cong \mathfrak{su}(2)\oplus\mathfrak{an}(2,\mathbb{R})_{\pm}\oplus i\,\mathfrak{su}(2)\oplus i\,\mathfrak{an}(2,\mathbb{R})_{\pm} \\ &\cong \mathfrak{su}(2)\oplus i\,\mathfrak{su}(2)\oplus\mathfrak{an}(2,\mathbb{R})_{\pm}\oplus i\,\mathfrak{an}(2,\mathbb{R})_{\pm} \\ &\cong \mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{an}(2,\mathbb{C})_{\pm} \cong \mathfrak{d}(\mathfrak{sl}(2,\mathbb{C}))\,. \end{aligned} \tag{A.1}$$

Thus, we have shown that the classical double of $\mathfrak{sl}(2,\mathbb{C})$, with respect to the r - matrix Eq.(3.42), is isomorphic to the complexification of $\mathfrak{so}(4)$, i.e.,

$$\mathfrak{d}(\mathfrak{sl}(2,\mathbb{C})) \cong \mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})^{*} \cong \mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{an}(2,\mathbb{C})_{\pm} \cong \mathfrak{so}(4,\mathbb{C})\ . \tag{A.2}$$

Note, that the correct way to show this isomorphism is indeed given by Eq.(A.1). The following, however, does not work. One might try to use a non-existent self-duality like $\mathfrak{sl}(2,\mathbb{C})^{*}\cong\mathfrak{sl}(2,\mathbb{C})$ and attempt something like

$$\mathfrak{d}(\mathfrak{sl}(2,\mathbb{C})) \cong \mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})^{*} \stackrel{\cdot}{\cong} \mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(4,\mathbb{C})\ . \tag{A.3}$$

However, we denoted by $\stackrel{\cdot}{\cong}$ the step where things go wrong. The reason being that the dualities used for $\mathfrak{su}(2)^{*}$ and $\mathfrak{sl}(2,\mathbb{C})^{*}$ are not the same. The duality for $\mathfrak{su}(2)^{*}$ uses the imaginary part of the Killing form of $\mathfrak{sl}(2,\mathbb{C})$, whereas $\mathfrak{sl}(2,\mathbb{C})^{*}$ uses the (whole) Killing form of $\mathfrak{sl}(2,\mathbb{C})$. This is important, because, even though we have $(\mathfrak{g}^{*})^{*}\cong\mathfrak{g}$ for finite dimensional Lie algebras, the following is not correct

$$\begin{aligned}\mathfrak{sl}(2,\mathbb{C})^{*} &\cong (\mathfrak{su}(2)\oplus\mathfrak{an}(2,\mathbb{R})_{\pm})^{*} \cong (\mathfrak{su}(2)\oplus\mathfrak{su}(2)^{*})^{*} \cong \mathfrak{su}(2)^{*}\oplus(\mathfrak{su}(2)^{*})^{*} \\ &\cong \mathfrak{su}(2)^{*}\oplus\mathfrak{su}(2) \cong \mathfrak{an}(2,\mathbb{R})_{\pm}\oplus\mathfrak{su}(2) \cong \mathfrak{sl}(2,\mathbb{C})\ .\end{aligned} \tag{A.4}$$

To see why, we should distinguish the duality with respect to the imaginary part of the $\mathfrak{sl}(2,\mathbb{C})$ Killing form, which is used in $\mathfrak{sl}(2,\mathbb{C})\cong\mathfrak{su}(2)\oplus\mathfrak{an}(2,\mathbb{R})_{\pm}\cong\mathfrak{su}(2)\oplus\mathfrak{su}(2)^{*}$ and the duality using the whole $\mathfrak{sl}(2,\mathbb{C})$ Killing form. Denoting the latter by $\star$, i.e., $\mathfrak{d}(\mathfrak{sl}(2,\mathbb{C}))\cong\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})^{\star}\cong\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{an}(2,\mathbb{C})_{\pm}$, we see where Eq.(A.4) goes wrong. If we write

$$\mathfrak{sl}(2,\mathbb{C})^{\star} \cong (\mathfrak{su}(2)\oplus\mathfrak{an}(2,\mathbb{R})_{\pm})^{\star} \cong (\mathfrak{su}(2)\oplus\mathfrak{su}(2)^{*})^{\star} \cong \mathfrak{su}(2)^{\star}\oplus(\mathfrak{su}(2)^{*})^{\star}\,, \tag{A.5}$$

we see that $\mathfrak{su}(2)^\star \ncong \mathfrak{an}(2,\mathbb{R})_\pm$, we only have $\mathfrak{su}(2)^* \cong \mathfrak{an}(2,\mathbb{R})_\pm$ (note the different stars / dualities !), and $(\mathfrak{su}(2)^*)^\star \ncong \mathfrak{su}(2)$, we only have $(\mathfrak{su}(2)^*)^* \cong \mathfrak{su}(2) \cong (\mathfrak{su}(2)^\star)^\star$. This shows that the isomorphism between $\mathfrak{d}(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathfrak{so}(4,\mathbb{C})$ does not simply work via $\mathfrak{sl}(2,\mathbb{C})^\star \cong \mathfrak{sl}(2,\mathbb{C})$ and the correct way is Eq.(A.1).

What do we learn from the isomorphism $\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{d}(\mathfrak{sl}(2,\mathbb{C}))$? Does this mean now that we can forget all our comments that we need to work with quasi-Lie bialgebras and quasi-Poisson manifolds? Certainly not. Like the real form $\mathfrak{so}(4)$, the Lie algebra $\mathfrak{so}(4,\mathbb{C})$ is not semi-simple, which means that due to topological, or rather Lie algebra cohomological reasons, there are other deformations besides simple 1-cocycle deformations. The fact that $\mathfrak{so}(4,\mathbb{C})$ is not semi-simple leads to the possibility of interesting quasi-Lie bialgebra structures, which, most likely, would go unnoticed, if we restrict ourselves to just Lie bialgebra structures. Most importantly, this isomorphism works only over $\mathbb{C}$. Hence, even though it is indeed true, that this isomorphism tells us that we can quantize $\mathfrak{so}(4,\mathbb{C})$ in the complex setting as a quantum double of $\mathcal{U}_\mathrm{q}(\mathfrak{sl}(2,\mathbb{C}))$, there is no way around the quasi-realm, when we want to go back to the real case $\mathfrak{so}(4)$, which is necessary in a physical setting, where we want to deal with real general relativity and not complex general relativity. We could also say that there is no equivalent of Eq.(A.1) that works over $\mathbb{R}$, unless $\mathfrak{d}(\mathfrak{su}(2))$ describes a quasi-Lie bialgebra double. In fact, despite the quantum double of $\mathcal{U}_\mathrm{q}(\mathfrak{sl}(2,\mathbb{C}))$, for a general complex deformation parameter q, is the correct, all encompassing quantum symmetry algebra for all the cases - Lorentzian, Euclidean and both signs of Λ - found in 3d quantum gravity, working over $\mathbb{C}$ poses the danger that one misses the interesting new structures provided by quasi-Lie bialgebras. Finally, as we saw in chapter 4, there is no way around the truncated tensor products in the quantum theory, when working with q a root of unity. This issue would remain not fully understood, if we would not consider quasi-Poisson structures in the classical setting.

A.2 Twisting of $\mathfrak{sl}(2,\mathbb{C})$ over $\mathbb{C}$

Now, let us investigate the notion of twisting of the classical double of $\mathfrak{sl}(2,\mathbb{C})$. We rewrite our basis $\{X^+, X^-, H\}$ of $\mathfrak{sl}(2,\mathbb{C})$ as

$$b_1 \equiv X^+ \quad , \quad b_2 \equiv X^- \quad , \quad b_3 \equiv H \tag{A.6}$$

and thus we have

$$[b_1, b_2] = b_3 \quad , \quad [b_3, b_1] = 2\, b_1 \quad , \quad [b_3, b_2] = -2\, b_2 \,, \tag{A.7}$$

from which we can deduce the structure constants

$$f_{12}{}^{3} = 1 \qquad , \qquad f_{31}{}^{1} = 2 \qquad , \qquad f_{32}{}^{2} = -2 \tag{A.8}$$

and recall that we have antisymmetry in the lower two indices $f_{ab}{}^{c} = -f_{ba}{}^{c}$. We choose the following basis for the "2-forms"

$$\bigwedge^{(2)}(\mathfrak{sl}(2,\mathbb{C})) = \mathrm{gen}_{\mathbb{C}}(b_2 \wedge b_3, b_3 \wedge b_1, b_1 \wedge b_2)\,, \tag{A.9}$$

which means that we can write for $Y \in \bigwedge^{(2)}(\mathfrak{sl}(2,\mathbb{C}))$

$$Y = Y_1\, b_2 \wedge b_3 + Y_2\, b_3 \wedge b_1 + Y_3\, b_1 \wedge b_2 = \varepsilon^{ijk}\, Y_i\, b_j \otimes b_k = \frac{1}{2}\,\varepsilon^{ijk}\, Y_i\, b_j \wedge b_k\,, \tag{A.10}$$

with $Y_i \in \mathbb{C}$. Furthermore, we choose

$$\bigwedge^{(3)}(\mathfrak{sl}(2,\mathbb{C})) = \mathrm{gen}_{\mathbb{C}}(b_1 \wedge b_2 \wedge b_3)\,, \tag{A.11}$$

where $\varepsilon^{ijk}\, b_i \wedge b_j \wedge b_k = 6\, b_1 \wedge b_2 \wedge b_3 = 6\, \varepsilon^{ijk}\, b_i \otimes b_j \otimes b_k$. Now, we want to investigate twisting of the (quasi-) Lie bialgebra $(\mathfrak{sl}(2,\mathbb{C}), \delta, \varphi)$ by an arbitrary element $Y \in \bigwedge^{(2)}(\mathfrak{sl}(2,\mathbb{C}))$. Recall Eq.(3.111), which was given by

$$\delta^{Y}(X) = \delta(X) + \mathrm{ad}_X^{(2)}(Y) \qquad , \qquad \varphi^{Y} = \varphi - \mathrm{Alt}(\delta \otimes \mathrm{id})(Y) + [\![Y, Y]\!]\,. \tag{A.12}$$

Let us consider first the standard quasi-Lie bialgebra structure for $\mathfrak{sl}(2,\mathbb{C})$, given by $\delta = 0$ and $\varphi = \lambda\, b_1 \wedge b_2 \wedge b_3$. Then Eq.(A.12) gives

$$\delta^{Y}(X) = \mathrm{ad}_X^{(2)}(Y) \qquad , \qquad \varphi^{Y} = \lambda\, b_1 \wedge b_2 \wedge b_3 + [\![Y, Y]\!]\,. \tag{A.13}$$

First, let us calculate

$$\begin{aligned} [\![Y, Y]\!] &= [Y_{12}, Y_{13}] + [Y_{12}, Y_{23}] + [Y_{13}, Y_{23}] \\ &= \varepsilon^{ijk}\, Y_i\, \varepsilon^{lmn}\, Y_l\, \Big(f_{jm}{}^{q}\, b_q \otimes b_k \otimes b_n + f_{km}{}^{q}\, b_j \otimes b_q \otimes b_n + f_{kn}{}^{q}\, b_j \otimes b_m \otimes b_q \Big) \\ &= \Big(4\, Y_1 Y_2 + (Y_3)^2 \Big)\, b_1 \wedge b_2 \wedge b_3\,. \end{aligned} \tag{A.14}$$

Thus, we see that the condition that the twisted coassociator should vanish, i.e., $\varphi^Y = 0$, is satisfied if

$$\lambda + 4\,Y_1 Y_2 + (Y_3)^2 = 0\,. \tag{A.15}$$

Next, we calculate the twisted cocycle. We find

$$\begin{aligned} \delta^Y(b_i) &= \mathrm{ad}^{(2)}_{b_i}(Y) = [\,b_i \otimes 1 + 1 \otimes b_i\,, Y] \\ &= \varepsilon^{jkl}\, Y_j\,(f_{ik}{}^m\, b_m \otimes b_l + f_{il}{}^m\, b_k \otimes b_m) \\ &= \varepsilon^{jkl}\, Y_j\, f_{ik}{}^m\, b_m \wedge b_l\,. \end{aligned} \tag{A.16}$$

This gives for the three basis vectors

$$\delta^Y(b_1) = 2\,Y_1\, b_1 \wedge b_2 \;+\; Y_3\, b_1 \wedge b_3\,, \tag{A.17}$$

$$\delta^Y(b_2) = -2\,Y_2\, b_1 \wedge b_2 \;+\; Y_3\, b_2 \wedge b_3\,, \tag{A.18}$$

$$\delta^Y(b_3) = -2\,Y_1\, b_2 \wedge b_3 \;-\; 2\,Y_2\, b_1 \wedge b_3\,. \tag{A.19}$$

Now, recall the “standard structure” from Eq.(3.38), i.e., $r = \kappa\, b_1 \wedge b_2$ for some $\kappa \in \mathbb{C}$, or, similarly, the r - matrix Eq.(3.41), (they have the same anti-symmetric part). With $r = \kappa\, b_1 \wedge b_2$ we get the following action of the cocycle on the basis vectors, cf. Eq.(3.43),

$$\delta^r(b_{1,2}) = \kappa\, b_{1,2} \wedge b_3 \quad , \quad \delta^r(b_3) = 0\,. \tag{A.20}$$

From the last relation we see that the twist Y must satisfy $Y_1 = 0 = Y_2$, if we want to connect Eq.(A.19) and $\delta^r(b_3) = 0$. Thus, we see that together with Eq.(A.15) we get the following solution for the twist Y, which transforms between the **standard quasi-Lie bialgebra structure** $(\mathfrak{sl}(2,\mathbb{C}), \delta = 0, \varphi = \lambda\, b_1 \wedge b_2 \wedge b_3)$ to the **standard Lie bialgebra structure** $(\mathfrak{sl}(2,\mathbb{C}), \delta^Y, \varphi^Y = 0)$, where $\delta^Y(X) = [X, r]$ and r - matrix $r = \pm\sqrt{-\lambda}\, b_1 \wedge b_2$,

$$Y_1 = 0 = Y_2 \quad , \quad Y_3 = \kappa = \pm\sqrt{-\lambda} \in \mathbb{C}\,. \tag{A.21}$$

Now, the important lesson here, especially when we compare with the real setting of (quasi-) Lie bialgebra structures of $\mathfrak{su}(2)$, is that we have a larger twist-freedom and, in particular, that for any λ in the standard quasi-Lie bialgebra structure $(\mathfrak{sl}(2,\mathbb{C}), \delta = 0, \varphi = \lambda\, b_1 \wedge b_2 \wedge b_3)$ we can find a twists such that we obtain a standard Lie bialgebra structure $(\mathfrak{sl}(2,\mathbb{C}), \delta^Y = \delta^r, \varphi^Y = 0)$, with $r = \pm\sqrt{-\lambda}\, b_1 \wedge b_2$. However, note that it seems that there are still disconnected twist equivalence classes. For example, from Eq.(A.15) and Eq.(A.17 - A.19) it is clear that there is no twist that would allow to transform between $(\mathfrak{sl}(2,\mathbb{C}),\delta=0,\varphi=\lambda\, b_1\wedge b_2\wedge b_3)$ and the flat, or trivial, structure $(\mathfrak{sl}(2,\mathbb{C}),\delta=0,\varphi=0)$.

Finally, we just would like to point out the following relations between the basis' in the $\mathfrak{sl}(2,\mathbb{C})$ and the $\mathfrak{su}(2)$ case. With

$$b_1 = i\,\tau_1+\tau_2 \qquad , \qquad b_2 = i\,\tau_1-\tau_2 \qquad , \qquad b_3=-2i\,\tau_3 \tag{A.22}$$

we get $b_1\wedge b_2\wedge b_3=(-4)\,\tau_1\wedge\tau_2\wedge\tau_3$ and

$$b_2\wedge b_3+b_3\wedge b_1=(4i)\,\tau_2\wedge\tau_3 \qquad , \qquad b_2\wedge b_3-b_3\wedge b_1=(-4)\,\tau_3\wedge\tau_1\,, \tag{A.23}$$

$$b_1\wedge b_2=(-2i)\,\tau_1\wedge\tau_2\,. \tag{A.24}$$

A.3 On quasi-Poisson structures related by equivariant maps

In this appendix we include some calculations that illustrate the concept of "F - relation". This was important to us to understand the definition of the quasi-Poisson structure presented in [44, 45]. It basically answered our question, why we use the standard left- and right- invariant vector fields in the definition of the quasi-Poisson structure, despite being given a non-standard group action Eq.(3.187), where by non-standard we mean not being the left- or right- translation.

The connection with the group action Eq.(3.187), or any other action on $D(\mathrm{SU}(2))$, is made via the notion of "F - relation", where F denotes an equivariant map with respect to the group action under consideration[1]. This notion states that, if we have two (or more) G - manifolds M_i, i.e., there are actions φ_i of G on M_i, and a G - equivariant map $F_{ij}:M_i\to M_j$, then the (fundamental) vector fields ξ_{M_i} and ξ_{M_j} are F_{ij} - related, which means that the map F_{ij} induces a G - equivariant map $F_{ij,*}:TM_i\to TM_j$ that relates ξ_{M_i} to ξ_{M_j}.

In the following we investigate this notion in more detail and show how the fundamental vector field associated with the group action Eq.(3.187) is indeed F - related with the fundamental vector fields of the left- and right- actions.

[1]We learned this concept from the lecture notes, `http://www.math.toronto.edu/mein/teaching/LectureNotes/action.pdf`, by one of the authors of [44, 45].

We can distinguish the following three (left) actions $\varphi_i : G \times M_i \to M_i$, where $M_1 = M_2 = M_3 = \mathrm{SO}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ in all three cases, and $G = \mathrm{SO}(4)$ as well,

$$\varphi_1(g,h) = \mu(g,h) = (g_1 h_1 g_2^{-1}, g_2 h_2 g_1^{-1}) \,, \tag{A.25}$$

$$\varphi_2(g,h) = \varphi_L(g,h) = hg^{-1} = (h_1 g_1^{-1}, h_2 g_2^{-1}) \,, \tag{A.26}$$

$$\varphi_3(g,h) = \varphi_R(g,h) = gh = (g_1 h_1, g_2 h_2) \,. \tag{A.27}$$

We found above the fundamental vector fields associated with those actions and learned that the left- and (minus of the) right- invariant vector fields ξ^L and $-\xi^R$ are associated with the fundamental vector fields of φ_2 and φ_3, respectively.

Now, let us first consider whether we can find G - equivariant maps F_{21} and F_{31} that would allow us to push-forward the left- and right- invariant vector fields from M_2 and M_3 to the corresponding F - related vector fields on M_1 associated with the μ - action. Those maps F_{21} and F_{31} are maps from SO(4) to SO(4) and we write them in components as

$$F_{i1} : \mathrm{SO}(4) = M_i \to \mathrm{SO}(4) = M_1 \quad , \quad F_{i1}(h_1,h_2) = (F^1_{i1}(h_1,h_2), F^2_{i1}(h_1,h_2)) \tag{A.28}$$

with $i \in \{2,3\}$. The equivariance condition means that those maps must satisfy

$$F_{i1}(\varphi_i(g,h)) = \varphi_1(g, F(h)) \quad , \quad \forall g \in G \quad , \quad h \in M_i \,. \tag{A.29}$$

Explicitly, for $i = 2$, with $g = (g_1, g_2)$ and $h = (h_1, h_2)$ this condition looks like

$$(F^1_{21}(h_1 g_1^{-1}, h_2 g_2^{-1}), F^2_{21}(h_1 g_1^{-1}, h_2 g_2^{-1})) \stackrel{!}{=} (g_1 F^1_{21}(h_1,h_2) g_2^{-1}, g_2 F^2_{21}(h_1,h_2) g_1^{-1}) \tag{A.30}$$

or

$$F^1_{21}(h_1 g_1^{-1}, h_2 g_2^{-1}) \stackrel{!}{=} g_1 F^1_{21}(h_1,h_2) g_2^{-1} \quad , \quad F^2_{21}(h_1 g_1^{-1}, h_2 g_2^{-1}) \stackrel{!}{=} g_2 F^2_{21}(h_1,h_2) g_1^{-1} \,. \tag{A.31}$$

One finds that this condition is satisfied with

$$F^1_{21}(h_1,h_2) = h_1^{-1} h_2 \qquad \text{and} \qquad F^2_{21}(h_1,h_2) = h_2^{-1} h_1 \,. \tag{A.32}$$

For $i = 3$, with $g = (g_1, g_2)$ and $h = (h_1, h_2)$ we get

$$(F^1_{31}(g_1 h_1, g_2 h_2), F^2_{31}(g_1 h_1, g_2 h_2)) \stackrel{!}{=} (g_1 F^1_{31}(h_1,h_2) g_2^{-1}, g_2 F^2_{31}(h_1,h_2) g_1^{-1}) \tag{A.33}$$

or

$$F_{31}^1(g_1h_1, g_2h_2) \stackrel{!}{=} g_1F_{31}^1(h_1, h_2)g_2^{-1} \quad , \quad F_{31}^2(g_1h_1, g_2h_2) \stackrel{!}{=} g_2F_{31}^2(h_1, h_2)g_1^{-1} \tag{A.34}$$

and this condition is satisfied with

$$F_{31}^1(h_1, h_2) = h_1h_2^{-1} \qquad \text{and} \qquad F_{31}^2(h_1, h_2) = h_2h_1^{-1} \,. \tag{A.35}$$

Now, how to those equivariant maps relate the vector fields associated with the different actions φ_i? We have to look at the differential (or push-forward) $(dF_{i1})_h$ at $h = (h_1, h_2)$, which is a map from T_hM_i to $T_{F(h)}M_1$. On M_2 or M_3 we had the left- and right- invariant vector fields at h defined via

$$\xi^L(h) = \frac{d}{dt}\bigg|_{t=0} \varphi_L(\exp(-t\xi), h) = h.\xi = (h_1\xi_1, h_2\xi_2)\,, \tag{A.36}$$

$$-\xi^R(h) = \frac{d}{dt}\bigg|_{t=0} \varphi_R(\exp(-t\xi), h) = -\xi.h = (-\xi_1h_1, -\xi_2h_2)\,. \tag{A.37}$$

This means that the push forward of $\xi^{L,R}$ is given via

$$(dF_{i1})_h(\xi^{L,R}) = \frac{d}{dt}\bigg|_{t=0} (F_{i1} \circ \gamma_h)(t)\,, \tag{A.38}$$

where $\gamma_h(t) = \varphi_{L,R}(\exp(-t\xi), h)$. We can calculate for example

$$
\begin{aligned}
(dF_{21})_h(\xi^L) &= \left.\frac{d}{dt}\right|_{t=0} (F_{21} \circ \varphi_L(\exp(-t\xi), h)) \qquad \text{(A.39)}\\
&= \left.\frac{d}{dt}\right|_{t=0} F_{21}(h.\exp(t\xi))\\
&= \left.\frac{d}{dt}\right|_{t=0} F_{21}(h_1\exp(t\xi_1), h_2\exp(t\xi_2))\\
&= \left.\frac{d}{dt}\right|_{t=0} (F^1_{21}(h_1\exp(t\xi_1), h_2\exp(t\xi_2)), F^2_{21}(h_1\exp(t\xi_1), h_2\exp(t\xi_2)))\\
&= \left.\frac{d}{dt}\right|_{t=0} (\exp(-t\xi_1)h_1^{-1}h_2\exp(t\xi_2), \exp(-t\xi_2)h_2^{-1}h_1\exp(t\xi_1))\\
&= (h_1^{-1}h_2\xi_2 - \xi_1 h_1^{-1}h_2, h_2^{-1}h_1\xi_1 - \xi_2 h_2^{-1}h_1)\\
&= (F^1_{21}(h)\xi_2 - \xi_1 F^1_{21}(h), F^2_{21}(h)\xi_1 - \xi_2 F^2_{21}(h)) = \xi_{\mathrm{SU}(2)\times\mathrm{SU}(2)}(F_{21}(h))\,, \qquad \text{(A.40)}
\end{aligned}
$$

which, indeed, F_{21} - relates to Eq.(3.192). Similarly, one finds for $i = 3$

$$
\begin{aligned}
(dF_{31})_h(\xi^R) &= \left.\frac{d}{dt}\right|_{t=0} (F_{31} \circ \varphi_R(\exp(-t\xi), h)) \qquad \text{(A.41)}\\
&= \left.\frac{d}{dt}\right|_{t=0} F_{31}(\exp(-t\xi_1)h_1, \exp(-t\xi_2)h_2)\\
&= \left.\frac{d}{dt}\right|_{t=0} (F^1_{31}(\exp(-t\xi_1)h_1, \exp(-t\xi_2)h_2), F^2_{31}(\exp(-t\xi_1)h_1, \exp(-t\xi_2)h_2))\\
&= \left.\frac{d}{dt}\right|_{t=0} (\exp(-t\xi_1)h_1h_2^{-1}\exp(t\xi_2), \exp(-t\xi_2)h_2h_1^{-1}\exp(t\xi_1))\\
&= (h_1h_2^{-1}\xi_2 - \xi_1 h_1h_2^{-1}, h_2h_1^{-1}\xi_1 - \xi_2 h_2h_1^{-1})\\
&= (F^1_{31}(h)\xi_2 - \xi_1 F^1_{31}(h), F^2_{31}(h)\xi_1 - \xi_2 F^2_{31}(h)) = \xi_{\mathrm{SU}(2)\times\mathrm{SU}(2)}(F_{31}(h))\,. \qquad \text{(A.42)}
\end{aligned}
$$

Hence, we found that those vector fields are indeed F_{i1} - related, which leads to the answer of our question from above, namely, that the correct definition for the Poisson bivector Eq.(3.189) is really just using the simple left- and right- invariant vector fields and not the fundamental vector field of the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ - action Eq.(3.187).

A.4 More calculations with quasi-Poisson brackets

Based on the results from [47], we knew that the quasi-Poisson brackets for SU(2) in the flux coordinates can be written as Eq.(3.181). Taking this as a starting point, we wanted to calculate the corresponding quasi-Poisson brackets of the group elements $f_{ij}(h)$ and indeed, we obtained the correct result, i.e., Eq.(3.164), via a different calculation. Let us start by considering the following SU(2) element

$$H = e^{a^i \tau_i} = \cos\left(\frac{a}{2}\right) \mathrm{id}_2 - i \sin\left(\frac{a}{2}\right) (\hat{n}^i \sigma_i) \in \mathrm{SU}(2)\,, \tag{A.43}$$

with $\tau_i = \frac{1}{2i}\sigma_i$, Pauli matrices σ_i, $a = \sqrt{a_i a^i}$ and $\hat{n} = \frac{\vec{a}}{a}$. Then, using Eq.(3.181), we want to calculate

$$\{H_{ij}, H_{kl}\} = \frac{a}{2} \tan\left(\frac{a}{2}\right)^{-1} a^r \, \varepsilon_r{}^{mn} \, (\partial_m H_{ij}) \, (\partial_n H_{kl}) \ , \tag{A.44}$$

where H_{ij} corresponds to the previous f_{ij} and $\partial_m = \frac{\partial}{\partial a^m}$. Note that we can write

$$\begin{aligned} (\hat{n}^i \sigma_i) &= \begin{pmatrix} (\hat{n}^i\sigma_i)_{11} & (\hat{n}^i\sigma_i)_{12} \\ (\hat{n}^i\sigma_i)_{21} & (\hat{n}^i\sigma_i)_{22} \end{pmatrix} = \begin{pmatrix} \hat{n}^3 & \hat{n}^1 - i\,\hat{n}^2 \\ \hat{n}^1 + i\,\hat{n}^2 & -\hat{n}^3 \end{pmatrix} \\ &= \frac{1}{a} \begin{pmatrix} a^3 & a^1 - i\,a^2 \\ a^1 + i\,a^2 & -a^3 \end{pmatrix} \end{aligned} \tag{A.45}$$

and thus we get

$$H_{11} = \cos\left(\frac{a}{2}\right) - i \sin\left(\frac{a}{2}\right) \hat{n}^3 \quad , \qquad H_{12} = -i \sin\left(\frac{a}{2}\right) (\hat{n}^1 - i\,\hat{n}^2)\,, \tag{A.46}$$

$$H_{21} = -i \sin\left(\frac{a}{2}\right) (\hat{n}^1 + i\,\hat{n}^2) \quad , \qquad H_{22} = \cos\left(\frac{a}{2}\right) + i \sin\left(\frac{a}{2}\right) \hat{n}^3\,. \tag{A.47}$$

Next, with

$$\partial_m \cos\left(\frac{a}{2}\right) = -\frac{1}{2}\sin\left(\frac{a}{2}\right)\,\hat{n}_m \quad , \quad \partial_m \sin\left(\frac{a}{2}\right) = \frac{1}{2}\cos\left(\frac{a}{2}\right)\,\hat{n}_m \tag{A.48}$$

and $\partial_m \hat{n}^k = \frac{1}{a}\left(\delta^k_m - \hat{n}^k\hat{n}_m\right)$ one finds

$$\begin{aligned}\partial_m H_{ij} &= \left(\partial_m \cos\left(\frac{a}{2}\right)\right)\delta_{ij} - i\,\left(\partial_m \sin\left(\frac{a}{2}\right)\right)(\hat{n}^k\sigma_k)_{ij} - i\,\sin\left(\frac{a}{2}\right)\left(\partial_m\hat{n}^k\right)(\sigma_k)_{ij} \\ &= -\frac{1}{2}\sin\left(\frac{a}{2}\right)\left[\hat{n}_m\delta_{ij} + \frac{2i}{a}(\sigma_m)_{ij}\right] - \frac{i}{2}\left(\cos\left(\frac{a}{2}\right) - \frac{2}{a}\sin\left(\frac{a}{2}\right)\right)\hat{n}_m\,(\hat{n}^k\sigma_k)_{ij}\,.\end{aligned} \tag{A.49}$$

Together with

$$\varepsilon_r{}^{mn}\hat{n}_m\hat{n}_n = 0 = \varepsilon_r{}^{mn}\left[\,\hat{n}_m(\sigma_n)_{kl}\delta_{ij} + \hat{n}_n(\sigma_m)_{ij}\delta_{kl}\,\right] \tag{A.50}$$

and

$$a^r\varepsilon_r{}^{mn}\left[\,\hat{n}_m(\sigma_n)_{kl}(\hat{n}\vec{\sigma})_{ij} + \hat{n}_n(\sigma_m)_{ij}(\hat{n}\vec{\sigma})_{kl}\,\right] = 0\,, \tag{A.51}$$

we finally find that

$$\begin{aligned}\{H_{ij}, H_{kl}\} &= -\frac{1}{4}\sin(a)\,\hat{n}^r\,\varepsilon_r{}^{mn}\,(\sigma_m)_{ij}(\sigma_n)_{kl} \\ &= -\frac{1}{2}\sin\left(\frac{a}{2}\right)\cos\left(\frac{a}{2}\right)\,\hat{n}^r\,\varepsilon_r{}^{mn}\,(\sigma_m)_{ij}(\sigma_n)_{kl}\,,\end{aligned} \tag{A.52}$$

which leads to

$$\{H_{ij}, H_{ij}\} = 0 \quad , \quad \{H_{11}, H_{22}\} = 0 \tag{A.53}$$

and together with Eqs. A.46 - A.47 to

$$\begin{aligned}\{H_{11}, H_{12}\} &= -\frac{i}{2}\sin\left(\frac{a}{2}\right)\cos\left(\frac{a}{2}\right)(\hat{n}^1 - i\,\hat{n}^2) = \frac{1}{2}\cos\left(\frac{a}{2}\right)\,H_{12}\,, \\ &= -\{H_{12}, H_{11}\} = \{H_{12}, H_{22}\} = -\{H_{22}, H_{12}\}\,,\end{aligned} \tag{A.54}$$

$$\begin{aligned}\{H_{11}, H_{21}\} &= \frac{i}{2}\sin\left(\frac{a}{2}\right)\cos\left(\frac{a}{2}\right)(\hat{n}^1 + i\,\hat{n}^2) = -\frac{1}{2}\cos\left(\frac{a}{2}\right)\,H_{21}\,, \\ &= -\{H_{21}, H_{11}\} = \{H_{21}, H_{22}\} = -\{H_{22}, H_{21}\}\end{aligned} \tag{A.55}$$

and

$$\{H_{12}, H_{21}\} = -i\, \sin\left(\frac{a}{2}\right) \cos\left(\frac{a}{2}\right)\, \hat{n}^3 = -\{H_{21}, H_{12}\} \tag{A.56}$$

$$= \frac{1}{2} \cos\left(\frac{a}{2}\right) (H_{11} - H_{22})\,,$$

where we used

$$H_{11} - H_{22} = -2i\, \sin\left(\frac{a}{2}\right)\, \hat{n}^3 \tag{A.57}$$

in the last equality. Those brackets can then be summarized with $H_1 \equiv H \otimes \mathrm{id}_2$ and $H_2 \equiv \mathrm{id}_2 \otimes H$ as

$$\{H_1 \overset{\otimes}{,} H_2\} = \frac{1}{2} \cos\left(\frac{a}{2}\right) \begin{pmatrix} 0 & H_{12} & -H_{12} & 0 \\ -H_{21} & 0 & H_{11} - H_{22} & H_{12} \\ H_{21} & H_{22} - H_{11} & 0 & -H_{12} \\ 0 & -H_{21} & H_{21} & 0 \end{pmatrix}. \tag{A.58}$$

Note, we use the following index structure

$$\{H_1 \overset{\otimes}{,} H_2\} = \begin{pmatrix} \{H_{11}, H_{11}\} & \{H_{11}, H_{12}\} & \{H_{12}, H_{11}\} & \{H_{12}, H_{12}\} \\ \{H_{11}, H_{21}\} & \{H_{11}, H_{22}\} & \{H_{12}, H_{21}\} & \{H_{12}, H_{22}\} \\ \{H_{21}, H_{11}\} & \{H_{21}, H_{12}\} & \{H_{22}, H_{11}\} & \{H_{22}, H_{12}\} \\ \{H_{21}, H_{21}\} & \{H_{21}, H_{22}\} & \{H_{22}, H_{21}\} & \{H_{22}, H_{22}\} \end{pmatrix}. \tag{A.59}$$

Using again Eq.(A.46) and Eq.(A.47), we can write

$$\cos\left(\frac{a}{2}\right) = \frac{1}{2} (H_{11} + H_{22})\,, \tag{A.60}$$

but most importantly, one finds that indeed, using Eq.(A.60), the brackets Eq.(A.58) are equal to the brackets in Eq.(3.164) for $\alpha = 1$.

Appendix B

Additional topics for chapter 4

B.1 Calculations for geometric operators

We present here a more detailed calculation of the length operators from section 4.4.1. We start again with

$$
\begin{aligned}
{}^{q}\vec{L}^{2}{}_{1} \triangleright |i_{1(23)}\rangle &= \sum_{k_1,k_2} {}_{q}C \begin{pmatrix} 1 & 1 & 0 \\ k_1 & k_2 & 0 \end{pmatrix} {}^{(1)}t^{1}{}_{k_1} \; {}^{(1)}t^{1}{}_{k_2} \triangleright |i_{1(23)}\rangle \\
&= \sum_{k_1,k_2} {}_{q}C \begin{pmatrix} 1 & 1 & 0 \\ k_1 & k_2 & 0 \end{pmatrix} {}^{(1)}t^{1}{}_{k_1} \; {}^{(1)}t^{1}{}_{k_2} \triangleright \\
&\qquad \left(\sum_{m_1,m_2,m_3} \frac{(-1)^{j_3+m_3} q^{-m_3}}{\sqrt{[2j_3+1]_q}} \, {}_{q}C \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} |j_1, m_1\rangle \left(|j_2, m_2\rangle |j_3, m_3\rangle\right) \right) \\
&= \sum_{k_1,k_2} {}_{q}C \begin{pmatrix} 1 & 1 & 0 \\ k_1 & k_2 & 0 \end{pmatrix} (t^{1}{}_{k_1} \otimes \mathbb{I} \otimes \mathbb{I}) \, (t^{1}{}_{k_2} \otimes \mathbb{I} \otimes \mathbb{I}) \triangleright \\
&\qquad \left(\sum_{m_1,m_2,m_3} \frac{(-1)^{j_3+m_3} q^{-m_3}}{\sqrt{[2j_3+1]_q}} \, {}_{q}C \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} |j_1, m_1\rangle \left(|j_2, m_2\rangle |j_3, m_3\rangle\right) \right) \\
&= \sum_{k_1,k_2} \sum_{m_1,m_2,m_3} {}_{q}C \begin{pmatrix} 1 & 1 & 0 \\ k_1 & k_2 & 0 \end{pmatrix} \frac{(-1)^{j_3+m_3} q^{-m_3}}{\sqrt{[2j_3+1]_q}} \, {}_{q}C \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} \\
&\qquad \times \quad t^{1}{}_{k_1} t^{1}{}_{k_2} \, |j_1, m_1\rangle \left(|j_2, m_2\rangle |j_3, m_3\rangle\right) \qquad \text{(B.1)}
\end{aligned}
$$

and from the Wigner-Eckart theorem we get

$$t^1{}_{k_1} t^1{}_{k_2} |j_1, m_1\rangle = \sum_{n_1,n_2=-j_1}^{j_1} N(j_1,j_1,1)^2 {}_qC \begin{pmatrix} 1 & j_1 & j_1 \\ k_1 & n_2 & n_1 \end{pmatrix} {}_qC \begin{pmatrix} 1 & j_1 & j_1 \\ k_2 & m_1 & n_2 \end{pmatrix} |j_1, n_1\rangle$$

$$= N(j_1,j_1,1)^2 \sum_{n_1,n_2=-j_1}^{j_1} (-1)^{-k_1} q^{-k_1} {}_qC \begin{pmatrix} 1 & j_1 & j_1 \\ k_1 & n_2 & n_1 \end{pmatrix} {}_qC \begin{pmatrix} 1 & j_1 & j_1 \\ k_1 & n_2 & m_1 \end{pmatrix} |j_1, n_1\rangle \,, \quad \text{(B.2)}$$

where we have used that the sum over k_1 and k_2 gives a $\delta_{k_1,-k_2}$ and we also used the following symmetries of the Clebsch-Gordan coefficients

$$ {}_qC \begin{pmatrix} 1 & j_1 & j_1 \\ k_2 & m_1 & n_2 \end{pmatrix} = {}_qC \begin{pmatrix} 1 & j_1 & j_1 \\ -k_1 & m_1 & n_2 \end{pmatrix} = {}_qC \begin{pmatrix} j_1 & 1 & j_1 \\ -m_1 & k_1 & -n_2 \end{pmatrix}$$

$$= (-1)^{-k_1} q^{-k_1} {}_qC \begin{pmatrix} j_1 & 1 & j_1 \\ -n_2 & -k_1 & -m_1 \end{pmatrix} = (-1)^{-k_1} q^{-k_1} {}_qC \begin{pmatrix} 1 & j_1 & j_1 \\ k_1 & n_2 & m_1 \end{pmatrix} . \quad \text{(B.3)}$$

Thus, we get with $k_1 \in \{-1, 0, 1\}$ and $(-1)^{-2k_1} = 1$, and the orthogonality relation for the Clebsch-Gordan coefficients

$${}^q\vec{L}^2{}_1 \triangleright |i_{1(23)}\rangle = \frac{N(j_1,j_1,1)^2}{\sqrt{[3]_q}} \sum_{k_1} \sum_{m_i} \sum_{n_1,n_2} (-1)^{1-2k_1} \frac{(-1)^{j_3+m_3} q^{-m_3}}{\sqrt{[2j_3+1]_q}} {}_qC \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}$$

$$\times \quad {}_qC \begin{pmatrix} 1 & j_1 & j_1 \\ k_1 & n_2 & n_1 \end{pmatrix} {}_qC \begin{pmatrix} 1 & j_1 & j_1 \\ k_1 & n_2 & m_1 \end{pmatrix} |j_1, n_1\rangle \left(|j_2, m_2\rangle |j_3, m_3\rangle\right)$$

$$= -\frac{N(j_1,j_1,1)^2}{\sqrt{[3]_q}} \sum_{m_i} \frac{(-1)^{j_3+m_3} q^{-m_3}}{\sqrt{[2j_3+1]_q}} {}_qC \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} |j_1, m_1\rangle \left(|j_2, m_2\rangle |j_3, m_3\rangle\right)$$

$$= -\frac{N(j_1,j_1,1)^2}{\sqrt{[3]_q}} |i_{1(23)}\rangle \,, \quad \text{(B.4)}$$

which is the results shown in section 4.4.1.

Now, consider ${}^q\vec{L}^2{}_1 \triangleright |i_{1^*(23)}\rangle$, which gives

$$
{}^q\vec{L}^2{}_1 \triangleright |i_{1^*(23)}\rangle = \sum_{k_1,k_2} {}_qC\begin{pmatrix} 1 & 1 & 0 \\ k_1 & k_2 & 0 \end{pmatrix} {}^{(1)}t^1{}_{k_1}\ {}^{(1)}t^1{}_{k_2} \triangleright |i_{1^*(23)}\rangle
$$

$$
= \sum_{k_1,k_2} {}_qC\begin{pmatrix} 1 & 1 & 0 \\ k_1 & k_2 & 0 \end{pmatrix} {}^{(1)}t^1{}_{k_1}\ {}^{(1)}t^1{}_{k_2} \triangleright
$$

$$
\left(\sum_{m_1,m_2,m_3} \frac{(-1)^{j_1+2m_1} q^{-2m_1}}{\sqrt{[2j_1+1]_q}}\ {}_qC\begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} \langle j_1, m_1| \left(|j_2,m_2\rangle |j_3,m_3\rangle\right) \right)
$$

$$
= \sum_{k_1,k_2} {}_qC\begin{pmatrix} 1 & 1 & 0 \\ k_1 & k_2 & 0 \end{pmatrix} (t^1{}_{k_1} \otimes \mathbb{I} \otimes \mathbb{I})\ (t^1{}_{k_2} \otimes \mathbb{I} \otimes \mathbb{I}) \triangleright
$$

$$
\left(\sum_{m_1,m_2,m_3} \frac{(-1)^{j_1+3m_1} q^{-3m_1}}{\sqrt{[2j_1+1]_q}}\ {}_qC\begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} |j_1, -m_1\rangle \left(|j_2,m_2\rangle |j_3,m_3\rangle\right) \right)
$$

$$
= \sum_{k_1,k_2} \sum_{m_1,m_2,m_3} {}_qC\begin{pmatrix} 1 & 1 & 0 \\ k_1 & k_2 & 0 \end{pmatrix} \frac{(-1)^{j_1+3m_1} q^{-3m_1}}{\sqrt{[2j_1+1]_q}}\ {}_qC\begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix}
$$

$$
\times \quad t^1{}_{k_1} t^1{}_{k_2}\, |j_1, -m_1\rangle \left(|j_2,m_2\rangle |j_3,m_3\rangle\right) . \tag{B.5}
$$

With the Wigner-Eckart theorem we get

$$
t^1{}_{k_1} t^1{}_{k_2}\, |j_1, -m_1\rangle = \sum_{n_i=-j_1}^{j_1} N(j_1,j_1,1)^2{}_qC\begin{pmatrix} 1 & j_1 & j_1 \\ k_1 & -n_2 & -n_1 \end{pmatrix} {}_qC\begin{pmatrix} 1 & j_1 & j_1 \\ k_2 & -m_1 & -n_2 \end{pmatrix} |j_1, -n_1\rangle
$$

$$
= N(j_1,j_1,1)^2 \sum_{n_i=-j_1}^{j_1} (-1)^{-k_1} q^{-k_1}{}_qC\begin{pmatrix} 1 & j_1 & j_1 \\ k_1 & -n_2 & -n_1 \end{pmatrix} {}_qC\begin{pmatrix} 1 & j_1 & j_1 \\ k_1 & -n_2 & -m_1 \end{pmatrix} |j_1, -n_1\rangle , \tag{B.6}
$$

where we have used again that the sum over k_1 and k_2 gives a $\delta_{k_1,-k_2}$. Thus, we get with $k_1 \in \{-1,0,1\}$ and $(-1)^{-2k_1} = 1$, and the orthogonality relation for the Clebsch-Gordan coefficients

$$
\begin{aligned}
{}^{q}\vec{L}^{2}{}_{1} \triangleright |i_{1^*(23)}\rangle = \frac{N(j_1,j_1,1)^2}{\sqrt{[3]_q}} \sum_{k_1}\sum_{m_i}\sum_{n_1,n_2} (-1)^{1-2k_1} \frac{(-1)^{j_1+3m_1} q^{-3m_1}}{\sqrt{[2j_1+1]_q}} \, {}_{q}C \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} \\
\times \quad {}_{q}C \begin{pmatrix} 1 & j_1 & j_1 \\ k_1 & -n_2 & -n_1 \end{pmatrix} {}_{q}C \begin{pmatrix} 1 & j_1 & j_1 \\ k_1 & -n_2 & -m_1 \end{pmatrix} \, |j_1,-n_1\rangle \left(|j_2,m_2\rangle |j_3,m_3\rangle\right) \\
= -\frac{N(j_1,j_1,1)^2}{\sqrt{[3]_q}} \sum_{m_i} \frac{(-1)^{j_1+3m_1} q^{-3m_1}}{\sqrt{[2j_1+1]_q}} \, {}_{q}C \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} \, |j_1,-m_1\rangle \left(|j_2,m_2\rangle |j_3,m_3\rangle\right) \\
= -\frac{N(j_1,j_1,1)^2}{\sqrt{[3]_q}} \sum_{m_i} \frac{(-1)^{j_1+2m_1} q^{-2m_1}}{\sqrt{[2j_1+1]_q}} \, {}_{q}C \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} \, \langle j_1,m_1| \left(|j_2,m_2\rangle |j_3,m_3\rangle\right) \\
= -\frac{N(j_1,j_1,1)^2}{\sqrt{[3]_q}} \, |i_{1^*(23)}\rangle \, . \qquad \text{(B.7)}
\end{aligned}
$$

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